Chapter 1

Logic

Definition 1.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

Definition 1.1.1. Definition of $\neg$.

p	$\neg p$
T	F
F	T

Definition 1.1.2. Definition of $\wedge$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.1.3. Definition of $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.1.4. Definition of $\iff$.

$$\begin{aligned} &p \iff q \\ \overset{\text{def}}{\iff} &((p \land q) \lor (\neg p \land \neg q)) \end{aligned}$$

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 1.1.5. Definition of $\implies$.

$$\begin{aligned} &p \implies q \\ \overset{\text{def}}{\iff} &(\neg p) \lor q \end{aligned}$$

1.2 Boolean algebra

Proposition 1.2.1. Associativity of $\lor$.

$$((P \lor Q) \lor R) \iff (P \lor (Q \lor R))$$

Proposition 1.2.2. Associativity of $\land$.

$$((P \land Q) \land R) \iff (P \land (Q \land R))$$

Proposition 1.2.3. Commutativity of $\lor$.

$$(P \lor Q) \iff (Q \lor P)$$

Proposition 1.2.4. Commutativity of $\land$.

$$(P \land Q) \iff (Q \land P)$$

Proposition 1.2.5. Identity of $\lor$.

$$(P \lor \text{False}) \iff P$$

Proposition 1.2.6. Identity of $\land$.

$$(P \land \text{True}) \iff P$$

Proposition 1.2.7. Annihilator of $\vee$.

$$(P \vee \text{True}) \iff \text{True}$$

Proposition 1.2.8. Annihilator of $\wedge$.

$$(P \wedge \text{False}) \iff \text{False}$$

Proposition 1.2.9. Idempotence of $\vee$.

$$(P \vee P) \iff P$$

Proposition 1.2.10. Idempotence of $\wedge$.

$$(P \wedge P) \iff P$$

Proposition 1.2.11. Complement of $\vee$.

$$(P \vee (\neg P)) \iff \text{True}$$

Proposition 1.2.12. Complement of $\wedge$.

$$(P \wedge (\neg P)) \iff \text{False}$$

Proposition 1.2.13. Absorption.

$$(P \vee (P \wedge Q)) \iff P$$

Proposition 1.2.14. Absorption.

$$(P \wedge (P \vee Q)) \iff P$$

Proposition 1.2.15. Distributivity.

$$(P \vee (Q \wedge R)) \iff ((P \vee Q) \wedge (P \vee R))$$

Proposition 1.2.16. Distributivity.

$$(P \wedge (Q \vee R)) \iff ((P \wedge Q) \vee (P \wedge R))$$

1.3 Quantifiers

Definition 1.3.1. Universal quantifier is denoted by $\forall$.

$$\begin{aligned}&\forall x, P(x)\\ \stackrel{\text{def}}{\iff}\ &(P(x_1) \land P(x_2) \land \dots)\end{aligned}$$

Definition 1.3.2. Existential quantifier is denoted by $\exists$.

$$\begin{aligned}&\exists x, P(x)\\ \stackrel{\text{def}}{\iff}\ &(P(x_1) \lor P(x_2) \lor \dots)\end{aligned}$$

Proposition 1.3.3.

$$(\forall x(P(x) \land Q(x))) \iff (\forall x, P(x)) \land (\forall x, Q(x))$$

Proposition 1.3.4.

$$(\exists x, P(x)) \lor (\exists x, Q(x)) \iff (\exists x, (P(x) \lor Q(x)))$$

Proposition 1.3.5.

$$(P \lor (\forall x, Q(x))) \iff (\forall x(P \lor Q(x)))$$

Proposition 1.3.6.

$$(P \land (\exists x, Q(x))) \iff (\exists x(P \land Q(x)))$$

Axiom 1.1. P does not depend on x.

$$(\forall x, P(y)) \iff P(y)$$

Axiom 1.2. P does not depend on x.

$$(\exists x, P(y)) \iff P(y)$$

Axiom 1.3. De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

Axiom 1.4. De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

Definition 1.3.7. Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned}&!\exists x, P(x)\\ \stackrel{\text{def}}{\iff}\ &(\exists x, P(x)) \land (\forall x \forall y(P(x) \land P(y) \implies x = y))\end{aligned}$$

Axiom 1.5. Axiom of Substitution

$$\forall x((\exists y((y = x) \land P(y))) \iff P(x))$$

1.4 Proof technique

Proposition 1.4.1.

$$(P \iff Q) \iff ((P \implies Q) \wedge (Q \implies P))$$

Proposition 1.4.2.

$$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$$

Proposition 1.4.3.

$$(x \implies y) \implies ((P(x,z)) \implies P(y,z))$$

Proposition 1.4.4.

$$(x \iff y) \implies ((P(x,z)) \iff P(y,z))$$

Proposition 1.4.5. contrapositive

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

Proposition 1.4.6.

$$(P \iff \text{True}) \iff P$$

1.5 Proposition

Let $P = P(x_1, x_2, \ldots, x_n)$. Let $Q = Q(x_1, x_2, \ldots, x_n)$. etc

Proposition 1.5.1. Double negation

$$\neg(\neg P) \iff P$$

Proposition 1.5.2. Reflexive property of iff.

$$P \iff P$$

Proof:

P	$P \iff P$
T	T
F	T

Proposition 1.5.3. Symmetric property of iff.

$$(P \iff Q) \iff (Q \iff P)$$

Proposition 1.5.4. Transitive property of iff.

$$((P \iff Q) \wedge (Q \iff R)) \implies (P \iff R)$$

Proposition 1.5.5. De Morgan's law

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

Proposition 1.5.6. De Morgan's law

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

Proposition 1.5.7.

$$(P \wedge Q) \implies (P \iff Q)$$

Proposition 1.5.8.

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

Proposition 1.5.9.

$$(P \wedge Q) \implies P$$

Lemma 1.5.10.

$$(P \wedge ((Q \wedge P) \implies R)) \implies (Q \implies R)$$

Proposition 1.5.11.

$$(P \wedge (P \implies Q)) \implies Q$$

Proposition 1.5.12.

$$(P \wedge (P \iff Q)) \implies Q$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

Definition 2.0.1. Definition of $\notin$.

$$
\begin{aligned}
& A \notin B \\
\stackrel{\text{def}}{\iff} & \neg(A \in B)
\end{aligned}
$$

Definition 2.0.2.

$$
\begin{aligned}
& \forall x \in S, P(x) \\
\stackrel{\text{def}}{\iff} & \forall x(x \in S \implies P(x))
\end{aligned}
$$

Definition 2.0.3.

$$
\begin{aligned}
& \exists x \in S, P(x) \\
\stackrel{\text{def}}{\iff} & \exists x(x \in S \land P(x))
\end{aligned}
$$

Proposition 2.0.4.

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\forall x \in S, P(x)) & \\
\iff & \neg(\forall x(x \in S \implies P(x))) & \text{Definition 2.0.2} \\
\iff & \neg(\forall x(\neg(x \in S) \vee P(x))) & \text{Definition 1.1.5} \\
\iff & \exists x, \neg(\neg(x \in S) \vee P(x)) & \text{Axiom 1.3} \\
\iff & \exists x, \neg(\neg(x \in S)) \wedge \neg(P(x)) & \text{Proposition 1.5.6} \\
\iff & \exists x, x \in S \wedge \neg(P(x)) & \text{Proposition 1.5.1} \\
\iff & \exists x \in S, \neg(P(x)) & \text{Definition 2.0.3}
\end{aligned}
$$

Proposition 2.0.5.

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\exists x \in S, P(x)) & \\
\iff & \neg(\exists x(x \in S \wedge P(x))) & \text{Definition 2.0.3} \\
\iff & \forall x, \neg(x \in S \wedge P(x)) & \text{Axiom 1.4} \\
\iff & \forall x, (\neg(x \in S)) \vee (\neg(P(x)) & \text{Proposition 1.5.5} \\
\iff & \forall x, x \in S \implies \neg(P(x)) & \text{Definition 1.1.5} \\
\iff & \forall x \in S, \neg(P(x)) & \text{Definition 2.0.2}
\end{aligned}
$$

2.1 Equality of sets

Definition 2.1.1. Definition of $=$.

$$
\begin{aligned}
& A = B \\
\overset{\text{def}}{\iff} & \forall x(x \in A \iff x \in B)
\end{aligned}
$$

Definition 2.1.2. Definition of $\neq$.

$$
\begin{aligned}
& A \neq B \\
\overset{\text{def}}{\iff} & \neg(A = B)
\end{aligned}
$$

Proposition 2.1.3. Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$\forall x($

	$x = x$	
$\iff$	$\forall y(y \in x \iff y \in x)$	Definition 2.1.1
$\iff$	True	Proposition 1.5.2

$)$

Proposition 2.1.4. Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$\forall x \forall y($

	$x = y$	
$\implies$	$\forall z(z \in x \iff z \in y)$	Definition 2.1.1
$\implies$	$\forall z(z \in y \iff z \in x)$	Proposition 1.5.3
$\implies$	$y = x$	Definition 2.1.1

$)$

Proposition 2.1.5. Transitive property of equality

$$\forall x \forall y \forall z((x = y) \land (y = z) \implies (x = z))$$

Proof:

$\forall x \forall y \forall z($

	$(x = y) \land (y = z)$	
$\implies$	$(\forall w(w \in x \iff w \in y)) \land (\forall w(w \in y \iff w \in z))$	Definition 2.1.1
$\implies$	$\forall w((w \in x \iff w \in y) \land (w \in y \iff w \in z))$	Proposition 1.3.3
$\implies$	$\forall w(w \in x \iff w \in z)$	Proposition 1.5.4
$\implies$	$x = z$	Definition 2.1.1

$)$

Axiom 2.1. Axiom of extensionality

$$\forall x \forall y($$
$$x = y \implies \forall A(x \in A \iff y \in A)$$
$$)$$

Axiom 2.2. Existence of empty set

$$\exists x \forall y (y \notin x)$$

Proposition 2.1.6. Uniqueness of empty set.

$$!\exists x \forall y (y \notin x)$$

Proof:
Let $P(x) = \forall y (y \notin x)$

$$\begin{aligned} & \exists x \forall y (y \notin x) && \text{Axiom 2.2} \\ \implies & \exists x, P(x) && \text{Definition of P(x)} \end{aligned}$$

$\forall x \forall y ($

$$\begin{aligned} & P(x) \wedge P(y) && \\ \implies \quad & (\forall z (z \notin x)) \wedge (\forall z (z \notin y)) && \text{Definition of P(x)} \\ \implies \quad & \forall z ((z \notin x) \wedge (z \notin y)) && \text{Proposition 1.3.3} \\ \implies \quad & \forall z (z \notin x \iff z \notin y) && \text{Proposition 1.5.7} \\ \implies \quad & \forall z (\neg (z \in x) \iff \neg (z \in y)) && \text{Definition 2.0.1} \\ \implies \quad & \forall z (z \in x \iff z \in y) && \text{Proposition 1.5.8} \\ \implies \quad & x = y && \text{Definition 2.1.1} \end{aligned}$$

$)$

$$\begin{aligned} & (\exists x, P(x)) \wedge \forall x \forall y ((P(x) \wedge P(y)) \implies (x = y)) && \\ \implies & !\exists x, P(x) && \text{Definition 1.3.7} \\ \implies & !\exists x \forall y (y \notin x) && \text{Definition of P(x)} \end{aligned}$$

Definition 2.1.7. The unique empty set is denoted by $\emptyset$.

$$\forall x (x \notin \emptyset)$$

Proof:

Let $P(x) = \forall y(y \notin x)$

$$
\begin{array}{ll}
\quad !\exists x \forall y(y \notin x) & \text{Proposition 2.1.6} \\
\implies !\exists x, P(x) & \text{Definition of P(x)} \\
\implies (\exists x, P(x)) \land \forall x \forall y((P(x) \land P(y)) \implies (x = y)) & \text{Definition 1.3.7} \\
\implies P(\emptyset) \land \forall x((P(x) \land P(\emptyset)) \implies (x = \emptyset)) & \text{Definition 2.1.7} \\
\implies P(\emptyset) & \text{Proposition 1.5.9} \\
\implies \forall y(y \notin \emptyset) & \text{Definition of P(x)}
\end{array}
$$

Proposition 2.1.8. Uniqueness of $\emptyset$

$$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$$

Proof:
Let $P(x) = \forall y(y \notin x)$

$$
\begin{array}{ll}
\quad !\exists x \forall y(y \notin x) & \text{Proposition 2.1.6} \\
\implies !\exists x, P(x) & \text{Definition of P(x)} \\
\implies (\exists x, P(x)) \land \forall x \forall y((P(x) \land P(y)) \implies (x = y)) & \text{Definition 1.3.7} \\
\implies P(\emptyset) \land \forall x((P(x) \land P(\emptyset)) \implies (x = \emptyset)) & \text{Definition 2.1.7} \\
\implies (\forall x, P(\emptyset)) \land \forall x((P(x) \land P(\emptyset)) \implies (x = \emptyset)) & \text{Axiom 1.1} \\
\implies \forall x(P(\emptyset) \land ((P(x) \land P(\emptyset)) \implies (x = \emptyset))) & \text{Proposition 1.3.3} \\
\implies \forall x(P(x) \implies (x = \emptyset)) & \text{Lemma 1.5.10} \\
\implies \forall x(\forall y(y \notin x) \implies (x = \emptyset)) & \text{Definition of P(x)}
\end{array}
$$

Proposition 2.1.9. Single choice

$$\forall x((x \neq \emptyset) \implies (\exists y, y \in x))$$

Proof:

$$
\begin{array}{ll}
\quad \forall x(\forall y(y \notin x) \implies (x = \emptyset)) & \\
\implies \forall x(\neg(x = \emptyset) \implies \neg(\forall y(y \notin x))) & \text{Proposition 1.4.5} \\
\implies \forall x((x \neq \emptyset) \implies \neg(\forall y(y \notin x))) & \text{Definition 2.1.2} \\
\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(y \notin x))) & \text{Axiom 1.3} \\
\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(\neg(y \in x)))) & \text{Definition 2.0.1} \\
\implies \forall x((x \neq \emptyset) \implies (\exists y, y \in x)) & \text{Proposition 1.5.1}
\end{array}
$$

Axiom 2.3. Axiom of pairing. Existence of pair set.

$$\forall x \forall y \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Proposition 2.1.10. Uniqueness of pairing set.

$$\forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Proof:
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$

$\forall x \forall y \forall A \forall B ($

$$\begin{aligned}
& P(A,x,y) \wedge P(B,x,y) & \\
\implies & (\forall z (z \in A \iff ((z = x) \vee (z = y)))) & \\
& \wedge (\forall z (z \in B \iff ((z = x) \vee (z = y)))) & \text{Definition of P(A,x,y)} \\
\implies & \forall z ((z \in A \iff ((z = x) \vee (z = y))) & \\
& \wedge (z \in B \iff ((z = x) \vee (z = y)))) & \text{Proposition 1.3.3} \\
\implies & \forall z (z \in A \iff z \in B) & \text{Proposition 1.5.4} \\
\implies & A = B & \text{Definition 2.1.1}
\end{aligned}$$

$)$

$$\begin{aligned}
& \forall x \forall y ! \exists A, P(A,x,y) & \text{Similar to the proof of the Proposition 2.1.6} \\
\implies & \forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y))) & \text{Definition of P(A,x,y)}
\end{aligned}$$

Definition 2.1.11. The unique pair set of x and y is denoted by $\{x, y\}$.
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$
Similar to the proof of Definition 2.1.7,

$$\forall x \forall y P(\{x, y\}, x, y)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x \forall y \forall A (P(A, x, y) \implies (A = \{x, y\}))$$

Proposition 2.1.12. Existence of singleton set.

$$\forall x \exists A \forall y (y \in A \iff (y = x))$$

Proof:

$$\begin{aligned}
& \forall x \exists A \forall y (y \in A \iff ((y = x) \vee (y = x))) & \text{Axiom 2.3} \\
\implies & \forall x \exists A \forall y (y \in A \iff (y = x)) & \text{Proposition 1.2.9}
\end{aligned}$$

Proposition 2.1.13. Uniqueness of singleton set.

$$\forall x!\exists A\forall y(y \in A \iff (x = y))$$

Let $P(A, x) = \forall y(y \in A \iff (x = y))$
The proof is similar to the proof of Proposition 2.1.10.

Definition 2.1.14. The unique singleton set of x is denoted by $\{x\}$.
Let $P(A, x) = \forall y(y \in A \iff (x = y))$
Similar to the proof of Definition 2.1.7,

$$\forall x P(\{x\}, x)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x\forall A(P(A, x) \implies (A = \{x\}))$$

Axiom 2.4. Axiom of union. Existence of union set.

$$\forall F\exists A\forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$$

Proposition 2.1.15. Uniqueness of union set.

$$\forall F!\exists A\forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$$

Proof:
Let $P(A, F) = \forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$

$\forall F\forall A\forall B($

$$\begin{aligned}
& P(A, F) \wedge P(B, F) & \\
\implies & (\forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))) & \\
& \wedge (\forall x(x \in B \iff (\exists Y((x \in Y) \wedge (Y \in F))))) & \text{Definition of P(A,F)} \\
\implies & \forall x((x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F)))) & \\
& \wedge (x \in B \iff (\exists Y((x \in Y) \wedge (Y \in F))))) & \text{Proposition 1.3.3} \\
\implies & \forall x(x \in A \iff x \in B) & \text{Proposition 1.5.4} \\
\implies & A = B & \text{Definition 2.1.1}
\end{aligned}$$

$)$

$$\begin{aligned}
& \forall F!\exists A, P(A, F) & \text{Similar to the proof of the Proposition 2.1.6} \\
\implies & \forall F!\exists A\forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F)))) & \text{Definition of P(A,F)}
\end{aligned}$$

Definition 2.1.16. The unique union set of F is denoted by $\bigcup F$.
Let $P(A,F) = \forall x(x \in A \iff (\exists Y((x \in Y) \land (Y \in F))))$
Similar to the proof of Definition 2.1.7,

$$\forall F P(\bigcup F, F)$$

Similar to the proof of Proposition 2.1.8,

$$\forall F \forall A(P(A,F) \implies (A = \bigcup F))$$

Definition 2.1.17. Definition of pairwise union $A \cup B$.

$$\begin{aligned} & A \cup B \\ \stackrel{\text{def}}{=} & \bigcup\{A, B\} \end{aligned}$$

Proposition 2.1.18. Property of pairwise union.

$$\forall A \forall B \forall x(x \in (A \cup B) \iff ((x \in A) \lor (x \in B)))$$

Proof:

$\forall A \forall B \forall x($

$$\begin{aligned}
& x \in (A \cup B) & \\
\iff & x \in \bigcup\{A, B\} & \text{Definition 2.1.1 and 2.1.17} \\
\iff & \exists Y((x \in Y) \land (Y \in \{A, B\})) & \text{Definition 2.1.16} \\
\iff & \exists Y((x \in Y) \land ((Y = A) \lor (Y = B))) & \text{Definition 2.1.11} \\
\iff & \exists Y(((x \in Y) \land (Y = A)) \lor ((x \in Y) \land (Y = B))) & \text{Proposition 1.2.16} \\
\iff & (\exists Y((x \in Y) \land (Y = A))) \lor (\exists Y((x \in Y) \land (Y = B))) & \text{Proposition 1.3.4} \\
\iff & ((x \in A) \lor (x \in B)) & \\
& \text{Proposition 1.4.4 and Axiom 1.5 with } P(A, x) = (x \in A) &
\end{aligned}$$

$)$

Proposition 2.1.19. Commutativity of $\cup$.

$$\forall x \forall y((x \cup y) = (y \cup x))$$

Proof:

$\forall x \forall y($

$$\begin{array}{lll} & (x \cup y) = (y \cup x) & \\ \iff & \forall z(z \in (x \cup y) \iff z \in (y \cup x)) & \text{Definition 2.1.1} \\ \iff & \forall z(((z \in x) \vee (z \in y)) \iff ((z \in y) \vee (z \in x))) & \text{Proposition 2.1.18} \\ \iff & \forall z(((z \in x) \vee (z \in y)) \iff ((z \in x) \vee (z \in y))) & \text{Proposition 1.2.3} \\ \iff & \text{True} & \text{Proposition 1.5.2} \end{array}$$

$)$

Proposition 2.1.20. Identity of $\cup$.

$$\forall x((x \cup \emptyset) = x)$$

Proof:

$\forall x($

$$\begin{array}{lll} & (x \cup \emptyset) = x & \\ \iff & \forall y(y \in (x \cup \emptyset) \iff (y \in x)) & \text{Definition 2.1.1} \\ \iff & \forall y(((y \in x) \vee (y \in \emptyset)) \iff (y \in x)) & \text{Proposition 2.1.18} \\ \iff & \forall y(((y \in x) \vee (\neg(\neg(y \in \emptyset)))) \iff (y \in x)) & \text{Proposition 1.5.1} \\ \iff & \forall y(((y \in x) \vee (\neg(y \notin \emptyset))) \iff (y \in x)) & \text{Definition 2.0.1} \\ \iff & \forall y(((y \in x) \vee (\neg(\text{True}))) \iff (y \in x)) & \text{Definition 2.1.7} \\ \iff & \forall y(((y \in x) \vee (\text{False})) \iff (y \in x)) & \text{Definition 1.1.1} \\ \iff & \forall y((y \in x) \iff (y \in x)) & \text{Proposition 1.2.5} \\ \iff & \text{True} & \text{Proposition 1.5.2} \end{array}$$

$)$

Definition 2.1.21. Definition of 0.

$$0 \stackrel{\text{def}}{=} \emptyset$$

Definition 2.1.22. Definition of successor $S(x)$.

$$S(x)$$
$$\stackrel{\text{def}}{=} x \cup \{x\}$$

Definition 2.1.23. Definition of 1.

$$
\begin{aligned}
1 &\overset{\text{def}}{=} S(0) && \\
&= 0 \cup \{0\} && \text{Definition 2.1.22} \\
&= \emptyset \cup \{\emptyset\} && \text{Definition 2.1.21} \\
&= \{\emptyset\} \cup \emptyset && \text{Proposition 2.1.19} \\
&= \{\emptyset\} && \text{Proposition 2.1.20}
\end{aligned}
$$