

# Chapter 1

## Logic

**Definition 1.0.0.1.** Proposition is a statement that is either true or false, but not both.

### 1.1 Logical operations

#### 1.1.1 Definition of $\neg$

**Definition 1.1.1.1.**

$$\neg(\text{True}) \\ \stackrel{\text{def}}{\iff} \text{False}$$

**Definition 1.1.1.2.**

$$\neg(\text{False}) \\ \stackrel{\text{def}}{\iff} \text{True}$$

#### 1.1.2 Definition of $\vee$

**Definition 1.1.2.1.**

$$(\text{True}) \vee (\text{True}) \\ \stackrel{\text{def}}{\iff} \text{True}$$

**Definition 1.1.2.2.**

$$(\text{True}) \vee (\text{False}) \\ \stackrel{\text{def}}{\iff} \text{True}$$

**Definition 1.1.2.3.**

$$\begin{aligned} & (\text{False}) \vee (\text{True}) \\ & \stackrel{\text{def}}{\iff} \text{True} \end{aligned}$$

**Definition 1.1.2.4.**

$$\begin{aligned} & (\text{False}) \vee (\text{False}) \\ & \stackrel{\text{def}}{\iff} \text{False} \end{aligned}$$

### 1.1.3 Definition of $\wedge$

**Definition 1.1.3.1.**

$$\begin{aligned} & (\text{True}) \wedge (\text{True}) \\ & \stackrel{\text{def}}{\iff} \text{True} \end{aligned}$$

**Definition 1.1.3.2.**

$$\begin{aligned} & (\text{True}) \wedge (\text{False}) \\ & \stackrel{\text{def}}{\iff} \text{False} \end{aligned}$$

**Definition 1.1.3.3.**

$$\begin{aligned} & (\text{False}) \wedge (\text{True}) \\ & \stackrel{\text{def}}{\iff} \text{False} \end{aligned}$$

**Definition 1.1.3.4.**

$$\begin{aligned} & (\text{False}) \wedge (\text{False}) \\ & \stackrel{\text{def}}{\iff} \text{False} \end{aligned}$$

### 1.1.4 Definition of $\iff$

**Definition 1.1.4.1.**

$$\begin{aligned} & \forall x \forall y ( \\ & \stackrel{\text{def}}{\iff} \begin{aligned} & x \iff y \\ & (x \wedge y) \vee ((\neg x) \wedge (\neg y)) \end{aligned} \\ & ) \end{aligned}$$

### 1.1.5 Definition of $\implies$

Definition 1.1.5.1.

$$\forall x \forall y ( \quad \quad \quad x \implies y \quad \quad \quad ) \\ \quad \quad \quad \Longleftrightarrow \quad \quad \quad (\neg x) \vee y$$

## 1.2 Boolean algebra

### 1.2.1 Associativity of $\vee$

Proposition 1.2.1.1.

$$((x \vee y) \vee z) \iff (x \vee (y \vee z))$$

### 1.2.2 Associativity of $\wedge$

Proposition 1.2.2.1.

$$((x \wedge y) \wedge z) \iff (x \wedge (y \wedge z))$$

### 1.2.3 Commutativity of $\vee$

Proposition 1.2.3.1.

$$(x \vee y) \iff (y \vee x)$$

### 1.2.4 Commutativity of $\wedge$

Proposition 1.2.4.1.

$$(x \wedge y) \iff (y \wedge x)$$

### 1.2.5 Identity of $\vee$

Proposition 1.2.5.1.

$$(x \vee (\text{False})) \iff x$$

Proposition 1.2.5.2.

$$((\text{False}) \vee x) \iff x$$

### 1.2.6 Identity of $\wedge$

Proposition 1.2.6.1.

$$(x \wedge (\text{True})) \iff x$$

Proposition 1.2.6.2.

$$((\text{True}) \wedge x) \iff x$$

### 1.2.7 Annihilator of $\vee$

Proposition 1.2.7.1.

$$(x \vee (\text{True})) \iff (\text{True})$$

Proposition 1.2.7.2.

$$((\text{True}) \vee x) \iff (\text{True})$$

### 1.2.8 Annihilator of $\wedge$

Proposition 1.2.8.1.

$$(x \wedge (\text{False})) \iff (\text{False})$$

Proposition 1.2.8.2.

$$((\text{False}) \wedge x) \iff (\text{False})$$

### 1.2.9 Idempotence of $\vee$

Proposition 1.2.9.1.

$$(x \vee x) \iff x$$

### 1.2.10 Idempotence of $\wedge$

Proposition 1.2.10.1.

$$(x \wedge x) \iff x$$

### 1.2.11 Complement of $\vee$

Proposition 1.2.11.1.

$$(x \vee (\neg x)) \iff (\text{True})$$

Proposition 1.2.11.2.

$$((\neg x) \vee x) \iff (\text{True})$$

### 1.2.12 Complement of $\wedge$

Proposition 1.2.12.1.

$$(x \wedge (\neg x)) \iff (\text{False})$$

Proposition 1.2.12.2.

$$((\neg x) \wedge x) \iff (\text{False})$$

### 1.2.13 Absorption of $\vee$ over $\wedge$

Proposition 1.2.13.1.

$$(x \vee (x \wedge y)) \iff x$$

Proposition 1.2.13.2.

$$(x \vee (y \wedge x)) \iff x$$

Proposition 1.2.13.3.

$$((x \wedge y) \vee x) \iff x$$

Proposition 1.2.13.4.

$$((y \wedge x) \vee x) \iff x$$

### 1.2.14 Absorption of $\wedge$ over $\vee$

Proposition 1.2.14.1.

$$(x \wedge (x \vee y)) \iff x$$

**Proposition 1.2.14.2.**

$$(x \wedge (y \vee x)) \iff x$$

**Proposition 1.2.14.3.**

$$((x \vee y) \wedge x) \iff x$$

**Proposition 1.2.14.4.**

$$((y \vee x) \wedge x) \iff x$$

## 1.2.15 Distributivity of $\vee$ over $\wedge$

**Proposition 1.2.15.1.**

$$(x \vee (y \wedge z)) \iff ((x \vee y) \wedge (x \vee z))$$

**Proposition 1.2.15.2.**

$$((x \wedge y) \vee z) \iff ((x \vee z) \wedge (y \vee z))$$

## 1.2.16 Distributivity of $\wedge$ over $\vee$

**Proposition 1.2.16.1.**

$$(x \wedge (y \vee z)) \iff ((x \wedge y) \vee (x \wedge z))$$

**Proposition 1.2.16.2.**

$$((x \vee y) \wedge z) \iff ((x \wedge z) \vee (y \wedge z))$$

## 1.2.17 Double negation

**Proposition 1.2.17.1.**

$$(\neg(\neg x)) \iff x$$

## 1.2.18 De Morgan's laws

**Proposition 1.2.18.1.**

$$(\neg(x \vee y)) \iff ((\neg x) \wedge (\neg y))$$

**Proposition 1.2.18.2.**

$$(\neg(x \wedge y)) \iff ((\neg x) \vee (\neg y))$$

## 1.3 Basic Proposition

**Proposition 1.3.0.1.**

$$((x \wedge (\neg y)) \vee y) \iff (x \vee y)$$

Proof:

$$\begin{aligned} & (x \wedge (\neg y)) \vee y \\ \iff & (x \vee y) \wedge ((\neg y) \vee y) && \text{Proposition 1.2.15.2} \\ \iff & (x \vee y) \wedge (\text{True}) && \text{Proposition 1.2.11.2} \\ \iff & x \vee y && \text{Proposition 1.2.6.1} \end{aligned}$$

## 1.4 Proof technique

**Proposition 1.4.0.1.**

$$(x \iff (\text{True})) \iff x$$

Proof:

$$\begin{aligned} & x \iff (\text{True}) \\ \iff & (x \wedge (\text{True})) \vee ((\neg x) \wedge (\neg(\text{True}))) && \text{Definition 1.1.4.1} \\ \iff & (x \wedge (\text{True})) \vee ((\neg x) \wedge (\text{False})) && \text{Definition 1.1.1.1} \\ \iff & x \vee ((\neg x) \wedge (\text{False})) && \text{Proposition 1.2.6.1} \\ \iff & x \vee (\text{False}) && \text{Proposition 1.2.8.1} \\ \iff & x && \text{Proposition 1.2.5.1} \end{aligned}$$

**Proposition 1.4.0.2.**

$$(x \implies y) \implies ((x \vee z) \implies (y \vee z))$$

Proof:

$(x \implies y) \implies ((x \vee z) \implies (y \vee z))$	
$\iff ((\neg x) \vee y) \implies ((x \vee z) \implies (y \vee z))$	Definition <a href="#">1.1.5.1</a>
$\iff ((\neg x) \vee y) \implies ((\neg(x \vee z)) \vee (y \vee z))$	Definition <a href="#">1.1.5.1</a>
$\iff (\neg((\neg x) \vee y)) \vee ((\neg(x \vee z)) \vee (y \vee z))$	Definition <a href="#">1.1.5.1</a>
$\iff ((\neg(\neg x)) \wedge (\neg y)) \vee ((\neg(x \vee z)) \vee (y \vee z))$	Proposition <a href="#">1.2.18.1</a>
$\iff (x \wedge (\neg y)) \vee ((\neg(x \vee z)) \vee (y \vee z))$	Proposition <a href="#">1.2.17.1</a>
$\iff (x \wedge (\neg y)) \vee (((\neg x) \wedge (\neg z)) \vee (y \vee z))$	Proposition <a href="#">1.2.18.1</a>
$\iff (x \wedge (\neg y)) \vee (((\neg x) \wedge (\neg z)) \vee (z \vee y))$	Proposition <a href="#">1.2.3.1</a>
$\iff (x \wedge (\neg y)) \vee (((\neg x) \wedge (\neg z)) \vee z) \vee y$	Proposition <a href="#">1.2.1.1</a>
$\iff (x \wedge (\neg y)) \vee (y \vee (((\neg x) \wedge (\neg z)) \vee z))$	Proposition <a href="#">1.2.3.1</a>
$\iff ((x \wedge (\neg y)) \vee y) \vee (((\neg x) \wedge (\neg z)) \vee z)$	Proposition <a href="#">1.2.1.1</a>
$\iff (x \vee y) \vee (((\neg x) \wedge (\neg z)) \vee z)$	Proposition <a href="#">1.3.0.1</a>
$\iff (x \vee y) \vee ((\neg x) \vee z)$	Proposition <a href="#">1.3.0.1</a>
$\iff ((x \vee y) \vee (\neg x)) \vee z$	Proposition <a href="#">1.2.1.1</a>
$\iff ((\neg x) \vee (x \vee y)) \vee z$	Proposition <a href="#">1.2.3.1</a>
$\iff (((\neg x) \vee x) \vee y) \vee z$	Proposition <a href="#">1.2.1.1</a>
$\iff ((\text{True}) \vee y) \vee z$	Proposition <a href="#">1.2.11.2</a>
$\iff (\text{True}) \vee z$	Proposition <a href="#">1.2.7.2</a>
$\iff \text{True}$	Proposition <a href="#">1.2.7.2</a>

**Proposition 1.4.0.3.**

$$(x \implies y) \implies ((x \wedge z) \implies (y \wedge z))$$

**Proposition 1.4.0.4.** Contrapositive

$$(x \implies y) \iff ((\neg y) \implies (\neg x))$$

**Proposition 1.4.0.5.** Transitive property of  $\implies$ .

$$((x \implies y) \wedge (y \implies z)) \implies (x \implies z)$$

**Proposition 1.4.0.6.**

$$(x \iff y) \iff ((x \implies y) \wedge (y \implies x))$$



**Proposition 1.4.0.7.**

$$(x \iff y) \implies ((x \vee z) \iff (y \vee z))$$

**Proposition 1.4.0.8.**

$$(x \iff y) \implies ((x \wedge z) \iff (y \wedge z))$$

**Proposition 1.4.0.9.** Symmetric property of  $\iff$  .

$$(x \iff y) \iff (y \iff x)$$

**Proposition 1.4.0.10.**

$$(x \iff y) \implies ((\neg x) \iff (\neg y))$$

**Proposition 1.4.0.11.** Transitive property of  $\iff$  .

$$((x \iff y) \wedge (y \iff z)) \implies (x \iff z)$$

**Proposition 1.4.0.12.** Reflexive property of  $\iff$  .

$$x \iff x$$

Proof:

$x \iff x$	
$\iff (x \wedge x) \vee ((\neg x) \wedge (\neg x))$	Definition <a href="#">1.1.4.1</a>
$\iff x \vee ((\neg x) \wedge (\neg x))$	Proposition <a href="#">1.2.10.1</a>
$\iff x \vee (\neg x)$	Proposition <a href="#">1.2.10.1</a>
$\iff \text{True}$	Proposition <a href="#">1.2.11.1</a>

## 1.5 Quantifiers

**Definition 1.5.0.1.** Universal quantifier is denoted by  $\forall$ .

$$\forall x, P(x) \\ \iff^{\text{def}} (P(x_1) \wedge P(x_2) \wedge \dots)$$

**Definition 1.5.0.2.** Existential quantifier is denoted by  $\exists$ .

$$\exists x, P(x) \\ \iff^{\text{def}} (P(x_1) \vee P(x_2) \vee \dots)$$

**Proposition 1.5.0.3.**

$$(\forall x(P(x) \wedge Q(x))) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

**Proposition 1.5.0.4.**

$$(\exists x, P(x)) \vee (\exists x, Q(x)) \iff (\exists x, (P(x) \vee Q(x)))$$

**Proposition 1.5.0.5.**

$$(P \vee (\forall x, Q(x))) \iff (\forall x(P \vee Q(x)))$$

**Proposition 1.5.0.6.**

$$(P \wedge (\exists x, Q(x))) \iff (\exists x(P \wedge Q(x)))$$

**Axiom 1.1.** P does not depend on x.

$$(\forall x, P(y)) \iff P(y)$$

**Axiom 1.2.** P does not depend on x.

$$(\exists x, P(y)) \iff P(y)$$

**Axiom 1.3.** De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

**Axiom 1.4.** De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

**Definition 1.5.0.7.** Uniqueness quantifier is denoted by  $!\exists$ .

$$\begin{aligned} & !\exists x, P(x) \\ \stackrel{\text{def}}{\iff} & (\exists x, P(x)) \wedge (\forall x \forall y (P(x) \wedge P(y) \implies x = y)) \end{aligned}$$

**Axiom 1.5.** Axiom of Substitution

$$\forall x((\exists y((y = x) \wedge P(y))) \iff P(x))$$

## 1.6 Proposition

**Proposition 1.6.0.1.**

$$(P \wedge Q) \implies (P \iff Q)$$

**Proposition 1.6.0.2.**

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

**Proposition 1.6.0.3.**

$$(P \wedge Q) \implies P$$

**Lemma 1.6.0.4.**

$$(P \wedge ((Q \wedge P) \implies R)) \implies (Q \implies R)$$

**Proposition 1.6.0.5.**

$$(P \wedge (P \implies Q)) \implies Q$$

**Proposition 1.6.0.6.**

$$(P \wedge (P \iff Q)) \implies Q$$

# Chapter 2

## Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by  $\in$ .

**Definition 2.0.0.1.** Definition of  $\notin$ .

$$\begin{aligned} A \notin B \\ \stackrel{\text{def}}{\iff} \neg(A \in B) \end{aligned}$$

**Definition 2.0.0.2.**

$$\begin{aligned} \forall x \in S, P(x) \\ \stackrel{\text{def}}{\iff} \forall x (x \in S \implies P(x)) \end{aligned}$$

**Definition 2.0.0.3.**

$$\begin{aligned} \exists x \in S, P(x) \\ \stackrel{\text{def}}{\iff} \exists x (x \in S \wedge P(x)) \end{aligned}$$

**Proposition 2.0.0.4.**

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$\neg(\forall x \in S, P(x))$	
$\iff \neg(\forall x(x \in S \implies P(x)))$	Definition <a href="#">2.0.0.2</a>
$\iff \neg(\forall x(\neg(x \in S) \vee P(x)))$	Definition <a href="#">1.1.5.1</a>
$\iff \exists x, \neg(\neg(x \in S) \vee P(x))$	Axiom <a href="#">1.3</a>
$\iff \exists x, \neg(\neg(x \in S)) \wedge \neg(P(x))$	Proposition <a href="#">1.2.18.1</a>
$\iff \exists x, x \in S \wedge \neg(P(x))$	Proposition <a href="#">1.2.17.1</a>
$\iff \exists x \in S, \neg(P(x))$	Definition <a href="#">2.0.0.3</a>

**Proposition 2.0.0.5.**

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$\neg(\exists x \in S, P(x))$	
$\iff \neg(\exists x(x \in S \wedge P(x)))$	Definition <a href="#">2.0.0.3</a>
$\iff \forall x, \neg(x \in S \wedge P(x))$	Axiom <a href="#">1.4</a>
$\iff \forall x, (\neg(x \in S)) \vee (\neg(P(x)))$	Proposition <a href="#">1.2.18.2</a>
$\iff \forall x, x \in S \implies \neg(P(x))$	Definition <a href="#">1.1.5.1</a>
$\iff \forall x \in S, \neg(P(x))$	Definition <a href="#">2.0.0.2</a>

## 2.1 Equality of sets

**Definition 2.1.0.1.** Definition of  $=$ .

$$A = B$$

$$\stackrel{\text{def}}{\iff} \forall x(x \in A \iff x \in B)$$

**Definition 2.1.0.2.** Definition of  $\neq$ .

$$A \neq B$$

$$\stackrel{\text{def}}{\iff} \neg(A = B)$$

**Proposition 2.1.0.3.** Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$\begin{aligned} & \forall x( \\ & \quad x = x \\ & \iff \forall y(y \in x \iff y \in x) \quad \text{Definition 2.1.0.1} \\ & \iff \text{True} \quad \text{Proposition 1.4.0.12} \\ & ) \end{aligned}$$

**Proposition 2.1.0.4.** Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$\begin{aligned} & \forall x \forall y( \\ & \quad x = y \\ & \implies \forall z(z \in x \iff z \in y) \quad \text{Definition 2.1.0.1} \\ & \implies \forall z(z \in y \iff z \in x) \quad \text{Proposition 1.4.0.9} \\ & \implies y = x \quad \text{Definition 2.1.0.1} \\ & ) \end{aligned}$$

**Proposition 2.1.0.5.** Transitive property of equality

$$\forall x \forall y \forall z((x = y) \wedge (y = z) \implies (x = z))$$

Proof:

$$\begin{aligned} & \forall x \forall y \forall z( \\ & \quad (x = y) \wedge (y = z) \\ & \implies (\forall w(w \in x \iff w \in y)) \wedge (\forall w(w \in y \iff w \in z)) \quad \text{Definition 2.1.0.1} \\ & \implies \forall w((w \in x \iff w \in y) \wedge (w \in y \iff w \in z)) \quad \text{Proposition 1.5.0.3} \\ & \implies \forall w(w \in x \iff w \in z) \quad \text{Proposition 1.4.0.11} \\ & \implies x = z \quad \text{Definition 2.1.0.1} \\ & ) \end{aligned}$$

**Axiom 2.1.** Axiom of extensionality

$$\begin{aligned} & \forall x \forall y( \\ & \quad x = y \implies \forall A(x \in A \iff y \in A) \\ & ) \end{aligned}$$

**Axiom 2.2.** Existence of empty set

$$\exists x \forall y (y \notin x)$$

**Proposition 2.1.0.6.** Uniqueness of empty set.

$$!\exists x \forall y (y \notin x)$$

Proof:

Let  $P(x) = \forall y (y \notin x)$

$$\begin{array}{ll} \exists x \forall y (y \notin x) & \text{Axiom 2.2} \\ \implies \exists x, P(x) & \text{Definition of } P(x) \end{array}$$

$$\begin{array}{llll} \forall x \forall y ( & & & \\ & P(x) \wedge P(y) & & \\ \implies & (\forall z (z \notin x) \wedge (\forall z (z \notin y))) & \text{Definition of } P(x) & \\ \implies & \forall z ((z \notin x) \wedge (z \notin y)) & \text{Proposition 1.5.0.3} & \\ \implies & \forall z (z \notin x \iff z \notin y) & \text{Proposition 1.6.0.1} & \\ \implies & \forall z (\neg(z \in x) \iff \neg(z \in y)) & \text{Definition 2.0.0.1} & \\ \implies & \forall z (z \in x \iff z \in y) & \text{Proposition 1.6.0.2} & \\ \implies & x = y & \text{Definition 2.1.0.1} & \\ ) & & & \end{array}$$

$$\begin{array}{ll} (\exists x, P(x)) \wedge \forall x \forall y ((P(x) \wedge P(y)) \implies (x = y)) & \\ \implies !\exists x, P(x) & \text{Definition 1.5.0.7} \\ \implies !\exists x \forall y (y \notin x) & \text{Definition of } P(x) \end{array}$$

**Definition 2.1.0.7.** The unique empty set is denoted by  $\emptyset$ .

$$\forall x (x \notin \emptyset)$$

Proof:

Let  $P(x) = \forall y(y \notin x)$

$\neg \exists x \forall y(y \notin x)$	Proposition 2.1.0.6
$\implies \neg \exists x, P(x)$	Definition of P(x)
$\implies (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y))$	Definition 1.5.0.7
$\implies P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Definition 2.1.0.7
$\implies P(\emptyset)$	Proposition 1.6.0.3
$\implies \forall y(y \notin \emptyset)$	Definition of P(x)

**Proposition 2.1.0.8.** Uniqueness of  $\emptyset$

$$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$$

Proof:

Let  $P(x) = \forall y(y \notin x)$

$\neg \exists x \forall y(y \notin x)$	Proposition 2.1.0.6
$\implies \neg \exists x, P(x)$	Definition of P(x)
$\implies (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y))$	Definition 1.5.0.7
$\implies P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Definition 2.1.0.7
$\implies (\forall x, P(\emptyset)) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Axiom 1.1
$\implies \forall x(P(\emptyset) \wedge ((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)))$	Proposition 1.5.0.3
$\implies \forall x(P(x) \implies (x = \emptyset))$	Lemma 1.6.0.4
$\implies \forall x(\forall y(y \notin x) \implies (x = \emptyset))$	Definition of P(x)

**Proposition 2.1.0.9.** Single choice

$$\forall x((x \neq \emptyset) \implies (\exists y, y \in x))$$

Proof:

$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$	
$\implies \forall x(\neg(x = \emptyset) \implies \neg(\forall y(y \notin x)))$	Proposition 1.4.0.4
$\implies \forall x((x \neq \emptyset) \implies \neg(\forall y(y \notin x)))$	Definition 2.1.0.2
$\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(y \notin x)))$	Axiom 1.3
$\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(\neg(y \in x))))$	Definition 2.0.0.1
$\implies \forall x((x \neq \emptyset) \implies (\exists y, y \in x))$	Proposition 1.2.17.1



**Axiom 2.3.** Axiom of pairing. Existence of pair set.

$$\forall x \forall y \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

**Proposition 2.1.0.10.** Uniqueness of pairing set.

$$\forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Proof:

$$\text{Let } P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

$$\forall x \forall y \forall A \forall B ($$

$$\begin{aligned} & P(A, x, y) \wedge P(B, x, y) \\ \implies & (\forall z (z \in A \iff ((z = x) \vee (z = y)))) \\ & \wedge (\forall z (z \in B \iff ((z = x) \vee (z = y)))) \quad \text{Definition of } P(A, x, y) \\ \implies & \forall z ((z \in A \iff ((z = x) \vee (z = y))) \\ & \wedge (z \in B \iff ((z = x) \vee (z = y)))) \quad \text{Proposition 1.5.0.3} \\ \implies & \forall z (z \in A \iff z \in B) \quad \text{Proposition 1.4.0.11} \\ \implies & A = B \quad \text{Definition 2.1.0.1} \end{aligned}$$

)

$$\begin{aligned} & \forall x \forall y ! \exists A, P(A, x, y) \quad \text{Similar to the proof of the Proposition 2.1.0.6} \\ \implies & \forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y))) \quad \text{Definition of } P(A, x, y) \end{aligned}$$

**Definition 2.1.0.11.** The unique pair set of  $x$  and  $y$  is denoted by  $\{x, y\}$ .

$$\text{Let } P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Similar to the proof of Definition 2.1.0.7,

$$\forall x \forall y P(\{x, y\}, x, y)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall x \forall y \forall A (P(A, x, y) \implies (A = \{x, y\}))$$

**Proposition 2.1.0.12.** Existence of singleton set.

$$\forall x \exists A \forall y (y \in A \iff (y = x))$$

Proof:

$$\begin{aligned} & \forall x \exists A \forall y (y \in A \iff ((y = x) \vee (y = x))) \quad \text{Axiom 2.3} \\ \implies & \forall x \exists A \forall y (y \in A \iff (y = x)) \quad \text{Proposition 1.2.9.1} \end{aligned}$$

**Proposition 2.1.0.13.** Uniqueness of singleton set.

$$\forall x! \exists A \forall y (y \in A \iff (x = y))$$

Let  $P(A, x) = \forall y (y \in A \iff (x = y))$

The proof is similar to the proof of Proposition 2.1.0.10.

**Definition 2.1.0.14.** The unique singleton set of  $x$  is denoted by  $\{x\}$ .

Let  $P(A, x) = \forall y (y \in A \iff (x = y))$

Similar to the proof of Definition 2.1.0.7,

$$\forall x P(\{x\}, x)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall x \forall A (P(A, x) \implies (A = \{x\}))$$

**Axiom 2.4.** Axiom of union. Existence of union set.

$$\forall F \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

**Proposition 2.1.0.15.** Uniqueness of union set.

$$\forall F! \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

Proof:

Let  $P(A, F) = \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$

$\forall F \forall A \forall B ($

$$\begin{aligned} & P(A, F) \wedge P(B, F) \\ \implies & (\forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) \\ & \wedge (\forall x (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) \quad \text{Definition of } P(A, F) \\ \implies & \forall x ((x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) \\ & \wedge (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) \quad \text{Proposition 1.5.0.3} \\ \implies & \forall x (x \in A \iff x \in B) \quad \text{Proposition 1.4.0.11} \\ \implies & A = B \quad \text{Definition 2.1.0.1} \end{aligned}$$

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$\forall F! \exists A, P(A, F)$  Similar to the proof of the Proposition 2.1.0.6  
 $\implies \forall F! \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$  Definition of  $P(A, F)$

**Definition 2.1.0.16.** The unique union set of  $F$  is denoted by  $\bigcup F$ .  
Let  $P(A, F) = \forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$   
Similar to the proof of Definition 2.1.0.7,

$$\forall F P(\bigcup F, F)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall F \forall A (P(A, F) \implies (A = \bigcup F))$$

**Definition 2.1.0.17.** Definition of pairwise union  $A \cup B$ .

$$\begin{aligned} & A \cup B \\ & \stackrel{\text{def}}{=} \bigcup \{A, B\} \end{aligned}$$

**Proposition 2.1.0.18.** Property of pairwise union.

$$\forall A \forall B \forall x (x \in (A \cup B) \iff ((x \in A) \vee (x \in B)))$$

Proof:

$\forall A \forall B \forall x ($

$$x \in (A \cup B)$$

$$\iff x \in \bigcup \{A, B\}$$

Definition 2.1.0.1 and 2.1.0.17

$$\iff \exists Y((x \in Y) \wedge (Y \in \{A, B\}))$$

Definition 2.1.0.16

$$\iff \exists Y((x \in Y) \wedge ((Y = A) \vee (Y = B)))$$

Definition 2.1.0.11

$$\iff \exists Y(((x \in Y) \wedge (Y = A)) \vee ((x \in Y) \wedge (Y = B)))$$

Proposition 1.2.16.1

$$\iff (\exists Y((x \in Y) \wedge (Y = A))) \vee (\exists Y((x \in Y) \wedge (Y = B)))$$

Proposition 1.5.0.4

$$\iff ((x \in A) \vee (x \in B))$$

$$\text{Axiom 1.5 with } P(A, x) = (x \in A)$$

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**Proposition 2.1.0.19.** Commutativity of  $\cup$ .

$$\forall x \forall y ((x \cup y) = (y \cup x))$$

Proof:

$\forall x \forall y ($

$$(x \cup y) = (y \cup x)$$

$$\iff \forall z (z \in (x \cup y) \iff z \in (y \cup x))$$

Definition [2.1.0.1](#)

$$\iff \forall z (((z \in x) \vee (z \in y)) \iff ((z \in y) \vee (z \in x)))$$

Proposition [2.1.0.18](#)

$$\iff \forall z (((z \in x) \vee (z \in y)) \iff ((z \in x) \vee (z \in y)))$$

Proposition [1.2.3.1](#)

$$\iff \text{True}$$

Proposition [1.4.0.12](#)

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**Proposition 2.1.0.20.** Identity of  $\cup$ .

$$\forall x ((x \cup \emptyset) = x)$$

Proof:

$\forall x ($

$$(x \cup \emptyset) = x$$

$$\iff \forall y (y \in (x \cup \emptyset) \iff (y \in x))$$

Definition [2.1.0.1](#)

$$\iff \forall y (((y \in x) \vee (y \in \emptyset)) \iff (y \in x))$$

Proposition [2.1.0.18](#)

$$\iff \forall y (((y \in x) \vee (\neg(\neg(y \in \emptyset)))) \iff (y \in x))$$

Proposition [1.2.17.1](#)

$$\iff \forall y (((y \in x) \vee (\neg(y \notin \emptyset))) \iff (y \in x))$$

Definition [2.0.0.1](#)

$$\iff \forall y (((y \in x) \vee (\neg(\text{True}))) \iff (y \in x))$$

Definition [2.1.0.7](#)

$$\iff \forall y (((y \in x) \vee (\text{False})) \iff (y \in x))$$

Definition [1.1.1.1](#)

$$\iff \forall y ((y \in x) \iff (y \in x))$$

Proposition [1.2.5.1](#)

$$\iff \text{True}$$

Proposition [1.4.0.12](#)

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**Definition 2.1.0.21.** Definition of  $0$ .

$$0 \stackrel{\text{def}}{=} \emptyset$$

**Definition 2.1.0.22.** Definition of successor  $S(x)$ .

$$S(x)$$

$$\stackrel{\text{def}}{=} x \cup \{x\}$$

**Definition 2.1.0.23.** Definition of 1.

$1 \stackrel{\text{def}}{=} S(0)$	
$= 0 \cup \{0\}$	Definition <a href="#">2.1.0.22</a>
$= \emptyset \cup \{\emptyset\}$	Definition <a href="#">2.1.0.21</a>
$= \{\emptyset\} \cup \emptyset$	Proposition <a href="#">2.1.0.19</a>
$= \{\emptyset\}$	Proposition <a href="#">2.1.0.20</a>