Chapter 1

Logic

Definition 1.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

Definition 1.1.1. Definition of $\neg$.

p	$\neg p$
T	F
F	T

Definition 1.1.2. Definition of $\wedge$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.1.3. Definition of $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.1.4. Definition of $\iff$.

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 1.1.5. Definition of $\implies$.

$$\begin{aligned} & p \implies q \\ \iff & (\neg p) \vee q \end{aligned}$$

1.2 Quantifiers

Definition 1.2.1. Universal quantifier is denoted by $\forall$.

$$\forall x, P(x)$$

Definition 1.2.2. Existential quantifier is denoted by $\exists$.

$$\exists x, P(x)$$

Axiom 1.1.

$$\forall x, (P(x) \wedge Q(x)) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

Axiom 1.2. De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

Axiom 1.3. De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

Definition 1.2.3. Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned} & !\exists x, P(x) \\ \iff & (\exists x, P(x)) \wedge (\forall y \forall z (P(y) \wedge P(z) \implies y = z)) \end{aligned}$$

1.3 Proposition

Let $P = P(x_1, x_2, \ldots, x_n)$. Let $Q = Q(x_1, x_2, \ldots, x_n)$. etc

Proposition 1.3.1. Double negation

$$\neg(\neg P) \iff P$$

Proposition 1.3.2. Reflexive property of iff.

$$P \iff P$$

Proof:

P	$P \iff P$
T	T
F	T

Proposition 1.3.3. Symmetric property of iff.

$$(P \iff Q) \iff (Q \iff P)$$

Proposition 1.3.4. Transitive property of iff.

$$((P \iff Q) \wedge (Q \iff R)) \implies (P \iff R)$$

Proposition 1.3.5. De Morgan's law

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

Proposition 1.3.6. De Morgan's law

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

Definition 2.0.1. Definition of $\notin$.

$$\begin{aligned} & A \notin B \\ \iff & \neg(A \in B) \end{aligned}$$

Definition 2.0.2.

$$\begin{aligned} & \forall x \in S, P(x) \\ \iff & \forall x(x \in S \implies P(x)) \end{aligned}$$

Definition 2.0.3.

$$\begin{aligned} & \exists x \in S, P(x) \\ \iff & \exists x(x \in S \land P(x)) \end{aligned}$$

Proposition 2.0.4.

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
&\neg(\forall x \in S, P(x)) && \\
\iff &\neg(\forall x(x \in S \implies P(x))) && \text{Definition 2.0.2} \\
\iff &\neg(\forall x(\neg(x \in S) \lor P(x))) && \text{Definition 1.1.5} \\
\iff &\exists x, \neg(\neg(x \in S) \lor P(x)) && \text{Axiom 1.2} \\
\iff &\exists x, \neg(\neg(x \in S)) \land \neg(P(x)) && \text{Proposition 1.3.6} \\
\iff &\exists x, x \in S \land \neg(P(x)) && \text{Proposition 1.3.1} \\
\iff &\exists x \in S, \neg(P(x)) && \text{Definition 2.0.3}
\end{aligned}
$$

Proposition 2.0.5.

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
&\neg(\exists x \in S, P(x)) && \\
\iff &\neg(\exists x(x \in S \land P(x))) && \text{Definition 2.0.3} \\
\iff &\forall x, \neg(x \in S \land P(x)) && \text{Axiom 1.3} \\
\iff &\forall x, (\neg(x \in S)) \lor (\neg(P(x)) && \text{Proposition 1.3.5} \\
\iff &\forall x, x \in S \implies \neg(P(x)) && \text{Definition 1.1.5} \\
\iff &\forall x \in S, \neg(P(x)) && \text{Definition 2.0.2}
\end{aligned}
$$

2.1 Equality of sets

Definition 2.1.1. Definition of $=$.

$$
\begin{aligned}
&A = B \\
\iff &\forall x(x \in A \iff x \in B)
\end{aligned}
$$

Definition 2.1.2. Definition of $\neq$.

$$
\begin{aligned}
&A \neq B \\
\iff &\neg(A = B)
\end{aligned}
$$

Proposition 2.1.3. Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$\begin{array}{lll} \forall x(& & \\ & x = x & \\ \iff & \forall y(y \in x \iff y \in x) & \text{Definition 2.1.1} \\ \iff & \text{True} & \text{Proposition 1.3.2} \\ ) & & \end{array}$$

Proposition 2.1.4. Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$\begin{array}{lll} \forall x \forall y(& & \\ & x = y & \\ \implies & \forall z(z \in x \iff z \in y) & \text{Definition 2.1.1} \\ \implies & \forall z(z \in y \iff z \in x) & \text{Proposition 1.3.3} \\ \implies & y = x & \text{Definition 2.1.1} \\ ) & & \end{array}$$

Proposition 2.1.5. Transitive property of equality

$$\forall x \forall y \forall z((x = y) \land (y = z) \implies (x = z))$$

Proof:

$$\begin{array}{lll} \forall x \forall y \forall z(& & \\ & (x = y) \land (y = z) & \\ \implies & (\forall w(w \in x \iff w \in y)) \land (\forall w(w \in y \iff w \in z)) & \text{Definition 2.1.1} \\ \implies & \forall w((w \in x \iff w \in y) \land (w \in y \iff w \in z)) & \text{Axiom 1.1} \\ \implies & \forall w(w \in x \iff w \in z) & \text{Proposition 1.3.4} \\ \implies & x = z & \text{Definition 2.1.1} \\ ) & & \end{array}$$

Axiom 2.1. Axiom of Substitution

$$\begin{array}{l} \forall x \forall y(\\ x = y \implies \forall A(x \in A \iff y \in A) \\ ) \end{array}$$