Chapter 1

Logic

Definition 1.0.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

1.1.1 Definition of $\neg$

Definition 1.1.1.1.

$$\begin{aligned} &\neg(\text{True}) \\ \overset{\text{def}}{\iff} &\text{False} \end{aligned}$$

Definition 1.1.1.2.

$$\begin{aligned} &\neg(\text{False}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

1.1.2 Definition of $\vee$

Definition 1.1.2.1.

$$\begin{aligned} &(\text{True}) \vee (\text{True}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

Definition 1.1.2.2.

$$\begin{aligned} &(\text{True}) \vee (\text{False}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

Definition 1.1.2.3.

$$\begin{aligned} & (\text{False}) \vee (\text{True}) \\ \overset{\text{def}}{\iff} & \text{True} \end{aligned}$$

Definition 1.1.2.4.

$$\begin{aligned} & (\text{False}) \vee (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

1.1.3 Definition of $\wedge$

Definition 1.1.3.1.

$$\begin{aligned} & (\text{True}) \wedge (\text{True}) \\ \overset{\text{def}}{\iff} & \text{True} \end{aligned}$$

Definition 1.1.3.2.

$$\begin{aligned} & (\text{True}) \wedge (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

Definition 1.1.3.3.

$$\begin{aligned} & (\text{False}) \wedge (\text{True}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

Definition 1.1.3.4.

$$\begin{aligned} & (\text{False}) \wedge (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

1.1.4 Definition of $\iff$

Definition 1.1.4.1.

$$\begin{aligned} & a \iff b \\ \overset{\text{def}}{\iff} & (a \wedge b) \vee ((\neg a) \wedge (\neg b)) \end{aligned}$$

1.1.5 Definition of $\implies$

Definition 1.1.5.1.

$$
\begin{aligned}
& a \implies b \\
\overset{\text{def}}{\iff} & (\neg a) \vee b
\end{aligned}
$$

1.2 Boolean algebra

1.2.1 Associativity of $\vee$

Proposition 1.2.1.1.

$$
\begin{aligned}
& (a \vee b) \vee c \\
\iff & a \vee (b \vee c)
\end{aligned}
$$

1.2.2 Associativity of $\wedge$

Proposition 1.2.2.1.

$$
\begin{aligned}
& (a \wedge b) \wedge c \\
\iff & a \wedge (b \wedge c)
\end{aligned}
$$

1.2.3 Commutativity of $\vee$

Proposition 1.2.3.1.

$$
\begin{aligned}
& a \vee b \\
\iff & b \vee a
\end{aligned}
$$

1.2.4 Commutativity of $\wedge$

Proposition 1.2.4.1.

$$
\begin{aligned}
& a \wedge b \\
\iff & b \wedge a
\end{aligned}
$$

1.2.5 Identity of $\vee$

Proposition 1.2.5.1.

$$\begin{aligned} & a \vee (\text{False}) \\ \iff & a \end{aligned}$$

Proposition 1.2.5.2.

$$\begin{aligned} & (\text{False}) \vee a \\ \iff & a \end{aligned}$$

1.2.6 Identity of $\wedge$

Proposition 1.2.6.1.

$$\begin{aligned} & a \wedge (\text{True}) \\ \iff & a \end{aligned}$$

Proposition 1.2.6.2.

$$\begin{aligned} & (\text{True}) \wedge a \\ \iff & a \end{aligned}$$

1.2.7 Annihilator of $\vee$

Proposition 1.2.7.1.

$$\begin{aligned} & a \vee (\text{True}) \\ \iff & \text{True} \end{aligned}$$

Proposition 1.2.7.2.

$$\begin{aligned} & (\text{True}) \vee a \\ \iff & \text{True} \end{aligned}$$

1.2.8 Annihilator of $\wedge$

Proposition 1.2.8.1.

$$\begin{aligned} & a \wedge (\text{False}) \\ \iff & \text{False} \end{aligned}$$

Proposition 1.2.8.2.

$$\begin{aligned} & (\text{False}) \wedge a \\ \iff & \text{False} \end{aligned}$$

1.2.9 Idempotence of $\vee$

Proposition 1.2.9.1.

$$\begin{aligned} & a \vee a \\ \iff & a \end{aligned}$$

1.2.10 Idempotence of $\wedge$

Proposition 1.2.10.1.

$$\begin{aligned} & a \wedge a \\ \iff & a \end{aligned}$$

1.2.11 Complement of $\vee$

Proposition 1.2.11.1.

$$\begin{aligned} & a \vee (\neg a) \\ \iff & \text{True} \end{aligned}$$

Proposition 1.2.11.2.

$$\begin{aligned} & (\neg a) \vee a \\ \iff & \text{True} \end{aligned}$$

1.2.12 Complement of $\wedge$

Proposition 1.2.12.1.

$$\begin{aligned} & a \wedge (\neg a) \\ \iff & \text{False} \end{aligned}$$

Proposition 1.2.12.2.

$$\begin{aligned} & (\neg a) \wedge a \\ \iff & \text{False} \end{aligned}$$

1.2.13 Absorption of $\vee$ over $\wedge$

Proposition 1.2.13.1.

$$\begin{aligned} & a \vee (a \wedge b) \\ \iff & a \end{aligned}$$

Proposition 1.2.13.2.

$$\begin{aligned} & a \vee (b \wedge a) \\ \iff & a \end{aligned}$$

Proposition 1.2.13.3.

$$\begin{aligned} & (a \wedge b) \vee a \\ \iff & a \end{aligned}$$

Proposition 1.2.13.4.

$$\begin{aligned} & (b \wedge a) \vee a \\ \iff & a \end{aligned}$$

1.2.14 Absorption of $\wedge$ over $\vee$

Proposition 1.2.14.1.

$$\begin{aligned} & a \wedge (a \vee b) \\ \iff & a \end{aligned}$$

Proposition 1.2.14.2.

$$\begin{aligned} & a \wedge (b \vee a) \\ \iff & a \end{aligned}$$

Proposition 1.2.14.3.

$$\begin{aligned} & (a \vee b) \wedge a \\ \iff & a \end{aligned}$$

Proposition 1.2.14.4.

$$\begin{aligned} & (b \vee a) \wedge a \\ \iff & a \end{aligned}$$

1.2.15 Distributivity of $\vee$ over $\wedge$

Proposition 1.2.15.1.

$$\begin{aligned} & a \vee (b \wedge c) \\ \iff & (a \vee b) \wedge (a \vee c) \end{aligned}$$

Proposition 1.2.15.2.

$$\begin{aligned} & (a \wedge b) \vee c \\ \iff & (a \vee c) \wedge (b \vee c) \end{aligned}$$

1.2.16 Distributivity of $\wedge$ over $\vee$

Proposition 1.2.16.1.

$$\begin{aligned} & a \wedge (b \vee c) \\ \iff & (a \wedge b) \vee (a \wedge c) \end{aligned}$$

Proposition 1.2.16.2.

$$\begin{aligned} & (a \vee b) \wedge c \\ \iff & (a \wedge c) \vee (b \wedge c) \end{aligned}$$

1.2.17 Double negation

Proposition 1.2.17.1.

$$\begin{aligned} & \neg(\neg a) \\ \iff & a \end{aligned}$$

1.2.18 De Morgan's laws

Proposition 1.2.18.1.

$$\begin{aligned} & \neg(a \vee b) \\ \iff & (\neg a) \wedge (\neg b) \end{aligned}$$

Proposition 1.2.18.2.

$$\begin{aligned} & \neg(a \wedge b) \\ \iff & (\neg a) \vee (\neg b) \end{aligned}$$

1.3 Basic Proposition

Proposition 1.3.0.1.

$$\begin{aligned} & (a \wedge (\neg b)) \vee b \\ \iff & a \vee b \end{aligned}$$

Proof of Proposition 1.3.0.1

$$\begin{aligned} & (a \wedge (\neg b)) \vee b && \\ \iff & (a \vee b) \wedge ((\neg b) \vee b) && \text{Proposition 1.2.15.2} \\ \iff & (a \vee b) \wedge (\text{True}) && \text{Proposition 1.2.11.2} \\ \iff & a \vee b && \text{Proposition 1.2.6.1} \end{aligned}$$

1.4 Proof technique

Proposition 1.4.0.1.

$$\begin{aligned} & a \iff (\text{True}) \\ \iff & a \end{aligned}$$

Proof of Proposition 1.4.0.1

$$\begin{aligned} & a \iff (\text{True}) && \\ \overset{\text{def}}{\iff} & (a \wedge (\text{True})) \vee ((\neg a) \wedge (\neg(\text{True}))) && \text{Definition 1.1.4.1} \\ \overset{\text{def}}{\iff} & (a \wedge (\text{True})) \vee ((\neg a) \wedge (\text{False})) && \text{Definition 1.1.1.1} \\ \iff & a \vee ((\neg a) \wedge (\text{False})) && \text{Proposition 1.2.6.1} \\ \iff & a \vee (\text{False}) && \text{Proposition 1.2.8.1} \\ \iff & a && \text{Proposition 1.2.5.1} \end{aligned}$$

Proposition 1.4.0.2.

$$\begin{aligned} & a \implies b \\ \implies & (a \vee c) \implies (b \vee c) \end{aligned}$$

Proof of Proposition 1.4.0.2

$$
\begin{aligned}
& (a \implies b) \implies ((a \vee c) \implies (b \vee c)) & \\
\overset{\text{def}}{\iff} & ((\neg a) \vee b) \implies ((a \vee c) \implies (b \vee c)) & \text{Definition 1.1.5.1} \\
\overset{\text{def}}{\iff} & ((\neg a) \vee b) \implies ((\neg (a \vee c)) \vee (b \vee c)) & \text{Definition 1.1.5.1} \\
\overset{\text{def}}{\iff} & (\neg ((\neg a) \vee b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Definition 1.1.5.1} \\
\iff & ((\neg (\neg a)) \wedge (\neg b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Proposition 1.2.18.1} \\
\iff & (a \wedge (\neg b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Proposition 1.2.17.1} \\
\iff & (a \wedge (\neg b)) \vee (((\neg a) \wedge (\neg c)) \vee (b \vee c)) & \text{Proposition 1.2.18.1} \\
\iff & (a \wedge (\neg b)) \vee (((\neg a) \wedge (\neg c)) \vee (c \vee b)) & \text{Proposition 1.2.3.1} \\
\iff & (a \wedge (\neg b)) \vee ((((\neg a) \wedge (\neg c)) \vee c) \vee b) & \text{Proposition 1.2.1.1} \\
\iff & (a \wedge (\neg b)) \vee (b \vee (((\neg a) \wedge (\neg c)) \vee c)) & \text{Proposition 1.2.3.1} \\
\iff & ((a \wedge (\neg b)) \vee b) \vee (((\neg a) \wedge (\neg c)) \vee c) & \text{Proposition 1.2.1.1} \\
\iff & (a \vee b) \vee (((\neg a) \wedge (\neg c)) \vee c) & \text{Proposition 1.3.0.1} \\
\iff & (a \vee b) \vee ((\neg a) \vee c) & \text{Proposition 1.3.0.1} \\
\iff & ((a \vee b) \vee (\neg a)) \vee c & \text{Proposition 1.2.1.1} \\
\iff & ((\neg a) \vee (a \vee b)) \vee c & \text{Proposition 1.2.3.1} \\
\iff & (((\neg a) \vee a) \vee b) \vee c & \text{Proposition 1.2.1.1} \\
\iff & ((\text{True}) \vee b) \vee c & \text{Proposition 1.2.11.2} \\
\iff & (\text{True}) \vee c & \text{Proposition 1.2.7.2} \\
\iff & \text{True} & \text{Proposition 1.2.7.2}
\end{aligned}
$$

Proposition 1.4.0.3.

$$
\begin{aligned}
& a \implies b \\
\implies & (a \wedge c) \implies (b \wedge c)
\end{aligned}
$$

Proposition 1.4.0.4. Contrapositive

$$
\begin{aligned}
& a \implies b \\
\iff & (\neg b) \implies (\neg a)
\end{aligned}
$$

Proposition 1.4.0.5. Transitive property of $\implies$.

$$
\begin{aligned}
& (a \implies b) \wedge (b \implies c) \\
\implies & a \implies c
\end{aligned}
$$

Proposition 1.4.0.6.

$$\begin{aligned}&a \iff b\\ \iff &(a \implies b) \wedge (b \implies a)\end{aligned}$$

Proposition 1.4.0.7.

$$\begin{aligned}&a \iff b\\ \implies &(a \vee c) \iff (b \vee c)\end{aligned}$$

Proposition 1.4.0.8.

$$\begin{aligned}&a \iff b\\ \implies &(a \wedge c) \iff (b \wedge c)\end{aligned}$$

Proposition 1.4.0.9. Symmetric property of $\iff$.

$$\begin{aligned}&a \iff b\\ \iff &b \iff a\end{aligned}$$

Proposition 1.4.0.10.

$$\begin{aligned}&a \iff b\\ \iff &(\neg a) \iff (\neg b)\end{aligned}$$

Proposition 1.4.0.11. Transitive property of $\iff$.

$$\begin{aligned}&(a \iff b) \wedge (b \iff c)\\ \implies &a \iff c\end{aligned}$$

Proposition 1.4.0.12. Reflexive property of $\iff$.

$$\begin{aligned}&a\\ \iff &a\end{aligned}$$

Proof of Proposition 1.4.0.12

$$\begin{aligned}&a \iff a\\ \overset{\text{def}}{\iff} &(a \wedge a) \vee ((\neg a) \wedge (\neg a)) && \text{Definition 1.1.4.1}\\ \iff &a \vee ((\neg a) \wedge (\neg a)) && \text{Proposition 1.2.10.1}\\ \iff &a \vee (\neg a) && \text{Proposition 1.2.10.1}\\ \iff &\text{True} && \text{Proposition 1.2.11.1}\end{aligned}$$

1.5 Quantifiers

Definition 1.5.0.1. Universal quantifier is denoted by $\forall$.

$$\begin{aligned}&\forall x(P(x))\\ \overset{\text{def}}{\iff}\ &(P(x_1) \land P(x_2) \land \dots)\end{aligned}$$

Definition 1.5.0.2. Existential quantifier is denoted by $\exists$.

$$\begin{aligned}&\exists x(P(x))\\ \overset{\text{def}}{\iff}\ &(P(x_1) \lor P(x_2) \lor \dots)\end{aligned}$$

Proposition 1.5.0.3.

$$\begin{aligned}&(\forall c(a)) \land (\forall c(b))\\ \iff\ &\forall c(a \land b)\end{aligned}$$

Proposition 1.5.0.4.

$$\begin{aligned}&(\exists c(a)) \lor (\exists c(b))\\ \iff\ &\exists c(a \lor b)\end{aligned}$$

Proposition 1.5.0.5.

$$\begin{aligned}&P \lor (\forall x(Q(x)))\\ \iff\ &\forall x(P \lor (Q(x)))\end{aligned}$$

Proposition 1.5.0.6.

$$\begin{aligned}&P \land (\exists x(Q(x)))\\ \iff\ &\exists x(P \land (Q(x)))\end{aligned}$$

Axiom 1.1.

$$\begin{aligned}&\forall x(P(y))\\ \iff\ &P(y)\end{aligned}$$

Axiom 1.2.

$$\begin{aligned}&\exists x(P(y))\\ \iff\ &P(y)\end{aligned}$$

Proposition 1.5.0.7. De Morgan's law

$$\begin{aligned} &\neg(\forall b(a)) \\ \iff &\exists b(\neg a) \end{aligned}$$

Proposition 1.5.0.8. De Morgan's law

$$\begin{aligned} &\neg(\exists b(a)) \\ \iff &\forall b(\neg a) \end{aligned}$$

Definition 1.5.0.9. Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned} &!\exists x(P(x)) \\ \stackrel{\text{def}}{\iff} &(\exists x(P(x))) \land (\forall x \forall y((P(x) \land P(y)) \implies (x = y))) \end{aligned}$$

Axiom 1.3. Axiom of Substitution

$$\forall x((\exists y((y = x) \land P(y))) \iff P(x))$$

1.6 Logic proposition

Proposition 1.6.0.1.

$$\begin{aligned} &a \land b \\ \implies &a \iff b \end{aligned}$$

Proposition 1.6.0.2.

$$\begin{aligned} &a \land b \\ \implies &a \end{aligned}$$

Proposition 1.6.0.3.

$$\begin{aligned} &a \land ((b \land a) \implies c) \\ \implies &b \implies c \end{aligned}$$

Proposition 1.6.0.4.

$$\begin{aligned} &a \land (a \implies b) \\ \implies &b \end{aligned}$$

Proposition 1.6.0.5.

$$\begin{aligned} &a \land (a \iff b) \\ \implies &b \end{aligned}$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

Definition 2.0.0.1. Definition of $\notin$.

$$\begin{aligned}
&\forall a \forall b(\\
&\qquad a \notin b \\
&\qquad \overset{\text{def}}{\iff} \neg(a \in b) \\
&)
\end{aligned}$$

Definition 2.0.0.2.

$$\begin{aligned}
&\forall a \in S, P(a) \\
\overset{\text{def}}{\iff} &\forall a((a \in S) \implies (P(a)))
\end{aligned}$$

Definition 2.0.0.3.

$$\begin{aligned}
&\exists a \in S, P(a) \\
\overset{\text{def}}{\iff} &\exists a((a \in S) \land (P(a)))
\end{aligned}$$

Proposition 2.0.0.4.

$$\begin{aligned}
&\forall a(\text{True}) \\
\iff &\text{True}
\end{aligned}$$

2.1 Equality of sets

Definition 2.1.0.1. Definition of $=$.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \overset{\text{def}}{\iff} \forall c((c \in a) \iff (c \in b)) \\
&)
\end{aligned}
$$

Definition 2.1.0.2. Definition of $\neq$.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a \neq b \\
&\qquad \overset{\text{def}}{\iff} \neg(a = b) \\
&)
\end{aligned}
$$

Proposition 2.1.0.3. Reflexive property of equality.

$$
\begin{aligned}
&\forall a(\\
&\qquad a = a \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.3

$$
\begin{aligned}
&\forall a(\\
&\qquad a = a \\
&\qquad \overset{\text{def}}{\iff} \forall b((b \in a) \iff (b \in a)) && \text{Definition 2.1.0.1} \\
&\qquad \iff \forall b(\text{True}) && \text{Proposition 1.4.0.12} \\
&\qquad \iff \text{True} && \text{Proposition 2.0.0.4} \\
&)
\end{aligned}
$$

Proposition 2.1.0.4. Symmetric property of equality.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \iff b = a \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.4

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \overset{\text{def}}{\iff} \forall c((c \in a) \iff (c \in b)) && \text{Definition 2.1.0.1} \\
&\qquad \iff \forall c((c \in b) \iff (c \in a)) && \text{Proposition 1.4.0.9} \\
&\qquad \overset{\text{def}}{\iff} b = a && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proposition 2.1.0.5. Transitive property of equality.

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad (a = b) \land (b = c) \\
&\qquad \implies a = c \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.5

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad (a = b) \land (b = c) \\
&\qquad \overset{\text{def}}{\iff} (\forall d((d \in a) \iff (d \in b))) \land (b = c) && \text{Definition 2.1.0.1} \\
&\qquad \overset{\text{def}}{\iff} (\forall d((d \in a) \iff (d \in b))) \land (\forall d((d \in b) \iff (d \in c))) && \text{Definition 2.1.0.1} \\
&\qquad \iff \forall d(((d \in a) \iff (d \in b)) \land ((d \in b) \iff (d \in c))) && \text{Proposition 1.5.0.3} \\
&\qquad \implies \forall d((d \in a) \iff (d \in c)) && \text{Proposition 1.4.0.11} \\
&\qquad \overset{\text{def}}{\iff} a = c && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Axiom 2.1. Axiom of extensionality

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \implies \forall c((a \in c) \iff (b \in c)) \\
&)
\end{aligned}
$$

Axiom 2.2. Substitution of $\in$.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \exists c((c = b) \land (a \in c)) \\
&\qquad \iff a \in b \\
&)
\end{aligned}
$$

Axiom 2.3. Substitution of $=$.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \iff \forall c((c = a) \iff (c = b)) \\
&)
\end{aligned}
$$

Axiom 2.4. Existence of empty set

$$
\begin{aligned}
&\forall a(\\
&\qquad a \notin \emptyset \\
&)
\end{aligned}
$$

Proposition 2.1.0.6. Uniqueness of $\emptyset$

$$
\begin{aligned}
&\forall a(\\
&\qquad \forall b(b \notin a) \\
&\qquad \iff a = \emptyset \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.6

$$
\begin{aligned}
&\forall a(\\
&\qquad \forall b(b \notin a) \\
&\qquad \iff \forall b((b \notin a) \iff (\text{True})) && \text{Proposition 1.4.0.1} \\
&\qquad \iff \forall b((b \notin a) \iff (b \notin \emptyset)) && \text{Axiom 2.4} \\
&\qquad \overset{\text{def}}{\iff} \forall b((\neg(b \in a)) \iff (b \notin \emptyset)) && \text{Definition 2.0.0.1} \\
&\qquad \overset{\text{def}}{\iff} \forall b((\neg(b \in a)) \iff (\neg(b \in \emptyset))) && \text{Definition 2.0.0.1} \\
&\qquad \iff \forall b((b \in a) \iff (b \in \emptyset)) && \text{Proposition 1.4.0.10} \\
&\qquad \overset{\text{def}}{\iff} a = \emptyset && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proposition 2.1.0.7. Single choice

$$
\begin{aligned}
&\forall a(\\
&\qquad a \neq \emptyset \\
&\qquad \iff \exists b(b \in a) \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.7

$$
\begin{aligned}
&\forall a(\\
&\qquad \text{True} \\
&\qquad \iff (\forall b(b \notin a)) \iff (a = \emptyset) && \text{Proposition 2.1.0.6} \\
&\qquad \iff (a = \emptyset) \iff (\forall b(b \notin a)) && \text{Proposition 1.4.0.9} \\
&\qquad \iff (\neg(a = \emptyset)) \iff (\neg(\forall b(b \notin a))) && \text{Proposition 1.4.0.10} \\
&\qquad \overset{\text{def}}{\iff} (a \neq \emptyset) \iff (\neg(\forall b(b \notin a))) && \text{Definition 2.1.0.2} \\
&\qquad \iff (a \neq \emptyset) \iff (\exists b(\neg(b \notin a))) && \text{Proposition 1.5.0.7} \\
&\qquad \overset{\text{def}}{\iff} (a \neq \emptyset) \iff (\exists b(\neg(\neg(b \in a)))) && \text{Definition 2.0.0.1} \\
&\qquad \iff (a \neq \emptyset) \iff (\exists b(b \in a)) && \text{Proposition 1.2.17.1} \\
&)
\end{aligned}
$$

Axiom 2.5. Existence of pair set

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad c \in \{a, b\} \\
&\qquad \iff (c = a) \vee (c = b) \\
&)
\end{aligned}
$$

Proposition 2.1.0.8. Uniqueness of pair set

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad \forall d((d \in c) \iff ((d = a) \vee (d = b))) \\
&\qquad \implies c = \{a, b\} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.8

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad \forall d((d \in c) \iff ((d = a) \vee (d = b))) \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \vee (d = b))) \wedge (\text{True})) && \text{Proposition 1.2.6.1} \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \vee (d = b))) \\
&\qquad\qquad \wedge ((d \in \{a, b\}) \iff ((d = a) \vee (d = b)))) && \text{Axiom 2.5} \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \vee (d = b))) \\
&\qquad\qquad \wedge (((d = a) \vee (d = b)) \iff (d \in \{a, b\}))) && \text{Proposition 1.4.0.9} \\
&\qquad \implies \forall d((d \in c) \iff (d \in \{a, b\})) && \text{Proposition 1.4.0.11} \\
&\qquad \overset{\text{def}}{\iff} c = \{a, b\} && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proposition 2.1.0.9. Substitution for pair set.

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad a = b \\
&\qquad \Longrightarrow \{a, c\} = \{b, c\} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.9

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad a = b \\
&\qquad \Longleftrightarrow \forall d((d = a) \iff (d = b)) && \text{Axiom 2.3} \\
&\qquad \Longrightarrow \forall d(((d = a) \vee (d = c)) \iff ((d = b) \vee (d = c))) && \text{Proposition 1.4.0.7} \\
&\qquad \Longleftrightarrow \forall d((d \in \{a, c\}) \iff ((d = b) \vee (d = c))) && \text{Axiom 2.5} \\
&\qquad \Longleftrightarrow \forall d((d \in \{a, c\}) \iff (d \in \{b, c\})) && \text{Axiom 2.5} \\
&\qquad \overset{\text{def}}{\Longleftrightarrow} \{a, c\} = \{b, c\} && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Definition 2.1.0.10. Definition of singleton set.

$$
\begin{aligned}
&\forall a(\\
&\qquad \{a\} \overset{\text{def}}{=} \{a, a\} \\
&)
\end{aligned}
$$

Proposition 2.1.0.11. Property of singleton set.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad b \in \{a\} \\
&\qquad \Longleftrightarrow b = a \\
&)
\end{aligned}
$$

Proposition (1)

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad b \in \{a\} \\
&\qquad \Longleftrightarrow b \in \{a, a\} \\
&)
\end{aligned}
$$

Proof of Proposition (1)

$$
\begin{aligned}
&\forall a(\\
&\qquad \text{True} \\
&\qquad \iff \{a\} = \{a, a\} && \text{Definition 2.1.0.10} \\
&\qquad \overset{\text{def}}{\iff} \forall b((b \in \{a\}) \iff (b \in \{a, a\})) && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.11

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad b \in \{a\} \\
&\qquad \iff b \in \{a, a\} && \text{Proposition (1)} \\
&\qquad \iff (b = a) \vee (b = a) && \text{Axiom 2.5} \\
&\qquad \iff b = a && \text{Proposition 1.2.9.1} \\
&)
\end{aligned}
$$

Proposition 2.1.0.12. Uniqueness of singleton set.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \forall c((c \in b) \iff (c = a)) \\
&\qquad \implies b = \{a\} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.12

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \forall c((c \in b) \iff (c = a)) \\
&\qquad \iff \forall c(((c \in b) \iff (c = a)) \wedge (\text{True})) && \text{Proposition 1.2.6.1} \\
&\qquad \iff \forall c(((c \in b) \iff (c = a)) \\
&\qquad\qquad \wedge ((c \in \{a\}) \iff (c = a))) && \text{Proposition 2.1.0.11} \\
&\qquad \iff \forall c(((c \in b) \iff (c = a)) \\
&\qquad\qquad \wedge ((c = a) \iff (c \in \{a\}))) && \text{Proposition 1.4.0.9} \\
&\qquad \implies \forall c((c \in b) \iff (c \in \{a\})) && \text{Proposition 1.4.0.11} \\
&\qquad \overset{\text{def}}{\iff} b = \{a\} && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proposition 2.1.0.13. Substitution for singleton set.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad a = b \\
&\qquad \Longrightarrow \{a\} = \{b\} \\
&)
\end{aligned}
$$

Axiom 2.6. Existence of union set.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad b \in (\bigcup a) \\
&\qquad \Longleftrightarrow \exists c((b \in c) \wedge (c \in a)) \\
&)
\end{aligned}
$$

Proposition 2.1.0.14. Uniqueness of union set.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \forall c((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
&\qquad \Longrightarrow b = (\bigcup a) \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.14

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \forall c((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
&\qquad \Longleftrightarrow \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \wedge (\mathrm{True})) && \text{Proposition 1.2.6.1} \\
&\qquad \Longleftrightarrow \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
&\qquad\qquad \wedge ((c \in (\bigcup a)) \iff (\exists d((c \in d) \wedge (d \in a))))) && \text{Axiom 2.6} \\
&\qquad \Longleftrightarrow \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
&\qquad\qquad \wedge ((\exists d((c \in d) \wedge (d \in a))) \iff (c \in (\bigcup a)))) && \text{Proposition 1.4.0.9} \\
&\qquad \Longrightarrow \forall c((c \in b) \iff (c \in (\bigcup a))) && \text{Proposition 1.4.0.11} \\
&\qquad \overset{\text{def}}{\Longleftrightarrow} b = (\bigcup a) && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Definition 2.1.0.15. Definition of pairwise union $a \cup b$.

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad (a \cup b) \overset{\text{def}}{=} (\bigcup \{a, b\}) \\
&)
\end{aligned}
$$

Proposition 2.1.0.16. Property of pairwise union.

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad c \in (a \cup b) \\
&\qquad \iff (c \in a) \lor (c \in b) \\
&)
\end{aligned}
$$

Proposition (1)

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad c \in (a \cup b) \\
&\qquad \iff c \in (\bigcup \{a, b\}) \\
&)
\end{aligned}
$$

Proof of Proposition (1)

$$
\begin{aligned}
&\forall a \forall b(\\
&\qquad \text{True} \\
&\qquad \iff (a \cup b) = (\bigcup \{a, b\}) && \text{Definition 2.1.0.15} \\
&\qquad \overset{\text{def}}{\iff} \forall c((c \in (a \cup b)) \iff (c \in (\bigcup \{a, b\}))) && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.16

$$
\begin{aligned}
&\forall a \forall b \forall c(\\
&\qquad c \in (a \cup b) \\
&\qquad \iff c \in (\bigcup \{a, b\}) && \text{Proposition (1)} \\
&\qquad \iff \exists d((c \in d) \land (d \in \{a, b\})) && \text{Axiom 2.6} \\
&\qquad \iff \exists d((c \in d) \land ((d = a) \lor (d = b))) && \text{Axiom 2.5} \\
&\qquad \iff \exists d(((c \in d) \land (d = a)) \lor ((c \in d) \land (d = b))) && \text{Proposition 1.2.16.1} \\
&\qquad \iff (\exists d((c \in d) \land (d = a))) \lor (\exists d((c \in d) \land (d = b))) && \text{Proposition 1.5.0.4} \\
&\qquad \iff (\exists d((d = a) \land (c \in d))) \lor (\exists d((c \in d) \land (d = b))) && \text{Proposition 1.2.4.1} \\
&\qquad \iff (\exists d((d = a) \land (c \in d))) \lor (\exists d((d = b) \land (c \in d))) && \text{Proposition 1.2.4.1} \\
&\qquad \iff (c \in a) \lor (\exists d((d = b) \land (c \in d))) && \text{Axiom 2.2} \\
&\qquad \iff (c \in a) \lor (c \in b) && \text{Axiom 2.2} \\
&)
\end{aligned}
$$

Proposition 2.1.0.17. Substitution for pairwise union.

$$\forall a \forall b \forall c($$

$$a = b$$

$$\Longrightarrow (a \cup c) = (b \cup c)$$

$$)$$

Proposition 2.1.0.18. Commutativity of $\cup$.

$$\forall a \forall b($$

$$(a \cup b) = (b \cup a)$$

$$)$$

Proof of Proposition 2.1.0.18

$\forall a \forall b($

$$\begin{aligned}
& (a \cup b) = (b \cup a) \\
& \overset{\text{def}}{\Longleftrightarrow} \forall c((c \in (a \cup b)) \iff (c \in (b \cup a))) && \text{Definition 2.1.0.1} \\
& \Longleftrightarrow \forall c(((c \in a) \vee (c \in b)) \iff (c \in (b \cup a))) && \text{Proposition 2.1.0.16} \\
& \Longleftrightarrow \forall c(((c \in a) \vee (c \in b)) \iff ((c \in b) \vee (c \in a))) && \text{Proposition 2.1.0.16} \\
& \Longleftrightarrow \forall c(((c \in a) \vee (c \in b)) \iff ((c \in a) \vee (c \in b))) && \text{Proposition 1.2.3.1} \\
& \Longleftrightarrow \forall c(\text{True}) && \text{Proposition 1.4.0.12} \\
& \Longleftrightarrow \text{True} && \text{Proposition 2.0.0.4}
\end{aligned}$$

$)$

Proposition 2.1.0.19. Identity of $\cup$.

$$\forall a($$

$$(a \cup \emptyset) = a$$

$$)$$

Proof of Proposition 2.1.0.19

$$
\begin{array}{lll}
\forall a(& & \\
& (a \cup \emptyset) = a & \\
& \stackrel{\text{def}}{\iff} \forall b((b \in (a \cup \emptyset)) \iff (b \in a)) & \text{Definition 2.1.0.1} \\
& \iff \forall b(((b \in a) \lor (b \in \emptyset)) \iff (b \in a)) & \text{Proposition 2.1.0.16} \\
& \iff \forall b(((b \in a) \lor (\neg(\neg(b \in \emptyset)))) \iff (b \in a)) & \text{Proposition 1.2.17.1} \\
& \stackrel{\text{def}}{\iff} \forall b(((b \in a) \lor (\neg(b \notin \emptyset))) \iff (b \in a)) & \text{Definition 2.0.0.1} \\
& \iff \forall b(((b \in a) \lor (\neg(\text{True}))) \iff (b \in a)) & \text{Axiom 2.4} \\
& \stackrel{\text{def}}{\iff} \forall b(((b \in a) \lor (\text{False})) \iff (b \in a)) & \text{Definition 1.1.1.1} \\
& \iff \forall b((b \in a) \iff (b \in a)) & \text{Proposition 1.2.5.1} \\
& \iff \forall b(\text{True}) & \text{Proposition 1.4.0.12} \\
& \iff \text{True} & \text{Proposition 2.0.0.4} \\
) & &
\end{array}
$$

Definition 2.1.0.20. Definition of 0.

$$0 \stackrel{\text{def}}{=} \emptyset$$

Definition 2.1.0.21. Definition of successor $S(x)$.

$$
\begin{array}{l}
\forall a(\\
\qquad (S(a)) \stackrel{\text{def}}{=} (a \cup \{a\}) \\
)
\end{array}
$$

Definition 2.1.0.22. Definition of 1.

$$1 \stackrel{\text{def}}{=} (S(0))$$

Proposition 2.1.0.23. Express 1 in term of $\emptyset$.

$$1 = \{\emptyset\}$$

Proof of Proposition 2.1.0.23

True	
$\iff 1 = (S(0))$	Definition 2.1.0.22
$\iff (1 = (S(0))) \wedge (\text{True})$	Proposition 1.2.6.1
$\iff (1 = (S(0))) \wedge ((S(0)) = (0 \cup \{0\}))$	Definition 2.1.0.21
$\implies 1 = (0 \cup \{0\})$	Proposition 2.1.0.5
$\iff (1 = (0 \cup \{0\})) \wedge (\text{True})$	Proposition 1.2.6.1
$\iff (1 = (0 \cup \{0\})) \wedge (0 = \emptyset)$	Definition 2.1.0.20
$\implies (1 = (0 \cup \{0\})) \wedge ((0 \cup \{0\}) = (\emptyset \cup \{0\}))$	Proposition 2.1.0.17
$\implies 1 = (\emptyset \cup \{0\})$	Proposition 2.1.0.5
$\iff (1 = (\emptyset \cup \{0\})) \wedge (\text{True})$	Proposition 1.2.6.1
$\iff (1 = (\emptyset \cup \{0\})) \wedge ((\emptyset \cup \{0\}) = (\{0\} \cup \emptyset))$	Proposition 2.1.0.18
$\implies 1 = (\{0\} \cup \emptyset)$	Proposition 2.1.0.5
$\iff (1 = (\{0\} \cup \emptyset)) \wedge (\text{True})$	Proposition 1.2.6.1
$\iff (1 = (\{0\} \cup \emptyset)) \wedge ((\{0\} \cup \emptyset) = \{0\})$	Proposition 2.1.0.19
$\implies 1 = \{0\}$	Proposition 2.1.0.5
$\iff (1 = \{0\}) \wedge (\text{True})$	Proposition 1.2.6.1
$\iff (1 = \{0\}) \wedge (0 = \emptyset)$	Definition 2.1.0.20
$\implies (1 = \{0\}) \wedge (\{0\} = \{\emptyset\})$	Proposition 2.1.0.13
$\implies 1 = \{\emptyset\}$	Proposition 2.1.0.5