# Chapter 1

# Logic

**Definition 1.0.1.** Proposition is a statement that is either true or false, but not both.

## 1.1 Logical operations

**Definition 1.1.1.** Definition of $\neg$.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

**Definition 1.1.2.** Definition of $\wedge$.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Definition 1.1.3.** Definition of $\vee$.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Definition 1.1.4.** Definition of $\iff$.

| $p$ | $q$ | $p \iff q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Definition 1.1.5.** Definition of $\implies$.

$$\begin{aligned} & p \implies q \\ \iff & (\neg p) \vee q \end{aligned}$$

## 1.2 Quantifiers

**Definition 1.2.1.** Universal quantifier is denoted by $\forall$.

$$\forall x, P(x)$$

**Definition 1.2.2.** Existential quantifier is denoted by $\exists$.

$$\exists x, P(x)$$

**Axiom 1.1.**

$$\forall x, (P(x) \wedge Q(x)) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

**Axiom 1.2.** De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

**Axiom 1.3.** De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

**Definition 1.2.3.** Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned} & !\exists x, P(x) \\ \iff & (\exists x, P(x)) \wedge (\forall x \forall y (P(x) \wedge P(y) \implies x = y)) \end{aligned}$$

## 1.3 Proof technique

## 1.4 Proposition

Let $P = P(x_1, x_2, \ldots, x_n)$. Let $Q = Q(x_1, x_2, \ldots, x_n)$. etc

**Proposition 1.4.1.** Double negation

$$\neg(\neg P) \iff P$$

**Proposition 1.4.2.** Reflexive property of iff.

$$P \iff P$$

Proof:

| $P$ | $P \iff P$ |
|---|---|
| T | T |
| F | T |

**Proposition 1.4.3.** Symmetric property of iff.

$$(P \iff Q) \iff (Q \iff P)$$

**Proposition 1.4.4.** Transitive property of iff.

$$((P \iff Q) \land (Q \iff R)) \implies (P \iff R)$$

**Proposition 1.4.5.** De Morgan's law

$$\neg(P \land Q) \iff (\neg P) \lor (\neg Q)$$

**Proposition 1.4.6.** De Morgan's law

$$\neg(P \lor Q) \iff (\neg P) \land (\neg Q)$$

**Proposition 1.4.7.**

$$(P \land Q) \implies (P \iff Q)$$

**Proposition 1.4.8.**

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

# Chapter 2

# Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

**Definition 2.0.1.** Definition of $\notin$.

$$\begin{aligned} & A \notin B \\ \iff & \neg(A \in B) \end{aligned}$$

**Definition 2.0.2.**

$$\begin{aligned} & \forall x \in S, P(x) \\ \iff & \forall x(x \in S \implies P(x)) \end{aligned}$$

**Definition 2.0.3.**

$$\begin{aligned} & \exists x \in S, P(x) \\ \iff & \exists x(x \in S \wedge P(x)) \end{aligned}$$

**Proposition 2.0.4.**

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\forall x \in S, P(x)) & \\
\iff & \neg(\forall x(x \in S \implies P(x))) & \text{Definition 2.0.2} \\
\iff & \neg(\forall x(\neg(x \in S) \lor P(x))) & \text{Definition 1.1.5} \\
\iff & \exists x, \neg(\neg(x \in S) \lor P(x)) & \text{Axiom 1.2} \\
\iff & \exists x, \neg(\neg(x \in S)) \land \neg(P(x)) & \text{Proposition 1.4.6} \\
\iff & \exists x, x \in S \land \neg(P(x)) & \text{Proposition 1.4.1} \\
\iff & \exists x \in S, \neg(P(x)) & \text{Definition 2.0.3}
\end{aligned}
$$

**Proposition 2.0.5.**

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\exists x \in S, P(x)) & \\
\iff & \neg(\exists x(x \in S \land P(x))) & \text{Definition 2.0.3} \\
\iff & \forall x, \neg(x \in S \land P(x)) & \text{Axiom 1.3} \\
\iff & \forall x, (\neg(x \in S)) \lor (\neg(P(x)) & \text{Proposition 1.4.5} \\
\iff & \forall x, x \in S \implies \neg(P(x)) & \text{Definition 1.1.5} \\
\iff & \forall x \in S, \neg(P(x)) & \text{Definition 2.0.2}
\end{aligned}
$$

## 2.1 Equality of sets

**Definition 2.1.1.** Definition of $=$.

$$
\begin{aligned}
& A = B \\
\iff & \forall x(x \in A \iff x \in B)
\end{aligned}
$$

**Definition 2.1.2.** Definition of $\neq$.

$$
\begin{aligned}
& A \neq B \\
\iff & \neg(A = B)
\end{aligned}
$$

**Proposition 2.1.3.** Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$\begin{array}{lll}
\forall x( & & \\
& x = x & \\
\iff & \forall y(y \in x \iff y \in x) & \text{Definition 2.1.1} \\
\iff & \text{True} & \text{Proposition 1.4.2} \\
) & &
\end{array}$$

**Proposition 2.1.4.** Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$\begin{array}{lll}
\forall x \forall y( & & \\
& x = y & \\
\implies & \forall z(z \in x \iff z \in y) & \text{Definition 2.1.1} \\
\implies & \forall z(z \in y \iff z \in x) & \text{Proposition 1.4.3} \\
\implies & y = x & \text{Definition 2.1.1} \\
) & &
\end{array}$$

**Proposition 2.1.5.** Transitive property of equality

$$\forall x \forall y \forall z((x = y) \wedge (y = z) \implies (x = z))$$

Proof:

$$\begin{array}{lll}
\forall x \forall y \forall z( & & \\
& (x = y) \wedge (y = z) & \\
\implies & (\forall w(w \in x \iff w \in y)) \wedge (\forall w(w \in y \iff w \in z)) & \text{Definition 2.1.1} \\
\implies & \forall w((w \in x \iff w \in y) \wedge (w \in y \iff w \in z)) & \text{Axiom 1.1} \\
\implies & \forall w(w \in x \iff w \in z) & \text{Proposition 1.4.4} \\
\implies & x = z & \text{Definition 2.1.1} \\
) & &
\end{array}$$

**Axiom 2.1.** Axiom of Substitution

$$\begin{array}{l}
\forall x \forall y( \\
x = y \implies \forall A(x \in A \iff y \in A) \\
)
\end{array}$$

**Axiom 2.2.** Existence of empty set

$$\exists x \forall y (y \notin x)$$

**Proposition 2.1.6.** Uniqueness of empty set

$$!\exists x \forall y (y \notin x)$$

Proof:
Let $P(x) = \forall y (y \notin x)$

$$\begin{aligned} & \exists x \forall y (y \notin x) && \text{Axiom 2.2} \\ \implies & \exists x, P(x) && \text{Definition of P(x)} \end{aligned}$$

$$\begin{aligned} \forall x \forall y ( \\ & & P(x) \wedge P(y) \\ & \implies & (\forall z (z \notin x)) \wedge (\forall z (z \notin y)) && \text{Definition of P(x)} \\ & \implies & \forall z ((z \notin x) \wedge (z \notin y)) && \text{Axiom 1.1} \\ & \implies & \forall z (z \notin x \iff z \notin y) && \text{Proposition 1.4.7} \\ & \implies & \forall z (\neg (z \in x) \iff \neg (z \in y)) && \text{Definition 2.0.1} \\ & \implies & \forall z (z \in x \iff z \in y) && \text{Proposition 1.4.8} \\ & \implies & x = y && \text{Definition 2.1.1} \\ ) \end{aligned}$$

$$\begin{aligned} & (\exists x, P(x)) \wedge \forall x \forall y ((P(x) \wedge P(y)) \implies (x = y)) \\ \implies & !\exists x, P(x) && \text{Definition 1.2.3} \\ \implies & !\exists x \forall y (y \notin x) && \text{Definition of P(x)} \end{aligned}$$