Chapter 1

Logic

Definition 1.0.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

1.1.1 Definition of $\neg$

Definition 1.1.1.1.

$$\begin{aligned} &\neg(\mathrm{True}) \\ \overset{\mathrm{def}}{\iff} &\mathrm{False} \end{aligned}$$

Definition 1.1.1.2.

$$\begin{aligned} &\neg(\mathrm{False}) \\ \overset{\mathrm{def}}{\iff} &\mathrm{True} \end{aligned}$$

1.1.2 Definition of $\vee$

Definition 1.1.2.1.

$$\begin{aligned} &(\mathrm{True}) \vee (\mathrm{True}) \\ \overset{\mathrm{def}}{\iff} &\mathrm{True} \end{aligned}$$

Definition 1.1.2.2.

$$\begin{aligned} &(\mathrm{True}) \vee (\mathrm{False}) \\ \overset{\mathrm{def}}{\iff} &\mathrm{True} \end{aligned}$$

Definition 1.1.2.3.

$$\begin{aligned} & (\text{False}) \vee (\text{True}) \\ \stackrel{\text{def}}{\iff} & \text{True} \end{aligned}$$

Definition 1.1.2.4.

$$\begin{aligned} & (\text{False}) \vee (\text{False}) \\ \stackrel{\text{def}}{\iff} & \text{False} \end{aligned}$$

1.1.3 Definition of $\wedge$

Definition 1.1.3.1.

$$\begin{aligned} & (\text{True}) \wedge (\text{True}) \\ \stackrel{\text{def}}{\iff} & \text{True} \end{aligned}$$

Definition 1.1.3.2.

$$\begin{aligned} & (\text{True}) \wedge (\text{False}) \\ \stackrel{\text{def}}{\iff} & \text{False} \end{aligned}$$

Definition 1.1.3.3.

$$\begin{aligned} & (\text{False}) \wedge (\text{True}) \\ \stackrel{\text{def}}{\iff} & \text{False} \end{aligned}$$

Definition 1.1.3.4.

$$\begin{aligned} & (\text{False}) \wedge (\text{False}) \\ \stackrel{\text{def}}{\iff} & \text{False} \end{aligned}$$

1.1.4 Definition of $\iff$

Definition 1.1.4.1.

$$\begin{aligned} & x \iff y \\ \stackrel{\text{def}}{\iff} & (x \wedge y) \vee ((\neg x) \wedge (\neg y)) \end{aligned}$$

1.1.5 Definition of $\implies$

Definition 1.1.5.1.

$$\begin{aligned} & x \implies y \\ \overset{\text{def}}{\iff} & (\neg x) \vee y \end{aligned}$$

1.2 Boolean algebra

1.2.1 Associativity of $\vee$

Proposition 1.2.1.1.

$$\begin{aligned} & (x \vee y) \vee z \\ \iff & x \vee (y \vee z) \end{aligned}$$

1.2.2 Associativity of $\wedge$

Proposition 1.2.2.1.

$$\begin{aligned} & (x \wedge y) \wedge z \\ \iff & x \wedge (y \wedge z) \end{aligned}$$

1.2.3 Commutativity of $\vee$

Proposition 1.2.3.1.

$$\begin{aligned} & x \vee y \\ \iff & y \vee x \end{aligned}$$

1.2.4 Commutativity of $\wedge$

Proposition 1.2.4.1.

$$\begin{aligned} & x \wedge y \\ \iff & y \wedge x \end{aligned}$$

1.2.5 Identity of $\lor$

Proposition 1.2.5.1.

$$\begin{aligned} & x \lor (\text{False}) \\ \iff & x \end{aligned}$$

Proposition 1.2.5.2.

$$\begin{aligned} & (\text{False}) \lor x \\ \iff & x \end{aligned}$$

1.2.6 Identity of $\land$

Proposition 1.2.6.1.

$$\begin{aligned} & x \land (\text{True}) \\ \iff & x \end{aligned}$$

Proposition 1.2.6.2.

$$\begin{aligned} & (\text{True}) \land x \\ \iff & x \end{aligned}$$

1.2.7 Annihilator of $\lor$

Proposition 1.2.7.1.

$$\begin{aligned} & x \lor (\text{True}) \\ \iff & \text{True} \end{aligned}$$

Proposition 1.2.7.2.

$$\begin{aligned} & (\text{True}) \lor x \\ \iff & \text{True} \end{aligned}$$

1.2.8 Annihilator of $\land$

Proposition 1.2.8.1.

$$\begin{aligned} & x \land (\text{False}) \\ \iff & \text{False} \end{aligned}$$

Proposition 1.2.8.2.

$$\begin{aligned} & (\text{False}) \land x \\ \iff & \text{False} \end{aligned}$$

1.2.9 Idempotence of $\vee$

Proposition 1.2.9.1.

$$\begin{aligned} & x \vee x \\ \iff & x \end{aligned}$$

1.2.10 Idempotence of $\wedge$

Proposition 1.2.10.1.

$$\begin{aligned} & x \wedge x \\ \iff & x \end{aligned}$$

1.2.11 Complement of $\vee$

Proposition 1.2.11.1.

$$\begin{aligned} & x \vee (\neg x) \\ \iff & \text{True} \end{aligned}$$

Proposition 1.2.11.2.

$$\begin{aligned} & (\neg x) \vee x \\ \iff & \text{True} \end{aligned}$$

1.2.12 Complement of $\wedge$

Proposition 1.2.12.1.

$$\begin{aligned} & x \wedge (\neg x) \\ \iff & \text{False} \end{aligned}$$

Proposition 1.2.12.2.

$$\begin{aligned} & (\neg x) \wedge x \\ \iff & \text{False} \end{aligned}$$

1.2.13 Absorption of $\vee$ over $\wedge$

Proposition 1.2.13.1.

$$\begin{aligned} & x \vee (x \wedge y) \\ \iff & x \end{aligned}$$

Proposition 1.2.13.2.

$$\begin{aligned} & x \vee (y \wedge x) \\ \iff & x \end{aligned}$$

Proposition 1.2.13.3.

$$\begin{aligned} & (x \wedge y) \vee x \\ \iff & x \end{aligned}$$

Proposition 1.2.13.4.

$$\begin{aligned} & (y \wedge x) \vee x \\ \iff & x \end{aligned}$$

1.2.14 Absorption of $\wedge$ over $\vee$

Proposition 1.2.14.1.

$$\begin{aligned} & x \wedge (x \vee y) \\ \iff & x \end{aligned}$$

Proposition 1.2.14.2.

$$\begin{aligned} & x \wedge (y \vee x) \\ \iff & x \end{aligned}$$

Proposition 1.2.14.3.

$$\begin{aligned} & (x \vee y) \wedge x \\ \iff & x \end{aligned}$$

Proposition 1.2.14.4.

$$\begin{aligned} & (y \vee x) \wedge x \\ \iff & x \end{aligned}$$

1.2.15 Distributivity of $\vee$ over $\wedge$

Proposition 1.2.15.1.

$$
\begin{aligned}
& x \vee (y \wedge z) \\
\iff & (x \vee y) \wedge (x \vee z)
\end{aligned}
$$

Proposition 1.2.15.2.

$$
\begin{aligned}
& (x \wedge y) \vee z \\
\iff & (x \vee z) \wedge (y \vee z)
\end{aligned}
$$

1.2.16 Distributivity of $\wedge$ over $\vee$

Proposition 1.2.16.1.

$$
\begin{aligned}
& x \wedge (y \vee z) \\
\iff & (x \wedge y) \vee (x \wedge z)
\end{aligned}
$$

Proposition 1.2.16.2.

$$
\begin{aligned}
& (x \vee y) \wedge z \\
\iff & (x \wedge z) \vee (y \wedge z)
\end{aligned}
$$

1.2.17 Double negation

Proposition 1.2.17.1.

$$
\begin{aligned}
& \neg(\neg x) \\
\iff & x
\end{aligned}
$$

1.2.18 De Morgan's laws

Proposition 1.2.18.1.

$$
\begin{aligned}
& \neg(x \vee y) \\
\iff & (\neg x) \wedge (\neg y)
\end{aligned}
$$

Proposition 1.2.18.2.

$$
\begin{aligned}
& \neg(x \wedge y) \\
\iff & (\neg x) \vee (\neg y)
\end{aligned}
$$

1.3 Basic Proposition

Proposition 1.3.0.1.

$$\begin{aligned} & (x \wedge (\neg y)) \vee y \\ \iff & x \vee y \end{aligned}$$

Proof of Proposition 1.3.0.1

$$\begin{aligned} & (x \wedge (\neg y)) \vee y && \\ \iff & (x \vee y) \wedge ((\neg y) \vee y) && \text{Proposition 1.2.15.2} \\ \iff & (x \vee y) \wedge (\text{True}) && \text{Proposition 1.2.11.2} \\ \iff & x \vee y && \text{Proposition 1.2.6.1} \end{aligned}$$

1.4 Proof technique

Proposition 1.4.0.1.

$$(x \iff (\text{True})) \iff x$$

Proof:

$$\begin{aligned} & x \iff (\text{True}) && \\ \iff & (x \wedge (\text{True})) \vee ((\neg x) \wedge (\neg(\text{True}))) && \text{Definition 1.1.4.1} \\ \iff & (x \wedge (\text{True})) \vee ((\neg x) \wedge (\text{False})) && \text{Definition 1.1.1.1} \\ \iff & x \vee ((\neg x) \wedge (\text{False})) && \text{Proposition 1.2.6.1} \\ \iff & x \vee (\text{False}) && \text{Proposition 1.2.8.1} \\ \iff & x && \text{Proposition 1.2.5.1} \end{aligned}$$

Proposition 1.4.0.2.

$$(x \implies y) \implies ((x \vee z) \implies (y \vee z))$$

Proof:

$$
\begin{aligned}
& (x \implies y) \implies ((x \lor z) \implies (y \lor z)) & \\
\iff & ((\neg x) \lor y) \implies ((x \lor z) \implies (y \lor z)) & \text{Definition 1.1.5.1} \\
\iff & ((\neg x) \lor y) \implies ((\neg(x \lor z)) \lor (y \lor z)) & \text{Definition 1.1.5.1} \\
\iff & (\neg((\neg x) \lor y)) \lor ((\neg(x \lor z)) \lor (y \lor z)) & \text{Definition 1.1.5.1} \\
\iff & ((\neg(\neg x)) \land (\neg y)) \lor ((\neg(x \lor z)) \lor (y \lor z)) & \text{Proposition 1.2.18.1} \\
\iff & (x \land (\neg y)) \lor ((\neg(x \lor z)) \lor (y \lor z)) & \text{Proposition 1.2.17.1} \\
\iff & (x \land (\neg y)) \lor (((\neg x) \land (\neg z)) \lor (y \lor z)) & \text{Proposition 1.2.18.1} \\
\iff & (x \land (\neg y)) \lor (((\neg x) \land (\neg z)) \lor (z \lor y)) & \text{Proposition 1.2.3.1} \\
\iff & (x \land (\neg y)) \lor ((((\neg x) \land (\neg z)) \lor z) \lor y) & \text{Proposition 1.2.1.1} \\
\iff & (x \land (\neg y)) \lor (y \lor (((\neg x) \land (\neg z)) \lor z)) & \text{Proposition 1.2.3.1} \\
\iff & ((x \land (\neg y)) \lor y) \lor (((\neg x) \land (\neg z)) \lor z) & \text{Proposition 1.2.1.1} \\
\iff & (x \lor y) \lor (((\neg x) \land (\neg z)) \lor z) & \text{Proposition 1.3.0.1} \\
\iff & (x \lor y) \lor ((\neg x) \lor z) & \text{Proposition 1.3.0.1} \\
\iff & ((x \lor y) \lor (\neg x)) \lor z & \text{Proposition 1.2.1.1} \\
\iff & ((\neg x) \lor (x \lor y)) \lor z & \text{Proposition 1.2.3.1} \\
\iff & (((\neg x) \lor x) \lor y) \lor z & \text{Proposition 1.2.1.1} \\
\iff & ((\text{True}) \lor y) \lor z & \text{Proposition 1.2.11.2} \\
\iff & (\text{True}) \lor z & \text{Proposition 1.2.7.2} \\
\iff & \text{True} & \text{Proposition 1.2.7.2}
\end{aligned}
$$

Proposition 1.4.0.3.

$$(x \implies y) \implies ((x \land z) \implies (y \land z))$$

Proposition 1.4.0.4. Contrapositive

$$(x \implies y) \iff ((\neg y) \implies (\neg x))$$

Proposition 1.4.0.5. Transitive property of $\implies$.

$$((x \implies y) \land (y \implies z)) \implies (x \implies z)$$

Proposition 1.4.0.6.

$$(x \iff y) \iff ((x \implies y) \land (y \implies x))$$

Proposition 1.4.0.7.

$$(x \iff y) \implies ((x \vee z) \iff (y \vee z))$$

Proposition 1.4.0.8.

$$(x \iff y) \implies ((x \wedge z) \iff (y \wedge z))$$

Proposition 1.4.0.9. Symmetric property of $\iff$.

$$(x \iff y) \iff (y \iff x)$$

Proposition 1.4.0.10.

$$(x \iff y) \implies ((\neg x) \iff (\neg y))$$

Proposition 1.4.0.11. Transitive property of $\iff$.

$$((x \iff y) \wedge (y \iff z)) \implies (x \iff z)$$

Proposition 1.4.0.12. Reflexive property of $\iff$.

$$\begin{aligned} & x \\ \iff & x \end{aligned}$$

Proof of Proposition 1.4.0.12

$$\begin{aligned}
& x \iff x & \\
\stackrel{\text{def}}{\iff} & (x \wedge x) \vee ((\neg x) \wedge (\neg x)) & \text{Definition 1.1.4.1} \\
\iff & x \vee ((\neg x) \wedge (\neg x)) & \text{Proposition 1.2.10.1} \\
\iff & x \vee (\neg x) & \text{Proposition 1.2.10.1} \\
\iff & \text{True} & \text{Proposition 1.2.11.1}
\end{aligned}$$

1.5 Quantifiers

Definition 1.5.0.1. Universal quantifier is denoted by $\forall$.

$$\begin{aligned} & \forall x, P(x) \\ \stackrel{\text{def}}{\iff} & (P(x_1) \wedge P(x_2) \wedge \dots) \end{aligned}$$

Definition 1.5.0.2. Existential quantifier is denoted by $\exists$.

$$\begin{aligned}&\exists x, P(x)\\ \overset{\text{def}}{\iff}\ &(P(x_1) \vee P(x_2) \vee \dots)\end{aligned}$$

Proposition 1.5.0.3.

$$(\forall x(P(x) \wedge Q(x))) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

Proposition 1.5.0.4.

$$(\exists x, P(x)) \vee (\exists x, Q(x)) \iff (\exists x, (P(x) \vee Q(x)))$$

Proposition 1.5.0.5.

$$(P \vee (\forall x, Q(x))) \iff (\forall x(P \vee Q(x)))$$

Proposition 1.5.0.6.

$$(P \wedge (\exists x, Q(x))) \iff (\exists x(P \wedge Q(x)))$$

Axiom 1.1. P does not depend on x.

$$(\forall x, P(y)) \iff P(y)$$

Axiom 1.2. P does not depend on x.

$$(\exists x, P(y)) \iff P(y)$$

Axiom 1.3. De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

Axiom 1.4. De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

Definition 1.5.0.7. Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned}&!\exists x, P(x)\\ \overset{\text{def}}{\iff}\ &(\exists x, P(x)) \wedge (\forall x \forall y(P(x) \wedge P(y) \implies x = y))\end{aligned}$$

Axiom 1.5. Axiom of Substitution

$$\forall x((\exists y((y = x) \wedge P(y))) \iff P(x))$$

1.6 Proposition

Proposition 1.6.0.1.

$$(P \wedge Q) \implies (P \iff Q)$$

Proposition 1.6.0.2.

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

Proposition 1.6.0.3.

$$(P \wedge Q) \implies P$$

Lemma 1.6.0.4.

$$(P \wedge ((Q \wedge P) \implies R)) \implies (Q \implies R)$$

Proposition 1.6.0.5.

$$(P \wedge (P \implies Q)) \implies Q$$

Proposition 1.6.0.6.

$$(P \wedge (P \iff Q)) \implies Q$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

Definition 2.0.0.1. Definition of $\notin$.

$$\begin{aligned} & A \notin B \\ \overset{\text{def}}{\iff} & \neg(A \in B) \end{aligned}$$

Definition 2.0.0.2.

$$\begin{aligned} & \forall x \in S, P(x) \\ \overset{\text{def}}{\iff} & \forall x(x \in S \implies P(x)) \end{aligned}$$

Definition 2.0.0.3.

$$\begin{aligned} & \exists x \in S, P(x) \\ \overset{\text{def}}{\iff} & \exists x(x \in S \land P(x)) \end{aligned}$$

Proposition 2.0.0.4.

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\forall x \in S, P(x)) & \\
\iff & \neg(\forall x(x \in S \implies P(x))) & \text{Definition 2.0.0.2} \\
\iff & \neg(\forall x(\neg(x \in S) \lor P(x))) & \text{Definition 1.1.5.1} \\
\iff & \exists x, \neg(\neg(x \in S) \lor P(x)) & \text{Axiom 1.3} \\
\iff & \exists x, \neg(\neg(x \in S)) \land \neg(P(x)) & \text{Proposition 1.2.18.1} \\
\iff & \exists x, x \in S \land \neg(P(x)) & \text{Proposition 1.2.17.1} \\
\iff & \exists x \in S, \neg(P(x)) & \text{Definition 2.0.0.3}
\end{aligned}
$$

Proposition 2.0.0.5.

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\exists x \in S, P(x)) & \\
\iff & \neg(\exists x(x \in S \land P(x))) & \text{Definition 2.0.0.3} \\
\iff & \forall x, \neg(x \in S \land P(x)) & \text{Axiom 1.4} \\
\iff & \forall x, (\neg(x \in S)) \lor (\neg(P(x)) & \text{Proposition 1.2.18.2} \\
\iff & \forall x, x \in S \implies \neg(P(x)) & \text{Definition 1.1.5.1} \\
\iff & \forall x \in S, \neg(P(x)) & \text{Definition 2.0.0.2}
\end{aligned}
$$

2.1 Equality of sets

Definition 2.1.0.1. Definition of $=$.

$$
\begin{aligned}
& A = B \\
\overset{\text{def}}{\iff} & \forall x(x \in A \iff x \in B)
\end{aligned}
$$

Definition 2.1.0.2. Definition of $\neq$.

$$
\begin{aligned}
& A \neq B \\
\overset{\text{def}}{\iff} & \neg(A = B)
\end{aligned}
$$

Proposition 2.1.0.3. Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$\begin{aligned}
&\forall x(\\
&\qquad\qquad x = x \\
&\iff \quad \forall y(y \in x \iff y \in x) \qquad \text{Definition 2.1.0.1} \\
&\iff \quad \text{True} \qquad \text{Proposition 1.4.0.12} \\
&)
\end{aligned}$$

Proposition 2.1.0.4. Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$\begin{aligned}
&\forall x \forall y(\\
&\qquad\qquad x = y \\
&\implies \quad \forall z(z \in x \iff z \in y) \qquad \text{Definition 2.1.0.1} \\
&\implies \quad \forall z(z \in y \iff z \in x) \qquad \text{Proposition 1.4.0.9} \\
&\implies \quad y = x \qquad \text{Definition 2.1.0.1} \\
&)
\end{aligned}$$

Proposition 2.1.0.5. Transitive property of equality

$$\forall x \forall y \forall z((x = y) \wedge (y = z) \implies (x = z))$$

Proof:

$$\begin{aligned}
&\forall x \forall y \forall z(\\
&\qquad\qquad (x = y) \wedge (y = z) \\
&\implies \quad (\forall w(w \in x \iff w \in y)) \wedge (\forall w(w \in y \iff w \in z)) \qquad \text{Definition 2.1.0.1} \\
&\implies \quad \forall w((w \in x \iff w \in y) \wedge (w \in y \iff w \in z)) \qquad \text{Proposition 1.5.0.3} \\
&\implies \quad \forall w(w \in x \iff w \in z) \qquad \text{Proposition 1.4.0.11} \\
&\implies \quad x = z \qquad \text{Definition 2.1.0.1} \\
&)
\end{aligned}$$

Axiom 2.1. Axiom of extensionality

$$\begin{aligned}
&\forall x \forall y(\\
&x = y \implies \forall A(x \in A \iff y \in A) \\
&)
\end{aligned}$$

Axiom 2.2. Existence of empty set

$$\exists x \forall y (y \notin x)$$

Proposition 2.1.0.6. Uniqueness of empty set.

$$!\exists x \forall y (y \notin x)$$

Proof:
Let $P(x) = \forall y (y \notin x)$

$$\begin{aligned} & \exists x \forall y (y \notin x) && \text{Axiom 2.2} \\ \implies & \exists x, P(x) && \text{Definition of P(x)} \end{aligned}$$

$\forall x \forall y ($

$$\begin{aligned} & P(x) \land P(y) && \\ \implies & (\forall z (z \notin x)) \land (\forall z (z \notin y)) && \text{Definition of P(x)} \\ \implies & \forall z ((z \notin x) \land (z \notin y)) && \text{Proposition 1.5.0.3} \\ \implies & \forall z (z \notin x \iff z \notin y) && \text{Proposition 1.6.0.1} \\ \implies & \forall z (\neg (z \in x) \iff \neg (z \in y)) && \text{Definition 2.0.0.1} \\ \implies & \forall z (z \in x \iff z \in y) && \text{Proposition 1.6.0.2} \\ \implies & x = y && \text{Definition 2.1.0.1} \end{aligned}$$

$)$

$$\begin{aligned} & (\exists x, P(x)) \land \forall x \forall y ((P(x) \land P(y)) \implies (x = y)) && \\ \implies & !\exists x, P(x) && \text{Definition 1.5.0.7} \\ \implies & !\exists x \forall y (y \notin x) && \text{Definition of P(x)} \end{aligned}$$

Definition 2.1.0.7. The unique empty set is denoted by $\emptyset$.

$$\forall x (x \notin \emptyset)$$

Proof:

Let $P(x) = \forall y(y \notin x)$

$$
\begin{aligned}
& !\exists x \forall y(y \notin x) && \text{Proposition 2.1.0.6} \\
\implies & !\exists x, P(x) && \text{Definition of P(x)} \\
\implies & (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y)) && \text{Definition 1.5.0.7} \\
\implies & P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Definition 2.1.0.7} \\
\implies & P(\emptyset) && \text{Proposition 1.6.0.3} \\
\implies & \forall y(y \notin \emptyset) && \text{Definition of P(x)}
\end{aligned}
$$

Proposition 2.1.0.8. Uniqueness of $\emptyset$

$$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$$

Proof:
Let $P(x) = \forall y(y \notin x)$

$$
\begin{aligned}
& !\exists x \forall y(y \notin x) && \text{Proposition 2.1.0.6} \\
\implies & !\exists x, P(x) && \text{Definition of P(x)} \\
\implies & (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y)) && \text{Definition 1.5.0.7} \\
\implies & P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Definition 2.1.0.7} \\
\implies & (\forall x, P(\emptyset)) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Axiom 1.1} \\
\implies & \forall x(P(\emptyset) \wedge ((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))) && \text{Proposition 1.5.0.3} \\
\implies & \forall x(P(x) \implies (x = \emptyset)) && \text{Lemma 1.6.0.4} \\
\implies & \forall x(\forall y(y \notin x) \implies (x = \emptyset)) && \text{Definition of P(x)}
\end{aligned}
$$

Proposition 2.1.0.9. Single choice

$$\forall x((x \neq \emptyset) \implies (\exists y, y \in x))$$

Proof:

$$
\begin{aligned}
& \forall x(\forall y(y \notin x) \implies (x = \emptyset)) && \\
\implies & \forall x(\neg(x = \emptyset) \implies \neg(\forall y(y \notin x))) && \text{Proposition 1.4.0.4} \\
\implies & \forall x((x \neq \emptyset) \implies \neg(\forall y(y \notin x))) && \text{Definition 2.1.0.2} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, \neg(y \notin x))) && \text{Axiom 1.3} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, \neg(\neg(y \in x)))) && \text{Definition 2.0.0.1} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, y \in x)) && \text{Proposition 1.2.17.1}
\end{aligned}
$$

Axiom 2.3. Axiom of pairing. Existence of pair set.

$$\forall x \forall y \exists A \forall z (z \in A \iff ((z = x) \lor (z = y)))$$

Proposition 2.1.0.10. Uniqueness of pairing set.

$$\forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \lor (z = y)))$$

Proof:
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \lor (z = y)))$

$\forall x \forall y \forall A \forall B ($

$$\begin{aligned}
& P(A,x,y) \land P(B,x,y) & \\
\implies & (\forall z (z \in A \iff ((z = x) \lor (z = y)))) & \\
& \land (\forall z (z \in B \iff ((z = x) \lor (z = y)))) & \text{Definition of P(A,x,y)} \\
\implies & \forall z ((z \in A \iff ((z = x) \lor (z = y))) & \\
& \land (z \in B \iff ((z = x) \lor (z = y)))) & \text{Proposition 1.5.0.3} \\
\implies & \forall z (z \in A \iff z \in B) & \text{Proposition 1.4.0.11} \\
\implies & A = B & \text{Definition 2.1.0.1}
\end{aligned}$$

$)$

$$\begin{aligned}
& \forall x \forall y ! \exists A, P(A, x, y) & \text{Similar to the proof of the Proposition 2.1.0.6} \\
\implies & \forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \lor (z = y))) & \text{Definition of P(A,x,y)}
\end{aligned}$$

Definition 2.1.0.11. The unique pair set of x and y is denoted by $\{x, y\}$.
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \lor (z = y)))$
Similar to the proof of Definition 2.1.0.7,

$$\forall x \forall y P(\{x, y\}, x, y)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall x \forall y \forall A (P(A, x, y) \implies (A = \{x, y\}))$$

Proposition 2.1.0.12. Existence of singleton set.

$$\forall x \exists A \forall y (y \in A \iff (y = x))$$

Proof:

$$\begin{aligned}
& \forall x \exists A \forall y (y \in A \iff ((y = x) \lor (y = x))) & \text{Axiom 2.3} \\
\implies & \forall x \exists A \forall y (y \in A \iff (y = x)) & \text{Proposition 1.2.9.1}
\end{aligned}$$

Proposition 2.1.0.13. Uniqueness of singleton set.

$$\forall x !\exists A \forall y (y \in A \iff (x = y))$$

Let $P(A, x) = \forall y (y \in A \iff (x = y))$
The proof is similar to the proof of Proposition 2.1.0.10.

Definition 2.1.0.14. The unique singleton set of x is denoted by $\{x\}$.
Let $P(A, x) = \forall y (y \in A \iff (x = y))$
Similar to the proof of Definition 2.1.0.7,

$$\forall x P(\{x\}, x)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall x \forall A (P(A, x) \implies (A = \{x\}))$$

Axiom 2.4. Axiom of union. Existence of union set.

$$\forall F \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

Proposition 2.1.0.15. Uniqueness of union set.

$$\forall F !\exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

Proof:
Let $P(A, F) = \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$

$$\begin{array}{lll}
\forall F \forall A \forall B (& & \\
& P(A, F) \wedge P(B, F) & \\
\implies & (\forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) & \\
& \wedge (\forall x (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) & \text{Definition of P(A,F)} \\
\implies & \forall x ((x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) & \\
& \wedge (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) & \text{Proposition 1.5.0.3} \\
\implies & \forall x (x \in A \iff x \in B) & \text{Proposition 1.4.0.11} \\
\implies & A = B & \text{Definition 2.1.0.1} \\
) & &
\end{array}$$

$$\begin{array}{ll}
\quad \forall F !\exists A, P(A, F) & \text{Similar to the proof of the Proposition 2.1.0.6} \\
\implies \forall F !\exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) & \text{Definition of P(A,F)}
\end{array}$$

Definition 2.1.0.16. The unique union set of F is denoted by $\bigcup F$.
Let $P(A,F) = \forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$
Similar to the proof of Definition 2.1.0.7,

$$\forall F P(\bigcup F, F)$$

Similar to the proof of Proposition 2.1.0.8,

$$\forall F \forall A(P(A,F) \implies (A = \bigcup F))$$

Definition 2.1.0.17. Definition of pairwise union $A \cup B$.

$$\begin{aligned} &A \cup B \\ \stackrel{\text{def}}{=} &\bigcup\{A, B\} \end{aligned}$$

Proposition 2.1.0.18. Property of pairwise union.

$$\forall A \forall B \forall x(x \in (A \cup B) \iff ((x \in A) \vee (x \in B)))$$

Proof:

$\forall A \forall B \forall x($

$$\begin{array}{lll} & x \in (A \cup B) & \\ \iff & x \in \bigcup\{A, B\} & \text{Definition 2.1.0.1 and 2.1.0.17} \\ \iff & \exists Y((x \in Y) \wedge (Y \in \{A, B\})) & \text{Definition 2.1.0.16} \\ \iff & \exists Y((x \in Y) \wedge ((Y = A) \vee (Y = B))) & \text{Definition 2.1.0.11} \\ \iff & \exists Y(((x \in Y) \wedge (Y = A)) \vee ((x \in Y) \wedge (Y = B))) & \text{Proposition 1.2.16.1} \\ \iff & (\exists Y((x \in Y) \wedge (Y = A))) \vee (\exists Y((x \in Y) \wedge (Y = B))) & \text{Proposition 1.5.0.4} \\ \iff & ((x \in A) \vee (x \in B)) & \\ & \text{Axiom 1.5 with } P(A, x) = (x \in A) & \end{array}$$

$)$

Proposition 2.1.0.19. Commutativity of $\cup$.

$$\forall x \forall y((x \cup y) = (y \cup x))$$

Proof:

$\forall x \forall y($

$$
\begin{array}{lll}
 & (x \cup y) = (y \cup x) & \\
\iff & \forall z(z \in (x \cup y) \iff z \in (y \cup x)) & \text{Definition 2.1.0.1} \\
\iff & \forall z(((z \in x) \vee (z \in y)) \iff ((z \in y) \vee (z \in x))) & \text{Proposition 2.1.0.18} \\
\iff & \forall z(((z \in x) \vee (z \in y)) \iff ((z \in x) \vee (z \in y))) & \text{Proposition 1.2.3.1} \\
\iff & \text{True} & \text{Proposition 1.4.0.12}
\end{array}
$$

$)$

Proposition 2.1.0.20. Identity of $\cup$.

$$\forall x((x \cup \emptyset) = x)$$

Proof:

$\forall x($

$$
\begin{array}{lll}
 & (x \cup \emptyset) = x & \\
\iff & \forall y(y \in (x \cup \emptyset) \iff (y \in x)) & \text{Definition 2.1.0.1} \\
\iff & \forall y(((y \in x) \vee (y \in \emptyset)) \iff (y \in x)) & \text{Proposition 2.1.0.18} \\
\iff & \forall y(((y \in x) \vee (\neg(\neg(y \in \emptyset)))) \iff (y \in x)) & \text{Proposition 1.2.17.1} \\
\iff & \forall y(((y \in x) \vee (\neg(y \notin \emptyset))) \iff (y \in x)) & \text{Definition 2.0.0.1} \\
\iff & \forall y(((y \in x) \vee (\neg(\text{True}))) \iff (y \in x)) & \text{Definition 2.1.0.7} \\
\iff & \forall y(((y \in x) \vee (\text{False})) \iff (y \in x)) & \text{Definition 1.1.1.1} \\
\iff & \forall y((y \in x) \iff (y \in x)) & \text{Proposition 1.2.5.1} \\
\iff & \text{True} & \text{Proposition 1.4.0.12}
\end{array}
$$

$)$

Definition 2.1.0.21. Definition of 0.

$$0 \overset{\text{def}}{=} \emptyset$$

Definition 2.1.0.22. Definition of successor $S(x)$.

$$S(x)$$
$$\overset{\text{def}}{=} x \cup \{x\}$$

Definition 2.1.0.23. Definition of 1.

$$
\begin{aligned}
1 &\stackrel{\text{def}}{=} S(0) \\
&= 0 \cup \{0\} && \text{Definition 2.1.0.22} \\
&= \emptyset \cup \{\emptyset\} && \text{Definition 2.1.0.21} \\
&= \{\emptyset\} \cup \emptyset && \text{Proposition 2.1.0.19} \\
&= \{\emptyset\} && \text{Proposition 2.1.0.20}
\end{aligned}
$$