Chapter 1

Logic

Definition 1.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

Definition 1.1.1. Definition of $\neg$.

p	$\neg p$
T	F
F	T

Definition 1.1.2. Definition of $\wedge$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.1.3. Definition of $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.1.4. Definition of $\iff$.

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 1.1.5. Definition of $\implies$.

$$\begin{aligned} & p \implies q \\ \iff & (\neg p) \vee q \end{aligned}$$

1.2 Quantifiers

Definition 1.2.1. Universal quantifier is denoted by $\forall$.

$$\forall x, P(x)$$

Definition 1.2.2. Existential quantifier is denoted by $\exists$.

$$\exists x, P(x)$$

Axiom 1.1.

$$\forall x, (P(x) \wedge Q(x)) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

Axiom 1.2. P does not depend on x.

$$(\forall x, P(y)) \iff P(y)$$

Axiom 1.3. De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

Axiom 1.4. De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

Definition 1.2.3. Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned} & !\exists x, P(x) \\ \iff & (\exists x, P(x)) \wedge (\forall x \forall y (P(x) \wedge P(y) \implies x = y)) \end{aligned}$$

1.3 Proof technique

Proposition 1.3.1.

$$(P \iff Q) \iff ((P \implies Q) \wedge (Q \implies P))$$

Proposition 1.3.2.

$$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$$

Proposition 1.3.3. contrapositive

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

1.4 Proposition

Let $P = P(x_1, x_2, \ldots, x_n)$. Let $Q = Q(x_1, x_2, \ldots, x_n)$. etc

Proposition 1.4.1. Double negation

$$\neg(\neg P) \iff P$$

Proposition 1.4.2. Reflexive property of iff.

$$P \iff P$$

Proof:

P	$P \iff P$
T	T
F	T

Proposition 1.4.3. Symmetric property of iff.

$$(P \iff Q) \iff (Q \iff P)$$

Proposition 1.4.4. Transitive property of iff.

$$((P \iff Q) \wedge (Q \iff R)) \implies (P \iff R)$$

Proposition 1.4.5. De Morgan's law

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

Proposition 1.4.6. De Morgan's law

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

Proposition 1.4.7.

$$(P \wedge Q) \implies (P \iff Q)$$

Proposition 1.4.8.

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

Proposition 1.4.9.

$$(P \wedge Q) \implies P$$

Lemma 1.4.10.

$$(P \wedge ((Q \wedge P) \implies R)) \implies (Q \implies R)$$

Proposition 1.4.11.

$$(P \vee P) \iff P$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

Definition 2.0.1. Definition of $\notin$.

$$\begin{aligned} &A \notin B \\ \iff &\neg(A \in B) \end{aligned}$$

Definition 2.0.2.

$$\begin{aligned} &\forall x \in S, P(x) \\ \iff &\forall x(x \in S \implies P(x)) \end{aligned}$$

Definition 2.0.3.

$$\begin{aligned} &\exists x \in S, P(x) \\ \iff &\exists x(x \in S \land P(x)) \end{aligned}$$

Proposition 2.0.4.

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\forall x \in S, P(x)) \\
\iff & \neg(\forall x(x \in S \implies P(x))) && \text{Definition 2.0.2} \\
\iff & \neg(\forall x(\neg(x \in S) \vee P(x))) && \text{Definition 1.1.5} \\
\iff & \exists x, \neg(\neg(x \in S) \vee P(x)) && \text{Axiom 1.3} \\
\iff & \exists x, \neg(\neg(x \in S)) \wedge \neg(P(x)) && \text{Proposition 1.4.6} \\
\iff & \exists x, x \in S \wedge \neg(P(x)) && \text{Proposition 1.4.1} \\
\iff & \exists x \in S, \neg(P(x)) && \text{Definition 2.0.3}
\end{aligned}
$$

Proposition 2.0.5.

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$$
\begin{aligned}
& \neg(\exists x \in S, P(x)) \\
\iff & \neg(\exists x(x \in S \wedge P(x))) && \text{Definition 2.0.3} \\
\iff & \forall x, \neg(x \in S \wedge P(x)) && \text{Axiom 1.4} \\
\iff & \forall x, (\neg(x \in S)) \vee (\neg(P(x)) && \text{Proposition 1.4.5} \\
\iff & \forall x, x \in S \implies \neg(P(x)) && \text{Definition 1.1.5} \\
\iff & \forall x \in S, \neg(P(x)) && \text{Definition 2.0.2}
\end{aligned}
$$

2.1 Equality of sets

Definition 2.1.1. Definition of $=$.

$$
\begin{aligned}
& A = B \\
\iff & \forall x(x \in A \iff x \in B)
\end{aligned}
$$

Definition 2.1.2. Definition of $\neq$.

$$
\begin{aligned}
& A \neq B \\
\iff & \neg(A = B)
\end{aligned}
$$

Proposition 2.1.3. Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$
\begin{array}{lll}
\forall x(& & \\
 & x = x & \\
\iff & \forall y(y \in x \iff y \in x) & \text{Definition 2.1.1} \\
\iff & \text{True} & \text{Proposition 1.4.2} \\
) & &
\end{array}
$$

Proposition 2.1.4. Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$
\begin{array}{lll}
\forall x \forall y(& & \\
 & x = y & \\
\implies & \forall z(z \in x \iff z \in y) & \text{Definition 2.1.1} \\
\implies & \forall z(z \in y \iff z \in x) & \text{Proposition 1.4.3} \\
\implies & y = x & \text{Definition 2.1.1} \\
) & &
\end{array}
$$

Proposition 2.1.5. Transitive property of equality

$$\forall x \forall y \forall z((x = y) \wedge (y = z) \implies (x = z))$$

Proof:

$$
\begin{array}{lll}
\forall x \forall y \forall z(& & \\
 & (x = y) \wedge (y = z) & \\
\implies & (\forall w(w \in x \iff w \in y)) \wedge (\forall w(w \in y \iff w \in z)) & \text{Definition 2.1.1} \\
\implies & \forall w((w \in x \iff w \in y) \wedge (w \in y \iff w \in z)) & \text{Axiom 1.1} \\
\implies & \forall w(w \in x \iff w \in z) & \text{Proposition 1.4.4} \\
\implies & x = z & \text{Definition 2.1.1} \\
) & &
\end{array}
$$

Axiom 2.1. Axiom of Substitution

$$
\begin{array}{l}
\forall x \forall y(\\
x = y \implies \forall A(x \in A \iff y \in A) \\
)
\end{array}
$$

Axiom 2.2. Existence of empty set

$$\exists x \forall y (y \notin x)$$

Proposition 2.1.6. Uniqueness of empty set.

$$!\exists x \forall y (y \notin x)$$

Proof:
Let $P(x) = \forall y (y \notin x)$

$$\begin{aligned} & \exists x \forall y (y \notin x) && \text{Axiom 2.2} \\ \implies & \exists x, P(x) && \text{Definition of P(x)} \end{aligned}$$

$$\begin{aligned} & \forall x \forall y (\\ & \qquad P(x) \land P(y) \\ & \implies \quad (\forall z (z \notin x)) \land (\forall z (z \notin y)) && \text{Definition of P(x)} \\ & \implies \quad \forall z ((z \notin x) \land (z \notin y)) && \text{Axiom 1.1} \\ & \implies \quad \forall z (z \notin x \iff z \notin y) && \text{Proposition 1.4.7} \\ & \implies \quad \forall z (\neg (z \in x) \iff \neg (z \in y)) && \text{Definition 2.0.1} \\ & \implies \quad \forall z (z \in x \iff z \in y) && \text{Proposition 1.4.8} \\ & \implies \quad x = y && \text{Definition 2.1.1} \\ &) \end{aligned}$$

$$\begin{aligned} & (\exists x, P(x)) \land \forall x \forall y ((P(x) \land P(y)) \implies (x = y)) \\ \implies & !\exists x, P(x) && \text{Definition 1.2.3} \\ \implies & !\exists x \forall y (y \notin x) && \text{Definition of P(x)} \end{aligned}$$

Definition 2.1.7. The unique empty set is denoted by $\emptyset$.

$$\forall x (x \notin \emptyset)$$

Proof:

Let $P(x) = \forall y(y \notin x)$

$$\begin{aligned}
& !\exists x \forall y(y \notin x) && \text{Proposition 2.1.6} \\
\implies & !\exists x, P(x) && \text{Definition of P(x)} \\
\implies & (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y)) && \text{Definition 1.2.3} \\
\implies & P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Definition 2.1.7} \\
\implies & P(\emptyset) && \text{Proposition 1.4.9} \\
\implies & \forall y(y \notin \emptyset) && \text{Definition of P(x)}
\end{aligned}$$

Proposition 2.1.8. Uniqueness of $\emptyset$

$$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$$

Proof:
Let $P(x) = \forall y(y \notin x)$

$$\begin{aligned}
& !\exists x \forall y(y \notin x) && \text{Proposition 2.1.6} \\
\implies & !\exists x, P(x) && \text{Definition of P(x)} \\
\implies & (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y)) && \text{Definition 1.2.3} \\
\implies & P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Definition 2.1.7} \\
\implies & (\forall x, P(\emptyset)) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)) && \text{Axiom 1.2} \\
\implies & \forall x(P(\emptyset) \wedge ((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))) && \text{Axiom 1.1} \\
\implies & \forall x(P(x) \implies (x = \emptyset)) && \text{Lemma 1.4.10} \\
\implies & \forall x(\forall y(y \notin x) \implies (x = \emptyset)) && \text{Definition of P(x)}
\end{aligned}$$

Proposition 2.1.9. Single choice

$$\forall x((x \neq \emptyset) \implies (\exists y, y \in x))$$

Proof:

$$\begin{aligned}
& \forall x(\forall y(y \notin x) \implies (x = \emptyset)) && \\
\implies & \forall x(\neg(x = \emptyset) \implies \neg(\forall y(y \notin x))) && \text{Proposition 1.3.3} \\
\implies & \forall x((x \neq \emptyset) \implies \neg(\forall y(y \notin x))) && \text{Definition 2.1.2} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, \neg(y \notin x))) && \text{Axiom 1.3} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, \neg(\neg(y \in x)))) && \text{Definition 2.0.1} \\
\implies & \forall x((x \neq \emptyset) \implies (\exists y, y \in x)) && \text{Proposition 1.4.1}
\end{aligned}$$

Axiom 2.3. Axiom of pairing. Existence of pair set.

$$\forall x \forall y \exists A \forall z (z \in A \iff ((z = x) \lor (z = y)))$$

Proposition 2.1.10. Uniqueness of pairing set.

$$\forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \lor (z = y)))$$

Proof:
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \lor (z = y)))$

$\forall x \forall y \forall A \forall B($

$$\begin{aligned}
& P(A, x, y) \land P(B, x, y) & \\
\implies & (\forall z (z \in A \iff ((z = x) \lor (z = y)))) & \\
& \land (\forall z (z \in B \iff ((z = x) \lor (z = y)))) & \text{Definition of P(A,x,y)} \\
\implies & \forall z ((z \in A \iff ((z = x) \lor (z = y))) & \\
& \land (z \in B \iff ((z = x) \lor (z = y)))) & \text{Axiom 1.1} \\
\implies & \forall z (z \in A \iff z \in B) & \text{Proposition 1.4.4} \\
\implies & A = B & \text{Definition 2.1.1}
\end{aligned}$$

$)$

$$\begin{aligned}
& \forall x \forall y ! \exists A, P(A, x, y) & \text{Similar to the proof of the Proposition 2.1.6} \\
\implies & \forall x \forall y ! \exists A \forall z (z \in A \iff ((z = x) \lor (z = y))) & \text{Definition of P(A,x,y)}
\end{aligned}$$

Definition 2.1.11. The unique pair set of x and y is denoted by $\{x, y\}$.
Let $P(A, x, y) = \forall z (z \in A \iff ((z = x) \lor (z = y)))$
Similar to the proof of Definition 2.1.7,

$$\forall x \forall y P(\{x, y\}, x, y)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x \forall y \forall A (P(A, x, y) \implies (A = \{x, y\}))$$

Proposition 2.1.12. Existence of singleton set.

$$\forall x \exists A \forall y (y \in A \iff (y = x))$$

Proof:

$$\begin{aligned}
& \forall x \exists A \forall y (y \in A \iff ((y = x) \lor (y = x))) & \text{Axiom 2.3} \\
\implies & \forall x \exists A \forall y (y \in A \iff (y = x)) & \text{Proposition 1.4.11}
\end{aligned}$$

Proposition 2.1.13. Uniqueness of singleton set.

$$\forall x ! \exists A \forall y (y \in A \iff (x = y))$$

Let $P(A, x) = \forall y (y \in A \iff (x = y))$
The proof is similar to the proof of Proposition 2.1.10.

Definition 2.1.14. The unique singleton set of x is denoted by $\{x\}$.
Let $P(A, x) = \forall y (y \in A \iff (x = y))$
Similar to the proof of Definition 2.1.7,

$$\forall x P(\{x\}, x)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x \forall A (P(A, x) \implies (A = \{x\}))$$