# Chapter 1

# Logic

**Definition 1.0.0.1.** Proposition is a statement that is either true or false, but not both.

## 1.1 Logical operations

### 1.1.1 Definition of $\neg$

**Definition 1.1.1.1.**

$$\begin{aligned} &\neg(\text{True}) \\ \overset{\text{def}}{\iff} &\text{False} \end{aligned}$$

**Definition 1.1.1.2.**

$$\begin{aligned} &\neg(\text{False}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

### 1.1.2 Definition of $\vee$

**Definition 1.1.2.1.**

$$\begin{aligned} &(\text{True}) \vee (\text{True}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

**Definition 1.1.2.2.**

$$\begin{aligned} &(\text{True}) \vee (\text{False}) \\ \overset{\text{def}}{\iff} &\text{True} \end{aligned}$$

**Definition 1.1.2.3.**

$$\begin{aligned} & (\text{False}) \vee (\text{True}) \\ \overset{\text{def}}{\iff} & \text{True} \end{aligned}$$

**Definition 1.1.2.4.**

$$\begin{aligned} & (\text{False}) \vee (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

### 1.1.3 Definition of $\wedge$

**Definition 1.1.3.1.**

$$\begin{aligned} & (\text{True}) \wedge (\text{True}) \\ \overset{\text{def}}{\iff} & \text{True} \end{aligned}$$

**Definition 1.1.3.2.**

$$\begin{aligned} & (\text{True}) \wedge (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

**Definition 1.1.3.3.**

$$\begin{aligned} & (\text{False}) \wedge (\text{True}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

**Definition 1.1.3.4.**

$$\begin{aligned} & (\text{False}) \wedge (\text{False}) \\ \overset{\text{def}}{\iff} & \text{False} \end{aligned}$$

### 1.1.4 Definition of $\iff$

**Definition 1.1.4.1.**

$$\begin{aligned} & a \iff b \\ \overset{\text{def}}{\iff} & (a \wedge b) \vee ((\neg a) \wedge (\neg b)) \end{aligned}$$

### 1.1.5 Definition of $\implies$

**Definition 1.1.5.1.**

$$\begin{aligned} & a \implies b \\ \overset{\text{def}}{\iff} & (\neg a) \lor b \end{aligned}$$

## 1.2 Boolean algebra

### 1.2.1 Associativity of $\lor$

**Proposition 1.2.1.1.**

$$\begin{aligned} & (a \lor b) \lor c \\ \iff & a \lor (b \lor c) \end{aligned}$$

### 1.2.2 Associativity of $\land$

**Proposition 1.2.2.1.**

$$\begin{aligned} & (a \land b) \land c \\ \iff & a \land (b \land c) \end{aligned}$$

### 1.2.3 Commutativity of $\lor$

**Proposition 1.2.3.1.**

$$\begin{aligned} & a \lor b \\ \iff & b \lor a \end{aligned}$$

### 1.2.4 Commutativity of $\land$

**Proposition 1.2.4.1.**

$$\begin{aligned} & a \land b \\ \iff & b \land a \end{aligned}$$

### 1.2.5 Identity of $\vee$

**Proposition 1.2.5.1.**

$$\begin{aligned} & a \vee (\text{False}) \\ \iff & a \end{aligned}$$

**Proposition 1.2.5.2.**

$$\begin{aligned} & (\text{False}) \vee a \\ \iff & a \end{aligned}$$

### 1.2.6 Identity of $\wedge$

**Proposition 1.2.6.1.**

$$\begin{aligned} & a \wedge (\text{True}) \\ \iff & a \end{aligned}$$

**Proposition 1.2.6.2.**

$$\begin{aligned} & (\text{True}) \wedge a \\ \iff & a \end{aligned}$$

### 1.2.7 Annihilator of $\vee$

**Proposition 1.2.7.1.**

$$\begin{aligned} & a \vee (\text{True}) \\ \iff & \text{True} \end{aligned}$$

**Proposition 1.2.7.2.**

$$\begin{aligned} & (\text{True}) \vee a \\ \iff & \text{True} \end{aligned}$$

### 1.2.8 Annihilator of $\wedge$

**Proposition 1.2.8.1.**

$$\begin{aligned} & a \wedge (\text{False}) \\ \iff & \text{False} \end{aligned}$$

**Proposition 1.2.8.2.**

$$\begin{aligned} & (\text{False}) \wedge a \\ \iff & \text{False} \end{aligned}$$

### 1.2.9 Idempotence of $\vee$

**Proposition 1.2.9.1.**

$$\begin{aligned} & a \vee a \\ \iff & a \end{aligned}$$

### 1.2.10 Idempotence of $\wedge$

**Proposition 1.2.10.1.**

$$\begin{aligned} & a \wedge a \\ \iff & a \end{aligned}$$

### 1.2.11 Complement of $\vee$

**Proposition 1.2.11.1.**

$$\begin{aligned} & a \vee (\neg a) \\ \iff & \text{True} \end{aligned}$$

**Proposition 1.2.11.2.**

$$\begin{aligned} & (\neg a) \vee a \\ \iff & \text{True} \end{aligned}$$

### 1.2.12 Complement of $\wedge$

**Proposition 1.2.12.1.**

$$\begin{aligned} & a \wedge (\neg a) \\ \iff & \text{False} \end{aligned}$$

**Proposition 1.2.12.2.**

$$\begin{aligned} & (\neg a) \wedge a \\ \iff & \text{False} \end{aligned}$$

### 1.2.13 Absorption of $\vee$ over $\wedge$

**Proposition 1.2.13.1.**

$$\begin{aligned} & a \vee (a \wedge b) \\ \iff & a \end{aligned}$$

**Proposition 1.2.13.2.**

$$\begin{aligned} & a \vee (b \wedge a) \\ \iff & a \end{aligned}$$

**Proposition 1.2.13.3.**

$$\begin{aligned} & (a \wedge b) \vee a \\ \iff & a \end{aligned}$$

**Proposition 1.2.13.4.**

$$\begin{aligned} & (b \wedge a) \vee a \\ \iff & a \end{aligned}$$

### 1.2.14 Absorption of $\wedge$ over $\vee$

**Proposition 1.2.14.1.**

$$\begin{aligned} & a \wedge (a \vee b) \\ \iff & a \end{aligned}$$

**Proposition 1.2.14.2.**

$$\begin{aligned} & a \wedge (b \vee a) \\ \iff & a \end{aligned}$$

**Proposition 1.2.14.3.**

$$\begin{aligned} & (a \vee b) \wedge a \\ \iff & a \end{aligned}$$

**Proposition 1.2.14.4.**

$$\begin{aligned} & (b \vee a) \wedge a \\ \iff & a \end{aligned}$$

### 1.2.15 Distributivity of $\vee$ over $\wedge$

**Proposition 1.2.15.1.**

$$\begin{aligned} & a \vee (b \wedge c) \\ \iff & (a \vee b) \wedge (a \vee c) \end{aligned}$$

**Proposition 1.2.15.2.**

$$\begin{aligned} & (a \wedge b) \vee c \\ \iff & (a \vee c) \wedge (b \vee c) \end{aligned}$$

### 1.2.16 Distributivity of $\wedge$ over $\vee$

**Proposition 1.2.16.1.**

$$\begin{aligned} & a \wedge (b \vee c) \\ \iff & (a \wedge b) \vee (a \wedge c) \end{aligned}$$

**Proposition 1.2.16.2.**

$$\begin{aligned} & (a \vee b) \wedge c \\ \iff & (a \wedge c) \vee (b \wedge c) \end{aligned}$$

### 1.2.17 Double negation

**Proposition 1.2.17.1.**

$$\begin{aligned} & \neg(\neg a) \\ \iff & a \end{aligned}$$

### 1.2.18 De Morgan's laws

**Proposition 1.2.18.1.**

$$\begin{aligned} & \neg(a \vee b) \\ \iff & (\neg a) \wedge (\neg b) \end{aligned}$$

**Proposition 1.2.18.2.**

$$\begin{aligned} & \neg(a \wedge b) \\ \iff & (\neg a) \vee (\neg b) \end{aligned}$$

## 1.3 Basic Proposition

**Proposition 1.3.0.1.**

$$\begin{aligned} &(a \wedge (\neg b)) \vee b \\ \iff &a \vee b \end{aligned}$$

Proof of Proposition 1.3.0.1

$$\begin{aligned} &(a \wedge (\neg b)) \vee b && \\ \iff &(a \vee b) \wedge ((\neg b) \vee b) && \text{Proposition 1.2.15.2} \\ \iff &(a \vee b) \wedge (\text{True}) && \text{Proposition 1.2.11.2} \\ \iff &a \vee b && \text{Proposition 1.2.6.1} \end{aligned}$$

## 1.4 Proof technique

**Proposition 1.4.0.1.**

$$\begin{aligned} &a \iff (\text{True}) \\ \iff &a \end{aligned}$$

Proof of Proposition 1.4.0.1

$$\begin{aligned} &a \iff (\text{True}) && \\ \overset{\text{def}}{\iff} &(a \wedge (\text{True})) \vee ((\neg a) \wedge (\neg(\text{True}))) && \text{Definition 1.1.4.1} \\ \overset{\text{def}}{\iff} &(a \wedge (\text{True})) \vee ((\neg a) \wedge (\text{False})) && \text{Definition 1.1.1.1} \\ \iff &a \vee ((\neg a) \wedge (\text{False})) && \text{Proposition 1.2.6.1} \\ \iff &a \vee (\text{False}) && \text{Proposition 1.2.8.1} \\ \iff &a && \text{Proposition 1.2.5.1} \end{aligned}$$

**Proposition 1.4.0.2.**

$$\begin{aligned} &a \implies b \\ \implies &(a \vee c) \implies (b \vee c) \end{aligned}$$

Proof of Proposition 1.4.0.2

$$
\begin{aligned}
& (a \implies b) \implies ((a \vee c) \implies (b \vee c)) & \\
\overset{\text{def}}{\iff} \; & ((\neg a) \vee b) \implies ((a \vee c) \implies (b \vee c)) & \text{Definition 1.1.5.1} \\
\overset{\text{def}}{\iff} \; & ((\neg a) \vee b) \implies ((\neg (a \vee c)) \vee (b \vee c)) & \text{Definition 1.1.5.1} \\
\overset{\text{def}}{\iff} \; & (\neg((\neg a) \vee b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Definition 1.1.5.1} \\
\iff \; & ((\neg(\neg a)) \wedge (\neg b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Proposition 1.2.18.1} \\
\iff \; & (a \wedge (\neg b)) \vee ((\neg (a \vee c)) \vee (b \vee c)) & \text{Proposition 1.2.17.1} \\
\iff \; & (a \wedge (\neg b)) \vee (((\neg a) \wedge (\neg c)) \vee (b \vee c)) & \text{Proposition 1.2.18.1} \\
\iff \; & (a \wedge (\neg b)) \vee (((\neg a) \wedge (\neg c)) \vee (c \vee b)) & \text{Proposition 1.2.3.1} \\
\iff \; & (a \wedge (\neg b)) \vee ((((\neg a) \wedge (\neg c)) \vee c) \vee b) & \text{Proposition 1.2.1.1} \\
\iff \; & (a \wedge (\neg b)) \vee (b \vee (((\neg a) \wedge (\neg c)) \vee c)) & \text{Proposition 1.2.3.1} \\
\iff \; & ((a \wedge (\neg b)) \vee b) \vee (((\neg a) \wedge (\neg c)) \vee c) & \text{Proposition 1.2.1.1} \\
\iff \; & (a \vee b) \vee (((\neg a) \wedge (\neg c)) \vee c) & \text{Proposition 1.3.0.1} \\
\iff \; & (a \vee b) \vee ((\neg a) \vee c) & \text{Proposition 1.3.0.1} \\
\iff \; & ((a \vee b) \vee (\neg a)) \vee c & \text{Proposition 1.2.1.1} \\
\iff \; & ((\neg a) \vee (a \vee b)) \vee c & \text{Proposition 1.2.3.1} \\
\iff \; & (((\neg a) \vee a) \vee b) \vee c & \text{Proposition 1.2.1.1} \\
\iff \; & ((\text{True}) \vee b) \vee c & \text{Proposition 1.2.11.2} \\
\iff \; & (\text{True}) \vee c & \text{Proposition 1.2.7.2} \\
\iff \; & \text{True} & \text{Proposition 1.2.7.2}
\end{aligned}
$$

**Proposition 1.4.0.3.**

$$
\begin{aligned}
& a \implies b \\
\implies & (a \wedge c) \implies (b \wedge c)
\end{aligned}
$$

**Proposition 1.4.0.4.** Contrapositive

$$
\begin{aligned}
& a \implies b \\
\iff & (\neg b) \implies (\neg a)
\end{aligned}
$$

**Proposition 1.4.0.5.** Transitive property of $\implies$.

$$
\begin{aligned}
& (a \implies b) \wedge (b \implies c) \\
\implies & a \implies c
\end{aligned}
$$

**Proposition 1.4.0.6.**

$$\begin{aligned} & a \iff b \\ \iff & (a \implies b) \land (b \implies a) \end{aligned}$$

**Proposition 1.4.0.7.**

$$\begin{aligned} & a \iff b \\ \implies & (a \lor c) \iff (b \lor c) \end{aligned}$$

**Proposition 1.4.0.8.**

$$\begin{aligned} & a \iff b \\ \implies & (a \land c) \iff (b \land c) \end{aligned}$$

**Proposition 1.4.0.9.** Symmetric property of $\iff$.

$$\begin{aligned} & a \iff b \\ \iff & b \iff a \end{aligned}$$

**Proposition 1.4.0.10.**

$$\begin{aligned} & a \iff b \\ \iff & (\neg a) \iff (\neg b) \end{aligned}$$

**Proposition 1.4.0.11.** Transitive property of $\iff$.

$$\begin{aligned} & (a \iff b) \land (b \iff c) \\ \implies & a \iff c \end{aligned}$$

**Proposition 1.4.0.12.** Reflexive property of $\iff$.

$$\begin{aligned} & a \\ \iff & a \end{aligned}$$

Proof of Proposition 1.4.0.12

$$\begin{aligned} & a \iff a && \\ \overset{\text{def}}{\iff} & (a \land a) \lor ((\neg a) \land (\neg a)) && \text{Definition 1.1.4.1} \\ \iff & a \lor ((\neg a) \land (\neg a)) && \text{Proposition 1.2.10.1} \\ \iff & a \lor (\neg a) && \text{Proposition 1.2.10.1} \\ \iff & \text{True} && \text{Proposition 1.2.11.1} \end{aligned}$$

## 1.5 Quantifiers

**Definition 1.5.0.1.** Universal quantifier is denoted by $\forall$.

$$\begin{aligned} &\forall x(P(x)) \\ \overset{\text{def}}{\iff} &(P(x_1) \land P(x_2) \land \dots) \end{aligned}$$

**Definition 1.5.0.2.** Existential quantifier is denoted by $\exists$.

$$\begin{aligned} &\exists x(P(x)) \\ \overset{\text{def}}{\iff} &(P(x_1) \lor P(x_2) \lor \dots) \end{aligned}$$

**Proposition 1.5.0.3.**

$$\begin{aligned} &(\forall c(a)) \land (\forall c(b)) \\ \iff &\forall c(a \land b) \end{aligned}$$

**Proposition 1.5.0.4.**

$$\begin{aligned} &\exists x(P(x) \lor Q(x)) \\ \iff &(\exists x(P(x))) \lor (\exists x(Q(x))) \end{aligned}$$

**Proposition 1.5.0.5.**

$$\begin{aligned} &P \lor (\forall x(Q(x))) \\ \iff &\forall x(P \lor (Q(x))) \end{aligned}$$

**Proposition 1.5.0.6.**

$$\begin{aligned} &P \land (\exists x(Q(x))) \\ \iff &\exists x(P \land (Q(x))) \end{aligned}$$

**Axiom 1.1.**

$$\begin{aligned} &\forall x(P(y)) \\ \iff &P(y) \end{aligned}$$

**Axiom 1.2.**

$$\begin{aligned} &\exists x(P(y)) \\ \iff &P(y) \end{aligned}$$

**Proposition 1.5.0.7.** De Morgan's law

$$\begin{aligned} &\neg(\forall b(a)) \\ \iff &\exists b(\neg a) \end{aligned}$$

**Proposition 1.5.0.8.** De Morgan's law

$$\begin{aligned} &\neg(\exists b(a)) \\ \iff &\forall b(\neg a) \end{aligned}$$

**Definition 1.5.0.9.** Uniqueness quantifier is denoted by $!\exists$.

$$\begin{aligned} &!\exists x(P(x)) \\ \overset{\text{def}}{\iff} &(\exists x(P(x))) \wedge (\forall x \forall y((P(x) \wedge P(y)) \implies (x = y))) \end{aligned}$$

**Axiom 1.3.** Axiom of Substitution

$$\forall x((\exists y((y = x) \wedge P(y))) \iff P(x))$$

## 1.6 Logic proposition

**Proposition 1.6.0.1.**

$$\begin{aligned} &a \wedge b \\ \implies &a \iff b \end{aligned}$$

**Proposition 1.6.0.2.**

$$\begin{aligned} &a \wedge b \\ \implies &a \end{aligned}$$

**Proposition 1.6.0.3.**

$$\begin{aligned} &a \wedge ((b \wedge a) \implies c) \\ \implies &b \implies c \end{aligned}$$

**Proposition 1.6.0.4.**

$$\begin{aligned} &a \wedge (a \implies b) \\ \implies &b \end{aligned}$$

**Proposition 1.6.0.5.**

$$\begin{aligned} &a \wedge (a \iff b) \\ \implies &b \end{aligned}$$

# Chapter 2

# Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by $\in$.

**Definition 2.0.0.1.** Definition of $\notin$.

$$
\begin{aligned}
&\forall a \forall b( \\
&\qquad a \notin b \\
&\qquad \overset{\text{def}}{\iff} \neg(a \in b) \\
&)
\end{aligned}
$$

**Definition 2.0.0.2.**

$$
\begin{aligned}
&\forall a \in S, P(a) \\
\overset{\text{def}}{\iff} &\forall a((a \in S) \implies (P(a)))
\end{aligned}
$$

**Definition 2.0.0.3.**

$$
\begin{aligned}
&\exists a \in S, P(a) \\
\overset{\text{def}}{\iff} &\exists a((a \in S) \wedge (P(a)))
\end{aligned}
$$

**Proposition 2.0.0.4.**

$$
\begin{aligned}
&\forall a(\text{True}) \\
\iff &\text{True}
\end{aligned}
$$

## 2.1 Equality of sets

**Definition 2.1.0.1.** Definition of $=$.

$$\forall a \forall b($$
$$a = b$$
$$\overset{\text{def}}{\iff} \forall c((c \in a) \iff (c \in b))$$
$$)$$

**Definition 2.1.0.2.** Definition of $\overset{\text{def}}{=}$.

$$\forall a \forall b($$
$$a \overset{\text{def}}{=} b$$
$$\overset{\text{def}}{\iff} \forall c((c \in a) \iff (c \in b))$$
$$)$$

**Definition 2.1.0.3.** Definition of $\neq$.

$$\forall a \forall b($$
$$a \neq b$$
$$\overset{\text{def}}{\iff} \neg(a = b)$$
$$)$$

**Proposition 2.1.0.4.** Reflexive property of equality.

$$\forall a($$
$$a = a$$
$$)$$

Proof of Proposition 2.1.0.4

$$\forall a($$
$$\begin{aligned} & a = a \\ \overset{\text{def}}{\iff} & \forall b((b \in a) \iff (b \in a)) && \text{Definition 2.1.0.1} \\ \iff & \forall b(\text{True}) && \text{Proposition 1.4.0.12} \\ \iff & \text{True} && \text{Proposition 2.0.0.4} \end{aligned}$$
$$)$$

**Proposition 2.1.0.5.** Symmetric property of equality.

$$\forall a \forall b( \quad a = b \iff b = a \quad )$$

Proof of Proposition 2.1.0.5

$$\begin{aligned}
&\forall a \forall b( \\
&\quad a = b \\
&\quad \overset{\text{def}}{\iff} \forall c((c \in a) \iff (c \in b)) && \text{Definition 2.1.0.1} \\
&\quad \iff \forall c((c \in b) \iff (c \in a)) && \text{Proposition 1.4.0.9} \\
&\quad \overset{\text{def}}{\iff} b = a && \text{Definition 2.1.0.1} \\
&)
\end{aligned}$$

**Proposition 2.1.0.6.** Transitive property of equality.

$$\forall a \forall b \forall c( \quad (a = b) \wedge (b = c) \implies a = c \quad )$$

Proof of Proposition 2.1.0.6

$$\begin{aligned}
&\forall a \forall b \forall c( \\
&\quad (a = b) \wedge (b = c) \\
&\quad \overset{\text{def}}{\iff} (\forall d((d \in a) \iff (d \in b))) \wedge (b = c) && \text{Definition 2.1.0.1} \\
&\quad \overset{\text{def}}{\iff} (\forall d((d \in a) \iff (d \in b))) \wedge (\forall d((d \in b) \iff (d \in c))) && \text{Definition 2.1.0.1} \\
&\quad \iff \forall d(((d \in a) \iff (d \in b)) \wedge ((d \in b) \iff (d \in c))) && \text{Proposition 1.5.0.3} \\
&\quad \implies \forall d((d \in a) \iff (d \in c)) && \text{Proposition 1.4.0.11} \\
&\quad \overset{\text{def}}{\iff} a = c && \text{Definition 2.1.0.1} \\
&)
\end{aligned}$$

**Axiom 2.1.** Axiom of extensionality

$$\forall a \forall b( \quad a = b \implies \forall c((a \in c) \iff (b \in c)) \quad )$$

**Axiom 2.2.** Existence of empty set

$$\forall a( \quad a \notin \emptyset \quad )$$

**Proposition 2.1.0.7.** Uniqueness of $\emptyset$

$$\forall a( \quad \forall b(b \notin a) \iff a = \emptyset \quad )$$

Proof of Proposition 2.1.0.7

$$
\begin{array}{lll}
\forall a( & & \\
& \forall b(b \notin a) & \\
& \iff \forall b((b \notin a) \iff (\mathrm{True})) & \text{Proposition } 1.4.0.1 \\
& \iff \forall b((b \notin a) \iff (b \notin \emptyset)) & \text{Axiom } 2.2 \\
& \overset{\text{def}}{\iff} \forall b((\neg(b \in a)) \iff (b \notin \emptyset)) & \text{Definition } 2.0.0.1 \\
& \overset{\text{def}}{\iff} \forall b((\neg(b \in a)) \iff (\neg(b \in \emptyset))) & \text{Definition } 2.0.0.1 \\
& \iff \forall b((b \in a) \iff (b \in \emptyset)) & \text{Proposition } 1.4.0.10 \\
& \overset{\text{def}}{\iff} a = \emptyset & \text{Definition } 2.1.0.1 \\
) & &
\end{array}
$$

**Proposition 2.1.0.8.** Single choice

$$\forall a( \quad a \neq \emptyset \iff \exists b(b \in a) \quad )$$

Proof of Proposition 2.1.0.8

$$
\begin{aligned}
&\forall a( \\
&\qquad \text{True} \\
&\qquad \iff (\forall b(b \notin a)) \iff (a = \emptyset) && \text{Proposition 2.1.0.7} \\
&\qquad \iff (a = \emptyset) \iff (\forall b(b \notin a)) && \text{Proposition 1.4.0.9} \\
&\qquad \iff (\neg(a = \emptyset)) \iff (\neg(\forall b(b \notin a))) && \text{Proposition 1.4.0.10} \\
&\qquad \overset{\text{def}}{\iff} (a \neq \emptyset) \iff (\neg(\forall b(b \notin a))) && \text{Definition 2.1.0.3} \\
&\qquad \iff (a \neq \emptyset) \iff (\exists b(\neg(b \notin a))) && \text{Proposition 1.5.0.7} \\
&\qquad \overset{\text{def}}{\iff} (a \neq \emptyset) \iff (\exists b(\neg(\neg(b \in a)))) && \text{Definition 2.0.0.1} \\
&\qquad \iff (a \neq \emptyset) \iff (\exists b(b \in a)) && \text{Proposition 1.2.17.1} \\
&)
\end{aligned}
$$

**Axiom 2.3.** Existence of pair set

$$
\begin{aligned}
&\forall a \forall b \forall c( \\
&\qquad c \in \{a, b\} \\
&\qquad \iff (c = a) \lor (c = b) \\
&)
\end{aligned}
$$

**Proposition 2.1.0.9.** Uniqueness of pair set

$$
\begin{aligned}
&\forall a \forall b \forall c( \\
&\qquad \forall d((d \in c) \iff ((d = a) \lor (d = b))) \\
&\qquad \implies c = \{a, b\} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.9

$$
\begin{aligned}
&\forall a \forall b \forall c( \\
&\qquad \forall d((d \in c) \iff ((d = a) \lor (d = b))) \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \lor (d = b))) \land (\text{True})) && \text{Proposition 1.2.6.1} \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \lor (d = b))) \\
&\qquad\qquad \land ((d \in \{a, b\}) \iff ((d = a) \lor (d = b)))) && \text{Axiom 2.3} \\
&\qquad \iff \forall d(((d \in c) \iff ((d = a) \lor (d = b))) \\
&\qquad\qquad \land (((d = a) \lor (d = b)) \iff (d \in \{a, b\}))) && \text{Proposition 1.4.0.9} \\
&\qquad \implies \forall d((d \in c) \iff (d \in \{a, b\})) && \text{Proposition 1.4.0.11} \\
&\qquad \overset{\text{def}}{\iff} c = \{a, b\} && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

**Definition 2.1.0.10.** Definition of singleton set.

$$\forall a($$
$$\{a\} \overset{\text{def}}{=} \{a, a\}$$
$$)$$

**Proposition 2.1.0.11.** Property of singleton set.

$$\forall a \forall b($$
$$\begin{aligned} & b \in \{a\} \\ \iff & b = a \end{aligned}$$
$$)$$

Proposition (1)

$$\forall a \forall b($$
$$\begin{aligned} & b \in \{a\} \\ \iff & b \in \{a, a\} \end{aligned}$$
$$)$$

Proof of Proposition (1)

$$\forall a($$
$$\begin{aligned} & \text{True} & \\ \iff & \{a\} \overset{\text{def}}{=} \{a, a\} & \text{Definition 2.1.0.10} \\ \overset{\text{def}}{\iff} & \forall b((b \in \{a\}) \iff (b \in \{a, a\})) & \text{Definition 2.1.0.2} \end{aligned}$$
$$)$$

Proof of Proposition 2.1.0.11

$$\forall a \forall b($$
$$\begin{aligned} & b \in \{a\} & \\ \iff & b \in \{a, a\} & \text{Proposition (1)} \\ \iff & (b = a) \vee (b = a) & \text{Axiom 2.3} \\ \iff & b = a & \text{Proposition 1.2.9.1} \end{aligned}$$
$$)$$

**Proposition 2.1.0.12.** Uniqueness of singleton set.

$$
\begin{aligned}
&\forall a \forall b( \\
&\qquad \forall c((c \in b) \iff (c = a)) \\
&\qquad \implies b = \{a\} \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.12

$$
\begin{aligned}
&\forall a \forall b( \\
&\qquad \forall c((c \in b) \iff (c = a)) \\
&\iff \forall c(((c \in b) \iff (c = a)) \land (\text{True})) && \text{Proposition 1.2.6.1} \\
&\iff \forall c(((c \in b) \iff (c = a)) \\
&\qquad \land ((c \in \{a\}) \iff (c = a))) && \text{Proposition 2.1.0.11} \\
&\iff \forall c(((c \in b) \iff (c = a)) \\
&\qquad \land ((c = a) \iff (c \in \{a\}))) && \text{Proposition 1.4.0.9} \\
&\implies \forall c((c \in b) \iff (c \in \{a\})) && \text{Proposition 1.4.0.11} \\
&\overset{\text{def}}{\iff} b = \{a\} && \text{Definition 2.1.0.1} \\
&)
\end{aligned}
$$

**Axiom 2.4.** Existence of union set.

$$
\begin{aligned}
&\forall a \forall b( \\
&\qquad b \in (\bigcup a) \\
&\qquad \iff \exists c((b \in c) \land (c \in a)) \\
&)
\end{aligned}
$$

**Proposition 2.1.0.13.** Uniqueness of union set.

$$
\begin{aligned}
&\forall a \forall b( \\
&\qquad \forall c((c \in b) \iff (\exists d((c \in d) \land (d \in a)))) \\
&\qquad \implies b = (\bigcup a) \\
&)
\end{aligned}
$$

Proof of Proposition 2.1.0.13

$\forall a \forall b($

$$
\begin{aligned}
& \forall c((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
\iff & \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \wedge (\text{True})) && \text{Proposition 1.2.6.1} \\
\iff & \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
& \quad \wedge ((c \in (\bigcup a)) \iff (\exists d((c \in d) \wedge (d \in a))))) && \text{Axiom 2.4} \\
\iff & \forall c(((c \in b) \iff (\exists d((c \in d) \wedge (d \in a)))) \\
& \quad \wedge ((\exists d((c \in d) \wedge (d \in a))) \iff (c \in (\bigcup a)))) && \text{Proposition 1.4.0.9} \\
\implies & \forall c((c \in b) \iff (c \in (\bigcup a))) && \text{Proposition 1.4.0.11} \\
\stackrel{\text{def}}{\iff} & b = (\bigcup a) && \text{Definition 2.1.0.1}
\end{aligned}
$$

$)$

**Definition 2.1.0.14.** Definition of pairwise union $A \cup B$.

$$
\begin{aligned}
& A \cup B \\
\stackrel{\text{def}}{=} & \bigcup \{A, B\}
\end{aligned}
$$

**Proposition 2.1.0.15.** Property of pairwise union.

$$\forall A \forall B \forall x(x \in (A \cup B) \iff ((x \in A) \vee (x \in B)))$$

Proof:

$\forall A \forall B \forall x($

$$
\begin{aligned}
& x \in (A \cup B) \\
\iff & x \in \bigcup \{A, B\} && \text{Definition 2.1.0.1 and 2.1.0.15} \\
\iff & \exists Y((x \in Y) \wedge (Y \in \{A, B\})) && \text{Definition 2.4} \\
\iff & \exists Y((x \in Y) \wedge ((Y = A) \vee (Y = B))) && \text{Definition 2.3} \\
\iff & \exists Y(((x \in Y) \wedge (Y = A)) \vee ((x \in Y) \wedge (Y = B))) && \text{Proposition 1.2.16.1} \\
\iff & (\exists Y((x \in Y) \wedge (Y = A))) \vee (\exists Y((x \in Y) \wedge (Y = B))) && \text{Proposition 1.5.0.4} \\
\iff & ((x \in A) \vee (x \in B)) \\
& \text{Axiom 1.3 with } P(A, x) = (x \in A)
\end{aligned}
$$

$)$

**Proposition 2.1.0.16.** Commutativity of $\cup$.

$$\forall x \forall y((x \cup y) = (y \cup x))$$

Proof:

$\forall x \forall y($

| | | |
|---|---|---|
| | $(x \cup y) = (y \cup x)$ | |
| $\iff$ | $\forall z(z \in (x \cup y) \iff z \in (y \cup x))$ | Definition 2.1.0.1 |
| $\iff$ | $\forall z(((z \in x) \lor (z \in y)) \iff ((z \in y) \lor (z \in x)))$ | Proposition 2.1.0.16 |
| $\iff$ | $\forall z(((z \in x) \lor (z \in y)) \iff ((z \in x) \lor (z \in y)))$ | Proposition 1.2.3.1 |
| $\iff$ | True | Proposition 1.4.0.12 |

$)$

**Proposition 2.1.0.17.** Identity of $\cup$.

$$\forall x((x \cup \emptyset) = x)$$

Proof:

$\forall x($

| | | |
|---|---|---|
| | $(x \cup \emptyset) = x$ | |
| $\iff$ | $\forall y(y \in (x \cup \emptyset) \iff (y \in x))$ | Definition 2.1.0.1 |
| $\iff$ | $\forall y(((y \in x) \lor (y \in \emptyset)) \iff (y \in x))$ | Proposition 2.1.0.16 |
| $\iff$ | $\forall y(((y \in x) \lor (\neg(\neg(y \in \emptyset)))) \iff (y \in x))$ | Proposition 1.2.17.1 |
| $\iff$ | $\forall y(((y \in x) \lor (\neg(y \notin \emptyset))) \iff (y \in x))$ | Definition 2.0.0.1 |
| $\iff$ | $\forall y(((y \in x) \lor (\neg(\text{True}))) \iff (y \in x))$ | Definition 2.2 |
| $\iff$ | $\forall y(((y \in x) \lor (\text{False})) \iff (y \in x))$ | Definition 1.1.1.1 |
| $\iff$ | $\forall y((y \in x) \iff (y \in x))$ | Proposition 1.2.5.1 |
| $\iff$ | True | Proposition 1.4.0.12 |

$)$

**Definition 2.1.0.18.** Definition of 0.

$$0 \stackrel{\text{def}}{=} \emptyset$$

**Definition 2.1.0.19.** Definition of successor $S(x)$.

$$S(x)$$
$$\stackrel{\text{def}}{=} x \cup \{x\}$$

**Definition 2.1.0.20.** Definition of 1.

$$
\begin{aligned}
1 &\overset{\text{def}}{=} S(0) \\
&= 0 \cup \{0\} && \text{Definition 2.1.0.20} \\
&= \emptyset \cup \{\emptyset\} && \text{Definition 2.1.0.19} \\
&= \{\emptyset\} \cup \emptyset && \text{Proposition 2.1.0.17} \\
&= \{\emptyset\} && \text{Proposition 2.1.0.18}
\end{aligned}
$$