

Chapter 1

Logic

Definition 1.0.1. Proposition is a statement that is either true or false, but not both.

1.1 Logical operations

Definition 1.1.1. Definition of \neg .

p	$\neg p$
T	F
F	T

Definition 1.1.2. Definition of \wedge .

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.1.3. Definition of \vee .

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.1.4. Definition of \Longleftrightarrow .

$$p \iff q$$

$$\stackrel{\text{def}}{\iff} ((p \wedge q) \vee (\neg p \wedge \neg q))$$

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 1.1.5. Definition of \implies .

$$p \implies q$$

$$\stackrel{\text{def}}{\iff} (\neg p) \vee q$$

1.2 Boolean algebra

Proposition 1.2.1. Associativity of \vee .

$$((P \vee Q) \vee R) \iff (P \vee (Q \vee R))$$

Proposition 1.2.2. Associativity of \wedge .

$$((P \wedge Q) \wedge R) \iff (P \wedge (Q \wedge R))$$

Proposition 1.2.3. Commutativity of \vee .

$$(P \vee Q) \iff (Q \vee P)$$

Proposition 1.2.4. Commutativity of \wedge .

$$(P \wedge Q) \iff (Q \wedge P)$$

Proposition 1.2.5. Identity of \vee .

$$(P \vee \text{False}) \iff P$$

Proposition 1.2.6. Identity of \wedge .

$$(P \wedge \text{True}) \iff P$$

Proposition 1.2.7. Annihilator of \vee .

$$(P \vee \text{True}) \iff \text{True}$$

Proposition 1.2.8. Annihilator of \wedge .

$$(P \wedge \text{False}) \iff \text{False}$$

Proposition 1.2.9. Idempotence of \vee .

$$(P \vee P) \iff P$$

Proposition 1.2.10. Idempotence of \wedge .

$$(P \wedge P) \iff P$$

Proposition 1.2.11. Complement of \vee .

$$(P \vee (\neg P)) \iff \text{True}$$

Proposition 1.2.12. Complement of \wedge .

$$(P \wedge (\neg P)) \iff \text{False}$$

Proposition 1.2.13. Absorption.

$$(P \vee (P \wedge Q)) \iff P$$

Proposition 1.2.14. Absorption.

$$(P \wedge (P \vee Q)) \iff P$$

Proposition 1.2.15. Distributivity.

$$(P \vee (Q \wedge R)) \iff ((P \vee Q) \wedge (P \vee R))$$

Proposition 1.2.16. Distributivity.

$$(P \wedge (Q \vee R)) \iff ((P \wedge Q) \vee (P \wedge R))$$

1.3 Quantifiers

Definition 1.3.1. Universal quantifier is denoted by \forall .

$$\forall x, P(x) \\ \stackrel{\text{def}}{\iff} (P(x_1) \wedge P(x_2) \wedge \dots)$$

Definition 1.3.2. Existential quantifier is denoted by \exists .

$$\exists x, P(x) \\ \stackrel{\text{def}}{\iff} (P(x_1) \vee P(x_2) \vee \dots)$$

Proposition 1.3.3.

$$(\forall x(P(x) \wedge Q(x))) \iff (\forall x, P(x)) \wedge (\forall x, Q(x))$$

Proposition 1.3.4.

$$(\exists x, P(x)) \vee (\exists x, Q(x)) \iff (\exists x, (P(x) \vee Q(x)))$$

Proposition 1.3.5.

$$(P \vee (\forall x, Q(x))) \iff (\forall x(P \vee Q(x)))$$

Proposition 1.3.6.

$$(P \wedge (\exists x, Q(x))) \iff (\exists x(P \wedge Q(x)))$$

Axiom 1.1. P does not depend on x.

$$(\forall x, P(y)) \iff P(y)$$

Axiom 1.2. P does not depend on x.

$$(\exists x, P(y)) \iff P(y)$$

Axiom 1.3. De Morgan's law

$$\neg(\forall x, P(x)) \iff \exists x, \neg(P(x))$$

Axiom 1.4. De Morgan's law

$$\neg(\exists x, P(x)) \iff \forall x, \neg(P(x))$$

Definition 1.3.7. Uniqueness quantifier is denoted by $!\exists$.

$$!\exists x, P(x) \\ \stackrel{\text{def}}{\iff} (\exists x, P(x)) \wedge (\forall x \forall y (P(x) \wedge P(y) \implies x = y))$$

Axiom 1.5. Axiom of Substitution

$$\forall x((\exists y((y = x) \wedge P(y))) \iff P(x))$$

1.4 Proof technique

Proposition 1.4.1.

$$(P \iff Q) \iff ((P \implies Q) \wedge (Q \implies P))$$

Proposition 1.4.2.

$$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$$

Proposition 1.4.3.

$$(P \implies Q) \implies ((P \vee R) \implies (Q \vee R))$$

Proposition 1.4.4.

$$(P \iff Q) \implies ((P \vee R) \iff (Q \vee R))$$

Proposition 1.4.5. contrapositive

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

1.5 Proposition

Let $P = P(x_1, x_2, \dots, x_n)$. Let $Q = Q(x_1, x_2, \dots, x_n)$. etc

Proposition 1.5.1. Double negation

$$\neg(\neg P) \iff P$$

Proposition 1.5.2. Reflexive property of iff.

$$P \iff P$$

Proof:

P	$P \iff P$
T	T
F	T

Proposition 1.5.3. Symmetric property of iff.

$$(P \iff Q) \iff (Q \iff P)$$

Proposition 1.5.4. Transitive property of iff.

$$((P \iff Q) \wedge (Q \iff R)) \implies (P \iff R)$$

Proposition 1.5.5. De Morgan's law

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

Proposition 1.5.6. De Morgan's law

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

Proposition 1.5.7.

$$(P \wedge Q) \implies (P \iff Q)$$

Proposition 1.5.8.

$$(\neg P \iff \neg Q) \iff (P \iff Q)$$

Proposition 1.5.9.

$$(P \wedge Q) \implies P$$

Lemma 1.5.10.

$$(P \wedge ((Q \wedge P) \implies R)) \implies (Q \implies R)$$

Proposition 1.5.11.

$$(P \wedge (P \implies Q)) \implies Q$$

Proposition 1.5.12.

$$(P \wedge (P \iff Q)) \implies Q$$

Chapter 2

Set theory

Set theory have one primitive notion, called set, and one binary relation, called set membership, denoted by \in .

Definition 2.0.1. Definition of \notin .

$$\begin{aligned} A \notin B \\ \iff \neg(A \in B) \end{aligned}$$

Definition 2.0.2.

$$\begin{aligned} \forall x \in S, P(x) \\ \iff \forall x (x \in S \implies P(x)) \end{aligned}$$

Definition 2.0.3.

$$\begin{aligned} \exists x \in S, P(x) \\ \iff \exists x (x \in S \wedge P(x)) \end{aligned}$$

Proposition 2.0.4.

$$\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg(P(x))$$

Proof:

$\neg(\forall x \in S, P(x))$	
$\iff \neg(\forall x(x \in S \implies P(x)))$	Definition 2.0.2
$\iff \neg(\forall x(\neg(x \in S) \vee P(x)))$	Definition 1.1.5
$\iff \exists x, \neg(\neg(x \in S) \vee P(x))$	Axiom 1.3
$\iff \exists x, \neg(\neg(x \in S)) \wedge \neg(P(x))$	Proposition 1.5.6
$\iff \exists x, x \in S \wedge \neg(P(x))$	Proposition 1.5.1
$\iff \exists x \in S, \neg(P(x))$	Definition 2.0.3

Proposition 2.0.5.

$$\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg(P(x))$$

Proof:

$\neg(\exists x \in S, P(x))$	
$\iff \neg(\exists x(x \in S \wedge P(x)))$	Definition 2.0.3
$\iff \forall x, \neg(x \in S \wedge P(x))$	Axiom 1.4
$\iff \forall x, (\neg(x \in S)) \vee (\neg(P(x)))$	Proposition 1.5.5
$\iff \forall x, x \in S \implies \neg(P(x))$	Definition 1.1.5
$\iff \forall x \in S, \neg(P(x))$	Definition 2.0.2

2.1 Equality of sets

Definition 2.1.1. Definition of $=$.

$$A = B$$

$$\stackrel{\text{def}}{\iff} \forall x(x \in A \iff x \in B)$$

Definition 2.1.2. Definition of \neq .

$$A \neq B$$

$$\stackrel{\text{def}}{\iff} \neg(A = B)$$

Proposition 2.1.3. Reflexive property of equality

$$\forall x(x = x)$$

Proof:

$$\begin{aligned} & \forall x(\\ & \quad x = x \\ & \iff \forall y(y \in x \iff y \in x) \quad \text{Definition 2.1.1} \\ & \iff \text{True} \quad \text{Proposition 1.5.2} \\ &) \end{aligned}$$

Proposition 2.1.4. Symmetric property of equality

$$\forall x \forall y((x = y) \implies (y = x))$$

Proof:

$$\begin{aligned} & \forall x \forall y(\\ & \quad x = y \\ & \implies \forall z(z \in x \iff z \in y) \quad \text{Definition 2.1.1} \\ & \implies \forall z(z \in y \iff z \in x) \quad \text{Proposition 1.5.3} \\ & \implies y = x \quad \text{Definition 2.1.1} \\ &) \end{aligned}$$

Proposition 2.1.5. Transitive property of equality

$$\forall x \forall y \forall z((x = y) \wedge (y = z) \implies (x = z))$$

Proof:

$$\begin{aligned} & \forall x \forall y \forall z(\\ & \quad (x = y) \wedge (y = z) \\ & \implies (\forall w(w \in x \iff w \in y)) \wedge (\forall w(w \in y \iff w \in z)) \quad \text{Definition 2.1.1} \\ & \implies \forall w((w \in x \iff w \in y) \wedge (w \in y \iff w \in z)) \quad \text{Proposition 1.3.3} \\ & \implies \forall w(w \in x \iff w \in z) \quad \text{Proposition 1.5.4} \\ & \implies x = z \quad \text{Definition 2.1.1} \\ &) \end{aligned}$$

Axiom 2.1. Axiom of extensionality

$$\begin{aligned} & \forall x \forall y(\\ & \quad x = y \implies \forall A(x \in A \iff y \in A) \\ &) \end{aligned}$$

Axiom 2.2. Existence of empty set

$$\exists x \forall y (y \notin x)$$

Proposition 2.1.6. Uniqueness of empty set.

$$!\exists x \forall y (y \notin x)$$

Proof:

Let $P(x) = \forall y (y \notin x)$

$$\begin{aligned} & \exists x \forall y (y \notin x) && \text{Axiom 2.2} \\ \implies & \exists x, P(x) && \text{Definition of } P(x) \end{aligned}$$

$$\begin{aligned} & \forall x \forall y (\\ & \quad P(x) \wedge P(y) \\ \implies & (\forall z (z \notin x)) \wedge (\forall z (z \notin y)) && \text{Definition of } P(x) \\ \implies & \forall z ((z \notin x) \wedge (z \notin y)) && \text{Proposition 1.3.3} \\ \implies & \forall z (z \notin x \iff z \notin y) && \text{Proposition 1.5.7} \\ \implies & \forall z (\neg(z \in x) \iff \neg(z \in y)) && \text{Definition 2.0.1} \\ \implies & \forall z (z \in x \iff z \in y) && \text{Proposition 1.5.8} \\ \implies & x = y && \text{Definition 2.1.1} \\ &) \end{aligned}$$

$$\begin{aligned} & (\exists x, P(x)) \wedge \forall x \forall y ((P(x) \wedge P(y)) \implies (x = y)) \\ \implies & !\exists x, P(x) && \text{Definition 1.3.7} \\ \implies & !\exists x \forall y (y \notin x) && \text{Definition of } P(x) \end{aligned}$$

Definition 2.1.7. The unique empty set is denoted by \emptyset .

$$\forall x (x \notin \emptyset)$$

Proof:

Let $P(x) = \forall y(y \notin x)$

$\neg \exists x \forall y(y \notin x)$	Proposition 2.1.6
$\implies \neg \exists x, P(x)$	Definition of P(x)
$\implies (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y))$	Definition 1.3.7
$\implies P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Definition 2.1.7
$\implies P(\emptyset)$	Proposition 1.5.9
$\implies \forall y(y \notin \emptyset)$	Definition of P(x)

Proposition 2.1.8. Uniqueness of \emptyset

$$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$$

Proof:

Let $P(x) = \forall y(y \notin x)$

$\neg \exists x \forall y(y \notin x)$	Proposition 2.1.6
$\implies \neg \exists x, P(x)$	Definition of P(x)
$\implies (\exists x, P(x)) \wedge \forall x \forall y((P(x) \wedge P(y)) \implies (x = y))$	Definition 1.3.7
$\implies P(\emptyset) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Definition 2.1.7
$\implies (\forall x, P(\emptyset)) \wedge \forall x((P(x) \wedge P(\emptyset)) \implies (x = \emptyset))$	Axiom 1.1
$\implies \forall x(P(\emptyset) \wedge ((P(x) \wedge P(\emptyset)) \implies (x = \emptyset)))$	Proposition 1.3.3
$\implies \forall x(P(x) \implies (x = \emptyset))$	Lemma 1.5.10
$\implies \forall x(\forall y(y \notin x) \implies (x = \emptyset))$	Definition of P(x)

Proposition 2.1.9. Single choice

$$\forall x((x \neq \emptyset) \implies (\exists y, y \in x))$$

Proof:

$\forall x(\forall y(y \notin x) \implies (x = \emptyset))$	
$\implies \forall x(\neg(x = \emptyset) \implies \neg(\forall y(y \notin x)))$	Proposition 1.4.5
$\implies \forall x((x \neq \emptyset) \implies \neg(\forall y(y \notin x)))$	Definition 2.1.2
$\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(y \notin x)))$	Axiom 1.3
$\implies \forall x((x \neq \emptyset) \implies (\exists y, \neg(\neg(y \in x))))$	Definition 2.0.1
$\implies \forall x((x \neq \emptyset) \implies (\exists y, y \in x))$	Proposition 1.5.1

Axiom 2.3. Axiom of pairing. Existence of pair set.

$$\forall x \forall y \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Proposition 2.1.10. Uniqueness of pairing set.

$$\forall x \forall y! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Proof:

$$\text{Let } P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

$$\forall x \forall y \forall A \forall B ($$

$$\begin{aligned} & P(A, x, y) \wedge P(B, x, y) \\ \implies & (\forall z (z \in A \iff ((z = x) \vee (z = y)))) \\ & \wedge (\forall z (z \in B \iff ((z = x) \vee (z = y)))) \quad \text{Definition of } P(A, x, y) \\ \implies & \forall z ((z \in A \iff ((z = x) \vee (z = y))) \\ & \wedge (z \in B \iff ((z = x) \vee (z = y)))) \quad \text{Proposition 1.3.3} \\ \implies & \forall z (z \in A \iff z \in B) \quad \text{Proposition 1.5.4} \\ \implies & A = B \quad \text{Definition 2.1.1} \end{aligned}$$

)

$$\begin{aligned} & \forall x \forall y! \exists A, P(A, x, y) \quad \text{Similar to the proof of the Proposition 2.1.6} \\ \implies & \forall x \forall y! \exists A \forall z (z \in A \iff ((z = x) \vee (z = y))) \quad \text{Definition of } P(A, x, y) \end{aligned}$$

Definition 2.1.11. The unique pair set of x and y is denoted by $\{x, y\}$.

$$\text{Let } P(A, x, y) = \forall z (z \in A \iff ((z = x) \vee (z = y)))$$

Similar to the proof of Definition 2.1.7,

$$\forall x \forall y P(\{x, y\}, x, y)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x \forall y \forall A (P(A, x, y) \implies (A = \{x, y\}))$$

Proposition 2.1.12. Existence of singleton set.

$$\forall x \exists A \forall y (y \in A \iff (y = x))$$

Proof:

$$\begin{aligned} & \forall x \exists A \forall y (y \in A \iff ((y = x) \vee (y = x))) \quad \text{Axiom 2.3} \\ \implies & \forall x \exists A \forall y (y \in A \iff (y = x)) \quad \text{Proposition 1.2.9} \end{aligned}$$

Proposition 2.1.13. Uniqueness of singleton set.

$$\forall x! \exists A \forall y (y \in A \iff (x = y))$$

Let $P(A, x) = \forall y (y \in A \iff (x = y))$

The proof is similar to the proof of Proposition 2.1.10.

Definition 2.1.14. The unique singleton set of x is denoted by $\{x\}$.

Let $P(A, x) = \forall y (y \in A \iff (x = y))$

Similar to the proof of Definition 2.1.7,

$$\forall x P(\{x\}, x)$$

Similar to the proof of Proposition 2.1.8,

$$\forall x \forall A (P(A, x) \implies (A = \{x\}))$$

Axiom 2.4. Axiom of union. Existence of union set.

$$\forall F \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

Proposition 2.1.15. Uniqueness of union set.

$$\forall F! \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$$

Proof:

Let $P(A, F) = \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$

$\forall F \forall A \forall B ($

$$\begin{aligned} & P(A, F) \wedge P(B, F) \\ \implies & (\forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) \\ & \wedge (\forall x (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) \quad \text{Definition of } P(A, F) \\ \implies & \forall x ((x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F)))) \\ & \wedge (x \in B \iff (\exists Y ((x \in Y) \wedge (Y \in F))))) \quad \text{Proposition 1.3.3} \\ \implies & \forall x (x \in A \iff x \in B) \quad \text{Proposition 1.5.4} \\ \implies & A = B \quad \text{Definition 2.1.1} \end{aligned}$$

)

$\forall F! \exists A, P(A, F)$ Similar to the proof of the Proposition 2.1.6
 $\implies \forall F! \exists A \forall x (x \in A \iff (\exists Y ((x \in Y) \wedge (Y \in F))))$ Definition of $P(A, F)$

Definition 2.1.16. The unique union set of F is denoted by $\bigcup F$.

Let $P(A, F) = \forall x(x \in A \iff (\exists Y((x \in Y) \wedge (Y \in F))))$

Similar to the proof of Definition 2.1.7,

$$\forall F P(\bigcup F, F)$$

Similar to the proof of Proposition 2.1.8,

$$\forall F \forall A (P(A, F) \implies (A = \bigcup F))$$

Definition 2.1.17. Definition of 0.

$$0 \stackrel{\text{def}}{=} \emptyset$$

Definition 2.1.18. Definition of successor $S(x)$.

$$\forall x (S(x) = \bigcup \{x, \{x\}\})$$

Proposition 2.1.19. Pairwise union.

$$\forall A \forall B \forall x (x \in \bigcup \{A, B\} \iff ((x \in A) \vee (x \in B)))$$

Proof:

$\forall A \forall B \forall x ($

$$x \in \bigcup \{A, B\}$$

$$\iff \exists Y ((x \in Y) \wedge (Y \in \{A, B\}))$$

Definition 2.1.16

$$\iff \exists Y ((x \in Y) \wedge ((Y = A) \vee (Y = B)))$$

Definition 2.1.11

$$\iff \exists Y (((x \in Y) \wedge (Y = A)) \vee ((x \in Y) \wedge (Y = B)))$$

Proposition 1.2.16

$$\iff (\exists Y ((x \in Y) \wedge (Y = A))) \vee (\exists Y ((x \in Y) \wedge (Y = B)))$$

Proposition 1.3.4

$$\iff ((x \in A) \vee (x \in B))$$

Proposition 1.4.4 and Axiom 1.5 with $P(A, x) = (x \in A)$

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Definition 2.1.20. Definition of 1.

$$1 \stackrel{\text{def}}{=} S(0)$$

$$= \bigcup \{0, \{0\}\}$$

Definition 2.1.18

$$= \bigcup \{\emptyset, \{\emptyset\}\}$$

Definition 2.1.17