Random Fact 13.1

The Limits of Computation

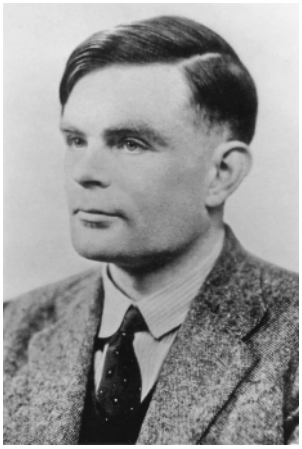
Have you ever wondered how your instructor or grader makes sure your programming homework is correct? In all likelihood, they look at your solution and perhaps run it with some test inputs. But usually they have a correct solution available. That suggests that there might be an easier way. Perhaps they could feed your program and their correct program into a "program comparator", a computer program that analyzes both programs and determines whether they both compute the same results. Of course, your solution and the program that is known to be correct need not be identical—what matters is that they produce the same output when given the same input.

How could such a program comparator work? Well, the Java compiler knows how to read a program and make sense of the classes, methods, and statements. So it seems plausible that someone could, with some effort, write a program that reads two Java programs, analyzes what they do, and determines whether they solve the same task. Of course, such a program would be very attractive to instructors, because it could automate the grading process. Thus, even though no such program exists today, it might be tempting to try to develop one and sell it to universities around the world.

However, before you start raising venture capital for such an effort, you should know that theoretical computer scientists have proven that it is impossible to develop such a program, *no matter how hard you try.*

There are quite a few of these unsolvable problems. The first one, called the *halting problem,* was discovered by the British researcher Alan Turing in 1936. Because his research occurred before the first actual computer was constructed, Turing had to devise a theoretical device, the *Turing machine,* to explain how computers could work. The Turing machine con-

Alan Turing

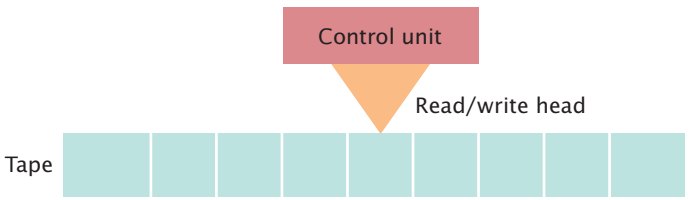


sists of a long magnetic tape, a read/write head, and a program that has numbered instructions of the form: "If the current symbol under the head is *x*, then replace it with *y*, move the head one unit left or right, and continue with instruction *n*" (see figure below). Interestingly enough, with only these instructions, you can program just as much as with Java, even though it is incredibly tedious to do so. Theoretical computer scientists like Turing machines because they can be described using nothing more than the laws of mathematics.

Expressed in terms of Java, the halting problem states: "It is impossible to write a program with two inputs, namely the source code of an arbitrary Java program P and a string I, and that decides whether the program *P*, when executed with the input *I*, will halt—that is, the program will not get into an infinite loop with the given input". Of course, for some

Program

Instruction number	If tape symbol is	Replace with	Then move head	Then go to instruction
1	0	2	right	2
1	1	1	left	4
2	0	0	right	2
2	1	1	right	2
2	2	0	left	3
3	0	0	left	3
3	1	1	left	3
3	2	2	right	1
4	1	1	right	5
4	2	0	left	4



A Turing Machine

kinds of programs and inputs, it is possible to decide whether the program halts with the given input. The halting problem asserts that it is impossible to come up with a single decision-making algorithm that works with all programs and inputs. Note that you can't simply run the program *P* on the input *I* to settle this question. If the program runs for 1,000 days, you don't know that the program is in an infinite loop. Maybe you just have to wait another day for it to stop.

Such a "halt checker", if it could be written, might also be useful for grading homework. An instructor could use it to screen student submissions to see if they get into an infinite loop with a particular input, and then stop checking them. However, as Turing demonstrated, such a program cannot be written. His argument is ingenious and quite simple.

Suppose a "halt checker" program existed. Let's call it *H*. From *H,* we will develop another program, the "killer" program *K*. *K* does the following computation. Its input is a string containing the source code for a program *R*. It then applies the halt checker on the input program *R* and the input string *R*. That is, it checks whether the program *R* halts if its input is its own source code. It sounds bizarre to feed a program to itself, but it isn't impossible. For example, the Java compiler is written in Java, and you can use it to compile itself. Or, as a simpler example, a word counting program can count the words in its own source code.

When *K* gets the answer from *H* that *R* halts when applied to itself, it is programmed to enter an infinite loop. Otherwise *K* exits. In Java, the program might look like this:

```
public class Killer
{
   public static void main(String[] args)
   {
      String r = read program input;
      HaltChecker checker = new HaltChecker();
      if (checker.check(r, r))
      {
         while (true) { } // Infinite loop
      }
      else
      {
         return;
      }
   }
}
```

Now ask yourself: What does the halt checker answer when asked whether *K* halts when given *K* as the input? Maybe it finds out that *K* gets into an infinite loop with such an input. But wait, that can't be right. That would mean that `checker.check(r, r)` returns `false` when `r` is the program code of *K*. As you can plainly see, in that case, the `killer` method returns, so K didn't get into an infinite loop. That shows that *K* must halt when analyzing itself, so `checker.check(r, r)` should return `true`. But then the `killer` method doesn't terminate—it goes into an infinite loop. That shows that it is logically impossible to implement a program that can check whether *every* program halts on a particular input.

It is sobering to know that there are *limits* to computing. There are problems that no computer program, no matter how ingenious, can answer.

Theoretical computer scientists are working on other research involving the nature of computation. One important question that remains unsettled to this day deals with problems that in practice are very time-consuming to solve. It may be that these problems are intrinsically hard, in which case it would be pointless to try to look for better algorithms. Such theoretical research can have important practical applications. For example, right now, nobody knows whether the most common encryption schemes used today could be broken by discovering a new algorithm. Knowing that no fast algorithms exist for breaking a particular code could make us feel more comfortable about the security of encryption.