COMPLUTENSE UNIVERSITY OF MADRID

FACULTY OF PHYSICAL SCIENCES

DEPARTMENT OF STRUCTURE OF MATTER, THERMAL PHYSICS AND ELECTRONICS

BACHELOR'S DEGREE THESIS

Code: ETE31

Chaos and Statistical Physics in Quantum Systems

Supervisor: Armando Relaño Pérez

Samuel Lozano Iglesias

Double Bachelor's Degree in Mathematics and Physics

Academic year 2023-2024

June call

Abstract:

A study of the transition between predictable (integrable) and chaotic behaviour in quantum systems is presented. The first section introduces Random Matrix Theory, highlighting the most important Gaussian ensembles for describing chaotic quantum systems. The characterisation of these ensembles by the distribution of the eigenvalues of their random matrices is also shown. This is used in the second section to distinguish integrable and chaotic situations in a quantum context, in addition to defining three statistics (s, r and *gap ratio*) that allow to analyse the spectrum of levels of the Hamiltonian of the system. In the last section, the analysis is applied to a Tavis-Cummings lattice, modifying the interaction strength between spins and photons and also breaking symmetries, in order to describe the transition. Finally, it is concluded that the transition is qualitatively monotonic and continuous, and the properties of its spectral distribution can be expressed as a convex combination of the properties of integrable and chaotic behaviour.

Key words: *Random matrices*, *statistic*, *chaotic system*, *integrable system*, *transition*.

Responsible Declaration on the Authorship and Ethical Use of Artificial Intelligence (AI) tools

I, Samuel Lozano Iglesias, with ID: 50250815-R, declare responsibly that this Bachelor's Degree Final Project entitled:

Chaos and statistical physics in quantum systems

is the result of my personal and creative intellectual work, and has been prepared in accordance with the ethical principles and standards of integrity in force in the academic community and, more specifically, at the Complutense University of Madrid.

I am, therefore, the author of the material included here and, when I have not done so and have taken the material from another source, I have either cited it or clearly stated its provenance -including, where appropriate, artificial intelligence tools-. The main ideas and contributions included in this work, and which demonstrate the acquisition of competences, are my own and do not come from other sources or have been rewritten using material from other sources.

I also assure that the data and resources used are legitimate, verifiable and have been obtained from reliable and authoritative sources. Furthermore, I have taken measures to guarantee the confidentiality and privacy of the data used, avoiding any type of bias or unfair discrimination in the treatment of the information.

In Madrid on 15 May 2024

Contents

1 Introduction

The primary motivation of physics, as well as that of philosophy and the other sciences, is to describe what the senses can detect. In this descriptive process, the need naturally arises to verify whether the description believed to be correct actually is, and it is this verification that leads to prediction: *if my expression adequately represents the dynamics of the system, I will be able to know what will happen in the future.* This, determinism, which seems nothing more than logical and common sense, puzzled the physics community at the beginning of the 20th century.

With the discovery of a new discipline, quantum physics, and the subsequent paradigm shift, physics went from being considered capable of understanding and characterizing any system with certainty, to recognizing the insurmountable obstacles it had to adapt to when studying quantum systems. So much so that science itself underwent a transformation, and concepts such as indeterminacy or randomness ceased to be associated with a lack of knowledge: *I am aware that I cannot describe the dynamics of a quantum system with total precision.*

Returning to the familiar and friendly classical physics, there is an area that challenges the concept of predictability: chaos. Characterized by sensitivity to initial conditions and non-periodicity, classical chaotic systems exhibit complex dynamics that, in the long term, are difficult to predict. Their analysis is based on the study of trajectories: *but then, what do I do if I want to study chaos and I don't have trajectories?*

This, among others, is the problem to be addressed in quantum chaotic systems. Starting from the assumption that an appropriate definition of quantum chaos exists, which is not trivial, and since the usual procedure $\hbar \to 0$ to relate classical and quantum analogs is not feasible due to the absence of trajectories, the study of a quantum chaotic system focuses on the matrix representation of its Hamiltonian. Yet, a new question arises: *if my system is chaotic and exhibits certain randomness, what matrices do I use to represent it?*

At this point, Random Matrix Theory is introduced, which is crucial for describing quantum chaos using the system's level spectra and Gaussian ensembles. This procedure allows for the characterization of an integrable system and a chaotic one, and thus to distinguish between them. However, it is possible for the same system, when varying certain parameters, to exhibit both types of behavior. Here lies the question on which this Bachelor's Thesis is developed: *how does the transition from integrable to chaotic behavior, and vice versa, occur in a quantum system?*

2 Random Matrix Theory

Random Matrix Theory (RMT) is a field of mathematics that arises from the intersection of probability theory and matrix algebra. This approach combines analytical and statistical tools to study properties of sets of matrices whose coefficients follow different probability distributions.

In many physical problems, the interactions between components exhibit a certain degree of randomness, and this must also be represented by the matrices that describe the system. The motivation behind RMT lies in the need to model complex systems where inherent variability plays a crucial role, as the introduction of random matrices allows the study to be approached from a perspective that is not purely deterministic.

In what follows, the subset of random matrices composed of finite Gaussian random matrices will be discussed. These will be formally and rigorously defined as $N \times N$ matrices with random variables following a joint Gaussian distribution. To deduce the properties of this set of matrices, we will start from the random variables that compose them and generalize to the arrangement of these variables in a matrix structure.

2.1 Gaussian Random Matrices

Before defining a Gaussian random matrix, it is necessary to define the concept of a Gaussian random variable [3].

Definition 2.1. *A real Gaussian random variable is a measurable function $X : \mathbb{R} \to \mathbb{R}$ whose probability distribution is Gaussian, with mean μ and variance σ^2:*

$$p(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(X-\mu)^2}{2\sigma^2}}$$

and is denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$.

A relevant parameter that characterizes a random variable is the expected value $\mathbb{E}[f(x)]$ of a function $f(x)$, which should be understood as the expectation of the experimental mean value of $f(x)$ if the experiment is repeated a sufficient number of times. In practice, for the Gaussian probability distribution, the most interesting expected values are those of polynomials, and in particular that of the polynomial x, which defines the expected value of the variable X itself.

Definition 2.2. *Given a function $f(x) : \mathbb{R} \to \mathbb{R}$, its expected value under the distribution $p(X)$ is defined as:*

$$\mathbb{E}[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot p(x) \, dx \, .$$

In particular, the expected value of the random variable X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot p(x) \, dx \, .$$

Before generalizing to the concept of a random matrix, a consideration must be made: the joint probability distribution of a finite number of independent random variables is the product of the individual distributions. Formally:

Theorem 2.3. *Let $X_1, ..., X_n$ be n independent random variables. The joint probability distribution of all of them can be written in terms of the individual probability distribution $p(X_i)$ of each one:*

$$p(X_1, ..., X_n) = p(X_1) \cdot ... \cdot p(X_n) =: \prod_{k=1}^{n} p(X_k) \, .$$

A Gaussian random matrix will then be an ordered set of $N \times N$ random variables whose probability distribution depends on the trace of the matrix [3].

Definition 2.4. *A Gaussian random matrix $H = (H_{jk})_{j,k=1}^{N}$ of dimension $N \times N$ is a matrix formed by a set of real random variables whose probability distribution is:*

$$p(H) := p(H_{11}, ..., H_{NN}) = \frac{1}{\mathcal{Z}_{N,A}} \cdot e^{-A \cdot tr(H^2)} \, ,$$

where A is a constant, $\mathcal{Z}_{N,A}$ is a normalization constant, and $tr(H^2)$ is the trace of the matrix H^2, given by

$$tr(H^2) = \sum_{k=1}^{N} H_{kk}^2 \, .$$

2.2 Classical Gaussian Ensembles, *β-ensembles*

Within Gaussian random matrices, three relevant ensembles can be identified[1]. Of these, only two will be studied in this document: the so-called GOE (from the English *Gaussian Orthogonal Ensemble*) and GUE (from the English *Gaussian Unitary Ensemble*), while the third ensemble is called GSE (from the English *Gaussian Symplectic Ensemble*) and its study involves greater complexity, in addition to not being as important in the physical analyses that will be carried out later.

2.2.1 *Gaussian Orthogonal Ensemble*, GOE

Definition 2.5. *The set of $N \times N$ Gaussian random matrices that are real and symmetric, i.e., $H_{jk} = H_{kj} \in \mathbb{R}$, and whose probability distribution is*

$$p(H) = \frac{1}{2^{N/2} \cdot \pi^{N(N+1)/4}} \cdot e^{-\frac{1}{4} \cdot tr(H^2)} \, ,$$

is called the Gaussian Orthogonal Ensemble (GOE). In short, for a matrix H belonging to this ensemble, it will be said that "H is GOE".

[1] An ensemble is understood as the subset of random matrices that share a probability distribution and are invariant under the same type of transformation.

The name does not come from the fact that GOE matrices are orthogonal, which they are not, but from their invariance under real and orthogonal transformations. Two useful characterizations of this type of matrices are the following, and their proof can be found in [7]:

Proposition 2.6. *Let $A = (a_{jk})_{j,k=1}^{N}$ be a random matrix formed by a_{jk} independent real Gaussian random variables. Then the matrix $H = \frac{A+A^T}{\sqrt{2}}$ is a GOE matrix.*

Proposition 2.7. *Let H be a real symmetric random matrix whose diagonal terms follow a normal distribution with mean 0 and variance 1, i.e., $H_{kk} \sim \mathcal{N}(0,1)$, and the off-diagonal terms follow a normal distribution with mean 0 but variance $1/2$, i.e., $H_{jk} \sim \mathcal{N}(0,1/2)$. Then the matrix H is GOE.*

2.2.2 *Gaussian Unitary Ensemble*, GUE

Definition 2.8. *The set of $N \times N$ Gaussian random matrices that are Hermitian, i.e., $H_{jk} = H_{kj}^* \in \mathbb{C}$, where $*$ denotes the complex conjugate, and whose probability distribution is*

$$p(H) = \frac{2^N}{\pi^{3N/2}} \cdot e^{-\frac{1}{2} \cdot tr(H^2)} \,,$$

is called the Gaussian Unitary Ensemble (GUE). In short, for a matrix H belonging to this ensemble, it will be said that "H is GUE".

In this case, the name comes from the invariance of these matrices under unitary transformations. Two characterizations analogous to the previous ones but for GUE matrices, with the proof presented in [7], are:

Proposition 2.9. *Let $A = (a_{jk})_{j,k=1}^{N}$ be a random matrix formed by a_{jk} independent complex Gaussian random variables. Then the matrix $H = \frac{A+A^*}{\sqrt{2}}$ is a GUE matrix.*

Proposition 2.10. *Let H be a Hermitian random matrix whose diagonal terms are real and follow a normal distribution with mean 0 and variance 1, i.e., $H_{kk} \sim \mathcal{N}(0,1)$, and the off-diagonal terms are such that $H_{jk} \sim (\mathcal{N}(0,1/2) + i\mathcal{N}(0,1/2))$. Then the matrix H is GUE.*

2.3 Characterization of the Probability Distribution by Eigenvalues

The determination of the eigenvalues of random matrices is essential in their characterization and identification, as they completely determine their probability distribution and, therefore, their ensemble. To do this, first, the spectrum of a matrix H is defined.

Definition 2.11. *The spectrum of a matrix H is the set $\mathcal{S} := \{\lambda \in \mathbb{C} : \det(H - \lambda I) = 0\}$ where λ is an eigenvalue of H and the matrix I is the identity.*

Thus, considering an ordered spectrum $\mathcal{S} = \{\lambda_1, ..., \lambda_N \in \mathbb{C} : \lambda_j < \lambda_k \text{ if } j < k\}$, the joint probability distribution of a random matrix can be characterized by the following theorem [3].

Theorem 2.12. *The joint probability distribution of a Gaussian random matrix H of dimension $N \times N$ with $\beta = 1, 2, 4$ is determined from its eigenvalues $\lambda_1, ..., \lambda_N$ as*

$$f(\lambda_1, ..., \lambda_N) = \frac{1}{\mathcal{Z}_{N,A}} \cdot e^{-A \cdot \sum\limits_{k=1}^{N} \lambda_k^2} \cdot \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta$$

where the normalization function $\mathcal{Z}_{N,A}$ can be written in terms of the Gamma function:

$$\mathcal{Z}_{N,A} = (4A)^{-N/2 - N(N-1) \cdot A} \cdot (2\pi)^{N/2} \cdot \prod_{k=0}^{N-1} \frac{\Gamma(1 + (k+1) \cdot 2A)}{\Gamma(1 + 2A)} \, .$$

The three possible values of β are characteristic of the three classical ensembles described earlier, with $\beta = 1$ for GOE, $\beta = 2$ for GUE, and $\beta = 4$ for GSE. This parameter β, called the Dyson index, is what gives the name *β-ensembles* to the studied ensembles.

From the above probability distribution, the intuition arises naturally to define a variable determined by the separation between consecutive eigenvalues. This, the statistic s, and another defined in a similar way, the statistic r, are two of the tools that allow classifying matrices into the different ensembles knowing only their eigenvalue spectrum.

3 Quantum Chaos

Chaotic motion is conceived as the opposite of predictable and deterministic motion. However, while the former is true, the latter is not: classical chaotic motion is neither random nor stochastic; it is deterministic. The key condition to discern whether a motion is chaotic is the absence of repetitive (i.e., predictable) patterns, as well as the fact that a small variation in the initial conditions greatly modifies the final outcome.

Returning to the concept of predictability, classical mechanics distinguishes between integrable (predictable) systems and chaotic systems, with many different types of the latter. This distinction is also maintained in quantum systems, but its definition is more complicated, requiring techniques still under development.

3.1 Integrable Systems - *Poisson*

Definition 3.1. *A dynamical system is said to be classically integrable if it has n independent and involutive constants of motion. That is, a system with Hamiltonian H is said to be integrable if it satisfies:*

(i) *There exist functions* $\{f_k\}_{k=1}^n$*, called constants of motion, satisfying* $\{f_k, H\} = 0$*,* $\forall\, k \in \{1, ..., n\}$*, where* $\{\cdot, \cdot\}$ *denotes the Poisson bracket, and in fact,* $\{f_k, H\} = \frac{df_k}{dt}$*.*

(ii) *The constants of motion satisfy* $\{f_j, f_k\} = 0$*,* $\forall j, k \in \{1, ...n\}$ *with* $j \neq k$*.*

(iii) *The constants of motion are pairwise independent.*

This definition, cited from [8], allows us to understand the concept and directly translate it into a quantum scenario, assuming that the classical limit emerges from the quantum description by taking $\hbar \to 0$. Thus, the "direct translation" of classical integrability to the quantum case is as follows:

Definition 3.2. *A dynamical system with Hamiltonian* H *is said to be quantum integrable if:*

(i) *There exist functions* $\{f_k\}_{k=1}^n$*, called constants of motion, satisfying* $[f_k, H] = 0$*, where* $[A, B] = AB - BA$ *is the commutator.*

(ii) *The constants of motion satisfy* $[f_j, f_k] = 0$*,* $\forall j, k \in \{1, ...n\}$ *with* $j \neq k$*.*

The key issue in this case lies in condition (III) of the classical case, since, according to von Neumann's theorem[2], this condition cannot be satisfied in quantum mechanics, making it difficult to separate quantum systems into two disjoint sets: integrable and non-integrable.

In fact, if one were to study a Hamiltonian in a finite system, the projectors would satisfy Definition 3.2, meaning that all finite systems would exhibit constants of motion. For this and other reasons, various alternative definitions of quantum integrability have been proposed, with a review available in [6].

To circumvent this issue, this document considers the study of the spectral levels of quantum systems. It can be rigorously shown [5], and is experimentally verified, that energy level crossings lead to a decorrelation between them: the correlation between levels in a system with a classically integrable analogue follows Poisson statistics.

3.2 Chaotic Systems - Gaussian Ensembles

Quantum chaotic systems start from the idea of generalizing the well-defined concept of classical chaotic systems. However, it is clear that the differences between these two types of systems outweigh their similarities. Although there is no universal definition of chaos, a mathematical proposal that encompasses its defining characteristics is given in [13]:

[2]Von Neumann's theorem, whose proof can be found in [2], states the following [14]: *Let* $\{O_k\}_{k=1}^n$ *be an arbitrary set of Hermitian operators such that* $[O_j, O_k] = 0$*,* $\forall\, j \neq k$*. Then, there exists a non-degenerate operator* $\mathcal{O}$ *such that* $O_k = O_k(\mathcal{O})$*,* $\forall\, k \in \{1, ..., n\}$*.*

Definition 3.3. *Let a dynamical system be defined by $\dot{x}(t) = F(X)$ with $x \in \mathcal{M}$ and $F : \mathcal{M} \to T(\mathcal{M})$ a vector field, where $\mathcal{M}$ is a differentiable manifold and $T(\mathcal{M})$ its tangent space. Then, the classical dynamical system is said to be chaotic if there exists a subset $\mathcal{N} \subset \mathcal{M}$ with the following properties:*

(i) *Any trajectory*[3] *starting in $\mathcal{N}$ remains in $\mathcal{N}$, $\forall t > 0$.*

(ii) *The trajectories do not tend to fixed points, periodic orbits, or quasi-periodic orbits: for $x(t) \in \mathcal{N}$, there does not exist any $t_1 > 0$ such that $\forall\, T > t_1$ and $\forall \varepsilon > 0$, there exists $t_2 > T$ satisfying $\|x(t_2) - x(t_1)\| < \varepsilon$.*

(iii) *Similar initial conditions lead to different trajectories for sufficiently long times: $\exists A > 0$ such that $\forall\, x \in \mathcal{N}$ and for any neighborhood $\mathcal{B}$ of that point, there exists a point $y \in \mathcal{B}$ satisfying $\|x - y\| > A$.*

From a quantum perspective, the corresponding classical description can be obtained by taking the limit $\hbar \to 0$, which suggests that a similar deduction might hold in the chaotic case. However, the classical definition of chaos relies on the fact that the trajectory is completely deterministic, verifiable, and comparable. This is the main difference from the quantum analogue, as the complete determination of quantum trajectories is fundamentally impossible due to Heisenberg's uncertainty principle.

Thus, a different approach is necessary for quantum chaotic systems, introducing Gaussian ensembles as defined in Section 2.1. These ensembles encapsulate certain characteristics that make them suitable for particular quantum systems, notably [11]:

- The GOE ensemble applies to systems invariant under time reversal and rotational symmetry, or those without rotational symmetry but with integer spin.
- The GUE ensemble applies to systems that are not invariant under time reversal.

These results are, to some extent, theoretically deducible from the properties of the ensembles, although many related findings are actually conjectures or ideas based on experimentation. A prominent example is the BGS (*Bohigas-Giannoni-Schmit*) conjecture[4], formulated for time-reversal symmetric systems but modifiable by replacing the GOE ensemble with the GUE for systems that are not:

Conjecture 3.4. *The energy level spectrum of any time-reversal invariant system whose classical analogue is chaotic exhibits the fluctuation properties associated with the GOE ensemble.*

[3] The solutions $x(t)$ of the differential equation $\dot{x}(t) = F(x)$ are called trajectories.

[4] This result is valid for systems with a positive Kolmogorov constant, such as those discussed in this document.

On this conjecture, which remains unproven to date[5], much of the study conducted in the following section will be focused, as well as on the transition between the so-called quantum integrable systems, associated with a Poisson statistics, and quantum chaotic systems, related to the Gaussian ensembles of random matrices.

3.3 Transitions from Integrable to Chaotic Systems

Following this preliminary study, it is interesting to consider what characteristics determine the transition from an integrable system to a chaotic one, and whether there is a way to understand and describe it based on its possible similarities with already known systems. Furthermore, it is relevant to understand how this transition can be detected and what makes it possible.

The last section of this document will address this topic, analyzing a particular case in some detail. For this analysis, the statistics presented in this section will be used, allowing us to approach the objective: identifying, categorizing, and characterizing the transition under study.

3.3.1 Statistic s

Considering the spacing between eigenvalues E_1 and E_2 of a 2×2 GOE matrix, i.e., the spacing $s = E_2 - E_1$, it is possible to demonstrate the *Wigner Hypothesis* [3], which states that the spacing distribution is given by:

$$P(s) = \frac{\pi s}{2} \cdot e^{-\frac{\pi s^2}{4}} .$$

If, instead, a Gaussian $N \times N$ matrix is considered, it can be verified that the hypothesis remains valid even for large values of N. In fact, assuming the ordered spectrum $\mathcal{S} = \{E_1, ..., E_N\}$, the spacing between consecutive eigenvalues $s_k = E_{k+1} - E_k$ for $k = 1, ..., N-1$ follows a distribution given by [12]:

$$P(s) = A(\beta) \cdot s^\beta \cdot e^{-B(\beta)s^2} ,$$

where $\beta = 1, 2, 4$ represents the different ensembles, and the values of $A(\beta)$ and $B(\beta)$ can be determined using the Gamma function:

$$A(\beta) = 2 \cdot \frac{\Gamma^{\beta+1}((\beta+2)/2)}{\Gamma^{\beta+2}((\beta+1)/2)} \quad \text{and} \quad B(\beta) = \frac{\Gamma^2((\beta+2)/2)}{\Gamma^2((\beta+1)/2)} .$$

The term s^β in the spacing distribution denotes the level repulsion, which can be linear, quadratic, or quartic depending on the ensemble.

Regardless of the details, the distribution $P(s) \sim s^\beta \cdot e^{-s^2}$ indicates that for very small ($s \to 0$) or very large spacings ($s \to \infty$), the distribution tends to 0, and it presents a maximum at a value that depends on β within the interval $(0, \infty)$. This explains why these ensembles can describe

[5]The numerous attempts to prove it have led to relatively successful demonstrations, among which the one by Michel Berry in 1985 stands out, available in [4]. Nevertheless, all attempts rely on approximations for $\hbar \to 0$.

chaotic systems: the level repulsion prevents two levels from having the same energy, causing the probability of overlap to tend to zero.

On the other hand, in the integrable limit, the level spacing follows the form $P(s) = e^{-s}$, which corresponds to a Poisson distribution representing non-chaotic integrable systems. In this case, $P(0) = 1$ and $P(s)$ decreases as s increases, indicating that the probability of degenerate levels (i.e., zero spacing) is higher than any other situation [4].

3.3.1.1 Level Rescaling

For the s statistic, it is necessary to rescale energy levels to ensure that the expected spacing is 1, i.e., $\langle s_k \rangle = \langle \epsilon_{k+1} - \epsilon_k \rangle = 1 \ \forall k \in \{1, ...n\}$, where ϵ_k is the rescaled energy of the k-th state. This is because the measurement scale is not common to all quantities, so it must be eliminated.

This requirement is satisfied by removing the smooth part $\bar{\rho}(E)$ of the density of states, given by $\rho(E) = \sum_{k=1}^{n} \delta(E - E_k)$, as it only indicates the locally appropriate energy scale, while the normalized fluctuating part $\tilde{\rho}(E)$ exhibits the properties of interest [1].

Using the cumulative level density $N(E) = \int_{E_0}^{E} \rho(E')\, dE'$, which simply takes the values $N(E_k) = k$, this distribution is fitted with a smooth function $\bar{N}(E)$, and the rescaled level $\epsilon_k = \bar{N}(E_k)$ is considered dimensionless.

For the particular case studied in this document, fitting the cumulative density with an eighth-degree polynomial is sufficient, as it appropriately separates the smooth and fluctuating components.

3.3.2 Statistic r

Another important statistic analyzed in various applications of RMT is the r statistic. It is defined based on the ordered eigenvalues $\{E_k\}_{k=1}^{N}$ of the Gaussian random matrix, without the need for rescaling:

$$r_k = \frac{E_{k+2} - E_{k+1}}{E_{k+1} - E_k} \qquad \text{for } k = 1, ..., N-2.$$

This statistic is useful because its distribution is well-defined and explicitly depends on the Dyson index β, but it does not require a rescaling method, allowing a parallel study to verify that rescaling does not introduce artificial properties. The probability distribution for Gaussian ensembles is [12]:

$$P(r) = C(\beta) \cdot \frac{(r + r^2)^{\beta}}{(1 + r + r^2)^{1+3\beta/2}} \qquad \text{with} \qquad C(\beta) = \frac{3^{3(1+\beta)/2} \cdot \Gamma(1 + \beta/2)^2}{2\pi \cdot \Gamma(1 + \beta)} \,.$$

In the integrable limit, the r statistic follows a Poisson-distributed ratio [12]:

$$P(r) = \frac{1}{(1 + r)^2} \,.$$

Since $P_{RMT}(0) = 0$ but $P_{Poisson}(0) = 1$, this statistic confirms that level repulsion is a defining property of chaotic systems.

3.3.3 *Gap ratio* $\tilde{r}$

Finally, it is worth considering a modification of the statistic r that is particularly interesting in the analysis of different collectives, as it exhibits well-defined expected values. This modification will be referred to as the *gap ratio*, denoted as $\tilde{r}_k$, to differentiate it from the statistic r. For $k = 1, \ldots, N-2$, it is defined as:

$$\tilde{r}_k = \frac{\min(E_{k+2} - E_{k+1}, E_{k+1} - E_k)}{\max(E_{k+2} - E_{k+1}, E_{k+1} - E_k)} \ .$$

The key feature in this case is that the support of the probability distribution of $\tilde{r}$ is the compact and bounded interval $[0, 1]$, and its probability distribution $\tilde{P}(\tilde{r})$ is directly related to that of the statistic r, $P(r)$ [10]:

$$\tilde{P}(\tilde{r}) = 2 \cdot P(\tilde{r}) \cdot \Theta(1 - \tilde{r}) \ ,$$

where $\Theta(x)$ is the Heaviside step function, defined as

$$\Theta(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

Thus, the expected value of the *gap ratio* $\tilde{r}$ can be computed for an integrable system following a Poisson distribution:

$$\langle \tilde{r} \rangle_{Poisson} = 2 \cdot \int_0^1 \frac{x}{(1+x)^2} \, dx = 2 \cdot \ln(|1+x|) + \left. \frac{2}{1+x} \right|_0^1 = 2 \ln(2) - 1 \approx 0.386 \qquad (1)$$

The expected value can also be computed for the GOE ensemble ($\beta = 1$):

$$\langle \tilde{r} \rangle_{GOE} = \frac{27 \cdot \Gamma(3/2)^2}{\pi \cdot \Gamma(2)} \int_0^1 \frac{x \cdot (x + x^2)}{(1 + x + x^2)^{5/2}} \, dx = -\frac{1}{2} \cdot \left. \frac{x^3 + 15x^2 + 12x + 8}{(x^2 + x + 1)^{3/2}} \right|_0^1 = 4 - 2\sqrt{3} \approx 0.536 \qquad (2)$$

Once again, the relevance of this statistic lies in the fact that its support is bounded, whereas the mean value of the Poisson statistic diverges for the statistic r, whose support is $\mathbb{R}^+$.

In summary, there are three possible statistics that can be used to determine the type of collectivity being studied and to analyze in detail the transition from an integrable to a chaotic system. This will be the focus of the next section: considering a particular quantum system and, once its eigenvalue spectrum is obtained, studying the characteristics that define it as a function of two varying parameters, either by analyzing the probability distribution of the eigenvalue spacings (statistics r and s) or by comparing the expected value of this probability distribution (statistic $\tilde{r}$).

4 Tavis-Cummings Network

The Tavis-Cummings (TCL) network is a model especially interesting from the perspective of quantum optics, as it incorporates several controllable quantum degrees of freedom. This network is relevant to the study at hand because it is composed of Tavis-Cummings (TC) integrable models that lose this property when coupled in the network: one can study the transition from a set of coupled integrable systems to a globally chaotic system.

For simplicity in treatment, we study an impurity model that consists of a single TC model to which a term is added that simulates the effect of coupling with its neighbors in the network. Additionally, this treatment allows the inclusion of a parameter μ, which breaks the symmetry and integrability of the TC model.

This model includes a set of N two-level atoms (possible spins $m_z = \pm 1/2$) that interact with, theoretically, infinitely many photons incident on them. To simulate this, a spin $N/2$ is considered, so the total angular momentum dimension is $N+1$. Ultimately, the Hamiltonian of the impurity model studied is as follows [10]:

$$H = \omega_c a^\dagger a + \omega_s S_z + \frac{\lambda}{\sqrt{S}} \cdot (a^\dagger S_- + a S_+) - \mu\sqrt{S} \cdot (a + a^\dagger)\ , \tag{3}$$

where the annihilation and creation operators are included: a and $a^\dagger$, respectively; the spin angular momentum operators S_z, S_+, and S_-. As for the frequencies ω_c and ω_s, a resonant regime is considered in which $\omega_c = \omega_s = \omega_0$ and the energy unit $\omega_0 = 1$ is taken.

We will therefore study the parameters λ and μ, the former as a measure of the strength of interaction between the spins and the photons, and the latter as a symmetry-breaking term. The construction and diagonalization of the matrices associated with the Hamiltonian, as well as the study presented below, can be found programmed in Python in [9].

4.1 Photon Truncation

For a practical study of the Hamiltonian (3), a finite number of incident photons n_{max} must be used, since the matrix to be diagonalized must have finite dimension. As theoretically, infinitely many photons would impinge on the network, a large number of photons must be considered relative to the number of spins, but determining the cutoff is not trivial.

To do this, it is understood and experimentally verified that the levels with lower energies remain at the same energy value as the number of photons increases, so all levels below a certain energy threshold can be considered real. This is observed in figure 1, where the evolution of the energy associated with the n-th state for different numbers of incident photons is shown for a model with $N = 2$.

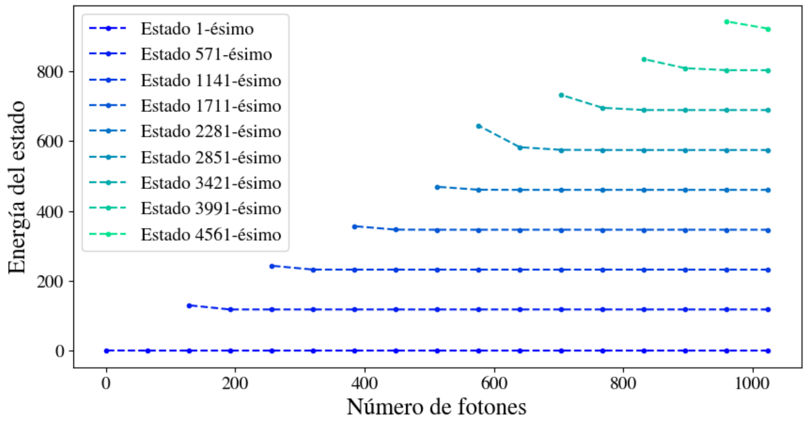


Figure 1: *Evolution of energy states as the number of incident photons increases.*

Thus, as a criterion for identifying real states, a comparison of the states obtained for the considered number of photons, n_{max}, with the states for $0.9 \cdot n_{max}$ is made: states associated with the n_{max} series will be assumed to be real if they coincide with those of the other series by more than 99%, and all their lower levels as well. Mathematically:

Remark 4.1. *Let* $\{E_{1,k}\}_{k=1}^{n_1}$ *and* $\{E_{2,k}\}_{k=1}^{n_2}$ *be the two series of states associated with* $0.9 \cdot n_{max}$ *and* n_{max}*, respectively. Real states* $\{\tilde{E}_k\}_{k=1}^{n}$ *will be taken such that, for* $k \in \{1, ..., n_1\}$*, with* $n_1 \leq n_2$*, if* $|E_{1,j} - E_{2,j}| \leq 0.01 \cdot E_{2,j}$ *,* $\forall\, j \leq k$ *then* $\tilde{E}_k = E_{2,k}$*.*

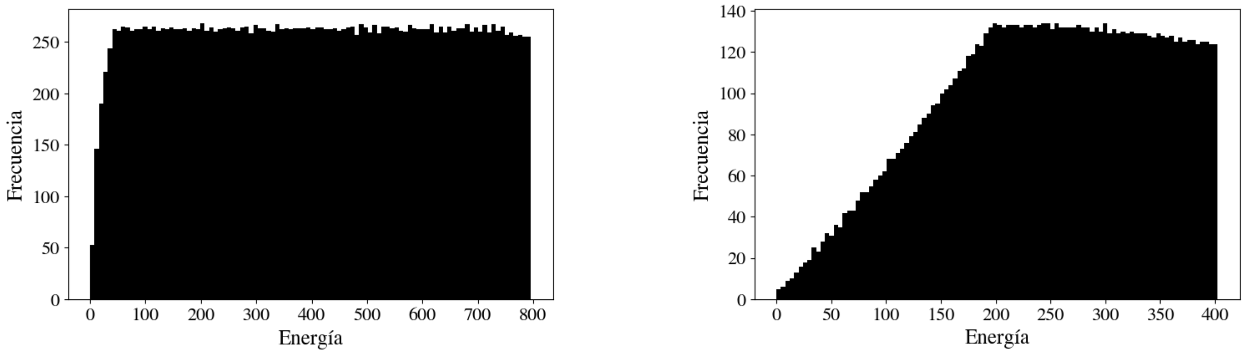
It is worth noting that photon truncation is more pronounced the higher the parameters λ and μ are, and also as the number of spins increases. This complicates obtaining results for values of $\lambda > 10$ and $\mu > 5$ in reasonable situations ($N \leq 32$), so the results presented in this document focus on intervals where truncation does not hinder the calculations.

4.2 Defined Situation: Integrable or Chaotic

The Tavis-Cummings network exhibits characteristics associated with integrable systems for low $\mu \geq 0$ and $\lambda > 0$. The energy spectra resemble those in figure 2a, keeping in mind that the ground state has been shifted to energy $E_0 = 0$, and the rest have moved accordingly. On the other hand, the network presents characteristics associated with chaotic systems for high $\mu \geq 0$ and $\lambda > 0$, with spectra resembling those in figure 2b.

As can be observed, for the chaotic case, the slope for low energies is less abrupt than for the integrable case, but this is not an extrapolable trend, as the cases where the slope is even smoother are characterized by presenting a mixed dynamics.

The differences between integrable and chaotic systems are present in the study of the s, r statistics, and the gap ratio, highlighting that the spectrum must be rescaled as indicated in section **??** for the s statistic.



(a) *Integrable system, $\mu = 0.2$.* (b) *Chaotic system, $\mu = 3$.*

Figure 2: *Comparison of extreme cases for $N = 32$ and $\lambda = 1$.*

These differences are visible in Figures 3a and 3b for the statistic s; 3c and 3d for the statistic r; and 3e and 3f for the *gap ratio*. In all of them, the theoretical distributions associated with Poisson, GOE, and GUE are plotted for comparison.

It is confirmed that the chosen rescaling method has been adequate, with the statistic s showing the same conclusions as the other statistics. Additionally, it can be seen that in the chaotic case, the probability distribution of all statistics approaches the one corresponding to the GOE ensemble, not the GUE, as the system is symmetric under time inversion.

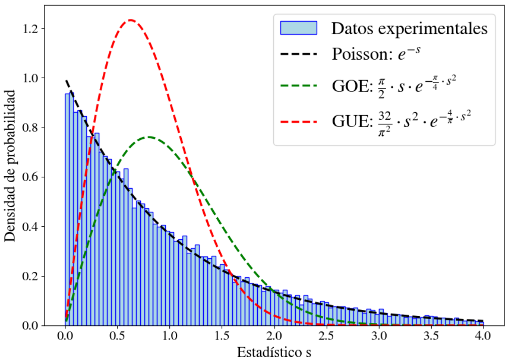
4.2.1 Pathological Case, $\lambda = 0$

It is instructive to study in some detail the case $\lambda = 0$ and $\mu \geq 0$. In this case, and following equation (3), we have a Hamiltonian with no coupling between the operators related to the spins (S_α) and the photons (a and $a^\dagger$), so the system should be equivalent to considering two independent systems.

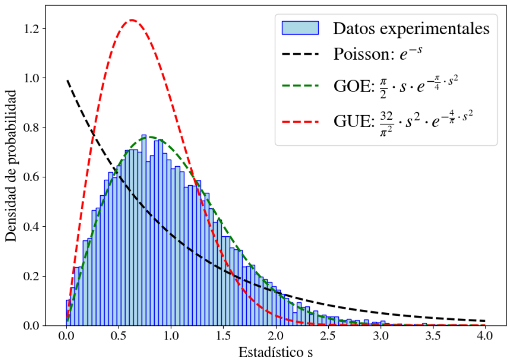
In practice, this is what is observed, with very close values ($s \to 0$) and equally spaced values ($r \to 1$), regardless of the parameter $\mu \geq 0$ considered, as shown in Figures 4a and 4b. The results for the *gap ratio* in Figure 4c are analogous.

4.3 Transition to Chaos as a Function of the Parameters

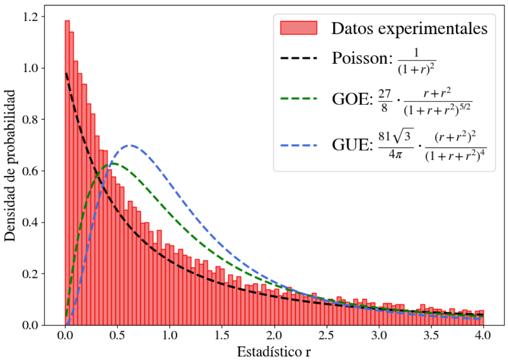
Once at least two situations are obtained where integrable and chaotic system characteristics are observed, respectively, it is natural to ask how the transition from one behavior to the other is characterized. Intuitively, one might think of a smooth transition, with the integrable and chaotic cases as extremes, but is the transition linear? Is it continuous? Is it monotonic? Once the system transitions from integrable to chaotic, if the symmetry continues to break, does chaos remain? Can the transition be characterized by intermediate systems between a chaotic one and an integrable one?



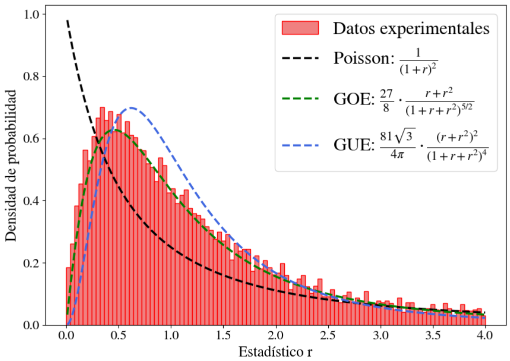
(a) *Statistic s for $\mu = 0.2$ and $\lambda = 1$.*



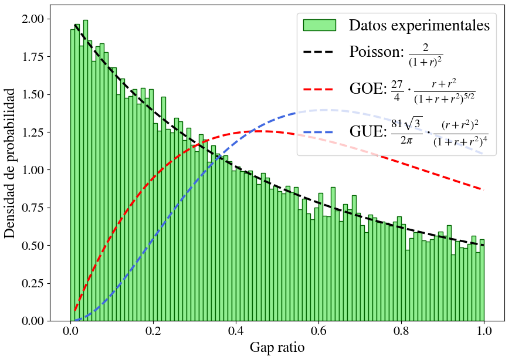
(b) *Statistic s for $\mu = 3.2$ and $\lambda = 0.5$.*



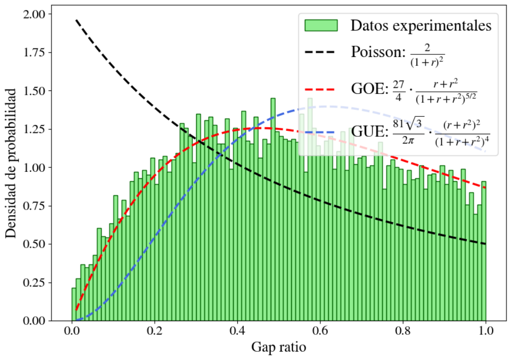
(c) *Statistic r for $\mu = 0.2$ and $\lambda = 1$.*



(d) *Statistic r for $\mu = 3.2$ and $\lambda = 0.5$.*

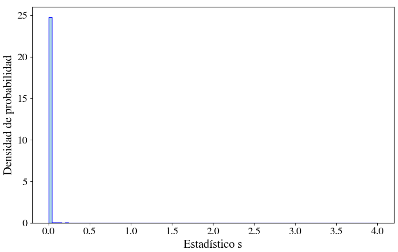


(e) *Gap ratio for $\mu = 0.2$ and $\lambda = 1$.*

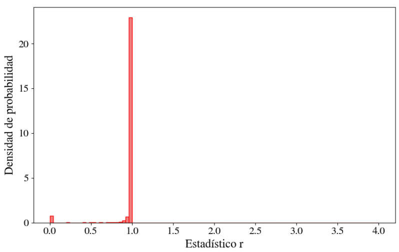


(f) *Gap ratio for $\mu = 3.2$ and $\lambda = 0.5$.*

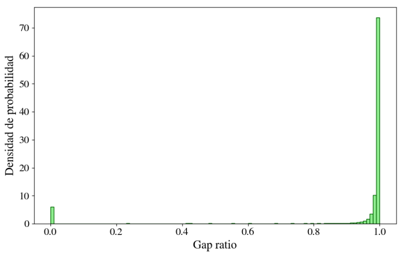
Figure 3: *Comparison of various statistics in integrable and chaotic cases for $N = 32$.*



(a) *Statistic s.*



(b) *Statistic r.*



(c) *Gap ratio.*

Figure 4: *Pathological case $\lambda = 0$ for $N = 32$, with $\mu \geq 0$.*

4.3.1 Comparison for Different Values of μ

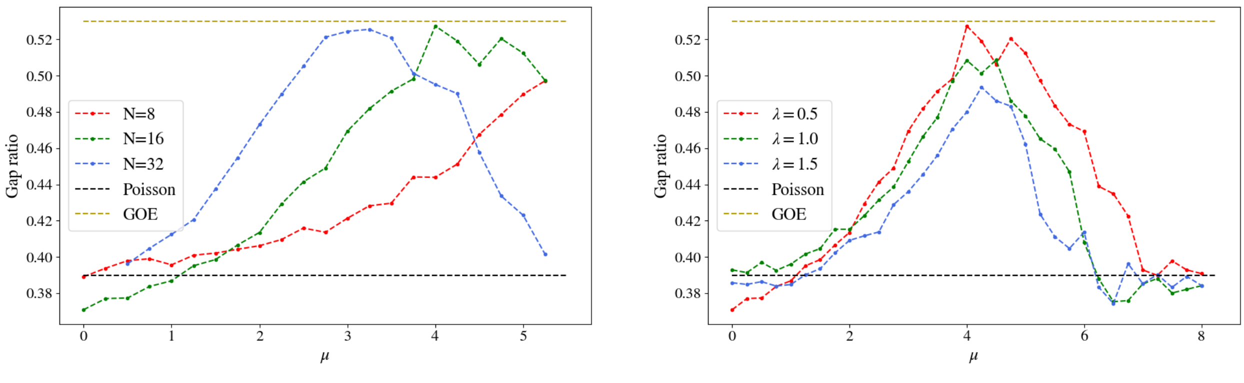
Considering a fixed value of $\lambda = 0.5$, a study is then conducted with μ variable in the range $0 \leq \mu \leq 5.5$, observing the different statistics. A noteworthy result is shown in Figure 5a, where the evolution of the *gap ratio* for different numbers of spins is compared.

Due to the difficulty of conducting such an analysis, and with the intention of extending the conclusions to larger values of μ and different values of λ, the study was repeated for $\lambda = 1$ and $\lambda = 1.5$ for a range of $\mu \in [0, 8]$, but only in the case $N = 16$. The comparative results are shown in Figure 5b, where the results from equations (1) and (2) for the *gap ratio* were taken into account. From these two figures, some relevant conclusions can be drawn:

1. As the number of spins increases, the symmetry breaking (parameter μ) required to observe chaotic behaviors becomes smaller.

2. After a series of μ values where chaotic behavior is observed, if μ continues to increase, i.e. greater symmetry breaking, the behavior returns to being integrable. This can be understood as there being values of μ that break symmetry but maintain the system coupled and chaotic, while for very high values of μ, the system behaves like fully decoupled integrable systems.

3. The greater the interaction between the spins and the photons, i.e. a higher λ, the less pure chaotic system (GOE) is obtained. This indicates that for situations with greater interaction between spins and photons, integrable or mixed-property systems tend to emerge.



(a) *Transitions for $\lambda = 0.5$ and variable N.*

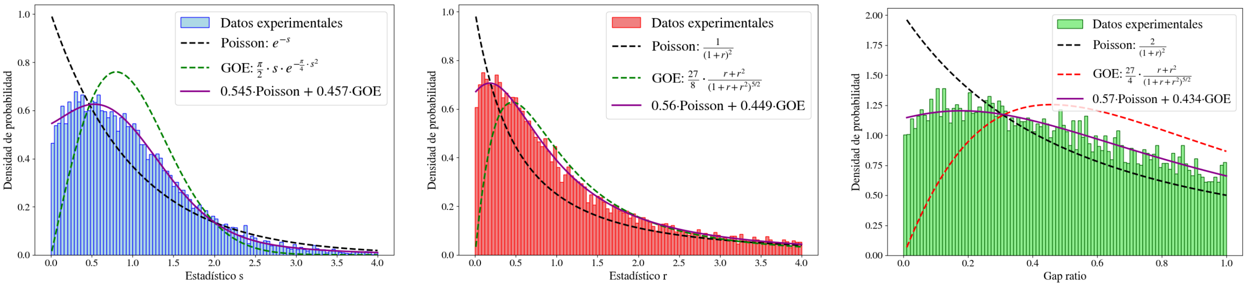
(b) *Transitions for $N = 16$ and variable λ.*

Figure 5: *Transitions between integrable and chaotic systems.*

With the intention of characterizing the transition observed in Figure 5a for $N = 32$ in more detail, a simultaneous analysis for the three statistics is conducted. For each of them, an attempt is made to fit the distribution obtained for $\lambda = 1$ and each value of μ as a linear combination of the theoretical Poisson and GOE distributions[6]. That is, for each statistic and each value of μ, we aim to obtain real values α and β such that:

$$P_\mu = \alpha \cdot P_{Poisson} + \beta \cdot P_{GOE} \tag{4}$$

A key feature in this case is that it is not imposed that $\alpha + \beta = 1$ or that $\alpha, \beta \geq 0$, allowing more freedom in the fit to identify the transition[7]. This procedure for each statistic with arbitrary values $\mu = 1.75$ and $\lambda = 1$ is shown in Figure 6.



(a) *Fit for the statistic s.* (b) *Fit for the statistic r.* (c) *Fit for the gap ratio.*

Figure 6: *Experimental fits for $\mu = 1.75$ and $\lambda = 1$ in the $N = 32$ case.*

Repeating the process for each value of $\mu \in [0, 4]$ and $\lambda = 1$, two curves can be obtained (one for the optimal values of α and the other for the optimal values of β) for each statistic, indicating how the system evolves from an integrable type to a chaotic type. A comparison of these curves for the three statistics and $N = 32$ can be seen in Figure 7.

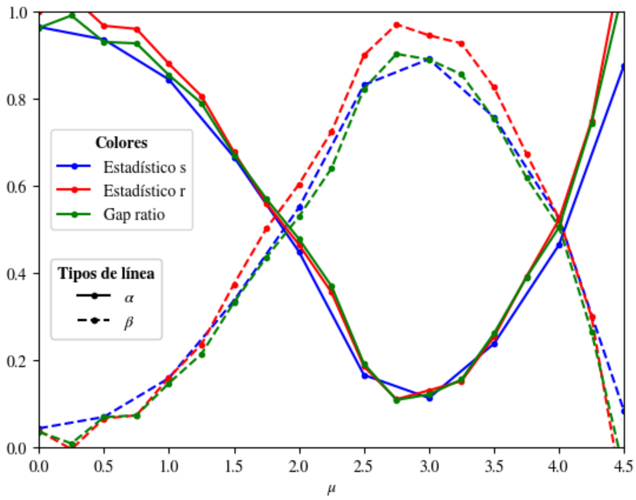


Figure 7: *Evolution of α and β for the three statistics as a function of μ for $N = 32$ and $\lambda = 1$.*

[6] In this case, $\lambda = 0.5$ in Figure 5a is replaced by $\lambda = 1$ for completeness and to show that the result does not depend on a specific λ value. Nevertheless, analogous results would be obtained for $\lambda = 0.5$.

[7] If the fit is adequate, since both distributions are normalized, it will trivially hold that $\alpha + \beta = 1$, but the condition $\alpha, \beta \geq 0$ does not necessarily have to be satisfied directly.

As seen in Figure 7, the three statistics follow practically identical evolutions, indicating that the studied properties are consistent. Moreover, a region around $\mu = 2$ is observed, where a completely mixed distribution is obtained, followed by an almost pure chaotic behavior, ending with a return to a mixed state and, then, integrability. This aligns with what was previously discussed.

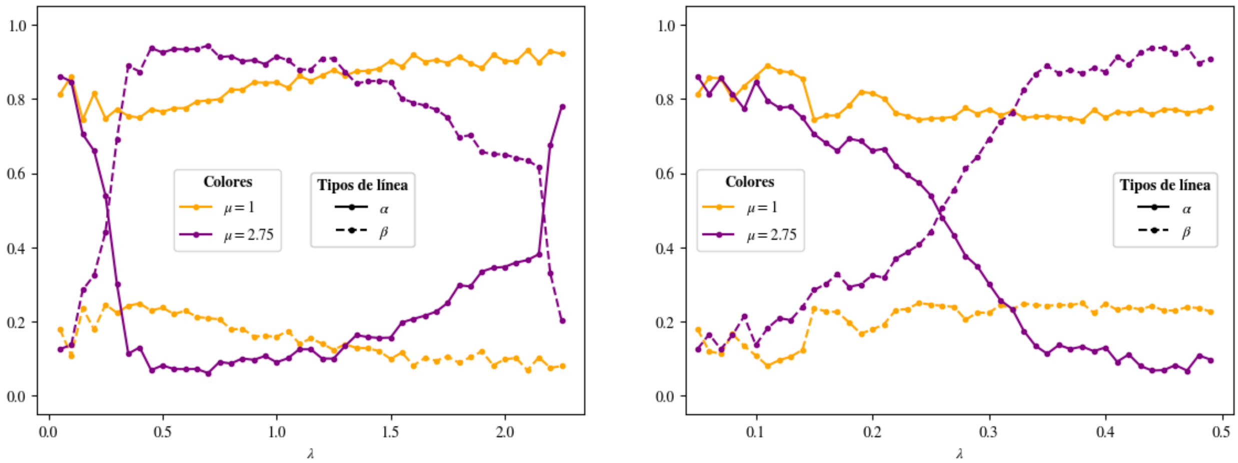
In this analysis, the most notable finding is that without previously imposing the linear combination to be convex, i.e. the coefficients being positive and summing to 1, the result obtained indicates that it is convex for all values of N, λ, and μ studied, allowing the following result to be extrapolated.

Conjecture 4.2. *Let us consider a time-reversal invariant system that undergoes a transition from a regime whose classical analogue is integrable to another whose classical analogue is chaotic. Then, the properties of its spectral distribution can be characterized as a convex combination of those of a Poisson distribution and those of the GOE ensemble.*

4.3.2 Comparison for different values of λ

Additionally, the impact of the parameter λ on the spectral distribution of the system is also analyzed. To this end, we compare the spectra obtained for $\mu = 1$ and $\mu = 2.75$, with $0.05 \leq \lambda \leq 2.25$ and $N = 32$, excluding the pathological case $\lambda = 0$.

Guided by Figures 5b and 7, for the case $\mu = 1$, one should not expect to observe a chaotic system for any value of λ, whereas for $\mu = 2.75$, GOE-related features are expected to emerge. However, does this hold for any value of λ?



(a) *Bidirectional abrupt transitions.* (b) *Zoom on the transition around $\lambda = 0.25$.*

Figure 8: *Evolution of α and β of the s statistic as a function of λ for $N = 32$, with $\mu = 1$ and $\mu = 2.75$.*

Performing an analogous analysis to the one presented in equation (4) for the s statistic, but now keeping μ fixed and varying λ, we observe the abrupt transition to chaos shown in Figure 8a for $\mu = 2.75$, while for $\mu = 1$, as expected, a chaotic regime is not reached.

It is now interesting to ask whether, given such an abrupt transition, it is monotonic. As can be discerned in Figure 8b, where the interval $0.05 \leq \lambda \leq 0.5$ has been studied with a very fine step $\Delta\lambda = 0.01$ to achieve higher resolution, the transition appears to be strictly monotonic. However, when considering values of λ with a smaller slope, it becomes only qualitatively monotonic. Nevertheless, one can argue that the small discrepancies might be due to errors associated with photon truncation.

Finally, it is worth noting that in Figure 8a, for $\mu = 2.75$, another abrupt transition is observed, this time towards an integrable system around $\lambda = 2.25$. This suggests that, despite not being symmetric, these types of transitions might exhibit some form of relationship in both directions, i.e., from integrability to chaos and vice versa.

5 Discussion

After the complete development of this document, it is possible to provide an answer, at least partially and with the caution inherent in not having conducted an exhaustive study of several systems, to the initial question: *how does the transition from integrable to chaotic behavior, and vice versa, occur in a quantum system?*

The transition between an integrable quantum system and a chaotic one is abrupt, qualitatively monotonic, and continuous. Furthermore, in intermediate situations, the behavior of the system can be characterized as a convex combination of integrable and chaotic behaviors, with the latter representing extreme behaviors of the system. The fact that the combination is convex is an intuitive concept but one that, a priori, does not necessarily have to occur: *the transition from an integrable quantum system to a chaotic quantum system, and vice versa, occurs through intermediate systems that can be characterized by a mixed behavior lying between the two aforementioned extremes.*

In summary, Random Matrix Theory has allowed for a rigorous and detailed study of a quantum system that, depending on two parameters, exhibited integrable, chaotic, and transitional behaviors. The use of appropriate statistical measures and the physical analysis of the Tavis-Cummings network have facilitated the characterization of the system in all situations, enabling conclusions to be drawn about the chaotic-integrable transition, which is the focus of this Bachelor's Thesis.

References

[1] A. A. Abul-Magd and A. Y. Abul-Magd. Unfolding of the spectrum for chaotic and mixed systems. *Physica A: Statistical Mechanics and its Applications*, 396:185–194, February 2014.

[2] P. Acuña. Von Neumann's theorem revisited. *Foundations of Physics*, 51(3):73, 2021.

[3] K. Andersson. *A Review of Gaussian Random Matrices.* Dissertation, Linköping University, 2020.

[4] M. V. Berry. Quantizing a classically ergodic system: Sinai's billiard and the KKR method. *Annals of Physics*, 131:163–216, 1981.

[5] M. V. Berry and M. Tabor. Level clustering in the regular spectrum. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 356(1686):375–394, 1977.

[6] J. S. Caux and J. Mossel. Remarks on the notion of quantum integrability. *Journal of Statistical Mechanics: Theory and Experiment*, 2011(02):P02023, February 2011.

[7] A. Edelman and N. Raj Rao. Random matrix theory. *Acta Numerica*, 14:233–297, May 2005.

[8] J. V. José and E. J. Saletan. *Classical Dynamics: A Contemporary Approach.* Cambridge University Press, 8th edition, 2012.

[9] S. Lozano. Caos y física estadística en sistemas cuánticos. `https://github.com/samuloza/Caos-y-fisica-estadistica-en-sistemas-cuanticos`, 2024.

[10] M. Prasad, H. K. Yadalam, M. Kulkarni, and C. Aron. Transition to chaos in extended systems and their quantum impurity models. *Journal of Physics A: Mathematical and Theoretical*, 57(1):015308, December 2023.

[11] H. J. Stöckmann. *Quantum Chaos: An Introduction.* Cambridge University Press, Cambridge, 2nd edition, 1999.

[12] S. H. Tekur. *Distribution of level spacing ratios in random matrix theory and chaotic quantum systems variants and applications.* PhD thesis, Indian Institute of Science Education and Research (IISER) Pune, 2019.

[13] J. T. Townsend. Chaos theory: A brief tutorial and discussion. In A. F. Healy, S. M. Kosslyn, and R. M. Shiffrin, editors, *Essays in Honor of William K. Estes*, volume 1, pages 65–96. Lawrence Erlbaum Associates, Inc., 1992.

[14] J. von Neumann. Über funktionen von funktionaloperatoren. *Annals of Mathematics*, 32(2):191–226, April 1931.