Econ 270 Lecture 5

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Intro to Probability

- Consider a fair coin
- Two options: $\{H, T\}$
- Frequentist view: if all options are equally likely, just need to count
- We'll start with counting, then work on expanding to general probability

Basics of Counting: Sets

- Suppose we flip a coin twice. How many ways are there to obtain 1 head?

Counting sets

- We can enumerate all possible outcomes: $\{HH, HT, TH, TT\}$. Each is equally likely
- 2 options with 1 head with 4 possible options
- Note that the order matters here

General power sets

- Suppose we instead toss 3 coins. How many possible options are there?

General Power Sets

- When we toss 1 additional coin, we have all of the previous options, plus either a heads or a tail at the end.
- So we end up with twice as many

General Power Sets

- Number of possible coin tosses for 10 tosses?
- Denominator is easy. But how would we count the number of ways to get 5 heads?

Simple counting example

- You roll two six sided dice. How many possible outcomes are there (assuming that order matters)?
- What's the probability that the sum of the two dice is a 4?

Permutations

- First consider the number of ways to order distinct items
- How could we arrange the letters A,B,C?
  - How does this differ from the number of ways to toss 3 coins?

Permutations: factorial

- To find the number of ways you can arrange n distinct items using every item exactly once, use recursion
- n choices for first placement, n-1 for second, n-2 for third , . . .
- $n(n-1)(n-2)...1 \equiv n!$
  - read “n factorial”
- Number of ways to arrange 4 distinct items?

Permutation: general

- Suppose we have 4 distinct items, but we only want to arrange 2 of them. How many ways are there to do this?
- Sometimes labeled as ${}_4P_2$
- Formula?

Permutations with Repetition

- We're now almost able to deal with counting coins. We just need a way to deal with repetition
- Consider the problem of how many ways we can arrange the letters A, B, A. We use each letter once
- If the As were different (e.g. one capital one lowercase), we would just have 3!

Permutations with Repetition

- Now consider the letters S, T, A, T, S
- There are 5! permutations if S and T are differentiated. How much double counting is there?
- General formula?

Permutations with Repetition

- Suppose we have k items, each appearing k_i (possibly 1) times, with $\sum k_i = n$
- Total permutations: $\frac{n!}{\Pi_{i=1}^{k} n_i!}$
  - Π is product
- Total permutations of the letters in banana?

Combinations

- Now we can get to counting heads. This is a special case of the prior formula
- Suppose you have 5 coin flips and want to how many ways to produce 3 heads * i.e. permutations of HHHTT?

Combinations

- When we only have 2 classes (a binary variable) this formula simplifies
- If we have n coin tosses, and k are heads, then there must be (n-k) tails
  - $= \frac{n!}{k!(n-k)!}$
- Usually denoted as $\binom{n}{k}$ and read "n choose k"

Coin Tossing

- You toss a coin 6 times. What is the probability you obtain 3 heads?

Poker

- Combinations generally work in cases where order doesn't matter
- How many possible 5 card hands are there in poker?

Birthday 'Paradox'

- Suppose there are 23 students in a room. What is the probability that two people share a birthday?

Probability Axioms

- With counting problems, we can calculate probabilities as long as every outcome is equally likely
- In general, we need a way to deal with outcomes that are not identical
- We start with some axioms of probability, then draw parallels where we can with the basic counting approach

Probability Axioms

- We start with a set of all possible outcomes, called a **sample space**
  - For tossing two coins, the sample space is $S = \{HH, HT, TH, TT\}$
- An **event** is a subset of the sample space
  - The event that you obtain 1 head is $\{HT, TH\}$
- We define a function P that measures the probability of any event. We require $P(S) = 1$ and $P(\emptyset) = 0$, where $\emptyset$ is the empty set

Probability Axioms

- In general, we assign a probability to each atom (individual element) of the sample space
  - $P(\{HH\}) = P(\{HT\}) = P(\{TH\}) = P(\{TT\}) = \frac{1}{4}$
- We then have a rule for combining elements in an event. If we have two **disjoint** (or **mutually exclusive**) events A and B, then the probability that either A or B occurs is $P(A) + P(B)$
  - "A or B" is sometimes written as $A \cup B$ for set union
  - This is formally called sigma subadditivity

Mutual Exclusion

- Two events are mutually exclusive if they don't share any elements. Mathematically: $A \cap B = \emptyset$
  - $A \cap B$ is the intersection of A and B, or more simply "A and B"
- $\{HT, TH\}$, and $\{HH\}$ are mutually exclusive.
  - Tossing exactly two heads and exactly 1 heads
- $\{HT, TH, HH\}$ and $\{HH\}$ are not mutually exclusive
  - Tossing at least one head and tossing exactly two heads have elements in common: $\{HT, TH, HH\} \cap \{HH\} = \{HH\}$

Example

- What is the probability that a randomly drawn card in a standard deck of 52 is either a club or a heart?
- What about the probability that it's either a club or a king?

General Addition

- If events aren't mutually exclusive, then we can't just add their probabilities
- The intersection of the two events ends up getting double counted in the process
- Easy correction: just subtract out the intersection
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

General Addition

- Suppose that 75% of students are econ majors, and 50% of students arrive to class late. 30% of students are econ students who arrive to class late. What percent of students are either econ students or arrive to class late?

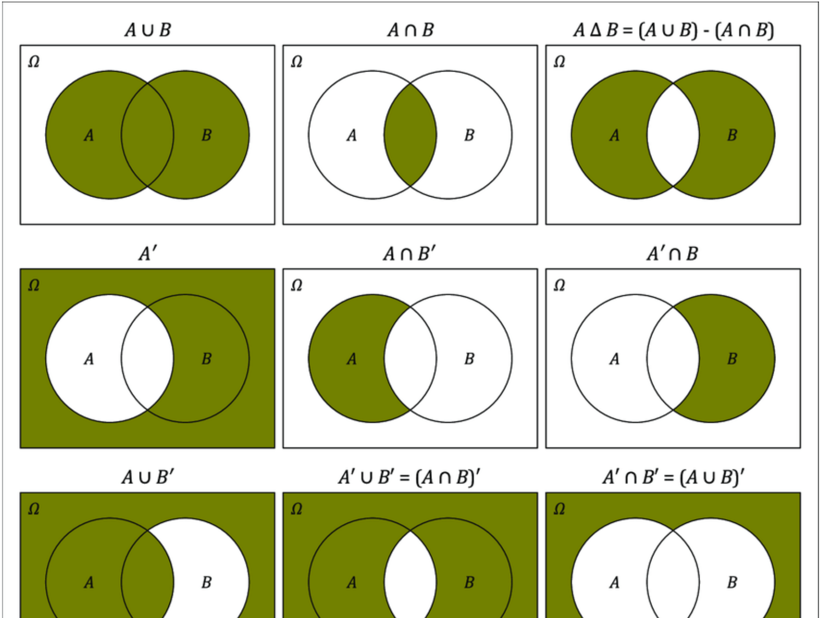
Complement

- Sometimes, it's easier to indirectly count probability
- What is the probability of obtaining 1 or more heads in 10 coin tosses?
  - $P(H = 1) + P(H = 2) + ... + P(H = 10)$
  - OR $1 - P(H = 0) = 1 - \frac{1}{1024} = \frac{1023}{1024}$
- Usually labeled as either $P(A')$ or $P(A^c)$

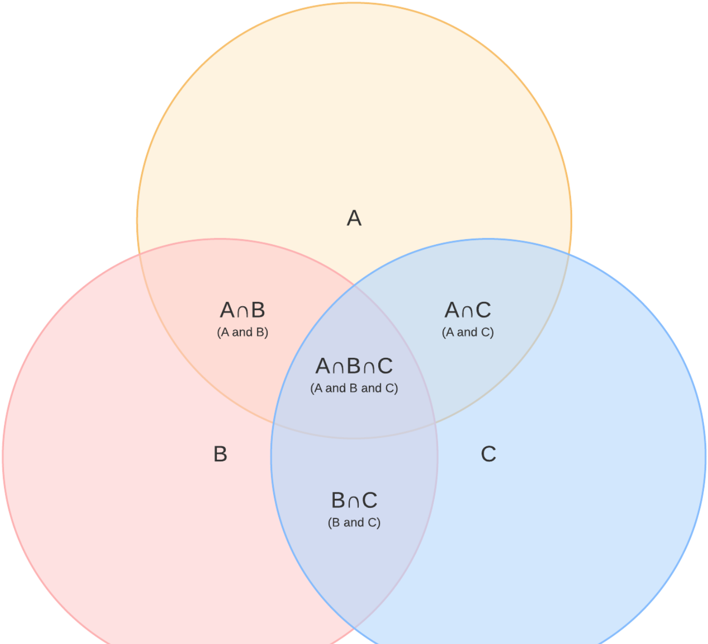
Basic Examples

- Roll a 6 sided die and let the events $A = \{2, 4, 6\}, B = \{4, 5, 6\}$
- What is $A \cup B$? $A \cap B$? A'?
- $P(A \cup B)$
- $P(A \cup A')$?

Venn Diagram



Venn Diagram



Exam P Question

SOCIETY OF ACTUARIES | Candidate Support

SOA Exam P Sample Exam

Question 5 of 30

A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

- A) 24
- B) 36
- C) 41
- D) 52

conditional probability/multiplication rule

- Sometimes we want to know the probability of an event when we're given additional information
- $P(A|B)$ is the probability of A given B
- We're just restricting our sample space to B first
  - $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - Or $P(A|B)P(B) = P(A \cap B)$

Conditional Probability

- Suppose 60% of people like peanut butter, 50% like jelly, and 40% like both. Given that a randomly sampled person likes peanut butter, what's the probability that they also likes jelly?
- Given that a randomly sampled person likes jelly, what's the probability they also like peanut butter?

Independence

- In the previous example, learning that someone likes peanut butter changes the probability that they like jelly.
  $P(J) = 50\%$, but $P(J|P) = 67\%$
  - Become peanut butter and jelly are complements, learning someone likes one means they're more likely to like the other
- Two events are **independent** if $P(A \cap B) = P(A)P(B)$
  - or $P(A|B) = P(A)$ and $P(B|A) = P(B)$
  - No information about A is gained when B is revealed

Independence

- Suppose $P(A) = 0.3$ and $P(B) = 0.7$. Can you calculate $P(A \cup B)$? $P(A|B)$?
  - What if they're independent?
- Examples of independent events?

Joint Probability Distributions

- If we can fill in missing information ($P(A \cap B)$ we can specify a probability distribution)
- Equivalent to being able to fill out a full venn diagram
- We can also convert to a table

Joint Probability Distributions

- Suppose $P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.1$
- $P(A \cup B)$?

	B	B'	Total
A			
A'			
Total			

Marginal Probabilities

- Sometimes we're given a joint distribution of A and B, but only want A
- Let A be the distribution of major, and B of year. What is the distribution of A?
- What percent of students are psych majors?

	Freshman	Sophomore	Junior	Senior	Total
Econ	0.1	0.1	0.05	0.05	
Math	0.05	0.05	0.05	0.1	
Psych	0.15	0.1	0.1	0.1	
Total					1

Marginal probabilities: Formula

- Getting marginals from a table is easy. But if we don't have a table?
- $P(X) = \sum_Y P(X, Y)$
  - We're just summing across a row or column
- $P(X)$ as calculated above always sums to 1 itself, so it is a valid distribution
  - $\sum_X \sum_Y P(X, Y) = 1$
- The formulas can look complicated, but they're based on simple principals

Law of Total Probability

- You have three bags of marbles. Bag 1 has 60 red and 40 blue marbles. Bag 2 has 50 red and 50 blue marbles. bag 3 has 5 red and 15 blue marbles.
- You choose a bag at random, then select a marble within that bag.
- What is the probability that the marble is red

Law of Total Probability

- Let $B_1, B_2, ...B_n$ be a partition of S. i.e. $B_i \cap B_j = \emptyset \ \forall i, j$ and $\bigcup_k B_k = S$
- $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ... + P(A|B_n)P(B_n)$
- Simple version: $P(A|B)P(B) + P(A|B')P(B')$
- Scary formula, easy intuition!

Law of Total Probability

- When the weather is good, the probability that you will arrive to class is 90%. But when the weather is bad, the probability that you arrive to class if only 50%
- Suppose there is a 25% chance of bad weather
- What is the probability you arrive to class?

Bayes Theorem

- You have three bags of marbles. Bag 1 has 60 red and 40 blue marbles. Bag 2 has 50 red and 50 blue marbles. bag 3 has 5 red and 15 blue marbles.
- You choose a bag at random, then select a marble within that bag.
- What is the probability that the marble came from bag 1, given that it's a red marble?

Bayes Theorem

- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$

Monty Hall Problem

- You choose between 1 of 3 doors, behind which there is a prize. One of these doors has the grand prize. After choosing, the gameshow host will reveal 1 door at random that you did not choose, but he will never reveal the grand prize in the process.
- After he reveals one of the doors, he gives you the option to switch doors. What is the probability that you win the grand prize if you switch doors?

Random Variables

- We've already learned how to calculate probability and defined events, but sometimes it makes sense to perform operations on an entire distribution
- A **Random Variable** contains all of the information about a probability distribution
  - Mathematically it just maps events to it's sample space
  - For this class, we care more about the operations we can perform on it than it's definition

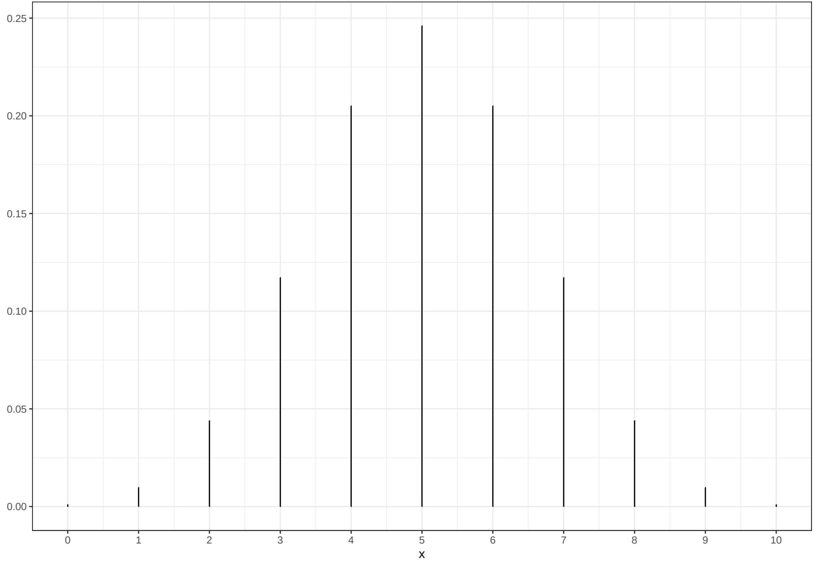
Random Variables

- Toss a coin 10 times and let the random variable X denote the number of heads that occur
- The shorthand $X = 5$ denotes the event that 5 heads occurs. $P(X = 5) = \binom{10}{5}(\frac{1}{2})^{10}$
- We can similarly denote $P(X = 0), P(X = 10), P(X \geq 3)$, etc
- Generically, we use $P(X = x)$ to denote the probability of obtaining x coin flips
  - $P(X = x) = \binom{10}{x}(\frac{1}{2})^{10}$

Random Variables: probability mass function

- For a discrete random variable we use the shorthand $p(x) = P(X = x)$ to denote the probability that an event occurs
- When graphed, the pmf is anologous to the empirical histogram

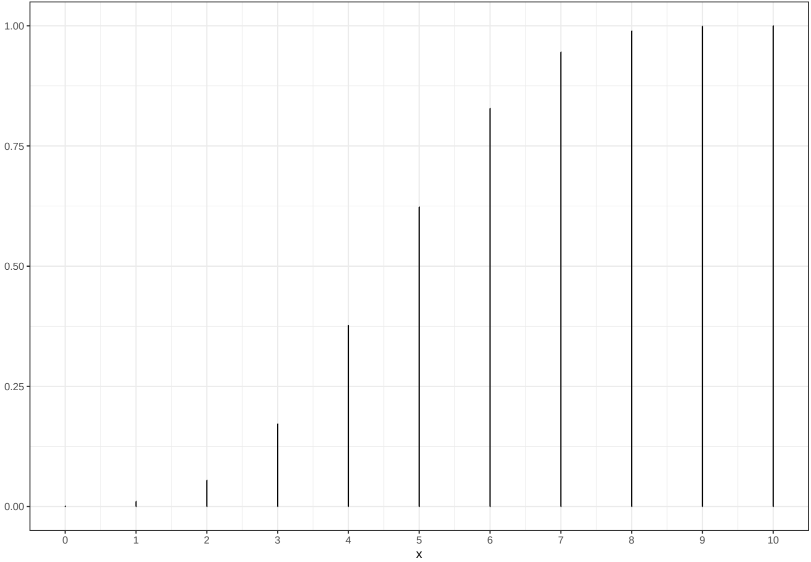
pmf: 10 coin flips



CDF

- The pmf gives us all of the information we need to calculate probabilities, but what if we need to calculate $P(3 \leq X \leq 7)$?
  - Need $p(3) + p(4) + p(5) + p(6) + p(7)$
- One convenient tool is the cumulative distribution function (CDF): $F(x) = P(X \leq x)$
  - The above is then $F(7) - F(2)$
- How would we get the median?

CDF



Expectation

- We can also calculate an average of a random variable. We call this the expectation and use the operator E
  - $\mu = E[X]$
- The mean is an average weighted by the probability:
  - $E[X] = \sum xp(x)$
- We can similarly calculate $var(X) = E[(X - E[X])^2)$
  - Expectation is a linear operator, so $E[aX + bY] = aE[X] + bE[Y]$
- Also $cov(X, Y) = E[(X - E[X])(Y - E[Y])]$

Random Variable Operations

- We partly define random variables to introduce the concepts of pmf/CDF and expectation, but mostly we want to do operations on random variables
- Let X_1 be the roll of a 6-sided die, and X_2 be the roll of a second 6-sided die. What is $X_1 + X_2$?
  - $Y = X_1 + X_2$ is its own random variable with a pmf, cdf, and expectation
  - Expectation is easy to calculate. Variance is slightly harder. The pmf is much harder.

Random Variable Addition: Expectation

- First, note that X_1 and X_2 are identical, so e.g. $E[X_1] = E[X_2]$
- We can get $E[X_1] = \sum xp(x) = \frac{1}{6}[1+2+3+4+5+6] = 3.5$
  - We have calculated this before, the only difference is the syntax
- To get $E[X_1 + X_2]$, use linearity. General formula: $E[\sum \alpha_i X_i] = \sum \alpha_i E[X_i]$
- So $E[X_1 + X_2] = E[X_1] + E[X_2] = 2E[X_1] = 7$

Random variable addition: Variance

- For variance, we have $E[(X - E[X])^2]$. First we can distribution the inner part, then we apply linearity of expectation:
  - $E[X^2 - 2XE[X] + E[X]^2] = E[X^2] - E[2xE[X]] + E[E[X^2]] = E[X^2] - 2E[X]E[X] + E[X^2] = E[X^2] - E[X]^2$
  - This means that $var(X + Y) \neq var(X) + var(Y)$
- General formula: $var(aX + bY) = a^2 var(X) + b^2 var(Y) + 2cov(X, Y)$
  - If the variables are independent, the last term is 0

Random variable addition: pmf

- In general, finding the pmf of the sum of random variables is a difficult problem
- It makes use of a convolution
  - $p(x+y) = P(X+Y=z) = \sum_x (z-x)P(X=x, Y=z-x) = \sum_x (z-x)p(x)p(z-x)$ if independent
- Sometimes the CDF is easier to calculate: $P(Z \leq z) = P(X+Y \leq z)$
- Conceptually this is easy, but in practice it'll be hard without some special tricks we'll introduce later.