1. **Derivation of the system model.**

- Forward Movement Model

Forward movement is the case when; $\alpha \in I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

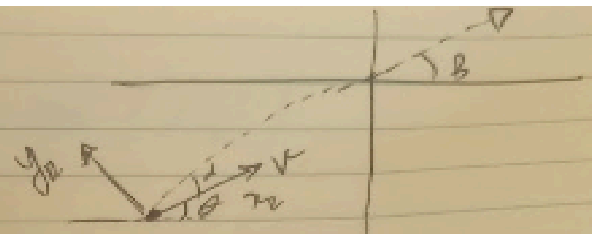


Fig 1

Given:

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)$$

$$\beta = -\theta - \alpha$$

§ Since the final goal pose is $(0, 0, \theta_f)$, $\Delta x = -x$, $\Delta y = -y$

$$\Rightarrow \rho^2 = x^2 + y^2 \text{ --- \#1}$$

$$\alpha = -\theta + \tan^{-1}(-y/-x) = -\theta + \tan^{-1}(y/x) \text{ --- \#2}$$

$$\beta = -\theta - \alpha \text{ --- \#3}$$

§ From #1 and fig.1 configuration

(I) $\frac{d}{dt}\left(\rho^2 = x^2 + y^2\right)$

$$2\rho \cdot \dot{\rho} = 2x \cdot \dot{x} + 2y \cdot \dot{y}$$

However,

$$\begin{cases} \dot{x} = V\cos\theta, & \dot{y} = V\sin\theta \\ x = -\rho\cos(\alpha+\theta), & y = -\rho\sin(\alpha+\theta) \end{cases}$$

$$\Rightarrow \dot{\rho} = -V\left[\cos(\alpha+\theta)\cos(\theta) + \sin(\alpha+\theta)\sin(\theta)\right]$$

$$\dot{\rho} = -V\cos\alpha$$

(II)

$$\frac{d}{dt}\left(\alpha = -\theta + \tan^{-1}(y/x)\right)$$

However: $\frac{d}{dt}\tan^{-1}(y/x) = \frac{\dot{y}x - \dot{x}y}{x^2+y^2}$

& By using the relations we derived in (I)

$$\frac{d}{dt}\left(\tan^{-1}(y/x)\right) = \frac{(v\cos\theta)(+\rho\sin(\alpha+\theta)) - (v\sin\theta)(+\rho\cos(\alpha+\theta))}{\rho^2}$$

$$= \frac{v}{\rho}\left[-\sin\theta\cos(\alpha+\theta) + \sin(\alpha+\theta)\cos\theta\right]$$

$$= \frac{v}{\rho}\sin(\theta+\alpha-\theta) = \frac{v}{\rho}\sin\alpha$$

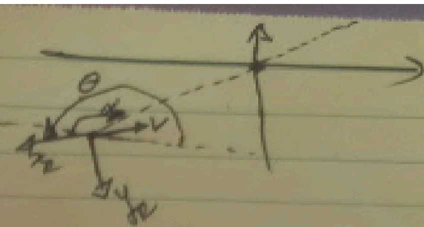
$$\therefore \frac{d}{dt}(\alpha) = -\frac{d\theta}{dt} + \frac{v}{\rho}\sin\alpha = \underline{-\omega + \frac{v}{\rho}\sin\alpha}$$

(III) $$\frac{d}{dt}(\beta) = \frac{d}{dt}(-\theta-\alpha) = -\omega - \left(-\omega + \frac{v}{\rho}\sin\alpha\right)$$

$$= -v/\rho\,\sin\alpha$$

- Backward Movement Model.

Backward movement is the case when; $\alpha \in I_2 = \left(-\pi, -\frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \pi\right)$



$\dot{x} = V\cos(\theta - \pi) = -V\cos\theta$

$\dot{y} = V\sin(\theta - \pi) = -V\sin\theta$

(I) $\dot{\rho} = \frac{1}{\rho}(x \cdot \dot{x} + y \cdot \dot{y})$

But $x = -\rho\cos(180 + \alpha + \theta - 180)$

⇒ B/se α is in clockwise

$x = -\rho\cos(\alpha + \theta)$ and $y = -\rho\sin(\alpha + \theta)$

⇒ $\dot{\rho} = \frac{1}{\rho}\left(-\rho\cos(\alpha+\theta) * -V\cos\theta - V\sin\theta * -\rho\sin(\alpha+\theta)\right)$

$$\boxed{\dot{\rho} = V\cos\alpha}$$

(II) $\dot{\alpha} = \frac{d}{dt}\left(-\theta + \tan^{-1}(y/x)\right)$, but $\frac{d\theta}{dt} = -\omega$

$$\Rightarrow \frac{d}{dt}\tan^{-1}(y/x) = \frac{\dot{y}x - y\dot{x}}{x^2+y^2} = \frac{-Vx\sin\theta + Vy\cos\theta}{\rho^2} = \frac{-V\sin\alpha}{\rho}$$

$$\therefore \boxed{\dot{\alpha} = \omega - \frac{V\sin\alpha}{\rho}}$$

(III)

$$\beta = -\theta - \alpha$$

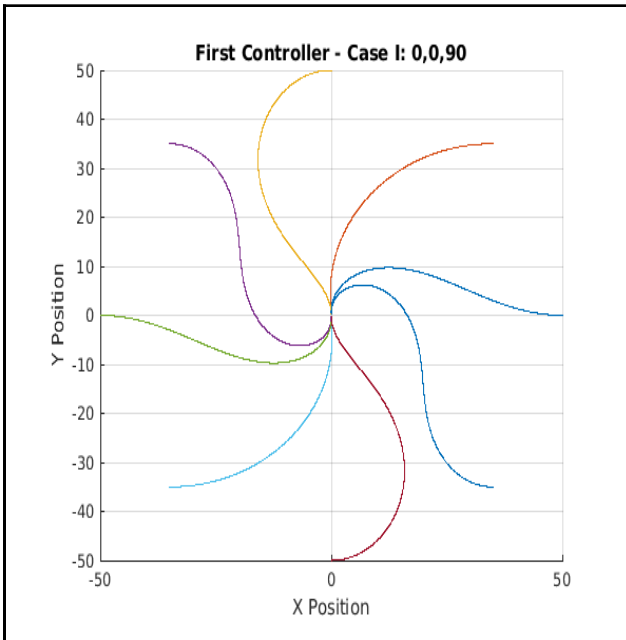
$$\frac{d}{dt}\beta = \frac{d}{dt}(-\theta-\alpha) = \omega - \left(\omega - \frac{v\sin\alpha}{\rho}\right)$$

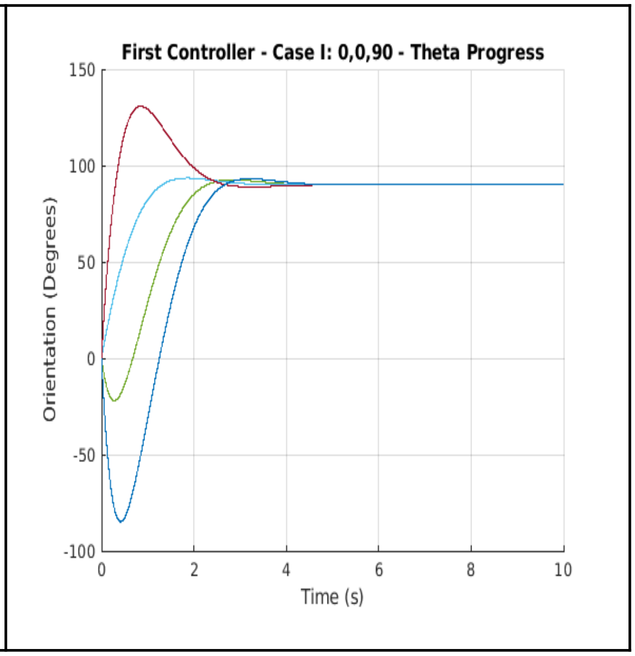
$$\boxed{\dot{\beta} = \frac{v\sin\alpha}{\rho}}$$

2. **Results of the simulation.**

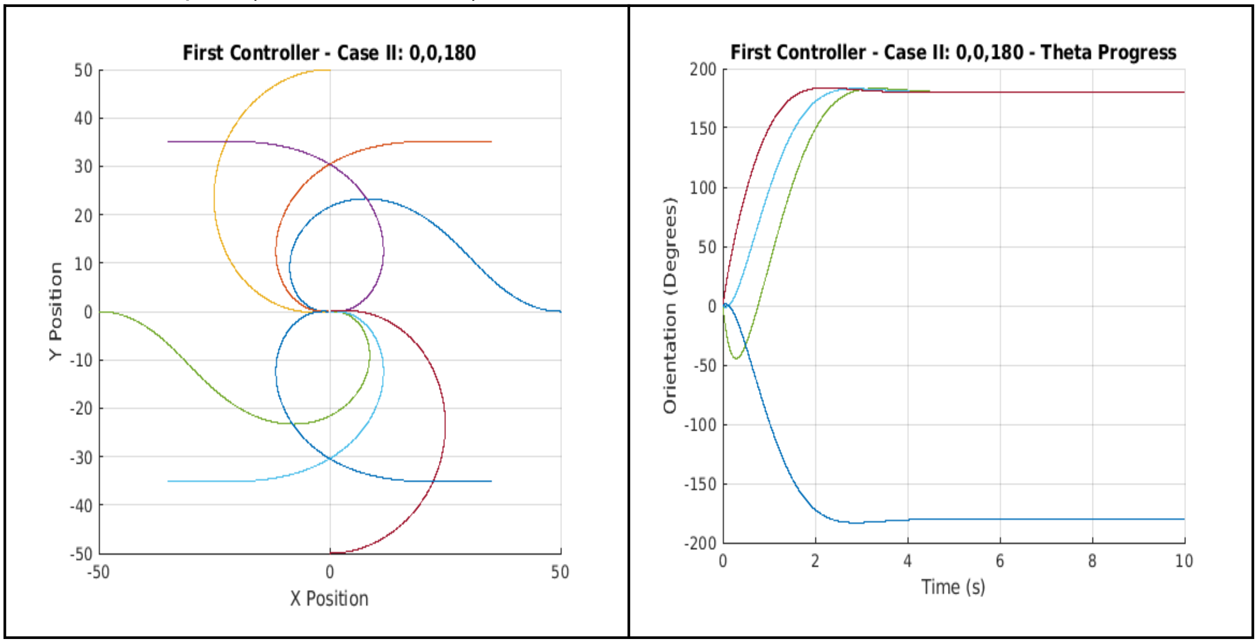
I. **The first controller(K_rho=2, K_alpha=5, K_beta=-2).**

Case I: Final pose(X=0,Y=0, θ=90)



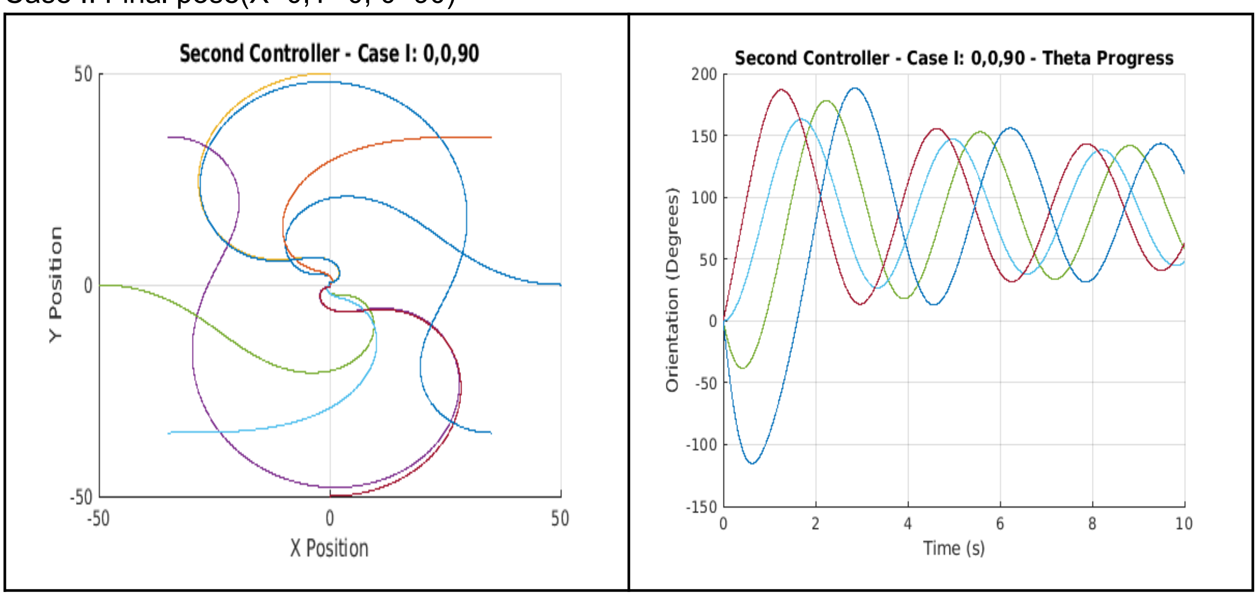


Case II: Final pose(X=0,Y=0, θ=180)

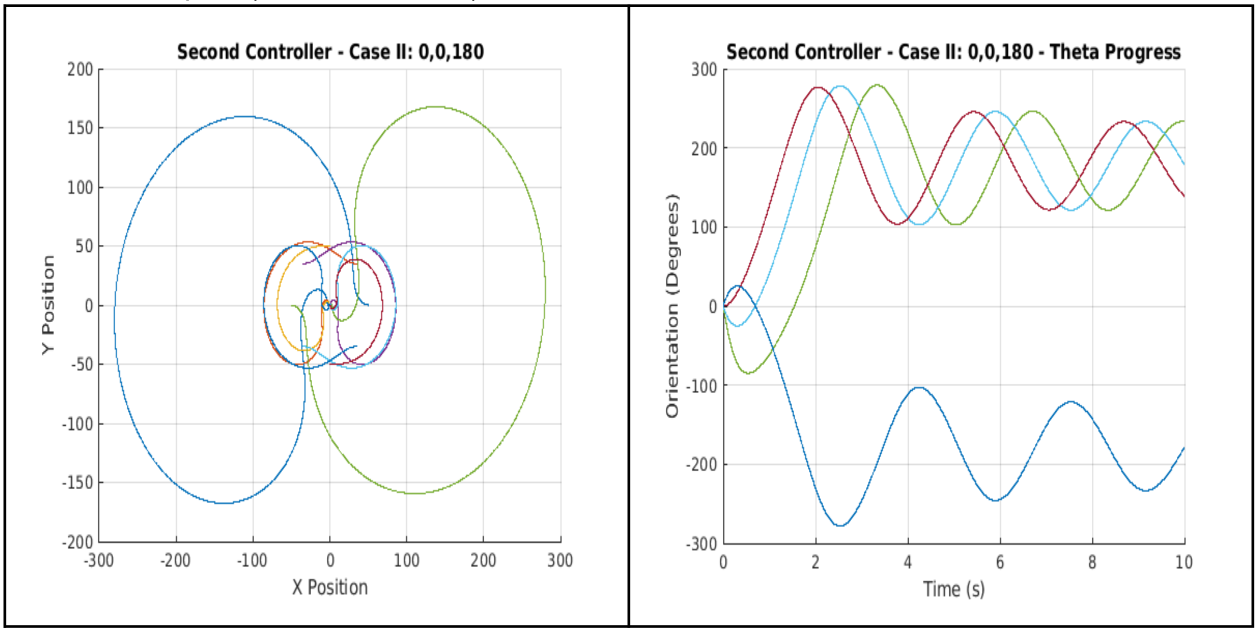


II. The second controller(K_rho=2, K_alpha=2, K_beta=-2).

Case I: Final pose(X=0,Y=0, θ=90)



Case II: Final pose(X=0,Y=0, θ=180)



3. Discussion

The results obtained align well with the theoretical predictions. Based on the stability analysis provided in the lecture material, the characteristic polynomial for the system model (Matrix A) is given by:

$$(\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha - k_\rho) - k_\rho k_\beta)$$

For the first controller(K_rho=2, K_alpha=5, K_beta=-2), the roots of the above polynomial reside in the left half plane. This confirms the expected stable behavior of the system. However, for the second controller(K_rho=2, K_alpha=2, K_beta=-2), the poles are -2, +√2j and -√2j. The presence of complex conjugate roots implies that the system is oscillatory around the steady state. This theoretical expectation matches the oscillatory behavior observed in our results.