LARCH

Livermore Analytical fiber Reinforced Composite Homogenizer

User's Manual

Version 1.0

Michael J. King

Table of Contents

Compatibility

The LLNL Composite Homogenizer is a Matlab script and should run on any system capable of running Matlab. It does not require any of the supplemental toolkits. It was developed under Matlab 2007.

Installation

Copy the master script (`Larch.m`) and the two graphics files (`FiberCoords.jpg` and `PlyCoords.jpg`) into any directory.

Running LARCH

Start Matlab and switch to the directory containing the master script `Larch.m`. Either type the name of the script at the Matlab command prompt, or open the script in the m-file editor and run it (using the appropriate button on the toolbar, or by pressing F5, or by selecting "Run Larch" under the Debug menu).

Note that the LARCH has no inherent system of units (except that ply orientations must be input in degrees). The output properties will have the same units as the input properties. It is up to the user to ensure that all properties are input in a consistent set of units.

Support

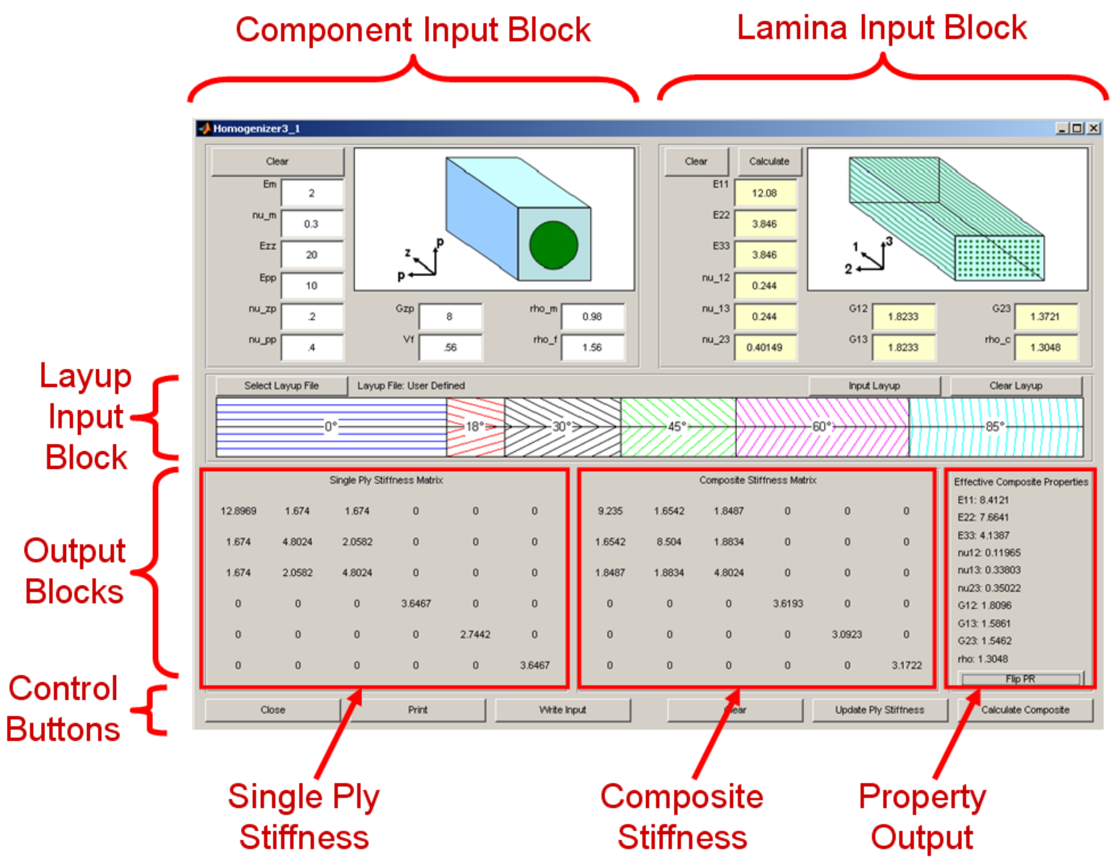
LARCH was developed by Michael J. King at Lawrence Livermore National Laboratory. Contact him at king74@llnl.gov to report any bugs or to receive the latest version.

Using LARCH

Introduction to LARCH

LARCH is intended to make estimates of the effective homogenized elastic properties of a composite lamina or a multi-ply laminate based on the constitutive properties of the components (the matrix and the fiber) or of a single lamina. It uses analytical homogenization theory to make these estimates. Refer to the Theory section for more details.

The LARCH interface has seven regions: three input blocks, three output blocks, and a set of control buttons, as shown below:



The user has the option of calculating the lamina properties (in the Lamina Input Block) from the component properties (as entered in the Component Input Block), or inputting some or all of them directly. Once the lamina properties are input, the stiffness matrix of a single ply can be calculated. The user can also input a layup using the Layup Input Block, and calculate the stiffness of the resulting composite laminate, and the effective elastic properties of the composite. Each of the functions of the seven blocks is described in detail below.

Component Input Block

This block gives the user the ability to input material properties for the composite components (the fiber and the matrix), so that the properties of a single lamina can be estimated. If the properties of a single lamina are already known, the Component Input Block need not be used.

The matrix is assumed to be isotropic and hence characterized by two elastic parameters. The fibers are assumed to be transversely isotropic (with the isotropic plane normal to the fiber axial direction), characterized by five elastic parameters. The effective lamina properties are calculated using a mixed homogenization methodology, where isostrain is assumed in the fiber direction and isostress is assumed in all other directions.

Clear – This button clears the Component Input Block. Note that this also causes the entries in the Lamina Input Block to turn red and the entries in the Composite Stiffness Matrix and the Property Output to become grey (indicating that they no longer reflect the current component properties).

Em – The Young's modulus of the matrix.

nu_m – The Poisson's ratio of the matrix.

Ezz – The axial modulus of the fiber. Note that LARCH assumes that the fiber has the same stiffness in tension as in compression.

Epp – The transverse modulus of the fiber.

nu_zp – The axial-transverse Poisson's ratio of the fiber, defined as the negative strain that develops in the transverse direction in response to unit axial strain.

nu_pp – The transverse Poisson's ratio of the fiber, defined as the negative strain that develops in one transverse direction in response to unit strain in the other transverse direction.

Gzp – The axial-transverse shear modulus of the fiber. Note that the transverse-transverse shear modulus of a transversely isotropic material is not independent; it can be calculated from the transverse modulus and the transverse-transverse Poisson's ratio.

Vf – The volume fraction of the fibers. Must be between zero and one.

rho_m and *rho_f* – The densities of the matrix and the fiber, respectively. These parameters are only used for calculating the density of the lamina (and of the composite) and are optional.

Lamina Input Block

The lamina input block is used to display the lamina properties calculated from the component properties, and also to allow the user to directly override these properties and input the lamina properties manually. The lamina are assumed to be orthotropic, and hence require nine elastic parameters.

The ten input boxes can be either red or yellow. Yellow indicates that the value was calculated from the component properties of the fiber and the matrix. Red indicates that the value is not consistent with the properties in the Component Input Block, either because the user has chosen to manually override a calculated value, or because the component property inputs have been changed since the lamina properties were last calculated.

If any yellow box is edited, it is turned red to indicate that the calculated value has been overridden, and the Single Ply Stiffness Matrix is turned grey to indicate that something has been changed since it was calculated.

Clear – This button clears all the input in the Lamina Input Block. Note that this also turns the entries red and changes the Single Ply Stiffness Matrix to grey zeros, indicating that the lamina properties no longer correspond to the component properties.

Calculate – This button calculates the effective properties of the lamina (i.e. a single ply) based on the properties in the Component Input Block. Checks are made to ensure that the values in the Component Input Block are valid (i.e. the component stiffness matrices are positive definite, the fiber volume fraction is between zero and one, etc.). If the properties are not valid, an error is issued and the user must input valid properties before the lamina properties can be calculated. If a negative stiffness was input, a warning is issued and the absolute value of the stiffness is used. Once the effective lamina properties are calculated, the boxes in the Lamina Input Block are updated and turned yellow, and the Single Ply Stiffness Matrix is updated and turned black.

E11, *E22*, and *E33* – The elastic moduli of the lamina in the fiber, in-plane transverse, and through-thickness directions, respectively.

nu_12, nu_13, and *nu_23* – The Poisson's ratios of the lamina. Note that nu_*ij* indicates the negative *j*-direction strain that results from unit strain in the *i*-direction.

G12, *G13*, and *G23* – The shear moduli of the lamina.

rho_c – The density of the lamina (and of the composite).

Layup Input Block

This block allows the composite layup to be input. Two methods for inputting a layup are available. Either the layup can be input manually through the use of dialog boxes, or it can be read in from a file.

If read from a file, the file should be a space- or comma-delimited ASCII file with two columns of data. The first column should give the orientation of each layer (in degrees); the second column should give the volume fraction (or the number of plies, or the thickness) associated with each layer. Note that if the volume fractions do not sum to unity, LARCH will automatically normalize them.

If the layup is to be input manually, the user must first indicate the number of layers, then input one angle for each layer, then input one volume fraction (or the number of plies, or the thickness) corresponding to each angle. Again, if the volume fractions do not sum to unity, LARCH will automatically normalize them. A maximum of twenty layers can be input.

Note that if the layup is changed or cleared after the composite stiffness has been calculated, the composite stiffness matrix and the effect properties will be changed from black to grey to indicate that they no longer reflect the current layup.

Important Note on Layup Input #1: LARCH was developed for calculating the properties of fiber wound composites. In a fiber-wound composite, every positively angled layer has a corresponding negative layer (e.g. for every +45° layer there is a -45° layer). If the user inputs an angle α other than 0° or 90°, either manually or through an input file, LARCH interprets that as being half a layer of angle α and half a layer of angle $-\alpha$. Hence the resulting composite stiffness matrix never has shear coupling terms (*i.e.* $C_{ij} = C_{ji} = 0$ for $i \in (1,2,3)$ and $j \in (4,5,6)$, and $C_{45} = C_{46} = C_{56} = 0$).

Important Note on Layup Input #2: LARCH was intended for homogenizing the bulk response, not the structural (*i.e.* plate/shell) response of the composite, and hence does not calculate the bending stiffnesses associated with the composite. Consequently, it does not matter where in a layup a given layer lies—for example, a (0°/90°/90°/0°) layup will produce the same homogenized stiffness as a (90°/0°/0°/90°) layup. For computational efficiency, if the same angle (or its negative) is input multiple times, the layers are combined into a single layer. Hence inputting a layup as (0°/90°/90°/0°) with four equal volume fractions would be equivalent to inputting a layup as 0°/90° with two equal volume fractions.

Select Layup File – This button opens a dialog box that allows the user to interactively select an ASCII file containing the layup data. After selection, the layup will be graphically displayed in the layup input block.

Input Layup – This button launches a series of dialog boxes that allows the user to manually specify the layup. After the layup has been specified, it will be graphically displayed in the layup input block.

Clear Layup – This button clears the currently stored layup, resetting the layup to its default configuration: a unidirectional composite with the fibers aligned at 0°.

Single Ply Stiffness Matrix

This block displays the thirty-six element symmetric stiffness matrix associated with the lamina properties. If the text is black, it represents the properties currently in the lamina input block. If the text is grey, it implies that the lamina properties have been changed since it was last calculated and it should be re-calculated (through the use of the "Update Ply Stiffness" or "Calculate Composite" buttons).

Important Notes on Stiffness Matrix Notation: The stiffness matrix gives the relationship between stress and ***tensorial*** strains (usually represented with the symbol ε), not ***engineering*** strains (where the shear components are usually represented with the symbol γ). Note that, for $i = j \in (4,5,6)$, $2\varepsilon_{ij} = \gamma_{ij}$. Hence the stiffness matrix values C_{44}, C_{55}, and C_{66} will correspond to ***twice*** the shear moduli of the lamina.

The stiffness matrix relates the stress vector $\{\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{13}\}^T$ to the strain vector $\{\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{23}\ \varepsilon_{13}\}^T$.

Composite Ply Stiffness Matrix

This block displays the homogenized stiffness matrix associated with the entire laminate. If the text is black, it represents the stiffness calculated from the properties currently in the lamina input block ***and*** the layup currently in the layup input block. If the text is grey, it implies that either the lamina properties or the layup has been changed since it was last calculated, and it should be re-calculated (through the use of the "Calculate Composite" button).

Important Note: The stiffness matrix gives the relationship between stress and ***tensorial*** strains (usually represented with the symbol ε), not ***engineering*** strains (where the shear components are usually represented with the symbol γ). Note that, for $i = j \in (4,5,6)$, $2\varepsilon_{ij} = \gamma_{ij}$. Hence the stiffness matrix values C_{44}, C_{55}, and C_{66} will correspond to ***twice*** the shear moduli of the composite.

The stiffness matrix relates the stress vector $\{\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{13}\}^T$ to the strain vector $\{\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{23}\ \varepsilon_{13}\}^T$.

Property Output

This block displays the effective composite properties that correspond to the stiffness matrix displayed in the composite ply stiffness matrix output block. E_{11} is the modulus in the in-plane direction corresponding to 0°. E_{22} is the modulus in the in-plane transverse direction (corresponding to 90°); E_{33} is the through-thickness modulus. *nu12*, *nu13*, and *nu23* are the Poisson's ratios corresponding to these directions; *G12*, *G13*, and *G23* are the shear moduli corresponding to these directions, and *rho* is the density. Note that, for an orthotropic material, $\nu_{ij} \neq \nu_{ji}$, and that different analysis codes require Poisson ratios to be in different formats. To convert the Poisson ratios to their alternate form, use the "Flip PR" button.

Like the Composite Ply Stiffness Matrix, the text in this block can be either black or grey. If it is black, it implies that the effective composite properties currently reflect the properties in the lamina input block ***and*** the layup currently in the layup input block. If the text is grey, it implies that either the lamina properties or the layup has been changed since they were last calculated, and they should be re-calculated (through the use of the "Calculate Composite" button).

Flip PR – This button converts the Poisson ratios from one form (ν_{12}, ν_{13}, ν_{23}) to the other (ν_{21}, ν_{31}, ν_{32}), through the use of the relation $\nu_{ij} = (E_{ii} / E_{jj})\ \nu_{ji}$.

Control Buttons
These buttons allow the user to control LARCH.

Close – Closes LARCH. All data in the current session will be lost. User confirmation is required.

Print – Opens a dialog to print LARCH as it currently appears, allowing the user to create a hard copy showing the current inputs and outputs.

Write Input – Nonfunctional, currently under development. Ultimately intended to write material property input decks for a variety of different code formats.

Clear – Clears all data stored in LARCH and resets everything to the default (starting) values. This will set all inputs in the component and lamina input blocks to zero (turning the lamina input boxes red), set the layup to its default (a single layer oriented at 0°), set all output quantities to zero, and change the text color in the three output blocks to grey.

Update Ply Stiffness – This button attempts to calculate the single ply stiffness matrix from the properties currently in the lamina input box. If successful, it will turn the single ply stiffness matrix black. If the properties in the lamina input box do not yield a positive definite stiffness matrix, an error message will be issued. Note that this button's function is automatically invoked when the "Calculate Composite" button (or the "Calculate" button in the lamina input box) is clicked. This button ***does not*** attempt to calculate the lamina properties from the component input; use the "Calculate" button in the lamina input box for that.

Calculate Composite – This button attempts to calculate the homogenized composite stiffness matrix and effective composite properties from the properties currently in the lamina input block and the layup in the layup input block. It first attempts to update the single ply stiffness, invoking the same function as the "Update Ply Stiffness" button. If this succeeds, it will then attempt to calculate the homogenized composite stiffness based on the layup. If it succeeds, it will turn the composite stiffness matrix and the effective composite properties black. This button ***does not*** attempt to calculate the lamina properties from the component input; use the "Calculate" button in the lamina input box for that.

Tutorial – Homogenization of an example Carbon-Fiber Epoxy Laminate

This example will walk the user through the process of estimating the homogenized properties of an example carbon fiber-epoxy laminate with the following characteristics:

Matrix modulus: 0.4 Msi
Matrix Poisson Ratio: 0.35
Matrix density: 1.3 g/cm^3
Fiber Axial Modulus: 40 Msi
Fiber density: 1.8 g/cm^3
Fiber Volume Fraction: 55%
Layup: (0° 90° 0° 90° -45° +45° 90° -12° +12° 90°)^{sym}

We will work in Msi units for stiffness and g/cm^3 units for density. Note that we are missing some of the fiber properties that are traditionally difficult to measure, such as the fiber Poisson ratios and shear modulus.

Start LARCH. The first step is to calculate the lamina properties from the component properties. Start by inputting all known component properties:

- enter the matrix modulus, 0.4, into the *Em* box
- enter the matrix poisson ratio, 0.35, into the *nu_m* box
- enter the matrix density, 1.3, into the *rho_m* box
- enter the fiber axial modulus, 40, into the *Ezz* box
- enter the fiber density, 1.8, into the *rho_f* box
- enter the fiber volume fraction, 0.55, into the *Vf* box

This is not enough information—the fiber Poisson ratios default to zero, which is probably incorrect, and if we attempt to proceed with the fiber transverse and shear moduli as zero, we will encounter an error. So we will have to make educated guesses for the remaining values. For a carbon fiber, the transverse moduli is often significantly smaller, but of the same order of magnitude, as its axial modulus, so let's guess 20 Msi. Enter 20 into the *Epp* box. Estimating the axial-transverse shear modulus to be ~1/3 of the axial modulus also seems reasonable, so enter 13 into the *Gzp* box. One third is also probably a decent guess for ν_{zp} and ν_{pp}, so enter 0.33 in the *nu_zp* and *nu_pp* boxes.

Now click the "Calculate" button in the lamina property box. The lamina properties will turn from red to yellow, and the single ply stiffness matrix will have black values filled in. We see that E_{11} ~ 22 Msi, $E_{22} = E_{33}$ ~ 0.98 Msi, $\nu_{12} = \nu_{13}$ ~ 0.339, ν_{23} ~ 0.53, $G_{12} = G_{13}$ ~ 0.4 Msi, G_{23} ~ 0.32 Msi, and ρ ~ 1.575 g/cm^3.

We can test how sensitive the lamina properties are to the properties we had to guess at. First, change the entry in the *Epp* box to 10. The lamina properties will turn red, indicating that these no longer represent the properties from the component input boxes.

Click “Calculate” again to re-calculate the lamina properties. As you can see, the transverse and through-thickness moduli, ν_{23}, and G_{23} all change, but only slightly. Similarly, changing the fiber axial-transverse shear modulus has very little effect. The lamina properties are not very sensitive to any of the fiber modiul save the axial modulus in this case, because the matrix is so compliant by comparison.

Go back to the properties we originally guessed and re-click “Calculate”. The component homogenization will always yield transversely isotropic lamina, but many real composite lamina are orthotropic—the in-plane transverse properties are often different from the through-thickness properties. Suppose we have data that indicates that, due to defects introduced in the manufacturing process, the through-thickness stiffnesses are only 75% as large as the in-plane transverse stiffnesses. We will override the properties estimated from the component homogenization to represent this. Change the value in the *E33* box from 0.98196 to 0.73647, and the value in the *G13* box from 0.40867 to 0.3065. These boxes will turn red to indicate that they no longer reflect the component properties.

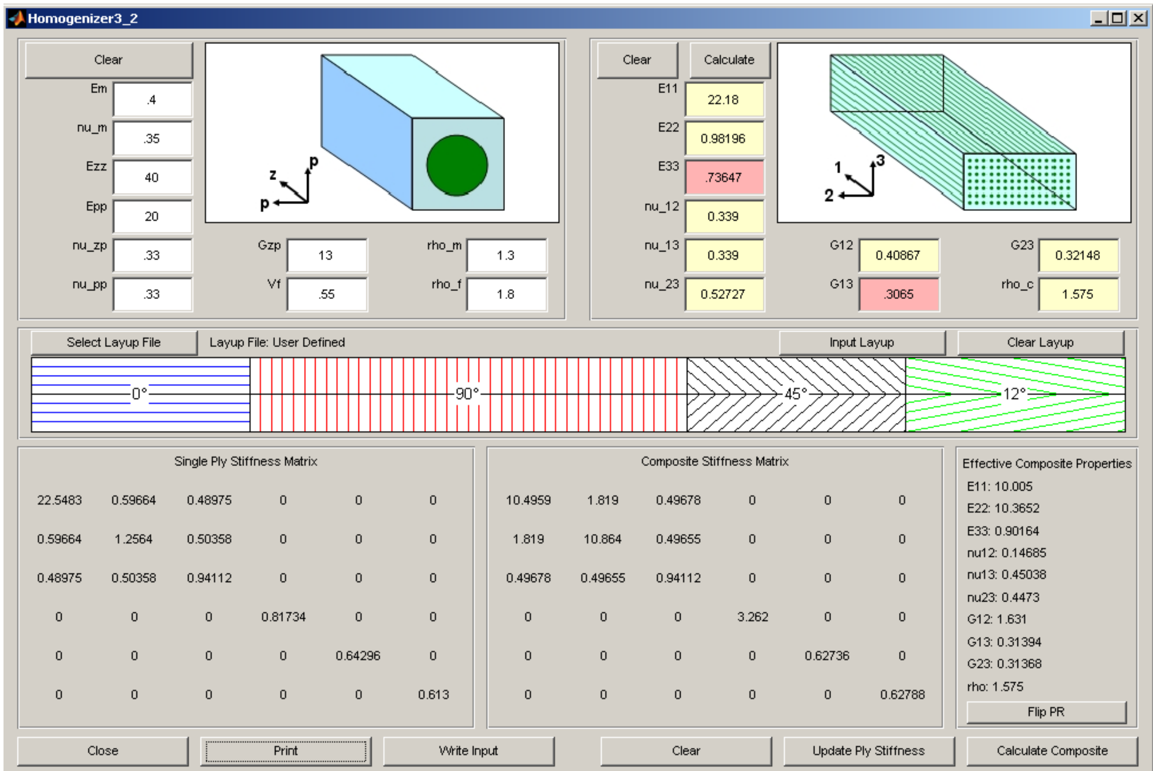
Before we enter the layup, click the “Calculate Composite” button to check that LARCH returns the correct properties for a uni-directional composite (the default layup). The three output boxes should be filled in with black text. The Single Ply Stiffness Matrix and the Composite Stiffness Matrix should be the same, and the Effective Composite Properties should be the same as the lamina input properties.

Now let’s enter the layup. We’ll be entering the layup manually, so click the “Input Layup” button. First, we’ll enter the layup literally. Our sample layup has twenty distinct layers (which happens to be the maximum number of layers LARCH can handle), so enter “20” in the dialog box that pops up and click “OK”. Now an angles dialog box will pop up. Enter the angles of each ply (in degrees): 0,90,0,90,-45,45,90,-12,12,90,90,12,-12,90,45,-45,90,0,90,0. Then click “OK”. Now another dialog box will appear requesting the thickness of each ply. Note that these are relative thicknesses—they will be normalized to unity—so they don’t have to equal the actual thickness of each layer. Every layer is a single ply, so we can leave the default of every layer having the same thickness. Just click “OK”.

LARCH will automatically combine any repeated layers, and assume that every orientation other than 0° or 90° is actually composed of positively and negatively oriented fibers. Hence our layup will be reduced to 20% 0° plies, 20% ±45° plies, 20% ±12° plies, and 40% 90° plies. This will be represented graphically in the Layup Input Block—sections of the box corresponding in size to these volume fractions will be labeled and shaded with lines oriented at the appropriate angles.

If we had wanted to save some typing (or if we had had more than twenty total plies in our sample layup), we could have entered this reduced layup more easily. Click the “Input Layup” button and indicate that there are just four layers (since there are four distinct families of orientations: 0°, 90°, ±45°, and ±12°). In the angles dialog, enter: 0,90,45,12. Now, in our sample layup, we have a total of four 0° plies, eight 90° plies, four plies whose orientation is either +45° or -45°, and four plies whose orientation is either +12° or -12°, so enter: 4,8,4,4 in the Thicknesses dialog box. When you click "OK", you will see that the calculated layup remains unchanged.

Now click the "Calculate Composite" button. LARCH will attempt to homogenize the current lamina properties for the chosen layup. It should appear like this:



Note that the Composite Stiffness Matrix will no longer be the same as the Single Ply Stiffness matrix. The homogenized properties will be displayed in the Effective Composite Properties box. If you need the Poisson's ratios in their alternate form (e.g. ν_{21} instead of ν_{12}), click the "Flip PR" button to convert them.

At this point you can experiment with the effects of changing the layup, the lamina input properties, the component properties, etc., on the homogenized properties. To create a hard copy of the composite for your records, click the "Print" button to open a print dialog, which will send an image of LARCH with the current inputs and outputs to your printer. Click "Close" and confirm with "Yes" when finished. Note that doing so will erase all data currently in the current LARCH session.

Theory

Conventions

LARCH uses different coordinate systems when homogenizing the lamina component properties to get the stiffness of a lamina, and when homogenizing the lamina to get the stiffness of the composite laminate.

A single lamina is assumed to be transversely isotropic, with the axis of isotropy aligned with the fiber direction. For the coordinate system used in the lamina homogenization, define the fiber direction to be the *z* direction, and the other two directions to be "*p*" directions (define the in-plane *p* direction to be *x* and the through-thickness *p* direction to be *y*). LARCH employs the following conventions:

$$\begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{zz} \\ \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{zx}/2 \\ \gamma_{xy}/2 \\ \gamma_{yz}/2 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_{zz} \\ \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma_{yz} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

This nonstandard ordering of the stress and strain components was chosen to be consistent with the convention that the axis of transverse isotropy is often represented as the *z*-direction, but the fiber direction in a composite lamina is often taken to be the *1*-direction.

For the coordinate system used in the laminate homogenization, define *x* and *y* to be the in-plane directions (where *x* corresponds to the direction of a 0°fiber) and *z* to be the through-thickness direction. Note that these are NOT the same as the *x, y*, and *z* directions defined above. LARCH employs the following numbering conventions and stiffness definition:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy}/2 \\ \gamma_{yz}/2 \\ \gamma_{xz}/2 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Material Behavior

Define the stiffnesses to be the coefficients that linearly relate the components of the stress vector to those of the strain vector:

$$C_{ij} \equiv \frac{\partial \sigma_i}{\partial \varepsilon_j}$$

Note that, because the shear strain terms correspond to tensorial shear strains, not engineering shear strains (which are twice as large), the shear terms of the stiffness matrices C_{44}, C_{55}, and C_{66} are twice the shear moduli (G_{xy}, $G_{yz,}$ and G_{xz}).

The composite matrix is assumed to be isotropic elastic. For a given modulus E and Poisson's ratio ν, its compliance matrix is:

$$\left[S_{ij}\right] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix}.$$

The fiber is assumed to be transversely isotropic, with five independent material properties: the modulus in the z-direction E_z, the modulus in the p-directions E_p, the axial-transverse Poisson's ratio ν_{zp}, the transverse-transverse Poisson's ratio ν_{pp}, and the axial-transverse shear modulus G_{zp} (the transverse-transverse shear modulus G_{pp} is not independent but can be calculated from E_p and ν_{pp}). The compliance matrix is:

$$\left[S_{ij}\right] = \begin{bmatrix} \frac{1}{E_z} & -\frac{\nu_{zp}}{E_z} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ & \frac{1}{E_p} & -\frac{\nu_{pp}}{E_p} & 0 & 0 & 0 \\ & & \frac{1}{E_p} & 0 & 0 & 0 \\ & & & \frac{1}{2G_{zp}} & 0 & 0 \\ & sym & & & \frac{1+\nu_{pp}}{E_p} & 0 \\ & & & & & \frac{1}{2G_{zp}} \end{bmatrix}$$

The lamina are characterized by an orthotropic material law, with nine material constants: E_x, E_y, E_z, ν_{xy}, ν_{yz}, ν_{xz}, G_{xy}, G_{xz}, and G_{yz}. These parameters can either be input directly or calculated from the fiber and matrix properties. The lamina compliance matrix is:

$$
\left[S_{ij}\right]=\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{xz}}{E_x} & 0 & 0 & 0 \\
 & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_y} & 0 & 0 & 0 \\
 & & \frac{1}{E_z} & 0 & 0 & 0 \\
 & & & \frac{1}{2G_{xy}} & 0 & 0 \\
 & sym & & & \frac{1}{2G_{yz}} & 0 \\
 & & & & & \frac{1}{2G_{xz}}
\end{bmatrix}
$$

For all materials, the stiffness matrix is the inverse of the compliance matrix.

Homogenizing the Component Properties

The component properties are homogenized using the methodology described by Zywicz (1), generalized for a fully three-dimensional stress and strain state. The basic assumptions are an isotropic linearly elastic matrix, and a transversely isotropic linearly elastic fiber. A representative volume element [RVE] is assumed consisting of a section of fiber, surrounded by an amount of matrix such that the desired fiber-matrix volume ratio is attained. A mixed homogenization scheme is used, rather than a purely isostrain (Voigt limit) or isostress (Reuss limit) homogenization scheme. Isostrain is assumed in the fiber direction, while iso-stress is assumed in all other directions:

$$
\begin{aligned}
&\sigma_1 = V^f\sigma_1^f + V^m\sigma_1^m \quad && \varepsilon_1 = \varepsilon_1^f = \varepsilon_1^m \\
&\sigma_\alpha = \sigma_\alpha^f = \sigma_\alpha^m && \varepsilon_\alpha = V^f\varepsilon_\alpha^f + V^m\varepsilon_\alpha^m \quad \text{with} \quad \alpha \in (2,3,4,5,6)
\end{aligned}
$$

Here V^f and V^m are the volume fractions of the fiber and matrix, respectively, and a superscript of an *f* or an *m* indicates that the quantity refers to the fiber or the matrix. Assume the following linearly elastic constitutive laws:

$$
\begin{aligned}
\sigma_i^m &= C_{ij}^m\varepsilon_j^m \Leftrightarrow \varepsilon_j^m = S_{ji}^m\sigma_i^m \quad \text{with} \quad S_{ji}^m = \left(C_{ij}^m\right)^{-1} \\
\sigma_i^f &= C_{ij}^f\varepsilon_j^f \Leftrightarrow \varepsilon_j^f = S_{ji}^f\sigma_i^f \quad \text{with} \quad S_{ji}^f = \left(C_{ij}^f\right)^{-1}
\end{aligned}
$$

After substantial algebraic manipulation, it can be shown that:

$$\sigma_i = C_{ij}^{eff} \varepsilon_j = \left(E_{ij} + G_{ik} D_{kl} B_{lj}\right)\varepsilon_j$$

with:

$$E_{ij} = \frac{1}{S_{11}^f} V^f \delta_{i1}\delta_{j1}$$

$$G_{ij} = V^f H_{ij} + V^m \delta_{ij}$$

$$H_{ij} = \begin{cases} 0 & j = 1 \\ q_j & i = 1, j > 1 \\ \delta_{ij} & i, j > 1 \end{cases} \qquad \text{with} \qquad q_j = -\frac{S_{1j}^f}{S_{11}^f}$$

$$D_{ij} = \left(J_{ji}\right)^{-1} \quad \text{with} \quad J_{ji} = S_{jk}^m \left(\delta_{ki} + \frac{V^f}{V^m} C_{kl}^m \hat{S}_{li}^f\right)$$

$$B_{ij} = \begin{cases} 1 & i, j = 1 \\ 0 & i = 1, j > 1 \\ \dfrac{V^f}{V^m} q_i & i > 1, j = 1 \\ \dfrac{1}{V^m} \delta_{ij} & i, j > 1 \end{cases}$$

The components of the matrix $\hat{S}_{ij}^f$ are determined in the following manner. The first row and first column are zero (*i.e.* $\hat{S}_{1j}^f = \hat{S}_{i1}^f = 0$). The remaining components are determined by inverting the submatrix formed by eliminating the first row and column of the fiber stiffness matrix:

$$\hat{S}_{\beta\alpha}^f = \left(C_{\alpha\beta}^f\right)^{-1} \text{ for } \alpha, \beta \in (2,3,4,5,6)$$

Once C^{eff} is known, it can be inverted to find the effective compliance matrix, and then the orthotropic lamina properties can be determined.

Homogenizing the Composite Properties

The individual lamina are also homogenized using a mixed homogenization scheme. However, in this scheme, isostrain is assumed in the plane of the laminate, while isostress is assumed in the through-thickness directions.

Define the following indicial conventions:

$$i, j, k \in (1,2,3,4,5,6)$$
$$a, b, c \in (1,2,4)$$
$$u, v, w \in (3,5,6)$$

The homogenization assumptions are:

$$
\begin{aligned}
\sigma_a &= \sum_n V^n \sigma_a^n & \qquad \varepsilon_a &= \varepsilon_a^n \\
\sigma_u &= \sigma_u^n & \qquad \varepsilon_u &= \sum_n V^n \varepsilon_u^n
\end{aligned}
$$

Here V^n is the volume fraction (or normalized thickness) of layer *n*, and a stress or strain quantity with a superscript of *n* refers to the stress or strain in that layer.

The goal is to express the total stress σ_i in terms of the total strain ε_j and an effective composite stiffness matrix $\bar{\bar{C}}_{ij}$. This first requires rotating the stiffness matrix associated with each layer into the global laminate coordinate frame. This gives stiffness matrices C_{ij}^n associated with each layer. It can be shown, again with significant algebraic manipulation, that the effective stiffness matrix is then:

$$
\begin{aligned}
\bar{\bar{C}}_{ab} &= \sum_n V^n \left\{ C_{ab}^n - C_{av}^n \tilde{S}_{vu}^n \left[C_{ub}^n - \bar{\hat{C}}_{uw} \bar{\tilde{C}}_{wb} \right] \right\} \\
\bar{\bar{C}}_{aw} &= \sum_n C_{av}^n \tilde{S}_{vu}^n \bar{\hat{C}}_{uw} \\
\bar{\bar{C}}_{ub} &= \bar{\hat{C}}_{uv} \bar{\tilde{C}}_{vb} \\
\bar{\bar{C}}_{uv} &= \bar{\hat{C}}_{uv}
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{\hat{C}}_{uv} &= \left[\bar{\tilde{S}}_{vu} \right]^{-1} \\
\bar{\tilde{S}}_{vu} &= \sum_n V^n \tilde{S}_{vu}^n \\
\bar{\tilde{C}}_{vb} &= \sum_n V^n \tilde{S}_{vu}^n C_{ub}^n \\
\tilde{S}_{vu}^n &= \left[C_{uv}^n \right]^{-1}
\end{aligned}
.
$$

References

1. Zywicz E., "A Tow-Level Progressive Damage Model for Simulating Carbon-Fiber Textile Composites: Final Report" Technical Report, Lawrence Livermore National Laboratory, 2003.