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On Arbitrage and Martingales

Bachelor thesis

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Abstract

This Bachelor thesis is based on the article "Martingales and arbitrage: a new look" by Alejandro Balbás and Pedro Jiménez-Guerra [2]. By introducing the concept of projective equivalence, this paper discusses how the existence of a projectively equivalent martingale measure is established under quite weak assumptions on the set of trading dates and the trajectory of the price process. The set of trading dates can be finite or countable, with bounded or unbounded time horizon and the price process does not have to be right-continuous. However, the projectively equivalent martingale measure is not defined on the initial probability space; instead, it is a martingale measure for the projective price process on the projective limit of a projective system of topological spaces. The existence is achieved by considering the projective limit of a projective system of Radon probability measures.

Contents

Chapter 1

Introduction

The concept of arbitrage is fundamental in Mathematical Finance. Intuitively, an arbitrage opportunity is a trading strategy that allows a profit without risk of loss. This should be impossible in any reasonable market. In finite discrete time, the absence of such arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. Under such a measure, the discounted price process of the traded assets is a martingale. But if the set of trading dates is infinite, the absence of arbitrage is not sufficient to build an equivalent martingale measure.

This paper discusses how the existence of a martingale measure is established under quite weak assumptions on the set of trading dates and the trajectory of the price process. The set of trading dates can be finite or countable, with bounded or unbounded time horizon and the price process does not have to be right-continuous. The martingale measure is established by introducing the concept of projective equivalence of measures. This is done by considering a projective system of topological spaces that is closely related to our price process. This projective system consists of topological spaces that each contain the restriction of the feasible trajectories of our price process to a finite subset of the set of trading dates. Two measures on the projective limit of this projective system of topological spaces are then called projectively equivalent if their projections, i.e. their image measures, are equivalent on all of the topological spaces that constitute the projective system. Next, we define a projective price process on the projective limit. The use of topological spaces suggests the use of Radon measures which have an interesting connection to linear functionals on the space of continuous functions with compact support. Finally, we exploit this connection and several other properties of Radon measures in order to construct a projective system of Radon measures whose projective limit will then be a projectively equivalent martingale measure for the projective price process under the assumption that our initial price process is finitely arbitrage-free.

The paper is structured as follows. Chapter 2 gives the description and probabilistic characterisation of a mathematical model for a financial market in which notions like price processes, trading strategies and arbitrage opportunities can be defined. Also, the first important result is given, namely the Fundamental Theorem of Asset Pricing. In Chapter 3, Radon measures are defined and their connection to linear functionals is shown with the Riesz representation theorem. Moreover, Polish and Radon spaces are defined, as well as projective systems of topological spaces and Radon measures. Then, results on the existence of projective limits of projective systems of Radon measures and an important example of a projective system of topological spaces are presented. With the help of this important example from Chapter 3, Chapter 4 transforms the mathematical model from Chapter 2 into a projective setting and introduces the projective price process and the concept of projective equivalence. Besides, a compactness result and the measurability of some functions are proved. In Chapter 5, we assume that our set of trading dates is countable and finally prove the existence of a projectively equivalent martingale measure in the main result Theorem 5.10. using some weak* compactness arguments.

This paper assumes basic knowledge in Functional Analysis, General Topology, Measure Theory and Probability Theory.

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Chapter 2

Arbitrage and martingale measures

This chapter gives the description and probabilistic characterisation of a mathematical model for a financial market in which notions like price processes, trading strategies and arbitrage opportunities can be defined. In Section 2.2., the Fundamental Theorem of Asset Pricing shows the characterisation of the absence of arbitrage in terms of equivalent martingale measures.

2.1 Arbitrage

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Suppose that $\mathcal{T} \subseteq [0, \infty)$ is a set of trading dates (finite or infinite, with bounded or unbounded time horizon) such that $0 \in \mathcal{T}$ (0 denoting the current date) and $\mathcal{T}$ contains at least two elements. The available information is provided by the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ over $\mathcal{T}$, i.e. an increasing family of σ-fields on Ω such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$ in $\mathcal{T}$. In addition, we assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\sigma(\bigcup_{t \in \mathcal{T}} \mathcal{F}_t) = \mathcal{F}$. The restriction of μ to $\mathcal{F}_t$ will be denoted by μ_t for all $t \in \mathcal{T}$.

We consider a financial market model in which $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ assets are traded. The asset prices are described by a price process S.

Definition 2.1. 1) A *price process* is an $\mathbb{R}^n$-valued, $\mathbb{F}$-adapted stochastic process $S = (S_t)_{t \in \mathcal{T}}$, where $S_t = (S_t^1, S_t^2, \ldots, S_t^n)$ and S_t^j is the price of asset j at time t for $j = 1, \ldots, n$ and $t \in \mathcal{T}$.

2) A *numéraire (asset)* is an asset j with $S^j > 0$, i.e. $S_t^j > 0$ for all $t \in \mathcal{T}$, μ-a.s.

There are other useful ways of thinking about the price process:

– a mapping $S : \Omega \times \mathcal{T} \to \mathbb{R}^n$, $(\omega, t) \mapsto S(\omega, t) := S_t(\omega)$ on the product space $\Omega \times \mathcal{T}$.

– a family of random functions $S_\cdot(\omega) : \mathcal{T} \to \mathbb{R}^n$, $t \mapsto S_\cdot(\omega)(t) := S_t(\omega)$, indexed by $\omega \in \Omega$. $S_\cdot(\omega)$ is called the *path* or *trajectory* of S for fixed $\omega \in \Omega$.

As usual, the first asset of the price process plays the role of a numéraire. For simplicity, we assume that

$$S_t^1(\omega) = 1 \quad \forall \omega \in \Omega \; \forall t \in \mathcal{T}. \tag{2.1}$$

Consequently, undiscounted and discounted quantities are equal, where discounting is effected by dividing all prices at all times in all states by the corresponding price of the numéraire asset.

Definition 2.2. 1) Consider a finite subset $V \subseteq \mathcal{T}$ containing 0, i.e. $V = \{0 = t_0 < t_1 < \ldots < t_T\}$, and a filtration $\mathbb{F}_V := (\mathcal{F}_t)_{t \in V}$. A *(trading) strategy* is an $\mathbb{R}^n$-valued process $\psi = (\psi_{t_k})_{k=1,\ldots,T}$ which is $\mathbb{F}_V$-predictable, i.e. ψ_{t_k} is $\mathcal{F}_{t_{k-1}}$-measurable for $k = 1, \ldots, T$.

2) A strategy ψ is called *self-financing* if $(\psi_{t_{k+1}} - \psi_{t_k}) \cdot S_{t_k} = 0$ μ-a.s. for $k = 1, \ldots, T-1$.

3) The *value/wealth process* of a strategy ψ is the $\mathbb{R}$-valued, $\mathbb{F}_V$-adapted process $W(\psi) = (W_{t_k}(\psi))_{k=0,\ldots,T}$ given by $W_0 := \psi_{t_1} \cdot S_0$ and $W_{t_k} := \psi_{t_k} \cdot S_{t_k}$ for $k = 1, \ldots, T$.

4) A strategy ψ is called *a-admissible* (for $S^V := (S_t)_{t \in V}$) if there exists $a \in \mathbb{R}$ such that $W(\psi) \geq -a$ μ-a.s. A strategy ψ is called *admissible* (for S^V) if it is a-admissible for some $a \in \mathbb{R}$.

Definition 2.3. An *arbitrage opportunity* is an admissible, self-financing strategy ψ with $-W_0(\psi) \in L^0_+(\mathcal{F}_0)$, $W_{t_T}(\psi) \in L^0_+(\mathcal{F}_{t_T})$ and not both $\equiv 0$. There are two kinds:

i) First kind: $W_{t_T}(\psi) \not\equiv 0$, i.e. $W_{t_T}(\psi) \geq 0$ μ-a.s. and $\mu[W_{t_T}(\psi) > 0] > 0$.

ii) Second kind: $W_0(\psi) \not\equiv 0$, i.e. $W_0(\psi) \leq 0$ μ-a.s. and $\mu[W_0(\psi) < 0] > 0$.

We call S^V *arbitrage-free* or S^V *satisfies NA* (no arbitrage) if there does not exist an arbitrage opportunity.

Remark 2.4. An arbitrage opportunity allows a profit without risk of loss. This should be impossible in any reasonable market.

Definition 2.5. S is called *finitely arbitrage-free* if S^V is arbitrage-free for all finite subsets $V \subseteq \mathcal{T}$ containing 0.

Hereafter, we assume bounded prices, i.e. $S^j_t \in L^\infty(\mathcal{F}_t)$ for $j = 1, \ldots, n$ and $t \in \mathcal{T}$.

Lemma 2.6. *For all $t \in \mathcal{T}$ there exists a μ-nullset $Z_t \in \mathcal{F}_t$ such that*

$$\forall j \in \{1, \ldots, n\}\ \forall \omega \in \Omega \setminus Z_t : \ |S^j_t(\omega)| \leq ||S^j_t||_\infty. \tag{2.2}$$

If we define

$$r_t := n^{\frac{1}{2}} \max_{j \in \{1,\ldots,n\}} ||S^j_t||_\infty, \tag{2.3}$$

then we also have

$$\forall \omega \in \Omega \setminus Z_t : \ ||S_t(\omega)|| \leq r_t, \tag{2.4}$$

where $||\cdot|| : \mathbb{R}^n \to [0, \infty)$ denotes the standard Euclidean norm.

Proof. By definition, we have

$$||S^j_t||_\infty := \inf\{c > 0 \mid \mu[|S^j_t| > c] = 0\} < \infty.$$

Now define the μ-nullsets $Z^j_t := \{|S^j_t| > ||S^j_t||_\infty\}$. If we take the μ-nullset

$$Z_t := \bigcup_{j \in \{1,\ldots,n\}} Z^j_t,$$

then (2.2) holds and (2.4) follows directly. □

2.2 Martingale measures

Definition 2.7. Consider two probability measures ν and μ on a measurable space $(\Omega, \mathcal{F})$. ν and μ are called *equivalent* on $\mathcal{F}$, $\nu \approx \mu$, if they have the same nullsets in $\mathcal{F}$, i.e. $\forall A \in \mathcal{F}: \ \nu(A) = 0 \iff \mu(A) = 0$.

Definition 2.8. Consider a probability space $(\Omega, \mathcal{A}, P)$ and a filtration in discrete time given by $\mathbb{A} = (\mathcal{A}_k)_{k \in \mathbb{N}_0}$. A stochastic process $M = (M_k)_{k \in \mathbb{N}_0}$ is called a *martingale* (with respect to P and $\mathbb{A}$) if

i) M is adapted to $\mathbb{A}$, i.e. M_k is $\mathcal{A}_k$-measurable $\forall k$.

ii) M is P-integrable, i.e. $E_P[|M_k|] < \infty \ \forall k$.

iii) M satisfies the martingale property, i.e.

$$E_P[M_\ell | \mathcal{F}_k] = M_k \qquad P\text{-a.s. } \forall k \leq \ell. \tag{2.5}$$

If we only look at a martingale $M = (M_k)_{k=0,1,\ldots,T}$ in finite discrete time for some $T \in \mathbb{N}$, then (2.5) is equivalent to

$$E_P[M_T | \mathcal{F}_k] = M_k \qquad P\text{-a.s. } \forall k = 0, 1, \ldots, T.$$

Definition 2.9. An *equivalent martingale measure (EMM)* for S^V and numéraire S^1 is a probability measure $\nu : \mathcal{F}_{t_T} \to [0,1]$ with $\nu \approx \mu_{t_T}$ on $\mathcal{F}_{t_T}$ such that $\frac{S^V}{S^1}$ is a martingale with respect to ν and $\mathbb{F}_V$, in particular

$$E_\nu[\frac{S_{t_T}}{S^1_{t_T}} | \mathcal{F}_{t_k}] = \frac{S_{t_k}}{S^1_{t_k}} \qquad \nu\text{-a.s. } \forall k = 0, 1, \ldots, T. \tag{2.6}$$

The set of all EMMs ν for S^V and numéraire S^1 is denoted by $\mathbb{P}(S^V; S^1)$ or simply $\mathbb{P}(S^V)$.

Because of our assumption (2.1), the above definition just requires S^V to be a martingale.

Remark 2.10. Due to the tower property of conditional expectations, we also have

$$E_\nu[S_{t_\ell}|\mathcal{F}_{t_k}] = S_{t_k} \quad \nu\text{-a.s. } \forall k, \ell = 0, 1, \ldots, T \text{ with } k \leq \ell.$$

Theorem 2.11 (Fundamental Theorem of Asset Pricing)**.** *Let $V \subseteq \mathcal{T}$ be a finite set of trading dates (containing* 0*) and S^1 a numéraire asset. Then we have that S^V is arbitrage-free if and only if there exists an equivalent martingale measure ν for S^V and numéraire S^1, i.e.*

$$NA \text{ for } S^V \iff \mathbb{P}(S^V; S^1) \neq \emptyset.$$

Proof. See Dalang et al. [7], Theorem 3.3., equivalence of (3.1) and (3.3). □

Despite the Fundamental Theorem of Asset Pricing, Back and Pliska [1] have shown with a counterexample that if the set of trading dates is not finite, absence of arbitrage in general does not imply the existence of an equivalent martingale measure. The goal of this paper is to show the existence of a martingale measure even if the set of trading dates is not finite. However, this will come at the cost of losing equivalence and transforming our mathematical model into another setting.

Chapter 3

Radon measures and projective systems

In this chapter, Radon measures are defined and their connection to linear functionals is shown with the Riesz representation theorem. Moreover, Polish and Radon spaces are defined, as well as projective systems of topological spaces and Radon measures. In the end, results on the existence of projective limits of projective systems of Radon measures and the important Example 3.21., that illustrates a projective system of topological spaces closely related to stochastic processes, are presented.

3.1 Radon measures

Definition 3.1. Let $(X, \mathcal{O})$ be a topological space.

1) Let $x \in X$. A *neighbourhood* of x is a subset $U \subseteq X$ such that there exists an open set $V \in \mathcal{O}$ with $x \in V \subseteq U$.

2) $(X, \mathcal{O})$ is called *Hausdorff space* if for all $x, y \in X$ with $x \neq y$ there exist neighbourhoods $U, V \subseteq X$ such that

$$x \in U,\ y \in V,\ U \cap V \neq \emptyset.$$

3) $(X, \mathcal{O})$ is called *locally compact* if for every point $x \in X$ there exists a compact neighbourhood.

4) $\mathcal{B}(X) := \sigma(\mathcal{O})$ is called the *Borel σ-field* on X.

Definition 3.2. Let $(X, \mathcal{O})$ be a locally compact Hausdorff space, $\mathcal{A} \subseteq 2^X$ a σ-field and $\nu : \mathcal{A} \to [0, \infty]$ a measure. Such a measure ν is called *Radon measure* if it has the following properties:

i) $\mathcal{B}(X) \subseteq \mathcal{A}$.

ii) Every compact subset $K \subseteq X$ has finite measure $\nu(K) < \infty$.

iii) For every measurable set $A \in \mathcal{A}$:

$$\nu(A) = \inf\{\nu(U) \mid U \subseteq X \text{ open}, A \subseteq U\}.$$

iv) For every open set $U \in \mathcal{O}$:

$$\nu(U) = \sup\{\nu(K) \mid K \subseteq X \text{ compact}, K \subseteq U\}. \tag{3.1}$$

Lemma 3.3. *Let $(X, \mathcal{O})$ be a locally compact Hausdorff space, $\mathcal{A} \subseteq 2^X$ a σ-field and $\nu : \mathcal{A} \to [0, \infty]$ a Radon measure. Then:*

1) For every measurable set $A \in \mathcal{A}$ with $\nu(A) < \infty$, we have

$$\nu(A) = \sup\{\nu(K) \mid K \subseteq X \text{ compact}, K \subseteq A\}.$$

2) There exists an open ν-nullset $G \subseteq X$ that contains all open ν-nullsets of X.

Proof. 1) See Salamon [12], Lemma 3.1.6.

2) Let $(U_i)_{i \in I}$ be the family of open ν-nullsets of X. We define

$$G := \bigcup_{i \in I} U_i.$$

Let $K \subseteq G$ be compact. Then there exists a finite subset $J \subseteq I$ such that $K \subseteq \bigcup_{j \in J} U_j$ and hence $\nu(K) = 0$. By (3.1), we have $\nu(G) = 0$.

□

Definition 3.4. The complement $X \setminus G := Sp(\nu)$ of the set G in Lemma 3.3. 2) is called the *support* of ν.

3.2 The Riesz representation theorem

Definition 3.5. 1) The space of continuous functions from X to $\mathbb{R}$ with compact support is

$$C_c(X) := \{f : X \to \mathbb{R} \mid f \text{ continuous, } \operatorname{supp}(f) \subseteq X \text{ compact}\}$$

where $\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$.

2) A linear functional $\Lambda : C_c(X) \to \mathbb{R}$ is called *positive* if

$$f \in C_c(X) \text{ and } f \geq 0 \quad \Rightarrow \quad \Lambda f \geq 0.$$

Theorem 3.6 (Riesz representation theorem)**.** *Let $(X, \mathcal{O})$ be a locally compact Hausdorff space and $\mathcal{B} \subseteq 2^X$ the Borel σ-field on X. Then for every positive, linear functional $\Lambda : C_c(X) \to \mathbb{R}$ there exists a unique Radon measure $\nu : \mathcal{B} \to [0, \infty]$ such that*

$$\Lambda f = \int_X f \, \mathrm{d}\nu \quad \forall f \in C_c(X). \tag{3.2}$$

Proof. See Salamon [12], Theorem 3.3.6. □

Corollary 3.7. *1) The positive, linear functional $\Lambda : C_c(X) \to \mathbb{R}$ given by (3.2) is continuous.*

2) If in addition $(X, \mathcal{O})$ is a compact Hausdorff space, then we can replace $C_c(X)$ by $C(X) := \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$ in Theorem 3.6.

Proof. 1) Λ is continuous if

$$\sup_{||f||_{C_c(X)}\leq 1,\, f\in C_c(X)} |\Lambda f| < \infty, \text{ where } ||f||_{C_c(X)} := \sup_{x\in X} |f(x)|.$$

For $f \in C_c(X)$ with $||f||_{C_c(X)} \leq 1$, we have

$$\begin{aligned}
|\Lambda f| &\overset{(3.2)}{=} |\int_X f \,\mathrm{d}\nu| = |\int_{\mathrm{supp}(f)} f \,\mathrm{d}\nu| \\
&\leq \int_{\mathrm{supp}(f)} |f| \,\mathrm{d}\nu \\
&\leq \underbrace{\sup_{x\in\mathrm{supp}(f)} |f(x)|}_{=||f||_{C_c(X)}\leq 1} \int_{\mathrm{supp}(f)} \mathrm{d}\nu < \infty.
\end{aligned}$$

In the last inequality, we used that compact subsets of X have finite measure.

2) Closed subsets of compact spaces are compact. Hence we get that $\mathrm{supp}(f) \subseteq X$ is compact for all continuous functions $f : X \to \mathbb{R}$ and thus $C_c(X) = C(X)$.

□

Theorem 3.8 (Banach-Alaoglu). *Let $(Y, ||\cdot||_Y)$ be a real normed vector space and $(Y^*, ||\cdot||_{Y^*})$ its dual space. Then the closed unit ball*

$$B := \{l \in Y^* \mid ||l||_{Y^*} \leq 1\}$$

in the dual space Y^* is weak compact.*

Proof. See Bühler and Salamon [6], Theorem 3.2.4. □

If $(X, \mathcal{O})$ is a locally compact Hausdorff space, then notions like weak* compactness or closedness for subsets A of the set $\mathcal{R}(X)$ of Radon measures on the Borel σ-field $\mathcal{B}(X)$ relate to $\varphi(A)$ via the bijection $\varphi : \mathcal{R}(X) \to C_c(X)^*$ from Theorem 3.6, where $C_c(X)^*$ is endowed with the weak* topology.

Corollary 3.9. *Let $(X, \mathcal{O})$ be a compact Hausdorff space, $\mathcal{B} \subseteq 2^X$ the Borel σ-field on X and $\mathcal{R}$ the set of Radon measures on $\mathcal{B}$. Then the set*

$$\mathcal{R}_1 := \{\nu \in \mathcal{R} \mid \nu(X) \leq 1\} \subseteq \mathcal{R}$$

is weak compact.*

Proof. By Corollary 3.7., we can identify $\mathcal{R}$ with the set of real-valued, continuous, positive, linear functionals on $C(X)$ via the bijection $\varphi : \mathcal{R} \to C(X)^*$ from Theorem 3.6. For $\nu \in \mathcal{R}_1$, we have $\nu(X) \leq 1$. For the corresponding linear functional $\Lambda : C(X) \to \mathbb{R}$, we then have $||\Lambda||_{C(X)^*} \leq 1$ because

$$|\Lambda f| = |\int_X f \,\mathrm{d}\nu| \leq ||f||_{C(X)} \underbrace{\int_X \mathrm{d}\nu}_{=\nu(X)\leq 1} \leq ||f||_{C(X)}$$

and

$$||\Lambda||_{C(X)^*} := \sup_{||f||_{C(X)}\leq 1\,,\, f\in C(X)} |\Lambda f|.$$

Hence $\varphi(\mathcal{R}_1) \subseteq B := \{\Lambda \in C(X)^* \mid ||\Lambda||_{C(X)^*} \leq 1\}$. Next, we show $B \subseteq \varphi(\mathcal{R}_1)$. For $\Lambda \in B$ there exists a unique Radon measure $\nu \in \mathcal{R}$ such that $\varphi(\nu) = \Lambda$. Then we have for $f \equiv 1 \in C(X)$

$$\Lambda f = \int_X f \,\mathrm{d}\nu = \int_X \mathrm{d}\nu = \nu(X).$$

So we get $\nu(X) \leq 1$ because otherwise $||\Lambda||_{C(X)^*} > 1$. Hence we have $B = \varphi(\mathcal{R}_1)$ and the assertion now follows from Theorem 3.8. with $Y := C(X)$. □

3.3 Polish and Radon spaces

Definition 3.10. 1) A locally compact Hausdorff space $(X, \mathcal{O})$ is called *Radon space* if every finite measure defined on the Borel σ-field $\mathcal{B}(X)$ is a Radon measure.

2) A *Polish space* is a separable topological space which is metrisable in

such a way that as a metric space, it is complete.

Example 3.11. $\mathbb{R}^C$ endowed with the product topology is a locally compact Polish space for all finite or countable sets C, where $\mathbb{R}^C$ is the space of functions from C to $\mathbb{R}$.

Theorem 3.12. *Every locally compact Polish space is a Radon space.*

Proof. See Klenke [9], Theorem 13.6. □

Consider a measure space $(\Omega, \mathcal{F}, \mu)$, a locally compact Hausdorff space $(X, \mathcal{O})$ with its Borel σ-field $\mathcal{B}$ and a measurable map $h : (\Omega, \mathcal{F}) \to (X, \mathcal{B})$. Then μ induces a measure $h(\mu)$ on $\mathcal{B}$ given by

$$h(\mu)(B) := \mu(h^{-1}(B)) \quad \forall B \in \mathcal{B}.$$

It is called *the image of* μ *through* h or simply *image measure.* Moreover, if μ is a finite measure and $(X, \mathcal{O})$ a Radon space, then $h(\mu)$ is a Radon measure.

Theorem 3.13. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space,* $(U, \mathcal{U})$ *a measurable space,* $h : (\Omega, \mathcal{F}) \to (U, \mathcal{U})$ *and* $g : (U, \mathcal{U}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *measurable mappings. Then* g *is integrable with respect to the image measure* $h(\mu)$ *if and only if* $g \circ h$ *is integrable with respect to the measure* μ*. In addition, one has*

$$\int_U g \,\mathrm{d}h(\mu) = \int_\Omega g \circ h \,\mathrm{d}\mu.$$

Proof. See Bogachev [4], Theorem 3.6.1. □

Lemma 3.14. *Let* $(X, \mathcal{O}^X)$ *and* $(Y, \mathcal{O}^Y)$ *be two compact Polish spaces and* $h : X \to Y$ *a continuous function. Then* h *induces a new function*

$$\mathcal{R}_f^X \to \mathcal{R}_f^Y, \ \nu \mapsto h(\nu) \tag{3.3}$$

where $\mathcal{R}_f^X$ *and* $\mathcal{R}_f^Y$ *are the sets of finite Radon measures on the Borel* σ*-fields of* X *and* Y *respectively. Moreover, this transformation is continuous if* $\mathcal{R}_f^X$ *and* $\mathcal{R}_f^Y$ *are endowed with their weak* topologies.*

Proof. Let $\nu \in \mathcal{R}_f^X$. By finiteness of ν and Theorem 3.12., $h(\nu)$ is a finite Radon measure and hence the transformation is well defined.

Let $\varphi_X : \mathcal{R}^X \to C(X)^*$ and $\varphi_Y : \mathcal{R}^Y \to C(Y)^*$ be the bijections from Theorem 3.6. for X and Y respectively, where $\mathcal{R}^X$ and $\mathcal{R}^Y$ denote the sets of Radon measures on $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively. We want to show that the transformation (3.3) is weak* continuous in the sense that the mapping

$$H : \varphi_X(\mathcal{R}_f^X) \to \varphi_Y(\mathcal{R}_f^Y),\ \varphi_X(\nu) \mapsto \varphi_Y(h(\nu)).$$

is weak* continuous. To that end, take a sequence $(\varphi_X(\nu_k))_{k\in\mathbb{N}}$ in $\varphi_X(\mathcal{R}_f^X)$ that weak* converges to some $\varphi_X(\nu) \in \varphi_X(\mathcal{R}_f^X)$, i.e. for all $f \in C(X)$

$$\varphi_X(\nu_k)(f) \to \varphi_X(\nu)(f) \quad (k \to \infty). \tag{3.4}$$

H is weak* continuous if $H(\varphi_X(\nu_k))$ weak* converges to $H(\varphi_X(\nu))$, i.e. for all $g \in C(Y)$

$$\varphi_Y(h(\nu_k))(g) \to \varphi_Y(h(\nu))(g) \quad (k \to \infty). \tag{3.5}$$

By Theorem 3.6., we have for all $g \in C(Y)$

$$\varphi_Y(h(\nu_k))(g) = \int_Y g \,\mathrm{d}h(\nu_k)$$

and by Theorem 3.13.

$$\int_Y g \,\mathrm{d}h(\nu_k) = \int_X \underbrace{g \circ h}_{\in C(X)} \,\mathrm{d}\nu_k.$$

But again by Theorem 3.6., we have

$$\varphi_X(\nu_k)(g \circ h) = \int_X g \circ h \,\mathrm{d}\nu_k$$

and so $\varphi_Y(h(\nu_k)) = \varphi_X(\nu_k)(\,\cdot \circ h)$ for all $k \in \mathbb{N}$ and analogously $\varphi_Y(h(\nu)) = \varphi_X(\nu)(\,\cdot \circ h)$. Hence (3.5) follows from (3.4) because $g \circ h \in C(X)$ for all $g \in C(Y)$. $\square$

3.4 Projective systems

Definition 3.15. 1) Consider a set P and a binary relation $\leq$ on P. Then $\leq$ is a *preorder* if for all $a, b, c \in P$

i) (Reflexivity) $a \leq a$,

ii) (Transitivity) $a \leq b$ and $b \leq c \implies a \leq c$.

2) A *directed set* is a non-empty set I together with a preorder such that every pair of elements has an upper bound, i.e.

$$\forall e, f \in I \ \exists g \in I : \quad e \leq g \text{ and } f \leq g.$$

Definition 3.16. Let $\leq$ be the ordering relation of a directed set I. Consider a family of Hausdorff spaces $(X_i, \mathcal{O}^i)_{i\in I}$ and continuous maps $\pi_{ij} : X_j \to X_i$ for $i, j \in I$ with $i \leq j$. We say that

$$\big((X_i, \pi_{ij})\big)_{i,j\in I,\, i\leq j}$$

is a *projective system of topological spaces* if

$$\pi_{ik} = \pi_{ij} \circ \pi_{jk} \text{ for all } i, j, k \in I \text{ with } i \leq j \leq k.$$

Its *projective limit* is

$$X = \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_j = \pi_{jk}(x_k) \text{ if } j, k \in I \text{ with } j \leq k\Big\}$$

endowed with the product topology $\mathcal{O}$.

Consider the canonical projections

$$\pi_i : X \to X_i,\ (x_j)_{j\in I} \mapsto x_i \text{ for } i \in I.$$

These maps are continuous and satisfy $\pi_i = \pi_{ij} \circ \pi_j$ for all $i, j \in I$ with $i \leq j$. Recall that the product topology is the coarsest topology on X which makes all the canonical projections continuous, i.e. $\mathcal{O} = \tau(\bigcup_{i\in I} \pi_i^{-1}(\mathcal{O}^i))$, where $\tau(\mathcal{A})$ denotes the coarsest topology on X containing the family of sets $\mathcal{A} \subseteq 2^X$.

X may be endowed with the Borel σ-field $\mathcal{B} := \sigma(\mathcal{O})$ or the *cylindrical σ-field* $\mathcal{B}_0 \subseteq \mathcal{B}$ which is defined by

$$\mathcal{B}_0 := \sigma(\bigcup_{i \in I} \pi_i^{-1}(\mathcal{O}^i)).$$

Note that $\mathcal{B}_0$ is the smallest σ-field which makes every canonical projection π_i $\mathcal{B}_0$-$\mathcal{B}^i$-measurable, where $\mathcal{B}^i := \sigma(\mathcal{O}^i)$ is the Borel σ-field on X_i.

Definition 3.17. 1) Let $\big((X_i, \pi_{ij})\big)_{i,j \in I,\, i \leq j}$ be a projective system of topological spaces. If ν_i is a finite Radon measure on $(X_i, \mathcal{B}^i)$ for all $i \in I$ and $\pi_{ij}(\nu_j) = \nu_i$ for all $i, j \in I$ with $i \leq j$, then $(\nu_i)_{i \in I}$ is called a *projective system of Radon measures.*

2) We say that $(\nu_i)_{i \in I}$ converges to the measure $\nu : \mathcal{B}_0 \to [0, \infty]$ or ν is the *projective limit* of $(\nu_i)_{i \in I}$ if $\pi_i(\nu) = \nu_i$ for all $i \in I$.

3.5 Existence of projective limits

The following two results on the existence of projective limits of projective systems of Radon measures are adapted from Bourbaki [5], §4.3., Theorem 2 and Schwartz [13], Part I, Chapter I.10., Theorem 21 respectively.

Theorem 3.18. *Let $(\nu_i)_{i \in I}$ be a projective system of Radon measures associated to a projective system of topological spaces $\big((X_i, \pi_{ij})\big)_{i,j \in I,\, i \leq j}$ with projective limit $(X, \mathcal{O})$ and $\mathcal{B}_0$ the cylindrical σ-field on X. If I is countable, then there exists a projective limit $\nu : \mathcal{B}_0 \to [0, \infty]$ of $(\nu_i)_{i \in I}$. Moreover, ν is unique and can be extended to a unique Radon measure defined on the Borel σ-field $\mathcal{B}$ on X.*

Theorem 3.19 (Prokhorov)**.** *Let $(\nu_i)_{i \in I}$ be a projective system of Radon measures associated to a projective system of topological spaces $\big((X_i, \pi_{ij})\big)_{i,j \in I,\, i \leq j}$ with projective limit $(X, \mathcal{O})$ and $\mathcal{B}$ the Borel σ-field on X. Then there exists a Radon measure $\nu : \mathcal{B} \to [0, \infty]$ such that $\pi_i(\nu) = \nu_i$ for all $i \in I$ if and only* *if for all $\epsilon > 0$ there exists a compact set $K \subseteq X$ with $\nu_i(X_i \setminus \pi_i(K)) < \epsilon$ for all $i \in I$. In the affirmitive case, ν is unique.*

Corollary 3.20. *If $Sp(\nu_i) \subseteq X_i$ is compact for all $i \in I$, then there exists a unique Radon measure $\nu : \mathcal{B} \to [0, \infty]$ such that $\pi_i(\nu) = \nu_i$ for all $i \in I$. Furthermore, ν has compact support and $Sp(\nu) \subseteq \prod_{i \in I} Sp(\nu_i)$.*

Proof. By Tychonoff's theorem, $\prod_{i \in I} Sp(\nu_i) \subseteq X$ is compact and hence the first assertion follows from Theorem 3.19. with $K := \prod_{i \in I} Sp(\nu_i)$ because $\nu_i(X_i \setminus \pi_i(K)) = \nu_i(X_i \setminus Sp(\nu_i)) = 0$ for all $i \in I$ by definition of the support of a Radon measure.

Compactness of $Sp(\nu)$ follows from the inclusion $Sp(\nu) \subseteq \prod_{i \in I} Sp(\nu_i)$ because it is a closed subset of a compact space. From the proof of Lemma 3.3. 2), we know that

$$Sp(\nu_i) = \bigcup_{F \in \mathcal{O}^i,\, \nu_i(F)=0} F.$$

Because $\mathcal{O} = \tau(\bigcup_{i \in I} \pi_i^{-1}(\mathcal{O}^i))$, we get that $Sp(\nu) \subseteq \prod_{i \in I} Sp(\nu_i)$. □

Example 3.21. Let $\mathcal{T} \subseteq [0, \infty)$ be the set of trading dates we considered in Chapter 2. There is a special type of projective system of topological spaces closely related to stochastic processes. Consider the set $\mathcal{P}_F(\mathcal{T})$ of finite subsets of $\mathcal{T}$ containing 0. Together with the order generated by the usual inclusion of sets, $\mathcal{P}_F(\mathcal{T})$ is a directed set. For every $V \in \mathcal{P}_F(\mathcal{T})$ we take the locally compact Polish and hence Radon space $(\mathbb{R}^n)^V$ of $\mathbb{R}^n$-valued functions on V endowed with the product topology. If $U, V \in \mathcal{P}_F(\mathcal{T})$ satisfy $V \subseteq U$, then $\pi_{VU} : (\mathbb{R}^n)^U \to (\mathbb{R}^n)^V,\ g \mapsto g|_V$ is the standard projection. Because we have

$$\pi_{WU} = \pi_{WV} \circ \pi_{VU} \text{ for all } U, V, W \in \mathcal{P}_F(\mathcal{T}) \text{ with } W \subseteq V \subseteq U,$$

we are facing a projective system of topological spaces

$$\big(((\mathbb{R}^n)^V, \pi_{VU})\big)_{V,U \in \mathcal{P}_F(\mathcal{T}),\, V \subseteq U} \tag{3.6}$$

whose projective limit can be identified with the space $(\mathbb{R}^n)^{\mathcal{T}}$ endowed with the product topology. Furthermore, $\pi_V : (\mathbb{R}^n)^{\mathcal{T}} \to (\mathbb{R}^n)^V,\ g \mapsto g|_V$ is also the standard projection for all $V \in \mathcal{P}_F(\mathcal{T})$. As in the general case, the projective limit $(\mathbb{R}^n)^{\mathcal{T}}$ can be endowed with its cylindrical or Borel σ-field denoted by $\mathcal{B}_0^{\mathcal{T}}$ and $\mathcal{B}^{\mathcal{T}}$ respectively.

Remark 3.22. Let $(\Omega, \mathcal{F})$ be a measurable space. When endowed with $\mathcal{B}_0^{\mathcal{T}}$, a mapping $f : \Omega \to (\mathbb{R}^n)^{\mathcal{T}}$ is $\mathcal{F}$-$\mathcal{B}_0^{\mathcal{T}}$-measurable if and only if $\pi_V \circ f : \Omega \to (\mathbb{R}^n)^V$ is $\mathcal{F}$-$\mathcal{B}^V$-measurable for all $V \in \mathcal{P}_F(\mathcal{T})$, where $\mathcal{B}^V$ denotes the Borel σ-field on $(\mathbb{R}^n)^V$.

The following result can be established by readapting some statements of Kopp [10], notably Theorem 0.1.7.

Theorem 3.23 (Daniell-Kolmogorov)**.** *Let $\mathcal{T}$ be a subset of $[0, \infty)$ such that $0 \in \mathcal{T}$ and $|\mathcal{T}| \geq 2$ and $\mathcal{P}_F(\mathcal{T})$ the set of finite subsets of $\mathcal{T}$ containing 0. Let $(\nu_V)_{V \in \mathcal{P}_F(\mathcal{T})}$ be a projective system of Radon measures associated with the projective system of of topological spaces*

$$\big(\big((\mathbb{R}^n)^V, \pi_{VU}\big)\big)_{V,U \in \mathcal{P}_F(\mathcal{T}),\, V \subseteq U}$$

whose projective limit can be identified with the space $(\mathbb{R}^n)^{\mathcal{T}}$ endowed with the product topology. Then there exists a unique projective limit $\nu_{\mathcal{T}} : \mathcal{B}_0^{\mathcal{T}} \to [0, \infty]$ of $(\nu_V)_{V \in \mathcal{P}_F(\mathcal{T})}$, where $\mathcal{B}_0^{\mathcal{T}}$ denotes the cylindrical σ-field on $(\mathbb{R}^n)^{\mathcal{T}}$.

Chapter 4

Projective system approach

With the help of Example 3.21., this chapter transforms the mathematical model from Chapter 2 into a projective setting by introducing the projective price process and the concept of projective equivalence. Moreover, we prove the measurability of some functions and the result in Lemma 4.6. on compactness which we will need in Chapter 5.

4.1 Measurability and compact support

Let us proceed with the same set-up as in Chapter 2.

Proposition 4.1. *Let $(X_i, \mathcal{O}^i)_{i \in I}$ be a countable family of topological spaces such that $\mathcal{O}^i$ has a countable basis for all $i \in I$. Then the cylindrical σ-field $\mathcal{B}_0$ on X is the same as the Borel σ-field $\mathcal{B}$ on X.*

Proof. See Elstrodt [8], Proposition 5.10. $\square$

Corollary 4.2. *Let $V \in \mathcal{P}_F(\mathcal{T})$. Then the cylindrical σ-field $\mathcal{B}_0^V$ on $(\mathbb{R}^n)^V$ is the same as the Borel σ-field $\mathcal{B}^V$ on $(\mathbb{R}^n)^V$.*

Proof. If we define $|V| =: m < \infty$, then we can identify $(\mathbb{R}^n)^V$ with $\mathbb{R}^{m \cdot n}$. Let $\mathcal{O}^{\mathbb{R}^n}$ be the Euclidean topology on $\mathbb{R}^n$. Consider the family of sets

$$\mathcal{E} := \{B(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}_+\},$$

where $B(x, r)$ denotes the open ball in $\mathbb{R}^n$ with centre x and radius r. Then every open set $U \subseteq \mathbb{R}^n$ can be written as a countable union of elements in $\mathcal{E}$. Hence $\mathcal{E}$ is a countable basis of $\mathcal{O}^{\mathbb{R}^n}$. The assertion now follows from Proposition 4.1. □

Definition 4.3. Let $V \in \mathcal{P}_F(\mathcal{T})$. We define S_V to be the function which maps $\omega \in \Omega$ to the restriction of the corresponding path of S to V, i.e.

$$S_V : \Omega \to (\mathbb{R}^n)^V, \ \omega \mapsto S_V(\omega) := S_\cdot(\omega)|_V$$

such that

$$S_V(\omega) : V \to \mathbb{R}^n, \ t \mapsto S_V(\omega)(t) := S_t(\omega).$$

In order to show measurability of functions, we often use the following basic result from Measure Theory.

Lemma 4.4. *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces and $\mathcal{E}' \subseteq 2^{\Omega'}$ a generator of $\mathcal{A}'$, i.e. $\sigma(\mathcal{E}') = \mathcal{A}'$. Then $\varphi : \Omega \to \Omega'$ is $\mathcal{A}$-$\mathcal{A}'$-measurable if and only if $\varphi^{-1}(\mathcal{E}') \subseteq \mathcal{A}$.*

Lemma 4.5. *Let $V \in \mathcal{P}_F(\mathcal{T})$, $v := \max(V)$ and $\mathcal{B}^V$ the Borel σ-field on $(\mathbb{R}^n)^V$. Then $S_V : \Omega \to (\mathbb{R}^n)^V$ is $\mathcal{F}_v$-$\mathcal{B}^V$-measurable.*

Proof. First, we define for all $t \in V$ the projections

$$\pi_t : (\mathbb{R}^n)^V \to \mathbb{R}^n, \ f \mapsto \pi_t(f) := f|_{\{t\}} = f(t).$$

By definition of the cylindrical σ-field, we have

$$\mathcal{B}_0^V = \sigma(\bigcup_{t \in V} \pi_t^{-1}(\mathcal{B}(\mathbb{R}^n))).$$

By considering

$$\pi_t \circ S_V : \Omega \to (\mathbb{R}^n)^V \to \mathbb{R}^n, \ \omega \mapsto S_\cdot(\omega)|_V \mapsto S_\cdot(\omega)|_{\{t\}} = S_t(\omega),$$

we see that $\pi_t \circ S_V = S_t$ which is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^n)$-measurable for all $t \in \mathcal{T}$ by definition. Hence for all $t \in V$

$$S_V^{-1}(\pi_t^{-1}(\mathcal{B}(\mathbb{R}^n))) = S_t^{-1}(\mathcal{B}(\mathbb{R}^n)) \subseteq \mathcal{F}_t \subseteq \mathcal{F}_v.$$

which implies

$$S_V^{-1}(\bigcup_{t \in V} \pi_t^{-1}(\mathcal{B}(\mathbb{R}^n))) \subseteq \mathcal{F}_v.$$

The assertion now follows from Lemma 4.4. and Corollary 4.2. □

Because $(\Omega, \mathcal{F}_v, \mu_v)$ is a probability space, the image measure $S_V(\mu_v) : \mathcal{B}^V \to [0,1]$ is a probability measure on the locally compact Polish space $(\mathbb{R}^n)^V$ and by Theorem 3.12., it is also a Radon probability measure. In order to simplify the notation, the previous probability measure will be represented by $S_V(\mu)$ and the subscript v will be omitted. Hence

$$(S_V(\mu))_{V \in \mathcal{P}_F(\mathcal{T})} \tag{4.1}$$

is a projective system of Radon probability measures associated with (3.6), i.e. for all $V, U \in \mathcal{P}_F(\mathcal{T})$ with $V \subseteq U$, we have that $\pi_{VU}(S_U(\mu)) = S_V(\mu)$. Indeed, $\pi_{VU} \circ S_U = S_V$.

Lemma 4.6. *For every $V \in \mathcal{P}_F(\mathcal{T})$, we have*

$$Sp(S_V(\mu)) \subseteq \prod_{t \in V} K_t \subseteq (\mathbb{R}^n)^V, \tag{4.2}$$

where K_t denotes the closed ball in $\mathbb{R}^n$ with centre 0 and radius r_t (see (2.3)) for all $t \in \mathcal{T}$. In particular, the support of $S_V(\mu)$ is compact.

Proof. Let $V \in \mathcal{P}_F(\mathcal{T})$. First, note that $K_t \subseteq \mathbb{R}^n$ is compact for all $t \in V$ and hence $\prod_{t \in V} K_t$ is compact as well. So compactness of $Sp(S_V(\mu))$ follows from (4.2) because it is a closed subset of a compact space.

Now we show (4.2) which is equivalent to

$$N := (\prod_{t \in V} K_t)^{\complement} \subseteq (Sp(S_V(\mu)))^{\complement}.$$

By definition of support (see Definition 3.4.), we know that this holds if and only if

$$N \text{ can be covered by open subsets } F \subseteq (\mathbb{R}^n)^V \text{ with } S_V(\mu)(F) = 0.$$

As the complement of a compact subset of a Hausdorff space, N is open. Next we show that N is a subset of a $S_V(\mu)$-nullset and by monotonicity a $S_V(\mu)$-nullset itself, i.e. $S_V(\mu)(N) = \mu(S_V^{-1}(N)) = 0$. By definition,

$$\prod_{t \in V} K_t = \{f : V \to \mathbb{R}^n \mid f(t) \in K_t, \forall t \in V\}$$

and

$$N = \{f : V \to \mathbb{R}^n \mid \exists t_0 \in V : f(t_0) \in (\mathbb{R}^n \setminus K_{t_0})\}$$

Now consider the pre-image of S_V

$$S_V^{-1} : 2^{(\mathbb{R}^n)^V} \to 2^{\Omega},\ B \mapsto S_V^{-1}(B) := \{\omega \in \Omega \mid S_V(\omega) \in B\}.$$

We know that

$$S_V(\omega) = S_.(\omega)|_V \in N \iff \exists f \in N : S_V(\omega)(t) = S_t(\omega) = f(t), \forall t \in V$$

but if $f \in N$, then

$$\exists t_0 \in V : ||S_{t_0}(\omega)|| = ||f(t_0)|| > r_{t_0},$$

which implies that $\omega \in Z_{t_0}$ by (2.4). Hence $S_V^{-1}(N) \subseteq \bigcup_{t_0 \in V} Z_{t_0}$ and by Lemma 2.6. $\mu(\bigcup_{t_0 \in V} Z_{t_0}) = 0$ because V is finite. □

Definition 4.7. We define $S_{\mathcal{T}}$ to be the function which maps $\omega \in \Omega$ to the corresponding path of S, i.e.

$$S_{\mathcal{T}} : \Omega \to (\mathbb{R}^n)^{\mathcal{T}},\ \omega \mapsto S_{\mathcal{T}}(\omega) := S_.(\omega).$$

Using the same argument as in the proof of Lemma 4.5., we get that $S_{\mathcal{T}}$ is $\mathcal{F}$-$\mathcal{B}_0^{\mathcal{T}}$-measurable, where $\mathcal{B}_0^{\mathcal{T}}$ denotes the cylindrical σ-field on $(\mathbb{R}^n)^{\mathcal{T}}$. Then we can consider its image measure $S_{\mathcal{T}}(\mu) : \mathcal{B}_0^{\mathcal{T}} \to [0,1]$ and get that the probability measure $S_{\mathcal{T}}(\mu)$ is the projective limit of (4.1), i.e. $\pi_V(S_{\mathcal{T}}(\mu)) = S_V(\mu)$ for all $V \in \mathcal{P}_F(\mathcal{T})$ because $\pi_V \circ S_{\mathcal{T}} = S_V$. Moreover, from Corollary 3.20. and Lemma 4.6. it follows that $S_{\mathcal{T}}(\mu)$ can be extended to a unique Radon measure (which we still denote by $S_{\mathcal{T}}(\mu)$) defined on the Borel σ-field $\mathcal{B}^{\mathcal{T}}$ on $(\mathbb{R}^n)^{\mathcal{T}}$ such that the support of $S_{\mathcal{T}}(\mu)$ satisfies

$$Sp(S_{\mathcal{T}}(\mu)) \subseteq \prod_{t \in \mathcal{T}} K_t. \tag{4.3}$$

4.2 The projective price process and projective equivalence

Definition 4.8. 1) The *projective price process* is defined as

$$S^* : (\mathbb{R}^n)^{\mathcal{T}} \times \mathcal{T} \to \mathbb{R}^n, \ (\overline{\omega}, t) \mapsto S^*(\overline{\omega}, t) := \overline{\omega}(t).$$

We use the notation $S_t^* := S^*(\,\cdot\,, t) : (\mathbb{R}^n)^{\mathcal{T}} \to \mathbb{R}^n$ for all $t \in \mathcal{T}$.

2) We consider the filtration $\mathbb{F}^* = (\mathcal{F}_t^*)_{t \in \mathcal{T}}$ on $(\mathbb{R}^n)^{\mathcal{T}}$ where $\mathcal{F}_t^*$ is defined as the smallest σ-field which makes $S_{t'}^*$ measurable for $t' \in \mathcal{T}$ with $t' \leq t$, i.e.

$$\mathcal{F}_t^* := \sigma(S_{t'}^*;\ t' \in \mathcal{T},\ t' \leq t) := \sigma\Big(\bigcup_{t' \in \mathcal{T},\, t' \leq t} (S_{t'}^*)^{-1}\big(\mathcal{B}(\mathbb{R}^n)\big)\Big).$$

Lemma 4.9. *1) S_t^* is $\mathcal{B}_0^{\mathcal{T}}$-$\mathcal{B}(\mathbb{R}^n)$-measurable for all $t \in \mathcal{T}$.*

2) $\mathcal{F}_t^* \subseteq \mathcal{B}_0^{\mathcal{T}}$ for all $t \in \mathcal{T}$.

Proof. 1) Define for all $t \in \mathcal{T}$ the projections

$$\pi_t : (\mathbb{R}^n)^{\mathcal{T}} \to \mathbb{R}^n, \ f \mapsto \pi_t(f) := f|_{\{t\}} = f(t).$$

By definition of the cylindrical σ-field, we have

$$\mathcal{B}_0^{\mathcal{T}} = \sigma(\bigcup_{t \in \mathcal{T}} \pi_t^{-1}(\mathcal{B}(\mathbb{R}^n))).$$

We note that $S_t^* = \pi_t$ and hence the assertion follows.

2) follows from 1).

□

Our major objective is to establish the existence of a probability measure $\nu : \mathcal{B}_0^{\mathcal{T}} \to [0,1]$ such that the projective price process $S^* = (S_t^*)_{t \in \mathcal{T}}$ is a martingale on the filtered probability space $((\mathbb{R}^n)^{\mathcal{T}}, \mathcal{B}_0^{\mathcal{T}}, \mathbb{F}^*, \nu)$, solving in this way the drawback pointed out after Theorem 2.11. In addition, ν and $S_{\mathcal{T}}(\mu)$ should satisfy some kind of equivalence. Balbás et al. [3] have shown that the complete equivalence between both probability measures does not necessarily hold in general, but we can introduce the weaker concept of *projective equivalence*.

Definition 4.10. Two probability measures $\nu : \mathcal{B}_0^{\mathcal{T}} \to [0,1]$ and $\xi : \mathcal{B}_0^{\mathcal{T}} \to [0,1]$ are said to be *projectively equivalent* if $\pi_V(\nu)$ and $\pi_V(\xi)$ are equivalent on $\mathcal{B}^V$ for all $V \in \mathcal{P}_F(\mathcal{T})$, where $\mathcal{B}^V$ denotes the Borel σ-field on $(\mathbb{R}^n)^V$.

Remark 4.11. Clearly, if ν and ξ are equivalent on $\mathcal{B}_0^{\mathcal{T}}$, then they are also projectively equivalent.

Definition 4.12. Let $V \in \mathcal{P}_F(\mathcal{T})$.

1) We define a *natural section* of the projective price process as a mapping

$$S^{V^*} : (\mathbb{R}^n)^V \times V \to \mathbb{R}^n, \ (\overline{\omega}, t) \mapsto S^{V^*}(\overline{\omega}, t) := \overline{\omega}(t).$$

We use the notation $S_t^{V^*} := S^{V^*}(\,\cdot\,, t) : (\mathbb{R}^n)^V \to \mathbb{R}^n$ for $t \in V$.

2) We define the filtration $\mathbb{F}_V^* = (\mathcal{F}_t^{V^*})_{t \in V}$ on the measurable space $((\mathbb{R}^n)^V, \mathcal{B}^V)$ where $\mathcal{F}_t^{V^*}$ is given by

$$\mathcal{F}_t^{V^*} := \sigma(S_{t'}^{V^*};\ t' \in V,\ t' \le t) \subseteq \mathcal{B}^V.$$

Note that the projective price process S^* is an $\mathbb{R}^n$-valued, $\mathbb{F}^*$-adapted stochastic process on the probability space $((\mathbb{R}^n)^\mathcal{T}, \mathcal{B}^\mathcal{T}, S_\mathcal{T}(\mu))$. Therefore the terms for S^V in Chapter 2 can be analogously defined for the natural sections S^{V^*} by replacing the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_V, \mu)$ with $((\mathbb{R}^n)^V, \mathcal{B}^V, \mathbb{F}^*_V, S_V(\mu))$.

The following standard result from Probability Theory is needed to prove further results.

Proposition 4.13. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(U,\mathcal{U})$ a measurable space, $h : (\Omega, \mathcal{A}) \to (U,\mathcal{U})$ a measurable map and Y a random variable on $(U,\mathcal{U})$. Then*

$$\int_U |Y| \, \mathrm{d}(h(P)) < \infty \iff \int_\Omega |Y \circ h| \, \mathrm{d}P < \infty, \tag{4.4}$$

and if (4.4) is satisfied, then

$$E_{h(P)}[Y] = E_P[Y \circ h].$$

In order to establish the existence of projectively equivalent martingale measures it is convenient to translate the absence of arbitrage in terms of the projective price process S^*.

Proposition 4.14. *Let $V = \{0 = t_0 < t_1 < \ldots < t_T\} \in \mathcal{P}_F(\mathcal{T})$ and $\nu : \mathcal{F}_{t_T} \to [0,1]$ be an equivalent martingale measure for S^V and filtration $\mathbb{F}_V$. Then $\nu^*_V := S_V(\nu)$ and $S_V(\mu)$ are equivalent on $\mathcal{B}^V$ and ν^*_V is an equivalent martingale measure for S^{V^*} and $\mathbb{F}^*_V$, in particular*

$$E_{\nu^*_V}[S^{V^*}_{t_T} | \mathcal{F}^{V^*}_{t_k}] = S^{V*}_{t_k} \quad \nu^*_V\text{-a.s. } \forall k = 0, 1, \ldots, T. \tag{4.5}$$

Moreover, the support of ν^*_V is compact and a subset of $\prod_{t \in V} K_t$.

Proof. $\nu^*_V = S_V(\nu)$ and $S_V(\mu)$ are equivalent on $\mathcal{B}^V$ because S_V is $\mathcal{F}_{t_T}$-$\mathcal{B}^V$-measurable by Lemma 4.5. and ν and μ are equivalent on $\mathcal{F}_{t_T}$.

By definition of the filtration $\mathbb{F}^*_V$, S^{V^*} is $\mathbb{F}^*_V$-adapted. Integrability follows from Proposition 4.13. with $\mathcal{A} := \mathcal{F}_{t_T}$, $P := \nu$, $U := (\mathbb{R}^n)^V, \mathcal{U} := \mathcal{B}^V, h := S_V$

and $Y := S_t^{V^*}$ for $t \in V$, because we have

$$S_t^{V^*} \circ S_V : \Omega \to (\mathbb{R}^n)^V \to \mathbb{R}^n,\ \omega \mapsto S_.(\omega)|_V \mapsto S_t(\omega)$$

and ν is an EMM for S^V and $\mathbb{F}_V$. It remains to show the martingale property for S^{V^*}. Let $k \in \{0, \ldots, T\}$ and $A \in \mathcal{F}_{t_k}^{V^*}$. By using Proposition 4.13 again, we get

$$E_{\nu_{V^*}}[S_{t_T}^{V^*} I_A] = E_\nu[S_{t_T} I_{S_V^{-1}(A)}] \quad \text{and} \quad E_\nu[S_{t_k} I_{S_V^{-1}(A)}] = E_{\nu_{V^*}}[S_{t_k}^{V^*} I_A].$$

If we can show that $S_V^{-1}(A) \in \mathcal{F}_{t_k}$, then the martingale property follows. Consider $\widetilde{V} := \{0 = t_0 < t_1 < \ldots < t_k\} \subseteq V$. The set $A \in \mathcal{F}_{t_k}^{V^*}$ has the form $A = A_1 \times A_2$, where $A_1 \subseteq (\mathbb{R}^n)^{\widetilde{V}}$, $A_2 = (\mathbb{R}^n)^{V \setminus \widetilde{V}}$ and $A_1 \in \mathcal{F}_{t_k}^{\widetilde{V}^*} \subseteq \mathcal{B}^{\widetilde{V}}$. Hence $S_V^{-1}(A) = S_{\widetilde{V}}^{-1}(A_1) \in \mathcal{F}_{t_k}$ because $S_{\widetilde{V}}$ is $\mathcal{F}_{t_k}$-$\mathcal{B}^{\widetilde{V}}$-measurable by Lemma 4.5.

Because $\nu_V^* = S_V(\nu)$ and $S_V(\mu)$ are equivalent, they have the same support and the last assertion follows directly from Lemma 4.6. □

Proposition 4.15. *If the price process S is finitely arbitrage-free, then the projective price process S^* is finitely arbitrage-free as well.*

Proof. Follows from Proposition 4.14. □

Chapter 5

Countable sets of trading dates

As a consequence of Proposition 4.14., if we were able to build a suitable projective system of Radon probability measures, then the projective limit, whose existence can be guaranteed by Corollary 3.20., could be an adequate candidate to be a projectively equivalent martingale measure for the projective price process $S^* = (S^*_t)_{t\in\mathcal{T}}$. The construction of this projective system of Radon probability measures is the goal of this chapter. This involves some arguments regarding the weak* compactness of a certain set of Radon probability measures.

5.1 Weak* compactness

We proceed with the same set-up as in Chaper 4. Let $\mathcal{T}$ be countable and $S = (S_t)_{t\in\mathcal{T}}$ finitely arbitrage-free.

Then there exists a bijection $\Phi : \mathbb{N}_0 \to \mathcal{T}$ such that $\Phi(0) = 0$. We use the notation $t_m := \Phi(m)$ for all $m \in \mathbb{N}_0$. Then the equality

$$\mathbb{N}_0 = \bigcup_{m\in\mathbb{N}_0} \{0, 1, \ldots, m\}$$

leads to

$$\mathcal{T} = \Phi(\mathbb{N}_0) = \bigcup_{m\in\mathbb{N}_0} \Phi(\{0, 1, \ldots, m\}) = \bigcup_{m\in\mathbb{N}_0} \mathcal{T}_m, \tag{5.1}$$

where $\mathcal{T}_m$ denotes the finite set $\{t_0, t_1, \ldots, t_m\}$ for all $m \in \mathbb{N}_0$.

We get the following result.

Lemma 5.1. *There exists an increasing sequence of $(\mathcal{T}_m)_{m\in\mathbb{N}_0}$ of finite subsets of $\mathcal{T}$ such that*

$$\mathcal{T} = \bigcup_{m\in\mathbb{N}_0} \mathcal{T}_m$$

and $0 \in \mathcal{T}_m$ for all $m \in \mathbb{N}_0$. Furthermore, $(\mathbb{R}^n)^{\mathcal{T}}$ can be identified with the projective limit of the projective system of topological spaces

$$\left(\left((\mathbb{R}^n)^{\mathcal{T}_r}, \pi_{rs}\right)\right)_{r,s\in\mathbb{N}_0,\, r\leq s}$$

where π_{rs} denotes the standard projection from $(\mathbb{R}^n)^{\mathcal{T}_s}$ to $(\mathbb{R}^n)^{\mathcal{T}_r}$.

Proof. Follows from (5.1) and Example 3.21. □

In the sequel, we will use the following notation for all $r \in \mathbb{N}_0$:

$$(\mathbb{R}^n)^r := (\mathbb{R}^n)^{\mathcal{T}_r},\ \mathcal{B}^r := \mathcal{B}^{\mathcal{T}_r},\ \mathcal{F}_t^{r^*} := \mathcal{F}_t^{\mathcal{T}_r^*},\ \mathbb{F}_r^* := \mathbb{F}_{\mathcal{T}_r}^*,$$
$$S^{r^*} := S^{\mathcal{T}_r^*},\ \nu_r^* := \nu_{\mathcal{T}_r}^*,\ \pi_r := \pi_{\mathcal{T}_r}.$$

Corollary 5.2. *Let $r \in \mathbb{N}_0$. Then there exists a Radon probability measure ν_r^* on $((\mathbb{R}^n)^r, \mathcal{B}^r)$ equivalent to $\pi_r(S_{\mathcal{T}}(\mu))$ and with support included in $\prod_{t\in\mathcal{T}_r} K_t$ such that the natural section $S^{r^*} = (S_t^{r^*})_{t\in\mathcal{T}_r}$ of the projective price process is a martingale with respect to ν_r^* and $\mathbb{F}_r^* = (\mathcal{F}_t^{r^*})_{t\in\mathcal{T}_r}$.*

Proof. Follows from Proposition 4.14. □

Definition 5.3. 1) For $r \in \mathbb{N}_0$, we define $\mathcal{M}_{rr}$ to be the set of Radon probability measures ν_r^* on $((\mathbb{R}^n)^r, \mathcal{B}^r)$ satisfying the properties mentioned in Corollary 5.2.

2) For $r, s \in \mathbb{N}_0$ with $r \leq s$, we define

$$\mathcal{M}_{rs} := \pi_{rs}(\mathcal{M}_{ss}) = \{\pi_{rs}(\nu_s^*) \mid \nu_s^* \in \mathcal{M}_{ss}\}.$$

3) For $r \in \mathbb{N}_0$, we define

$$\mathcal{M}_r := \bigcap_{s \geq r,\, s \in \mathbb{N}_0} \mathcal{M}_{rs}.$$

Lemma 5.4. *For $r, s \in \mathbb{N}_0$ with $r \leq s$, we have $\mathcal{M}_{rs} \subseteq \mathcal{M}_{rr}$ and in particular $\mathcal{M}_r \subseteq \mathcal{M}_{rr}$.*

Proof. Let $\nu_s^* \in \mathcal{M}_{ss}$. First, we check that $\pi_{rs}(\nu_s^*) \approx \pi_r(S_{\mathcal{T}}(\mu))$ on $((\mathbb{R}^n)^r, \mathcal{B}^r)$. By assumption, we know that $\nu_s^* \approx \pi_s(S_{\mathcal{T}}(\mu))$ on $((\mathbb{R}^n)^s, \mathcal{B}^s)$, i.e.

$$\forall B \in \mathcal{B}^s : \nu_s^*(B) = 0 \iff \pi_s(S_{\mathcal{T}}(\mu))(B) = 0.$$

Now let $A \in \mathcal{B}^r$ with $\pi_{rs}(\nu_s^*)(A) = 0$, i.e. $\nu_s^*(\pi_{rs}^{-1}(A)) = 0$. But $\pi_{rs} : (\mathbb{R}^n)^s \to (\mathbb{R}^n)^r$ is $\mathcal{B}^s$-$\mathcal{B}^r$-measurable and so $\pi_{rs}^{-1}(A) \in \mathcal{B}^s$ and hence

$$\begin{aligned}
&\pi_{rs}(\nu_s^*)(A) = \nu_s^*(\pi_{rs}^{-1}(A)) = 0 \iff \\
&0 = \pi_s(S_{\mathcal{T}}(\mu))(\pi_{rs}^{-1}(A)) = \pi_{rs}(\pi_s(S_{\mathcal{T}}(\mu)))(A) = \pi_r(S_{\mathcal{T}}(\mu))(A)
\end{aligned}$$

where we have used $\pi_{rs} \circ \pi_s = \pi_r$.

Equivalent measures have the same support and hence $Sp(\pi_{rs}(\nu_s^*)) \subseteq \prod_{t \in \mathcal{T}_r} K_t$ follows from Lemma 4.6.

It remains to show that the natural section $S^{r^*} = (S_t^{r^*})_{t \in \mathcal{T}_r}$ of the projective price process is a martingale with respect to ν_r^* and $\mathbb{F}_r^* = (\mathcal{F}_t^{r^*})_{t \in \mathcal{T}_r}$. Using the same argument as in the proof of Proposition 4.14., this follows from Proposition 4.13. with $(\Omega, \mathcal{A}, P) := ((\mathbb{R}^n)^s, \mathcal{B}^s, \nu_s^*)$, $(U, \mathcal{U}) := ((\mathbb{R}^n)^r, \mathcal{B}^r)$, $h := \pi_{rs}$ and $Y := S_t^{r^*}$ because for all $t \in \mathcal{T}_r \subseteq \mathcal{T}_s$

$$S_t^{s^*} = S_t^{r^*} \circ \pi_{rs} : (\mathbb{R}^n)^s \to (\mathbb{R}^n)^r \to \mathbb{R}^n, \overline{\omega} \mapsto \overline{\omega}|_{\mathcal{T}_r} \mapsto \overline{\omega}(t). \qquad \square$$

Corollary 5.5. *Let $(X, \mathcal{O})$ be a compact Polish space and R a Radon measure on $(X, \mathcal{B}(X))$. Then for all $p \in [1, \infty)$, the set of continuous functions $C(X)$ is dense in $L^p(X, \mathcal{B}(X), R)$.*

Proof. Follows from Makarov and Podkorytov [11], Theorem 13.3.3. $\square$

Theorem 5.6. *Let Y be a separable Banach space and $K \subseteq Y^*$. Then the following are equivalent.*

i) K is weak compact.*

ii) K is bounded and weak closed.*

iii) K is sequentially weak compact, i.e. every sequence in K has a weak* convergent subsequence with limit in K.*

iv) K is bounded and sequentially weak closed, i.e. if $y^* \in Y^*$ is the weak* limit of a sequence in K, then $y^* \in K$.*

Proof. See Bühler and Salamon [6], Theorem 3.2.5. □

Lemma 5.7. *The sets $\mathcal{M}_{rr}$ and $\mathcal{M}_r$ are non-empty and weak* compact for all $r \in \mathbb{N}_0$.*

Proof. By Corollary 5.2., the set $\mathcal{M}_{rr}$ is non-empty for all $r \in \mathbb{N}_0$.

For all $s \in \mathbb{N}_0$ and $Q \in \mathcal{M}_{ss}$, we have that $Q : \mathcal{B}^s \to [0, 1]$ is a Radon probability measure on the space $(\mathbb{R}^n)^s$ and $Sp(Q) \subseteq \prod_{t\in\mathcal{T}_s} K_t$. Hence $Q : \mathcal{B}(X^s) \to [0, 1]$ can be considered a Radon measure on the compact Polish space $(X^s, \mathcal{O}^{X^s})$ where $X^s := \prod_{t\in\mathcal{T}_s} K_t$ and $\mathcal{O}^{X^s}$ denotes the subspace topology. Let $\mathcal{R}^s$ be the set of Radon measures on $\mathcal{B}(X^s)$. Then we get that $\mathcal{M}_{ss} \subseteq \mathcal{R}^s$ and for $Q \in \mathcal{M}_{ss}$ that $Q(X^s) = 1$. Hence $\mathcal{M}_{ss}$ is a subset of $\mathcal{R}^s_1 := \{\nu \in \mathcal{R}^s \mid \nu(X^s) \leq 1\}$ and thus bounded. By Theorem 5.6. with $Y := C(X^s)$, $\mathcal{M}_{ss}$ is weak* compact if we can show that $\mathcal{M}_{ss}$ is sequentially weak* closed. Let $(Q_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{M}_{ss}$ that weak* converges to some $Q \in \mathcal{R}^s$, i.e. for all $f \in C(X^s)$

$$\int_{X^s} f \,\mathrm{d}Q_k \to \int_{X^s} f \,\mathrm{d}Q \qquad (k \to \infty). \tag{5.2}$$

Because $\mathcal{R}^s_1$ is weak* compact, we get $Q \in \mathcal{R}^s_1$. We clearly have $Q(X^s) = 1$ and hence Q is a Radon probability measure on $(X^s, \mathcal{B}(X^s))$. By Theorem 3.12., Q is also a Radon probability measure on $((\mathbb{R}^n)^s, \mathcal{B}^s)$.

Next, we show that Q is equivalent to $\pi_r(S_{\mathcal{T}}(\mu))$ on $((\mathbb{R}^n)^s, \mathcal{B}^s)$. It suffices to show that $Q \approx Q_k$ on $(X^s, \mathcal{B}(X^s))$ for some $k \in \mathbb{N}$, because for all $A \in \mathcal{B}^s$, we have $Q[A \cap ((\mathbb{R}^n)^s \setminus X^s)] = 0 = Q_k[A \cap ((\mathbb{R}^n)^s \setminus X^s)]$ and for all $k \in \mathbb{N}$, $Q_k \approx \pi_r(S_{\mathcal{T}}(\mu))$ on $((\mathbb{R}^n)^s, \mathcal{B}^s)$. Let $A \in \mathcal{B}(X^s)$. Then $I_A \in L^1(X^s, \mathcal{B}(X^s), Q)$ and $I_A \in L^1(X^s, \mathcal{B}(X^s), Q_k)$ for all $k \in \mathbb{N}$. By Corollary 5.5., there exist a sequence $(f_\ell)_{\ell \in \mathbb{N}} \subseteq C(X^s)$ such that

$$\int_{X^s} f_\ell \, \mathrm{d}Q \to \int_{X^s} I_A \, \mathrm{d}Q \quad \text{and} \quad \int_{X^s} f_\ell \, \mathrm{d}Q_k \to \int_{X^s} I_A \, \mathrm{d}Q_k \qquad (\ell \to \infty)$$

uniformly in $k \in \mathbb{N}$ and in Q. Let $\epsilon > 0$. Then there exists $L \in \mathbb{N}$ such that

$$\Big|\int_{X^s} f_L \, \mathrm{d}Q - \int_{X^s} I_A \, \mathrm{d}Q\Big| \leq \epsilon \quad \text{and} \quad \Big|\int_{X^s} f_L \, \mathrm{d}Q_k - \int_{X^s} I_A \, \mathrm{d}Q_k\Big| \leq \epsilon \tag{5.3}$$

for all $k \in \mathbb{N}$. By (5.2), there exists $K \in \mathbb{N}$ such that

$$\Big|\int_{X^s} f_L \, \mathrm{d}Q_K - \int_{X^s} f_L \, \mathrm{d}Q\Big| \leq \epsilon. \tag{5.4}$$

By combining (5.3) and (5.4) and using the triangle inequality, we get

$$\Big|\int_{X^s} I_A \, \mathrm{d}Q_K - \int_{X^s} I_A \, \mathrm{d}Q\Big| \leq 3\epsilon$$

which implies that $Q \approx Q_k$ on $(X^s, \mathcal{B}(X^s))$ for some $k \in \mathbb{N}$.

Next, we show that Q is an EMM for S^{s^*} and $\mathbb{F}^*_s$ on $((\mathbb{R}^n)^s, \mathcal{B}^s)$. Let $B' \in \mathcal{F}^{s^*}_{t_r}$ for $r \in \{0, 1, \ldots, s\}$. Define $B := B' \cap X^s \in \mathcal{B}(X^s)$ and note that for all $k \in \mathbb{N}$

$$\int_{X^s} S^{s^*}_{t_s} I_B \, \mathrm{d}Q_k = \int_{(\mathbb{R}^n)^s} S^{s^*}_{t_s} I_{B'} \, \mathrm{d}Q_k = \int_{(\mathbb{R}^n)^s} S^{s^*}_{t_r} I_{B'} \, \mathrm{d}Q_k = \int_{X^s} S^{s^*}_{t_r} I_B \, \mathrm{d}Q_k.$$

Using the same argument as above, we can approximate I_B by continuous functions in $C(X^s)$ and get

$$\int_{X^s} S_{t_r}^{s^*} I_B \,\mathrm{d}Q = \int_{X^s} S_{t_r}^{s^*} I_B \,\mathrm{d}Q.$$

Hence $Q \in \mathcal{M}_{ss}$ and $\mathcal{M}_{ss}$ is weak* compact.

By Lemma 3.14., the transformation $\mathcal{M}_{ss} \to \mathcal{M}_{rr},\ Q \mapsto \pi_{rs}(Q)$ is weak* continuous for all $r, s \in \mathbb{N}_0$ with $s \geq r$. Hence the image $\pi_{rs}(\mathcal{M}_{ss}) = \mathcal{M}_{rs}$ is non-empty and weak* compact, in particular, it is weak* closed by Theorem 5.6. Now let $r \in \mathbb{N}_0$. $\mathcal{M}_r := \bigcap_{s \geq r,\, s \in \mathbb{N}_0} \mathcal{M}_{rs}$ is weak* closed and hence weak* compact because $\mathcal{M}_r$ is a subset of $\mathcal{R}_1^r$ and $\mathcal{R}_1^r$ is weak* compact. Let $s_0, s_1 \in \mathbb{N}_0$ with $s_1 \geq s_0 \geq r$. Then

$$\mathcal{M}_{rs_1} = \pi_{rs_1}(\mathcal{M}_{s_1 s_1}) = \underbrace{\pi_{rs_0} \circ \overbrace{\pi_{s_0 s_1}}_{=\pi_{rs_1}}(\mathcal{M}_{s_1 s_1})}^{=\mathcal{M}_{s_0 s_1} \subseteq \mathcal{M}_{s_0 s_0}} \subseteq \pi_{rs_0}(\mathcal{M}_{rs_0}) = \mathcal{M}_{rs_0}.$$

Hence $(\mathcal{M}_{rs})_{s \geq r,\, s \in \mathbb{N}_0}$ is a decreasing sequence of non-empty, closed subsets of a compact space and therefore it has a non-empty intersection $\mathcal{M}_r = \bigcap_{s \geq r,\, s \in \mathbb{N}_0} \mathcal{M}_{rs} \neq \emptyset$. □

5.2 Projectively equivalent martingale measures

Lemma 5.8. *Let $r \in \mathbb{N}_0$ and define $s := r + 1$. Then $\nu_s^* \in \mathcal{M}_s$ implies $\pi_{rs}(\nu_s^*) \in \mathcal{M}_r$.*

Proof. Let $\nu_s^* \in \mathcal{M}_s$. By definition,

$$\nu_s^* \in \mathcal{M}_s \iff \forall t \in \mathbb{N}_0 \text{ with } t \geq s\ \exists \nu_t^* \in \mathcal{M}_{tt} :\ \pi_{st}(\nu_t^*) = \nu_s^*$$

and analogously

$$\pi_{rs}(\nu_s^*) \in \mathcal{M}_r \iff \forall t \in \mathbb{N}_0 \text{ with } t \geq r\ \exists \nu_t^* \in \mathcal{M}_{tt} :\ \pi_{rt}(\nu_t^*) = \pi_{rs}(\nu_s^*).$$

So we get that for all $t \in \mathbb{N}_0$ with $t \geq s = r+1$ there exists $\nu_t^* \in \mathcal{M}_{tt}$ such that

$$\pi_{rs}(\nu_s^*) = \pi_{rs}(\pi_{st}(\nu_t^*)) = \pi_{rt}(\nu_t^*).$$

For $t = r$, we can simply take $\nu_r^* := \pi_{rs}(\nu_s^*) \in \mathcal{M}_{rs} \subseteq \mathcal{M}_{rr}$ to get

$$\pi_{rr}(\nu_r^*) = \nu_r^* = \pi_{rs}(\nu_s^*).$$

Hence $\pi_{rs}(\nu_s^*) \in \mathcal{M}_r$. □

Definition 5.9. 1) An *inductive set* is a partially ordered set X in which every totally ordered subset $L \subseteq X$ has an upper bound in X, i.e. it is a set fulfilling the assumptions of Zorn's lemma.

2) Let A be a set and let $\leq$ be a binary relation on A. Then a subset $B \subseteq A$ is said to be *cofinal* if it satisfies the following condition:

$$\forall a \in A\ \exists b \in B: \quad a \leq b.$$

Finally, we present the main result of this paper.

Theorem 5.10. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\mathcal{T}$ a countable subset of $[0,\infty)$ containing 0 and $S = (S_t)_{t\in\mathcal{T}}$ a price process. If S is finitely arbitrage-free, then there exists a Radon probability measure ν^* on $((\mathbb{R}^n)^{\mathcal{T}}, \mathcal{B}^{\mathcal{T}})$ such that:*

i) $Sp(\nu^*) \subseteq \prod_{t\in\mathcal{T}} K_t$, where K_t denotes the closed ball in $\mathbb{R}^n$ with centre 0 and radius r_t (see (2.3)) for all $t \in \mathcal{T}$.

ii) ν^* and $S_{\mathcal{T}}(\mu)$ are projectively equivalent.

iii) The projective price process S^* is a martingale with respect to ν^* and $\mathbb{F}^* = (\mathcal{F}_t^*)_{t\in\mathcal{T}}$.

Proof. First, we define $\mathcal{H}$ to be the set

$$\{(\nu_r^*)_{r\in N} \mid N \subseteq \mathbb{N}_0, \nu_r^* \in \mathcal{M}_r\ \forall r \in N, \pi_{rs}(\nu_s^*) = \nu_r^* \text{ if } r,s \in N \text{ and } r \leq s\}.$$

By Lemma 5.7., $\mathcal{H}$ is non-empty. Indeed, there exists $\nu_r^* \in \mathcal{M}_r$ for all $r \in \mathbb{N}_0$ and hence $(\nu_r^*)_{r\in N} \in \mathcal{H}$ for $N = \{r\}$. Furthermore, for all $s \in \mathbb{N}$ we can consider $N = \{r_s < r_{s-1} < \ldots < r_1 < r_0\}$ where $r_i := s - i$ for $i \in \{0, 1, \ldots, s\}$. There exists $\nu_s^* \in \mathcal{M}_s$ and by Lemma 5.8., we can iteratively define $\nu_{r_i}^* := \pi_{r_i r_{i-1}}(\nu_{r_{i-1}^*})$ for $i \in \{1, \ldots, s\}$ such that $(\nu_r^*)_{r\in N} \in \mathcal{H}$. Next, we consider the natural order $\preceq$ on $\mathcal{H}$, i.e. for $(\nu_r^*)_{r\in N}, (\nu_{r'}^*)_{r'\in N'} \in \mathcal{H}$, we have

$$(\nu_r^*)_{r\in N} \preceq (\nu_{r'}^*)_{r'\in N'} \;:\Longleftrightarrow\; N \subseteq N' \text{ and } \nu_r^* = \nu_{r'}^* \text{ for all } r \in N \text{ with } r = r'.$$

Endowed with this order, $\mathcal{H}$ is an inductive set and hence Zorn's lemma ensures the existence of a maximal element

$$(\nu_r^*)_{r\in P} \in \mathcal{H}. \tag{5.5}$$

If $P \subseteq \mathbb{N}_0$ is cofinal, then (5.5) is a projective system of Radon measures whose associated projective system of topological spaces has a projective limit that can be identified with $(\mathbb{R}^n)^{\mathcal{T}}$. Moreover, Theorem 3.18. ensures the existence of the projective limit ν^* of $(\nu_r^*)_{r\in P}$, i.e. a Radon probability measure $\nu^* : \mathcal{B}^{\mathcal{T}} \to [0, 1]$. By definition, we have that for all $r \in P$, ν_r^* is equivalent to $\pi_r(S_{\mathcal{T}}(\nu))$ and by definition of the projective limit, we have that $\pi_r(\nu^*) = \nu_r^*$. Hence ν^* and $S_{\mathcal{T}}(\nu)$ are projectively equivalent. By Corollary 5.2. and the definition of the set $\mathcal{M}_{rr}$, we have that $Sp(\nu_r^*) \subseteq \prod_{t\in\mathcal{T}_r} K_t$. Hence by Corollary 3.20., property ii) is satisfied. Using the same argument as in the proof of Proposition 4.14., property iii) follows from Proposition 4.13. with $(\Omega, \mathcal{A}, P) := ((\mathbb{R}^n)^{\mathcal{T}}, \mathcal{B}^{\mathcal{T}}, \nu^*)$, $(U, \mathcal{U}) := ((\mathbb{R}^n)^r, \mathcal{B}^r)$, $h := \pi_r$ and $Y := S_t^*$ because for all $t \in \mathcal{T}_r \subseteq \mathcal{T}$

$$S_t^{r^*} = S_t^* \circ \pi_r : (\mathbb{R}^n)^{\mathcal{T}} \to (\mathbb{R}^n)^r \to \mathbb{R}^n,\; \overline{\omega} \mapsto \overline{\omega}|_{\mathcal{T}_r} \mapsto \overline{\omega}(t)$$

and $\pi_r(\nu^*) = \nu_r^*$ is an EMM for S^{r^*}.

If $P \subseteq \mathbb{N}_0$ is not cofinal, then there exists a maximum value $\max(P) =: r < \infty$. Since $\nu_r^* \in \mathcal{M}_r = \bigcap_{s\geq r,\, s\in\mathbb{N}_0} \mathcal{M}_{rs}$, there exists a sequence $(\nu_s^*)_{s\geq r,\, s\in\mathbb{N}_0}$

such that for all $s \in \mathbb{N}_0$ with $s \geq r$, we have $\nu_s^* \in \mathcal{M}_{ss}$ and

$$\pi_{rs}(\nu_s^*) = \nu_r^*. \tag{5.6}$$

Define

$$\nu_{r+1,s}^* := \pi_{r+1,s}(\nu_s^*)$$

for all $s \in \mathbb{N}_0$ with $s \geq r+1$. It follows that

$$\pi_{r,r+1}(\nu_{r+1,s}^*) = \nu_r^* \tag{5.7}$$

for all $s \in \mathbb{N}_0$ with $s \geq r+1$. Now consider the sequence $(\nu_{r+1,s}^*)_{s\geq r+1,\, s\in\mathbb{N}_0} \subseteq \mathcal{M}_{r+1,r+1}$. By Lemma 5.7., $\mathcal{M}_{r+1,r+1}$ is weak* compact and by Theorem 5.6. with $Y := C(X)$ where $X := \prod_{t\in\mathcal{T}_{r+1}} K_t$, $\mathcal{M}_{r+1,r+1}$ is also sequentially weak* compact. Hence the sequence $(\nu_{r+1,s}^*)_{s\geq r+1,\, s\in\mathbb{N}_0}$ possesses an accumulation point $\nu_{r+1}^* \in \mathcal{M}_{r+1,r+1}$, i.e. there exists an unbounded subset $J \subseteq \{s \in \mathbb{N}_0 \mid s \geq r+1\}$ such that

$$\nu_{r+1}^* = \overset{w^*}{\lim_{J\ni s\to\infty}} \nu_{r+1,s}^*$$

where $\lim^{w^*}$ denotes the weak* limit. From (5.7) we get that

$$\pi_{r,r+1}(\nu_{r+1}^*) = \pi_{r,r+1}(\overset{w^*}{\lim_{J\ni s\to\infty}} \nu_{r+1,s}^*) = \overset{w^*}{\lim_{J\ni s\to\infty}} \pi_{r,r+1}(\nu_{r+1,s}^*) = \nu_r^*$$

and thus for all $s \in \mathbb{N}_0$ with $s \geq r+1$

$$\pi_{r,r+1}(\nu_{r+1}^*) = \nu_r^*. \tag{5.8}$$

Combining (5.6) and (5.8), we get that

$$\pi_{r,r+1}(\nu_{r+1}^*) = \nu_r^* = \pi_{rs}(\nu_s^*) = \pi_{r,r+1}(\pi_{r+1,s}(\nu_s^*))$$

and hence for all $s \in \mathbb{N}_0$ with $s \geq r+1$

$$\pi_{r+1,s}(\nu_s^*) = \nu_{r+1}^*$$

which implies that $\nu^*_{r+1} \in \mathcal{M}_{r+1}$. But this is a contradiction because (5.5) is a maximal element. Thus $P \subseteq \mathbb{N}_0$ must be cofinal. $\square$

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