# Operators

### Linear Operator

$$F(a\mathbf{\Phi}_1 + b\mathbf{\Phi}_2) = a \cdot F\mathbf{\Phi}_1 + b \cdot F\mathbf{\Phi}_2 \quad \forall \mathbf{\Phi}_1, \mathbf{\Phi}_2$$

### Eigenvalue, Eigenfunction

$$Fu_n = f_n u_n$$

$u_n$ is a eigen function to the operator $F$ with corresponding eigenvalue $f_n$.

### Hermitian Operator

$$\int (Hu) * v \, d^3r = \int u * Hv \, d^3r, \quad \forall u, v$$

A hermitian operator has real eigenvalues and corresponding eigenfunctions can be choosen to be orthonormal. Practically all operators in quantum mechanics are linear and hermitian.

### Eigenfunction Expansion

$$\boldsymbol{\psi}(\boldsymbol{r}) = \sum_n a_n m \cdot u_n(\boldsymbol{r}), \quad a_n = \int u_n * \cdot \boldsymbol{\psi} \cdot d^3r$$

### Expansion Postulate

At a meassurement of an observable $F$ on a system described by a wavefunction $\boldsymbol{\psi}$ only eigenvalues of the operator $F$ can be found. The probability of the result $F = f_n$ is given by

$$P(F = f_n) = \left| \int u_n * \boldsymbol{\psi} \, d^3r \right|^2, \quad Fu_n = f_n u_n$$

### Momentum Operators

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) \right]$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial\varphi}$$

$L^2$ and $L_z$ have normalized eigenfunctions $\Upsilon_l^m(\theta, \varphi)$ for which it holds that:

$$L^2 \Upsilon_l^m = \hbar^2 l(l+1) \Upsilon_l^m$$

$$L_z \Upsilon_l^m = m\hbar \Upsilon_l^m$$

| $l$ | $m$ | $\Upsilon_l^m(\theta, \varphi)$ |
|---|---|---|
| 0 | 0 | $\Upsilon_0^0 = \frac{1}{\sqrt{4\pi}}$ |
| 1 | 0 | $\Upsilon_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$ |
| 1 | $\pm 1$ | $\Upsilon_1^{\pm 1} = \pm\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi}$ |
| 2 | 0 | $\Upsilon_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$ |
| 2 | $\pm 1$ | $\Upsilon_2^{\pm 1} = \pm\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\varphi}$ |
| 2 | $\pm 2$ | $\Upsilon_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\varphi}$ |

### Commutators and Momentum Operators

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ even} \\ -1 & ijk \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$[x_i, p_j] = i\hbar \cdot \delta_{ij}$$

$$[x_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot x_k$$

$$[L_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot L_k$$

$$[x_i, x_j] = [p_i, p_j] = 0$$

$$[p_i, L_j] = i\hbar \cdot \epsilon_{ijk} \cdot p_k$$

$$J_+ = J_x + iJ_y$$
$$J_- = J_x - iJ_y$$

$$J_\pm J_\mp = J^2 - J_z^2 \pm \hbar \cdot J_z$$

$$[J_+, J_-] = 2\hbar \cdot J_z$$

$$[J_z, J_\pm] = \pm\hbar \cdot J_\pm$$

$$J_+ \phi_{j,m} = \sqrt{(j-m)(j+m+1} \cdot \hbar \cdot \phi_{j,m+1}$$

$$J_- \phi_{j,m} = \sqrt{(j+m)(j-m+1} \cdot \hbar \cdot \phi_{j,m-1}$$

$$\Upsilon_l^l(\theta, \varphi) = (-1)^l \sqrt{\frac{2l+1)}{4\pi} \frac{(2l)!}{2^{2l}(l!)^2}} \cdot \sin^l \theta \cdot e^{il\varphi}$$