Electromagnetic field theory

Statics

Coulombs law

The force F on a point charge q_1 at the point $\mathbf{r_1}$ caused by a point charge q_2 at the point $\mathbf{r_2}$

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 |r_1 - r_2|^2} \frac{(r_1 - r_2)}{|r_1 - r_2|}$$

Electric Field Strength

From a point charge q in $\boldsymbol{r'}$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 R^2}\boldsymbol{e}_R$$

From charge distribution

$$\boldsymbol{E}(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0 R^2}\boldsymbol{e}_R dq(\boldsymbol{r'}),$$

$$dq(\boldsymbol{r'}) = \begin{cases} \rho_{tot}(\boldsymbol{r'})dv' = \rho(\boldsymbol{r'}) + \rho_p(\boldsymbol{r'}))dv' \\ \rho_{tot,s}(\boldsymbol{r'})dS' = \rho_s(\boldsymbol{r'}) + \rho_{p,s}(\boldsymbol{r'}))dS' \\ \rho_l(\boldsymbol{r'})dl' \end{cases}$$

From point dipole $\boldsymbol{p} = p\boldsymbol{e}_z$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos(\theta)\boldsymbol{e}_r + \sin(\theta)\boldsymbol{e}_\theta)$$

From line charge ρ_l

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_l}{2\pi\epsilon_0 r_c}\boldsymbol{e}_{r_c}$$

From dipole line $\boldsymbol{p}_l = p_l\boldsymbol{e}_x$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{p_l}{2\pi\epsilon_0 r_c^2}(\cos(\varphi)\boldsymbol{e}_{r_c} + \sin(\varphi)\boldsymbol{e}_\varphi)$$

Electrical Potential

$$\boldsymbol{E} = -\boldsymbol{\nabla} V$$

From pointsource q in $\boldsymbol{r'}$

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 R}$$

From charge distribution

$$V(\boldsymbol{r}) = \int \frac{1}{4\pi\epsilon_0 R} dq(\boldsymbol{r'})$$

From point dipole $\boldsymbol{p} = p\boldsymbol{e}_z$

$$V(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} = \frac{p\cos(\theta)}{4\pi\epsilon_0 r^2}$$

From line charge ρ_l

$$V(\boldsymbol{r}) = \frac{\rho_l}{2\pi\epsilon_0}\ln\left(\frac{1}{r_c}\right)$$

From dipole line $\boldsymbol{p}_l = p_l\boldsymbol{e}_x$

$$V(\boldsymbol{r}) = \frac{p_l}{2\pi\epsilon_0}\frac{\cos(\varphi)}{r_c}$$

Electrical flow density

Where $\boldsymbol{D}$ is defined by $\boldsymbol{\nabla D} = \rho$

Gauss law, where $\boldsymbol{e}_n$ is the unit normal to the volume surface pointing outwards $\boldsymbol{P}, \boldsymbol{E}$ and $\boldsymbol{D}$:

$$\begin{cases} \boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} & \text{(valid generally)} \\ \boldsymbol{D} = \epsilon_r\epsilon_0\boldsymbol{E} \end{cases}$$

Polarization Charge

$$\rho_p = -\boldsymbol{\nabla}\cdot\boldsymbol{P} \qquad \text{space charge density}$$

$$\rho_{p,s} = \boldsymbol{e}_{n1}\cdot(\boldsymbol{P}_1 - \boldsymbol{P}_2) \qquad \text{surface charge density}$$

where the unit normal $\boldsymbol{e}_{n1}$ is directed from 1 to 2.

Boundary Conditions

$$\begin{cases} E_t \text{ continuous} \\ \rho_s = \boldsymbol{e}_{n2}\cdot(\boldsymbol{D}_1 - \boldsymbol{D}_2) \end{cases}$$

where ρ_s is free surface charge density and $\boldsymbol{e}_{n2}$ is directed from volume 2 to volume 1.

Electrostatic Energy

$$W_e = \frac{1}{2}\sum_i Q_i V_i$$

$$W_e = \frac{1}{2}\int \rho V \, dv$$

$$W_e = \frac{1}{2}\int \boldsymbol{E}\cdot\boldsymbol{D}\, dv$$

Maxwell's voltage

$$|\boldsymbol{T}| = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D} \qquad \boldsymbol{E} \text{ is a bisector to } \boldsymbol{e}_n \text{ and } \boldsymbol{T}$$

Torque on Electrical Dipole

$$\boldsymbol{T}_e = \boldsymbol{p}\times\boldsymbol{E}$$

DC Current

Current Density

$$I = \int \boldsymbol{J}\cdot e_n \, dS$$

Conservation Equation

$$\boldsymbol{\Delta}\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0$$

$$\oint \boldsymbol{J}\cdot\boldsymbol{e}_n \, dS = -\frac{dQ}{dt}$$

Conductivity

$$\boldsymbol{J} = \sigma\boldsymbol{E}$$

Effect

$$P = \int \boldsymbol{J}\cdot\boldsymbol{E}\, dv$$

Boundary Conditions

$$\begin{cases} \boldsymbol{e}_{n2}\cdot(\boldsymbol{J}_1 - \boldsymbol{J}_2) = 0 & \text{(no surface current)} \\ \boldsymbol{E}_{t1} = \boldsymbol{E}_{t2} \end{cases}$$

Time Constant

$$RC = \frac{\epsilon_r\epsilon_0}{\sigma}$$

Analogy Elektrostatics - DC Current

$\boldsymbol{E}, V$	$\boldsymbol{E}, V$
$\boldsymbol{D}$	$\boldsymbol{J}$
$\epsilon_r\epsilon_0$	σ
Q	I
C	G

Magnetostatics

Magnetic Flow Density

From point dipole $\boldsymbol{m} = m\boldsymbol{e}_z$:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\boldsymbol{e}_r + \sin\theta\boldsymbol{e}_\theta)$$

From current density $\boldsymbol{J}_{tot}(\boldsymbol{r'})$:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}_{tot}(\boldsymbol{r'})\times\boldsymbol{e}_R}{R^2}\, dv'$$

where $\boldsymbol{J}_{tot} = \boldsymbol{J} + \boldsymbol{J}_m$. From current line:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{I\, d\boldsymbol{l'}\times\boldsymbol{e}_R}{R^2}$$

From circular thread loop:

$$\boldsymbol{B}(x=0, y=0, z) = \frac{\mu_0 I}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\boldsymbol{e}_z$$

From coil:

$$\boldsymbol{B} = \frac{\mu_0 N I}{\ell}\frac{\cos(\alpha_2) - \cos(\alpha_1)}{2}\boldsymbol{e}_z$$

From long straight current path:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r_c}\boldsymbol{e}_\varphi$$

Vector Potential

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

From current density $\boldsymbol{J}_{tot}(\boldsymbol{r}')$:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{tot}(\boldsymbol{r}')}{R}\, dv'$$

From current line:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\boldsymbol{l}'}{R}$$

From long straight current path:

$$\boldsymbol{A} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{r}\right) \boldsymbol{e}_z$$

From point dipole :

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}$$

Magnetic Flow

$$\Phi = \int \boldsymbol{B} \cdot \boldsymbol{e}_n\, dS = \oint \boldsymbol{A} \cdot dl$$

Linked Flow

$$\Lambda = N\Phi$$

Self-Inductance and Mutual Inductance

$$\Lambda_1 = L_1 I_1 + M I_2$$
$$\Lambda_2 = L_2 I_2 + M I_1$$

Magnetic Field Strength

Amperes Law:

$$\oint \boldsymbol{H} \cdot d\ell = \int \boldsymbol{J} \cdot \boldsymbol{e}_n\, dS = I_{\text{inside}}$$

Connection between magnetization $\boldsymbol{M}, \boldsymbol{B}$ and $\boldsymbol{H}$:

$$\begin{cases} \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) & \text{(holds generally)} \\ \boldsymbol{B} = \mu_r \mu_0 \boldsymbol{H} \end{cases}$$

Equivalent Current Density

$$\boldsymbol{J}_m = \boldsymbol{\nabla} \times \boldsymbol{M} \qquad \text{volume current density}$$

$$\boldsymbol{J}_m = \boldsymbol{\nabla} \times \boldsymbol{M} \qquad \text{surface current density}$$

Boundary Conditions

$$\begin{cases} \boldsymbol{e}_{n2} \times (\boldsymbol{H}_1 - H_2) = \boldsymbol{J}_s \\ \boldsymbol{B}_n \text{ Continuous} \end{cases}$$

Scalar Potential

From magnetic dipole $\boldsymbol{m}$:

$$V_m = \frac{1}{4\pi} \frac{\boldsymbol{m} \cdot \boldsymbol{e}_R}{R^2}$$

Magnetic Pole Density

$$\begin{cases} \rho_m = -\boldsymbol{\nabla} \cdot \boldsymbol{M} & \text{volume pole density} \\ \rho_{m,s} = e_{n1} \cdot (\boldsymbol{M}_1 - \boldsymbol{M}_2) & \text{surface pole density} \end{cases}$$

Magnetic Force Law

$$d\boldsymbol{F}_m = I d\boldsymbol{l} \times \boldsymbol{B}$$

Magetic moment for current loop

$$\boldsymbol{m} = \int I \boldsymbol{e}_n\, dS$$

Torque on Magnetic Moment

$$\boldsymbol{T}_m = \boldsymbol{m} \times \boldsymbol{B}$$

Maxwell's Voltage

$$|\boldsymbol{T}| = \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{H} \qquad \boldsymbol{B} \text{ is a bisector to } \boldsymbol{e}_n \text{ and } \boldsymbol{T}$$

Magnetic Energy

$$W_m = \frac{1}{2} \int \boldsymbol{J} \cdot \boldsymbol{A} dv = \frac{1}{2} \int \boldsymbol{B} \cdot \boldsymbol{H} dv = \frac{1}{2} \sum_i \sum_j L_{ij} I_i I_j$$

Two coils:

$$W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2$$

Reluctance

$$R = \frac{1}{\mu_r \mu_0 S}$$

Electromagnetic Fields

Induced emk

$$\mathcal{E} = \oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\ell$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathcal{E} = -\frac{d\Lambda}{dt} \quad \text{(coil with multiple turns)}$$

Maxwell's equations

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot D = \rho$$

$$\boldsymbol{\nabla} \boldsymbol{B} = 0$$

The Conservation Equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

Potentials

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho\left(\boldsymbol{r}', t - \frac{R}{c}\right)}{R} \, dv' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_{ret}}{R} \, dv'$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}\left(\boldsymbol{r}', t - \frac{R}{c}\right)}{R} \, dv' = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{ret}}{R} \, dv'$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$$

Magnetic Flow Density

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}_{ret} \times \boldsymbol{e}_R}{R^2} \, dv' + \frac{\mu_0}{4\pi c} \int \frac{\boldsymbol{J}'_{ret} \times \boldsymbol{e}_R}{R} \, dv'$$

Filamentuos Antenna

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \frac{i(z, t - R/c) d\boldsymbol{l} \times \boldsymbol{e}_R}{R^2} + \frac{\mu_0}{4\pi c} \int \frac{i(z, t - R/c) d\boldsymbol{l} \times \boldsymbol{e}_R}{R}$$

Oscillating Electric Dipole

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{p}'(t - R/c) \times \boldsymbol{e}_R}{R^2} + \frac{\mu_0}{4\pi c} \frac{\boldsymbol{p}''(t - R/c) \times \boldsymbol{e}_R}{R}$$

Oscillating Magnetic Dipole

$$\boldsymbol{B} = -\frac{\mu_0}{4\pi} \frac{\boldsymbol{m}'(t - R/c) \times \boldsymbol{e}_R}{R^2} - \frac{\mu_0}{4\pi c} \frac{\boldsymbol{m}''(t - R/c) \times \boldsymbol{e}_R}{R}$$

Pointing's Vector

$$\boldsymbol{P}_S(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t) \times \boldsymbol{H}(\boldsymbol{r}, t)$$

Time Harmonic Fields

Planar Sinusoidal Wave

$$\boldsymbol{E} = \hat{E} \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{r} + \phi) \boldsymbol{e}_E \quad \text{instantanious value}$$

$$\boldsymbol{E} = E_0 e^{-j\boldsymbol{k} \cdot \boldsymbol{r}} \boldsymbol{e}_E \quad \text{complex value}$$

$$E_0 = \hat{E} e^{j\phi} \quad \text{top value scale}$$

$$E_0 = \frac{\hat{E}}{\sqrt{2}} e^{j\phi} \quad \text{effective value scale}$$

Propagation Rate

$$v = \frac{1}{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}} \quad v = \frac{\omega}{k} \quad k = |\boldsymbol{k}|$$

Wave Impedance Non-Conductive Space

$$\eta = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}$$

Rule of Right-Hand Systems

$\boldsymbol{e}_k = \boldsymbol{e}_E \times \boldsymbol{e}_H \quad E = \eta H \qquad \boldsymbol{e}_k = \boldsymbol{e}_E \times \boldsymbol{e}_B \quad E = vB$

Planar Wave in Space with Condctivity

$$\boldsymbol{E} = E_0 e^{\gamma z} \boldsymbol{e}_x$$

Complex Propagation Constant

$$\gamma = \sqrt{j\omega\mu_r\mu_0(\sigma + j\omega\epsilon_r\epsilon_0)} \qquad \gamma = \alpha j\beta$$

Waveinpedance, Space With Given Conductivity

$$\eta = \sqrt{\frac{j\omega\mu_r\mu_0}{\sigma + j\omega\epsilon_r\epsilon_0}}$$

Penetration Depth

$$\delta = \sqrt{\frac{2}{\omega\mu_r\mu_0\sigma}}$$

Derivatives

Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$