

Fourier Analysis

Fourier Analysis Introduction

Fourier Sum

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

Fourier Coefficients

$$a_0 = \frac{2}{T} \int_0^T f(t) dt$$

$$a_m = \frac{2}{T} \int_0^T f(t) \cos(m\omega t) dt$$

$$b_m = \frac{2}{T} \int_0^T f(t) \sin(m\omega t) dt$$

Rewriting with Eulers Formula

$$f(t) = \sum_{m=-\infty}^{m=\infty} c_m e^{-im\omega t}$$

$$c_m = \frac{1}{T} \int_0^T f(t) e^{im\omega t} dt$$

For Non-periodic Functions

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega$$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Fourier s integral theorem

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} A(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 k$$

$$A(\mathbf{k}) = \int_{-\infty}^{\infty} f(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3 r$$

Periodic boundaries

If the function $f(\mathbf{r})$ is such that

$$f(\mathbf{r}) = f(\mathbf{r} + L\mathbf{R})$$

[For some positive integer L and lattice vector \mathbf{R} .
Then/Får något positivt heltal L och gittervektor \mathbf{R} .
Då håller att]]

$$f(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=\frac{\mathbf{G}}{L}} c_{\mathbf{k}} \cdot e^{i\mathbf{k} \cdot \mathbf{r}}$$

$$c_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int_V f(\mathbf{r}) \cdot e^{-i\mathbf{k} \cdot \mathbf{r}} d^3 r$$

Where \mathbf{G} is the reciprocal lattice vector and $V = L^3 |\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)| = L^3 V_a$. The functions $\frac{1}{\sqrt{V}} e^{i\mathbf{k} \cdot \mathbf{r}}$ is a complete orthonormal basis in V . If the volume V is large, the sum can be replaced by an integral:

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3 k$$

Dirac Delta Function

$$\int_A^B f(x) \delta(x - x_0) dx = \begin{cases} f(x_0) & \text{if } A < x_0 < B \\ 0 & \text{otherwise} \end{cases}$$

If $f(x)$ is a "nice" function.

$$\delta(f(x)) = \sum_{\forall i; f(x_i)=0, f'(x_i) \neq 0} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk$$

Kronecker Delta

$$\delta_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\phi(n-m)} d\phi = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$