Integrals and Identities

Some integrals

Indefinite Integrals

$$\int \frac{x}{ax+b}dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b|$$

$$\int \frac{1}{x(ax+b)}dx = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|$$

$$\int \frac{ax+b}{fx+g}dx = \frac{ax}{f} + \frac{bf-ag}{f^2}\ln|fx+g|$$

$$\int \frac{x}{(ax+b)(fx+g)}dx = \frac{1}{bf-ag}\left[\frac{b}{a}\ln|ax+b| - \frac{g}{f}\ln|fx+g|\right]$$

Definition: $\chi = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & \text{if } 4ac > b^2 \\ \frac{1}{a(p-q)}\ln\left|\frac{x-p}{x-q}\right| & \text{if } 4ac - b^2 \end{cases}$

Where p and q are the roots of $ax^2+bx+c=0$.

$$\int \frac{1}{ax^2+bx+c}dx = \chi$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\chi$$

$$\int \frac{x^2}{ax^2+bx+c}dx = \frac{x}{a} - \frac{b}{2a^2}\ln\left|ax^2+bx+c\right| + \frac{b^2-2ac}{2a^2}\chi$$

$$\int \frac{1}{(ax^2+bx+c)^2}dx = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{(4ac-b^2)}\chi$$

$$\int \frac{x}{(ax^2+bx+c)^2}dx = -\frac{bx+2c}{(4ac-b^2)(ax^2+bx+c)} - \frac{b}{(4ac-b^2)}\chi$$

$$\int \sqrt{ax+b}dx = \frac{2}{3a}(ax+b)^{3/2}$$

$$\int x\sqrt{ax+b}dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2}$$

$$\int \frac{1}{\sqrt{ax+b}}dx = \frac{2\sqrt{ax+b}}{a}$$

$$\int \frac{x}{\sqrt{ax+b}}dx = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

$$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

$$\int x\sqrt{a^2-x^2}dx = -\frac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

$$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x+\sqrt{x^2+a^2}\right|$$

$$\int \frac{1}{\sin ax}dx = \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right|$$

$$\int \frac{1}{\cos ax}dx = \frac{1}{a}\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right|$$

$$\int \tan ax\, dx = -\frac{1}{a}\ln|\cos ax|$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax|$$

$$\int x\sin x\,dx = \sin x - x\cos x$$

$$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$$

$$\int \ln x\,dx = x\ln|x| - x$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Definite Integrals

$$\int_0^\infty x^{2n}\cdot e^{-ax^2}\,dx = \frac{(2n-1)!!}{2(2a)^n}\sqrt{\frac{\pi}{a}}$$

Where $a > 0$, $n! =$ negative integer.

$$\int_0^\infty x^{2n+1}\cdot e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$$

$$\int_0^\infty x^k\cdot e^{-ax}\,dx = \Gamma(k+1)\cdot a^{-(k+1)}$$

$$I_k = \int_0^\infty \frac{x^k}{e^x-1}\,dx = \Gamma(k+1)\cdot\zeta(k+1)$$

$$J_k = \int_0^\infty \frac{x^k}{e^x+1}\,dx = (1-2^{-k})\cdot I_k$$

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$$

$$\int_0^{\pi/2}\sin^{2a+1}x\cdot\cos^{2b+1}x\,dx = \frac{\Gamma(a+1)\cdot\Gamma(b+1)}{2\Gamma(a+b+2)}$$

Stirling's approximation

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \qquad N \gg 1$$

Error Function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\,d\xi$$

$$\mathrm{erf}(\infty) = 1$$

Power Series

Power Series

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

$$\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - ...$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{1}{15!}x^5 + ...|x| < \frac{\pi}{2}$$

$$\ln(1+x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ...|x| < 1$$

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + ...$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + ...$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ...$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - ...|x| < 1$$

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + ...|x| < 1$$

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ...$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ...$$

Trigonometric Functions

Trigonometric Functions

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$$

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1 - \cos\alpha)$$

$$\cos^2\frac{\alpha}{2} = \frac{1}{2}(1 + \cos\alpha)$$

$$\sin\alpha + \cos\beta = 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$$

$$\sin\alpha - \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$$

$$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$$

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$$

$$\sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$$

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$$

Hyperbolic Functions

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$