

# Vector Analysis

## Vector Analysis

### Vector Products

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where  $\hat{x}, \hat{y}, \hat{z}$  are unit vectors.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c}((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}) - \mathbf{d}((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})$$

### Gradient, Divergence, Curl and The Laplace Operator

$$\text{grad} f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left( \frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) \text{ Where } dV \text{ in polar coordinates are } r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\text{div} \mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 a_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta a_\theta) + \frac{1}{r \sin \theta} \frac{\partial a_\varphi}{\partial \varphi}$$

$$\text{rota} = \nabla \times \mathbf{a} = \left( \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)$$

$$= \left( \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta a_\varphi) - \frac{\partial a_\theta}{\partial \varphi} \right), \frac{1}{r \sin \theta} \frac{\partial a_r}{\partial \varphi} - \frac{1}{r} \frac{\partial}{\partial r} (r a_\varphi), \frac{1}{r} \frac{\partial}{\partial r} (r a_\theta) - \frac{1}{r} \frac{\partial a_r}{\partial \theta} \right)$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$$

$$\Delta f(r) = \frac{1}{r} \frac{d^2}{dr^2} (rf), \quad r \neq 0$$

$$\nabla \times (\nabla \mathcal{U}) = 0$$

$$\nabla \cdot (\nabla \mathcal{U}) = \nabla^2 \mathcal{U}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$$

$$\nabla \cdot (\mathcal{U} \mathbf{V}) = \mathcal{U} \Delta \mathbf{V} + 2 \nabla \mathcal{U} \cdot \nabla \mathbf{V} + \mathbf{V} \Delta \mathcal{U}$$

$$\nabla \cdot (\mathcal{U} \mathbf{V}) = \mathcal{U} \Delta \mathbf{V} + 2(\nabla \mathcal{U} \cdot \nabla) \mathbf{V} + \mathbf{V} \Delta \mathcal{U}$$

$$\nabla \times (\mathcal{U} \mathbf{V}) = \mathcal{U} \nabla \times \mathbf{V} + (\nabla \mathcal{U}) \times \mathbf{V}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{B}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

### Gauss sats

$$\oint_{S(V)} \mathbf{a} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{a}) dV$$

### Stokes sats

$$\oint_{C(S)} \mathbf{a} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$$

Where  $S$  is an arbitrary surface with border  $C(S)$

### Greens sats

$$\oint_{S(V)} (\Psi \nabla \varphi - \varphi \nabla \Psi) \cdot d\mathbf{S} = \int_V (\Psi \Delta \varphi - \varphi \Delta \Psi) dV$$