Fourier Analysis

Fourier Analysis Introduction

Fourier Sum

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

Fourier Coefficients

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)dt$$

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega t)dt$$

Rewriting with Eulers Formula

$$f(t) = \sum_{m=-\infty}^{m=\infty} c_n e^{-im\omega t}$$

$$c_m = \frac{1}{T}\int_0^T f(t)e^{im\omega t}dt$$

For Non-periodic Functions

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

Fourier s integral theorem

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} A(\boldsymbol{k})e^{i\boldsymbol{k}\cdot r}d^3k$$

$$A(\boldsymbol{k}) = \int_{-\infty}^{\infty} f(\boldsymbol{r})e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r$$

Periodic boundaries

If the function $f(\boldsymbol{r})$ is such that

$$f(\boldsymbol{r}) = f(\boldsymbol{r} + L\boldsymbol{R})$$

[For some positive integer L and lattice vector $\boldsymbol{R}$. Then/Får något positivt heltal L och gittervektor $\boldsymbol{R}$. Då håller att]]

$$f(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}=\frac{\boldsymbol{G}}{L}} c_{\boldsymbol{k}} \cdot e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$c_{\boldsymbol{k}} = \frac{1}{\sqrt{V}}\int_V f(\boldsymbol{r})\cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r$$

Where $\boldsymbol{G}$ is the reciprocal lattice vector and $V = L^3|\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)| = L^3V_a$. The functions $\frac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is a complete orthonormal basis in V.. If the volume V is large, the sum can be replaced by an integral:

$$\sum_{\boldsymbol{k}} \to \frac{V}{(2\pi)^3}\int d^3k$$

Dirac Delta Function

$$\int_A^B f(x)\delta(x-x_0)\, dx = \begin{cases} f(x_0) & \text{if } A < x_0 < B \\ 0 & \text{otherwise} \end{cases}$$

If $f(x)$ is a "nice" function.

$$\delta(f(x)) = \sum_{\forall i; f(x_i)=0, f'(x_i)\neq 0} \frac{1}{|f'(x_i)|}\delta(x-x_i)$$

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\, dk$$

Kronecker Delta

$$\delta_{n,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\phi(n-m)}\, d\phi = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$