

Bordered Heegaard Floer homology

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ABSTRACT. We construct Heegaard Floer theory for 3-manifolds with connected boundary. The theory associates to an oriented, parametrized two-manifold a differential graded algebra. For a three-manifold with parametrized boundary, the invariant comes in two different versions, one of which (type D) is a module over the algebra and the other of which (type A) is an \mathcal{A}_∞ module. Both are well-defined up to chain homotopy equivalence. For a decomposition of a 3-manifold into two pieces, the \mathcal{A}_∞ tensor product of the type D module of one piece and the type A module from the other piece is \widehat{HF} of the glued manifold.

As a special case of the construction, we specialize to the case of three-manifolds with torus boundary. This case can be used to give another proof of the surgery exact triangle for \widehat{HF} . We relate the bordered Floer homology of a three-manifold with torus boundary with the knot Floer homology of a filling.

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Contents

Chapter 1. Introduction	1
1.1. Background	1
1.2. The bordered Floer homology package	2
1.3. On gradings	5
1.4. The case of three-manifolds with torus boundary	5
1.5. Previous work	6
1.6. Further developments	6
1.7. Organization	7
Acknowledgements	8
Chapter 2. \mathcal{A}_∞ structures	9
2.1. \mathcal{A}_∞ algebras and modules	9
2.2. \mathcal{A}_∞ tensor products	14
2.3. Type D structures	16
2.4. Another model for the \mathcal{A}_∞ tensor product	21
2.5. Gradings by non-commutative groups	24
Chapter 3. The algebra associated to a pointed matched circle	29
3.1. The strands algebra $\mathcal{A}(n, k)$	29
3.2. Matched circles and their algebras	34
3.3. Gradings	38
Chapter 4. Bordered Heegaard diagrams	45
4.1. Bordered Heegaard diagrams: definition, existence, and uniqueness	45
4.2. Examples of bordered Heegaard diagrams	51
4.3. Generators, homology classes and spin ^c structures	52
4.4. Admissibility criteria	56
4.5. Closed diagrams	57
Chapter 5. Moduli spaces	59
5.1. Overview of the moduli spaces	59
5.2. Holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$	61
5.3. Holomorphic curves in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$	67
5.4. Compactifications via holomorphic combs	70
5.5. Gluing results for holomorphic combs	79
5.6. Degenerations of holomorphic curves	83
5.7. More on expected dimensions	93
Chapter 6. Type D modules	103
6.1. Definition of the type D module	103

6.2. $\partial^2 = 0$	105
6.3. Invariance	110
6.4. Twisted coefficients	134
Chapter 7. Type A modules	137
7.1. Definition of the type A module	137
7.2. Compatibility with algebra	139
7.3. Invariance	142
7.4. Twisted coefficients	147
Chapter 8. Pairing theorem via nice diagrams	149
Chapter 9. Pairing theorem via time dilation	153
9.1. Moduli of matched pairs	154
9.2. Dilating time	157
9.3. Dilating to infinity	162
9.4. Completion of the proof of the pairing theorem	173
9.5. A twisted pairing theorem	174
9.6. An example	175
Chapter 10. Gradings	179
10.1. Algebra review	179
10.2. Domains	180
10.3. Type A structures	183
10.4. Type D structures	186
10.5. Refined gradings	188
10.6. Tensor product	191
Chapter 11. Bordered manifolds with torus boundary	195
11.1. Torus algebra	195
11.2. Surgery exact triangle	198
11.3. Preliminaries on knot Floer homology	199
11.4. From \widehat{CFD} to HFK^-	202
11.5. From CFK^- to \widehat{CFD} : Statement of results	206
11.6. Generalized coefficient maps and boundary degenerations	209
11.7. From CFK^- to \widehat{CFD} : Basis-free version	214
11.8. Proof of Theorem 11.26	232
11.9. Satellites revisited	237
Appendix A. Bimodules and change of framing	241
A.1. Statement of results	241
A.2. Sketch of the construction	242
A.3. Computations for 3-manifolds with torus boundary	246
A.4. From HFK to \widehat{CFD} for arbitrary integral framings	251
Index of Notation	255
Index of Definitions	267
Bibliography	275

List of Figures

1.1	Example of cutting a Heegaard diagram	3
3.1	Splittings and shuffles of Reeb chords.	33
3.2	Pointed matched circles for the genus 2 surface	35
3.3	Orientation of a pointed matched circle.	35
4.1	3-manifold represented by a bordered Heegaard diagram.	48
4.2	Bordered Heegaard diagram for the trefoil complement.	51
4.3	Heegaard diagrams for handlebodies.	52
4.4	Another Heegaard diagram for a genus 2 handlebody	52
5.1	Examples of three kinds of codimension 1 degenerations.	61
5.2	A capping operation	66
5.3	Sources of curves at east ∞	69
5.4	Visualizing shuffle curves, I.	70
5.5	Visualizing shuffle curves, II.	70
5.6	Schematic of a height 2 holomorphic comb.	72
5.7	$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ may be a proper subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, I.	74
5.8	$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ may be a proper subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, II.	75
5.9	1-parameter family of maps v from an odd shuffle component.	84
5.10	Annulus at east ∞ that does not exist.	86
5.11	Expected and unexpected behavior of $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \subset \overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^\triangleright)$	90
5.12	Degenerations discussed in Theorem 5.61.	91
5.13	Domains of model curves used to compute inv'	98
5.14	Jittered intersection number.	99
5.15	Parts of the proof of Lemma 5.73.	99
6.1	Local illustrations of possible contributions to ∂^2 in \widehat{CFD}	109
6.2	Labeling of triangle.	113
6.3	Curves α and α^H	114
6.4	A triangle is a strip with a boundary puncture.	117
6.5	A degeneration at v_{12}	124
6.6	Straightening α -curves	125

6.7	Illustration of a boundary bigon degenerating, in proof of invariance.	128
6.8	Curves α , α^H and α' in Σ	131
6.9	Two compositions of triangles, as degenerations of a rectangle.	133
7.1	Local illustrations of terms occurring the \mathcal{A}_∞ relation for \widehat{CFA}	140
7.2	Domain where m_3 is necessary for associativity.	141
8.1	First step in making a Heegaard diagram with boundary nice.	150
9.1	Schematic of an ideal-matched comb	163
9.2	Illustration of the fibered product in the proof of Proposition 9.40	172
9.3	Degenerating the hexagon.	175
11.1	Bordered Heegaard diagrams for torus boundary.	195
11.2	Labeling of regions around $\partial\overline{\Sigma}$ (torus case).	196
11.3	From bordered diagrams to doubly-pointed diagrams.	203
11.4	Diagram for longitude of solid torus.	205
11.5	Reconstructing \widehat{CFD} from CFK	210
11.6	Boundary degeneration as end of a one-dimensional moduli space.	212
11.7	Sketch of some of the maps from Theorem 11.36.	216
11.8	Twisting, in proof of Theorem 11.36.	217
11.9	Homology classes in the winding region.	222
11.10	Homology classes in the winding region.	223
11.11	Notation for the local multiplicities in the winding region.	224
11.12	Pinching Ω along C_2	228
11.13	Illustration of the proof of Theorem 11.26	236
11.14	Diagram for the $(2, 1)$ cable.	238
11.15	Cable of the trefoil.	239
A.1	Bordered Heegaard diagrams for diffeomorphisms.	243
A.2	Heegaard diagrams for Dehn twists of the torus.	247
A.3	The DA bimodules for generators of the genus-one mapping class group	249
A.4	Deformed Heegaard diagram for identity map of torus	250
A.5	The bimodule $\widehat{CFAA}(\mathbb{I}, 0)$	251
A.6	The bimodule $\widehat{CFDD}(\mathbb{I}, 0)$	252

CHAPTER 1

Introduction

1.1. Background

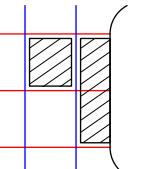
Since the pioneering work of Simon Donaldson, techniques from gauge theory have taken a central role in the study of smooth four-manifold topology [DK90]. His numerical invariants, associated to closed, smooth four-manifolds with $b_2^+ > 1$, have shed much light on our understanding of differential topology in dimension four. Moreover, these invariants, and the subsequent closely-related Seiberg-Witten invariants [Wit94] and Heegaard Floer invariants [OSz04d] all fit into a formal framework reminiscent of the “topological quantum field theories” proposed by Witten [Wit88]. Crudely speaking, these theories have the following form. To a closed three-manifold Y one associates a (suitably graded) abelian group, the *Floer homology* of Y , and, to a four-manifold W with boundary identified with Y , a homology class in the Floer homology of the boundary. If a closed, smooth four-manifold X decomposes along Y into a union of two four-manifolds with boundary X_1 and X_2 , then the numerical (Donaldson, Seiberg-Witten, or Heegaard Floer) invariant of the closed four-manifold is obtained as a suitable pairing between the relative invariants coming from X_1 and X_2 in a corresponding version of Floer homology of Y .

As the name suggests, the first such construction was proposed by Andreas Floer (for a restricted class of three-manifolds) as a tool for studying Donaldson’s theory. A complete construction of the corresponding three-dimensional Floer theory for Seiberg-Witten invariants was given by Kronheimer and Mrowka [KM07]. Heegaard Floer homology was defined by Zoltán Szabó and the second author.

The aim of the present work is to perform a corresponding construction one dimension lower. Specifically, we produce an invariant which, loosely speaking, associates to a parametrized, closed, oriented surface F a differential graded algebra $\mathcal{A}(F)$ and associates to a three-manifold whose boundary is identified with F a differential graded module over $\mathcal{A}(F)$. When a closed, oriented three-manifold can be decomposed along F into two pieces Y_1 and Y_2 , a suitable variant of Floer homology is gotten as a pairing of the differential graded modules associated to Y_1 and Y_2 .

We give now a slightly more detailed version of this picture, starting with some more remarks about Heegaard Floer homology, and then a more precise sketch of the invariants constructed in the present book.

Recall that there are several variants of Heegaard Floer homology stemming from the fact that, in its most basic form, the Heegaard Floer homology of a three-manifold is the homology of a chain complex defined over a polynomial ring in an indeterminant U . The full theory can be promoted to construct invariants for closed, smooth four-manifolds [OSz06], similar in character (and conjecturally equal) to Seiberg-Witten invariants. In this book, we focus on the specialization of Heegaard Floer homology for three-manifolds in the case where $U = 0$, giving the three-manifold invariant denoted $\widehat{HF}(Y)$. The corresponding simplified theory is not rich enough to construct interesting closed four-manifold invariants, but it



does already contain interesting geometric information about the underlying three-manifold including, for example, full information about the minimal genera of embedded surfaces in Y [OSz04a, Ni09].

With this background in place, we proceed as follows. Let F be a closed, oriented two-manifold. Equip F with some additional data, in the form of a minimal handle decomposition and a basepoint, data which we refer to as a *pointed matched circle* (see Definition 3.16) and denote by \mathcal{Z} . This handle decomposition can be thought of as giving a parameterization of F . We associate to \mathcal{Z} a differential graded algebra $\mathcal{A}(\mathcal{Z})$.

Next, fix a three-manifold with boundary Y together with an orientation-preserving diffeomorphism $\phi: F \rightarrow \partial Y$. To this data (Y, ϕ) we associate a left differential module over $\mathcal{A}(-\mathcal{Z})$, the *type D module of Y* , denoted $\widehat{\text{CFD}}(Y, \phi: F \rightarrow Y)$ or, less precisely, $\widehat{\text{CFD}}(Y)$. To (Y, ϕ) we also associate the *type A module of Y* , denoted $\widehat{\text{CFA}}(Y, \phi: F \rightarrow Y)$ or $\widehat{\text{CFA}}(Y)$. The type A module of Y invariant is a more general type of object: it is a right \mathcal{A}_∞ module over $\mathcal{A}(\mathcal{Z})$. Both invariants are defined by counting holomorphic curves. The definitions of both of these modules depend on a number of auxiliary choices, including compatible Heegaard diagrams and associated choices of almost complex structures. However, as is typical of Floer homology theories, the underlying (\mathcal{A}_∞) homotopy equivalence types of the modules are independent of these choices, giving rise to topological invariants of our three-manifold with boundary. When speaking informally about these invariants, we refer to them collectively as the *bordered Heegaard Floer invariants* of Y .

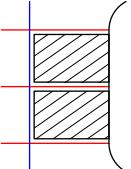
The relationship with closed invariants is encapsulated as follows. Suppose that Y_1 and Y_2 are two three-manifolds with boundary, where ∂Y_1 is identified with F and ∂Y_2 is identified with $-F$. Then we can form a closed three-manifold Y by identifying Y_1 and Y_2 along their boundaries. The pairing theorem for bordered Heegaard Floer homology states that the Heegaard Floer complex \widehat{HF} of Y is homotopy equivalence to the \mathcal{A}_∞ tensor product of $\widehat{\text{CFA}}(Y_1)$ with $\widehat{\text{CFD}}(Y_2)$.

Recall that \widehat{HF} can be calculated algorithmically for any closed three-manifold, thanks to the important work of Sarkar and Wang [SW10]. Nevertheless, the present work (and forthcoming extensions) makes it possible to compute it in infinite families: by cutting a 3-manifold into simpler pieces along surfaces and computing an invariant for each piece, we can reduce the computation to a computation for each piece and an algebraic computation. Indeed, we expect this to lead to a more efficient algorithm for computation in general.

Moreover, bordered Heegaard Floer homology provides a conceptual framework for organizing the structure of \widehat{HF} . We give a few of these properties in this book, and return to further applications elsewhere (see Section 1.6).

1.2. The bordered Floer homology package

The formal algebraic setting for the constructions herein lie outside the working toolkit for the typical low-dimensional topologist: \mathcal{A}_∞ modules over differential graded algebras, and their \mathcal{A}_∞ tensor products (though, we should point out, these objects are now familiar in symplectic topology, cf. [Fuk93, Kon95, Sei02]). We will recall the basics before giving precise statements of our results. But before doing this, we sketch the outlines of the geometry which underpins the constructions, giving informal statements of the basic results of this package.



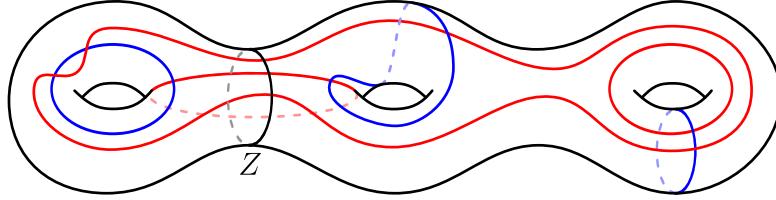


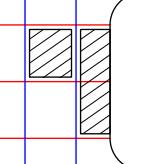
FIGURE 1.1. **An example of cutting a Heegaard diagram.** The Heegaard diagram shown may be degenerated by cutting along (or pinching) the curve marked Z . Each half of the result will be a bordered Heegaard diagram.

Central to this geometric picture is the cylindrical reformulation of Heegaard Floer homology [Lip06a]. Recall that Heegaard Floer homology in its original incarnation [OSz04d] is an invariant associated to a Heegaard diagram for a three-dimensional manifold Y . Start from a Heegaard diagram for Y , specified by an oriented surface Σ , equipped with two g -tuples of attaching circles $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$, as well as an additional basepoint $z \in \Sigma$ in the complement of these circles. One then considers a version of Lagrangian Floer homology in the g -fold symmetric product of Σ . In the cylindrical reformulation, disks in the g -fold symmetric product are reinterpreted as holomorphic curves in the four-manifold $\Sigma \times [0, 1] \times \mathbb{R}$, with coordinates (x, s, t) , satisfying certain constraints (dictated by the attaching circles) at $s \in \{0, 1\}$ and asymptotic constraints as t approaches $\pm\infty$.

Starting from this cylindrical reformulation, suppose that our Heegaard diagram for Y is equipped with a closed, separating curve Z in Σ . We suppose that Z meets the following combinatorial requirements: it is disjoint from the β -curves—so cutting Σ along Z and the β -curves results in a surface with two components—and although we do not require that Z be disjoint from the α -curves, we still require that the result of cutting Σ along Z and the α -curves results in a surface with two components. The constructions in this book emerge when one considers limits of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ as the complex structure on Σ is pinched along Z .

In the limit as the circle Z is pinched to a node, the holomorphic curves limit to holomorphic curves in $((\Sigma_1 \amalg \Sigma_2) \times [0, 1] \times \mathbb{R})$, where here Σ_1 and Σ_2 are the two components of $\Sigma - Z$ equipped with a cylindrical ends. The holomorphic curves from these two halves have constrained limiting behavior as they enter the ends of Σ_1 and Σ_2 : there are finitely many values of t where our holomorphic curves are asymptotic to arcs in the ideal boundaries $\partial\Sigma_i$. The data of the asymptotics (the values of t and the arcs in $\partial\Sigma_i$) must match in the limit. Thus, we may reconstruct holomorphic curve counts for $\Sigma \times [0, 1] \times \mathbb{R}$ from holomorphic curve counts in $\Sigma_i \times [0, 1] \times \mathbb{R}$.

The data on the Heegaard surface specified by Z has a more intrinsic, three-dimensional interpretation. The α -circles meet Z generically in a collection of points, which come in pairs (those points which belong to the same α_i). This is the structure of a *matched circle* mentioned above. It specifies a closed surface F . This can be seen in terms of Morse functions, as follows. Think about the Heegaard diagram as the intersections of the ascending and descending disks of a self-indexing Morse function on Y with the middle level in the usual way. We then construct F from Z by flowing Z backwards and forwards under the Morse flow. Concretely, F is obtained from a disk with boundary Z by attaching one-handles along the matched pairs of points, and then filling in the remaining disk with a two-handle. (Our assumptions guarantee there is only one two-handle to be added in the end.) The two



halves of a Heegaard diagram specify three-manifolds Y_1 and Y_2 meeting along their common boundary F .

In general terms, the differential graded algebra $\mathcal{A}(\mathcal{Z})$ is constituted from the Reeb chords of a corresponding matched circle \mathcal{Z} . (The reader should not be thrown by the use of the term “Reeb chord” here: the ambient contact manifold here is the circle, and hence the Reeb chords are simply arcs in a circle, with boundaries on the feet of the 1-handles. That said, the terminology is not entirely pointless; see Chapter 5.) The product on the algebra and its differential are defined combinatorially, but rigged to agree with (relatively simple) holomorphic curve counts in the cylinder which interpolates between Reeb chords. The precise algebra is defined in Chapter 3.

The type D module for the component Y_2 is defined as a chain complex over $\mathcal{A}(\mathcal{Z})$, generated by g_2 -tuples of intersection points of various of the α_i and β_j contained in Σ_2 , with a differential given as a weighted count of rigid holomorphic curves, where the coefficient in $\mathcal{A}(\mathcal{Z})$ of a rigid holomorphic curve measures the asymptotics of its Reeb chords at its boundary. The precise definition is given in Chapter 6. In that chapter, we also prove a more precise version (Theorem 6.16) of the following:

THEOREM 1.1. *Let Y_2 be a three-manifold equipped with an orientation-preserving diffeomorphism $\phi: F \rightarrow -\partial Y_2$. The homotopy equivalence class of the associated differential module $\widehat{CFD}(Y_2, \phi: F \rightarrow -\partial Y_2)$ is a topological invariant of the three-manifold Y_2 with boundary parameterized by $-F$.*

For the type A module for the component Y_1 , on the other hand, the algebraic operations are defined by counting holomorphic curves that are rigid when subject to certain height constraints on their Reeb chords. Specifically, we require that certain clumps of these Reeb chords occur at the same height. The proper algebraic set-up for these operations is that of an \mathcal{A}_∞ module over $\mathcal{A}(\mathcal{Z})$, and the precise definition of the type A module is given in Chapter 7, where we also prove a more precise version (Theorem 7.17) of the following:

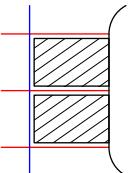
THEOREM 1.2. *Let Y_1 be a three-manifold equipped with an orientation-preserving diffeomorphism $\phi: F \rightarrow \partial Y_1$. The \mathcal{A}_∞ homotopy equivalence class of the \mathcal{A}_∞ module $\widehat{CFA}(Y_1, \phi: F \rightarrow \partial Y_1)$ is a topological invariant of the three-manifold Y_1 with boundary parameterized by F .*

Of course, differential graded modules over a differential graded algebra are special cases of \mathcal{A}_∞ modules. Moreover, there is a pairing, the \mathcal{A}_∞ tensor product, between two \mathcal{A}_∞ modules M_1 and M_2 , giving rise to a chain complex $M_1 \tilde{\otimes} M_2$, whose chain homotopy type depends only on the chain homotopy types of the two factors; this generalizes the derived tensor product of ordinary modules.

The Heegaard Floer homology of Y can be reconstituted from the bordered Heegaard Floer homology of its two components according to the following pairing theorem:

THEOREM 1.3. *Let Y_1 and Y_2 be two three-manifolds with parameterized boundary $\partial Y_1 = F = -\partial Y_2$, where F is specified by the pointed matched circle \mathcal{Z} . Fix corresponding bordered Heegaard diagrams for Y_1 and Y_2 . Let Y be the closed three-manifold obtained by gluing Y_1 and Y_2 along F . Then $\widehat{HF}(Y)$ is homotopy equivalent to the \mathcal{A}_∞ tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$. In particular,*

$$\widehat{HF}(Y) \cong H_* \left(\widehat{CFA}(Y_1) \tilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y_2) \right).$$



We give two proofs of the above theorem. One proof (in Chapter 8) makes use of the powerful technique of Sarkar and Wang [SW10]: we construct convenient Heegaard diagrams for Y_1 and Y_2 , where the holomorphic curve counts can be calculated combinatorially. For such diagrams, the \mathcal{A}_∞ structure of the type A module simplifies immensely (higher multiplications all vanish), \mathcal{A}_∞ tensor products coincide with traditional tensor products, and the proof of the pairing theorem becomes quite simple. The other proof (in Chapter 9) involves a rescaling argument to identify $\widehat{CF}(Y)$ with another model for the \mathcal{A}_∞ tensor product. This second proof gives geometric insight into the connection between the analysis and the algebra. It also leads to pairing theorems for triangle and polygon maps [LOT14, LOT].

1.3. On gradings

One further surprising aspect of the theory that deserves further comment is the structure of gradings.

Gradings in the Heegaard Floer theory of closed 3-manifolds have an unconventional form: for each spin^c structure there is a relative grading taking values in a cyclic abelian group. Alternatively, this can be viewed as an absolute grading valued in some \mathbb{Z} -set. Indeed, for the case of Floer homology of Seiberg-Witten monopoles [KM07], this \mathbb{Z} -set has an elegant formulation as the set of isotopy classes of non-vanishing vector fields on the 3-manifold.

Gradings in bordered Floer homology have a correspondingly even less conventional form. The algebra of a surface is graded by the Heisenberg group associated to the surface's intersection form. (This group also has a geometric interpretation; see Remark 3.48.) The gradings for bordered Heegaard Floer modules take values in sets with actions of this Heisenberg group. This is explained in Chapter 10.

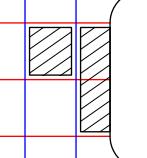
1.4. The case of three-manifolds with torus boundary

As explained in Chapter 11, for three-manifolds with boundary the torus T , the (appropriate summand of the) algebra $\mathcal{A}(T)$ is particularly simple: it is finite-dimensional (an eight-dimensional subalgebra of 4×4 upper-triangular matrices) and has vanishing differential.

The bordered Heegaard Floer homology of a solid torus bounded by T can be easily calculated; see also [Lip06b, Section 5.3]. A fundamental result in Heegaard Floer homology, the *surgery exact triangle*, gives a long exact sequence relating the Heegaard Floer homology groups of three three-manifolds which are obtained as different fillings of the same three-manifold with torus boundary. (The result was first proved in [OSz04c], inspired by a corresponding result of Floer for the case of his instanton homology, see [Flo95], compare also [KMOSz07] for a Seiberg-Witten analogue.) This result can now be seen as a consequence of the pairing theorem, together with a short exact sequence relating three type D modules associated to three different solid tori with boundary T .

In a related vein, recall [OSz04b, Ras03] that there is a construction of Heegaard Floer homology theory for knots K in a three-manifold Y . Information about this knot Floer homology turns out to determine the bordered Floer homology of the knot complement. This is stated precisely and worked out in Theorem 11.26 in Section 11.7 for large framings and in Theorem A.11 in Appendix A for general framings, using some results from [LOT10a]. This result provides a number of explicit, non-trivial examples of bordered Floer homology.

Indeed, using this connection between knot Floer homology and bordered Heegaard Floer homology, and combining it with an adaptation of Theorem 1.3 (stated in Theorem 11.19



below), we are able to calculate knot Floer homology groups of satellite knots in terms of the filtered chain homotopy type of the knot filtration of the pattern knot, together with some information (a type A module) associated to the pattern. We illustrate this in some concrete examples, where the type A module can be calculated explicitly. These ideas are further pursued in [Pet09, Lev12b, Lev12a].

1.5. Previous work

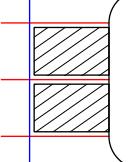
Some of the material in this book, especially from Chapters 4 and 5, first appeared in the first author's Ph.D. thesis [Lip06b]. As in that work, we construct here an invariant of bordered manifolds that lives in a suitable category of chain complexes. By modifying the algebra and modules introduced in that work appropriately we are able to reconstruct the invariant of a closed 3-manifold. Related constructions in the case of manifolds with torus boundary have been worked out by Eftekhary [Eft08]. For the case of satellite knots, there is extensive work by Hedden, including [Hed05a], which we used as a check of some of our work. In a different direction, an invariant for manifolds-with-boundary equipped with sutures on the boundary has been given by András Juhász [Juh06].

An introduction to some of the structures used in this book, in the form of a toy model illustrating key features, is given in a separate paper [LOT09].

1.6. Further developments

The theory presented here is in some ways rather limited as a theory of 3-manifolds with boundary: we deal only with a single, connected boundary component, for the somewhat limited \widehat{HF} theory. We return to these points in other papers:

- For a 3-manifold M with two boundary components, together with parametrizations of these boundary components and a framed arc connecting them we associate several types of bimodules [LOT10a]. These bimodules are well-defined up to homotopy equivalence, and satisfy appropriate pairing theorems. Some parts of this theory are sketched in Appendix A.
- As a special case, these bimodules allow us to change the parametrization of the boundary. (For instance, if F is a torus this corresponds to changing the framing on a knot.) This gives a representation of the mapping class group of F on the derived category of modules over $\mathcal{A}(\mathcal{Z})$. This mapping class group action is faithful [LOT13]. (Bimodules for generators of the mapping class group of the torus are given explicitly in Appendix A.)
- The type A and D versions of the bordered invariant of a 3-manifold with boundary are also related by certain dualizing bimodules [LOT10a]. (Again, this is sketched in a little more detail in Appendix A.) They are also related by another kind of duality, using the Hom-functor [LOT11]. In particular, $\widehat{CFA}(Y)$ can be calculated explicitly from $\widehat{CFD}(Y)$, and vice-versa.
- Decomposing a three-manifold into simple pieces, and calculating the (bi)modules associated to the pieces, one can calculate the Heegaard Floer invariant \widehat{HF} for any 3-manifold, as well as the bordered invariants of any bordered 3-manifold [LOT10c]. Similar techniques, together with a pairing theorem for polygon maps, allow one to compute the spectral sequence from Khovanov homology to \widehat{HF} of the branched double cover and, more generally, the link surgery spectral sequence [LOT10b],



LOT14]. These methods also allow one to compute the maps on \widehat{HF} associated to 4-dimensional cobordisms **[LOT]**.

- András Juhász has constructed an invariant of sutured manifolds, called sutured Floer homology **[Juh06]**, which has been remarkably effective at studying geometric questions **[Juh08]**. Rumen Zarev has developed a *bordered-sutured Floer theory* generalizing both bordered Floer homology and sutured Floer homology. This theory allows one to recover the sutured invariants by tensoring the bordered Floer invariants with a module recording the sutures **[Zar09]**. In turn, it also shows that much of bordered Floer theory can be captured in terms of sutured Floer homology **[Zar10]**.

1.7. Organization

In Chapter 2 we briefly recall the language of \mathcal{A}_∞ modules. Most of the material in that chapter is standard, with the possible exception of a particular model, denoted \boxtimes , for the \mathcal{A}_∞ tensor product. This model is applicable when one of the two factors has a particular simple algebraic structure, which we call a *type D structure*; the modules \widehat{CFD} have this form.

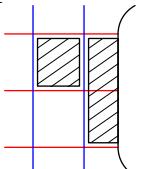
In Chapter 3 we construct the differential graded algebra associated to a closed, oriented surface. Chapter 4 collects a number of basic properties of Heegaard diagrams for three-manifolds with (parameterized) boundary, including when two such diagrams represent the same 3-manifold. The generators of the bordered Floer modules are also introduced there, and we see how to associate spin^c-structures to these and discuss domains connecting them (when they represent the same spin^c-structure). In Chapter 5, we collect the technical tools for moduli spaces of holomorphic curves which are counted in the algebraic structure underlying bordered Floer homology.

With this background in place, we proceed in Chapter 6 to the definition of the type *D* module, and establish its invariance properties sketched in Theorem 1.1. In Chapter 7 we treat the case of type *A* modules, establishing a precise version of Theorem 1.2. We turn next to the pairing theorem. Chapter 8 gives one proof of Theorem 1.3, using so-called nice diagrams (in the sense of **[SW10]**). As a bi-product, this gives an algorithm to calculate the bordered Floer homology of a three-manifold. We give a second, more analytic proof of Theorem 1.3 in Chapter 9.

While gradings on the algebra are discussed in Chapter 3, discussion of gradings on the invariants of bordered 3-manifolds and a graded version of the pairing theorem are deferred to Chapter 10,

We conclude the book proper with a chapter on the torus boundary case, Chapter 11. After reviewing the algebra associated to the torus, we use a computation of the bordered Floer homology for solid tori to give a quick proof of the surgery exact triangle for \widehat{HF} . In this chapter, we also relate this invariant with knot Floer homology. We start by showing how to recapture much of knot Floer homology from the bordered Floer invariants of a knot complement. We then go the other direction, showing that the bordered Floer homology of knot complements in S^3 (with suitable framings) can be computed from knot Floer homology; this is Theorem 11.26.

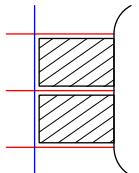
In Appendix A, we give a quick introduction to **[LOT10a]**, which gives a generalization of the present work to the case of three-manifolds with two boundary components. Using an important special case, one can describe the dependence of the bordered invariants on



the parameterization of the boundary. We give explicit answers for this dependence in the case of torus boundary. Using these computations, we are able to extend Theorem 11.26 to arbitrary framings.

Acknowledgements

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CHAPTER 2

\mathcal{A}_∞ structures

In this work, we extensively use of the notion of an \mathcal{A}_∞ module. Although \mathcal{A}_∞ notions, first introduced by Stasheff in the study of H -spaces [Sta63], have become commonplace now in symplectic geometry (see for example [Fuk93, Kon95, Sei02]), they might not be so familiar to low-dimensional topologists. In Sections 2.1 and 2.2, we review the notions of primary importance to us now, in particular sketching the \mathcal{A}_∞ tensor product. Keller has another pleasant exposition of this material in [Kel01].

The type D module $\widehat{CFD}(\mathcal{H})$ defined in Chapter 6 is an ordinary differential graded module, not an \mathcal{A}_∞ module, so the reader may wish to skip this chapter and concentrate on the type D structure at first. Similarly, the type A module $\widehat{CFA}(\mathcal{H})$ of a nice diagram \mathcal{H} (in the technical sense, see Definition 8.1, which in turn is motivated by [SW10]) is an ordinary differential graded module. However, the notion of grading by a non-commutative group (Section 2.5) is used in all cases.

In Section 2.3, we introduce a further algebraic structure which naturally gives rise to a module, which we call a *type D structure*. In Section 2.4, we also give an explicit and smaller construction of the \mathcal{A}_∞ tensor product when one of the two factors is induced from a type D structure. Although type D structures have appeared in various guises elsewhere (see Example 2.19 and Remark 2.25), this tensor product is apparently new. As the name suggests, the type D module of a bordered three-manifold comes from a type D structure.

Finally, in Section 2.5 we introduce gradings of differential graded algebras or \mathcal{A}_∞ algebras with values in a non-commutative group.

2.1. \mathcal{A}_∞ algebras and modules

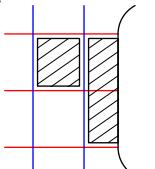
Although we will be working over differential graded (*dg*) algebras (which are less general than \mathcal{A}_∞ algebras) we recall the definition of \mathcal{A}_∞ algebras here as it makes a useful warm-up for defining for \mathcal{A}_∞ modules, which we will need in the sequel.

To avoid complicating matters at first, we will work in the category of \mathbb{Z} -graded complexes over a fixed ground ring \mathbf{k} , which we assume to have characteristic two. For our purposes, this ground ring \mathbf{k} will typically be a direct sum of copies of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. In particular, we consider modules M over \mathbf{k} which are graded by the integers, so that

$$M = \bigoplus_{d \in \mathbb{Z}} M_d.$$

If M is a graded module and $m \in \mathbb{Z}$, we define $M[n]$ to be the graded module defined by $M[n]_d = M_{d-n}$.

Note that since we are working over characteristic two and don't need to worry about signs, the following discussion also carries over readily to the ungraded setting. The modules in our present application are graded, but not by \mathbb{Z} . We return to this point in Section 2.5.



DEFINITION 2.1. Fix a ground ring \mathbf{k} with characteristic two. An \mathcal{A}_∞ algebra \mathcal{A} over \mathbf{k} is a graded \mathbf{k} -module A , equipped with \mathbf{k} -linear multiplication maps

$$\mu_i: A^{\otimes i} \rightarrow A[2 - i]$$

defined for all $i \geq 1$, satisfying the compatibility conditions

$$(2.2) \quad \sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i(a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \underbrace{\mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1})}_{i \text{ strands}} \otimes a_{\ell+j} \otimes \cdots \otimes a_n) = 0$$

for each $n \geq 1$. Here, $A^{\otimes i}$ denotes the \mathbf{k} -module $\overbrace{A \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} A}^i$. Note that throughout this chapter, all tensor products are over \mathbf{k} unless otherwise specified. We use \mathcal{A} for the \mathcal{A}_∞ algebra and A for its underlying \mathbf{k} -module. An \mathcal{A}_∞ algebra \mathcal{A} is *strictly unital* if there is an element $1 \in \mathcal{A}$ with the property that $\mu_2(a, 1) = \mu_2(1, a) = a$ and $\mu_i(a_1, \dots, a_i) = 0$ if $i \neq 2$ and $a_j = 1$ for some j .

In particular, an \mathcal{A}_∞ algebra is a chain complex over \mathbf{k} , with differential μ_1 . In the case where all $\mu_i = 0$ for $i > 2$, an \mathcal{A}_∞ algebra is just a differential graded algebra over \mathbf{k} , with differential μ_1 and (associative) multiplication μ_2 . We have assumed that \mathbf{k} has characteristic two; in the more general case, the compatibility equation must be taken with signs (see, e.g., [Kel01]).

We can think of the algebraic operations graphically as follows. The module $A^{\otimes i}$ is denoted by drawing i parallel, downward-oriented strands. The multiplication operation μ_i is represented by an oriented, planar tree with one vertex, i incoming strands and one outgoing strand. With this convention, the compatibility relation with n inputs can be visualized as follows. Fix a tree T with one vertex, n incoming edges, and one outgoing one. Consider next all the planar trees S (with two vertices) with the property that if we contract one edge in S , we obtain T . Each such tree S represents a composition of multiplication maps. The compatibility condition states that the sum of all the maps gotten by these resolutions S vanishes. For example,

$$\begin{array}{ccccccccccccc} \downarrow & & \downarrow \\ \mu_1 & & \mu_1 & & \mu_1 & & \mu_3 & & \mu_1 & & \mu_2 & & \mu_2 \\ \downarrow & & \downarrow \\ \mu_3 & + & \mu_3 & + & \mu_3 & + & \mu_1 & + & \mu_2 & + & \mu_2 & + & \mu_2 \\ \downarrow & & \downarrow \\ & & & & & & & & & & & & & = 0. \end{array}$$

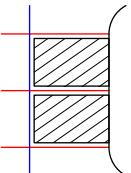
Note that the last two terms are the usual associativity relation, and the remaining terms say that associativity holds only up to a homotopy.

Another way to think of the compatibility conditions (Equation (2.2)) uses the tensor algebra

$$\mathcal{T}^*(A[1]) := \bigoplus_{n=0}^{\infty} A^{\otimes n}[n].$$

The maps μ_i can be combined into a single map

$$\mu: \mathcal{T}^*(A[1]) \rightarrow A[2]$$



with the convention that $\mu_0 = 0$. We can also construct a natural degree 1 endomorphism $\overline{D}: \mathcal{T}^*(A[1]) \rightarrow \mathcal{T}^*(A[1])$ by

$$(2.3) \quad \overline{D}(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n \sum_{\ell=1}^{n-j+1} a_1 \otimes \cdots \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_n.$$

Then the compatibility condition is that

$$\mu \circ \overline{D} = 0$$

or equivalently

$$\overline{D} \circ \overline{D} = 0.$$

These conditions can also be expressed graphically as

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \overline{D} & & \overline{D} \\ \Downarrow & \text{or} & \Downarrow . \\ \mu & & \overline{D} \\ \Downarrow & & \Downarrow \end{array}$$

Here, the doubled arrows indicate elements of $\mathcal{T}^*(A[1])$, while the single arrows indicate elements of A .

At some points later we will want to allow only finitely many nonzero μ_i . We give this condition a name:

DEFINITION 2.4. An \mathcal{A}_∞ algebra $(A, \{\mu_i\}_{i=1}^\infty)$ is *operationally bounded* if $\mu_i = 0$ for i sufficiently large.

DEFINITION 2.5. A (*right*) \mathcal{A}_∞ module \mathcal{M} over \mathcal{A} is a graded \mathbf{k} -module M , equipped with operations

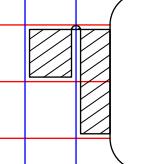
$$m_i: M \otimes A^{\otimes(i-1)} \rightarrow M[2-i],$$

defined for all $i \geq 1$, satisfying the compatibility conditions

$$(2.6) \quad \begin{aligned} 0 &= \sum_{i+j=n+1} m_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

An \mathcal{A}_∞ module \mathcal{M} over a strictly unital \mathcal{A}_∞ algebra is said to be *strictly unital* if for any $\mathbf{x} \in M$, $m_2(\mathbf{x} \otimes 1) = \mathbf{x}$ and $m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ if $i > 2$ and some $a_j = 1$. The module \mathcal{M} is said to be *bounded* if $m_i = 0$ for all sufficiently large i .

Graphically, the module $M \otimes A^{\otimes i-1}$ is represented by i parallel strands, where the leftmost strand is colored with M (while the others are colored by A). The multiplication map m_i is represented by the one-vertex tree with i incoming strands, the leftmost of which is M -colored, and whose output is also M -colored. The compatibility condition can be thought of the same as before, except now we have distinguished the leftmost strands with M . Thus, the compatibility conditions can be thought of as stating the vanishing of maps induced by sums of planar trees, with a fixed number of inputs (and one output).



As before, we can also write this condition in terms of the tensor algebra. The maps m_i (and the μ_i) can be combined to form a degree 1 map $\overline{m}: M \otimes \mathcal{T}^*(A[1]) \rightarrow M \otimes \mathcal{T}^*(A[1])$, using the following analogue of Equation (2.3):

$$(2.7) \quad \begin{aligned} \overline{m}(\mathbf{x}_1 \otimes a_2 \otimes \cdots \otimes a_n) &= \sum_{\ell=1}^n m_\ell(\mathbf{x} \otimes a_2 \otimes \cdots \otimes a_\ell) \otimes \cdots \otimes a_n \\ &+ \sum_{j=2}^n \sum_{\ell=1}^{n-j+1} a_1 \otimes \cdots \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_n. \end{aligned}$$

The \mathcal{A}_∞ relation is equivalent to the condition that $\overline{m} \circ \overline{m} = 0$.

The \mathcal{A}_∞ relation in tensor form can also be expressed graphically. To do this conveniently, let $\Delta: \mathcal{T}^*A \rightarrow \mathcal{T}^*A \otimes \mathcal{T}^*A$ be the canonical comultiplication

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{m=0}^n (a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_n).$$

If we use a dashed line to denote an element of \mathcal{M} then the compatibility condition for an \mathcal{A}_∞ module can be drawn as

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \downarrow \\ m \\ \text{---} \\ \downarrow \\ m \\ \text{---} \\ \downarrow \\ \text{---} \\ \Delta \\ \text{---} \\ \downarrow \\ \text{---} \\ \downarrow \\ \text{---} \\ \text{---} \\ + \\ \text{---} \\ \downarrow \\ m \\ \text{---} \\ \downarrow \\ \text{---} \\ \overline{D} \\ \text{---} \\ \text{---} \\ = \\ 0. \end{array}$$

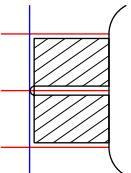
EXAMPLE 2.8. Let \mathcal{M} be a right \mathcal{A}_∞ module over \mathcal{A} . Suppose moreover that, for all $i > 2$, $m_i = 0$ and $\mu_i = 0$. Then not only is \mathcal{A} a differential graded algebra, but also \mathcal{M} is a module over \mathcal{A} (in the traditional sense), equipped with a differential satisfying the Leibniz rule with respect to the algebra action on the module, i.e., a *differential graded module*.

DEFINITION 2.9. Let \mathcal{M} and \mathcal{M}' be strictly unital right \mathcal{A}_∞ modules over a strictly unital \mathcal{A}_∞ algebra. A *strictly unital homomorphism* f of \mathcal{A}_∞ modules, or simply an \mathcal{A}_∞ homomorphism, is a collection of maps

$$f_i: M \otimes A^{\otimes(i-1)} \rightarrow M'[1-i]$$

indexed by $i \geq 1$, satisfying the compatibility conditions

$$\begin{aligned} 0 &= \sum_{i+j=n+1} m'_i(f_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} f_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}) \end{aligned}$$



and the unital condition

$$f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$$

if $i > 1$ and some $a_j = 1$.

We call a strictly unital homomorphism $\{f_i\}$ of \mathcal{A}_∞ modules *bounded* if $f_i = 0$ for i sufficiently large.

The compatibility condition can be formulated in terms of the tensor algebra as follows. Promote the maps $\{f_i\}_{i=1}^\infty$ to a degree 0 map

$$\bar{f}: M \otimes \mathcal{T}^*(A[1]) \rightarrow M' \otimes \mathcal{T}^*(A[1])$$

by the formula:

$$(2.10) \quad \bar{f}(\mathbf{x}_1 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{\ell=0}^n f_{\ell+1}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_\ell) \otimes \cdots \otimes a_n.$$

The \mathcal{A}_∞ relation now is equivalent to the condition that the map \bar{f} induces a chain map from the chain complex $M \otimes \mathcal{T}^*(A[1])$ with its induced differential \bar{m} (Equation (2.7)) to the chain complex $M' \otimes \mathcal{T}^*(A[1])$ with its induced differential \bar{m}' .

The compatibility condition for \mathcal{A}_∞ homomorphisms can be drawn as

$$\begin{array}{c} \text{---} \\ | \\ \Delta \quad + \quad \bar{D} \\ | \quad | \\ m \quad f \quad f \\ \text{---} \\ f \\ \downarrow \end{array} = 0.$$

Here, we use color to distinguish between \mathcal{M} and \mathcal{M}' .

For example, for any \mathcal{A}_∞ module \mathcal{M} , the *identity homomorphism* \mathbb{I} is the map with

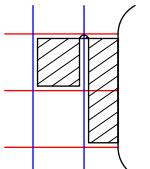
$$\begin{aligned} \mathbb{I}_1(\mathbf{x}) &:= \mathbf{x} \\ \mathbb{I}_i(\mathbf{x} \otimes A^{\otimes i-1}) &:= 0 \quad (i > 0). \end{aligned}$$

If f is an \mathcal{A}_∞ homomorphism from \mathcal{M} to \mathcal{M}' , and g is an \mathcal{A}_∞ homomorphism from \mathcal{M}' to \mathcal{M}'' , we can form their *composite* $g \circ f$, defined by

$$(g \circ f)_n(\mathbf{x} \otimes (a_1 \otimes \cdots \otimes a_{n-1})) := \sum_{i+j=n+1} g_j(f_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_{n-1}).$$

DEFINITION 2.11. Given any collection of maps

$$h_i: M \otimes A^{\otimes i-1} \rightarrow M'[-i]$$



with $h_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ if $i > 1$ and some $a_j = 1$, we can construct a strictly unital \mathcal{A}_∞ homomorphism f by the expression

$$\begin{aligned} f_n(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-1}) &= \\ &\sum_{i+j=n+1} h_i(m_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} m'_i(h_j(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes \cdots \otimes a_{n-1}) \\ &+ \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} h_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

If an \mathcal{A}_∞ homomorphism f can be obtained from a map h in this way, we call f *null homotopic*. Two \mathcal{A}_∞ homomorphisms $f, g: \mathcal{M} \rightarrow \mathcal{M}'$ are *homotopic* if their difference is null homotopic.

Two \mathcal{A}_∞ modules \mathcal{M} and \mathcal{M}' are (\mathcal{A}_∞) *homotopy equivalent* if there are \mathcal{A}_∞ homomorphisms f from \mathcal{M} to \mathcal{M}' and g from \mathcal{M}' to \mathcal{M} such that $f \circ g$ and $g \circ f$ are homotopic to the identity.

Suppose that $f, g: \mathcal{M} \rightarrow \mathcal{M}'$ are homotopic via a homotopy h . Then, we can promote the components of h , $\{h_i\}_{i=1}^\infty$ to a map

$$\bar{h}: M \otimes \mathcal{T}^*(A[1]) \rightarrow M' \otimes \mathcal{T}^*(A[1])$$

following Equation (2.10). The condition that h gives the homotopy between f and g is equivalent to the more familiar condition

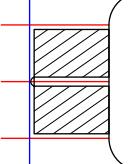
$$\bar{h} \circ \bar{m} + \bar{m}' \circ \bar{h} = \bar{f} + \bar{g}.$$

Graphically, this equation is

There are analogous definitions for left \mathcal{A}_∞ modules, maps, and homotopies.

2.2. \mathcal{A}_∞ tensor products

Consider the category of strictly unital \mathcal{A}_∞ modules over a *dg* algebra A , whose morphism sets are the strictly unital \mathcal{A}_∞ homomorphisms. In this category, the usual naïve tensor product $M \otimes_A N$ of a right (\mathcal{A}_∞) module M and a left module N is not available: the relation $(m \cdot a) \otimes n \sim m \otimes (a \cdot n)$ used in the definition of the tensor product is not transitive in the \mathcal{A}_∞ setting. In fact, even in the original setting of differential graded modules, this tensor product is often not the right one: if M and M' are quasi-isomorphic, then $M \otimes_A N$ and $M' \otimes_A N$ are not necessarily quasi-isomorphic.



To produce a functor which works for \mathcal{A}_∞ modules, we pass instead to the \mathcal{A}_∞ tensor product, a generalization of the derived tensor product.

DEFINITION 2.12. Let \mathcal{A} be an \mathcal{A}_∞ algebra over \mathbf{k} , \mathcal{M} be a right \mathcal{A}_∞ module over \mathcal{A} and \mathcal{N} be a left \mathcal{A}_∞ module over \mathcal{A} . Then their \mathcal{A}_∞ *tensor product* is the chain complex

$$\mathcal{M} \tilde{\otimes}_{\mathcal{A}} \mathcal{N} := M \otimes \mathcal{T}^*(A[1]) \otimes N$$

equipped with the boundary operator

$$\begin{aligned} \partial(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{y}) := & \\ & \sum_{i=1}^{n+1} m_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) \otimes \cdots \otimes a_n \otimes \mathbf{y} \\ & + \sum_{i=1}^n \sum_{\ell=1}^{n-i+1} \mathbf{x} \otimes a_1 \otimes \cdots \otimes \mu_i(a_\ell \otimes \cdots \otimes a_{\ell+i-1}) \otimes \cdots \otimes a_n \otimes \mathbf{y} \\ & + \sum_{i=1}^{n+1} \mathbf{x} \otimes a_1 \otimes \cdots \otimes m_i(a_{n-i+2} \otimes \cdots \otimes a_n \otimes \mathbf{y}). \end{aligned}$$

LEMMA 2.13. *The operator ∂ on $\mathcal{M} \tilde{\otimes} \mathcal{N}$ defined above satisfies $\partial^2 = 0$.*

PROOF. This is a straightforward consequence of the relations defining \mathcal{A}_∞ algebras and modules. \square

An important case is when $\mathcal{N} = \mathcal{A}$. The resulting complex is called the *bar resolution* $\overline{\mathcal{M}}$ of \mathcal{M} . As a \mathbf{k} -module it is

$$\overline{\mathcal{M}} \cong M \otimes \mathcal{T}^*(A[1]) \otimes A \cong (M \otimes \mathcal{T}^+(A[1]))[-1]$$

where $\mathcal{T}^+(A[1]) := \bigoplus_{i=1}^\infty (A[1])^{\otimes i}$ is $\mathcal{T}^*(A[1])$ without the first summand of \mathbf{k} . $\overline{\mathcal{M}}$ has the further structure of a right \mathcal{A}_∞ module over \mathcal{A} , with multiplications $m_1 = \partial$ as defined above and, for $i \geq 2$,

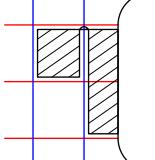
$$\begin{aligned} m_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := & \\ & \sum_{\ell=1}^n \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{n-\ell} \otimes \mu_{i+\ell-1}(a_{n-\ell+1} \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}). \end{aligned}$$

Note the range of summation: the multiplication $\mu_{i+\ell-1}$ is applied to all of the b_j 's and at least one a_j .

The following lemma is again straightforward and standard.

LEMMA 2.14. *$\overline{\mathcal{M}}$ is an \mathcal{A}_∞ module over \mathcal{A} .*

Notice that if \mathcal{A} is an honest differential graded algebra then $\overline{\mathcal{M}}$ is an honest differential graded module, i.e., all of the higher m_i vanish. Furthermore, if \mathcal{M} is also an honest differential graded module, this is the usual bar resolution. If in addition \mathcal{N} is an honest differential graded module, then we could also define $\mathcal{M} \tilde{\otimes} \mathcal{N}$ as $\overline{\mathcal{M}} \otimes_A \mathcal{N}$, the naïve tensor product with the bar resolution, but this is not possible in general.



REMARK 2.15. One way to understand the products on $\overline{\mathcal{M}}$ and Lemma 2.14 is via \mathcal{A}_∞ bimodules. If \mathcal{A} and \mathcal{B} are two \mathcal{A}_∞ algebras, an \mathcal{A} - \mathcal{B} bimodule \mathcal{N} is a \mathbf{k} -module N with maps

$$m_{i,j}: A^{\otimes i} \otimes N \otimes B^{\otimes j} \rightarrow N$$

satisfying a natural version of the compatibility conditions. In particular, \mathcal{A} is an \mathcal{A} - \mathcal{A} bimodule in a natural way, and $\overline{\mathcal{M}}$ is just $\mathcal{M} \tilde{\otimes}_{\mathcal{A}} \mathcal{A}$. A more general version of Lemma 2.14 is that if \mathcal{M} is a \mathcal{A} module and \mathcal{N} is an \mathcal{A} - \mathcal{B} bimodule, then $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ is a right \mathcal{B} module.

PROPOSITION 2.16. *If \mathcal{A} and \mathcal{M} are strictly unital, then $\overline{\mathcal{M}}$ is homotopy equivalent to \mathcal{M} .*

PROOF. We define an \mathcal{A}_∞ module map ϕ from $\overline{\mathcal{M}}$ to \mathcal{M} by

$$\phi_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := m_{i+n}(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1}).$$

Similarly, we define an \mathcal{A}_∞ module map ψ from \mathcal{M} to $\overline{\mathcal{M}}$ by

$$\psi_i(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1}) := \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1.$$

Also define maps for a homotopy h from $\overline{\mathcal{M}}$ to $\overline{\mathcal{M}}$ by

$$h_i((\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n) \otimes b_1 \otimes \cdots \otimes b_{i-1}) := \mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1.$$

It is straightforward to verify that $\phi \circ \psi$ is the identity map, while $\psi \circ \phi$ is homotopic, via h , to the identity map. \square

PROPOSITION 2.17. *An \mathcal{A}_∞ map $f: \mathcal{M} \rightarrow \mathcal{M}'$ induces a chain map $f \tilde{\otimes} \mathbb{I}: \mathcal{M} \tilde{\otimes} \mathcal{N} \rightarrow \mathcal{M}' \tilde{\otimes} \mathcal{N}$. If f is null homotopic, so is $f \tilde{\otimes} \mathbb{I}_N$. In particular, if \mathcal{M}_1 and \mathcal{M}_2 are homotopy equivalent right \mathcal{A}_∞ modules and \mathcal{N}_1 and \mathcal{N}_2 are homotopy equivalent left \mathcal{A}_∞ modules, then $\mathcal{M}_1 \tilde{\otimes} \mathcal{N}_1$ is homotopy equivalent to $\mathcal{M}_2 \tilde{\otimes} \mathcal{N}_2$.*

PROOF. For f a collection of maps $f_i: M \otimes A^{\otimes(i-1)} \rightarrow M[c-i]$ for a fixed $c \in \{0, 1\}$, define $f \tilde{\otimes} \mathbb{I}_N$ by

$$(f \tilde{\otimes} \mathbb{I}_N)(\mathbf{x} \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{y}) := \sum_{i=1}^{n+1} f_i(\mathbf{x} \otimes a_1 \cdots \otimes a_{i-1}) \otimes a_i \otimes \cdots \otimes a_n \otimes \mathbf{y}.$$

If f is an \mathcal{A}_∞ map, so is $f \tilde{\otimes} \mathbb{I}_N$, and if h is a null homotopy of f , then $h \tilde{\otimes} \mathbb{I}_N$ is a null homotopy of $f \tilde{\otimes} \mathbb{I}_N$. The same construction works, *mutatis mutandis*, for chain maps and homotopies on the right. \square

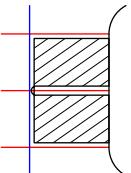
2.3. Type D structures

There is a smaller model for the derived tensor product in the case where one of the two factors has a special type: where it is the \mathcal{A}_∞ module associated to a type *D* structure (Definitions 2.18). In this section, we introduce these type *D* structures and develop their basic properties.

We first specialize to the case where \mathcal{A} is a differential graded algebra, which is all that we need for our present purposes.

DEFINITION 2.18. Fix a *dg* algebra \mathcal{A} . Let N be a graded \mathbf{k} -module, equipped with a map

$$\delta_N^1: N \rightarrow (A \otimes N)[1],$$



satisfying the compatibility condition that

$$(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathbb{I}_N) \circ \delta^1: N \rightarrow A \otimes N$$

vanishes. (When the module N is fixed, we drop the subscript N from the notation for δ^1 .) We call the pair (N, δ_N^1) a *type D structure over A with base ring \mathbf{k}* . Let $(N_1, \delta_{N_1}^1)$ and $(N_2, \delta_{N_2}^1)$ be two type D structures. A \mathbf{k} -module map $\psi^1: N_1 \rightarrow A \otimes N_2$ is a *D-structure homomorphism* if

$$(\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes \psi^1) \circ \delta_{N_1}^1 + (\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes \delta_{N_2}^1) \circ \psi^1 + (\mu_1 \otimes \mathbb{I}_{N_2}) \circ \psi^1 = 0.$$

Given type D homomorphisms $\phi^1: N_1 \rightarrow A \otimes N_2$ and $\psi^1: N_2 \rightarrow A \otimes N_3$, their *composite* $(\phi^1 \circ \psi^1)$ is defined by $(\mu_2 \otimes \mathbb{I}_{N_3}) \circ (\mathbb{I}_A \otimes \psi^1) \circ \phi^1$. A *homotopy* h between two D -structure homomorphisms ψ^1 and ϕ^1 from $(N_1, \delta_{N_1}^1)$ to $(N_2, \delta_{N_2}^1)$ is a \mathbf{k} -module map $h: N_1 \rightarrow (A \otimes N_2)[-1]$ with

$$(\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes h) \circ \delta_{N_1}^1 + (\mu_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes \delta_{N_2}^1) \circ h + (\mu_1 \otimes \mathbb{I}_{N_2}) \circ h = \psi^1 - \phi^1.$$

EXAMPLE 2.19. Assume A is a *dg* algebra over $\mathbf{k} = \mathbb{F}_2$. Let M be a *dg* module which is free as an A -module. For a fixed basis of M over A , let X denote the \mathbf{k} -span of that basis. The restriction of the boundary operator to X gives a map

$$\delta^1: X \rightarrow \mathcal{A} \otimes X = M$$

which satisfies the relations of a type D structure. Maps between such modules are D -structure homomorphisms.

LEMMA 2.20. *If (N, δ^1) is a type D structure, then $A \otimes N$ can be given the structure of a left A module, with*

$$\begin{aligned} m_1(a \otimes \mathbf{y}) &:= [(\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta^1) + \mu_1 \otimes \mathbb{I}_N](a \otimes \mathbf{y}) \\ m_2(a_1 \otimes (a \otimes \mathbf{y})) &:= \mu_2(a_1 \otimes a) \otimes \mathbf{y}. \end{aligned}$$

Moreover, if $\psi^1: N_1 \rightarrow A \otimes N_2$ is a D -structure homomorphism, then there is an induced map of differential graded modules from $A \otimes N_1$ to $A \otimes N_2$, defined by

$$(a \otimes \mathbf{y}) \mapsto (m_2 \otimes \mathbb{I}_{N_2}) \circ (\mathbb{I}_A \otimes \psi^1)$$

Similarly, a homotopy between two type D -structure homomorphisms induces a chain homotopy between the associated chain maps.

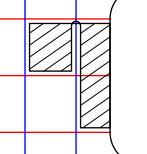
PROOF. The proof is straightforward. □

Given a type D structure (N, δ^1) over a *dg* algebra A , we denote the A -module from Lemma 2.20 by \mathcal{N} .

Note that the structure equation for a type D structure (N, δ^1) is equivalent to the condition that m_1 on \mathcal{N} is, indeed, a differential.

As a partial converse to Lemma 2.20, we have the following lemma. Recall that if N_1 and N_2 are two *dg* modules over a *dg* algebra, then a *dg* homomorphism is a chain map $\psi: N_1 \rightarrow N_2$ which commutes with the algebra action. Equivalently, it is an \mathcal{A}_∞ homomorphism $\{\psi_i\}$ whose components ϕ_i vanish for all $i > 1$.

LEMMA 2.21. *Let N_1 and N_2 be two type D structures over a *dg* algebra A , and \mathcal{N}_1 and \mathcal{N}_2 be their associated *dg* modules. The correspondence from Lemma 2.20 gives an isomorphism between the space of type D homomorphisms from N_1 to N_2 with the space of *dg**



homomorphisms from \mathcal{N}_1 to \mathcal{N}_2 . Moreover, two type D homomorphisms are homotopic if and only if the corresponding dg homomorphisms are homotopic by an A -equivariant homotopy.

PROOF. Given $\psi: \mathcal{N}_1 \rightarrow \mathcal{N}_2$, we can realize ψ as the homomorphism associated to the map $\psi^1: N_1 \rightarrow \mathcal{A} \otimes N_2$ determined by $\psi^1(\mathbf{x}) = \phi(1 \otimes \mathbf{x})$. Similarly, if $h: \mathcal{N}_1 \rightarrow \mathcal{N}_2$ is an A -equivariant homotopy (i.e., $h(a \cdot x) = a \cdot h(x)$ for all $a \in A$ and $x \in \mathcal{N}_1$, then we can define $h^1(\mathbf{x}) = h(1 \otimes \mathbf{x})$ for any $\mathbf{x} \in N_1$. It is straightforward to verify that these maps satisfy the required properties. \square

DEFINITION 2.22. Let A be an \mathcal{A}_∞ algebra and N a graded left \mathbf{k} -module. A map $\delta^1: N \rightarrow (A \otimes N)[1]$ can be iterated to construct maps

$$\delta^k: N \rightarrow (A^{\otimes k} \otimes N)[k]$$

inductively by

$$\begin{aligned}\delta^0 &= \mathbb{I}_N \\ \delta^i &= (\mathbb{I}_{A^{\otimes(i-1)}} \otimes \delta^1) \circ \delta^{i-1}.\end{aligned}$$

By construction, these maps satisfy the basic equation

$$(\mathbb{I}_{A^{\otimes j}} \otimes \delta^i) \circ \delta^j = \delta^{i+j}$$

for all $i, j \geq 0$, or graphically

$$\begin{array}{ccc} \begin{array}{c} \downarrow \\ \delta \\ \swarrow \quad \searrow \\ \Delta \\ \downarrow \end{array} & = & \begin{array}{c} \downarrow \\ \delta \\ \swarrow \quad \searrow \\ \Delta \\ \downarrow \end{array} \end{array}$$

We say that δ^1 is *bounded* if for all $\mathbf{x} \in N$, there is an integer n so that for all $i \geq n$, $\delta^i(\mathbf{x}) = 0$.

We call a type D structure *bounded* if its structure map δ^1 is.

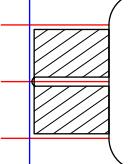
For a bounded map δ^1 , we can think of the δ^k as fitting together to form a map

$$\begin{aligned}\delta: N &\rightarrow \mathcal{T}^*(A[1]) \otimes N, \\ \delta(\mathbf{x}) &:= \sum_{k=0}^{\infty} \delta^k(\mathbf{x}).\end{aligned}$$

If δ^1 is not bounded then, instead, the δ^k fit together to form a map to the completed tensor algebra:

$$\begin{aligned}\delta: N &\rightarrow \overline{\mathcal{T}}^*(A[1]) \otimes N := \left(\prod_{i=0}^{\infty} (A[1])^{\otimes i} \right) \otimes N, \\ \delta(\mathbf{x}) &:= \prod_{k=0}^{\infty} \delta^k(\mathbf{x}).\end{aligned}$$

We can also give a tensor algebra version of Definition 2.18. This also allows us to generalize it to the \mathcal{A}_∞ setting.



DEFINITION 2.23. Fix an \mathcal{A}_∞ algebra \mathcal{A} and a pair (N, δ^1) as above. Assume that either (N, δ^1) is bounded or \mathcal{A} is operationally bounded. Then we say (N, δ^1) defines a type D structure if

$$(\mu \otimes \mathbb{I}_N) \circ \delta = 0,$$

or equivalently

$$(\overline{D} \otimes \mathbb{I}_N) \circ \delta = 0,$$

or graphically

$$\begin{array}{c} | \\ \downarrow \\ \delta \\ \swarrow \quad \searrow \\ \overline{D} \\ \swarrow \quad \searrow \\ | \\ \downarrow \end{array} = 0.$$

For two type D structures $(N_1, \delta_{N_1}^1)$ and $(N_2, \delta_{N_2}^1)$, a map $\psi^1: N_1 \rightarrow A \otimes N_2$ can be used to construct maps

$$\begin{aligned} \psi^k: N_1 &\rightarrow (A^{\otimes k} \otimes N_2)[k-1] \\ \psi^k(\mathbf{x}) &:= \sum_{i+j=k-1} (\mathbb{I}_{A^{\otimes(i+1)}} \otimes \delta_{N_2}^j) \circ (\mathbb{I}_{A^{\otimes i}} \otimes \psi^1) \circ \delta_{N_1}^i. \end{aligned}$$

The map ψ^1 is *bounded* if $\psi^k = 0$ for k sufficiently large. (This is automatic if N_1 and N_2 are bounded.) We can define

$$\psi(\mathbf{x}) := \sum_{k=1}^{\infty} \psi^k(\mathbf{x}),$$

which maps to $\mathcal{T}^*(A[1]) \otimes N$ if ψ^1 is bounded and $\overline{\mathcal{T}}^*(A[1]) \otimes N$ in general. Assuming either ψ^1 is bounded or \mathcal{A} is operationally bounded, we say ψ is a *D -structure homomorphism* if

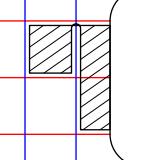
$$(\mu \otimes \mathbb{I}_{N_2}) \circ \psi = 0.$$

Similarly, ψ is *homotopic to ϕ* if

$$(\mu \otimes \mathbb{I}_{N_2}) \circ h = \psi - \phi$$

for h constructed from some map $h^1: N \rightarrow (A \otimes N_2)[-1]$ in the analogous way (and with h^1 bounded if \mathcal{A} is not operationally bounded).

The main complication with type D structures over an \mathcal{A}_∞ algebra is that the category of type D structures over an \mathcal{A}_∞ algebra is an \mathcal{A}_∞ category, with composition given graphically



as follows:

$$\circ_k(f_1, \dots, f_k) = \begin{array}{c} \vdots \\ \delta^{M_0} \\ \downarrow \\ (f_1)^1 \\ \downarrow \\ \delta^{M_1} \\ \parallel \downarrow \\ (f_2)^1 \\ \downarrow \\ \vdots \\ \parallel \downarrow \\ (f_k)^1 \\ \downarrow \\ \delta^{M_k} \\ \parallel \downarrow \\ \mu \\ \downarrow \end{array}$$

Note that the condition $\circ_1(f) = 0$ is exactly the condition of being a type D homomorphism.

Specializing Definition 2.23 to the case of a dg algebra, we obtain Definition 2.18.

If $(N_1, \delta_{N_1}^1)$ and $(N_2, \delta_{N_2}^1)$ are both bounded then any homomorphism $\psi^1: N_1 \rightarrow A \otimes N_2$ between them is automatically bounded.

There is a generalization of Lemma 2.20.

LEMMA 2.24. *Let \mathcal{A} be an operationally bounded \mathcal{A}_∞ algebra, and (N, δ^1) a type D structure over it. Then $A \otimes N$ can be given the structure of a left \mathcal{A}_∞ module \mathcal{N} , with*

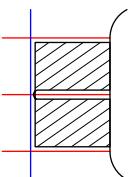
$$m_i := \sum_{k=0}^{\infty} (\mu_{i+k} \otimes \mathbb{I}_N) \circ (\mathbb{I}_{A^{\otimes i}} \otimes \delta^k).$$

Moreover, if $\psi^1: N_1 \rightarrow A \otimes N_2$ is a D -structure homomorphism, then there is an induced \mathcal{A}_∞ homomorphism from \mathcal{N}_1 to \mathcal{N}_2 , whose i^{th} component is given by

$$\sum_{k=1}^{\infty} (\mu_{i+k+1} \otimes \mathbb{I}_{N_1}) \circ \psi^k.$$

Similarly, a homotopy between two D -structure homomorphisms induces a homotopy between the associated \mathcal{A}_∞ maps.

As in the case for differential graded algebras, we will denote the \mathcal{A}_∞ module associated to a type D structure (N, δ^1) constructed in Lemma 2.20 by \mathcal{N} .



REMARK 2.25. Type D structures have appeared elsewhere in various guises. For example, let \mathcal{C} be a small dg category. Define an algebra $\mathcal{A}(\mathcal{C})$ by

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a,b \in \text{Ob}(\mathcal{C})} \text{Mor}(a, b),$$

with multiplication given by composition if defined and zero otherwise. Then twisted complexes (see [BK90]) over the split closure of \mathcal{C} are the same as type D structures over $\mathcal{A}(\mathcal{C})$. (Note that this differs slightly from the definition in [Sei08a] in that we have no condition about being lower-triangular.)

Alternatively, type D structures can be viewed as differential comodules over the bar resolution, thought of as a coalgebra. See [LH03, Kel].

2.4. Another model for the \mathcal{A}_∞ tensor product

Having introduced type D structures, we now introduce a pairing \boxtimes between \mathcal{A}_∞ modules and type D structures. When defined, this pairing is homotopy equivalent to the derived tensor product with the \mathcal{A}_∞ module associated to the type D structure, but is smaller; for instance, it is finite-dimensional if the modules are. As we will see in Chapter 9, this new tensor product also matches better with the geometry and helps give a proof of the pairing theorem.

Throughout this section, we restrict attention to the case where $\mathcal{A}_\infty = (A, \{\mu_i\}_{i=1}^\infty)$ is operationally bounded. Note that this includes the case of dg algebras.

DEFINITION 2.26. Let \mathcal{M} be a right \mathcal{A}_∞ module over \mathcal{A} and (N, δ^1) a type D structure. Suppose moreover that either \mathcal{M} is a bounded \mathcal{A}_∞ module or (N, δ^1) is a bounded type D structure. Then, we can form the \mathbf{k} -module $\mathcal{M} \boxtimes N = M \otimes_{\mathbf{k}} N$, equipped with the endomorphism

$$\partial^\boxtimes(\mathbf{x} \otimes \mathbf{y}) := \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N)(\mathbf{x} \otimes \delta^k(\mathbf{y})).$$

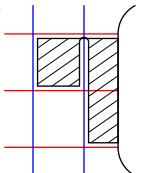
We call $(M \otimes N, \partial^\boxtimes)$ the *box tensor product* of \mathcal{M} and (N, δ^1) .

Graphically, the differential on the box tensor product is given by:

$$\partial^\boxtimes = \begin{array}{c} | \\ \downarrow \\ \delta \\ \swarrow \searrow \\ m \\ \downarrow \end{array}.$$

EXAMPLE 2.27. If \mathcal{A} is a dg algebra and \mathcal{M} is a differential module, then $\mathcal{M} \boxtimes N$ is the ordinary tensor product $\mathcal{M} \otimes_{\mathcal{A}} N$.

EXAMPLE 2.28. Let \mathcal{A} be a finite-dimensional dg algebra over a ring $\mathbf{k} = \mathbb{F}_2 e_1 \oplus \cdots \oplus \mathbb{F}_2 e_m$, a direct sum of copies of \mathbb{F}_2 . Choose a \mathbb{F}_2 -basis $\{a_1, \dots, a_n\}$ for \mathcal{A} which is homogeneous with respect to the action by \mathbf{k} , in the sense that for each a_i , there are elements e_{ℓ_i} and e_{r_i}



with $a_i = e_{\ell_i} a_i e_{r_i}$. Given a type D structure (N, δ^1) over \mathcal{A} , we have maps $D_i: e_{\ell_i} N \rightarrow e_{r_i} N$ for $i = 1, \dots, n$ uniquely determined by the formula

$$\delta^1 = \sum_i a_i \otimes D_i.$$

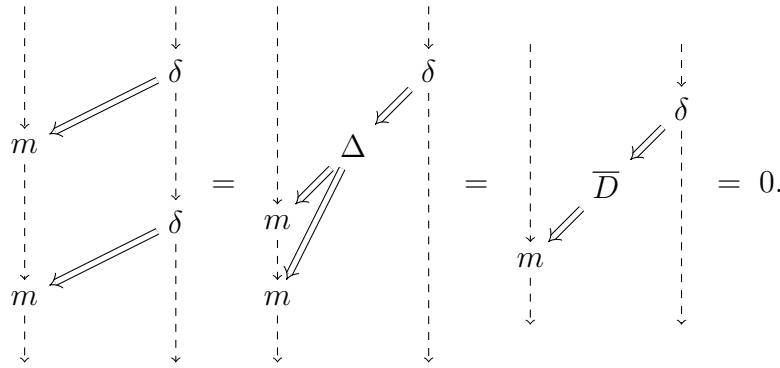
If M is a right \mathcal{A}_∞ module over A then we can write the differential ∂^\boxtimes more explicitly as

$$(2.29) \quad \partial^\boxtimes(\mathbf{x} \otimes \mathbf{y}) = \sum m_{k+1}(\mathbf{x}, a_{i_1}, \dots, a_{i_k}) \otimes (D_{i_k} \circ \dots \circ D_{i_1})(\mathbf{y}),$$

where the sum is taken over all sequences i_1, \dots, i_k of elements in $\{1, \dots, n\}$ (including the empty sequence with $k = 0$).

LEMMA 2.30. *Let \mathcal{M} be an \mathcal{A}_∞ module and N a type D structure. Assume that either M or N is bounded. Then the \mathbf{k} -module $M \otimes N$ equipped with the endomorphism ∂^\boxtimes is a chain complex.*

PROOF. The proof is easiest to understand in the graphical notation:

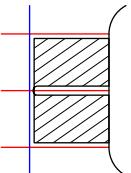


The first equality uses the definition of δ (or that δ is a comodule map), the second the structure equation for an \mathcal{A}_∞ module, and the third the structure equation for a type D structure. \square

EXAMPLE 2.31. Fix an \mathcal{A}_∞ algebra \mathcal{A} and a left type D structure N over \mathcal{A} . We can view \mathcal{A} also as a right \mathcal{A}_∞ module over \mathcal{A} . In this case, the chain complex $\mathcal{A} \boxtimes N$ constructed above coincides with the chain complex underlying the \mathcal{A}_∞ module associated to (N, δ^1) from Lemma 2.20 (in the case where \mathcal{A} is a dg algebra) or 2.24 (in the general case).

LEMMA 2.32. *Let $f: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a homomorphism of \mathcal{A}_∞ modules and $\phi^1: N_1 \rightarrow A \otimes N_2$ a homomorphism of type D structures. Then:*

- (1) *If either \mathcal{M}_1 , \mathcal{M}_2 and f are bounded or $(N_1, \delta_{N_1}^1)$ is bounded then f induces a chain map $f \boxtimes \mathbb{I}_{N_1}: \mathcal{M}_1 \boxtimes N_1 \rightarrow \mathcal{M}_2 \boxtimes N_1$.*
- (2) *If either $(N_1, \delta_{N_1}^1)$ and $(N_2, \delta_{N_2}^1)$ are bounded or \mathcal{M}_1 is bounded then ϕ^1 induces a chain map $\mathbb{I}_{\mathcal{M}_1} \boxtimes \phi^1: \mathcal{M}_1 \boxtimes N_1 \rightarrow \mathcal{M}_1 \boxtimes N_2$.*
- (3) *Suppose $f': \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is another homomorphism of \mathcal{A}_∞ modules which is homotopic to f . Suppose further that either $(N_1, \delta_{N_1}^1)$ is bounded or \mathcal{M}_1 , f , f' and the homotopy between them are all bounded. Then $f \boxtimes \mathbb{I}_{N_1}$ is homotopic to $f' \boxtimes \mathbb{I}_{N_1}$.*
- (4) *Suppose $\tilde{\psi}^1: N_1 \rightarrow A \otimes N_2$ is another homomorphism of type D structures which is homotopic to ψ^1 . Suppose further that either \mathcal{M}_1 is bounded or ψ^1 , $\tilde{\psi}^1$ and the homotopy between them are all bounded. Then $\mathbb{I}_{\mathcal{M}_1} \boxtimes \psi^1$ and $\mathbb{I}_{\mathcal{M}_1} \boxtimes \tilde{\psi}^1$ are homotopic.*



- (5) Under boundedness conditions so that all of the \boxtimes products are defined, given a third map $g: \mathcal{M}_2 \rightarrow \mathcal{M}_3$, then $(g \circ f) \boxtimes \mathbb{I}_{N_1}$ is homotopic to $(g \boxtimes \mathbb{I}_{N_1}) \circ (f \boxtimes \mathbb{I}_{N_1})$.
- (6) Under boundedness conditions so that all of the \boxtimes products are defined, given a third map $\psi^1: (N_2, \delta_{N_2}^1) \rightarrow (N_3, \delta_{N_3}^1)$, then $\mathbb{I}_{\mathcal{M}_1} \boxtimes (\psi^1 \circ \phi^1)$ is homotopic to $(\mathbb{I}_{\mathcal{M}_1} \boxtimes \psi^1) \circ (\mathbb{I}_{\mathcal{M}_1} \boxtimes \phi^1)$.
- (7) Under boundedness conditions so that all of the \boxtimes products are defined, $(f \boxtimes \mathbb{I}_{N_1}) \circ (\mathbb{I}_{\mathcal{M}_1} \boxtimes \phi^1)$ is homotopic to $(\mathbb{I}_{\mathcal{M}_1} \boxtimes \phi^1) \circ (f \boxtimes \mathbb{I}_{N_1})$.

PROOF. Given f , we define the induced map $f \otimes \mathbb{I}_N$ by

$$(f \otimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) := \sum_{k=0}^{\infty} (f_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta^k(\mathbf{y})),$$

where f_k are the components of f . Similarly, given a homotopy h with components h_k from ϕ to ϕ' , we define a homotopy $h \boxtimes \mathbb{I}_N$ from $f \boxtimes \mathbb{I}_N$ to $f' \boxtimes \mathbb{I}_N$ by

$$(h \boxtimes \mathbb{I}_N)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (h_{k+1} \otimes \mathbb{I}_N) \circ (\mathbf{x} \otimes \delta^k(\mathbf{y})).$$

Graphically, the maps $f \boxtimes \mathbb{I}_N$ and $h \boxtimes \mathbb{I}_N$ are given by

$$\begin{array}{ccc} \begin{array}{c} \vdots \\ \downarrow \\ \text{---} \end{array} & \quad \text{and} \quad & \begin{array}{c} \vdots \\ \downarrow \\ \text{---} \end{array} \\ \begin{array}{c} \delta \\ \swarrow \\ f \\ \downarrow \\ \vdots \end{array} & & \begin{array}{c} \delta \\ \swarrow \\ h \\ \downarrow \\ \vdots \end{array} \end{array}$$

Given ϕ^1 , we define $\mathbb{I}_{\mathcal{M}} \boxtimes \phi^1$ by

$$(\mathbb{I}_{\mathcal{M}} \boxtimes \phi^1)(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=1}^{\infty} (m_{k+1} \otimes \mathbb{I}_{N'}) \circ (\mathbf{x} \otimes \phi^k(\mathbf{y})),$$

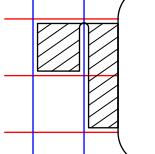
with ϕ^k constructed from the map $\phi^1: N \rightarrow A \otimes N'$ as in Definition 2.23. A homotopy h from ϕ^1 to $\tilde{\phi}^1$ is promoted to a homotopy from $\mathbb{I}_{\mathcal{M}} \boxtimes \phi^1$ to $\mathbb{I}_{\mathcal{M}} \boxtimes \tilde{\phi}^1$ by a similar formula.

Verification of the properties of these maps is a straightforward modification of the argument that $\partial^\boxtimes \circ \partial^\boxtimes = 0$. \square

EXAMPLE 2.33. Continuing from Example 2.31, if $\psi^1: N_1 \rightarrow A \otimes N_2$ is a homomorphism of type D structures, then $\mathbb{I}_{\mathcal{A}} \boxtimes \psi^1: \mathcal{A} \boxtimes N_1 \rightarrow \mathcal{A} \boxtimes N_2$ is the chain map underlying the \mathcal{A}_∞ homomorphism from \mathcal{N}_1 to \mathcal{N}_2 from Lemmas 2.20 or 2.24.

PROPOSITION 2.34. Let \mathcal{M} be an \mathcal{A}_∞ module over an operationally bounded \mathcal{A}_∞ algebra \mathcal{A} and (N, δ^1) be a type D structure. Suppose that either \mathcal{M} is a bounded \mathcal{A}_∞ module or (N, δ^1) is bounded. In the former case, suppose moreover that (N, δ^1) is homotopy equivalent to a bounded type D structure. Then the product $\mathcal{M} \boxtimes N$ is homotopy equivalent to the \mathcal{A}_∞ tensor product $\mathcal{M} \widetilde{\otimes} N$.

PROOF. The boundedness assumption implies that $\mathcal{M} \boxtimes N$ is defined. Moreover, since \boxtimes respects homotopy equivalences (by Lemma 2.32), we may assume that N is bounded. By Proposition 2.16, \mathcal{M} is \mathcal{A}_∞ homotopy equivalent to $\overline{\mathcal{M}}$, and hence by Lemma 2.32,



$\mathcal{M} \boxtimes N \simeq \overline{\mathcal{M}} \boxtimes N$. But this is the same complex with the same boundary operator as $\mathcal{M} \tilde{\otimes} \mathcal{N}$. For instance, as \mathbf{k} -modules we have

$$\overline{\mathcal{M}} \boxtimes \mathcal{N} \cong (M \otimes \mathcal{T}^+(A[1])[-1] \otimes N \cong M \otimes \mathcal{T}^*(A[1]) \otimes (A \otimes N) \cong \mathcal{M} \tilde{\otimes} \mathcal{N}.$$

(Recall that the underlying space of \mathcal{N} is $A \otimes N$. Also, the \mathcal{A}_∞ homotopy equivalence between \mathcal{M} and $\overline{\mathcal{M}}$ is not bounded, so we need the boundedness assumption on N .) \square

The following consequence of Lemma 2.32 will be used in Section 11.2:

DEFINITION 2.35. Let (N_1, δ_1) , (N_2, δ_2) , and (N_3, δ_3) be three type D structures over some fixed dg algebra A , and let $\phi^1: N_1 \rightarrow A \otimes N_2$ and $\psi^1: N_2 \rightarrow A \otimes N_3$ be two homomorphisms of type D structures. We say these form a *short exact sequence* if the induced modules \mathcal{N}_1 , \mathcal{N}_2 , \mathcal{N}_3 equipped with homomorphisms associated to between them $\mathbb{I}_A \otimes \phi^1$ and $\mathbb{I}_A \otimes \psi^1$ fit into a short exact sequence of differential graded A -modules. We abbreviate this, writing

$$0 \longrightarrow N_1 \xrightarrow{\phi^1} N_2 \xrightarrow{\psi^1} N_3 \longrightarrow 0.$$

PROPOSITION 2.36. *If $0 \longrightarrow N_1 \xrightarrow{\phi^1} N_2 \xrightarrow{\psi^1} N_3 \longrightarrow 0$ is a short exact sequence of type D structures over a dg algebra and \mathcal{M} is a bounded \mathcal{A}_∞ module, then there is an exact sequence in homology*

$$H_*(\mathcal{M} \boxtimes N_1) \longrightarrow H_*(\mathcal{M} \boxtimes N_2) \longrightarrow H_*(\mathcal{M} \boxtimes N_3) \longrightarrow H_*(\mathcal{M} \boxtimes N_1)[-1].$$

PROOF. The short exact sequence can be viewed as a 3-step filtered type D structure N . Since $A \boxtimes N_3$ is a projective A -module, this short exact sequence is split. The splitting can be viewed as a null-homotopy of the 3-step filtered complex.

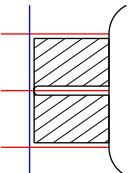
Next, form $\mathcal{M} \boxtimes N$ to get a three-step filtered complex, whose associated graded complex is $\bigoplus_{i=1}^3 \mathcal{M} \boxtimes N_i$. (It is perhaps worth noting that the differential of an element in $\mathcal{M} \boxtimes N_1 \subset \mathcal{M} \boxtimes N$ can have a component in $\mathcal{M} \boxtimes N_3$. In fact, the $\mathcal{M} \boxtimes N_3$ component of the differential from $\mathcal{M} \boxtimes N_1$ is precisely the null-homotopy of $(\mathbb{I}_\mathcal{M} \boxtimes \psi^1) \circ (\mathbb{I}_\mathcal{M} \boxtimes \phi^1)$.) By Lemma 2.32, this complex $\mathcal{M} \boxtimes N$ is homotopy equivalent to the trivial complex; in particular, $\mathcal{M} \boxtimes N$ is acyclic. It follows that there is a long exact sequence on the homology groups of the terms in its associated graded complex. (This latter point is a small generalization of the fact that a short exact sequence of chain complexes induces a long exact sequence in homology: it is a degenerate case of the Leray spectral sequence.) \square

REMARK 2.37. Proposition 2.36 is an explicit form of the statement that $\mathcal{M} \boxtimes \cdot$ is a triangulated functor. Since $\mathcal{M} \boxtimes N \simeq \mathcal{M} \tilde{\otimes} \mathcal{N}$ in a natural way, this follows from the fact that the \mathcal{A}_∞ tensor product is a triangulated functor; over a dg algebra, this, in turn, follows from the fact that the derived tensor product is a triangulated functor.

2.5. Gradings by non-commutative groups

In earlier sections, we considered the familiar case of \mathbb{Z} -graded modules. Here we record basic definitions of \mathcal{A}_∞ algebras (including differential algebras) that are graded over a possibly non-commutative group G . The definitions are straightforward generalizations of standard definitions. As before, we assume that our ground ring \mathbf{k} has characteristic 2 so that we need not worry about signs.

We start with ordinary differential graded algebras.



DEFINITION 2.38. Let (G, λ) be a pair of a group G (written multiplicatively) with a distinguished element λ in the center of G . A *differential algebra graded by (G, λ)* is, first of all, a differential algebra A with a grading gr of A by G (as a set), i.e., a decomposition $A = \bigoplus_{g \in G} A_g$. We say an element a in A_g is *homogeneous of degree g* and write $\text{gr}(a) = g$. For homogeneous elements a and b , we further require the grading to be

- compatible with the product, i.e., $\text{gr}(a \cdot b) = \text{gr}(a) \text{gr}(b)$, and
- compatible with the differential, i.e., $\text{gr}(\partial a) = \lambda^{-1} \text{gr}(a)$.

For an ordinary \mathbb{Z} -grading, we would take $(G, \lambda) = (\mathbb{Z}, 1)$. (We choose the differential to lower the grading to end up with homology rather than cohomology.) We require λ to be central because otherwise the identity

$$\partial(ab) = \partial a \cdot b + a \cdot \partial b$$

would not be homogeneous in general. The map $k \mapsto \lambda^k$ is a homomorphism from \mathbb{Z} to G , but even if λ is of infinite order in G , there may not be a map from G to \mathbb{Z} taking λ to 1, so we do not in general get an ordinary \mathbb{Z} -grading.

REMARK 2.39. When working over a field whose characteristic is not 2, we need a $\mathbb{Z}/2$ grading (i.e., a homomorphism from G to $\mathbb{Z}/2$ taking λ to 1) in order to, for instance, specify the signs in the Leibniz relation. We will not consider this further here.

There is a natural extension of this definition to \mathcal{A}_∞ algebras. In general, if V and W are G -graded \mathbf{k} -modules, we can turn $V \otimes W$ into a G -graded \mathbf{k} -module by setting

$$(2.40) \quad \text{gr}(v \otimes w) := \text{gr}(v) \text{gr}(w)$$

for homogeneous elements v and w of V and W , respectively. If V is a G -graded module and $n \in \mathbb{Z}$, define $V[n]$ to be the graded module with

$$(2.41) \quad V[n]_g := V_{\lambda^{-n}g}.$$

Since λ is central, we have $(V[n]) \otimes W \cong (V \otimes W)[n] \cong V \otimes (W[n])$. Now the previous definitions of the multiplications maps m_i and their compatibility conditions carry over as written.

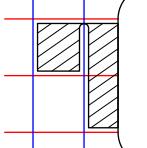
We now turn to differential modules.

DEFINITION 2.42. Let A be a differential algebra graded by G , as in Definition 2.38. Let S be a set with a right G action. A *right differential A -module graded by S* , or, simply, a right S -graded module, is a right differential A -module M with a grading gr of M by S (as a set) so that, for homogeneous elements $a \in A$ and $x \in M$,

- $\text{gr}(xa) = \text{gr}(x) \text{gr}(a)$ and
- $\text{gr}(\partial x) = \text{gr}(x)\lambda^{-1}$.

If the action of G on S is transitive, S is determined by the stabilizer G_s of any point $s \in S$, and we can identify S with $G_s \backslash G$, the space of right cosets of G_s .

More generally, if V is graded by S and W is graded by G , then Equation (2.40) gives a natural grading of $V \otimes W$ by S and Equation (2.41) defines a shift functor; then the previous definitions for \mathcal{A}_∞ modules, homomorphisms, and homotopies carry through. (For homomorphisms and homotopies, the two modules should be graded by the same set.) For example, for \mathcal{A}_∞ modules M over a graded \mathcal{A}_∞ algebra \mathcal{A} , if x is a homogeneous element of M



and a_1, \dots, a_ℓ are homogeneous elements of \mathcal{A} , then $y = m_{\ell+1}(x, a_1, \dots, a_\ell)$ is homogeneous of degree

$$(2.43) \quad \text{gr}(y) = \lambda^{\ell-1} \text{gr}(x) \cdot \text{gr}(a_1) \cdots \text{gr}(a_\ell).$$

Left modules are defined similarly, but with gradings by a set T with a left action by G .

Suppose the pair (G, λ) is such that λ has infinite order. Then there is a partial order on G given by $g < h$ if $h = g\lambda^n$ for some $n > 0$. So, it sometimes makes sense to say that $\text{gr}(a) < \text{gr}(b)$ for homogeneous elements a, b in a group-graded algebra. Similarly, for a G -set S , if the subgroup $\langle \lambda \rangle$ acts freely on S then there is a partial order on S given by $s < t$ if $t = \lambda^n s$ for some $n > 0$. Thus, for a set-graded module M , it sometimes makes sense to write $\text{gr}(x) < \text{gr}(y)$ for homogeneous elements $x, y \in M$.

Finally we turn to tensor products.

DEFINITION 2.44. Let A be a differential algebra graded by (G, λ) , let M be a right A -module graded by a right G -set S , and let N be a left A -module graded by a left G -set T . Define $S \times_G T$ by

$$S \times_G T := (S \times T) / \{ (s, gt) \sim (sg, t) \mid s \in S, t \in T, g \in G \}.$$

Define a $(S \times_G T)$ grading on $M \otimes_{\mathbf{k}} N$ by

$$(2.45) \quad \text{gr}(m \otimes n) := [\text{gr}(m) \times \text{gr}(n)].$$

By inspection, this descends to the naïve tensor product

$$M \otimes_A N := M \otimes_{\mathbf{k}} N / \langle m \otimes an - ma \otimes n \mid m \in M, n \in N, a \in A \rangle.$$

Furthermore, there is a natural action of λ (or any element in the center of G) on $S \times_G T$ via

$$\lambda \cdot [s \times t] := [s\lambda \times t] = [s \times \lambda t].$$

(This definition fails to descend to the quotient for non-central elements in place of λ .) Again by inspection, the boundary operator

$$\partial(m \otimes_A n) := (\partial m) \otimes_A n + m \otimes_A (\partial n)$$

acts by λ^{-1} on the gradings. Thus $M \otimes_A N$ is a chain complex with a grading by $S \times_G T$, which is a set with a \mathbb{Z} -action.

Using Equation (2.45) and the same shift functor as before, we can extend the definitions of both versions of the tensor product, chain maps and homotopies to this setting as well.

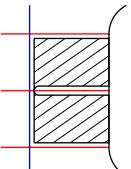
We now see a genuinely new feature of dg algebras where the grading is non-commutative: even if λ has infinite order in G , S , and T , it may not have infinite order on $S \times_G T$. We illustrate this with a simple, and pertinent, example.

EXAMPLE 2.46. Let H be a variant of the Heisenberg group:

$$H = \langle \alpha, \beta, \lambda \mid \alpha\beta = \lambda^2\beta\alpha, \lambda \text{ is central} \rangle.$$

(The usual Heisenberg group over the integers has λ instead of λ^2 in the first relation; with the exponent of 2, the group supports a $\mathbb{Z}/2$ grading.) Let L be the subgroup of H generated by α . Then λ has infinite order on H and on H/L and $L \backslash H$, the spaces of left and right cosets of L . However, the reduced product is

$$\begin{aligned} (L \backslash H) \times_H (H/L) &\simeq L \backslash H/L \\ &= \{ L\beta^k \lambda^l L \mid k, l \in \mathbb{Z} \}. \end{aligned}$$

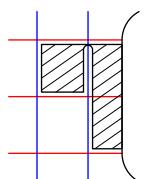


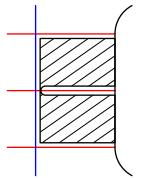
These double cosets are not all distinct. In particular,

$$\begin{aligned} L\beta^k \lambda^l L &= L\alpha\beta^k \lambda^l L \\ &= L\beta^k \lambda^{l+2k} \alpha L \\ &= L\beta^k \lambda^{l+2k} L. \end{aligned}$$

Equalities like this generate all equalities between the double cosets. Thus the order of λ on $L\beta^k \lambda^l L$ is $2k$, which is not free when $k \neq 0$.

This example is exactly what we will see when we do 0-surgery on a knot in S^3 . (See Section 10.6.)





CHAPTER 3

The algebra associated to a pointed matched circle

The aim of the present chapter is to define the algebra $\mathcal{A}(\mathcal{Z})$ associated to the boundary of a three-manifold, parametrized by the pointed matched circle \mathcal{Z} . In Section 3.1, we start by defining the strands algebra $\mathcal{A}(n)$, which is a simplified analogue of this algebra. The desired algebra $\mathcal{A}(\mathcal{Z})$, introduced in Section 3.2, is described as a subalgebra of $\mathcal{A}(n)$. In Section 3.3, we explain the grading on $\mathcal{A}(\mathcal{Z})$.

Although the algebras are defined combinatorially, the motivation for their definitions comes from the behavior of holomorphic curves. The reader might find it helpful to consult Examples 6.13–6.15 and 7.13–7.16, as well as [LOT09], in conjunction with this chapter.

Our algebras, by construction, have characteristic 2. With a little more work, the algebras could be lifted to give a construction over \mathbb{Z} ; for the algebra $\mathcal{A}(n)$ this is done in [Kho10]. Indeed, with considerably more work, one should be able to define bordered Floer homology over \mathbb{Z} .

3.1. The strands algebra $\mathcal{A}(n, k)$

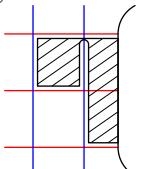
We will describe an algebra $\mathcal{A}(n, k)$ in three different ways, which will be useful for different purposes.

3.1.1. Algebraic definition. For non-negative integers n and k with $n \geq k$, let $\mathcal{A}(n, k)$ be the \mathbb{F}_2 -vector space generated by partial permutations (S, T, ϕ) , where S and T are k -element subsets of the set $\{1, \dots, n\}$ (hereafter denoted $[n]$) and ϕ is a bijection from S to T , with $i \leq \phi(i)$ for all $i \in S$. For brevity, we will adopt a 2-line notation for these generators, so that, for instance, $\langle \begin{smallmatrix} 1 & 2 & 4 \\ 5 & 3 & 4 \end{smallmatrix} \rangle$ denotes the algebra element $(\{1, 2, 4\}, \{3, 4, 5\}, \{1 \mapsto 5, 2 \mapsto 3, 4 \mapsto 4\})$ inside $\mathcal{A}(3, 5)$.

For $a = (S, T, \phi)$ a generator of $\mathcal{A}(n, k)$, let $\text{inv}(a)$ or $\text{inv}(\phi)$ be the number of *inversions* of ϕ : the number of pairs $i, j \in S$ with $i < j$ and $\phi(j) < \phi(i)$. We make $\mathcal{A}(n, k)$ into an algebra using composition. For $a, b \in \mathcal{A}(n, k)$, with $a = (S, T, \phi)$, $b = (T, U, \psi)$, and $\text{inv}(\psi \circ \phi) = \text{inv}(\phi) + \text{inv}(\psi)$, define $a \cdot b$ to be $(S, U, \psi \circ \phi)$. In all other cases, (i.e., if the range of ϕ is not the domain of ψ , or if $\text{inv}(\psi \circ \phi) \neq \text{inv}(\phi) + \text{inv}(\psi)$), the product of two generators is zero.

There is one idempotent $I(S)$ in $\mathcal{A}(n, k)$ for each k -element subset S of $[n]$; it is (S, S, \mathbb{I}_S) . It is easy to see that the $I(S)$ are all the minimal idempotents in $\mathcal{A}(n, k)$. (*Minimal idempotents* are the idempotents that can not be further decomposed as sums of orthogonal idempotents). Let $\mathcal{I}(n, k) \subset \mathcal{A}(n, k)$ be the subalgebra generated by the idempotents.

We also define a (co)differential on $\mathcal{A}(n, k)$. For $a = (S, T, \phi)$ in $\mathcal{A}(n, k)$, let $\text{Inv}(\phi)$ be the set counted in $\text{inv}(\phi)$: the set $\{(i, j) \mid i < j, \phi(j) < \phi(i)\}$. For $\sigma \in \text{Inv}(\phi)$, let ϕ_σ be



defined by

$$\phi_\sigma(k) = \begin{cases} \phi(j) & k = i \\ \phi(i) & k = j \\ \phi(k) & \text{otherwise.} \end{cases}$$

Note that $\text{inv}(\phi_\sigma) < \text{inv}(\phi)$, since the inversion (i, j) was removed from $\text{Inv}(\phi)$ and none are added. It is also possible that $\text{inv}(\phi_\sigma) < \text{inv}(\phi) - 1$, since more inversions might be removed. Define

$$\partial a := \sum_{\substack{\sigma \in \text{Inv}(\phi) \\ \text{inv}(\phi_\sigma) = \text{inv}(\phi) - 1}} (S, T, \phi_\sigma).$$

LEMMA 3.1. *The set $\mathcal{A}(n, k)$, equipped with a product induced by composition of partial permutations and with the above-defined endomorphism ∂ , is a differential algebra.*

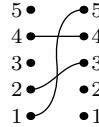
We defer the proof until Section 3.1.2, in which we develop a convenient geometrical language for $\mathcal{A}(n, k)$.

DEFINITION 3.2. The differential algebra $\mathcal{A}(n, k)$ is called the *strands algebra with k strands and n places*. The direct sum $\mathcal{A}(n) = \bigoplus_k \mathcal{A}(n, k)$ is called the *strands algebra with n places*.

REMARK 3.3. The function inv provides a \mathbb{Z} -grading on $\mathcal{A}(n)$.

REMARK 3.4. Setting the product in $\mathcal{A}(n, k)$ to be zero if the number of inversions in the composition $\psi \circ \phi$ decreases is reminiscent of the nilCoxeter algebra [FS94, Kho01].

3.1.2. Geometric definition. There is an alternate description of the algebra $\mathcal{A}(n)$, in terms of *strands diagrams*. We represent a triple (S, T, ϕ) by a diagram with n dots on both the left and right (numbered from the bottom up), with a set of strands representing ϕ connecting the subsets S on the left and T on the right. The condition that $i \leq \phi(i)$ means that strands stay horizontal or move up when read from left to right. For instance, the diagram

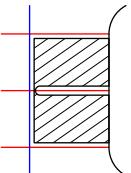
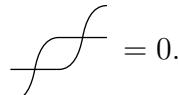


represents the algebra element $\langle \frac{1}{5} \frac{2}{3} \frac{4}{4} \rangle$.

It is easy to see that $\text{inv}(a)$ is the minimal number of crossings in any strand diagram. Accordingly, we may view basis elements of $\mathcal{A}(n)$ as strand diagrams as above, with no pair of strands crossing more than once, considered up to homotopy. The summand $\mathcal{A}(n, k) \subset \mathcal{A}(n)$ corresponds to strand diagrams with exactly k strands.

We can define the product on $\mathcal{A}(n, k)$ in these pictures using horizontal juxtaposition. The product $a \cdot b$ of two algebra elements is 0 if the right side of a does not match the left side of b ; otherwise, we place the two diagrams next to each other and join them together. If in this juxtaposition two strands cross each other twice, we set the product to be 0:

$$(3.5) \quad \text{Diagram showing two strands crossing twice, resulting in } 0.$$



This corresponds to the condition that, with $a = (T, U, \phi)$ and $b = (S, T, \psi)$, we set $a \cdot b = 0$ when $\text{inv}(\phi \circ \psi) \neq \text{inv}(\phi) + \text{inv}(\psi)$. If there are no double crossings then the juxtaposition is the product. For instance, the algebra element above can be factored as

$$\begin{array}{ccc}
 5\bullet & \bullet 5 & 5\bullet \\
 4\bullet \text{---} \bullet 4 & 4\bullet \text{---} \bullet 4 & 4\bullet \text{---} \bullet 4 \\
 3\bullet & \bullet 3 & 3\bullet \\
 2\bullet \text{---} \bullet 2 & 2\bullet \text{---} \bullet 2 & 2\bullet \text{---} \bullet 2 \\
 1\bullet & \bullet 1 & 1\bullet \\
 \end{array} = \begin{array}{c}
 5\bullet \\
 4\bullet \\
 3\bullet \\
 2\bullet \\
 1\bullet
 \end{array} \quad \begin{array}{c}
 5\bullet \\
 4\bullet \\
 3\bullet \\
 2\bullet \\
 1\bullet
 \end{array}$$

The boundary operator ∂a also has a natural graphical description. For any crossing in a strand diagram, we can consider *smoothing* it:

$$\rightarrow \rightarrow$$

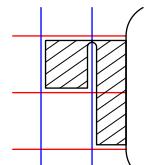
The boundary of a strand diagram is the sum over all ways of smoothing one crossing of the diagram, where we remove those diagrams from the sum for which there are two strands which cross each other twice (compare Equation (3.5)). For instance, we have

$$(3.6) \quad \partial \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

It is straightforward to check that this agrees with the earlier description.

PROOF OF LEMMA 3.1. Consider the larger algebra $\overline{\mathcal{A}}(n, k)$ generated over \mathbb{F}_2 by strands diagrams but where we do not set double crossings to 0. (We also do not allow isotoping away double crossings.) It is obvious that $\overline{\mathcal{A}}(n, k)$ is a differential algebra, i.e., that the product is associative and the differential satisfies the Leibniz rule and $\partial^2 = 0$. Let $\overline{\mathcal{A}}_d(n, k) \subset \overline{\mathcal{A}}(n, k)$ be the sub-vector space of $\mathcal{A}(n, k)$ generated by all strands diagrams with at least one double crossing. We claim that $\overline{\mathcal{A}}_d(n, k)$ is a differential ideal in $\overline{\mathcal{A}}(n, k)$; since $\mathcal{A}(n, k) \cong \overline{\mathcal{A}}(n, k)/\overline{\mathcal{A}}_d(n, k)$, the result follows. Clearly, $\overline{\mathcal{A}}(n, k)$ is an ideal; and if $s \in \overline{\mathcal{A}}_d(n, k)$ is a strand diagram with at least two double crossings then $\partial(s) \in \overline{\mathcal{A}}_d(n, k)$. If s is a strand diagram with a single double crossing, there are two terms in $\partial(s)$ which do not lie in $\overline{\mathcal{A}}_d(n, k)$, but these terms cancel. \square

REMARK 3.7. In combinatorial terms, the terms in the boundary of an algebra element a correspond to the edges in the Hasse diagram of the Bruhat order on the symmetric group that start from the permutation corresponding to a . More precisely, the *Bruhat order* on the symmetric group may be defined by setting $w > w'$ if, when we write $w = s_1 \dots s_r$ as a reduced product of adjacent transpositions, we can delete some of the s_i to obtain a product representing w' . This turns out not to depend on the reduced expression for w that we picked. (See, e.g., [BB05, Section 2.2].) If we represent a reduced expression for w by drawing strands crossing each other at each adjacent transposition, then deleting an s_i corresponds to smoothing a crossing. The *Hasse diagram* of a partially ordered set is the graph connecting two elements w, w' if one covers the other, i.e., if $w > w'$ and there is no other element w'' with $w > w'' > w'$. In the case of Bruhat order on the symmetric group, this is equivalent to $w > w'$ and $\text{inv}(w) = \text{inv}(w') + 1$ (see [BB05, Theorem 2.2.6], *inter alia*), just as in $\mathcal{A}(n, k)$.



3.1.3. Reeb chords. There is one more description of $\mathcal{A}(n, k)$, relating it to the Reeb chords we will see from the moduli spaces. Fix an oriented circle Z with a set of n distinct points $\mathbf{a} = \{a_1, \dots, a_n\} \subset Z$, together with a basepoint $z \in Z \setminus \mathbf{a}$. We view Z as a contact 1-manifold, and \mathbf{a} as a Legendrian submanifold of Z . A *Reeb chord* ρ in $(Z \setminus z, \mathbf{a})$ is an immersed (and consequently embedded) arc in $Z \setminus z$ with endpoints in \mathbf{a} , with orientation induced by the orientation on Z . The initial endpoint of ρ is denoted ρ^- and the final endpoint ρ^+ . A set $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_j\}$ of j Reeb chords is said to be *consistent* if $\boldsymbol{\rho}^- := \{\rho_1^-, \dots, \rho_j^-\}$ and $\boldsymbol{\rho}^+ := \{\rho_1^+, \dots, \rho_j^+\}$ are both sets with j elements (i.e., no two ρ_i share initial or final endpoints). We adopt a two-line notation for consistent sets of Reeb chords so that, for instance, $\left\{ \begin{smallmatrix} 1 & 2 \\ 5 & 3 \end{smallmatrix} \right\}$ is the set of chords $\{[1, 5], [2, 3]\}$.

DEFINITION 3.8. Let $\boldsymbol{\rho}$ be a consistent set of Reeb chords. The *strands algebra element associated to $\boldsymbol{\rho}$* , denoted $a_0(\boldsymbol{\rho}) \in \mathcal{A}(n)$, is defined by

$$a_0(\boldsymbol{\rho}) := \sum_{\{S \mid S \cap (\boldsymbol{\rho}^- \cup \boldsymbol{\rho}^+) = \emptyset\}} (S \cup \boldsymbol{\rho}^-, S \cup \boldsymbol{\rho}^+, \phi_S)$$

where $\phi_S|_S = \mathbb{I}$ and $\phi_S(\rho_i^-) = \rho_i^+$. Geometrically, this corresponds to taking the sum over all ways of consistently adding horizontal strands to the set of strands connecting each ρ_i^- to ρ_i^+ . For convenience, define $a_0(\boldsymbol{\rho})$ to be 0 if $\boldsymbol{\rho}$ is not consistent.

The strands algebra $\mathcal{A}(n)$ is generated as an algebra by the elements $a_0(\boldsymbol{\rho})$ and idempotents $I(S)$, with $S \subset \mathbf{a}$. In fact, the basis for $\mathcal{A}(n)$ over \mathbb{F}_2 that we constructed in Section 3.1.1 are the non-zero elements of the form $I(S)a_0(\boldsymbol{\rho})$, where $S \subset \mathbf{a}$ is a k -element subset, and $\boldsymbol{\rho}$ is a consistent set of Reeb chords.

We give a description of the product on $\mathcal{A}(n, k)$ in terms of this basis, after introducing two definitions.

DEFINITION 3.9. A pair of Reeb chords (ρ, σ) is said to be *interleaved* if $\rho^- < \sigma^- < \rho^+ < \sigma^+$. They are said to be *nested* if $\rho^- < \sigma^- < \sigma^+ < \rho^+$. An unordered pair of Reeb chords is said to be interleaved (respectively nested) if either possible ordering is interleaved (respectively nested).

DEFINITION 3.10. If ρ and σ are Reeb chords such that $\rho^+ = \sigma^-$, we say that ρ and σ *abut* and define their *join* $\rho \uplus \sigma$ to be the Reeb chord $[\rho^-, \sigma^+]$ obtained by concatenating the two chords.

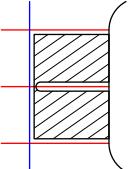
Note that abutting is not a symmetric relation.

DEFINITION 3.11. For two consistent sets of Reeb chords $\boldsymbol{\rho}, \boldsymbol{\sigma}$, define $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ to be obtained from the union $\boldsymbol{\rho} \cup \boldsymbol{\sigma}$ by replacing every abutting pair $\rho_i \in \boldsymbol{\rho}$ and $\sigma_j \in \boldsymbol{\sigma}$ by their join $\rho_i \uplus \sigma_j$. If $\rho_i \in \boldsymbol{\rho}$, let

$$\rho_i^{++} := \begin{cases} \sigma_j^+ & \rho_i, \sigma_j \text{ abut for some } \sigma_j \in \boldsymbol{\sigma} \\ \rho_i^+ & \text{otherwise.} \end{cases}$$

Similarly define σ_j^{--} for $\sigma_j \in \boldsymbol{\sigma}$.

For $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ consistent sets of Reeb chords, we say that $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are *composable* if $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ is consistent and has no double crossings; precisely, if $\boldsymbol{\rho} \uplus \boldsymbol{\sigma}$ is consistent and there is no pair $\rho_i \in \boldsymbol{\rho}$ and $\sigma_j \in \boldsymbol{\sigma}$ so that $\rho_i^- < \sigma_j^{--}$, $\sigma_j^- < \rho_i^+$, and $\rho_i^{++} < \sigma_j^+$. Finally, we iterate this notion to sequences of consistent sets of Reeb chords $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$: we say such a sequence is *composable* if for all $i = 1, \dots, n - 1$, the sets $\bigcup_{j=1}^i \boldsymbol{\rho}_j$ and $\boldsymbol{\rho}_{i+1}$ are composable.



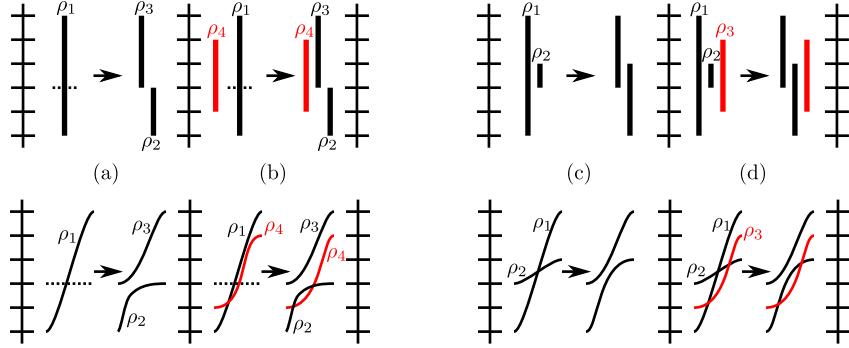


FIGURE 3.1. **Splittings and shuffles of Reeb chords.** (a) A splitting. (b) A weak splitting which is not a splitting. In both cases, the position at which the splitting occurs is marked with a dotted line. (c) A shuffle. (d) A weak shuffle which is not a shuffle. In all cases, the chords are shown on the top line, and the associated strands diagrams on the bottom line.

LEMMA 3.12. *Given consistent sets of Reeb chords ρ and σ and a subset $S \subset [4k]$ with $I(S)a_0(\rho) \neq 0$, we have*

$$I(S)a_0(\rho)a_0(\sigma) = \begin{cases} I(S)a_0(\rho \uplus \sigma) & \rho \text{ and } \sigma \text{ composable} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. This follows directly from the definitions. \square

We now describe the differential on $\mathcal{A}(n)$ in terms of Reeb chords. Observe that there are two ways a term in the boundary operator can appear, depending on whether we smooth a crossing between two moving strands or between a moving strand and a horizontal strand.

DEFINITION 3.13. For a Reeb chord ρ_1 , a *splitting of ρ_1* is a pair of abutting chords (ρ_2, ρ_3) so that $\rho_1 = \rho_2 \uplus \rho_3$. For a set ρ of Reeb chords, a *weak splitting of ρ* is a set obtained from ρ by replacing a chord $\rho_1 \in \rho$ with two chords $\{\rho_2, \rho_3\}$, where (ρ_2, ρ_3) is a splitting of ρ_1 . A *splitting of ρ* is a weak splitting of ρ , replacing ρ_1 with $\{\rho_2, \rho_3\}$, which is consistent and does not introduce double crossings, i.e., there is no chord $\rho_4 \in \rho$ nested in ρ_1 and with $\rho_4^- < \rho_2^- = \rho_3^- < \rho_4^+$. See Figure 3.1.

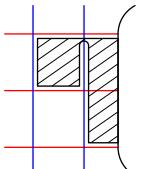
For a set of Reeb chords ρ , a *weak shuffle of ρ* is a set obtained by replacing a nested pair (ρ_1, ρ_2) of chords in ρ with the corresponding interleaved pair $([\rho_1^-, \rho_2^+], [\rho_2^-, \rho_1^+])$. A *shuffle of ρ* is a weak shuffle with nested pair (ρ_1, ρ_2) that does not introduce double crossings, i.e., there is no chord $\rho_3 \in \rho$ with ρ_2 nested in ρ_3 and ρ_3 nested in ρ_1 . Again, see Figure 3.1.

LEMMA 3.14. *We have*

$$\partial a_0(\rho) = \sum_{\substack{\rho' \text{ a splitting} \\ \text{of } \rho}} a_0(\rho') + \sum_{\substack{\rho' \text{ a shuffle} \\ \text{of } \rho}} a_0(\rho').$$

PROOF. This follows directly from the definitions. \square

See also Lemma 5.78 for alternate characterizations of compositability, splittings, and shuffles in terms of a grading on the algebra.



REMARK 3.15. In the type A module, these two types of terms in the boundary, splittings and shuffles, appear respectively as join curve ends and odd shuffle curve ends, as defined in Definition 5.58; see the proof of Proposition 7.12.

3.2. Matched circles and their algebras

We now turn to the algebra associated to a parametrized surface. We represent our boundary surfaces by *matched circles*.

DEFINITION 3.16. A *matched circle* is a triple (Z, \mathbf{a}, M) of an oriented circle Z , $4k$ points $\mathbf{a} = \{a_1, \dots, a_{4k}\}$ in Z (as in Section 3.1.3), and a *matching*: a 2-to-1 function $M: \mathbf{a} \rightarrow [2k]$. We require that performing surgery along the $2k$ pairs of points in Z (viewed as 0-spheres) yields a single circle, rather than several disjoint circles.

A *pointed matched circle* \mathcal{Z} is a matched circle together with a basepoint $z \in Z \setminus \mathbf{a}$.

DEFINITION 3.17. To each matched circle \mathcal{Z} we associate an oriented surface $F(\mathcal{Z})$, or just F , of genus k , by taking a disk with boundary Z (respecting the orientation), attaching oriented 2-dimensional 1-handles along the pairs specified by M , and filling the resulting boundary circle with another disk. Alternately, F may be obtained from a polygon with $4k$ sides, with the points \mathbf{a} in the middle of the sides, by gluing pairs of sides specified by M in an orientation-preserving way. The last disk glued in is a neighborhood of the unique resulting vertex. We call $F(\mathcal{Z})$ the *surface associated to \mathcal{Z}* .

If $-\mathcal{Z}$ denotes the orientation reverse of \mathcal{Z} (i.e., the same data except with the orientation on Z reversed) then $F(-\mathcal{Z})$ is the orientation reverse $-F(\mathcal{Z})$ of $F(\mathcal{Z})$, canonically.

In the other direction from Definition 3.17, we have the following:

DEFINITION 3.18. Given a closed, orientable surface F of genus k , let $f: F \rightarrow \mathbb{R}$ be a Morse function with a unique index 0 critical point and a unique index 2 critical point, and such that $f(p_i) = 1$ for all index 1 critical points p_1, \dots, p_{2k} . Suppose that $f^{-1}(3/2)$ is non-empty. Fix also a Riemannian metric g on F . Let $Z = f^{-1}(3/2)$, and let $\mathbf{a} \subset Z$ denote the ascending spheres of the index 1 critical points. Define $M(a_j) = i$ if a_j is in the ascending sphere of p_i . Choose also a point $z \in Z \setminus \mathbf{a}$. Then $\mathcal{Z} = (Z, \mathbf{a}, M, z)$ is a pointed matched circle, and $F(\mathcal{Z}) \cong F$. We say that (f, g) is *compatible with the pointed matched circle \mathcal{Z}* .

The points \mathbf{a} in a matched circle (Z, \mathbf{a}, M) inherit a cyclic ordering from the orientation of Z ; in a pointed matched circle, this cyclic ordering is lifted to an honest ordering, which we denote by \lessdot .

EXAMPLE 3.19. Let Z be a circle with four points, which we number $0, 1, 2, 3$ in their cyclic ordering, and let $M: \{1, \dots, 4\} \rightarrow \{1, 2\}$ be characterized by $M(i) \equiv i \pmod{2}$. Then Z is a matched circle which represents the torus. Indeed, it is the unique matched circle which represents the torus.

EXAMPLE 3.20. Let Z be a circle with eight marked points, which we number $\{0, \dots, 7\}$ in their cyclic ordering. Let $M: \{1, \dots, 8\} \rightarrow \{1, \dots, 4\}$ be characterized by $M(i) \equiv i \pmod{4}$. Then Z is a matched circle which represents a surface of genus 2, called the *antipodal* pointed matched circle. A different matched circle representing the same surface can be obtained by forming the connected sum (in the obvious way) of two pointed matched circles for the torus; this is called the *split* pointed matched circle. See Figure 3.2.

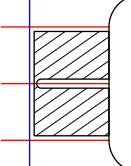




FIGURE 3.2. **Two pointed matched circles for the genus 2 surface.**
Left: the split pointed matched circle. Right: the antipodal pointed matched circle.

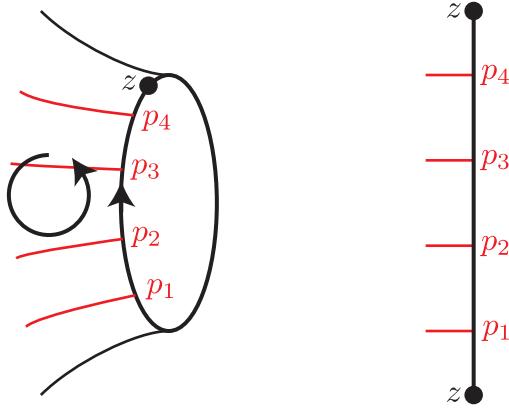


FIGURE 3.3. **Orientation of a pointed matched circle.** Left: the orientation of a Heegaard surface, and the induced orientation of its boundary. Right: the induced ordering of \mathbf{a} : in this figure, $p_1 \lessdot p_2 \lessdot p_3 \lessdot p_4$.

DEFINITION 3.21. Fix a pointed matched circle \mathcal{Z} . For each subset \mathbf{s} of $[2k]$ we have one idempotent $I(\mathbf{s})$ in the strands algebra $\mathcal{A}(4k)$, given by

$$(3.22) \quad I(\mathbf{s}) := \sum_{\substack{S \subset [2k] \\ S \text{ a section of } M \text{ over } \mathbf{s}}} I(S),$$

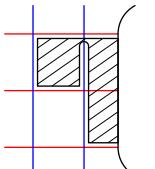
where by a *section* of M over \mathbf{s} we mean a subset of S that maps bijectively to \mathbf{s} via M . That is, each element of \mathbf{s} defines a pair of points in \mathbf{a} ; $I(\mathbf{s})$ is the sum over the $2^{|\mathbf{s}|}$ minimal idempotents in $\mathcal{A}(4k, |\mathbf{s}|)$ that pick one point out of each of these pairs. The *ring of idempotents associated to the pointed matched circle* is the algebra $\mathcal{I}(\mathcal{Z})$ generated by the elements $I(\mathbf{s})$ associated to all the subsets $\mathbf{s} \subset [2k]$. Define

$$\mathbf{I} := \sum_{\mathbf{s} \subset [2k]} I(\mathbf{s})$$

to be the unit in $\mathcal{I}(\mathcal{Z})$.

We will now define $\mathcal{A}(\mathcal{Z})$, our main algebra of interest.

DEFINITION 3.23. For a given pointed matched circle \mathcal{Z} , the *algebra associated to \mathcal{Z}* , denoted $\mathcal{A}(\mathcal{Z})$, is the subalgebra of $\bigoplus_{i=0}^{2k} \mathcal{A}(4k, i)$ generated (as an algebra) by $\mathcal{I}(\mathcal{Z})$ and by $\mathbf{I}_{a_0}(\boldsymbol{\rho})\mathbf{I}$ for every set of Reeb chords $\boldsymbol{\rho}$. The *algebra element associated to $\boldsymbol{\rho}$* , denoted $a(\boldsymbol{\rho})$, is the projection $\mathbf{I} \cdot a_0(\boldsymbol{\rho}) \cdot \mathbf{I}$ of $a_0(\boldsymbol{\rho})$ to $\mathcal{A}(\mathcal{Z})$.



Set

$$\begin{aligned}\mathcal{A}(\mathcal{Z}, i) &= \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(4k, k+i) \\ \mathcal{I}(\mathcal{Z}, i) &= \mathcal{I}(\mathcal{Z}) \cap \mathcal{A}(4k, k+i),\end{aligned}$$

and call $\mathcal{A}(\mathcal{Z}, i)$ the part of $\mathcal{A}(\mathcal{Z})$ with *weight* i .

It is easy to verify that $\mathcal{A}(\mathcal{Z})$ is closed under multiplication and ∂ .

In Chapter 2, we considered algebras (and modules) defined over a ground ring \mathbf{k} . From now on, we will view $\mathcal{A}(\mathcal{Z})$ as defined over the ground ring $\mathbf{k} = \mathcal{I}(\mathcal{Z})$.

The algebra $\mathcal{A}(\mathcal{Z})$ decomposes (as a direct sum of differential graded algebras) according to the weights of the algebra elements,

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^k \mathcal{A}(\mathcal{Z}, i).$$

There is a convenient basis for $\mathcal{A}(\mathcal{Z})$ over \mathbb{F}_2 , consisting of the non-zero elements of the form $I(\mathbf{s})a(\boldsymbol{\rho})$, where $\mathbf{s} \subset [2k]$ and $\boldsymbol{\rho}$ is a consistent set of Reeb chords. In two-line notation, we express these *basic generators* $I(\mathbf{s})a(\boldsymbol{\rho})$ as

$$\left[\begin{smallmatrix} x_1 & \cdots & x_k & z_1 & \cdots & z_l \\ y_1 & \cdots & y_k \end{smallmatrix} \right] := I(\{M(x_1), \dots, M(x_k), M(z_1), \dots, M(z_l)\})a(\{ \begin{smallmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{smallmatrix} \}).$$

Graphically, we denote the matching M by drawing dotted arcs on the left and right of a strand diagram. We draw the strands in an element $\mathcal{A}(\mathcal{Z})$ as we did for elements of $\mathcal{A}(n)$ in Section 3.1.2, except that we draw horizontal strands (which, if we expand out the sum for $I(\mathbf{s})$, appear in half the terms) with dashed lines. For instance, if \mathcal{Z} is the split genus-two pointed matched circle, one basic generator is

$$(3.24) \quad \text{Diagram showing a split genus-two pointed matched circle with strands labeled by dots. It is equated to the sum of two diagrams: one with strands labeled } \begin{smallmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{smallmatrix} \text{ and one with strands labeled } \begin{smallmatrix} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{smallmatrix}.$$

which, in two-line notation, can be denoted $\left[\begin{smallmatrix} 1 & 2 \\ 6 \end{smallmatrix} \right]$ or $\left[\begin{smallmatrix} 1 & 4 \\ 6 \end{smallmatrix} \right]$. In particular, note that in a non-zero basic element of $\mathcal{A}(\mathcal{Z})$, no initial (respectively terminal) ends of strands can be matched with each other.

EXAMPLE 3.25. Let \mathcal{Z} be the unique pointed matched circle representing the torus. The algebra $\mathcal{A}(\mathcal{Z}, 0)$ is described in detail in Section 11.1. The algebra $\mathcal{A}(\mathcal{Z}, -1)$ is isomorphic to \mathbb{F}_2 , with generator $(\emptyset, \emptyset, \phi)$ where ϕ is the unique bijection from \emptyset to \emptyset . The algebra $\mathcal{A}(\mathcal{Z}, 1)$ has \mathbb{F}_2 -basis

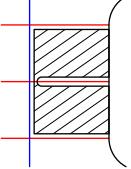
$$\left[\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} 2 & 1 \\ 4 \end{smallmatrix} \right], \left[\begin{smallmatrix} 2 & 3 \\ 3 & 4 \end{smallmatrix} \right], \left[\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix} \right], \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix} \right].$$

The non-trivial differentials are given by

$$\partial \left[\begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix} \right], \quad \partial \left[\begin{smallmatrix} 2 & 1 \\ 4 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 2 & 3 \\ 3 & 4 \end{smallmatrix} \right], \quad \partial \left[\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix} \right].$$

In particular, the homology $H_*(\mathcal{A}(\mathcal{Z}, 1))$ is \mathbb{F}_2 , generated by the idempotent. Therefore, the algebra $\mathcal{A}(\mathcal{Z}, 1)$ is formal (quasi-isomorphic to its homology).

It is not a coincidence that $H_*(\mathcal{A}(\mathcal{Z}, 1)) \cong H_*(\mathcal{A}(\mathcal{Z}, -1))$. In general, for a pointed matched circle \mathcal{Z} for a surface of genus k , $\mathcal{A}(\mathcal{Z}, -k)$ is 1-dimensional while $\mathcal{A}(\mathcal{Z}, k)$ is formal and has 1-dimensional homology. (The first fact is trivial; the second takes a little work.)



More generally, there is a duality relating $\mathcal{A}(\mathcal{Z}, i)$ and $\mathcal{A}(\mathcal{Z}', -i)$ for appropriate pairs of pointed matched circles \mathcal{Z} and \mathcal{Z}' ; see [LOT11]. The fact that the algebra is one-dimensional in the extremal weight relates to the fact that $\widehat{\text{HFK}}$ is monic for fibered knots; see [LOT10a, Theorem 7] for further discussion.

Finally, we give interpretations of $\mathcal{A}(\mathcal{Z})$ directly in terms of Reeb chords. For any pointed matched circle \mathcal{Z} and set of Reeb chords ρ , if $a_0(\rho) \neq 0$, then $a(\rho) \neq 0$ if and only if $|M(\rho^-)| = |M(\rho^+)| = |\rho|$, i.e., if and only if no two ρ_i have matched initial or final endpoints.

LEMMA 3.26. *Let ρ and σ be a pair of sets of Reeb chords so that $a_0(\rho) \cdot a_0(\sigma) \neq 0$. Then $a(\rho) \cdot a(\sigma) = 0$ if and only if there is a matched pair $\{p, q\}$ in \mathcal{Z} satisfying one of the following conditions:*

- (1) $\rho \uplus \sigma$ has a chord starting at p and another one starting at q ;
- (2) $\rho \uplus \sigma$ has a chord terminating at p and another one terminating at q ; or
- (3) ρ has a chord which terminates in p while σ has a chord which starts in q .

Further, if none of Conditions (1)–(3) are satisfied then for any s such that $I(s) \cdot a(\rho) \neq 0$ and $I(s) \cdot a(\rho \uplus \sigma) \neq 0$ it follows that $I(s) \cdot a(\rho) \cdot a(\sigma) \neq 0$.

PROOF. Any of the three conditions implies the vanishing of $a(\rho) \cdot a(\sigma)$: in view of Lemma 3.12, the first is the case where $\mathbf{I} \cdot a_0(\rho) \cdot a_0(\sigma) = 0$, the second is where $a_0(\rho) \cdot a_0(\sigma) \cdot \mathbf{I} = 0$, and the third is where $a_0(\rho) \cdot \mathbf{I} \cdot a_0(\sigma) = 0$.

Conversely, suppose none of Conditions (1)–(3) hold. Let $S := (\rho \uplus \sigma)^-$ and $T := (\rho \uplus \sigma)^+$. The failure of Condition (1) (respectively (2)) implies that the restriction of M to S (respectively T) is injective, so $I(S)$ (respectively $I(T)$) is a summand in \mathbf{I} . Let $U := (S \setminus \rho^-) \cup \rho^+$; observe that $I(S)a_0(\rho) = I(S)a_0(\rho)I(U)$. It follows from Lemma 3.12 (and the definitions) that $U = (T \setminus \sigma^+) \cup \sigma^-$. By the failure of Condition (3), the restriction of M to U is injective, so U is a summand of \mathbf{I} . Since $I(S) \cdot a_0(\rho) \cdot I(U) \cdot a_0(\sigma) \cdot I(T) \neq 0$, it follows that $a(\rho) \cdot a(\sigma) \neq 0$.

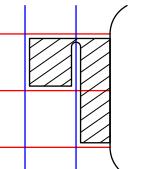
For the last part of the statement, since $I(s) \cdot a(\rho \uplus \sigma) \neq 0$ there is a subset $S \subset [4k]$ such that $M(S) = s$, $M|_S$ is injective and $(\rho \uplus \sigma)^- \subset S$. Moreover, by the failure of Condition (3), $U := (S \setminus \rho^-) \cup \rho^+$ contains σ^- , and since $I(s) \cdot a(\rho) \neq 0$, $M|_U$ is injective. Similarly, $T := (U \setminus \sigma^+) \cup \sigma^+ = (S \setminus (\rho \uplus \sigma)^-) \cup (\rho \uplus \sigma)^+$ has $M|_T$ injective since $I(s) \cdot a(\rho \uplus \sigma) \neq 0$. So, setting $t := M(T)$ and $u := M(U)$, we have

$$I(s) \cdot a_0(\rho) \cdot I(u) \cdot a_0(\sigma) \cdot I(t) \neq 0,$$

as desired. \square

REMARK 3.27. One might wonder to what extent $\mathcal{A}(\mathcal{Z})$ is a topological invariant of the surface specified by \mathcal{Z} . In fact, if \mathcal{Z} and \mathcal{Z}' are two different pointed matched circles representing diffeomorphic surfaces, then neither $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$ nor their homologies $H_*(\mathcal{A}(\mathcal{Z}))$ and $H_*(\mathcal{A}(\mathcal{Z}'))$ are necessarily isomorphic; indeed, they need not have the same rank. (See [LOT10a, Theorem 9].) However, it turns out that the module categories of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z}')$ are derived equivalent [LOT10a, Theorem 1].

REMARK 3.28. The summands $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(\mathcal{Z})$ for $i \neq 0$ act trivially on the modules $\widehat{\text{CFD}}(Y)$ and $\widehat{\text{CFA}}(Y)$ defined in Chapters 6 and 7. So, in the present work, only the subalgebra $\mathcal{A}(\mathcal{Z}, 0) \subset \mathcal{A}(\mathcal{Z})$ will be of interest. The other summands of $\mathcal{A}(\mathcal{Z})$ do play an important role in the case of three-manifolds with disconnected boundary; see [LOT10a], and also Appendix A.



REMARK 3.29. One can think of the index i in $\mathcal{A}(\mathcal{Z}, i)$ as corresponding to spin c -structures on the surface $F(\mathcal{Z})$, compare [LOT10a]. For a 3-manifold Y with connected boundary ∂Y , only the middle spin c -structure on ∂Y extends over Y .

REMARK 3.30. We get pointed matched circles by splitting a pointed Heegaard diagram in two along a circle Z , satisfying a suitable combinatorial hypothesis; see Lemma 4.31. The points \mathbf{a} are the intersections of Z with the α -circles and the matching M records which intersection points are on the same α -circle. We declare one of the two components of the complement of Z in the Heegaard surface to be the “left” (west, or type A) side, and let Z inherit the orientation as the boundary of this side. See Figure 3.3.

3.3. Gradings

Although the algebra $\mathcal{A}(n, k)$ has a \mathbb{Z} -grading, for instance by the crossing number (i.e., number of inversions), it turns out that the subalgebra $\mathcal{A}(\mathcal{Z})$ has no \mathbb{Z} -grading.

EXAMPLE 3.31. For \mathcal{Z} the pointed matched circle representing a torus, consider the two algebra elements in $\mathcal{A}(\mathcal{Z}, 1)$

$$x = \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix} = \text{Diagram } x \quad \text{and} \quad y = \begin{bmatrix} 2 & 1 \\ 4 & \end{bmatrix} = \text{Diagram } y.$$

On one hand, we have

$$y \cdot x = \text{Diagram } y \cdot x$$

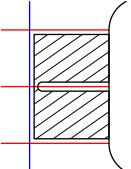
but on the other hand,

$$\begin{aligned} \partial x &= \text{Diagram } \partial x \\ (\partial x) \cdot y &= \text{Diagram } (\partial x) \cdot y \end{aligned}$$

Hence $\partial((\partial x) \cdot y) = y \cdot x$, which is inconsistent with any \mathbb{Z} -grading in which x and y are homogeneous.

Instead, we grade $\mathcal{A}(\mathcal{Z})$ by a non-commutative group. A grading of a differential algebra A by a non-commutative group G is a straightforward, but not very standard, notion. Briefly, it is a grading of A as an algebra by G together with a distinguished central element $\lambda \in G$ so that for any homogeneous elements $a, b \in A$, we have $\text{gr}(a \cdot b) = \text{gr}(a) \text{gr}(b)$ and $\text{gr}(\partial a) = \lambda^{-1} \text{gr}(a)$. See Section 2.5 for the details of the algebraic setting.

In fact, we give two gradings of $\mathcal{A}(\mathcal{Z})$ by non-commutative groups. In Section 3.3.1 we construct a canonical grading of $\mathcal{A}(\mathcal{Z})$ by a group $G'(n)$. In Section 3.3.2 we refine this grading to a grading by a subgroup $G(\mathcal{Z}) \subset G'(n)$. The subgroup $G(\mathcal{Z})$ has a more obvious topological interpretation and behaves better with respect to the operation of gluing Heegaard diagrams (see Section 10.6), but the definition of the refined grading on $\mathcal{A}(\mathcal{Z})$ depends on some additional choices.



3.3.1. Unrefined grading on $\mathcal{A}(\mathcal{Z})$. We start by giving a grading gr' of $\mathcal{A}(n, k)$ with values in a non-commutative group $G'(n)$. We then show that this grading has the property that gr' is unchanged by adding horizontal strands, and so descends to a grading on $\mathcal{A}(\mathcal{Z})$.

Let $Z' = Z \setminus \{z\}$. The group $G'(n)$ can be abstractly defined as a central extension by \mathbb{Z} of the relative homology group $H_1(Z', \mathbf{a})$ (which is isomorphic to \mathbb{Z}^{n-1}). In particular, there is an exact sequence of groups

$$\mathbb{Z} \xrightarrow{\lambda} G'(n) \xrightarrow{[\cdot]} H_1(Z', \mathbf{a})$$

where in the first map 1 maps to λ .

To define the central extension concretely we make some more definitions. For $p \in \mathbf{a}$ and $\alpha \in H_1(Z', \mathbf{a})$, define the *multiplicity* $m(\alpha, p)$ of p in α to be the average multiplicity with which α covers the regions on either side of p . Extend m to a map $H_1(Z', \mathbf{a}) \times H_0(\mathbf{a}) \rightarrow \frac{1}{2}\mathbb{Z}$ bilinearly.

For $\alpha_1, \alpha_2 \in H_1(Z', \mathbf{a})$, define

$$(3.32) \quad L(\alpha_1, \alpha_2) := m(\alpha_2, \partial\alpha_1)$$

where ∂ is the connecting homomorphism $\partial: H_1(Z', \mathbf{a}) \rightarrow H_0(\mathbf{a})$ from the long exact sequence for a pair. The map L can be thought of as the linking number of $\partial\alpha_1$ and $\partial\alpha_2$. Also, for $\alpha \in H_1(Z', \mathbf{a})$, define $\epsilon(\alpha) \in (\frac{1}{2}\mathbb{Z})/\mathbb{Z}$ by

$$\epsilon(\alpha) := \frac{1}{4} \#(\text{parity changes in } \alpha) \pmod{1}.$$

Here, a parity change in α is a point $p \in \mathbf{a}$ so that the multiplicity of α to the left of p has different parity than the multiplicity of α to the right of p , or, equivalently, so that $m(\alpha, p)$ is a half-integer. (Note that there are always an even number of such points.)

DEFINITION 3.33. Define $G'(n)$ to be the group generated by pairs (j, α) , where $j \in \frac{1}{2}\mathbb{Z}$, $\alpha \in H_1(Z', \mathbf{a})$ and $j \equiv \epsilon(\alpha) \pmod{1}$, with multiplication given by

$$(3.34) \quad (j_1, \alpha_1) \cdot (j_2, \alpha_2) := (j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

(Thus the map L is a 2-cocycle on $H_1(Z', \mathbf{a})$ which defines the central extension $G'(n)$.) In an element $g = (j, \alpha)$ of $G'(n)$, we refer to j as the *Maslov component* of g and α as the *spin^c component* of g . The distinguished central element of $G'(n)$ is $\lambda = (1, 0)$.

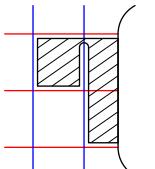
To see that $G'(n)$ is, in fact, a group we make L more explicit and prove some elementary properties.

LEMMA 3.35. If $\alpha = \sum_{i=1}^{n-1} u_i[p_i, p_{i+1}]$ and $\beta = \sum_{i=1}^{n-1} v_i[p_i, p_{i+1}]$, then

$$L(\alpha, \beta) = \sum_{i=1}^{n-2} \frac{u_i v_{i+1} - u_{i+1} v_i}{2}.$$

In particular, $L(\beta, \alpha) = -L(\alpha, \beta)$.

Note that $u_i v_{i+1} - u_{i+1} v_i$ can be interpreted as the determinant of a 2×2 submatrix of the matrix $\begin{pmatrix} u_1 & u_2 & \cdots & u_{n-1} \\ v_1 & v_2 & \cdots & v_{n-1} \end{pmatrix}$.



PROOF. Recall that $\partial[p_i, p_{i+1}] = p_{i+1} - p_i$, from which it follows that $\partial\alpha = \sum_{i=1}^n (u_{i-1} - u_i)p_i$, where we set $u_0 = u_n = 0$. Then

$$\begin{aligned} L(\alpha, \beta) &= \sum_{i=1}^n \frac{(u_{i-1} - u_i)(v_{i-1} + v_i)}{2} \\ &= \sum_{i=1}^n \frac{u_{i-1}v_i - u_iv_{i-1}}{2} + \sum_{i=1}^n \left(\frac{u_{i-1}v_{i-1}}{2} - \frac{u_iv_i}{2} \right) \\ &= \sum_{i=1}^n \frac{u_{i-1}v_i - u_iv_{i-1}}{2} \end{aligned}$$

since the second sum in the second line telescopes. The first and last terms in the final sum vanish, proving the result. \square

LEMMA 3.36. If $\alpha, \beta \in H_1(Z', \mathbf{a})$, then

$$\epsilon(\alpha + \beta) \equiv \epsilon(\alpha) + \epsilon(\beta) + L(\alpha, \beta) \pmod{1}.$$

PROOF. Consider the terms in $L(\alpha, \beta) = m(\beta, \partial\alpha)$ corresponding to the different points $p \in \mathbf{a}$. A particular p contributes a half-integer to $L(\alpha, \beta)$ when p appears an odd number of times in $\partial\alpha$ and $m(\beta, p)$ is a half-integer, i.e., p appears an odd number of times in $\partial\beta$. This case (when p is odd in both $\partial\alpha$ and $\partial\beta$) is exactly when the contribution of p to $\epsilon(\alpha + \beta) - \epsilon(\alpha) - \epsilon(\beta)$ is $1/2 \pmod{1}$. \square

PROPOSITION 3.37. $G'(n)$ is a group.

PROOF. Lemma 3.36 guarantees that if (j_1, α_1) and (j_2, α_2) are in $G'(n)$, then the product is as well:

$$\begin{aligned} j_1 + j_2 + L(\alpha_1, \alpha_2) &\equiv \epsilon(\alpha_1) + \epsilon(\alpha_2) + L(\alpha_1, \alpha_2) \pmod{1} \\ &\equiv \epsilon(\alpha_1 + \alpha_2) \pmod{1}. \end{aligned}$$

The inverse of (j, α) is easily seen to be $(-j, -\alpha)$. We must also check that the product is associative:

$$\begin{aligned} ((j_1, \alpha_1) \cdot (j_2, \alpha_2)) \cdot (j_3, \alpha_3) &= (j_1 + j_2 + j_3 + L(\alpha_1, \alpha_2) + L(\alpha_1 + \alpha_2, \alpha_3), \alpha_1 + \alpha_2 + \alpha_3) \\ &= (j_1 + j_2 + j_3 + L(\alpha_1, \alpha_2) + L(\alpha_1, \alpha_3) + L(\alpha_2, \alpha_3), \alpha_1 + \alpha_2 + \alpha_3) \end{aligned}$$

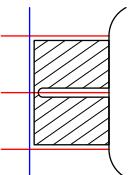
where we use bilinearity of L . Associating in the opposite order gives the same result. \square

DEFINITION 3.38. The grading $\text{gr}'(a)$ of an element $a \in \mathcal{A}(n, k)$ with starting idempotent S and ending idempotent T is defined as follows. The spin^c ($H_1(Z', \mathbf{a})$) component of the grading of an element $a = (S, T, \phi)$, denoted $[a]$, is defined to be the element of the relative homology group $H_1(Z', \mathbf{a})$ given by summing up the intervals corresponding to the strands:

$$[a] := \sum_{s \in S} [s, \phi(s)].$$

The Maslov component of the grading is given by

$$\iota(a) := \text{inv}(a) - m([a], S).$$



So, the grading of an element a is given by

$$\text{gr}'(a) := (\iota(a), [a]).$$

Note that $\partial[a] = T - S \in H_0(\mathbf{a})$, so by Lemma 3.35, $m([a], S) = m([a], T)$.

PROPOSITION 3.39. *The function gr' defines a grading (in the sense of Definition 2.38) on $\mathcal{A}(n, k)$, with $\lambda = (1, 0)$.*

PROOF. Let a be an algebra element with starting idempotent S and ending idempotent T . We first check that $\text{gr}'(a) \in G'(n)$. Modulo 1, we have

$$\begin{aligned} \iota(a) &\equiv m([a], S) \\ &\equiv \frac{1}{2}\#(S \setminus T) \equiv \frac{1}{2}\#(T \setminus S) \\ &\equiv \frac{1}{4}\#(S \Delta T) \equiv \epsilon([a]), \end{aligned}$$

where $S \Delta T$ is the symmetric difference of S and T :

$$S \Delta T := (S \setminus T) \cup (T \setminus S).$$

For the behavior of gr' under the action of ∂ , observe that $\iota(\partial a) = \iota(a) - 1$: neither S nor $[a]$ is changed by ∂ , so $m([a], S)$ is unchanged, and in the definition of ι only the $\text{inv}(a)$ term changes. Thus with $\lambda = (1, 0)$, we have $\text{gr}'(\partial a) = \lambda^{-1} \text{gr}'(a)$, as desired.

We also check that gr' is compatible with multiplication: if a is given by (S, T, ϕ_1) and b is given by (T, U, ϕ_2) with $a \cdot b \neq 0$, then

$$\begin{aligned} [a \cdot b] &= [a] + [b] \\ \iota(a \cdot b) &= \text{inv}(a \cdot b) - m([a] + [b], S) \\ &= \text{inv}(a) + \text{inv}(b) - m([a], S) - m([b], T) + m([b], T - S) \\ &= \iota(a) + \iota(b) + m([b], \partial[a]) \end{aligned}$$

and so $\text{gr}'(a \cdot b) = \text{gr}'(a) \cdot \text{gr}'(b)$. □

PROPOSITION 3.40. *For a consistent set ρ of Reeb chords, $a_0(\rho)$ is homogeneous with respect to gr' . In particular, the grading gr' descends to a grading on the subalgebra $\mathcal{A}(\mathcal{Z})$.*

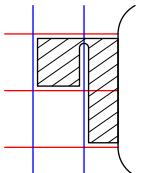
PROOF. Observe that adding a horizontal strand at $p \in \mathbf{a}$ to an algebra element $a = (S, T, \phi)$ does not change $[a]$ and changes $\text{inv}(a)$ and $m([a], S)$ by the same amount, $m([a], p)$. Thus $\text{gr}'(a)$ is unchanged by adding a horizontal strand (moving from $\mathcal{A}(n, k)$ to $\mathcal{A}(n, k+1)$). In particular the terms in the definition of $a_0(\rho)$ all have the same grading, proving the first statement. Since $\mathcal{A}(\mathcal{Z})$ is generated by elements of the form $I(\mathbf{s})a_0(\rho)$, and $I(\mathbf{s})$ is homogeneous with grading $(0, 0)$, the second statement follows. □

As in Section 2.5, there is a partial order on $G'(n)$ by $(j, \alpha) < (j', \alpha')$ if $\alpha = \alpha'$ and $j < j'$.

3.3.2. Refined grading on $\mathcal{A}(\mathcal{Z}, i)$. We can also construct a refined grading gr on $\mathcal{A}(\mathcal{Z}, i)$ with values in a smaller group $G(\mathcal{Z})$.

The group $G(\mathcal{Z})$ can be abstractly defined as a central extension of $H_1(F) \cong \mathbb{Z}^{2k}$. Again, there is a sequence of groups

$$(3.41) \quad \mathbb{Z} \xrightarrow{\lambda} G(\mathcal{Z}) \xrightarrow{[\cdot]} H_1(F)$$



where $1 \in \mathbb{Z}$ maps to $\lambda \in G(\mathcal{Z})$. This central extension is defined as an abstract group as follows. Let λ denote the generator of the center of $G(\mathcal{Z})$ and for $g \in G(\mathcal{Z})$, let $[g]$ be its image in $H_1(F)$, as indicated above. Then, for $g, h \in G(\mathcal{Z})$, we have

$$(3.42) \quad gh = hg\lambda^{2([g] \cap [h])}.$$

Equation (3.42) determines $G(\mathcal{Z})$ uniquely up to isomorphism.

In order to describe gr , first suppose that we have an element α of $H_1(Z', \mathbf{a})$ for which $M_*(\partial\alpha) = 0$; that is, for each pair p, q of points in \mathbf{a} identified by M , the sum of the multiplicities of $\partial\alpha$ at p and at q is 0. Then we can construct an element $\bar{\alpha}$ of $H_1(F(\mathcal{Z}))$ from α by embedding Z in $F(\mathcal{Z})$ to make α a chain in $F(\mathcal{Z})$, and then adding multiples of the cores of the 1-handles we attached in the construction of $F(\mathcal{Z})$ to form a cycle $\bar{\alpha}$. Indeed, we can identify $H_1(F)$ as $\ker(M_* \circ \partial)$ inside $H_1(Z', \mathbf{a})$.

With this setup, we define $G(\mathcal{Z})$ to be the subgroup $\{(j, \alpha) \in G'(4k) \mid M_*(\partial(\alpha)) = 0\}$ of $G'(4k)$.

LEMMA 3.43. $G(\mathcal{Z})$ is a group that fits in the exact sequence (3.41) and satisfies equation (3.42).

PROOF. The fact that $[\cdot]: G'(4k) \rightarrow H_1(Z', \mathbf{a})$ is a homomorphism guarantees that $G(\mathcal{Z})$ is a group. The image of $G(\mathcal{Z})$ under $[\cdot]$ is, by definition, $\ker(M_* \circ \partial) \simeq H_1(F)$ and the kernel of $[\cdot]$ is $\{(j, 0) \mid j \in \mathbb{Z}\}$, so we have the desired exact sequence. Finally, for $\alpha, \beta \in \ker(M_* \circ \partial)$, the corresponding elements $\bar{\alpha}, \bar{\beta}$ in $H_1(F)$ satisfy

$$\bar{\alpha} \cap \bar{\beta} = m(\alpha, \beta),$$

which implies Equation (3.42). \square

To define the grading gr on $\mathcal{A}(\mathcal{Z}, i)$, pick an arbitrary base idempotent $I(\mathbf{s}_0) \in \mathcal{A}(\mathcal{Z}, i)$ and, for each other idempotent $I(\mathbf{s})$, pick a group element $\psi(\mathbf{s}) = \psi(I(\mathbf{s})) \in G'(4k)$ so that $M_*\partial([\psi(\mathbf{s})]) = \mathbf{s} - \mathbf{s}_0$. Then define

$$(3.44) \quad \text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})) := \psi(\mathbf{s}) \text{gr}'(a(\boldsymbol{\rho}))\psi(\mathbf{t})^{-1}.$$

We will call the choices $\mathbf{s}_0, \psi(\mathbf{s})$ the choice of *grading refinement data*.

LEMMA 3.45. The function gr is a grading with values in $G(\mathcal{Z}) \subset G'(4k)$.

PROOF. To check the values are in $G(\mathcal{Z})$, we compute:

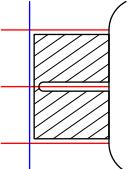
$$\begin{aligned} M_*\partial[\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t}))] &= M_*\partial[\psi(\mathbf{s})] + M_*\partial[a(\boldsymbol{\rho})] - M_*\partial[\psi(\mathbf{t})] \\ &= (\mathbf{s} - \mathbf{s}_0) + (I(\mathbf{t}) - I(\mathbf{s})) - (I(\mathbf{t}) - \mathbf{s}_0) \\ &= 0. \end{aligned}$$

It is straightforward to check that gr is a grading. \square

REMARK 3.46. Different choices of grading refinement data give gradings on $\mathcal{A}(\mathcal{Z}, i)$ that are conjugate in an idempotent-dependent way, in the following sense. For any two choices (\mathbf{s}_0, ψ) and (\mathbf{s}'_0, ψ') giving gradings gr_1 and gr_2 respectively, there is a function $\xi: \mathcal{I}(\mathcal{Z}, i) \rightarrow G(\mathcal{Z})$ so that, if $a \in \mathcal{A}(\mathcal{Z})$ is homogeneous and satisfies $a = I(\mathbf{s})aI(\mathbf{t})$,

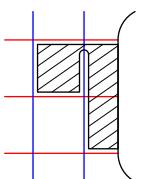
$$\text{gr}_2(a) = \xi(\mathbf{s}) \cdot \text{gr}_1(a) \cdot \xi(\mathbf{t})^{-1}.$$

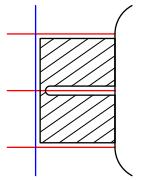
(The map ξ is given by $\xi(\mathbf{s}) = \psi'(\mathbf{s})\psi(\mathbf{s})^{-1}$.) As a result, the categories of $\mathcal{A}(\mathcal{Z}, i)$ -modules that are graded with respect to these different gradings are equivalent; see [LOT10a, Proposition 3.10] for more details.



REMARK 3.47. An alternate way to construct the refined grading, which we employ in Chapter 11 for the case where F is a torus, is to use a homomorphism from $G'(4k)$ to $G(\mathcal{Z})$ that fixes $G(\mathcal{Z})$ as a subset of $G'(4k)$ (after extending scalars from \mathbb{Z} to \mathbb{Q}). For $k > 1$, however, there is no homomorphism $G'(4k) \rightarrow G(\mathcal{Z})$ fixing $G(\mathcal{Z})$.

REMARK 3.48. $G(\mathcal{Z})$ has a topological interpretation as the group of homotopy classes of non-vanishing vector fields on $F \times [0, 1]$ with fixed behavior on the boundary. Precisely, fix a vector field V_0 on $F \times \mathbb{R}$ that is invariant by translation in \mathbb{R} . Then the homotopy classes of vector fields on $F \times [0, 1]$ that agree with V_0 on neighborhoods of $F \times \{0\}$ and $F \times \{1\}$ form a group by concatenation; this group is isomorphic to $G(\mathcal{Z})$. This is unsurprising in light of the fact that Seiberg-Witten Floer homology on Y is graded by non-vanishing vector fields on Y [KM07, Section 28].





CHAPTER 4

Bordered Heegaard diagrams

Just as closed 3-manifolds can be represented by Heegaard diagrams, bordered 3-manifolds can be described by a suitable type of Heegaard diagrams, which we call *bordered Heegaard diagrams*. Just as Heegaard Floer homology of closed 3-manifolds is associated to closed Heegaard diagrams, bordered Floer invariants of 3-manifolds are associated to bordered Heegaard diagrams. This chapter is devoted to defining bordered Heegaard diagrams and discussing their combinatorial / topological properties.

In Section 4.1, we define bordered 3-manifolds and bordered Heegaard diagrams (Definitions 4.2 and 4.3, respectively), and discuss the relevant existence and uniqueness properties for bordered diagrams. In Section 4.2, we give some examples of bordered Heegaard diagrams, to give the reader a concrete picture. In Section 4.3, we describe the generators of the bordered Heegaard Floer complex, and discuss the spin^c structures associated to generators, and the homology classes connecting them. As in the closed case, one cannot use an arbitrary bordered Heegaard diagram to define bordered Heegaard Floer homology: rather, one needs to use bordered diagrams satisfying certain combinatorial conditions ensuring that the sums encountered in the differentials are finite. In Section 4.4, we describe these criteria. Finally, in Section 4.5, we discuss how to glue bordered Heegaard diagrams and their generators to obtain Heegaard diagrams and generators for closed manifolds.

4.1. Bordered Heegaard diagrams: definition, existence, and uniqueness

We recall first the diagrams relevant for Heegaard Floer homology of closed three-manifolds.

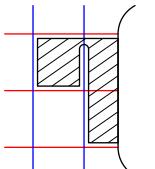
DEFINITION 4.1. A *pointed Heegaard diagram* \mathcal{H} is a quadruple $(\Sigma, \alpha, \beta, z)$ consisting of

- a compact, oriented surface Σ with no boundary, of some genus g ;
- two g -tuples $\alpha = \{\alpha_1, \dots, \alpha_g\}$ and $\beta = \{\beta_1, \dots, \beta_g\}$ of circles in Σ , with the circles in each tuple disjoint; and
- a point z in $\Sigma \setminus (\alpha \cup \beta)$,

such that the α -circles and β -circles intersect transversally and $\Sigma \setminus \alpha$ and $\Sigma \setminus \beta$ are each connected.

We will also sometimes refer to pointed Heegaard diagrams as in Definition 4.1 as *closed* Heegaard diagrams, to distinguish them from the bordered Heegaard diagrams introduced soon.

When discussing pointed Heegaard diagrams (and their bordered variants), we slightly abuse notation, letting α denote both the set or tuple of α -curves and the union of the α -curves, and similarly for β . Note that the α -circles and β -circles are each maximal collections of embedded circles with connected complement. An equivalent requirement to connectivity of the complement is that the α_i be homologically linearly independent in $H_1(\Sigma)$, spanning a Lagrangian subspace, and likewise for the β_i .



Thus fortified, we turn to the bordered case.

DEFINITION 4.2. A *bordered 3-manifold* is a triple (Y, \mathcal{Z}, ϕ) where Y is a compact, oriented 3-manifold with connected boundary ∂Y , \mathcal{Z} is a pointed matched circle, and $\phi: F(\mathcal{Z}) \rightarrow \partial Y$ is an orientation-preserving homeomorphism.

Two bordered 3-manifolds (Y, \mathcal{Z}, ϕ) and $(Y', \mathcal{Z}', \phi')$ are called *equivalent* if there is an orientation-preserving homeomorphism $\psi: Y \rightarrow Y'$ so that $\phi' = \psi \circ \phi$.

Note that if $\phi: F(\mathcal{Z}) \rightarrow \partial Y$ and $\phi': F(\mathcal{Z}) \rightarrow \partial Y$ are isotopic homeomorphisms, then, by viewing the isotopy as a homeomorphism on a collar of ∂Y , we see that the bordered 3-manifolds (Y, \mathcal{Z}, ϕ) and (Y, \mathcal{Z}, ϕ') are equivalent.

We will often abuse notation and refer to a bordered 3-manifold Y , suppressing \mathcal{Z} and ϕ . Also, by a bordered 3-manifold we will implicitly mean an equivalence class of bordered 3-manifolds.

DEFINITION 4.3. A *bordered Heegaard diagram* is a quadruple $\mathcal{H} = (\bar{\Sigma}, \bar{\alpha}, \beta, z)$ consisting of

- a compact, oriented surface $\bar{\Sigma}$ with one boundary component, of some genus g ;
- a g -tuple of pairwise-disjoint circles $\beta = \{\beta_1, \dots, \beta_g\}$ in the interior of Σ ;
- a $(g+k)$ -tuple of pairwise-disjoint curves $\bar{\alpha}$ in $\bar{\Sigma}$, split into $g-k$ circles $\alpha^c = (\alpha_1^c, \dots, \alpha_{g-k}^c)$ in the interior of $\bar{\Sigma}$ and $2k$ arcs $\bar{\alpha}^a = (\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a)$ in $\bar{\Sigma}$ with boundary on $\partial \bar{\Sigma}$ (and transverse to $\partial \bar{\Sigma}$); and
- a point z in $(\partial \bar{\Sigma}) \setminus (\bar{\alpha} \cap \partial \bar{\Sigma})$,

such that the intersections are transverse and $\bar{\Sigma} \setminus \bar{\alpha}$ and $\bar{\Sigma} \setminus \beta$ are connected.

The conditions on the numbers of β -circles and α -circles and arcs again come naturally from requiring that the tuples be maximal collections of curves of the appropriate type with connected complement. Again, an equivalent requirement to connectivity of the complement is that the β_i be homologically linearly independent in $H_1(\bar{\Sigma})$ and that the α_i be homologically linearly independent in $H_1(\bar{\Sigma}, \partial \bar{\Sigma})$.

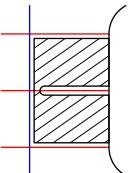
The boundary of a bordered Heegaard diagram naturally has the structure of a pointed matched circle:

LEMMA 4.4. Let $(\bar{\Sigma}, \bar{\alpha}, \beta, z)$ be a bordered Heegaard diagram. Let $Z = \partial \bar{\Sigma}$ and $\mathbf{a} = \bar{\alpha} \cap \partial \bar{\Sigma}$. Define a matching M on \mathbf{a} by $M(\alpha_i^a \cap \partial \bar{\Sigma}) = i$. Then (Z, \mathbf{a}, M, z) is a pointed matched circle, in the sense of Definition 3.16.

PROOF. We need to verify that the result Z' of performing surgery on the points in \mathbf{a} according to the matching M gives a (connected) circle. Let $\bar{\Sigma}'$ be the result of performing surgery on $\bar{\Sigma}$ along each α -circle, i.e., deleting a tubular neighborhood of each α -circle and gluing in two disks to the result. The circle Z' is homeomorphic to the boundary B of a regular neighborhood of $\partial \bar{\Sigma} \cup \bigcup_{i=1}^{2k} \alpha_i^a$. But B , in turn, is homeomorphic to $\partial(\bar{\Sigma}' \setminus \text{nbd}(\bar{\alpha}))$, and $\bar{\Sigma}' \setminus \text{nbd}(\bar{\alpha})$ is a disk. \square

DEFINITION 4.5. If \mathcal{H} is a bordered Heegaard diagram, we call the pointed matched circle appearing on its boundary (in the sense of Lemma 4.4) the *boundary of \mathcal{H}* and denote it $\partial \mathcal{H}$.

CONSTRUCTION 4.6. Let \mathcal{H} be a bordered Heegaard diagram. We will show how \mathcal{H} specifies a bordered 3-manifold, with boundary parameterized by $F(\partial \mathcal{H})$.



The construction is illustrated in Figure 4.1. Write $\mathcal{H} = (\bar{\Sigma}, \bar{\alpha}, \beta, z)$, and write $\mathcal{Z} = \partial\mathcal{H}$. Let $[-\epsilon, 0] \times Z$ denote a closed collar neighborhood of $\partial\bar{\Sigma}$, so that $0 \times Z$ is identified with $\partial\bar{\Sigma}$. Choose also a closed tubular neighborhood $Z \times [0, 1]$ of Z in $F(\mathcal{Z})$.

Let Y_0 denote the three-manifold obtained by gluing $\bar{\Sigma} \times [0, 1]$ to $[-\epsilon, 0] \times F(\mathcal{Z})$, by identifying $([-\epsilon, 0] \times Z) \times [0, 1] \subset \bar{\Sigma} \times [0, 1]$ with $[-\epsilon, 0] \times (Z \times [0, 1]) \subset [-\epsilon, 0] \times F(\mathcal{Z})$.

Next, attach a 3-dimensional 2-handle to each $\beta_i \times \{1\} \subset \bar{\Sigma} \times [0, 1] \subset Y_0$ and to each $\alpha_i^c \times \{0\} \subset \bar{\Sigma} \times [0, 1] \subset Y_0$. Call the result Y_1 . The manifold Y_1 has three boundary components:

- an S^2 which meets $\bar{\Sigma} \times \{1\}$,
- a surface Σ' of genus $2k$ which meets $\bar{\Sigma} \times \{0\}$, and
- a surface identified with $F(\mathcal{Z})$, given by $\{0\} \times F(\mathcal{Z}) \subset [-\epsilon, 0] \times F(\mathcal{Z})$.

Glue a 3-ball to the S^2 boundary component of Y_1 . Join each $\alpha_i^a \times \{0\} \cap \Sigma'$ to the co-core of the corresponding handle in $\{-\epsilon\} \times F(\mathcal{Z})$ to form a closed curve, and attach a 3-dimensional 2-handle along each of these circles. Call the result of these attachings Y_2 . The manifold Y_2 has one S^2 boundary component which meets $\bar{\Sigma} \times \{0\} \subset Y_0$ and a boundary component identified with $F(\mathcal{Z})$. Glue a 3-ball to the S^2 boundary component of Y_2 and call the result Y . Let ϕ be the obvious identification of $F(\mathcal{Z})$ with ∂Y . Then (Y, \mathcal{Z}, ϕ) is the desired bordered 3-manifold.

DEFINITION 4.7. Given a bordered Heegaard diagram we call the bordered 3-manifold given by Construction 4.6 the *bordered 3-manifold represented by \mathcal{H}* and denote it $Y(\mathcal{H})$.

Construction 4.6 introduces a few extra pairs of canceling handles. This is done to make it fit well with the Morse theory below. A quicker (and equivalent) construction is the following: $Y(\mathcal{H})$ is obtained by thickening the Heegaard surface $[0, 1] \times \bar{\Sigma}$, attaching three-dimensional two-handle to each $\alpha_i^c \times \{0\} \times \bar{\Sigma}$, and a three-dimensional two-handle to each $\beta_i \times \{1\} \times \bar{\Sigma}$. The boundary of the result is naturally identified with $F(\mathcal{Z})$.

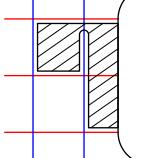
Conversely, any 3-manifold with connected boundary is specified by some bordered Heegaard diagram. One way to see this is in terms of Morse theory, following [Lip06b, Section 2.2]:

DEFINITION 4.8. Fix a bordered 3-manifold (Y, \mathcal{Z}, ϕ) . We say that a pair consisting of a Riemannian metric g on Y and a self-indexing Morse function f on Y are *compatible with* (Y, \mathcal{Z}, ϕ) if

- (1) the boundary of Y is geodesic,
- (2) the gradient vector field $\nabla f|_{\partial Y}$ is tangent to ∂Y ,
- (3) f has a unique index 0 and a unique index 3 critical point, both of which lie on ∂Y , and are the unique index 0 and 2 critical points of $f|_{\partial Y}$, respectively,
- (4) the index 1 critical points of $f|_{\partial Y}$ are also index 1 critical points of f , and
- (5) the pair $(f \circ \phi, \phi^* g)$ on $F(\mathcal{Z})$ are compatible with the pointed matched circle \mathcal{Z} in the sense of Definition 3.18.

LEMMA 4.9. Any bordered 3-manifold (Y, \mathcal{Z}, ϕ) is represented by some bordered Heegaard diagram \mathcal{H} .

PROOF. The first step is to find a Morse function f and metric g compatible with (Y, \mathcal{Z}, ϕ) as in Definition 4.8. Choose a Morse function f_F and metric g_F on $F(\mathcal{Z})$ compatible with \mathcal{Z} in the sense of Definition 3.18 which is self-indexing except that f_F takes is 3 on the index 2



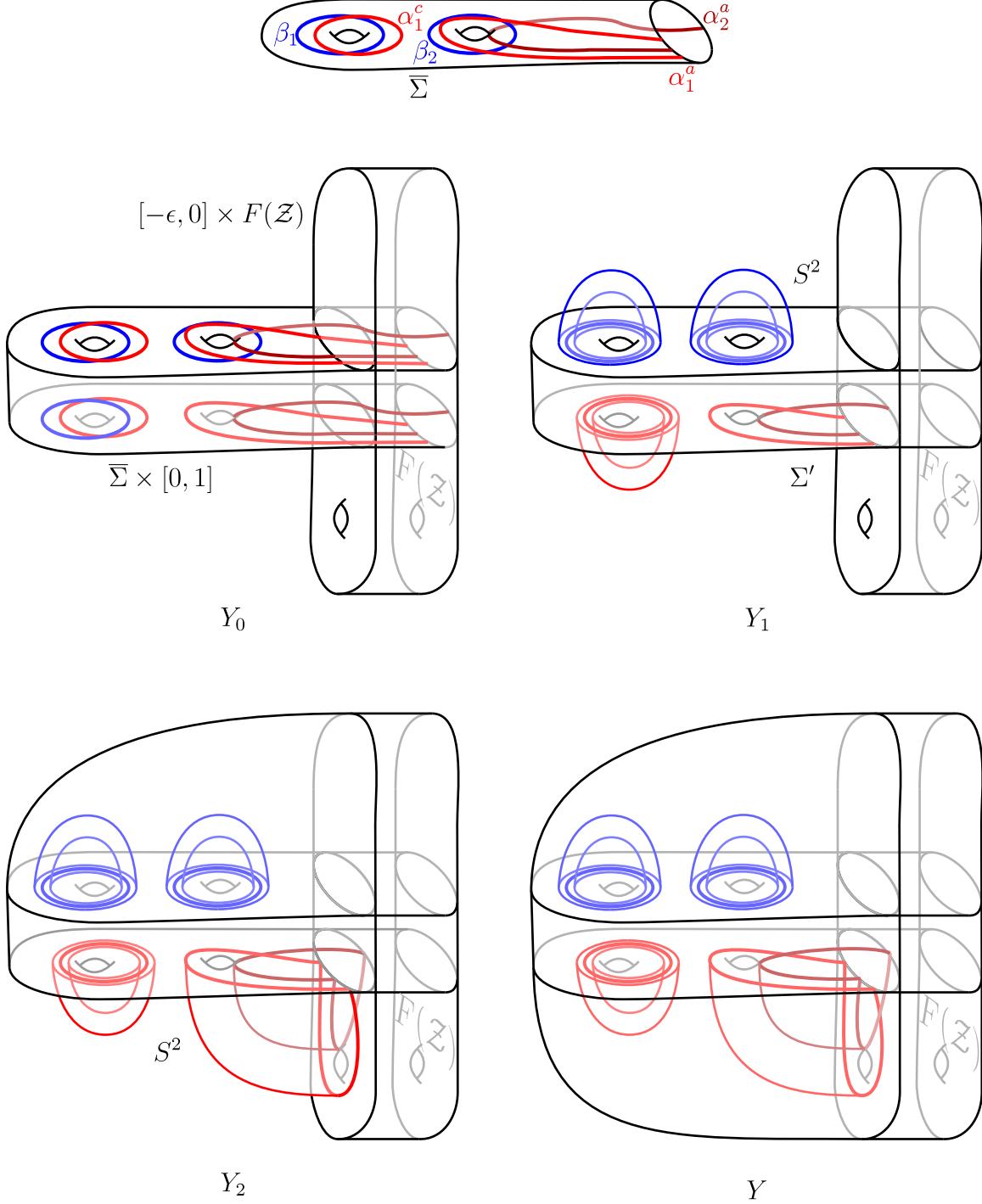
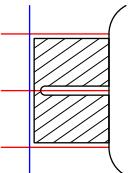


FIGURE 4.1. The 3-manifold represented by a bordered Heegaard diagram. The bordered Heegaard diagram \mathcal{H} is shown at the top; the four steps in Construction 4.6 are illustrated below.

critical point. Extend $f_F \circ \phi^{-1}$ and $\phi_* g_F$ to f and g on a collar neighborhood of ∂Y satisfying Conditions (1), (2) and (4), and so that the index 0 and 2 critical points of $f_F \circ \phi^{-1}$ are index 0 and 3 critical points of f . (Note that Condition (5) is automatically satisfied.) Extend f and g arbitrarily to the rest of Y . Now consider the graph formed by flows between the



index 0 and 1 critical points, which is connected. Since the descending flows from boundary index 1 critical points remain on the boundary, we can and do cancel every interior index 0 critical point with an interior index 1 critical point. Similarly cancel interior index 3 critical points with index 2 critical points, so that Condition (3) is satisfied. Modify the function in the interior so the result is self-indexing.

Now, given the boundary compatible pair (f, g) , construct a bordered Heegaard diagram $(\bar{\Sigma}, \bar{\alpha}, \beta, z)$ as follows. Let the Heegaard surface $\bar{\Sigma}$ be $f^{-1}(3/2)$, the curves $\bar{\alpha}$ be the intersection of the ascending disks of the index 1 critical points of f (including those on ∂Y) with $\bar{\Sigma}$, and the curves β be the intersection of the descending disks of the index 2 critical points of f with $\bar{\Sigma}$. \square

The Morse theory description also implies that any two bordered Heegaard diagrams for the same bordered 3-manifold can be connected by (bordered) Heegaard moves, as specified in the following:

PROPOSITION 4.10. *Any pair of bordered Heegaard diagrams for equivalent bordered 3-manifolds can be made diffeomorphic by a sequence of*

- isotopies of the α -curves and β -circles, not crossing $\partial\bar{\Sigma}$,
- handleslides of α -curves over α -circles and β -circles over β -circles, and
- stabilizations and destabilizations in the interior of $\bar{\Sigma}$.

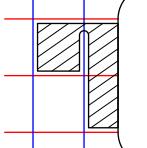
Before giving the proof of Proposition 4.10, for which we follow closely the proof of [OSz04d, Proposition 2.2], we include two lemmas which will be useful for excluding stabilizing by index zero (or three) critical points (compare [OSz04d, Lemmas 2.3 and 2.4]).

LEMMA 4.11. *Let F be a compact surface of genus h with a single boundary component. Let $\gamma = \{\gamma_1, \dots, \gamma_h\}$ be an h -tuple of embedded, pairwise disjoint, simple closed curves so that $F \setminus \gamma$ is a punctured disk. Suppose moreover that δ is a simple closed curve in F which is disjoint from the γ . Then either δ is null-homologous or there is some γ_i with the property that δ is isotopic to a curve obtained by handlesliding γ_i across some collection of the γ_j with $j \neq i$.*

PROOF. If we surger out $\gamma_1, \dots, \gamma_h$, we replace F by the disk D with $2h$ marked points $\{p_1, q_1, \dots, p_h, q_h\}$ (where the pair of points $\{p_i, q_i\}$ corresponds to the zero-sphere which replaced the circle γ_i). Now, δ can be viewed as a Jordan curve in the disk D . If δ separates some p_i from its corresponding q_i , then it is easy to see that γ is isotopic to the curve gotten by handlesliding γ_i over some collection of the γ_j with $j \neq i$. Otherwise, δ was null-homologous. \square

LEMMA 4.12. *Let F be a compact surface of genus h with a single boundary component and $\{\gamma_1, \dots, \gamma_d\}$ a collection of embedded, pairwise disjoint, simple closed curves. Then any two h -tuples of linearly-independent γ_i 's are related by a sequence of isotopies and handleslides.*

PROOF. This is proved by induction on h . The case where $h = 0$ is vacuously true. Next, suppose we have two subsets with an element in common, which we label γ_1 . Surger out γ_1 to obtain a surface F' with one lower genus. Isotopies in F' which cross the two surgery points correspond to handleslides in F across γ_1 . Thus, in this case, the proof follows from the induction hypothesis. Finally, consider the case where we have two disjoint subsets $\{\gamma_1, \dots, \gamma_h\}$ and $\{\gamma_{h+1}, \dots, \gamma_{2h}\}$. The curve γ_{h+1} cannot be null-homologous, so,



after renumbering, we can obtain γ_{h+1} by handlesliding γ_1 over some of the $\{\gamma_2, \dots, \gamma_h\}$, according to Lemma 4.11. Thus, we have reduced to the case where the two subsets are not disjoint. \square

PROOF OF PROPOSITION 4.10. The proof involves adapting standard handle calculus as in [OSz04d, Section 2.1].

Any two bordered Heegaard diagrams \mathcal{H}_0 and \mathcal{H}_1 for equivalent bordered manifolds $(Y_i, \mathcal{Z}_i, \phi_i)$ come from Morse functions and metrics (f_0, g_0) and (f_1, g_1) on Y_i compatible with the bordering (\mathcal{Z}, ϕ) of Y_i , in the sense of Definition 4.8. Using the equivalence between the two bordered manifolds, we can view f_1 and g_1 as being defined on $Y = Y_1$. Moreover, we may choose f_0, g_0, f_1 and g_1 so that for some neighborhood $N(\partial Y)$ of ∂Y , $f_1|_{N(\partial Y)} = f_2|_{N(\partial Y)}$ and $g_1|_{N(\partial Y)} = g_2|_{N(\partial Y)}$. We may connect f_0 and f_1 by a generic path f_t of smooth functions which are Morse functions for generic t and so that the family is constant on $N(\partial Y)$.

Although the f_t will not in general be self-indexing, there are still no flow lines from higher-index critical points to lower-index. To f_t at generic t we can still associate a generalized Heegaard diagram \mathcal{H}_t , possibly with extra α - and β -circles.

While f_t remains a Morse function, the Heegaard diagram changes by isotopy of the α - and β -curves. When f_t fails to be a Morse function, there is either a birth-death singularity involving a pair of critical points of adjacent index, or there is a flow line between two critical points of the same index. The possibilities, and the way the generalized Heegaard diagram changes, are:

Index 0 and 1 birth–death: A disjoint α -circle is added or deleted, without changing the span of the α -circles.

Index 1 and 2 birth–death: The Heegaard diagram is stabilized or destabilized.

Index 2 and 3 birth–death: A disjoint β -circle is added or deleted, without changing the span of the β -circles.

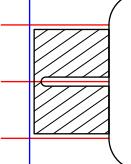
Flow line between index 1 critical points: A handleslide of an α -circle or α -arc over an α -circle. Since our conditions imply that the descending disk of the index 1 critical points corresponding to α -arcs lies entirely on the boundary, we cannot have a handleslide over an α -arc.

Flow line between index 2 critical points: A handleslide of a β -circle over a β -circle.

Note also that the critical points on the boundary cannot be involved in any cancellations, since f was fixed on all of $N(\partial Y)$.

Now, given any two bordered Heegaard diagrams \mathcal{H}_0 and \mathcal{H}_1 for Y , in view of the above remarks, we can stabilize \mathcal{H}_0 and \mathcal{H}_1 to obtain a pair of bordered Heegaard diagrams \mathcal{H}'_0 and \mathcal{H}'_1 with the same genus and with the following property. Each of \mathcal{H}'_0 and \mathcal{H}'_1 can be extended by adding new α - and β -circles if necessary to form a pair of generalized bordered Heegaard diagrams \mathcal{H}''_0 and \mathcal{H}''_1 respectively, which in turn can be connected by isotopies and handleslides (subject to the constraints from Proposition 4.10, i.e., α -arcs are allowed to slide over α -circles, but not vice versa). The proposition is proved then once we show that for each generalized Heegaard diagram \mathcal{H}'' of genus g , if \mathcal{H}_1 and \mathcal{H}_2 are any two ordinary bordered Heegaard diagrams obtained from \mathcal{H}'' by picking out $g - k$ of the α -circles and g of the β -circles, each homologically linearly independent, then \mathcal{H}_1 and \mathcal{H}_2 can be connected by handleslides and isotopies.

To this end, consider any two $(g - k)$ -tuples of the α -circles of \mathcal{H}'' chosen so that they, along with the $2k$ α -arcs, span a $g + k$ -dimensional subspace of $H_1(\overline{\Sigma}, \partial \overline{\Sigma})$. The fact that



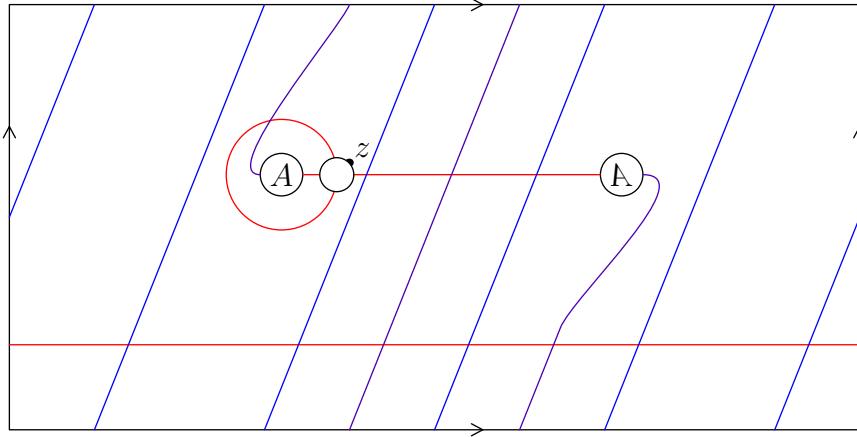


FIGURE 4.2. **Bordered Heegaard diagram for the trefoil complement.**

This is based on the observation that +5 surgery on the trefoil is the lens space $L(5, -1)$, with the knot forming the core of the solid torus a Berge knot. The diagram is on a genus 2 surface, represented as a torus with a handle attached (at A).

these two $(g - k)$ -tuples can be connected by isotopies and handleslides follows directly from Lemma 4.12, applied to the surface $F = \bar{\Sigma} \setminus \{\alpha_1^a, \dots, \alpha_{2k}^a\}$. It follows similarly that any two g -tuples of the β circles which are linearly independent in $H_1(\bar{\Sigma})$ can be connected by handleslides and isotopies. This completes the proof. \square

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be the result of attaching a *cylindrical end* to $(\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}, z)$, in the sense of symplectic field theory [EGH00]. Topologically $\Sigma = \bar{\Sigma} \setminus \partial\bar{\Sigma}$, and $\alpha_i^a = \bar{\alpha}_i^a \setminus (\partial\bar{\alpha}_i^a)$. In due course, we will choose a conformal structure on Σ making it a punctured Riemann surface and so that each α_i^a is radial near the puncture. Abusing terminology, we will often also refer to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ as a bordered Heegaard diagram.

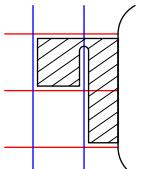
4.2. Examples of bordered Heegaard diagrams

Here are a few examples of bordered Heegaard diagrams.

Fix an oriented surface Σ , equipped with a g -tuple of pairwise disjoint, homologically independent curves, $\boldsymbol{\beta}$, and a $(g - 1)$ -tuple of pairwise disjoint, homologically independent curves $\boldsymbol{\alpha}^c = \{\alpha_1^c, \dots, \alpha_{g-1}^c\}$. Then $(\Sigma, \boldsymbol{\alpha}^c, \boldsymbol{\beta})$ is a Heegaard diagram for a three-manifold with torus boundary, and indeed any such three-manifold Y can be described by a Heegaard diagram of this type. To turn such a diagram into a bordered Heegaard diagram, we proceed as follows. Fix an additional pair of circles γ_1 and γ_2 in Σ so that:

- γ_1 and γ_2 intersect, transversally, in a single point p and
- both of the homology classes $[\gamma_1]$ and $[\gamma_2]$ are homologically independent from $[\alpha_1^c], \dots, [\alpha_{g-1}^c]$.

Let D be a disk around p which is disjoint from all the above curves, except for γ_1 and γ_2 , each of which it meets in a single arc. Then, the complement of D specifies a bordered Heegaard diagram for Y , for some parametrization of ∂Y . A bordered Heegaard diagram for the trefoil complement is illustrated in Figure 4.2. (See also Chapter 11 for a further discussion of bordered three-manifolds with torus boundary.)



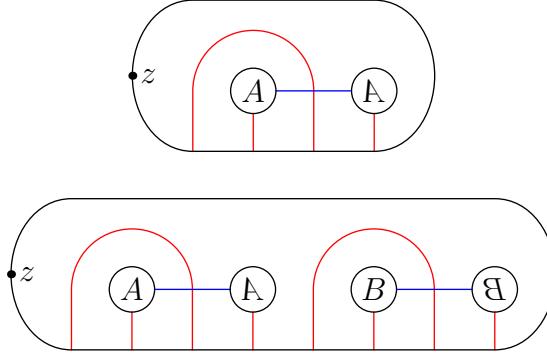


FIGURE 4.3. **Heegaard diagrams for handlebodies.** Top: a bordered genus one handlebody. Bottom: the boundary connect sum of two copies of the handlebody on top.

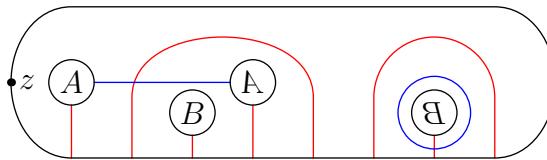


FIGURE 4.4. **Another bordered Heegaard diagram for a genus two handlebody,** using a different parameterization of the boundary.

As another example, consider boundary connect sums. Fix bordered Heegaard diagrams $\mathcal{H}_i = (\overline{\Sigma}_i, \overline{\alpha}_i, \overline{\beta}_i, z_i)$ for Y_i with $i = 1, 2$. Take the boundary connect sum of $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ along z_1 and z_2 , i.e., attach a rectangle to $\overline{\Sigma}_1 \cup \overline{\Sigma}_2$ by gluing two opposite sides to $\partial\overline{\Sigma}_1$ and $\partial\overline{\Sigma}_2$ along intervals containing z_1 and z_2 . Introduce a new basepoint z on one of the two remaining boundary components of the rectangle. (This involves a choice.) This procedure gives a bordered Heegaard diagram for the boundary connect sum of Y_1 with Y_2 . See Figure 4.3, where we form the boundary connect sum of two bordered Heegaard diagrams for genus-one handlebodies to obtain a bordered Heegaard diagram for a genus-two handlebody.

Any bordered Heegaard diagram in which there are no α circles necessarily represents a genus g handlebody. In Figure 4.4, we have illustrated a genus two handlebody with respect to a parameterization of the genus two surface which is different from the one used in Figure 4.3.

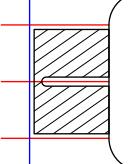
4.3. Generators, homology classes and spin^c structures

Fix a bordered Heegaard diagram \mathcal{H} , specifying some three-manifold Y .

DEFINITION 4.13. By a *generator* of \mathcal{H} we mean a g -element subset $\mathbf{x} = \{x_1, \dots, x_g\}$ so that

- exactly one x_i lies on each β -circle,
- exactly one x_i lies on each α -circle and
- at most one x_i lies on each α -arc.

Let $\mathfrak{G}(\mathcal{H})$ or $\mathfrak{G}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the set of generators. Given a generator \mathbf{x} , let $o(\mathbf{x})$ denote the set of α -arcs which are occupied by \mathbf{x} , i.e., $o(\mathbf{x}) := \{i \mid \mathbf{x} \cap \alpha_i^a \neq \emptyset\} \subset [2k]$.



Soon (Chapter 5 and later) we will be interested in holomorphic curves in $\Sigma \times I_s \times \mathbb{R}_t$, where $I_s = [0, 1]$ is the unit interval with parameter s and \mathbb{R}_t is \mathbb{R} with parameter t . These curves have boundary on $\alpha \times \{1\} \times \mathbb{R}_t$ and $\beta \times \{0\} \times \mathbb{R}_t$. They will be asymptotic to g -tuples of chords $\mathbf{x} \times I_s$ and $\mathbf{y} \times I_s$ at $\pm\infty$, where \mathbf{x} and \mathbf{y} are generators. Each such curve carries a relative homology class, and we let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of these relative homology classes. More precisely, we make the following definition.

DEFINITION 4.14. Fix generators \mathbf{x} and \mathbf{y} . Let $I_s = [0, 1]$ and $I_t = [-\infty, \infty]$ be intervals. We work in the relative homology group

$$H_2(\overline{\Sigma} \times I_s \times I_t, ((S_\alpha \cup S_\beta \cup S_\partial) \times I_t) \cup (G_{\mathbf{x}} \times \{-\infty\}) \cup (G_{\mathbf{y}} \times \{\infty\})),$$

where

$$\begin{aligned} S_\alpha &= \alpha \times \{1\} \\ S_\beta &= \beta \times \{0\} \\ S_\partial &= (\partial \overline{\Sigma} \setminus z) \times I_s \\ G_{\mathbf{x}} &= \mathbf{x} \times I_s \\ G_{\mathbf{y}} &= \mathbf{y} \times I_s. \end{aligned}$$

Let $\pi_2(\mathbf{x}, \mathbf{y})$, the *homology classes connecting \mathbf{x} to \mathbf{y}* , denote those elements of this group which map to the relative fundamental class of $\mathbf{x} \times I_s \cup \mathbf{y} \times I_s$ under the composition of the boundary homomorphism and collapsing the remainder of the boundary.

DEFINITION 4.15. Given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, projecting to $\overline{\Sigma}$ gives a well-defined element of $H_2(\overline{\Sigma}, \alpha \cup \beta \cup \partial \overline{\Sigma})$, i.e., a linear combination of the components of $\overline{\Sigma} \setminus (\alpha \cup \beta)$, which we sometimes call *regions*. We call the linear combination of regions corresponding to B the *domain* of B .

It is not hard to see that a homology class B is uniquely determined by its domain.

DEFINITION 4.16. Given any point $p \in \Sigma \setminus (\alpha \cup \beta)$, the *local multiplicity* of B at p is the local degree of the two-chain at the point p .

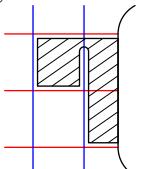
By hypothesis, the coefficient in the domain of B of the region D_z adjacent to $z \in \partial \overline{\Sigma}$ is 0.

REMARK 4.17. We use the notation π_2 for this object to agree with the original conventions [OSz04d]. In the cylindrical setting of Heegaard Floer homology this is a homology class rather than a homotopy class. Also, note that, unlike the closed case, we require the multiplicity of a domain to be 0 at z ; this is because we are only concerned with \widehat{HF} .

Notice that concatenation (in the t factor) gives a product $*: \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{w}) \rightarrow \pi_2(\mathbf{x}, \mathbf{w})$; this operation corresponds to addition of domains. If $\pi_2(\mathbf{x}, \mathbf{y})$ is non-empty then concatenation makes $\pi_2(\mathbf{x}, \mathbf{y})$ into a free, transitive $\pi_2(\mathbf{x}, \mathbf{x})$ -set. Elements of $\pi_2(\mathbf{x}, \mathbf{x})$, or their domains, are called *periodic domains*.

LEMMA 4.18. *There is a natural group isomorphism $\pi_2(\mathbf{x}, \mathbf{x}) \cong H_2(Y, \partial Y)$.*

PROOF. The proof is a simple adaptation from the closed case ([Lip06a, Lemma 2.1] in this cylindrical setting); the present case is also proved in [Lip06b, Lemma 2.6.1], which we



repeat here. Let z' be a point in the interior of $\bar{\Sigma}$ near z . Let $\Sigma' = (\bar{\Sigma}/\partial\bar{\Sigma}) \setminus \{z'\}$. There is an isomorphism

$$\pi_2(\mathbf{x}, \mathbf{x}) \cong H_2(\Sigma' \times [0, 1], (\bar{\alpha} \times \{1\}) \cup (\beta \times \{0\}))$$

gotten by adding (or subtracting) a copy of $\mathbf{x} \times [0, 1] \times \mathbb{R}$. From the long exact sequence for the pair $(\Sigma' \times [0, 1], (\bar{\alpha} \times \{1\}) \cup (\beta \times \{0\}))$, we have

$$0 \rightarrow H_2(\Sigma' \times [0, 1], (\bar{\alpha} \times \{1\}) \cup (\beta \times \{0\})) \rightarrow H_1((\bar{\alpha} \times \{1\}) \cup (\beta \times \{0\})) \rightarrow H_1(\Sigma').$$

(Note that $H_2(\Sigma' \times [0, 1]) = 0$.) Consequently,

$$\pi_2(\mathbf{x}, \mathbf{x}) \cong \ker(H_1(\bar{\alpha}/\partial\bar{\alpha}) \oplus H_1(\beta) \rightarrow H_1(\bar{\Sigma}/\partial\bar{\Sigma})).$$

But it follows by analyzing Construction 4.6 that this kernel is exactly $H_2(Y, \partial Y)$. \square

We sometimes denote the image of $B \in \pi_2(\mathbf{x}, \mathbf{x})$ in $H_2(Y, \partial Y)$ by $[B]$.

We split up the boundary of the domain of B into three pieces, $\partial^\alpha(B)$, contained in α , $\partial^\beta(B)$, contained in β , and $\partial^\partial(B)$, contained in $\partial\bar{\Sigma}$. We orient them so that the oriented boundary of the domain of B is $\partial^\alpha(B) + \partial^\beta(B) + \partial^\partial(B)$; in particular, this implies that $-\partial(\partial^\beta B) = \partial(\partial^\alpha B) + \partial(\partial^\partial B) = \mathbf{y} - \mathbf{x}$. We will think of $\partial^\partial B$ as an element of $H_1(\partial\bar{\Sigma}, \mathbf{a})$.

DEFINITION 4.19. A homology class B connecting \mathbf{x} to \mathbf{y} is called *provincial* if $\partial^\partial B = 0$. Let $\pi_2^\partial(\mathbf{x}, \mathbf{y})$ denote the set of provincial homology classes connecting \mathbf{x} to \mathbf{y} .

In other words, a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ belongs to $\pi_2^\partial(\mathbf{x}, \mathbf{y})$ if and only if the domain of B does not include any of the regions adjacent to $\partial\bar{\Sigma}$.

The concatenation maps obviously restrict to maps $*: \pi_2^\partial(\mathbf{x}, \mathbf{y}) \times \pi_2^\partial(\mathbf{y}, \mathbf{w}) \rightarrow \pi_2^\partial(\mathbf{x}, \mathbf{w})$, again making $\pi_2^\partial(\mathbf{x}, \mathbf{y})$ into a free, transitive $\pi_2^\partial(\mathbf{x}, \mathbf{x})$ -set (when non-empty). Elements of $\pi_2^\partial(\mathbf{x}, \mathbf{x})$, or their domains, are called *provincial periodic domains*.

LEMMA 4.20. *There is a natural group isomorphism $\pi_2^\partial(\mathbf{x}, \mathbf{x}) \cong H_2(Y)$.*

PROOF. Again, the proof is a simple adaptation of the closed case, and is given in [Lip06b, Lemma 2.6.4]. There is an isomorphism

$$\pi_2^\partial(\mathbf{x}, \mathbf{y}) \cong H_2(\bar{\Sigma} \times [0, 1], (\alpha \times \{1\}) \cup (\beta \times \{0\})).$$

Using the long exact sequence for the pair $(\bar{\Sigma} \times [0, 1], (\alpha \times \{1\}) \cup (\beta \times \{0\}))$, it follows that

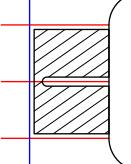
$$H_2(\bar{\Sigma} \times [0, 1], (\alpha \times \{1\}) \cup (\beta \times \{0\})) \cong \ker(H_1(\alpha) \oplus H_1(\beta) \rightarrow H_1(\bar{\Sigma})).$$

The Mayer-Vietoris sequence for the decomposition of Y along $\bar{\Sigma}$ identifies this kernel with $H_2(Y)$. \square

Abusing notation, we may also sometimes denote the image of $B \in \pi_2^\partial(\mathbf{x}, \mathbf{x})$ in $H_2(Y)$ by $[B]$.

Next we ask when $\pi_2(\mathbf{x}, \mathbf{y})$ and $\pi_2^\partial(\mathbf{x}, \mathbf{y})$ are non-empty. As in the closed case [OSz04d, Section 2.6], the answer relates to spin^c structures on Y . Also as in the closed case, our treatment is close to Turaev's [Tur97].

Choose a self-indexing Morse function f and metric g on Y inducing the bordered Heegaard diagram $(\bar{\Sigma}, \bar{\alpha}, \beta)$. Let \mathbf{x} be a generator. Using ∇f , z and \mathbf{x} we will construct a non-vanishing vector field $\vec{v}_z(\mathbf{x})$; this reduces the structure group of TY from $SO(3)$ to $SO(2) = U(1)$, and composing with the inclusion $U(1) \hookrightarrow U(2) = \text{spin}^c(3)$ given by $(e^{i\theta}) \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$ defines a spin^c structure on Y .



Each $x_i \in \mathbf{x}$ lies on a flow line γ_i of ∇f from an index 1 critical point of f to an index 2 critical point of f . The point z lies on a flow line γ from the index 0 critical point of f to the index 3 critical point of f . Let B denote a regular neighborhood of $\gamma_1 \cup \dots \cup \gamma_g \cup \gamma$. Since each flow line γ_i or γ runs between critical points of opposite parity, it is possible to extend $\nabla f|_{Y \setminus B}$ to a vector field $\vec{v}_z^0(\mathbf{x})$ on Y so that $\vec{v}_z^0(\mathbf{x})$ is non-vanishing on B . The vector field $\vec{v}_z^0(\mathbf{x})$ still has exactly k zeroes, one for each $\bar{\alpha}_i^a$ not containing an x_i . All of these zeroes lie on ∂Y , however, so one can modify $\vec{v}_z^0(\mathbf{x})$ in a neighborhood of these zeroes to produce a non-vanishing vector field $\vec{v}_z(\mathbf{x})$. This modification can be done in a standard way, depending only on $o(\mathbf{x})$, and not on the generator \mathbf{x} itself. The vector field $\vec{v}_z(\mathbf{x})$ induces the spin^c structure $\mathfrak{s}_z(\mathbf{x})$. It is routine to verify that $\mathfrak{s}_z(\mathbf{x})$ does not depend on the choices made in its construction.

Notice that the restriction of $\mathfrak{s}_z(\mathbf{x})$ to a collar neighborhood of ∂Y depends only on the set of occupied α -arcs $o(\mathbf{x})$, and not on \mathbf{x} itself. That is, for o a k -element subset of $\{\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a\}$, there is an induced spin^c structure $\mathfrak{s}_z(o)$ on a collar neighborhood of ∂Y . Given \mathbf{x} , let $\mathfrak{s}_z^{\text{rel}}(\mathbf{x})$ denote the relative spin^c structure on $(Y, \text{nbd}(\partial Y))$ induced by (z, \mathbf{x}) , relative to $\mathfrak{s}_z(o)$.¹

LEMMA 4.21. *Given generators \mathbf{x} and \mathbf{y} , $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$ if and only if $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$. Further, $\pi_2^\partial(\mathbf{x}, \mathbf{y}) \neq \emptyset$ if and only if $o(\mathbf{x}) = o(\mathbf{y})$ and $\mathfrak{s}_z^{\text{rel}}(\mathbf{x}) = \mathfrak{s}_z^{\text{rel}}(\mathbf{y})$.*

PROOF. Again, this is a straightforward adaptation of the closed case, and is also proved in [Lip06b, Lemmas 2.6.2 and 2.6.5].

On the one hand, the obstruction $\epsilon(\mathbf{x}, \mathbf{y})$ to finding a homology class connecting \mathbf{x} to \mathbf{y} is gotten as follows: join \mathbf{x} to \mathbf{y} by a union of paths $\gamma_\alpha \subset \boldsymbol{\alpha} \cup (\partial \bar{\Sigma} \setminus z)$ and by a union of paths $\gamma_\beta \subset \boldsymbol{\beta}$. Then \mathbf{x} and \mathbf{y} can be connected by a homology class if and only if $\gamma_\alpha - \gamma_\beta$ can be made null-homologous in $\bar{\Sigma}$ by adding or subtracting some α -curves and β -circles, i.e., if and only if the image $\epsilon(\mathbf{x}, \mathbf{y})$ of $\gamma_\alpha - \gamma_\beta$ in $H_1(\bar{\Sigma} \times [0, 1], \bar{\alpha} \times \{0\} \cup \boldsymbol{\beta} \times \{1\} \cup (\partial \bar{\Sigma} \setminus z) \times [0, 1]) \cong H_1(Y, \partial Y)$ vanishes.

On the other hand, the difference class $\mathfrak{s}_z(\mathbf{y}) - \mathfrak{s}_z(\mathbf{x})$ is gotten as follows. As in the definition of $\mathfrak{s}_z(\mathbf{x})$, after fixing a Morse function and metric inducing the Heegaard diagram, the generator \mathbf{x} corresponds to g flows $\gamma_{x_1}, \dots, \gamma_{x_g}$ in Y , and the generator \mathbf{y} corresponds to g flows $\gamma_{y_1}, \dots, \gamma_{y_g}$ in Y . The difference class is given by

$$\mathfrak{s}_z(\mathbf{y}) - \mathfrak{s}_z(\mathbf{x}) = [\gamma_{y_1} + \dots + \gamma_{y_g} - \gamma_{x_1} - \dots - \gamma_{x_g}] \in H_1(Y, \partial Y) \cong H^2(Y).$$

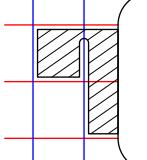
The chain $\gamma_{y_1} + \dots + \gamma_{y_g} - \gamma_{x_1} - \dots - \gamma_{x_g}$ is homologous in $(Y, \partial Y)$ to the chain $\gamma_\alpha - \gamma_\beta$ defining $\epsilon(\mathbf{x}, \mathbf{y})$.

The case for $\mathfrak{s}_z^{\text{rel}}$ is exactly analogous, except that now $\gamma_\alpha \subset \boldsymbol{\alpha}$, the group $H_1(\bar{\Sigma} \times [0, 1], \bar{\alpha} \times \{0\} \cup \boldsymbol{\beta} \times \{1\} \cup (\partial \bar{\Sigma} \setminus z) \times [0, 1]) \cong H_1(Y, \partial Y)$ is replaced by $H_1(\bar{\Sigma} \times [0, 1], \bar{\alpha} \times \{0\} \cup \boldsymbol{\beta} \times \{1\})$, and the group $H_1(Y, \partial Y)$ is replaced by $H_1(Y) \cong H^2(Y, \partial Y)$. \square

For \mathfrak{s} a spin^c structure on Y , define $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \subset \mathfrak{S}(\mathcal{H})$ to be those generators \mathbf{x} so that $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}$.

¹A spin^c-structure on Y is a lift of the classifying map $[TY]: Y \rightarrow BSO(3)$ to a map $\mathfrak{s}: Y \rightarrow \text{Bspin}^c(3)$. Given a spin^c-structure \mathfrak{s}_V on a subset $V \subset Y$, a relative spin^c-structure on Y relative to \mathfrak{s}_V is a lift $\mathfrak{s}: Y \rightarrow \text{Bspin}^c(3)$ with $\mathfrak{s}|_V = \mathfrak{s}_V$. The relative spin^c-structures on (Y, V) form an affine copy of $H^2(Y, V)$.

While we normally consider spin^c-structures up to homotopy (and relative spin^c-structures up to homotopy relative to V) to make sense of relative spin^c-structures on (Y, V) relative to \mathfrak{s}_V , it is important that the lift \mathfrak{s}_V over V be given on the nose, not just up to homotopy.



4.4. Admissibility criteria

As in [OSz04d], we need certain additional hypothesis to ensure the definitions of the differentials involve only finite sums. The idea is as follows. Given a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, in Chapter 5 we will construct a moduli space of holomorphic curves in the homology class B . In Chapters 6 and 7, we will use counts these moduli spaces, when they are 0-dimensional, to define the invariants of bordered 3-manifolds. For each B , such a count will be finite. But we will sum over all B (perhaps with given ∂^∂), so we must ensure that only finitely many such B can contribute. We will use the fact (Lemma 5.4) that if the moduli space of curves in class B is non-empty then B must be positive, in the following sense:

DEFINITION 4.22. A homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ is called *positive* if the coefficient in the domain of B of every component of $\bar{\Sigma} \setminus (\bar{\alpha} \cup \beta)$ is non-negative.

We will have two different analogues of the “weak admissibility” criterion of [OSz04d, Definition 4.10], one of which we call provincial admissibility and the other of which we call simply admissibility. We will see in Lemmas 4.28 and 4.29 that these conditions ensure that the set of positive homology classes with given ∂^∂ (respectively all positive homology classes) is finite.

In Chapters 6 and 7, provincial admissibility will be sufficient to define the invariants of bordered 3-manifolds. However, in order for the tensor product involved in gluing two bordered 3-manifolds to make sense, one of the two manifolds has to be admissible and not just provincially admissible.

DEFINITION 4.23. A bordered Heegaard diagram is called *provincially admissible* if every provincial periodic domain has both positive and negative coefficients.

DEFINITION 4.24. A bordered Heegaard diagram is called *admissible* if every periodic domain has both positive and negative coefficients.

PROPOSITION 4.25. *Every bordered Heegaard diagram is isotopic to an admissible bordered Heegaard diagram. Further, any two admissible bordered Heegaard diagrams can be connected through a sequence of Heegaard moves in which every intermediate Heegaard diagram is admissible. The same statements hold if “admissible” is replaced by “provincially admissible.”*

PROOF. This follows by winding transverse to the β -circles, as in the case of closed manifolds [OSz04d, Section 5]. \square

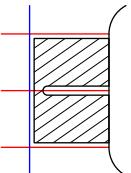
As in the closed case, there are reformulations of the two admissibility criteria:

LEMMA 4.26. *A bordered Heegaard diagram (Σ, α, β) is admissible if and only if there is an area function A on Σ such that $A(P) = 0$ for any periodic domain P . A bordered Heegaard diagram (Σ, α, β) is provincially admissible if and only if there is an area function A on Σ such that $A(P) = 0$ for any provincial periodic domain P .*

PROOF. The “only if” direction is immediate. For the “if” direction, the proof follows exactly as in the closed case, see [OSz04d, Lemma 4.12]. For the reader’s benefit, we recast the result here in terms of Farkas’s lemma from the theory of convex sets [Far02]:

LEMMA 4.27. *Let V be a vector space and $\{p_i\}_{i=1}^N \subset V$ a finite set of vectors. Then either*

- *there is a nonzero linear functional $\ell \in V^*$ such that $\ell(p_i) \geq 0$ for all i or*



- there are constants $c_i > 0$ such that $\sum_{i=1}^N c_i p_i = 0$.

Now, to prove the first statement, enumerate the regions R_1, \dots, R_N of Σ . Take V^* to be the vector space generated by the periodic domains in Σ , and $p_i \in V = V^{**}$ to be evaluation at the i^{th} region R_i . By assumption, there is no nonzero $P \in V^*$ such that $p_i(V) \geq 0$ for all i . Consequently, there are constants $c_i > 0$ such that $\sum_{i=1}^N c_i p_i = 0$. But then setting the area of R_i to be c_i is the desired area function.

The second statement follows by just the same reasoning, using only the provincial periodic domains. \square

PROPOSITION 4.28. *Suppose that \mathcal{H} is provincially admissible, as in Definition 4.23. Fix generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and an element $h \in H_1(Z', \mathbf{a})$. Then there are finitely many $B \in \pi_2(\mathbf{x}, \mathbf{y})$ such that $\partial^\partial B = h$ and all coefficients of B are non-negative.*

PROOF. Let A be an area form as guaranteed by Lemma 4.26. If $B, B' \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\partial^\partial B = \partial^\partial B' = h$ then $B - B'$ is a provincial periodic domain. Consequently the areas $A(B)$ and $A(B')$ are equal. But there are only finitely many positive domains of any given area. \square

PROPOSITION 4.29. *Suppose that \mathcal{H} is admissible, as in Definition 4.24. There are finitely many generators $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$ such that all coefficients of B are non-negative.*

PROOF. There are only finitely many generators, so we may first fix \mathbf{x} and \mathbf{y} . Let A be an area form as guaranteed by Lemma 4.26. If $B, B' \in \pi_2(\mathbf{x}, \mathbf{y})$ then $B - B'$ is a periodic domain. Consequently the areas $A(B)$ and $A(B')$ are equal. But there are only finitely many positive domains of any given area. \square

4.5. Closed diagrams

In this section we discuss the behavior of (bordered) Heegaard diagrams, generators and homology classes under cutting and gluing.

We will use minus signs to denote orientation reversal, so, for instance, $-Y$ is Y with its orientation reversed and $-\mathcal{H}$ is $\mathcal{H} = (\bar{\Sigma}, \bar{\alpha}, \beta, z)$ with the orientation of $\bar{\Sigma}$ reversed. Note that $Y(-\mathcal{H}) \cong -Y(\mathcal{H})$ and $F(-\mathcal{Z}) \cong -F(\mathcal{Z})$, canonically.

Gluing bordered Heegaard diagrams corresponds to gluing bordered 3-manifolds:

LEMMA 4.30. *Let $\mathcal{H}_1 = (\bar{\Sigma}_1, \bar{\alpha}_1, \beta_1, z_1)$ and $\mathcal{H}_2 = (\bar{\Sigma}_2, \bar{\alpha}_2, \beta_2, z_2)$ be bordered Heegaard diagrams for bordered 3-manifolds $(Y_1, \mathcal{Z}, \phi_1)$ and $(Y_2, \mathcal{Z}, \phi_2)$. Obtain a closed Heegaard diagram $-\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ by gluing $-\mathcal{H}_1$ and \mathcal{H}_2 along their boundaries, according to the markings from their pointed matched circles.*

$$-\mathcal{H}_1 \cup_\partial \mathcal{H}_2 = (\bar{\Sigma}_1 \cup_\partial \bar{\Sigma}_2, \bar{\alpha}_1 \cup_\partial \bar{\alpha}_2, \beta_1 \cup \beta_2, z_1 = z_2).$$

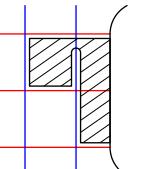
Then $-\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ represents $-Y_1 \cup_\partial Y_2 = -Y_1 \cup_{\phi_2 \circ \phi_1^{-1}} Y_2$.

PROOF. This is immediate from the Morse theory description. \square

Cutting bordered Heegaard diagrams corresponds to cutting bordered 3-manifolds:

LEMMA 4.31. *Let $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ be a (closed) Heegaard diagram and Z a separating curve such that*

- Z is disjoint from the β -circles (so that $\Sigma \setminus (\beta \cup Z)$ has exactly two components),



- $\Sigma \setminus (\alpha \cup Z)$ has exactly two components, and
- Z passes through the basepoint.

Write $\Sigma = \bar{\Sigma}_L \cup_Z \bar{\Sigma}_R$. Then $\mathcal{H}_L = (\bar{\Sigma}_L, \alpha \cap \bar{\Sigma}_L, \beta \cap \bar{\Sigma}_L, z)$ and $\mathcal{H}_R = (\bar{\Sigma}_R, \alpha \cap \bar{\Sigma}_R, \beta \cap \bar{\Sigma}_R, z)$ are bordered Heegaard diagrams. Further, there is a separating surface $F \subset Y(\mathcal{H})$ so that $Y(\mathcal{H}_L)$ and $Y(\mathcal{H}_R)$ are the two components of $Y(\mathcal{H}) \setminus F$.

PROOF. It is immediate from the hypotheses that \mathcal{H}_L and \mathcal{H}_R are bordered Heegaard diagrams. The fact that there is a surface $F \subset Y(\mathcal{H})$ as in the statement of the lemma follows from Lemma 4.30. Alternatively, F is the union of the flow lines through Z with respect to a Morse function and metric inducing \mathcal{H} . \square

Next we discuss how generators and homology classes behave under these cutting operations. Let \mathcal{H}_1 and \mathcal{H}_2 be bordered Heegaard diagrams with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$. Fix generators $\mathbf{x}_1 \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_2)$. If the corresponding sets of occupied α -arcs are disjoint, i.e., $o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset$, then the union $\mathbf{x}_1 \cup \mathbf{x}_2$ can be viewed as a generator in $\mathfrak{S}(\mathcal{H})$ for the Heegaard Floer complex of the closed manifold Y . We will call pairs of generators $(\mathbf{x}_1, \mathbf{x}_2)$ such that $o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset$ compatible pairs.

LEMMA 4.32. *Let $(\mathcal{H}_1, z) = (\Sigma_1, \alpha_1, \beta_1, z)$ and $(\mathcal{H}_2, z) = (\Sigma_2, \alpha_2, \beta_2, z)$ be two bordered Heegaard diagrams for the bordered three-manifolds $(Y_1, -\mathcal{Z}, \phi_1)$ and $(Y_2, \mathcal{Z}, \phi_2)$, with compatibly marked boundaries. Given $\mathbf{x}_1, \mathbf{y}_1 \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{x}_2, \mathbf{y}_2 \in \mathfrak{S}(\mathcal{H}_2)$, we have a natural identification of $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$ with the subset of $\pi_2(\mathbf{x}_1, \mathbf{y}_1) \times \pi_2(\mathbf{x}_2, \mathbf{y}_2)$ consisting of pairs (B_1, B_2) with $\partial^\partial(B_1) + \partial^\partial(B_2) = 0$. Moreover, given $p \in \Sigma_2 \setminus (\alpha \cup \beta)$ and $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$ for $i = 1, 2$ which agree along the boundary, the local multiplicity of B_2 at p coincides with the local multiplicity of the glued homology class in $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$ (under the above identification) at the point p , now thought of as a point in $\Sigma = \Sigma_1 \cup \Sigma_2$.*

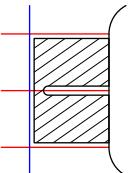
PROOF. The proof is straightforward. \square

For B_1 and B_2 which agree along the boundary, let $B_1 \natural B_2$ be the homology class in $\pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$ guaranteed by Lemma 4.32.

Finally, we give a criterion for when a closed Heegaard diagram is admissible.

LEMMA 4.33. *Let \mathcal{H}_1 and \mathcal{H}_2 be two bordered Heegaard diagrams with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$. If \mathcal{H}_1 is admissible and \mathcal{H}_2 is provincially admissible, then $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ is admissible.*

PROOF. Let $B_1 \natural B_2$ be a positive periodic domain for \mathcal{H} . Since \mathcal{H}_1 is admissible, $B_1 = 0$ and so $\partial^\partial B_2 = 0$. Then since \mathcal{H}_2 is provincially admissible, $B_2 = 0$. \square



CHAPTER 5

Moduli spaces

Let Y be a bordered 3-manifold. In order to define invariants of Y we will choose a bordered Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \beta, z)$ for Y and count J -holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ with boundary conditions coming from $\overline{\alpha}$ and β . In this chapter we give the technical results concerning the moduli spaces of holomorphic curves that we use to construct the invariants $\widehat{CFD}(\mathcal{H})$ and $\widehat{CFA}(\mathcal{H})$. We begin in Section 5.1 with an outline of the main results that we will need and the ideas in the proof; the precise mathematics begins in Section 5.2 below.

5.1. Overview of the moduli spaces

In this chapter, we will discuss J -holomorphic curves¹

$$u: (S, \partial S) \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R}))$$

where S is a Riemann surface with boundary and punctures. Here we view Σ , the interior of $\overline{\Sigma}$, as a Riemann surface with one puncture, denoted p . As in Section 4.3, the coordinate on the $[0, 1]$ factor is denoted s and the coordinate on the \mathbb{R} factor is denoted t .

We view the manifold $\Sigma \times [0, 1] \times \mathbb{R}$ as having three different infinities: $\Sigma \times [0, 1] \times \{+\infty\}$, $\Sigma \times [0, 1] \times \{-\infty\}$ and $p \times [0, 1] \times \mathbb{R}$, which we refer to as $+\infty$, $-\infty$ and $e\infty$ respectively. (Here, e stands for “east”.) The asymptotics of the holomorphic curves u we consider are:

- At $\pm\infty$, u is asymptotic to a g -tuple of chords of the form $x_i \times [0, 1] \times \{\pm\infty\}$, where $\mathbf{x} = \{x_i\}_{i=1}^g$ is a generator in the sense of Definition 4.13.
- At $e\infty$, u is asymptotic to a finite collection of Reeb chords $\{\rho_i \times \{(1, t_i)\}\}$. Here ρ_i is a Reeb chord in the ideal contact boundary $\partial\overline{\Sigma}$ (also called Z) with endpoints on $\mathbf{a} = \alpha \cap Z$ (see Section 3.1.3). We refer to t_i as the *height* of the chord ρ_i .

The set of east punctures of u is partially ordered by the heights of the corresponding Reeb chords. This induces an *ordered partition* $\vec{P} = (P_1, P_2, \dots, P_\ell)$ of the east punctures, where each P_i consists of those punctures occurring at one particular height. (For $\widehat{CFD}(\mathcal{H})$ we only need to consider discrete partitions where each P_i has only one puncture.)

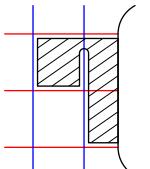
With this background, we consider the moduli space

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$$

(Definition 5.10), which consists of curves from the *decorated source* S^\triangleright , asymptotic to \mathbf{x} at $-\infty$ and \mathbf{y} and $+\infty$, in the homology class B (as in Section 4.3) and respecting the ordered partition \vec{P} of the Reeb chords. Here the source surface S is decorated with labels describing its asymptotics (Definition 5.2). We also consider the reduced moduli space, its quotient

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) := \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) / \mathbb{R}$$

¹With respect to an appropriate J ; see Definition 5.1 below.



by translation in the t coordinate.

These moduli spaces are smooth manifolds (Proposition 5.6) whose dimensions are easy to compute (Equation (5.9)). In fact, for an *embedded* curve the dimension is independent of the source S and depends only on the asymptotics and homology class of the curve (Section 5.7); this gives rise to the gradings on the algebra and on the modules.

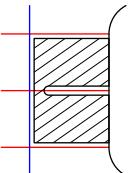
As is typical in symplectic geometry, our invariants are defined by looking only at *rigid* moduli spaces, those $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ which are 0-dimensional. To show that $\partial^2 = 0$ and for proofs of invariance, we also need to consider 1-dimensional moduli spaces, and, in particular, we need appropriate compactifications. These are provided by the spaces $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$, which include *holomorphic combs*. The idea is that, as in symplectic field theory, some parts of a holomorphic curve may go to infinity relative to other parts. In our case, this can result in a degeneration at $\pm\infty$ or at $e\infty$. When a curve degenerates at $\pm\infty$, the result is two or more curves of the kind we have already discussed. A degeneration at $e\infty$ creates a new kind of object: a curve in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$. Section 5.3 is devoted to discussing these curves at east infinity. This compactification is defined in Definition 5.22, and a schematic illustration of a holomorphic comb can be found in Figure 5.6.

The main result we need for our purposes is a classification of the codimension 1 degenerations (Section 5.6). Proposition 5.43 and Lemma 5.56 say that for the curves we consider, these degenerations come in four kinds:

- Degenerating into a two-story building at $\pm\infty$. These are exactly the kinds of degenerations that occur in Heegaard Floer homology for closed three-manifolds.
- A boundary branch point of $\pi_\Sigma \circ u$ can go off to $e\infty$, in the process splitting a Reeb chord in two. Below, this is referred to as degenerating a “join curve” at $e\infty$. (From the point of view of the curve at $e\infty$, two Reeb chords come together and merge as we travel further away from the main surface.)
- A collapse in the ordering of \vec{P} , i.e., the heights of two parts of \vec{P} coming together. In the process, some “split curves” can degenerate at $e\infty$. These again correspond to boundary branch points of $\pi_\Sigma \circ u$ approaching $e\infty$, but in a combinatorially different way from join curves. (In the curve at $e\infty$, one (or more) Reeb chord splits apart as we travel further away from the main surface.)
- Degenerating a “shuffle curve” at east ∞ . This degeneration involves either an interior branch point or two boundary branch points approaching east ∞ , and is explained in more detail at the end of Section 5.3; see in particular Figure 5.5 for a domain in which a shuffle curve degenerates. This type of degeneration is not relevant for \widehat{CFD} , only for \widehat{CFA} . (In the curve at $e\infty$, two Reeb chords recombine with each other to create two different Reeb chords as we travel further away from the main surface.)

Figure 5.1 shows examples in which the first three kinds of degenerations occur.

Our next goal is to obtain an identity relating the number of degenerations of the four types above. The usual way to do this is to show that if the expected dimension $\text{ind}(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is 2, then the space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is a compact one-manifold with boundary given by all holomorphic combs with the correct combinatorics to occur as one of these degenerations. The identity would then follow from the fact that a compact one-manifold has an even number of endpoints.



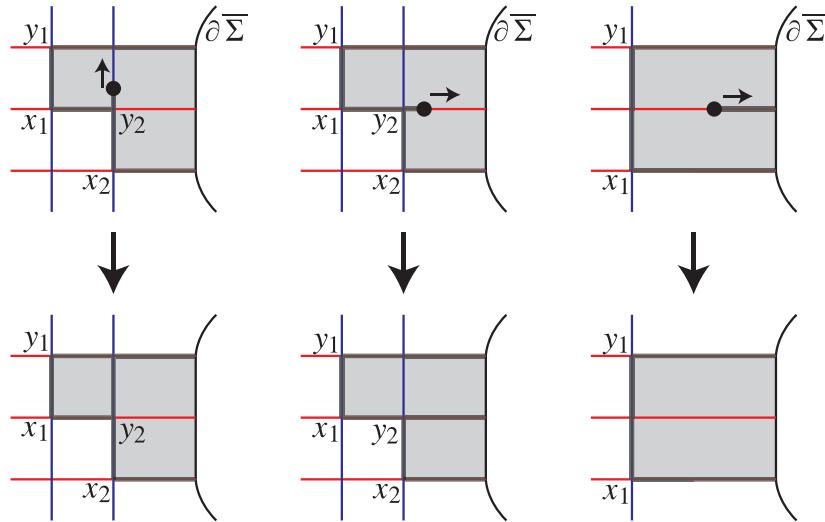


FIGURE 5.1. **Examples of three kinds of codimension 1 degenerations.**

The large black dot represents a boundary branch point of $\pi_\Sigma \circ u$. Left: degenerating into a two-story building. Center: degenerating a join curve. Right: degenerating a split curve. The diagrams show the projection of the curve in $\Sigma \times [0, 1] \times \mathbb{R}$ to Σ .

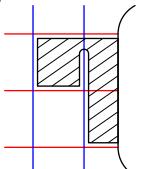
Unfortunately, available gluing technology seems ill equipped for this approach, the difficulty being with split and shuffle curve degenerations. (See Remark 5.41 and the beginning of Section 5.6.4 for more discussion of this point.) Instead, we use a somewhat indirect argument to show that there are an even number of holomorphic combs of the types that may appear as degenerations in an index 2 moduli space (Theorem 5.61). This is all that we will need later to define the invariants.

This chapter is organized as follows. Section 5.2 defines the moduli spaces of curves in $\Sigma \times [0, 1] \times \mathbb{R}$ of interest to us, and Section 5.3 the moduli spaces for curves at east infinity. In Section 5.4, we put these moduli spaces together to define the compactified moduli spaces of so-called holomorphic combs and prove that they form compactifications. Next, in Section 5.5 we prove the gluing results for combs needed for the rest of the paper. In Section 5.6 we restrict what kind of degenerations can occur in codimension-one; in particular, we show that only those degenerations for which we proved gluing results are possible, leading to Theorem 5.61. Finally, in Section 5.7 we study the index of the $\bar{\partial}$ -operator at an embedded curve, and use this to give further restrictions on the codimension-one boundaries of moduli spaces of embedded curves.

Fortified by this overview, we turn to the mathematics itself.

5.2. Holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$

In this section, we will define various moduli spaces of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ with names like $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$, $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ and $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. In Chapters 6 and 7 we will use the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ to define our invariants of bordered 3-manifolds. These are, consequently, the spaces in which we will ultimately be interested. However, it



is easier to formulate several technical results by describing the spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ as subspaces of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ and $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, which we introduce first.

As before, let \mathcal{H} be a bordered Heegaard diagram of genus g representing a manifold Y with boundary of genus k . Choose a symplectic form ω_Σ on Σ , with respect to which the boundary of Σ is a cylindrical end. Let j_Σ be a complex structure on Σ compatible with ω_Σ . We will further assume that the α -arcs α^a are cylindrical near p in the following sense. Fix a punctured neighborhood U_p of p , and a symplectic identification $\phi : U_p \xrightarrow{\cong} S^1 \times (0, \infty) \subset T^*S^1$. Then we assume that both $j_\Sigma|_{U_p}$ and $\phi(\alpha_i^a \cap U_p)$ are invariant with respect to the \mathbb{R} -translation action in $S^1 \times (0, \infty)$. With respect to j_Σ , the boundary is a puncture, which we denote p .

Let $\Sigma_{\bar{e}}$ be the result of filling in the puncture of Σ , so j_Σ induces a complex structure on $\Sigma_{\bar{e}}$.

We consider moduli spaces of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$. Let

$$\begin{aligned}\pi_\Sigma &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \Sigma, \\ \pi_D &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}, \\ s &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow [0, 1], \text{ and} \\ t &: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}\end{aligned}$$

denote the projections. We equip $[0, 1] \times \mathbb{R}$ with the symplectic form $\omega_D := ds \wedge dt$ and the complex structure j_D with $j_D\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial t}$, and equip $\Sigma \times [0, 1] \times \mathbb{R}$ with the split symplectic form $\pi_\Sigma^*(\omega_\Sigma) + \pi_D^*(\omega_D)$. Note also that there is an \mathbb{R} -action on $\Sigma \times [0, 1] \times \mathbb{R}$ by translation on the t coordinate.

DEFINITION 5.1. We say an almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ is *admissible* if the following conditions are satisfied:

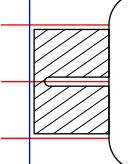
- (J-1) The projection map π_D is J -holomorphic.
- (J-2) For $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ the vector fields tangent to the fibers of π_Σ induced by s and t respectively, $J\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$ (and consequently the fibers of π_Σ are J -holomorphic).
- (J-3) The \mathbb{R} -action is J -holomorphic.
- (J-4) The complex structure is split, i.e., $J = j_\Sigma \times j_D$, over some (fixed, \mathbb{R} -invariant) neighborhood of $p \times [0, 1] \times \mathbb{R}$.

For example, the split complex structure $j_\Sigma \times j_D$ is admissible in the above sense. Fix now an admissible J ; later, we will assume J is generic in certain senses.

DEFINITION 5.2. By a *decorated source* S^\triangleright we mean:

- (1) a smooth (not nodal) Riemann surface S with boundary and punctures on the boundary (i.e., a compact Riemann surface with boundary minus finitely many points on the boundary),
- (2) a labeling of each puncture of S by one of $+$, $-$, or e and
- (3) a labeling of each e puncture of S by a Reeb chord in (Z, \mathbf{a}) , as defined in Section 3.1.3.

We view two decorated sources as equivalent if there is an orientation-preserving diffeomorphism between them which respects the labelings of the punctures.



Given a decorated source S^\triangleright , let $S_{\bar{e}}$ denote the result of filling in the e punctures of S . We will consider maps

$$u: (S, \partial S) \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R}))$$

such that

- (M-1) The map u is (j, J) -holomorphic with respect to some almost complex structure j on S .
- (M-2) The map $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ is proper.
- (M-3) The map u extends to a proper map $u_{\bar{e}}: S_{\bar{e}} \rightarrow \Sigma_{\bar{e}} \times [0, 1] \times \mathbb{R}$.
- (M-4) The map $u_{\bar{e}}$ has finite energy in the sense of Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder [BEH⁺03] (see also Section 5.4).
- (M-5) $\pi_{\mathbb{D}} \circ u$ is a g -fold branched cover. In particular, $\pi_{\mathbb{D}} \circ u$ is non-constant on every component of S .
- (M-6) At each $-$ -puncture q of S , $\lim_{z \rightarrow q} (t \circ u)(z) = -\infty$.
- (M-7) At each $+$ -puncture q of S , $\lim_{z \rightarrow q} (t \circ u)(z) = +\infty$.
- (M-8) At each e puncture q of S , $\lim_{z \rightarrow q} (\pi_\Sigma \circ u)(z)$ is the Reeb chord ρ labeling q .
- (M-9) $\pi_\Sigma \circ u$ does not cover the region of Σ adjacent to z .
- (M-10) For each $t \in \mathbb{R}$ and each $i = 1, \dots, g$, $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of exactly one point. Similarly, for each $t \in \mathbb{R}$ and each $i = 1, \dots, g - k$, $u^{-1}(\alpha_i^c \times \{1\} \times \{t\})$ consists of exactly one point.

We call Condition (M-10) *weak boundary monotonicity*. Sometimes later we will impose the following additional condition, which we call *strong boundary monotonicity*:

- (M-11) For each $t \in \mathbb{R}$ and each $i = 1, \dots, 2k$, $u^{-1}(\alpha_i^a \times \{1\} \times \{t\})$ consists of at most one point.

However, our moduli spaces are easiest to define if we do not initially impose this condition.

It follows from the Conditions (M-1)–(M-10) that at $-\infty$, u is asymptotic to a g -tuple of chords of the form $x_i \times [0, 1]$, where $x_i \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$. By weak boundary monotonicity, the set $\mathbf{x} = \{x_i\}$ has the property that exactly one point of \mathbf{x} lies on each α -circle, and one point on each β -circle. However, more than one point of \mathbf{x} may lie on the same α -arc. We call such a g -tuple a *generalized generator*. If the strong boundary monotonicity condition is also satisfied then at most one x_i lies on each α -arc, and hence \mathbf{x} is a generator in the sense of Section 4.3. Exactly analogous statements hold at $+\infty$.

Any curve u satisfying Conditions (M-1)–(M-10) belongs to some homology class $[u] \in \pi_2(\mathbf{x}, \mathbf{y})$. (Here, $\pi_2(\mathbf{x}, \mathbf{y})$ is the obvious generalization of the notion from Section 4.3 to the case of generalized generators.) We collect these curves into moduli spaces:

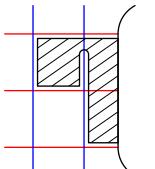
DEFINITION 5.3. Given generalized generators \mathbf{x} and \mathbf{y} and a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, let

$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$$

denote the moduli space of curves satisfying Conditions (M-1)–(M-10), with decorated source S^\triangleright , asymptotic to \mathbf{x} at $-\infty$ and \mathbf{y} at $+\infty$ and in the homology class B .

The domains of holomorphic curves (in the sense of Definition 4.15) are always positive.

LEMMA 5.4. *If the moduli space $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is non-empty then the homology class B is positive (in the sense of Definition 4.22).*



PROOF. Let u be an element of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$. The multiplicity of B at a region R is given by the intersection number $u \cdot (\{p\} \times [0, 1] \times \mathbb{R})$. By Condition (J-2) of Definition 5.1, the fiber $\{p\} \times [0, 1] \times \mathbb{R}$ is J -holomorphic, so $u \cdot (\{p\} \times [0, 1] \times \mathbb{R})$ is the intersection number between two J -holomorphic curves, which is positive by [McD94] or [MW95]. \square

Given $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$, for each puncture q of S labeled by e , let $ev_q(u) = t \circ u_{\bar{e}}(q)$. This gives an evaluation map $ev_q: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}$. Set

$$ev = \prod_{q \in E(S^\triangleright)} ev_q: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^{E(S^\triangleright)},$$

where $E(S^\triangleright)$ (or just E , when unambiguous) is the set of east punctures of S^\triangleright .

We next use the evaluation maps to define certain subspaces of the $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$. Fix a partition $P = \{P_i\}$ of E . By the *partial diagonal* Δ_P in \mathbb{R}^E we mean the subspace of \mathbb{R}^E defined by the set of equations $\{x_p = x_q \mid P_i \in P, p, q \in P_i\}$. That is, Δ_P is obtained by requiring all coordinates in each part P_i of P to be equal. Now, we use the cycle Δ_P to cut down the moduli spaces:

DEFINITION 5.5. Given generalized generators \mathbf{x} and \mathbf{y} , $B \in \pi_2(\mathbf{x}, \mathbf{y})$, a decorated source S^\triangleright , and a partition P of the e punctures of S^\triangleright , let

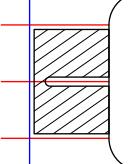
$$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := ev^{-1}(\Delta_P) \subset \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright).$$

Notice that $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is the special case of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ when P is the discrete partition, with one element per part.

We turn now to the meaning of genericity for J .

PROPOSITION 5.6. *In the space of (C^r or C^∞) admissible almost complex structures there is a dense set of J with the property that the moduli spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ are transversally cut out by the $\bar{\partial}$ -equations. Indeed, for any countable set $\{M_i\}$ of submanifolds of \mathbb{R}^E , there is a dense set of admissible J which satisfy the further property that $ev: \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^E$ is transverse to all of the M_i .*

This result follows from standard arguments; see [MS04, Chapter 3] for a nice explanation in a slightly different context, or [Lip06a, Proposition 3.8] for the proof of the first half of the statement in a closely related setting. Briefly, one considers the Banach manifold \mathcal{B} of triples (j_S, J, u) where j_S is a point in the Teichmüller space of complex structures on S , J is a C^ℓ complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying Conditions (J-1)–(J-4), and $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ is a $W_\delta^{1,p}$ map satisfying Conditions (M-2)–(M-10). (Here, $W_\delta^{1,p}$ denotes a weighted Sobolev space; see, for example, [Sei08a, Section (8h)].) The $\bar{\partial}$ map defines a section of a vector bundle \mathcal{E} over \mathcal{B} , whose fiber is the space of $(0, 1)$ -forms on S valued in $u^*T(\Sigma \times [0, 1] \times \mathbb{R})$. Applying elliptic regularity one sees that this section $\bar{\partial}$ is transverse to the 0-section of \mathcal{E} , so $\bar{\partial}^{-1}(0)$ is a Banach manifold. An infinite-dimensional version of Sard's theorem guarantees that a generic complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ (satisfying Conditions (J-1)–(J-4)) is a regular value of the projection $\bar{\partial}^{-1}(0) \rightarrow \mathcal{J}$ to the space \mathcal{J} of almost complex structures satisfying Conditions (J-1)–(J-4). For any such regular value, the moduli spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ are transversally cut out. (Note that Condition (M-5) guarantees that, even after we quotient by automorphisms, the moduli space has no orbifold points.)



For the second half of the statement, one verifies further that as a map from $\overline{\partial}^{-1}(0)$ to \mathbb{R}^E , evaluation at the marked points is a submersion; the result then follows. See [MS04, Section 3.4] for a more detailed discussion of this part in a closely related setting.

In particular, by Proposition 5.6, we can choose J satisfying the following definition:

DEFINITION 5.7. An admissible almost complex structure is called *sufficiently generic* if the moduli spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ are transversely cut out for all choices of \mathbf{x} , \mathbf{y} , B , S^\triangleright and P .

From now on, we will always work with sufficiently generic almost complex structures, unless we explicitly say otherwise.

The expected dimension of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is not hard to calculate:

PROPOSITION 5.8. The expected dimension $\text{ind}(B, S^\triangleright, P)$ of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is

$$(5.9) \quad \text{ind}(B, S^\triangleright, P) = g - \chi(S) + 2e(B) + |P|.$$

Here $|P|$ denotes the number of parts in P and $e(B)$ is the Euler measure of the domain of B .

Here the *Euler measure* of a region in $\Sigma \setminus (\alpha \cup \beta)$ is its Euler characteristic minus $1/4$ the number of corners (intersections of α -curves and β -curves or α -arcs and $\partial\Sigma$), and is defined to be additive under union. The Euler measure can be viewed as the integral of the Gaussian curvature of the region with respect to any Riemannian metric for which the boundary, the α -curves and the β -circles are all geodesics, meeting at right angles at the corners.

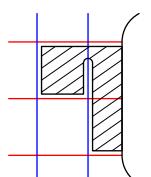
PROOF OF PROPOSITION 5.8. This proof is given in [Lip06b, Proposition 4.5.1]; we repeat it here for the reader’s convenience. In brief, the Proposition follows from the corresponding index formula for closed Heegaard Floer homology, [Lip06a, Formula (6)], by a standard doubling argument at the east punctures, cf. [Bou06, Section 5]. (The formula [Lip06a, Formula (6)] itself follows from the Riemann-Roch theorem by another doubling argument.)

Let $\bar{\Sigma}^\cap$ denote the result of gluing a collar $\partial[0, \epsilon) \times \bar{\Sigma}$ to $\partial\bar{\Sigma}$. Inside $[0, \epsilon) \times (\partial\bar{\Sigma} \setminus z)$, choose arcs connecting every pair of (ends of) α -arcs, as shown in Figure 5.2. Call the new arcs $\bar{\alpha}_0^\cap$, and let $\alpha^\cap = \bar{\alpha} \cup \bar{\alpha}_0^\cap$. (This “caps off” the Reeb chords at east infinity.) We call the result the *capped diagram*. By attaching cylindrical ends to them, we can think of the $\bar{\alpha}_0^\cap$ as arcs α_0^\cap lying in $\mathbb{R} \times \partial\bar{\Sigma}$.

For each Reeb chord ρ_i there is a holomorphic disk D_{ρ_i} in $(\mathbb{R} \times \partial\bar{\Sigma} \times [0, 1] \times \mathbb{R}, \alpha_0^\cap \times \{1\} \times \mathbb{R})$ asymptotic to γ_i . Given a map $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ where the e punctures of S^\triangleright are decorated by Reeb chords $\rho_{i_1}, \dots, \rho_{i_n}$ one can preglue u to $\bigcup_{j=1}^n D_{\rho_{i_j}}$ to obtain an approximately holomorphic map $u^\cap: S^\cap \rightarrow \Sigma^\cap \times [0, 1] \times \mathbb{R}$.

Each height constraint decreases both $|P|$ and the expected dimension by one. So, it suffices to prove the formula in the case that P is the discrete partition. Recall from [Lip06a, Formula (6), Section 4.1] that in the closed case the index at a map $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ is given by $g - \chi(S) + 2e(B)$. The closed formula applies to the linearization of the $\bar{\partial}$ -operator at u^\cap , giving $\text{ind}(u^\cap) = g - \chi(S^\cap) + 2e(B^\cap) = g - \chi(S) + 2e(B^\cap)$, where B^\cap is the domain in $\bar{\Sigma}^\cap$ corresponding to the domain B in $\bar{\Sigma}$. The index of $D\bar{\partial}$ at any of the disks D_ρ is 1. It follows that the index at u is

$$\text{ind}(u) = g - \chi(S) + 2(e(B) + |P|/2) - |P| + |P|,$$



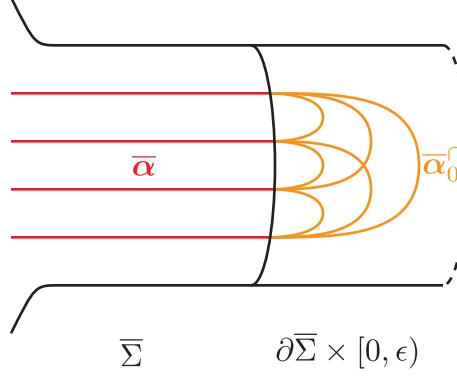


FIGURE 5.2. **A capping operation.** Part of the surface $\bar{\Sigma}^\cap$ and the arcs $\bar{\alpha}^\cap$ are shown; on the left side is $\bar{\Sigma}$ and $\bar{\alpha}$, and on the right side is $[0, \epsilon) \times \partial\bar{\Sigma}$ and $\bar{\alpha}_0^\cap$.

where the $|P|/2$ comes from the effect of capping on the Euler measure, the $-|P|$ from the fact that $\text{ind}(D_\rho) = 1$, and the $+|P|$ comes from the matching conditions at the punctures (because of the Morse-Bott nature of the gluing). This proves the proposition. \square

Suppose $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Then the t -coordinate of u induces a partial ordering on the partition P . This combinatorial data allows us to divide $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ into different strata.

DEFINITION 5.10. Given \mathbf{x} , \mathbf{y} , and S^\triangleright as above and an ordered partition \vec{P} of the e punctures of S^\triangleright , with P the corresponding unordered partition, let $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ denote the (open) subset of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ consisting of those holomorphic curves for which the ordering of P induced by t agrees with the ordering given in \vec{P} .

That is, for $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ and $\vec{P} = (P_1, \dots, P_m)$, we have $u \in \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ if and only if for all $i < i'$, $q \in P_i$ and $q' \in P_{i'}$ implies $t \circ u(q) < t \circ u(q')$.

Given a partition P of $E(S^\triangleright)$, there is an associated set $[P]$ of multi-sets of Reeb chords, given by replacing each puncture of S^\triangleright in P by the associated Reeb chord. Similarly, given an ordered partition \vec{P} of $E(S^\triangleright)$ there is an associated sequence $[\vec{P}]$ of multi-sets of Reeb chords.

The \mathbb{R} -action on $\Sigma \times [0, 1] \times \mathbb{R}$ by translation on \mathbb{R} induces an \mathbb{R} -action on each $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ and $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. We denote this action by τ_R for $R \in \mathbb{R}$. The action is free except in the trivial case that S^\triangleright consists of g disks with two boundary punctures each, and $B = 0$. We say that a curve is *stable* if it is not this trivial case. We are interested primarily in the quotient by this action:

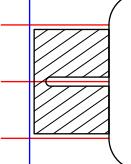
DEFINITION 5.11. Given \mathbf{x} , \mathbf{y} , B , S^\triangleright , and P or \vec{P} as above, let

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)/\mathbb{R}$$

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) := \widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})/\mathbb{R}.$$

denote the reduced moduli spaces.

The expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ (except in the trivial case $B = 0$) is given by $\text{ind}(B, S^\triangleright, P) - 1$.



Although the evaluation maps ev_q (for $q \in E$) do not descend to the quotient by translation, the difference between any two of them, $ev_p - ev_q$, does descend. We denote this difference by

$$(5.12) \quad ev_{p,q}: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \rightarrow \mathbb{R}.$$

We can also combine all the evaluation maps into a single map

$$ev: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \rightarrow \mathbb{R}^E / \mathbb{R}$$

where \mathbb{R} acts diagonally by translation on \mathbb{R}^E .

5.3. Holomorphic curves in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$

In order to discuss the compactifications of the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ and $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$, we will need to consider certain holomorphic curves “at east ∞ ,” i.e., in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$. We endow $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ with the obvious split symplectic form. Inside this space, we have $4k$ Lagrangian planes $\mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R}$. The space $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ has four different ends: the ends $\pm\infty$ in the first \mathbb{R} -factor, which we call east and west infinity respectively, and the ends $\pm\infty$ in the second \mathbb{R} -factor, which we call $\pm\infty$. Note that there is an $(\mathbb{R} \times \mathbb{R})$ -action on $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$, by translation in the two \mathbb{R} -factors. There are projection maps π_Σ , π_D , s , and t , just as before.

Fix a split complex structure $J = j_\Sigma \times j_D$ on $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$.

DEFINITION 5.13. By a *bi-decorated source* T^\diamondsuit we mean:

- (1) A topological type of smooth (not nodal) Riemann surface T with boundary and punctures on the boundary.
- (2) A labeling of each puncture of T by either e or w (east or west), and
- (3) A labeling of each puncture of T by a Reeb chord ρ in (Z, \mathbf{a}) .

Given a bi-decorated source T^\diamondsuit , we will consider maps

$$v: (T, \partial T) \rightarrow (\mathbb{R} \times (Z \setminus z) \times [0, 1] \times \mathbb{R}, \mathbf{a} \times \mathbb{R} \times \{1\} \times \mathbb{R})$$

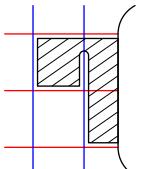
satisfying the following conditions:

- (1) The map v is (j, J) -holomorphic with respect to some almost complex structure j on S .
- (2) The map v is proper.
- (3) The map $s \circ v$ maps ∂T to 1 .
- (4) At each west puncture q of S labeled by a Reeb chord ρ , $\lim_{z \rightarrow q} \pi_\Sigma \circ u(z)$ is $\rho \subset \{-\infty\} \times Z$.
- (5) At each east puncture q of S labeled by a Reeb chord ρ , $\lim_{z \rightarrow q} \pi_\Sigma \circ u(z)$ is $\rho \subset \{+\infty\} \times Z$.

Note that each component of such a holomorphic curve necessarily maps to a single point under π_D by the maximum principle, since its boundary maps entirely to $s = 1$.

Again, we collect these holomorphic curves into moduli spaces:

DEFINITION 5.14. For T^\diamondsuit a bi-decorated source let $\tilde{\mathcal{N}}(T^\diamondsuit)$ denote the moduli space of holomorphic maps from T satisfying properties (1)–(5) above.



A holomorphic map v is called *stable* if there are no infinitesimal automorphisms of v . (If $\pi_\Sigma \circ v$ is non-constant on each component of T , one can verify that T is stable if and only if it has a component which is not a twice-punctured disk.) We shall be interested in $\tilde{\mathcal{N}}(T^\diamond)$ only in the cases that the corresponding maps are stable.

Suppose that q is a puncture of T^\diamond . Then there is an evaluation map

$ev_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}$ given by $ev_q(v) = \lim_{z \rightarrow q} t \circ v(z)$. (Here, writing \lim is somewhat silly, since $t \circ v$ is constant on each connected component of T^\diamond .) There are, consequently, evaluation maps

$$ev_w := \prod_{q \in W(S^\diamond)} ev_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}^{W(T^\diamond)} \quad \text{and}$$

$$ev_e := \prod_{q \in E(S^\diamond)} ev_q: \tilde{\mathcal{N}}(T^\diamond) \rightarrow \mathbb{R}^{E(T^\diamond)}$$

where $W(T^\diamond)$ or W is the set of west punctures of T^\diamond and $E(T^\diamond)$ or E is the set of east punctures.

We define sub-moduli spaces of $\tilde{\mathcal{N}}(T^\diamond)$ by cutting-down by partial diagonals.

DEFINITION 5.15. Given a bi-decorated source T^\diamond and partitions P_w and P_e of the west and east punctures of T^\diamond , respectively, let

$$\tilde{\mathcal{N}}(T^\diamond; P_w, P_e) = (ev_w \times ev_e)^{-1}(\Delta_{P_w} \times \Delta_{P_e}).$$

If P_w is the discrete partition, we denote $\tilde{\mathcal{N}}(T^\diamond; P_w, P_e)$ by $\tilde{\mathcal{N}}(T^\diamond; P_e)$.

The moduli spaces $\tilde{\mathcal{N}}(T^\diamond)$ can be understood concretely. Indeed, they are determined by the t -coordinates of the components of T , and a holomorphic map from T^\diamond to Σ , which gives $\pi_\Sigma \circ v$ (which in turn is determined by local branching data over Σ). However, in general the moduli spaces $\tilde{\mathcal{N}}(T^\diamond)$ are not transversally cut out. One situation in which they are—and indeed this is the case which is relevant for our applications—is the following:

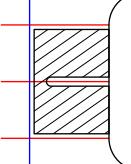
PROPOSITION 5.16. *Suppose T^\diamond is a decorated source such that all components of T are topological disks. Then $\tilde{\mathcal{N}}(T^\diamond)$ is transversally cut out by the $\bar{\partial}$ equation for any split complex structure on $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$.*

PROOF. This is proved by explicit computation of the cokernel of the $\bar{\partial}$ -operator, which is identified with a certain sheaf cohomology group. See McDuff and Salamon [[MS04](#), Section 3.3] for a nice exposition of these ideas in the absolute case. (The relative case can be deduced from the absolute case via a doubling argument; see, for instance, Hofer, Lizan, and Sikorav [[HLS97](#), Section 4].) More details can also be found in [[Lip06b](#), Lemma 4.1.2]. \square

The $(\mathbb{R} \times \mathbb{R})$ -translation action on $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ induces an $(\mathbb{R} \times \mathbb{R})$ -action on $\tilde{\mathcal{N}}(T^\diamond)$ for any T^\diamond ; if $\tilde{\mathcal{N}}(T^\diamond)$ is stable then (almost by definition) this $(\mathbb{R} \times \mathbb{R})$ -action is free. As for the earlier moduli spaces, we are primarily interested in the quotient space:

DEFINITION 5.17. Given a stable decorated source T^\diamond , let $\mathcal{N}(T^\diamond) = \tilde{\mathcal{N}}(T^\diamond)/(\mathbb{R} \times \mathbb{R})$.

Certain holomorphic curves in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ will play a special role below. By a *trivial component* we mean a component of T^\diamond which is a topological disk with exactly two boundary punctures, one east and one west, and both labeled by the same Reeb chord.



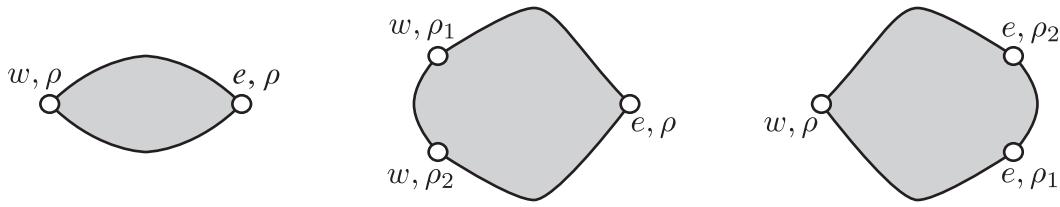


FIGURE 5.3. **Sources of curves at east ∞ .** Left: a trivial component. Center: a join component. Right: a split component.

Holomorphic maps from such components are uninteresting. In particular, they are preserved by the \mathbb{R} -translation action on $\mathbb{R} \times Z$.

A more interesting kind of component is a *join component*. This is a component of T^\diamond which is a topological disk with two west punctures and one east puncture. In counter-clockwise order, suppose the punctures are labeled by (e, ρ_e) , (w, ρ_1) and (w, ρ_2) . (We will sometimes refer to the (w, ρ_1) puncture as the *top puncture* of the join component, and the (w, ρ_2) puncture as the *bottom puncture*.) Then there is a holomorphic map from this component if and only if $\rho_e = \rho_2 \uplus \rho_1$. If such a holomorphic map exists, it is unique up to translation. A curve consisting entirely of one join component and some number of trivial components is called a *join curve*.

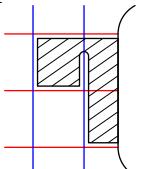
Symmetrically, a *split component* is a component of T^\diamond which is a topological disk with two punctures labeled e and one labeled w . In counterclockwise order, suppose the punctures are labeled by (w, ρ_w) , (e, ρ_1) and (e, ρ_2) . (We will sometimes refer to the (e, ρ_2) puncture as the *top puncture* of the split component, and the (e, ρ_1) puncture as the *bottom puncture*.) Then there is a holomorphic map from such this component if and only if $\rho_w = \rho_1 \uplus \rho_2$. Again, if such a holomorphic map exists, it is unique up to translation. A stable curve consisting entirely of some number of split components and some number of trivial components is called a *split curve*. (Note the definition is not symmetric with that of join curves.)

The sources of these components are illustrated in Figure 5.3. Split and join components can arise from the degenerations in Figure 5.1.

Finally, there is one more complicated kind of curve which we must consider, a *shuffle curve*. A *shuffle component* is a topological disk with two e punctures and two w punctures, with cyclic ordering (e, w, e, w) around the boundary. (A shuffle component is pictured in Figure 5.4, and an example where one degenerates off is shown in Figure 5.5.) The map from a shuffle component to $\mathbb{R} \times Z$ has either one interior branch point or two boundary branch points. Suppose that in counterclockwise order, the punctures of a shuffle component are labeled $(e, \rho_1), (w, \rho_2), (e, \rho_3), (w, \rho_4)$. Then the moduli space is non-empty if and only if $\rho_1^+ = \rho_2^+, \rho_2^- = \rho_3^-, \rho_3^+ = \rho_4^+$ and $\rho_4^- = \rho_1^-$. If a join curve admits a representative, exactly one of the following cases occurs:

- $\{\rho_1, \rho_3\}$ is nested and $\{\rho_2, \rho_4\}$ is interleaved, or
- $\{\rho_2, \rho_4\}$ is nested and $\{\rho_1, \rho_3\}$ is interleaved.

(Recall the definition of interleaved and nested pairs of Reeb chords from Definition 3.9.) In fact, we will see in Proposition 5.42 that the two cases are *not* symmetric for our purposes: when the first occurs in a degeneration, it occurs an odd number of times. When the second



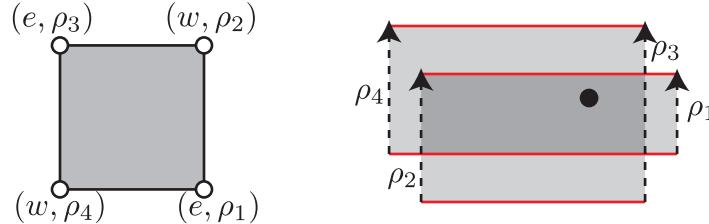


FIGURE 5.4. **Visualizing shuffle curves.** Left: a shuffle-component. Right: a schematic of the image of an odd shuffle-component projected to $\partial\bar{\Sigma} \times \mathbb{R}$.

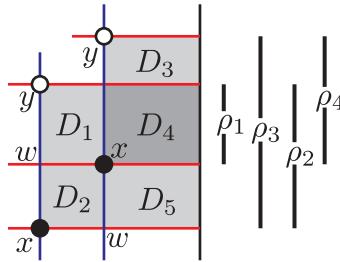


FIGURE 5.5. **A domain in a Heegaard diagram where a shuffle component degenerates off.** The domain is $B = D_1 + D_2 + D_3 + 2D_4 + D_5 \in \pi_2(\mathbf{x}, \mathbf{y})$, as indicated by the shading. There is a 1-parameter family of holomorphic curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ where P is the partition $\{\{\rho_1, \rho_3\}\}$. One end of this moduli space consists of a two-story holomorphic building connecting \mathbf{x} to \mathbf{w} via D_2 and \mathbf{w} to \mathbf{y} via $D_1 + D_3 + 2D_4 + D_5$. At the other end, either an interior branch point of $\pi_\Sigma \circ u$ (in region D_4) or two boundary branch points of $\pi_\Sigma \circ u$ (between D_3 and D_4 or between D_4 and D_5) approach $\partial\bar{\Sigma}$, resulting in the degeneration of a shuffle curve.

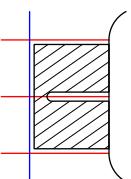
occurs, it occurs an even number of times. We will call the first case an *odd shuffle component* and the second case an *even shuffle component*, to remember the distinction.

A holomorphic curve in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ consisting of one shuffle component and some number of trivial components is called a *shuffle curve*; a shuffle curve can be odd or even, depending on the type of its shuffle component.

5.4. Compactifications via holomorphic combs

Just as the compactifications of moduli spaces in Morse theory involve not just flow lines but broken flow lines, in order to discuss the compactifications of the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ we need to introduce a more general object than holomorphic curves. The idea is that, because of the non-compactness of the target space $\Sigma \times [0, 1] \times \mathbb{R}$, holomorphic curves can degenerate into “holomorphic buildings” with several stories, as described by Eliashberg, Givental, and Hofer [EGH00]. In our setting, however, there are two different kinds of infinities: $\pm\infty$ and $e\infty$. This leads us to the notion of *holomorphic combs*.

By a *simple holomorphic comb* we mean a pair of holomorphic maps (u, v) where u maps to $\Sigma \times [0, 1] \times \mathbb{R}$ and v maps to $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$, and such that the asymptotics of u at east ∞ match with the asymptotics of v at west ∞ . More precisely:



DEFINITION 5.18. A *simple holomorphic comb* is a pair (u, v) with $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ and $v \in \mathcal{N}(T^\diamondsuit)$ for some sources S^\triangleright and T^\diamondsuit , together with a one-to-one correspondence between $E(S^\triangleright)$ and $W(T^\diamondsuit)$, preserving the labeling by Reeb chords, and such that $\text{ev}(u) = \text{ev}_w(v)$ inside $\mathbb{R}^{E(S^\triangleright)} / \mathbb{R} \cong \mathbb{R}^{W(T^\diamondsuit)} / \mathbb{R}$.

More generally, we also want to allow the components at east ∞ to degenerate further:

DEFINITION 5.19. A *holomorphic story* is a sequence (u, v_1, \dots, v_k) (for some $k \geq 0$) where, for some B , S^\triangleright , and T_i^\diamondsuit ,

- $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$,
- $v_i \in \mathcal{N}(T_i^\diamondsuit)$,
- (u, v_1) is a simple holomorphic comb (if $k \geq 1$),
- there is a correspondence between $E(T_i^\diamondsuit)$ and $W(T_{i+1}^\diamondsuit)$ for $i = 1, \dots, k-1$ which preserves the labelings by Reeb chords, and
- $\text{ev}_e(v_i) = \text{ev}_w(v_{i+1})$ in $\mathbb{R}^{E(T_i^\diamondsuit)} / \mathbb{R} \cong \mathbb{R}^{W(T_{i+1}^\diamondsuit)} / \mathbb{R}$.

We refer to (u, v_1, \dots, v_k) as the *horizontal levels* of the holomorphic story.

We also need to allow degeneration at $\pm\infty$:

DEFINITION 5.20. A *holomorphic comb* of height N is a sequence $(u_j, v_{j,1}, \dots, v_{j,k_j})$, $j = 1, \dots, N$, of holomorphic stories with u_j a stable curve in $\mathcal{M}^{B_j}(\mathbf{x}_j, \mathbf{x}_{j+1}; S_j^\triangleright)$ for some sequence of generalized generators $\mathbf{x}_1, \dots, \mathbf{x}_{N+1}$. We admit the case $N = 0$, the *trivial holomorphic comb* from \mathbf{x}_1 to \mathbf{x}_1 , which corresponds to a trivial (unstable) holomorphic curve. We refer to the index j as the *vertical level*, so the comb $(u_j, v_{j,1}, \dots, v_{j,k_j})$, $j = 1, \dots, N$ has N vertical levels (or *stories*).

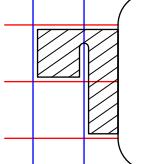
See Figure 5.6 for a schematic illustration of a height 2 holomorphic comb.

We will call a holomorphic comb *toothless* if it has no components at $e\infty$. The *spine* of a holomorphic comb is the sub-comb of components mapped to $\Sigma \times [0, 1] \times \mathbb{R}$.

Finally, to compactify our moduli spaces, we will also need to drop Condition (1) of Definition 5.2, and allow our holomorphic curves to have nodal sources. If we want to explicitly allow nodes, we will refer to such combs as *nodal holomorphic combs*; if we want to explicitly rule out nodal combs, we will refer to *smooth combs*. When the distinction is unimportant, we will simply talk about holomorphic combs.

Note that a (nodal) holomorphic comb $U = \{(u_j, v_{j,1}, \dots, v_{j,k_j})\}_{j=1}^N$ naturally represents a homology class B in $\pi_2(\mathbf{x}_1, \mathbf{x}_{N+1})$, where \mathbf{x}_1 and \mathbf{x}_{N+1} are the first and last generators involved. (Specifically, $B = B_1 * \dots * B_N$, where B_j is the domain of u_j . The $v_{i,j}$ are irrelevant to this homology class.) Further, U is asymptotic to a well-defined set of Reeb chords at (far) east ∞ : these are the asymptotics at east infinity of the v_{j,k_j} .

Next, we turn to what it means for a sequence of holomorphic curves to converge to a holomorphic comb or, more generally, for a sequence of holomorphic combs to converge to another holomorphic comb. This is a simple adaptation of definitions for holomorphic curves converging to holomorphic buildings in [EGH00, BEH⁺03]. First, adapting the definition of convergence in [BEH⁺03, Section 7.3], say, to the relative case of holomorphic buildings in a symplectic manifold W with cylindrical ends with boundary on a Lagrangian $L \subset W$ with cylindrical ends involves only trivial notational changes; we leave this to the reader. Next, call a component C of U Σ -*stable* (respectively \mathbb{D} -*stable*) if $(\pi_\Sigma \circ U)|_C$ (respectively $(\pi_\mathbb{D} \circ U)|_C$) is a stable map (i.e., has a finite automorphism group), and Σ -*unstable* (respectively \mathbb{D} -*unstable*)



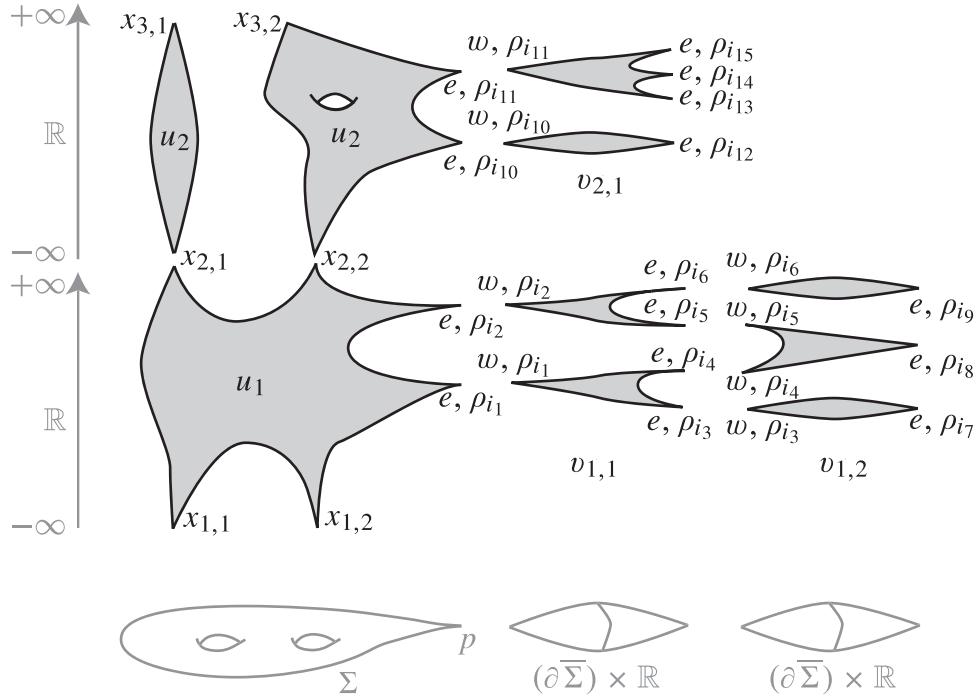


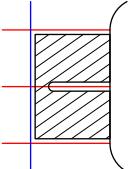
FIGURE 5.6. A schematic of a height 2 holomorphic comb.

otherwise. (Here, we view trivial strips as stable maps to \mathbb{D} , i.e., we do not quotient by automorphisms of the target, so every component of U must be either Σ -stable or \mathbb{D} -stable or both.) With this terminology in hand we are ready to define convergence to a holomorphic comb:

DEFINITION 5.21. We say a sequence u_n of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ converges to a holomorphic comb U if the following conditions hold:

- (C-1) Let S_Σ be the result of collapsing all Σ -unstable components of U . Then $\{\pi_\Sigma \circ u_n\}$ converges to $\pi_\Sigma \circ U|_{S_\Sigma}$ as buildings (i.e., in the sense of [BEH⁺03, Section 7.3]). (Note that, for a non-split almost complex structure, the maps $\pi_\Sigma \circ u_n$ and $\pi_\Sigma \circ U$ are not holomorphic, but the topology defined in [BEH⁺03] makes sense for smooth buildings, not just holomorphic ones.)
- (C-2) Let $S_\mathbb{D}$ be the result of collapsing all \mathbb{D} -unstable components of U . Then $\{\pi_\mathbb{D} \circ u_n\}$ converges to $\pi_\mathbb{D} \circ U|_{S_\mathbb{D}}$ as holomorphic buildings (i.e., in the sense of [BEH⁺03, Section 7.3]).
- (C-3) Let $\tau_t: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ denote translation by t units in the \mathbb{R} direction. Then for each smooth point q in the spine of U there is a neighborhood $V \ni q$, a sequence of points q_n in the source of u_n , neighborhoods $V_n \ni q_n$, diffeomorphisms $V_n \cong V$, and numbers $t_n \in \mathbb{R}$ so that $\tau_{t_n} \circ u|_{V_n}$ converges in the C_{loc}^∞ topology to $u|_V$.
- (C-4) All u_n for n sufficiently large represent the same homology class B , and U represents B as well.

Convergence of a sequence of holomorphic combs to a holomorphic comb is defined similarly; we leave this extension to the reader.



Given a simple holomorphic comb (u, v) with S^\triangleright and T^\diamondsuit the decorated sources of u and v respectively, there is a natural way to (pre)glue S^\triangleright and T^\diamondsuit to form a decorated source $S^\triangleright \natural T^\diamondsuit$: the surface $S \natural T$ is obtained by identifying small neighborhoods of the east punctures of S with neighborhoods of the corresponding west punctures of T . (Note that this is a purely topological operation, not involving any differential equations. We use the term “preglue” to distinguish from gluing of holomorphic curves.) The ∞ punctures of T^\diamondsuit and $\pm\infty$ punctures of S^\triangleright carry over to decorate $S^\triangleright \natural T^\diamondsuit$. The pregluing of sources extends in an obvious way to general smooth holomorphic combs: for any smooth holomorphic comb there is a corresponding (preglued) smooth surface. The construction also extends to nodal holomorphic combs: there is a canonical way to deform away the nodes in the source to produce a smooth surface.

From these constructions we can produce a compactification of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$.

DEFINITION 5.22. We make the following definitions.

- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is the space of all (possibly nodal) holomorphic combs whose preglued surfaces are S^\triangleright , in the homology class B , with asymptotics \mathbf{x} at $-\infty$, \mathbf{y} at $+\infty$.
- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$.
- For $p, q \in E(S^\triangleright)$, $\overline{ev}_{p,q}: \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, S^\triangleright) \rightarrow [-\infty, \infty]$ is the extension of the map $ev_{p,q}$ (see Equation (5.12)).
- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is the space of all holomorphic combs respecting the partition P . Formally, we set

$$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) := \bigcap_{\substack{P_i \in P \\ p, q \in P_i}} \overline{ev}_{p,q}^{-1}(0).$$

- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$.
- $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$.

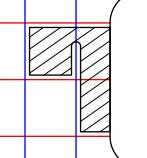
The space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is often a proper subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, as the next example illustrates. (Because of difficulties with transversality at east ∞ , $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ may also be a proper subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$.)

EXAMPLE 5.23. Consider the portion of a Heegaard diagram shown in the left of Figure 5.7, and let S^\triangleright be the decorated source shown on the right of Figure 5.7. (We number the east punctures of S^\triangleright for convenience in referring to them.) Let P denote the partition $\{\{1, 3\}, \{2, 4\}\}$. The space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is then homeomorphic to $\mathbb{R} \times (0, \infty) \times \mathbb{R} \times (0, \infty)$, via the map

$$(ev_1, ev_1 - ev_2, ev_3, ev_3 - ev_4).$$

The space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is the subspace $x_1 = x_3, x_2 = x_4$ of $\mathbb{R} \times (0, \infty) \times \mathbb{R} \times (0, \infty)$.

Now, consider the compactifications. In the compactification, when $(ev_1 - ev_2) \rightarrow 0$ the curve degenerates a split component at east ∞ , and similarly when $(ev_3 - ev_4) \rightarrow 0$. If $(ev_1 - ev_2) \rightarrow 0$ and $(ev_3 - ev_4) \rightarrow 0$, there are two split components at east ∞ . There is a one-parameter family of these curves at east ∞ , given by the relative \mathbb{R} -coordinate of the branch point under π_Σ . Consequently the part of the space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ with $(ev_1 - ev_2)$



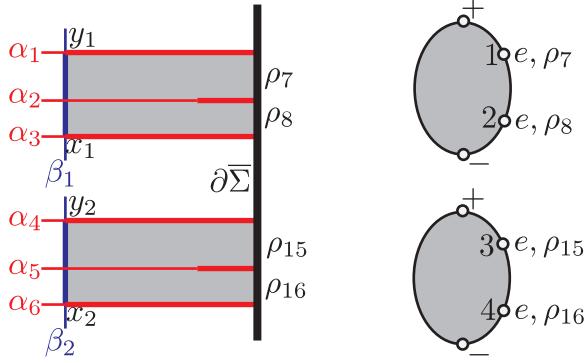


FIGURE 5.7. An example illustrating that $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ may be a proper subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Left: a portion of a Heegaard diagram. The domain of interest is shaded in gray. Right: the decorated source S^\triangleright .

and $(\text{ev}_3 - \text{ev}_4)$ small has the form shown in Figure 5.8, and $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is the subspace pictured. By contrast, $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ contains the entire stratum in which two split components have degenerated.

Next, we turn to the technical results justifying the definitions of the moduli spaces $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$.

PROPOSITION 5.24. *The spaces $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ are compact. That is, suppose that $\{U_n\}$ is a sequence of holomorphic combs in a fixed homology class, with a fixed preglued topological source. Then $\{U_n\}$ has a subsequence which converges to a (possibly nodal) holomorphic comb U , in the same homology class as the U_n . Similarly, $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ and $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ are compact.*

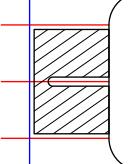
We will deduce Proposition 5.24 from two existing compactness results. First, there is a local compactness result for holomorphic curves:

THEOREM 5.25. [MS04, Theorem 4.1.1] *Let (M, J) be a compact almost complex manifold, $L \subset M$ a compact totally real submanifold, and J_ν a sequence of almost complex structures on M that converges in the C^∞ topology to J . Let $(\Sigma, j_\Sigma, d\text{vol}_\Sigma)$ be a (possibly non-compact) Riemann surface with boundary, $\Omega_\nu \subset \Sigma$ an increasing sequence of open sets that exhaust Σ , and $u_\nu: (\Omega_\nu, \Omega_\nu \cap \partial\Sigma) \rightarrow (M, L)$ a sequence of J_ν -holomorphic curves such that*

$$\sup_\nu \|du_\nu\|_{L^\infty(K)} < \infty$$

for every compact subset $K \subset \Sigma$. Then u_ν has a subsequence which converges uniformly with all derivatives on compact subsets of Σ to a J -holomorphic curve $u: (\Sigma, \partial\Sigma) \rightarrow (M, L)$.

Recall from [BEH⁺03, Section 2], that if V is an odd-dimensional manifold, then an almost-complex structure J on $\mathbb{R} \times V$ is called *cylindrical* if it is invariant under translation in the \mathbb{R} -factor and $\mathbf{R} := J(\frac{\partial}{\partial t})$ is tangent to V . In this case there is a one-form λ on V which is characterized by the properties that $\lambda|_{JTV \cap TV} \equiv 0$ and $\lambda(\mathbf{R}) \equiv 1$. The cylindrical structure J is called *symmetric* if the Lie derivative of λ in the direction \mathbf{R} vanishes identically. The almost complex structure J is *adjusted* to a closed, maximal-rank 2-form ω on V if ω is



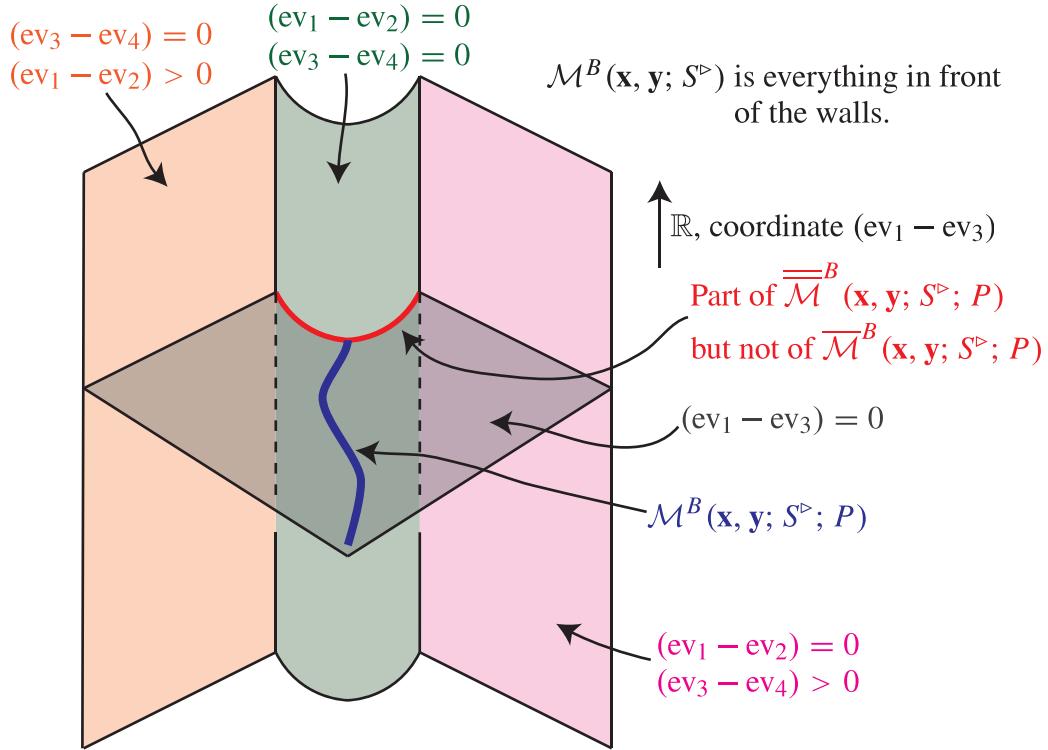


FIGURE 5.8. An illustration of various moduli spaces for the Heegaard diagram from Figure 5.7. The colored “walls” form the boundary of $\overline{\mathcal{M}}^B(x, y; S^D)$, the open part of which one visualizes as the empty space in front of the walls. The subspace where $ev_1 = ev_3$ is the gray horizontal plane. The moduli space $\mathcal{M}^B(x, y; S^D; P)$ is a thick, squiggly line segment; the space $\overline{\mathcal{M}}^B(x, y; S^D; P)$ is a “T” formed by that line segment and an arc in $\partial\overline{\mathcal{M}}^B(x, y; S^D)$.

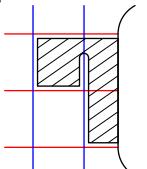
J -invariant; $\omega(v, Jv)$ is positive for all $v \in JTV \cap TV$; and the Lie derivative of ω in the direction \mathbf{R} is zero. Given a map $u: S \rightarrow \mathbb{R} \times V$, the ω -energy of u is $\int_S (\pi_V \circ u)^* \omega$; the λ -energy of u is $\sup_\phi \int_S [(\phi \circ t \circ u) dt] \wedge (\pi_V \circ u)^* \lambda$, where the supremum is over all functions $\phi: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ with compact support and integral 1; and the energy $E(u)$ of u is the λ -energy plus the ω -energy [BEH⁺03, Section 5.3].

The following degenerate case will play a role in the proof of Proposition 5.24:

EXAMPLE 5.26. If V is S^1 then the ω -energy (and, in fact, ω) vanishes for dimension reasons. The λ -energy is given by the degree of the map $u: S \rightarrow \mathbb{R} \times S^1$.

With these definitions in hand, we have the compactness theorem in symplectic field theory:

THEOREM 5.27. ([BEH⁺03, Theorem 10.1], see also [Abb14]) Let $(\mathbb{R} \times V, J)$ be a symmetric cylindrical almost complex manifold. Suppose that the almost complex structure J is adjusted to a 2-form ω . Then for every $E > 0$, the space $\overline{\mathcal{M}}_{g,\mu}(V) \cap \{E(F) \leq E\}$ is compact.



Here, $\overline{\mathcal{M}}_{g,\mu}(V)$ denotes the moduli space of holomorphic buildings in $\mathbb{R} \times V$ with source a surface of genus g with μ marked points, and $E(F)$ denotes the energy of a building F . (Punctures are viewed as marked points, but there may be extra marked points as well.)

OBSERVATION 5.28. *Theorem 5.27 implies that any sequence of holomorphic maps*

$$u_n: (S, \partial S) \rightarrow ([0, 1] \times \mathbb{R}, \{0, 1\} \times \mathbb{R})$$

(with bounded energy) has a convergent subsequence: we can double the source S across its boundary to obtain a sequence of holomorphic maps

$$S \cup_{\partial} (-S) \rightarrow S^1 \times \mathbb{R}$$

and apply Theorem 5.27 to the result.

Finally, we will use another relative version of symplectic field theory compactness in the special case of maps between Riemann surfaces:

THEOREM 5.29. *Let (W, j) be a punctured Riemann surface and $L \subset W$ a Lagrangian submanifold (union of curves) which is cylindrical near the punctures of W . Assume that L is embedded away from finitely many transverse double points. Then for every $E > 0$ the space $\overline{\mathcal{M}}_{S,\mu}(W, L, J) \cap \{E(F) \leq E\}$ is compact, where $\overline{\mathcal{M}}_{S,\mu}(W, L, J) \cap \{E(F) \leq E\}$ is the space of holomorphic buildings in W with boundary on L , topological type S , μ marked points, and energy $\leq E$.*

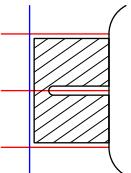
In this simple case, the energy of a map $F: S \rightarrow \Sigma$ between Riemann surfaces is the integral over S of the pullback of a chosen area form on Σ .

See [BEH⁺03, Theorem 10.2] for the analogous result in the closed case, without the dimension restriction. We will not prove Theorem 5.29, but there are at least three ways one could go about doing so:

- (1) Imitate the arguments in [BEH⁺03] but in the relative case. This approach should yield Theorem 5.29, where W is an arbitrary symplectic manifold, not just a Riemann surface.
- (2) Apply Gromov's doubling trick to deduce the relative case from the closed case. Again this would yield Theorem 5.27, again without the dimension restriction, but with the restriction that the Lagrangians are real analytic, J is integrable near the Lagrangians, and the Lagrangians are embedded. (This last restriction does not suit our purposes.)
- (3) Observe that a map between Riemann surfaces is determined by its branch points, so Theorem 5.29 can be deduced from the fact that the Deligne-Mumford moduli space of Riemann surfaces with boundary and marked points is compact.

See also [BEH⁺03, Section 11.3] for a little more discussion of compactness in relative symplectic field theory and [Abb14] for further discussion.

PROOF OF PROPOSITION 5.24. In the case that the complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ is split, $J = j_\Sigma \times j_{\mathbb{D}}$, the result follows by considering the projections $\pi_{\mathbb{D}} \circ u_i$ and $\pi_\Sigma \circ u_i$ separately and applying Theorems 5.27 and 5.29, respectively. For general J , we use this projection argument in a neighborhood of east ∞ (using Property (J-4) of J). Away from this neighborhood we use a combination of Theorem 5.27 applied to $\pi_{\mathbb{D}} \circ u$ (which is holomorphic by Property J-2) and Theorem 5.25. We spell this out in detail below; see also [Lip06b, Proposition 4.2.1] for a slightly different argument.



To ease notation, we will restrict our attention to the case of a sequence $\{u_n: S_n \rightarrow \Sigma \times [0, 1] \times \mathbb{R}\}_{n=1}^\infty$ of holomorphic curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^D)$, rather than a sequence of arbitrary holomorphic combs. It is routine to adapt the argument to the general case.

Step 1. *Extracting the vertical level structure on the limit.* Fix a point p_r in each region r of Σ . For generically chosen points p_r , the points p_r are regular values of $\pi_\Sigma \circ u_n$ for all n . Let $\{q_{r,i,n}\} = (\pi_\Sigma \circ u_n)^{-1}(p_r)$ be the preimages of p_r . We view the $q_{r,i,n}$ as extra marked points in S_n . Now, apply Theorem 5.27 (in light of Observation 5.28) to $\{\pi_D \circ u_n\}$ (with these extra marked points) to extract a convergent subsequence. (Note that the energy bound required by Theorem 5.27 is trivially satisfied: the ω -energy is 0 (for dimension reasons) while the λ -energy is equal to g , the genus of Σ ; compare Example 5.26.) Relabel so that $\{u_n\}$ refers to this subsequence, and let $(\pi_D \circ U)$ denote the limit of this sequence, which is a (vertical) holomorphic building in $[0, 1] \times \mathbb{R}$. Let S_∞ denote the source of $(\pi_D \circ U)$. (We have not yet defined U , just $\pi_D \circ U$.)

Let $V_p \subset \Sigma$ be a closed disk neighborhood of east ∞ over which J is split, $J|_{V_p \times [0,1] \times \mathbb{R}} = j_\Sigma \times j_D$. Let W_p be the complement of a slightly smaller closed disk around east ∞ . We will extract convergent subsequences over V_p and W_p separately, and then use unique continuation to conclude that they agree on the overlap.

Step 2. *Convergence over V_p .* Let $T_n = (\pi_\Sigma \circ u_n)^{-1}(V_p)$. We have holomorphic maps

$$(\pi_\Sigma \circ u_n)|_{T_n}: (T_n, \partial T_n) \rightarrow (V_p, \partial V_p \cup \boldsymbol{\alpha}).$$

By Theorem 5.29 the sequence $(\pi_\Sigma \circ u_n)|_{T_n}$ has a convergent subsequence. (Here, the energy bound comes from the fact that we are working in a fixed homology class B , so all holomorphic curves have the same area.) Replace $\{u_n\}$ by the corresponding subsequence, so $(\pi_\Sigma \circ u_n)|_{T_n}$ converges. Let $(\pi_\Sigma \circ U)|_T$ denote the limit of this subsequence; this is a (horizontal) holomorphic building in V_p (with components extending into east ∞ , $\mathbb{R} \times Z$). Let

$$T' = S_\infty \setminus [(\pi_\Sigma \circ U)|_T]^{-1}(W_p).$$

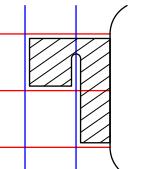
Step 3. *Convergence over W_p , i.e., to T' .* From Step 1, the sequence of decorated sources S_n is converging (in the Deligne-Mumford moduli space) to S_∞ . This means that (for large n) we can view each of the S_n as a complex structure on a single smooth surface (with boundary and punctures) S , and as $n \rightarrow \infty$ these complex structures converge in the C^∞ topology away from some finite collection of pairwise-disjoint, embedded *collapsing curves*. Near these collapsing curves, the complex structures are collapsing (or, equivalently, forming long necks).

Let $q \in T'$ be a smooth point. Choose a neighborhood $X \ni q$ whose closure is compact (i.e., which does not touch the punctures in S_∞) and contained in T' . We want to define $\pi_D \circ U$ on X . To this end, note that since

$$\pi_2(\Sigma \times [0, 1] \times \mathbb{R}) = \pi_2(\Sigma \times [0, 1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})) = 0,$$

Gromov's bubbling lemma (see, e.g., [BEH⁺03, Lemma 5.11]) implies that $\|d(u_n)\|_{L^\infty}$ is bounded on X . The facts that the curves $\pi_D \circ u_n$ converge and that the curves u_n all represent the homology class B gives a bound on the energy of the curves u_n . Consequently, Theorem 5.25 applies with $\Sigma = X$ to give a subsequence of the $u_n|_X$ converging to a holomorphic curve

$$U|_X: X \rightarrow W_p \times [0, 1] \times \mathbb{R} \subset \Sigma \times [0, 1] \times \mathbb{R}.$$



(The reader concerned about the non-compactness of $W_p \times [0, 1] \times \mathbb{R}$ should note that by assumption the holomorphic curves $u_n|_X$ map to a compact subset Y of $W_p \times [0, 1] \times \mathbb{R}$. Let $\Sigma_{\bar{e}}$ be the result of filling in the puncture e to get a closed Riemann surface. One can embed Y in $\Sigma_{\bar{e}} \times S^2$, the product of Σ and the two-sphere, in such a way that the almost complex structure J extends to an almost complex structure on all of $\Sigma_{\bar{e}} \times S^2$ which is tamed by the split symplectic form.)

Choose a countable collection of such X which covers the smooth part of T' and take a diagonal subsequence. This subsequence then converges in C_{loc}^∞ away from the collapsing curves.

Step 4. Nodes and punctures. The argument in Step 3 defines U on all of T' except the nodes. The energy bound (coming from the fact that the curves u_n all represent B) and the removable singularities theorem (see, e.g., [MS04, Theorem 4.1.2]) imply that U extends to the nodes, as well. To see that U is continuous across the nodes (i.e., approaches the same value from both sides of the nodes) requires a slight further argument, for which we refer the reader to [MS04, Section 4.7].

Next, we claim that near the punctures of $U|_{T'}$, $\pi_\Sigma \circ U$ converges to points in $\alpha \cap \beta$. This again follows from the fact that the maps u_n all represent the homology class B , so the energy $\int (\pi_\Sigma \circ U)^* \omega_\Sigma$ is bounded. (Since $\pi_\Sigma \circ U$ is not holomorphic with respect to a single complex structure on Σ but rather with respect to a family parameterized by $[0, 1]$, this claim uses a convergence result for strip-like ends, such as [Flo88, Theorem 2].)

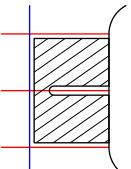
Step 5. Piecing together T and T' . On $T \cap T'$ we have defined U twice: once in Steps 1 and 2 and once in Steps 1, 3 and 4, but it follows from the fact that C_{loc}^∞ is Hausdorff that these definitions coincide. So, the maps $U|_T$ and $U|_{T'}$ glue to give a map on all of the components of S_∞ mapped to $\Sigma \times [0, 1] \times \mathbb{R}$.

Note that it follows from the presence of the marked points we added in Step 1 that the curve U represents the homology class B .

Step 6. Components at east ∞ . Finally, it remains to define $\pi_D \circ U$ on the components of S_∞ mapped to east ∞ (i.e., $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$); these components of S_∞ , and the restriction of $\pi_\Sigma \circ U$ from them to $\mathbb{R} \times Z$, appeared in Step 2. So, fix such a component C of S_∞ . There are two cases: either the component C appeared already in Step 1 or it did not. (Roughly, in the language of Definition 5.21, these two cases correspond to C being D -stable or D -unstable, respectively.) In the first case, Step 1 defines a map $C \rightarrow [0, 1] \times \mathbb{R}$. Recall that the surface S_∞ is obtained by collapsing some arcs in S , which we called collapsing curves. Let C_0 be the preimage in S of $C \subset S_\infty$. Then, since C is mapped to $e\infty$, the boundary of C_0 consists of some collapsing arcs and some arcs in S mapped by $\pi_\Sigma \circ u_n$ to the α -curves. It follows that $\pi_D \circ U$ maps ∂C to $\{1\} \times \mathbb{R}$. The open mapping principle then implies that $(\pi_D \circ U)|_C$ is constant.

In the second case we have not yet defined $\pi_D \circ U|_C$. We can add extra marked points to the curves $\pi_D \circ u_n$ so that C (with some extra marked points) does appear in the limit. Dropping these extra marked points makes $(\pi_D \circ U)|_C$ unstable (or else we would have been in the previous case), so in particular $(\pi_D \circ U)|_C$ is constant.

We have now defined U completely. Moreover, it is immediate from the construction that the sequence u_n converges to U in the sense of Definition 5.21. \square



5.5. Gluing results for holomorphic combs

We next turn to gluing results, which show that the space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is well-behaved near some of the simplest holomorphic combs.

PROPOSITION 5.30. *Let (u_1, u_2) be a two-story holomorphic building with $u_1 \in \mathcal{M}^{B_1}(\mathbf{x}, \mathbf{y}; S_1^\triangleright; P_1)$ and $u_2 \in \mathcal{M}^{B_2}(\mathbf{y}, \mathbf{w}; S_2^\triangleright; P_2)$. Then for sufficiently small open neighborhoods U_1 of u_1 and U_2 of u_2 , there is an open neighborhood of (u_1, u_2) in $\overline{\mathcal{M}}^{B_1 * B_2}(\mathbf{x}, \mathbf{w}; S_1^\triangleright \natural S_2^\triangleright; P_1 \cup P_2)$ which is homeomorphic to $U_1 \times U_2 \times [0, 1]$.*

PROPOSITION 5.31. *Let (u, v) be a simple holomorphic comb with $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ and $v \in \mathcal{N}(T^\diamondsuit; P_e)$. Let $m = |E(S^\triangleright)| = |W(T^\diamondsuit)|$. Assume that the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ and $\mathcal{N}(T^\diamondsuit; P_e)$ are transversally cut out at u and v respectively, and that $\text{ev}: \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \rightarrow \mathbb{R}^m/\mathbb{R}$ and $\text{ev}_w: \mathcal{N}(T^\diamondsuit; P_e) \rightarrow \mathbb{R}^m/\mathbb{R}$ are transverse at (u, v) . Then, for sufficiently small open neighborhoods U_u of u and U_v of v , there is an open neighborhood of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright \natural T^\diamondsuit; P_e)$ which is homeomorphic to $(U_u \times_{\text{ev}} U_v) \times [0, 1]$.*

PROOF OF PROPOSITIONS 5.30 AND 5.31. The results follow by standard arguments. Indeed, since each puncture of a holomorphic curve in our setting is mapped either to $\pm\infty$ or to east or west ∞ , the fact that we have two kinds of infinities is irrelevant to a local statement of this kind. The only complication to the usual arguments, then, is that the asymptotics at east and west infinity are degenerate (i.e., Morse-Bott). Thus, the proof can be obtained by adapting [Bou06, Section 5.3] to the relative case or [Lip06a, Proposition A.1] to the Morse-Bott case. \square

We will also use the following generalization of Proposition 5.31 to non-simple one-story holomorphic combs in the proof of Proposition 5.39.

PROPOSITION 5.32. *Consider a height 1 holomorphic comb (u, v_1, \dots, v_k) with $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$, $v_i \in \mathcal{N}(T_i^\diamondsuit)$ for $i = 1, \dots, k-1$ and $v_k \in \mathcal{N}(T_k^\diamondsuit; P_e)$. Let $m_i = |W(T_i^\diamondsuit)|$. Assume that all the moduli spaces are transversally cut out, and that the map*

$$\begin{aligned} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) \times \mathcal{N}(T_1^\diamondsuit) \times \cdots \times \mathcal{N}(T_k^\diamondsuit; P_e) &\rightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k} \times \mathbb{R}^{m_k} \\ (u, v_1, \dots, v_k) &\mapsto (\text{ev}(u), \text{ev}_w(v_1), \text{ev}_e(v_1), \text{ev}_w(v_2), \dots, \text{ev}_e(v_{k-1}), \text{ev}_w(v_k)) \end{aligned}$$

is transverse to the diagonal $\{(x_1, x_1, \dots, x_k, x_k) \mid x_i \in \mathbb{R}^{k_i}\}$. Then, for sufficiently small open neighborhoods U_u of u and U_{v_i} of v_i , there is an open neighborhood of (u, v_1, \dots, v_k) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright \natural T_1^\diamondsuit \natural \cdots \natural T_k^\diamondsuit; P_e)$ which is homeomorphic to $(U_u \times_{\text{ev}} U_{v_1} \times_{\text{ev}} \cdots \times_{\text{ev}} U_{v_k}) \times [0, 1]^k$.

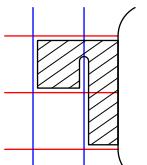
Again, the proof is essentially standard.

As mentioned at the end of Section 5.1, we cannot prove that the compactified moduli spaces are manifolds with corners in general. We will get around this difficulty by proving that certain evaluation maps are degree one, in an appropriate sense, at the corners; this will be enough for the results in Section 5.6. Before stating the propositions, we make some definitions.

DEFINITION 5.33. Let (u, v) be a simple holomorphic comb with $v \in \mathcal{N}(T^\diamondsuit)$. A *smeared neighborhood* of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is an open neighborhood of

$$\{(u, v') \mid v' \in \mathcal{N}(T^\diamondsuit), (u, v') \in \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)\}.$$

There is an exactly analogous notion of a *smeared neighborhood* of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$.



REMARK 5.34. The notion of a smeared neighborhood of (u, v) depends only on the combinatorics of (the decorated source of) v , not the map v itself.

DEFINITION 5.35. Given a continuous map $f: X \rightarrow Y$ of topological spaces and a point $p \in Y$, we say f is *proper near* p if there is an open neighborhood $U \ni p$ such that $f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$ is proper.

We will use the following weak notion of stratified spaces.

DEFINITION 5.36. An *n -dimensional stratified space* is a topological space X written as a union of strata $\{X_i\}_{i=0}^n$ where X_i is a smooth i -dimensional manifold and the closure of X_k is contained in $\bigcup_{i \leq k} X_i$. We typically suppress the strata from the notation. Let X and Y be stratified spaces. A *stratified map* $f: X \rightarrow Y$ is a continuous map with the property that the preimage of any stratum in $Y_i \subset Y$ is a union of connected components of strata X_j of X , and the restriction of f to each stratum of X_j , thought of as a map into Y_i , is a smooth map.

DEFINITION 5.37. Let X be a stratified space so that the top stratum is a smooth m -manifold, and let $f: X \rightarrow \mathbb{R}_+^m$ be a stratified map so that $f^{-1}((0, \infty)^m)$ is the top stratum of X . Let $p \in \mathbb{R}_+^m$, and assume f is proper near p . We say that f is *odd degree near* p if there is an open neighborhood U of p such that for any regular value $p' \in U \cap (0, \infty)^m$, $f^{-1}(p')$ consists of an odd number of points.

LEMMA 5.38. Let X be a stratified space such that the union of the top two strata of X forms an m -manifold with boundary. Let $f: X \rightarrow \mathbb{R}_+^m$ be a stratified map which is proper near 0. Assume that $f^{-1}((0, \infty)^m)$ is the top stratum of X . Then

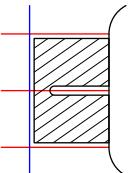
- (1) If $p, p' \in (0, \infty)^m$ near 0 are regular values of f then $|f^{-1}(p)| \equiv |f^{-1}(p')| \pmod{2}$.
- (2) If the restriction of f to the preimage of some facet \mathbb{R}_+^{m-1} of \mathbb{R}_+^m is odd degree near $0 \in \mathbb{R}_+^{m-1}$ then f is odd degree near $0 \in \mathbb{R}_+^m$.

PROOF. Let U be a neighborhood of 0 over which f is proper. Let V be $f^{-1}(U \cap \mathbb{R}_+^m)$. Then $f|_V: V \rightarrow (U \cap (0, \infty)^m)$ is a proper map. So part (1) follows from standard degree theory.

For part (2), let $p \in \mathbb{R}_+^{m-1}$ be a regular value of $f|_V$, and let $B \subset \mathbb{R}_+^{m-1}$ be a ball neighborhood of p small enough that $f|_{f^{-1}(B)}$ is a covering map. Write $f^{-1}(B) = B_1 \amalg \dots \amalg B_{2\ell+1}$; each B_i is an $(m-1)$ -ball. For each i choose an $(m-1)$ -ball B'_i whose interior lies in the top stratum of X and with $\partial B'_i = \partial B_i$. Note that by our assumptions on f , $f(\text{Int}(B'_i))$ is entirely contained in $(0, \infty)^m$. So $f(B_i \cup B'_i)$ is a (singular) $(m-1)$ -sphere in \mathbb{R}_+^m , and for $p' \in (0, \infty)^m$ sufficiently close to p , the sphere $f(B_i \cup B'_i)$ has winding number ± 1 around p' . It follows that f has odd degree at regular values near p . Choosing p close enough to 0 gives the result. \square

PROPOSITION 5.39. Suppose that (u, v) is a simple holomorphic comb in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Assume that v is a split curve and that there are two parts P_1 and P_2 of P such that, for each split component of T^\diamond , its bottom puncture belongs to P_1 and its top puncture belongs to P_2 . Assume also that $\text{ind}(B, S^\triangleright, P) = 2$ so $\dim \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) = 1$.

Let $q_1 \in P_2$ and $q_2 \in P_1$ be the top and bottom punctures, respectively, on one of the split components of T^\diamond . Then, for generic J , there is a smeared neighborhood U of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ so that the evaluation map $\overline{\text{ev}}_{q_1, q_2}: U \rightarrow \mathbb{R}_+$ is proper near 0 and odd degree near 0.



PROOF. To see the $\bar{\text{ev}}$ is proper near 0 on U , note that $\bar{\text{ev}}_{q_1, q_2}$ is clearly proper on the closure \overline{U} of U . By examining the topology of S^\triangleright , we see that $\overline{U} \setminus U$ is mapped to strictly positive values by $\bar{\text{ev}}_{q_1, q_2}$.

Let the split components of T^\diamondsuit be T_i^\diamondsuit for $i = 1, \dots, N$. For notational convenience, we will assume that T^\diamondsuit has no trivial strips, so that the only parts of P are P_1 and P_2 . Let P' be the partition P with P_2 split into the discrete partition; that is, P' has $N + 1$ parts, one of which is P_1 and all others consisting of a single puncture. We will show that there is a smeared neighborhood U' of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$ so that the evaluation map $\bar{\text{ev}}: U' \rightarrow \mathbb{R}_+^N$ is odd degree near 0 in the sense of Definition 5.37. (We see that $\bar{\text{ev}}$ is proper near 0 as before.) This suffices, since for generic J , the map ev from $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$ is transverse to the diagonal in $(0, \infty)^N$ by Proposition 5.6.

By Proposition 5.32, we can make a smeared neighborhood U of (u, v) inside $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$ into a stratified space of a particularly nice kind; in particular, the union of the top two strata is an N -manifold with boundary. The codimension 1 strata of U (of dimension $N - 1$) consist of simple holomorphic combs (u', v') , where v' is a split curve with no more than N split components. Higher codimension strata of U are more complicated one-story holomorphic combs.

We call such a neighborhood in an N -dimensional compactified moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P')$ a *split neighborhood*: namely, a split neighborhood is an open set containing the entire stratum of simple combs (u, v) , where v is a split curve with N split components, with all bottom punctures of v belonging to the same part of P' and all top punctures belonging to different parts.

We will show by induction on N that $\bar{\text{ev}}$ on a split neighborhood has odd degree near 0. For $N = 1$, by Proposition 5.31 we have a map from $[0, 1)$ to \mathbb{R}_+ mapping 0 to 0 which is proper near 0; such a map automatically has odd degree near 0. For $N > 1$, note that the preimage of the interior of a facet of \mathbb{R}_+^N consists of holomorphic curves (u', v') , where v' has a single split component; thus, u' lives in a split neighborhood U' of a smaller moduli space, of dimension $N - 1$. By induction, $\bar{\text{ev}}$ on U' has odd degree near 0, and so by Lemma 5.38, $\bar{\text{ev}}$ on U also has odd degree near 0. \square

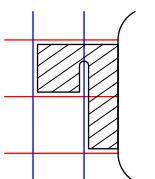
In Chapter 9, we will also need the following generalization of Proposition 5.39. To state it, we need a little more terminology. A *generalized split curve* is a union of holomorphic disks in $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$ each of which has exactly one w puncture. Each component of a generalized split curve is a *generalized split component*.

PROPOSITION 5.40. *Let (u, v) be a simple holomorphic comb in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$, where v is a generalized split curve. Write $\vec{P} = (P_1, \dots, P_k)$, label the components of v as T_1, \dots, T_m and label the east punctures of T_i by $q_{i,1}, \dots, q_{i,n_i}$. Assume that there are integers $0 = \ell_1, \ell_2, \dots, \ell_k, \ell_{k+1} = m$ such that $P_j = \{q_{\ell_j+1,1}, \dots, q_{\ell_{j+1}, n_{\ell_{j+1}}}\}$. (That is, P_j consists of the punctures on components $T_{\ell_j+1}, \dots, T_{\ell_{j+1}}$.)*

Suppose that $\text{ind}(B, S^\triangleright, P) = 1$.

Then there is a smeared neighborhood U of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ so that the evaluation map

$$\text{ev}: U \rightarrow \mathbb{R}_{+}^{n_1-1} \times \mathbb{R} \times \mathbb{R}_{+}^{n_2-1} \times \mathbb{R} \times \cdots \times \mathbb{R}_{+}^{n_{\ell_2}-1} \times \mathbb{R}_{+}^{n_{\ell_2+1}-1} \times \mathbb{R} \times \cdots \times \mathbb{R}_{+}^{n_{\ell_3}-1} \times \cdots \cdots \times \mathbb{R}_{+}^{n_m-1}$$



defined by

$$\text{ev}(u) = (\underbrace{\text{ev}_{q_{1,2}}(u) - \text{ev}_{q_{1,1}}(u), \dots, \text{ev}_{q_{1,n_1}}(u) - \text{ev}_{q_{1,n_1-1}}(u)}_{\in \mathbb{R}_+^{n_1-1}}, \underbrace{\text{ev}_{q_{2,1}}(u) - \text{ev}_{q_{1,n_1}}(u)}_{\in \mathbb{R}}, \\ \underbrace{\text{ev}_{q_{2,2}}(u) - \text{ev}_{q_{2,1}}(u), \dots, \text{ev}_{q_{2,n_2}}(u) - \text{ev}_{q_{2,n_2-1}}(u)}_{\in \mathbb{R}_+^{n_2-1}}, \underbrace{\text{ev}_{q_{3,1}}(u) - \text{ev}_{q_{2,n_2}}(u)}_{\in \mathbb{R}}, \dots)$$

is proper near 0 and odd degree near 0.

PROOF. The proof is similar to the proof of Proposition 5.39. The moduli space of generalized split curves has ends where one degenerates off a split curve at (far) east or west infinity; the lowest-dimensional stratum consists of sequences of (ordinary) split curves. Using this, one can inductively reduce to the case of gluing on split curves. Details are left to the reader. \square

REMARK 5.41. It would be convenient to extend Proposition 5.6 to the compactified moduli spaces, and in particular to assert that for height 1 holomorphic combs, say, the extension of ev to $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ is transverse to the partial diagonals Δ_P . In particular, this would eliminate the need for the cumbersome Propositions 5.39, 5.40 and 5.42. Such a statement would require a smooth structure on the $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$, which we have not constructed. Moreover, the validity of the statement would depend strongly on the smooth structure: in the language of Hofer [Hof06], it depends on the *gluing profile*. (In fact, for the exponential gluing profile advocated there, the extension of ev would generally not be transverse to Δ_P .)

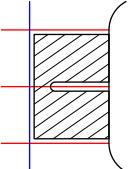
We conclude this section with a result on shuffle curves. It states that, when they occur, odd shuffle curves appear in the boundary of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ (with $\text{ind}(B, S^\triangleright, P) = 2$) an odd number of times, and even shuffle curves appear an even number of times.

PROPOSITION 5.42. Fix a generic J . Let $(u, v) \in \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ be a simple holomorphic comb with v a shuffle curve. Assume that the two e punctures q_i and q_j of the shuffle component belong to the same part P_i of P . Assume also that $\text{ind}(B, S^\triangleright, P) = 2$.

- (1) If v is an odd shuffle curve then there is a smeared neighborhood U of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ so that $\overline{U} \setminus \partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is homeomorphic to $\coprod_{i=1}^{2m-1} [0, 1]$ (for some $m \in \mathbb{N}$).
- (2) If v is an even shuffle curve then there is a smeared neighborhood U of (u, v) in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ so that $\overline{U} \setminus \partial \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is homeomorphic to $\coprod_{i=1}^{2m-2} [0, 1]$ (for some $m \in \mathbb{N}$).

PROOF. First some notation: let a_1, \dots, a_4 be the endpoints of the Reeb chords on the shuffle component, ordered so that $a_1 \lessdot \dots \lessdot a_4$. Let ρ_{ij} (with $i < j$ and $i, j \in \{1, \dots, 4\}$) denote the Reeb chord from a_i to a_j . Let T^\diamond denote the source of v .

The two cases are similar; we first prove case (1). In this case, the e punctures of the shuffle component of T^\diamond are labeled by ρ_{14} and ρ_{23} . The w punctures are labeled by ρ_{13} and ρ_{24} . The moduli space $\mathcal{N}(T^\diamond)$ has a natural compactification $\overline{\mathcal{N}}(T^\diamond)$ which is homeomorphic to $[0, 1]$. (The parameter is obtained by moving the branch point(s); see Figure 5.9 for a schematic illustration.) By our transversality (Propositions 5.6 and 5.16) and gluing (Proposition 5.31) results, any curve in $\mathcal{N}(T^\diamond)$ can be glued to u to obtain a curve in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$.



To see when there exist such curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ we consider the ends of the moduli space $\overline{\mathcal{N}}(T^\diamondsuit)$. The two ends correspond to holomorphic combs (u, v_1, v_2) and (u, v_3, v_4) . The curves v_1, \dots, v_4 are as follows:

- v_1 is a split curve with one split component, labeled by (w, ρ_{13}) , (e, ρ_{12}) and (e, ρ_{23}) , together with a trivial component labeled by (e, ρ_{24}) and (w, ρ_{24}) .
- v_2 is a join curve, with a join component labeled by (w, ρ_{24}) , (w, ρ_{12}) and (e, ρ_{14}) , together with a trivial component labeled by (e, ρ_{23}) and (w, ρ_{23}) .
- v_3 is a split curve with one split component, labeled by (w, ρ_{24}) , (e, ρ_{23}) and (e, ρ_{34}) , together with a trivial component labeled by (e, ρ_{13}) and (w, ρ_{13}) .
- v_4 is a join curve, with a join component labeled by (w, ρ_{34}) , (w, ρ_{13}) and (e, ρ_{14}) , together with a trivial component labeled by (e, ρ_{23}) and (w, ρ_{23}) .

See Figure 5.9.

Suppose one glues v_1 to u , with some gluing parameter ϵ_1 . Then in the resulting curve the Reeb chord ρ_{23} will be above ρ_{12} . If one then glues v_2 to the result, with some gluing parameter $\epsilon_2 \ll \epsilon_1$, in the resulting curve the Reeb chord ρ_{23} will be above ρ_{14} .

By contrast, suppose one glues v_3 to u , with some gluing parameter ϵ_1 . Then in the resulting curve the Reeb chord ρ_{34} will be above ρ_{23} . If one then glues v_2 to the result, with some gluing parameter $\epsilon_2 \ll \epsilon_1$, in the resulting curve the Reeb chord ρ_{14} will be above ρ_{23} .

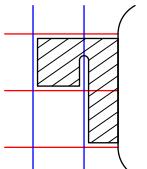
So, since the two Reeb chords switched order as we moved from one end of $\overline{\mathcal{N}}(T^\diamondsuit)$ to the other, for given ϵ there is algebraically one curve $v' \in \mathcal{N}$ so that gluing u with v' with gluing parameter ϵ gives a curve with ρ_{14} and ρ_{23} at the same height—i.e., an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Part (1) of the proposition follows.

Part (2) follows by a similar analysis. In this case, the e punctures of the shuffle component are labeled by ρ_{13} and ρ_{24} , and the w punctures are labeled by ρ_{14} and ρ_{23} . A schematic illustration of $\overline{\mathcal{N}}(T^\diamondsuit)$ may be obtained by turning Figure 5.9 upside down. An analysis as above shows that at both ends of $\overline{\mathcal{N}}$, the Reeb chord ρ_{24} is above ρ_{13} . This implies that, algebraically, there are zero curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ for a given gluing parameter. \square

5.6. Degenerations of holomorphic curves

To construct our invariants we study degenerations of 1-dimensional moduli spaces. The compactness theorem, Proposition 5.24, permits two kinds of degenerations to occur in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$:

- The source S can degenerate to a point in the boundary of the moduli space of Riemann surfaces. Such degenerations correspond to the conformal structure pinching along some circles and/or arcs, resulting in a surface with some nodes q_i . There are three possible behaviors of the limit near each q_i :
 - The \mathbb{R} -coordinate of the map may converge to $\pm\infty$ at q_i , so q_i corresponds to a level-splitting in the resulting multi-story holomorphic comb. This is a case of *splitting at $\pm\infty$* .
 - The holomorphic map may converge to the puncture p in Σ on one side of q_i , so the puncture q_i corresponds to a splitting at east ∞ ; or the analogous behavior for combs with several east ∞ levels may occur. We call this *splitting at east ∞* .



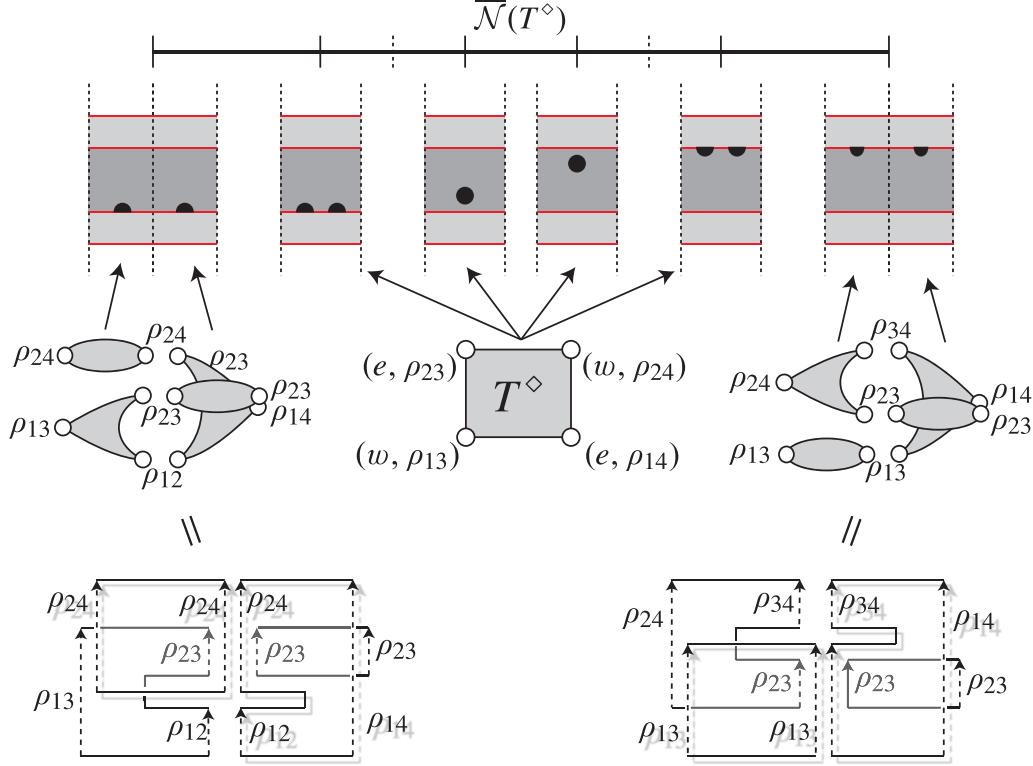
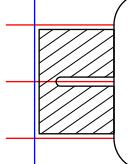


FIGURE 5.9. **The one-parameter family of maps v from an odd shuffle component.** The dark dots denote interior branch points, and the dark semi-dots denote boundary branch points. The two ends of the moduli space are shown on the left and the right. At the bottom, the two ends are sketched in the style of Figure 5.4. The corresponding pictures for an even shuffle component are obtained by reflection.

- The holomorphic map may extend continuously over q_i , sending q_i to some point in $\Sigma \times [0, 1] \times \mathbb{R}$ (or perhaps $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$), resulting in a nodal holomorphic comb. We call this *becoming nodal*.
- The source S may not degenerate, but the map could become singular. In this case, the derivative blows up at some point or points. (In the proof of Proposition 5.24, we added marked points in this case, so that it became a special case of the previous point.) There are three sub-cases.
 - The derivative may blow up at a puncture of S . If the puncture is a \pm puncture then this is the other case of *splitting at $\pm\infty$* , where the curve that splits off has a non-stable source. If it were an e puncture then this would be a case of splitting or east ∞ , but there are no non-trivial curves at $e\infty$ with unstable sources, so this does not occur.
 - The derivative may blow up in the interior of S , which results in *bubbling a holomorphic sphere*.
 - The derivative may blow up on the boundary of S , which results in *bubbling a holomorphic disk*.

The goal of this section is to restrict which of these degenerations can occur in codimension 1. Ultimately, we will rule out bubbling disks and spheres and becoming nodal, and will



restrict strongly which curves can split off at $\pm\infty$ and $e\infty$. In Section 5.7.3, we show that a few more degenerations can not occur for embedded curves. The remaining degenerations will be incorporated in the algebra of our invariants, in Chapters 6 and 7.

5.6.1. First restrictions on codimension-one degenerations.

Let

$$\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright) := \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P) \setminus \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P).$$

Some restrictions follow easily from the index formula and other simple considerations:

PROPOSITION 5.43. *Suppose that $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is 1-dimensional. Then for generic J every holomorphic comb in $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ has one of the following forms:*

- (1) *a two-story holomorphic building* (u_1, u_2);
- (2) *a simple holomorphic comb* (u, v) where v is a join curve;
- (3) *a simple holomorphic comb* (u, v) where v is a shuffle curve;
- (4) *a one-story holomorphic comb* (u, v_1, \dots, v_k) such that each v_i is a split curve, and, furthermore, the result $T_1^\triangleright \natural \dots \natural T_k^\triangleright$ of pregluing the sources of v_1, \dots, v_k is also a split curve; or
- (5) *a nodal holomorphic curve, obtained by degenerating some arcs with boundary on ∂S .*

(Recall the definitions of join curves, split curves and shuffle curves from the end of Section 5.3; in particular, a join curve or a shuffle curve has only one non-trivial component, while a split curve may have arbitrarily many.)

PROOF OF PROPOSITION 5.43. We begin with some general remarks. Since $\pi_2(\Sigma) = 0$, $\pi_2(\Sigma, \alpha) = 0$ and $\pi_2(\Sigma, \beta) = 0$, no disks or spheres can bubble off. As usual, Deligne-Mumford type degenerations, in which a closed circle or circles degenerate, are codimension 2, and hence do not occur in these moduli spaces. By contrast, it is possible for an arc with boundary on ∂S to get pinched in codimension 1 (and become a node); this is case (5).

Next, suppose we have a simple holomorphic comb (u, v) in $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Let $S^{\triangleright'}$ and $T^{\triangleright'}$ denote the sources of u and v respectively, and let m be $|W(T^{\triangleright'})|$.

The curve v induces a partition P' of $W(T^{\triangleright'})$, by specifying that p and q belong to the same part if they have the same t -coordinate in v . Then, $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^{\triangleright'}; P')$ (using the identification of $E(S^{\triangleright'})$ and $W(T^{\triangleright'})$).

Notice that $|P'| \leq |P|$. Also notice that $\chi(S') = \chi(S) - \chi(T') + m$. Since every component of T' must have a west puncture, $\chi(T') \leq m$, with equality if and only T' is a disjoint union of topological disks with one west puncture per component.

Since, by hypothesis, the expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ is 1, by Formula (5.9) we have

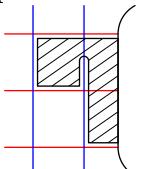
$$\text{ind}(B, S^\triangleright, P) = g - \chi(S) + 2e(B) + |P| = 2.$$

Now,

$$\begin{aligned} \text{ind}(B, S^{\triangleright'}, P') &= g - \chi(S') + 2e(B) + |P'| \\ &= g - \chi(S) + \chi(T') - m + 2e(B) + |P'| \\ &= 2 - |P| + \chi(T') - m + |P'|. \end{aligned}$$

Since $|P'| \leq |P|$ and $\chi(T') \leq m$, we therefore have $\text{ind}(B, S^{\triangleright'}, P') \leq 2$.

Suppose that $\text{ind}(B, S^{\triangleright'}, P') = 2$. Then $\chi(T') = m$ and $|P'| = |P'|$. Consequently, T' consists entirely of topological disks, each with a single puncture labeled w . It follows from



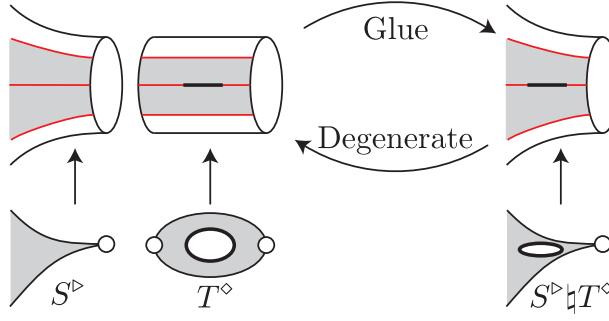


FIGURE 5.10. **A putative annulus at east ∞ .** Any near enough curve has a boundary component with no punctures, which violates the maximum modulus principle.

Propositions 5.16 and 5.31 that v can be glued to u , and so $\text{ind}(B, S^>, P) > \text{ind}(B, S'^>, P')$, contradicting the hypothesis.

Next, suppose that $\text{ind}(B, S'^>, P') = 1$. There are now two cases:

- $|P| = |P'|$ and $\chi(T') = m - 1$. In this case, either T' consists of $m - 1$ topological disks and one topological annulus, or just $m - 1$ topological disks. It is easy to see that the former does not occur: the image in $\mathbb{R} \times \mathbb{Z}$ of the annular component of T' is a strip with a slit in the middle. (See Figure 5.10.) Any curve close enough to this comb will still have this slit and so would be forced, by the maximum modulus theorem, to have a component on which $\pi_{\mathbb{D}} \circ u$ is constant.

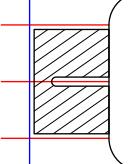
By contrast, the case that T' consists of $m - 1$ topological disks can occur. In this case, all but one of these disks has one w puncture, and one has two w punctures. We claim that none of the disks has two consecutive e punctures (not separated by a w puncture). Indeed, any two consecutive e punctures will have different heights in any curve with source $S'^> \sqcup T'^>$. But this contradicts the fact that $|P| = |P'|$. It follows that $m - 2$ of the disks in $T'^>$ are trivial components and the remaining disk is either a join component or a shuffle component.

- $|P'| = |P| - 1$ and $\chi(T') = m$. In this case, T' consists of m topological disks, each with a single west puncture. On first glance, it appears each disk may have an arbitrary number of east punctures. However, by Propositions 5.6, 5.16 and 5.31, we can glue u to v in $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^>)$ (though perhaps not in $\overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^>; P)$). It is then easy to see that the east punctures on a single component of T' will have different t -coordinates in all of the glued curves. It follows that if any component of T' has more than two east punctures then $|P| - |P'| > 1$, a contradiction. Therefore $T'^>$ is a split curve, as in Case (4) of the statement.

If $\text{ind}(B, S'^>, P') \leq 0$ then, by Proposition 5.6, the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S'^>; P')$ is empty.

Next, consider a general one-story holomorphic comb (u, v_1, \dots, v_ℓ) . The arguments above then imply that the bi-decorated source $T_1^> \sqcup \dots \sqcup T_\ell^>$ obtained by pregluing the sources of v_1, \dots, v_ℓ is a join curve or a split curve.

Finally, consider a general holomorphic comb U in $\partial \overline{\mathcal{M}}(\mathbf{x}, \mathbf{y}; S^>; P)$. It follows from Formula (5.9) that each story of U drops the expected dimension by 1, so there are at most



two stories. Further, if there are two stories, each story individually has index 1, so by the arguments above cannot have any components at east ∞ . \square

The first four kinds of degenerations will be involved in the definitions of the invariants, in Chapters 6 and 7. So we now show that nodal degenerations can not occur in certain parts of the moduli spaces. We give names to three different kinds of nodal degenerations:

DEFINITION 5.44. A *boundary degeneration* is a nodal holomorphic comb with the property that some irreducible component of the source has no + or - punctures and is mapped via a non-constant map to Σ .

DEFINITION 5.45. A *boundary double point* is a holomorphic comb whose source is a nodal curve and has a node p on the boundary with the property that the projection to $[0, 1] \times \mathbb{R}$ is not constant near either preimage point p_1 or p_2 of p in the normalization of the nodal curve.

DEFINITION 5.46. A *ghost component* of a holomorphic comb u is a component S_0 of the source S of u such that $u|_{S_0}$ is constant. If u has a ghost component then we say u is *haunted*.

LEMMA 5.47. *Every nodal degeneration results in a boundary degeneration, a curve with boundary double point, or a haunted curve.*

PROOF. If a nodal curve has no ghost components and no boundary degenerations then $\pi_{\mathbb{D}} \circ u$ is non-trivial on every component, and hence u has a boundary double point. \square

We rule out boundary degenerations immediately:

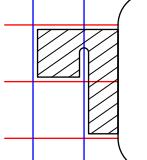
LEMMA 5.48. *Boundary degenerations do not exist.*

PROOF. Consider an irreducible component S of the source with no + or - punctures; and consider its corresponding map to $\Sigma \times [0, 1] \times \mathbb{R}$. Projecting further to \mathbb{R} , we obtain a harmonic function on S which is bounded on the boundary of S . Thus, this map is constant. It follows that the map u maps into a single fiber Σ . Since we assumed that the map was non-trivial, its image gives a non-trivial relative homology class $B \in H_2(\Sigma, \alpha; \mathbb{Z})$ with $n_z(B) = 0$. Such a two-chain violates homological linear independence of the α -curves which our bordered diagrams are required to satisfy, Definition 4.3. \square

REMARK 5.49. The non-existence of boundary degenerations is an artifact of our definition of bordered diagrams—in particular, the homological linear independence of the α - (respectively β -) curves. In more general settings, one may wish to consider diagrams with an excess of α - or β -curves. In those settings, boundary degenerations do play a role (and the definitions of the invariants need to be enhanced to take them into account).

REMARK 5.50. In the proof of Proposition 5.24 we observed that holomorphic curves at ∞ can be viewed as ghost components of a curve in $\Sigma_{\bar{e}} \times [0, 1] \times \mathbb{R}$. Since Definition 5.46 is about curves in $\Sigma \times [0, 1] \times \mathbb{R}$, it does not include such ghosts.

5.6.2. More on strong boundary monotonicity. A key tool in eliminating more nodal degenerations will be strong boundary monotonicity, to which we now turn. The definition of strong boundary monotonicity in Section 5.2 is a condition on curves; but we will show in Lemma 5.53 that the condition in fact depends only on the asymptotics of the curve. We will also see, in Lemma 5.55, that the strong boundary monotonicity condition is closed. We start with a lemma about the boundaries of our holomorphic curves.



LEMMA 5.51. Let $u \in \mathcal{M}(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. Let a be an arc in ∂S mapped by u to $\alpha_i \times \{1\} \times \mathbb{R}$ for some i . Then $t \circ u|_a: a \rightarrow \mathbb{R}$ is either strictly monotone or constant.

PROOF. The map $\pi_{\mathbb{D}} \circ u$ is holomorphic and has image contained in $[0, 1] \times \mathbb{R}$, and its restriction to the arc a is contained in $\{1\} \times \mathbb{R}$. Thus, if $t \circ u|_a$ is non-constant then it cannot have any critical points on the arc. \square

DEFINITION 5.52. Let \mathbf{s} be a k -element multi-set of elements of $[2k]$ (i.e., a formal linear combination of elements of $[2k]$ with coefficients in $\mathbb{N} \cup \{0\}$, so that the sum of the coefficients is k), and $\vec{\rho} = (\rho_1, \dots, \rho_n)$ be a sequence of non-empty multi-sets of Reeb chords. Define

$$o(\mathbf{s}, \vec{\rho}) := (\mathbf{s} \cup M(\vec{\rho}^+)) \setminus M(\vec{\rho}^-)$$

where $\vec{\rho}^+ := \bigcup_i \rho_i^+$ and the union and difference are interpreted in terms of multi-sets. Let $\vec{\rho}_{[i,j]}$ be the subsequence (ρ_i, \dots, ρ_j) of $\vec{\rho}$. (If $i > j$ the subsequence is empty.) We say that the pair $(\mathbf{s}, \vec{\rho})$ is *strongly boundary monotone* if the following three conditions are satisfied:

- (SB-1) The multi-set \mathbf{s} is actually a subset of $[2k]$ (i.e., \mathbf{s} has no repeated entries),
- (SB-2) $M(\rho_{i+1}^-) \subset o(\mathbf{s}, \vec{\rho}_{[1,i]})$ and
- (SB-3) $M(\rho_{i+1}^+)$ is disjoint from $o(\mathbf{s}, \vec{\rho}_{[1,i]}) \setminus M(\rho_{i+1}^-)$.

This is equivalent to the following conditions:

- (SB-1') For each $i = 0, \dots, n$, the multi-set $o(\mathbf{s}, \vec{\rho}_{[1,i]})$ is a k -element subset of $[2k]$ with no repeated elements.
- (SB-2') For each i , $M(\rho_i^-)$ and $M(\rho_i^+)$ (as multi-sets) have no elements with multiplicity bigger than 1.

We also extend the definition to pairs $(\mathbf{x}, \vec{\rho})$, where \mathbf{x} is a generalized generator, which we call strongly boundary monotone if $(o(\mathbf{x}), \vec{\rho})$ is strongly boundary monotone.

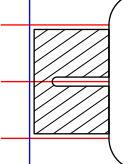
LEMMA 5.53. If $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone and $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ with $[\vec{P}] = \vec{\rho}$, then u is strongly boundary monotone. Conversely, if $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ contains a strongly boundary monotone holomorphic curve u then $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone.

PROOF. The fact that $(\mathbf{x}, \vec{\rho})$ being strongly boundary monotone implies u is strongly boundary monotone is immediate from Lemma 5.51.

To see that $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ strongly boundary monotone implies $(\mathbf{x}, \vec{\rho})$ strongly boundary monotone, we also use the fact that there are no boundary degenerations, as follows. Condition (SB-1) is immediate from the fact that \mathbf{x} is a generator. To prove Condition (SB-2), suppose that at some stage $M(\rho_{i+1}^-) \notin o(\mathbf{s}, \vec{\rho}_{[1,i]})$, so there is some chord $\rho_j \in \rho_{i+1}$ with $\rho_j^- \notin o(\mathbf{s}, \vec{\rho}_{[1,i]})$. Then there must also be a chord $\rho_\ell \in \rho_{i+1}$ with $M(\rho_\ell^+) = M(\rho_j^-)$, and an arc A on ∂S so that u is asymptotic to ρ_ℓ at one end of A , ρ_j at the other end of A , and $t \circ u|_A$ is constant. But by the open mapping principle applied to $\pi_{\mathbb{D}} \circ u$, this implies that the component of S containing A is a boundary degeneration, violating Lemma 5.48. A similar argument implies Condition (SB-3). \square

We will sometimes say that a moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is strongly boundary monotone if $(\mathbf{x}, [\vec{P}])$ is strongly boundary monotone; this is justified by Lemma 5.53.

We next show that strong boundary monotonicity is a closed condition on the compactified moduli space. To make sense of this, we need to define strong boundary monotonicity for holomorphic combs. The following, somewhat weak definition will be sufficient for our purposes:



DEFINITION 5.54. A holomorphic comb U is *strongly boundary monotone* if, after deleting any ghost components, every story of the spine of U is strongly boundary monotone.

LEMMA 5.55. *The strong boundary monotonicity condition is closed. That is, if $\{U_j\}$ is a convergent sequence of strongly boundary monotone holomorphic combs then the limit U of $\{U_j\}$ is strongly boundary monotone.*

PROOF. This follows from the monotonicity of $t \circ u$ on arcs (Lemma 5.51) and the non-existence of boundary degenerations (Lemma 5.48). \square

5.6.3. No nodal curves. In this section, we rule out boundary double points and haunted curves for moduli spaces satisfying the strong boundary monotonicity condition. We start with boundary double points.

LEMMA 5.56. *Suppose that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone. Then the moduli spaces $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ with $[\vec{P}] = \vec{\rho}$ do not contain any curves with boundary double points.*

PROOF. Suppose that $u_j \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ converges to a holomorphic comb u with source S containing a boundary double point. We have a node p in the source S whose pre-images p_1 and p_2 in the normalization \tilde{S} of S both get mapped non-trivially to $[0, 1] \times \mathbb{R}$. Since p is on the boundary of S , it follows that p is mapped to some α - or β -curve. Now, since the projection $t \circ \tilde{u}$ is a non-trivial harmonic function near both p_1 and p_2 , by the open mapping principle, we can find disjoint neighborhoods N_i of $p_i \in \partial \tilde{S}$ (for $i = 1, 2$), with the property that both $t \circ \tilde{u}(N_1)$ and $t \circ \tilde{u}(N_2)$ are the same open interval I around $t \circ u(p)$. It follows that, for large enough j , u_j is not strongly boundary monotone. \square

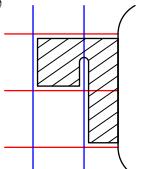
Next we show that ghosts are vanishingly rare.

LEMMA 5.57. *Suppose that $(\mathbf{x}, [\vec{P}])$ is strongly boundary monotone and $\text{ind}(B, S^\triangleright, \vec{P}) \leq 2$. Then $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright, \vec{P})$ does not contain any haunted curves.*

PROOF. This follows from Lemma 5.56 and a standard argument, a version of which we give. Suppose that arcs and circles A_1, \dots, A_n in S collapse to form some ghost components. Let S_1, \dots, S_m be the ghost components of the limit and S_0 the union of the other components of the limit, S_0^\triangleright decorated source corresponding to S_0 and $u: S_0 \cup \dots \cup S_m \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ the limiting holomorphic curve. The arcs A_i come in two types: arcs which separate a ghost-component from a non-ghost component, which we will call *important arcs*; and arcs separating two ghost components, a ghost component from itself, two non-ghost components, or a non-ghost component from itself.

Let G be the *ghost graph* with one vertex for each ghost component and one edge for each node connecting two ghost components. Let G_1, \dots, G_p be the connected components of G and T_i , $i = 1, \dots, p$, the union of the ghost components corresponding to G_i . We claim that each T_i has exactly one node corresponding to an important arc. Certainly T_i has at least one node corresponding to an important arc, and if some T_i has more than one node corresponding to an important arc then $u|_{S_0}$ would have a double point, violating Lemma 5.56. This argument also implies that smoothing the nodes in T_i gives a surface with a single boundary component.

Each S_i must be stable; in particular, no S_i is a disk or sphere with 2 or fewer nodes. Since exactly one node of T_i corresponds to an important arc, it follows that for each $i = 1, \dots, p$,



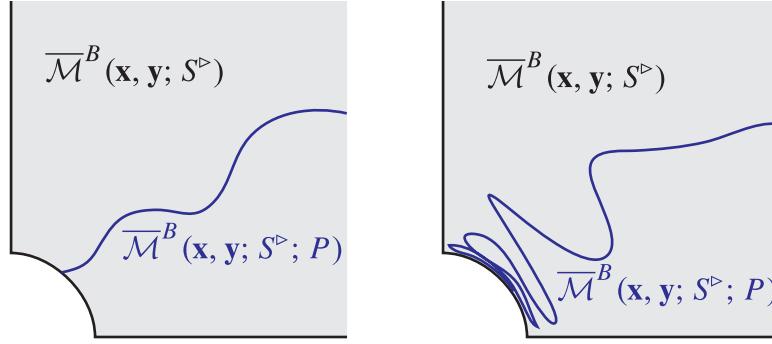


FIGURE 5.11. A picture of the expected behavior of $\bar{\mathcal{M}}(x, y; S^\triangleright; \vec{P}) \subset \bar{\mathcal{M}}(x, y; S^\triangleright)$ (left) and a behavior of $\bar{\mathcal{M}}(x, y; S^\triangleright; \vec{P}) \subset \bar{\mathcal{M}}(x, y; S^\triangleright)$ which we have not ruled out (right). The reader might compare with Figure 5.8.

either G_i is not a tree or some component of T_i is not a disk. Since the smoothing of T_i is a surface with connected boundary, it follows that $\chi(S_0) \geq \chi(S) + 2$. Thus,

$$\begin{aligned} \text{ind}(B, S_0^\triangleright, \vec{P}) &= g - \chi(S_0) + 2e(B) + |\vec{P}| \\ &\leq g - \chi(S) + 2e(B) + |\vec{P}| - 2 \\ &= \text{ind}(B, S^\triangleright, \vec{P}) - 2 \\ &\leq 0. \end{aligned}$$

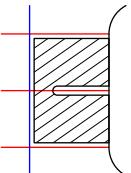
Hence, $u|_{S_0}$ belongs to a negative-dimensional moduli space, and so does not exist. \square

5.6.4. The codimension-one boundary of the moduli space of curves with a given source. We have now ruled out bubbling and nodal degenerations, so the only ends of the moduli spaces that remain are two-story buildings, collisions of levels and degenerating join, split and shuffle curves at $e\infty$. At this point, we would like to assert that if strong boundary monotonicity is satisfied then $\bar{\mathcal{M}}^B(x, y; S^\triangleright; \vec{P})$ is a compact 1-manifold with boundary, and $\partial\bar{\mathcal{M}}^B(x, y; S^\triangleright; \vec{P})$ consists of exactly those pieces. The difficulty with this statement is the case of shuffle curves and split curves with many split components: in these cases, the transversality needed by Proposition 5.31 is absent. One would like to assert, for instance, that simple combs (u', v') in $\bar{\mathcal{M}}^B(x, y; S^\triangleright; \vec{P}) \setminus \mathcal{M}^B(x, y; S^\triangleright; \vec{P})$ are isolated. The maps u' are, indeed, isolated, but this is not obvious for the components v' at east ∞ . (This is related to the fact that the evaluation map from $\bar{\mathcal{M}}^B(x, y; S^\triangleright; \vec{P})$ collapses strata coming from non-trivial moduli at east ∞ .)

However, all we will need later is that the sum of the number of elements of the moduli spaces occurring as the boundaries of $\bar{\mathcal{M}}^B(x, y; S^\triangleright; \vec{P})$ (when the latter space is one-dimensional) is zero modulo two. This weaker statement is Theorem 5.61, below.

Before stating this theorem, we define formally various moduli spaces that can appear at the end of one-dimensional spaces, as illustrated in Figure 5.12, and place one more restriction on collisions of levels.

DEFINITION 5.58. Fix a one dimensional moduli space $\mathcal{M}^B(x, y; S^\triangleright; \vec{P})$ (or \mathcal{M} for short) satisfying strong boundary monotonicity.



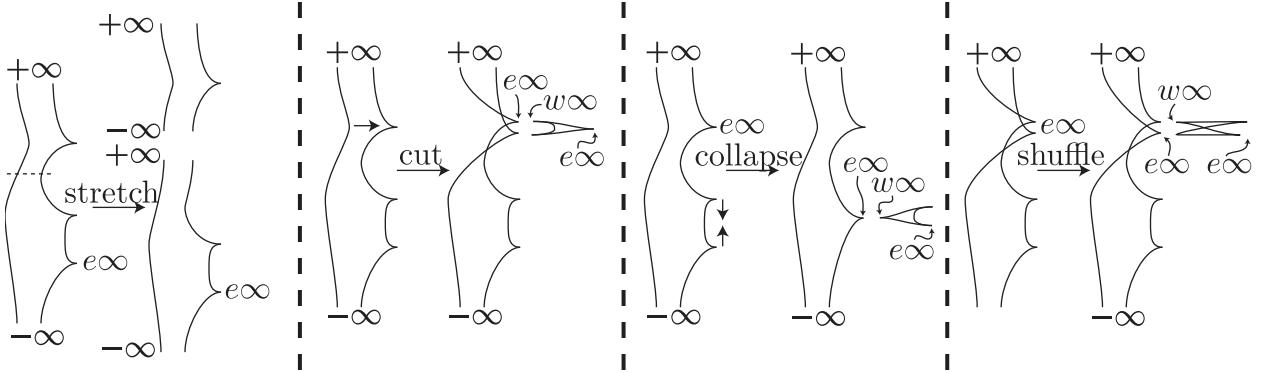


FIGURE 5.12. **Schematics and examples of the four kinds of degenerations of Theorem 5.61.** Far left: degenerating into a two-story building. Center left: degenerating a join curve. Center right: degenerating a split curve. Far right: degenerating a shuffle curve.

A *two-story end* of \mathcal{M} is an element of $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; \vec{P}_2)$, where $\mathbf{w} \in \mathfrak{S}(\mathcal{H})$, $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ and $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$ are such that $B = B_1 * B_2$, and $S^\triangleright = S_1^\triangleright \sqcup S_2^\triangleright$ is a way of splitting S^\triangleright in two which divides the ordered partition \vec{P} as $\vec{P}_1 < \vec{P}_2$.

A *join curve end* of \mathcal{M} at level i is an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright'; \vec{P}')$, where S^\triangleright' and \vec{P}' are obtained in the following way. Pick an east puncture q of S^\triangleright in the i^{th} part of \vec{P} and a decomposition $\rho_q = \rho_a \uplus \rho_b$. Then the decorated source S^\triangleright' is any source with a pair of punctures a and b labeled by ρ_a and ρ_b , and such that S^\triangleright is obtained from S^\triangleright' by pregluing a join component to S^\triangleright' at the punctures a and b . The partition \vec{P}' is obtained from \vec{P} by replacing q with $\{a, b\}$ in the i^{th} part.

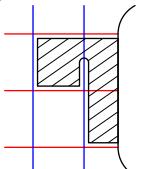
An *odd* (respectively *even*) *shuffle curve end* of \mathcal{M} at level i is an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright'; \vec{P}')$, where S^\triangleright' and \vec{P}' are obtained in the following way. Pick east punctures q_1 and q_2 of S^\triangleright contained in the i^{th} part of \vec{P} , and so that the corresponding Reeb chords are nested (respectively interleaved). Then S^\triangleright' is any source together with punctures q'_1 and q'_2 such that S^\triangleright is obtained from S^\triangleright' by pregluing an odd (respectively even) shuffle component T^\diamondsuit to S^\triangleright' at the punctures q'_1 and q'_2 with q_1 and q_2 labeling the punctures of T^\diamondsuit . The partition \vec{P}' is obtained from \vec{P} by replacing $\{q_1, q_2\}$ with $\{q'_1, q'_2\}$ in the i^{th} part.

A *collision of levels i and $i+1$* in \mathcal{M} is an element of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright'; (P_1, \dots, P_i \uplus P_{i+1}, \dots, P_n))$ with i satisfying $1 \leq i < n$ and S^\triangleright' obtained from S^\triangleright by contracting arcs on $\partial S^\triangleright$ connecting punctures labeled by abutting pairs of Reeb chords from P_i and P_{i+1} (and replacing those pairs of Reeb chords ρ_i, ρ_j by their joins $\rho_i \uplus \rho_j$), and $P_i \uplus P_{i+1}$ denotes the Reeb chords in S^\triangleright' coming from P_i and P_{i+1} . (The notions of abutting and \uplus are given in Definition 3.10.)

We extend this definition inductively to a collision of levels $i, \dots, i+j$: a collision of levels $i, \dots, i+j$ is a collision of levels i and $i+1$ in the result of a collision of levels $i+1, \dots, i+j$.

We will see (Lemma 5.60) that homological linear independence of the α -curves often prevents collisions of levels. First, a definition:

DEFINITION 5.59. We say that two sets of Reeb chords ρ_i and ρ_{i+1} are *weakly composable* if, for all $\rho_j \in \rho_i$ and $\rho_k \in \rho_{i+1}$, if $M(\rho_j^+) = M(\rho_k^-)$ then $\rho_j^+ = \rho_k^-$. More generally, a sequence



$\vec{\rho} = (\rho_1, \dots, \rho_n)$ is said to be weakly composable if for all $i = 1, \dots, n-1$, the sets $\rho_1 \uplus \dots \uplus \rho_i$ and ρ_{i+1} are weakly composable.

Note that if two sets of Reeb chords are composable in the sense of Definition 3.11 then they are weakly composable, but the converse is false.

LEMMA 5.60. *Suppose that $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is strongly boundary monotone and $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ contains a point corresponding to a collision of levels i and $i+1$ in \vec{P} . Then $[P_i]$ and $[P_{i+1}]$ are weakly composable.*

More generally, suppose that $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ contains a point corresponding to a collision of levels $i, i+1, \dots, j$ in \vec{P} . Then $([P_i], [P_{i+1}], \dots, [P_j])$ is weakly composable.

PROOF. Let $\{u_j\}$ be a sequence of curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ whose spine converges to a collision of levels. Fix chords $\rho \in [P_i]$ and $\sigma \in [P_{i+1}]$ with the property that $M(\rho^+) = M(\sigma^-)$. Our goal is to show that $\rho^+ = \sigma^-$.

Consider the arc A on ∂S along which u_j leaves ρ . The image $\pi_\Sigma \circ u_j(A)$ is contained in some curve α_i . Similarly, for the arc B on $\partial S^\triangleright$ along which u_j enters σ , $\pi_\Sigma \circ u_j(B)$ is contained in the same curve α_i . By hypothesis, the sequences $t(u_j(\rho))$ and $t(u_j(\sigma))$ converge to the same point.

We must have $A = B$ by strong boundary monotonicity on u_j . Therefore, by Lemma 5.51, the restriction of $t \circ u$ to A is constant, and hence u must be constant on the entire component of S containing A . Lemmas 5.48 and 5.57 imply that this component must be mapped to $\text{east } \infty$. It follows that the terminal point of ρ coincides with the initial point of σ , as desired.

The more general case is similar, and is left to the reader. \square

We are now ready to state the theorem we will use to prove all the algebraic properties of our modules.

THEOREM 5.61. *Suppose that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone. Fix \mathbf{y} , $B \in \pi_2(\mathbf{x}, \mathbf{y})$, S^\triangleright , and \vec{P} so that $[\vec{P}] = \vec{\rho}$ and $\text{ind}(B, S^\triangleright, P) = 2$. Let $\mathcal{M} = \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. Then the total number of*

- (ME-1) two-story ends of \mathcal{M} ,
- (ME-2) join curve ends of \mathcal{M} ,
- (ME-3) odd shuffle curve ends of \mathcal{M} and
- (ME-4) collision of levels i and $i+1$ in \mathcal{M}

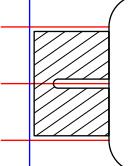
is even. Moreover, in Case (ME-4) the parts P_i and P_{i+1} are weakly composable.

PROOF. Let $U_{<\epsilon}$ denote the open subset of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ where two parts of \vec{P} have height difference $< \epsilon$. Let U_{shuf} denote the union of a smeared neighborhood of each shuffle curve end in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. It follows from the results above that

$$\overline{\mathcal{M}}_{\text{cropped}}^B = \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \setminus (U_{<\epsilon} \cup U_{\text{shuf}})$$

is a compact 1-manifold with boundary (smoothness following from Proposition 5.6, compactness from Proposition 5.24), the boundary of which consists of the following pieces:

- Two-story holomorphic buildings (u_1, u_2) . For ϵ small enough, the number of these correspond exactly to the number of two-story ends of \mathcal{M} . (This is ensured by Proposition 5.30. The fact that splittings must occur at generators, not just generalized generators, follows from strong boundary monotonicity.)



- Simple holomorphic combs (u, v) with v a join curve. For ϵ small enough the number of these boundary components is the number of join curve ends of \mathcal{M} . (This is an application of Proposition 5.31, with transversality hypotheses ensured by Propositions 5.6 and 5.16)
- An even number of points for each even shuffle curve end, and an odd number of points for each odd shuffle curve end. (This is an application of Proposition 5.42.)
- The subspace of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{P})$ where two parts differ in height by ϵ . By Propositions 5.39 (if at least one split component degenerates) and 5.6 (if no split components degenerate), for ϵ small enough the number of boundary points of this form agrees, modulo 2, with the number of collisions of two levels from S^\triangleright . The fact that the levels must be weakly composable follows from Lemma 5.60.

(In the first case, we take ϵ and U_{shuf} small enough that no two-story buildings occur in $\mathcal{M}^B \setminus \overline{\mathcal{M}}_{\text{cropped}}^B$, and in the second case we take them small enough that no join curve degenerations occur in $\overline{\mathcal{M}}^B \setminus \overline{\mathcal{M}}_{\text{cropped}}^B$. In the third case, ϵ should be small enough that $U_{<\epsilon}$ is disjoint from U_{shuf} . In the fourth case, ϵ should be small enough for Proposition 5.39 to apply.)

We have accounted for all the boundary components, in view of Proposition 5.43 combined with Lemma 5.56. \square

REMARK 5.62. We will see in Lemma 5.77 that for moduli spaces of embedded curves, the only collisions which occur are ones in which the colliding parts are composable (as defined in Section 3.1.3) and not merely weakly composable, as well as similar restrictions on the join curve and shuffle curve ends.

5.7. More on expected dimensions

Recall from Proposition 5.8 that the expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ depends on the topology of the source curve. Our aim here is to study an index formula which depends on the source curve only through its homology class B and its asymptotics $[\vec{P}]$. We establish two key properties of this index:

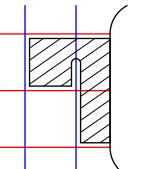
- In cases where we have an embedded curve representing the homology class, this formula agrees with our earlier index formula (Proposition 5.69 in Section 5.7.1).
- The index is additive under juxtapositions (Proposition 5.75 in Section 5.7.2).

A fundamental consequence of the index formula is the following:

- For collisions of levels occurring in the boundaries of two-dimensional, strongly boundary monotone moduli spaces, the colliding levels are composable, rather than just weakly composable (Lemma 5.77 in Section 5.7.3).

This last property is the justification for setting double crossings of strands to zero in the algebra associated to a surface. There are also restrictions on join curve and odd shuffle curve ends, also proved in Section 5.7.3.

5.7.1. The index at an embedded curve. Before stating the index formula, we recall and extend some definitions from Section 3.3.1. For $\alpha_1, \alpha_2 \in H_1(Z', \mathbf{a})$, recall that $L(\alpha_1, \alpha_2) = m(\alpha_2, \partial\alpha_1)$ is the “linking” of α_1 and α_2 . Concretely, if the α_i are represented



by single Reeb chords ρ_i , we have

$$(5.63) \quad L([\rho_1], [\rho_2]) = \begin{cases} 1/2 & \text{if } \rho_1^+ = \rho_2^- \text{ or } \rho_1^- \lessdot \rho_2^- \lessdot \rho_1^+ = \rho_2^+ \text{ or } \rho_1^- = \rho_2^- \lessdot \rho_1^+ \lessdot \rho_2^+ \\ -1/2 & \text{if } \rho_2^+ = \rho_1^- \text{ or } \rho_2^- \lessdot \rho_1^- \lessdot \rho_2^+ = \rho_1^+ \text{ or } \rho_1^- = \rho_2^- \lessdot \rho_2^+ \lessdot \rho_1^+ \\ 0 & \text{if } \rho_1 \cap \rho_2 = \emptyset \text{ or } \rho_1 \subset \text{Int}(\rho_2) \text{ or } \rho_2 \subset \text{Int}(\rho_1) \text{ or } \rho_1 = \rho_2 \\ 1 & \text{if } \rho_1^- \lessdot \rho_2^- \lessdot \rho_1^+ \lessdot \rho_2^+ \\ -1 & \text{if } \rho_2^- \lessdot \rho_1^- \lessdot \rho_2^+ \lessdot \rho_1^+. \end{cases}$$

(The notation \lessdot is defined in Section 3.2.) Recall that for $a \in \mathcal{A}(n, k)$ with starting idempotent S , the quantity $\iota(a) = \text{inv}(a) - m([a], S)$ is the Maslov component of the grading of a . We can again express this more concretely for in terms of Reeb chords.

LEMMA 5.64. *For ρ a set of Reeb chords so that $a(\rho) \neq 0$,*

$$\iota(a(\rho)) = - \sum_{\{\rho_1, \rho_2\} \subset \rho} |L([\rho_1], [\rho_2])| - \frac{|\rho|}{2}.$$

PROOF. Recall from the proof of Proposition 3.40 that $\iota(a)$ is unchanged under adding horizontal strands; thus we may pretend that the only strands in $a(\rho)$ are those coming from the Reeb chords in ρ . Both terms in $\iota(a(\rho))$, namely $\text{inv}(a(\rho))$ and $m([\rho], \rho^-)$, can be written as sums over pairs of Reeb chords or single chords. A single chord contributes $-1/2$ to $m([\rho], \rho^-)$, while a straightforward case analysis shows that the contribution to the sum from a pair of chords is -1 if they are interleaved, $-1/2$ if the endpoints abut, and 0 otherwise, as in the statement. \square

We will shorten notation and write $L(\rho_1, \rho_2)$ (respectively $L(\rho_1, \rho_2)$) for $L([\rho_1], [\rho_2])$ (respectively $L([\rho_1], [\rho_2])$) and $\iota(\rho)$ for $\iota(a(\rho))$. Also, for a strongly boundary monotone sequence of sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_\ell)$, define $\iota(\vec{\rho})$ by

$$(5.65) \quad \iota(\vec{\rho}) := \sum_i \iota(\rho_i) + \sum_{i < j} L(\rho_i, \rho_j).$$

LEMMA 5.66. *The product of a sequence of elements in $G'(n)$ is given by*

$$(k_1, \alpha_1) \cdots (k_n, \alpha_n) = \left(\sum_i k_i + \sum_{i < j} L(\alpha_i, \alpha_j), \sum_i \alpha_i \right).$$

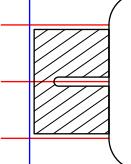
PROOF. As in the proof of Proposition 3.37, expand the right-associated product on the left-hand side of the equation and use bilinearity of L . \square

LEMMA 5.67. *For $\vec{\rho}$ a strongly boundary monotone sequence of Reeb chords, $\iota(\vec{\rho})$ is the Maslov component of $\text{gr}'(a(\rho_1)) \cdots \text{gr}'(a(\rho_\ell))$.*

PROOF. Clear from the definitions and Lemma 5.66. \square

For $x \in \alpha \cap \beta$ and B a domain, define $n_x(B)$ to be the average of the local multiplicities of B in the four regions surrounding x , and for $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, define $n_{\mathbf{x}}(B) := \sum_{x \in \mathbf{x}} n_x(B)$.

DEFINITION 5.68. We say that a pair $(B, \vec{\rho})$ with $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\vec{\rho}$ a sequence of non-empty sets of Reeb chords is *compatible* if the homology classes on the boundary agree,



$[\vec{\rho}] = \partial^\partial B$, and $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone. For such a pair, define the *embedded Euler characteristic*, *embedded index*, and *embedded moduli space* by

$$\begin{aligned}\chi_{\text{emb}}(B, \vec{\rho}) &:= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\rho}) \\ \text{ind}(B, \vec{\rho}) &:= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\rho}| + \iota(\vec{\rho}) \\ \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}) &:= \bigcup_{\substack{\chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\rho}) \\ [\vec{P}] = \vec{\rho}}} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}).\end{aligned}$$

Define $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ similarly.

Justification for this terminology is given by the following propositions.

PROPOSITION 5.69. *Suppose that $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ admits a holomorphic representative u . Then*

$$(5.70) \quad \chi(S) = \chi_{\text{emb}}(B, [\vec{P}])$$

if and only if u is embedded. When there is such an embedded, holomorphic u , the expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is given by

$$(5.71) \quad \text{ind}(B, S^\triangleright, \vec{P}) = \text{ind}(B, [\vec{P}]).$$

If $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ is strongly boundary monotone and admits a non-embedded holomorphic representative then $\text{ind}(B, S^\triangleright, \vec{P}) \leq \text{ind}(B, [\vec{P}]) - 2$.

PROOF. We imitate the proof of [Lip06a, Proposition 4.2], to which the reader is referred for a more leisurely account. We prove that if u is embedded, the Euler characteristic is as stated; the other direction and Formula (5.71) are immediate consequences; we will return to the inequality for non-embedded curves at the end of the proof.

Fix an embedded holomorphic curve $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. Let $\text{br}(u)$ denote the ramification number of $\pi_\Sigma \circ u$. (Here, boundary branch points contribute $1/2$.) By the Riemann-Hurwitz formula,

$$\chi(S) = e(S) + g/2 + \sum_i |P_i|/2 = e(B) - \text{br}(u) + g/2 + \sum_i |P_i|/2.$$

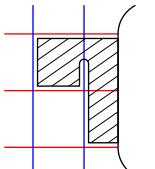
So, to compute $\chi(S)$ we only need to compute $\text{br}(u)$.

Let $\tau_R(u)$ denote a copy of u translated by R units in the \mathbb{R} -direction. We will prove the result by comparing $u \cdot \tau_R(u)$ for R small and R large.

Assume now that the partition \vec{P} is discrete, i.e., each part P_i consists of a single puncture labeled by ρ_i ; we will return to the general case at the end of the proof. Since u is embedded, for small ϵ the curves u and $\tau_\epsilon(u)$ intersect only near where u is tangent to $\partial/\partial t$, i.e., at branch points of u . Since u and $\tau_\epsilon(u)$ are J -holomorphic, their algebraic intersection number is $\text{br}(u)$. (Here and later, intersections along the boundary count for $1/2$.)

On the other hand, for R large, from the vantage point of u , $\tau_R(u)$ looks like g trivial strips $\mathbf{x} \times [0, 1] \times \mathbb{R}$; and from the vantage point of $\tau_R(u)$, u looks like g trivial strips $\mathbf{y} \times [0, 1] \times \mathbb{R}$. Thus, the number of intersections of u and $\tau_R(u)$ is $u \cdot \tau_R(u) = n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) - g/2$; the $g/2$ comes from the $2g$ corners. We will show that

$$(5.72) \quad u \cdot \tau_\epsilon(u) - u \cdot \tau_R(u) = \sum_{i < j} L(\rho_i, \rho_j).$$



and so

$$\begin{aligned}
\chi(S) &= e(B) - \text{br}(u) + g/2 + |\vec{P}|/2 \\
&= e(B) - u \cdot \tau_\epsilon(u) + g/2 + |\vec{P}|/2 \\
&= e(B) - u \cdot \tau_R(u) + g/2 - \sum_{i < j} L(\rho_i, \rho_j) + |\vec{P}|/2 \\
&= g + e(B) - n_x(B) - n_y(B) - \iota([\vec{P}])
\end{aligned}$$

as desired.

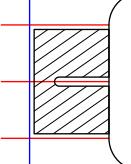
It follows from a simple Schwartz reflection argument that the intersection number $u \cdot \tau_r(u)$ can only change when an e puncture of $\tau_r(u)$, mapped to some Reeb chord ρ_1 , passes an e puncture of u mapped to some Reeb chord ρ_2 . Call this change $-L'(\rho_1, \rho_2)$; that is, $L'(\rho_1, \rho_2)$ is the number of double points which disappear when ρ_1 goes from below ρ_2 to above ρ_2 . We claim that $L'(\rho_1, \rho_2)$ is universal, depending only on ρ_1 and ρ_2 , and not on u itself. Equation (5.72) will then follow from a few model computations.

To see that $L'(\rho_1, \rho_2)$ is universal we employ a doubling argument, which also computes $L'(\rho_1, \rho_2)$. That is, we construct a new Heegaard diagram with boundary $(\Sigma', \alpha', \beta', z)$, which we glue to Σ along the boundary. We also construct holomorphic curves u_2 and u_1 which can be glued to u and $\tau_r(u)$ respectively, yielding curves uu_2 and $\tau_r(u)u_1$, respectively. Since the number of intersections between uu_2 and $\tau_r(u)u_1$ is topological, remaining constant under deformations of uu_2 and $\tau_r(u)u_1$, the number $L'(\rho_1, \rho_2)$ is equal to the number of double points which appear when u_1 is slid up relative to u_2 .

In fact, it suffices to construct Σ' , u_1 , u_2 and so on locally near $\rho_1 \cup \rho_2$. The constructions of Σ' , u_1 and u_2 can then be made to depend only on ρ_1 and ρ_2 , not on the curve u itself. This implies that $L'(\rho_1, \rho_2)$ depends only on ρ_1 and ρ_2 .

Our model curves will have the domains shown in Figure 5.13. We distinguish 10 different cases of how ρ_1 and ρ_2 can overlap.

- (1) $\rho_1^+ = \rho_2^-$. In this case, $L'(\rho_1, \rho_2) = 1/2$. We choose the curves u_1 and u_2 to consist of one strip each, as indicated in Figure 5.13 (upper left). When sliding u_1 up past u_2 , one boundary double point appears.
- (2) $\rho_1^- = \rho_2^+$. In this case, $L'(\rho_1, \rho_2) = -1/2$. This case is the same as case (1), with the roles of u_1 and u_2 exchanged.
- (3) $\rho_1^- \lessdot \rho_2^- \lessdot \rho_1^+ = \rho_2^+$. In this case, $L'(\rho_1, \rho_2) = 1/2$. We again choose u_1 and u_2 to consist of one strip each, though in this case they overlap, as indicated in Figure 5.13 (top right). When sliding u_1 up past u_2 , one boundary double point appears. (The fact that $u_1 \cup u_2$ is not a valid holomorphic curve—its $-\infty$ asymptotics are not a generator—is irrelevant: we are only interested in its behavior near $\rho_1 \cup \rho_2$.)
- (4) $\rho_1^- = \rho_2^- \lessdot \rho_1^+ \lessdot \rho_2^+$. In this case, $L'(\rho_1, \rho_2) = 1/2$. Similarly to case (3), we choose u_1 and u_2 to consist of one strip each, as indicated in Figure 5.13 (center left). Again, when sliding u_1 up past u_2 , one boundary double point appears.
- (5) $\rho_2^- \lessdot \rho_1^- \lessdot \rho_1^+ = \rho_2^+$. In this case $L'(\rho_1, \rho_2) = -1/2$. This case is the same as case (4) with the roles of u_1 and u_2 exchanged.
- (6) $\rho_1^- = \rho_2^- \lessdot \rho_2^+ \lessdot \rho_1^+$. In this case $L'(\rho_1, \rho_2) = -1/2$. This case is the same as case (3) with the roles of u_1 and u_2 exchanged.
- (7) ρ_1 is nested inside ρ_2 or vice-versa. In this case, $L' = 0$. Indeed, taking u_1 and u_2 to consist of one strip each, as indicated in Figure 5.13 (upper right), there are no



- intersection points between u_1 and u_2 no matter how they are slid. (A degenerate sub-case is when $\rho_1 = \rho_2$.)
- (8) ρ_1 and ρ_2 are disjoint. In this case, $L'(\rho_1, \rho_2) = 0$. Taking u_1 and u_2 to each consist of a single strip, as in Figure 5.13 (lower left) again gives no intersection points, no matter how they are slid.
 - (9) (ρ_1, ρ_2) is interleaved, with $\rho_1^- \lessdot \rho_2^- \lessdot \rho_1^+ \lessdot \rho_2^+$. In this case, $L'(\rho_1, \rho_2) = 1$. Take u_1 and u_2 to consist of a single strip each, as in Figure 5.13 (lower right). Then if u_2 is slid much higher than u_1 , there are no intersection points, while if u_2 is much lower than u_1 , there is a single interior intersection point.
 - (10) (ρ_2, ρ_1) is interleaved, with $\rho_2^- \lessdot \rho_1^- \lessdot \rho_2^+ \lessdot \rho_1^+$. In this case, $L' = -1$. This is the same as case the previous case, with the roles of u_1 and u_2 reversed.

This completes the proof in the case that \vec{P} is a discrete partition.

In the general case, it is not true that $\text{br}(u)$ is equal to the intersection number of u with $\tau_\epsilon(u)$ for ϵ small. That is, there may be intersection points of u and $\tau_\epsilon(u)$ which run off to east ∞ as $\epsilon \rightarrow 0$. This phenomenon is a simple variant of the change in number of double points studied above, and it follows from the model computations used above that

$$u \cdot \tau_\epsilon(u) - \text{br}(u) = \sum_i \sum_{\{\rho_a, \rho_b\} \subset [P_i]} |L(\rho_a, \rho_b)|.$$

Otherwise, the argument proceeds as before. The result follows from Lemma 5.64.

Finally, if $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ admits a non-embedded holomorphic representative then

$$u \cdot \tau_\epsilon(u) - \text{br}(u) = 2 \text{sing}(u) + \sum_i \sum_{\{\rho_a, \rho_b\} \subset [P_i]} |L(\rho_a, \rho_b)|,$$

where $\text{sing}(u)$ denotes the order of singularity of u (i.e., number of double-points in an equivalent singularity; compare [MW95]). The rest of the argument goes through, giving

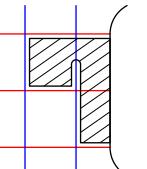
$$\begin{aligned} \chi(S) &= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota([\vec{P}]) + 2 \text{sing}(u) \\ \text{ind}(u) &= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\rho}| + \iota(\vec{\rho}) - 2 \text{sing}(u) \\ &= \text{ind}(B, \vec{\rho}) - 2 \text{sing}(u). \end{aligned}$$

Strong boundary monotonicity rules out boundary double points in u (which contribute $1/2$ to $\text{sing}(u)$), so $\text{sing}(u)$ is a positive integer. Thus, $\text{ind}(u) \leq \text{ind}(B, \vec{\rho}) - 2$, as desired. \square

5.7.2. The embedded index formula is additive. Since Proposition 5.69 assumed the existence of a holomorphic curve, it is not obvious that $\text{ind}(B, \vec{\rho})$ is additive in cases when there is no holomorphic representative. To prove this, we will adapt Sarkar's proof of the same fact in the closed case [Sar11, Theorem 3.2] to our situation. We start with a definition and some lemmas.

Suppose a is an oriented sub-arc of α , with endpoints in $\alpha \cap \beta$, and b is an oriented sub-arc of β , with endpoints in $\alpha \cap \beta$. We define the *jittered intersection number* $a \cdot b$ as follows. Let a_{SW} , a_{SE} , a_{NW} , and a_{NE} denote the four translates of a shown in Figure 5.14. The endpoints of these translates are disjoint from β , and the translates do not intersect $\alpha \cap \beta$. The translates inherit orientations from the orientations of a and b . Consequently, the intersection numbers $a_{SW} \cdot b, \dots, a_{NE} \cdot b$ are well defined. Define

$$a \cdot b = \frac{1}{4} (a_{SW} \cdot b + a_{SE} \cdot b + a_{NW} \cdot b + a_{NE} \cdot b).$$



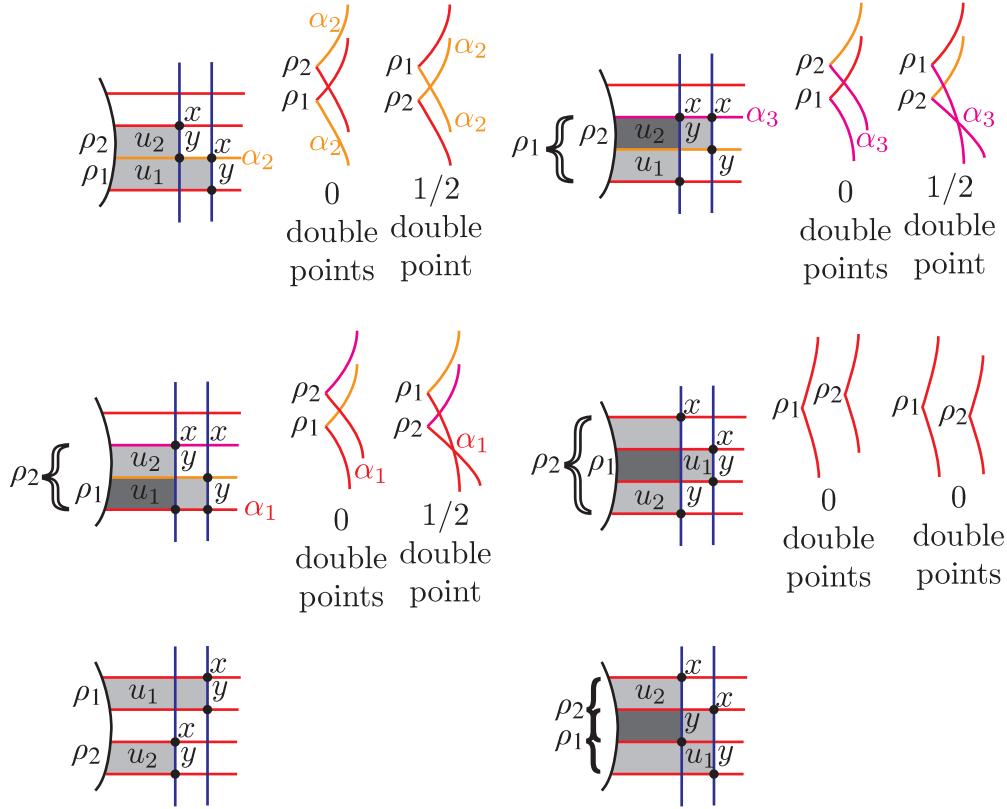


FIGURE 5.13. **The domains of model curves used to compute inv' .** Top left: case (1). Case (2) is obtained by switching ρ_1 and ρ_2 . Top right: case (3). Center left: case (4). Center right: case (7). Bottom left: case (8). Bottom right: case (9). In the top four pictures, a schematic for $u_1(\partial S_1), u_2(\partial S_2) \subset \boldsymbol{\alpha} \times \{1\} \times \mathbb{R}$ is also shown.

Note that instead of jittering a we could have jittered b with the same result. We extend the definition of $a \cdot b$ bilinearly to cellular 1-chains a contained in $\boldsymbol{\alpha}$ and b contained in $\boldsymbol{\beta}$. We define $b \cdot a$ analogously. We also define $a \cdot a'$ for a and a' both contained in $\boldsymbol{\alpha}$ (or $\boldsymbol{\beta}$) in exactly the same way. (This definition is analogous to the definitions of $n_{\mathbf{x}}(B)$ for Proposition 5.69 and $m(\alpha, P)$ in Section 3.3.1.)

The following lemma is due to Sarkar [Sar11, Section 3]. For the reader's convenience we repeat the proof here.

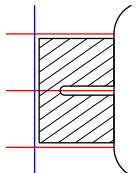
LEMMA 5.73. *The jittered intersection number $a \cdot b$ has the following properties:*

- (1) $a \cdot b = -b \cdot a$.
- (2) If a and a' are both contained in $\boldsymbol{\alpha}$ then $a \cdot a' = 0$.

For the remaining items, let $B \in \pi_2(\mathbf{x}, \mathbf{y})$, $a = \partial^{\boldsymbol{\alpha}}(B)$, $b = \partial^{\boldsymbol{\beta}}(B)$, $B' \in \pi_2(\mathbf{y}, \mathbf{w})$, $a' = \partial^{\boldsymbol{\alpha}}(B')$ and $b' = \partial^{\boldsymbol{\beta}}(B')$.

- (3) $n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) = a \cdot b$.
- (4) $n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B') = a' \cdot b$ and $n_{\mathbf{y}}(B) - n_{\mathbf{w}}(B) = a \cdot b'$.
- (5) If B and B' are provincial then $a \cdot b' + b \cdot a' = 0$.

PROOF. Properties (1) and (2) are obvious.



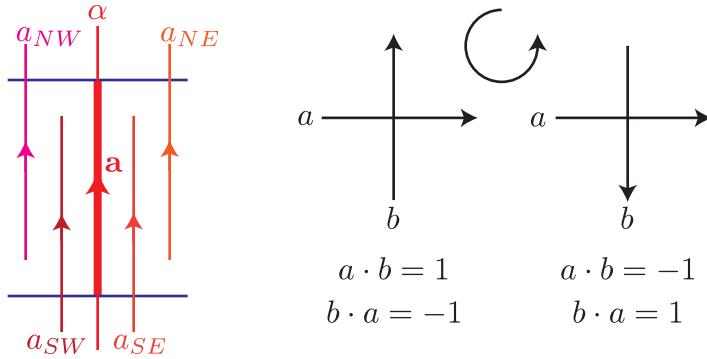


FIGURE 5.14. Jittered intersection number. Left: the four translates a_{SW} , a_{SE} , a_{NW} , and a_{NE} of an arc a . Right: conventions for intersection numbers.

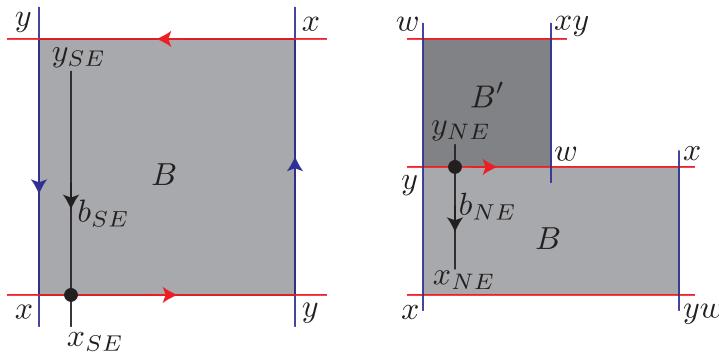
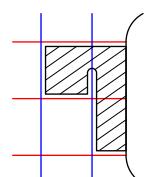


FIGURE 5.15. Parts (3) and (4) of the proof of Lemma 5.73. Left: part (3). The arrows show the orientations of a and b from ∂B . Notice that $a \cdot b_{SE} = -1$ and $n_{y_{SE}}(B) = n_{x_{SE}}(B) - (a \cdot b_{SE})$. Right: part (4). The horizontal arrow indicates the orientation induced by $\partial B'$, the vertical arrow the orientation induced by ∂B . Notice that $a' \cdot b_{NE} = -1 = n_{x_{NE}}(B') - n_{y_{NE}}(B')$.

To understand Property (3), assume for simplicity that a and b are embedded arcs, and consider the path b_{SE} . The multiplicity of B at the starting point of b_{SE} is $n_{y_{SE}}(B)$. Each time b_{SE} crosses a , the multiplicity of B increases by 1, for intersections contributing positively to $a \cdot b$, or decreases by 1, for intersections contributing negatively to $a \cdot b$. Consequently, at the ending point of b_{SE} , the multiplicity $n_{x_{SE}}$ is $n_{y_{SE}} + a \cdot b_{SE}$ and so $n_{x_{SE}}(B) - n_{y_{SE}}(B) = a \cdot b_{SE}$. Averaging over the four possible directions gives the result. Property (4) is proved similarly to Property (3), using b' instead of b . See Figure 5.15 for an example of these two properties.

Finally, Property (5) follows from the fact that $a + b$ and $a' + b'$ are null-homologous 1-chains, so $(a + b) \cdot (a' + b') = 0$. But by bilinearity and Property (2), this reduces to $a \cdot b' + b \cdot a' = 0$. (An alternate way to see Property (5), and see that the signs are write, is to use $\mathbf{x} - \mathbf{y} = \partial(\partial^\beta B)$ to compute $n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B') = B' \cdot (\partial(\partial^\beta B)) = (\partial B') \cdot \partial^\beta B$, using an extension of the jittered intersection number to 2-chains and 0-chains.) \square

We can extend Property (5) of Lemma 5.73 to non-provincial domains, as well.



LEMMA 5.74. *With notation as in Lemma 5.73, for arbitrary B and B' in $\pi_2(\mathbf{x}, \mathbf{y})$ and $\pi_2(\mathbf{y}, \mathbf{w})$, respectively, we have $a \cdot b' + b \cdot a' = L(\partial^\partial B, \partial^\partial B')$.*

PROOF. We now have that $a + b + \partial^\partial B$ and $a' + b' + \partial^\partial B'$ are null-homologous 1-chains, so their cap product in homology is 0. We can relate this to an extension of the jittered intersection product by, for instance, treating $\partial\bar{\Sigma}$ as an extra β -circle. Computing this intersection number and dropping terms that always vanish gives:

$$(a + b + \partial^\partial B) \cdot (a' + b' + \partial^\partial B') = a \cdot b' + b \cdot a' + a \cdot \partial^\partial B' + \partial^\partial B \cdot a' = 0.$$

Almost by definition, $a \cdot \partial^\partial B' = -\frac{1}{2}m(\partial^\partial B', \partial(\partial^\partial B))$ and $\partial^\partial B \cdot a' = \frac{1}{2}m(\partial^\partial B, \partial(\partial^\partial B'))$. Both of these terms are equal to $-\frac{1}{2}L(\partial^\partial B, \partial^\partial B')$, as desired. \square

PROPOSITION 5.75. *The embedded index is additive in the following sense. Let $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $B' \in \pi_2(\mathbf{y}, \mathbf{w})$, and let $\vec{\rho}$ and $\vec{\rho}'$ be sequences of non-empty sets of Reeb chords with $[\vec{\rho}] = \partial^\partial B$ and $[\vec{\rho}'] = \partial^\partial B'$. Then $\text{ind}(B, \vec{\rho}) + \text{ind}(B', \vec{\rho}') = \text{ind}(B * B', (\vec{\rho}, \vec{\rho}'))$*

(Here, $(\vec{\rho}, \vec{\rho}')$ denotes the sequence obtained by simply concatenating $\vec{\rho}$ and $\vec{\rho}'$.)

PROOF. It is clear that terms $e(B)$ and $|\vec{\rho}|$ of Definition 5.68 are additive. Thus, we must show that

$$n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + n_{\mathbf{y}}(B') + n_{\mathbf{w}}(B') + \iota(\vec{\rho}) + \iota(\vec{\rho}') = n_{\mathbf{x}}(B + B') + n_{\mathbf{w}}(B + B') + \iota((\vec{\rho}, \vec{\rho}')).$$

On the other hand, we have

$$\begin{aligned} [n_{\mathbf{x}}(B') - n_{\mathbf{y}}(B')] - [n_{\mathbf{y}}(B) - n_{\mathbf{w}}(B)] &= a' \cdot b - a \cdot b' \\ &= a' \cdot b + b \cdot a' - L(\partial^\partial B, \partial^\partial B') \\ &= \iota(\vec{\rho}) + \iota(\vec{\rho}') - \iota((\vec{\rho}, \vec{\rho}')), \end{aligned}$$

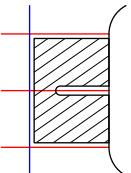
where the first equality uses property (4) of Lemma 5.73, the second uses Lemma 5.74, and the third uses property (1) and the definition of ι . This is equivalent to what we were trying to show. \square

5.7.3. The codimension-one boundary of the moduli spaces of embedded curves. We finish this chapter with two lemmas relating to which holomorphic curves occur in the boundaries of moduli spaces of embedded curves. We start by showing that when embedded holomorphic curves converge to holomorphic buildings, the stories are embedded.

LEMMA 5.76. *For a generic almost complex structure J and any pair $(B, \vec{\rho})$ with $\text{ind}(B, \vec{\rho}) = 2$, the two-story buildings which occur in the boundary of the embedded moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ (as in Theorem 5.61, say) are embedded. Equivalently, if (u_1, u_2) is such a two-story building, with $S_i^>$ the source of u_i , B_i the homology class of u_i and \tilde{P}_i the ordered partition associated to u_i , then:*

- $\chi(S_i^>) = \chi_{\text{emb}}(B_i, \tilde{P}_i)$
- $\text{ind}(B_i, P_i) = 1$.

PROOF. This follows from the inequality $\text{ind}(u) \leq \text{ind}(B, \vec{\rho})$ when u is non-embedded in Proposition 5.69 and the fact that moduli spaces of negative expected dimension ($\text{ind} \leq 0$) are empty (Proposition 5.6). \square



The next lemma states that when two levels collide in a sequence of embedded curves, the corresponding parts of a partition must be composable (as in Section 3.1.3), and not merely weakly composable (Definition 5.59). Similar statements hold for splittings and shuffles.

LEMMA 5.77. *Let $(\mathbf{x}, \vec{\rho})$ satisfy the strong boundary monotonicity condition, and $B \in \pi_2(\mathbf{x}, \mathbf{y})$ be compatible with $\vec{\rho}$. Suppose that $\text{ind}(B, \vec{\rho}) = 2$. Suppose there is a holomorphic comb (u, v) in $\partial\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$, and let $\vec{\rho}'$ be the sequence of sets of Reeb chords appearing as asymptotics of u . Then*

- (1) *if two levels ρ_i and ρ_{i+1} collide in u , ρ_i and ρ_{i+1} are composable;*
- (2) *if u is a join curve end at level i , ρ'_i is a splitting of ρ_i ; and*
- (3) *if u is an odd shuffle curve end at level i , ρ'_i is a shuffle of ρ_i .*

As a preliminary, we first state a characterization of compositability, splittings, and shuffles in terms of gradings. Recall that there is a partial order on $G'(n)$ by comparing the Maslov components: $(k_1, \alpha_1) < (k_2, \alpha_2)$ if $\alpha_1 = \alpha_2$ and $k_1 < k_2$.

LEMMA 5.78. *Let ρ and σ be sets of Reeb chords such that $a(\rho)$, $a(\sigma)$ and $a(\rho \uplus \sigma)$ are non-zero. Then:*

- (1) *The pair (ρ, σ) is composable if and only if $\text{gr}'(a(\rho \uplus \sigma)) = \text{gr}'(a(\rho)) \cdot \text{gr}'(a(\sigma))$.*
- (2) *A weak splitting ρ' of ρ is a splitting if and only if $\text{gr}'(a(\rho')) = \lambda^{-1} \text{gr}'(a(\rho))$.*
- (3) *A weak shuffle ρ' of ρ is a shuffle if and only if $\text{gr}'(a(\rho')) = \lambda^{-1} \text{gr}'(a(\rho))$.*

In all cases, if equality does not hold then the left-hand side has lower grading.

PROOF. Clear. □

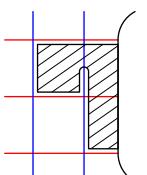
PROOF OF LEMMA 5.77. By Lemma 5.76, u is embedded.

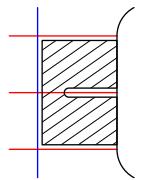
For the first statement, by Theorem 5.61 we know that ρ_i and ρ_{i+1} are weakly composable. By Lemma 5.78, if ρ_i and ρ_{i+1} are weakly composable but not composable, then $\text{gr}'(a(\rho_i \uplus \rho_{i+1})) < \text{gr}'(a(\rho_i)) \text{gr}'(a(\rho_{i+1}))$, so, by Lemma 5.67, $\iota(\vec{\rho}') < \iota(\vec{\rho})$. Therefore $\text{ind}(B, \vec{\rho}') < \text{ind}(B, \vec{\rho}) + (|\vec{\rho}'| - |\vec{\rho}|) = 1$. Thus the embedded moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$ has negative expected dimension and is therefore empty.

Similarly, a join curve end at level i gives a weak splitting of ρ_i by definition. By Lemma 5.78, for a weak splitting that is not a splitting we have $\text{gr}'(a(\rho'_i)) < \text{gr}'(a(\rho_i)) - 1$. This implies that $\text{ind}(B, \vec{\rho}') < 1$, so again $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$ has negative expected dimension.

The case of odd shuffle curve ends is similar. □

REMARK 5.79. A similar analysis shows that *even* shuffle curve ends on the boundary of a 1-dimensional embedded moduli space all have negative expected dimension. We do not use that, as we already showed in case (2) of Proposition 5.42 that such curves appear an even number of times on the boundary of any moduli space.





CHAPTER 6

Type D modules

We now turn to defining the type D module \widehat{CFD} of a bordered Heegaard diagram, using the moduli spaces developed in Chapter 5. The definition, in Section 6.1, is fairly straightforward. More work goes into proving that $\partial^2 = 0$ (Section 6.2) and that the module is independent of the choices made in its definition (Section 6.3). We also introduce a version of \widehat{CFD} with twisted coefficients, in Section 6.4. We defer the construction of the grading on \widehat{CFD} to Chapter 10.

6.1. Definition of the type D module

Let \mathcal{H} be a bordered Heegaard diagram (in the sense of Definition 4.3) which is provincially admissible (in the sense of Definition 4.23) for a bordered three-manifold Y , and let \mathcal{Z} be the *reverse* of the pointed matched circle (in the sense of Definition 3.16) appearing on the boundary of $\mathcal{H} = (\Sigma, \alpha, \beta, z)$. That is, \mathcal{Z} is $(Z, \bar{\alpha} \cap \partial\bar{\Sigma}, M, z)$, where $Z = -\partial\bar{\Sigma}$ has the opposite of the boundary orientation. Let $\mathcal{A}(\mathcal{Z})$ be the associated algebra, as in Chapter 3. We will abbreviate $\mathcal{A}(\mathcal{Z})$ by \mathcal{A} when it causes no confusion.

Fix a spin^c structure \mathfrak{s} over Y . The goal of this section is to define a left dg module $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ over \mathcal{A} .

Let $\mathfrak{S}(\mathcal{H})$ be the set of generators of \mathcal{H} , as defined in Definition 4.13. That is, $\mathfrak{S}(\mathcal{H})$ consists of subsets of $\alpha \cap \beta$ with

- g elements in all,
- exactly one element on each β circle,
- exactly one element on each α circle, and
- at most one element on each α arc.

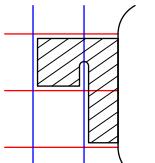
As in Section 4.3, let $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \subset \mathfrak{S}(\mathcal{H})$ denote the subset of generators representing the (absolute) spin^c structure \mathfrak{s} . Let $X(\mathcal{H}, \mathfrak{s})$ be the \mathbb{F}_2 -vector space spanned by $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $X(\mathcal{H})$ the \mathbb{F}_2 -vector spanned by $\mathfrak{S}(\mathcal{H})$, so that $X(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y)} X(\mathcal{H}, \mathfrak{s})$. Recall from Section 4.3 that, for $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, $o(\mathbf{x}) \subset [2k]$ denotes the set of α -arcs occupied by \mathbf{x} . Define $I_D(\mathbf{x})$ to be $I([2k] \setminus o(\mathbf{x}))$; that is, $I_D(\mathbf{x})$ is the idempotent corresponding to the complement of $o(\mathbf{x})$. Now, define an action on $X(\mathcal{H}, \mathfrak{s})$ of the sub-algebra of idempotents $\mathcal{I} \subset \mathcal{A}$ via

$$(6.1) \quad I(\mathbf{s}) \cdot \mathbf{x} := \begin{cases} \mathbf{x} & I(\mathbf{s}) = I_D(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

(Here, \mathbf{s} is a k -element subset of $[2k]$, as in Formula (3.22).)

As an \mathcal{A} -module, $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is defined by

$$\widehat{CFD}(\mathcal{H}, \mathfrak{s}) := \mathcal{A} \otimes_{\mathcal{I}} X(\mathcal{H}, \mathfrak{s}).$$



So, the module structure on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is quite simple; it is given by

$$(6.2) \quad a \cdot (b \otimes \mathbf{x}) := (ab) \otimes \mathbf{x}.$$

In particular, the summands $\mathcal{A}(\mathcal{Z}, i)$ of $\mathcal{A}(\mathcal{Z})$ for $i \neq 0$ act trivially on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$.

By contrast, the differential on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ involves counting moduli spaces of holomorphic curves $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$, where $\vec{\rho}$ is a sequence (ρ_1, \dots, ρ_n) of Reeb chords. (Here we are using $\vec{\rho}$ both for a sequence of Reeb chords and for the corresponding sequence of one-element sets of Reeb chords.)

Recall from Section 3.2 that $a(\rho_i)$ denotes the element of \mathcal{A} associated to ρ_i . For $\vec{\rho}$ a sequence of Reeb chords, let $a(\vec{\rho})$ be the product $a(\rho_1) \cdots a(\rho_n)$. (The reader is cautioned that this typically does not coincide with the algebra element $a(\boldsymbol{\rho})$ associated to a set of Reeb chords $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_n\}$.) Let $-\vec{\rho}$ be the sequence $(-\rho_1, \dots, -\rho_n)$ of chords with reversed orientation. (If ρ is a Reeb chord in $\partial\mathcal{H}$, $-\rho$ is a Reeb chord in $\mathcal{Z} = -\partial\mathcal{H}$.)

DEFINITION 6.3. Fix an almost-complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ which is admissible in the sense of Definition 5.1 and sufficiently generic in the sense of Definition 5.7. For $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, define

$$a_{\mathbf{x}, \mathbf{y}}^B := \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=1\}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})) a(-\vec{\rho}).$$

Here there is also an implicit condition that $(B, \vec{\rho})$ is compatible (i.e., that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone and $\partial^\partial B = [\vec{\rho}]$); we will often omit such conditions from the notation. Compactness of the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ implies that the sum defining $a_{\mathbf{x}, \mathbf{y}}^B$ is finite.

Define the differential on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ by

$$\partial(\mathbf{I} \otimes \mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} a_{\mathbf{x}, \mathbf{y}}^B \otimes \mathbf{y}.$$

We will verify in Lemma 6.5 that if \mathcal{H} is provincially admissible then the sums defining ∂ are finite. Extend ∂ by linearity and the Leibniz rule to all of $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$.

Let $\widehat{CFD}(\mathcal{H}) := \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y)} \widehat{CFD}(\mathcal{H}, \mathfrak{s})$.

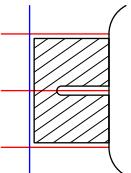
We will often abbreviate elements $a \otimes \mathbf{x} \in \widehat{CFD}(\mathcal{H}, \mathfrak{s})$ writing, simply $a\mathbf{x}$. In particular, we blur the distinction between generators $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and their associated elements $\mathbf{I} \otimes \mathbf{x} \in \widehat{CFD}(\mathcal{H}, \mathfrak{s})$

REMARK 6.4. Consider the \mathcal{I} -module $X(\mathcal{H}, \mathfrak{s})$, equipped with the map $\delta^1: X(\mathcal{H}, \mathfrak{s}) \rightarrow \mathcal{A} \otimes_{\mathcal{I}} X(\mathcal{H}, \mathfrak{s})$ given by

$$\delta^1(\mathbf{x}) := \partial(\mathbf{I} \otimes \mathbf{x}).$$

This defines a type D structure over \mathcal{A} with base ring \mathcal{I} on $X(\mathcal{H}, \mathfrak{s})$, in the sense of Definition 2.18. (The compatibility relation is equivalent to Proposition 6.7 below.) $\widehat{CFD}(\mathcal{H})$, then, is the differential module associated to this type D structure, as in Lemma 2.20. Further, Theorem 6.16, below, and Lemma 2.21 imply that this type D structure is invariant up to homotopy as a type D structure, not just as a module.

LEMMA 6.5. *If \mathcal{H} is provincially admissible, then the sum in the definition of $\partial\mathbf{x}$ on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is finite for every $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$. If \mathcal{H} is admissible, the map δ^1 (for the underlying type D structure, as in Remark 6.4) is bounded in the sense of Definition 2.22.*



PROOF. By Lemma 5.4, if $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ is non-empty then B has a positive domain. From Proposition 4.28 we then see that if \mathcal{H} is provincially admissible, then for a given \mathbf{x}, \mathbf{y} and $\vec{\rho}$ the union over B of 0-dimensional moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ is finite. There are only finitely many possible generators \mathbf{y} and algebra elements $a \in \mathcal{A}$; furthermore, for any given a there are only finitely many ways to write it as a product of Reeb chords, and so only finitely many possible $\vec{\rho}$. The first statement then follows.

If \mathcal{H} is admissible, then by Proposition 4.29, there are only finitely many $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with non-negative coefficients, and hence only finitely many possible terms in any $\delta^k(\mathbf{x})$. More precisely, terms in $\delta^k(\mathbf{x})$ count points in a product space $\prod_{i=1}^k \mathcal{M}^{B_i}(\mathbf{x}_i, \mathbf{x}_{i+1}; \vec{\rho}_i)$, where we sum over all $k+1$ -tuples $\{\mathbf{x}_i\}_{i=1}^{k+1}$ in \mathfrak{S} with $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_{k+1} = \mathbf{y}$, k -tuples $\{B_i \in \pi_2(\mathbf{x}_i, \mathbf{x}_{i+1})\}_{i=1}^k$ and compatible sequences of Reeb chords $\{\vec{\rho}_i\}_{i=1}^k$. Thus, for each non-zero $\delta^k(\mathbf{x})$ we can find such sequences $\{\mathbf{x}_i\}_{i=1}^{k+1}$ and $\{B_i \in \pi_2(\mathbf{x}_i, \mathbf{x}_{i+1})\}_{i=1}^k$ with the property that each B_i is positive. Adding the B_i , we obtain a corresponding $B \in \pi_2(\mathbf{x}, \mathbf{x}_k)$ which is also positive. Moreover, if we define $|B|$ to be the sum of the coefficients of B then $|B| \geq k$. Since there are only finitely many options for B , this gives an upper bound on k , as desired. \square

REMARK 6.6. In the definition of the boundary operator, the requirement that $\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ is redundant: if $a_{\mathbf{x}, \mathbf{y}}^B \neq 0$, then $B \in \pi_2(\mathbf{x}, \mathbf{y})$ is non-empty and hence, by Lemma 4.21, $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{y})$. Thus, we could have defined $\widehat{CFD}(\mathcal{H})$ without reference to spin^c structures, and then noted that the complex splits as a direct sum over $\text{spin}^c(Y)$.

6.2. $\partial^2 = 0$

This section is devoted to proving that $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is a differential module, i.e.:

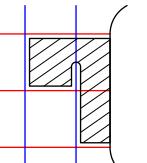
PROPOSITION 6.7. *The boundary operator ∂ on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ satisfies $\partial^2 = 0$.*

The proof of Proposition 6.7 involves a number of pieces, some of which we give as lemmas. The impatient reader is encouraged to skip to the examples after the proof, which illustrate the most important cases which arise.

In our next two lemmas, we look at which ordered lists of Reeb chords $\vec{\rho}$ actually contribute to the boundary operator. The second lemma follows from the first, which is more general than our immediate needs, but will be useful in Chapter 9. The first lemma is phrased in terms of sequences of sets of Reeb chords (rather than merely sequences of Reeb chords). In particular, if $\vec{\rho} = (\rho_1, \dots, \rho_n)$ is a sequence of sets of Reeb chords, the notation $-\vec{\rho}$ means the corresponding sequence of sets of Reeb chords where each Reeb chord has its orientation reversed (but with the ordering unchanged). Similarly, for a sequence $\vec{\rho} = (\rho_1, \dots, \rho_n)$ of Reeb chords, $-\vec{\rho}$ denotes $(-\rho_1, \dots, -\rho_n)$.

LEMMA 6.8. *Let $\mathbf{s} \subset [2k]$ be a subset and $\vec{\rho} = (\rho_1, \dots, \rho_n)$ be a sequence of non-empty sets of Reeb chords. Let $o(\mathbf{s}, \vec{\rho}_{[1,i]})$ be as in Definition 5.52. Suppose that for $i = 0, \dots, n$, $M(\rho_{i+1}^-) \subset o(\mathbf{s}, \vec{\rho}_{[1,i]})$ and $M((-\rho_{i+1})^-) \subset o([2k] \setminus \mathbf{s}, -\vec{\rho}_{[1,i]})$. (That is, $(\mathbf{s}, \vec{\rho})$ and $([2k] \setminus \mathbf{s}, -\vec{\rho})$ satisfy Conditions (SB-1) and (SB-2) of the strong boundary monotonicity condition, Definition 5.52.) Then:*

- (1) *Both $(\mathbf{s}, \vec{\rho})$ and $([2k] \setminus \mathbf{s}, -\vec{\rho})$ are strongly boundary monotone.*
- (2) *For $i = 1, \dots, n$, the sets $o(\mathbf{s}, \vec{\rho}_{[1,i]})$ and $o([2k] \setminus \mathbf{s}, -\vec{\rho}_{[1,i]})$ are disjoint.*
- (3) *For $i = 1, \dots, n$, $M(\rho_i^-)$ and $M(\rho_i^+)$ are disjoint.*



PROOF. We prove Conclusions (2) and (3) simultaneously by induction on i . The case where $i = 0$ is the statement that \mathbf{s} and $[2k] \setminus \mathbf{s}$ are disjoint. Assume the result for i . By hypothesis, $M((-\boldsymbol{\rho}_{i+1})^-) \subset o([2k] \setminus \mathbf{s}, -\vec{\boldsymbol{\rho}}_{[1,i]})$, which, by the inductive hypothesis, is disjoint from $o(\mathbf{s}, \vec{\boldsymbol{\rho}}_{[1,i]})$, which in turn contains $M(\boldsymbol{\rho}_{i+1}^+)$. Since reversing orientation switches the sets of endpoints $\boldsymbol{\rho}_{i+1}^-$ and $\boldsymbol{\rho}_{i+1}^+$, $M(-\boldsymbol{\rho}_{i+1}^-) = M(\boldsymbol{\rho}_{i+1}^+)$. Thus, we have verified Conclusion (3). Thus, since $M(-\boldsymbol{\rho}_{i+1}^+) = M(\boldsymbol{\rho}_{i+1}^-)$ and $M(\boldsymbol{\rho}_{i+1}^+)$ are disjoint, it follows that $o(\mathbf{s}, \vec{\boldsymbol{\rho}}_{[1,i+1]})$ and $o([2k] \setminus \mathbf{s}, -\vec{\boldsymbol{\rho}}_{[1,i+1]})$ are disjoint, as well, verifying Conclusion (2).

Finally, we turn to Conclusion (1). To see that $(\mathbf{s}, \vec{\boldsymbol{\rho}})$ is boundary monotone, it suffices to prove that $M(\boldsymbol{\rho}_{i+1}^+)$ is disjoint from $o(\mathbf{s}, \vec{\boldsymbol{\rho}}_{[1,i]}) \setminus M(\boldsymbol{\rho}_{i+1}^-)$. But $M(\boldsymbol{\rho}_{i+1}^+) = M(-\boldsymbol{\rho}_{i+1}^-) \subset o([2k] \setminus \mathbf{s}, \vec{\boldsymbol{\rho}}_{[1,i]})$, which by Conclusion (2) is disjoint from $o(\mathbf{s}, \vec{\boldsymbol{\rho}}_{[1,i]})$. By symmetry, $([2k] \setminus \mathbf{s}, -\vec{\boldsymbol{\rho}})$ is boundary monotone, as well. \square

Lemma 6.8 has the following consequence which is relevant for our present purposes:

LEMMA 6.9. *Suppose that $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ is strongly boundary monotone and $I_D(\mathbf{x})a(-\vec{\boldsymbol{\rho}}) \neq 0$. Then for each i , $M(\rho_i^-) \neq M(\rho_i^+)$.*

PROOF. The condition that $I_D(\mathbf{x})a(-\vec{\boldsymbol{\rho}}) \neq 0$ implies that $(I_D(\mathbf{x}), -\vec{\boldsymbol{\rho}}) = ([2k] \setminus o(\mathbf{x}), -\vec{\boldsymbol{\rho}})$ is strongly boundary monotone. Since by hypothesis, $(o(\mathbf{x}), \vec{\boldsymbol{\rho}})$ is strongly boundary monotone as well, Lemma 6.8 applies, and gives the desired conclusion. \square

The next two lemmas identify how some of the terms in Theorem 5.61 relate to the algebraic framework. Specifically, a term in $\partial^2 \mathbf{x}$ can come either from $a_1 \mathbf{y}$ appearing in $\partial \mathbf{x}$ and $a_2 \mathbf{z}$ appearing in $\partial \mathbf{y}$, contributing $a_1 a_2 \mathbf{z}$ to $\partial^2 \mathbf{x}$, or from $a \mathbf{y}$ appearing in $\partial \mathbf{x}$ and a' appearing in ∂a , contributing $a' \mathbf{y}$ to $\partial^2 \mathbf{x}$. Each type of term has its own interpretation in terms of moduli spaces.

LEMMA 6.10. *For $\mathbf{x}, \mathbf{w}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$, $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$ and $B_2 \in \pi_2(\mathbf{w}, \mathbf{y})$, we have*

$$a_{\mathbf{x}, \mathbf{w}}^{B_1} a_{\mathbf{w}, \mathbf{y}}^{B_2} = \sum_{\{\vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\rho}}_2 \mid \text{ind}(B_1 * B_2, (\vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\rho}}_2))=2\}} \# (\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\boldsymbol{\rho}}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\boldsymbol{\rho}}_2)) a(-(\vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\rho}}_2)).$$

(Here, $(\vec{\boldsymbol{\rho}}_1, \vec{\boldsymbol{\rho}}_2)$ denotes the sequence obtained by concatenating $\vec{\boldsymbol{\rho}}_1$ and $\vec{\boldsymbol{\rho}}_2$.)

PROOF. Clear. \square

LEMMA 6.11. *For $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, we have*

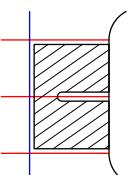
$$\partial a_{\mathbf{x}, \mathbf{y}}^B = \sum_{\{\vec{\boldsymbol{\rho}} \mid \text{ind}(B, \vec{\boldsymbol{\rho}})=2\}} \sum_{\left\{ i \mid \begin{array}{l} \rho_i, \rho_{i+1} \\ \text{abut} \end{array} \right\}} \# \mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_i \uplus \rho_{i+1}, \dots, \rho_n)) a(-\vec{\boldsymbol{\rho}}).$$

PROOF. By definition,

$$a_{\mathbf{x}, \mathbf{y}}^B = \sum_{\{\vec{\boldsymbol{\rho}} \mid \text{ind}(B, \vec{\boldsymbol{\rho}})=1\}} \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})) a(-\rho_1) \cdots a(-\rho_n).$$

and so

$$\begin{aligned} \partial a_{\mathbf{x}, \mathbf{y}}^B &= \sum_{\{\vec{\boldsymbol{\rho}} \mid \text{ind}(B, \vec{\boldsymbol{\rho}})=1\}} \sum_i \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})) a(-\rho_1) \cdots (\partial(a(-\rho_i))) \cdots a(-\rho_n) \\ &= \sum_{\{\vec{\boldsymbol{\rho}} \mid \text{ind}(B, \vec{\boldsymbol{\rho}})=1\}} \sum_i \sum_{\{\rho_j, \rho_k \mid \rho_i = \rho_j \uplus \rho_k\}} \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})) a(-\rho_1) \cdots a(-\rho_j) a(-\rho_k) \cdots a(-\rho_n). \end{aligned}$$



To see the last step, note that if ρ_j and ρ_k abut (in that order), then $-\rho_j$ and $-\rho_k$ do not abut; thus, for $\rho_i = \rho_j \uplus \rho_k$,

$$\partial(a(-\rho_i)) = \sum_{\{\rho_j, \rho_k \mid \rho_i = \rho_j \uplus \rho_k\}} a(\{-\rho_j, -\rho_k\}) = \sum_{\{\rho_j, \rho_k \mid \rho_i = \rho_j \uplus \rho_k\}} a(-\rho_j)a(-\rho_k).$$

Now, from the definition of $\text{ind}(B, \vec{\rho})$ (Definition 5.68),

$$\text{ind}(B, (\rho_1, \dots, \rho_j, \rho_k, \dots, \rho_n)) = \text{ind}(B, \vec{\rho}) + 1 = 2.$$

(By contrast, $\text{ind}(B, (\rho_1, \dots, \rho_k, \rho_j, \dots, \rho_n)) = \text{ind}(B, \vec{\rho})$.) The result follows by reindexing the sum to run over $\vec{\rho}' = (\rho_1, \dots, \rho_j, \rho_k, \dots, \rho_n)$. \square

PROOF OF PROPOSITION 6.7. Roughly, the proof proceeds by considering the boundaries of the index 1 moduli spaces. More precisely, we appeal to Theorem 5.61.

Observe that

$$\begin{aligned} \partial^2(a\mathbf{x}) &= \partial \left[(\partial a)\mathbf{x} + a \left(\sum_{\mathbf{w}} a_{\mathbf{x}, \mathbf{w}} \mathbf{w} \right) \right] \\ &= (\partial^2 a)\mathbf{x} + 2(\partial a)(\partial \mathbf{x}) + a \left[\left(\sum_{\mathbf{w}} (\partial a_{\mathbf{x}, \mathbf{w}}) \mathbf{w} \right) + \left(\sum_{\mathbf{w}} \sum_{\mathbf{y}} a_{\mathbf{x}, \mathbf{w}} a_{\mathbf{w}, \mathbf{y}} \mathbf{y} \right) \right], \end{aligned}$$

where $a_{\mathbf{x}, \mathbf{y}} = \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} a_{\mathbf{x}, \mathbf{y}}^B$. So it suffices to show that for all \mathbf{x} and \mathbf{y} ,

$$(6.12) \quad \partial(a_{\mathbf{x}, \mathbf{y}}) + \sum_{\mathbf{w}} a_{\mathbf{x}, \mathbf{w}} a_{\mathbf{w}, \mathbf{y}} = 0.$$

(This is, of course, equivalent to the compatibility condition for the underlying type D structure, Definition 2.18.)

Fix generators \mathbf{x} and \mathbf{y} . Fix also $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and a non-zero algebra element a . Fix an algebra element a and a sequence of Reeb chords $\vec{\rho}$ so that $\text{ind}(B, \vec{\rho}) = 2$. (The index condition depends only on B and a , not the particular sequence $\vec{\rho}$; see Lemma 5.67 and the definition of $\text{ind}(B, \vec{\rho})$.) By Theorem 5.61, applied to $(\mathbf{x}, \mathbf{y}, B, \vec{\rho})$ and the union, over all sources S^\triangleright with $\chi(S) = \chi_{\text{emb}}(B, \vec{\rho})$, of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$, the sum of the following terms is equal to zero:

- (1) The number of two-story ends, i.e., the number of elements of

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2)$$

where $B = B_1 * B_2$ and $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$.

- (2) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_j, \rho_k\}, \rho_{i+1}, \dots, \rho_n))$$

where $\rho_i = \rho_j \uplus \rho_k$.

- (3) The number of split curve ends (necessarily with one split component), i.e., the number of elements of

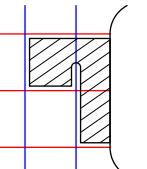
$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \rho_i \uplus \rho_{i+1}, \rho_{i+2}, \dots, \rho_n))$$

where $\rho_i^+ = \rho_{i+1}^-$.

- (4) The number of other collisions of levels, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_i, \rho_{i+1}\}, \rho_{i+2}, \dots, \rho_n))$$

where $\rho_i^+ \neq \rho_{i+1}^-$.



(Both the third and fourth contributions come from Case (ME-4) of Theorem 5.61. Shuffle curve ends cannot happen when the partition is discrete.)

Next, we sum over all choices of vector $\vec{\rho}$ with $a = a(-\vec{\rho})$. By Lemma 6.10, the sum of terms of type (1) corresponds to the coefficient in a of $\sum_{\{\mathbf{w}, B_1, B_2 \mid B_1 * B_2 = B\}} a_{\mathbf{x}, \mathbf{w}}^{B_1} a_{\mathbf{w}, \mathbf{y}}^{B_2}$, and by Lemma 6.11 the sum of the terms of type (3) corresponds to the coefficient of a in $\partial a_{\mathbf{x}, \mathbf{y}}^B$. We will show that the remaining two types of terms cancel against each other.

For the terms of type (4), there are several cases that cannot contribute:

- $M(\rho_i^-) = M(\rho_{i+1}^-)$ or $M(\rho_i^+) = M(\rho_{i+1}^+)$. This never occurs by Lemma 6.9 and strong boundary monotonicity.
- $M(\rho_i^+) = M(\rho_{i+1}^-)$ and $\rho_i^+ \neq \rho_{i+1}^-$. This is ruled out by Lemma 5.60.
- $M(\rho_i^-) = M(\rho_{i+1}^+)$ and $\rho_i^- \neq \rho_{i+1}^+$. In this case, $a(-\rho_i)a(-\rho_{i+1}) = 0$.
- (ρ_i, ρ_{i+1}) are interleaved (in that order). This degeneration does not occur in codimension one for embedded curves. More precisely, then the sets $\{\rho_i\}$ and $\{\rho_{i+1}\}$ are not composable by Definition 3.11, so by Lemma 5.77 this degeneration cannot occur.
- (ρ_{i+1}, ρ_i) are interleaved (in that order). Then $a(-\rho_i)a(-\rho_{i+1}) = 0$ as there are double crossing strands.

The remaining possibilities are

- (4a) $\rho_i^- = \rho_{i+1}^+$. These are exactly the moduli spaces which occur in Case (2), for the factorization $a(-\rho_1) \cdots a(-(\rho_i \uplus \rho_{i+1})) \cdots a(-\rho_n)$ of a . So, these two degenerations cancel.
- (4b) $\{M(\rho_i^-), M(\rho_i^+)\} \cap \{M(\rho_{i+1}^-), M(\rho_{i+1}^+)\} = \emptyset$, with $\{\rho_i, \rho_{i+1}\}$ nested (in either order) or disjoint. In this case, the same degeneration also occurs for the factorization $a(-\rho_1) \cdots a(-\rho_{i+1})a(-\rho_i) \cdots a(-\rho_n)$ of a (also in Case (4b)). So, these two degenerations cancel.

It follows that the coefficient of a in $\partial a_{\mathbf{x}, \mathbf{y}}^B + \sum_{\{\mathbf{w}, B_1, B_2 \mid B_1 * B_2 = B\}} a_{\mathbf{x}, \mathbf{w}}^{B_1} a_{\mathbf{w}, \mathbf{y}}^{B_2}$ vanishes. Summing over B and noting that the choice of algebra element a was arbitrary, Equation (6.12) follows. \square

The most interesting points in the proof of Proposition 6.7 are seen in the following examples, illustrated in Figure 6.1; see also [LOT09]. We put the boundary of $\bar{\Sigma}$ on the left to indicate visually that the orientation of $\partial\bar{\Sigma}$ is reversed in the algebra.

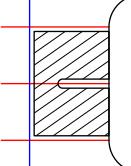
EXAMPLE 6.13. On the left of Figure 6.1 is a piece of a diagram with four generators. The complex \widehat{CFD} of this piece satisfies the relations

$$\begin{aligned} \partial\{a, c\} &= \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \{a, d\} + \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \right] \{c, e\} \\ \partial\{c, e\} &= \{b, d\} \\ \partial\{a, d\} &= \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right] \{b, d\}. \end{aligned}$$

The fact that $\partial^2 = 0$ follows from the relations in the algebra, more specifically

$$\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] \cdot \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \right].$$

This illustrates Case (1) in the proof of Proposition 6.7 (at the two two-story ends of the moduli space), as well as the cancellation of Case (2) and Case (4a) (when the boundary branch point goes out to $\partial\bar{\Sigma}$). The reader may also want to compare with Figure 5.1.



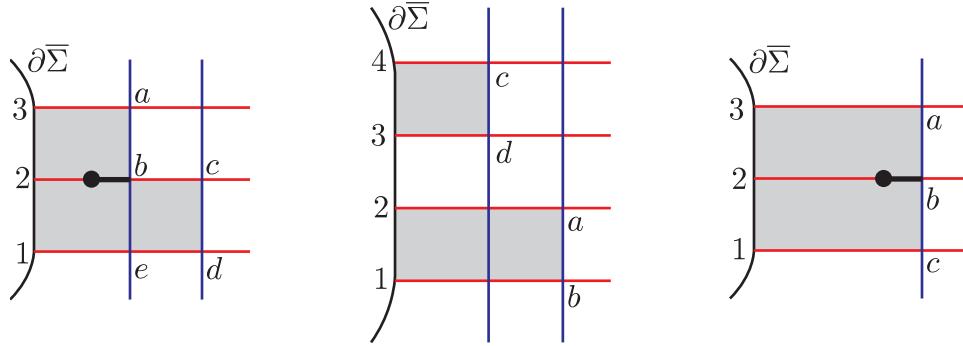


FIGURE 6.1. **Local illustrations of possible contributions to ∂^2 in the type D module.** Left: a family of curves with one end a two-story building and the other end a join curve end. Center: domain of a family of curves with a two-story building at one end and a collision of two levels not resulting in any split curves at the other. Right: a family of curves with a two-story building at one end and a collision of two levels, degenerating a split curve, at the other. (The thick lines indicate cuts; the boundary branch points at the ends of the cuts are shown as black dots.)

EXAMPLE 6.14. In the center of Figure 6.1, we have

$$\begin{aligned}\partial\{a, c\} &= \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right]\{b, c\} + \left[\begin{smallmatrix} 3 & 1 \\ 4 & \end{smallmatrix}\right]\{a, d\} \\ \partial\{b, c\} &= \left[\begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix}\right]\{b, d\} \\ \partial\{a, d\} &= \left[\begin{smallmatrix} 1 & 4 \\ 2 & \end{smallmatrix}\right]\{b, d\}.\end{aligned}$$

The fact that $\partial^2 = 0$ follows from the fact that $a(\{\frac{3}{4}\})$ and $a(\{\frac{1}{2}\})$ commute in the algebra—or, more precisely, because

$$\left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 & 1 \\ 4 & \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 4 \\ 2 & \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}\right].$$

This is the self-cancellation of Case (4b) of the proof of Proposition 6.7.

EXAMPLE 6.15. On the right of Figure 6.1, we have

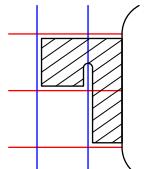
$$\begin{aligned}\partial\{a\} &= \left[\begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix}\right]\{b\} + \left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\right]\{c\} \\ \partial\{b\} &= \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right]\{c\}.\end{aligned}$$

Here, $\partial^2 = 0$ because of a differential in the algebra:

$$\partial\left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}\right].$$

This illustrates Case (3) of the proof of Proposition 6.7, or equivalently Lemma 6.11.

The above examples are local: they are, of course, not the modules associated to actual bordered Heegaard diagrams. We refer the reader wishing to see global examples to Section 11.2, where we compute the invariants of genus one handlebodies with certain borderings.



6.3. Invariance

THEOREM 6.16. *Up to homotopy equivalence, the differential module $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is independent of the choice of admissible, sufficiently generic almost complex structure and depends on the provincially admissible Heegaard diagram \mathcal{H} with $\mathcal{Z} = -\partial\mathcal{H}$ only through its induced (equivalence class of) bordered three-manifold $(Y, \phi: -F(\mathcal{Z}) \rightarrow \partial Y)$. That is, if \mathcal{H} and \mathcal{H}' are provincially admissible bordered Heegaard diagrams for $(Y, \phi: -F(\mathcal{Z}) \rightarrow \partial Y)$; J and J' are admissible, sufficiently generic almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$ and $\Sigma' \times [0, 1] \times \mathbb{R}$; and \mathfrak{s} is a spin^c structure on Y , then $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ (computed with respect to J) and $\widehat{CFD}(\mathcal{H}', \mathfrak{s})$ (computed with respect to J') are homotopy equivalent differential $\mathcal{A}(\mathcal{Z})$ -modules.*

The proof of Theorem 6.16 is modeled on the invariance proof for the closed case [OSz04d] (as modified for the cylindrical case in [Lip06a], say). In particular, the proof proceeds by showing that each of the three kinds of Heegaard moves from Proposition 4.10 induces a homotopy equivalence. (We also use Proposition 4.25 to ensure that all of the intermediate diagrams are provincially admissible.)

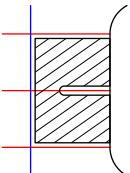
There are two essentially new issues. The first, and easier, is that we must keep track of the algebra elements which occur as coefficients in the maps. We will illustrate how this is done by defining the chain map induced by deformation of the almost complex structure. The second, more serious, issue is that the traditional proof of handleslide invariance uses “triangle maps,” the general definitions of which are subtle for bordered Heegaard diagrams. In the special case of handleslides the difficulties which come up in triangle maps in general can be circumvented, however. We illustrate this in a proof of invariance under handleslides which carry one α -arc over an α -circle (the hardest case). These two issues having been explained, the reader should have no difficulty adapting the rest of the invariance proof to our setting. For instance, the proof of isotopy invariance is similar to the proof of invariance under change of complex structure, but notationally slightly more complicated.

6.3.1. The chain map for change of complex structure. Recall that the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ from Chapter 5 depended on a choice of a generic, admissible almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$, and consequently the differential module $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ also depends on this choice; to make this choice explicit, we will write $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J)$ in this section. As is usual in Floer homology theories, one proves independence of the almost complex structure by constructing “continuation maps”, and proving that they are chain homotopy equivalences; see, for example, [Flo89, Theorem 4]. More specifically, let J_0 and J_1 be two generic, admissible almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$, and let ∂_0 and ∂_1 denote the boundary operators on $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_0)$ and $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_1)$, respectively. Call a (smooth) path $\{J_r \mid r \in [0, 1]\}$ of almost complex structures from J_0 to J_1 (constant near 0 and 1) *admissible* if each J_r is admissible. To a generic (in a sense to be made precise presently) admissible path J_r between J_0 and J_1 we will associate a “continuation map”

$$F^{J_r}: \widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_0) \rightarrow \widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_1).$$

To the path J_r we can associate a single almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ by

$$(6.17) \quad J|_{(x,s,t)} = \begin{cases} J_1|_{(x,s,t)} & \text{if } t \geq 1 \\ J_t|_{(x,s,t)} & \text{if } 0 \leq t \leq 1 \\ J_0|_{(x,s,t)} & \text{if } t \leq 0. \end{cases}$$



(Here $J|_{(x,s,t)}$ means the map J induces on the tangent space at $x \in \Sigma$, $s \in [0, 1]$, and $t \in \mathbb{R}$.)

For $B \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P; J)$ denote the moduli spaces of holomorphic curves defined in Definition 5.10 with respect to the almost complex structure J . (Note that here there is no \mathbb{R} -action, since J is not \mathbb{R} -invariant.) For generic J , these moduli spaces are all transversely cut out, as in Proposition 5.6. This is the genericity requirement we impose on $\{J_r\}$. For $\vec{\rho}$ a sequence of non-empty sets of Reeb chords so that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone and B is compatible with $\vec{\rho}$, define the embedded moduli space by

$$(6.18) \quad \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J) := \bigcup_{\chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\rho})} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J).$$

Since we do not divide by \mathbb{R} -translation, we will primarily be interested in index 0 moduli spaces, not the index 1 ones.

Then define the map F^{J_r} on $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ by

$$(6.19) \quad F^{J_r}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho}) = 0\}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)) a(-\vec{\rho}) \mathbf{y},$$

and extend F^{J_r} to all of $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_0)$ by $F^{J_r}(a\mathbf{x}) = aF^{J_r}(\mathbf{x})$. Compactness and provincial admissibility imply that this sum is finite, as in Lemma 6.5.

It is immediate from Lemma 4.21 that $F^{J_r}(\mathbf{x})$ respects the decomposition into spin^c structures; see also Remark 6.6.

The next task is to show that F^{J_r} is a chain map. We will use the following analogue of Theorem 5.61 for almost complex structures which are not translation invariant. The statement is more general than we need for present purposes; we will also use it to prove invariance of the type A module in Section 7.3.

PROPOSITION 6.20. *Suppose that $(\mathbf{x}, \vec{\rho})$ satisfies the strong boundary monotonicity condition. Fix $\mathbf{y} \in \mathfrak{S}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$, S^\triangleright , and \vec{P} such that $[\vec{P}] = \vec{\rho}$ and $\text{ind}(B, S^\triangleright, P) = 1$. Consider the parameterized moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J)$. Then the total number of the following ends of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J)$ is even:*

- (1) two-story ends, with either
 - (a) a J_0 -holomorphic curve followed by a J -holomorphic curve or
 - (b) a J -holomorphic curve followed by a J_1 -holomorphic curve;
- (2) join curve ends;
- (3) odd shuffle curve ends; and
- (4) collision of levels P_i and P_{i+1} from \vec{P} , where $[P_i]$ and $[P_{i+1}]$ are weakly composable.

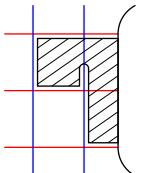
(The definition of two-story ends, join curve ends, odd or even shuffle curve ends, and collisions of levels are given in Definition 5.58. As indicated, two-story ends come in two types: elements of $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1; J_0) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; J)$ and elements of $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; S_1^\triangleright; \vec{P}_1; J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; S_2^\triangleright; J_1)$.)

PROOF. The proof is essentially the same as the proof of Theorem 5.61. \square

PROPOSITION 6.21. *The map F^{J_r} is a chain map, i.e., $\partial_1 \circ F^{J_r} + F^{J_r} \circ \partial_0 = 0$.*

PROOF. The proof is essentially the same as the proof of Proposition 6.7. First, observe that

$$\partial_1(F^{J_r}(a\mathbf{x})) + F^{J_r}(\partial_0(a\mathbf{x})) = (\partial a)F^{J_r}(\mathbf{x}) + a\partial_1(F^{J_r}(\mathbf{x})) + aF^{J_r}(\partial_0(\mathbf{x})) + (\partial a)F^{J_r}(\mathbf{x}).$$



So, it suffices to prove that $\partial_1(F^{J_r}(\mathbf{x})) + F^{J_r}(\partial_0(\mathbf{x})) = 0$ for any generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$.

Now, as in the proof of Proposition 6.7, for given \mathbf{x} , \mathbf{y} , B and $\vec{\rho} = (\rho_1, \dots, \rho_n)$ with $\text{ind}(B, \vec{\rho}) = 1$, Proposition 6.20 implies that the sum of the following terms is zero:

- (1) The number of two-story buildings of the form

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1; J_0) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2; J)$$

where $B = B_1 * B_2$ and $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$.

- (2) The number of two-story buildings of the form

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1; J) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2; J_1)$$

where $B = B_1 * B_2$ and $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$.

- (3) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_j, \rho_k\}, \rho_{i+1}, \dots, \rho_n); J)$$

where $\rho_i = \rho_j \uplus \rho_k$.

- (4) The number of split curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \rho_i \uplus \rho_{i+1}, \rho_{i+2}, \dots, \rho_n); J)$$

where $\rho_i^+ = \rho_{i+1}^-$.

- (5) The number of other collisions of levels, i.e., the number of elements of

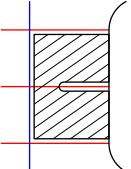
$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_{i-1}, \{\rho_i, \rho_{i+1}\}, \rho_{i+2}, \dots, \rho_n); J)$$

where $\rho_i^+ \neq \rho_{i+1}^-$.

The first term corresponds to $F^{J_t}(\partial_0(\mathbf{x}))$. The second corresponds to the terms in $\partial_1(F^{J_t}(\mathbf{x}))$ coming from applying ∂_1 to the resulting generators $\mathbf{w} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$. The fourth term corresponds to the terms in $\partial_1(F^{J_t}(\mathbf{x}))$ coming from the differential on the algebra \mathcal{A} . The third and fifth terms cancel in pairs, as in the proof of Proposition 6.7. \square

One can similarly use a path of almost complex structures from J_1 to J_0 to define a chain map $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_1) \rightarrow \widehat{CFD}(\mathcal{H}, \mathfrak{s}; J_0)$. The proof that these maps are mutually inverse chain homotopy equivalences is obtained by adapting the standard arguments in analogous manners to the above, and we leave this to the interested reader. (Some more details are given in the type A case, in Proposition 7.19.)

6.3.2. Handlesliding an α -arc over an α -circle. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ denote a bordered Heegaard diagram, which we assume for convenience to be admissible; this can be arranged by performing isotopies of the α -curves. Write $\boldsymbol{\alpha} = \{\alpha_1^a, \dots, \alpha_{2k}^a, \alpha_1^c, \dots, \alpha_{g-k}^c\}$. Provisionally, let $\alpha_1^{a,H}$ denote the result of performing a handleslide of α_1^a over α_1^c , and $\boldsymbol{\alpha}^H = \{\alpha_1^{a,H}, \alpha_2^a, \dots, \alpha_{g-k}^c\}$. Assume that the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$ is also admissible. We want to show that $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$ are chain homotopy equivalent.



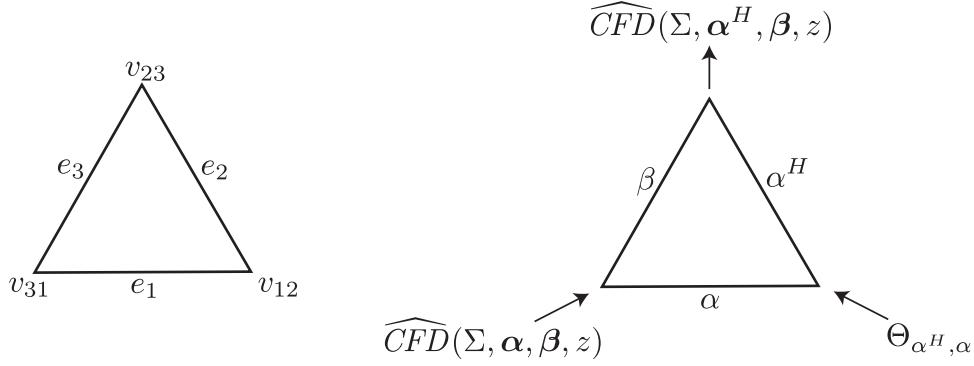


FIGURE 6.2. The three-punctured disk (triangle) Δ (left), and the labeling inducing $F_{\alpha, \alpha^H, \beta}$ (right). In counter-clockwise cyclic order, the edges are labeled α^H, β .

6.3.2.1. *Moduli spaces of triangles: basic definitions.* As is traditional, we will produce a chain map $F_{\alpha, \alpha^H, \beta}: \widehat{CFD}(\Sigma, \alpha^H, \beta, z) \rightarrow \widehat{CFD}(\Sigma, \alpha, \beta, z)$ by counting holomorphic triangles in $(\Sigma, \alpha, \alpha^H, \beta, z)$. That is, let Δ denote a disk with three boundary punctures. Label the edges of Δ by e_1 , e_2 and e_3 , counterclockwise, and let v_{ij} denote the puncture between e_i and e_j ; see Figure 6.2. To define $F_{\alpha, \alpha^H, \beta}$ we will count holomorphic maps

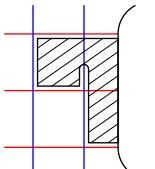
$$u: (T, \partial T) \rightarrow (\Sigma \times \Delta, (\alpha \times e_1) \cup (\alpha^H \times e_2) \cup (\beta \times e_3))$$

which do not cover z .

The α^H we use are actually a slight modification of the above, as shown in Figure 6.3. Obtain $\alpha_i^{c,H}$ from α_i^c by performing a small Hamiltonian perturbation, so that $\alpha_i^{c,H}$ intersects α_i^c transversely in two points, is disjoint from all other α -curves, and intersects β_j transversally close to $\beta_j \cap \alpha_i^c$. For $i = 2, \dots, 2k$, $\alpha_i^{a,H}$ is an isotopic translate of α_i^a , intersecting α_i^a transversely in a single point, and such that there are two short Reeb chords in $\partial\bar{\Sigma}$ running from $\alpha_i^{a,H}$ to α_i^a (rather than the other way). The isotopy is chosen small enough that $\alpha_i^{a,H}$ is disjoint from all other α -curves and intersects β_j transversally close to $\beta_j \cap \alpha_i^a$. Obtain $\alpha_1^{a,H}$ by handlesliding α_1^a over α_1^c and then performing an isotopy so that $\alpha_1^{a,H}$ intersects α_1^a transversely in a single point, and such that there are two short Reeb chords in $\partial\bar{\Sigma}$ running from $\alpha_1^{a,H}$ to α_1^a . We also arrange that $\alpha_1^{a,H}$ is disjoint from all other α -curves and intersects the β_j transversally at points close to $\beta_j \cap (\alpha_1^a \cup \alpha_1^c)$. Also define \mathbf{a}^H to be the set $\alpha^H \cap \partial\bar{\Sigma}$. There is a natural bijection between \mathbf{a} and \mathbf{a}^H .

Most of the notions from Chapter 5 generalize in straightforward manners to the setting of triangles. Following Definition 5.1, we fix an almost complex structure J on $\Sigma \times \Delta$ such that:

- (1) The projection map $\pi_\Delta: \Sigma \times \Delta \rightarrow \Delta$ is (J, j_Δ) -holomorphic.
- (2) The fibers of the projection map $\pi_\Sigma: \Sigma \times \Delta \rightarrow \Sigma$ are J -holomorphic.
- (3) The almost complex structure is split near $p \times \Delta$. (Recall that p is the puncture of Σ).
- (4) For each puncture $v_{i,j}$ of Δ , choose a neighborhood $U_{i,j}$ which is biholomorphic to $[0, 1] \times [0, \infty)$, and such a biholomorphism. This identification induces an action of \mathbb{R}_+ on $\Sigma \times U_{i,j}$. We require that these actions be J -holomorphic.



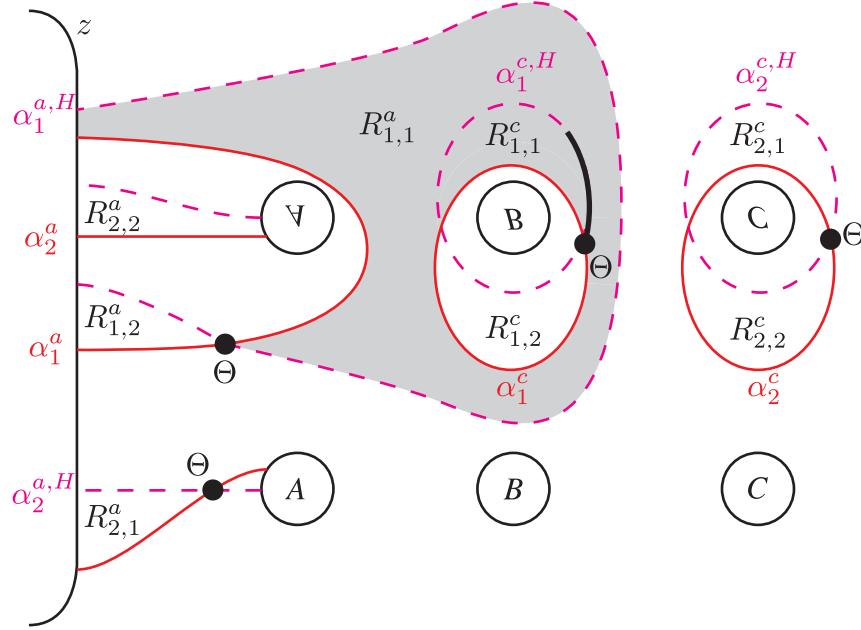


FIGURE 6.3. **The curves α and α^H in Σ .** The α -curves are solid while the α^H -curves are dashed. The intersection points which belong to generators Θ_\circ are marked. The annulus $R_{1,1}^a + R_{1,1}^c$ is shaded. The branch cut corresponding to a possible holomorphic curve is also indicated by a thick arc.

- (5) Identify Δ with $([0, 1] \times \mathbb{R}) \setminus \{(1, 0)\}$ so that v_{12} is identified with $(1, 0)$ (see Figure 6.4), and let $\tau_R: ([0, 1] \times \mathbb{R}) \setminus \{(1, 0), (1, -R)\} \rightarrow ([0, 1] \times \mathbb{R}) \setminus \{(1, R), (1, 0)\}$ denote translation in the \mathbb{R} -factor. We require that the map τ_R is J -holomorphic for each R .

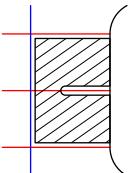
We call such a J *admissible*. Notice that the restrictions of an admissible almost complex structure J on $\Sigma \times \Delta$ to $\Sigma \times U_{i,j}$ induce three admissible structures $J_{i,j}$ on $\Sigma \times [0, 1] \times \mathbb{R}$. Condition 5 can be thought of as a global version of Condition 4, and in particular implies Condition 4 at the punctures $v_{3,1}$ and $v_{2,3}$; together, Conditions 4 and 5 imply that $J_{1,2}$ is split. Condition 5 is only needed for the computation of the embedded index in Proposition 6.25.

Similarly, we can define decorated sources for maps to $\Sigma \times \Delta$: a decorated source T^Δ is an oriented surfaces with boundary and punctures, together with a labeling of each puncture by an element of $\{v_{1,2}, v_{2,3}, v_{3,1}, e\}$ and a further labeling of each e puncture either by a Reeb chord with endpoints in \mathbf{a} or by a Reeb chord with endpoints in \mathbf{a}^H .

Given a decorated source T^Δ we consider maps

$$u: (T, \partial T) \rightarrow (\Sigma \times \Delta, (\boldsymbol{\alpha} \times e_1) \cup (\boldsymbol{\alpha}^H \times e_2) \cup (\boldsymbol{\beta} \times e_3))$$

satisfying the obvious analogues of Conditions (M-1)–(M-11) from Section 5.2. Such a holomorphic curve is asymptotic to a generator \mathbf{x} for $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ at $v_{3,1}$ and a generator \mathbf{y} for $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$ at $v_{2,3}$. The asymptotics at $v_{1,2}$, however, are more subtle. At each puncture of T^Δ mapped to $v_{1,2}$, u is either asymptotic to a point in $\boldsymbol{\alpha}^H \cap \boldsymbol{\alpha}$ (in the interior of Σ), or is asymptotic to a Reeb chord in $\partial \bar{\Sigma}$ running from \mathbf{a}^H to \mathbf{a} . In defining the map $F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ we will use curves with only the former kind of asymptotics, but in proving that $F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is



a chain map we will need to consider curves with the latter asymptotics. Such asymptotics present certain technical issues, as we will discuss below.

For each k -element subset o of $\{1, \dots, 2k\}$ there is a distinguished g -element subset of $\boldsymbol{\alpha}^H \cap \boldsymbol{\alpha}$, denoted Θ_o . The set Θ_o contains the unique intersection point between α_i^a and $\alpha_i^{a,H}$ for $i \in o$, as well as the higher-graded intersection point between $\alpha_i^{c,H}$ and α_i^c for each $i = 1, \dots, g - k$. (The higher-graded intersection point x between $\alpha_i^{c,H}$ and α_i^c is the one such that there are two holomorphic maps $([0, 1] \times \mathbb{R}, \{1\} \times \mathbb{R}, \{0\} \times \mathbb{R}) \rightarrow (\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\alpha})$ mapping $-\infty$ to x .) We are thinking of Θ_o as a generator for counts of holomorphic curves in $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\alpha}, z)$, although we will not define an object $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\alpha}, z)$.

Like curves in $\Sigma \times [0, 1] \times \mathbb{R}$, curves in $\Sigma \times \Delta$ carry (relative) homology classes, in a manner analogous to Definition 4.14. Let $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ denote the set of homology classes connecting \mathbf{x} , \mathbf{y} and Θ_o (not covering z , as before). For a homology class B in $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ we can consider its *domain*, the multiplicities with which $\pi_\Sigma(B)$ covers each component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\alpha}^H \cup \boldsymbol{\beta})$. The map from homology classes to domains is injective. Let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$ denote the moduli space of holomorphic curves asymptotic to \mathbf{x} at $v_{3,1}$, \mathbf{y} at $v_{2,3}$ and Θ_o at $v_{1,2}$, with decorated source T^Δ and in the homology class B .

Along e_1 (respectively e_2) a curve $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$ is asymptotic to a collection of Reeb chords in $(\partial\bar{\Sigma}, \mathbf{a})$ (respectively $(\partial\bar{\Sigma}, \mathbf{a}^H)$). The orientation on e_1 (respectively e_2) induces a partial order of these collections of Reeb chords. Thus, given ordered partitions \vec{P} and \vec{P}^H of Reeb chords in $(\partial\bar{\Sigma}, \mathbf{a})$ and $(\partial\bar{\Sigma}, \mathbf{a}^H)$ respectively, we may consider the subspace

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H) \subset \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$$

consisting of those holomorphic curves so that the induced ordered partitions of the Reeb chords along e_1 and e_2 are given by \vec{P} and \vec{P}^H respectively.

LEMMA 6.22. *If the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ is non-empty then, in the notation of Definition 5.52, $o = o(\mathbf{x}, [\vec{P}])$.*

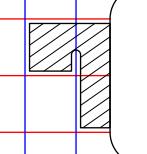
PROOF. This is immediate from the definitions. □

The arguments in Proposition 5.24—essentially, considering the maps $\pi_\Sigma \circ u$ and $\pi_\Delta \circ u$ separately—show that the moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ admit natural compactifications $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$. Other than degenerations at v_{12} , which we will discuss presently, the limit curves are natural analogues of the holomorphic combs introduced in Section 5.4. As for bigons in Chapter 5, we let $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ denote the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$.

6.3.2.2. Homology classes and expected dimensions of triangles. The expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ is

$$(6.23) \quad \text{ind}(B, T^\Delta, P, P^H) = \frac{g}{2} - \chi(T) + 2e(B) + |P| + |P^H|.$$

(The proof is similar to the proof of Proposition 5.8. See [Lip06a, p. 1018] for the closed case.) Like Formula (5.9), Formula (6.23) depends on the topology of T^Δ , and not just on the homology class and asymptotics. However, as was the case for embedded curves in $\Sigma \times [0, 1] \times \mathbb{R}$, the Euler characteristic of an embedded curve in $\Sigma \times \Delta$ is determined by the homology class and asymptotics. To state the formula in a convenient form, first note that for any generator $\mathbf{x} \in \mathfrak{S}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, there is a nearby generator $\mathbf{x}' \in \mathfrak{S}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta})$ (but not vice



versa), as well as a canonical homology class $T_{\mathbf{x}}$ in $\pi_2(\mathbf{x}, \mathbf{x}', \Theta_o(\mathbf{x}))$, the union of a number of small triangles (contained in the isotopy region) and possibly an annulus with boundary on α_1^a , $\alpha_1^{a,H}$ and α_1^c .

LEMMA 6.24. *Any homology class $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ can be written uniquely as*

$$(T_{\mathbf{x}} *_{12} B_{\alpha^H, \alpha}) *_{23} B_{\alpha^H, \beta}$$

for some $B_{\alpha^H, \alpha} \in \pi_2(\Theta_o(\mathbf{x}), \Theta_o)$ and $B_{\alpha^H, \beta} \in \pi_2(\mathbf{x}', \mathbf{y})$.

Here $*_{12}$ is the natural operation of attaching a homology class in $(\Sigma, \alpha^H, \alpha, z)$ to a homology class in $(\Sigma, \alpha, \alpha^H, \beta, z)$ at the corner v_{12} of Δ , and likewise $*_{23}$ is the operation of attaching a homology class in $(\Sigma, \alpha^H, \beta, z)$ at the corner v_{23} of Δ .

PROOF. The space $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ has a free and transitive action by the space of *triply-periodic domains*, the space of 2-chains in Σ which have local degree zero at z and whose boundary is a formal linear combination of elements of $\alpha \cup \alpha^H \cup \beta$. The space of triply-periodic domains in $(\Sigma, \alpha, \alpha^H, \beta, z)$ (and therefore $\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$) is isomorphic to the kernel of the map

$$H_1(\beta) \oplus H_1(\alpha/\partial\alpha) \oplus H_1(\alpha^H/\partial\alpha^H) \rightarrow H_1(\bar{\Sigma}/\partial\bar{\Sigma}).$$

Since $H_1(\alpha/\partial\alpha)$ and $H_1(\alpha^H/\partial\alpha^H)$ are the same subspace of $H_1(\bar{\Sigma}/\partial\bar{\Sigma})$, this kernel is isomorphic to

$$\begin{aligned} \ker(H_1(\beta) \oplus H_1(\alpha/\partial\alpha) \rightarrow H_1(\bar{\Sigma}/\partial\bar{\Sigma})) &\oplus \ker(H_1(\alpha/\partial\alpha) \oplus H_1(\alpha^H/\partial\alpha^H) \rightarrow H_1(\bar{\Sigma}/\partial\bar{\Sigma})) \\ &\cong \pi_2(\mathbf{x}, \mathbf{x}) \times \pi_2(\Theta_o, \Theta_o). \end{aligned}$$

Thus, the action by $\pi_2(\mathbf{x}, \mathbf{x}) \times \pi_2(\Theta_o, \Theta_o)$ on the set of triply-periodic domains is free and transitive. The result follows. \square

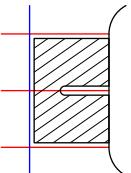
For $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$, let $b(B) \in \pi_2(\mathbf{x}', \mathbf{y})$ be the domain $B_{\alpha^H, \beta} \in \pi_2(\mathbf{x}', \mathbf{y})$ whose existence is guaranteed by Lemma 6.24.

Next, we prove an analogue of Sarkar's index theorem for triangles [Sar11]. Given a domain $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$, let $\partial^\alpha(B)$ (respectively $\partial^{\alpha^H}(B)$) denote the part of ∂B contained in the α -curves (respectively α^H -curves), oriented as ∂B . Let $\partial^{\alpha^H}(B) \cdot \partial^\alpha(B)$ denote the jittered intersection number of $\partial^{\alpha^H}(B)$ and $\partial^\alpha(B)$, as in Section 5.7.2. As for bigons in Section 5.7, $n_{\mathbf{x}}(B)$ denotes the point measure of B at \mathbf{x} , and $e(B)$ denotes the Euler measure of B . Also analogous to the bigon case, define

$$\begin{aligned} \iota(\vec{\rho}, \vec{\rho}^H) &:= \iota(\vec{\rho}) + \iota(\vec{\rho}^H) + L(\vec{\rho}, \vec{\rho}^H) \\ &= - \sum_i \frac{|\rho_i|}{2} - \sum_i \frac{|\rho_i^H|}{2} - \sum_i \sum_{\{\rho_a, \rho_b\} \subset \rho_i} |L(\rho_a, \rho_b)| - \sum_i \sum_{\{\rho_a, \rho_b\} \subset \rho_i^H} |L(\rho_a, \rho_b)| \\ &\quad + \sum_{\substack{i < j \\ \rho_i \in \rho_i, \rho_j \in \rho_j}} L(\rho_i, \rho_j) + \sum_{\substack{i < j \\ \rho_i \in \rho_i^H, \rho_j \in \rho_j^H}} L(\rho_i, \rho_j) + \sum_{\rho_i \in \rho_i, \rho_j \in \rho_j^H} L(\rho_i, \rho_j). \end{aligned}$$

PROPOSITION 6.25. *If $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ contains an embedded holomorphic curve u then the expected dimension of \mathcal{M}^B near u is given by*

$$(6.26) \quad \text{ind}(u) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \partial^{\alpha^H}(B) \cdot \partial^\alpha(B) - g/2 + \iota([\vec{P}], [\vec{P}^H]).$$



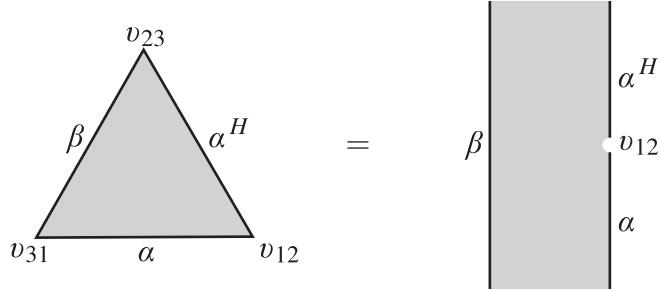


FIGURE 6.4. **A triangle is biholomorphic to a strip with a boundary puncture.** The edges are labeled by the corresponding arcs.

PROOF. The proof is similar to the proof of Proposition 5.69. Identifying the triangle Δ with $[0, 1] \times \mathbb{R} \setminus \{(1, 0)\}$, so that v_{12} is identified with $(1, 0)$, we can view u as a map to $\Sigma \times [0, 1] \times \mathbb{R}$. (See Figure 6.4.) Let S denote the source of u . Then

$$\chi(S) = e(B) - \text{br}(\pi_\Sigma \circ u) + \frac{3g}{4} + \sum_i |P_i|/2 + \sum_i |P_i^H|/2,$$

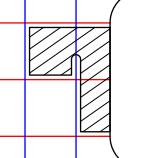
where br denotes the order of branching and $\frac{3g}{4}$ comes from the corners. Let $\tau_r: \Sigma \times [0, 1] \times \mathbb{R} \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ denote translation in the \mathbb{R} direction. Then for ϵ sufficiently small,

$$\text{br}(\pi_\Sigma \circ u) = u \cdot \tau_\epsilon(u) - \sum_i \sum_{\{\rho_a, \rho_b\} \subset [P_i]} |L(\rho_a, \rho_b)| - \sum_i \sum_{\{\rho_a, \rho_b\} \subset [P_i^H]} |L(\rho_a, \rho_b)|.$$

Translating farther, we see that for R sufficiently large,

$$(6.27) \quad \begin{aligned} u \cdot \tau_\epsilon(u) &= u \cdot \tau_R(u) + \partial^{\alpha^H}(B) \cdot \partial^\alpha(B) + \frac{g}{4} + \sum_{\substack{i < j \\ \rho_i \in [P_i], \rho_j \in [P_j]}} L(\rho_i, \rho_j) \\ &\quad + \sum_{\substack{i < j \\ \rho_i \in [P_i^H], \rho_j \in [P_j^H]}} L(\rho_i, \rho_j) + \sum_{\substack{\rho_i \in [P_i], \rho_j \in [P_j^H]}} L(\rho_i, \rho_j). \end{aligned}$$

The $L(\cdot, \cdot)$ terms in Equation (6.27) arise exactly as in the proof of Proposition 5.69; note that this uses Condition (5), so that $\tau_r(u)$ is holomorphic for any $r \in \mathbb{R}$. The term $\partial^{\alpha^H}(B) \cdot \partial^\alpha(B) + g/4$ comes from the number of intersection points that appear (or disappear) at the boundary, as follows. For finitely many $r \in (0, \infty)$, the α^H -boundary of u intersects the α -boundary of $\tau_r(u)$. Generically, these intersections will be transverse, so it suffices to consider that case. We also assume that the complex structure on $\Sigma \times \Delta$ is chosen so that α and α^H meet at right angles. Let $\partial^\alpha u = u(\partial S) \cap \alpha$ and $\partial^{\alpha^H} u = u(\partial S) \cap \alpha^H$ denote the subset of $u(\partial S)$. Consider a point $q \in (\partial^{\alpha^H} u) \cap (\partial^\alpha \tau_r(u))$, and let $q_1 = u^{-1}(q)$, $q_2 = (\tau_r(u))^{-1}(q)$. We can choose an identification of a neighborhood of q with a neighborhood of $(0, 1)$ in $\mathbb{C} \times \{x + iy \mid x \leq 1\}$ and identifications of neighborhoods of q_1 and q_2 with neighborhoods of 0 in \mathbb{H} so that, to first order, on these neighborhoods the maps u and $\tau_r(u)$ are given by



either

$$\begin{aligned} u(x + iy) &= (x + iy, 1 - y + ix) \\ \tau_r(u)(x' + iy') &= (y' - ix', 1 - \lambda y' + i\lambda x') \end{aligned}$$

or

$$\begin{aligned} u(x + iy) &= (x + iy, 1 - y + ix) \\ \tau_r(u)(x' + iy') &= (-y' + ix', 1 - \lambda y' + i\lambda x'), \end{aligned}$$

depending on whether q corresponds to a positive or negative intersection in $(\partial^{\alpha^H} u) \cap (\partial^\alpha \tau_r(u))$. (Again, we are using the fact that, by Condition (5), $\tau_r(u)$ is holomorphic.) Here, λ is a positive real number. Consider the first case. The points in the domains of u and $\tau_{r+\delta}(u)$ that map to the same point are, respectively,

$$\begin{aligned} x + iy &= \frac{\lambda}{1 + \lambda^2}(\delta + i\lambda\delta) \\ x' + iy' &= \frac{\lambda}{1 + \lambda^2}(-\lambda\delta + i\delta). \end{aligned}$$

These points lie in the domains of u and $\tau_{r+\delta}(u)$ if $\delta > 0$, and not if $\delta < 0$. So, for this local model, one intersection point appears as we translate u up. Similarly, in the second case, the points in the domain mapping to the intersection are

$$\begin{aligned} x + iy &= \frac{\lambda}{1 + \lambda^2}(\delta - i\lambda\delta) \\ x' + iy' &= \frac{\lambda}{1 + \lambda^2}(-\lambda\delta - i\delta). \end{aligned}$$

These points lie in the domains of u and $\tau_{r+\delta}(u)$ if $\delta < 0$, and not if $\delta > 0$. So, for this local model, one intersection point disappears as we translate u up. This explains the term $\partial^{\alpha^H}(B) \cdot \partial^\alpha(B)$, except for the part of $\partial^{\alpha^H}(B) \cdot \partial^\alpha(B)$ coming from intersections at the corners at Θ_o . For contributions coming from a corner of u intersecting an edge of $\tau_r(u)$ or vice-versa, the analysis is the same as before except that the intersection point that appears or disappears is on the boundary. The expression $\partial^{\alpha^H}(B) \cdot \partial^\alpha(B)$ also picks up $-1/4$ from each of the g points in Θ_o , coming from the intersections of the corners of the domain with themselves. These terms do not correspond to points appearing or disappearing; this is the reason for the $g/4$ term in Equation (6.27).

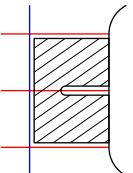
As in the bigon case, $u \cdot \tau_R(u) = n_x(B) + n_y(B) - g/2$. So,

$$\begin{aligned} \chi(S) &= e(B) - \text{br}(\pi_\Sigma \circ u) + \frac{3g}{4} + \sum_i \frac{|P_i|}{2} + \sum_i \frac{|P_i^H|}{2} \\ &= e(B) + n_x(B) + n_y(B) + \partial^{\alpha^H}(B) \cdot \partial^\alpha(B) - g/2 + \iota([\vec{P}], [\vec{P}^H]). \end{aligned}$$

(The last line uses Formula (6.23)). This proves the result. \square

For notational convenience, let

$$\begin{aligned} \text{ind}(B, \vec{\rho}, \vec{\rho}^H) &= e(B) + n_x(B) + n_y(B) + \partial^{\alpha^H}(B) \cdot \partial^\alpha(B) - g/2 + \iota(\vec{\rho}, \vec{\rho}^H) \\ \chi_{\text{emb}}(B, \vec{\rho}, \vec{\rho}^H) &= e(B) - n_x(B) - n_y(B) + g - \partial^{\alpha^H}(B) \cdot \partial^\alpha(B) - \iota(\vec{\rho}, \vec{\rho}^H). \end{aligned}$$



LEMMA 6.28. *The function $\text{ind}(B, \vec{\rho}, \vec{\rho}^H)$ is additive with respect to gluing domains at v_{23} or v_{31} , in the following sense. If $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$, $B_{\alpha, \beta} \in \pi_2(\mathbf{x}', \mathbf{x})$ and $B_{\alpha^H, \beta} \in \pi_2(\mathbf{y}, \mathbf{y}')$, and $(\vec{\rho}, \vec{\rho}^H)$, $\vec{\rho}_{\alpha, \beta}$ and $\vec{\rho}_{\alpha^H, \beta}$ are sequences of non-empty sets of Reeb chords compatible with B , $B_{\alpha, \beta}$ and $B_{\alpha^H, \beta}$, respectively, then*

$$\begin{aligned} \text{ind}(B_{\alpha, \beta} *_{31} B *_{23} B_{\alpha^H, \beta}, (\vec{\rho}_{\alpha, \beta}, \vec{\rho}), (\vec{\rho}^H, \vec{\rho}_{\alpha^H, \beta})) \\ = \text{ind}(B_{\alpha, \beta}, \vec{\rho}_{\alpha, \beta}) + \text{ind}(B, \vec{\rho}, \vec{\rho}^H) + \text{ind}(B_{\alpha^H, \beta}, \vec{\rho}_{\alpha^H, \beta}). \end{aligned}$$

PROOF. The proof is essentially the same as in the closed case [Sar11, Theorem 3.3] (which also inspired the proof of Proposition 5.75). \square

Next we prove an analogue of Lemma 6.28 for gluing at v_{12} :

LEMMA 6.29. *Let $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$ and let $(\vec{\rho}, \vec{\rho}^H)$ be a pair of sequences of non-empty sets of Reeb chords compatible with B . Write $\vec{\rho} = (\rho_1, \dots, \rho_j)$ and $\vec{\rho}^H = (\rho_{j+1}^H, \dots, \rho_l^H)$. Let ρ_j^H be the set of α^H -Reeb chords corresponding to ρ_j . Let $B_{\alpha, \alpha^H} \in \pi_2(\Theta_{o'}, \Theta_o)$ (for some o') be a domain so that $B *_{12} B_{\alpha, \alpha^H}$ is compatible with $((\rho_1, \dots, \rho_{j-1}), (\rho_j^H, \rho_{j+1}^H, \dots, \rho_l^H))$. Then*

$$(6.30) \quad \text{ind}(B *_{12} B_{\alpha, \alpha^H}, (\rho_1, \dots, \rho_{j-1}), (\rho_j^H, \rho_{j+1}^H, \dots, \rho_l^H)) = \text{ind}(B, \vec{\rho}, \vec{\rho}^H).$$

PROOF. The difference between the two sides of Equation (6.30) is:

$$\begin{aligned} & e(B_{\alpha, \alpha^H}) + n_{\mathbf{x}}(B_{\alpha, \alpha^H}) + n_{\mathbf{y}}(B_{\alpha, \alpha^H}) \\ & + \partial^{\alpha^H}(B) \cdot \partial^{\alpha}(B_{\alpha, \alpha^H}) + \partial^{\alpha^H}(B_{\alpha, \alpha^H}) \cdot \partial^{\alpha}(B) + \partial^{\alpha^H}(B_{\alpha, \alpha^H}) \cdot \partial^{\alpha}(B_{\alpha, \alpha^H}) \\ (6.31) \quad & + \sum_{i=1}^{j-1} (L(\rho_i, \rho_j^H) - L(\rho_i, \rho_j)) + \sum_{i=j+1}^l (L(\rho_j^H, \rho_i^H) - L(\rho_j, \rho_i^H)) \\ & + \iota(\rho_j^H) - \iota(\rho_j). \end{aligned}$$

By inspection, $\iota(\rho_j^H) = \iota(\rho_j)$. Since $\partial^{\partial} B_{\alpha, \alpha^H} = \rho_j^H - \rho_j$, by multi-linearity of L ,

$$\begin{aligned} \sum_{i=1}^{j-1} (L(\rho_i, \rho_j^H) - L(\rho_i, \rho_j)) &= L(\rho_1 + \dots + \rho_{j-1}, \partial^{\partial} B_{\alpha, \alpha^H}) \\ \sum_{i=j+1}^l (L(\rho_j^H, \rho_i^H) - L(\rho_j, \rho_i^H)) &= L(\partial^{\partial} B_{\alpha, \alpha^H}, \rho_{j+1}^H + \dots + \rho_l^H). \end{aligned}$$

We next use the techniques of part (4) of Lemma 5.73 and Lemma 5.74. We have $\Theta_o - \mathbf{x} = \partial(\partial^{\alpha} B + \rho_1 + \dots + \rho_j)$, so $n_{\Theta_o}(B_{\alpha, \alpha^H}) - n_{\mathbf{x}}(B_{\alpha, \alpha^H}) = \partial B_{\alpha, \alpha^H} \cdot (\partial^{\alpha} B + \rho_1 + \dots + \rho_j)$, which yields

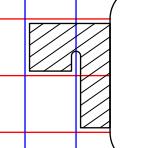
$$\partial^{\alpha^H}(B_{\alpha, \alpha^H}) \cdot \partial^{\alpha}(B) + L(\rho_1 + \dots + \rho_j, \partial^{\partial} B_{\alpha, \alpha^H}) + n_{\mathbf{x}}(B_{\alpha, \alpha^H}) = n_{\Theta_o}(B_{\alpha, \alpha^H}).$$

Similarly,

$$\partial^{\alpha^H}(B) \cdot \partial^{\alpha}(B_{\alpha, \alpha^H}) + L(\partial^{\partial} B_{\alpha, \alpha^H}, \rho_{j+1}^H + \dots + \rho_l^H) + n_{\mathbf{y}}(B_{\alpha, \alpha^H}) = n_{\Theta_o}(B_{\alpha, \alpha^H}).$$

So, Formula (6.31) reduces to

$$e(B_{\alpha, \alpha^H}) + 2n_{\Theta_o}(B_{\alpha, \alpha^H}) + \partial^{\alpha^H}(B_{\alpha, \alpha^H}) \cdot \partial^{\alpha}(B_{\alpha, \alpha^H}) + L(\rho_j, \partial^{\partial} B_{\alpha, \alpha^H}).$$



Since $[\boldsymbol{\rho}_j^H] - [\boldsymbol{\rho}_j] = \partial^\partial B_{\alpha, \alpha^H}$, we have

$$L(\boldsymbol{\rho}_j, \partial^\partial B_{\alpha, \alpha^H}) = L(\boldsymbol{\rho}_j, \boldsymbol{\rho}_j^H).$$

Similarly to part (3) of Lemma 5.73,

$$n_{\Theta_o}(B_{\alpha, \alpha^H}) + \partial^{\alpha^H}(B_{\alpha, \alpha^H}) \cdot \partial^\alpha(B_{\alpha, \alpha^H}) + L(\boldsymbol{\rho}_j, \boldsymbol{\rho}_j^H) = n_{\Theta_{o'}}(B_{\alpha, \alpha^H}).$$

By inspection of the possible domains for B_{α, α^H} ,

$$n_{\Theta_o}(B_{\alpha, \alpha^H}) + n_{\Theta_{o'}}(B_{\alpha, \alpha^H}) + e(B_{\alpha, \alpha^H}) = 0.$$

This proves the result. \square

COROLLARY 6.32. *If $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ contains an embedded curve u then the expected dimension of \mathcal{M}^B near u is given by*

$$\text{ind}(u) = \text{ind}(b(B), \vec{\rho}),$$

where $\vec{\rho}$ is the sequence of Reeb chords gotten by replacing each α^H -chord in $([\vec{P}], [\vec{P}^H])$ with the corresponding α -chord.

PROOF. This is immediate from Proposition 6.25 and Lemmas 6.24, 6.28 and 6.29 and the observation that, in the notation of Lemma 6.24, the small triangles $T_\mathbf{x}$ have index 0. \square

6.3.2.3. *Definition of the homomorphism associated to a handleslide.* For $\mathbf{x} \in \mathfrak{S}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathbf{y} \in \mathfrak{S}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta})$, $B \in \pi_2(\mathbf{x}', \mathbf{y})$ and $\vec{\rho}$ a sequence of non-empty sets of Reeb chords so that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone and $(B, \vec{\rho})$ is compatible, define

$$\begin{aligned} \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}_1, \vec{\rho}_2) &:= \bigcup_{\substack{[\vec{P}] = \vec{\rho}_1, [\vec{P}^H] = \vec{\rho}_2 \\ b(B') = B \\ \chi(T^\Delta) = \chi_{\text{emb}}(B', \vec{\rho}_1, \vec{\rho}_2)}} \mathcal{M}^{B'}(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}_1, \vec{P}_2) \\ \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \vec{\rho}) &:= \bigcup_{\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)} \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o(\mathbf{x}, \vec{\rho}_1); \boldsymbol{\rho}_1, \boldsymbol{\rho}_2). \end{aligned}$$

The triangle map from $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$ is defined by

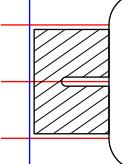
$$(6.33) \quad F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(\mathbf{x}) := \sum_{\mathbf{y}} \sum_{B \in \pi_2(\mathbf{x}', \mathbf{y})} \sum_{\{\vec{\rho}\} \mid \text{ind}(B, \vec{\rho}) = 0} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \vec{\rho})) a(-\vec{\rho}) \mathbf{y}$$

$$F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(a\mathbf{x}) := aF_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(\mathbf{x}).$$

As in the definition of the boundary operator, if both $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta})$ are provinicially admissible this sum is finite, as the 0-dimensional moduli spaces are compact and only finitely many of them are non-empty.

The map $F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is defined on all of $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. However, Lemma 4.21 and the fact that $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_z(\mathbf{x}')$ imply that if we restrict to the summand $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z; \mathfrak{s})$ of $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ representing a fixed spin^c structure \mathfrak{s} then the image of $F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is contained in the corresponding summand $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z; \mathfrak{s})$ of $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$.

Our next goal is to prove that $F_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is a chain map. The new technical point relates to degenerations at v_{12} , to which we turn next.



6.3.2.4. *Holomorphic curves at v_{12} .* We start by completely analyzing what curves can degenerate at v_{12} ; but first some more notation.

There are $2(g+k)$ components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\alpha}^H)$ not adjacent to z .

- For each α -circle α_i^c ($i = 1, \dots, g-k$) there is a pair of bigons $R_{i,1}^c$ and $R_{i,2}^c$ contained entirely in the interior of $\bar{\Sigma}$, with boundaries in $\alpha_i^c \cup \alpha_i^{c,H}$. (The numbering 1 and 2 is arbitrary.)
- For each α -arc α_i^a with $i \geq 2$ there is a pair of bigons $R_{i,1}^a$ and $R_{i,2}^a$ with boundary in $\alpha_i^a \cup \alpha_i^{a,H}$. (Again, the numbering 1 and 2 is arbitrary.) These bigons are adjacent to $\partial\bar{\Sigma}$; one of the two punctures of each $R_{i,j}^a$ corresponds to $e\infty$ in Σ .
- There is an annulus $R_{1,1}^a$ with boundary in $\alpha_1^a \cup \alpha_1^{a,H} \cup \alpha_1^c \cup \alpha_1^{c,H}$ and a bigon $R_{1,2}^a$ with boundary in $\alpha_1^a \cup \alpha_1^{a,H}$. Both $R_{1,1}^a$ and $R_{1,2}^a$ are adjacent to $\partial\bar{\Sigma}$.

We call the regions $R_{i,1}^a$ and $R_{i,2}^a$ (respectively $R_{i,1}^c$ and $R_{i,2}^c$) *twin regions*. See Figure 6.3.

LEMMA 6.34. *Let*

$$u: (S, \partial S) \rightarrow (\Sigma \times [0, 1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{0\} \times \mathbb{R}) \cup (\boldsymbol{\alpha}^H \times \{1\} \times \mathbb{R}))$$

be a holomorphic curve asymptotic to a generator Θ_o at $-\infty$, with multiplicity 0 at z . Then S is a disjoint union of bigons and possibly an annulus. Moreover:

- (c-1) *If S_i and S_j are distinct components of S then $\pi_\Sigma(u(S_i)) \cap \pi_\Sigma(u(S_j)) = \emptyset$.*
- (c-2) *If S_i is a bigon then either $u|_{S_i}$ is a trivial strip (i.e., $(\pi_\Sigma \circ u)|_{S_i}$ is constant and $(\pi_D \circ u)|_{S_i}$ is a diffeomorphism) or $(\pi_\Sigma \circ u)|_{S_i}$ is a diffeomorphism onto its image, which is one of the distinguished bigons $R_{i,j}^c$ or $R_{i,j}^a$.*
- (c-3) *The image of the annular component of S (if it exists) is one of $R_{1,1}^a$, $R_{1,1}^a + R_{1,1}^c$, $R_{1,1}^a + R_{1,2}^c$, $R_{1,1}^a + R_{1,1}^c + R_{1,2}^c$, $R_{1,1}^a + 2R_{1,1}^c$ or $R_{1,1}^a + 2R_{1,2}^c$.*
- (c-4) *If a region $R_{i,1}^a$ (respectively $R_{i,2}^a$, $R_{i,1}^c$, $R_{i,2}^c$) is in the image of $\pi_\Sigma \circ u$ then its twin region $R_{i,2}^a$ (respectively $R_{i,1}^a$, $R_{i,2}^c$, $R_{i,1}^c$) is not in the image of $\pi_\Sigma \circ u$, with the possible exception that both $R_{i,1}^c$ and $R_{i,2}^c$ may be in the image of $\pi_\Sigma \circ u$ on an annular component.*

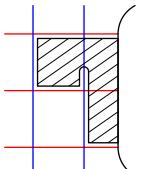
PROOF. Suppose first that $\pi_\Sigma \circ u$ does not cover the annulus $R_{1,1}^a$. Since u is asymptotic to the generator Θ_o , each bigon $R_{i,j}^a$ or $R_{i,j}^c$ appears with multiplicity 0 or 1, and at most one of each pair of twin bigons appears in the image of $\pi_\Sigma \circ u$. The result follows.

Next, suppose that $\pi_\Sigma \circ u$ covers the annulus $R_{1,1}^a$. Then, as before, the domain of u is a disjoint union of bigons away from a neighborhood of the annulus. The asymptotics at $-\infty$ imply that near the annulus, the domain has one of the forms $R_{1,1}^a$, $R_{1,1}^a + R_{1,1}^c$, $R_{1,1}^a + R_{1,2}^c$, $R_{1,1}^a + R_{1,1}^c + R_{1,2}^c$, $R_{1,1}^a + 2R_{1,1}^c$ or $R_{1,1}^a + 2R_{1,2}^c$. The result follows. \square

6.3.2.5. *Gluing results at v_{12} .* Next we prove two gluing results for these curves at v_{12} . (See Figure 6.7 for a typical example of this gluing.) Again, we need a little more terminology. The holomorphic curves in Lemma 6.34 are asymptotic to a generator Θ_o at $-\infty$. At $+\infty$, they are asymptotic to g -tuples of the form

$$\mathbf{w}_{pd}^+ = (\{w_1, \dots, w_\ell\} \times [0, 1] \times \{+\infty\}) \cup (\{\rho_1, \dots, \rho_{g-\ell}\} \times [0, 1] \times \{+\infty\})$$

where each w_i is a point in $\boldsymbol{\alpha}^H \cap \boldsymbol{\alpha}$ and each ρ_i is a Reeb chord in $\partial\bar{\Sigma}$ from some $\alpha_j^{a,H}$ to some α_k^a . We call tuples \mathbf{w}_{pd}^+ of this form *positive partially-diagonal generators*. (In fact, for the curves in Lemma 6.34, $j = k$, but we do not require this condition for partially diagonal generators.) We will abuse notation and also refer to the set $\mathbf{w}_{pd}^+ = \{w_1, \dots, w_\ell, \rho_1, \dots, \rho_{g-\ell}\}$



as a positive partially-diagonal generator. *Negative partially-diagonal generators* are defined similarly, but with Reeb chords running from some α_j^a to some $\alpha_k^{a,H}$, and are denoted by \mathbf{w}_{pd}^- .

We collect the holomorphic curves at v_{12} into moduli spaces. Specifically, for each generator Θ_o , positive partially-diagonal generator \mathbf{w}_{pd}^+ and domain $B \in \pi_2^{\alpha^H, \alpha}(\Theta_o, \mathbf{w}_{pd}^+)$, there is a corresponding moduli space of (stable) holomorphic curves $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$. (As usual, we mod out by the action of \mathbb{R} by translation.) Lemma 6.34 shows that the topological type of the source of curves in $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ is determined by B and \mathbf{w}_{pd}^+ , but when we want to name the source explicitly we will also write this moduli space as $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+; S^\triangleright)$.

It follows from Lemma 6.34 that the moduli spaces $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ are typically high-dimensional:

COROLLARY 6.35. *Fix a generic split almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. Let $\mathbf{w}_{pd}^+ = \{w_1, \dots, w_\ell, \rho_1, \dots, \rho_{g-\ell}\}$ be a positive partially-diagonal generator. Then the moduli space $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ for generic J is either empty or has dimension at least $\max\{g - \ell - 1, 0\}$. Further:*

- (1) *If $g - \ell = 0$ and $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ is non-empty and has dimension 0 then B is the bigon $R_{i,j}^c$ for some c (and $\mathbf{w}_{pd}^+ = \Theta_o$).*
- (2) *If $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ is non-empty and has dimension $g - \ell - 1$ then every coefficient in B is 0 or 1 and each connected component of the support of B is one of $R_{i,j}^a$ for some $i > 1$; $R_{1,2}^a$; $R_{1,1}^a \cup R_{1,1}^c$; or $R_{1,1}^a \cup R_{1,2}^c$.*

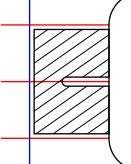
Conversely, each bigon has a unique holomorphic representative, and the sum of the number of holomorphic representatives of $R_{1,1}^a + R_{1,1}^c$ and $R_{1,1}^a + R_{1,2}^c$ is odd.

PROOF. This is a straightforward analysis of the moduli spaces corresponding to the domains in Lemma 6.34, as follows.

Property (c-1) ensures that the moduli space $\widetilde{\mathcal{M}}^B(\Theta_o, \mathbf{w}_{pd}^+)$ (i.e., before we divide out by the \mathbb{R} action) splits as a product of moduli spaces associated to the component curves. By the classification from Lemma 6.34, these factors correspond to moduli spaces representing the domains in the following table, listed with their number of chords and expected dimension (before dividing out by \mathbb{R}):

Homology class	Number of chords	Expected dimension of moduli space
0	0	0
$R_{i,j}^c$	0	1
$R_{i,j}^a, i > 1$	1	1
$R_{1,2}^a$	1	1
$R_{1,1}^a$	1	0
$R_{1,1}^a + R_{1,1}^c$	1	1
$R_{1,1}^a + R_{1,2}^c$	1	1
$R_{1,1}^a + 2R_{1,1}^c$	1	2
$R_{1,1}^a + R_{1,1}^c + R_{1,2}^c$	1	2

The only homology class where the number of chords exceeds the dimension is $R_{1,1}^a$; but in this case, the dimension is 0, so (for generic J) the corresponding moduli space is empty, and



hence cannot appear as a factor of a non-empty moduli space. The desired inequality now follows from the fact that $\dim \mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$ is the sum of the dimensions of the constituent moduli spaces minus one (for the \mathbb{R} action), while $g - \ell$ is the total number of chords.

The condition that $g - \ell = 0$ is equivalent to the condition that each constituent piece has no chords in it, in which case the homology class must be a sum of domains $R_{i,j}^c$. Moreover, each component contributes 1 to the dimension (before dividing out by \mathbb{R}). This establishes Point (1).

The condition that $g - \ell - 1$ coincides with the dimension ensures that each constituent piece must have exactly as many chords as its dimension: thus, each constituent piece must be one of $R_{i,j}^a$ for some $i > 1$, $R_{1,2}^a$, $R_{1,1}^a + R_{1,1}^c$ or $R_{1,1}^a + R_{1,2}^c$. This establishes Point (2). \square

Given a negative partially-diagonal generator \mathbf{w}_{pd}^- and generators $\mathbf{x} \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathbf{y} \in \mathfrak{S}(\boldsymbol{\alpha}^H, \boldsymbol{\beta})$ we can consider the moduli space

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; T^\Delta; \vec{P}, \vec{P}^H)$$

of holomorphic triangles asymptotic to the generators \mathbf{x} , \mathbf{y} and \mathbf{w}_{pd}^- at v_{31} , v_{23} and v_{12} respectively and with ordered partitions of Reeb chords \vec{P} and \vec{P}^H along e_1 and e_2 , respectively. We can also consider the moduli space

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; \vec{\rho}, \vec{\rho}^H)$$

of embedded holomorphic triangles asymptotic to the sequences of non-empty sets of Reeb chords $\vec{\rho}$ and $\vec{\rho}^H$ along e_1 and e_2 , respectively.

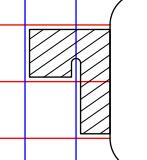
We call a pair of partially-diagonal generators $\mathbf{w}_{pd}^+ = \{w_1^+, \dots, w_\ell^+, \rho_1, \dots, \rho_{g-\ell}\}$ and $\mathbf{w}_{pd}^- = \{w_1^-, \dots, w_m^-, \rho'_1, \dots, \rho'_{g-m}\}$ of opposite polarity *compatible* if $\{w_1^+, \dots, w_\ell^+\} = \{w_1^-, \dots, w_m^-\}$ (so, in particular, $m = \ell$) and there is a bijection $\mu: \{\rho_1, \dots, \rho_{g-\ell}\} \rightarrow \{\rho'_1, \dots, \rho'_{g-m}\}$ so that ρ_i and $\mu(\rho_i)$ are adjacent in the sense that either $\rho_i^+ = \mu(\rho_i)^-$ or $\rho_i^- = \mu(\rho_i)^+$. We will call the map μ a *matching* between \mathbf{w}_{pd}^+ and \mathbf{w}_{pd}^- .

Given a compatible pair of partially-diagonal generators \mathbf{w}_{pd}^+ and \mathbf{w}_{pd}^- and a matching μ between them there are two associated sets of Reeb chords $\boldsymbol{\rho}_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu)$ and $\boldsymbol{\rho}_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu)$, given by

$$\begin{aligned} \boldsymbol{\rho}_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu) &:= \bigcup_{\rho_i^+ = \mu(\rho_i)^-} \rho_i \uplus \mu(\rho_i) \\ \boldsymbol{\rho}_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu) &:= \bigcup_{\rho_i^- = \mu(\rho_i)^+} \mu(\rho_i) \uplus \rho_i. \end{aligned}$$

When the matching is implicit or determined by the rest of the data, as will usually be the case below, we will drop it from the notation, writing simply $\boldsymbol{\rho}_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-)$ and $\boldsymbol{\rho}_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-)$.

If \mathbf{w}_{pd}^+ and \mathbf{w}_{pd}^- are compatible partially-diagonal generators then the product of moduli spaces $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; \vec{\rho}, \vec{\rho}^H)$ sits inside the compactified moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}', \vec{\rho}^{H'})$, where $\vec{\rho}'$ is obtained from $\vec{\rho}$ by adjoining the set $\boldsymbol{\rho}_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu)$ at the end and $\vec{\rho}^{H'}$ is obtained from $\vec{\rho}$ by adjoining the set $\boldsymbol{\rho}_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-, \mu)$ at the beginning. (In principle, this construction depends on a choice of matching μ , but boundary monotonicity of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; \vec{\rho}, \vec{\rho}^H)$ implies that this matching is determined by the rest of the data.)



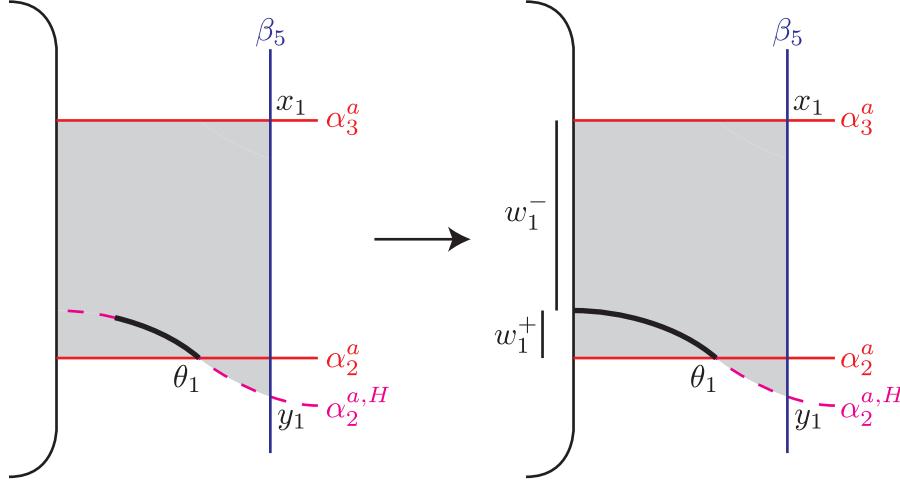


FIGURE 6.5. **Example of a degeneration at v_{12} .** At the left is a 1-parameter family of holomorphic curves connecting x_1 , y_1 and θ_1 ; the parameter is the length of the cut. One end of this moduli space corresponds to a degeneration at v_{12} into a curve connecting θ_1 and w_1^+ and a triangle connecting x_1 , y_1 and w_1^- .

In other words, there is an obvious notion of a sequence of curves in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}', \vec{\rho}^{H,\prime})$ converging to a pair $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; \vec{\rho}, \vec{\rho}^H)$. See Figure 6.5 for an example.

In order to understand the structure of $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}', \vec{\rho}^{H,\prime})$ near the boundary stratum $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; \vec{\rho}, \vec{\rho}^H)$ we will study some auxiliary moduli spaces. Consider the product $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H)$. There is an obvious way to preglue S^\triangleright and T^Δ at the punctures corresponding to w_{pd}^\pm , giving a source $S^\triangleright \natural T^\Delta$. The preglued surface $S^\triangleright \natural T^\Delta$ has punctures r_1, \dots, r_a labeled by the Reeb chords in $\rho_\alpha(\mathbf{w}_{\text{pd}}^+, \mathbf{w}_{\text{pd}}^-)$ and r_1^H, \dots, r_b^H labeled by the Reeb chords in $\rho_{\alpha^H}(\mathbf{w}_{\text{pd}}^+, \mathbf{w}_{\text{pd}}^-)$, where $a + b = g - \ell$ is the number of chords in w_{pd}^\pm . For any permutations $\sigma_a \in S_a$ and $\sigma_b \in S_b$ we can consider the moduli space

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma_a(1)}, \dots, r_{\sigma_a(a)})), ((r_{\sigma_b(1)}^H, \dots, r_{\sigma_b(b)}^H) \vec{P}^H)).$$

(Recall that for sequences s and t , (s, t) denotes the concatenation of s and t .) There is an obvious notion for a sequence of curves in this moduli space to converge to a curve in $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H)$.

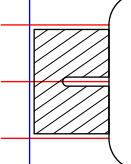
With this terminology in place, we have the following gluing results:

PROPOSITION 6.36. *A neighborhood of $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H)$ in*

$$\bigcup_{\sigma_a \in S_a, \sigma_b \in S_b} \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma_a(1)}, \dots, r_{\sigma_a(a)})), ((r_{\sigma_b(1)}^H, \dots, r_{\sigma_b(b)}^H) \vec{P}^H))$$

is homeomorphic to $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H) \times [0, \epsilon]$.

PROOF. At first glance, it seems that this requires a new kind of gluing argument. Specifically, suppose that $u_{\alpha^H, \alpha} \in \mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright)$ and $u_{\alpha, \alpha^H, \beta} \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H)$.



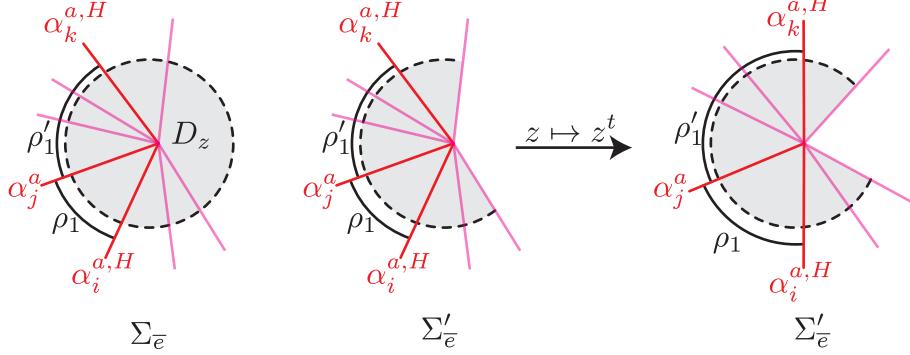


FIGURE 6.6. **Straightening-out α -arcs.** The conformal map giving this straightening is locally the map $z \mapsto z^t$ for some appropriate $t > 0$.

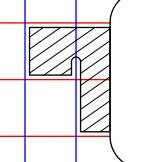
Write $\mathbf{w}_{\text{pd}}^+ = \{w_1^+, \dots, w_\ell^+, \rho_1, \dots, \rho_{g-\ell}\}$ and $\mathbf{w}_{\text{pd}}^- = \{w_1^-, \dots, w_\ell^-, \rho'_1, \dots, \rho'_{g-\ell}\}$. At a puncture q_i of $u_{\alpha^H, \alpha}$ corresponding to ρ_i , $\lim_{q \rightarrow q_i} t(u_{\alpha^H, \alpha}(q)) = \infty$ and $\lim_{q \rightarrow q_i} \pi_\Sigma(u_{\alpha^H, \alpha}(q)) = p$, the puncture in Σ (so the curve goes to infinity in two directions at once). Similarly, at a puncture q'_i of $u_{\alpha, \alpha^H, \beta}$ corresponding to ρ'_i , $\lim_{q \rightarrow q'_i} \pi_\Delta(u_{\alpha, \alpha^H, \beta}(q)) = v_{12}$ and $\lim_{q \rightarrow q'_i} \pi_\Sigma(u_{\alpha, \alpha^H, \beta}(q)) = p$. Such diagonal asymptotics seem not to have been treated in the literature.

Because of the simple form of the curves in $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+)$, we can avoid the new analysis by observing that our situation is biholomorphic to a familiar one, as follows.

Consider first the case that $\ell = g - 1$, so that each of $\mathbf{w}_{\text{pd}}^\pm$ contains a single Reeb chord. Recall that $\Sigma_{\bar{e}}$ denotes the result of filling in the puncture on Σ . The various $\bar{\alpha}_i^a$ and $\bar{\alpha}_i^{a,H}$ meet at the puncture p in $\Sigma_{\bar{e}}$, in general with a corner. To fix notation, suppose that the chord ρ_1 runs from $\alpha_i^{a,H}$ to α_j^a and that ρ'_1 runs from α_j^a to $\alpha_k^{a,H}$; the case that ρ'_1 runs from α_k^a to $\alpha_i^{a,H}$ is similar. Let D_z be the component of $\Sigma \setminus (\alpha \cup \beta)$ adjacent to z and let $\Sigma'_{\bar{e}} = \Sigma_{\bar{e}} \setminus D_z$. We can choose a complex structure j'_Σ (and smooth structure) on $\Sigma'_{\bar{e}}$ so that j'_Σ agrees with the fixed complex structure j_Σ from Chapter 5 on $\Sigma'_{\bar{e}} \setminus \{e\}$ and so that the chord $\rho_1 \sqcup \rho'_1$ corresponds to an 180° angle with respect to j'_Σ . (See Figure 6.6. Note that while the total angle at the corner of $\Sigma'_{\bar{e}}$ may be more than 360° , the holomorphic curves are contained in a subset with total angle 180° .) Let J' be an almost complex structure on $\Sigma'_{\bar{e}} \times \Delta$ which agrees with J away from p and with $j'_\Sigma \times j_\Delta$ near p .

The holomorphic curve $u_{\alpha, \alpha^H, \beta}$ has image in $\Sigma'_{\bar{e}} \times \Delta$ and is holomorphic with respect to J' . Similarly, the holomorphic curve $u_{\alpha^H, \alpha}$ has image in $\Sigma'_{\bar{e}} \times [0, 1] \times \mathbb{R}$ and is holomorphic with respect to the complex structure on $\Sigma'_{\bar{e}} \times [0, 1] \times \mathbb{R}$ induced by J' . Moreover, the gluing problem for $u_{\alpha, \alpha^H, \beta}$ and $u_{\alpha^H, \alpha}$ in $\Sigma'_{\bar{e}} \times [0, 1] \times \mathbb{R}$ has the same form as the usual gluing of holomorphic curves in Heegaard Floer theory at a generator. So, in this case the result follows from the usual gluing theorem, [Lip06a, Proposition A.2], say.

The general case, where $\mathbf{w}_{\text{pd}}^\pm$ contains more than one Reeb chord, follows from the above argument with one more observation. The almost complex structure J' that we used above depended on the endpoints of the chords ρ_1 and ρ'_1 being glued; it may not be possible to choose a single almost complex structure which straightens out all of the relevant arcs. However, we may choose the complex structure J' on $\Sigma \times \Delta$ to depend on the point in the source of $u_{\alpha, \alpha^H, \beta}$. (In other words, for curves with source S , we are now considering holomorphic sections of a topologically trivial bundle $\Sigma \times \Delta \times S \rightarrow S$, with a complex



structure on $\Sigma \times \Delta \times S$ given by $J'_s \times j_S$, where j_S is the complex structure on S and J'_s is an almost complex structure on $\Sigma \times \Delta$; compare [Sei08b, Section 3].) In particular, we can choose $J' = J'(s, d, p)$ ($s \in \Sigma$, $d \in \Delta$, $p \in S$ the source of $u_{\alpha, \alpha^H, \beta}$) so that for p near the puncture p'_i , the angle corresponding to $\rho_i \sqcup \rho'_i$ or $\rho'_i \sqcup \rho_i$ (as appropriate) is 180° . The result then follows as before. \square

It follows from Proposition 6.36 that if $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho})$ is a 1-dimensional moduli space of holomorphic curves, where $\vec{\rho}$ is the discrete partition, then, in degenerations at v_{12} , the resulting curve for $(\boldsymbol{\alpha}^H, \boldsymbol{\alpha})$ has index 1, and hence at most one non-trivial component. In particular, by Lemma 6.34, at most one α - α^H -Reeb chord is involved.

PROPOSITION 6.37. *Suppose that $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and sequences of non-empty sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_m)$, $\vec{\rho}^H = (\rho_1^H, \dots, \rho_n^H)$ are such that $\text{ind}(B, (\vec{\rho}, \vec{\rho}^H)) = 1$. For a sufficiently generic, admissible almost complex structure J , the boundary of the moduli space $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$ (where $o = o(\mathbf{x}, \vec{\rho})$) consists of curves of the following types:*

(MT-1) *Two-story buildings corresponding to breaks at v_{31} , i.e., curves in*

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{u}; (\rho_1, \dots, \rho_i)) \times \mathcal{M}^{B_2}(\mathbf{u}, \mathbf{y}, \Theta_o; (\rho_{i+1}, \dots, \rho_m), \vec{\rho}^H)$$

*where $B_1 \in \pi_2^{\alpha, \beta}(\mathbf{x}, \mathbf{u})$ and $B_2 \in \pi_2(\mathbf{u}, \mathbf{y})$ are such that $B_1 * B_2 = B$, and $1 \leq i \leq m$.*

(MT-2) *Two-story buildings corresponding to breaks at v_{23} , i.e., curves in*

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{u}, \Theta_o; \vec{\rho}, (\rho_1^H, \dots, \rho_i^H)) \times \mathcal{M}^{B_2}(\mathbf{u}, \mathbf{y}; (\rho_{i+1}^H, \dots, \rho_n^H))$$

*where $B_1 \in \pi_2(\mathbf{x}, \mathbf{u})$ and $B_2 \in \pi_2^{\alpha^H, \beta}(\mathbf{u}, \mathbf{y})$ are such that $B_1 * B_2 = B$, and $1 \leq i \leq n$.*

(MT-3) *Join curve ends.*

(MT-4) *Split curve ends.*

(MT-5) *Other collisions of levels among $\vec{\rho}$ or $\vec{\rho}^H$.*

(MT-6) *Shuffle curve ends.*

(MT-7) *Degenerations at v_{12} where the resulting partially diagonal generator involves only intersection points (not Reeb chords), i.e., curves in*

$$\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; \vec{\rho}, \vec{\rho}^H)$$

where $\mathbf{w}_{pd}^+ = \mathbf{w}_{pd}^- = \{w_1, \dots, w_g\} \subset \boldsymbol{\alpha} \cap \boldsymbol{\alpha}^H$. Moreover, the homology class B_1 (and generator \mathbf{w}_{pd}^+) is such that $\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+)$ is 0-dimensional.

(MT-8) *Degenerations where ρ_m approaches v_{12} , i.e., curves in*

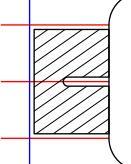
$$\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; (\rho_1, \dots, \rho_{m-1}), \vec{\rho}^H)$$

where \mathbf{w}_{pd}^\pm are compatible partially-diagonal generators, $\rho_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-) = \rho_m$ and $\rho_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-) = \emptyset$. Moreover, the homology class B_1 (and generator \mathbf{w}_{pd}^+) is such that moduli space $\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+)$ is $(|\rho_m| - 1)$ -dimensional (compare Corollary 6.35).

(MT-9) *Degenerations where ρ_1^H approaches v_{12} , i.e., curves in*

$$\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{y}, \mathbf{w}_{pd}^-; \vec{\rho}, (\rho_2^H, \dots, \rho_n^H))$$

where \mathbf{w}_{pd}^\pm are compatible partially-diagonal generators, $\rho_\alpha(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-) = \emptyset$ and $\rho_{\alpha^H}(\mathbf{w}_{pd}^+, \mathbf{w}_{pd}^-) = \rho_1^H$. Moreover, the homology class B_1 (and generator \mathbf{w}_{pd}^+) is such that moduli space $\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{pd}^+)$ is $(|\rho_1^H| - 1)$ -dimensional (compare Corollary 6.35).



Conversely, each curve of type (MT-1), (MT-2), (MT-3), (MT-4), (MT-5), (MT-7) and the odd case of (MT-6) corresponds to an odd number of ends of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$. (By contrast, we will see in the proof of Proposition 6.39 that not all curves of types (MT-8) and (MT-9) correspond to ends of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$.)

PROOF. That any degeneration has one of forms (MT-1)–(MT-6) or corresponds to a degeneration at v_{12} follows from an argument analogous to the proof of Theorem 5.61. That the degenerations at v_{12} have the specified forms follows from Proposition 6.36. For example, for case (MT-8), Proposition 6.36 implies that without the height constraints imposed by ρ_m , the glued moduli space has dimension

$$\dim(\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{\text{pd}}^+)) + \dim(\mathcal{M}^{B_2}(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; (\rho_1, \dots, \rho_{m-1}), \vec{\rho}^H)) + 1.$$

So, the dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$ is

$$\dim(\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{\text{pd}}^+)) + \dim(\mathcal{M}^{B_2}(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; (\rho_1, \dots, \rho_{m-1}), \vec{\rho}^H)) + 1 - |\rho_m| + 1.$$

By Corollary 6.35, this is at least

$$|\rho_m| - 1 + 0 + 1 - |\rho_m| + 1 = 1,$$

with equality only if $\dim(\mathcal{M}^{B_1}(\Theta_o, \mathbf{w}_{\text{pd}}^+)) = |\rho_m| - 1$.

The converse follows from gluing arguments similar to those used to prove Theorem 5.61 (and Proposition 6.36). \square

As a warm-up to our next proposition, we have the following:

LEMMA 6.38. *For any B , $\vec{\rho}$ and $\vec{\rho}^H$ so that $\text{ind}(B, (\vec{\rho}, \vec{\rho}^H)) = 1$, the moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$ has an even number of ends of type (MT-7).*

PROOF. By Corollary 6.35, each end of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}, \vec{\rho}^H)$ of type (MT-7) corresponds to a pair of curves $(u_{\alpha^H, \alpha}, u_{\alpha, \alpha^H, \beta})$ where $u_{\alpha^H, \alpha}$ is a bigon with domain some $R_{i,j}^c$. There is a corresponding end $(u'_{\alpha^H, \alpha}, u_{\alpha, \alpha^H, \beta})$ where $u'_{\alpha^H, \alpha}$ is a bigon with domain $R_{i,j \pm 1}^c$. The result follows. \square

The following proposition is similar in spirit to Lemma 6.38, but for the bigons with domains $R_{i,j}^a$ and the annulus with domain $R_{1,1}^a + R_{1,1}^c$ or $R_{1,1}^a + R_{1,2}^c$.

PROPOSITION 6.39. *Fix $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and a sequence of non-empty sets of Reeb chords (ρ_1, \dots, ρ_m) such that $\text{ind}(B, (\rho_1, \dots, \rho_n)) = 1$. Let $o_i = o(\vec{\mathbf{x}}, (\rho_1, \dots, \rho_i))$. For a sufficiently generic almost complex structure J , the number of ends of the moduli space*

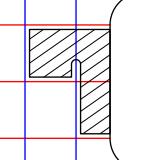
$$\mathcal{M}_{i,i+1}^B := \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_{o_i}; (\rho_1, \dots, \rho_i), (\rho_{i+1}^H, \dots, \rho_m^H))$$

where ρ_i approaches v_{12} has the same parity as the number of ends of

$$\mathcal{M}_{i-1,i}^B := \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_{o_{i-1}}; (\rho_1, \dots, \rho_{i-1}), (\rho_i^H, \dots, \rho_m^H))$$

where ρ_i^H approaches v_{12} .

(Since we have fairly weak control over the behavior of \mathcal{M}^B near the boundary, the parity of the number of ends might deserve some explanation. For each $\epsilon > 0$, consider $\{u \in \mathcal{M}_{i,i+1}^B \mid d(\pi_\Delta \circ u(P_i), v_{12}) \geq \epsilon\}$, where d denotes the distance in Δ , viewed as the unit disk in \mathbb{C} with three boundary punctures, and P_i denotes the punctures of the source of u mapped to ρ_i . For generic ϵ , this is a manifold with boundary. The number of boundary



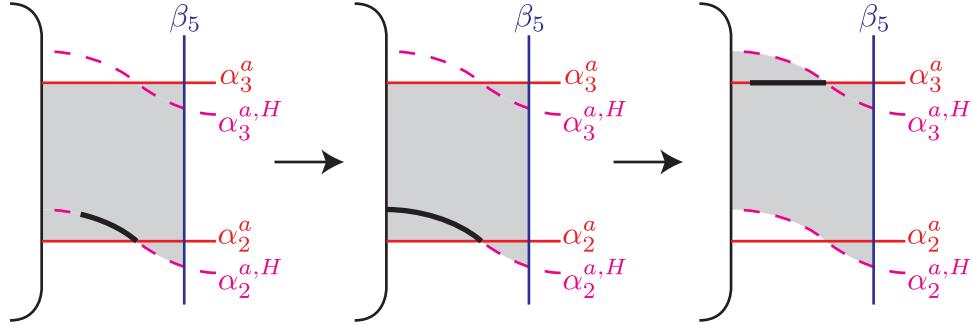


FIGURE 6.7. An illustration of a boundary bigon degenerating, and a different one being glued on the other side. The domain (shaded) changes slightly in the process.

components of this manifold modulo 2 is independent of ϵ if ϵ is generic and sufficiently small. This is the (modulo 2) number of ends of $\mathcal{M}_{i,i+1}^B$ where ρ_i approaches v_{12} .)

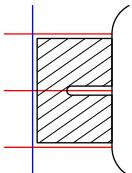
PROOF. We start with the case that ρ_i consists of a single Reeb chord ρ ; the argument in this case is illustrated schematically in Figure 6.7. By Proposition 6.37, ends of $\mathcal{M}_{i,i+1}^B$ at which ρ_i approaches v_{12} correspond to a pair of curves $(u_{\alpha^H,\alpha}, u_{\alpha,\alpha^H,\beta})$. Proposition 6.37 and Corollary 6.35 imply that the curve $u_{\alpha^H,\alpha}$ is either a bigon with domain $R_{i,j}^a$ (for some i, j) or an annulus with domain $R_{1,1}^a + R_{1,1}^c$ or $R_{1,1}^a + R_{1,2}^c$. In either case, there is algebraically one such curve $u_{\alpha^H,\alpha}$ in the moduli space $\mathcal{M}^B(\Theta_o, \mathbf{w}_{pd}^+)$; this is obvious in the case of bigons, and follows from [OSz04d, Lemma 9.4] in the case of annuli. Which domain $R_{i,j}^a$ occurs in the domain is determined by the chord ρ : the region $R_{i,j}^a$ has the point ρ^+ in its boundary. Proposition 6.36 implies that near $(u_{\alpha^H,\alpha}, u_{\alpha,\alpha^H,\beta})$, the compactified moduli space $\overline{\mathcal{M}}_{i,i+1}^B$ is homeomorphic to $[0, \epsilon)$.

There is a corresponding curve $u'_{\alpha^H,\alpha}$, again a bigon $R_{i,j}^a$ or annulus $R_{1,1}^a + R_{1,1}^c$ or $R_{1,1}^a + R_{1,2}^c$, with ρ^- in the boundary of the domain of $u'_{\alpha^H,\alpha}$. The pair $(u_{\alpha^H,\alpha}, u_{\alpha,\alpha^H,\beta})$ is a point in $\partial\overline{\mathcal{M}}_{i-1,i}^B$. Again, Proposition 6.36 implies that near $(u_{\alpha^H,\alpha}, u_{\alpha,\alpha^H,\beta})$ the moduli space $\overline{\mathcal{M}}_{i-1,i}^B$ is homeomorphic to $[0, \epsilon)$. The result, in the case that ρ_i consists of a single chord, follows.

As a warm-up to the general case, we consider the case that ρ_i contains exactly two Reeb chords, ρ_1 and ρ_2 . Our argument is analogous to the proof of Proposition 5.39. By Proposition 6.37, when ρ_i approaches the puncture, the curve degenerates a curve $u_{\alpha^H,\alpha}$ in $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\alpha}, z)$ with two non-trivial components. By Corollary 6.35, each of these components maps either to the bigon $R_{i,j}^a$ or to the annulus $R_{1,1}^a + R_{1,1}^c$ or $R_{1,1}^a + R_{1,2}^c$. There is also a leftover component $u_{\alpha,\alpha^H,\beta}$ in $\Sigma \times \Delta$. Let $\mathcal{M}(u_{\alpha^H,\alpha})$ denote the moduli space of curves in $(\Sigma \times [0, 1] \times \mathbb{R}, \boldsymbol{\alpha} \times \{0\} \times \mathbb{R} \cup \boldsymbol{\alpha}^H \times \{1\} \times \mathbb{R})$ of which $u_{\alpha^H,\alpha}$ is a member. The moduli space $\mathcal{M}(u_{\alpha^H,\alpha})$ is homeomorphic to (an odd number of copies of) \mathbb{R} ; different points are gotten by sliding the two non-trivial components of $u_{\alpha^H,\alpha}$ relative to each other.

By Proposition 6.36, a neighborhood \mathcal{U} of $\mathcal{M}(u_{\alpha^H,\alpha}) \times \{u_{\alpha,\alpha^H,\beta}\}$ in

$$\begin{aligned} \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_{o_i}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}, \{\rho_1\}, \{\rho_2\}), (\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H)) \\ \cup \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_{o_i}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}, \{\rho_2\}, \{\rho_1\}), (\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H)) \end{aligned}$$



is homeomorphic to $\mathbb{R} \times [0, \epsilon]$. Choose this homeomorphism so that $\delta \in (0, \epsilon)$ is the distance in Δ between v_{12} and the closer of $\pi_\Delta(u(P_1))$ and $\pi_\Delta(u(P_2))$, where P_1 and P_2 are the punctures corresponding to ρ_1 and ρ_2 , respectively. Given $\delta \in (0, \epsilon)$, for t sufficiently small the point (t, δ) lies in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}, \{\rho_1\}, \{\rho_2\}), (\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H))$, say, while for t sufficiently large the point (t, δ) lies in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1}, \{\rho_2\}, \{\rho_1\}), (\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H))$. It follows that $\mathcal{M}_{i,i+1}^B$ has an odd number of ends in $\mathcal{M}(u_{\alpha^H, \alpha}) \times \{u_{\alpha, \alpha^H, \beta}\}$.

For each of the components of $u_{\alpha^H, \alpha}$ there is a matching bigon or annulus, as in the single Reeb chord case; denote the union of the matching components by $u'_{\alpha^H, \alpha}$. (That is, $u'_{\alpha^H, \alpha}$ has two non-trivial components whose domains contain the points ρ_1^- and ρ_2^- , respectively.) A similar argument to the previous paragraph shows that $\mathcal{M}_{i-1,i}^B$ has an odd number of ends in $\mathcal{M}(u'_{\alpha^H, \alpha}) \times \{u_{\alpha, \alpha^H, \beta}\}$. The result follows.

For the general case, write $\boldsymbol{\rho}_i = \{\rho_1, \dots, \rho_m\}$. Consider a pair of curves $(u_{\alpha^H, \alpha}, u_{\alpha, \alpha^H, \beta}) \in \partial \overline{\mathcal{M}}_{i,i+1}^B$, as above. Let S^\triangleright denote the source of $u_{\alpha^H, \alpha}$ and T^Δ the source of $u_{\alpha, \alpha^H, \beta}$. Let \vec{P}, \vec{P}^H be the ordered partitions of punctures on $u_{\alpha, \alpha^H, \beta}$ along e_1 and e_2 induced by $(\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{i-1})$ and $(\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H)$. Let r_1, \dots, r_m denote the punctures of $S^\triangleright \natural T^\Delta$ mapped to ρ_1, \dots, ρ_m . Consider the moduli space

$$\bigcup_{\sigma \in S_m} \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma(1)}, \dots, r_{\sigma(m)})), \vec{P}^H)$$

There is an evaluation map

$$\text{ev}: \bigcup_{\sigma \in S_m} \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma(1)}, \dots, r_{\sigma(m)})), \vec{P}^H) \rightarrow (e_1)^m$$

given by

$$\text{ev}(u) = (\pi_\Delta \circ u)(r_1, \dots, r_m).$$

The moduli space $\mathcal{M}_{i,i+1}^B$ is the preimage under ev of the total diagonal $\{(x, x, \dots, x) \mid x \in e_1\}$. We can also evaluate at only some of the punctures; let

$$\text{ev}_{[1,j]}(u) = (\pi_\Delta \circ u)(r_1, \dots, r_j)$$

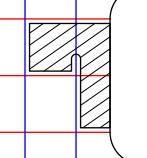
Let

$$\begin{aligned} \Gamma: \mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright) \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H) \times [0, \epsilon) \\ \rightarrow \bigcup_{\sigma \in S_m} \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma(1)}, \dots, r_{\sigma(m)})), \vec{P}^H) \end{aligned}$$

denote the gluing map from Proposition 6.36. Recall from Corollary 6.35 that there is a bijection between the nontrivial components of $u_{\alpha^H, \alpha}$ and the punctures r_i . Gluing at the punctures r_1, \dots, r_j (i.e., translating the components corresponding to r_{j+1}, \dots, r_m to $-\infty$ and then gluing at v_{12}) gives a gluing map

$$\begin{aligned} \Gamma_{[1,j]}: V_{[1,j]} \times \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H) \times [0, \epsilon) \\ \rightarrow \bigcup_{\sigma \in S_m} \overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; S^\triangleright \natural T^\Delta; (\vec{P}, (r_{\sigma(1)}, \dots, r_{\sigma(m)})), \vec{P}^H) \end{aligned}$$

for the corresponding stratum $V_{[1,j]}$ of $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright)$ (where the curve at v_{12} has at least two stories, and the punctures r_1, \dots, r_j are on a higher story than the punctures r_{j+1}, \dots, r_m). Write $V_{[1,j]} = W_{[1,j]} \times W_{[j+1,m]}$ where $W_{[1,j]}$ is the moduli space corresponding



to the punctures r_1, \dots, r_j and $W_{[j+1, m]}$ is the moduli space corresponding to the punctures r_{j+1}, \dots, r_m .

We will prove the following statement by induction on m :

- Given any $(u_{\alpha^H, \alpha}, u_{\alpha, \alpha^H, \beta})$ as above there is an open neighborhood U in $(e_1)^j$ containing $\{(v_{12}, v_{12}, \dots, v_{12})\}$ and an $\epsilon > 0$ so that

$$\text{ev}_{[1,j]} \circ \Gamma_{[1,j]} : W_{[1,j]} \times \{u_{\alpha, \alpha^H, \beta}\} \times (0, \epsilon) \rightarrow (e_1)^j$$

maps with odd degree onto U .

Given this statement for the case $j = m$, taking the preimage of the diagonal implies the proposition.

The case $j = 1$ is clear from (the proof of) Proposition 6.36.

The inductive step follows from Lemma 5.38 with $X = W_{[1,j]}$ and $f = \text{ev}_{[1,j]} \circ \Gamma_{[1,j]}$. The fact that the union of the top two strata of $W_{[1,j]}$ is a manifold with boundary is clear from the structure of $W_{[1,j]}$ (analyzed completely in Corollary 6.35) or, alternatively, follows from gluing theory. That f is proper near 0 follows from the gluing construction. The inductive hypothesis gives the fact that the restriction of f to the (codimension 1) facet where r_1 is on the bottom story and r_2, \dots, r_m are on the top story (adjacent to v_{12}), with image in $(e_1)^{m-1}$, has odd degree near 0. \square

6.3.2.6. *The map associated to a handleslide is a chain map.*

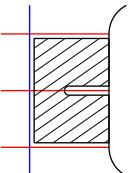
PROPOSITION 6.40. $F_{\alpha, \alpha^H, \beta}$ is a chain map, i.e., $F \circ \partial + \partial \circ F = 0$.

PROOF. This follows from Proposition 6.37, Lemma 6.38 and Proposition 6.39, similarly to the proof of Proposition 6.7. Ends of type (MT-1) correspond to $F_{\alpha, \alpha^H, \beta} \circ \partial(\mathbf{x})$. Ends of type (MT-2) correspond to part of $\partial \circ F_{\alpha, \alpha^H, \beta}(\mathbf{x})$. Ends of type (MT-4) correspond to the rest of $\partial \circ F_{\alpha, \alpha^H, \beta}(\mathbf{x})$. Ends of type (MT-5) cancel with each other and with ends of type (MT-3), as in the proof of Proposition 6.7. Ends of type (MT-7) cancel in pairs by Lemma 6.38. Ends of type (MT-8) cancel with ends of type (MT-9) by Proposition 6.39. The result follows. \square

To show that $F_{\alpha, \alpha^H, \beta}$ is an isomorphism, we wish to construct a chain map the other direction and a chain homotopy between their composition and the identity map. A count of triangles in $(\Sigma, \alpha^H, \alpha, \beta)$ does not work as desired, as the small bigons $R_{i,j}^a$ (and annuli $R_{1,1}^a + R_{1,1}^c$ and $R_{1,1}^a + R_{1,2}^c$) are not holomorphic: their orientations are consistent with $(\alpha, \alpha^H, \beta)$, not $(\alpha^H, \alpha, \beta)$. Instead we define a third set of circles, which we denote α' . Obtain $\alpha_i^{c'}$ from α_i^c by performing a small Hamiltonian perturbation, so that $\alpha_i^{c'}$ intersects each of α_i^c and $\alpha_i^{c,H}$ transversely in two points and is disjoint from all other curves in $\alpha \cup \alpha^H$. Let $\alpha_i^{a'}$ be an isotopic translate of α_i^a , intersecting each of α_i^a and $\alpha_i^{a,H}$ in a single point, and such that there are two short Reeb chords in $\partial \bar{\Sigma}$ running from $\alpha_i^{a'}$ to $\alpha_i^{a,H}$ (and hence two slightly longer Reeb chords running from $\alpha_i^{a'}$ to α_i^a). The isotopy is chosen small enough that $\alpha_i^{a'}$ is disjoint from all other curves in $\alpha \cup \alpha^H$. See Figure 6.8. Now there is a chain map $F_{\alpha^H, \alpha', \beta}$ defined using $(\Sigma, \alpha^H, \alpha', \beta, z)$, in exactly the same way that $F_{\alpha, \alpha^H, \beta}$ was defined using $(\Sigma, \alpha, \alpha^H, \beta, z)$.

6.3.2.7. *Completion of the proof of handleslide invariance.* To prove handleslide invariance, we also need to consider the map $F_{\alpha, \alpha', \beta}$ defined in the same way as $F_{\alpha, \alpha^H, \beta}$ but with α' in place of α^H .

PROPOSITION 6.41. *The map $F_{\alpha, \alpha', \beta}$ is an isomorphism of differential \mathcal{A} -modules.*



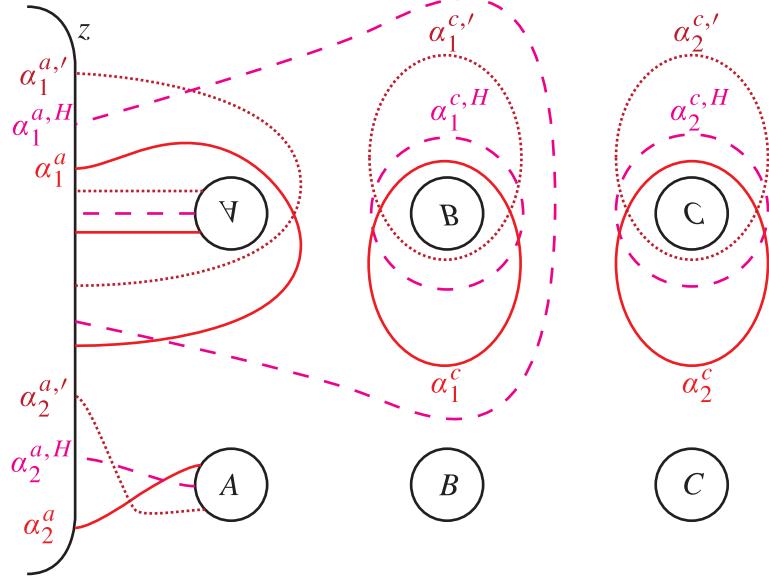


FIGURE 6.8. **The curves α , α^H and α' in Σ .** The curves in α are solid, those in α^H are dashed, and those in α' are dotted.

PROOF. The proof that $F_{\alpha, \alpha', \beta}$ is a chain map is parallel to, but marginally easier than, the proof of Proposition 6.40, the only difference being that the discussion of annuli is irrelevant here.

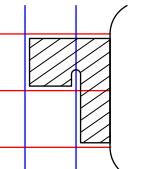
Next we argue that $F_{\alpha, \alpha', \beta}$ is in fact an isomorphism. The proof, which makes use of the energy filtration, is a straightforward analogue of the proof for closed 3-manifolds [OSz04d, Proposition 9.8]. We will use the following standard lemma.

LEMMA 6.42. *Let $F: A \rightarrow B$ be a map of filtered groups which is decomposed as a sum $F = F_0 + \ell$ where F_0 is a filtration-preserving isomorphism and ℓ has strictly lower order than F_0 . Suppose that the filtration on B is bounded below. Then F is an isomorphism of groups.*

Choose an area form Area on Σ such that if D is a periodic domain in (Σ, α, β) then $\text{Area}(D) = 0$. (The existence of Area is guaranteed by Lemma 4.26.) Arrange also that periodic domains in $(\Sigma, \alpha', \alpha)$ have area 0, and that the boundary bigons all have the same area. It follows that for D a periodic domain in (Σ, α', β) , $\text{Area}(D) = 0$ as well.

For each spin^c structure \mathfrak{s} on Y , we define a map \mathcal{F} from $\mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$ to \mathbb{R} . Choose one generator $\mathbf{x}_0 \in \mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$ and declare $\mathcal{F}(\mathbf{x}_0) = 0$. Then, for any other $\mathbf{x} \in \mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$ pick $A_{\mathbf{x}_0, \mathbf{x}} \in \pi_2(\mathbf{x}_0, \mathbf{x})$ and define $\mathcal{F}(\mathbf{x}) = -\text{Area}(A_{\mathbf{x}_0, \mathbf{x}})$. Since periodic domains have area 0, $\mathcal{F}(\mathbf{x})$ is independent of the choice of $A_{\mathbf{x}_0, \mathbf{x}}$. More generally, for $a \in \mathcal{A}$ such that $a\mathbf{x} \neq 0$, define $\mathcal{F}(a\mathbf{x}) = \mathcal{F}(\mathbf{x})$. Since holomorphic curves have positive domains, the map \mathcal{F} induces a filtration on $\widehat{\text{CFD}}(\Sigma, \alpha, \beta)$.

There is an obvious identification between $\mathfrak{S}(\Sigma, \alpha, \beta, \mathfrak{s})$ and $\mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s})$. As before, let \mathbf{x}' denote the generator corresponding to \mathbf{x} , and let $T_{\mathbf{x}} \in \pi_2(\mathbf{x}, \mathbf{x}', \Theta_{o(\mathbf{x})})$ be the canonical small triangle. Define a map $\mathcal{F}'_0: \mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s}) \rightarrow \mathbb{R}$ in the same way as \mathcal{F} , still using the area form Area but using \mathbf{x}'_0 in place of \mathbf{x}_0 . Define $\mathcal{F}': \mathfrak{S}(\Sigma, \alpha', \beta, \mathfrak{s}) \rightarrow \mathbb{R}$ by $\mathcal{F}' = \mathcal{F}'_0 - \text{Area}(T_{\mathbf{x}_0})$. By extending \mathcal{F}' by $\mathcal{F}'(ay) = \mathcal{F}'(y)$, we get a filtration \mathcal{F}' on $\widehat{\text{CFD}}(\Sigma, \alpha', \beta)$.



To see that $F_{\alpha, \alpha', \beta}$ respects the filtrations \mathcal{F} and \mathcal{F}' , suppose there is a term $a\mathbf{y}$ in $F_{\alpha, \alpha', \beta}(\mathbf{x})$, and thus a positive domain $B \in \pi_2(\mathbf{x}, \mathbf{y}', \Theta_o)$, and pick a domain $A_{\mathbf{x}_0, \mathbf{x}} \in \pi_2(\mathbf{x}_0, \mathbf{x})$. Then, for an appropriate $o' = o(\mathbf{x}_0)$ and domain $A_{o, o'} \in \pi_2(\Theta_o, \Theta_{o'})$, the domain $A_{\mathbf{x}_0, \mathbf{x}} + B - T_{\mathbf{x}_0} - A_{o, o'}$ connects \mathbf{x}'_0 and \mathbf{y}' . Our assumptions guarantee that $\text{Area}(A_{o, o'}) = 0$, so

$$\mathcal{F}'(\mathbf{y}') = -\text{Area}(A_{\mathbf{x}_0, \mathbf{x}} + B - T_{\mathbf{x}_0} - A_{o, o'}) - \text{Area}(T_{\mathbf{x}_0}) = -\text{Area}(A_{\mathbf{x}_0, \mathbf{x}}) - \text{Area}(B) < \mathcal{F}(\mathbf{x})$$

as desired.

Now, if we choose Area so that for all \mathbf{x} the triangle $T_{\mathbf{x}}$ is the unique triangle of minimal area connecting \mathbf{x} , \mathbf{y}' and Θ_o for any \mathbf{y}' and o (this is easily accomplished), then the top order part of $F_{\alpha, \alpha', \beta}$ with respect to \mathcal{F} and \mathcal{F}' is simply the map $\mathbf{x} \mapsto \mathbf{x}'$. In particular, the top order part of $F_{\alpha, \alpha', \beta}$ is a group isomorphism.

Consequently, it follows from Lemma 6.42 that the \mathcal{A} -module homomorphism $F_{\alpha, \alpha', \beta}$ is an \mathbb{F}_2 vector space isomorphism $\widehat{\text{CFD}}(\Sigma, \alpha, \beta, z) \rightarrow \widehat{\text{CFD}}(\Sigma, \alpha', \beta, z)$, and hence also a \mathcal{A} -module isomorphism. \square

REMARK 6.43. We could weaken our assumptions and prove handleslide invariance for Heegaard diagrams \mathcal{H} that are provincially admissible (rather than admissible) by being slightly more clever in the choice of filtration \mathcal{F} , as follows. Pick a map $f: H_1(\partial\bar{\Sigma}, \mathbf{a}) \rightarrow \mathbb{R}$ so that for any periodic domain B for (Σ, α, β) , $f(\partial^\partial B) + \text{Area}(B) = 0$. (This is possible for any provincially admissible diagram.) Then define $\mathcal{F}(\mathbf{x}) = \text{Area}(A_{\mathbf{x}_0, \mathbf{x}}) + f(\partial^\partial A_{\mathbf{x}_0, \mathbf{x}})$, and make similar other adjustments. Alternatively, we can do an isotopy to make \mathcal{H} admissible, do the handleslide, and then do an inverse isotopy.

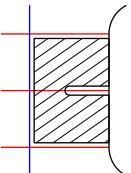
PROPOSITION 6.44. *The map $F_{\alpha, \alpha', \beta}$ is chain homotopic to $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$.*

PROOF. The proof is in two steps. First, the composition $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$ is a count of pairs of triangles, in $(\Sigma, \alpha, \alpha^H, \beta, z)$ and $(\Sigma, \alpha^H, \alpha', \beta, z)$. This map is chain homotopic to counting pairs of triangles in $(\Sigma, \alpha, \alpha', \beta, z)$ and $(\Sigma, \alpha, \alpha^H, \alpha', z)$, via a chain homotopy counting (index -1) holomorphic curves in a path of almost complex structures on the product of Σ and a rectangle. (See Figure 6.9 for a schematic illustration.) The reader unfamiliar with such arguments is directed to any of [OSz04d, Theorem 8.16], [Lip06a, Proposition 10.29] or [Lip06b, Lemma 7.2.5]. The only new types of degeneration are curves in either $(\Sigma, \alpha^H, \alpha, z)$ or $(\Sigma, \alpha', \alpha^H, z)$ touching $\partial\bar{\Sigma}$, either small bigons or annuli. As in Proposition 6.40, these degenerations cancel in pairs.

Second, a direct computation shows that the only rigid triangles in $(\Sigma, \alpha, \alpha^H, \alpha', z)$ asymptotic to $\Theta_{o, \alpha^H, \alpha}$ and $\Theta_{o, \alpha', \alpha^H}$ at two of the corners are asymptotic to $\Theta_{o, \alpha', \alpha}$ at the third corner—and that there is algebraically a single such triangle. Again, the reader is directed to [OSz04d, Lemma 9.7] for details—the proof there applies without change in our case. \square

PROPOSITION 6.45. *The \mathcal{A} -modules $\widehat{\text{CFD}}(\Sigma, \alpha, \beta, z)$ and $\widehat{\text{CFD}}(\Sigma, \alpha^H, \beta, z)$ are homotopy equivalent as dg modules.*

PROOF. If we arrange that α' and α are close enough then the obvious identification of generators of $\widehat{\text{CFD}}(\Sigma, \alpha, \beta, z)$ and $\widehat{\text{CFD}}(\Sigma, \alpha', \beta, z)$ induces an isomorphism of \mathcal{A} -modules (because all of the moduli spaces counted in the definitions of ∂ are the same). Hence, by Propositions 6.41 and 6.44, the composition of $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$ and the evident identification of generators is chain homotopic to an automorphism of $\widehat{\text{CFD}}(\Sigma, \alpha, \beta, z)$.



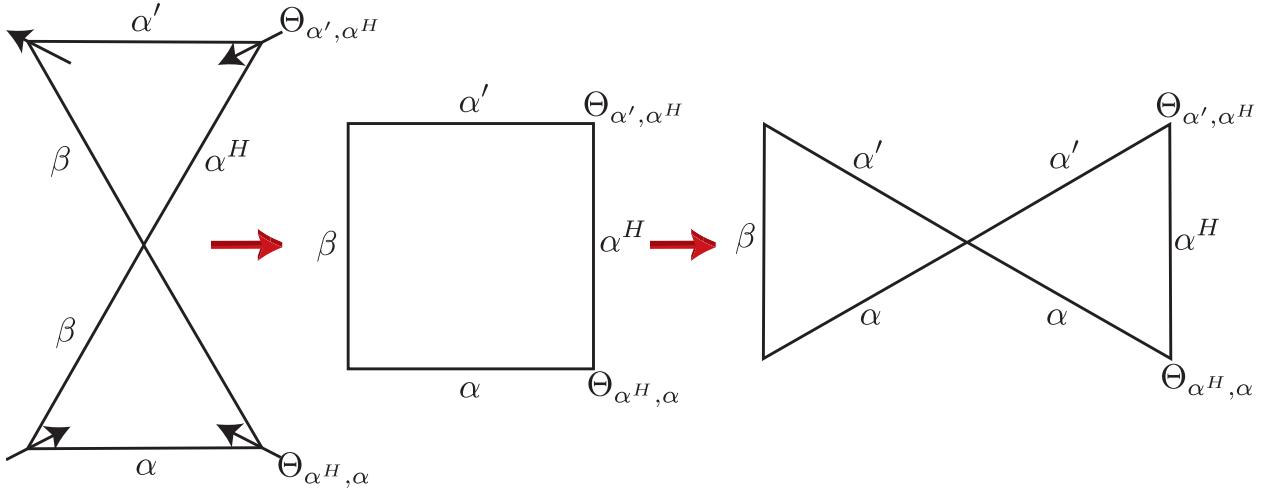
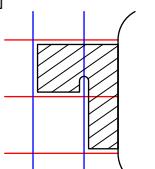


FIGURE 6.9. **Two compositions of triangles, as degenerations of a rectangle.** The bold arrows indicate a path in the compactified moduli space of rectangles, starting at a configuration giving $F_{\alpha^H, \alpha', \beta} \circ F_{\alpha, \alpha^H, \beta}$ and ending at a configuration giving $F_{\alpha, \alpha', \beta}$.

Choose a fourth set of curves $\alpha^{H'}$ isotopic to α^H and define another map $F_{\alpha', \alpha^{H'}, \beta}$: $\widehat{CFD}(\Sigma, \alpha', \beta, z) \rightarrow \widehat{CFD}(\Sigma, \alpha^{H'}, \beta, z)$. The same argument shows that the composition of $F_{\alpha', \alpha^{H'}, \beta} \circ F_{\alpha^H, \alpha', \beta}$ and the evident identification of generators is chain homotopic to an automorphism of $\widehat{CFD}(\Sigma, \alpha^H, \beta, z)$. The result follows. \square

6.3.3. Completion of the invariance proof.

PROOF OF THEOREM 6.16. In Section 6.3.1, we saw that $\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J)$ is independent of the choice of generic almost-complex structure J in its definition. Suppose now that \mathcal{H} and \mathcal{H}' are two Heegaard diagrams which represent equivalent bordered three-manifolds (and in particular $-\partial\mathcal{H} = \mathcal{Z} = -\partial\mathcal{H}'$). Then, according to Proposition 4.10, we can pass from \mathcal{H} to \mathcal{H}' by a sequence of Heegaard moves. Invariance under isotopies has not been spelled out, but this can be done by writing down the continuation maps for isotopies modeled on the continuation maps for variation of almost-complex structures as in Equation (6.19). Invariance under handleslides of an arc over a circle was verified in Proposition 6.45 in the case that the diagrams are admissible; in the general case, first do an isotopy to make the diagrams admissible, then a handleslide, and then undo the isotopy. The case of handleslides of a circle over a circle is similar but easier, and is left to the reader. Invariance under stabilization is straightforward: by stabilizing near the basepoint, one can see that the chain complexes before and after the stabilization are isomorphic (just as in the closed case for \widehat{HF} , [OSz04d, Theorem 10.1]). \square



In view of Theorem 6.16, we can now refer to \widehat{CFD} without reference to a Heegaard diagram, defining $\widehat{CFD}(Y, \phi, \mathfrak{s})$ (where $\phi: F(-\mathcal{Z}) \rightarrow \partial Y$) to be $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$, where \mathcal{H} is any bordered Heegaard diagram representing the bordered three-manifold (Y, ϕ) . We will sometimes drop the map ϕ from the notation, writing $\widehat{CFD}(Y, \mathfrak{s})$ to denote $\widehat{CFD}(Y, \phi: F(-\mathcal{Z}) \rightarrow \partial Y, \mathfrak{s})$.

To obtain a topological invariant of the bordered three-manifold without reference to a spin^c structure, we simply sum over all spin^c structures, defining

$$\widehat{CFD}(Y) = \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y)} \widehat{CFD}(Y, \mathfrak{s}).$$

PROOF OF THEOREM 1.1. With the above definition, this follows immediately from Theorem 6.16. \square

6.4. Twisted coefficients

As with the original Heegaard Floer homology [OSz04c, Section 8], there are versions of $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ with twisted coefficients. We describe the (maximal) twist by $H_2(Y, \partial Y)$. (Recall from Lemma 4.18 that, for any $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$, $\pi_2(\mathbf{x}, \mathbf{x}) \cong H_2(Y, \partial Y)$.) Besides its intrinsic interest, we explain this version because, as we shall see in Chapter 10, the grading on the twisted theory is easier to understand than the grading on the untwisted theory.

Fix a spin^c structure \mathfrak{s} over Y , and pick a base generator $\mathbf{x}_0 = \mathbf{x}_0(\mathfrak{s}) \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ representing \mathfrak{s} . (For spin^c structures \mathfrak{s} without any generators, the twisted chain complex is 0.) Let $\underline{\mathfrak{S}}(\mathcal{H}, \mathbf{x}_0)$ be the set of elements of the form $e^{B_0}\mathbf{x}$, where $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$, $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$, and e is a formal variable. Let $\underline{X}(\mathcal{H}, \mathbf{x}_0)$ be the \mathbb{F}_2 vector space spanned by $\underline{\mathfrak{S}}(\mathcal{H}, \mathbf{x}_0)$. There is an action of $\mathcal{I}(\mathcal{Z})$ on $\underline{X}(\mathcal{H}, \mathbf{x}_0)$, where

$$I(\mathbf{s}) \cdot e^{B_0}\mathbf{x} := \begin{cases} e^{B_0}\mathbf{x} & I(\mathbf{s}) = I_D(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

as in Equation (6.1).

As a module over $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ (where $\mathcal{Z} = -\partial\mathcal{H}$), define the twisted chain complex $\underline{\widehat{CFD}}(\mathcal{H}, \mathbf{x}_0)$ to be $\mathcal{A} \otimes_{\mathcal{I}} \underline{X}(\mathcal{H}, \mathbf{x}_0)$.

There is a natural action of $H_2(Y, \partial Y)$ on $\underline{\widehat{CFD}}(\mathcal{H}, \mathbf{x}_0)$, by composition with the corresponding periodic domain: for $\alpha \in H_2(Y, \partial Y)$ corresponding to $B_\alpha \in \pi_2(\mathbf{x}_0, \mathbf{x}_0)$, define

$$\alpha * e^{B_0}\mathbf{x} := e^{B_\alpha * B_0}\mathbf{x}.$$

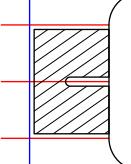
As in the untwisted case, the action of \mathcal{A} on $\underline{\widehat{CFD}}(\mathcal{H}, \mathfrak{s})$ is given by left multiplication:

$$a \cdot (b \otimes e^{B_0}\mathbf{x}) := (ab) \otimes e^{B_0}\mathbf{x}.$$

For the boundary operator, the contributions of the holomorphic curves connecting \mathbf{x} to \mathbf{y} are separated according to their homology classes:

$$\partial(e^{B_0}\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} a_{\mathbf{x}, \mathbf{y}}^B e^{B_0 * B}\mathbf{y}.$$

It is convenient to think of the action by $H_2(Y, \partial Y)$ as a right action of the group-ring $\mathbb{F}_2[H_2(Y, \partial Y)]$ on $\underline{\widehat{CFD}}(\mathcal{H}, \mathbf{x}_0, \mathfrak{s})$. Writing the group-ring as sums $\sum_{\alpha \in H_2(Y, \partial Y)} n_\alpha e^\alpha$, where



$n_\alpha \in \mathbb{Z}/2\mathbb{Z}$ is zero for all but finitely $\alpha \in H_2(Y, \partial Y)$, we let

$$e^{B_0} \mathbf{x} \cdot \left(\sum_{\alpha} n_{\alpha} \cdot e^{\alpha} \right) = \sum_{\alpha} n_{\alpha} e^{B_{\alpha}*B_0} \mathbf{x}.$$

PROPOSITION 6.46. $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0)$ is a differential \mathcal{A} - $\mathbb{F}_2[H_2(Y, \partial Y)]$ bimodule.

PROOF. Here, we are thinking of $\mathbb{F}_2[H_2(Y, \partial Y)]$ as a dg algebra with trivial differential, so that the statement means that

- $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}, \mathfrak{s})$ is a chain complex, and
 - for $a \in \mathcal{A}$, $m \in \widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0, \mathfrak{s})$, and $c \in \mathbb{F}_2[H_2(Y, \partial Y)]$,
- $$(a \cdot m) \cdot c = a \cdot (m \cdot c)$$
- $$\partial(a \cdot m \cdot c) = (\partial a)m \cdot c + a \cdot (\partial m) \cdot c.$$

The proof of Proposition 6.7 readily adapts to show that $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0, \mathfrak{s})$ is a chain complex. The action by $\mathbb{F}_2[H_2(Y, \partial Y)]$ commutes both with the differential and with the action by \mathcal{A} . The result follows. \square

THEOREM 6.47. Up to homotopy equivalence, the differential module $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0)$ depends on the generator \mathbf{x}_0 only through its induced spin^c structure $\mathfrak{s}_z(\mathbf{x})$ and on the Heegaard diagram only through its (equivalence class of) underlying bordered three-manifold $(Y, \phi: -F(\mathcal{Z}) \rightarrow \partial Y)$ (with $\mathcal{Z} = -\partial \mathcal{H}$). That is, if $(\mathcal{H}, \mathbf{x}_0)$ and $(\mathcal{H}', \mathbf{x}'_0)$ are provincially admissible bordered Heegaard diagram for Y with $-\partial \mathcal{H} = \mathcal{Z} = -\partial \mathcal{H}'$, and $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $\mathbf{x}'_0 \in \mathfrak{S}(\mathcal{H}', \mathfrak{s})$ are base generators representing the same spin^c structure over Y , then $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0)$ and $\widehat{\underline{CFD}}(\mathcal{H}', \mathbf{x}'_0)$ are homotopy equivalent $\mathcal{A}(\mathcal{Z})$ - $\mathbb{F}_2[H_2(Y, \partial Y)]$ bimodules.

PROOF. First, we address the dependence of $\widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0)$ on the base generator \mathbf{x}_0 . Suppose that \mathcal{H} is an admissible diagram, and that \mathbf{x}_0 and \mathbf{x}_1 are two different generators which represent the same spin^c structure over Y . Then, by Lemma 4.21, we can find some $b \in \pi_2(\mathbf{x}_0, \mathbf{x}_1)$. We use b to induce a map $\underline{X}(\mathcal{H}, \mathbf{x}_1) \rightarrow \underline{X}(\mathcal{H}, \mathbf{x}_0)$ sending the generator $e^B \mathbf{x}$ to $e^{b*B} \mathbf{x}$. To see that this respects the action by $H_2(Y, \partial Y)$, note that for $\alpha \in H_2(Y, \partial Y)$, $b * B_{\alpha} = B_{\alpha} * b$ (the map $\alpha \rightarrow B_{\alpha}$ from $H_2(Y; \mathbb{Z})$ to $\pi_2(\mathbf{x}_0, \mathbf{x}_0)$ gives a domain which does not depend on the basepoint \mathbf{x}_0). The map $\underline{X}(\mathcal{H}, \mathbf{x}_1) \rightarrow \underline{X}(\mathcal{H}, \mathbf{x}_0)$ extends to a bimodule map $\Phi_b: \widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_1) \rightarrow \widehat{\underline{CFD}}(\mathcal{H}, \mathbf{x}_0)$. If $\mathbf{x}_0 = \mathbf{x}_1$ and $b \in \pi_2(\mathbf{x}_0, \mathbf{x}_1)$ is the trivial homology class then Φ_b is the identity. Moreover, given $b_0 \in \pi_2(\mathbf{x}_0, \mathbf{x}_1)$ and $b_1 \in \pi_2(\mathbf{x}_1, \mathbf{x}_2)$, $\Phi_{b_0 * b_1} = \Phi_{b_1} \circ \Phi_{b_0}$. It follows that for any $b \in \pi_2(\mathbf{x}_0, \mathbf{x}_1)$, Φ_b is an isomorphism.

To adapt the proof of Theorem 6.16 to the twisted setting, we must give twisted analogues of the continuation and triangle maps. For example, for a suitable path of almost complex structures J_r , we define

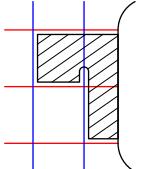
$$\underline{F}^{J_r}: \widehat{\underline{CFD}}(\mathcal{H}, \mathfrak{s}; J_0) \rightarrow \widehat{\underline{CFD}}(\mathcal{H}, J_1)$$

by adapting the map defined in Equation (6.19):

$$\underline{F}^{J_r}(e^{B_0} \mathbf{x}) = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{B \in \pi_2(\mathbf{x}, \mathbf{y})} \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=0\}} \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)) a(-\vec{\rho}) e^{B_0 * B} \mathbf{y}.$$

We modify the triangle map from Equation (6.33) similarly, writing

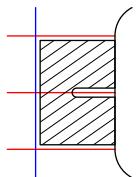
$$\underline{F}_{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(e^{B_0} \mathbf{x}) := \sum_{\mathbf{y}} \sum_{B \in \pi_2(\mathbf{x}', \mathbf{y})} \sum_{\{\vec{\rho} \mid \text{ind}(B, \vec{\rho})=0\}} \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \vec{\rho})) a(-\vec{\rho}) e^{B'_0 * B} \mathbf{y},$$



where $B'_0 \in \pi_2(\mathbf{x}'_0, \mathbf{x})$ is determined (via Lemma 6.24) by

$$T_{\mathbf{x}} *_{13} B_0 = T_{\mathbf{x}_0} *_{12} B_{\alpha, \alpha^H} *_{23} B'_0.$$

With the maps in place, the proof Theorem 6.16 adapts easily to the twisted setting. \square



CHAPTER 7

Type A modules

We now turn to defining the type A module \widehat{CFA} associated to a (provincially admissible) bordered Heegaard diagram \mathcal{H} . This uses more of the moduli spaces from Chapter 5 than the type D module does; in particular, we must consider moduli spaces of curves where the (ordered) partition of Reeb chords has more than one chord per part.

7.1. Definition of the type A module

Fix a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$, satisfying the provincial admissibility criterion of Definition 4.23, and a spin^c structure \mathfrak{s} over Y . The module $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ is a right (\mathcal{A}_∞) $\mathcal{A}(\mathcal{Z})$ -module, where $\mathcal{Z} = \partial\mathcal{H}$. Unlike \widehat{CFD} , which is essentially free (projective) over $\mathcal{A}(\mathcal{Z})$, much of the data of the Heegaard diagram is encoded in the multiplication on $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$, as well as in certain higher products.

The module $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ is generated over \mathbb{F}_2 by the set of generators $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$. Given $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$, define $I_A(\mathbf{x}) := I(o(\mathbf{x}))$. (Recall that $o(\mathbf{x}) \subset [2k]$ is the set of α -arcs occupied by \mathbf{x} .) We define a right action of $\mathcal{I}(\mathcal{Z})$ on $\widehat{CFA}(\mathcal{H})$ by

$$(7.1) \quad \mathbf{x} \cdot I(\mathbf{s}) := \begin{cases} \mathbf{x} & I_A(\mathbf{x}) = I(\mathbf{s}) \\ 0 & \text{otherwise.} \end{cases}$$

Again, the summands $\mathcal{A}(\mathcal{Z}, i)$ of $\mathcal{A}(\mathcal{Z})$ act trivially on $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ for $i \neq 0$.

Recall from Section 3.1.3 that to a set $\boldsymbol{\rho}$ of Reeb chords we associate an algebra element $a(\boldsymbol{\rho})$.

LEMMA 7.2. *For $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $\vec{\boldsymbol{\rho}}$ a sequence of non-empty sets of Reeb chords, $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ is strongly boundary monotone (Definition 5.52) if and only if the tensor product*

$$I_A(\mathbf{x}) \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_n)$$

is not zero.

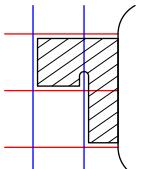
PROOF. In the notation of Definition 5.52, if $I_A(\mathbf{x}) \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_i) \neq 0$ then the right idempotent of $I_A(\mathbf{x}) \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_i)$ is $I(o(o(\mathbf{x}), \vec{\boldsymbol{\rho}}_{[1,i]}))$, i.e.,

$$(I_A(\mathbf{x}) \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_i))I(o(o(\mathbf{x}), \vec{\boldsymbol{\rho}}_{[1,i]})) = I_A(\mathbf{x}) \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_1) \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} a(\boldsymbol{\rho}_i),$$

and then $I(o(o(\mathbf{x}), \vec{\boldsymbol{\rho}}_{[1,i]}))a(\boldsymbol{\rho}_{i+1}) \neq 0$ if and only if $M(\boldsymbol{\rho}_{i+1}^-) \subset o(o(\mathbf{x}), \vec{\boldsymbol{\rho}}_{[1,i]})$ and $M(\boldsymbol{\rho}_{i+1}^+)$ is disjoint from $o(o(\mathbf{x}), \vec{\boldsymbol{\rho}}_{[1,i]}) \setminus M(\boldsymbol{\rho}_{i+1}^-)$, which is exactly the condition of strong boundary monotonicity. \square

As explained in Definition 2.5, an \mathcal{A}_∞ module structure on $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ is a family of maps

$$m_{n+1}: \widehat{CFA}(\mathcal{H}, \mathfrak{s}) \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} \cdots \otimes_{\mathcal{I}} \mathcal{A} \rightarrow \widehat{CFA}(\mathcal{H}, \mathfrak{s})$$



satisfying Equation 2.6. In particular, in view of Lemma 7.2, since the tensor products are over $\mathcal{I}(\mathcal{Z})$, it suffices to define

$$m_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n))$$

when $(\mathbf{x}, \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)$ is strongly boundary monotone. (In the following sections we will use commas to separate the tensor factors in the argument to m_{n+1} and similar maps.)

DEFINITION 7.3. Fix an admissible (in the sense of Definition 5.1), sufficiently generic (in the sense of Definition 5.7) almost-complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. For $(\mathbf{x}, \vec{\boldsymbol{\rho}})$ strongly boundary monotone (and $\boldsymbol{\rho}_i \neq \emptyset$ for each i), define

$$(7.4) \quad m_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \vec{\boldsymbol{\rho}})=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)) \mathbf{y}$$

$$(7.5) \quad m_2(\mathbf{x}, \mathbf{I}) := \mathbf{x}$$

$$(7.6) \quad m_{n+1}(\mathbf{x}, \dots, \mathbf{I}, \dots) := 0, \quad n > 1,$$

where \mathbf{I} is the unit in $\mathcal{A}(\mathcal{Z})$. (Equations (7.5) and (7.6) state that $\widehat{\text{CFA}}(\mathcal{H})$ is strictly unital.)

We will sometimes use the alternate notation $m_1(\mathbf{x}) = \partial(x)$ and $m_2(\mathbf{x}, a) = \mathbf{x} \cdot a$.

$$\text{Let } \widehat{\text{CFA}}(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y)} \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}).$$

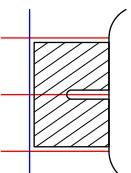
LEMMA 7.7. *If \mathcal{H} is provincially admissible, then for any fixed generators \mathbf{x} and \mathbf{y} and a consistent sequence $\vec{\boldsymbol{\rho}}$ of non-empty sets of Reeb chords, there are at most finitely many $B \in \pi_2(\mathbf{x}, \mathbf{y})$ such that $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\boldsymbol{\rho}})$ is non-empty; in particular, the sums appearing in the definition of m_k are finite. Moreover, if \mathcal{H} is admissible, then there are only finitely many non-zero maps m_k (i.e., in the language of Section 2.1, $(\widehat{\text{CFA}}(\mathcal{H}), \{m_k\})$ is bounded).*

PROOF. The first statement is immediate from Proposition 4.28 and the fact that if a domain admits a holomorphic representative then all of its coefficients must be non-negative (Lemma 5.4). For the second, observe that if we let $|B|$ be the sum of all local multiplicities, then m_{n+1} counts domains with $|B| \geq n$. Thus, Proposition 4.29 gives the desired result. \square

REMARK 7.8. Notice that the operation $m_1: \widehat{\text{CFA}} \rightarrow \widehat{\text{CFA}}$ is the differential obtained by counting only provincial holomorphic curves, i.e., curves which do not approach $\partial\bar{\Sigma}$.

REMARK 7.9. The value of m_{n+1} when one of the arguments is \mathbf{I} can be made more natural by allowing the corresponding $\boldsymbol{\rho}_i$ to be empty (since $a(\emptyset) = 1$). The corresponding moduli spaces should be interpreted as having a choice of height t_i , and no other conditions at t_i . Except in the case of trivial strips, these moduli spaces cannot be 0-dimensional, and hence they do not contribute to m_{n+1} .

REMARK 7.10. In Chapter 8 we will show that for special kinds of diagrams (the obvious analogues of “nice diagrams” from [SW10]) the m_i vanish for $i > 2$, so $\widehat{\text{CFA}}$ is an ordinary (not \mathcal{A}_∞) differential module. However, there are reasons for considering more general diagrams. First, it is not clear how to prove invariance if one restricts only to nice diagrams. Further, in spite of the added analytic and algebraic complications, it is often easier to compute with a diagram which is not nice, as general diagrams often have a much smaller set of generators $\mathfrak{S}(\mathcal{H})$.



7.2. Compatibility with algebra

In this section we will prove that $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ is an \mathcal{A}_∞ module over $\mathcal{A}(\mathcal{Z})$. Recall from Definition 2.5 that the basic relation for an \mathcal{A}_∞ module (over an \mathcal{A}_∞ algebra over \mathbb{F}_2) is

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{i+j=n+1} \sum_{\ell=1}^{n-j} m_i(\mathbf{x}, a_1, \dots, a_{\ell-1}, \mu_j(a_\ell, \dots, a_{\ell+j-1}), \dots, a_{n-1}). \end{aligned}$$

For the algebra $\mathcal{A}(\mathcal{Z})$, the μ_i vanish for $i > 2$. Therefore, we must check that

$$\begin{aligned} (7.11) \quad 0 = & \sum_{i+j=n+1} m_i(m_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-1} m_n(\mathbf{x}, a_1, \dots, \partial a_\ell, \dots, a_{n-1}) \\ & + \sum_{\ell=1}^{n-2} m_{n-1}(\mathbf{x}, a_1, \dots, a_\ell a_{\ell+1}, \dots, a_{n-1}). \end{aligned}$$

PROPOSITION 7.12. *The data $(\widehat{CFA}(\mathcal{H}, \mathfrak{s}), \{m_i\}_{i=1}^\infty)$ forms an \mathcal{A}_∞ module over $\mathcal{A}(\mathcal{Z})$.*

We give the proof shortly. The most interesting points in the proof of Proposition 7.12 can be seen in the following examples. (The reader may wish to compare Examples 7.13–7.15 with Examples 6.13–6.15.)

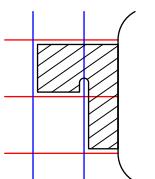
EXAMPLE 7.13. On the left of Figure 7.1 is a piece of a diagram with four generators. The non-trivial multiplications in the corresponding module \widehat{CFA} are

$$\begin{aligned} \partial\{b, d\} &= \{c, e\} \\ \{a, d\} \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \right] &= \{a, c\} \\ \{c, e\} \cdot \left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right] &= \{a, c\} \\ \{b, d\} \cdot \left[\begin{smallmatrix} 2 & 1 \\ 3 & \end{smallmatrix} \right] &= \{a, d\} \\ \{b, d\} \cdot \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix} \right] &= \{a, c\} \end{aligned}$$

There is an \mathcal{A}_∞ compatibility equation

$$(\partial\{b, d\}) \cdot \left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right] + \{b, d\} \cdot \partial \left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right] = 0.$$

(Recall that $\partial \left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix} \right]$.) Geometrically, this corresponds to considering the two-dimensional moduli space of curves from $\{b, d\}$ to $\{a, c\}$ with a Reeb chord from 1 to 3. This moduli space has two ends, one of which is a two-story building, and the other of which is a join curve end.



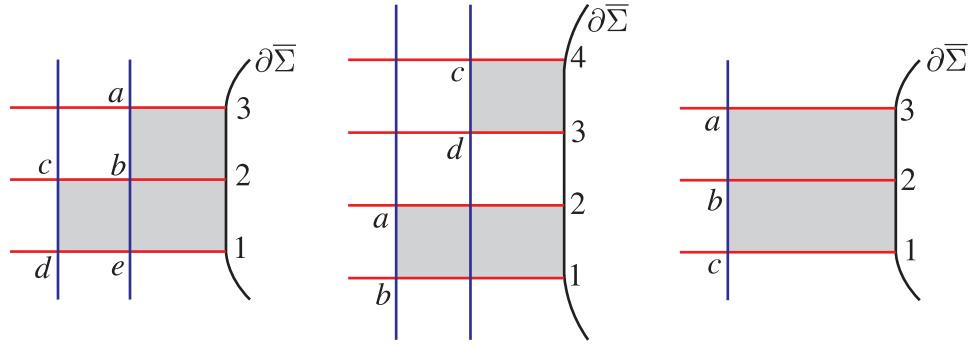


FIGURE 7.1. Local illustrations of terms occurring the \mathcal{A}_∞ relation for \widehat{CFA} . The moduli spaces are the same as the moduli spaces in Figure 6.1 in Section 6.1, but the algebraic interpretations of the phenomena are different for \widehat{CFA} than for \widehat{CFD} .

EXAMPLE 7.14. In the center of Figure 7.1 is a piece of a diagram with four generators. The non-trivial products are

$$\begin{aligned} \{b, d\} \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \right] &= \{a, d\} \\ \{b, c\} \cdot \left[\begin{smallmatrix} 1 & 4 \\ 2 & \end{smallmatrix} \right] &= \{a, c\} \\ \{a, d\} \cdot \left[\begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix} \right] &= \{a, c\} \\ \{b, d\} \cdot \left[\begin{smallmatrix} 3 & 1 \\ 4 & \end{smallmatrix} \right] &= \{b, c\} \\ \{b, d\} \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix} \right] &= \{a, c\}. \end{aligned}$$

Here, associativity follows from the fact that

$$(\{b, d\} \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \right]) \cdot \left[\begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix} \right] = \{b, d\} \cdot \left[\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix} \right] = (\{b, d\} \cdot \left[\begin{smallmatrix} 3 & 1 \\ 4 & \end{smallmatrix} \right]) \cdot \left[\begin{smallmatrix} 1 & 4 \\ 2 & \end{smallmatrix} \right].$$

The second equality above can be seen geometrically as follows. Consider the two-dimensional moduli space connecting $\{b, d\}$ to $\{a, c\}$ where the Reeb chord from 3 to 4 is encountered before the one from 1 to 2. Then this moduli space has two ends, one of which is a two story building, while at the other end the heights of two Reeb chords are the same.

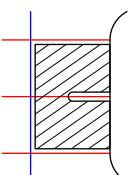
EXAMPLE 7.15. On the right in Figure 7.1 is a piece of a diagram with three generators. The non-trivial products are

$$\begin{aligned} \{c\} \cdot \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right] &= \{b\} \\ \{c\} \cdot \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \right] &= \{a\} \\ \{b\} \cdot \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right] &= \{a\}. \end{aligned}$$

Associativity follows from the fact that

$$(\{c\} \cdot \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right]) \cdot \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right] = \{c\} \cdot \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \right].$$

Geometrically, at one end of the moduli space is a two story building. At the other end, two levels collide and a split curve degenerates.



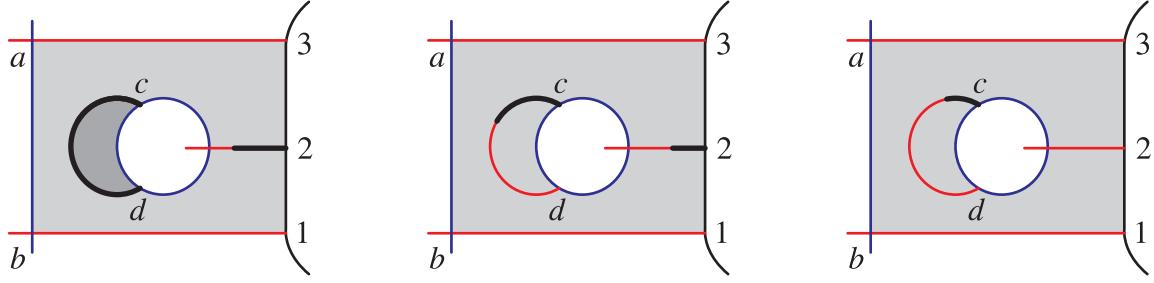


FIGURE 7.2. **A domain where m_3 is necessary for associativity.** The domain is shaded; the dark arcs denote cuts, whose ends are boundary branched points of $\pi_\Sigma \circ u$. A typical curve in the one-parameter family is indicated in the center. On the left is a two-story end of the moduli space. On the right is the other end, corresponding to degenerating a split curve.

EXAMPLE 7.16. Figure 7.2 shows an example in which the higher product m_3 is needed for \mathcal{A}_∞ associativity. The diagram has four generators, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$ and $\{b, d\}$. The differential is given by $\partial\{a, c\} = \{a, d\}$ and $\partial\{b, c\} = \{b, d\}$. For an appropriate choice of complex structure J , there is a non-trivial product (m_2) given by $\{b, c\} \cdot [\frac{1}{3}] = \{a, c\}$; this corresponds to the domain and branch cuts indicated in the right of the figure. One can factor $[\frac{1}{3}] = [\frac{1}{2}] \cdot [\frac{2}{3}]$. Obviously $\{b, c\} \cdot [\frac{1}{2}] = 0$, so ordinary associativity fails. There is, however, for the same choice of complex structure J as above, a higher product

$$m_3 (\{b, d\}, [\frac{1}{2}], [\frac{2}{3}]) = \{a, c\},$$

with domain given by the lighter-colored region in the middle of Figure 7.2. Thus, the \mathcal{A}_∞ associativity condition is satisfied:

$$m_3 ((\partial\{b, c\}), [\frac{1}{2}], [\frac{2}{3}]) + (\{b, c\} \cdot [\frac{1}{2}]) \cdot [\frac{2}{3}] + \{b, c\} \cdot [\frac{1}{3}] = 0.$$

Geometrically, this corresponds to the one-parameter moduli space of curves asymptotic to two Reeb chords (at different heights), shown at the left of Figure 7.2. One end (pictured in the center) is a two-story building where one of the stories contains both Reeb chords. At the other end (pictured at the right), the two Reeb chords collide and a split curve degenerates.

PROOF OF PROPOSITION 7.12. Morally, the proof proceeds by considering the ends of the index 2 moduli spaces. As in the type D module case, however, we will instead appeal to Theorem 5.61.

Fix generators \mathbf{x} and \mathbf{y} , a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and a sequence of non-empty sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_n)$ so that $(\mathbf{x}, \vec{\rho})$ is strongly boundary monotone, B is compatible with $\vec{\rho}$, and $\text{ind}(B, \vec{\rho}) = 2$. Set $\rho_i = \{\rho_{i,j}\}$. Theorem 5.61 applied to the data $(\mathbf{x}, \mathbf{y}, B, \vec{\rho})$ (for all sources S^\triangleright with $\chi(S^\triangleright) = \chi_{\text{emb}}(B, \vec{\rho})$) shows that the sum of the following numbers is zero:

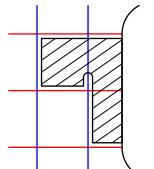
- (1) The number of two-story ends, i.e., the number of elements of

$$\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\rho}_1) \times \mathcal{M}^{B_2}(\mathbf{w}, \mathbf{y}; \vec{\rho}_2)$$

where $B = B_1 * B_2$ and $\vec{\rho} = (\vec{\rho}_1, \vec{\rho}_2)$.

- (2) The number of join curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_1, \dots, \rho_i^{a,b}, \dots, \rho_n))$$



where $\rho_{i,j} = \rho_a \uplus \rho_b$ and $\rho_i^{a,b}$ is obtained from ρ_i by replacing $\rho_{i,j}$ with $\{\rho_a, \rho_b\}$.

- (3) The number of odd shuffle curve ends, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}')$$

where $\vec{\rho}'$ is obtained from $\vec{\rho}$ by performing a weak shuffle on one ρ_i .

- (4) The number of collisions of levels, i.e., the number of elements of

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \rho_1, \dots, \rho_i \uplus \rho_{i+1}, \dots, \rho_n)$$

where ρ_i and ρ_{i+1} are weakly composable.

(We are also using Lemma 5.76 to ensure that the limiting curves are embedded.)

The two-story ends correspond to the first term in Formula (7.11).

Next we discuss the second term in Formula (7.11). Lemma 3.14 implies that, for a consistent set ρ of Reeb chords,

$$\partial a(\rho) = \sum_{\substack{\rho' \text{ a splitting} \\ \text{of } \rho}} a(\rho') + \sum_{\substack{\rho' \text{ a shuffle} \\ \text{of } \rho}} a(\rho').$$

On the other hand, sums (2) and (3) above correspond, respectively, to the sum over all weak splittings and weak shuffles one of the sets of Reeb chords in $\vec{\rho}$. By Lemma 5.77, only honest (not just weak) splittings or shuffles contribute, and therefore the second term in Formula (7.11) corresponds to sums (2) and (3) above.

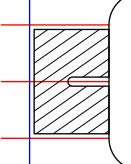
Finally, sum (4) corresponds to the third term in Formula (7.11), as follows. Since, by Lemma 3.12, $a(\rho_i)a(\rho_{i+1})$ is $a(\rho_i \uplus \rho_{i+1})$ if ρ_i and ρ_{i+1} are composable, and is zero otherwise, the coefficient of \mathbf{y} in $m_{n-1}(\mathbf{x}, a(\rho_1), \dots, a(\rho_i)a(\rho_{i+1}), \dots, a(\rho_n))$ is equal to sum (4) if ρ_i and ρ_{i+1} are composable, and is 0 otherwise. But by Lemma 5.77, sum (4) is also 0 if ρ_i and ρ_{i+1} are not composable, and otherwise gives the third term of Formula (7.11). \square

7.3. Invariance

THEOREM 7.17. *Up to \mathcal{A}_∞ homotopy equivalence, the \mathcal{A}_∞ module $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ is independent of the almost complex structure used to define it, and depends on the provincially admissible Heegaard diagram \mathcal{H} with $\mathcal{Z} = \partial\mathcal{H}$ only through its induced equivalence class of bordered three-manifold $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$. That is, if \mathcal{H} and \mathcal{H}' are provincially admissible bordered Heegaard diagram for $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$ and \mathfrak{s} is a spin^c structure on Y then $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ and $\widehat{CFA}(\mathcal{H}', \mathfrak{s})$ are \mathcal{A}_∞ homotopy equivalent $\mathcal{A}_\infty(\mathcal{Z})$ -modules.*

As for invariance of \widehat{CFD} , the proof of Theorem 7.17 differs from the standard arguments in essentially two ways: the additional algebraic complications in dealing with \mathcal{A}_∞ maps and homotopies, and the additional analytic complications in the case of handleslide invariance because of the presence of east ∞ . Both of these issues are slightly more involved for \widehat{CFA} than for \widehat{CFD} . We will take a similar approach as for \widehat{CFD} : first we will explain succinctly the proof of invariance under change in almost complex structure, and then we will discuss the (mild) new complications in the triangle maps used to prove handleslide invariance.

7.3.1. Change of complex structure. In this section, we address invariance of \widehat{CFA} under change in almost complex structure. In the course of this analysis, we will relate the algebra of \mathcal{A}_∞ maps and their \mathcal{A}_∞ homotopies to holomorphic curve counts; because



this is somewhat more involved than the algebra of differential graded algebras required in Section 6.3.1, we will explain more steps in the proof.

Fix a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ and sufficiently generic, admissible almost complex structures J_0 and J_1 . The complex structures J_0 and J_1 lead to potentially different moduli spaces, and hence to two potentially different \mathcal{A}_∞ modules $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0)$ and $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$. We will show that these two \mathcal{A}_∞ modules are \mathcal{A}_∞ homotopy equivalent.

As in Section 6.3.1, call a (smooth) path $\{J_r \mid r \in [0, 1]\}$ of almost complex structures from J_0 to J_1 *admissible* if each J_r is admissible. To the path J_r we can associate a single almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ by Formula (6.17), and define embedded moduli spaces $\mathcal{M}(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)$ by Formula (6.18).

Recall from Definition 2.9 that an \mathcal{A}_∞ morphism between \mathcal{A}_∞ modules \mathcal{M}_1 and \mathcal{M}_2 is a compatible family of maps

$$f_n: M_1 \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{n-1} \rightarrow M_2,$$

for $n \in \mathbb{N}$. So define an \mathcal{A}_∞ map $f^{J_r} = \{f_n^{J_r}\}_{n=1}^\infty$ from $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0)$ to $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$ by setting

$$\begin{aligned} f_n^{J_r}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{n-1})) &:= \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \vec{\rho})=0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1}; J)) \mathbf{y} \\ f_n^{J_r}(\mathbf{x}, \dots, \mathbf{I}, \dots) &:= 0 \end{aligned}$$

and extending f^{J_r} to all of $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0) \otimes \mathcal{A}^{\otimes(n-1)}$ multilinearly over $\mathcal{I}(\mathcal{Z})$. The above map is defined over all of $\widehat{\text{CFA}}(\mathcal{H}; J_0)$; but, in view of Lemma 4.21, it clearly maps $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0)$ to $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$.

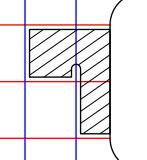
Our first task is to show that f^{J_r} is an \mathcal{A}_∞ morphism. We again use Proposition 6.20.

LEMMA 7.18. *The map $f^{J_r} = \{f_n^{J_r}\}_{n \in \mathbb{N}}: \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0) \rightarrow \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$ is a strictly unital morphism of \mathcal{A}_∞ modules, as defined in Definition 2.9.*

PROOF. The map f^{J_r} is strictly unital by definition. It remains to check the \mathcal{A}_∞ relation for morphisms, which we recall from Definition 2.9 (specialized for $\mu_i = 0$ for $i > 2$):

$$\begin{aligned} 0 &= \sum_{i+j=n+1} m_i^{J_1}(f_j^{J_r}(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ &\quad + \sum_{i+j=n+1} f_i^{J_r}(m_j^{J_0}(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ &\quad + \sum_{\ell=1}^{n-1} f_n^{J_r}(\mathbf{x}, a_1, \dots, a_{\ell-1}, \partial a_\ell, a_{\ell+1}, \dots, a_{n-1}) \\ &\quad + \sum_{\ell=1}^{n-2} f_{n-1}^{J_r}(\mathbf{x}, a_1, \dots, a_{\ell-1}, a_\ell a_{\ell+1}, a_{\ell+2}, \dots, a_{n-1}). \end{aligned}$$

This follows in a similar manner to Proposition 7.12. The first term corresponds to the two-story ends with a J -holomorphic curve followed by a J_1 -holomorphic curve. The second term corresponds to two-story ends with a J_0 -holomorphic curve followed by a J -holomorphic



curve. The third term corresponds to join curve ends and odd shuffle curve ends. The fourth term corresponds to collisions of two levels. \square

Similarly, fix a generic path J'_r of admissible almost complex structures with $J'_0 = J_1$ and $J'_1 = J_0$. Let J' denote the associated complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. Then J'_r defines an \mathcal{A}_∞ morphism $f^{J'_r} : \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1) \rightarrow \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0)$. We will show:

PROPOSITION 7.19. *The maps f^{J_r} and $f^{J'_r}$ are homotopy inverses to each other. In particular, $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0)$ is \mathcal{A}_∞ homotopy equivalent to $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$.*

The proof is again essentially standard, but as the algebra may be unfamiliar we will give a few details.

For $R \gg 0$, let $J \natural_R J'$ denote the almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ given by

$$J \natural_R J' |_{(x,s,t)} := \begin{cases} J_0|_{(x,s,t)} & \text{if } R+1 \leq t \\ J'_{t-R}|_{(x,s,t-R)} & \text{if } R \leq t \leq R+1 \\ J_1|_{(x,s,t)} & \text{if } 1 \leq t \leq R \\ J_t|_{(x,s,t)} & \text{if } 0 \leq t \leq 1 \\ J_0|_{(x,s,t)} & \text{if } t \leq 0. \end{cases}$$

Associated to $J \natural_R J'$ is an \mathcal{A}_∞ morphism $f^{J \natural_R J'} : \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_0) \rightarrow \widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}; J_1)$.

LEMMA 7.20. *For R sufficiently large the map $f^{J \natural_R J'}$ is the \mathcal{A}_∞ composite $f^{J'_r} \circ f^{J_r}$.*

PROOF. Recall from Section 2.1 that $f^{J'_r} \circ f^{J_r}$ has

$$(f^{J'_r} \circ f^{J_r})_n(\mathbf{x}, a_1, \dots, a_{n-1}) = \sum_{i+j=n+1} f_j^{J'_r}(f_i^{J_r}(\mathbf{x}, a_1, \dots, a_{i-1}), \dots, a_{n-1}).$$

The coefficient of \mathbf{y} in

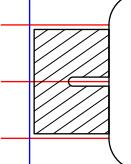
$$f_j^{J'_r}(f_i^{J_r}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{i-1})), \dots, a(\boldsymbol{\rho}_{n-1}))$$

counts holomorphic curves in $\mathcal{M}^{B_1}(\mathbf{x}, \mathbf{w}; \vec{\boldsymbol{\rho}}_{[1,i-1]}; J) \times \mathcal{M}^{B_2}(\mathbf{x}, \mathbf{w}; \vec{\boldsymbol{\rho}}_{[i,n-1]}; J')$ where $\mathbf{w} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$, $B_1 \in \pi_2(\mathbf{x}, \mathbf{w})$, $B_2 \in \pi_2(\mathbf{w}, \mathbf{x})$, $\text{ind}(B_1, \vec{\boldsymbol{\rho}}_{[1,i-1]}) = 0$ and $\text{ind}(B_2, \vec{\boldsymbol{\rho}}_{[i,n-1]}) = 0$. It follows from obvious non- \mathbb{R} -invariant analogues of Propositions 5.24 and 5.30 (compactness and gluing) that for R large enough such curves correspond to curves in

$$\bigcup_{B \in \pi_2(\mathbf{x}, \mathbf{y})} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1}; J \natural_R J').$$

The result follows. \square

We next define an \mathcal{A}_∞ homotopy h between $f^{J \natural_R J'}$ and the identity map. Fix a path of almost complex structures $\{J^w\}_{w \in [0,1]}$ on $\Sigma \times [0, 1] \times \mathbb{R}$ with $J^0 = J \natural_R J'$ and $J^1 = J_0$, and such that each J^w satisfies conditions (J-1), (J-2) and (J-4) of Definition 5.1—i.e., J^w is admissible except for not being \mathbb{R} -invariant. For a generic path $\{J^w\}_{w \in [0,1]}$ and any B, S^\triangleright, P with $\text{ind}(B, S^\triangleright, P) = -1$, there are finitely many $w \in (0, 1)$ for which $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P; J^w)$



is non-empty. So set

$$\begin{aligned}\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \{J^w\}) &:= \bigcup_{w \in [0,1]} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J^w) \\ h_n(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{n-1})) &:= \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \vec{\rho}) = -1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \{J^w\})) \mathbf{y}.\end{aligned}$$

In this context, the analogue of Theorem 5.61 is:

PROPOSITION 7.21. *Suppose that $(\mathbf{x}, \vec{\rho})$ satisfies the strong boundary monotonicity condition. Fix \mathbf{y} , $B \in \pi_2(\mathbf{x}, \mathbf{y})$, S^\triangleright , and \vec{P} such that both $[P] = \vec{\rho}$ and $\text{ind}(B, S^\triangleright, \vec{P}) = 0$. Then, for a generic path $\{J^w\}_{w \in [0,1]}$, in the moduli space $\bigcup_{w \in [0,1]} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^w)$, the total number of all*

- (1) elements of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^0)$;
- (2) elements of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}; J^1)$;
- (3) two-story ends, with either
 - (a) a J_0 -holomorphic curve followed by a J^w -holomorphic curve or
 - (b) a J^w -holomorphic curve followed by a J_0 -holomorphic curve;
- (4) join curve ends;
- (5) odd shuffle curve ends; and
- (6) collisions of two levels P_i and P_{i+1} from \vec{P} , where $[P_i]$ and $[P_{i+1}]$ are composable,

is even.

LEMMA 7.22. *The map h is an \mathcal{A}_∞ homotopy between $f^{J \natural_R J'}$ and the identity map on $\widehat{CFA}(\mathcal{H}, J_0)$.*

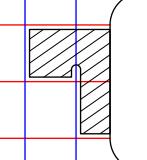
PROOF. We must check that

$$(7.23) \quad f_1^{J \natural_R J'}(\mathbf{x}) - \mathbf{x} = h_1(m_1^{J_0}(\mathbf{x})) - m_1^{J_0}(h_1(\mathbf{x}))$$

and that for $n > 1$,

$$\begin{aligned}(7.24) \quad f_n^{J \natural_R J'}(\mathbf{x}, a_1, \dots, a_{n-1}) &= \sum_{i+j=n+1} h_i(m_j^{J_0}(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ &\quad + \sum_{i+j=n+1} m_i^{J_0}(h_j(\mathbf{x}, a_1, \dots, a_{j-1}), \dots, a_{n-1}) \\ &\quad + \sum_{\ell=1}^{n-1} h_n(\mathbf{x}, a_1, \dots, \partial a_\ell, \dots, a_{n-1}) \\ &\quad + \sum_{\ell=1}^{n-2} h_{n-1}(\mathbf{x}, a_1, \dots, a_\ell a_{\ell+1}, \dots, a_{n-1}).\end{aligned}$$

As expected, this follows from Proposition 7.21. The $f^{J \natural_R J'}$ terms in Formulas (7.23) and (7.24) correspond to item (1) of Proposition 7.21. The $-\mathbf{x}$ term of Formula (7.23) corresponds to item (2), since $J^1 = J_0$ and for an \mathbb{R} -invariant complex structure the only index 0 holomorphic curves are trivial strips. As usual, two-story ends correspond to the remaining two terms in Formula (7.23) as well as the first two sums in Formula (7.24). The



join curve and shuffle curve ends account for the third sum in Formula (7.24). Collisions of two levels account for the final sum in Formula (7.24). \square

PROOF OF PROPOSITION 7.19. This is immediate from Lemmas 7.20 and 7.22, together with the obvious analogues for the composition $f^{J_r} \circ f^{J'_r}$. \square

7.3.2. Handlesliding an α -arc over an α -circle. The purpose of the present subsection is to prove handleslide invariance:

PROPOSITION 7.25. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a bordered Heegaard diagram and $(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}^H, z)$ the result of performing a handleslide of one of the α -curves over an α -circle or one β -curve over another β -circle. Then there is a strictly unital \mathcal{A}_∞ homotopy equivalence between $\widehat{CFA}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $\widehat{CFA}(\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$.*

The most subtle case of handleslide invariance is handlesliding an α -arc over an α -circle; we focus on this case. We will be brief, as most of the argument is the same as the type D case, in Section 6.3.2.

Fix a Heegaard triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta})$ as in Section 6.3.2. There are associated moduli spaces of holomorphic curves in $\Sigma \times \Delta$

$$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n) := \bigcup_{i=0}^n \mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_{o_i}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_i), (\boldsymbol{\rho}_{i+1}^H, \dots, \boldsymbol{\rho}_n^H)) \mathbf{y}.$$

Define the \mathcal{A}_∞ morphism $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}} = \{f_n^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}\}_{n=1}^\infty$ associated to the handleslide by

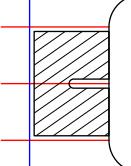
$$(7.26) \quad f_n^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_{n-1})) := \sum_{\mathbf{y}} \sum_{\substack{B \in \pi_2(\mathbf{x}', \mathbf{y}) \\ \text{ind}(B, \vec{\rho})=0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{n-1})) \mathbf{y}$$

and extending to all of $\widehat{CFA}(\mathcal{H}, \mathfrak{s}, J_0)$ by multi-linearity over $\mathcal{I}(\mathcal{Z})$.

PROOF OF PROPOSITION 7.25. This follows along the same lines as the proof of Proposition 6.45. That $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is a strictly unital \mathcal{A}_∞ morphism follows from Propositions 6.37 and 6.39, similarly to how Proposition 7.12 follows from Theorem 5.61. To prove that $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is invertible we consider its composition with another triangle map, $f^{\boldsymbol{\alpha}^H, \boldsymbol{\alpha}', \boldsymbol{\beta}}$, where $\boldsymbol{\alpha}'$ is a small Hamiltonian perturbation of $\boldsymbol{\alpha}$, as in Section 6.3.2.6. Associativity for holomorphic triangle maps, the type A analogue of Proposition 6.44, shows that $f^{\boldsymbol{\alpha}^H, \boldsymbol{\alpha}', \boldsymbol{\beta}} \circ f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is homotopic to $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\beta}}$. It follows from Lemma 6.42 that $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\beta}}$ is an isomorphism (noting that an \mathcal{A}_∞ homomorphism $\{f_i\}$ is an isomorphism if and only if f_1 is an isomorphism). Thus, $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is injective on homology (or, better, monic in the homotopy category). That $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ is surjective on homology (or, better, epic in the homotopy category) follows from a similar argument using a fourth set of circles $\boldsymbol{\alpha}^{H'}$, as in the proof of Proposition 6.44. \square

7.3.3. Completion of the invariance proof.

PROOF OF THEOREM 7.17. Independence from the complex structure was verified in Section 7.3.1. Handleslide invariance was outlined in Section 7.3.2. Stabilization invariance is obvious, by stabilizing in the region containing the basepoint. Isotopy invariance is left to the reader. In light of Proposition 4.10 this verifies topological invariance. \square



In view of Theorem 7.17, if $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$ is a bordered three-manifold and \mathfrak{s} is a spin^c structure over Y , define $\widehat{\text{CFA}}(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$ to be $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, where \mathcal{H} is any bordered diagram representing the bordered three-manifold. Let

$$\widehat{\text{CFA}}(Y) := \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y)} \widehat{\text{CFA}}(Y, \mathfrak{s}).$$

PROOF OF THEOREM 1.2. This follows at once from Theorem 7.17. \square

7.4. Twisted coefficients

We can also define a twisted theory $\widehat{\text{CFA}}(\mathcal{H}, \mathbf{x}_0)$, for \mathbf{x}_0 a generator representing the $\mathfrak{s} \in \text{spin}^c(Y)$, in an analogous way to the definition of $\widehat{\text{CFD}}(\mathcal{H}, \mathbf{x}_0)$ in Section 6.4.

Consider the generating set $\underline{\mathfrak{S}}(\mathcal{H}, \mathbf{x}_0)$, consisting of elements of the form $e^{B_0}\mathbf{x}$, where $\mathbf{x} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$. $\widehat{\text{CFA}}(\mathcal{H}, \mathbf{x}_0)$ is the \mathbb{F}_2 vector space spanned by $\underline{\mathfrak{S}}(\mathcal{H}, \mathfrak{s})$. It is given an action of $\mathcal{I}(\mathcal{Z})$ generalizing Equation (7.1), via

$$I(\mathbf{s}) \cdot e^{B_0}\mathbf{x} := \begin{cases} \mathbf{x} & I(\mathbf{s}) = I_A(\mathbf{x}) \\ 0 & \text{otherwise} \end{cases}$$

There is an action of $H_2(Y, \partial Y)$ on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, by composition with the corresponding periodic domain: For $\alpha \in H_2(Y, \partial Y)$ corresponding to $B \in \pi_2(\mathbf{x}_0, \mathbf{x}_0)$, define

$$\alpha * e^{B_0}\mathbf{x} := e^{B * B_0}\mathbf{x}.$$

We can view this as a left action by the group ring $\mathbb{F}_2[H^2(Y, \partial Y)]$.

The \mathcal{A}_∞ action by $\mathcal{A}(\mathcal{Z})$ is specified by:

$$\begin{aligned} m_{n+1}(e^{B_0}\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)) &:= \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B, \boldsymbol{\rho})=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)) e^{B_0 * B} \cdot \mathbf{y} \\ m_2(e^{B_0}\mathbf{x}, \mathbf{I}) &:= \mathbf{x} \\ m_{n+1}(e^{B_0}\mathbf{x}, \dots, \mathbf{I}, \dots) &:= 0, \quad n > 1, \end{aligned}$$

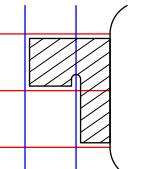
where here \mathbf{I} is the unit in $\mathcal{A}(\mathcal{Z})$. Extend this to an \mathcal{A}_∞ bimodule action of $\mathbb{F}_2[H^2(Y, \partial Y)]$ and $\mathcal{A}(\mathcal{Z})$ by setting

$$\begin{aligned} m_{1,1,0}(\alpha, e^{B_0}\mathbf{x}) &:= \alpha * e^{B_0}\mathbf{x} \\ m_{m,1,n}(\alpha_1, \dots, \alpha_m, e^{B_0}\mathbf{x}, a_1, \dots, a_n) &:= 0 \quad \text{if } nm > 0 \text{ or } m > 1. \end{aligned}$$

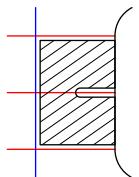
PROPOSITION 7.27. *The data $(\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s}), \{m_i\}_{i=1}^\infty)$ forms an \mathcal{A}_∞ bimodule over the rings $\mathbb{F}_2[H_2(Y, \partial Y)]$ and $\mathcal{A}(\mathcal{Z})$.*

PROOF. This is a simple adaptation of the proof of Proposition 7.12. \square

THEOREM 7.28. *Up to \mathcal{A}_∞ homotopy equivalence, the \mathcal{A}_∞ bimodule $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ over $\mathbb{F}_2[H_2(Y, \partial Y)]$ and $\mathcal{A}(\mathcal{Z})$ is independent of the almost complex structure used to define it, and depends on the Heegaard diagram \mathcal{H} only through its (equivalence class of) underlying bordered three-manifold $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$ and on the generator \mathbf{x}_0 only through its induced spin^c structure $\mathfrak{s}_z(\mathbf{x}_0)$.*



PROOF. The proof of Theorem 7.17 can be adapted easily to the twisted context, following the model of Theorem 6.47. Note once again that all the maps commute with the $\mathbb{F}_2[H_2(Y, \partial)]$ -action (rather than merely commuting up to homotopy). \square



CHAPTER 8

Pairing theorem via nice diagrams

We now give a proof of the pairing theorem (Theorem 1.3), adapting methods from Sarkar and Wang [SW10].

DEFINITION 8.1. A bordered Heegaard diagram $(\Sigma, \alpha, \beta, z)$ is called *nice* if every region in $\bar{\Sigma}$ except for the region D_z adjacent to z is a (topological) disk with at most 4 corners. That is, each region in the interior of Σ is either a bigon or a quadrilateral and any region at the boundary of Σ except for D_z has two α -arcs, one β -arc and one arc of $\partial\bar{\Sigma}$ in its boundary.

PROPOSITION 8.2. *Any bordered Heegaard diagram $(\Sigma, \alpha, \beta, z)$ can be turned into a nice bordered Heegaard diagram $(\Sigma, \alpha, \beta', z)$ via*

- *isotopies of the β_i not crossing $\partial\Sigma$ and*
- *handleslides among the β_i .*

PROOF. We start by doing a finger move as in Figure 8.1. That is, let a_1, \dots, a_{4k} denote the ends of the α -arcs intersecting $\partial\Sigma$, enumerated (say) counter-clockwise around $\partial\Sigma$, with a_1 and a_{4k} forming part of ∂D_z . Let b be the β -arc on the boundary of D_z which intersects a_1 closest to $\partial\bar{\Sigma}$. Do a finger move pushing b consecutively through a_1, \dots, a_{4k} . In the resulting diagram, the regions R_1, \dots, R_{4k-1} adjacent to $\partial\bar{\Sigma}$ other than D_z are all rectangles.

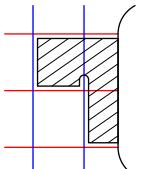
Now apply the Sarkar-Wang algorithm [SW10], which we recall briefly, to the resulting diagram. Assign to each region R its *distance*, which is the minimum number of β -arcs which a path in $\Sigma \setminus \alpha$ connecting R and D_z must cross. A region is called *bad* if it is not a bigon or rectangle. The algorithm is inductive. At each step, choose a bad region R of maximal distance and with minimal badness (i.e., maximal Euler measure) among such regions. Let b_R be a β -arc separating R from a region of smaller distance. Do a finger move, pushing b_R through one of the α -edges of R , and continue the finger move until reaching either a lower distance region, a bigon, another bad region, or the region R again. In the last case, the algorithm tells us to either choose a different edge through which to push b_R or perform an appropriate handleslide; otherwise, one repeats the process. The crucial point for us is that the finger move will never reach any of the regions R_1, \dots, R_{2n-1} . The proof in the closed case thus implies that the process eventually yields a nice bordered Heegaard diagram. \square

Nice diagrams are automatically admissible:

LEMMA 8.3. *If $\mathcal{H} = (\Sigma, \alpha, \beta, z)$ is nice then \mathcal{H} is admissible.*

PROOF. The proof is exactly the same as in the closed case ([LMW08, Corollary 3.2]). One shows that the region containing z occurs both to the left and to the right of each β -circle: if all the regions to the left (say) of some β_i are bigons or rectangles then one sees that the β -circles are homologically linearly dependent. \square

Sarkar and Wang apply the index formula to show that for nice Heegaard diagrams (in the closed case), the differential counts only bigons and rectangles. This fact has the following analogue for nice bordered Heegaard diagrams.



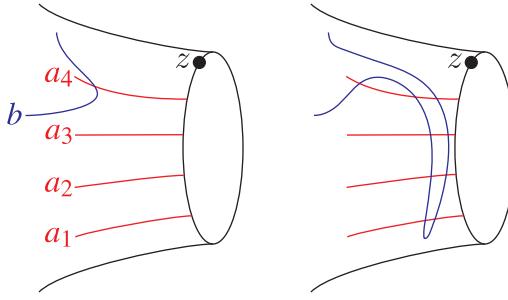


FIGURE 8.1. First step in making a Heegaard diagram with boundary nice: perform a finger-move to separate off the boundary.

PROPOSITION 8.4. Let \mathcal{H} be a nice bordered Heegaard diagram. Suppose that $\text{ind}(B, \vec{\rho}) = 1$.

1. Then any holomorphic curve $u \in \mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ has one of the following three forms:
 - (1) The source S of u consists of g bigons. Each bigon is either a trivial strip or has a unique puncture mapped to east ∞ . Further, $\vec{\rho}$ has only one part.
 - (2) The source S consists of g bigons. All but one of the bigons are trivial strips, and the other is mapped by $\pi_\Sigma \circ u$ to the interior of Σ .
 - (3) The source S consists of $g - 1$ bigons and one quadrilateral. All of the bigons are trivial strips, and the quadrilateral is mapped by $\pi_\Sigma \circ u$ to the interior of Σ .

Conversely, suppose the homology class B and asymptotic data $\vec{\rho}$ admit a topological map satisfying the condition of Case (1), (2), or (3), and $\text{ind}(B, \vec{\rho}) = 1$. Then there is a holomorphic map in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$, unique up to translation.

PROOF. By Formula (5.9), for a rigid holomorphic curve we have

$$1 = g - \chi(S) + 2e(D(u)) + |P|.$$

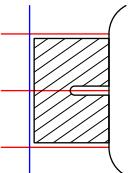
Since S has at most g connected components, $\chi(S) \leq g$. Since the Heegaard diagram is nice, $e(D(u)) \geq 0$. Consequently, one of the following three cases holds:

- $g = \chi(S)$, $e(D(u)) = 0$ and $|P| = 1$. This corresponds to Case 1, above.
- $g = \chi(S)$, $e(D(u)) = 1/2$ and $|P| = 0$. This corresponds to Case 2, above.
- $g - 1 = \chi(S)$, $e(D(u)) = 0$ and $|P| = 0$. This corresponds to Case 3, above.

It is clear that in each case the holomorphic representative is unique. The converse, that such domains admit holomorphic representatives, follows from the Riemann mapping theorem. \square

REMARK 8.5. In the closed case, Sarkar and Wang show that for a rigid holomorphic curve u in a nice diagram, the image of $\pi_\Sigma \circ u$ is an empty, embedded bigon or rectangle [SW10, Theorem 3.3]. The analogue in the bordered case is that the domain of a rigid curve is an empty embedded bigon, rectangle or half strip (rather than, say, merely immersed). This follows immediately from the Proposition 8.4 and the result in the closed case, by doubling the Heegaard diagram along the boundary and considering the doubled domain.

We do not need this stronger result. Proposition 8.4 already gives an algorithm for counting holomorphic curves in a nice diagram: one checks which domains have index 1, and are represented by orientation-preserving maps of a bigon, rectangle or half-strip. (This is



analogous to the approach taken in [LMW08] for holomorphic triangles.) Proposition 8.4 also suffices to prove the pairing theorem, our main goal.

COROLLARY 8.6. *Let \mathcal{H} be a nice Heegaard diagram with boundary, and $\widehat{\text{CFA}} = \widehat{\text{CFA}}(\mathcal{H})$ the associated type A module. Then the higher \mathcal{A}_∞ operations m_i ($i > 2$) vanish on $\widehat{\text{CFA}}$.*

PROOF. The higher multiplications m_i ($i > 2$) count rigid holomorphic curves in moduli spaces $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ with $|\vec{\rho}| > 1$. But by Proposition 8.4, in a nice diagram there are no such curves. \square

As a consequence of these remarks, we have a quick proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Let $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z)$ and $\mathcal{H}_2 = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z)$ be nice Heegaard diagrams with boundary for Y_1 and Y_2 respectively. Their union $\mathcal{H} = (\overline{\Sigma}_1 \cup_\partial \overline{\Sigma}_2, \overline{\boldsymbol{\alpha}}_1 \cup_\partial \overline{\boldsymbol{\alpha}}_2, \overline{\boldsymbol{\beta}}_1 \cup \overline{\boldsymbol{\beta}}_2, z)$ is a Heegaard diagram for $Y = Y_1 \cup_F Y_2$. We want to show that $\widehat{\text{CF}}(\mathcal{H}) \cong \widehat{\text{CFA}}(\mathcal{H}_1) \widetilde{\otimes} \widehat{\text{CFD}}(\mathcal{H}_2)$.

It is easy to see that \mathcal{H} is a nice diagram, since \mathcal{H}_1 and \mathcal{H}_2 are. Sarkar and Wang [SW10] showed that the differential for $\widehat{\text{CF}}(\mathcal{H})$ counts index 1 rectangles and bigons. (This is also a special case of Proposition 8.4.) We compare this complex with $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ from Section 2.4, which by Proposition 2.34 serves as a model for the derived tensor product $\widetilde{\otimes}$. (Note, in particular, that since \mathcal{H}_2 is admissible, $\widehat{\text{CFD}}(\mathcal{H}_2)$ is bounded.)

There is an obvious identification of generators of $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ and $\widehat{\text{CF}}(\mathcal{H})$. Let $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$. (Recall that the tensor products \otimes are over the ground ring $\mathbf{k} = \mathcal{I}(\mathcal{Z})$, the subring of idempotents in $\mathcal{A}(\mathcal{Z})$.)

By Proposition 8.4, the differential $\partial(\mathbf{x}_2)$ can be written as $\sum_{\mathbf{y}_2} a_{\mathbf{x}_2, \mathbf{y}_2} \otimes \mathbf{y}_2$, where each term in $a_{\mathbf{x}_2, \mathbf{y}_2}$ is either 1—coming from a provincial rectangle or bigon—or an algebra element in which a single strand moves. Applying Corollary 8.6, the differential in $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ is given by

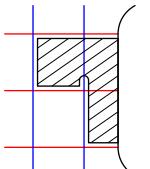
$$\partial^\boxtimes(\mathbf{x}_1 \otimes \mathbf{x}_2) = (m_1(\mathbf{x}_1)) \otimes \mathbf{x}_2 + \sum_{\mathbf{y}_2} m_2(\mathbf{x}_1, a_{\mathbf{x}_2, \mathbf{y}_2}) \otimes \mathbf{y}_2.$$

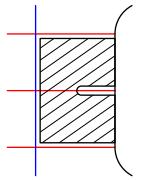
The first term in the sum counts holomorphic bigons and rectangles contained entirely in $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z)$. The second sum counts both rectangles and bigons contained entirely in $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z)$ (corresponding to terms where $a_{\mathbf{x}_2, \mathbf{y}_2}$ is 1) and rectangles in $(\overline{\Sigma}_1 \cup_\partial \overline{\Sigma}_2, \overline{\boldsymbol{\alpha}}_1 \cup \overline{\boldsymbol{\alpha}}_2, \overline{\boldsymbol{\beta}}_1 \cup \overline{\boldsymbol{\beta}}_2, z)$ crossing $\partial\overline{\Sigma}_1$ (corresponding to terms with a single moving strand in $a_{\mathbf{x}_2, \mathbf{y}_2}$). But, since \mathcal{H} is a nice diagram, this is exactly the differential on $\widehat{\text{CF}}(\mathcal{H})$. \square

REMARK 8.7. The idea to use the Sarkar-Wang algorithm to prove a gluing result is partly inspired by work of Juhász [Juh08].

REMARK 8.8. Since for \mathcal{H}_1 a nice diagram $\widehat{\text{CFA}}(\mathcal{H}_1)$ is an ordinary \mathcal{A} -module, we can also consider the ordinary tensor product $\widehat{\text{CFA}}(\mathcal{H}_1) \otimes \widehat{\text{CFD}}(\mathcal{H}_2)$. It is easy to see that this agrees, on the one hand, with $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$. On the other hand, it is straightforward to check that $\widehat{\text{CFD}}(\mathcal{H}_2)$ is a projective \mathcal{A} -module, and thus $\widehat{\text{CFA}}(\mathcal{H}_1) \otimes \widehat{\text{CFD}}(\mathcal{H}_2) \cong \widehat{\text{CFA}}(\mathcal{H}_1) \widetilde{\otimes} \widehat{\text{CFD}}(\mathcal{H}_2)$. This gives an alternative to the use of Proposition 2.34.

With a little extra work, one can establish a graded version of the pairing theorem along these lines; see Section 10.6.





CHAPTER 9

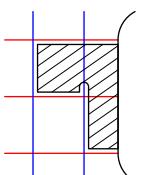
Pairing theorem via time dilation

The aim of the present section is to give an alternate proof of Theorem 1.3, which describes \widehat{HF} of a three-manifold which is separated by a surface in terms of the bordered invariants of the two pieces. Whereas the proof from Chapter 8 used special Heegaard diagrams, the current proof starts from any (suitably admissible) diagrams for the two pieces. Our motivation for including this second proof is twofold. First, the proof illuminates the connection between the analysis of holomorphic curves and the algebra underlying the definitions of the bordered Floer modules and their \mathcal{A}_∞ tensor products. Second, the proof applies to other contexts, such as counting higher polygon maps; see [LOT14].

In slightly more detail, let F be a parameterized surface, and let Y_1 and Y_2 be three-manifolds with compatibly parameterized boundaries, $\partial Y_1 = F(\mathcal{Z}) = -\partial Y_2 = -F(\mathcal{Z})$. Let $Y = Y_1 \cup_F Y_2$. Any two (suitably admissible) diagrams \mathcal{H}_1 for Y_1 and \mathcal{H}_2 for Y_2 can be glued together to form an admissible Heegaard diagram \mathcal{H} for Y . Standard gluing theory describes the moduli spaces counted in the differential for $\widehat{CF}(\mathcal{H})$ in terms of fibered products of moduli spaces from \mathcal{H}_1 and \mathcal{H}_2 . Starting from this description, the differential for $\widehat{CF}(\mathcal{H})$ is then deformed to the differential on the derived tensor product by a procedure we call *time dilation*.

More precisely, we construct a family of chain complexes $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ from fibered products of the moduli spaces for \mathcal{H}_1 and \mathcal{H}_2 , indexed by a real parameter $T > 0$ which, informally, speeds up the time parameter t on the type D side (i.e., \mathcal{H}_2), as compared to time on the type A side (\mathcal{H}_1). Standard gluing theory identifies $\widehat{CF}(1; \mathcal{H}_1, \mathcal{H}_2)$ with $\widehat{CF}(\mathcal{H}_1 \cup_\partial \mathcal{H}_2)$. A continuation argument shows that the chain homotopy type of the complex $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ is independent of T . A further compactness and gluing argument shows that for all sufficiently large T , the behavior of the complex stabilizes. In the large T limit, the complex is then identified with the \boxtimes product, in the sense of Definition 2.26, of the chain complexes $\widehat{CFA}(\mathcal{H}_1)$ and $\widehat{CFD}(\mathcal{H}_2)$. Since this in turn is identified with the derived tensor product (Proposition 2.34), Theorem 1.3 follows.

This chapter is organized as follows. In Section 9.1 we explain how, via a neck-stretching argument, holomorphic curves for $\mathcal{H}_1 \cup_\partial \mathcal{H}_2$ correspond to certain pairs of holomorphic curves for \mathcal{H}_1 and \mathcal{H}_2 (for suitable almost complex structures). In Section 9.2 we introduce the chain complexes $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ and show that they are all chain homotopy equivalent. In Section 9.3 we study the behavior of holomorphic curves as the dilation parameter T goes to ∞ . This is the most technically involved part of the chapter. In Section 9.4 we combine the results from the previous sections to prove Theorem 1.3. Finally, in Section 9.6 we give a local example of the steps in the argument; the reader may want to read this simple example before delving into the details of the proof.



9.1. Moduli of matched pairs

Let $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1, z)$ and $\mathcal{H}_2 = (\Sigma_2, \alpha_2, \beta_2, z)$ be provincially admissible Heegaard diagrams for Y_1 and Y_2 of genus g_1 and g_2 , respectively, inducing compatible markings of F ; i.e., $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$. Let $\Sigma = \overline{\Sigma}_1 \cup_{\partial} \overline{\Sigma}_2$, $\alpha = \overline{\alpha}_1 \cup_{\partial} \overline{\alpha}_2$, and $\beta = \beta_1 \cup \beta_2$. Assume that one of \mathcal{H}_1 or \mathcal{H}_2 is admissible. Then, by Lemma 4.33, $(\Sigma, \alpha, \beta, z)$ is an admissible Heegaard diagram (denoted \mathcal{H}) for Y . Recall from Section 4.5 that generators $\mathfrak{S}(\mathcal{H})$ of $\widehat{CF}(\mathcal{H})$ correspond to certain elements $(\mathbf{x}_1, \mathbf{x}_2)$ of $\mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$, called compatible pairs of generators; these are the pairs $(\mathbf{x}_1, \mathbf{x}_2)$ so that $I_A(\mathbf{x}_1) = I_D(\mathbf{x}_2)$. Moreover, by Lemma 4.32, $\pi_2(\mathbf{x}, \mathbf{y})$ is naturally a subset of $\pi_2(\mathbf{x}_1, \mathbf{y}_1) \times \pi_2(\mathbf{x}_2, \mathbf{y}_2)$.

DEFINITION 9.1. By a *compatible pair* of decorated sources we mean decorated sources S_1^\triangleright and S_2^\triangleright together with a bijection φ between $E(S_1^\triangleright)$ and $E(S_2^\triangleright)$, such that for each puncture q_i of S_1^\triangleright , the Reeb chord labeling $\varphi(q_i)$ is the orientation reverse of the Reeb chord labeling q_i . A compatible pair of sources S_1^\triangleright and S_2^\triangleright can be *preglued* to form a surface $S^\diamond := S_1^\triangleright \natural S_2^\triangleright$, by identifying a neighborhood of q_i with a neighborhood of $\varphi(q_i)$ for each i .

DEFINITION 9.2. Fix two compatible pairs of generators $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ ($i = 1, 2$), and homology classes $B_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$ and $B_2 \in \pi_2(\mathbf{x}_2, \mathbf{y}_2)$ inducing a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$. Fix a compatible pair of sources S_1^\triangleright and S_2^\triangleright connecting \mathbf{x}_1 to \mathbf{y}_1 and \mathbf{x}_2 to \mathbf{y}_2 respectively. Then define the *moduli space of matched pairs* to be the fibered product

$$\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright) := \widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright) \times_{\text{ev}_1=\text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright).$$

That is, the moduli space of matched pairs consists of pairs (u_1, u_2) with $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$ such that $\text{ev}(u_1) = \text{ev}(u_2)$, under the correspondence induced by φ .

Define the index of a matched pair by

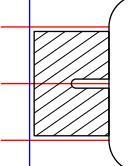
$$(9.3) \quad \begin{aligned} \text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) &:= \text{ind}(B_1, S_1^\triangleright, P) + \text{ind}(B_2, S_2^\triangleright, P) - m \\ &= g_1 + g_2 + 2e(B_1) + 2e(B_2) - \chi(S_1) - \chi(S_2) + m, \end{aligned}$$

where P is the discrete partition on $E(S_1^\triangleright)$ (or equivalently $E(S_2^\triangleright)$) and $m = |P| = |E(S_1^\triangleright)|$. This is the expected dimension of the moduli space of matched pairs coming from its description as a fibered product of two moduli spaces over \mathbb{R}^m .

LEMMA 9.4. *For generic, admissible almost complex structures on $\Sigma_i \times [0, 1] \times \mathbb{R}$, the moduli spaces of matched pairs $\widetilde{\mathcal{MM}}^{B_1 \natural B_2}(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ are transversely cut out by the $\bar{\partial}$ -equation and the evaluation map. Thus, the moduli space is a manifold whose dimension is given by $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright)$.*

PROOF. For the transversality statement, by Proposition 5.6, for a generic almost complex structure J_1 on Σ_1 , the moduli spaces of holomorphic curves $\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright)$ are transversally cut out for all (countably many) choices of $\mathbf{x}_1, \mathbf{y}_1, B_1$ and S_1^\triangleright . Again by Proposition 5.6, for a generic almost complex structure J_2 on Σ_2 , the moduli spaces of holomorphic curves $\widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright)$, for all choices of $B_2, \mathbf{x}_2, \mathbf{y}_2$ and S_2^\triangleright , are transversally cut out and the evaluation maps $\text{ev}: \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright) \rightarrow \mathbb{R}^m$ are transverse to the images $\text{ev}(\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright))$ for all choices of $B_1, \mathbf{x}_1, \mathbf{y}_1$ and S_1^\triangleright .

The statement about the expected dimension is immediate from the definitions and the index formula, Formula (5.9). \square



LEMMA 9.5. Fix a compatible pair of generators \mathbf{x}_1 and \mathbf{x}_2 with $\mathbf{x}_i \in \mathfrak{S}(\mathcal{H}_i)$. Let (u_1, u_2) be any matched pair, i.e., $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$ such that $\text{ev}(u_1) = \text{ev}(u_2)$, under the correspondence induced by φ . Then, both u_1 and u_2 are strongly boundary monotone.

PROOF. The condition that \mathbf{x}_1 and \mathbf{x}_2 can be combined to give a generator of the glued diagram is equivalent to the condition that $o(\mathbf{x}_1) \amalg o(\mathbf{x}_2) = [2k]$. Now, let $\vec{\rho} = (\{\rho_1\}, \dots, \{\rho_n\})$ denote the sequence of (singleton sets of) Reeb chords in order on the u_1 side. Of course, we encounter the same sequence of chords, only with orientations reversed, in the same order on the u_2 side. Continuity of u_1 and u_2 ensures that $M(\rho_{i+1}^-) \subset o(\mathbf{x}_1, \vec{\rho}_{[1,i]})$ and $M(-\rho_{i+1}^-) \subset o(\mathbf{x}_2, -\vec{\rho}_{[1,i]})$. So, Lemma 6.8 implies the result. \square

Translation induces an \mathbb{R} -action on $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$. This action is free except where both sides of the matching are trivial strips. When the action is free we say the curves in the moduli space are *stable*, and denote the quotient $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)/\mathbb{R}$ by $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$.

Given a compatible pair S_1^\triangleright and S_2^\triangleright with $S^\diamond = S_1^\triangleright \natural S_2^\triangleright$, consider the moduli space of holomorphic curves $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\diamond)$ for the diagram \mathcal{H} . The expected dimension of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\diamond)$ is given by

$$\text{ind}(B, S^\diamond) := g - \chi(S) + 2e(B).$$

(This is [Lip06a, Formula (6)], which is a special case of Equation (5.9).) If $|E(S_1^\triangleright)| = m$, we have $\chi(S_1 \natural S_2) = \chi(S_1) + \chi(S_2) - m$. It follows from Formula (9.3) that the index of the moduli space of curves from \mathbf{x} to \mathbf{y} represented by the pre-glued source S^\diamond coincides with the index of the moduli space of matched pairs, i.e.,

$$\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = \text{ind}(B_1 \natural B_2, S_1^\triangleright \natural S_2^\triangleright).$$

A stronger identification occurs at the level of moduli spaces.

PROPOSITION 9.6. There are generic admissible complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$ and on $\Sigma_i \times [0, 1] \times \mathbb{R}$ for $i = 1, 2$ with the property that for all $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, homology classes $B \in \pi_2(\mathbf{x}, \mathbf{y})$, and sources S^\diamond with $\text{ind}(B, S^\diamond) = 1$, the number of elements in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^\diamond)$ is equal (modulo 2) to the number of elements in

$$\bigcup_{S^\diamond = S_1^\triangleright \natural S_2^\triangleright} \mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright),$$

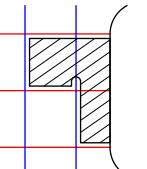
where $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 \cup \mathbf{y}_2$.

PROOF. The result follows from standard compactness and gluing techniques, in a completely analogous way to Propositions 5.24 (but using [BEH⁺03, Theorem 10.3] in place of [BEH⁺03, Theorem 10.2]) and 5.31 (using a Morse-Bott analogue of [Lip06a, Proposition A.2], say). \square

As always, the curves of primary importance to us are the embedded ones. For a Heegaard diagram \mathcal{H} for a closed manifold with genus g , $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$, and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, define

$$\begin{aligned} \chi_{\text{emb}}(B) &:= g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) \\ \text{ind}(B) &:= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B). \end{aligned}$$

(This is a special case of Definition 5.68.)



DEFINITION 9.7. The *embedded matched moduli space* $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ is the union of $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ over all pairs $S_1^\triangleright, S_2^\triangleright$ with $\chi(S_1^\triangleright \natural S_2^\triangleright) = \chi_{\text{emb}}(B)$. (Note there are only finitely many terms in this union.) Also define $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ to be the quotient of $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ by the \mathbb{R} -action (assuming $B \neq 0$, so the \mathbb{R} -action is free).

It follows from Lemma 9.4 that for generic J_1 and J_2 , $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ is a manifold of dimension $\text{ind}(B) - 1$.

LEMMA 9.8. *A matched pair of curves (u_1, u_2) is in the corresponding embedded matched moduli space $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ if and only if both u_i are embedded and every pair of Reeb chords appearing at the same height in the u_i are either nested or disjoint.*

PROOF. Let (u_1, u_2) be any matched pair of curves. Let S_i^\triangleright be the source of u_i , and let $\vec{\rho}$ be sequence of sets of Reeb chords on u_1 . Let $m = |E(S_1^\triangleright)|$ and $B_1 \natural B_2 = B$. By Proposition 5.69, $\chi(S_1) \geq \chi_{\text{emb}}(B_1, \vec{\rho})$ and $\chi(S_2) \geq \chi_{\text{emb}}(B_2, -\vec{\rho})$.

We claim that

$$\iota(\vec{\rho}) + \iota(-\vec{\rho}) \leq -m,$$

as follows. The $L(\rho_i, \rho_j)$ terms in the definition of ι (Formula (5.65)) contribute oppositely to $\iota(\vec{\rho})$ and $\iota(-\vec{\rho})$, as the orientation on the circle is reversed, so the only contributions to $\iota(\vec{\rho}) + \iota(-\vec{\rho})$ are from the $\iota(\rho_i)$ terms. From Lemma 5.64 we see that each chord contributes -1 , and each pair of chords ρ_1, ρ_2 at the same level has a further negative contribution of $-2|L(\rho_1, \rho_2)|$, from which the claim follows.

Thus

$$\begin{aligned} \chi(S_1^\triangleright \natural S_2^\triangleright) &= \chi(S_1) + \chi(S_2) - m \\ &\geq \chi_{\text{emb}}(B_1, \vec{\rho}) + \chi_{\text{emb}}(B_2, -\vec{\rho}) - m \\ &= g + e(B) - n_x(B) - n_y(B) - \iota(\vec{\rho}) - \iota(-\vec{\rho}) - m \\ &\geq \chi_{\text{emb}}(B). \end{aligned}$$

The pair (u_1, u_2) is in the embedded moduli space if and only if we have equality at each step, which is equivalent to both u_1 and u_2 being embedded and all $|L(\rho_1, \rho_2)|$ terms vanishing. These last terms vanish if and only if for all ρ_1 and ρ_2 on the same level, ρ_1 and ρ_2 are nested or disjoint, as desired. \square

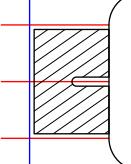
DEFINITION 9.9. Let $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$ be the \mathbb{F}_2 -vector space with basis those $\mathbf{x}_1 \times \mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$ so that $\mathbf{x}_1 \cup \mathbf{x}_2$ is in $\mathfrak{S}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$. Define a boundary operator ∂_1 on $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$ by

$$\partial_1(\mathbf{x}_1 \times \mathbf{x}_2) := \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B \mid \text{ind}(B)=1\}} \#(\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$

Since, by Lemma 4.33, $\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ is admissible, the argument from Lemma 6.5 shows that this sum is finite.

THEOREM 9.10. $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$ is a chain complex isomorphic to the complex $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$, for suitable choices of almost-complex structures.

PROOF. According to [Lip06a, Theorem 2], $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$ can be calculated by the chain complex whose differential counts points in $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, S^\diamond)$, where we take the union over all



choices of $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and compatible S^\diamond so that $\text{ind}(S^\diamond, B) = \text{ind}(B) = 1$. According to Proposition 9.6, these counts agree with ∂_1 . \square

REMARK 9.11. Theorem 9.10 shows, in particular, that $\partial_1^2 = 0$. This also follows as a special case of Proposition 9.19, below.

9.2. Dilating time

We now generalize the notion of matched pairs by inserting a real parameter T .

DEFINITION 9.12. Fix a real number $T > 0$. Fix two compatible pairs of generators $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ ($i = 1, 2$). Fix compatible sources S_1^\diamond and S_2^\diamond connecting \mathbf{x}_1 to \mathbf{y}_1 and \mathbf{x}_2 to \mathbf{y}_2 respectively. The *moduli space of T -matched pairs* $\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$ is defined to be the fibered product

$$\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\diamond) \times_{T \cdot \text{ev}_1 = \text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\diamond).$$

That is, the moduli space of T -matched pairs consists of pairs (u_1, u_2) with $u_i \in \widetilde{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\diamond)$ and $T \cdot \text{ev}(u_1) = \text{ev}(u_2)$. Let

$$\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) := \bigcup_{\{S_1^\diamond, S_2^\diamond | \chi(S_1^\diamond \natural S_2^\diamond) = \chi_{\text{emb}}(B)\}} \widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$$

denote the embedded moduli space of T -matched pairs.

The index of a T -matched pair does not depend on the T -parameter, and so is still given by $\text{ind}(B_1, S_1^\diamond; B_2, S_2^\diamond)$ as defined in Equation (9.3).

DEFINITION 9.13. The moduli space of T -matched pairs has an \mathbb{R} -action given by $\tau_t(u_1, u_2) := (\tau_{T \cdot t}(u_1), \tau_t(u_2))$. Let

$$\begin{aligned} \mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond) &:= \widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)/\mathbb{R} \\ \mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) &:= \widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)/\mathbb{R}. \end{aligned}$$

LEMMA 9.14. For generic admissible almost complex structures J_i on $\Sigma_i \times [0, 1] \times \mathbb{R}$, for all choices of $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ and $B \in \pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$ so that $B \neq 0$,

- for generic values of T , $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ is a manifold of dimension $\text{ind}(B) - 1$, and
- $\bigcup_{T>0} \mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ is a manifold of dimension $\text{ind}(B)$.

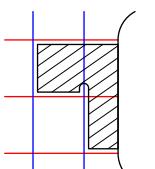
PROOF. As in Lemma 9.4, the first statement follows from Proposition 5.6 and Equation (5.9). The proof of the second statement proceeds in a standard way; see, for instance, [MS04, Section 3.4] for a nice explanation of this type of argument. \square

DEFINITION 9.15. For $T \in (0, \infty)$, let $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ be the vector space with the same basis as $\widehat{CF}(\mathcal{H}_1, \mathcal{H}_2)$ from Definition 9.9 and with boundary operator

$$\partial_T: \widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2) \rightarrow \widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$$

defined by

$$\partial_T(\mathbf{x}_1 \times \mathbf{x}_2) := \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B | \text{ind}(B) = 1\}} \#(\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$



The next goal is to show that $(\partial_T)^2 = 0$. As usual, this requires considering the ends of $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ when $\text{ind}(B) = 2$. To this end, we consider a compactification of $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ by holomorphic combs, following Section 5.4.

DEFINITION 9.16. Given $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ for $i = 1, 2$, a *T-matched story* from $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ to $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ is a sequence $(u_1, v_1, \dots, v_k, u_2)$ where

- $u_1 \in \mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright)$, $u_2 \in \mathcal{M}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright)$, and $v_j \in \mathcal{N}(T_j^\diamond)$ for some sources S_i^\triangleright and T_j^\diamond ;
- the sources are equipped with bijective correspondences between $E(S_1^\triangleright)$ and $W(T_1^\diamond)$, $E(T_i^\diamond)$ and $W(T_{i+1}^\diamond)$, and $E(T_k^\diamond)$ and $E(S_2^\triangleright)$;
- the correspondences preserve the labelings by Reeb chords, with orientation reversal between $E(T_k^\diamond)$ and $E(S_2^\triangleright)$; and
- the following compatibility conditions hold:

$$\begin{aligned}\text{ev}(u_1) &= \text{ev}_w(v_1), \\ \text{ev}_e(v_i) &= \text{ev}_w(v_{i+1}) \text{ and} \\ T \cdot \text{ev}_e(v_k) &= \text{ev}(u_2).\end{aligned}$$

We call u_1 the *west-most level* of the *T-matched story* and u_2 the *east-most level* of the *T-matched story*. A *T-matched story* is *stable* if either u_1 or u_2 is stable. (In particular, if $|E(S_1^\triangleright)| > 0$, the comb is automatically stable.)

A *T-matched comb* of height N is a sequence of stable *T-matched stories* running from $\mathbf{x}^j = (\mathbf{x}_1^j, \mathbf{x}_2^j)$ to $\mathbf{x}^{j+1} = (\mathbf{x}_1^{j+1}, \mathbf{x}_2^{j+1})$ for some sequences of generators \mathbf{x}_1^j in \mathcal{H}_1 and \mathbf{x}_2^j in \mathcal{H}_2 (for $j = 1, \dots, N$).

As in Section 5.4, there is a natural compactification $\overline{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ of $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ inside the space of stable *T-matched holomorphic combs*. The following is the analogue of Proposition 5.43, and the proof is similar.

PROPOSITION 9.17. Suppose that $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 2$. Then for generic J , every *T-matched comb* in $\partial \overline{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ has one of the following forms:

- (1) a two-story *T-matched comb*;
- (2) a *T-matched comb* (u_1, v, u_2) where the only non-trivial component of v is a join component; or
- (3) a *T-matched comb* (u_1, v, u_2) where the only non-trivial component of v is a split component.

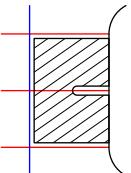
PROOF. The proof of Lemma 9.5 applies to show that the components of a *T-matched pair* are strongly monotone, as well. Thus, boundary double points cannot appear on the boundaries of the *T-matched moduli spaces*, owing to the strong boundary monotonicity of the moduli spaces, Lemma 5.56. Boundary degenerations are ruled out by Lemma 5.48, and ghost curves have codimension 2 as in Lemma 5.57.

Now suppose that we have a one-story *T-matched comb* $(u_1, v_1, \dots, v_\ell, u_2)$ in the boundary of the moduli space, with source $(S_1^{\triangleright'}, T_1^\diamond, \dots, T_\ell^\diamond, S_2^{\triangleright'})$. Let $T^\diamond = T_1^\diamond \natural \dots \natural T_\ell^\diamond$. By the index hypothesis,

$$2 = g - \chi(S_1^\triangleright) - \chi(S_2^\triangleright) + 2e(B) + m_0,$$

where $m_0 = |E(S_1^\triangleright)| = |E(S_2^\triangleright)|$. We also have $S_1^\triangleright \natural S_2^\triangleright \cong S_1^{\triangleright'} \natural T^\diamond \natural S_2^{\triangleright'}$ and so

$$\chi(S_1^\triangleright) + \chi(S_2^\triangleright) - m_0 = \chi(S_1^{\triangleright'}) + \chi(T^\diamond) + \chi(S_2^{\triangleright'}) - m_1 - m_2$$



where $m_1 = |E(S_1^{\triangleright'})|$ and $m_2 = |E(S_2^{\triangleright'})|$. Let k be the number of components of T^{\diamond} . Let P_i be the partition of $E(S_i^{\triangleright'})$ according to the components of T^{\diamond} , so that $|P_i| = k$. Then an upper bound for the dimension of the space of limit curves for generic J is

$$\begin{aligned} \text{ind}(B_1, S_1^{\triangleright'}, P_1) + \text{ind}(B_2, S_2^{\triangleright'}, P_2) - k - 1 \\ = g - \chi(S_1^{\triangleright'}) - \chi(S_2^{\triangleright'}) + 2e(B) + k - 1 \\ = \chi(S_1^{\triangleright}) - \chi(S_1^{\triangleright'}) + \chi(S_2^{\triangleright}) - \chi(S_2^{\triangleright'}) - m_0 + k + 1 \\ = (\chi(T^{\diamond}) - k) + (k - m_1) + (k - m_2) + 1. \end{aligned}$$

All three terms in parentheses are non-positive, so if the space of limit curves is non-empty, at most one can be negative. If all are equal to 0, T^{\diamond} consists of trivial strips and there is no degeneration. If $\chi(T^{\diamond}) < k$, one component of T^{\diamond} is an annulus, which is ruled out as in Proposition 5.43. If $m_1 = k + 1$ or $m_2 = k + 1$, then T^{\diamond} is a join or split curve, respectively.

The arguments above also show that the index of a story in the degenerate curve is at least 1, and that if the index is 1 there are no components at east infinity. Thus the only other possibility for a degeneration is a two-story matched curve. \square

In our setting, the matched combs with components at east infinity cancel in pairs, as boundary branch points move between Σ_1 and Σ_2 . Roughly speaking, the next proposition states that rigid T -matched combs (u_1, v, u_2) appear with multiplicity one in the boundaries of their corresponding one-dimensional moduli spaces.

PROPOSITION 9.18. *Let (u_1, v, u_2) be a T -matched comb of index two with source $(S_1^{\triangleright}, T^{\diamond}, S_2^{\triangleright})$, where T^{\diamond} has one non-trivial component, which is either a join or split component. Then there are arbitrarily small open neighborhoods U_1 of (u_1, v, u_2) in*

$$\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright} \sqcup T^{\diamond}) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$$

with the property that $\partial\overline{U_1}$ meets $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright} \sqcup T^{\diamond}; \mathbf{x}_2, \mathbf{y}_2, S_2^{\triangleright})$ in an odd number of points. Similarly, there are arbitrarily small open neighborhoods U_2 of (u_1, v, u_2) in

$$\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2, T^{\diamond} \sqcup S_2^{\triangleright})$$

with the property that $\partial\overline{U_2}$ meets $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^{\triangleright}; \mathbf{x}_2, \mathbf{y}_2, T^{\diamond} \sqcup S_2^{\triangleright})$ in an odd number of points.

PROOF. Suppose that v is a split curve. We label the east punctures of $S_1^{\triangleright} \sqcup T^{\diamond}$ by $\{e_i\}_{i=0}^{\ell}$ so that e_0 and e_1 are the two east punctures which are in the same component in the split curve. Let $\{w_i\}_{i=0}^{\ell}$ be the corresponding punctures of S_2^{\triangleright} . There are open neighborhoods $N_1 \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^{\triangleright} \sqcup T^{\diamond})$ of (u_1, v) and $N_2 \subset \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^{\triangleright})$ of u_2 which admit evaluation maps

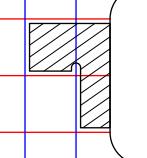
$$\text{ev}^1: N_1 \rightarrow \mathbb{R}^{\ell} \quad \text{ev}^2: N_2 \rightarrow \mathbb{R}^{\ell},$$

where the i^{th} coordinate of ev^2 is ev_{w_i, w_0} (as in Equation (5.12)), and the i^{th} coordinate of ev^1 is $T \cdot \text{ev}_{e_i, e_0}$. Let

$$f_j: N_j \rightarrow \mathbb{R}$$

be the first component of ev^j for $j = 1, 2$, i.e., $f_2 = \text{ev}_{w_1} - \text{ev}_{w_0}$ and $f_1 = \text{ev}_{e_1} - \text{ev}_{e_0}$.

By the assumptions on the index, $\dim N_1 + \dim N_2 = \ell + 1$, so if we knew that the maps ev^1 and ev^2 were smooth (and transverse) it would follow that the images intersect in a



1-manifold, which would prove the result. The difficulty is that we do not know that ev^1 is smooth on the boundary. Instead, we will use linking number considerations.

Let U be a small neighborhood of u_1 inside $\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright)$. Gluing (Proposition 5.31) gives a map

$$\gamma: U \times [0, \epsilon] \rightarrow N_1 \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright \natural T^\diamond)$$

which is a homeomorphism onto a neighborhood of (u_1, v) . Shrinking U slightly, we may assume that γ is defined on $\overline{U} \times [0, \epsilon]$. Note that $\gamma(\overline{U} \times \{0\})$ consists of pairs (u'_1, v') where v' is a split curve whose height is determined by u'_1 . Choose U so that

- (1) \overline{U} is a ball contained in $\mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright) \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright)$ and
- (2) U is sufficiently small that $\text{ev}^1(\gamma(\overline{U} \times \{0\})) \cap \text{ev}^2(N_2)$ consists of the single point $\text{ev}^1((u_1, v)) = \text{ev}^1(\gamma(u_1 \times 0))$.

To see that the second requirement can be met, note that $\text{ev}^1(\gamma(\overline{U} \times \{0\})) = \{0\} \times \text{ev}^1(\overline{U})$; and by Proposition 5.6 we may assume that ev^2 is transverse to this submanifold (compare Lemma 9.4).

The rest of the argument is similar to the proof of Proposition 5.39. The intersection $\text{ev}^1(\gamma(\partial\overline{U} \times \{0\})) \cap \text{ev}^2(N_2)$ is empty and hence, for sufficiently small ϵ :

- (3) The intersection $\text{ev}^1(\gamma(\partial\overline{U} \times [0, \epsilon])) \cap \text{ev}^2(N_2)$ is empty.

Since $\gamma(\overline{U} \times \{\epsilon\})$ is contained in $\mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright \natural T^\diamond) \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright \natural T^\diamond)$, f_1 is positive on $\gamma(\overline{U} \times \{\epsilon\})$. Therefore:

- (4) For sufficiently small $\delta > 0$, $\text{ev}^2(f_2^{-1}((-\infty, \delta]))$ is disjoint from $\text{ev}^1(\gamma(U \times \{\epsilon\}))$.

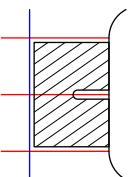
Consider now the ball $D_1 = \overline{U} \times [0, \epsilon]$ and choose a closed ball $D_2 \subset N_2$ which is a neighborhood of u_2 and contained in $f_2^{-1}((-\infty, \delta])$. Combining (2), (3) and (4), $\text{ev}^1(\partial D_1)$ intersects $\text{ev}^2(D_2)$ only in the single point $\text{ev}^1(u_1, v)$, which is in the image $\text{ev}^2(D_2^\circ)$ of the interior D_2° of D_2 . Therefore linking number considerations show that $\text{ev}^1(D_1) \cap \text{ev}^2(\partial D_2)$ consists of an odd number of points. We can therefore take U_1 in the statement of the proposition to be $\gamma(U \times [0, \epsilon]) \times D_2^\circ$: then, the intersection of $\partial\overline{U}_1$ with $\overline{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright \natural T^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ consists of the point (u_1, v, u_2) , from $(\partial\gamma(\overline{U} \times [0, \epsilon])) \times D_2^\circ$, which is not in $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright \natural T^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$, and an odd number of other points, from $\gamma(U \times [0, \epsilon]) \times (\partial D_2)$.

A similar analysis holds for the other moduli space or if v has a join component. \square

PROPOSITION 9.19. *The map ∂_T is a differential.*

PROOF. Fix $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ and a homology class $B \in \pi_2(\mathbf{x}_1 \cup \mathbf{x}_2, \mathbf{y}_1 \cup \mathbf{y}_2)$ with $\text{ind}(B) = 2$. Consider the ends of the moduli space $\overline{\mathcal{M}}^B(T; \mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2)$ of embedded T -matched holomorphic curves representing B . According to Proposition 9.17, these ends correspond either to simple matched combs, which cancel in pairs according to Proposition 9.18, or to two-story holomorphic buildings, with each story an index one T -matched curve. Indeed, each possible two-story building appears as an end an odd number of times, according to an extension of Proposition 5.30 along the lines of Proposition 9.18. Hence, the number of such two-story buildings must be even. But the number of such terms (over all homology classes B with index two) is the $\mathbf{y}_1 \times \mathbf{y}_2$ coefficient of $(\partial_T)^2(\mathbf{x}_1 \times \mathbf{x}_2)$. \square

The chain complex $\widehat{CF}(1; \mathcal{H}_1, \mathcal{H}_2)$ is $\widehat{CF}(\mathcal{H}_1 \cup_{\partial} \mathcal{H}_2)$ by Theorem 9.10. We will next show that the homotopy type of $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ is independent of the choice of T .



DEFINITION 9.20. Let T_1 and T_2 be positive real numbers, and fix a smooth function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with positive derivative so that

$$\psi(t) = \begin{cases} T_1 \cdot t & t \leq -1 \\ T_2 \cdot t & t \geq 1. \end{cases}$$

There are induced maps $\psi^m: \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined by $\psi^m(t_1, \dots, t_m) = (\psi(t_1), \dots, \psi(t_m))$. Fix two compatible pairs of generators $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ ($i = 1, 2$). Fix compatible sources S_1^\triangleright and S_2^\triangleright connecting \mathbf{x}_1 to \mathbf{y}_1 and \mathbf{x}_2 to \mathbf{y}_2 respectively. The *moduli space of ψ -matched pairs*, denoted

$$\mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$$

is defined as the fibered product

$$\mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright) \times_{\psi^m \circ \text{ev}_1 = \text{ev}_2} \mathcal{M}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright),$$

where $m = |E(S_1^\triangleright)|$. That is, the moduli space consists of pairs (u_1, u_2) with $u_i \in \mathcal{M}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright)$ such that $\psi^m \circ \text{ev}(u_1) = \text{ev}(u_2)$. Let

$$\mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) := \bigcup_{\{S_1^\triangleright, S_2^\triangleright \mid \chi(S_1^\triangleright \natural S_2^\triangleright) = \chi_{\text{emb}}(B)\}} \mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$$

denote the space of embedded ψ -matched curves.

The space of ψ -matched pairs has a natural compactification by *ψ -matched combs*, defined analogously to Definition 9.16, except using the map induced by ψ rather than simply rescaling on the left side.

The moduli spaces of ψ -matched pairs do not usually have a natural action by \mathbb{R} .

For ψ as above, define

$$F_{T_1, T_2}: \widehat{\mathit{CF}}(T_1; \mathcal{H}_1, \mathcal{H}_2) \rightarrow \widehat{\mathit{CF}}(T_2; \mathcal{H}_1, \mathcal{H}_2)$$

by

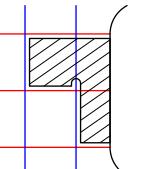
$$F_{T_1, T_2}(\mathbf{x}_1 \times \mathbf{x}_2) = \sum_{\mathbf{y}_1, \mathbf{y}_2} \sum_{\{B \mid \text{ind}(B)=0\}} \#(\mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)) \cdot (\mathbf{y}_1 \times \mathbf{y}_2).$$

We sketch now familiar arguments (generalizing the above proof of Proposition 9.19) which show that F_{T_1, T_2} is a chain map.

LEMMA 9.21. Suppose that $\mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ is 1-dimensional. Then for a generic almost complex structure J , every ψ -matched comb in $\partial \overline{\mathcal{M}}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ has one of the following forms:

- (1) a two-story matched curve $(u'_1, u'_2) * (u''_1, u''_2)$, where (u'_1, u'_2) is a ψ -matched curve and (u''_1, u''_2) is a T_2 -matched curve;
- (2) a two-story matched curve $(u'_1, u'_2) * (u''_1, u''_2)$, where (u'_1, u'_2) is a T_1 -matched curve and (u''_1, u''_2) is a ψ -matched curve;
- (3) a ψ -matched comb (u_1, v, u_2) where the only non-trivial component of v is a join component; or
- (4) a ψ -matched comb (u_1, v, u_2) where the only non-trivial component of v is a split component.

PROOF. This follows along the same lines of Proposition 9.17 above. (Observe, once again, that the proof of Lemma 9.5 applies to show that ψ -matched combs also consist of boundary monotone factors.) \square



PROPOSITION 9.22. F_{T_1, T_2} induces a chain homotopy equivalence from $\widehat{CF}(T_1; \mathcal{H}_1, \mathcal{H}_2)$ to $\widehat{CF}(T_2; \mathcal{H}_1, \mathcal{H}_2)$. In particular, for any $T > 0$, $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ is homotopy equivalent to $\widehat{CF}(\mathcal{H})$.

PROOF. For fixed $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ and homology class B with $\text{ind}(B) = 1$, consider the moduli space $\mathcal{M}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$. Lemma 9.21 accounts for the ends of these moduli spaces. A straightforward adaptation of Proposition 9.18 shows that the ends enumerated in Lemma 9.21 which do not *a priori* cancel in pairs are the two-story ψ -matched curves. Gluing as in the proof of Proposition 9.19 shows that those ends are in one-to-one correspondence with the types of two-story buildings enumerated in Lemma 9.21. But these counts, which must add up to zero, are precisely the coefficients of $\mathbf{y}_1 \times \mathbf{y}_2$ in $\partial_{T_2} \circ F_{T_1, T_2}(\mathbf{x}_1 \times \mathbf{x}_2) + F_{T_1, T_2} \circ \partial_{T_1}(\mathbf{x}_1 \times \mathbf{x}_2)$.

Showing that F_{T_1, T_2} is a chain homotopy equivalence also follows along familiar lines in Floer homology. The homotopy inverse is provided by F_{T_2, T_1} defined with a suitable function ψ_{21} , and the chain homotopy to the identity is provided by counting curves matched by a suitable family of functions, depending on a parameter c , interpolating between multiplication by T_1 and a function which agrees with $\psi(t)$ for t sufficiently small, and $\psi_{21}(-t + c)$ for t sufficiently large.

The second statement is immediate from this and Theorem 9.10. \square

9.3. Dilating to infinity

Now that we have established suitable independence of $\widehat{CF}(T; \mathcal{H}_1, \mathcal{H}_2)$ from T , we turn to the large T behavior of this complex. The work, of course, is in understanding the behavior of T -matched curves as $T \rightarrow \infty$. In Section 9.3.1 we prove a compactness result, showing that T -matched curves converge to so-called ideal-matched curves as $T \rightarrow \infty$. While general ideal-matched curves can be fairly complicated, we show in Section 9.3.2 that for rigid moduli spaces they are quite simple; this special class of ideal-matched curves are called simple ideal-matched curves. It turns out that simple ideal-matched curves contain too much information at east ∞ to correspond one-to-one with T -matched curves. In Section 9.3.3 we trim this extra information and prove the desired gluing result.

9.3.1. Ideal limits of T -matched curves.

DEFINITION 9.23. Let $\overline{\mathbb{R}} := [-\infty, +\infty]$ be the standard compactification of \mathbb{R} . For U a holomorphic comb of height N , define its *time-parameter space* $T(U)$ to be the union of N copies of $\overline{\mathbb{R}}$ modulo identifying $+\infty$ in the i^{th} copy with $-\infty$ in the $(i+1)^{\text{st}}$ copy:

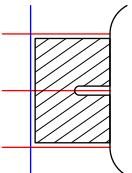
$$T(U) := \left(\bigcup_{i=1}^N \overline{\mathbb{R}}_i \right) / ((+\infty)_i \sim (-\infty)_{i+1}).$$

When U is the trivial comb ($N = 0$), let $T(U)$ be a single point.

DEFINITION 9.24. An *ideal-matched holomorphic comb* consists of

- a pair (U_1, U_2) of strongly boundary monotone, stable holomorphic combs, for \mathcal{H}_1 and \mathcal{H}_2 , respectively;
- an order-preserving map Φ from the set of stories of U_2 to $T(U_1)$; and
- a correspondence φ from $E(U_2)$ to $E(U_1)$,

so that, for every east-most puncture p of U_2 ,



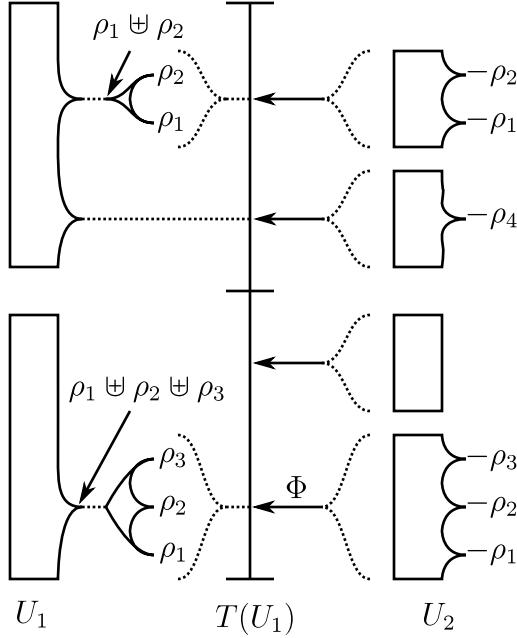


FIGURE 9.1. **Schematic illustration of an ideal-matched comb.** On the left is a two-story comb U_1 . In the center is the time-parameter space $T(U_1)$. On the right is a 4-story comb U_2 . The arrows indicate the map Φ in the definition of an ideal-matched comb.

- if p is labeled by the Reeb chord ρ , then $\varphi(p)$ is labeled by $-\rho$ and
- if p is a puncture on the story C of U_2 , the t -coordinate of $\varphi(p)$ is $\Phi(C)$.

See Figure 9.1 for a schematic illustration of an ideal-matched comb. As usual, we will often suppress φ from the notation. Let the *total west source* S_1^\triangleright and *total east source* S_2^\triangleright of (U_1, U_2) be the result of pregluing all the sources of U_1 and U_2 , respectively. We view such an ideal-matched comb as a point in

$$\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright; \vec{P}_1) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; P_2)$$

where \mathbf{x}_i , \mathbf{y}_i , and B_i are as usual, \vec{P}_1 is the induced partition on $E(S_1^\triangleright)$, and P_2 is the discrete partition on $E(S_2^\triangleright)$.

Two ideal-matched combs are called *equivalent* if they differ only by translating some of the components of U_2 and/or some of the components of U_1 together with the map Φ .

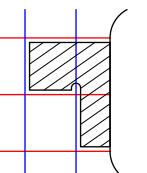
Let

$$\overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$$

denote the moduli space of equivalence classes of ideal-matched combs in the homology class B connecting $\mathbf{x}_1, \mathbf{x}_2$ to $\mathbf{y}_1, \mathbf{y}_2$ and with total west and east sources S_1^\triangleright and S_2^\triangleright . Let

$$\overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) := \bigcup_{\{S_1^\triangleright, S_2^\triangleright \mid \chi(S_1^\triangleright \natural S_2^\triangleright) = \chi_{\text{emb}}(B)\}} \overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright).$$

In the definition, either U_1 or U_2 may be of height 0, in which case the other is an ordinary provincial comb.



DEFINITION 9.25. Let $\{U^j\}_{j=1}^\infty$ be a sequence of T_j -matched combs, where $T_j \rightarrow \infty$. Let (U_1, U_2) be an ideal-matched holomorphic comb. From each T_j -matched comb U^j extract a pair of combs u_1^j and u_2^j by first taking u_2^j to consist of the east-most levels of each story and u_1^j to be the rest of each story and then throwing out any stories consisting of trivial strips. We say that the sequence $\{U^j\}$ converges to (U_1, U_2) if the following conditions are satisfied:

- The sequences $\{u_1^j\}$ and $\{u_2^j\}$ converge to U_1 and U_2 , respectively.
- For sufficiently large j the identification between the east-most punctures of u_1^j and the east punctures of u_2^j corresponds to the identification between the east-most punctures of U_1 and the east-most punctures of U_2 .
- For each story C of U_2 , $\Phi(C) \in T(U_1)$ is determined as follows. Place marked points p_j on the source of u_2^j so that the sequence $\{p_j\}$ converges to a point on the source of C . Choose marked points q_j in the corresponding story of the source of u_1^j so that

$$T_j \cdot t(u_1^j(q_j)) = t(u_2^j(p_j)).$$

Then $\Phi(C)$ is the limit of $t(u_1^j(q_j))$ (which may be $\pm\infty$).

It is clear that if the third condition of Definition 9.25 is satisfied for one choice of sequences p_j and q_j for the story C then it holds for any choice of sequences p_j and q_j . If C has an $e\infty$ puncture then $\Phi(C)$ is given by the \mathbb{R} -coordinate of the corresponding $e\infty$ puncture of U_1 , so Φ is determined by the rest of the data in this case. By contrast, if C is provincial then $\Phi(C)$ is not determined by the rest of the data. Also, $\Phi(C) = \infty$ is only possible for provincial stories C of U_2 .

PROPOSITION 9.26. Fix decorated sources S_1^\triangleright and S_2^\triangleright and homology classes B_1 and B_2 . Let $\{(U_1^j, U_2^j)\}_{j=1}^\infty$ be a sequence of T_j -matched holomorphic combs with total source $(S_1^\triangleright, S_2^\triangleright)$ in the homology class $B_1 \sqcup B_2$. Suppose that $T_j \rightarrow \infty$. Then $\{(U_1^j, U_2^j)\}_{j=1}^\infty$ has a subsequence which converges to an ideal-matched holomorphic comb, with total source $(S_1^\triangleright, S_2^\triangleright)$ and in the homology class $B_1 \sqcup B_2$.

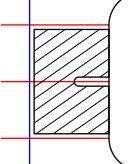
PROOF. The key point is that, by Proposition 5.24, the sequence $\{(U_1^j, U_2^j)\}_{j=1}^\infty$ limits to some pair of holomorphic combs. Strong boundary monotonicity follows from Lemma 5.55. Moreover, since the evaluation maps are continuous on the compactified moduli space, the limit is an ∞ -matched (i.e., ideal-matched) holomorphic comb. More details follow.

We prove the claim in the case that each U_i^j is a holomorphic curve; the general case is similar, but the notation more complicated. So, let $\{(u_1^j, u_2^j)\}$ be a sequence of T_j -matched holomorphic curves connecting $\mathbf{x}_1 \otimes \mathbf{x}_2$ to $\mathbf{y}_1 \otimes \mathbf{y}_2$ with west source S_1^\triangleright and east source S_2^\triangleright . Replacing the sequence with a subsequence if necessary, we may assume that there is an ordering $q_{i,1}, \dots, q_{i,n}$ of the punctures of S_i^\triangleright so that if $a < b$ then for any j , we have $\text{ev}_{q_{i,b}, q_{i,a}}(u_1^j) \geq 0$ (and so also $\text{ev}_{q_{2,b}, q_{2,a}}(u_2^j) \geq 0$).

Consider the space

$$\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright; \vec{P}_1) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; \vec{P}_2) \times [1, \infty].$$

where \vec{P}_i is the sequence of sets of punctures ($\{q_{i,1}\}, \dots, \{q_{i,n}\}$). By Proposition 5.24, this space is compact. On the open part, there are evaluation maps $\text{ev}^i: \mathcal{M}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright; \vec{P}_i) \rightarrow \mathbb{R}^{E(S_i^\triangleright)} / \mathbb{R}$. We wish to extend these maps to $\overline{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright; \vec{P}_i)$. To this end, identify



$\mathbb{R}^{E(S_i^\triangleright)} / \mathbb{R}$ with \mathbb{R}^{n-1} by using the ordering on the east punctures $q_{i,1}, \dots, q_{i,n}$ and considering differences of successive coordinates. Thus the map ev^i is given by

$$\begin{aligned}\text{ev}^i(u_i) &= (\text{ev}_{q_{i,2}}(u_i) - \text{ev}_{q_{i,1}}(u_i), \text{ev}_{q_{i,3}}(u_i) - \text{ev}_{q_{i,2}}(u_i), \dots, \text{ev}_{q_{i,n}}(u_i) - \text{ev}_{q_{i,n-1}}(u_i)) \\ &= (\text{ev}_{q_{i,2}, q_{i,1}}(u_i), \text{ev}_{q_{i,3}, q_{i,2}}(u_i), \dots, \text{ev}_{q_{i,n}, q_{i,n-1}}(u_i)).\end{aligned}$$

The image of ev^i lies inside $\mathbb{R}_+^{n-1} = [0, \infty)^{n-1}$, and the map ev^i has a continuous extension to a map $\overline{\text{ev}}^i: \overline{\mathcal{M}}^{B_i}(\mathbf{x}_i, \mathbf{y}_i; S_i^\triangleright; \vec{P}_i) \rightarrow [0, \infty]^{n-1}$.

Now, consider the map

$$f: \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright; \vec{P}_1) \times \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; \vec{P}_2) \times [1, \infty] \rightarrow [0, \infty]^{n-1}$$

defined by

$$f(u_1, u_2, T) = T \cdot \overline{\text{ev}}^1(u_1) - \overline{\text{ev}}^2(u_2).$$

The preimage $f^{-1}(0)$ is the union over all $T \in [1, \infty)$ of the moduli space of T -matched holomorphic combs (with total sources S_1^\triangleright and S_2^\triangleright , in the homology class $B_1 \sqsubset B_2$, with the partial order on the Reeb chords consistent with the order $q_{i,1}, \dots, q_{i,n}$), together with the fiber over $T = \infty$, which we will discuss presently.

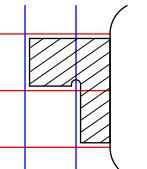
The space $f^{-1}(0)$ is a closed subspace of a compact space, and hence compact. Consequently, the sequence of points $\{(u_1^j, u_2^j, T_j)\}_{j=1}^\infty$ in $f^{-1}(0)$ has a convergent subsequence. The limit (U_1, U_2) of this subsequence lies in $f^{-1}(0) \cap \{T = \infty\}$. The limit (U_1, U_2) can be endowed with the structure of an ideal-matched holomorphic comb as follows. The correspondence φ is inherited from the φ on each (U_1^j, U_2^j) . For each story of U_2 with at least one e puncture, the map Φ is induced from the rest of the structure. For each story u_i of U_2 with no e puncture, place a marked point p_i on the source of u_i ; a sequence of marked points $p_{i,j}$ on the source of u_2^j converging to p_i ; and a sequence of points $q_{i,j}$ on u_1^j with $T_j \cdot t(u_1^j(q_{i,j})) = t(u_2^j(p_{i,j}))$. Since the time-parameter space $T(U_1)$ is compact, we may take a further subsequence of the $\{(u_1^j, u_2^j)\}$ so that for each i , the sequence $\{t(u_1^j(q_{i,j}))\}$ converges. Then Φ takes the i^{th} story of U_2 to $\lim_{j \rightarrow \infty} t(q_{i,j})$. It is immediate from the definitions that our subsequence of $\{(u_1^j, u_2^j)\}$ converges to (U_1, U_2, ϕ, Φ) . \square

REMARK 9.27. We have not claimed that the topology corresponding to Definition 9.25 is Hausdorff.

9.3.2. Simple ideal-matched curves, the limits of rigid T -matched curves. We will see in this section that there are strong restrictions on which ideal-matched combs come from rigid T -matched curves.

DEFINITION 9.28. A *simple ideal-matched curve* is an ideal-matched comb (U_1, U_2) in which

- U_1 has at most one story;
- U_2 has no components at east infinity;
- every component of U_1 at east infinity is a disk with a single west puncture; and
- exactly one of the following two conditions holds:
 - (1) $E(S_1^\triangleright) = E(S_2^\triangleright) = \emptyset$, one of U_1 or U_2 is the trivial comb of height 0 and the other of U_1 or U_2 is a height 1 holomorphic comb with index 1, or
 - (2) each story of U_2 has at least one e puncture and has index 1 (with the discrete partition), the map Φ is injective, and U_1 has index 1 (with the induced partition).



In the above definition, the east infinity components of U_1 are like split components, except that there may be any number of east punctures; in the terminology of Section 5.5 they are generalized split components. Note that curves with generalized split components that are not split components are not transversally cut with respect to the induced partition.

LEMMA 9.29. *Simple ideal-matched curves have $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 1$.*

PROOF. In Case (1) in the definition of simple ideal-matched curves the statement is obvious. To prove Case (2), let $\vec{P} = (P_1, \dots, P_n)$ denote the induced ordered partition of the e punctures of S_1^\triangleright (by their \mathbb{R} -coordinates), and let Q^i denote the discrete partition of the e punctures of S_i^\triangleright . By definition,

$$\text{ind}(B_1, S_1^\triangleright, \vec{P}) = 1 \quad \text{ind}(B_2, S_2^\triangleright, Q^2) = n.$$

Note that $|Q^1| = |Q^2| = \sum |P_i|$. By Formula (9.3),

$$\begin{aligned} \text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) &= \text{ind}(B_1, S_1^\triangleright, Q^1) + \text{ind}(B_2, S_2^\triangleright, Q^2) - |Q^1| \\ &= \text{ind}(B_1, S_1^\triangleright, \vec{P}) + \text{ind}(B_2, S_2^\triangleright, Q^2) - n \\ &= 1 + n - n = 1. \end{aligned}$$

□

LEMMA 9.30. *With respect to generic almost complex structures J_1 and J_2 , if $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) < 1$ then $\overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ is empty, and if $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 1$ then $\overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ consists of simple ideal-matched curves (U_1, U_2) .*

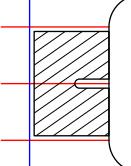
PROOF. This is similar to the proofs of Propositions 5.43 and 9.17. The case with no east punctures is immediate. Otherwise, suppose first that U_1 has a single story, and for notational simplicity assume that U_2 also has a single story, so that all east punctures of U_1 appear at a single height. Let $S_1^{\triangleright'}$ and $S_2^{\triangleright'}$ be the sources of the main components of U_1 and U_2 , respectively, and similarly let T_1^\diamondsuit and T_2^\diamondsuit be the result of gluing together the horizontal levels of the sources of the east components of U_1 and U_2 . Let k_2 be the number of components of T_2^\diamondsuit , $m_0 = |E(S_1^\triangleright)| = |E(S_2^\triangleright)|$, and $m_i = |E(S_i^{\triangleright'})|$ for $i = 1, 2$. Let P'_i be the induced partition on $E(S_i^{\triangleright'})$. For generic J , an upper bound for the dimension of $\overline{\mathcal{MM}}^B(\infty; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ is

$$\begin{aligned} &\text{ind}(B_1, S_1^{\triangleright'}, P'_1) + \text{ind}(B_2, S_2^{\triangleright'}, P'_2) - 2 \\ &= g + 2e(B) - \chi(S_1^{\triangleright'}) - \chi(S_2^{\triangleright'}) + k_2 - 1 \\ &= g + 2e(B) - (\chi(S_1^\triangleright) - \chi(T_1^\diamondsuit) + m_1) - (\chi(S_2^\triangleright) - \chi(T_2^\diamondsuit) + m_2) + k_2 - 1 \\ &= (\chi(T_1^\diamondsuit) - m_1) + (\chi(T_2^\diamondsuit) - m_0) + (k_2 - m_2) + (\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) - 1). \end{aligned}$$

When $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) < 1$, the sum is negative and so there are no ideal-matched combs. When $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright) = 1$, the three other terms in parentheses must be 0, which implies that (U_1, U_2) is a simple ideal-matched curve as stated.

The case when U_2 has multiple stories is similar. If U_1 has more than one story, each story must contribute at least one to $\text{ind}(B_1, S_1^\triangleright; B_2, S_2^\triangleright)$ by the above analysis, contradicting the index assumption. □

9.3.3. Trimmed curves and gluing. We showed in Section 9.3.1 that T -matched curves converge to ideal-matched curves as $T \rightarrow \infty$. It is not true, however, that every ideal-matched curve arises this way. For instance, the split curves that normally occur at $e\infty$ for a simple ideal-matched curve have moduli, so the moduli spaces of simple ideal-matched



curves of expected dimension 0 are not, in fact, 0-dimensional. The philosophically correct solution is probably to remember some microscopic information at the curves in $e\infty$ (e.g., maps to the upper half-plane rather than $[0, 1] \times \mathbb{R}$). For our purposes, however, it is enough to simply trim off the components at $e\infty$:

DEFINITION 9.31. A *trimmed simple ideal-matched curve* is a pair of holomorphic combs (w_1, w_2) connecting two compatible pairs of generators $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$ ($i = 1, 2$) such that either:

- (T-A) One of w_1 or w_2 is trivial and the other is a index 1 holomorphic curve with no e punctures or
- (T-B) (w_1, w_2) has the following properties:
 - (T-B1) The comb w_1 is a holomorphic curve for \mathcal{H}_1 (one-story building with no components at east infinity) which is asymptotic to the non-trivial sequence of non-empty sets of Reeb chords $\vec{\rho} = (\rho_1, \dots, \rho_n)$.
 - (T-B2) The curve w_1 has index 1 with respect to $\vec{\rho}$.
 - (T-B3) The comb w_2 is an n -story holomorphic building for \mathcal{H}_2 with no components at east infinity.
 - (T-B4) Each story of w_2 has index one.
 - (T-B5) Each of w_1 and w_2 is strongly boundary monotone.
 - (T-B6) For each $i = 1, \dots, n$, the east punctures of the i^{th} story of w_2 are labeled, in order, by a non-empty sequence of Reeb chords $(-\rho_1^i, \dots, -\rho_{\ell_i}^i)$, which have the property that the sequence of singleton sets of chords $\vec{\rho}^i = (\{\rho_1^i\}, \dots, \{\rho_{\ell_i}^i\})$ is composable. We will let $\vec{\rho}$ be the concatenation of the sequences $\vec{\rho}^1, \dots, \vec{\rho}^n$.
 - (T-B7) The composition of the sequence of singleton sets of Reeb chords $\vec{\rho}^i$ on the i^{th} story of w_2 (with reversed orientation) coincides with the i^{th} set of Reeb chords ρ_i , in the partition for w_1 ; i.e.,

$$\rho_i = \bigoplus_{j=1}^{\ell_i} \{\rho_j^i\}.$$

LEMMA 9.32. *When Conditions (T-B1) through (T-B5) hold, then Conditions (T-B6) and (T-B7) hold if and only if the following holds:*

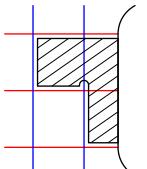
- (T-B6') For all $i = 1, \dots, n$, the east punctures of the i^{th} story of w_2 are labeled, in order, by a non-empty sequence of Reeb chords $(-\rho_1^i, \dots, -\rho_{\ell_i}^i)$ which has the property that

$$(9.33) \quad I(o(\mathbf{x}_1, \vec{\rho}_{[1, i-1]}))a(\rho_i) = I(o(\mathbf{x}_1, \vec{\rho}_{[1, i-1]})) \prod_{j=1}^{\ell_i} a(\rho_j^i) \neq 0.$$

Here, $o(\mathbf{x}_1, \vec{\rho}_{[1, i-1]})$ denotes the set of α -arcs occupied by w_1 between ρ_{i-1} and ρ_i ; see Definition 5.52.

PROOF. Suppose that (w_1, w_2) is a trimmed simple ideal-matched comb. Let \mathbf{y}^i denote the initial point of the i^{th} story of w_2 (so that $\mathbf{y}^1 = \mathbf{x}_2$).

Claim 1. *The sets $o(\mathbf{x}_1, \vec{\rho}_{[1, i-1]})$ and $o(\mathbf{y}^i)$ are disjoint.* In the case where $i = 1$, this is the fact that \mathbf{x}_1 and \mathbf{y}_1 are a compatible pair of generators. The case where $i > 1$ follows from Conclusion (2) of Lemma 6.8, applied to $\vec{\rho}$, using Condition (T-B7).



Let $o_i = o(\mathbf{x}_1, \vec{\rho}_{[1,i-1]})$.

Claim 2. For each $i = 1, \dots, n$,

$$(9.34) \quad I(o_i)a_0(\boldsymbol{\rho}_i) = I(o_i) \prod_{j=1}^{\ell_i} a_0(\rho_j^i) \neq 0.$$

This follows from Claim 1, the composability of the $\vec{\rho}^i$ (Condition (T-B6)) and Lemma 3.12.

We wish to conclude that Equation (9.34) holds with a in place of a_0 .

For the next three claims, pick some $p, q \in \mathbf{a}$ with $M(p) = M(q)$ and $p \neq q$. Let $\{\rho_{j_1}^i, \dots, \rho_{j_m}^i\}$ be the subsequence of chords on the i^{th} level which start or end in $\{p, q\}$.

Claim 3. There is no ρ_j^i such that $(\rho_j^i)^- = p$ and $(\rho_j^i)^+ = q$. We argue this as follows. Let $\rho_{j_k}^i$ be the first ρ_j^i such that $(\rho_{j_k}^i)^- = p$ and $(\rho_{j_k}^i)^+ = q$. By strong boundary monotonicity of w_2 (Condition (T-B5)), for $\ell = 1, \dots, k-1$, the chords $\rho_{j_\ell}^i$ alternate between starting at one of $\{p, q\}$ and terminating at one of $\{p, q\}$. Similarly, since $\rho_{j_k}^i$ terminates at q , strong boundary monotonicity of w_2 implies that $\rho_{j_{k-1}}^i$ starts at one of $\{p, q\}$. Now, there are two cases, according to the parity of k . If k is even, then there are at least two strands starting in $\{p, q\}$ in $\biguplus_{j=1}^{j_k} \{\rho_j^i\}$, and hence at least two strands starting at p and q in $\boldsymbol{\rho}_i$, violating boundary monotonicity of w_1 (Condition (T-B5)). If k is odd, there is at least one strand starting in $\{p, q\}$ in $\biguplus_{j=1}^{j_k} \{\rho_j^i\}$, and hence $\{p, q\}$ appears in o_i . But in this case, $\rho_{j_1}^i$ terminates at one of $\{p, q\}$, and hence $\{p, q\}$ appears in $o(\mathbf{y}^i)$. This violates Claim 1.

Claim 4. For $k = 1, \dots, m$, $\rho_{j_k}^i$ alternates between starting at one of $\{p, q\}$ and ending at one of $\{p, q\}$. This follows immediately from boundary monotonicity of w_2 , together with Claim 3.

Claim 5. There is no k such that $(\rho_{j_k}^i)^+$ and $(\rho_{j_{k+1}}^i)^-$ are both in $\{p, q\}$ but not equal to each other. Suppose that there is such a k . From Claim 4, it follows that if k is odd then $\boldsymbol{\rho}_i$ has a chord starting at one of $\{p, q\}$, and $\rho_{j_1}^i$ is a chord terminating at one of $\{p, q\}$; hence $\{p, q\}$ is in the initial idempotent for both w_1 and w_2 , contradicting Conclusion (2) from Proposition 6.8. If k is even, there are at least two strands leaving one of p or q in $\boldsymbol{\rho}_i$, contradicting boundary monotonicity of w_1 .

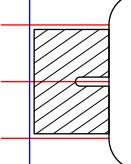
Claim 4 ensures that for $k = 1, \dots, m$, $\rho_{j_k}^i$ and $\rho_{j_{k+1}}^i$ do not have matched initial or terminal points. Together with Claim 5 (using a straightforward inductive application of Lemma 3.26), this ensures that not only is $I(o_i) \prod_{j=1}^{\ell_i} a_0(\rho_j^i) \neq 0$, but in fact $I(o_i) \prod_{j=1}^{\ell_i} a(\rho_j^i) \neq 0$.

Conversely, suppose that Conditions (T-B1)–(T-B5) hold, along with Condition (T-B6'). Then, the non-vanishing of $I(o_i) \prod_{j=1}^{\ell_i} a(\rho_j^i)$ implies the non-vanishing of $I(o_i) \prod_{j=1}^{\ell_i} a_0(\rho_j^i)$, which, by Lemma 3.12, ensures both the composability of $\vec{\rho}^i$ (i.e., Condition (T-B6)) and the property that

$$I(o_i) \prod_{j=1}^{\ell_i} a_0(\rho_j^i) = I(o_i) a_0 \left(\biguplus_{j=1}^{\ell_i} \{\rho_j^i\} \right).$$

Equation (9.33) now gives Conclusion (T-B7). \square

Let $\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)$ denote the moduli space of trimmed simple ideal-matched curves connecting $\mathbf{x}_1, \mathbf{x}_2$ to $\mathbf{y}_1, \mathbf{y}_2$ with homology class B and east/west sources



S_1^\triangleright and S_2^\triangleright , modulo \mathbb{R} -translation of w_1 and each story of w_2 . Let

$$\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) := \bigcup_{\substack{\chi(S_1^\triangleright) = \chi_{emb}(B_1, \vec{\rho}) \\ \chi(S_2^\triangleright) = \chi_{emb}(B_2, -\vec{\rho})}} \mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright).$$

LEMMA 9.35. *The spine of a simple ideal-matched curve is a trimmed simple ideal-matched curve. That is, if (U_1, U_2) is a simple ideal-matched curve, where $U_1 = (w_1, v_1)$ (with v_1 a generalized split curve) and U_2 is toothless comb w_2 , then (w_1, w_2) is an embedded trimmed simple ideal-matched curve.*

PROOF. Case (T-A) is trivial. In case (T-B), properties (T-B1), (T-B2), (T-B3), (T-B4), and (T-B5) are all immediate from the fact that (U_1, U_2) is a simple ideal-matched curve.

Properties (T-B6) and (T-B7) are verified as follows.

First, weak composability follows from Property (T-B5) and Lemma 5.53.

Next, let $-\vec{\rho}$ denote the sequence of Reeb chords coming from w_2 , and $\vec{\rho}$ be the sequence of sets of Reeb chords on w_1 . Then $\vec{\rho}$ is the partition of Reeb chords at west infinity for the generalized split curve v_1 , while $\vec{\rho}$ is the collection of Reeb chords at east infinity of v_1 , with a discrete partition and an ordering compatible with $\vec{\rho}$. It follows that $\vec{\rho}$ is obtained from $\vec{\rho}$ by contracting various consecutive sequences of Reeb chords.

The stronger condition of composability follows from the following index considerations (akin to Lemma 5.77). By Lemma 5.78, we have an inequality:

$$\prod_{i=1}^{|\vec{\rho}|} \text{gr}'(a(\rho_i)) \leq \prod_{i=1}^{|\vec{\rho}|} \text{gr}'(a(\rho_i)),$$

with strict inequality if and only if some sequence in $\vec{\rho}$ which is contracted to get an element ρ_i is not composable. Equivalently, by Lemma 5.67, if ι is the function from Equation (5.65) then $\iota(\vec{\rho}) \leq \iota(\vec{\rho})$ with strict inequality if and only if there is a non-composable sequence which is contracted.

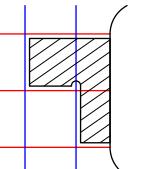
It follows from Definition 5.68 that:

$$\begin{aligned} \text{ind}(B_1, \vec{\rho}) &= \text{ind}(B_1, \vec{\rho}) + |\vec{\rho}| - |\vec{\rho}| + (\iota(\vec{\rho}) - \iota(\vec{\rho})) \\ &= \text{ind}(B_1, \vec{\rho}) + \text{ind}(B_2, \vec{\rho}) - |\vec{\rho}| + (\iota(\vec{\rho}) - \iota(\vec{\rho})) + (|\vec{\rho}| - \text{ind}(B_2, \vec{\rho})) \\ &= \text{ind}(B_1 \natural B_2) + (\iota(\vec{\rho}) - \iota(\vec{\rho})) + (|\vec{\rho}| - \text{ind}(B_2, \vec{\rho})). \end{aligned}$$

By Lemma 9.29, $\text{ind}(B_1 \natural B_2) = 1$, and we just verified that $(\iota(\vec{\rho}) - \iota(\vec{\rho})) \leq 0$. Since $|\vec{\rho}|$ is equal to the number of stories of U_2 and each story has index 1, $(|\vec{\rho}| - \text{ind}(B_2, \vec{\rho})) = 0$. Since there is a holomorphic curve, $\text{ind}(B_1, \vec{\rho}) \geq 1$ by Proposition 5.69, so $\iota(\vec{\rho}) = \iota(\vec{\rho})$ and $|\vec{\rho}| = \text{ind}(B_2, \vec{\rho})$. It follows that w_1 and w_2 are embedded and the Reeb chords in $\vec{\rho}$ which are collapsed in $\vec{\rho}$ are composable. \square

It turns out that there is a combinatorially unique way to un-trim curves. To this end, it helps to have the following:

DEFINITION 9.36. Let $\vec{\rho} = (\rho_1, \dots, \rho_n)$ be a sequence of Reeb chords. A subsequence $(\rho_{j_1}, \dots, \rho_{j_m})$ is called *abutting* if $\rho_{j_k}^+ = \rho_{j_{k+1}}^-$ for each $k = 1, \dots, m-1$. An *amalgamation* of $\vec{\rho}$ is a partition of $\vec{\rho}$ into abutting sub-sequences. Given an amalgamation of $\vec{\rho}$, there is an associated set of Reeb chords ρ whose elements are formed by joining together the chords in the abutting sub-sequences specified by the amalgamation. We call such a set ρ an *amalgam* of $\vec{\rho}$.



LEMMA 9.37. *If $\vec{\rho} = (\rho_1, \dots, \rho_n)$ is a sequence of Reeb chords with $a_0(\rho_1) \cdots a_0(\rho_n) = a_0(\rho) \neq 0$, then there is a unique amalgamation of $\vec{\rho}$ with resulting amalgam ρ .*

PROOF. For existence of the amalgamation, let $j_1 = 1$, let j_2 be the first index after j_1 so that $\rho_{j_2}^- = \rho_{j_1}^+$, and so on. This gives an abutting subsequence $(\rho_{j_1}, \dots, \rho_{j_m})$. Since $\rho = \bigcup_j \{\rho_j\}$, one of the chords in ρ is $\rho_{j_1} \uplus \cdots \uplus \rho_{j_m}$. We take $(\rho_{j_1}, \dots, \rho_{j_m})$ to be one part of the desired amalgamation. The other parts of the amalgamation are gotten by iteratively repeating this process with $\vec{\rho}$ replaced by $\vec{\rho} \setminus (\rho_{j_1}, \dots, \rho_{j_m})$. We call the resulting amalgamation the *maximal amalgamation* of $\vec{\rho}$; by construction the corresponding amalgam is ρ .

For uniqueness, we argue as follows. By the non-vanishing of $a_0(\rho_1) \cdots a_0(\rho_n)$, given $p \in \mathcal{Z}$, the subsequence of Reeb chords in $\vec{\rho}$ which start or end at p alternates between starting at p and ending at p . In the maximal amalgamation, a chord ending at p always is joined to the next chord starting at p . For an amalgamation which does not always join these pairs, the number of chords starting or ending at p is greater than for the maximal amalgamation. It follows that the only amalgamation whose amalgam is ρ is the maximal amalgamation: any other amalgamation results in an amalgam with more chords in it. \square

LEMMA 9.38. *Let (w_1, w_2) be a trimmed simple ideal-matched curve. Then there is a generalized split curve $v: T^\diamond \rightarrow \partial \bar{\Sigma} \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ such that (w_1, v, w_2) is a simple ideal-matched curve. Moreover, the combinatorial type of T^\diamond is determined by (w_1, w_2) .*

PROOF. Again Case (T-A) is trivial, so we consider Case (T-B). Observe that each generalized split curve v with e punctures labeled by $\vec{\rho}$ gives an amalgamation of $\vec{\rho}$, depending only on the combinatorial type of the source T^\diamond : the parts of the amalgamation correspond to the generalized split components making up v . Furthermore, the w punctures of v are labeled by the corresponding amalgam of $\vec{\rho}$. In addition, in any simple ideal-matched curve, v must match the i^{th} level of w_1 with the i^{th} story of w_2 . So it suffices to show that for a trimmed simple ideal-matched curve with w_1 asymptotic to $\vec{\rho}$ and the i^{th} level of w_2 asymptotic to $-\vec{\rho}^i$, there is a unique amalgamation of $\vec{\rho}^i$ whose amalgam is ρ_i . But this follows from Property (T-B6') and Lemma 9.37. \square

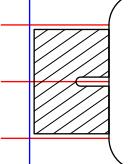
LEMMA 9.39. *For a generic choice of almost complex structures J_1 and J_2 and a homology class $B = B_1 \natural B_2$, the moduli space $\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$ consists of a finite number of points.*

PROOF. Consider a curve $(w_1, w_2) \in \mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$. There are only finitely many sequences of sets of Reeb chords $\vec{\rho}$ compatible with B_1 . Thus, by Condition (T-B2), w_1 lies in one of finitely many index 1 moduli spaces which, by Propositions 5.6 and 5.24 and Theorem 5.61 (transversality, compactness, and the analysis of the boundary strata), is a compact 0-manifold. Similarly, there are a finite number of ways to decompose B_2 as a sum $B_2 = B_{2,1} + \cdots + B_{2,n}$ with each $B_{2,i}$ a positive domain, and a finite number of sequences of sets of Reeb chords compatible with each $B_{2,j}$. Thus, each story of w_2 lies in one of finitely many index 1 moduli spaces, each of which is a compact 0-manifold. \square

PROPOSITION 9.40. *For $\mathbf{x}_i, \mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i)$; B_1, B_2 so that $\text{ind}(B_1 \natural B_2) = 1$; generic J_1 and J_2 ; and generic T sufficiently large,*

$$\#\mathcal{MM}^{B_1 \natural B_2}(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) = \#\mathcal{MM}_{tsic}^{B_1 \natural B_2}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2).$$

PROOF. Let $B = B_1 \natural B_2$. By Lemma 9.39, the moduli space $\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ consists of finitely many points; we label these points (w_1^i, w_2^i) . Let (w_1^i, v^i, w_2^i) be an untrimming of (w_1^i, w_2^i) , as given by Lemma 9.38.



Consider the union of moduli spaces

$$\mathcal{MM}^B(\geq T_0; \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright) := \bigcup_{T \geq T_0} (T, \mathcal{MM}^B(T, \mathbf{x}_1, \mathbf{y}_1, S_1^\triangleright; \mathbf{x}_2, \mathbf{y}_2, S_2^\triangleright)).$$

Our goal is to find disjoint neighborhoods \mathcal{W}^i of each (w_1^i, v^i, w_2^i) in $\overline{\mathcal{MM}}^{B_1 \natural B_2}(\geq T_0; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ so that, for T sufficiently large, $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) \subset \bigcup_i \mathcal{W}^i$ and, for each i , $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ intersects \mathcal{W}^i in an odd number of points. The neighborhoods \mathcal{W}^i will have the form

$$\mathcal{W}^i = (\mathcal{U}_1^i \times \mathcal{U}_2^i) \cap \overline{\mathcal{MM}}^{B_1 \natural B_2}(\geq T_0; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$$

where $\mathcal{U}_1^i \subset \overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_{i,1}^\triangleright \natural T_i^\diamondsuit)$ is a smeared neighborhood (Definition 5.33) of (w_1^i, v^i) and $\mathcal{U}_2^i \subset \overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_{i,2}^\triangleright)$ is a neighborhood of w_2^i .

The proof involves two steps, of course: compactness and gluing. The compactness step is easier, and we start there. Fix any collection of neighborhoods \mathcal{W}^i as above. By Proposition 9.26, any sequence of holomorphic curves in $\mathcal{MM}^B(T_j; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ with $T_j \rightarrow \infty$ converges to an ideal-matched holomorphic comb, which by Lemma 9.30 must be a simple ideal-matched holomorphic curve. By Lemma 9.35 and the uniqueness part of Lemma 9.38, this limit lies in some \mathcal{W}^i . It follows that $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) \subset \bigcup_i \mathcal{W}^i$ for T sufficiently large.

We turn next to the gluing step. Fix a trimmed simple ideal-matched curve $(w_1, w_2) \in \mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$, and let (w_1, v, w_2) be the un-trimming of (w_1, w_2) . Let S_i^\triangleright denote the source of w_i and T^\diamondsuit the source of v . Let $\vec{P} = (p_1, \dots, p_n)$ be the sequence of e punctures of w_2 , ordered by the story in which they occur and, within each story, by their \mathbb{R} -coordinates. Via the correspondence ϕ (from Definition 9.24), we can also view \vec{P} as a list of the e punctures of v . We can also group the p_i according to the stories of w_2 or, equivalently, according to their heights in v ; write this grouping as

$$\vec{Q} = (\underbrace{\{p_1, \dots, p_{\ell_1}\}}_{Q_1}, \underbrace{\{p_{\ell_1+1}, \dots, p_{\ell_2}\}}_{Q_2}, \dots, \underbrace{\{p_{\ell_{m-1}+1}, \dots, p_{\ell_m}\}}_{Q_m}).$$

Thus, n is the total number of e punctures of w_2 , m is the number of stories of w_2 and $\ell_i - \ell_{i-1}$ is the number of e punctures on the i^{th} story of m , so $n = \ell_m$.

Observe that, by the proof of Lemma 9.38, the ordering of the elements of Q_i induced by P is compatible with the induced ordering around the boundary of each generalized split-component of v .

View (w_1, v) as an element of $\overline{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright \natural T^\diamondsuit; \vec{P})$ and w_2 as an element of $\overline{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; \vec{P})$. There are evaluation maps

$$\text{ev}^1: \mathcal{M}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\triangleright \natural T^\diamondsuit; \vec{P}) \rightarrow \mathbb{R}_+^{n-1} \quad \text{ev}^2: \mathcal{M}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\triangleright; \vec{P}) \rightarrow \mathbb{R}_+^{n-1}$$

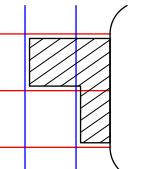
given by

$$\text{ev}^i(u_i) = (\text{ev}_{p_2}(u_i) - \text{ev}_{p_1}(u_i), \text{ev}_{p_3}(u_i) - \text{ev}_{p_2}(u_i), \dots, \text{ev}_{p_n}(u_i) - \text{ev}_{p_{n-1}}(u_i)).$$

Define $\pi_1: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-m}$ and $\pi_2: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{m-1}$ by

$$\begin{aligned} \pi_1(t_1, \dots, t_n) &= (t_1, \dots, t_{\ell_1-1}, t_{\ell_1+1}, \dots, t_{\ell_2-1}, t_{\ell_2+1}, \dots, t_{\ell_{m-1}-1}, t_{\ell_m}) \\ \pi_2(t_1, \dots, t_n) &= (t_{\ell_1}, t_{\ell_2}, \dots, t_{\ell_{m-1}}). \end{aligned}$$

We can find a smeared neighborhood \mathcal{U}_1 of (w_1, v) and a neighborhood \mathcal{U}_2 of w_2 with the following properties:



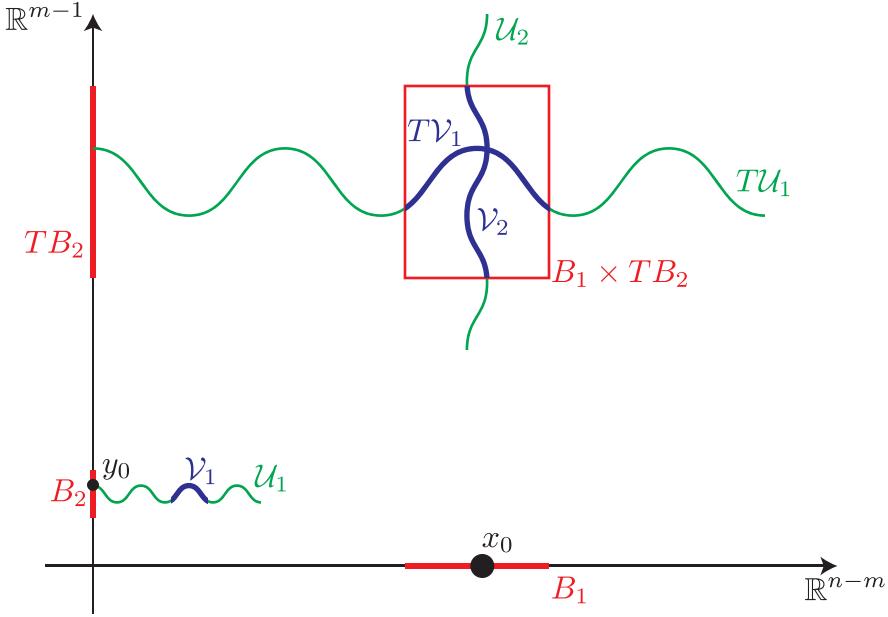


FIGURE 9.2. Illustration of the fibered product in the proof of Proposition 9.40.

- (i-1) The map $\pi_2 \circ ev^1$ sends \mathcal{U}_1 to a neighborhood of some vector y_0 in the interior of \mathbb{R}_+^{m-1} (disjoint from $\partial\mathbb{R}_+^{m-1}$). This uses continuity of the relative evaluation maps on the compactified moduli space. The vector y_0 records the height differences of the punctures on w_1 .
- (i-2) The map $\pi_1 \circ ev^2$ sends \mathcal{U}_2 to a neighborhood of some vector x_0 in the interior of \mathbb{R}_+^{n-m} . Again, this uses continuity of the evaluation maps: the vector x_0 records the height differences of the punctures within each of the stories of w_2 .
- (i-3) For all sufficiently small δ , $\pi_1 \circ ev^1$ maps \mathcal{U}_1 properly and with odd degree to $\{x \in \mathbb{R}_+^{n-m} \mid \|x\| < \delta\}$. This uses Proposition 5.40, and our observation about the ordering of the points in each Q_i (to see that the image of the punctures on each generalized split component lands in $\mathbb{R}_+^{n-m} \subset \mathbb{R}^{n-m}$).
- (i-4) For all sufficiently large R , $\pi_2 \circ ev^2$ maps \mathcal{U}_2 properly and with odd degree onto $\{x \in \mathbb{R}_+^{m-1} \mid \|x\| > R\} \cup \{\infty\}$ (so that w_2 is mapped to ∞). This uses Proposition 5.30 (or rather, an n -story analogue thereof).
- (i-5) The closure of \mathcal{U}_1 intersects $\{(w_1^i, v^i)\}$ in the one point (w_1, v) .
- (i-6) The closure of \mathcal{U}_2 intersects $\{w_2^i\}$ in the one point w_2 .

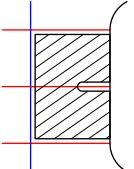
It follows from these observations and an intersection theory argument that for any generic, large choice of T , the fibered product

$$(9.41) \quad \mathcal{U}_1 \times \mathcal{U}_2 \cap \overline{\mathcal{MM}}^{B_1 \sqcup B_2}(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) = \mathcal{U}_1 \times_{ev^1} \mathcal{U}_2$$

consists of an odd number of points. We spell this out presently. First, we rephrase Conditions (i-1) and (i-2) more concretely:

- (i-1') There is a $y_0 \in \mathbb{R}_+^{m-1}$ and $\epsilon > 0$ so that $\pi_2 \circ ev^1$ sends \mathcal{U}_1 into $B_2 := B_\epsilon(y_0) \subset \mathbb{R}_+^{m-1}$.
- (i-2') There is an $x_0 \in \mathbb{R}_+^{n-m}$ and $\eta > 0$ so that $\pi_1 \circ ev^2$ sends \mathcal{U}_2 into $B_1 := B_\eta(x_0) \subset \mathbb{R}_+^{n-m}$.

See Figure 9.2.



Think of the fibered product $\mathcal{U}_1 \times_{T\text{ev}^1} \mathcal{U}_2$ as a subset of $\mathbb{R}_+^{n-m} \times \mathbb{R}_+^{m-1}$. By the above two conditions, it follows that the fibered product is contained in the subset $B_1 \times TB_2$. Choose T large enough so that $T^{-1}B_1$ is contained inside the ball of radius δ about the origin in \mathbb{R}_+^{n-m} , and so that TB_2 lies outside a ball of radius R around the origin in \mathbb{R}_+^{m-1} , where δ and R are constants from Conditions (i-3) and (i-4), respectively. Then let $\bar{\mathcal{V}}_1$ denote the preimage under $T\text{ev}^1$ of $\bar{B}_1 \times T\bar{B}_2$, and let $\bar{\mathcal{V}}_2$ denote the preimage under ev^2 of $\bar{B}_1 \times T\bar{B}_2$. Both are compact, because $\pi_1 \circ \text{ev}^1$ is proper on the preimage of $T^{-1}B_1$ and $\pi_2 \circ \text{ev}^2$ is proper on the preimage of B_2 .

Now, Condition (i-1') shows that $T\text{ev}^1: \bar{\mathcal{V}}_1 \rightarrow \bar{B}_1 \times T\bar{B}_2$ can be viewed as a relative cycle, representing some homology class in $H_{n-m}(\bar{B}_1 \times TB_2, (\partial\bar{B}_1) \times TB_2; \mathbb{F}_2) \cong \mathbb{F}_2$. Indeed, Condition (i-3) ensures that $[\bar{\mathcal{V}}_1]$ is the non-trivial class: the degree of $\pi_1 \circ T\text{ev}^1: \bar{\mathcal{V}}_1 \rightarrow \bar{B}_1$ can be interpreted as the intersection pairing of $[\bar{\mathcal{V}}_1]$ with an element of $H_{m-1}(B_1 \times T\bar{B}_2, B_1 \times (\partial T\bar{B}_2); \mathbb{F}_2) \cong \mathbb{F}_2$.

Symmetrically, by Conditions (i-2') and (i-4), $\text{ev}^2: \bar{\mathcal{V}}_2 \rightarrow \bar{B}_1 \times T\bar{B}_2$ represents a generator of $H_{m-1}(B_1 \times T\bar{B}_2, B_1 \times (\partial T\bar{B}_2); \mathbb{F}_2) \cong \mathbb{F}_2$. The count of points in the fibered product $\bar{\mathcal{V}}_1 \times_{T\text{ev}^1} \bar{\mathcal{V}}_2 = \mathcal{U}_1 \times_{T\text{ev}^1} \mathcal{U}_2$ can now be interpreted as the intersection pairing of $[\bar{\mathcal{V}}_1]$ with $[\bar{\mathcal{V}}_2]$ under the non-trivial intersection pairing

$$H_{n-m}(\bar{B}_1 \times TB_2, (\partial\bar{B}_1) \times TB_2) \otimes H_{m-1}(B_1 \times T\bar{B}_2, B_1 \times (\partial T\bar{B}_2)) \rightarrow H_0(B_1 \times TB_2) \cong \mathbb{F}_2,$$

as desired. \square

9.4. Completion of the proof of the pairing theorem

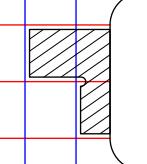
We are now in a position to give our second proof of the pairing theorem, Theorem 1.3.

PROOF OF THEOREM 1.3. Fix bordered Heegaard diagrams \mathcal{H}_1 and \mathcal{H}_2 for $(Y_1, \phi_1: F(\mathcal{Z}) \rightarrow \partial Y_1)$ and $(Y_2, \phi_2: -F(\mathcal{Z}) \rightarrow \partial Y_2)$. By Proposition 4.25, we may assume that both \mathcal{H}_1 and \mathcal{H}_2 are provincially admissible, and at least one of \mathcal{H}_1 or \mathcal{H}_2 is admissible. (We could assume both diagrams are admissible, but we will not need this.) By Lemma 4.33, this implies that $\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ is a Heegaard diagram for Y which is weakly admissible for all spin^c -structures. By Lemmas 6.5 and 7.7, our assumption also implies that at least one of $\widehat{\text{CFA}}(\mathcal{H}_1)$ or $\widehat{\text{CFD}}(\mathcal{H}_2)$ is bounded.

By Proposition 2.34, the \boxtimes -product $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ (Definition 2.26) is a model for $\widehat{\text{CFA}}(Y_1) \widetilde{\otimes} \widehat{\text{CFD}}(Y_2)$. By Proposition 9.22 (which uses Theorem 9.10), $\widehat{\text{CF}}(Y)$ is homotopy equivalent to $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ for any $T > 0$. So, it remains to identify $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ with $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ for T sufficiently large.

On the level of \mathbb{F}_2 -vector spaces, the identification is straightforward. As a vector space, $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ is given by $X(\mathcal{H}_1) \otimes_{\mathcal{I}(\mathcal{Z})} X(\mathcal{H}_2)$, which has a basis $\{(\mathbf{x}_1, \mathbf{x}_2) \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2) \mid o(\mathbf{x}_1) \cap o(\mathbf{x}_2) = \emptyset\}$. This is exactly the set of generators for $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ (and for $\widehat{\text{CF}}(\mathcal{H})$).

Next we identify the differentials. It follows from the admissibility criteria that there are only finitely many homology classes B which can contribute to the differential on $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ (compare Lemmas 6.5 and 7.7). So, for large enough T , Proposition 9.40 implies that the differential on $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ counts trimmed simple ideal-matched curves.



Let $\{a_i\}$ be the set of basic generators for the algebra $\mathcal{A}(\mathcal{Z})$. Define operators $D_i: X(\mathcal{H}_2) \rightarrow X(\mathcal{H}_2)$ by

$$\partial_2(\mathbf{x}_2) = \sum_i a_i \otimes D_i(\mathbf{x}_2)$$

as in Example 2.28. Then, for large T , the differential on $\widehat{\text{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$ is given by

$$(9.42) \quad \partial_T(\mathbf{x}_1 \otimes \mathbf{x}_2) = \sum m_{n+1}(\mathbf{x}_1, a_{i_1}, \dots, a_{i_n})(D_{i_n} \circ \dots \circ D_{i_1})(\mathbf{x}_2),$$

where the sum is taken over all finite sequences a_{i_1}, \dots, a_{i_n} of basic generators for $\mathcal{A}(\mathcal{Z})$ (including the empty sequence, where $n = 0$). To see this, consider a trimmed simple ideal-matched curve (w_1, w_2) connecting $\mathbf{x}_1 \otimes \mathbf{x}_2$ to $\mathbf{y}_1 \otimes \mathbf{y}_2$. The holomorphic building w_2 corresponds to a sequence of differentials $D_{i_n} \circ \dots \circ D_{i_1}$ on $\widehat{\text{CFD}}(\mathcal{H}_2)$, where n is the number of stories of w_2 . (This uses one direction of Lemma 9.32 to guarantee that the algebra elements coming from w_2 are nonzero and agree with the algebra elements coming from w_1 .) The curve w_1 corresponds to a higher product $m_{n+1}(\mathbf{x}_1, a_{i_1}, \dots, a_{i_n})$ on $\widehat{\text{CFA}}(\mathcal{H}_1)$ of \mathbf{x}_1 with the corresponding algebra elements a_{i_1}, \dots, a_{i_n} . Every non-zero term on the right of Equation (9.42) arises this way. (This uses the other direction of Lemma 9.32, to guarantee that the corresponding curves w_1 and w_2 satisfy conditions (T-B6) and (T-B7) of Definition 9.31.)

Equation (9.42) is exactly the differential on $\widehat{\text{CFA}}(\mathcal{H}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2)$ (compare Equation (2.29)). This concludes the proof. \square

9.5. A twisted pairing theorem

Next, we give a version of the pairing theorem for the twisted-coefficient variants of $\widehat{\text{CFD}}$ and $\widehat{\text{CFA}}$. Loosely speaking, it says that the derived tensor product of twisted bordered invariants computes the twisted Heegaard Floer complex of the glued-up three-manifold. Before stating the precise version, we introduce a little algebra.

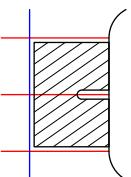
LEMMA 9.43. *The left action on $\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1)$ by $\mathbb{F}_2[H_2(Y_1, \partial Y_1)]$ and the right action of $\widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2)$ by $\mathbb{F}_2[H_2(Y_2, \partial Y_2)]$ induce a left $\mathbb{F}_2[H_2(Y, F)]$ -action on the tensor product $\widehat{\text{CFD}}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{\text{CFA}}(\mathcal{H}_2, \mathfrak{s}_2)$.*

PROOF. The two actions give $\widehat{\text{CFD}}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{\text{CFA}}(\mathcal{H}_2, \mathfrak{s}_2)$ the structure of a bimodule over $\mathbb{F}_2[H_2(Y_1, F)]$ and $\mathbb{F}_2[H_2(Y_2, F)]$; since $\mathbb{F}_2[H_2(Y_2, F)]$ is commutative, the right action by $\mathbb{F}_2[Y_2, F]$ can be viewed as a left action, giving the product complex the structure of a left $\mathbb{F}_2[H_2(Y_1, F)] \otimes \mathbb{F}_2[H_2(Y_2, F)] \cong \mathbb{F}_2[H_2(Y, F)]$ -module. \square

The quotient map $H_2(Y) \rightarrow H_2(Y, F)$ induces a ring map $i: \mathbb{F}_2[H_2(Y)] \rightarrow \mathbb{F}_2[H_2(Y, F)]$. Note that i is injective. As in [OSz04c, Section 8], the twisted Heegaard Floer complex $\widehat{\text{CF}}(Y)$ is a chain complex of free $\mathbb{F}_2[H_2(Y)]$ -modules, and hence induces a chain complex of free $\mathbb{F}_2[H_2(Y, F)]$ -modules $i_*(\widehat{\text{CF}}(Y)) = \widehat{\text{CF}}(Y) \otimes_{\mathbb{F}_2[H_2(Y)]} \mathbb{F}_2[H_2(Y, F)]$.

THEOREM 9.44. *There is a homotopy equivalence of $\mathbb{F}_2[H_2(Y, F)]$ -modules.*

$$\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2) \simeq i_* \left(\bigoplus_{\substack{\mathfrak{s} \in \text{spin}^c(Y) \\ \mathfrak{s}|_{Y_i} = \mathfrak{s}_i, i=1,2}} \widehat{\text{CF}}(\mathcal{H}, \mathfrak{s}) \right).$$



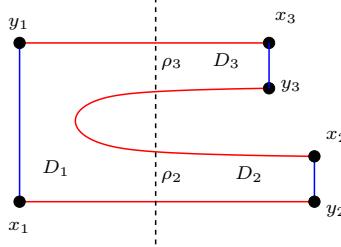


FIGURE 9.3. **Degenerating the hexagon.** A hexagon in the Heegaard diagram (giving a flow from $\mathbf{x} = \{x_1, x_2, x_3\}$ to $\mathbf{y} = \{y_1, y_2, y_3\}$) is divided into three pieces D_1 , D_2 , and D_3 , grouped as D_1 and $D_2 \cup D_3$.

PROOF. Lemma 9.43 implies that as modules the two sides agree. It remains to show that the differentials agree (up to homotopy). For this we can use either proof of the pairing theorem, and only need to check that the group-ring coefficients on the two sides agree. Let \mathbf{x}_i , $i = 1, 2$, be the chosen base generator for \mathcal{H}_i in the spin^c-structure \mathfrak{s}_i . Fix generators $\mathbf{y} = \mathbf{y}_1 \otimes \mathbf{y}_2$ and $\mathbf{w} = \mathbf{w}_1 \otimes \mathbf{w}_2$ and domains $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$. Given domains $C_i \in \pi_2(\mathbf{y}_i, \mathbf{w}_i)$, the group-ring coefficient of $\mathbf{w}_1 \otimes \mathbf{w}_2$ in the differential of $e^{B_1} \mathbf{x}_1 \otimes \mathbf{x}_2 e^{B_2}$ coming from (C_1, C_2) is $e^{B_1 * C_1} \mathbf{x}_1 \otimes \mathbf{x}_2 e^{B_2 * C_2}$. The group-ring coefficient of \mathbf{w} in $\partial(\mathbf{x})$ coming from $C_1 \natural C_2$ is $e^{(B_1 \natural B_2) * (C_1 \natural C_2)}$. With respect to the identification from Lemma 9.43, these coefficients agree. \square

REMARK 9.45. We can recover $\widehat{\text{CF}}(Y)$ from $i_*(\widehat{\text{CF}}(Y))$ as follows. Since $H_1(F)$ is a free abelian group, so is the cokernel of the projection map $i: H_2(Y) \rightarrow H_2(Y, F)$. So, we can choose a splitting $\ell: H_2(Y, F) \rightarrow H_2(Y)$, with $\ell \circ i = \mathbb{I}_{H_2(F)}$. Then $\widehat{\text{CF}}(Y) \cong \ell_*(i_*(\widehat{\text{CF}}(Y)))$.

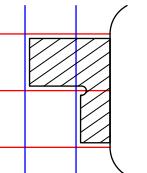
9.6. An example

We conclude this chapter with a local example. Consider a hexagon connecting the generator $\mathbf{x} = \{x_1, x_2, x_3\}$ to $\mathbf{y} = \{y_1, y_2, y_3\}$ in a genus three Heegaard diagram \mathcal{H} for some three-manifold Y . Here, the hexagon is a domain D in Σ whose boundary, as we traverse it, contains arcs in three distinct α -curves and arcs in three distinct β -curves, so that if the arc $\partial D \cap \alpha_i$ (with its boundary orientation) goes from x_i to y_i . This domain corresponds to an element $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\text{ind}(B) = 1$. In fact, it is not difficult to see that the domain B always contributes \mathbf{y} to the differential of \mathbf{x} , i.e., $\#\mathcal{M}(\mathbf{x}, \mathbf{y}; B) = 1$; see, for instance, [Ras03, Lemma 9.11]. We wish to see this contribution from the point of view of time dilation in a bordered setting where the dividing curve Z , which separates \mathcal{H} into two bordered diagrams, crosses the hexagon in two arcs. Such a curve divides the hexagon into three regions which we denote $\{D_i\}_{i=1}^3$ with the convention that D_i connects x_i to y_i . We label so that D_1 meets both arcs in $Z \cap D$, whereas D_2 and D_3 each meet only one apiece. We label the Reeb chords corresponding to these arcs ρ_2 and ρ_3 , numbered so that ρ_i meets D_i for $i = 2, 3$. See Figure 9.3 for an illustration.

We start, as a warm-up, by considering the problem from the point of view of matched pairs (i.e., using the complex $\widehat{\text{CF}}(\mathcal{H}_1, \mathcal{H}_2)$ from Theorem 9.10). We verify that

$$\#(\mathcal{MM}^B(\{x_1\}, \{y_1\}; \{x_2, x_3\}, \{y_2, y_3\})) = 1.$$

Now, $\mathcal{M}(\{x_1\}, \{y_1\}; D_1)$ consists of a single curve u . Then $\text{ev}_{\rho_3, \rho_2}(u) = \text{ev}_{\rho_3}(u) - \text{ev}_{\rho_2}(u)$ is some fixed real number t_0 , the height difference between the Reeb chords ρ_3 and ρ_2 .



The number t_0 is necessarily positive, but the precise value of t_0 depends on the conformal parameters in the picture.

The moduli space on the other side, $\mathcal{M}(\{x_2, x_3\}, \{y_2, y_3\}; D_2 + D_3)$ consists of a pair of disjoint disks, which can be moved relative to each other subject to the constraint that ρ_2 is below ρ_3 . (The constraint comes from strong boundary monotonicity.) In particular, the evaluation map $\text{ev}_{\rho_3, \rho_2}$ sends this moduli space homeomorphically to $\mathbb{R}_{\geq 0}$.

It follows that there is a unique point in the fibered product, as claimed.

We compare this against two different computations which arise by time dilation. The two different computations arise by thinking of D_1 as the part on the A side or on the D side.

We start by placing D_1 on the D side. In this case, the type A module for $D_2 \cup D_3$ has four generators $x_2x_3 = \{x_2, x_3\}$, $x_2y_3 = \{x_2, y_3\}$, $y_2x_3 = \{y_2, x_3\}$, and $y_2y_3 = \{y_2, y_3\}$ and the following operations:

$$\begin{aligned} m_2(x_2x_3, \rho_2) &= y_2x_3 \\ m_2(x_2y_3, \rho_2) &= y_2y_3 \\ m_2(x_2x_3, \rho_3) &= x_2y_3 \\ m_2(y_2x_3, \rho_3) &= y_2y_3 \\ m_2(x_2x_3, \rho_2\rho_3) &= y_2y_3. \end{aligned}$$

The first four of these are gotten by looking the union of a bigon with a trivial strip. The last is a union of the two non-trivial bigons $D_2 + D_3$, and comes from the element of $\mathcal{M}(\{x_2, x_3\}, \{y_2, y_3\}, D_2 + D_3)$ with the property that the $\text{ev}_{\rho_2, \rho_3} = 0$. We have already seen that there is a unique such element.

The type D module of D_1 has two generators x_1 and y_1 , and the bigon D_1 is interpreted as giving a differential

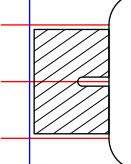
$$\partial x_1 = \rho_2\rho_3 \cdot y_1.$$

The \boxtimes product of these two complexes has two generators, $\mathbf{x} = x_2x_3 \otimes x_1$ and $\mathbf{y} = y_2y_3 \otimes y_1$, and a differential single from \mathbf{x} to \mathbf{y} , consistent our earlier computation. Algebraically, this differential pairs the operation $m_2(x_2x_3, \rho_2 \cdot \rho_3) = y_2y_3$ with the (unbroken) differential on the type D side. In the graphical notation of Chapter 2, this operation is given by

$$\begin{array}{ccc} x_2x_3 & & x_1 \\ \downarrow & & \downarrow \\ m_2 & \nearrow \rho_2\rho_3 & \delta^1 \\ \downarrow & & \downarrow \\ y_2y_3 & & y_1 \end{array}.$$

Analytically, the moduli spaces $\mathcal{MM}^B(T; x_2x_3, y_2y_3; x_1, y_1)$ consist of the curve u in \mathcal{H}_1 and the pair of disks in \mathcal{H}_2 with punctures at height difference t_0/T . Taking $T \rightarrow \infty$, the separation between these heights goes to 0; this corresponds to the stated m_2 .

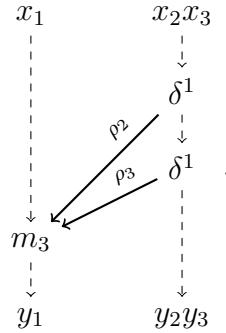
Next, place D_1 on the A side. Now, the type A module has two generators x_1 and y_1 , and a single non-trivial operation $m_3(x_1, \rho_2, \rho_3) = y_1$. The type D module for $D_2 \cup D_3$ is



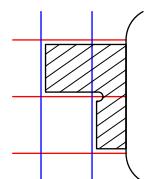
generated by the four generators x_2x_3 , x_2y_3 , y_2x_3 , and y_2y_3 , with differentials

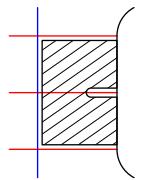
$$\begin{aligned}\partial x_2x_3 &= \rho_2 \cdot y_2x_3 + \rho_3 \cdot x_2y_3 \\ \partial x_2y_3 &= \rho_2 \cdot y_2y_3 \\ \partial y_2x_3 &= \rho_3 \cdot y_2y_3.\end{aligned}$$

On the \boxtimes product of these two complexes, we once again get two generators and a single differential. The generators are now labeled $x_1 \otimes x_2x_3$ and $y_1 \otimes y_2y_3$. Algebraically, the differential in this case is gotten by pairing the m_3 on the type A side with the differential from x_2x_3 to $\rho_2 \cdot y_2x_3$ followed by the differential from y_2x_3 to $\rho_3 \cdot y_2y_3$. In the graphical notation of Chapter 2, this operation is given by



Analytically, the moduli spaces $\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; x_2x_3, y_2y_3)$ consist of the curve u in \mathcal{H}_1 and the pair of disks in \mathcal{H}_2 with punctures at height difference $T \cdot t_0$. Taking $T \rightarrow \infty$, the separation between these heights goes to ∞ ; this corresponds to the pair of δ^1 's.





CHAPTER 10

Gradings

We now turn to gradings on the structures we have already constructed in Chapters 6–9: Type D invariants $\widehat{CFD}(\mathcal{H})$; Type A invariants $\widehat{CFA}(\mathcal{H})$; and their tensor products $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$. This extends the gradings on the algebra from Section 3.3, and we will use the material on the dimension of the moduli spaces $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ from Section 5.7.

As described in Definition 2.42, the gradings gr' on $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ and $\widehat{CFA}(\mathcal{H}, \mathfrak{s})$ take values in sets $S'_D(\mathcal{H}, \mathfrak{s})$ and $S'_A(\mathcal{H}, \mathfrak{s})$ (depending on the spin^c structure \mathfrak{s}) with actions of the grading group $G'(4k)$; a left action in the case of $S'_D(\mathcal{H}, \mathfrak{s})$ (Section 10.4) and a right action in the case of $S'_A(\mathcal{H}, \mathfrak{s})$ (Section 10.3). We will also define refinements gr of these grading taking values in sets with actions of the smaller group $G(\mathcal{Z})$ (Section 10.5). As discussed in Section 10.6, the tensor product takes values in the product over $G(\mathcal{Z})$ of these two grading sets. Before constructing these gradings, we collect some properties of the grading groups and relationships with the expected dimensions of moduli spaces from earlier chapters (Section 10.1) and give a useful grading on domains (Section 10.2).

10.1. Algebra review

For reference, we recall briefly the relevant material from Sections 3.3 and 5.7.

In Section 3.3 we introduced two grading groups:

- $G'(4k)$, which is a central extension of $H_1(Z', \mathbf{a})$ by \mathbb{Z} , and
- $G(\mathcal{Z})$, which is a subgroup of $G'(4k)$ isomorphic to a central extension of $H_1(F(\mathcal{Z}))$ by \mathbb{Z} .

We review $G'(4k)$ here, and delay reviewing $G(\mathcal{Z})$ until Section 10.5 where we construct the refined gradings.

Elements of $G'(4k)$ are written (j, α) , where $j \in \frac{1}{2}\mathbb{Z}$ and $\alpha \in H_1(Z', \mathbf{a})$. We impose the additional restriction that $j \equiv \epsilon(\alpha) \pmod{1}$, where $\epsilon(\alpha)$ is one quarter the number of parity changes in α . The multiplication on $G'(4k)$ is given by

$$(j_1, \alpha_1) \cdot (j_2, \alpha_2) := (j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

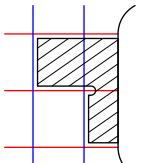
where $L(\alpha_1, \alpha_2)$ is the linking of the boundaries of α_1 and α_2 , given by

$$L(\alpha_1, \alpha_2) := m(\alpha_2, \partial\alpha_1)$$

or, concretely by Formula (5.63). We call j the *Maslov component* of (j, α) and α the *spin^c component* of (j, α) . The distinguished central element of $G'(4k)$ is the element $\lambda = (1, 0)$; thus the differential lowers the first component of the grading by 1.

More generally, we have (Lemma 5.66)

$$(j_1, \alpha_1) \cdots (j_n, \alpha_n) = \left(\sum_i j_i + \sum_{i < j} L(\alpha_i, \alpha_j), \sum_i \alpha_i \right).$$



The grading $\text{gr}'(a)$ of an element $a \in \mathcal{A}(\mathcal{Z})$ with starting idempotent S is given by

$$\begin{aligned}\iota(a) &:= \text{inv}(a) - m([a], S) \\ \text{gr}'(a) &:= (\iota(a), [a])\end{aligned}$$

(Definition 3.38), where $\text{inv}(a)$ denotes the number of inversions of a . For convenience, if $\vec{\rho}$ is a sequence of Reeb chords, let $\text{gr}'(\vec{\rho})$ be the product $\prod_{i=1}^n \text{gr}'(a(\rho_i))$. (If $a(\vec{\rho}) \neq 0$, then $\text{gr}'(\vec{\rho}) = \text{gr}'(a(\vec{\rho}))$.) Similarly set $\text{gr}'(\vec{\rho})$ to be $\prod_{i=1}^n \text{gr}'(a(\rho_i))$.

The function ι can also be written, for $\boldsymbol{\rho}$ a set of Reeb chords with $a(\boldsymbol{\rho}) \neq 0$, as

$$\iota(a(\boldsymbol{\rho})) = - \sum_{\{\rho_1, \rho_2\} \subset \boldsymbol{\rho}} |L(\rho_1, \rho_2)| - \frac{|\boldsymbol{\rho}|}{2}$$

(Lemma 5.64). We also define

$$\begin{aligned}\iota(\boldsymbol{\rho}) &:= \iota(a(\boldsymbol{\rho})) \\ \iota(\vec{\rho}) &:= \sum_i \iota(\boldsymbol{\rho}_i) + \sum_{i < j} L(\boldsymbol{\rho}_i, \boldsymbol{\rho}_j)\end{aligned}$$

so that $\iota(\vec{\rho})$ is the Maslov component of $\text{gr}'(\vec{\rho})$ (Lemma 5.67).

These quantities are related to the dimensions of the embedded moduli space as follows. For $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\vec{\rho}$ a compatible sequence of sets of Reeb chords, $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$ has expected dimension

$$\text{ind}(B, \vec{\rho}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\vec{\rho}| + \iota(\vec{\rho})$$

(Definition 5.68 and Proposition 5.69).

10.2. Domains

Before defining a grading on modules, we define a grading on domains.

DEFINITION 10.1. For $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, define $g'(B) \in \frac{1}{2}\mathbb{Z} \times H_1(Z', \mathbf{a})$ by

$$(10.2) \quad g'(B) := (-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^\partial(B)).$$

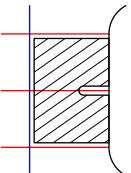
LEMMA 10.3. For any $B \in \pi_2(\mathbf{x}, \mathbf{y})$, the element $g'(B)$ is in $G'(4k)$.

A special case of Lemma 10.3 is that for provincial domains B , the expected dimension $e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B)$, which is *a priori* in $\frac{1}{4}\mathbb{Z}$, is actually in \mathbb{Z} .

PROOF. We must show that $-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) \equiv \epsilon(\partial^\partial(B)) \pmod{1}$. To do so, we follow [OSz04d, Lemma 2.17] to construct a surface related to B . First, consider $\tilde{B} := \ell[\Sigma] + B$ where ℓ is large enough that all local multiplicities of \tilde{B} are non-negative. Here, \tilde{B} is an element of $\widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$, the set of homology classes of curves connecting \mathbf{x} to \mathbf{y} which are allowed to cross the basepoint z . (This is denoted $\pi_2(\mathbf{x}, \mathbf{y})$ in [OSz04d], but in this book we have reserved π_2 for classes which have local multiplicity zero at z .) Since $e([\Sigma]) + n_{\mathbf{x}}([\Sigma]) + n_{\mathbf{y}}([\Sigma]) = 1$, we have $e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) \equiv e(\tilde{B}) + n_{\mathbf{x}}(\tilde{B}) + n_{\mathbf{y}}(\tilde{B})$ modulo 1.

Label the components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ by R_i for $i = 1, \dots, N$, and write $\tilde{B} = \sum_i m(R_i)R_i$ with $i \geq 0$. Then we will construct our surface by identifying copies of R_i :

$$F := \left(\coprod_{i=1}^N \coprod_{j=1}^{m(R_i)} R_i^{(j)} \right) / \sim$$



where each $R_i^{(j)}$ is a diffeomorphic copy of R_i . To describe the identification \sim , first note that the α -curves are divided by the β -circles into shorter pieces, which we call α -segments, and similarly the β -circles are divided by the α -curves into β -segments. For each α - or β -segment c , let $R_1(c)$ and $R_2(c)$ be the two regions adjoining c , with $m(R_1(c)) \leq m(R_2(c))$. For $x \in c$, let $x_1^{(j)}$ and $x_2^{(j)}$ be the lifts of x to $R_1^{(j)}(c)$ and $R_2^{(j)}(c)$, respectively. Let $\delta_c := m(R_2(c)) - m(R_1(c))$. Then the equivalence relation \sim is generated by

$$\begin{aligned} x_1^{(j)} &\sim x_2^{(j+\delta_a)} & \text{for } x \text{ in an } \alpha\text{-segment } a, j = 1, \dots, m(R_1(a)) \\ x_1^{(j)} &\sim x_2^{(j)} & \text{for } x \text{ in an } \beta\text{-segment } b, j = 1, \dots, m(R_1(b)). \end{aligned}$$

Let Φ be the evident projection from F to Σ , so $\tilde{B} = \Phi_*[F]$. Topologically, F is a manifold with boundary; requiring the map Φ to be a local isometry makes F into a cone manifold.

We next analyze the local behavior of F around a point $x \in \alpha_i \cap \beta_j$. Let the four regions around x be $R_1(x), \dots, R_4(x)$ in cyclic order, arranged so that $m(R_1(x)) \leq m(R_i(x))$ for $i = 2, 3, 4$. The fact that $\tilde{B} \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$ means that

$$m(R_1(x)) + m(R_3(x)) = m(R_2(x)) + m(R_4(x)) + C(x),$$

where $C(x) \in \{0, -1, 1\}$. Specifically, $C(x) = 0$ when x is in neither \mathbf{x} nor \mathbf{y} or when x is in both \mathbf{x} and \mathbf{y} . For $x \in \Sigma$ let $D(x)$ be a small disk neighborhood of x and let $N(x) = \Phi^{-1}(D(x))$. The geometry of $N(x)$ depends on $C(x)$ as follows:

- If $C(x) = 0$ then $N(x)$ is a union of $m(R_1(x))$ disks and some half-disks, with the boundary of the half-disks lying above either α_i or β_j ; in this case, $N(x)$ is a manifold with boundary but no corners.
- If $C(x) = 1$ then one component of $N(x)$ has a cone point at x with total angle $2\pi m(R_1(x)) + \pi/2$. The other components of $N(x)$ are half-disks as in the $C = 0$ case.
- If $C(x) = -1$, then one component of $N(x)$ has a cone point at x with total angle $2\pi(m(R_1(x)) + 1) - \pi/2$. Again, the other components of $N(x)$ are half-disks.

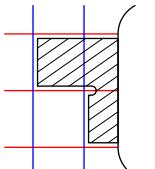
Let $n_+ := \#\{x \in \alpha \cap \beta \mid C(x) = 1\}$ and $n_- := \#\{x \in \alpha \cap \beta \mid C(x) = -1\}$, and let $h(x)$ be the number of half-disks in $N(x)$.

Recall that $\partial^\partial \tilde{B}$ denotes the part of $\partial \tilde{B}$ contained in $\partial \bar{\Sigma}$. For each $a_i \in \mathbf{a}$, let $\delta_i := m(\partial^\partial \tilde{B}, a_i + \epsilon) - m(\partial^\partial \tilde{B}, a_i - \epsilon)$ be the difference in multiplicities with which $\partial^\partial \tilde{B}$ covers the regions on the two sides of a_i . The surface F has corners at intersection points with $C(x) = \pm 1$ and at points above $\partial \Sigma$, and the number of corners of F over $\partial \bar{\Sigma}$ is $\frac{1}{4} \sum_i |\delta_i|$. Thus,

$$e(\tilde{B}) = e(F) \equiv \frac{1}{4} \left(-n_+ + n_- + \sum_{i=1}^{4k} |\delta_i| \right) \pmod{1}.$$

On the other hand,

$$\begin{aligned} n_{\mathbf{x}}(\tilde{B}) + n_{\mathbf{y}}(\tilde{B}) &= n_{\mathbf{x}+\mathbf{y}}(\tilde{B}) \equiv \sum_{x \in \mathbf{x}+\mathbf{y}} \left(\frac{1}{4} C(x) + \frac{1}{2} h(x) \right) \\ &\equiv \frac{1}{4} (n_+ - n_-) + \frac{1}{2} \sum_{x \in [(\mathbf{x}+\mathbf{y}) \cap \alpha^a]} h(x). \end{aligned}$$



For the second congruence, observe that on each β -circle β_j there is one $x_j \in \mathbf{x}$ and one $y_j \in \mathbf{y}$, and $h(x_j) = h(y_j)$. A similar statement holds for each α -circle, so it suffices to sum $h(x)$ only when x is in an α -arc.

It follows that

$$e(\tilde{B}) + n_{\mathbf{x}}(\tilde{B}) + n_{\mathbf{y}}(\tilde{B}) \equiv \frac{1}{4} \sum_{i=1}^{4k} |\delta_i| + \frac{1}{2} \sum_{x \in [(\mathbf{x}+\mathbf{y}) \cap \alpha^a]} h(x).$$

It remains to show that the right-hand side is equal to $\epsilon(\partial^\partial \tilde{B})$. Consider the contribution to the right-hand side from a particular α -arc α_ℓ^a and its endpoints a_i and a_j .

- If ℓ is not in $o(\mathbf{x})$ or $o(\mathbf{y})$, the set of α -arcs occupied by \mathbf{x} and \mathbf{y} , respectively, then there is no contribution to the $h(x)$ term and $\delta_i = -\delta_j$, so the net contribution is $\frac{1}{2}$ if δ_i is odd and 0 otherwise, agreeing with the contribution to $\epsilon(\partial^\partial \tilde{B})$.
- If ℓ is in both $o(\mathbf{x})$ and $o(\mathbf{y})$, then we have points $x_\ell \in \mathbf{x} \cap \alpha_\ell^a$ and $y_\ell \in \mathbf{y} \cap \alpha_\ell^a$. We have $h(x_\ell) = h(y_\ell)$, so the same argument as in the previous case applies.
- If ℓ is in exactly one of $o(\mathbf{x})$ and $o(\mathbf{y})$, we have $\delta_j = \delta_i \pm 1$. Assume without loss of generality that $|\delta_j| = |\delta_i| + 1$. Let $x_\ell \in (\mathbf{x} + \mathbf{y}) \cap \alpha_\ell^a$ be the unique corner on this arc. Then $h(x_\ell) = |\delta_i|$, so

$$\frac{1}{4}(|\delta_i| + |\delta_j|) + \frac{1}{2}h(x_\ell) \equiv \frac{1}{4}(2|\delta_i| + 1) + \frac{1}{2}|\delta_i| \equiv \frac{1}{4}.$$

Since exactly one of δ_i and δ_j is odd, the contribution of a_i and a_j to $\epsilon(\partial^\partial \tilde{B})$ is also $\frac{1}{4}$, as desired. \square

LEMMA 10.4. *For $B_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $B_2 \in \pi_2(\mathbf{y}, \mathbf{w})$, we have*

$$g'(B_1)g'(B_2) = g'(B_1 * B_2).$$

PROOF. This is obvious for the spin^c component. For the Maslov component, the proof is similar to Proposition 5.75. If we set $a_i = \partial^\alpha(B_i)$ and $b_i = \partial^\alpha(B_i)$, the Maslov component of $g'(B_1 * B_2)$ is

$$\begin{aligned} & (-n_{\mathbf{x}} - n_{\mathbf{w}} - e)(B_1 * B_2) \\ &= (-n_{\mathbf{x}} - n_{\mathbf{y}} - e)(B_1) + (-n_{\mathbf{y}} - n_{\mathbf{w}} - e)(B_2) + (n_{\mathbf{y}} - n_{\mathbf{w}})(B_1) + (n_{\mathbf{y}} - n_{\mathbf{x}})(B_2) \\ &= (-n_{\mathbf{x}} - n_{\mathbf{y}} - e)(B_1) + (-n_{\mathbf{y}} - n_{\mathbf{w}} - e)(B_2) + a_1 \cdot b_2 - a_2 \cdot b_1 \\ &= (-n_{\mathbf{x}} - n_{\mathbf{y}} - e)(B_1) + (-n_{\mathbf{y}} - n_{\mathbf{w}} - e)(B_2) + L_{\partial\mathcal{H}}(\partial^\partial(B_1), \partial^\partial(B_2)). \end{aligned}$$

where the second equality uses Lemma 5.73 and the third equality uses Lemma 5.74. The last formula is the Maslov component of $g'(B_1)g'(B_2)$, as desired. \square

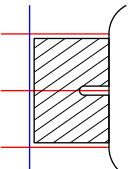
DEFINITION 10.5. For a generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, let $P'(\mathbf{x})$ be the subset of $G'(4k)$ given by $\{g'(B) \mid B \in \pi_2(\mathbf{x}, \mathbf{x})\}$.

COROLLARY 10.6. *$P'(\mathbf{x})$ is a subgroup of $G'(4k)$.*

This grading on domains is closely related to $\text{ind}(B, \vec{\rho})$.

LEMMA 10.7. *For a compatible pair $(B, \vec{\rho})$ where $\vec{\rho} = (\rho_1, \rho_2, \dots, \rho_\ell)$, we have*

$$g'(B) = \lambda^{\ell - \text{ind}(B, \vec{\rho})} \text{gr}'(\vec{\rho}).$$



PROOF. Applying Lemma 5.67, Definition 5.68 and Definition 10.1, in turn, gives

$$\begin{aligned}\text{gr}'(\vec{\rho}) &= (\iota(\vec{\rho}), \partial^\partial B) \\ &= (\text{ind}(B, \vec{\rho}) - \ell - e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^\partial B) \\ &= \lambda^{\text{ind}(B, \vec{\rho}) - \ell} g'(B).\end{aligned}$$
□

10.3. Type A structures

In this section we describe the (unrefined) grading on $\widehat{\text{CFA}}(\mathcal{H})$. Per Definition 2.42, we are looking for a right $G'(4k)$ -set $S'_A(\mathcal{H})$ and a map $\text{gr}' : \mathfrak{S}(\mathcal{H}) \rightarrow S'_A(\mathcal{H})$ so that

$$\begin{aligned}\text{gr}'(\partial \mathbf{x}) &= \lambda^{-1} \text{gr}'(\mathbf{x}) \\ \text{gr}'(\mathbf{x}a) &= \text{gr}'(\mathbf{x}) \text{gr}(a)\end{aligned}$$

and similarly, for higher products,

$$\text{gr}'(m_{\ell+1}(\mathbf{x}, a_1, \dots, a_\ell)) = \lambda^{-1+\ell} \text{gr}'(\mathbf{x}) \text{gr}(a_1) \cdots \text{gr}(a_\ell).$$

Since the different spin^c structures do not interact in any algebra action, we will construct one grading set $S'_A(\mathcal{H}, \mathfrak{s})$ for each spin^c structure \mathfrak{s} on Y ; then $S'_A(\mathcal{H})$ is defined as $\coprod_{\mathfrak{s} \in \text{spin}^c(Y)} S'_A(\mathcal{H}, \mathfrak{s})$.

We first give the grading on the module with twisted coefficients.

DEFINITION 10.8. To define the grading on the module $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, define $\underline{\text{S}}'_A(\mathcal{H}, \mathfrak{s})$ to be $G'(4k)$ (independent of \mathcal{H} and \mathfrak{s}), and set $\underline{\text{gr}}'(e^{B_0} \mathbf{x}) := g'(B_0)$.

PROPOSITION 10.9. *The map $\underline{\text{gr}}'$ defines a grading on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$. Furthermore, for $B \in \pi_2(\mathbf{x}, \mathbf{y})$, $\underline{\text{gr}}'$ satisfies*

$$(10.10) \quad \underline{\text{gr}}'(e^{B_0} \mathbf{x}) \cdot g'(B) = \underline{\text{gr}}'(e^{B_0 * B} \mathbf{y}),$$

and any other grading $\widetilde{\text{gr}}$ with this property is related to $\underline{\text{gr}}'$ by left translation, in the sense that there is an element $g_0 \in G'(4k)$ so that for all \mathbf{x} ,

$$\widetilde{\text{gr}}(\mathbf{x}) = g_0 \cdot \underline{\text{gr}}'(\mathbf{x}).$$

PROOF. Equation (10.10) is immediate from Lemma 10.4:

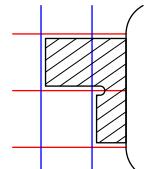
$$\underline{\text{gr}}'(e^{B_0 * B} \mathbf{y}) = g'(B_0 * B) = g'(B_0) g'(B) = \underline{\text{gr}}'(e^{B_0} \mathbf{x}) \cdot g'(B).$$

To check that $\underline{\text{gr}}'$ is a grading, from Definition 2.5 (in the form in Equation (2.43)), we want to show that for $\vec{\rho} = (\rho_1, \dots, \rho_\ell)$ any sequence of sets of Reeb chords, $m_{\ell+1}(e^{B_0} \mathbf{x}, a(\rho_1), \dots, a(\rho_\ell))$ is homogeneous of degree $\lambda^{\ell-1} \underline{\text{gr}}'(e^{B_0} \mathbf{x}) \text{gr}'(\vec{\rho})$. Let $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and let $\vec{\rho} = (\rho_1, \dots, \rho_\ell)$ be a sequence of sets of Reeb chords compatible with B . If $e^{B_0 * B} \mathbf{y}$ appears as a term in $m_{\ell+1}(e^{B_0} \mathbf{x}, a(\rho_1), \dots, a(\rho_\ell))$, then $\text{ind}(B, \vec{\rho}) = 1$, so by Equation (10.10) and Lemma 10.7,

$$\underline{\text{gr}}'(e^{B_0 * B} \mathbf{y}) = \underline{\text{gr}}'(e^{B_0} \mathbf{x}) \cdot g'(B) = \lambda^{\ell-1} \underline{\text{gr}}'(e^{B_0} \mathbf{x}) \cdot \text{gr}'(\vec{\rho}).$$

For any other grading $\widetilde{\text{gr}}$ satisfying Equation (10.10), let $g_0 = \widetilde{\text{gr}}(e^{I_0} \mathbf{x}_0)$, where $I_0 \in \pi_2(\mathbf{x}_0, \mathbf{x}_0)$ is the trivial domain. Then for any generator \mathbf{x} and any $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$,

$$\widetilde{\text{gr}}(e^{B_0} \mathbf{x}) = \widetilde{\text{gr}}(e^{I_0} \mathbf{x}) \cdot g'(B_0) = g_0 \cdot \underline{\text{gr}}'(e^{B_0} \mathbf{x}).$$
□



For the untwisted theory, the presence of periodic domains means that we can not just define the relative grading of \mathbf{x} and \mathbf{y} to be $g'(B) \in G'(4k)$ for some $B \in \pi_2(\mathbf{x}, \mathbf{y})$, as the analogue of Equation (10.10) would fail for any periodic domain B for which $g'(B)$ is non-trivial. Instead, we look for a right $G'(4k)$ -set $S'_A(\mathcal{H}, \mathfrak{s})$ satisfying an analogue of Equation (10.10). The $G'(4k)$ action on $S'_A(\mathcal{H}, \mathfrak{s})$ will be transitive but not in general free; this generalizes the fact that for closed 3-manifolds the gradings in a non-torsion spin^c structure take values in a quotient of \mathbb{Z} .

DEFINITION 10.11. Suppose that $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq \emptyset$. Pick any $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and define $S'_A(\mathcal{H}, \mathfrak{s})$ to be the coset space

$$(10.12) \quad S'_A := P'(\mathbf{x}_0) \backslash G'(4k).$$

Define $\text{gr}' : \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S'_A(\mathcal{H}, \mathfrak{s})$ on \mathbf{x} by picking any $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ and setting

$$(10.13) \quad \text{gr}'(\mathbf{x}) := P'(\mathbf{x}_0) \cdot g'(B_0).$$

The value of $\text{gr}'(\mathbf{x})$ is independent of B_0 : If $B_1 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ is any other element, then $g'(B_1 * (B_0)^{-1}) \in P'(\mathbf{x}_0)$. Thus, $\text{gr}'(\mathbf{x}) = P'(\mathbf{x}_0) \cdot g'(B_1)$.

For completeness, suppose next that $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \emptyset$. Choose a diagram \mathcal{H}' isotopic to \mathcal{H} and so that $\mathfrak{S}(\mathcal{H}', \mathfrak{s}) \neq \emptyset$ (by winding transverse to the β -circles; compare [OSz04d, Lemma 5.2]). Define $S'_A(\mathcal{H}, \mathfrak{s}) = S'_A(\mathcal{H}', \mathfrak{s})$, and let $\text{gr}' : \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S'_A(\mathcal{H}, \mathfrak{s})$ be the unique map from the empty set to $S'_A(\mathcal{H}', \mathfrak{s})$.

LEMMA 10.14. *The function $\text{gr}' : \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S'_A(\mathcal{H}, \mathfrak{s})$ above satisfies, for $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$,*

$$(10.15) \quad \text{gr}'(\mathbf{x}) \cdot g'(B) = \text{gr}'(\mathbf{y}).$$

Furthermore, if $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq \emptyset$ then $S'_A(\mathcal{H}, \mathfrak{s})$ is universal, in the sense that for any other $G'(4k)$ -set \tilde{S} and grading function $\tilde{\text{gr}}$ on $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$ satisfying (10.15), there is a unique $G'(4k)$ -set map $f : S'_A(\mathcal{H}, \mathfrak{s}) \rightarrow \tilde{S}$ so that for any generator \mathbf{x} , $\tilde{\text{gr}}(\mathbf{x}) = f(\text{gr}'(\mathbf{x}))$.

PROOF. To check Equation (10.15), pick $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$. Then $B_0 * B \in \pi_2(\mathbf{x}_0, \mathbf{y})$, so we have

$$\text{gr}'(\mathbf{y}) = P'(\mathbf{x}_0) \cdot g'(B_0) \cdot g'(B) = \text{gr}'(\mathbf{x}) \cdot g'(B).$$

(Note that we used the freedom to pick an arbitrary element of $\pi_2(\mathbf{x}_0, \mathbf{y})$ to define $\text{gr}'(\mathbf{y})$.)

For universality, let v_0 be $\tilde{\text{gr}}(\mathbf{x}_0)$. Then Equation (10.15) implies that $P'(\mathbf{x}_0)$ stabilizes v_0 . We can therefore set, for $h \in G'(4k)$,

$$f(P'(\mathbf{x}_0) \cdot h) := v_0 \cdot h.$$

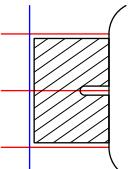
It is immediate that $\tilde{\text{gr}}(\mathbf{x}) = f(\text{gr}'(\mathbf{x}))$. □

Observe that any right $G'(4k)$ -set map from $G'(4k)$ to itself is left translation as in Proposition 10.9.

COROLLARY 10.16. *If we use a different base generator $\mathbf{x}'_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ to define $S'_A(\mathcal{H}, \mathfrak{s})$, the grading of an element \mathbf{x} with respect to \mathbf{x}'_0 is $g'(B_0) \cdot \text{gr}'(\mathbf{x})$ for any $B_0 \in \pi_2(\mathbf{x}'_0, \mathbf{x}_0)$.*

PROOF. This follows from noticing that $P'(\mathbf{x}'_0) = g'(B_0) \cdot P'(\mathbf{x}_0) \cdot g'(B_0)^{-1}$. □

Left translation by $g'(B_0)$ is an isomorphism of right $G'(4k)$ -sets from $P'(\mathbf{x}_0) \backslash G'(4k)$ to $P'(\mathbf{x}'_0) \backslash G'(4k)$.



PROPOSITION 10.17. *For the grading gr' of Definition 10.11, $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ is a graded right \mathcal{A}_∞ module in the sense of Definition 2.42.*

PROOF. This follows from Lemma 10.14 as in the proof of Proposition 10.9. \square

We now have the following graded version of Theorems 7.17 and 7.28 (and indeed Theorem 1.2):

THEOREM 10.18. *Up to graded \mathcal{A}_∞ homotopy equivalence, the $G'(4k)$ -graded \mathcal{A}_∞ module $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ over $\mathcal{A}(\partial\mathcal{H})$ is independent of the choices made in its definition. That is, for any choices of provincially admissible Heegaard diagrams \mathcal{H}_1 and \mathcal{H}_2 , sufficiently generic admissible almost complex structures J_i for \mathcal{H}_i and base generators $\mathbf{x}_0^i \in \mathfrak{S}(\mathcal{H}_i, \mathfrak{s})$, there is an isomorphism of $G'(4k)$ -sets*

$$\phi: S'_A(\mathcal{H}_1, \mathbf{x}_0^1) \rightarrow S'_A(\mathcal{H}_2, \mathbf{x}_0^2)$$

and an \mathcal{A}_∞ homotopy equivalence of \mathcal{A}_∞ modules

$$f = \{f_{1+i}: \widehat{\text{CFA}}(\mathcal{H}_1, J_1) \otimes \mathcal{A}(\partial\mathcal{H})^{\otimes i} \rightarrow \widehat{\text{CFA}}(\mathcal{H}_2, J_2)\}$$

such that, for $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_1)$ and homogeneous elements $a_1, \dots, a_\ell \in \mathcal{A}(\partial\mathcal{H})$,

$$\text{gr}'(f_{1+\ell}(\mathbf{x}, a_1, \dots, a_\ell)) = \phi(\text{gr}'(\mathbf{x})) \text{gr}'(a_1) \dots \text{gr}'(a_\ell) \lambda^\ell.$$

The graded \mathcal{A}_∞ bimodule $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ over $\mathbb{F}_2[H_2(Y, \partial Y)]$ and $\mathcal{A}(\partial\mathcal{H})$ is similarly invariant.

PROOF. We prove invariance of the untwisted theory; the proof for the twisted theory is similar.

Independence of the grading from the choice of base generator (assuming $\mathcal{H}_1 = \mathcal{H}_2$ and $J_1 = J_2$) is immediate from Corollary 10.16: the map g is the identity map (i.e., $g_1 = \mathbb{I}$ and $g_i = 0$ for $i > 1$) and the map ϕ is given by $\phi(s) = g'(B_0)s$, where $B_0 \in \pi_2(\mathbf{x}_0^2, \mathbf{x}_0^1)$. (Left translation is an isomorphism of right $G'(4k)$ -sets.)

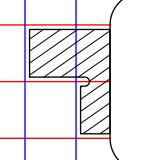
The rest of the proof follows the proof of the ungraded version, Theorem 7.17, noting that the identifications induced by changes of complex structure, isotopies, handleslides, and stabilizations all respect gradings.

Stabilization invariance is immediate: stabilizing near the basepoint, the \mathcal{A}_∞ modules before and after stabilization are identified, along with all the grading data. For changes of complex structure, the fact that the continuation maps preserve gradings is a straightforward adaptation of proof that $\widehat{\text{CFA}}$ is graded (Proposition 10.17). The key point is that now, if \mathbf{y} appears in $f_{1+\ell}^{J_r}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_\ell))$, then there is a domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\text{ind}(B, \bar{\boldsymbol{\rho}}) = 0$ (rather than 1, as it would be for $m_{1+\ell}$ in place of $f_{1+\ell}^{J_r}$), and hence

$$\text{gr}'(\mathbf{y}) = \text{gr}'(\mathbf{x}) \cdot \text{gr}'(a_1) \dots \text{gr}'(a_\ell) \lambda^\ell.$$

This is precisely what is needed for $f^{J_r} = \{f_n^{J_r}\}_{n=1}^\infty$ to be a graded \mathcal{A}_∞ homomorphism.

The proof of isotopy invariance is the same as the proof of invariance under change of complex structure except that the base generator may disappear during the isotopy. If there are no generators in $\mathfrak{S}(\mathcal{H}_1, \mathfrak{s})$ or $\mathfrak{S}(\mathcal{H}_2, \mathfrak{s})$ then the result is obvious. Otherwise, we can choose the isotopy so that there is an intermediate Heegaard diagram $\mathcal{H}_{3/2}$ so that there is some generator \mathbf{x}_0^1 in both $\mathfrak{S}(\mathcal{H}_1, \mathfrak{s})$ and $\mathfrak{S}(\mathcal{H}_{3/2}, \mathfrak{s})$, and a generator \mathbf{x}_0^2 in both $\mathfrak{S}(\mathcal{H}_{3/2}, \mathfrak{s})$ and $\mathfrak{S}(\mathcal{H}_2, \mathfrak{s})$. The same argument as for independence of complex structures gives a graded



homotopy equivalence between $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s})$ and $\widehat{CFA}(\mathcal{H}_{3/2}, \mathfrak{s})$, using \mathbf{x}_0^1 to define the grading, and a graded homotopy equivalence between $\widehat{CFA}(\mathcal{H}_{3/2}, \mathfrak{s})$ and $\widehat{CFA}(\mathcal{H}_2, \mathfrak{s})$, using \mathbf{x}_0^2 to define the grading. Together with the independence of the grading from the base generator, which we already verified, this gives the invariance result for isotopies.

For handleslides, we must take into account the fact that the grading sets for the two diagrams are different. Specifically, suppose that \mathcal{H}_2 is gotten from \mathcal{H}_1 by a handleslide; as in Section 7.3.2, suppose this is a handleslide of an α -arc over an α -circle, and write $\mathcal{H}_2 = (\Sigma, \boldsymbol{\alpha}^H, \boldsymbol{\beta}, z)$, $\mathcal{H}_1 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Since we have already verified independence from the basepoint, we may choose \mathbf{x}_0^2 to be the generator connected to \mathbf{x}_0^1 by a union of small triangles T_x (see Section 6.3.2). Then, the grading set for \mathcal{H}_1 is $P'(\mathbf{x}_0^1) \setminus G'(4k)$, whereas the grading set for \mathcal{H}_2 is $P'(\mathbf{x}_0^2) \setminus G'(4k)$. There is a map $\Phi: P'(\mathbf{x}_0) \rightarrow P'(\mathbf{x}_0')$ defined by sending $B \in \pi_2(\mathbf{x}_0^1, \mathbf{x}_0^1)$ to the element $\Phi(B) \in \pi_2(\mathbf{x}_0^2, \mathbf{x}_0^2)$ uniquely determined by

$$T_x *_{13} B = T_x *_{12} *B_{\alpha, \alpha^H} *_{23} \Phi(B),$$

using Lemma 6.24. (Here, $B_{\alpha, \alpha^H} \in \pi_2(\Theta_o, \Theta_{o'})$ for appropriate o and o' .) By inspection, $g'(\Phi(B)) = g'(B)$. So, $P'(\mathbf{x}_0^1) = P'(\mathbf{x}_0^2)$, and $S'_A(\mathcal{H}_1, \mathbf{x}_0^1) = S'_A(\mathcal{H}_2, \mathbf{x}_0^2)$.

Next we check that the handleslide map $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$ respects this identification of grading sets. By Corollary 6.32 (and the definition of the map $f^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}$, Equation (7.26)),

$$f_{1+\ell}^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_\ell))$$

is given by a count of triangles in homology classes b so that $\text{ind}(b(B), (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_\ell)) = 0$. So, as for the map associated to changes of almost complex structure, if \mathbf{y} occurs in $f_{1+\ell}^{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H, \boldsymbol{\beta}}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_\ell))$ then

$$\text{gr}'(\mathbf{y}) = \text{gr}'(\mathbf{x}) \cdot \text{gr}'(a_1) \cdots \text{gr}'(a_\ell) \lambda^\ell.$$

This completes the proof. \square

10.4. Type D structures

We turn now to defining a grading on $\widehat{CFD}(\mathcal{H})$. One point that merits attention is that the relevant grading groups $G'(4k)$ and $G(\mathcal{Z})$ depend on the orientation of \mathcal{Z} . In particular, in the definition of the product on $G'(4k)$,

$$(j_1, \alpha_1) \cdot (j_2, \alpha_2) := (j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

(from Equation (3.34)), the correction term $L(\cdot, \cdot)$ depends on whether we are in $\partial\mathcal{H}$ or $-\partial\mathcal{H}$. To avoid confusion, let $r: Z \rightarrow -Z$ be the identity map, which is orientation-reversing, and let $R: G'(4k) \rightarrow G'(4k)$ or $G(\mathcal{Z}) \rightarrow G(-\mathcal{Z})$ be the induced map:

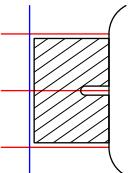
$$(10.19) \quad R(j, \alpha) = (j, r_*(\alpha)).$$

It follows from the form of the product that R is a group anti-homomorphism, so, for instance,

$$R(g'(B_1 * B_2)) = R(g'(B_2))R(g'(B_1)).$$

We have an analogue of Lemma 10.7 for the moduli spaces relevant for \widehat{CFD} .

LEMMA 10.20. *For a compatible pair $(B, \vec{\rho})$, we have $\text{gr}'(-\vec{\rho})R(g'(B)) = (-\text{ind}(B, \vec{\rho}), 0)$.*



PROOF. This can be deduced from Lemma 10.7, but we find it clearer to give a direct proof. The spin^c component of $R(g'(B))$ is the negative of the spin^c component of $\underline{\text{gr}}'(-\vec{\rho})$, since all the boundary intervals are reversed in the latter. Thus the spin^c component of the result is 0, and Maslov component of the product is obtained by adding the Maslov components of the factors. The Maslov component of $R(g'(B))$ is, by definition, $-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B)$. On the other hand, if $\vec{\rho} = (\rho_1, \dots, \rho_n)$ then

$$\begin{aligned}
\underline{\text{gr}}'(-\vec{\rho}) &= \left(-\frac{1}{2}, -[\rho_1]\right) \cdots \left(-\frac{1}{2}, -[\rho_n]\right) \\
&= \left(-\frac{n}{2} + \sum_{i < j} L(r_*(-[\rho_i]), r_*(-[\rho_j])), -\sum_i [\rho_i]\right) \\
(10.21) \quad &= \left(-\frac{n}{2} - \sum_{i < j} L([\rho_i], [\rho_j]), -\sum_i [\rho_i]\right) \\
&= \left(-n - \iota(\vec{\rho}), -\sum_i [\rho_i]\right),
\end{aligned}$$

where we use Lemma 5.66, bilinearity of L , and the definition of ι (Equation (5.65)). Thus, by Definition 5.68, the Maslov component of the product is $-\text{ind}(B, \vec{\rho})$. \square

DEFINITION 10.22. For $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$ as for $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, define $S'_D(\mathcal{H}, \mathfrak{s})$ to be $G'(4k)$, and set $\underline{\text{gr}}'(e^{B_0}\mathbf{x}) := R(g'(B_0))$.

PROPOSITION 10.23. *The map $\underline{\text{gr}}'$ above defines a grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$. Furthermore, for $B \in \pi_2(\mathbf{x}, \mathbf{y})$, $\underline{\text{gr}}'$ satisfies*

$$(10.24) \quad R(g'(B)) \cdot \underline{\text{gr}}'(e^{B_0}\mathbf{x}) = \underline{\text{gr}}'(e^{B_0 * B}\mathbf{y}),$$

and any other grading $\widetilde{\text{gr}}$ with this property is related to $\underline{\text{gr}}'$ by right translation, in the sense that there is an element $g_0 \in G'(4k)$ so that for all \mathbf{x} ,

$$\widetilde{\text{gr}}(\mathbf{x}) = \underline{\text{gr}}'(\mathbf{x}) \cdot g_0.$$

PROOF. If there is a term of the form $a(-\vec{\rho})e^{B_0 * B}\mathbf{y}$ occurring in $\partial(e^{B_0}\mathbf{x})$ then $\text{ind}(B, \vec{\rho}) = 1$, and so

$$\begin{aligned}
\underline{\text{gr}}'(a(-\vec{\rho})e^{B_0 * B}\mathbf{y}) &= \underline{\text{gr}}'(a(-\vec{\rho}))R(g'(B))\underline{\text{gr}}'(e^{B_0}\mathbf{x}) \\
&= \lambda^{-1}\underline{\text{gr}}'(e^{B_0}\mathbf{x}).
\end{aligned}$$

The universality statement is analogous to the case for $\widehat{\text{CFA}}$, Proposition 10.9. \square

Again, for the untwisted theory, we need to find an appropriate left $G'(4k)$ -set satisfying a version of Equation (10.24).

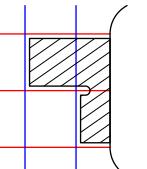
DEFINITION 10.25. To define the grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$, take $S'_D(\mathcal{H}, \mathfrak{s})$ to have the same underlying set as $S'_A(\mathcal{H}, \mathfrak{s})$, with the left action of g on $S'_D(\mathcal{H}, \mathfrak{s})$ being given by the right action of $R(g)$ on $S'_A(\mathcal{H}, \mathfrak{s})$. Concretely, if $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq \emptyset$,

$$S'_D(\mathcal{H}, \mathfrak{s}) := G'(4k)/R(P'(\mathbf{x}_0)),$$

and define $\text{gr}' : \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S'_D(\mathcal{H}, \mathfrak{s})$ on \mathbf{x} by picking any $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ and setting

$$(10.26) \quad \text{gr}'(\mathbf{x}) := R(g'(B_0)) \cdot R(P'(\mathbf{x}_0)).$$

(If $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \emptyset$ we proceed as in this degenerate case of Definition 10.11.)



LEMMA 10.27. *The function $\text{gr}' : \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S'_D(\mathcal{H}, \mathfrak{s})$ above satisfies, for $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$,*

$$(10.28) \quad R(g'(B)) \cdot \text{gr}'(\mathbf{x}) = \text{gr}'(\mathbf{y}).$$

Furthermore, $S'_D(\mathcal{H}, \mathfrak{s})$ is universal, in the sense that for any other $G'(4k)$ -set \tilde{S} and grading function $\tilde{\text{gr}}$ on $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$ satisfying (10.28), there is a unique $G'(4k)$ -set map $f : S'_D(\mathcal{H}, \mathfrak{s}) \rightarrow \tilde{S}$ so that for any generator \mathbf{x} , $\tilde{\text{gr}}(\mathbf{x}) = f(\text{gr}'(\mathbf{x}))$.

PROOF. This follows from Lemma 10.14 by reversal of all products. \square

PROPOSITION 10.29. *For the grading gr' given by Definition 10.25, $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ is a graded left differential module*

PROOF. This is similar to the proof of Proposition 10.23, and is left to the reader. \square

THEOREM 10.30. *Up to graded homotopy equivalence, the $G'(4k)$ -set graded differential module $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ over $\mathcal{A}(-\partial\mathcal{H})$ is independent of the choices made in its definition. That is, for any choices of provincially admissible Heegaard diagrams \mathcal{H}_1 and \mathcal{H}_2 , sufficiently generic admissible almost complex structures J_i for \mathcal{H}_i and base generators $\mathbf{x}_0^i \in \mathfrak{S}(\mathcal{H}_i, \mathfrak{s})$, there is an isomorphism of $G'(4k)$ -sets*

$$\phi : S'_D(\mathcal{H}_1, \mathbf{x}_0^1) \rightarrow S'_D(\mathcal{H}_2, \mathbf{x}_0^2)$$

and a homotopy equivalence of differential modules

$$f : \widehat{CFD}(\mathcal{H}_1, J_1) \rightarrow \widehat{CFD}(\mathcal{H}_2, J_2)$$

such that, for each $\mathbf{x} \in \mathfrak{S}(\mathcal{H}_1)$ and each term $a\mathbf{y}$ in $f(\mathbf{x})$ (where $\mathbf{y} \in \mathfrak{S}(\mathcal{H}_2, \mathfrak{s})$ and $a \in \mathcal{A}(-\partial\mathcal{H})$),

$$\text{gr}'(a) \text{gr}'(\mathbf{y}) = \phi(\text{gr}'(\mathbf{x})).$$

The dg bimodule $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ over $\mathbb{F}_2[H_2(Y, \partial Y)]$ and $\mathcal{A}(-\partial\mathcal{H})$ is similarly invariant.

PROOF. This is similar to the proof of Theorem 10.18, and is left to the reader. \square

10.5. Refined gradings

We turn now to a refined grading gr on these modules, building on the refined grading of the algebra. We start with a review of the small grading group $G(\mathcal{Z})$. The group $G(\mathcal{Z})$ can be viewed either abstractly as a central extension or concretely as a subgroup of $G'(4k)$. Abstractly, it is the central extension of $H_1(F(\mathcal{Z}))$, with a sequence of groups

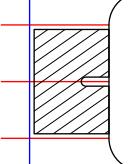
$$\mathbb{Z} \xrightarrow{\lambda} G(\mathcal{Z}) \xrightarrow{[\cdot]} H_1(F)$$

satisfying the relation

$$gh = hg\lambda^{2([g] \cap [h])}$$

(Equation (3.42)). Concretely, the group $G(\mathcal{Z})$ is the subgroup of $G'(4k)$ consisting of those elements (j, α) where $M_*\partial(\alpha) = 0$, that is, for each pair p, q of points in \mathbf{a} identified by the matching M defining \mathcal{Z} , the sum of the multiplicities of $\partial\alpha$ at p and at q is 0.

The refined grading on $\mathcal{A}(\mathcal{Z}, 0)$ is built from grading refinement data: a choice of a base idempotent $I(\mathbf{s}_0) \in \mathcal{A}(\mathcal{Z}, 0)$ and, for every other minimal idempotent $I(\mathbf{s})$, an element



$\psi(\mathbf{s}) = \psi(I(\mathbf{s})) \in G'(4k)$ satisfying $M_*\partial([\psi(\mathbf{s})]) = \mathbf{s} - \mathbf{s}_0$. The grading on the algebra is then defined by

$$\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})I(\mathbf{t})) := \psi(\mathbf{s}) \text{gr}'(a(\boldsymbol{\rho}))\psi(\mathbf{t})^{-1}$$

(Equation (3.44)).

We now turn to refined gradings on domains. Recall first that every generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ has an associated set $o(\mathbf{x})$ of occupied α -arcs (Definition 4.13) and associated idempotents $I_A(\mathbf{x}) = I(o(\mathbf{x}))$ and $I_D(\mathbf{x}) = I([2k] \setminus o(\mathbf{x}))$. Then, for $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and $B \in \pi_2(\mathbf{x}, \mathbf{y})$, define

$$(10.31) \quad g(B) := \psi(I_A(\mathbf{x}))g'(B)\psi(I_A(\mathbf{y}))^{-1}$$

$$(10.32) \quad P(\mathbf{x}) := \{g(B) \mid B \in \pi_2(\mathbf{x}, \mathbf{x})\}.$$

LEMMA 10.33. *The function g satisfies $g(B) \in G(\mathcal{Z})$ and $g(B_1 * B_2) = g(B_1) \cdot g(B_2)$.*

PROOF. To verify that $g(B) \in G(\mathcal{Z})$, compute

$$\begin{aligned} M_*\partial([g(B)]) &= M_*\partial([\psi(I_A(\mathbf{x}))]) + M_*\partial([g'(B)]) - M_*(\partial([\psi(I_A(\mathbf{y}))])) \\ &= (o(\mathbf{x}) - \mathbf{s}_0) + (o(\mathbf{y}) - o(\mathbf{x})) - (o(\mathbf{y}) - \mathbf{s}_0) \\ &= 0. \end{aligned}$$

The fact that $g(B_1 * B_2) = g(B_1) \cdot g(B_2)$ is immediate from the definitions and Lemma 10.4. \square

In particular, $P(\mathbf{x})$ is a subgroup of $G(\mathcal{Z})$.

DEFINITION 10.34. To define the refined grading on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, define $\underline{S}_A(\mathcal{H}, \mathfrak{s})$ to be $G(\partial\mathcal{H})$ (independent of \mathcal{H} and \mathfrak{s}), and set $\text{gr}(e^{B_0}\mathbf{x}) := g(B_0)$.

To define the refined grading on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$, suppose first that $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq \emptyset$. Pick any $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$. Define $S_A(\mathcal{H}, \mathfrak{s})$ by

$$S_A(\mathcal{H}, \mathfrak{s}) := P(\mathbf{x}_0) \setminus G(\partial\mathcal{H})$$

and define $\text{gr}(\mathbf{x})$ by picking any $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ and setting

$$\text{gr}(\mathbf{x}) := P(\mathbf{x}_0) \cdot g(B).$$

For completeness, if $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \emptyset$ choose a diagram \mathcal{H}' isotopic to \mathcal{H} and so that $\mathfrak{S}(\mathcal{H}', \mathfrak{s}) \neq \emptyset$, let $S_A(\mathcal{H}, \mathfrak{s}) = S_A(\mathcal{H}', \mathfrak{s})$ and let $\text{gr}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S_A(\mathcal{H}, \mathfrak{s})$ be the unique map from the empty set to $S_A(\mathcal{H}', \mathfrak{s})$.

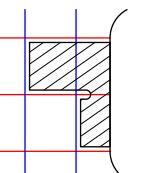
PROPOSITION 10.35. *The function $\text{gr}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow \underline{S}_A(\mathcal{H}, \mathfrak{s})$ above defines a grading on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$. Likewise, $\text{gr}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S_A(\mathcal{H}, \mathfrak{s})$ defines a grading on $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$.*

PROOF. The proof follows the proofs of Propositions 10.9 and 10.17, noting that if $m_{n+1}(\mathbf{x}, a_1, \dots, a_n) \neq 0$, the idempotent of \mathbf{x} agrees with the left idempotent of a_1 and the right idempotent of a_i agrees with the left idempotent of a_{i+1} , so the corresponding ψ 's cancel. \square

To define the refined gradings on $\widehat{\text{CFD}}$ and $\widehat{\text{CFD}}$, we first need to fix grading refinement data for $\mathcal{A}(-\partial\mathcal{H}, 0)$. We choose it as follows. Fix a choice of $(\mathbf{s}_0)_{\partial\mathcal{H}}$ and $\psi_{\partial\mathcal{H}}$ determining a grading refinement for $\mathcal{A}(\partial\mathcal{H}, 0)$ and define

$$(\mathbf{s}_0)_{-\partial\mathcal{H}} := [2k] \setminus (\mathbf{s}_0)_{\partial\mathcal{H}}$$

$$\psi_{-\partial\mathcal{H}}(\mathbf{s}) := R(\psi([2k] \setminus \mathbf{s}))^{-1}.$$



With this choice,

$$M_*\partial([\psi_{-\partial\mathcal{H}}(\mathbf{s})]) = -M_*\partial([\psi_{\partial\mathcal{H}}([2k] \setminus \mathbf{s})]) = -(([2k] - \mathbf{s}) - (\mathbf{s}_0)_{\partial\mathcal{H}}) = \mathbf{s} - (\mathbf{s}_0)_{-\partial\mathcal{H}},$$

so $(\mathbf{s}_0)_{-\partial\mathcal{H}}$ and $\psi_{-\partial\mathcal{H}}(\mathbf{s})$ define a grading refinement on $\mathcal{A}(-\partial\mathcal{H}, 0)$; we call this the *reverse* of the grading refinement $(\mathbf{s}_0)_{\partial\mathcal{H}}$ and $\psi_{\partial\mathcal{H}}$. Then, for $B \in \pi_2(\mathbf{x}, \mathbf{y})$,

$$\begin{aligned} R(g(B)) &= R(\psi_{\partial\mathcal{H}}(I_A(\mathbf{x}))g'(B)\psi_{\partial\mathcal{H}}(I_A(\mathbf{y}))^{-1}) \\ (10.36) \quad &= \psi_{-\partial\mathcal{H}}([2k] \setminus I_A(\mathbf{y}))R(g'(B))\psi_{-\partial\mathcal{H}}([2k] \setminus I_A(\mathbf{x}))^{-1} \\ &= \psi_{-\partial\mathcal{H}}(I_D(\mathbf{y}))R(g'(B))\psi_{-\partial\mathcal{H}}(I_D(\mathbf{x}))^{-1}. \end{aligned}$$

DEFINITION 10.37. To define the refined grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$, define $\underline{S}_D(\mathcal{H}, \mathfrak{s})$ to be $G(-\partial\mathcal{H})$ (independent of \mathcal{H} and \mathfrak{s}), and set $\underline{\text{gr}}(e^{B_0}\mathbf{x}) := R(g(B_0))$.

To define the refined grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$, suppose first that $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) \neq \emptyset$. Pick any $\mathbf{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$. Define $S_D(\mathcal{H}, \mathfrak{s})$ by

$$S_D(\mathcal{H}, \mathfrak{s}) := G(-\partial\mathcal{H})/R(P(\mathbf{x}_0))$$

and define $\text{gr}(\mathbf{x})$ by picking any $B_0 \in \pi_2(\mathbf{x}_0, \mathbf{x})$ and setting

$$\text{gr}(\mathbf{x}) := R(g(B)) \cdot R(P(\mathbf{x}_0)).$$

For completeness, if $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \emptyset$ choose a diagram \mathcal{H}' isotopic to \mathcal{H} and so that $\mathfrak{S}(\mathcal{H}', \mathfrak{s}) \neq \emptyset$, let $S_D(\mathcal{H}, \mathfrak{s}) = S_D(\mathcal{H}', \mathfrak{s})$ and let $\text{gr}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S_D(\mathcal{H}, \mathfrak{s})$ be the unique map from the empty set to $S_D(\mathcal{H}', \mathfrak{s})$.

PROPOSITION 10.38. *The function $\underline{\text{gr}}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow \underline{S}_D(\mathcal{H}, \mathfrak{s})$ above defines a grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$. Likewise, $\text{gr}: \mathfrak{S}(\mathcal{H}, \mathfrak{s}) \rightarrow S_D(\mathcal{H}, \mathfrak{s})$ defines a grading on $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$.*

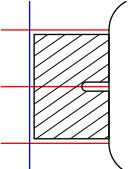
PROOF. The proof follows the proofs of Propositions 10.23 and 10.29, noting that if $a\mathbf{x} \neq 0$, the right idempotent of a agrees with the idempotent of \mathbf{x} , so the intermediate ψ cancels. \square

We have the following analogue of the graded invariance theorems (Theorems 10.30 and 10.18) with respect to the refined gradings.

THEOREM 10.39. *Up to graded \mathcal{A}_∞ homotopy equivalence, the $G(\partial\mathcal{H})$ -set graded \mathcal{A}_∞ module $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ over $\mathcal{A}(\partial\mathcal{H})$ is independent of the choices of provincially admissible Heegaard diagram, sufficiently generic admissible almost complex structure and base generator used to define it; and up to graded homotopy equivalence, the $G(-\partial\mathcal{H})$ -set graded differential module $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$ over $\mathcal{A}(-\partial\mathcal{H})$ is independent of the choices of provincially admissible Heegaard diagram, sufficiently generic admissible almost complex structure and base generator used to define it. Similar statements hold for $\mathbb{F}_2[H_2(Y, \partial Y)]-\mathcal{A}(\partial\mathcal{H})$ and $\mathbb{F}_2[H_2(Y, \partial Y)]-\mathcal{A}(-\partial\mathcal{H})$ bimodules $\widehat{\text{CFA}}(\mathcal{H}, \mathfrak{s})$ and $\widehat{\text{CFD}}(\mathcal{H}, \mathfrak{s})$.*

PROOF. This is immediate from the definitions and Theorems 10.30 and 10.18. \square

REMARK 10.40. Theorem 10.39 does not assert that the refined gradings are independent of the grading refinement data. For some discussion of this point, see Remark 3.46.



10.6. Tensor product

We now turn our attention to a graded version of the pairing theorem. Let $(Y_1, \phi_1: F(\mathcal{Z}) \rightarrow \partial Y_1)$ and $(Y_2, \phi_2: F(-\mathcal{Z}) \rightarrow \partial Y_2)$ be bordered three-manifolds which agree along their boundary. As discussed in Section 4.5, given bordered Heegaard diagrams \mathcal{H}_i for the (Y_i, ϕ_i) , gluing \mathcal{H}_1 and \mathcal{H}_2 along their boundary gives a Heegaard diagram $\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ for $Y = Y_1 \cup_{\partial} Y_2$.

For each spin^c structure \mathfrak{s}_1 over Y_1 , $\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1)$ is graded by the set $P_1(\mathbf{x}_1) \setminus G(\mathcal{Z})$, where $\mathbf{x}_1 \in \mathfrak{S}(\mathcal{H}_1)$ is a chosen base generator representing \mathfrak{s}_1 , and $P_1(\mathbf{x}_1)$ denotes the image of $\pi_2(\mathbf{x}_1, \mathbf{x}_1)$ in $G(\mathcal{Z})$. Similarly, for each spin^c structure \mathfrak{s}_2 over Y_2 , $\widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2)$ is graded by the set $G(\mathcal{Z}) / R(P_2(\mathbf{x}_2))$, where $\mathbf{x}_2 \in \mathfrak{S}(\mathcal{H}_2)$ is a chosen base generator representing \mathfrak{s}_2 , and $P_2(\mathbf{x}_2)$ denotes the image of $\pi_2(\mathbf{x}_2, \mathbf{x}_2)$ in $G(-\mathcal{Z})$. (Here the grading refinement on $G(-\mathcal{Z})$ is the reverse of the grading refinement on $G(\mathcal{Z})$, so that the grading refinement used in the definition of the grading of $\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1)$ is the same as the grading refinement used in the definition of $\widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2)$.) Correspondingly, the tensor product $\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2)$ inherits a grading gr^{\boxtimes} with values in the double-coset space

$$P_1(\mathbf{x}_1) \setminus G(\mathcal{Z}) / R(P_2(\mathbf{x}_2)).$$

We choose the base generators \mathbf{x}_1 and \mathbf{x}_2 to occupy complementary α -arcs, i.e., so that $\mathbf{x}_1 \times \mathbf{x}_2 \in \mathfrak{S}(\mathcal{H})$. Let $\mathbf{x} := \mathbf{x}_1 \times \mathbf{x}_2$.

The above double-coset space inherits an action by \mathbb{Z} (acting on the Maslov component). The quotient by this action is naturally identified with the quotient of $H_1(F)$ by the image of $H_2(Y_1, F) \oplus H_2(Y_2, F)$ under the boundary homomorphism, or, equivalently, with the image of $H_1(F)$ in $H_1(Y)$. More succinctly, we have

$$\begin{array}{ccc} P_1(\mathbf{x}_1) \setminus G / R(P_2(\mathbf{x}_2)) & & \\ \downarrow & \searrow [\cdot] & \\ (P_1(\mathbf{x}_1) \setminus G / R(P_2(\mathbf{x}_2))) / \mathbb{Z} & \xrightarrow{\cong} & \text{Im}(H_1(F) \rightarrow H_1(Y)), \end{array}$$

where $\gamma \mapsto [\gamma]$ denotes the map from the double-coset space to $H_1(Y)$. A more invariant version is

$$\begin{array}{ccc} P_1(\mathbf{x}_1) \setminus G / R(P_2(\mathbf{x}_2)) & & \\ \downarrow & \searrow \Pi & \\ (P_1(\mathbf{x}_1) \setminus G / R(P_2(\mathbf{x}_2))) / \mathbb{Z} & \xrightarrow{\cong} & \{\mathfrak{s} \in \text{spin}^c(Y) \mid \mathfrak{s}|_{Y_i} \cong \mathfrak{s}_i, i = 1, 2\}, \end{array}$$

where the map Π is defined by $\Pi(\gamma) := \mathfrak{s}(\mathbf{x}) + [\gamma]$. We are implicitly using Poincaré duality to think of $\text{spin}^c(Y)$ as an affine space over $H_1(Y)$ (rather than the usual $H^2(Y)$).

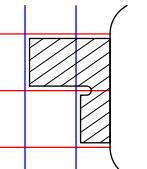
Let $\text{div}(c_1(\mathfrak{s}'))$ be the divisibility of the first Chern class of \mathfrak{s}' , i.e.,

$$H^1(Y; \mathbb{Z}) \cup c_1(\mathfrak{s}') = \text{div}(c_1(\mathfrak{s}')) \cdot H^3(Y; \mathbb{Z}).$$

LEMMA 10.41. *For $\mathfrak{s}' \in \text{spin}^c(Y)$, $\Pi^{-1}(\mathfrak{s}')$ is isomorphic as a \mathbb{Z} -set to $\mathbb{Z}/\text{div}(c_1(\mathfrak{s}'))$.*

PROOF. Suppose that $\gamma \in G(\mathcal{Z})$ is a group element with the property that γ and $\lambda^t \cdot \gamma$ represent the same double coset. This means that there are $B_i \in \pi_2(\mathbf{x}_i, \mathbf{x}_i)$ with the property that

$$\gamma = \lambda^t \cdot g(B_1) \cdot \gamma \cdot R(g(B_2)).$$



Let $\psi_0 = \psi(I_A(\mathbf{x}_1)) = \psi(I_D(\mathbf{x}_2))$. Then, by (10.31) and (10.36),

$$\gamma = \lambda^t \cdot \psi_0 g'(B_1) \psi_0^{-1} \cdot \gamma \cdot \psi_0 R(g'(B_2)) \psi_0^{-1}$$

or, setting $\gamma' = \psi_0^{-1} \gamma \psi_0$,

$$(10.42) \quad \gamma' = \lambda^t \cdot g'(B_1) \cdot \gamma' \cdot R(g'(B_2)).$$

Note that $[\gamma]$ and $[\gamma']$ are the same element of $H_1(F)$.

This implies that $\partial^\partial B_1 + \partial^\partial B_2 = 0$, so B_1 and B_2 glue to a domain $B = B_1 \sqcup B_2 \in \pi_2(\mathbf{x}, \mathbf{x})$ (Lemma 4.32). Using the formula $e(B) + 2n_{\mathbf{x}}(B) = \langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle$ ([OSz04c, Proposition 7.4]), Equation (10.42) implies

$$\begin{aligned} 1 &= \lambda^t \cdot \lambda^{-e(B)-2n_{\mathbf{x}}(B)} (0, \partial^\partial B_1) \cdot \gamma' \cdot (0, -\partial^\partial B_1) \cdot \gamma'^{-1} \\ &= \lambda^{t-\langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle - 2[\gamma] \cap [\partial^\partial B_1]} \\ &= \lambda^{t-\langle c_1(\mathfrak{s}(\mathbf{x})), [B] \rangle - 2[\gamma] \cap [B]} \\ &= \lambda^{t-\langle c_1(\Pi(\gamma)), [B] \rangle}, \end{aligned}$$

where $[\gamma] \cap [\partial^\partial B_1]$ denotes the algebraic intersection number of $[\gamma]$ and $[\partial^\partial B_1]$ in $H_1(F)$, while $[\gamma] \cap [B]$ denotes the algebraic intersection number of $[\gamma] \in H_1(Y)$ and $[B] \in H_2(Y)$. The second equality uses Equation (3.42) and the last uses the identity $c_1(\Pi(\gamma)) = c_1(\mathfrak{s}(\mathbf{x})) + 2[\gamma]$. From this (and the fact that any periodic domain B on Y can be decomposed into periodic domains B_1 and B_2 (Lemma 4.32)), the result follows. \square

THEOREM 10.43. *Fix $\mathfrak{s}_i \in \text{spin}^c(Y_i)$. There is a homotopy equivalence*

$$\Phi: \widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1) \boxtimes \widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2) \rightarrow \bigoplus_{\substack{\mathfrak{s} \in \text{spin}^c(Y) \\ \mathfrak{s}|_{Y_i} = \mathfrak{s}_i, i=1,2}} \widehat{\text{CF}}(\mathcal{H}, \mathfrak{s})$$

which respects the identification between grading sets in the following sense:

- (1) If $m_1 \boxtimes m_2$ is a homogeneous element with grading $\text{gr}^\boxtimes(m_1 \boxtimes m_2) \in P_1(\mathbf{x}_1) \setminus G/P_2(\mathbf{x}_2)$, then $\Phi(m_1 \boxtimes m_2)$ lies in the $\Pi(\text{gr}^\boxtimes(m_1 \boxtimes m_2)) \in \text{spin}^c(Y)$ summand of $\widehat{\text{CF}}(Y)$.
- (2) If $m_1 \boxtimes m_2$ and $n_1 \boxtimes n_2$ are homogeneous elements whose gradings are related by

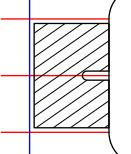
$$\text{gr}^\boxtimes(m_1 \boxtimes m_2) = \lambda^t \text{gr}^\boxtimes(n_1 \boxtimes n_2),$$

so that $\Pi(\text{gr}^\boxtimes(m_1 \boxtimes m_2)) = \Pi(\text{gr}^\boxtimes(n_1 \boxtimes n_2)) = \mathfrak{s}' \in \text{spin}^c(Y)$ then $\Phi(m_1 \boxtimes m_2)$ and $\Phi(n_1 \boxtimes n_2)$ are elements whose relative Maslov grading is given by t modulo $\text{div}(c_1(\mathfrak{s}'))$.

PROOF. The map Φ is the homotopy equivalence from Theorem 1.3. We must prove that Φ respects the grading sets.

To define the identification of grading sets we chose complementary generators \mathbf{x}_1 and \mathbf{x}_2 with $\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}_i$. (This is always possible after modifying \mathcal{H}_1 and \mathcal{H}_2 by an isotopy, which changes $\widehat{\text{CFA}}(\mathcal{H}_1, \mathfrak{s}_1)$ and $\widehat{\text{CFD}}(\mathcal{H}_2, \mathfrak{s}_2)$ by a graded homotopy equivalence.) Assume for simplicity that $I_A(\mathbf{x}_1) = I_D(\mathbf{x}_2)$ is \mathfrak{s}_0 , the base idempotent used to define the grading refinement. (Again, we may need to modify \mathcal{H}_1 and \mathcal{H}_2 by an isotopy to achieve this.) In this case, some formulas simplify: $P(\mathbf{x}_i) = P'(\mathbf{x}_i)$, and, for $\mathbf{y}_i \in \mathfrak{S}(\mathcal{H}_i, \mathfrak{s}_i)$,

$$\text{gr}(\mathbf{y}_1) = \text{gr}'(\mathbf{y}_1) \cdot \psi(I_A(\mathbf{y}_1))^{-1} \quad \text{and} \quad \text{gr}(\mathbf{y}_2) = \psi(I_D(\mathbf{y}_2)) \cdot \text{gr}'(\mathbf{y}_2).$$



Let $\mathbf{y}_1 \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{y}_2 \in \mathfrak{S}(\mathcal{H}_2)$ be generators with $\mathfrak{s}(\mathbf{y}_i) = \mathfrak{s}_i$ and such that $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$ is a generator for \mathcal{H} . Let $B_i \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$. Then

$$\begin{aligned}\text{gr}(\mathbf{y}_1) &= P_1(\mathbf{x}_1) \cdot g(B_1) \in S_A(\mathcal{H}_1) = P_1(\mathbf{x}_1) \setminus G(\mathcal{Z}) \\ \text{gr}(\mathbf{y}_2) &= R(g(B_2)) \cdot R(P_2(\mathbf{x}_2)) \in S_D(\mathcal{H}_2) = G(\mathcal{Z}) / R(P_2(\mathbf{x}_2)).\end{aligned}$$

Consequently,

$$\begin{aligned}\text{gr}(\mathbf{y}_1 \times \mathbf{y}_2) &= \text{gr}'(\mathbf{y}_1) \cdot \psi(I_A(\mathbf{y}_1))^{-1} \cdot \psi(I_D(\mathbf{y}_2)) \cdot \text{gr}'(\mathbf{y}_2) \\ &= \text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2),\end{aligned}$$

where we have used $I_A(\mathbf{y}_1) = I_D(\mathbf{y}_2)$. We wish to use this to calculate the difference element between the two spin^c structures associated to $\mathbf{x} = \mathbf{x}_1 \times \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$. The spin^c component of $\text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2)$ is $\partial^\partial B_1 + \partial^\partial B_2$. This represents a homology class γ in $H_1(F)$. Moreover, the image of γ in $H_1(Y)$ agrees with the difference element $\epsilon(\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{y}_1 \times \mathbf{y}_2)$ defined in [OSz04d, Section 2.6] (see also Lemma 4.21), whose Poincaré dual is $\mathfrak{s}_z(\mathbf{x}_1 \times \mathbf{x}_2) - \mathfrak{s}_z(\mathbf{y}_1 \times \mathbf{y}_2)$. It follows that $\Pi(\text{gr}'(\mathbf{y}_1) \cdot \text{gr}'(\mathbf{y}_2))$ represents the same spin^c structure as $\mathbf{y} = \mathbf{y}_1 \times \mathbf{y}_2$. This verifies Part 1.

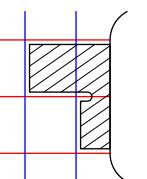
For Part 2, let $\mathbf{y}'_1 \in \mathfrak{S}(\mathcal{H}_1)$ and $\mathbf{y}'_2 \in \mathfrak{S}(\mathcal{H}_2)$ be another pair of generators with $\mathfrak{s}(\mathbf{y}') = \mathfrak{s}_i$ and such that $\mathbf{y}' = \mathbf{y}'_1 \times \mathbf{y}'_2$ is a generator for \mathcal{H} in the same spin^c structure as \mathbf{y} . Fix some $B' \in \pi_2(\mathbf{y}, \mathbf{y}')$, where $B' = B'_1 \sqcup B'_2$. Thus $B_i * B'_i \in \pi_2(\mathbf{x}_i, \mathbf{y}'_i)$. Now, the grading of \mathbf{y}' is given by

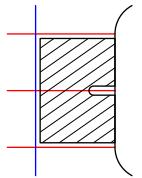
$$\begin{aligned}\text{gr}(\mathbf{y}') &= P_1(\mathbf{x}_1) \cdot g'(B_1 * B'_1) R(g'(B_2 * B'_2)) \cdot P_2(\mathbf{x}_2) \\ &= P_1(\mathbf{x}_1) \cdot g'(B_1) g'(B'_1) R(g'(B'_2)) R(g'(B_2)) \cdot P_2(\mathbf{x}_2) \\ &= P_1(\mathbf{x}_1) \cdot g'(B_1) \lambda^{-e(B') - n_{\mathbf{y}}(B') - n_{\mathbf{y}'}(B')} R(g'(B_2)) \cdot P_2(\mathbf{x}_2) \\ &= \lambda^{-e(B') - n_{\mathbf{y}}(B') - n_{\mathbf{y}'}(B')} \cdot \text{gr}(\mathbf{y}).\end{aligned}$$

In the second equality, we use Lemma 10.4 and the fact that R is an anti-homomorphism. This proves Part 2. \square

REMARK 10.44. To get the correct grading set on the tensor product, we need to use the refined gradings: the double-coset space $P'_1(\mathbf{x}_1) \setminus G'(4k) / R(P'_2(X_2))$ is larger than the double-coset space $P_1(\mathbf{x}_1) \setminus G(\mathcal{Z}) / R(P_2(X_2))$ in Theorem 10.43.

REMARK 10.45. An alternative to the refined grading would be to consider a “grading groupoid”, with objects corresponding to the different idempotents $I(\mathbf{s})$ and $\text{Mor}(\mathbf{s}_1, \mathbf{s}_2)$ being those elements $g \in G'(4k)$ with $M_*\partial([g]) = \mathbf{s}_1 - \mathbf{s}_2$. Then S_A and S_D are equipped with right and left actions of this grading groupoid, and the grading set for the tensor product is the (suitably defined) product over the grading groupoid.





CHAPTER 11

Bordered manifolds with torus boundary

In this chapter, we specialize to the case of bordered manifolds with torus boundary. In particular, we show that the bordered Floer invariant of a manifold with torus boundary is closely related to its knot Floer homology. As a warm-up, after introducing some notation in Section 11.1, we use the pairing theorem to give a quick proof of a version of the surgery exact triangle for Heegaard Floer homology, in Section 11.2. We then turn to the relationship between bordered Floer homology for manifolds with torus boundary and knot Floer homology. In Section 11.3 we recall the conventions on knot Floer homology. The easier task of extracting (associated graded) knot Floer homology from bordered Floer homology is addressed in Section 11.4. This follows quickly from a suitable adaptation of the pairing theorem and a simple model calculation for the solid torus.

Following this, Section 11.5 explains how to extract the type D module for a knot complement in S^3 from the (filtered) knot Floer homology. After further developing the holomorphic curve machinery in Section 11.6, this result is proved in Sections 11.7 and 11.8. Finally, in Section 11.9, we use bordered Floer homology to study satellite knots. (The knot Floer homology groups of various families of satellite knots have also been studied using a more direct analysis. See [Eft05], [Hed07], and [Hed05b].)

11.1. Torus algebra

A genus g bordered Heegaard diagram \mathcal{H} for a 3-manifold Y with torus boundary has two α -arcs, $g - 1$ α -circles, and $g \beta$ -circles. The α -arcs are necessarily arranged as in Figure 11.1. As a result, the parametrization of ∂Y is specified by a pair of a meridian and a longitude, intersecting once, and the matching on $\partial\mathcal{H}$ is always the same. Let $\mathcal{A}(\mathbb{T}) = \mathcal{A}(\mathbb{T}, 0)$ denote the $i = 0$ part of the corresponding algebra. Since we will mostly be interested in $\widehat{CFD}(\mathcal{H})$, we label the regions between the arcs in the opposite order to the induced orientation on the boundary, as in Figure 11.2. (From the type A side, the regions would be labeled in

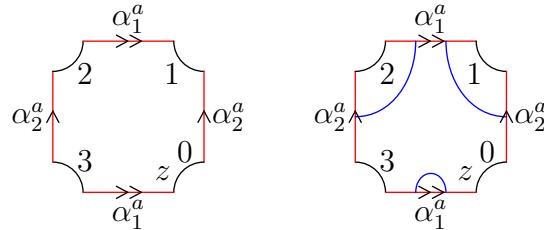
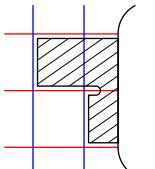


FIGURE 11.1. **Bordered Heegaard diagrams for torus boundary.** Left: the necessary arrangement of the α -arcs. Opposite sides are identified as indicated. There may be additional handles attached with corresponding α -circles. Right: a β -circle to create an admissible diagram for a solid torus.



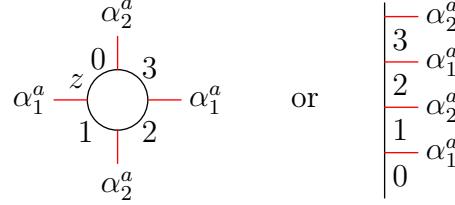
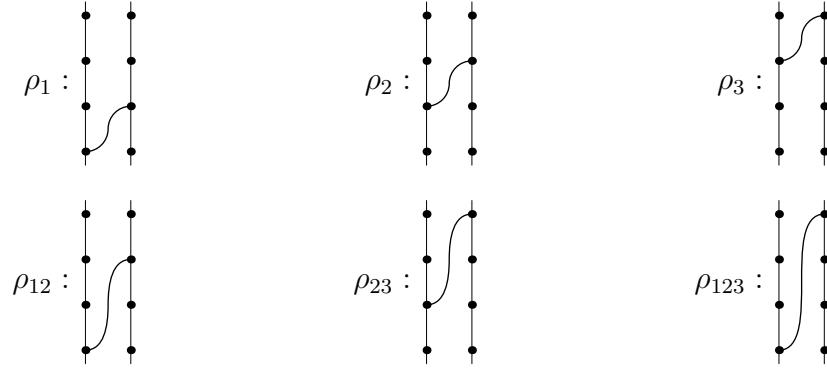


FIGURE 11.2. **Labeling of the regions around $\partial\bar{\Sigma}$.** With respect to the boundary orientation, they are labeled 0, 3, 2, 1 with z in the region labeled 0.

the opposite order around the puncture.) The point z is in the region labeled 0. Label the α -arcs so that regions 0 and 1 are separated from regions 2 and 3 by α_2^a .

The algebra $\mathcal{A}(\mathbb{T})$ has two minimal idempotents ι_0 and ι_1 , corresponding respectively to α_1^a and α_2^a being occupied on the type A side, and 6 other basic generators, denoted graphically as follows:



The differential is zero, and the non-zero products are

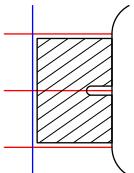
$$\rho_1\rho_2 = \rho_{12} \quad \rho_2\rho_3 = \rho_{23} \quad \rho_1\rho_{23} = \rho_{123} \quad \rho_{12}\rho_3 = \rho_{123}.$$

(All other products of two non-idempotent basic generators vanish identically.) There are also compatibility conditions with the idempotents:

$$\begin{aligned} \rho_1 &= \iota_0\rho_1\iota_1 & \rho_2 &= \iota_1\rho_2\iota_0 & \rho_3 &= \iota_0\rho_3\iota_1 \\ \rho_{12} &= \iota_0\rho_{12}\iota_0 & \rho_{23} &= \iota_1\rho_{23}\iota_1 & \rho_{123} &= \iota_0\rho_{123}\iota_1. \end{aligned}$$

The unrefined grading takes values in the group $G'(\mathbb{T})$, which we denote G' in this chapter. It is generated by quadruples $(j; a, b, c)$ with $j \in \frac{1}{2}\mathbb{Z}$, $a, b, c \in \mathbb{Z}$ and j is an integer if all of a, b, c are even or if b is even and a and c are odd, and is a half-integer otherwise. The group law is

$$(j_1; a_1, b_1, c_1) \cdot (j_2; a_2, b_2, c_2) = \left(j_1 + j_2 + \frac{1}{2} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \frac{1}{2} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; a_1 + a_2, b_1 + b_2, c_1 + c_2 \right).$$



The G' -grading on the algebra is given by

$$\begin{aligned}\text{gr}'(\rho_1) &= \left(-\frac{1}{2}; 1, 0, 0\right) \\ \text{gr}'(\rho_2) &= \left(-\frac{1}{2}; 0, 1, 0\right) \\ \text{gr}'(\rho_3) &= \left(-\frac{1}{2}; 0, 0, 1\right).\end{aligned}$$

The distinguished central element is $\lambda = (1; 0, 0, 0)$.

There is also a refined grading, as in Section 3.3.2, with values in the group $G(\mathbb{T})$. We will instead give a grading with values in a somewhat larger group G given explicitly by triples $(j; p, q)$ where $j, p, q \in \frac{1}{2}\mathbb{Z}$ and $p + q \in \mathbb{Z}$. The group law is

$$(j_1; p_1, q_1) \cdot (j_2; p_2, q_2) = \left(j_1 + j_2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}; p_1 + p_2, q_1 + q_2\right)$$

and the distinguished central element is $\lambda = (1; 0, 0)$. (The group $G(\mathbb{T})$ from Section 3.3.2 would have $p, q \in \mathbb{Z}$ and $2j \equiv p + q \pmod{2}$.) Let $G'_{\mathbb{Q}} \supset G'$ be the group defined in the same way as G' but where $j, a, b, c \in \mathbb{Q}$ (so $G'_{\mathbb{Q}}$ is a \mathbb{Q} -central extension of $H_1(Z, \mathbf{a}; \mathbb{Q})$). There is a homomorphism $G \rightarrow G'_{\mathbb{Q}}$ defined by $(j; p, q) \mapsto (j; p, p + q, q)$. To define the refined grading, choose ι_0 as the base idempotent and set $\psi(\iota_1) := (0; \frac{1}{2}, 0, \frac{1}{2}) \in G'_{\mathbb{Q}}$. From Equation (3.44) we then find

$$\begin{aligned}(11.1) \quad \text{gr}(\rho_1) &= \left(-\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right) \\ \text{gr}(\rho_2) &= \left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right) \\ \text{gr}(\rho_3) &= \left(-\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\right).\end{aligned}$$

Note that for any algebra element a , the gradings $\text{gr}'(a)$ and $\text{gr}(a)$ are related by the homomorphism from G' to G defined by

$$(11.2) \quad (j; a, b, c) \mapsto \left(j; \frac{a+b-c}{2}, \frac{-a+b+c}{2}\right).$$

We find it convenient to have the following concrete reformulation of type D structures over the torus algebra. (This will be helpful in Section 11.7, where we understand how to construct the type D module of a knot complement from the knot Floer complex.)

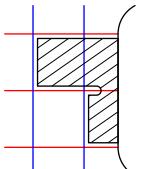
DEFINITION 11.3. Let $V = V^0 \oplus V^1$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. A collection of (torus) *coefficient maps* is a collection of maps

$$D_I: V^{[i_0-1]} \rightarrow V^{[i_n]}$$

indexed by increasing sequences of consecutive integers $I = \{i_0, \dots, i_n\} \subset \{1, 2, 3\}$, where $[i]$ denotes the reduction of $i \pmod{2}$, satisfying the compatibility equation stated below. (The empty sequence also has a corresponding map D_\emptyset or just D , which maps V^i to V^i for $i = 0, 1$.) For any increasing sequence of consecutive integers $I \subset \{1, 2, 3\}$ we require that

$$(11.4) \quad \sum_{\{I=J \cup K | J < K\}} D_K \circ D_J = 0,$$

where $J < K$ means that each element of J is smaller than every element of K .



Explicitly, Equation (11.4) in the case where $I = \emptyset$ says that D is a differential; similarly, $I = \{1\}$ says that D_1 is a chain map with respect to the differential D ; $I = \{1, 2\}$ says that D_{12} is a null-homotopy of the composite $D_2 \circ D_1$; $I = \{123\}$ gives the following:

$$D \circ D_{123} + D_3 \circ D_{12} + D_{23} \circ D_1 + D_{123} \circ D = 0.$$

LEMMA 11.5. *A type D structure in the sense of Definition 2.18 over $\mathcal{A}(\mathbb{T})$ with base ring $\mathcal{I}(\mathbb{T})$ corresponds to a pair of $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces equipped with coefficient maps in the sense of Definition 11.3.*

PROOF. Let V be a vector space underlying a type D structure over $\mathcal{A}(\mathbb{T})$. The projection $V \rightarrow V^0$ is induced by multiplication by the idempotent ι_0 , while the projection $V \rightarrow V^1$ is induced by multiplication by the idempotent ι_1 . The coefficient maps can be extracted from the map δ^1 by the formula

$$\delta^1 = \mathbf{I} \otimes D + \sum_i \rho_i \otimes D_i + \sum_{\{i,j|j=i+1\}} \rho_{ij} \otimes D_{ij} + \rho_{123} \otimes D_{123}.$$

Equation (11.4) gives the components of the compatibility condition for δ^1 . \square

In particular, thinking of $\widehat{\text{CFD}}(Y)$ as a type D structure, as in Remark 6.4, we can talk about its coefficient maps.

REMARK 11.6. The coefficient maps depend on the type D structure itself, not just on its corresponding differential module.

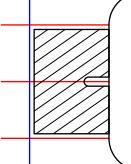
11.2. Surgery exact triangle

Recall that Heegaard Floer homology admits a surgery exact triangle [OSz04c]. Specifically, for a pair (M, K) of a 3-manifold M and a framed knot K in M , there is an exact triangle

$$(11.7) \quad \begin{array}{ccc} \widehat{HF}(M_{-1}) & \longrightarrow & \widehat{HF}(M_0) \\ & \searrow & \swarrow \\ & \widehat{HF}(M_\infty) & \end{array}$$

where M_{-1} , M_0 , and M_∞ are -1 , 0 , and infinity surgery on K , respectively. As a simple application of bordered Floer theory, we prove a version of this result. A similar computation was carried out in [Lip06b, Section 5.3]. We do not, however, verify that the maps in our present triangle agree with those in the original version; see [LOT14] or [LOT].

We prove this exact triangle by constructing three provincially admissible bordered Heegaard diagrams \mathcal{H}_{-1} , \mathcal{H}_0 , and \mathcal{H}_∞ for the three solid tori filled at the corresponding slopes, and exhibiting a short exact sequence relating the chain complexes $\widehat{\text{CFD}}(\mathcal{H}_\bullet)$. For any knot complement with (admissible) bordered Heegaard diagram \mathcal{H}' , we can then take the derived tensor product of $\widehat{\text{CFA}}(\mathcal{H}')$ with this short exact sequence. The exact triangle in Equation (11.7) is the induced exact triangle on homology (see Proposition 2.36).



The three diagrams we use are

$$(11.8) \quad \begin{array}{c} \text{Diagram } \mathcal{H}_\infty : \text{ A rectangle with red boundary. Inside, there is a blue vertical line segment from bottom to top. The top arc is labeled } r, \text{ the right arc } 2, \text{ the top-right corner } 1, \text{ the bottom-right corner } z, \text{ the bottom arc } 3, \text{ and the left arc } r. \\ \text{Diagram } \mathcal{H}_{-1} : \text{ A rectangle with red boundary. Inside, there is a blue curve that starts at the bottom-left corner } a, \text{ goes up to the top-left corner } 2, \text{ then turns right to the top-right corner } 1, \text{ then turns down to the bottom-right corner } z, \text{ then turns left to the bottom-left corner } b, \text{ and finally turns up to the top-left corner } a. \\ \text{Diagram } \mathcal{H}_0 : \text{ A rectangle with red boundary. Inside, there is a blue horizontal line segment from left to right. The top arc is labeled } n, \text{ the right arc } 2, \text{ the top-right corner } 1, \text{ the bottom-right corner } z, \text{ the bottom arc } 3, \text{ and the left arc } n. \end{array}$$

A generator for $\widehat{CFD}(\mathcal{H}_\bullet)$ consists of a single intersection point between the β -circle in \mathcal{H}_\bullet and an α -arc. These intersections are labeled above.

The boundary operators on the $\widehat{CFD}(\mathcal{H}_\bullet)$ (and the support of the relevant domains) are given by

$$\begin{array}{llll} \partial r = \rho_{23}r & \partial a = \rho_3b + \rho_1b & \partial b = 0 & \partial n = \rho_{12}n \\ \text{Diagram: } \text{A rectangle with a shaded diagonal band from bottom-left to top-right. Labels: } z \text{ at bottom-right.} & \text{Diagrams: } \text{Two rectangles. The first has a shaded band from bottom-left to top-right, with a blue arc from } a \text{ to } 2. \text{ The second has a shaded band from bottom-left to top-right, with a blue arc from } b \text{ to } 1. \text{ Labels: } z \text{ at bottom-right.} & & \text{Diagram: } \text{A rectangle with a shaded band from bottom-left to top-right. Labels: } z \text{ at bottom-right.} \end{array}$$

There is a short exact sequence

$$0 \longrightarrow \widehat{CFD}(\mathcal{H}_\infty) \xrightarrow{\varphi} \widehat{CFD}(\mathcal{H}_{-1}) \xrightarrow{\psi} \widehat{CFD}(\mathcal{H}_0) \longrightarrow 0$$

where the maps φ and ψ are given by

$$\begin{array}{ll} \varphi(r) = b + \rho_2a & \psi(a) = n \\ \text{Diagram: } \text{A rectangle with a green diagonal band from bottom-left to top-right. Labels: } z \text{ at bottom-right.} & \text{Diagram: } \text{A rectangle with a blue diagonal band from bottom-left to top-right. Labels: } z \text{ at bottom-right.} \\ \psi(b) = \rho_2n . & \text{Diagram: } \text{A rectangle with a blue diagonal band from bottom-left to top-right. Labels: } z \text{ at bottom-right.} \end{array}$$

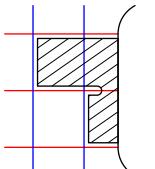
Indeed, one can guess the maps φ and ψ by considering holomorphic triangles, as indicated above. It is straightforward to verify that $\psi \circ \varphi = 0$, that φ is injective, that ψ is surjective, and that $\ker \psi = \text{im } \varphi$.

11.3. Preliminaries on knot Floer homology

We focus now on the relationship between Heegaard Floer invariants of three-manifolds with torus boundary and knot Floer homology. Before delving into the details, we recall some of the basics of knot Floer homology, with a special emphasis on the bigradings. For simplicity of notation, we restrict attention to the case where the ambient three-manifold is S^3 . More detailed accounts may be found in the original papers [**OSz04b, Ras03**].

The knot Floer invariant of a knot is the filtered chain homotopy type of a filtered chain complex over $\mathbb{F}_2[U]$. Specifically, a knot in S^3 can be specified by a suitable Heegaard diagram $\mathcal{H}_K = (\Sigma, \alpha, \beta, w, z)$, where w and z are a pair of basepoints in the complement of the α - and β -circles. The *knot Floer complex* is defined as a chain complex $CFK^-(K)$ generated over $\mathbb{F}_2[U]$ by \mathfrak{S}_K , the usual generators of Heegaard Floer homology with respect to the Heegaard diagram \mathcal{H}_K , endowed with the differential

$$(11.9) \quad \partial^-(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}_K} \sum_{\substack{B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B)=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y})) U^{n_w(B)} \cdot \mathbf{y}.$$



In the above equation, $\tilde{\pi}_2(\mathbf{x}, \mathbf{y})$ refers to the set of homology classes of curves connecting \mathbf{x} to \mathbf{y} which are allowed to cross both w and z ; this is denoted $\pi_2(\mathbf{x}, \mathbf{y})$ in [OSz04b, Ras03], but in this book we have reserved π_2 for classes which have local multiplicity zero at z .

The complex is endowed with an *Alexander filtration*. The Alexander depth (filtration degree) A of the generators is characterized, up to an overall translation, by

$$(11.10) \quad \begin{aligned} A(\mathbf{x}) - A(\mathbf{y}) &= n_z(B) - n_w(B) \\ A(U \cdot \mathbf{x}) &= A(\mathbf{x}) - 1 \end{aligned}$$

for any $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$ and homology class $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$. The boundary operators do not increase A . Furthermore, the boundary operators decrease (by one) the *Maslov grading* M , defined up to overall translation by

$$(11.11) \quad \begin{aligned} M(\mathbf{x}) - M(\mathbf{y}) &= \text{ind}(B) - 2n_w(B) \\ M(U \cdot \mathbf{x}) &= M(\mathbf{x}) - 2. \end{aligned}$$

(Note that the knot enters the definition only through the Alexander filtration.) We will explain how to remove the indeterminacy in the Alexander and Maslov gradings below, in Equations (11.14) and in Equations (11.13) and (11.15), respectively.

In some contexts, it is natural to consider a different grading N , the *z -normalized Maslov grading*, characterized up to an overall translation, by

$$(11.12) \quad \begin{aligned} N(\mathbf{x}) - N(\mathbf{y}) &= \text{ind}(B) - 2n_z(B) \\ N(U \cdot \mathbf{x}) &= N(\mathbf{x}). \end{aligned}$$

Any two of A , M , and N determine the remaining one, by the relation

$$(11.13) \quad N = M - 2A.$$

Let $gCFK^-(K)$ denote the associated graded object (with respect to the Alexander filtration). Explicitly, this is a bigraded complex with a differential

$$\partial^-(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}_K} \sum_{\substack{B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B)=1, n_z(B)=0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y})) U^{n_w(B)} \cdot \mathbf{y}.$$

The functions A and M induce Alexander and Maslov gradings on $gCFK^-(K)$; the Alexander grading is preserved by the differential, and the Maslov grading is decreased by one by the differential.

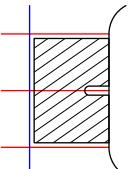
The homology groups of $gCFK^-(K)$ are the *knot Floer homology groups* of K ,

$$HFK^-(K) = \bigoplus_{r,d} HFK_d^-(K, r).$$

The index r is the Alexander grading and d is the Maslov grading.

The above constructions can be specialized to $U = 0$, where we have a different version of knot Floer homology. The $U = 0$ specialization of $CFK^-(K)$ is a filtered chain complex over \mathbb{F}_2 denoted $\widehat{CFK}(K)$. We denote the associated graded complex to $\widehat{CFK}(K)$ by $\widehat{gCFK}(K)$. Explicitly, $\widehat{gCFK}(K)$ is a bigraded complex generated over \mathbb{F}_2 by \mathfrak{S}_K , and endowed with the differential

$$\widehat{\partial}(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}_K} \sum_{\substack{B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B)=1, n_z(B)=n_w(B)=0}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y})) \mathbf{y}.$$



(Sometimes, when we wish to call attention to a fixed doubly-pointed Heegaard diagram, we denote $\widehat{gCFK}(K)$ by $\widehat{gCFK}(\mathcal{H}, w, z)$.) The homology of $\widehat{gCFK}(K)$ is denoted

$$\widehat{HFK}(K) = \bigoplus_{r,d} \widehat{HFK}_d(K, r).$$

The indeterminacy (up to an overall translation) of the Alexander grading can be removed by requiring that the function A have the following symmetry property for all $r \in \mathbb{Z}$:

$$(11.14) \quad \#\{\mathbf{x} \in \mathfrak{S}_K \mid A(\mathbf{x}) = r\} \equiv \#\{\mathbf{x} \in \mathfrak{S}_K \mid A(\mathbf{x}) = -r\} \pmod{2}.$$

The graded Euler characteristic of \widehat{HFK} coincides with the (symmetrized) Alexander polynomial Δ_K in the sense that

$$\sum_{d,r} (-1)^d \operatorname{rank} \widehat{HFK}_d(K, r) \cdot T^r \doteq \Delta_K(T),$$

where \doteq indicates that the two Laurent polynomials agree up to multiplication by $\pm T^k$ (for some k). The symmetry property from Equation (11.14), which pins down the indeterminacy in the Alexander grading, can be reformulated as the condition that

$$\sum_{d,r} (-1)^d \operatorname{rank} \widehat{HFK}_d(K, r) \cdot T^r = \pm \Delta_K(T).$$

The indeterminacy in the Maslov grading can be removed by first removing the indeterminacy in the z -normalized Maslov grading, and demanding that Equation (11.13) continue to hold. In turn, to pin down the algebraic grading N , consider the chain complex $gCFK^-(K)/(U = 1)$. Note that neither the Alexander nor the Maslov gradings descend to this complex; but the z -normalized Maslov grading does. The homology of the resulting complex has rank one. The algebraic grading is then normalized by requiring that this homology is supported in grading equal to zero; i.e.,

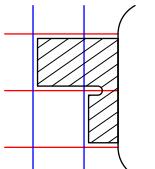
$$(11.15) \quad H_* \left(\frac{gCFK^-(K)}{U = 1} \right) = \begin{cases} \mathbb{F}_2 & * = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $*$ denotes the algebraic grading N .

It is more traditional to define the Maslov gradings in terms of holomorphic disks crossing the basepoint z . This is normalized by considering $\widehat{CF}(S^3)$, defined as holomorphic disks which do not cross the basepoint w (i.e., set $U = 0$ in the chain complex defined in Equation (11.9) and ignore its Alexander filtration), and require that the resulting homology is supported in degree zero with respect to the Maslov grading. Our present formulation defines gradings in terms of disks which never cross z . The equivalence of the two points of view follows from the fact that $gCFK^-(K)/(U = 1)$ is the chain complex $\widehat{CF}(S^3)$, using the basepoint z in place of w .

Knot Floer homology naturally gives rise to an integral valued concordance invariant τ for knots [OSz03]. One construction of τ comes from considering the filtered chain homotopy type of $\widehat{CFK}(K)$. From this point of view, $\tau(K)$ is the minimal s for which the generator of $H_*(\widehat{CF}(S^3))$ can be represented as a sum of generators in Alexander grading less than or equal to s . In an alternative formulation, τ has an interpretation from the associated graded object $HFK^-(K)$, namely,

$$(11.16) \quad \tau(K) = -\max\{s \mid \forall d \geq 0, U^d \cdot HFK^-(K, s) \neq 0\}.$$



(See, for example, [OSzT08, Lemma A.2].)

The above definition of the Alexander grading carries over with little change to the more general case where $K \subset Y$ is a null-homologous knot in an arbitrary closed, oriented three-manifold (rather than just S^3). The Maslov grading is a little more subtle; but that is because Maslov gradings on closed manifolds are more complicated. Our present applications, however, primarily concern the case where $Y = S^3$.

11.4. From \widehat{CFD} to HFK^-

Let K be a null-homologous knot in a 3-manifold Y . Recall from Section 4.2 that we can construct a bordered Heegaard diagram for $Y \setminus \text{nbd}(K)$ as follows. Fix a Heegaard diagram $(\Sigma_g, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$ for $Y \setminus \text{nbd}(K)$, in the classical sense. Let $\mu \subset \Sigma$ and $\lambda \subset \Sigma$ denote a meridian and longitude for K , respectively. Arrange that λ and μ intersect in a single point p , and that they are disjoint from all α_i^c . Set $\alpha_1^a := \lambda \setminus \{p\}$ and $\alpha_2^a := \mu \setminus \{p\}$. Then

$$(\Sigma \setminus \{p\}, \alpha_1^a, \alpha_2^a, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$$

is a bordered Heegaard diagram for $Y \setminus \text{nbd}(K)$.

Number the four regions in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ adjacent to p , in counter-clockwise cyclic order (i.e., the opposite of the induced boundary orientation), so that the meridian α_2^a separates regions 0 and 3.. Let z and w be basepoints in regions 0 and 2, respectively. (See Figure 11.8.) Then

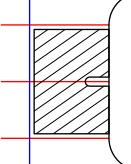
$$(\Sigma, \mu, \alpha_1^c, \dots, \alpha_{g-1}^c, \beta_1, \dots, \beta_g)$$

is a doubly-pointed Heegaard diagram for K in Y . So, with notation as in Section 11.1, for the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ it is immediate that (V^0, D) is the knot Floer chain complex $\widehat{gCFK}(Y, K)$. (The chain complex (V^1, D) is the *longitude Floer complex* studied by Eftekhary [Eft05].)

It follows that for any bordered Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for $Y \setminus \text{nbd}(K)$, with framing such that α_2^a is a meridian of K , the complex (V^0, D) is homotopy equivalent to $\widehat{gCFK}(Y, K)$. To see this, define an \mathcal{A} -module W_0 to be one-dimensional over \mathbb{F}_2 with ι_0 acting as the identity and all other basic generators of \mathcal{A} acting by 0. Then the tensor product $W_0 \boxtimes \widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is exactly (V^0, D) . By Lemma 2.32, the homotopy type of this tensor product depends only on the homotopy type of $\widehat{CFD}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Thus, we have the following.

PROPOSITION 11.17. *Let (\mathcal{H}, z) be any bordered Heegaard diagram for $Y \setminus \text{nbd}(K)$, with framing such that α_2^a is a meridian of K . Then, with notation as in Section 11.1, (V^0, D) is homotopy equivalent to $\widehat{gCFK}(Y, K)$. In particular, $\widehat{gCFK}(Y, K)$ is determined by $\widehat{CFD}(\mathcal{H}, z)$.*

The relative Maslov grading on $\widehat{gCFK}(Y, K)$ within each spin^c structure (on the knot complement) is induced from the grading on $\widehat{CFD}(\mathcal{H}, z)$ as follows. Two generators represent the same spin^c-structure on $Y \setminus K$ if and only if they can be connected by a provincial domain. Consequently, the gradings of two such generators differ by a power of λ , and this λ -power is their Maslov grading difference. One can also determine the Alexander grading, the relative Maslov grading between generators in different Alexander gradings and, in the case that $Y = S^3$, the absolute Maslov grading; we return to these points later.



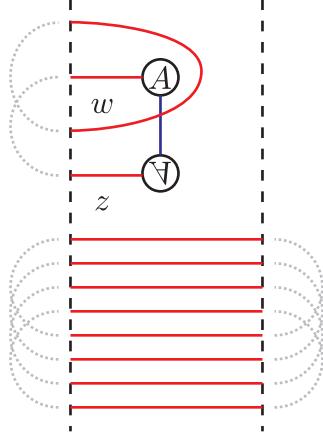


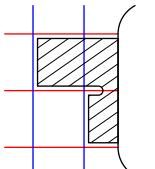
FIGURE 11.3. **Turning a bordered diagram for $Y' = Y \setminus \text{nbd}(K \cup A)$ into a doubly-pointed bordered diagram for (Y, K) .** This diagram can be thought of as a bordered Heegaard diagram with two basepoints for the standard 1-handle attachment cobordism M from a surface of genus g to a surface of genus $g+1$. Gluing the left boundary of this diagram, or an obvious analogue, to a bordered Heegaard diagram for $(Y', \partial Y' = T^2 \# \Sigma_g)$, so that the top portion of this diagram is glued to the T^2 summand in $\partial Y'$, gives a doubly-pointed Heegaard diagram for $(Y' \cup_{\partial} M, K)$, where K is the core of the 1-handle in M .

There is a simple geometric interpretation of W_0 , which we explain after a more general discussion.

A *doubly-pointed bordered Heegaard diagram* is a bordered Heegaard diagram (Σ, α, β) for $(Y, \partial Y)$, together with basepoints z and w in $\bar{\Sigma} \setminus (\bar{\alpha} \cup \bar{\beta})$. We further assume $z \in \partial \bar{\Sigma}$. A doubly-pointed bordered Heegaard diagram specifies a knot K in Y : let γ_α (respectively γ_β) denote a path in $\Sigma \setminus \alpha$ (respectively $\Sigma \setminus \beta$) from z to w . Let $\tilde{\gamma}_\alpha$ denote the result of pushing the interior of γ_α into the α -handlebody and $\tilde{\gamma}_\beta$ the result of pushing the interior of γ_β into the β -handlebody. Then $K = \tilde{\gamma}_\alpha \cup \tilde{\gamma}_\beta$. We orient K as $\tilde{\gamma}_\alpha - \tilde{\gamma}_\beta$.

Every knot in a bordered three-manifold can be realized by a doubly-pointed bordered Heegaard diagram. Specifically, let Y be a bordered three-manifold with genus g boundary, and let $K \subset Y$ be a knot. By connecting K to the boundary by an arc A and deleting a regular neighborhood of $K \cup A$, we can write Y as the union of a three-manifold Y' with genus $g+1$ boundary and a standard cobordism M (a two-handle attachment) from the surface of genus $g+1$ to a surface with genus g . Gluing the (partial) diagram shown in Figure 11.3 to a bordered Heegaard diagram for Y' gives a doubly-pointed bordered Heegaard diagram for K in Y .

To a doubly-pointed bordered Heegaard diagram (\mathcal{H}, z, w) , we can associate type D and type A modules as before, now working over a ground ring of $\mathbb{F}_2[U]$, with the understanding that holomorphic disks crossing w with multiplicity n contribute a factor of U^n . (Holomorphic disks crossing z are still forbidden.) We denote the resulting modules by $CFA^-(\mathcal{H}, z, w)$ and $CFD^-(\mathcal{H}, z, w)$. Setting $U = 1$ recaptures $\widehat{CFA}(\mathcal{H}, z)$ and $\widehat{CFD}(\mathcal{H}, z)$, while setting $U = 0$ gives a pair of modules denoted $\widehat{CFA}(\mathcal{H}, z, w)$ and $\widehat{CFD}(\mathcal{H}, z, w)$ where we count only those holomorphic disks with multiplicity zero at w .



REMARK 11.18. These new modules $CFD^-(\mathcal{H}, z, w)$, etc., are not invariants of the pair (Y, K) . Because the proofs of invariance in Sections 6.3 and 7.3 require one basepoint, z , to remain on the boundary in the course of the handleslides and isotopies, the module $CFD^-(\mathcal{H}, z, w)$ is only an invariant of the graph in $(Y, \partial Y)$ obtained by attaching a point in K to ∂Y ; see also [LOT10a, Example ??].

For a doubly-pointed bordered Heegaard diagram with boundary \mathcal{Z} , we can enhance the grading set to include an Alexander grading. Specifically, the enhanced grading group has the form $\tilde{G}(\mathcal{Z}) = G(\mathcal{Z}) \times \mathbb{Z}$. (This is a direct product.) We call the new \mathbb{Z} summand the *Alexander factor*. Given $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ and $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$, define

$$\tilde{g}(B) := (g(B), n_w(B) - n_z(B)),$$

where $g(B)$ is the grading from Equation (10.31). We can use this enhanced grading group to define an enhanced grading on $CFD^-(\mathcal{H}, z, w)$. Following the discussion from Section 10.4, a spin^c structure \mathfrak{s} on Y gives rise to a grading on the type D module with values in the coset space

$$\tilde{S}_D(\mathcal{H}, \mathfrak{s}) := (\tilde{G}/\tilde{P}(\mathbf{x}_0)),$$

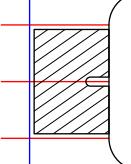
where \mathbf{x}_0 is any intersection point representing \mathfrak{s} , and $\tilde{P}(\mathbf{x}_0)$ denotes the image of the space of periodic domains in \tilde{G} under the homomorphism \tilde{g} . Namely, we define $\tilde{g}(\mathbf{x}) := \tilde{g}(B) \cdot \tilde{P}(\mathbf{x}_0)$, where $B \in \pi_2(\mathbf{x}, \mathbf{x}_0)$. Note that when we extend this grading to $CFD^-(\mathcal{H}, z, w)$, we take the convention that multiplication by U drops the Alexander factor of the grading by one (and preserves the other components of the grading).

Suppose next that (\mathcal{H}_1, z) is a bordered Heegaard diagram for $(Y_1, \partial Y_1 = F)$ and that (\mathcal{H}_2, z, w) is a doubly-pointed bordered Heegaard diagram for $(Y_2, \partial Y_2 = -F)$, equipped with the knot K . Then, the grading set of the tensor product $\widehat{CFA}(\mathcal{H}_1, z) \boxtimes CFD^-(\mathcal{H}_2, z, w)$ is naturally the double-coset space $\tilde{P}(\mathbf{x}_1) \backslash \tilde{G} / \tilde{P}(\mathbf{x}_2)$. (In the definition of $\tilde{P}(\mathbf{x}_1)$, we take the convention that $n_w(B) = 0$ for any domain B since w is thought of as located on the Y_2 side of the Heegaard diagram.)

Projecting onto the Alexander factor in the double-coset space induces a map from the grading set to a transitive \mathbb{Z} -set. Indeed, the transitive \mathbb{Z} -set is isomorphic to $\mathbb{Z}/n_w(\pi_2(\mathbf{x}, \mathbf{x}))$ for any generator \mathbf{x} of $\mathcal{H}_1 \cup_Z \mathcal{H}_2$. For any $B \in \pi_2(\mathbf{x}, \mathbf{x})$ representing the homology class $A \in H_2(Y_1 \cup Y_2)$, the quantity $n_w(B)$ is the intersection of K with A . In particular, if K is null-homologous, the \mathbb{Z} -set is (non-canonically) isomorphic to \mathbb{Z} . In this way, when Y is an integer homology sphere, the tensor product $\widehat{CFA}(\mathcal{H}_1, z) \boxtimes CFD^-(\mathcal{H}_2, z, w)$ inherits a (relative) \mathbb{Z} grading, which we call its *Alexander grading*. Corresponding remarks also apply to $CFA^-(\mathcal{H}_2, z, w) \boxtimes \widehat{CFD}(\mathcal{H}_1, z)$.

We now have a pairing theorem for knot Floer homology.

THEOREM 11.19. *Let $(\mathcal{H}_1, z, w) = (\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z, w)$ be a doubly-pointed bordered Heegaard diagram for $(Y_1, \partial Y_1 = F, K)$ and $(\mathcal{H}_2, z) = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z)$ be a bordered diagram for*



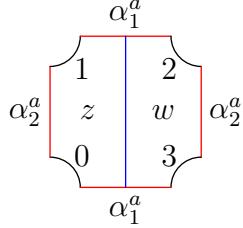


FIGURE 11.4. **The doubly-pointed Heegaard diagram \mathcal{H}_{st} for the longitude in the solid torus.** This is a genus 1 Heegaard diagram with boundary, with α -arcs α_1^a and α_2^a , and a β -circle β_1 intersecting α_1^a in a single point and disjoint from α_2^a . Note that the boundary markings follow the conventions of a type A module.

$(Y_2, \partial Y_2 = -F)$. Then, there are homotopy equivalences

$$\begin{aligned} gCFK^-(Y_1 \cup_F Y_2, K) &\simeq CFA^-(\mathcal{H}_1, w, z) \boxtimes \widehat{CFD}(\mathcal{H}_2, z) \\ &\simeq \widehat{CFA}(\mathcal{H}_2, z) \boxtimes CFD^-(\mathcal{H}_1, w, z) \\ g\widehat{CFK}(Y_1 \cup_F Y_2, K) &\simeq \widehat{CFA}(\mathcal{H}_1, w, z) \boxtimes \widehat{CFD}(\mathcal{H}_2, z) \\ &\simeq \widehat{CFA}(\mathcal{H}_2, z) \boxtimes \widehat{CFD}(\mathcal{H}_1, w, z), \end{aligned}$$

which respect gradings in an obvious sense. In particular, when K is null-homologous for $Y_1 \cup_F Y_2$, the above homotopy equivalences respect the relative z -normalized Maslov and Alexander gradings. (The Alexander and z -normalized Maslov gradings on the knot Floer homology appearing on the left-hand side are as recalled in Section 11.3; the relative Alexander grading on the right-hand side is defined immediately above, and the relative Maslov grading is induced by projecting onto the Maslov component of the double-coset space).

PROOF. The identification of complexes follows from an extension of either of the two proofs of Theorem 1.3 to the doubly-pointed context, without any significant change. The statement about Alexander gradings now follows from Lemma 4.32. \square

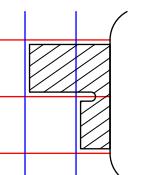
Proposition 11.17 can be seen as a special case of Theorem 11.19, as follows. Consider the doubly-pointed Heegaard diagram \mathcal{H}_{st} with boundary for the solid torus shown in Figure 11.4. If we restrict to holomorphic curves which cross neither z nor w , the resulting type A module $\widehat{CFA}(\mathcal{H}_{st})$ has a single generator over \mathbb{F}_2 , no differential, and indeed is isomorphic to the module W_0 described before the statement of Proposition 11.17. Thus, Proposition 11.17 follows from the $U = 0$ version of Theorem 11.19 (with the solid torus on the type A side).

To reconstruct CFK^- of a knot from \widehat{CFD} , we must describe $CFA^-(\mathcal{H}_{st})$. But this is easily done.

LEMMA 11.20. *The type A module $CFA^-(\mathcal{H}_{st})$ for \mathcal{H}_{st} , representing the solid torus equipped with a longitudinal unknot, has a single generator \mathbf{x}_0 and higher products given by*

$$m_{3+i}(\mathbf{x}_0, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2) = U^{i+1} \cdot \mathbf{x}_0$$

for all $i \geq 0$.



PROOF. The only curves which can contribute are disks covering the region containing w a total of k times. There is exactly one such disk for each k , and it contributes the term in the statement with $k = i + 1$. \square

Thus, according to Theorem 11.19, $HFK^-(Y, K)$ can be calculated as a derived tensor product of $\widehat{CFD}(Y \setminus K)$ with $CFA^-(\mathcal{H}_{st})$, generalizing Proposition 11.17.

In a different direction, Theorem 11.19 gives a technique for studying the knot Floer homology of satellite knots. We return to this point in Section 11.9 (where we give also some examples), contenting ourselves for now with the following corollary.

COROLLARY 11.21. *Suppose that K_1 and K_2 are knots in S^3 such that $\widehat{CFD}(S^3 \setminus K_1) \simeq \widehat{CFD}(S^3 \setminus K_2)$ (with respect to the 0-framing on $S^3 \setminus K_1$ and $S^3 \setminus K_2$, say). Let K_1^C (respectively K_2^C) denote the satellite of K_1 (respectively K_2) with pattern C . Then $HFK^-(K_1^C) \cong HFK^-(K_2^C)$.*

PROOF. According to Theorem 11.19, the knot Floer homology of the satellite knot is obtained as the derived tensor product of the type D module of the companion knot (K_1 or K_2) with the type A module of a doubly-pointed Heegaard diagram for the pattern knot in the solid torus. \square

In light of Theorem 11.26 (and Proposition 11.38), Corollary 11.21 implies that if $CFK^-(K_1)$ is filtered homotopy equivalent to $CFK^-(K_2)$ then $HFK^-(K_1^C)$ is isomorphic to $HFK^-(K_2^C)$.

11.5. From CFK^- to \widehat{CFD} : Statement of results

We now turn to the opposite direction, computing $\widehat{CFD}(S^3 \setminus K)$ from $CFK^-(K)$, Theorem 11.26 below. The statement will refer to some general algebraic properties of the chain complex $CFK^-(K)$, which is a free, \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$; and in particular, it refers to certain convenient bases for this chain complex. After setting up the needed algebra, we give the statement of the theorem.

DEFINITION 11.22. Let $\dots \subset V_i \subset V_{i+1} \subset \dots$ be an (ascending) filtration of a vector space V , so $V = \bigcup_{i=-\infty}^{\infty} V_i$. Assume that $\bigcap_{i=-\infty}^{\infty} V_i = 0$. Given $v \in V$, define $A(v) := \inf\{i \in \mathbb{Z} \mid v \in V_i\}$; we call $A(v)$ the *filtration level* of v . Let $\text{gr}(V)$ denote the associated graded vector space, i.e.,

$$\text{gr}(V) := \bigoplus_{i=-\infty}^{\infty} \text{gr}_i(V) \quad \text{gr}_i(V) := V_i / V_{i-1}.$$

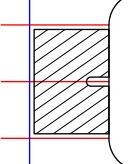
Composing the projection $V_i \rightarrow V_i / V_{i-1}$ and the inclusion $V_i / V_{i-1} \rightarrow \text{gr}(V)$ gives a map $\iota_i: V_i \rightarrow \text{gr}(V)$. For $v \in V$ define $[v] := \iota_{A(v)}(v)$.

A *filtered basis* of a filtered vector space V is a basis $\{v_1, \dots, v_n\}$ for V such that $\{[v_1], \dots, [v_n]\}$ is a basis for $\text{gr}(V)$. Similarly, we say $\{v_1, \dots, v_m\}$ is *filtered linearly-independent* if $\{[v_1], \dots, [v_m]\}$ is linearly independent.

There is a filtered analogue of the basis extension theorem: if $\{v_1, \dots, v_m\}$ is filtered linearly-independent then there is a filtered basis for V containing $\{v_1, \dots, v_m\}$.

We now look at filtered $\mathbb{F}_2[U]$ modules in general. Let C be a finitely-generated free $\mathbb{F}_2[U]$ module equipped with

- an integer-valued grading, which we refer to as the *Maslov grading* M ,
- an integer-valued ascending filtration, the *Alexander filtration* A , and



- a differential, denoted ∂ .

Assume these data satisfy the following compatibility conditions:

- The differential ∂ drops the Maslov grading by one: for homogeneous $\xi \in C$, $M(\partial\xi) = M(\xi) - 1$.
- The differential ∂ respects the Alexander filtration: $A(\partial\xi) \leq A(\xi)$.
- Multiplication by U is compatible with the Maslov grading, in the sense that $M(U \cdot \xi) = M(\xi) - 2$.
- Multiplication by U drops the Alexander filtration by at least 1, in the sense that $A(U \cdot \xi) \leq A(\xi) - 1$.

Given such a C , there are two naturally associated chain complexes. One complex, which we call the *vertical complex* $C^v := C/(U \cdot C)$, is a filtered complex which inherits the Alexander filtration. Its homology is called the *vertical homology*, denoted $H^v(C)$. Next define $C^\infty := C \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, U^{-1}]$. This has two filtrations, namely the Alexander filtration and the filtration by powers of U^{-1} . (Precisely, the s^{th} filtered part of the second filtration contains generators $U^t \mathbf{x}$, where $t \geq -s$ and \mathbf{x} is a generator of C .) Then the *horizontal complex* C^h is defined to be the degree 0 part of the associated graded complex to C^∞ with respect to the Alexander filtration. (Note that C^v is the degree 0 part of the associated graded space to C^∞ with respect to the filtration by U powers.) The complex C^h still has the filtration by powers of U^{-1} and inherits a differential, whose homology is called the *horizontal homology*, denoted $H^h(C)$.

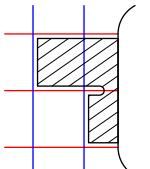
We illustrate C^∞ graphically by picking a filtered basis ξ_i over $\mathbb{F}_2[U]$ of C and plotting a generator over \mathbb{F}_2 of $C \otimes \mathbb{F}_2[U, U^{-1}]$ of the form $U^{-x} \cdot \xi_i$ with Alexander depth y on the plane at the position (x, y) . Then the differential of a generator at (x, y) can be graphically represented by arrows connecting the point at (x, y) with the coordinates of other generators. These arrows necessarily point (non-strictly) to the left and down. The complex C^v is obtained by restricting to one vertical column, and C^h is obtained by restricting to one horizontal row. (See the left-hand side of Figure 11.5.)

DEFINITION 11.23. We call C *reduced* if ∂ strictly drops either the Alexander or U power filtration; i.e., if for all \mathbf{x} , we have $\partial\mathbf{x} = \mathbf{y}_1 + U \cdot \mathbf{y}_2$ where $A(\mathbf{y}_1) < A(\mathbf{x})$.

We call a filtered basis $\{\xi_i\}$ over $\mathbb{F}_2[U]$ for C *vertically simplified* if for each basis vector ξ_i , either $\partial\xi_i \in U \cdot C$ or $\partial\xi_i \equiv \xi_{i+1} \pmod{U \cdot C}$. In the latter case we say that there is a *vertical arrow from ξ_i to ξ_{i+1}* ; so, in a vertically simplified basis, there is at most one vertical arrow starting or ending at each basis element. (The fact that $\partial^2 = 0$ says that there cannot be a vertical arrow both starting and ending at a single element.) The *length* of the vertical arrow is the difference $A(\xi_i) - A(\xi_{i+1})$.

Similarly, we call a filtered basis $\{\eta_i\}$ over $\mathbb{F}_2[U]$ for C *horizontally simplified* if for each basis vector η_i , either $A(\partial\eta_i) < A(\eta_i)$ or there is an m so that $\partial\eta_i = U^m \cdot \eta_{i+1} + \epsilon$ where $A(\eta_i) = A(U^m \cdot \eta_{i+1})$ and $A(\epsilon) < A(\eta_i)$. In the latter case we say that there is a *horizontal arrow from η_i to η_{i+1}* ; so, in a horizontally simplified basis, there is at most one horizontal arrow starting or ending at each basis element. The *length* of the horizontal arrow is the integer m .

REMARK 11.24. These notions could alternatively be formulated without reference to a basis. Vertical (respectively horizontal) arrows correspond to non-vanishing differentials in the spectral sequence belonging to the filtered complex C^v (respectively C^h).



Every finitely generated chain complex C as above is homotopy equivalent to one which is reduced; moreover, for a reduced complex, we can find a vertically or horizontally simplified basis. See Proposition 11.57.

In particular, these constructions apply to knot Floer homology $CFK^-(K)$. The complexes $C^v(CFK^-(K))$ and $C^h(CFK^-(K))$ (ignoring the filtration) are both canonically isomorphic to $\widehat{CF}(S^3)$ and so if K is a knot in S^3 , then $H^v(CFK^-(K)) \cong H^h(CFK^-(K)) \cong \widehat{HF}(S^3) \cong \mathbb{F}_2$. As such, after passing to a reduced model for $CFK^-(K)$, a vertically (respectively horizontally) simplified basis ξ_i (respectively η_i) has a preferred element which (after reordering) we label ξ_0 (respectively η_0), characterized by the property that it has no in-coming or out-going vertical (respectively horizontal) arrows.

Observe that

$$(11.25) \quad A(\xi_0) = \tau(K) \quad M(\xi_0) = 0 \quad A(\eta_0) = -\tau(K) \quad M(\eta_0) = -2\tau(K).$$

The first equation follows from the definition of $\tau(K)$; the second from the fact that $\widehat{HF}(S^3)$ is supported in degree $M = 0$; the third follows from the reformulation of $\tau(K)$ (Equation (11.16)); and the fourth from the normalization of $N = M - 2A$.

THEOREM 11.26. *Let $CFK^-(K)$ be a model for a chain complex for a knot $K \subset S^3$ which is reduced. Let Y be the bordered three-manifold $S^3 \setminus \text{nbd } K$. Given any sufficiently large integer n , we consider Y with framing $-n$. The associated type D module $\widehat{CFD}(Y)$ can be extracted from $CFK^-(K)$ by the following procedure.*

In the idempotent ι_0 , $\widehat{CFD}(Y)$ is isomorphic as an \mathbb{F}_2 -module to $\widehat{HFK}(K)$.¹ Let $\{\xi_i\}$ be a vertically simplified basis for $CFK^-(K)$. To each vertical arrow of length ℓ from ξ_i to ξ_{i+1} , we introduce a string of basis elements $\kappa_1^i, \dots, \kappa_\ell^i$ for $\iota_1 \widehat{CFD}(Y)$ and differentials

$$\xi_i \xrightarrow{D_1} \kappa_1^i \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \kappa_k^i \xleftarrow{D_{23}} \kappa_{k+1}^i \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \kappa_\ell^i \xleftarrow{D_{123}} \xi_{i+1}$$

where ξ_i and ξ_{i+1} are viewed as elements of $\widehat{CFD}(Y)$. (The reader is cautioned to note the directions of the above arrows.)

Similarly, endow $CFK^-(K)$ with a horizontally simplified basis $\{\eta_i\}$. To each horizontal arrow from η_i to η_{i+1} of length ℓ , we introduce a string of basis elements $\lambda_1^i, \dots, \lambda_\ell^i$ connected by a string of differentials

$$\eta_i \xrightarrow{D_3} \lambda_1^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_k^i \xrightarrow{D_{23}} \lambda_{k+1}^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_\ell^i \xrightarrow{D_2} \eta_{i+1}.$$

(As before, η_i and η_{i+1} are viewed as elements of $\widehat{CFD}(Y)$, via the identification of $\iota_0 \widehat{CFD}(Y)$ with $CFK^-(K)$.)

There is a final string, the unstable chain, of generators μ_1, \dots, μ_m connecting ξ_0 and η_0 , connected by a string of differentials,

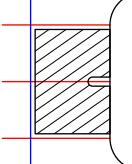
$$\xi_0 \xrightarrow{D_1} \mu_1 \xleftarrow{D_{23}} \mu_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \mu_m \xleftarrow{D_3} \eta_0.$$

The length m of this string is $n + 2\tau(K)$.

The gradings are determined as follows:

- The grading set is $G/\lambda^{-1} \text{gr}(\rho_{23})^n \text{gr}(\rho_{12})^{-1} = G/\left(\frac{n-1}{2}; -1, n\right)$.

¹That is, the submodule $\iota_0 \widehat{CFD}(Y)$ is identified with $\widehat{HFK}(K)$ as an \mathbb{F}_2 -module.



- Recall that any element \mathbf{x}_0 of V^0 is represented by a generator \mathbf{x} for the knot Floer complex. The grading of \mathbf{x}_0 in the above grading set is determined by the Maslov grading M and Alexander grading A of \mathbf{x} by the formula

$$(11.27) \quad \text{gr}(\mathbf{x}_0) = \lambda^{M(\mathbf{x})-2A(\mathbf{x})} (\text{gr}(\rho_{23}))^{-A(\mathbf{x})} = (M - \frac{3}{2}A; 0, -A).$$

REMARK 11.28. The normalization in Formula 11.27 is chosen so that if one tensors the module with $\widehat{\text{CFA}}(\mathcal{H}_\infty)$, the invariant of the ∞ -framed solid torus, and declares that the generator r for the solid torus has grading 0, then the generator $r \otimes \eta_0$ of $\widehat{\text{HF}}(S^3)$ has (Maslov) grading 0, as well. Although we do not specify the gradings of the elements of V^1 , these can be readily worked out, since they are connected to elements of V^0 by coefficient maps. See Equation (11.55) for an example.

As stated, Theorem 11.26 only computes $\widehat{\text{CFD}}$ for sufficiently negative framing parameters. In [LOT10a], we give bimodules which allow us to deduce the type D module for arbitrary framings; this is summarized in Appendix A. It turns out that most of the statement of Theorem 11.26 goes through for arbitrary integral framings; the only part which depends on the framing is the unstable chain, connecting ξ_0 and η_0 , which can be explicitly described. In particular, for framing $2\tau(K)$, this entire chain can be replaced by a single map

$$(11.29) \quad \xi_0 \xrightarrow{D_{12}} \eta_0.$$

See Theorem A.11 for the complete statement.

The procedure for translating knot Floer complexes to type D modules given by Theorem 11.26 is illustrated in Figure 11.5, with one proviso: to keep the pictures small, we have illustrated type D modules with framing parameter $n = 2\tau(K)$, for which one needs to refer to Theorem A.11.

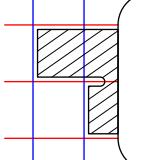
It is interesting to note that the above construction of $\widehat{\text{CFD}}$ uses only the vertical and horizontal arrows, while CFK^- in general can contain diagonal arrows. In concrete terms, the vertical arrows correspond to counting holomorphic disks ϕ with $n_w(\phi) = 0$, while horizontal ones correspond to counts of holomorphic disks with $n_z(\phi) = 0$. Diagonal arrows come from disks crossing both z and w .

We will deduce Theorem 11.26 from a more complicated statement, Theorem 11.36, which does not make reference to a choice of basis, and which comes more directly from a Heegaard diagram. This basis-free version is stated and proved in Section 11.7, and we deduce Theorem 11.26 from it in Section 11.8. Before turning giving these proofs, we discuss some further counts of holomorphic curves, in Section 11.6.

11.6. Generalized coefficient maps and boundary degenerations

So far, the holomorphic curves we count have not covered the basepoint, and in particular have not involved the Reeb chords that contain the basepoint. In this section, we explain a certain further structure on the type D module coming from counting certain holomorphic curves which contain ρ_0 in their boundary as well. These additional maps will be used in Section 11.7 to construct $\widehat{\text{CFD}}$ from CFK^- . This further structure exists for arbitrary (connected) boundary, but for notational simplicity we will restrict our attention to the torus boundary case.

Fix a bordered Heegaard diagram (\mathcal{H}, z) for (Y^3, T^2) , and label the regions around $\partial\Sigma$ as in Section 11.1. Recall that for $I = \{i_0, \dots, i_n\}$ an interval in $\{1, 2, 3\}$ there is an associated



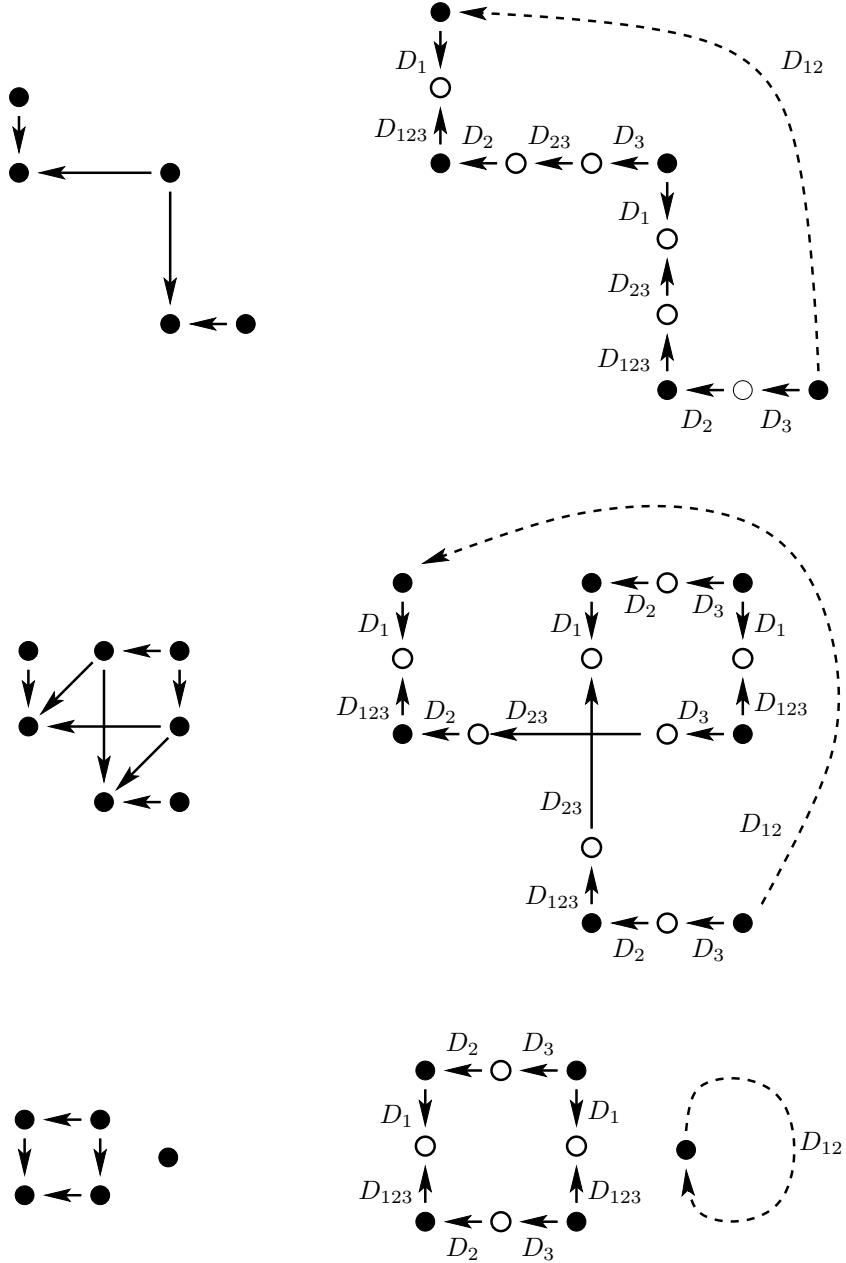
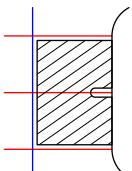


FIGURE 11.5. **Reconstructing the type D module.** A table with data from knot Floer homology on the left, where vertical differentials correspond to ∂_z and horizontal ones corresponding to ∂_w ; the length of the arrow represents the change in Alexander grading. Top: the left-handed $(3, 4)$ torus knot. Middle: the $(2, -1)$ cable of the left-handed trefoil. Bottom: the figure eight knot. The corresponding type D modules are on the right, with black dots for generators in the idempotent i_0 and white ones for generators in the idempotent i_1 . The unstable chain is denoted by the dotted arrow. Here we take framing parameter $2\tau(K)$ as in Equation (11.29). Explicitly, these are framings -6 , -4 and 0 , respectively.



map $D_I: V^{[i_0-1]} \rightarrow V^{[i_n]}$. This is induced by the differential on \widehat{CFD} , which in turn is defined by counting holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$, cf. Chapter 6. By moving the basepoint z from the region labeled 0 to one of the regions labeled 1, 2 or 3, we obtain maps $D_I: V \rightarrow V$ for intervals I in $\{0, 1, 2, 3\}$ with respect to the cyclic ordering, of length at most 3. (For instance, putting z in region 1 the new maps we obtain are D_0 , D_{30} and D_{230} .) We will call these new maps *generalized coefficient maps*.

We define four more generalized coefficient maps, D_{0123} , D_{1230} , D_{2301} and D_{3012} as follows. By dropping Condition (M-9) on holomorphic maps from Section 5.2, we can consider moduli spaces of holomorphic curves asymptotic to Reeb chords ρ_0 , ρ_{01} and so on. Collecting these into moduli spaces as in Section 6.1, for \mathbf{x} a generator of $\widehat{CFD}(\mathcal{H}, z)$ in idempotent ι_0 , we set

$$D_{0123}(\mathbf{x}) := \sum_{\substack{\mathbf{y} \in \mathfrak{S}(\mathcal{H}) \\ B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y}) \\ \vec{\rho} \mid \text{ind}(B, \vec{\rho}) = 1}} \# (\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})) \mathbf{y}$$

where the sum over $\vec{\rho}$ is over $\{(\rho_0, \rho_1, \rho_2, \rho_3), (\rho_{012}, \rho_3), (\rho_0, \rho_{123})\}$. (These are the three sequences of Reeb chords for which $\prod a(-\rho_\alpha) = a(-\rho_{0123})$ in a suitable generalization of \mathcal{A} .) The maps D_{1230} , D_{2301} and D_{3012} are defined similarly.

By moving the basepoint z , it follows from the discussion in Section 11.1 that these maps satisfy the compatibility conditions

$$\sum_{\{I=J \cup K \mid J < K\}} D_K \circ D_J = 0$$

where $<$ now denotes the cyclic ordering on $\{0, 1, 2, 3\}$ and I is any *proper* cyclic interval in $\{0, 1, 2, 3\}$. The compatibility equations involving all four Reeb chords are, perhaps, more surprising, as there is now a right hand side.

PROPOSITION 11.30. *The maps D_{0123} and D_{2301} satisfy*

$$(11.31) \quad D \circ D_{0123} + D_3 \circ D_{012} + D_{23} \circ D_{01} + D_{123} \circ D_0 + D_{0123} \circ D = \mathbb{I}: V^1 \rightarrow V^1$$

$$(11.32) \quad D \circ D_{2301} + D_1 \circ D_{230} + D_{01} \circ D_{23} + D_{301} \circ D_2 + D_{2301} \circ D = \mathbb{I}: V^1 \rightarrow V^1.$$

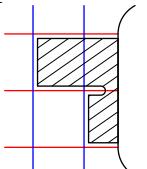
The maps D_{1230} and D_{3012} satisfy similar formulas.

(Here, \mathbb{I} denotes the identity map.)

REMARK 11.33. The grading shifts of the new coefficient maps may be computed directly. Alternatively, they are uniquely specified by the grading shifts for the original grading shift maps, together with the above formulas, which are homogeneous.

The reason that the right hand sides of Equations (11.31) and (11.32) are the identity map rather than zero is the existence of certain boundary degenerations. We can see these boundary degenerations explicitly in simple cases. For example, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a genus 1 Heegaard diagram for a solid torus—for instance, the diagram in Figure 11.6. Then $\Sigma \setminus \boldsymbol{\alpha}$ is a rectangle (disk with four boundary punctures). There is a holomorphic map $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$, where S is the union of a trivial strip and a rectangle, and the rectangle is mapped in the obvious way to $\Sigma \setminus \boldsymbol{\alpha}$, and by a constant map to $[0, 1] \times \mathbb{R}$.

Such curves look quite strange in the cylindrical setting. (In particular, it is not clear when they are transversely cut out.) We therefore invoke, for the first and only time in



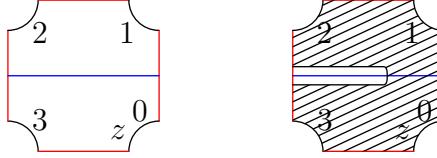


FIGURE 11.6. **Boundary degeneration as end of a one-dimensional moduli space.** Left: a genus 1 Heegaard diagram (Σ, α, β) for the solid torus. Here, β_1 is disjoint from α_1^a and intersects α_2^a in a single point. Right: one point in a one-parameter family of holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$, with one end a two-story building and the other end a boundary degeneration.

this book, the *tautological correspondence* between holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ and holomorphic disks in $\text{Sym}^g(\Sigma)$. That is, recall that there is a bijective correspondence

$$\begin{array}{ccc} \text{Holomorphic maps} & & \text{Holomorphic curves} \\ \phi: [0, 1] \times \mathbb{R} \rightarrow \text{Sym}^g(\Sigma) & \longleftrightarrow & u \text{ in } \Sigma \times [0, 1] \times \mathbb{R} \\ \text{with } \phi(\{0\} \times \mathbb{R}) \subset \prod \beta_i & & \text{with boundary in } \beta \times \{0\} \times \mathbb{R} \\ \text{and } \phi(\{1\} \times \mathbb{R}) \subset \prod \alpha_i & & \text{and in } \alpha \times \{1\} \times \mathbb{R} \\ & & \text{so that } \pi_{\mathbb{D}} \circ u \text{ is a } g\text{-fold branched cover.} \end{array}$$

(In the simplest case, one uses a complex structure $\text{Sym}^g(j_\Sigma)$ on $\text{Sym}^g(\Sigma)$ and a split complex structure $j_\Sigma \times j_{\mathbb{D}}$ on $\Sigma \times [0, 1] \times \mathbb{R}$. More generally, admissible almost complex structures J on $\Sigma \times [0, 1] \times \mathbb{R}$ correspond to working with paths of complex structures on $\text{Sym}^g(\Sigma)$.) See, for instance, [Lip06a, Section 13] for a detailed explanation.

To extend this to the bordered setting, recall that $\Sigma_{\bar{e}}$ denotes the result of filling-in the puncture p of Σ . Viewing $\text{Sym}^g(\Sigma)$ as the complement of the divisor $p \times \text{Sym}^{g-1}(\Sigma_{\bar{e}})$ in $\text{Sym}^g(\Sigma_{\bar{e}})$, we see that $\text{Sym}^g(\Sigma)$ is a symplectic manifold with a cylindrical end, in a way compatible with the complex structure $\text{Sym}^g(j_\Sigma)$. The end of $\text{Sym}^g(\Sigma)$ is modeled on the unit normal bundle $U\nu(\text{Sym}^{g-1}(\Sigma_{\bar{e}}))$ to $p \times \text{Sym}^{g-1}(\Sigma_{\bar{e}})$, an S^1 -bundle over $\text{Sym}^{g-1}(\Sigma_{\bar{e}})$; compare [BEH⁺03, Example 2.2 and Remark 5.9]. There is a Reeb-like vector field \vec{R} on $U\nu(\text{Sym}^{g-1}(\Sigma_{\bar{e}}))$ tangent to the S^1 -fibers.

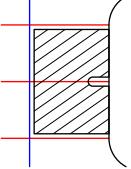
The Lagrangian subspace $\alpha_i^a \times \alpha_1^c \times \cdots \times \alpha_g^c$ ($i = 1, 2$) intersects the ideal boundary of $\text{Sym}^g(\Sigma)$ in the subspace $(\partial\bar{\alpha}_i^a) \times \alpha_1^c \times \cdots \times \alpha_g^c$. So, for instance, the Reeb chord ρ_1 corresponds to a $(g-1)$ -dimensional family of \vec{R} -chords between $\partial\bar{\alpha}_1^a$ and $\partial\bar{\alpha}_2^a$, parameterized by the points in $\alpha_1^c \times \cdots \times \alpha_g^c$; and similarly for the other Reeb chords.

The tautological correspondence extends to a correspondence between holomorphic maps $u: S \rightarrow \Sigma \times [0, 1] \times \mathbb{R}$ asymptotic to a sequence of Reeb chords (ρ_1, \dots, ρ_n) at $e\infty$ and generators \mathbf{x} and \mathbf{y} at $\pm\infty$ and holomorphic maps

$$\phi: (([0, 1] \times \mathbb{R}) \setminus \{(1, t_1), \dots, (1, t_n)\}) \rightarrow \text{Sym}^g(\Sigma)$$

asymptotic to a chord of the form $\rho_i \times \{(a_{i,1}, \dots, a_{i,g-1}\}$ at the puncture $(1, t_i)$ and to \mathbf{x} and \mathbf{y} at $\pm\infty$. Here, the values of the t_i and the points $a_{i,j} \in \alpha_j$ are allowed to vary, with the restriction that $t_1 < t_2 < \cdots < t_n$. In particular, the \mathbb{R} -coordinates of the e punctures of u correspond to the values of the t_i .

With respect to this correspondence, the boundary degeneration mentioned above comes from a disk in $\text{Sym}^g(\Sigma)$ with boundary entirely in the α -tori; we will view this as a map from the upper half-plane \mathbb{H} with some boundary punctures. There is nothing strange about this



disk. In particular, for generic J on the symmetric product $\text{Sym}^g(\Sigma)$, the moduli spaces of these disks will be transversely cut out.

We collect the boundary degenerations in $\text{Sym}^g(\Sigma)$ into moduli spaces:

DEFINITION 11.34. Given $i = 1$ or 2 , $\mathbf{x} \in \alpha_i^a \times \alpha_1^c \times \cdots \times \alpha_{g-1}^c$, a sequence $\vec{\rho} = (\rho_1, \dots, \rho_n)$ of Reeb chords, and an almost complex structure J on $\text{Sym}^g(\Sigma)$, let $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$ denote the moduli space of J -holomorphic maps

$$\phi: \mathbb{H} \setminus \{(t_1, \dots, t_n)\} \rightarrow \text{Sym}^g(\Sigma),$$

where the t_i are real numbers (points in $\partial\mathbb{H}$) which are allowed to vary with the given order, such that the boundary of \mathbb{H} is mapped into $\{\alpha_1^a \cup \alpha_2^a\} \times \alpha_2^c \times \cdots \times \alpha_{g-1}^c$, the homology class of ϕ is $[\Sigma]$, ϕ is asymptotic to \mathbf{x} at ∞ and to a chord of the form $\rho_i \times \{(a_{i,1}, \dots, a_{i,g-1})\}$ at t_i .

PROPOSITION 11.35. *There is a non-empty open subset of nearly-symmetric almost complex structures J on $\text{Sym}^g(\Sigma)$, as in [OSz04d, Section 3], such that the moduli spaces $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$, where $\vec{\rho}$ is a cyclic permutation of $(\rho_0, \rho_1, \rho_2, \rho_3)$ and $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, are transversely cut out and have an odd number of points. Further, for these almost complex structures, all $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$ for other $\vec{\rho}$ are empty.*

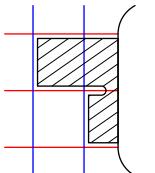
PROOF. As already discussed, it is standard that $\mathcal{M}^{[\Sigma]}(\mathbf{x}; \vec{\rho}; J)$ is transversely cut out for generic J . To prove the moduli space has an odd number of points, we start by studying a model case. Consider the Heegaard diagram $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ shown in Figure 11.6. In this diagram, there is a single generator \mathbf{x} . With respect to any almost complex structure j on Σ_1 , there is a one-parameter family of holomorphic disks in $\Sigma_1 = \text{Sym}^{g_1}(\Sigma_1)$, in the homology class $[\Sigma_1]$, with asymptotics $(\rho_0, \rho_1, \rho_2, \rho_3)$. One end of this moduli space is a broken holomorphic curve and the other end is the boundary degeneration under discussion. Since the domain of these curves has multiplicity 1 somewhere (in fact, everywhere), for a generic j on Σ , this moduli space of $\text{Sym}^g(j)$ -holomorphic curves is transversely cut out, including at its boundary. So, the count of boundary degenerations in $\text{Sym}^g(\Sigma_1) = \Sigma_1$ is 1.

Now, for the general case, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be any bordered Heegaard diagram with torus boundary. Then $(\Sigma, \boldsymbol{\alpha})$ can be decomposed as a connect sum $(\Sigma_1, \boldsymbol{\alpha}_1) \# (\Sigma_2, \boldsymbol{\alpha}_2)$ where $(\Sigma_1, \boldsymbol{\alpha}_1)$ is the α -curves of the diagram discussed above and $\boldsymbol{\alpha}_2$ consists of closed curves. A simple adaptation of the arguments showing stabilization invariance of HF of closed 3-manifolds [OSz04d, Section 10] shows that for almost complex structures with a long neck between $(\Sigma_1, \boldsymbol{\alpha}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2)$, there are an odd number of holomorphic curves in $\mathcal{M}^{[\Sigma]}(\mathbf{x}; (\rho_0, \rho_1, \rho_2, \rho_3); J)$. Similar arguments apply for other cyclic orderings of $(\rho_0, \rho_1, \rho_2, \rho_3)$. This proves the first claim in the proposition.

Curves with other asymptotics (e.g., (ρ_0, ρ_{123})) cannot occur in $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ by inspection. Hence, if there is a long neck between $(\Sigma_1, \boldsymbol{\alpha}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2)$, they do not occur for $(\Sigma, \boldsymbol{\alpha})$. This proves the second claim in the proposition. \square

PROOF OF PROPOSITION 11.30. We will prove Relation (11.31); the other relations follow similarly. Consider the moduli space

$$\bigcup_{\text{ind}(B, (\rho_0, \rho_1, \rho_2, \rho_3))=2} \mathcal{M}^B(\mathbf{x}, \mathbf{y}; (\rho_0, \rho_1, \rho_2, \rho_3)).$$



For $\mathbf{x} \neq \mathbf{y}$, the ends of this moduli space all correspond to terms on the left side of Formula (11.31). By contrast, for $\mathbf{x} = \mathbf{y}$ it follows from Proposition 11.35, together with a standard gluing argument (compare [OSz08, Theorem 5.1]) and the tautological correspondence, that an odd number of ends of the moduli space correspond to boundary degenerations. The result is immediate. \square

11.7. From CFK^- to \widehat{CFD} : Basis-free version

In this section we state and prove a basis-free version of Theorem 11.19; this is Theorem 11.36 below. We deduce Theorem 11.19 from Theorem 11.36 in the next section. Before stating the basis-free version, we introduce a little more notation for knot Floer homology.

Fix a doubly-pointed Heegaard diagram \mathcal{H} for K , with basepoints w and z . We write C for the chain complex $\widehat{gCFK}(\mathcal{H}, w, z)$ computing $\widehat{HFK}(K)$ and $C(r)$ for its summand in Alexander grading r . There are maps $\partial^i: C(r) \rightarrow C(r+i)$ defined by counting holomorphic disks ϕ with $n_z(\phi) = -i$ and $n_w(\phi) = 0$ if $i \leq 0$, and $n_z(\phi) = 0$ and $n_w(\phi) = i$ if $i \geq 0$. Set $\partial_w := \sum_{i \geq 0} \partial^i$ and $\partial_z := \sum_{i \leq 0} \partial^i$. The maps ∂_w and ∂_z each give differentials on C . Furthermore, \widehat{C} has two filtrations,

$$C(\geq s) := \bigoplus_{r \geq s} C(r) \quad \text{and} \quad C(\leq s) := \bigoplus_{r \leq s} C(r),$$

preserved by ∂_w and ∂_z , respectively. We will principally use not the complex $(C(\geq s), \partial_w)$, but the complex $(C(\leq s), \partial_w)$, defined to be the quotient of (C, ∂_w) by the subcomplex $(C(\geq s+1), \partial_w)$. We likewise need the quotient complex $(C(\geq s), \partial_z)$, with a similar definition.

The above constructions can be interpreted using the notions from Section 11.5. The complex $CFK^-(K)$ is a finitely generated, free, \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$ satisfying compatibility conditions spelled out in that section. The complex (C, ∂_w) is the horizontal complex associated to $CFK^-(K)$, while (C, ∂_z) is the vertical complex. In particular, $(C(\geq s), \partial_w)$ is identified with the s^{th} summand of $\widehat{gCFK}^-(\mathcal{H}, s)$ with respect to the Alexander grading, while $(C(\leq s), \partial_z)$ is identified with the s^{th} filtered complex of $\widehat{CFK}(\mathcal{H})$. Thus $(C(\geq s), \partial_w)$ and $(C(\geq s), \partial_z)$ for s sufficiently small both have homology isomorphic to \mathbb{F}_2 ; likewise for $(C(\leq s), \partial_z)$ and $(C(\leq s), \partial_w)$ for s sufficiently large.

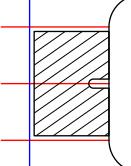
THEOREM 11.36. *Let $K \subset S^3$ be a knot with meridian μ and 0-framed longitude λ , and let C , ∂_w , and ∂_z be the data associated as above. Given a positive integer n , let*

$$V^0 := \bigoplus_{s \in \mathbb{Z}} V_s^0 \quad \text{and} \quad V^1 := \bigoplus_{s \in \mathbb{Z} + \frac{n+1}{2}} V_s^1$$

where

$$\begin{aligned} V_s^0 &:= C(s) \\ V_s^1 &:= \begin{cases} C(\leq s + \frac{n-1}{2}) & s \leq -\frac{n}{4} \\ \mathbb{F}_2 & |s| < \frac{n}{4} \\ C(\geq s - \frac{n-1}{2}) & s \geq \frac{n}{4}. \end{cases} \end{aligned}$$

Let $V := V^0 \oplus V^1$. Then, for n large enough, the following maps D_I satisfy the compatibility condition (11.4):



- The differential D restricted to $V_s^0 = C(s)$ is the differential ∂^0 on the knot complex; for $s \leq -\frac{n}{4}$, D restricted to $V_s^1 = C(\leq s + \frac{n-1}{2})$ is the differential ∂_w ; for $s \geq \frac{n}{4}$, D restricted to $V_s^1 = C(\geq s - \frac{n-1}{2})$ is the differential ∂_z ; for $|s| < \frac{n}{4}$, D restricted to $V_s^1 = \mathbb{F}_2$ is identically zero.

- The map

$$D_1: V_s^0 = C(s) \rightarrow V_{s+\frac{n-1}{2}}^1 = C(\geq s)$$

is the obvious inclusion of the subcomplex.

- The map D_2 is nonzero only on V_s^1 for $s \leq -\frac{n}{4}$, in which case

$$D_2: V_s^1 = C(\leq s + \frac{n-1}{2}) \rightarrow V_{s+\frac{n+1}{2}}^0 = C(s + \frac{n+1}{2})$$

is given by

$$\pi \circ \partial_w: C(\leq s + \frac{n-1}{2}) \rightarrow C(s + \frac{n+1}{2})$$

where $\pi: C \rightarrow C(s + \frac{n+1}{2})$ is the projection.

- The map

$$D_3: V_s^0 = C(s) \rightarrow V_{s-\frac{n-1}{2}}^1 = C(\leq s)$$

is the obvious inclusion of the subcomplex.

- The map D_{12} is identically zero.
- For $s < -\frac{n}{4}$, the map

$$D_{23}: V_s^1 = C(\leq s + \frac{n-1}{2}) \rightarrow V_{s+1}^1 = C(\leq s + \frac{n+1}{2})$$

is the obvious inclusion map; for the s with $s \leq -\frac{n}{4} < s + 1$,

$$D_{23}: V_s^1 = (C, \partial_w) \rightarrow V_{s+1}^1 = \mathbb{F}_2$$

is a chain map inducing an isomorphism in homology; for $\frac{-n+2}{4} < s < \frac{n-2}{4}$, $D_{23}: V_s^1 = \mathbb{F}_2 \rightarrow V_{s+1}^1 = \mathbb{F}_2$ is the isomorphism; for the s with $s < \frac{n}{4} \leq s + 1$,

$$D_{23}: V_s^1 = \mathbb{F}_2 \rightarrow V_{s+1}^1 = (C, \partial_z)$$

is a chain map inducing an isomorphism on homology; for $s > \frac{n}{4}$,

$$D_{23}: V_s^1 = C(\geq s - \frac{n-1}{2}) \rightarrow V_{s+1}^1 = C(\geq s - \frac{n+1}{2})$$

is the obvious projection map.

- The map

$$D_{123}: V_s^0 \rightarrow V_{s+\frac{n+1}{2}}^1$$

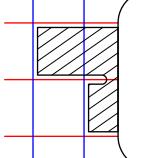
is the composite of one component of ∂_w ,

$$\partial_w^1: C(s) \rightarrow C(s + 1),$$

with the inclusion of $C(s + 1)$ in $(C(\geq s + 1), \partial_z) = V_{s+\frac{n+1}{2}}^1$.

Moreover, the associated type D module is homotopy equivalent to $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$, marked by the curves μ and $-n \cdot \mu + \lambda$.

Let $S_D(K)$ denote the grading set for $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$, so $S_D(K) \cong G/\langle R(g(P)) \rangle$ (where P is a primitive periodic domain). The type D structure $V^0 \oplus V^1$ has a grading by $S_D(K)$ so that the homotopy equivalence with $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$ respects the gradings. Further, there is a projection map $p: G \rightarrow (\frac{1}{2}\mathbb{Z})^2$ given by $p(m; i, j) = (i, j)$. The map p induces a map $S_D(K) \rightarrow (\frac{1}{2}\mathbb{Z})^2 / \langle p(R(g(P))) \rangle \rightarrow \frac{1}{2}\mathbb{Z}$, giving a grading on $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$



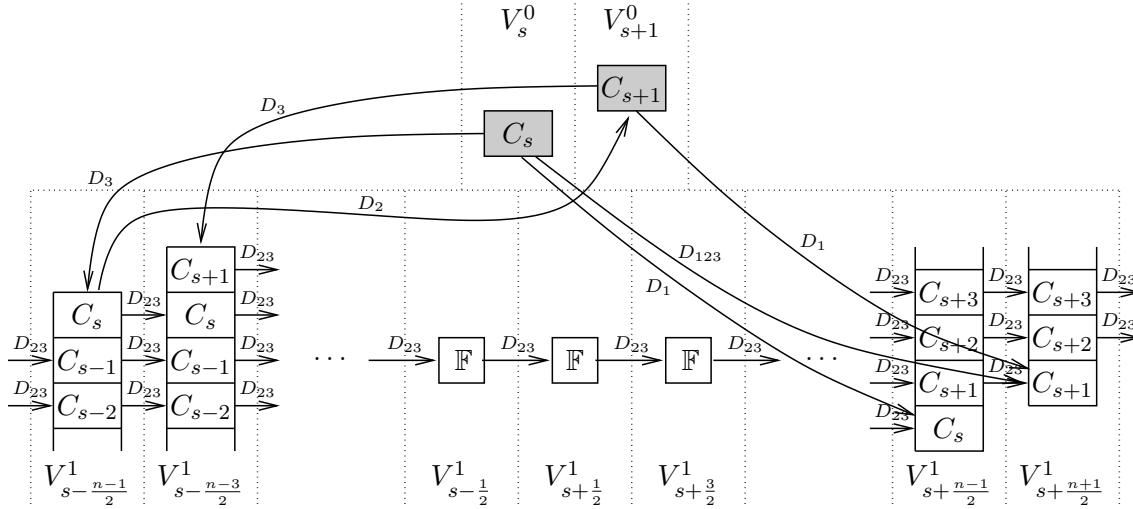


FIGURE 11.7. **A sketch of some of the maps from Theorem 11.36.** The horizontal coordinate represents the grading by relative spin^c structures. The shaded squares correspond to V^0 , while the unshaded ones represent V^1 .

by $\frac{1}{2}\mathbb{Z}$. (Explicitly, this map sends $(m; i, j)$ to $-ni - j$.) Then the homotopy equivalence identifies this grading with the s grading above.

Figure 11.7 sketches some of the maps from Theorem 11.36.

REMARK 11.37. The complexes $C(\geq s)$ appear in Hedden's cabling and Whitehead doubling formulas [Hed05b, Hed07].

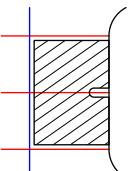
In effect, Theorem 11.36 describes how to associate a type D module to a suitable chain complex C , equipped with boundary operators ∂_w and ∂_z and filtrations. In that statement, we apply it to $C = CFK^-(K)$; but the association is functorial, in the following sense:

PROPOSITION 11.38. *Up to homotopy equivalence, for sufficiently large n the type D module associated in Theorem 11.36 depends only on the \mathbb{Z} -filtered, \mathbb{Z} -graded chain homotopy type of the $\mathbb{F}_2[U]$ -chain complex $CFK^-(K)$.*

PROOF. A homomorphism ϕ of type D structures (over $\mathcal{A}(\mathbb{T})$) can be thought of as a collection of coefficient maps Φ_I , $I \subset \{1, 2, 3\}$, satisfying the following straightforward analogue of Equation 11.4:

$$(11.39) \quad \sum_{\{I=J \cup K | J < K\}} D'_K \circ \Phi_J + \Phi_K \circ D_J = 0.$$

Let $\phi: C \rightarrow C'$ be a \mathbb{Z} -filtered, \mathbb{Z} -graded homotopy equivalence of $\mathbb{F}_2[U]$ -chain complexes. The map ϕ induces homotopy equivalences on the associated complexes $C(s)$, $C(\geq s)$, and $C(\leq s)$, which in turn induces homotopy equivalences between the associated V^0 and V^1 's. Call these homotopy equivalences Φ_\emptyset . The map ϕ also induces maps $C_{\leq s} \rightarrow C'_{s+1}$ which fit together to give maps $\Phi_{\{2\}}: V_{s-\frac{n-1}{2}}^1 \rightarrow V_{s+1}^{0,1}$. Define all other Φ_I to be 0. It is straightforward to verify that Φ satisfies Equation (11.39). A homotopy of \mathbb{Z} -filtered, \mathbb{Z} -graded maps induces a homotopy of maps of type D structures in the same way. This proves the result. \square



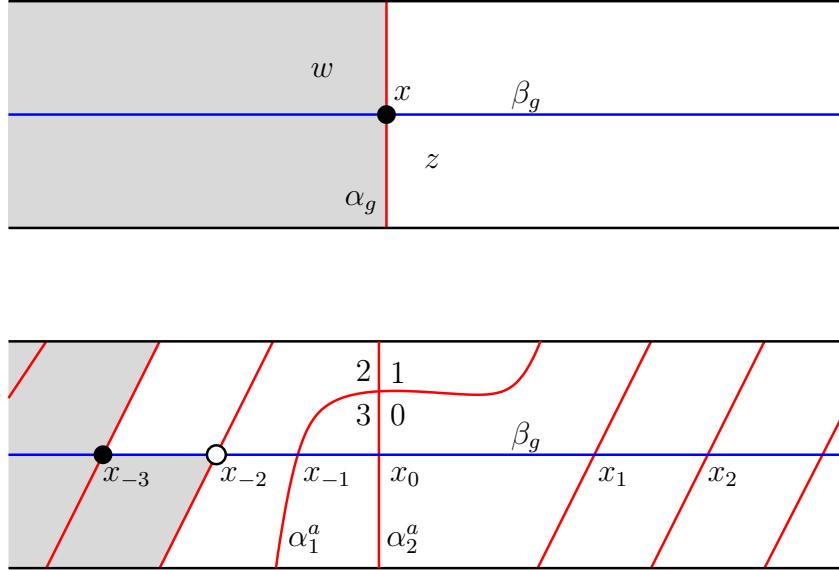


FIGURE 11.8. **Twisting.** At the top, we have pictured a portion of a doubly-pointed Heegaard diagram for a knot. Below it, we have a bordered diagram for the knot complement, with suitably twisted framing. The gray shading represents a domain from $\mathbf{a} \times x$ to $\mathbf{b} \times x$ (for some \mathbf{a} and \mathbf{b}) which, in the top diagram, crosses w with multiplicity one and z with multiplicity zero. There is a corresponding (provincial) domain below connecting $\mathbf{a} \times x_i$ to $\mathbf{b} \times x_{i+1}$, provided that $i \leq -2$.

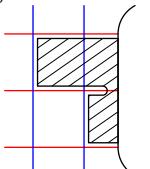
We prove Theorem 11.36 by considering a bordered Heegaard diagram $\mathcal{H}(n)$ for the knot complement with sufficiently large negative framing $-n$, in a way analogous to the surgery formula for Heegaard Floer homology [OSz04b]. For an analogous choice of bordered Heegaard diagram \mathcal{H} , the corresponding type D module, $\widehat{CFD}(\mathcal{H}) = W^0 \oplus W^1$, has generators which correspond to the generators of $V = V^0 \oplus V^1$ as stated in the theorem, except for W_s^1 with $|s| < \frac{n}{4}$. Moreover, again except for W_s^1 with $|s| < \frac{n}{4}$, the coefficient maps are as stated in the theorem. We then construct a homotopy equivalence which fixes the cases with $|s| < \frac{n}{4}$. We give the proof after some preliminary lemmas.

To describe our bordered Heegaard diagram $\mathcal{H}(n)$ we start with a doubly-pointed Heegaard diagram $(\Sigma_0, \alpha_0, \beta_0, z, w)$ for the knot K . We stabilize this diagram to obtain a new diagram $\mathcal{H}_K := (\Sigma, \alpha, \beta, z, w)$. Here, Σ is gotten by attaching a two-dimensional one-handle to Σ_0 , with feet near the markings z and w ; α is gotten from α_0 by introducing a new circle α_g , contained in the new one-handle; and β is gotten from β_0 by introducing a new circle β_g , which meets α_g in a single transverse intersection point, and is disjoint from the other β -circles. In particular, the circle α_g is a meridian for the knot.

Identify a neighborhood \mathcal{W} of α_g in Σ with $[-1, 1] \times S^1$ so that α_g is identified with $\{0\} \times S^1$ and $\beta_g \cap \mathcal{W}$ is identified with $[-1, 1] \times \{e^{-\pi i/2}\}$. Let λ denote a 0-framed longitude of K in Σ , disjoint from the α_i for $i \neq g$, and with $\lambda \cap \mathcal{W} = [-1, 1] \times \{1\}$. Let

$$\bar{\alpha}_1^a = (\lambda \cap (\Sigma \setminus \mathcal{W})) \cup \{(t, e^{\pi i n(t+1)}) \in \mathcal{W} \mid t \in [-1, 1]\}.$$

That is, $\bar{\alpha}_1^a$ is obtained from the 0-framed longitude λ by winding n times around the meridian α_g inside \mathcal{W} . We refer to \mathcal{W} as the *winding region*. Note that $\bar{\alpha}_1^a$ intersects β_g in n points



in \mathcal{W} , and intersects α_g in a single point p . Define $\alpha_1^a := \overline{\alpha}_1^a \setminus \{p\}$ and $\alpha_2^a := \alpha_g \setminus \{p\}$. Then, with notation as in Section 4.2,

$$\mathcal{H}(n) := (\Sigma, \alpha_1^a, \alpha_2^a, \alpha_1, \dots, \alpha_{g-1}, \beta)$$

is a bordered Heegaard diagram for $S^3 \setminus \text{nbd}(K)$, with framing $-n$. We will also write \mathcal{H} for $\mathcal{H}(n)$ when the framing is clear. We label the four quadrants around p by $0, \dots, 3$, arranged in a counterclockwise order (i.e., the order induced by $\partial \text{nbd}(p)$), so that, if you forget α_1^a , 0 and 1 are in the region which contains z in \mathcal{H}_K . As usual, we place the basepoint in region 0. See Figure 11.8 for an illustration. For convenience of notation, we will henceforth assume that n is divisible by 4.

We now turn to the grading on $\widehat{CFD}(\mathcal{H})$. Following Section 10.4, the grading takes values in the coset space $G/P(\mathbf{x}_0)$, where $P(\mathbf{x}_0)$ is the span of $g(B)$ over all periodic domains $B \in \pi_2(\mathbf{x}_0, \mathbf{x}_0)$. For a space with the same homology groups as a solid torus (such as a knot complement), the group of periodic domains is isomorphic to \mathbb{Z} . Let B_0 be a generator of $\pi_2(\mathbf{x}_0, \mathbf{x}_0)$. Since the knot has framing $-n$, we can choose the sign of B_0 so that B_0 has local multiplicities $(0, 1, 1-n, -n)$ in the regions $(0, 1, 2, 3)$, respectively. Thus $P(\mathbf{x}_0) \subset G$ is $\langle (v; 1, -n) \rangle$ for some $v \in \frac{1}{2}\mathbb{Z}$ (depending on \mathbf{x}_0). For present purposes it suffices to consider only the spin^c component of the grading on $\widehat{CFD}(\mathcal{H})$, which we can recover with the homomorphism from G to $\frac{1}{2}\mathbb{Z}$ given by

$$(11.40) \quad (j; p, q) \mapsto -np - q.$$

(Note that $g(P(\mathbf{x}_0))$ is in the kernel of this map.)

More generally, let $S: \mathfrak{S}(\mathcal{H}) \rightarrow \frac{1}{2}\mathbb{Z}$ be a function characterized up to overall translation by the formula

$$(11.41) \quad S(\mathbf{x}) - S(\mathbf{y}) = \left(\frac{n+1}{2}\right) \cdot n_0(B) - \left(\frac{n-1}{2}\right) \cdot n_1(B) - \left(\frac{n+1}{2}\right) \cdot n_2(B) + \left(\frac{n-1}{2}\right) \cdot n_3(B),$$

where B is any homology class in $\widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$ (i.e., classes possibly covering the basepoint z).

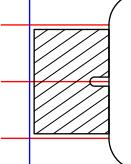
LEMMA 11.42. *Let Y be a homology solid torus, i.e., a three-manifold with torus boundary and $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$. Pick $\mu, \lambda \in H_1(\partial Y)$ so that λ generates the kernel of the map on H_1 induced by inclusion of the boundary while the image of μ generates $H_1(Y)$. Let $\widehat{CFD}(Y) \cong W^0 \oplus W^1$ be the decomposition by idempotents of the type D invariant of Y , with boundary marked by μ and $-n \cdot \mu + \lambda$. Then a function S satisfying (11.41) gives a decomposition*

$$W^i \cong \bigoplus_{s \in \frac{1}{2}\mathbb{Z}} W_s^i,$$

well-defined up to overall translation. The coefficient maps respect this grading in the following sense:

$$\begin{array}{lll} D_1: W_s^0 \rightarrow W_{s+\frac{n-1}{2}}^1 & D_2: W_s^1 \rightarrow W_{s+\frac{n+1}{2}}^0 & D_3: W_s^0 \rightarrow W_{s-\frac{n-1}{2}}^1 \\ D_{12}: W_s^0 \rightarrow W_{s+n}^0 & D_{23}: W_s^1 \rightarrow W_{s+1}^1 & D_{123}: W_s^0 \rightarrow W_{s+\frac{n+1}{2}}^1. \end{array}$$

PROOF. We first check that there is a function S satisfying Equation (11.41). To see this, observe that there is a natural splitting $\widetilde{\pi}_2(\mathbf{x}, \mathbf{y}) = \pi_2(\mathbf{x}, \mathbf{y}) \oplus \mathbb{Z}[\Sigma]$. Now, the right-hand side of Equation (11.41) remains unchanged when we add a copy of $[\Sigma]$ to the domain B ; so it suffices to show that the right-hand side of Equation (11.41) depends only on the generators \mathbf{x} and \mathbf{y} , when B is chosen to lie in $\pi_2(\mathbf{x}, \mathbf{y})$ (i.e., when B satisfies $n_0(B) = 0$). For such B , $S(B)$ is obtained by taking $g(B)$ and applying first the map in Formula (11.2) followed



by the map in Formula (11.40). Since Formula (11.40) was chosen to annihilate periodic domains, it follows that the result is independent of the choice of B .

The fact that the coefficient maps change grading as stated is immediate from the definition of S , Formula (11.41). \square

Let \mathfrak{S}_K denote the set of generators for the knot Floer complex $\widehat{CFK}(\mathcal{H}_K)$, and $\mathfrak{S}(n)$ denote the generators for $\widehat{CFD}(\mathcal{H}(n))$. Order the intersection points of $(\alpha_1 \cup \alpha_2)$ with β_g in the winding region in the order they are encountered along β_g , $\{x_i\}_{i=-\frac{n}{2}}^{\frac{n}{2}}$, so that x_0 is the intersection point of α_2 with β_g . (Recall that we assume for notational convenience that n is divisible by 4.) For each generator $\mathbf{x} \in \mathfrak{S}_K$ there are $n+1$ corresponding generators in $\mathfrak{S}(n)$, gotten by adding a point in $(\alpha_1 \cup \alpha_2) \cap \beta_g$ in the winding region. That is, we let $\{\mathbf{x}_i\}_{i=-\frac{n}{2}}^{\frac{n}{2}}$ denote the sequence of generators with the property that the $(\alpha_1 \cup \alpha_2) \cap \beta_g$ -component of \mathbf{x}_i is x_i and, outside the winding region, \mathbf{x}_i agrees with \mathbf{x} .

LEMMA 11.43. *For a given n , we can choose a function S on $\mathfrak{S}(n)$ as above so that it is related to the Alexander grading $A: \mathfrak{S}_K \rightarrow \mathbb{Z}$ of elements in \mathfrak{S}_K by the formula*

$$(11.44) \quad S(\mathbf{x}_k) = A(\mathbf{x}) - k + \left(\frac{(n+1) \cdot \text{sgn}(k)}{2} \right),$$

where $\text{sgn}(k) = -1, 0$, or 1 if $k < 0$, $k = 0$, or $k > 0$ respectively.

PROOF. The region 3 is a domain connecting \mathbf{x}_0 to \mathbf{x}_{-1} , and for $k < 0$ there is a domain connecting \mathbf{x}_k to \mathbf{x}_{k-1} with multiplicities $(0, 0, 1, 1)$ in regions $(0, 1, 2, 3)$. (See Figure 11.8 and the middle of Figure 11.10.) Similarly, region 1 connects \mathbf{x}_0 to \mathbf{x}_1 and, for $k > 0$, the generators \mathbf{x}_k and \mathbf{x}_{k+1} are connected by a domain with multiplicities $(1, 1, 0, 0)$. It follows from Equation (11.41) that

$$S(\mathbf{x}_k) = S(\mathbf{x}_0) - k + \left(\frac{(n+1) \cdot \text{sgn}(k)}{2} \right).$$

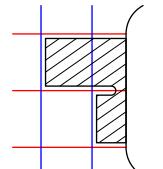
According to Equation (11.10), the Alexander grading on knot Floer homology is characterized (up to translation) by the property that if $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$ and $\phi \in \widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$ is any domain in \mathcal{H}_K , then $A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi)$. We can convert ϕ into a domain ϕ' in $\mathcal{H}(n)$ that connects \mathbf{x}_0 to \mathbf{y}_0 with $n_0(\phi) = n_1(\phi) = n_z(\phi)$ and $n_2(\phi) = n_3(\phi) = n_w(\phi)$ (as is evident from Figure 11.8). So,

$$S(\mathbf{x}_0) = A(\mathbf{x}) + c.$$

for some constant independent of the intersection point $\mathbf{x} \in \mathfrak{S}_K$. We can modify S by an overall translation to take $c = 0$. \square

We now have a collection of generators in $\mathfrak{S}(n)$ which have corresponding elements of \mathfrak{S}_K . The remaining generators (coming from intersections of α_2^a with the β -curves outside of the winding region) are called *exterior generators*. Note that the non-exterior generators are the generators of the form \mathbf{x}_i for some $\mathbf{x} \in \mathfrak{S}_K$, $i \in \{-n/2, -n/2 + 1, \dots, n/2\}$.

LEMMA 11.45. *For any doubly-pointed Heegaard diagram \mathcal{H}_K of a knot in S^3 , there is a constant c (independent of the framing parameter $-n$) with the following property. For the function $S: \mathfrak{S}(n) \rightarrow \mathbb{Q}$ as defined in Lemma 11.43, the generators $\mathbf{y} \in \mathfrak{S}(n)$ with $|S(\mathbf{y})| \geq c$ satisfy the following properties:*



- (c-1) \mathbf{y} is not an exterior generator, so $\mathbf{y} = \mathbf{x}_i$ for some $\mathbf{x} \in \mathfrak{S}_K$, $i \in \mathbb{Z}$; and
- (c-2) the sign of $S(\mathbf{y})$ agrees with the sign of i .

PROOF. First, we find c_1 so that generators \mathbf{y} with $|S(\mathbf{y})| \geq c_1$ satisfy Property (c-1). Note that the set of exterior generators is independent of the framing parameter $-n$. Start with some initial framing parameter $-m$, fix some $\mathbf{x} \in \mathfrak{S}_K$ and consider $\mathbf{x}_{-1} \in \mathfrak{S}(m)$. For each exterior generator \mathbf{y} , we can find a domain $B_0 \in \pi_2(\mathbf{y}, \mathbf{x}_{-1})$ with $n_0(B_0) = n_1(B_0) = 0$ and $n_2(B_0) = n_3(B_0) = c(\mathbf{y})$. (Start with an arbitrary domain in $\tilde{\pi}_2(\mathbf{y}, \mathbf{x}_{-1})$ and add copies of $[\Sigma]$ and the periodic domain to make $n_0 = n_1 = 0$; since \mathbf{y} and \mathbf{x}_{-1} are in the same idempotent, we will then have $n_2 = n_3$.) Now, in $\mathcal{H}(n)$, the domain B_0 can be used to construct a new domain $B_n \in \tilde{\pi}_2(\mathbf{y}, \mathbf{x}_{\frac{m-n}{2}-1})$ with $n_0(B_n) = n_1(B_n) = 0$ and $n_2(B_n) = n_3(B_n) = c(\mathbf{y})$. Since there are only finitely many possible exterior generators, we see that $\{|S(\mathbf{y}) - S(\mathbf{x}_{\frac{m-n}{2}})| \mid \mathbf{y} \text{ an exterior generator}\}$ is bounded independently of n . It follows from Lemma 11.43 that $|S(\mathbf{y})|$ for exterior generators y is bounded by some constant c_1 independent of n , giving Property (c-1).

Next, we claim that there is a constant c_2 with the property that if $S(\mathbf{x}_i) > c_2$, then $i > 0$. Specifically, choose $c_2 \geq \max_{\mathbf{x} \in \mathfrak{S}_K} A(\mathbf{x})$. Then Equation (11.44) shows that $S(\mathbf{x}_0) \leq c_2$. Similarly, if \mathbf{x}_{-i} is a non-exterior generator with $i > 0$ then, since $i \leq \frac{n}{2}$, Equation (11.44) guarantees that

$$S(\mathbf{x}_{-i}) = A(\mathbf{x}) + i - \frac{n+1}{2} \leq A(\mathbf{x}) - \frac{1}{2} \leq c_2 - \frac{1}{2} < c_2.$$

This guarantees Property (c-2) for $S(\mathbf{y}) > 0$. A symmetric argument ensures the property also in the case where $S(\mathbf{y}) < 0$. \square

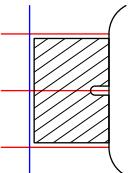
Lemmas 11.46–11.49 below state that for an appropriate class of almost complex structures, the coefficient maps have the desired forms from Theorem 11.36. The boundary of the winding region \mathcal{W} consists of two circles C_1 and C_2 parallel to the meridional α -circle; we label these so that C_1 is on the 2, 3 side of \mathcal{W} and C_2 is on the 0, 1 side of winding; see Figure 11.9. Fix an admissible complex structure J_0 on $\Sigma \times [0, 1] \times \mathbb{R}$. We now consider a two-parameter family of complex structures J_{T_1, T_2} on $\Sigma \times [0, 1] \times \mathbb{R}$ gotten by inserting a necks of lengths T_1 and T_2 along C_1 and C_2 , respectively. We say that a result *holds for sufficiently pinched almost complex structures* if there is a K_1 so that for any $T_1 > K_1$ there is a K_2 so that for any $T_2 > K_2$ the result holds for all generic almost complex structures of the form J_{T_1, T_2} .

LEMMA 11.46. *Let $K \subset S^3$ be a knot with Heegaard diagram \mathcal{H}_K , and let c be the constant from Lemma 11.45. For any sufficiently large n the following holds on $\mathcal{H}(n)$. Given any sufficiently pinched almost complex structure, for each generator $\mathbf{x} \in \mathfrak{S}_K$, with corresponding generators $\{\mathbf{x}_k\}$ in $\mathfrak{S}(n)$, we have:*

- $D_{23}(\mathbf{x}_k) = \mathbf{x}_{k-1}$ if $k < 0$ and $S(\mathbf{x}_k) \leq -c - 1$;
- $D_{01}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ if $k > 0$ and $S(\mathbf{x}_k) \geq c + 1$;
- $D_3(\mathbf{x}_0) = \mathbf{x}_{-1}$; and
- $D_1(\mathbf{x}_0) = \mathbf{x}_1$.

That is, for each $\mathbf{x} \in \mathfrak{S}_K$, we have a string of generators connected by coefficient maps as follows:

$$\dots \xleftarrow{D_{23}} \mathbf{x}_{-3} \xleftarrow{D_{23}} \mathbf{x}_{-2} \xleftarrow{D_{23}} \mathbf{x}_{-1} \xleftarrow{D_3} \mathbf{x}_0 \xrightarrow{D_1} \mathbf{x}_1 \xrightarrow{D_{01}} \mathbf{x}_2 \xrightarrow{D_{01}} \mathbf{x}_3 \xrightarrow{D_{01}} \dots$$



PROOF. The domains realizing the maps displayed in the statement can be found inside the winding region. For example, the domain realizing D_3 carrying \mathbf{x}_0 to \mathbf{x}_{-1} is the region marked by 3 in Figure 11.8. Domains from \mathbf{x}_{-k} to \mathbf{x}_{-k-1} are annuli ϕ_k which have a boundary component in α_1^a as in the middle of Figure 11.10.

The case of D_{23} . We analyze the map D_{23} , verifying it has the form stated, in the following eight steps.

Step 1. *The term \mathbf{x}_{-k-1} occurs in $D_{23}(\mathbf{x}_{-k})$.* By cutting ϕ_K along α_1^a at x_{-k} , we obtain a bigon realizing a non-zero coefficient of \mathbf{x}_{-k-1} in $D_{23}(\mathbf{x}_{-k})$. The rest of the proof (for the operator D_{23}) consists in showing that this is the only term in $D_{23}(\mathbf{x}_{-k})$.

Step 2. *If \mathbf{y}_{-k-1} occurs as a term in $D_{23}(\mathbf{x}_{-k})$ then $\mathbf{y} = \mathbf{x}$.* Suppose that \mathbf{y}_{-k-1} occurs in $D_{23}(\mathbf{x}_{-k})$. Considering the corners shows that any domain $\phi \in \widetilde{\pi}_2(\mathbf{x}_{-k}, \mathbf{y}_{-k-1})$ is a disjoint union of ϕ_k with a different positive domain ϕ' . As such, the corresponding moduli space naturally has an $(\mathbb{R} \oplus \mathbb{R})$ -action (translating the two components independently). If this moduli space is generic and one-dimensional, it follows that the other component ϕ' is trivial, so $\mathbf{y}_{-k-1} = \mathbf{x}_{-k-1}$.

Step 3. *No exterior generators occur in $D_{23}(\mathbf{x}_{-k})$.* If an exterior generator \mathbf{y} appears in $D_{23}(\mathbf{x}_{-k})$, then $S(\mathbf{y}) = S(\mathbf{x}_{-k}) + 1 \leq -c$, in view of Lemma 11.42 and our assumption on $S(\mathbf{x}_{-k})$. By Lemma 11.45 (and specifically Property (c-1)), it follows that \mathbf{y} is not an exterior generator.

Step 4. *No terms of the form \mathbf{y}_j where $j > 0$ occur in $D_{23}(\mathbf{x}_{-k})$.* This is same argument as in Step 3, using Property (c-2) this time.

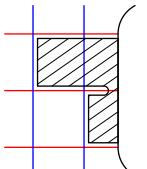
In view of Steps 3 and 4, only terms of the form \mathbf{y}_{-k+i} with $-k + i < 0$ can appear in $D_{23}(\mathbf{x}_{-k})$. Suppose now that such a term does, indeed, appear, and let $B \in \widetilde{\pi}_2(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$ be a corresponding homology class with a pseudo-holomorphic representative. Our goal now is to study properties of this domain.

Step 5. *Pinching Σ induces a decomposition of the domain B .*

The boundary of the winding region \mathcal{W} consists of two circles C_1 and C_2 parallel to the meridional α -circle, where C_1 is the component on the side of the regions 2 and 3. Fix a complex structure J and, as in the definition of “sufficiently pinched,” consider degenerating J (and so Σ) along $\partial\mathcal{W}$. Suppose that there is a $\varphi \in \widetilde{\mathcal{M}}(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$ with multiplicity +1 in regions 2 and 3 and 0 in regions 0 and 1. We will show by positivity considerations that $i \geq -1$; in fact, we will see that the homology class B can be constructed as a kind of connected sum of a domain $B_0 \in \widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$ with $n_w(B_0) = i + 1$ and $n_z(B_0) = 0$ with a homology class $B_1 \in \widetilde{\pi}_2(x_{-k}, x_{-k+i})$.

More precisely, let $B \in \widetilde{\pi}_2(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$ be a homology class giving a term in $D_{23}(\mathbf{x}_{-k})$. The domain B has multiplicity zero at the regions 0 and 1, no corners outside of the winding region on β_g or α_1^a , and in fact, both of its corners on β_g (and on α_1^a) occur in the negative half of the winding region. (Specifically the corners occur at x_{-k} and x_{-k+i} , where both $-k$ and $-k + i$ are negative; here we are using Step 4.) It follows that B induces a two-chain B_0 on the destabilized surface Σ_0 , having local multiplicity zero at z , and some multiplicity j at the point w (which in turn corresponds to C_1 , shrunk to a point).

The domain at the winding region induces another two-chain, supported in a disk. Specifically, let \mathcal{W}_L be the part of the winding region to the left of α_2^a , so \mathcal{W}_L contains the regions 2 and 3. As we degenerate our curve C_1 to a point, \mathcal{W}_L becomes a disk Σ'_1 whose outer boundary is α_2^a , the meridian of the knot from Figure 11.8, with a preferred central point δ_1 , the node obtained by degenerating C_1 . See Figure 11.10. The two-chain B_1 induced from B



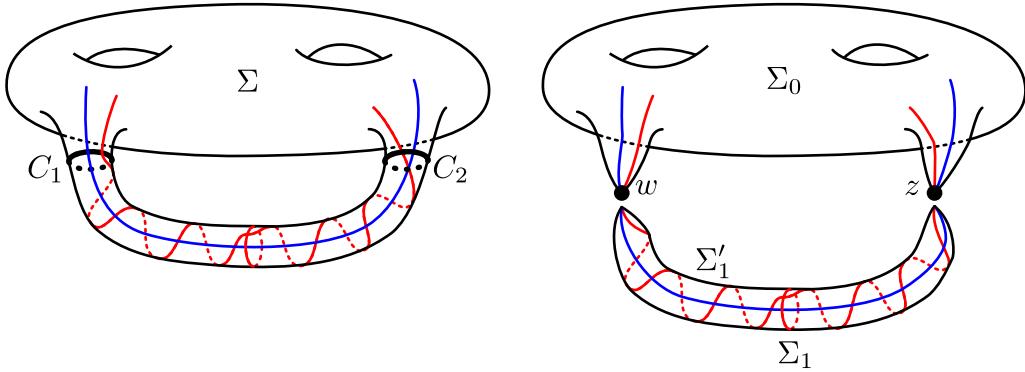
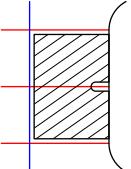


FIGURE 11.9. **Pinching off the winding region.** The Heegaard surface is degenerated at the two brown curves to give a nodal surface with two components: one of these is the de-stabilized knot diagram Σ_0 (equipped with a pair of arcs α_g and β_g connecting the w and z punctures) and the other as a sphere, which contains the winding region.

must have local multiplicity j at δ_1 . There are two distinguished arcs in Σ'_1 connecting δ_1 to the boundary. One is a portion of β_g . The other is a portion of α_1^a which winds around, meeting β_g in a sequence of intersection points $\{x_k\}_{k=-n/2}^0$. Recall that for each k there is a homology class ϕ_k connecting x_k to x_{k-1} , with multiplicity +1 at the regions 2 and 3, and multiplicity zero at δ_1 . There is also a homology class ψ_k connecting x_{k-1} to x_k , which can be thought of as the complement of ϕ_k . It is easy to see that a positive domain in this disk region which connects intersections x_k and x_{k+i} of the α - and β -arcs and has multiplicity one in regions 2 and 3 necessarily decomposes as a composite $\phi_k * \psi_k * \psi_{k+1} * \dots * \psi_{k+i}$. In particular, for our initial $B \in \tilde{\pi}_2(\mathbf{x}_{-k}, \mathbf{y}_{-k+i})$, we get $B_1 \in \tilde{\pi}_2(x_{-k}, x_{-k+i})$, which we decompose as $\phi_k * \psi_k * \psi_{k+1} * \dots * \psi_{k+i}$. Consequently, the multiplicities of B_1 near the boundary of the winding region are all equal (and indeed equal to $i+1$). Thus, the homology class B_0 on Σ_0 has multiplicity $j = i+1$ at w .

Step 6. *Holomorphic curves have constrained behavior under the degeneration.* We turn next to the behavior of holomorphic curves as we stretch the neck along $\partial\mathcal{W}$. By an argument analogous to the proof of Proposition 9.6, a sequence of curves u_r holomorphic with respect to neck lengths $r \rightarrow \infty$ converges to a pair (u_0, u_1) consisting of a holomorphic curve (or comb) u_0 in $\Sigma_0 \times [0, 1] \times \mathbb{R}$ and a holomorphic curve (or comb) u_1 in $\mathcal{W} \times [0, 1] \times \mathbb{R}$, where we view \mathcal{W} as a sphere with three punctures, corresponding to the two circles where \mathcal{W} is joined to the rest of Σ and the puncture p in \mathcal{W} (i.e., $e\infty$ of Σ). (For the purposes of some of the arguments here, it is useful to bear in mind that Σ_0 is equipped with a little extra structure now, in the form of two arcs, remnants of α_g and β_g , which connect the punctures w and z .) For notational convenience, write Σ_1 to denote \mathcal{W} viewed as a three-punctured sphere. Let δ_1 denote the puncture of Σ_1 corresponding to the curve C_1 .

The maps u_i are analogous to the matched discussed in Section 9.1 (Definition 9.2), but can have Reeb orbits in addition to Reeb chords. More precisely, in the case that u_0 and u_1 are curves rather than combs,



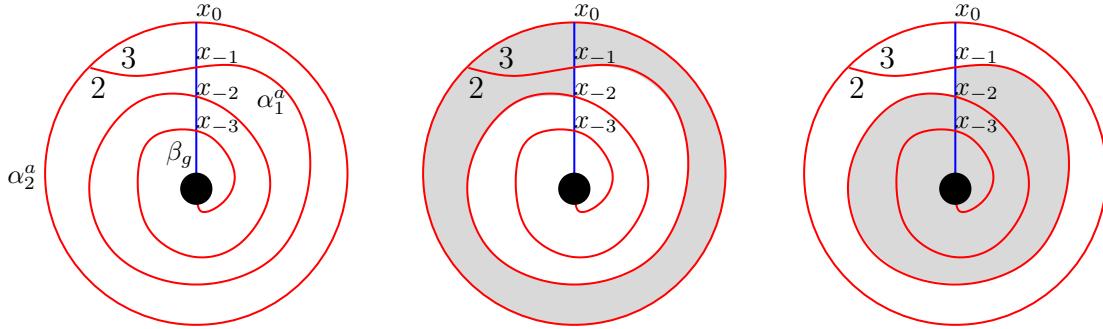


FIGURE 11.10. **Homology classes in the winding region.** The winding region to the left of the meridian, from Figure 11.8, is drawn inverted so that the outer boundary of the winding region corresponds to the dark marked point, and the meridional arc corresponds to the outer boundary of the picture. In the center and right we illustrate two homology classes: $\phi_1 \in \tilde{\pi}_2(\mathbf{x}_{-1}, \mathbf{x}_{-2})$ and its complement, $\psi_1 \in \tilde{\pi}_2(\mathbf{x}_{-2}, \mathbf{x}_{-1})$ from the proof of Lemma 11.46.

- u_0 is a holomorphic map $S_0 \rightarrow \Sigma_0 \times [0, 1] \times \mathbb{R}$ (with respect to an appropriate almost complex structure J on $\Sigma_0 \times [0, 1] \times \mathbb{R}$), where S_0 is a surface with boundary, boundary punctures, and interior punctures. At each boundary puncture, u_0 is asymptotic to one of:
 - a chord in $\mathbf{x} \times [0, 1]$ at $-\infty$ or $\mathbf{y} \times [0, 1]$ at $+\infty$,
 - a chord in C_1 starting and ending on $\alpha_1^a \cup C_1$,
 - a chord in C_1 starting and ending on $\beta_g \cap C_1$, or
 - a chord in C_1 starting on β_g and ending on α_1^a or starting on α_1^a and ending on β_g .

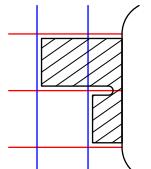
We call punctures of these types $\pm\infty$ punctures, α boundary punctures, β boundary punctures and mixed boundary punctures, respectively.

At each interior puncture, u_0 is asymptotic to the circle C_1 at some point $(x, y) \in [0, 1] \times \mathbb{R}$ (i.e., $\pi_\Sigma \circ u \rightarrow \delta_1$ and $\pi_D \circ u \rightarrow (x, y)$); or possibly to a multiple cover of C_1 .

- u_1 is a map $S_1 \rightarrow \Sigma_1 \times [0, 1] \times \mathbb{R}$ (with respect to an appropriate almost complex structure J on $\Sigma_1 \times [0, 1] \times \mathbb{R}$), where S_1 is a surface with boundary, boundary punctures, and interior punctures. At each boundary puncture, u_1 is asymptotic to one of:
 - a chord in $x_i \times [0, 1]$ at $\pm\infty$,
 - a chord ρ at $e\infty$ (at some point $(1, t) \in [0, 1] \times \mathbb{R}$).
 - a chord in C_1 starting and ending on $\alpha_1^a \cup C_1$,
 - a chord in C_1 starting and ending on $\beta_g \cap C_1$, or
 - a chord in C_1 starting on β_g and ending on α_1^a or starting on α_1^a and ending on β_g .

We call punctures of these types $\pm\infty$ punctures, $e\infty$ punctures, α boundary punctures, β boundary punctures and mixed boundary punctures, respectively.

At each interior puncture, u_0 is asymptotic to the circle C_1 at some point $(x, y) \in [0, 1] \times \mathbb{R}$ (i.e., $\pi_\Sigma \circ u \rightarrow \delta_1$ and $\pi_D \circ u \rightarrow (x, y)$); or possibly to a multiple cover of C_1 .



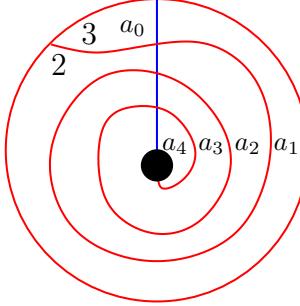


FIGURE 11.11. **Notation for the local multiplicities in the winding region.** The multiplicities are denoted a_0, \dots, a_4 , in order of distance from the region containing ρ_3 .

- There is a bijection ϕ_a between the α boundary punctures of S_0 and the α boundary punctures of S_1 so that for each boundary puncture q_i^a of S_0 , $u_0 t(u_0(q_i^a)) = t(u_1(\phi_a(q_i^a)))$; and similarly for the β boundary punctures.
- There is a bijection ϕ_c between the interior punctures of S_0 and the interior punctures of S_1 so that for each interior puncture q_i^c of S_0 , $\pi_{\mathbb{D}}(u_0(q_i^c)) = \pi_{\mathbb{D}}(u_1(\phi_c(q_i^c)))$.

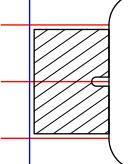
More generally, u_0 and u_1 can be combs—the obvious analogues of combs from Section 5.4, but allowing Reeb orbits. (Indeed, in the presence of mixed boundary punctures, u_1 is forced to be a holomorphic comb with at least two vertical levels.)

Step 7. Holomorphic curves with Reeb chords do not occur in the limit. We show next that the limiting curves (u_0, u_1) have no mixed boundary punctures, α boundary punctures or β boundary punctures.

If u_1 has a mixed boundary puncture then u_1 necessarily has at least two stories. Let $u_{1,1}$ be the first story of u_1 that contains a mixed boundary puncture, so $u_{1,1}$ is asymptotic to some $x_{-i} \times [0, 1]$ at $-\infty$ and to a chord γ_1 connecting α_1^a to β_g at $+\infty$. Number the local multiplicities of $u_{1,1}$ in the various regions $\{a_i\}_{i=0}^{n/2+1}$, ordered by distance from the region containing the Reeb chord ρ_3 , as indicated in Figure 11.11. Note that a_0 and a_1 coincide with the local multiplicities of $u_{1,1}$ around ρ_3 and ρ_2 , respectively. In particular, $a_0, a_1 \in \{0, 1\}$. We rule out the four possibilities for a_0 and a_1 in turn.

If $a_0 = 1$ and $a_1 = 0$ then the domain $u_{1,1}$ consists entirely of the region 3. This contradicts the fact that $u_{1,1}$ is asymptotic to a mixed chord at $+\infty$. If $a_0 = 0$ and $a_1 = 1$ then $a_j = j$ for $j = 1, \dots, i$ and $a_j = i$ for $j > i$ (since $u_{1,1}$ has a $-\infty$ corner at x_{-i} and no other corners in the interior of \mathcal{W}). But the fact that $u_{1,1}$ is asymptotic to a mixed chord at $+\infty$ implies that $a_{n/2} = a_{n/2-1} - 1$, a contradiction. If $a_0 = a_1 = 0$ then $0 = a_0 = a_1 = \dots = a_i$. Since x_{-i} is an initial corner, it follows that $a_{i+1} = -1$, a contradiction to the fact that $u_{1,1}$ is holomorphic. Finally, if $a_0 = a_1 = 1$ then $a_j = 1$ for $j \leq i$, $a_{i+1} = 0$ and $a_{i+2} = -1$, which again contradicts the fact that $u_{1,1}$ is holomorphic. (Here we use the assumption that n is sufficiently large, to guarantee that $k + 2 < n/2$ from the assumption on $S(\mathbf{x}_i)$.) It follows that u has no mixed chords.

Next, suppose u_0 has an α boundary puncture. Then the source S_0 of u_0 has some boundary component which is mapped to α_1^a . Since \mathbf{x} and \mathbf{y} have no components in $\alpha_1^a \cap \Sigma_0$, $t \circ u_0$ remains bounded on this component of ∂S_0 . By the maximum principle, it follows that $\pi_{\mathbb{D}} \circ u_0$ is constant on this component of S_0 . In particular, the boundary of this component



of S_0 is contained entirely in α . Thus, $\pi_{\Sigma_0} \circ u_0$ gives a homological relation between $\alpha \cap \Sigma_0$. Since these curves and arcs are homologically linearly independent, and the two-chain has multiplicity 0 at $z \in \Sigma_0$ (as established in Step 5), it follows that all its local multiplicities vanish. (See also Lemma 5.48 for a similar argument.) Thus, u_0 (and hence u_1) has no α boundary punctures. A similar argument shows that u_0 and u_1 have no β boundary punctures.

Step 8. *Dimension constraints on holomorphic curves with Reeb orbits rule out all holomorphic curves which are not supported in the winding region.* In sum, the holomorphic curves u_0 and u_1 are asymptotic to some collection of Reeb orbits (copies of $C_1 \times (x, y)$, or multiple covers of these) at δ_1 . Generically, all of these Reeb orbits will be simple (not multiply covered), so it suffices to consider this case. As in Step 5, we decompose B_1 as $B_1 = \phi_k * \psi_k * \dots * \psi_{k+i-1}$, where now i is the number of Reeb orbits of u_0 and u_1 . (In the case $i = 0$ there are no ψ_j in the decomposition and no Reeb orbits.) The (expected) dimension of the moduli space of curves containing u_1 is $\text{ind}(u_1) = 2i + 1$, and the (expected) dimension of the moduli space of curves near u_0 is given by $\text{ind}(B_0)$. Finally, each Reeb orbit contributes a two-dimensional matching condition between u_0 and u_1 , so

$$\begin{aligned}\text{ind}(B, (\rho_2, \rho_3)) &= \text{ind}(u_0) + \text{ind}(u_1) - 2i \\ &= \text{ind}(B_0) + 1.\end{aligned}$$

Thus, since $\text{ind}(B, (\rho_2, \rho_3)) = 1$, $\text{ind}(B_0) = 0$, so u_0 is a trivial holomorphic curve. It follows that $i = 0$, and B is exactly the class ϕ_k , showing that the only possible curve is the one analyzed in Step 1.

This completes the proof for D_{23} .

The case of D_{01} . This case is symmetric to the case of D_{23} , via a rotation of the winding region (which also reverses the sign of the indexing of the x_i and the sign of the function S).

The case of D_3 . Suppose that $\phi \in \widetilde{\pi}_2(\mathbf{x}_0, \mathbf{y})$ is a positive domain with local multiplicity one at the region marked with 3 and zero at the regions marked by 0, 1 and 2. Considering local multiplicities around the intersection point x_{-1} , we see that x_{-1} is forced to be a corner point, and indeed that the intersection of ϕ with the winding region \mathcal{W} is forced to be the bigon domain from x_0 to x_{-1} . It follows (as in Step 2 of the analysis of D_{23}) that for generic almost-complex structures, the only positive, index one $\phi \in \widetilde{\pi}_2(\mathbf{x}_0, \mathbf{y})$ which admits a pseudo-holomorphic representative is this bigon, connecting \mathbf{x}_0 to $\mathbf{y} = \mathbf{x}_{-1}$.

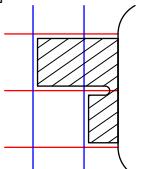
The case of D_1 . This is symmetric to the case of D_3 via a rotation of the winding region. \square

LEMMA 11.47. *For any sufficiently large n , if $\mathbf{y} \in \mathfrak{S}(n)$ satisfies $|S(\mathbf{y})| < \frac{n}{4}$ then $D_{230}(\mathbf{y}) = D_2(\mathbf{y}) = D_0(\mathbf{y}) = D_{012}(\mathbf{y}) = 0$.*

PROOF. By the same argument as used to prove Lemma 11.42, the grading shifts of these maps are given by

$$\begin{array}{ll} D_0: W_s^1 \rightarrow W_{s-\frac{n+1}{2}}^0 & D_2: W_s^1 \rightarrow W_{s+\frac{n+1}{2}}^0 \\ D_{230}: W_s^1 \rightarrow W_{s-\frac{n-1}{2}}^0 & D_{012}: W_s^1 \rightarrow W_{s+\frac{n-1}{2}}^0. \end{array}$$

Thus, if $|S(\mathbf{y})| < \frac{n}{4}$ then, writing D_* for any of the above maps, we find $|S(D_*(\mathbf{y}))| \geq \frac{n-2}{4}$. By Equation (11.43), the S -grading of any generator \mathbf{x}_0 of W^0 coincides with the Alexander grading of the corresponding $\mathbf{x} \in \mathfrak{S}_K$. Thus, taking $n \geq 4 \max_{\mathbf{x} \in \mathfrak{S}_S} |A(\mathbf{x})| + 2$, $D_*(\mathbf{y})$ lies in the zero group. \square



LEMMA 11.48. *The following holds for any sufficiently pinched almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. For each positive domain $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$ in \mathcal{H}_K with $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$, $n_w(B) = k \geq 0$, and $n_z(B) = 0$, there is a corresponding sequence of domains $B_i \in \tilde{\pi}_2(\mathbf{x}_{i-k}, \mathbf{y}_i)$ for $k - \frac{n}{2} \leq i \leq 0$ such that:*

- B_i is positive.
- If $i < 0$ or $k = 0$ then B_i is provincial, and if $i = 0$ and $k > 0$ then B_i crosses region 2 with multiplicity one and has multiplicity 0 at regions 0, 1 and 3.
- If $i < 0$ or $k = 0$ then $\text{ind}(B_i, ()) = \text{ind}(B)$, and if $i = 0$ and $k > 0$ then $\text{ind}(B_i, (\{\rho_2\})) = \text{ind}(B)$.
- Outside the winding region, B_i agrees with B (in the obvious sense).

In the other direction, suppose $B' \in \tilde{\pi}_2(\mathbf{x}_{i-k}, \mathbf{y}_i)$ satisfies $n_0(B') = n_1(B') = n_3(B') = 0$ and $n_2(B') \in \{0, 1\}$. Let $\vec{\rho}(B')$ be the empty sequence of sets of Reeb chords if $n_2(B') = 0$ and the sequence $(\{\rho_2\})$ if $n_2(B') = 1$. Suppose that

$$\begin{aligned}\text{ind}(B', \vec{\rho}(B')) &= 1 \\ \mathcal{M}(B', \vec{\rho}(B')) &\neq \emptyset.\end{aligned}$$

Then $B' = B_i$ for some $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$, and $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$.

Similarly, for each positive domain $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$ in \mathcal{H}_K with $\mathbf{x}, \mathbf{y} \in \mathfrak{S}_K$, $n_z(B) = k \geq 0$, and $n_w(B) = 0$, there is a corresponding sequence of domains $B_i \in \tilde{\pi}_2(\mathbf{x}_{i+k}, \mathbf{y}_i)$ for $0 \leq i \leq \frac{n}{2} - k$ such that:

- B_i is positive.
- If $i > 0$ or $k = 0$ then B_i is provincial, and if $i = 0$ and $k > 0$ then B_i crosses region 0 with multiplicity one and has multiplicity 0 at regions 1, 2 and 3.
- If $i > 0$ or $k = 0$ then $\text{ind}(B_i, ()) = \text{ind}(B)$, and if $i = 0$ and $k > 0$ then $\text{ind}(B_i, (\{\rho_0\})) = \text{ind}(B)$.
- Outside the winding region, B_i agrees with B (in the obvious sense).

In the other direction, suppose $B' \in \tilde{\pi}_2(\mathbf{x}_{i+k}, \mathbf{y}_i)$ satisfies $n_1(B') = n_2(B') = n_3(B') = 0$ and $n_0(B') \in \{0, 1\}$. Let $\vec{\rho}(B')$ be the empty sequence of sets of Reeb chords if $n_0(B') = 0$ and the sequence $(\{\rho_0\})$ if $n_0(B') = 1$. Suppose that

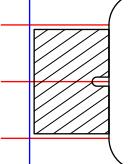
$$\begin{aligned}\text{ind}(B', \vec{\rho}(B')) &= 1 \\ \mathcal{M}(B', \vec{\rho}(B')) &\neq \emptyset.\end{aligned}$$

Then $B' = B_i$ for some $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$, and $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$.

PROOF. Given B , the domain B_i is constructed using the procedure illustrated by the shaded region in Figure 11.8; B_i is uniquely determined by its corners $x_{i\pm k}$ and x_i , its local multiplicities at the regions 0, 1, 2, 3, and the condition that B_i agrees with B outside the winding region. It is immediate from the index formulas that B_i has the same index as B .

Suppose next that $B' \in \tilde{\pi}_2(\mathbf{x}_{-s}, \mathbf{y}_{-s+t})$ for $s, t \geq 0$ is an index one homology class with $n_0(B') = n_1(B') = n_3(B') = 0$ which has a holomorphic representative. We claim that $B' = B_{-s}$ for some $B \in \tilde{\pi}_2(\mathbf{x}, \mathbf{y})$ with $n_w(B) = k$, $n_z(B) = 0$, and $\#\mathcal{M}(B', \vec{\rho}(B')) = \#\mathcal{M}(B)$.

By stretching along the boundary of the winding region, as in the proof of Lemma 11.46, we see that B' can be written as a connected sum of a homology class B_0 with a standard chain $\psi_{s,t} \in \tilde{\pi}_2(x_{-s}, x_{-s+t})$. We have that $\text{ind}(\psi_{s,t}, \vec{\rho}(B')) = 2t$, and $\text{ind}(B') = \text{ind}(B_0) + \text{ind}(\psi_{s,t}) - 2t$. Thus, $\text{ind}(B_0) = 1$.



We claim the moduli space is obtained as a fibered product $\widetilde{\mathcal{M}}(\varphi_0) \times_{\text{Sym}^t(\mathbb{D})} \widetilde{\mathcal{M}}(\psi_{s,t})$. As in the proof of Lemma 11.46, this follows from compactness and gluing results, together with the following argument to rule out curves asymptotic to Reeb chords at the boundary of \mathcal{W} (so the only asymptotics at $\partial\mathcal{W}$ are Reeb orbits). Chords with both endpoints on α or both endpoints on β are ruled out exactly as in the proof of Lemma 11.46: such chords force components of the curve in Σ_0 to be constant. Mixed Reeb chords are ruled out by considering the local multiplicities at the regions in \mathcal{W} , similarly to the proof of Lemma 11.46; for the bottom-most story containing a mixed puncture the local multiplicities a_i are all 1 for i sufficiently large, which is inconsistent with being asymptotic to a mixed Reeb chord at $+\infty$.

Having identified our moduli space with the fiber product $\widetilde{\mathcal{M}}(B_0) \times_{\text{Sym}^t(\mathbb{D})} \widetilde{\mathcal{M}}(\psi_{s,t})$, we next claim that the map $\widetilde{\mathcal{M}}(\psi_{s,t}) \rightarrow \text{Sym}^t(\mathbb{D})$, gotten by projecting the preimage of the connect sum point q , has degree one (onto $\text{Sym}^t(\mathbb{D})$), so that the moduli space has the same count modulo two as $\widetilde{\mathcal{M}}(B_0)$, as desired.

In the case where $t = 1$, this can be seen directly. Consider the moduli space $\psi_{s,1}$, which, in the notation of the proof of Lemma 11.46, is ψ_{s-1} ; see also the right of Figure 11.10. Our claim is that the map of $\widetilde{\mathcal{M}}(\psi_{s-1})$ to $[0, 1] \times \mathbb{R}$ has degree one or, equivalently, after we divide out the space $\widetilde{\mathcal{M}}(\psi_{s-1})$ by translations, the preimage of q is mapped with degree one onto the interval $[0, 1]$. But the quotient of $\widetilde{\mathcal{M}}(\psi_{s-1})$ by translations can be parameterized by depth of the cut at x_{-s} . In the limit as the cut goes out to the marked point along the α -circle, the preimage of q is mapped (under projection to $[0, 1]$) to $\{1\}$; if the cut goes out along the β -circle, then the preimage of q is mapped to $\{0\}$. Hence the evaluation map has degree one when $t = 1$, as stated.

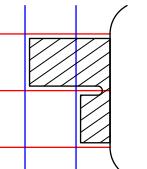
For $t > 1$ we show that the evaluation map has degree one as follows. By a straightforward analysis of domains in the winding region (see Step 5 in the proof of Lemma 11.46), $\psi_{s,t}$ can be written as a juxtaposition of flowlines with Maslov index two. In fact, any decomposition of $\psi_{s,t}$ consisting of non-negative and non-constant homology classes must necessarily be a decomposition into homology classes each of which has non-zero multiplicity at q . It follows that the map $\widetilde{\mathcal{M}}(\psi_{s,t}) \rightarrow \text{Sym}^t(\mathbb{D})$ is proper. To calculate its degree, consider the fibers of points in $\text{Sym}^t(\mathbb{D})$ where one of the components has \mathbb{R} coordinate much larger than all the others. Such holomorphic disks are in the image of the gluing map of a moduli space with multiplicity 1 with another moduli space with multiplicity $t - 1$. By induction, it follows that the evaluation map has degree one. \square

LEMMA 11.49. *For any sufficiently pinched almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$, for $s \geq \frac{n}{4}$ the map $D_{23}: W_s^1 \rightarrow W_{s+1}^1$ is as given by Theorem 11.36.*

PROOF. We introduce some preliminary notation. A curve w representing D_{23} has local multiplicity zero at the regions marked by 0 and 1. Thus, if we let $\Omega \subset \Sigma$ denote the complement in Σ of the regions marked by 0 and 1, the source of w maps to $\Omega \times [0, 1] \times \mathbb{R}$.

We prove the lemma in nine steps.

Step 1. $D_{23}(\mathbf{x}_1) = 0$ for any $\mathbf{x} \in \mathfrak{S}_K$, since there are no domains which can represent such a map. On the one hand, any domain with initial corner at x_1 and vanishing local multiplicities at 0 and 1 must in fact vanish at all four regions near x_1 . On the other hand,



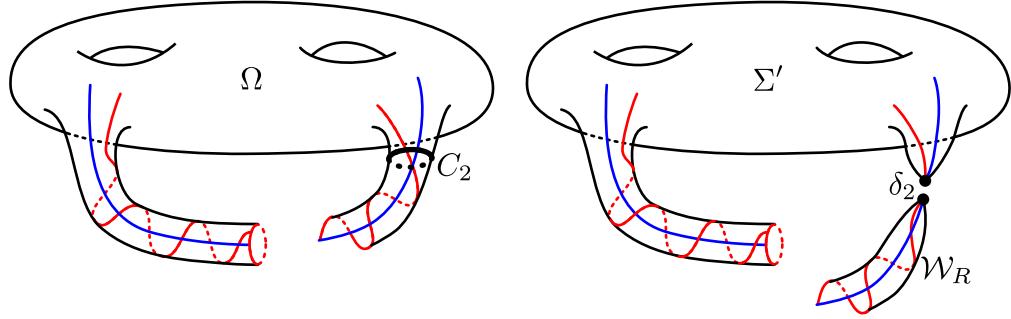


FIGURE 11.12. **Pinching Ω along C_2 .** This is half the pinching from Figure 11.9 (done on $\Omega \subset \Sigma$).

a domain which represents D_{23} must have a cut (along α_1^a) which goes out the boundary (separating regions 2 and 3). Thus, it cannot have vanishing multiplicities around x_1 .

Step 2. *Degenerate the surface to extract a boundary degeneration.* Consider $D_{23}(\mathbf{x}_i)$ for some $1 < i$. We are interested in terms \mathbf{y}_j in $D_{23}(\mathbf{x}_i)$ where $j > 0$. We use a degeneration argument similar to the proof of Lemma 11.46. As in the definition of sufficiently pinched almost complex structures let C_2 be the boundary component of the winding region on the 0, 1 side (i.e., the right side as shown in Figure 11.9). Pinch the surface $\Omega \subset \Sigma$ along C_2 . The resulting nodal surface has two irreducible components: a topological disk which we denote \mathcal{W}_R corresponding to part of the winding region; and another component Σ' corresponding to most of the diagram and containing the regions 2 and 3. See Figure 11.12. Let δ_2 denote the node (or puncture) corresponding to C_2 after the degeneration. Holomorphic curves giving terms in $D_{23}(\mathbf{x}_i)$ degenerate to pairs of holomorphic curves (u, v) where u is a curve in $\mathcal{W}_R \times [0, 1] \times \mathbb{R}$ and v is a curve whose interior lies in $\Sigma' \times [0, 1] \times \mathbb{R}$.

By hypothesis, the components of \mathbf{x}_i and \mathbf{y}_j on α_1^a both lie in \mathcal{W}_R , and v has punctures mapped to the Reeb chords ρ_2 and ρ_3 . Let S_0 be the component of (the source of) v part of the boundary of which is mapped to α_1^a . Then one component of $\partial \overline{S_0}$ is mapped entirely to α_1^a , and the t -coordinate of v remains bounded on this component. Consequently, as in the proof of Lemma 5.51, $\pi_{\mathbb{D}} \circ v$ is constant on S_0 , so $v|_{S_0}$ is a boundary degeneration.

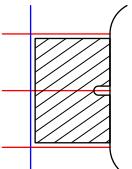
Write the source of v as $S = S_0 \amalg S_1$.

Step 3. *The curve v consists of the boundary degeneration and trivial strips.* By Proposition 5.69, D_{23} counts domains B with

$$(11.50) \quad 0 = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B).$$

Let B_2 be the domain in \mathcal{W}_R determined by u and B_1 the domain in Σ' determined by $v|_{S_1}$. Let $n_{\delta_2}(B_i)$ denote the local multiplicity of B_i at δ_2 . Then,

$$(11.51) \quad \begin{aligned} n_{\delta_2}(B_2) &= n_{\delta_2}(B_1) + 1 \\ n_{x_i}(B_2) + n_{\mathbf{x}}(B_1) &= n_{\mathbf{x}}(B) - (g-1) \\ n_{x_j}(B_2) + n_{\mathbf{y}}(B_1) &= n_{\mathbf{y}}(B) - (g-1) \\ e(B_2) + e(B_1) &= e(B) - 2 + 2g + 2n_{\delta_2}(B_2). \end{aligned}$$



(In these formulas, for the purposes of the Euler measure e , the point δ_2 is treated as a node, not a puncture.) Combining Equations (11.51) and (11.50) gives

$$(11.52) \quad e(B_1) + n_{\mathbf{x}}(B_1) + n_{\mathbf{y}}(B_1) + e(B_2) + n_{x_i}(B_2) + n_{x_j}(B_2) = 2n_{\delta_2}(B_1) + 2.$$

A direct analysis of the domains in the winding region shows that

$$e(B_2) + n_{x_i}(B_2) + n_{x_j}(B_2) = 2n_{\delta_2}(B_2).$$

It follows that $e(B_1) + n_{\mathbf{x}}(B_1) + n_{\mathbf{y}}(B_1) = 0$. This is the expected dimension of the parameterized moduli space of holomorphic curves in the domain B_1 . It follows that $v|_{S_1}$ consists of trivial strips, so $B_1 = 0$ and $n_{\delta_2}(B_2) = 1$. In particular, $\mathbf{y} = \mathbf{x}$.

Step 4. *Analyzing u .* The homology class of u is uniquely determined by the properties that its local multiplicities at the regions 0 and 1 are zero, x_i is a corner, and its local multiplicity δ_2 is 1. It follows that u connects x_i to x_{i-1} . Combining this with the previous step, it follows that $\mathbf{y}_j = \mathbf{x}_{i-1}$. It also follows from the above considerations that there is a unique holomorphic curve u in this homology class.

At this point, we have shown that either $D_{23}(\mathbf{x}_i) = \mathbf{x}_{i-1}$ or $D_{23}(\mathbf{x}_i) = 0$. It remains to exclude the latter case; this involves a gluing argument.

Step 5. *Relaxing the almost complex structure.* We would like to assert that there is a unique holomorphic curve in the moduli space of v with the Reeb chord in C_2 at a given t -coordinate, and that we can glue this curve v to u to get a holomorphic curve in Σ . Unfortunately, if $g > 1$ then the curve v is not transversally cut out for any choice of admissible almost complex structure in the sense of Definition 5.1.

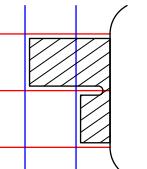
Instead, we relax Condition (J-1) of the definition of admissible complex structures, to [Lip06a, Condition (J5')], which we restate here as:

- (J'-1) There is a 2-plane distribution ξ on $\Sigma \times [0, 1]$ such that the restriction of ω to ξ is non-degenerate, J preserves ξ , and the restriction of J to ξ is compatible with ω . We further assume that ξ is tangent to Σ near $(\alpha \cup \beta) \times [0, 1]$.

This is sufficient for [BEH⁺03] to give compactness for the moduli spaces of holomorphic curves. We call such almost-complex structures *relaxed*.

Given an almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$, let $\mathcal{M}(v; \beta; J)$ denote the moduli space of J -holomorphic curves containing v . For a generic almost complex structure satisfying (J'-1) rather than (J-1), the moduli spaces $\mathcal{M}(v; \beta; J)$ is transversally cut out (and, in particular, does not contain any boundary degenerations).

Step 6. *Independence of $\mathcal{M}(v, \beta; J)$ from the β -curves.* Let β' be another set of β -curves that agree with β near \mathbf{x} and let $\mathcal{M}(v; \beta'; J)$ denote the moduli space of holomorphic curves with respect to β' corresponding to $\mathcal{M}(v; \beta; J)$. We claim that for J a generic almost complex structure satisfying condition (J'-1) close enough to an almost complex structure J_0 satisfying condition (J-1), $\#\mathcal{M}(v; \beta'; J) = \#\mathcal{M}(v; \beta; J)$. Indeed, taking $J \rightarrow J_0$, elements of $\mathcal{M}(v; \beta'; J)$ converge to the unions of α -boundary degenerations and trivial strips. Let $U_{\mathbf{x}}$ be a small neighborhood of \mathbf{x} so that $\beta \cap U_{\mathbf{x}} = \beta' \cap U_{\mathbf{x}}$. It follows that for J close enough



to J_0 and $w \in \mathcal{M}(v; \beta'; J)$, $w(\partial S) \subset \alpha \cup (\beta \cap U_x)$; and similarly for elements of $\mathcal{M}(v; \beta; J)$. Thus, for J close enough to J_0 , $\mathcal{M}(v; \beta; J) = \mathcal{M}(v; \beta'; J)$.

Step 7. *A model computation when $g = 1$.* In the special case where $g = 1$, the moduli space $\mathcal{M}(v; \beta; j_\Sigma \times j_{\mathbb{D}})$ is transversally cut out, and consists of a single point. This is analogous to Proposition 5.16.

Step 8. Stabilizing the model computation. Let $\beta' = \{\beta'_1, \dots, \beta'_{g-1}\}$ be a $(g-1)$ -tuple of curves so that $\beta'_i \cap \alpha_i$ is a single point and $\beta'_i \cap \alpha_j = \emptyset$ for $i \neq j$ (and the β'_j are disjoint from α_1^a and α_2^a). Choose a curve γ so that $\Sigma' \setminus \gamma$ has two connected components, one containing β' and the other containing α_1^a . Stretching the neck along γ , $\mathcal{M}(v; \beta'; J)$ breaks up as a fibered product of two moduli spaces over a disk. One is the moduli space from the model computation in Step 7, which has a single, transversally cut-out point. The other maps degree 1 onto the disk, by the same argument used to prove stabilization invariance in the cylindrical setting [Lip06a, Proposition A.3].

Step 9. Conclusion of the proof. Steps 6 and 8 imply that $\mathcal{M}(v; \beta; J)$ is transversally cut out, and consists of an odd number of points, for some relaxed almost-complex structure J , which we can take to be arbitrarily close to some fixed, admissible J_0 (sufficiently pinched). By a standard gluing argument, this moduli space can be glued to u to give an odd number of holomorphic curves contributing to $D_{23}(\mathbf{x}_i)$. Thus, the number of points in the moduli space of J -holomorphic curves $\mathcal{M}(\mathbf{x}_i, \mathbf{x}_{i-1}; (\rho_2, \rho_3); J)$ is odd. Let J_0 be a sufficiently pinched, generic almost complex structure satisfying the more restrictive Condition (J-1). Since the moduli space of J_0 -holomorphic curves $\mathcal{M}(\mathbf{x}_i, \mathbf{x}_{i-1}; (\rho_2, \rho_3); J_0)$ is transversally cut out, it follows that $\#\mathcal{M}(\mathbf{x}_i, \mathbf{x}_{i-1}; (\rho_2, \rho_3); J_0)$ is odd, as well. Thus, $D_{23}(\mathbf{x}_i) = \mathbf{x}_{i-1}$, as desired. \square

PROOF OF THEOREM 11.36. We choose n larger than $4c$, where c is the constant from Lemma 11.45, so that all exterior generators \mathbf{x} have $|S(\mathbf{x})| < \frac{n-2}{4}$, and large enough that Lemmas 11.46 and 11.47 hold.

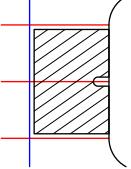
Let $W = W^0 \oplus W^1$ be the type D module for $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$. We see from Lemma 11.43 that $W_s^0 = C(s)$, and its differential D is clearly identified with the differential on knot Floer homology.

Consider next generators \mathbf{x} representing elements of W^1 . In the case where $|S(\mathbf{x})| \geq \frac{n-2}{4}$, according to Lemmas 11.43 and 11.45, all generators with the same S -grading are of the form \mathbf{x}_t for $\mathbf{x} \in \mathfrak{S}_K$, and t uniquely determined by (the Alexander grading of) \mathbf{x} . This gives the identification of W_s^1 with V_s^1 for $|s| > \frac{n}{4}$ as stated in the theorem. For instance, for $s < -\frac{n}{4}$, W_s^1 contains \mathbf{x}_k for $S(\mathbf{x}) = s + \frac{n+1}{2} - k$ and can therefore be identified with $C(\leq s + \frac{n-1}{2})$, as desired. Moreover, Lemma 11.48 then identifies the differential D on W_s^1 with the stated differential D on V_s^1 . We return to studying W^1 with $|S(\mathbf{x})| < \frac{n}{4}$ later in the proof. We say what we can about the other coefficient maps first, though.

Any domain representing D_1 must start at a generator of the form \mathbf{x}_0 . By Lemma 11.46, the map D_1 is represented by the bigon from \mathbf{x}_0 to \mathbf{x}_1 . Similarly, any domain representing D_3 is represented by the bigon from \mathbf{x}_0 to \mathbf{x}_{-1} . We conclude that D_1 and D_3 are the maps from \mathbf{x}_0 to \mathbf{x}_1 and \mathbf{x}_{-1} , respectively, as desired.

Next, any domain representing D_2 must start at a generator of the form \mathbf{x}_s with $s < 0$ and end at a generator of the form \mathbf{y}_0 . By Lemma 11.48, such a domain must be induced by some domain $\phi \in \widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$ with $n_z(\phi) = 0$ and $n_w(\phi) = k \geq 1$.

An analogous argument shows that D_{123} has the stated form, except that we necessarily have $n_w(\phi) = 1$.



There are no domains which could represent a map of the form D_{12} , for such a domain would need to have a cut going out to the intersection point x_0 , which has the wrong idempotents.

The fact that D_{23} has the stated form on W_s^1 for $s < -\frac{n}{4}$ was established in Lemma 11.46. The fact that D_{23} has the given form for $s > \frac{n}{4}$ follows from Lemma 11.49.

We have now shown that all the coefficient maps have the right form for W_s^1 with $|s| > \frac{n}{4}$ and for W_s^0 . We claim next that for all $|s| < \frac{n}{4}$, $H_*(W_s^1, D) \cong \mathbb{F}_2$, and in fact

$$D_{23}: W_s^1 \rightarrow W_{s+1}^1$$

is a chain map (with respect to D) inducing an isomorphism on homology. We investigate the terms in Equation (11.31) and Equation (11.32), bearing in mind Lemma 11.47. Specifically, for \mathbf{x} with $|S(\mathbf{x})| \leq \frac{n}{4}$, $D_3 \circ D_{012}(\mathbf{x}) = 0$, $D_1 \circ D_{230}(\mathbf{x}) = 0$, and $D_{301} \circ D_2(\mathbf{x}) = 0$ for degree reasons (comparing Lemma 11.42 with the observation that $W_s^0 = 0$ for all $|s| \geq \frac{n}{4}$). Similarly, since

$$D_0: W_s^1 \rightarrow W_{s-\frac{n+1}{2}}^0,$$

(by the same considerations as in Lemma 11.42), it follows that $D_{123} \circ D_0(\mathbf{x}) = 0$ for $|S(\mathbf{x})| \leq \frac{n}{4}$. Thus D_{01} and D_{23} are homotopy inverses in this range, and so the map induced on homology by $D_{23}: W_s^1 \rightarrow W_{s+1}^1$ is an isomorphism. Earlier in the proof we identified $W_{-\frac{n+2}{4}}^1$ with $C(\leq n) = (C, \partial_w)$, a chain complex with homology isomorphic to \mathbb{F}_2 . We can conclude that

$$H_*(W_s^1, D) \cong \mathbb{F}_2$$

for all $|s| < \frac{n}{4}$. Let $\phi: W_s^1 \rightarrow \mathbb{F}_2$ denote a (graded) quasi-isomorphism.

It follows that for $|s| < \frac{n}{4}$, we have a chain map $\phi: W_s^1 \rightarrow \mathbb{F}_2$ inducing an isomorphism on homology. The map

$$\Phi: W \rightarrow V$$

given by

$$\Phi(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & \text{if } \mathbf{x} \in W_s^1 \text{ with } |s| < \frac{n}{4} \\ \mathbf{x} & \text{otherwise} \end{cases}$$

induces a homotopy equivalence between the type D module determined by W and the one determined by V .

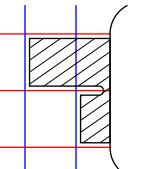
The rest of the statement about gradings is clear from the construction. \square

REMARK 11.53. Most of the grading on V can be read off immediately from the Heegaard diagram for $S^3 \setminus \text{nbd}(K)$: there is a one-to-one correspondence between V_s^0 and W_s^0 and between V_s^1 and W_s^1 as long as $|s| \geq \frac{n}{4}$.

PROPOSITION 11.54. *There is an element s_0 of the grading set with the property that for any generator \mathbf{y} for $\widehat{HFK}(K)$, the corresponding generator \mathbf{y}_0 for $\widehat{CFD}(S^3 \setminus K)$ has grading given by*

$$\text{gr}(\mathbf{y}_0) = \lambda^{M(\mathbf{y}) - 2A(\mathbf{y})} \text{gr}(\rho_{23})^{-A(\mathbf{y})} \cdot s_0.$$

PROOF. Let \mathbf{x} and \mathbf{y} be any generators for $\widehat{HFK}(K)$. After switching the roles of \mathbf{x} and \mathbf{y} , we can assume that $A(\mathbf{x}) \leq A(\mathbf{y})$. We can find some $B \in \pi_2(\mathbf{x}, \mathbf{y})$ (with $n_z(B) = 0$).



By Equation (11.10), $n_w(B) = A(\mathbf{y}) - A(\mathbf{x}) \geq 0$; also, by Equation (11.11), followed by the formula for the Maslov index [Lip06a, Corollary 4.3],

$$\begin{aligned} M(\mathbf{x}) - M(\mathbf{y}) &= \text{ind}(B) - 2n_w(B) \\ &= e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) - 2n_w(B). \end{aligned}$$

The domain B induces a domain $\tilde{B} \in \pi_2(\mathbf{x}_0, \mathbf{y}_0)$ with $n_z(\tilde{B}) = 0$ and

$$\begin{aligned} n_{\mathbf{x}_0}(\tilde{B}) &= n_{\mathbf{x}}(B) & \partial^\partial \tilde{B} &= (0, n_w(B), n_w(B)) \\ n_{\mathbf{y}_0}(\tilde{B}) &= n_{\mathbf{y}}(B) & e(\tilde{B}) &= e(B) - \frac{1}{2}n_w(B); \end{aligned}$$

so

$$\begin{aligned} g(\tilde{B}) &= (-e(\tilde{B}) - n_{\mathbf{x}_0}(\tilde{B}) - n_{\mathbf{y}_0}(\tilde{B}); \partial^\partial \tilde{B}) \\ &= \left(-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) + \frac{1}{2}n_w(B); \partial^\partial \tilde{B} \right) \\ &= \left(M(\mathbf{y}) - M(\mathbf{x}) - \frac{3}{2}n_w(B); \partial^\partial \tilde{B} \right) \\ &= \left(M(\mathbf{y}) - \frac{3}{2}A(\mathbf{y}) - M(\mathbf{x}) + \frac{3}{2}A(\mathbf{x}); 0, n_w(B) \right). \end{aligned}$$

(In the expression for $\partial^\partial \tilde{B}$ we list the multiplicities of $[\rho_1]$, $[\rho_2]$ and $[\rho_3]$. In the final equation, we are writing the group element in coordinates for G as in Equation (11.1).) So,

$$\begin{aligned} \text{gr}(\mathbf{y}) &= \text{gr}(R(\tilde{B})) \cdot \text{gr}(\mathbf{x}) \\ (11.55) \quad &= \left(M(\mathbf{y}) - \frac{3}{2}A(\mathbf{y}) - M(\mathbf{x}) + \frac{3}{2}A(\mathbf{x}); 0, -n_w(B) \right) \cdot \text{gr}(\mathbf{x}) \\ &= \lambda^{M(\mathbf{y})-2A(\mathbf{y})-M(\mathbf{x})+2A(\mathbf{x})} \text{gr}(\rho_{23})^{A(\mathbf{x})-A(\mathbf{y})} \cdot \text{gr}(\mathbf{x}). \end{aligned}$$

Thus,

$$\lambda^{-M(\mathbf{y})+2A(\mathbf{y})} \text{gr}(\rho_{23})^{A(\mathbf{y})} \cdot \text{gr}(\mathbf{y}) = \lambda^{-M(\mathbf{x})+2A(\mathbf{x})} \text{gr}(\rho_{23})^{A(\mathbf{x})} \cdot \text{gr}(\mathbf{x}).$$

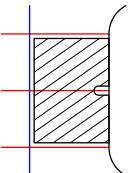
Since \mathbf{x} and \mathbf{y} were arbitrary, the result follows. \square

11.8. Proof of Theorem 11.26

We next deduce Theorem 11.26 from Theorem 11.36. But first, we show that $CFK^-(K)$ can be horizontally or vertically reduced, as in Definition 11.23 (and as needed for the statement of Theorem 11.26). To this end, we have the following preliminary lemma:

LEMMA 11.56. *Let V and W be finite-dimensional filtered vector spaces and $f: V \rightarrow W$ be a (possibly un-filtered) isomorphism between the underlying vector spaces. Then there is a filtered basis for V that maps under f to a filtered basis for W .*

PROOF. The filtration $W_j \subset W$ of W induces a second filtration on V , by the subsets $f^{-1}W_j$, which, together with the original filtration on V gives a $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration on V by subcomplexes $V_{s,t} = V_s \cap f^{-1}W_t$. Correspondingly, the *filtration level* of an element $v \in V$ is defined to be the pair of integers (s, t) where $s = \inf\{i \in \mathbb{Z} \mid v \in V_i\}$, $t = \inf\{j \in \mathbb{Z} \mid f(v) \in W_j\}$. Let $\llbracket v \rrbracket$ denote the projection of v to $V_{s,t}/(V_{s-1,t} + V_{s,t-1})$. We can form the associated



$\mathbb{Z} \oplus \mathbb{Z}$ -graded vector space $\text{gr}_{\mathbb{Z} \oplus \mathbb{Z}}(V) = \bigoplus_{s,t} V_{s,t}/(V_{s-1,t} + V_{s,t-1})$. A $(\mathbb{Z} \oplus \mathbb{Z})$ -filtered basis is a basis for V whose projection under $\llbracket \cdot \rrbracket$ is a basis for $\text{gr}_{\mathbb{Z} \oplus \mathbb{Z}}(V)$.

Now, pick a $(\mathbb{Z} \oplus \mathbb{Z})$ -filtered basis for V (which can be obtained by taking a basis for $\text{gr}_{\mathbb{Z} \oplus \mathbb{Z}} V$ and lifting it to a basis for V). Such a basis is *a fortiori* a filtered basis for either the filtration by V or the filtration induced by W ; i.e., it can be thought of as a (\mathbb{Z}) -filtered basis for V whose image under f is a filtered basis of W , as needed. \square

PROPOSITION 11.57. *Let C be a \mathbb{Z} -filtered, \mathbb{Z} -graded, finitely-generated chain complex over $\mathbb{F}_2[U]$ which is free as an $\mathbb{F}_2[U]$ -module. Then C is \mathbb{Z} -filtered, \mathbb{Z} -graded homotopy equivalent to a chain complex C' which is reduced. Further, one can choose a basis for C' over $\mathbb{F}_2[U]$ which is vertically simplified or, if one prefers, a basis which is horizontally simplified instead.*

PROOF. We show that C can be reduced by induction on its rank. If C is not reduced, by definition it admits a non-trivial differential which does not change the Alexander filtration, so there is some $\xi \in C$ with $\partial\xi \neq 0$ and $A(\xi) = A(\partial\xi)$. Then ξ and $\partial\xi$ generate a subcomplex of C whose quotient complex Q is filtered homotopy equivalent to C with rank two less than the rank of C .

Next, we argue that C can be vertically simplified. Consider $C^\vee = C/(U \cdot C)$ with its differential ∂^\vee . By Lemma 11.56 applied to the restriction of ∂^\vee to a map $C^\vee/\text{Ker}(\partial^\vee) \rightarrow \text{Im}(\partial^\vee)$, we can find a filtered basis $\{\xi_2, \xi_4, \dots, \xi_{2m}\}$ for $\text{Im}(\partial^\vee)$ and vectors $\{\xi_1, \xi_3, \dots, \xi_{2m-1}\}$ in C^\vee so that $\partial^\vee(\xi_{2i-1}) = \xi_{2i}$ and $\{\xi_1, \xi_3, \dots, \xi_{2m-1}\}$ descends to a filtered basis of $C^\vee/\text{Ker}(\partial^\vee)$. The set $\{\xi_1, \xi_2, \dots, \xi_{2m}\}$ is linearly independent, since ∂^\vee is a differential. Extend $\{\xi_2, \xi_4, \dots, \xi_{2m}\}$ to a filtered basis $\{\xi_2, \xi_4, \dots, \xi_{2m}, \xi_{2m+1}, \dots, \xi_n\}$ for $\text{Ker}(\partial^\vee)$. Then $\{\xi_1, \dots, \xi_n\}$ is a filtered basis for C^\vee , and can be thought of as a basis for C over $\mathbb{F}_2[U]$, as well. As a basis for C , $\{\xi_1, \dots, \xi_n\}$ is vertically simplified.

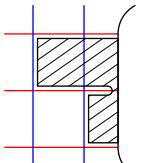
A similar argument applies for $C^h = \text{gr}_0(C \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, U^{-1}])$ (where we are now taking the associated graded object with respect to the Alexander grading) to give a filtered basis $\{y_1, \dots, y_n\}$ for C^h so that $\partial^h(y_{2i-1}) = y_{2i}$ for $1 \leq i \leq m$ and $\partial^h(y_i) = 0$ for $i > 2m$. Pick (arbitrary) lifts \hat{y}_i of y_i to $C^\infty = C \otimes_{\mathbb{F}_2[U]} \mathbb{F}_2[U, U^{-1}]$; by assumption $A(\hat{y}_i) = 0$, and each \hat{y}_i can be written uniquely as $\hat{y}_i = U^{k_i} \eta_i$ for some $\eta_i \in C$ with non-zero projection to C/UC . Then the η_i are the desired horizontally simplified basis. \square

PROOF OF THEOREM 11.26. Let C be a reduced complex homotopy equivalent to $CFK^-(K)$, and let $\{\xi_i\}$ and $\{\eta_i\}$ be bases of C which are vertically and horizontally simplified, respectively. These bases are related by

$$(11.58) \quad \xi_i = \sum_j \Lambda_i^j \eta_j$$

for a suitable change of basis matrix Λ . Let \widehat{CFD} be $\widehat{CFD}(S^3 \setminus \text{nbd}(K))$ with a suitably large negative framing $-n$, as computed in Theorem 11.36. In fact, we will think of \widehat{CFD} as associated to C (rather than $CFK^-(K)$), in view of Proposition 11.38. As in Theorem 11.36, $\{\xi_i\}$ and $\{\eta_i\}$ can also be thought of as two different bases for V_s^0 over \mathbb{F}_2 or, alternatively, two bases for $\iota_0 \widehat{CFD}$ over $\mathcal{A}(\mathbb{T})$. We will modify and extend these to two bases for all of \widehat{CFD} which

- on $\iota_0 \widehat{CFD}$ are still related by Formula (11.58),
- agree on $\iota_1 \widehat{CFD}$, and



- split as a direct sum of a contractible complex and a complex with the coefficient maps from Theorem 11.26.

Here, to define the coefficient maps, we view \widehat{CFD} as a type D structure as in Remark 6.4.

To give an initial basis for $\iota_1 \widehat{CFD}$, recall that all ξ_i except for ξ_0 come in pairs ξ_j and ξ_{j+1} , where $\partial_z \xi_j = \xi_{j+1}$. Fix i so that there is a height ℓ vertical arrow from ξ_i to ξ_{i+1} . Then the generators for V_s^1 for $s > 0$ coming from this pair have coefficient maps like

That is, the only time we get a contribution from this pair to the D_\emptyset -homology of V_s^1 for $s > 0$ is for $A(\xi_{i+1}) + \left(\frac{n-1}{2}\right) < s \leq A(\xi_i) + \left(\frac{n-1}{2}\right)$. We call these generators κ_k^i as in the statement of the theorem, where $k = A(\xi_i) - s + \left(\frac{n-1}{2}\right) + 1$. For $s = A(\xi_{i+1}) + \left(\frac{n-1}{2}\right)$, we have a canceling pair of generators α^i (from ξ_i) and β^i (from ξ_{i+1}). We also have canceling pairs for smaller s .

Similarly, for a horizontal arrow from η_i to η_{i+1} of width ℓ , we get generators and coefficient maps like

$$(11.60) \quad \eta_i \xrightarrow{D_3} \lambda_1^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_k^i \xrightarrow{D_{23}} \lambda_{k+1}^i \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \lambda_\ell^i \xrightarrow{D_{23}} \gamma^i \xrightarrow{D_{23}} \dots$$

$\downarrow D_2$ $\downarrow D_\emptyset$ $\downarrow D_\emptyset$
 $\eta_{i+1} \xrightarrow{D_3} \delta^i \xrightarrow{D_{23}} \dots$

(The λ_k^i are elements of V_s^1 with $s < 0$.) In particular, there is an undesirable term in $D_3(\eta_{i+1})$. In addition, we usually have $D_{123}(\eta_i) = 0$; the exception is when the arrow has width equal to one, when

$$(11.61) \quad D_{123}(\eta_i) = D_1(\eta_{i+1}).$$

Consider next ξ_0 , which has no vertical differential either into or out of it. We see that its corresponding element comes from a chain

$$\dots \xrightarrow{D_{23}} \varphi_3 \xrightarrow{D_{23}} \varphi_2 \xrightarrow{D_{23}} \varphi_1 \xleftarrow{D_1} \xi_0,$$

where the chain of elements φ_i extends back to V_s^1 where s is small in magnitude (and hence V_s^1 is one-dimensional). Similarly, consider the element η_0 which has no horizontal differential either into or out of it. (Remember that ξ_0 and η_0 can coincide.) Associated to η_0 is a similar string

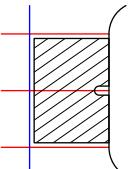
$$\eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \mu_3 \xrightarrow{D_{23}} \dots$$

where the chain of elements μ_i extends forward to V_s^1 where s is sufficiently small in magnitude that V_s^1 is one-dimensional). Thus, for some sufficiently large a and b , we must have that $\varphi_a = \mu_b$; i.e., we can connect the two chains to a single one of the form

$$\eta_0 \xrightarrow{D_3} \mu_1 \xrightarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \mu_3 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} \mu_m \xleftarrow{D_1} \xi_0$$

We have now constructed bases for all the V_s^1 :

- λ_k^i and μ_k for $s \leq -n/4$,
 - $\phi_k = \mu_k$ for $-n/4 < s < n/4$, and



- κ_k^i and ϕ_k for $s \geq n/4$.

However, we have extra generators for the modules and extra terms in the coefficient maps. To bring the module to the desired form, we must change basis so as to

- split off the chain of canceling generators in (11.60) by arranging for $D_3(\eta_{i+1})$ and $D_{23}(\lambda_\ell^i)$ to vanish;
- eliminate the generators with canceling differentials in (11.59), in particular α^j and β^j ; and
- bring D_{123} to the desired form, by eliminating the non-trivial terms belonging to width one arrows (Equation (11.61)).

We do this by successively changing basis (and, at some point, dropping out β^j).

We start by modifying the base to eliminate the unwanted terms in D_3 (from η_{i+1} when there is a horizontal arrow from η_i). To this end, we change basis by

$$\eta'_{i+1} := \eta_{i+1} + \rho_3 \cdot \gamma^i.$$

With this substitution, we have that $D_3(\eta'_{i+1}) = 0$, $D_2(\lambda_\ell^i) = \eta'_{i+1}$, and $D_{23}(\lambda_\ell^i) = 0$. Let $\{\xi'_j\}$ denote the corresponding change of basis applied to the $\{\xi_j\}$ so that Formula (11.58) remains true. Note that the other coefficient maps are unchanged; in particular $D_1(\xi'_j) = D_1(\xi_j)$, and there are no coefficient maps other than D_2 which enter η'_{i+1} . Also, the chain of canceling generators from this horizontal arrow now splits off as a direct sum, so we can drop them.

We pass now to a submodule to eliminate α^i and β^i . To do this, for each vertical arrow from ξ_i to ξ_{i+1} of height ℓ (and corresponding new elements ξ'_i and ξ'_{i+1}), we replace ξ'_{i+1} by

$$\xi''_{i+1} := \xi'_{i+1} + \rho_1 \cdot \alpha^i.$$

Now we have $D_{123}(\xi''_{i+1}) = \kappa_\ell^i + D_{123}(\xi_{i+1})$. Again, there is a corresponding modification to the other generators, replacing η'_i by η''_i . Now the submodule without the chain of canceling generators from (11.59) is easily seen to be homotopy equivalent to the original module.

Finally, we wish to bring D_{123} to the desired form, eliminating the terms from horizontal arrows of length one. This is done as follows. Suppose that ξ_i is the initial point of some vertical arrow. Replace κ_1^i by

$$\tilde{\kappa}_1^i := \kappa_1^i + \rho_{23} \cdot D_{123}(\xi_i).$$

With this new basis, then, $D_{123}(\xi''_i) = 0$ (as its contribution is absorbed into D_1). Moreover, this change does not affect any of the other coefficient maps.

It remains to make one last change of basis, corresponding to terminal points ξ_{i+1} of vertical arrows, so that $D_{123}(\xi_{i+1})$ is the element taking the place of κ_ℓ^i . *A priori* we have $D_{123}(\xi''_{i+1}) = \kappa_\ell^i + D_{123}(\xi_{i+1})$. In the case where the length of the arrow from ξ_i to ξ_{i+1} is greater than one, we make the change by letting

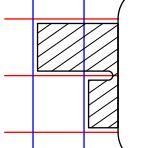
$$\tilde{\kappa}_\ell^i := \kappa_\ell^i + D_{123}(\xi_{i+1}).$$

Since $D_{123}(\xi_{i+1})$ is in the kernel of D_{23} (from the description in Theorem 11.36), all other coefficient maps remain unchanged.

In the case where the length of the arrow from ξ_i to ξ_{i+1} is one, proceed as follows. First, we find an element ζ_i so that

$$\partial \zeta_i = \rho_{23} \cdot D_{123}(\xi_{i+1}).$$

In particular, $D_\emptyset(\zeta_i) = 0$ and $D_{23}(\zeta_i) = D_{123}(\xi_{i+1})$. This element is found as follows. Recall that ξ_i and ξ_{i+1} correspond to elements \mathbf{x}^i and \mathbf{x}^{i+1} in $C(s)$ and $C(s-1)$, respectively. Let $\mathbf{y} := \partial^1(\mathbf{x}^i) \in C(s+1)$, i.e., the image of \mathbf{x}^i under the map which counts holomorphic



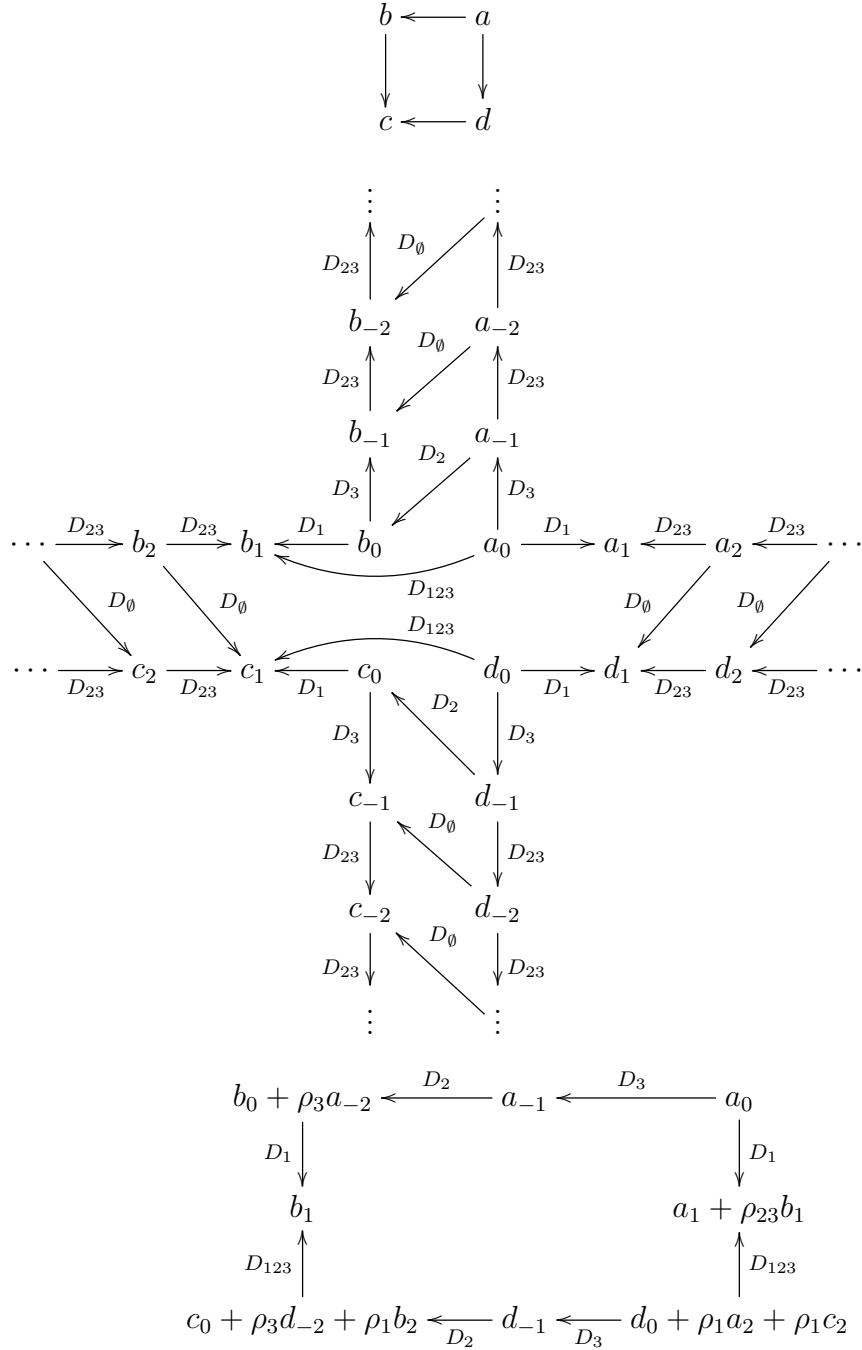
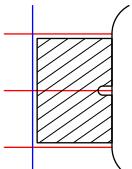


FIGURE 11.13. **An illustration of the proof of Theorem 11.26.** The top illustrates a summand in the knot complex; the middle illustrates a corresponding piece coming from Theorem 11.36; the bottom is obtained by making the substitutions coming from the proof of Theorem 11.26 and dropping acyclic summands.



disks which cross w exactly once. (Recall from Theorem 11.36 that $D_{123}(\xi_{i+1})$ is $\partial^1(\mathbf{x}^{i+1})$ considered as an element of $C(\geq s)$.) We then let $\zeta_i = \partial_z(\mathbf{y})$, where we consider \mathbf{y} as a chain in $C(\geq s - 1)$. Then ζ_i , thought of as a generator of the type D module, has the desired properties. Making the substitution

$$\xi''_{i+1} := \xi''_{i+1} + \rho_1 \cdot \zeta_i,$$

we see that $D_1(\xi'''_{i+1}) = D_1(\xi''_{i+1})$ and $D_{123}(\xi'''_{i+1})$ has the desired form.

This module now has the form promised in Theorem 11.26; except that we must relate the length m of the chain of μ_i to the framing parameter n and the knot invariant $\tau(K)$, and we must compute the gradings. We do both of these, as follows. On the one hand, the stable chain of length m ensures that

$$\lambda^{-m+1} \text{gr}(\eta_0) = \text{gr}(\rho_3) \cdot \text{gr}(\rho_{23})^{m-1} \cdot \text{gr}(\rho_1)^{-1} \cdot \text{gr}(\xi_0).$$

On the other hand, by Equation (11.25), Proposition 11.54 guarantees that there is some element of the grading set \mathbf{s} so that

$$\begin{aligned} \text{gr}(\xi_0) &= \lambda^{-2\tau(K)} \text{gr}(\rho_{23})^{-\tau(K)} \mathbf{s} \\ \text{gr}(\eta_0) &= \text{gr}(\rho_{23})^{\tau(K)} \mathbf{s}. \end{aligned}$$

Substituting these into the relation from the stable chain (and making a straightforward computation in G), we see that

$$\begin{aligned} \mathbf{s} &= \lambda^{m-1-2\tau(K)} \text{gr}(\rho_{23})^{-\tau(K)} \text{gr}(\rho_3) \cdot \text{gr}(\rho_{23})^{m-1} \cdot \text{gr}(\rho_1)^{-1} \text{gr}(\rho_{23})^{-\tau(K)} \cdot \mathbf{s} \\ &= \lambda^{-1} \cdot \text{gr}(\rho_{23})^{m-2\tau(K)} \text{gr}(\rho_{12})^{-1} \cdot \mathbf{s}. \end{aligned}$$

This gives a periodic domain with local multiplicities $(0, -1, m - 2\tau - 1, m - 2\tau)$ at the four regions $(0, 1, 2, 3)$ at the boundary. We conclude that $n = m - 2\tau$ (see the discussion leading to Formula (11.40)), and

$$G \cdot \mathbf{s} \cong G/\lambda^{-1} \text{gr}(\rho_{23})^n \text{gr}(\rho_{12})^{-1}.$$

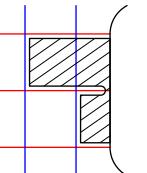
The rest of the grading statement follows from Proposition 11.54. \square

In Figure 11.13 we have illustrated some of the changes of basis from Theorem 11.26 in the case where the knot complex has a summand which is a square, as illustrated on the top of the figure. (Such a summand appears, e.g., in the knot Floer complex for the figure eight knot, on the bottom of Figure 11.5.) Initially, Theorem 11.36 gives a corresponding summand as illustrated in the middle of Figure 11.13. Going through the cancellations prescribed in the proof of Theorem 11.26, we end up with the complex on the bottom of Figure 11.13.

11.9. Satellites revisited

In this section, we illustrate Theorems 11.19 and 11.26 by showing how to compute HFK^- of a knot K obtained as a cable of the left-handed trefoil T in S^3 . (Specifically, we compute HFK^- of the $(2, -3)$ -cable of T .) Although the knot Floer homology groups of this particular knot were already known [Hed05a], the techniques here extend to give a formula for HFK^- of any particular satellite in terms of CFK^- of the pattern.

Figure 11.14 is a Heegaard diagram for the $(2, 1)$ cabling operation. Since this diagram has genus equal to one, the holomorphic curves are straightforward to count. The type A



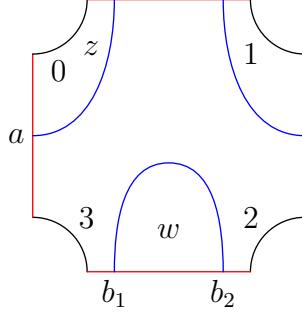


FIGURE 11.14. **Heegaard diagram for computing the type A module for the $(2, 1)$ cable.** This is a doubly-pointed Heegaard diagram for the $(2, 1)$ cable (of the unknot), thought of as a knot in the solid torus. The basepoint z lies in the region marked with a 0.

module $CFA^-(C)$ associated to this diagram is given as follows. It has three generators a , b_1 and b_2 , and the following relations:

$$(11.62) \quad \begin{aligned} m_1(b_1) &= U \cdot b_2 \\ m_2(a, \rho_1) &= b_2 \\ m_{3+i}(a, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2, \rho_1) &= U^{2i+2} \cdot a, \quad i \geq 0 \\ m_{4+i}(a, \rho_3, \overbrace{\rho_{23}, \dots, \rho_{23}}^i, \rho_2, \rho_1) &= U^{2i+1} \cdot b_1, \quad i \geq 0. \end{aligned}$$

Letting $u = gr(U) = (0; 0, 0; -1)$ in $\tilde{G} = G(\mathcal{Z}) \times \mathbb{Z}$, we find that the grading set is isomorphic to

$$gr(a) \cdot \tilde{G} \cong u^{-2} \lambda gr(\rho_3) gr(\rho_2) \setminus \tilde{G} = u^{-2} gr(\rho_{23}) \setminus \tilde{G};$$

and

$$gr(a) = e, \quad gr(b_1) = \lambda u gr(\rho_1) \quad gr(b_2) = gr(\rho_1);$$

or, in coordinates, the grading set is $(\frac{1}{2}; 0, 1; -2) \setminus \tilde{G}$ and

$$gr(a) = (0; 0, 0; 0), \quad gr(b_1) = (\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; -1) \quad gr(b_2) = (-\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; 0).$$

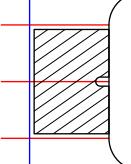
Here, the first three components parameterize the usual grading group (G) and the last component represents the additional \mathbb{Z} enhancement.

Consider next the type D module for the left-handed trefoil knot T with framing -2 , represented graphically in Figure 11.15. This module has generators x_1, x_2, x_3 in the ι_0 idempotent and y_1, y_2 in the ι_1 idempotent, and coefficient maps:

$$\begin{aligned} D_3(x_1) &= y_1 & D_{12}(x_1) &= x_3 & D_2(y_1) &= x_2 \\ D_{123}(x_2) &= y_2 & D_1(x_3) &= y_2. \end{aligned}$$

(Although this framing is not large enough for Theorem 11.26 to apply, $\widehat{CFD}(T)$ may still be computed; see Theorem A.11, whose proof uses results from [LOT10a].)

Gradings of the elements x_i are computed in Equation (11.27), with the exception of y_1 and y_2 . But the gradings of these are related to the gradings of the x_i according to the coefficient maps. Explicitly, the generators x_1, x_2 , and x_3 correspond to knot Floer generators



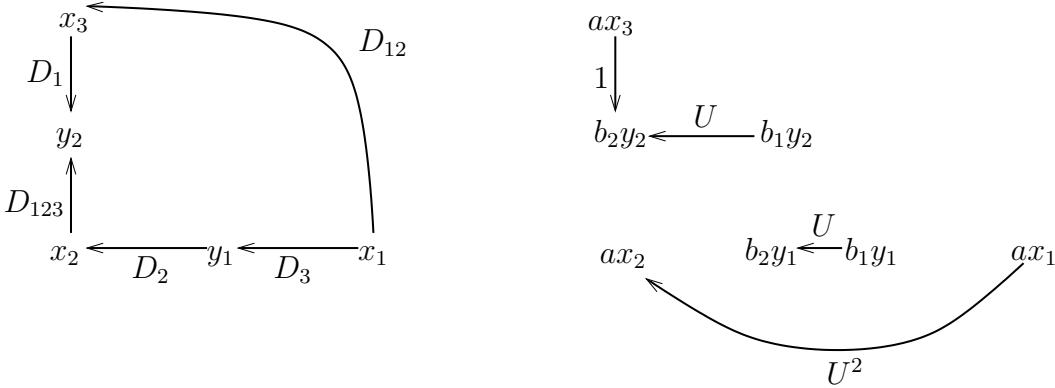


FIGURE 11.15. **Cable of the trefoil.** On the left, the coefficient maps for the left-handed trefoil. On the right, the result of cabling. Arrows here represent differentials; they are labeled by their corresponding coefficients (powers of U).

with (M, A) -bigradings given by $(0, -1)$, $(1, 0)$, and $(2, 1)$ respectively. Since $\text{gr}(x_2) = \lambda$ and $\partial y_1 = \rho_2 \cdot x_2$, it follows that

$$\text{gr}(y_1) = \lambda^2 \text{gr}(\rho_2).$$

Indeed, in coordinates:

$$\begin{aligned} \text{gr}(x_1) &= (\frac{3}{2}; 0, 1) & \text{gr}(x_2) &= (1; 0, 0) & \text{gr}(x_3) &= (\frac{1}{2}; 0, -1) \\ \text{gr}(y_1) &= (\frac{3}{2}; \frac{1}{2}, \frac{1}{2}) & \text{gr}(y_2) &= (\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}). \end{aligned}$$

Let K denote the $(2, -3)$ -cable of the $(0$ -framed) left-handed trefoil (which is also the $(2, 1)$ -cable of the -2 -framed left-handed trefoil). By Theorem 11.19, $gCFK^-(K)$ is computed by the tensor product $CFA^-(C) \boxtimes \widehat{CFD}(T)$, which has generators $a \otimes x_i$ and $b_k \otimes y_j$, which we abbreviate by dropping the tensor product sign. The grading set is identified with the double-coset space

$$u^{-2} \text{gr}(\rho_{23}) \setminus \widetilde{G} / \lambda^{-1} \text{gr}(\rho_{23})^2 \text{gr}(\rho_{12})^{-1}.$$

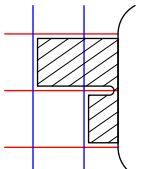
Every element of the double-coset space is equivalent to a uniquely specified element of the form $\lambda^a u^b$ for $a, b \in \mathbb{Z}$: use right translation to eliminate $\text{gr}(\rho_{12})$ from the spin c component and then left translation to eliminate $\text{gr}(\rho_{23})$. For example,

$$\begin{aligned} \text{gr}(b_1 y_1) &= \text{gr}(b_1) \text{gr}(y_1) = \lambda^3 u \text{gr}(\rho_{12}) \\ &\sim (u^4 \text{gr}(\rho_{23})^{-2})(\lambda^3 u \text{gr}(\rho_{12}))(\lambda^{-1} \text{gr}(\rho_{23})^2 \text{gr}(\rho_{12})^{-1}) = \lambda^6 u^{-5}. \end{aligned}$$

Proceeding in this manner, we find:

$$\begin{aligned} \text{gr}(ax_1) &= (2; 0, 0; -2) & \text{gr}(ax_2) &= (1; 0, 0; 0) & \text{gr}(ax_3) &= (0; 0, 0; 2) \\ (11.63) \quad \text{gr}(b_1 y_1) &= (6; 0, 0; -5) & \text{gr}(b_2 y_1) &= (5; 0, 0; -4) & \text{gr}(b_1 y_2) &= (0; 0, 0; 1) \\ \text{gr}(b_2 y_2) &= (-1; 0, 0; 2) \end{aligned}$$

The first component here is the z -normalized Maslov grading N of Equation (11.13), *a priori* up to an overall translation. Similarly, the last component is the Alexander grading, again up to an overall translation. In fact, we graded both the knot complement and the cabling piece so that the z -normalized Maslov grading N is computed absolutely; see Remark 11.28.



The differentials are readily computed; they are:

$$\begin{aligned}\partial(a \otimes x_1) &= U^2 \cdot a \otimes x_2 & \partial(a \otimes x_2) &= 0 & \partial(a \otimes x_3) &= b_2 \otimes y_2 \\ \partial(b_1 \otimes y_1) &= U \cdot b_2 \otimes y_1 & \partial(b_2 \otimes y_1) &= 0 & \partial(b_1 \otimes y_2) &= U \cdot b_2 \otimes y_2 \\ \partial(b_2 \otimes y_2) &= 0.\end{aligned}$$

To pin down the indeterminacy in the Alexander grading, we proceed as follows. Observe that the Poincaré polynomial for the tensor product, after setting $U = 0$, is given by

$$\sum_{d,r} q_N^d \cdot t^r \cdot \text{rank } H_{(A,N)=(r,d)}(CFA^-(C) \boxtimes \widehat{CFD}(T)) = t^1 q_N^0 + t^0 q_N + t^{-2} q_N^2 + t^{-4} q_N^5 + t^{-5} q_N^6.$$

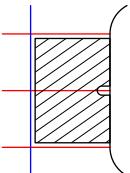
Here, the formal variable t records the Alexander factor of the grading, while the formal variable q_N records the z -normalized grading N of Equation (11.13), which is the first coordinate of Equation (11.63), up to an additive constant. Further specializing to $q_N = (-1)$ gives a polynomial in t which is not symmetric under exchanging t and t^{-1} , but t^2 times it is. So, by Equation (11.14), the absolute Alexander gradings are 2 less than the last components in Equation (11.63). With respect to this translation, the Poincaré polynomial is

$$\sum_{d,r} q_N^d \cdot t^r \cdot \text{rank } H_{(A,N)=(r,d)}(CFA^-(C) \boxtimes \widehat{CFD}(T)) = t^{-3} q_N^6 + t^{-2} q_N^5 + q_N^2 + t^2 q_N + t^3.$$

Since $N = M - 2A$, we can recover the Poincaré polynomial of K in terms of the usual Alexander and Maslov gradings:

$$\sum_{d,r} q^d \cdot t^r \cdot \text{rank } H_d(CFA^-(C) \boxtimes \widehat{CFD}(T), r) = t^{-3} + t^{-2} q + q^2 + t^2 q^5 + t^3 q^6.$$

Here, the formal variable t records Alexander gradings as before, while q records the usual Maslov gradings. This answer is in agreement with [Hed05a, Theorem 1.0.6].



APPENDIX A

Bimodules and change of framing

The modules $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ depend (up to homotopy equivalence) not only on the 3-manifold Y but also on the parametrization of ∂Y by a reference surface F ; this dependence can already be seen for the modules associated to solid tori computed in Section 11.2. The result of reparametrization (e.g., change of framing on a knot complement) is captured by tensoring with certain bimodules. Furthermore, the modules $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ are related to each other by certain dualizing bimodules. These topics are explored in detail in [LOT10a]. In this chapter we present highlights from that paper, and then exhibit the bimodules relevant to the case of torus boundary. When combined with Theorem 11.26, this gives a complete computation of the bordered Floer invariants of knot complements in S^3 , with any framing.

We start by briefly stating the main results on bimodules in Section A.1. We then sketch the definitions of the bimodules in Section A.2. We give explicitly the bimodules for a set of generators of the mapping class group of the torus in Section A.3 and use these bimodules in Section A.4 to finish the computation of \widehat{CFD} of knot complements in terms of the knot Floer homology.

A.1. Statement of results

Fix a pointed matched circle \mathcal{Z} . Recall from Definition 4.2 that a bordered 3-manifold with boundary $F(\mathcal{Z})$ is a triple (Y, \mathcal{Z}, ϕ) where Y is a compact, oriented 3-manifold with boundary, \mathcal{Z} is a pointed matched circle and $\phi: F(\mathcal{Z}) \rightarrow \partial Y$ is an orientation-preserving diffeomorphism. We have associated to (Y, ϕ) homotopy equivalence classes of modules $\widehat{CFD}(Y, -\phi)$ (Chapter 6) and $\widehat{CFA}(Y, \phi)$ (Chapter 7) over $\mathcal{A}(-\mathcal{Z})$ and $\mathcal{A}(\mathcal{Z})$, respectively. Here, $-\phi$ is the map ϕ but viewed as a map $-F(\mathcal{Z}) \rightarrow -\partial Y$; compare Theorem 1.1.

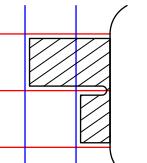
THEOREM A.1. *Given a diffeomorphism $\psi: F(\mathcal{Z}) \rightarrow F(\mathcal{Z})$, there is an \mathcal{A}_∞ bimodule ${}_{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\psi)_{\mathcal{A}(\mathcal{Z})}$, well-defined up to (\mathcal{A}_∞) homotopy equivalence, such that for any bordered three-manifold $(Y, \phi: F(\mathcal{Z}) \rightarrow \partial Y)$,*

$$\begin{aligned}\widehat{CFA}(Y, \phi \circ \psi^{-1}) &\simeq \widehat{CFA}(Y, \phi) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\psi) \\ \widehat{CFD}(-Y, \phi \circ \psi) &\simeq \widehat{CFDA}(\psi) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(-Y, \phi).\end{aligned}$$

Moreover, given another map $\xi: F(\mathcal{Z}) \rightarrow F(\mathcal{Z})$,

$$\widehat{CFDA}(\psi \circ \xi) \simeq \widehat{CFDA}(\xi) \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\psi).$$

As we will discuss in Section A.2, $\widehat{CFDA}(\psi)$ is defined in terms of a Heegaard diagram with two boundary components, one of which is treated in type D fashion and the other of which is treated in type A fashion.



Next we turn to the duality between \widehat{CFD} and \widehat{CFA} . By $M_{R,S}$ we mean a bimodule M with commuting right actions by rings R and S ; similarly, by ${}_{R,S}M$ we mean a bimodule M with commuting left actions by R and S .

THEOREM A.2. *For any pointed matched circle \mathcal{Z} there are differential graded bimodules $\widehat{CFAA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})}$ and ${}_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I})$ such that for any bordered (Y, ϕ) ,*

$$\begin{aligned}\widehat{CFD}(Y, -\phi) &\simeq \widehat{CFA}(Y, \phi) \tilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFDD}(\mathbb{I}) \\ \widehat{CFA}(Y, \phi) &\simeq \widehat{CFAA}(\mathbb{I}) \tilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \widehat{CFD}(Y, -\phi).\end{aligned}$$

As the notation suggests, these dualizing bimodules correspond to the identity map $\mathbb{I}: F \rightarrow F$. In fact, for any mapping class $\phi: F \rightarrow F$ there are associated bimodules $\widehat{CFDD}(\phi)$ and $\widehat{CFAA}(\phi)$; see [LOT10a].

A.2. Sketch of the construction

In this section we outline the construction of the bimodules discussed above. Details can be found in [LOT10a], although the constructions are close enough to the construction of \widehat{CFD} and \widehat{CFA} that the reader might find it amusing to fill them in on his or her own. We start by discussing bordered Heegaard diagrams representing (mapping cylinders of) diffeomorphisms (Section A.2.1). We then give the definitions of the bimodules $\widehat{CFDA}(\phi)$ (Section A.2.2) and $\widehat{CFDD}(\mathbb{I})$ and $\widehat{CFAA}(\mathbb{I})$ (Section A.2.3).

A.2.1. Bordered Heegaard diagrams for diffeomorphisms. The bimodule for a diffeomorphism ϕ , $\widehat{CFDA}(\phi)$, is associated to a bordered Heegaard diagram for ϕ . Such a diagram is a slight generalization of the notion from Chapter 4 to allow more than one boundary component. (Such diagrams already made a brief appearance in Section 11.4.)

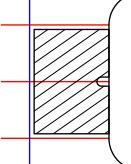
Specifically, an *arced bordered Heegaard diagram with two boundary components* is a quadruple $\mathcal{H} = (\bar{\Sigma}, \bar{\alpha}, \beta, z)$ where

- $\bar{\Sigma}$ is an oriented surface of genus g , with two boundary components $\partial_L \bar{\Sigma}$ and $\partial_R \bar{\Sigma}$;
- $\bar{\alpha} = \{\bar{\alpha}_1^{a,L}, \dots, \bar{\alpha}_{2k}^{a,L}, \bar{\alpha}_1^{a,R}, \dots, \bar{\alpha}_{2k}^{a,R}, \alpha_1^c, \dots, \alpha_{g-2k}^c\}$ where the $\bar{\alpha}_i^{a,L}$ are embedded arcs with endpoints on $\partial_L \bar{\Sigma}$, the $\bar{\alpha}_i^{a,R}$ are embedded arcs with endpoints on $\partial_R \bar{\Sigma}$, and the α_i^c are embedded circles in Σ ; and all of the $\bar{\alpha}_i^{a,L}, \bar{\alpha}_i^{a,R}$ and α_i^c are pairwise disjoint, and $\Sigma \setminus \bar{\alpha}$ is connected;
- $\beta = \{\beta_1, \dots, \beta_g\}$ are circles in Σ such that $\Sigma \setminus \beta$ is connected; and
- z is an arc in $\bar{\Sigma} \setminus (\bar{\alpha} \cup \beta)$ connecting $\partial_L \bar{\Sigma}$ and $\partial_R \bar{\Sigma}$.

As for ordinary bordered Heegaard diagrams, we let $\Sigma = \bar{\Sigma} \setminus \partial \bar{\Sigma}$; we will be sloppy about the distinction between Σ and $\bar{\Sigma}$.

The data \mathcal{H} specifies a 3-manifold Y with two boundary components in a manner exactly analogous to Section 4.1. Moreover, setting \mathcal{Z}_i to be the pointed matched circle specified by $\partial_i \Sigma$ ($i \in \{L, R\}$), the diagram \mathcal{H} specifies a homeomorphism ϕ_i from $F_i = F(\mathcal{Z}_i)$ to $\partial_i Y$, as well as a framed arc connecting $\partial_L Y$ and $\partial_R Y$. We call \mathcal{H} a bordered Heegaard diagram for (Y, ϕ_L, ϕ_R) .

Assume that Y is homeomorphic to $[0, 1] \times F_R$. There is a homeomorphism $\Phi: [0, 1] \times F_R \xrightarrow{\cong} Y$ extending $\phi_R: F_R \rightarrow Y$. Moreover, this homeomorphism is unique up to homeomorphisms $\Xi: [0, 1] \times F_R \rightarrow [0, 1] \times F_R$ such that $\Xi|_{\{1\} \times F_R}$ is the identity. It follows that



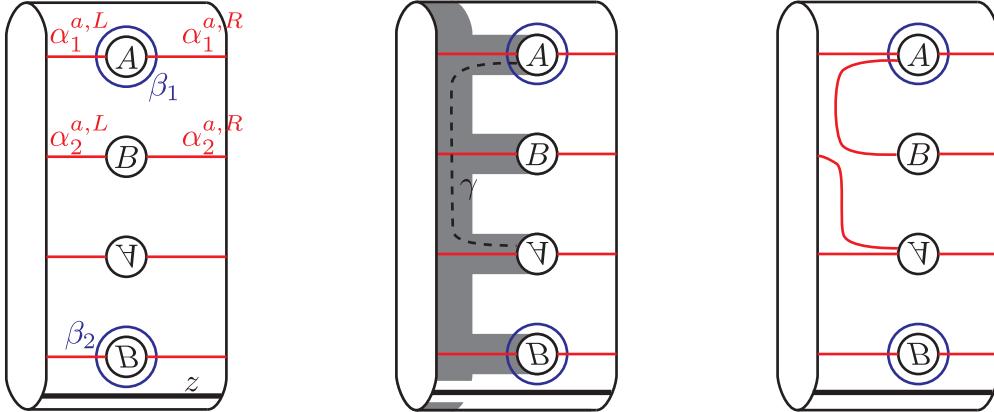


FIGURE A.1. **Bordered Heegaard diagrams for diffeomorphisms.** Left: the identity bordered Heegaard diagram for the genus 1 surface. Center: the same diagram, with a regular neighborhood of $\alpha_1^{a,L} \cup \dots \cup \alpha_{2k}^{a,L} \cup \partial_L \bar{\Sigma} \setminus \{z\}$ in dark gray and a closed curve γ as a dashed line. Right: a bordered Heegaard diagram for the Dehn twist around the curve γ .

$\Phi|_{\{0\} \times F_R}$ is well-defined up to homotopy, and hence (because F_R is a surface) up to isotopy. (Note that $\Phi|_{\{0\} \times F_R}$ is orientation-reversing.) Consequently, $(\Phi|_{\{0\} \times F_R})^{-1} \circ \phi_L$ is an orientation-reversing homeomorphism $F_L \rightarrow F_R$, well-defined up to isotopy. If we assume that $F_L = -F_R$ then $(\Phi|_{\{0\} \times F_R})^{-1} \circ \phi_L$ can be regarded as an element $\Upsilon(\mathcal{H})$ of the mapping class group $MCG(F_R)$. Paying attention to the arc z , we obtain a map

$$\Upsilon: \{\text{bordered Heegaard diagrams for } ([0, 1] \times F, -F, F)\} \rightarrow MCG_0(F),$$

where MCG_0 denotes the strongly based mapping class group of any surface $F = F(\mathcal{Z})$; here, “strongly based” refers to the mapping class group of diffeomorphisms fixing a small disk in F .

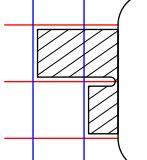
Every element of $MCG_0(F)$ arises this way. Indeed, given a diffeomorphism $\psi: F \rightarrow F$, we can explicitly construct a bordered Heegaard diagram realizing ψ as follows. Given a pointed matched circle \mathcal{Z} , let $\mathbb{I}_{\mathcal{Z}} = (\bar{\Sigma}_k, \bar{\alpha}, \beta, z)$ be the bordered Heegaard diagram with $-\partial_L \mathbb{I}_{\mathcal{Z}} = \partial_R \mathbb{I}_{\mathcal{Z}} = \mathcal{Z}$, with no α -circles and where $\bar{\alpha}_i^{a,L}$ and $\bar{\alpha}_i^{a,R}$ each intersect β_i in a single point and are disjoint from β_j for $i \neq j$. (See Figure A.1.) Then $\mathbb{I}_{\mathcal{Z}}$ is a bordered Heegaard diagram for the identity map of $F(\mathcal{Z})$. Further, a regular neighborhood of

$$\bar{\alpha}_1^{a,L} \cup \dots \cup \bar{\alpha}_{2k}^{a,L} \cup \partial_L \bar{\Sigma}$$

in $\bar{\Sigma}$ is canonically homeomorphic to F minus two disks. By removing the arc z from this neighborhood, we obtain F minus one disc. Under this identification, let $\bar{\psi}: \bar{\Sigma} \rightarrow \bar{\Sigma}$ be the result of extending $\psi \in MCG_0(F)$ by the identity. Then, $(\bar{\Sigma}, \bar{\psi}(\bar{\alpha}), \beta, z)$ is a bordered Heegaard diagram for ψ . (Equivalently, we could have applied $\bar{\psi}^{-1}$ to β .)

The hypothesis that $F_L = -F_R$ in the above construction is not crucial. In the more general case (where F_L and $-F_R$ are different), one obtains Heegaard diagrams corresponding to elements of a *mapping class groupoid*. See [LOT10a] for further details.

A.2.2. The definition of \widehat{CFDA} . Let $\mathcal{H} = (\bar{\Sigma}_g, \bar{\alpha}, \beta, z)$ be a bordered Heegaard diagram for $\psi \in MCG_0(F_R)$. The bimodule $\widehat{CFDA}(\psi) = \widehat{CFDA}(\mathcal{H})$ is defined by counting



holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$. As usual, we choose an almost complex structure so that the two boundary components of $\bar{\Sigma}$ become cylindrical ends of Σ , and hence of $\Sigma \times [0, 1] \times \mathbb{R}$. We call the end corresponding to $\partial_L \bar{\Sigma}$ the *left boundary* and the end corresponding to $\partial_R \bar{\Sigma}$ *right boundary*.

The bimodule $\widehat{CFDA}(\mathcal{H})$ is defined by treating the left boundary as a type D end (i.e., analogously to Chapter 6) and the right boundary a type A end (i.e., analogously to Chapter 7).

More precisely, let $\mathfrak{S}(\mathcal{H})$ denote the set of g -tuples of points \mathbf{x} in Σ so that exactly one x_i lies on each α - and each β -circle, and no two x_i lie on the same α -arc. Given $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$, let $o^L(\mathbf{x})$ denote the set of α^L -arcs occupied by \mathbf{x} and $o^R(\mathbf{x})$ the set of α^R -arcs occupied by \mathbf{x} . Let $X(\mathcal{H})$ be the \mathbb{F}_2 -vector space spanned by $\mathfrak{S}(\mathcal{H})$. We define left and right actions of $\mathcal{I}(F_R)$ on $\mathfrak{S}(\mathcal{H})$ by

$$I(\mathbf{s}) \cdot \mathbf{x} \cdot I(\mathbf{t}) := \begin{cases} \mathbf{x} & \mathbf{s} = [2k] \setminus o^L(\mathbf{x}) \text{ and } \mathbf{t} = o^R(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases}$$

where \mathbf{s} and \mathbf{t} are subsets of $[2k]$, as in Formula (3.22).

As a left module, $\widehat{CFDA}(\mathcal{H})$ is just $\mathcal{A}(F_L) \otimes_{\mathcal{I}} X(\mathcal{H})$. Note that, for the first time in this book, we have a module on which $\mathcal{A}(\mathcal{Z}, i)$ may act non-trivially for $i \neq 0$.

Our next task is to define the \mathcal{A}_∞ bimodule structure on $\widehat{CFDA}(\mathcal{H})$. (See, for instance, [Kel01, Section 3] for basic notions relating to \mathcal{A}_∞ bimodules.) The left and right actions strongly commute, i.e.,

$$m_{i,1,j}(a_1, \dots, a_i, \mathbf{x}, b_1, \dots, b_j) = 0$$

if $i, j > 0$, it only remains to describe the right action. The bimodule $\widehat{CFDA}(\mathcal{H})$ is also strictly unital, i.e.,

$$\begin{aligned} m_{0,1,1}(\mathbf{x}, \mathbf{I}) &= \mathbf{x} \\ m_{0,1,n}(\mathbf{x}, \dots, \mathbf{I}, \dots) &= 0, \quad n > 1. \end{aligned}$$

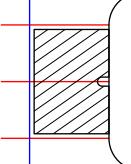
The definition of the operations $m_{0,1,n}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n))$ uses holomorphic curves. Let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^D; \vec{\rho}^A)$ denote the moduli space of embedded holomorphic curves in the homology class B , asymptotic to the sequence of Reeb chords in $\vec{\rho}^D$ at west ∞ (with no height constraints), and to the ordered partition $\vec{\rho}^A = (\boldsymbol{\rho}_1^A, \dots, \boldsymbol{\rho}_n^A)$ of Reeb chords at east ∞ . The rest of the bimodule structure is determined by

$$m_{0,1,n}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\vec{\rho}^D \mid \text{ind}(B; \vec{\rho}^D; \vec{\rho}^A) = 1\}}} (\#\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^D; \vec{\rho}^A)) a(-\vec{\rho}^D) \mathbf{y}.$$

(Compare Definitions 6.3 and 7.3. The case $n = 0$ is essentially the differential on \widehat{CFD} .)

We prove in [LOT10a] that this does, indeed, define an \mathcal{A}_∞ bimodule. Invariance—the first part of Theorem A.1—is proved similarly to invariance of \widehat{CFD} or \widehat{CFA} . Both of the proofs of the Pairing Theorem (Chapters 8 and 9) extend easily to prove the second and third parts of Theorem A.1. Again, this is explained in more detail in [LOT10a].

REMARK A.3. Just as \widehat{CFD} and \widehat{CFA} have been graded by left or right G -sets, bimodules are graded by sets with left and right actions by G . The bimodule $\widehat{CFDA}(\psi)$ is graded by $G(F_R)$, thought of as a set with commuting left and right actions by $G(F_R) \cong G(F_L)$.



The construction of the grading is a simple modification of the discussions in Sections 10.4 and 10.3.

A.2.3. The definition of $\widehat{CFDD}(\mathbb{I})$ and $\widehat{CFAA}(\mathbb{I})$. The bimodules $\widehat{CFDD}(\mathbb{I})$ and $\widehat{CFAA}(\mathbb{I})$ are both defined in terms of holomorphic curves with respect to the identity Heegaard diagram $\mathbb{I}_{\mathcal{Z}} = (\bar{\Sigma}, \bar{\alpha}, \beta, \mathbf{z})$. The bimodule $\widehat{CFDD}(\mathbb{I})$ is defined by treating both boundary components as type D boundary components, and the bimodule $\widehat{CFAA}(\mathbb{I})$ is defined by treating both boundary components as type A boundary.

More precisely, $\widehat{CFDD}(\mathbb{I})$ is defined as follows. Recall that $-\mathcal{Z}$ is the pointed matched circle obtained by reversing the orientation on \mathcal{Z} . The bimodule $\widehat{CFDD}(\mathbb{I})$ is generated over $\mathcal{A}(\mathcal{Z}) \times \mathcal{A}(-\mathcal{Z})$ by the $2^{2k} = \sum_i \binom{2k}{i}$ ways of choosing i points from $\{\bar{\alpha}_1^{L,a} \cap \beta_1, \bar{\alpha}_2^{L,a} \cap \beta_2, \dots, \bar{\alpha}_{2k}^{L,a} \cap \beta_{2k}\}$ (and the $2k - i$ points on the complementary $\alpha_i^{R,a}$); we can identify this with the set $\mathfrak{S}(\mathbb{I}_{\mathcal{Z}})$ from Section A.2.2 by looking at o^L , the set of strands occupied on the left. There are left actions of $\mathcal{I}(-\mathcal{Z})$ and $\mathcal{I}(\mathcal{Z})$ on $X(\mathbb{I}_{\mathcal{Z}})$ by, respectively,

$$\begin{aligned} I(\mathbf{s}) \bullet \mathbf{x} &:= \begin{cases} \mathbf{x} & \mathbf{s} = [2k] \setminus o^L(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases} \\ I(\mathbf{s}) \circ \mathbf{x} &:= \begin{cases} \mathbf{x} & \mathbf{s} = [2k] \setminus o^R(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(Note that $o^R(\mathbf{x}) = [2k] \setminus o^L(\mathbf{x})$.) These actions commute. Consequently, extending scalars from the idempotents to the whole algebra gives commuting left actions of $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$; this is the bimodule structure on $\widehat{CFDD}(\mathbb{I})$.

The differential on $\widehat{CFDD}(\mathbb{I})$ is given as follows. Let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L, \vec{\rho}^R)$ denote the moduli space of embedded holomorphic curves in the homology class B , asymptotic to the sequence of Reeb chords $\vec{\rho}^L$ on the left and $\vec{\rho}^R$ on the right. (We do not constrain the relative ordering of Reeb chords on the left and right.) Then

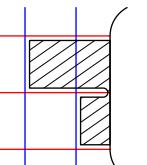
$$(A.4) \quad \partial \mathbf{x} := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\rho^L, \rho^R \mid \text{ind}(B; \rho^L, \rho^R) = 1\}}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L, \vec{\rho}^R)) a(-\vec{\rho}^L) \bullet a(-\vec{\rho}^R) \circ \mathbf{y}.$$

Next we turn to $\widehat{CFAA}(\mathbb{I})$. As an \mathbb{F}_2 -vector space, the bimodule $\widehat{CFAA}(\mathbb{I})$ is just $X(\mathbb{I}_{\mathcal{Z}})$. There are commuting right actions of $\mathcal{I}(-\mathcal{Z})$ and $\mathcal{I}(\mathcal{Z})$ on $X(\mathbb{I}_{\mathcal{Z}})$ by

$$\begin{aligned} \mathbf{x} \bullet I(\mathbf{s}) &:= \begin{cases} \mathbf{x} & \mathbf{s} = o^L(\mathbf{x}) \\ 0 & \text{otherwise,} \end{cases} \\ \mathbf{x} \circ I(\mathbf{s}) &:= \begin{cases} \mathbf{x} & \mathbf{s} = o^R(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The bimodule $\widehat{CFAA}(\mathbb{I})$ is strictly unital, i.e.,

$$\begin{aligned} m_{1,1,0}(\mathbf{x}; \mathbf{I};) &:= m_{1,0,1}(\mathbf{x}; ; \mathbf{I}) := \mathbf{x} \\ m_{1,m,n}(\mathbf{x}; \dots, \mathbf{I}, \dots; \dots) &:= m_{1,m,n}(\mathbf{x}; \dots; \dots, \mathbf{I}, \dots) := 0, \quad m + n > 1. \end{aligned}$$



The other higher multiplications are defined using holomorphic curves. Let $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}^L; \vec{\eta}^R)$ denote the moduli space of embedded holomorphic curves in the homology class B , asymptotic to the ordered partition $\vec{\rho} = (\rho_1, \dots, \rho_m)$ at $(\partial_L \bar{\Sigma}) \times [0, 1] \times \mathbb{R}$ and asymptotic to the ordered partition $\vec{\eta} = (\eta_1, \dots, \eta_n)$ at $(\partial_R \bar{\Sigma}) \times [0, 1] \times \mathbb{R}$. Then define

$$m_{1,m,n}(\mathbf{x}; a(\rho_1), \dots, a(\rho_m); a(\eta_1), \dots, a(\eta_n)) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \text{ind}(B; \vec{\rho}; \vec{\eta})=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; \vec{\eta})) \mathbf{y}.$$

Theorem A.2 now follows from a version of the pairing theorem, which can be proved using either the nice diagrams technique of Chapter 8 or the time dilation argument of Chapter 9.

REMARK A.5. The gradings on $\widehat{CFAA}(\mathbb{I})$ and $\widehat{CFDD}(\mathbb{I})$ are defined analogously to Sections 10.3 and 10.4 respectively. In the case of $\widehat{CFAA}(\mathbb{I})$ (respectively $\widehat{CFDD}(\mathbb{I})$) the G -grading set is $G(\mathcal{Z})$, viewed as having two right (respectively left) actions, one by $G(\mathcal{Z})$ and one by $G(-\mathcal{Z}) \cong G(\mathcal{Z})^{\text{op}}$. In fact, both of these actions are the standard action of the group on itself by translation.

REMARK A.6. For each of $\widehat{CFDA}(\mathcal{H})$, $\widehat{CFDD}(\mathbb{I})$ and $\widehat{CFAA}(\mathbb{I})$, the summands $\mathcal{A}(\mathcal{Z}, i)$ inside $\mathcal{A}(\mathcal{Z})$ act non-trivially even for $i \neq 0$, and the bimodules decompose as direct sums

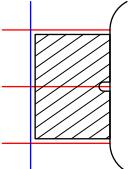
$$\begin{aligned} \widehat{CFDA}(\mathcal{H}) &= \bigoplus_{i=-k}^k \widehat{CFDA}(\mathcal{H}, i) \\ \widehat{CFDD}(\mathbb{I}) &= \bigoplus_{i=-k}^k \widehat{CFDD}(\mathbb{I}, i) \\ \widehat{CFAA}(\mathbb{I}) &= \bigoplus_{i=-k}^k \widehat{CFAA}(\mathbb{I}, i). \end{aligned}$$

Of course, only the summands corresponding to $i = 0$ contribute to tensor products with \widehat{CFD} or \widehat{CFA} as in Theorem A.1. The other summands do arise naturally if one studies self-gluing of Heegaard diagrams, which in turn relates to the Floer homology of open books.

A.3. Computations for 3-manifolds with torus boundary

In this section we state the relevant bimodules for three-manifolds with torus boundary. These are used in Section A.4 to deduce a version of Theorem 11.26 for arbitrary integral framings of knots, Theorem A.11. (In fact, the data from this section can be used to calculate the result for arbitrary rational framings; but that will be left to the interested reader.) The computations of the bimodules are given in [LOT10a].

There is a unique pointed matched circle representing the surface of genus one. Let $\mathcal{A} = \mathcal{A}(\mathcal{Z}, 0)$ denote the $i = 0$ summand of the corresponding algebra. Label the generators of \mathcal{A} as in Chapter 11.



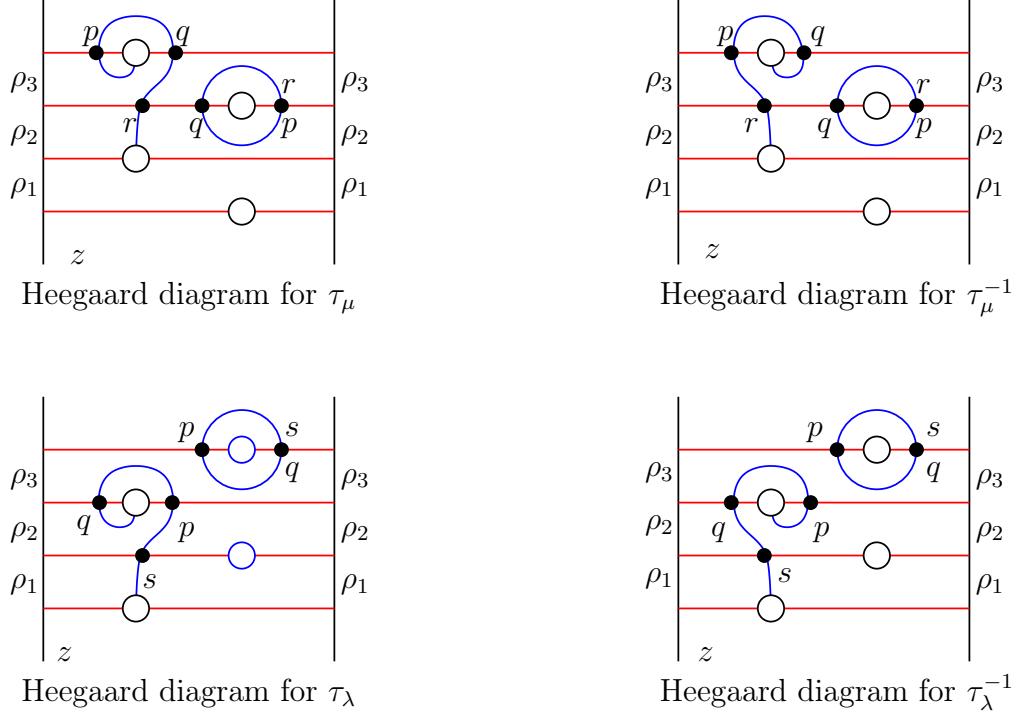


FIGURE A.2. **Heegaard diagrams for mapping class group elements.**

Genus 2 diagrams for τ_μ , τ_μ^{-1} , τ_λ and τ_λ^{-1} are shown. In each of the four diagrams, there are three generators in the $i = 0$ summand.

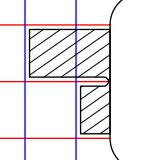
A.3.1. Type DA invariants of Dehn twists. The mapping class group is generated by Dehn twists τ_μ and τ_λ along a meridian and a longitude, respectively (i.e., τ_μ takes an n -framed knot complement to an $n + 1$ -framed knot complement). Recall from Section 11.4 that α_2^a is a meridian. Thus, τ_μ corresponds to a Dehn twist around a curve μ dual to the curve α_1^a from Figure 11.1.

Most of this section is devoted to describing the $i = 0$ part $\widehat{CFDA}(\cdot, 0)$ of the type DA bimodules for these Dehn twists and their inverses. First, we dispense with the $i = \pm 1$ parts. The summand $\mathcal{A}(\mathbb{T}, -1)$ of $\mathcal{A}(\mathbb{T})$ is isomorphic to \mathbb{F}_2 , and $\widehat{CFDA}(\phi, -1) \cong \mathbb{F}_2$ for any mapping class ϕ . The summand $\mathcal{A}(\mathbb{T}, 1)$ of $\mathcal{A}(\mathbb{T})$ is 7-dimensional, but is quasi-isomorphic to \mathbb{F}_2 . Thus, $\mathcal{A}(\mathbb{T}, 1)$ -bimodules are determined (up to quasi-isomorphism) by corresponding \mathbb{F}_2 -bimodules. For any mapping class ϕ , the \mathbb{F}_2 -bimodule corresponding to $\widehat{CFDA}(\phi, 1)$ is, again, \mathbb{F}_2 .

Heegaard diagrams for τ_μ , τ_μ^{-1} , τ_λ and τ_λ^{-1} are illustrated in Figure A.2. Each of the four bimodules $\widehat{CFDA}(\tau_\mu, 0)$, $\widehat{CFDA}(\tau_\mu^{-1}, 0)$, $\widehat{CFDA}(\tau_\lambda, 0)$ and $\widehat{CFDA}(\tau_\lambda^{-1}, 0)$ has the form $\mathcal{A} \otimes_{\mathcal{I}} X$ where X is 3-dimensional over \mathbb{F}_2 . The vector space X has a basis consisting of \mathbf{p} , \mathbf{q} and either \mathbf{r} or \mathbf{s} . The generators are compatible with the idempotents as follows:

$$(A.7) \quad \iota_0 \cdot \mathbf{p} \cdot \iota_0 = \mathbf{p} \quad \iota_1 \cdot \mathbf{q} \cdot \iota_1 = \mathbf{q} \quad \iota_1 \cdot \mathbf{r} \cdot \iota_0 = \mathbf{r} \quad \iota_0 \cdot \mathbf{s} \cdot \iota_1 = \mathbf{s}.$$

As in Section A.2.2, the left module structure is induced from the isomorphism $\widehat{CFDA} \cong \mathcal{A} \otimes_{\mathcal{I}} X$. Higher products of the form $m_{i,1,j}(a_1, \dots, a_i, \mathbf{x}, b_1, \dots, b_j)$ vanish if $ij > 0$. So, it remains to give the higher products of the form $m_{0,1,j}$ for $j \geq 0$. We do this for each of the four bimodules in turn; see also Figure A.3.



The bimodule $\widehat{CFDA}(\tau_\mu, 0)$ is generated by \mathbf{p} , \mathbf{q} and \mathbf{r} . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{array}{ll} m_{0,1,1}(\mathbf{p}, \rho_1) = \rho_1 \otimes \mathbf{q} & m_{0,1,1}(\mathbf{p}, \rho_{12}) = \rho_{123} \otimes \mathbf{r} \\ m_{0,1,1}(\mathbf{p}, \rho_{123}) = \rho_{123} \otimes \mathbf{q} & m_{0,1,2}(\mathbf{p}, \rho_3, \rho_2) = \rho_3 \otimes \mathbf{r} \\ m_{0,1,2}(\mathbf{p}, \rho_3, \rho_{23}) = \rho_3 \otimes \mathbf{q} & m_{0,1,1}(\mathbf{q}, \rho_2) = \rho_{23} \otimes \mathbf{r} \\ m_{0,1,1}(\mathbf{q}, \rho_{23}) = \rho_{23} \otimes \mathbf{q} & m_{0,1,0}(\mathbf{r}) = \rho_2 \otimes \mathbf{p} \\ m_{0,1,1}(\mathbf{r}, \rho_3) = \mathbf{q}. & \end{array}$$

The bimodule $\widehat{CFDA}(\tau_\mu^{-1}, 0)$ is generated by \mathbf{p} , \mathbf{q} and \mathbf{r} . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{array}{ll} m_{0,1,0}(\mathbf{p}) = \rho_3 \otimes \mathbf{r} & m_{0,1,1}(\mathbf{p}, \rho_1) = \rho_1 \otimes \mathbf{q} \\ m_{0,1,1}(\mathbf{p}, \rho_{12}) = \rho_1 \otimes \mathbf{r} & m_{0,1,1}(\mathbf{p}, \rho_{123}) = \rho_{123} \otimes \mathbf{q} \\ m_{0,1,2}(\mathbf{p}, \rho_{123}, \rho_2) = \rho_{12} \otimes \mathbf{p} & m_{0,1,1}(\mathbf{q}, \rho_2) = \mathbf{r} \\ m_{0,1,1}(\mathbf{q}, \rho_{23}) = \rho_{23} \otimes \mathbf{q} & m_{0,1,2}(\mathbf{q}, \rho_{23}, \rho_2) = \rho_2 \otimes \mathbf{p} \\ m_{0,1,1}(\mathbf{r}, \rho_3) = \rho_{23} \otimes \mathbf{q} & m_{0,1,2}(\mathbf{r}, \rho_3, \rho_2) = \rho_2 \otimes \mathbf{p}. \end{array}$$

The bimodule $\widehat{CFDA}(\tau_\lambda, 0)$ is generated by \mathbf{p} , \mathbf{q} and \mathbf{s} . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{array}{ll} m_{0,1,2}(\mathbf{q}, \rho_2, \rho_1) = \rho_2 \otimes \mathbf{s} & m_{0,1,2}(\mathbf{q}, \rho_2, \rho_{12}) = \rho_2 \otimes \mathbf{p} \\ m_{0,1,2}(\mathbf{q}, \rho_2, \rho_{123}) = \rho_{23} \otimes \mathbf{q} & m_{0,1,1}(\mathbf{p}, \rho_1) = \rho_{12} \otimes \mathbf{s} \\ m_{0,1,1}(\mathbf{p}, \rho_{12}) = \rho_{12} \otimes \mathbf{p} & m_{0,1,1}(\mathbf{p}, \rho_{123}) = \rho_{123} \otimes \mathbf{q} \\ m_{0,1,1}(\mathbf{p}, \rho_3) = \rho_3 \otimes \mathbf{q} & m_{0,1,0}(\mathbf{s}) = \rho_1 \otimes \mathbf{q} \\ m_{0,1,1}(\mathbf{s}, \rho_2) = \mathbf{p} & m_{0,1,1}(\mathbf{s}, \rho_{23}) = \rho_3 \otimes \mathbf{q}. \end{array}$$

The bimodule $\widehat{CFDA}(\tau_\lambda^{-1}, 0)$ is generated by \mathbf{p} , \mathbf{q} and \mathbf{s} . The idempotents act as above; the other non-trivial algebra actions are given as follows:

$$\begin{array}{ll} m_{0,1,0}(\mathbf{q}) = \rho_2 \otimes \mathbf{s} & m_{0,1,1}(\mathbf{p}, \rho_1) = \mathbf{s} \\ m_{0,1,1}(\mathbf{p}, \rho_{12}) = \rho_{12} \otimes \mathbf{p} & m_{0,1,1}(\mathbf{p}, \rho_{123}) = \rho_{123} \otimes \mathbf{q} \\ m_{0,1,1}(\mathbf{p}, \rho_3) = \rho_3 \otimes \mathbf{q} & m_{0,1,2}(\mathbf{p}, \rho_{12}, \rho_1) = \rho_1 \otimes \mathbf{q} \\ m_{0,1,1}(\mathbf{s}, \rho_2) = \rho_{12} \otimes \mathbf{p} & m_{0,1,1}(\mathbf{s}, \rho_{23}) = \rho_{123} \otimes \mathbf{q} \\ m_{0,1,2}(\mathbf{s}, \rho_2, \rho_1) = \rho_1 \otimes \mathbf{q}. & \end{array}$$

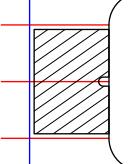
REMARK A.8. The computations leading to the results above are eased by noting that

- $\widehat{CFDA}(\tau_\mu^{-1}, 0)$ is inverse to $\widehat{CFDA}(\tau_\mu, 0)$, i.e.,

$$\widehat{CFDA}(\tau_\mu^{-1}, 0) \boxtimes \widehat{CFDA}(\tau_\mu, 0) \simeq \widehat{CFDA}(\mathbb{I}, 0);$$

- the Heegaard diagram for τ_λ is obtained from that for τ_μ^{-1} by a horizontal reflection (across the x -axis); and
- the Heegaard diagram for τ_λ^{-1} is obtained from that for τ_μ by a horizontal reflection.

Thus there is essentially only one computation to do.



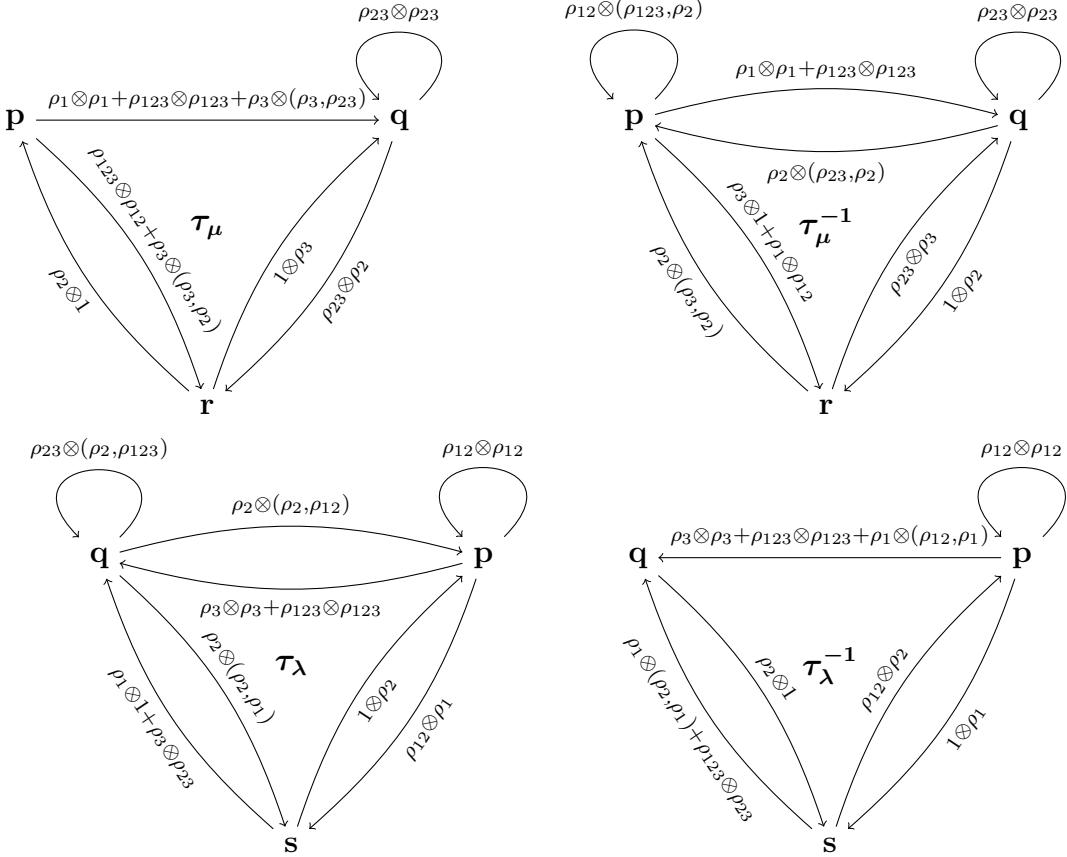
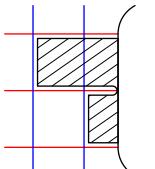


FIGURE A.3. Type *DA* bimodules for torus mapping class group action. These are the bimodules associated to τ_μ , τ_μ^{-1} , τ_λ , τ_λ^{-1} respectively. The notation is as follows. Consider the bimodule for τ_μ . The label $\rho_1 \otimes \rho_1$ on the horizontal arrow indicates that $m_{0,1,1}(\mathbf{p}, \rho_1)$ contains a term of the form $\rho_1 \otimes \mathbf{q}$. Similarly, the label $\rho_3 \otimes (\rho_3, \rho_{23})$ on that arrow indicates that $m_{0,1,2}(\mathbf{p}, \rho_3, \rho_{23})$ contains a term of the form $\rho_3 \otimes \mathbf{q}$.

REMARK A.9. As the reader may well have expected, up to homotopy equivalence the type *DA* bimodule $\widehat{CFDA}(\mathbb{I}, 0)$ associated to the identity map is isomorphic to \mathcal{A} as an \mathcal{A} - \mathcal{A} bimodule.

A.3.2. The type *AA* invariant of the identity. To compute the type *AA* and *DD* identity bimodules for the torus, it is convenient to use the Heegaard diagram pictured in Figure A.4. Note that the boundary segments on both sides have been labeled to be consistent with the type *A* conventions.



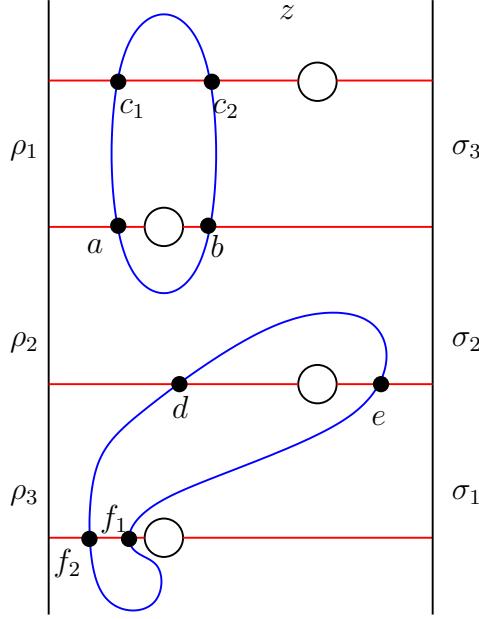


FIGURE A.4. **A Heegaard diagram for the genus 1 identity map.** This diagram has four more intersection points in $\alpha \cap \beta$ than the standard Heegaard diagram for the identity map, introduced by two finger moves. As a result of these finger moves, the diagram is admissible, not merely provincially admissible, and is useful for computing $\widehat{CFAA}(\mathbb{I})$.

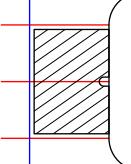
The resulting type AA bimodule has six generators, denoted $bf_1, bf_2, c_1e, c_2e, ae$ and bd . Its non-trivial \mathcal{A}_∞ -operations are given by:

$$\begin{array}{ll}
 m_{1,1,0}(bf_1; \sigma_1;) = ae & m_{1,0,0}(bf_1; ;) = bf_2 \\
 m_{1,1,1}(bf_1; \sigma_{12}; \rho_2) = bd & m_{1,1,1}(bf_1; \sigma_{12}; \rho_{23}) = bf_2 \\
 m_{1,1,1}(bf_1; \sigma_{123}; \rho_2) = c_2e & m_{1,3,1}(bf_1; \sigma_3, \sigma_2, \sigma_1; \rho_2) = c_2e \\
 m_{1,0,0}(c_1e; ;) = c_2e & m_{1,0,1}(c_1e; ; \rho_1) = ae \\
 m_{1,1,1}(c_1e; \sigma_2; \rho_{12}) = bd & m_{1,1,1}(c_1e; \sigma_{23}; \rho_{12}) = c_2e \\
 m_{1,1,1}(c_1e; \sigma_2; \rho_{123}) = bf_2 & m_{1,1,1}(ae; \sigma_2; \rho_{23}) = bf_2 \\
 m_{1,1,1}(ae; \sigma_2; \rho_2) = bd & m_{1,1,1}(ae; \sigma_{23}; \rho_2) = c_2e \\
 m_{1,0,1}(bd; ; \rho_3) = bf_2 & m_{1,1,0}(bd; \sigma_3;) = c_2e.
 \end{array}$$

Here, to avoid confusion, we denote algebra elements acting on one side (which we think of as $\mathcal{A}(-\mathcal{Z}, 0)$) by σ_i while on the other side (which we think of as $\mathcal{A}(\mathcal{Z}, 0)$) they are denoted by ρ_i (as indicated in Figure A.4).

REMARK A.10. The type AA identity bimodule of the torus has a model which is 2-dimensional over \mathbb{F}_2 . However, this model has infinitely many non-trivial higher products. These can be described explicitly; see [LOT10c, Section 8.4].

A.3.3. The type DD invariant of the identity. The type DD bimodule can also be calculated using the same Heegaard diagram. The result, which is given graphically in



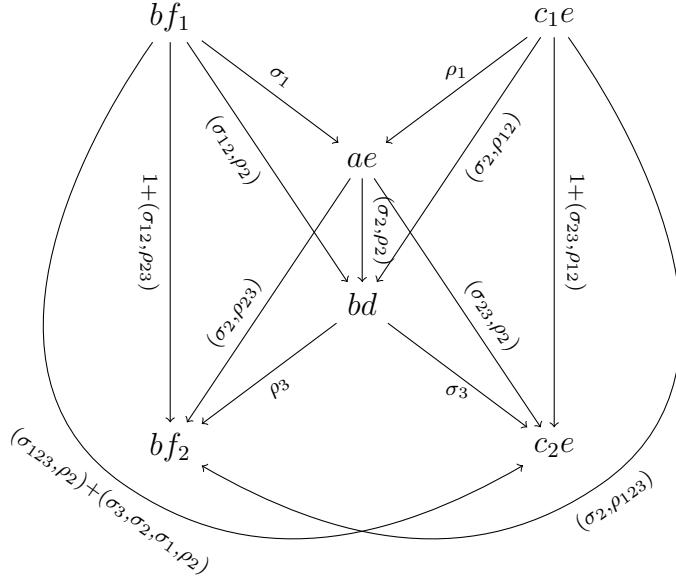


FIGURE A.5. **The type AA bimodule $\widehat{CFAA}(\mathbb{I}, 0)$.** The labels on the arrows indicate the \mathcal{A}_∞ operations; for instance, the label (σ_{23}, ρ_2) on the arrow from ae to c_2e means that $m_{1,1,1}(ae, \sigma_{23}, \rho_2)$ contains a term c_2e .

Figure A.6, is given as follows:

$$\begin{aligned} \partial bf_1 &= (\sigma_3 \otimes 1) \otimes ae + bf_2 & \partial c_1 e &= (1 \otimes \rho_3) \otimes ae + c_2 e + (\sigma_2 \otimes \rho_{123}) \otimes bf_2 \\ \partial ae &= (\sigma_2 \otimes \rho_2) \otimes bd & \partial bd &= (1 \otimes \rho_1) \otimes bf_2 + (\sigma_1 \otimes 1) \otimes c_2 e \\ \partial bf_2 &= 0 & \partial c_2 e &= 0. \end{aligned}$$

(We have relabeled the ρ_i and σ_i to conform to our conventions for type D boundary.) This simplifies to a bimodule with two generators x and y with

$$\partial x = (\rho_1 \sigma_3 + \rho_3 \sigma_1 + \rho_{123} \sigma_{123}) \otimes y \quad \partial y = (\rho_2 \sigma_2) \otimes x.$$

A.4. From HFK to \widehat{CFD} for arbitrary integral framings

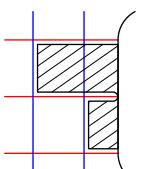
We turn now to the generalization of Theorem 11.26 for arbitrary (integral) framings.

THEOREM A.11. *Let $K \subset S^3$ be a knot and let $CFK^-(K)$ be a reduced model for the knot Floer complex of K . Let Y be the bordered three-manifold $S^3 - \text{nbd } K$, given any integral framing n . The associated type D module $\widehat{CFD}(Y)$ can be extracted from $CFK^-(K)$ using the procedure described in Theorem 11.26, except for the unstable chain, whose precise form depends on the framing parameter n . There are three cases for the form of the unstable chain. When $n < 2\tau(K)$, the unstable chain has the form*

$$\xi_0 \xrightarrow{D_1} \mu_1 \xleftarrow{D_{23}} \mu_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} \mu_m \xleftarrow{D_3} \eta_0,$$

where $m = 2\tau(K) - n$. When $n = 2\tau(K)$, the unstable chain has the form

$$\xi_0 \xrightarrow{D_{12}} \eta_0.$$



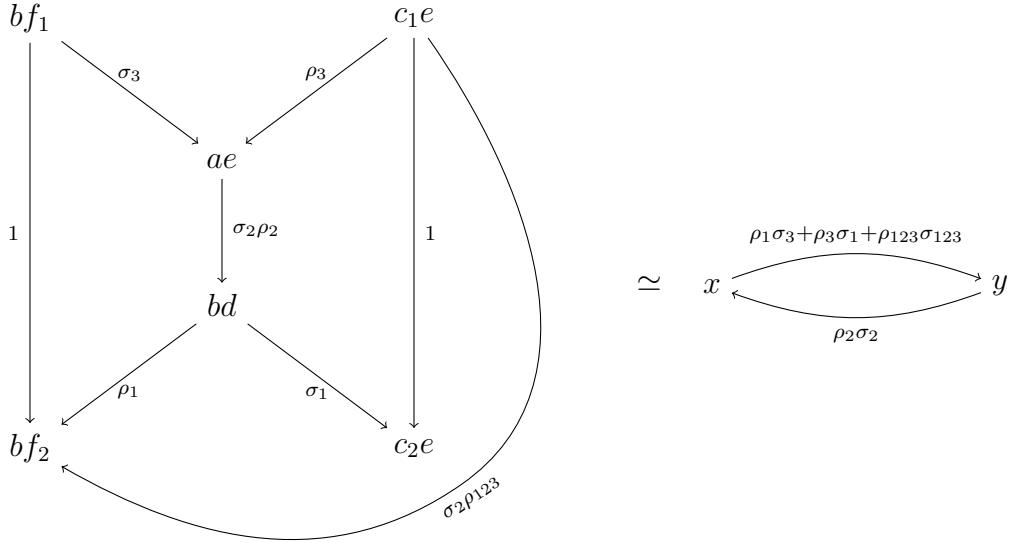


FIGURE A.6. **The type DD bimodule $\widehat{CFDD}(\mathbb{I}, 0)$.** The labels on the arrows indicate differentials; for instance, the arrow from ae to bd signifies a term $(\rho_2 \cdot \sigma_2) \otimes bd$ in ∂ae . The form on the left comes from the Heegaard diagram in Figure A.4, and is bounded in an analogous sense to Definition 2.22. The form on the right is homotopy equivalent, but no longer bounded.

Finally, when $n > 2\tau(K)$ the unstable chain has the form

$$\xi_0 \xrightarrow{D_{123}} \mu_1 \xrightarrow{D_{23}} \mu_2 \cdots \xrightarrow{D_{23}} \mu_m \xrightarrow{D_2} \eta_0,$$

where $m = n - 2\tau(K)$.

The gradings are determined as follows:

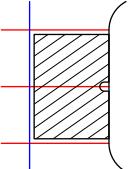
- The grading set is $G/\lambda^{-1} \text{gr}(\rho_{23})^{-n} \text{gr}(\rho_{12})^{-1}$.
- Recall that any element \mathbf{x}_0 of V^0 is represented by a generator \mathbf{x} for the knot Floer complex. The grading of \mathbf{x}_0 in the above grading set is determined by the Maslov grading M and Alexander grading A of \mathbf{x} by the formula

$$(A.12) \quad \text{gr}(\mathbf{x}_0) = \lambda^{M(\mathbf{x})-2A(\mathbf{x})} (\text{gr}(\rho_{23}))^{-A(\mathbf{x})} = (M - \frac{3}{2}A; 0, -A).$$

(The reader is warned that n here has the opposite sign from Theorem 11.26, since the framing there is $-n$.)

PROOF. In view of Theorem A.1, we need only see what happens as we iteratively tensor the D module for a knot complement with sufficiently small framing parameter, as calculated in Theorem 11.26, with the bimodule $\widehat{CFDA}(\tau_\mu, 0)$.

We denote the generators in the idempotent ι_0 for $\widehat{CFD}(Y)$ by x_i , and the generators in idempotent ι_1 by y_j . In particular, we write x_0 and x_1 for the elements ξ_0 and η_0 in the statement of the above theorem.



In the tensor product, each generator x_i gives rise to a pair of generators $\mathbf{p} \boxtimes x_i$ and $\mathbf{r} \boxtimes x_i$, while each generator y_j gives rise to a generator $\mathbf{q} \boxtimes y_j$. Note that the generators $\mathbf{p} \boxtimes x_i$ are the generators in the ι_0 -idempotent for the tensor product.

If we think of the x_i as the generators in a horizontally-simplified basis in the sense of Definition 11.23, then if there are $2k + 1$ basis elements, k of them are initial points of horizontal arrows. For each initial point of a horizontal arrow x_i , we can cancel $\mathbf{r} \boxtimes x_i$ with some corresponding element of the form $\mathbf{q} \boxtimes y_1$; i.e., if we have a chain

$$x_i \xrightarrow{D_3} y_1 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} y_\ell \xrightarrow{D_2} x_j;$$

this is transformed to

$$\begin{array}{ccccccccc} \mathbf{p} \boxtimes x_i & \xrightarrow{D_3} & \mathbf{q} \boxtimes y_2 & \xrightarrow{D_{23}} & \mathbf{q} \boxtimes y_3 & \xrightarrow{D_{23}} & \dots & \xrightarrow{D_{23}} & \mathbf{q} \boxtimes y_\ell & \xrightarrow{D_{23}} & \mathbf{r} \boxtimes x_j \\ D_2 \uparrow & & D_{23} \uparrow & & & & & & & & \downarrow D_2. \\ \mathbf{r} \boxtimes x_i & \xrightarrow{D_\emptyset} & \mathbf{q} \boxtimes y_1 & & & & & & & & \mathbf{p} \boxtimes x_j \end{array}$$

Setting $y'_i = \mathbf{q} \boxtimes y_{i+1}$ for $i = 1 \dots \ell - 1$ and $y'_\ell = \mathbf{r} \boxtimes x_j$ and dropping the canceling pair $\mathbf{r} \boxtimes x_i$ and $\mathbf{q} \boxtimes y_1$, our new chain has the same form as the original chain.

Vertical chains are taken to chains of the same form (without any need for cancellation).

Finally, we must observe the behavior of the “unstable chain” under tensor product. To this end, recall the generator x_0 which is neither the initial nor the final intersection point of a horizontal arrow. There is then a chain of the form

$$x_0 \xrightarrow{D_3} y_1 \xrightarrow{D_{23}} y_2 \xrightarrow{D_{23}} y_3 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} y_m \xleftarrow{D_1} x_1$$

with $m > 1$. Under the tensor product, canceling a differential from $\mathbf{r} \boxtimes x_0$ to $\mathbf{q} \boxtimes y_1$ as before, this chain is carried to a similar chain from with length one less, from $\mathbf{p} \boxtimes x_0$ to $\mathbf{p} \boxtimes x_1$.

In the case where $m = 1$, we have the chain

$$x_0 \xrightarrow{D_3} y_1 \xleftarrow{D_1} x_1.$$

This chain is transformed to another configuration where we can once again cancel the pair $\mathbf{r} \boxtimes x_0$ and $\mathbf{q} \boxtimes y_1$, to get the chain

$$\mathbf{p} \boxtimes x_0 \xleftarrow{D_{12}} \mathbf{p} \boxtimes x_1.$$

Now if we start with a chain of the form

$$x_0 \xleftarrow{D_{12}} x_1,$$

this in turn is transformed into

$$\mathbf{p} \boxtimes x_0 \xleftarrow{D_2} \mathbf{r} \boxtimes x_0 \xleftarrow{D_{123}} \mathbf{p} \boxtimes x_1 \xleftarrow{D_2} \mathbf{r} \boxtimes x_1.$$

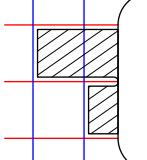
The arrow from $\mathbf{r} \boxtimes x_1$ should be viewed as part of another chain.

The remaining elements lie in an unstable chain of the form

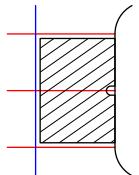
$$x_0 \xleftarrow{D_2} y_1 \xleftarrow{D_{23}} y_2 \dots \xleftarrow{D_{23}} y_m \xleftarrow{D_{123}} x_1$$

with $m = 1$. Now it is easy to see inductively that tensoring with our bimodule takes such an unstable chain with parameter m to one with parameter $m + 1$.

We leave verification of the statement about gradings, which requires the gradings on the bimodules from [LOT10a], to the interested reader. The precise relationship between



the length (and type) of the unstable chain, the framing parameter, and τ follows as in the proof of Theorem 11.26. \square



Index of Notation

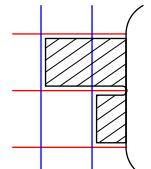
General	255
\mathcal{A}_∞ structures	255
Algebra associated to a pointed matched circle	256
Gradings	257
Heegaard diagrams	258
Moduli spaces	259
Type D modules	260
Invariance of \widehat{CFD}	261
Type A modules	262
The pairing theorem	262
Manifolds with torus boundary	263
Bimodules and change of framing	264

General

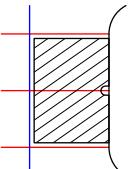
\mathbb{I}	Identity morphism. 13
$-$	Orientation reversal. 57
$[n]$	The set $\{1, \dots, n\}$. 29
Δ	The symmetric difference of two sets; $S \Delta T := (S \setminus T) \cup (T \setminus S)$. 41

\mathcal{A}_∞ structures

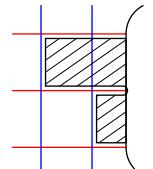
\mathcal{A}	An \mathcal{A}_∞ algebra. Also used for algebra associated to a surface. 10
A	The \mathbf{k} -module underlying the \mathcal{A}_∞ algebra \mathcal{A} . 10
$A^{\otimes i}$	The i^{th} tensor power of A . 10
\bar{D}	Differential on bar (tensor) algebra of an \mathcal{A}_∞ algebra. 11
δ^1	Structure map of type D structure. 16
δ^k	Iterate of structure map δ^1 on a type D structure. 18
δ	Sum of all δ^k for a type D structure. Also used for connecting homomorphism on H_* . 18
ψ^k	Analogue of δ^k for a homomorphism of type D structures. 19
ψ	Analogue of δ for a homomorphism of type D structures. 19
$\{f_i\}$	A morphism of \mathcal{A}_∞ modules. 13
\mathbf{k}	Ground ring, a sum of \mathbb{F}_2 's. 9
\mathcal{M}	An \mathcal{A}_∞ module. 11
M	The \mathbf{k} -module underlying an \mathcal{A}_∞ module \mathcal{M} . 11
$\bar{\mathcal{M}}$	Bar resolution of \mathcal{M} . 15
$\overline{\mathcal{M}}$	Module underlying bar resolution of \mathcal{M} . 15
μ_i	The (higher) products on an \mathcal{A}_∞ algebra. 10
m_i	The (higher) products on an \mathcal{A}_∞ module. 11
\overline{m}	Differential on bar resolution of an \mathcal{A}_∞ module \mathcal{M} gotten by considering all m_i at once. 12
(N, δ^1)	A type D structure. 17
\mathcal{N}	dg (resp. \mathcal{A}_∞) module associated to a type D structure N over a dg (resp. \mathcal{A}_∞) algebra. 17, 20
$\mathcal{T}^*(A)$	Tensor (bar) algebra on A . 10
$\mathcal{T}^+(A)$	Augmentation ideal in tensor algebra $\mathcal{T}^*(A)$. 15



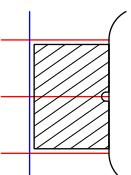
$\tilde{\otimes}$	The \mathcal{A}_∞ tensor product. 15
$f \tilde{\otimes} \mathbb{I}$	The \mathcal{A}_∞ tensor product of f with the identity map. 16
\boxtimes	Box tensor product, a smaller model for \mathcal{A}_∞ tensor product. 21
∂^\boxtimes	Differential on \boxtimes . 21
$\phi \boxtimes \mathbb{I}_N$	Box tensor product of a homomorphism with the identity on N . 22
Algebra associated to a pointed matched circle	
\mathbf{a}	n points in an oriented circle. 34
$\mathcal{A}(n, k)$	Strands algebra on n places and k strands. 29
$\mathcal{A}(n)$	Strands algebra on n places, $\mathcal{A}(n) = \bigoplus_k \mathcal{A}(n, k)$. 30
$\mathcal{A}(\mathcal{Z})$	Algebra associated to pointed matched circle \mathcal{Z} . $\mathcal{A}(\mathcal{Z}) \subset \mathcal{A}(4k)$. 35
$\mathcal{A}(\mathcal{Z}, i)$	Summand of $\mathcal{A}(\mathcal{Z})$ with weight i . $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(4k, k+i)$. 36
$a_0(\rho)$	Element of $\mathcal{A}(n)$ associated to consistent set of Reeb chords ρ . 32
$a(\rho)$	Element of $\mathcal{A}(\mathcal{Z})$ associated to set ρ of Reeb chords. 35
$a(\vec{\rho})$	Product $a(\rho_1) \cdots a(\rho_n)$ where $\vec{\rho} = (\rho_1, \dots, \rho_n)$. 104
$a(-\vec{\rho})$	Product $a(-\rho_1) \cdots a(-\rho_n)$ where $\vec{\rho} = (\rho_1, \dots, \rho_n)$. 104
$F(\mathcal{Z})$	Surface associated to pointed matched circle \mathcal{Z} . 34
$I(S)$	Idempotent in $\mathcal{A}(n, k)$ associated to set $S \subset [n]$, $ S = k$. 29
$I(\mathbf{s})$	Idempotent of $\mathcal{A}(\mathcal{Z})$ associated to set \mathbf{s} of matched pairs. 35
$\mathcal{I}(n, k)$	Ring of idempotents in $\mathcal{A}(n, k)$. 29
$\mathcal{I}(\mathcal{Z})$	Ring of idempotents of $\mathcal{A}(\mathcal{Z})$. 35
$\mathcal{I}(\mathcal{Z}, i)$	Summand of $\mathcal{I}(\mathcal{Z})$ with weight i . 36
\mathbf{I}	Unit of $\mathcal{A}(\mathcal{Z})$. 35, 138
$\text{inv}(a)$	Number of inversions of element a of $\mathcal{A}(n, k)$. 29
$\text{Inv}(\phi)$	Set of inversions of partial permutation ϕ . 29
$\text{inv}(\phi)$	Number of inversions of partial permutation ϕ . 29
\lessdot	Ordering on \mathbf{a} induced by orientation on Z and basepoint z . 34
M	Matching function of a matched circle. 34
$\left\{ \begin{smallmatrix} 1 & 2 \\ 5 & 3 \end{smallmatrix} \right\}$	Consistent set of Reeb chords $\{[1, 5], [2, 3]\}$. 32
$\left[\begin{smallmatrix} 1 & 4 \\ 6 \end{smallmatrix} \right]$	Element of $\mathcal{A}(\mathcal{Z})$ shown in Equation (3.24). 36
$\left\langle \begin{smallmatrix} 1 & 2 & 4 \\ 5 & 3 & 4 \end{smallmatrix} \right\rangle$	Element of $\mathcal{A}(n, k)$ given by partial permutation $1 \mapsto 5, 2 \mapsto 3, 4 \mapsto 4$. 29
	Graphical notation for the algebra element $\left\langle \begin{smallmatrix} 1 & 2 & 4 \\ 5 & 3 & 4 \end{smallmatrix} \right\rangle$. 30
ϕ_σ	Result of resolving crossing in ϕ corresponding to σ . 29
ρ	A Reeb chord in Z . 32
ρ^-	Initial endpoint of Reeb chord ρ . 32
ρ^{--}	Initial endpoint of chord being glued to left of ρ . 32
ρ^+	Terminal endpoint of Reeb chord ρ . 32
ρ^{++}	Terminal endpoint of chord being glued to right of ρ . 32
ρ	A set of Reeb chords. 32
ρ^-	Initial endpoints of set of Reeb chords ρ . 32
ρ^+	Terminal endpoints of set of Reeb chords ρ . 32
$\vec{\rho}$	A sequence (ρ_1, \dots, ρ_n) of sets of Reeb chords. 88
$(\vec{\rho}, \vec{\rho}')$	Concatenation of the sequences $\vec{\rho}$ and $\vec{\rho}'$. 100
$\vec{\rho}_{[i,j]}$	Subsequence (ρ_i, \dots, ρ_j) of $\vec{\rho}$. 88
$-\vec{\rho}$	Sequence obtained by reversing the orientation of each Reeb chord in $\vec{\rho}$ (but keeping the same order). 105
$\vec{\rho}$	A sequence (ρ_1, \dots, ρ_n) of Reeb chords or the associated sequence $(\{\rho_1\}, \dots, \{\rho_n\})$ of singleton sets of Reeb chords. 104
$-\vec{\rho}$	The sequence $(-\rho_1, \dots, -\rho_n)$. 105
\boxplus	Join of abutting Reeb chords. 32
\boxplus	Join of composable sets of Reeb chords. 32



Z	An oriented circle. 32
\mathcal{Z}	Pointed matched circle. 34
z	Basepoint for pointed matched circle. 34
Gradings	
$[a]$	Homology class in $H_1(Z', \mathbf{a})$ of algebra element a . 40
$\bar{\alpha}$	Element of $H_1(F(\mathcal{Z}))$ associated to appropriate element of $H_1(Z', \mathbf{a})$. 42
A_g	Part of A in grading g . 25
$\text{div}(c_1(\mathfrak{s}))$	Divisibility of $c_1(\mathfrak{s})$. 191
∂	Connecting homomorphism $H_1(Z', \mathbf{a}) \rightarrow H_0(\mathbf{a})$. 39
$\epsilon(\alpha)$	One quarter the number of parity changes in α . 39, 179
G	A grading group. 25
$G'(n)$	Big grading group on n positions; $\mathbb{Z} \hookrightarrow G'(n) \twoheadrightarrow H_1(Z', \mathbf{a})$. 39, 179
$G(\mathcal{Z})$	Small grading group for \mathcal{Z} ; $\mathbb{Z} \hookrightarrow G(n) \twoheadrightarrow H_1(F(\mathcal{Z}))$. 41, 188
$g'(B)$	$(-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), \partial^{\partial}(B))$. Also given by $\lambda^{\ell-\text{ind}(B, \vec{\rho})} \prod_{i=1}^{\ell} \text{gr}'(a(\rho_i))$ if $(B, \vec{\rho})$ is compatible. 180, 182
(G, λ)	Group with distinguished central element. 25
gr	Grading on group-graded differential algebra or module.
$\underline{\text{gr}}$	$G(\mathcal{Z})$ -grading on $\mathcal{A}(\mathcal{Z})$; or $G(\mathcal{Z})$ -set grading on $\widehat{\text{CFA}}$ or $\widehat{\text{CFD}}$. 42, 189, 190
gr'	Grading in $G'(n)$ of element of $\mathcal{A}(\mathcal{Z})$; or $G'(4k)$ -set grading on $\widehat{\text{CFA}}$ or $\widehat{\text{CFD}}$. 41, 180, 184, 187
$\underline{\text{gr}'}$	$G'(4k)$ -set grading on $\widehat{\text{CFA}}$ or $\widehat{\text{CFD}}$. 183, 187
$\text{gr}'(\rho)$	$\prod_{i=1}^n \text{gr}'(a(\rho_i))$. 180
$\text{gr}'(\vec{\rho})$	$\prod_{i=1}^n \text{gr}'(a(\rho_i))$. 180
$\underline{\text{gr}}$	$G(\mathcal{Z})$ -set grading on $\widehat{\text{CFA}}$ or $\widehat{\text{CFD}}$. 189, 190
$\text{gr}(m \otimes n)$	Grading on tensor product of modules. 26
gr^{\boxtimes}	Grading on $\widehat{\text{CFA}} \boxtimes \widehat{\text{CFD}}$. 191
$g(B)$	$\psi(I_A(\mathbf{x}))g'(B)\psi(I_A(\mathbf{y}))^{-1}$. 189
G_s	Stabilizer of s . 25
$G_s \backslash G$	Right cosets of G_s . 25
$\iota(a)$	Maslov component of grading of algebra element a . 40, 94, 180
$\iota(\rho)$	Shorthand for $\iota(a(\rho))$. Given explicitly by $\sum_{\{\rho_1, \rho_2\} \subset \rho} L(\rho_1, \rho_2) - \frac{ \rho }{2}$. 94, 180
$\iota(\vec{\rho})$	The Maslov component of $\text{gr}'(a(\rho_1)) \cdots \text{gr}'(a(\rho_\ell))$, i.e., $\sum_i \iota(\rho_i) + \sum_{i < j} L(\rho_i, \rho_j)$. 94, 180
(j, α)	Element of $G'(n)$; $j \in \frac{1}{2}\mathbb{Z}$ is the Maslov component; $\alpha \in H_1(Z, \mathbf{a})$ is the spin ^c component. 39, 179 = $(j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$. 179
λ	Distinguished central element of a grading group or, in particular, of $G'(n)$ or $G'(\mathcal{Z})$. 25, 39
$<$	Partial order on a grading group or set given by $a < b$ if $b = \lambda^n a$ for some $n \in \mathbb{N}$. 26, 41
$L(\alpha_1, \alpha_2)$	$m(\alpha_2, \partial\alpha_1)$. 39, 94, 179
$L(\rho_1, \rho_2)$	$L([\rho_1], [\rho_2])$. 94
$L(\rho_1, \rho_2)$	$L([\rho_1], [\rho_2])$. 94
$m(\alpha, p)$	Multiplicity of p in α . 39
$\Pi(\gamma)$	Projection $G/R(P_2(\mathbf{x}_2)) \rightarrow \{\mathfrak{s} \in \text{spin}^c(Y) \mid \mathfrak{s} _{Y_i} \cong \mathfrak{s}_i, i = 1, 2\}$ gotten by modding-out \mathbb{Z} -action. 191
$P(\mathbf{x})$	Subgroup of $G(\mathcal{Z})$ given by $\{g(B) \mid B \in \pi_2(\mathbf{x}, \mathbf{x})\}$. 189
$P'(\mathbf{x})$	Subgroup of $G'(4k)$ given by $\{g'(B) \mid B \in \pi_2(\mathbf{x}, \mathbf{x})\}$. 182
$S \times_G T$	Fiber product of G -sets. 26
$\psi(\mathbf{s}) = \psi(I(\mathbf{s}))$	Elements of $G'(4k)$ used to refine $G'(4k)$ -grading to $G(\mathcal{Z})$ -grading. 42, 188
r	Identity map $Z \rightarrow -Z$ (orientation-reversing). 186



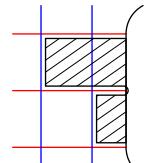
R	Map $G'(4k) \rightarrow G'(4k)$ or $G(\mathcal{Z}) \rightarrow G(-\mathcal{Z})$ given by $R(j, \alpha) = (j, r_*(\alpha))$. 186
S'_A	$G'(4k)$ -grading set for $\widehat{\text{CFA}}$. 184
S_A	$G(\mathcal{Z})$ -grading set for $\widehat{\text{CFA}}$. 189
S_D	$G(\mathcal{Z})$ -grading set for $\widehat{\text{CFD}}$. 190
$V[n]$	Shift functor applied to V (\mathbb{Z} -graded or G -graded sense). 9, 25
Heegaard diagrams	
$\overline{\alpha}$	Attaching circles and arcs in $\overline{\Sigma}$ for a bordered Heegaard diagram corresponding to index 1 critical points. 46
α	Curves in non-compact surface Σ corresponding to $\overline{\alpha}$. 51
$\overline{\alpha}_i^a$	The i^{th} curve in $\overline{\alpha}^a$. 46
$\overline{\alpha}^a$	Attaching arcs among $\overline{\alpha}$. 46
α^a	Curves in Σ corresponding to $\overline{\alpha}^a$. 51, 62
α^c	Attaching circles among $\overline{\alpha}$. 46
α_i^c	The i^{th} curve in α^c . 46
$a_{SW}, a_{SE}, a_{NW}, a_{NE}$	Small translates in $\Sigma \setminus (\alpha \cup \beta)$ of $a \in \alpha \cap \beta$. 97
$a \cdot b$	Intersection number or jittered intersection number of a and b . 97
B	A homology class in $\pi_2(\mathbf{x}, \mathbf{y})$. 53, 59
$\partial^\alpha(B)$	Portion of the boundary of B contained in $\overline{\alpha}$. 54
$\partial^\beta(B)$	Portion of the boundary of B contained in β . 54
$\partial^\partial(B)$	Portion of the boundary of B contained in $\partial\overline{\Sigma}$. 54
β	Attaching circles in a closed or bordered Heegaard diagram corresponding to index 2 critical points. 45, 46
β_i	The i^{th} curve in β . 46
$[B]$	Element of $H_2(Y)$ represented by $B \in \pi_2(\mathbf{x}, \mathbf{x})$. 54
*	Concatenation of homology classes corresponding to addition of domains. 53, 54
$\partial\mathcal{H}$	Pointed matched circle appearing at the boundary of a bordered Heegaard diagram \mathcal{H} (Lemma 4.4). 46
D_z	Region (component of $\overline{\Sigma} \setminus (\overline{\alpha} \cup \beta)$) adjacent to $z \in \partial\overline{\Sigma}$. 53, 149
e	Formal variable used in twisted coefficients. 134
f	A Morse function. 34, 47
g	A Riemannian metric. 34, 47
$\mathfrak{S}(\mathcal{H})$	The set of generators for the Heegaard diagram \mathcal{H} . 52
$\mathfrak{S}(\mathcal{H}, \mathfrak{s})$	The set of generators for the Heegaard diagram \mathcal{H} in the spin ^c -structure \mathfrak{s} . 55, 134
$\mathfrak{S}(\Sigma, \alpha, \beta)$	Same as $\mathfrak{S}(\mathcal{H})$ for $\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \beta, z)$. 52
$\underline{\mathfrak{S}}(\mathcal{H}, \mathbf{x}_0)$	The set of generators for the twisted invariants for the Heegaard diagram \mathcal{H} based at the generator \mathbf{x}_0 . 134, 147
\cup_∂	Gluing 3-manifolds, Heegaard diagrams, etc. along their boundaries. 57
\natural	Operation of gluing homology classes; also used for pregluing holomorphic curves. 58
\mathcal{H}	A bordered Heegaard diagram or pointed Heegaard diagram. 45, 46
$n_x(B)$	Average multiplicity of B near x . 94
$n_{\mathbf{x}}(B)$	$\sum_{x_i \in \mathbf{x}} n_{x_i}(B)$. 94
$o(\mathbf{x})$	Set of occupied α -arcs in the generator \mathbf{x} . 52, 189
$o(\mathbf{s}, \vec{\rho})$	Multi-set of α -arcs obtained from multi-set \mathbf{s} by deleting initial points of $\vec{\rho}$, adding terminal points of $\vec{\rho}$. 88
$o(\mathbf{s}, \vec{\rho}_{[1,i]})$	Same as $o(\mathbf{s}, \vec{\rho})$ but for initial segment (ρ_1, \dots, ρ_i) of $\vec{\rho}$. 88
$\pi_2(\mathbf{x}, \mathbf{y})$	Set of homology classes from \mathbf{x} to \mathbf{y} (with local multiplicity 0 at z). 53
$\widetilde{\pi}_2(\mathbf{x}, \mathbf{y})$	Set of homology classes from \mathbf{x} to \mathbf{y} with possibly non-zero local multiplicity at z ; $\widetilde{\pi}_2(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \times \pi_2(\mathbf{x}, \mathbf{y})$. 180, 200
$\pi_2^\partial(\mathbf{x}, \mathbf{y})$	Set of provincial homology classes from \mathbf{x} to \mathbf{y} . 54
$\overline{\Sigma}$	Compact Heegaard surface with one boundary component. 46
Σ	Result of attaching a cylindrical end to $\overline{\Sigma}$. 51, 62
$\Sigma_{\bar{e}}$	Closed Riemann surface obtained by filling puncture of Σ . 62, 78



$\epsilon(\mathbf{x}, \mathbf{y})$	Obstruction to connecting \mathbf{x} and \mathbf{y} by a homology class; equal to $\mathfrak{s}_z(\mathbf{y}) - \mathfrak{s}_z(\mathbf{x})$. 55, 193
$\mathfrak{s}_z^{\text{rel}}(\mathbf{x})$	Relative spin ^c -structure on $(Y, \partial Y)$ induced by the generator \mathbf{x} and basepoint z . 55
$\mathfrak{s}_z(\mathbf{x})$	spin ^c -structure on Y induced by the generator \mathbf{x} and basepoint z . 55
\mathbf{x}	A generator, a set of g points in $\alpha \cap \beta$, one on each circle and at most one on each arc. 52
x_i	An element of the generator \mathbf{x} . 52
$X(\mathcal{H})$	\mathbb{F}_2 -vector space spanned by $\mathfrak{S}(\mathcal{H})$. 103
$X(\mathcal{H}, \mathfrak{s})$	\mathbb{F}_2 -vector space spanned by $\mathfrak{S}(\mathcal{H}, \mathfrak{s})$. 103
$\underline{X}(\mathcal{H}, \mathbf{x}_0)$	\mathbb{F}_2 -vector space spanned by $\underline{\mathfrak{S}}(\mathcal{H}, \mathbf{x}_0)$. 134
(Y, \mathcal{Z}, ϕ)	A bordered 3-manifold. 46, 241
$Y(\mathcal{H})$	The bordered 3-manifold specified by a bordered Heegaard diagram \mathcal{H} . 47
z	Basepoint on $\partial\bar{\Sigma}$ for a bordered Heegaard diagram or basepoint in Σ for a closed Heegaard diagram. 45, 46

Moduli spaces

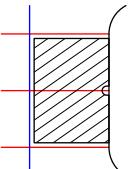
$\chi_{\text{emb}}(B, \vec{\rho})$	Euler characteristic of embedded curve in homology class B with asymptotics $\vec{\rho}$, given by $\chi_{\text{emb}}(B, \vec{\rho}) = g + e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B) - \iota(\vec{\rho})$. 95
Δ_P	Partial diagonal $\{x_p = x_q \mid P_i \in P, p, q \in P_i\}$ associated to partition P . 64
$e(B)$	Euler measure of domain B . 65
ev	Evaluation map $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \rightarrow \mathbb{R}^E/\mathbb{R}$. 64, 67
ev_q	Evaluation map at the puncture q , a map $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P}) \rightarrow \mathbb{R}$ or $\widetilde{\mathcal{N}}^B(T^\diamondsuit) \rightarrow \mathbb{R}$. 64, 67, 68
ev_e	Evaluation map at the east punctures of T^\diamondsuit , a map $\widetilde{\mathcal{N}}^B(T^\diamondsuit) \rightarrow \mathbb{R}^{E(T^\diamondsuit)}$. 68
$ev_{p,q}$	Difference $ev_p - ev_q$. 67
$\overline{ev}_{p,q}$	Extension of $ev_{p,q}$ to $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$. 73
ev_w	Evaluation map at the west punctures of T^\diamondsuit , a map $\widetilde{\mathcal{N}}^B(T^\diamondsuit) \rightarrow \mathbb{R}^{W(T^\diamondsuit)}$. 68
$\text{ind}(B, \vec{\rho})$	Expected dimension of moduli space of embedded curves in homology class B with asymptotics $\vec{\rho}$; $\text{ind}(B, \vec{\rho}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + \vec{\rho} + \iota(\vec{\rho})$. 95, 180
$\text{ind}(B, S^\triangleright, P)$	Expected dimension of $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. 65
$e\infty$	East infinity: end of $\Sigma \times [0, 1] \times \mathbb{R}$ at the puncture of Σ . 59
$-\infty$	End as $t \rightarrow -\infty$ of $\Sigma \times [0, 1] \times \mathbb{R}$. 59
$+\infty$	End as $t \rightarrow \infty$ of $\Sigma \times [0, 1] \times \mathbb{R}$. 59
$j_{\mathbb{D}}$	Standard complex structure on \mathbb{D} . 62
j_Σ	Complex structure on Σ (fixed). 62
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$	Open moduli space of embedded curves in homology class B with asymptotics \mathbf{x}, \mathbf{y} and $\vec{\rho}$; counts of these moduli spaces are used to define \widehat{CFD} and \widehat{CFA} . 95
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; \vec{\rho})$	Compactified moduli space of embedded curves in homology class B with asymptotics \mathbf{x}, \mathbf{y} and $\vec{\rho}$. 95
$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$	Moduli space of curves with decorated source S^\triangleright asymptotic to \mathbf{x} and \mathbf{y} . 63
$\overline{\widetilde{\mathcal{M}}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$	Space of holomorphic combs with preglued source S^\triangleright connecting \mathbf{x} to \mathbf{y} in homology class B . Naïve compactification of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$. 73
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$	Closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright)$. 73
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$	Moduli space of curves with decorated source S^\triangleright asymptotic to \mathbf{x} at $-\infty$ and \mathbf{y} at $+\infty$, with partition of $e\infty$ punctures P . 64
$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$	Quotient $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)/\mathbb{R}$. 66
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$	Space of holomorphic combs with preglued source S^\triangleright connecting \mathbf{x} to \mathbf{y} in homology class B , respecting partition P of east punctures. Naïve compactification of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. 73
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$	Closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$. 73



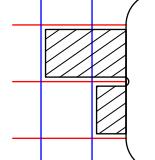
$\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$	Moduli space of curves with decorated source S^\triangleright asymptotic to \mathbf{x} at $-\infty$ and \mathbf{y} at $+\infty$, with ordered partition \vec{P} of $e\infty$ punctures. 59, 66
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$	Quotient $\widetilde{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})/\mathbb{R}$. 59, 66
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$	Compactification of the space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$, via holomorphic combs; the closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; \vec{P})$. 60, 73
$\widetilde{\mathcal{N}}(T^\diamond)$	Moduli space of curves in $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ with bi-decorated source T^\diamond . 67
$\mathcal{N}(T^\diamond)$	Quotient $\widetilde{\mathcal{N}}(T^\diamond)/(\mathbb{R} \times \mathbb{R})$. 68
$\widetilde{\mathcal{N}}(T^\diamond; P_w, P_e)$	Subspace of $\widetilde{\mathcal{N}}(T^\diamond)$ with partitions P_w / P_e of west / east punctures. 68
$\widetilde{\mathcal{N}}(T^\diamond; P_e)$	Special case of $\widetilde{\mathcal{N}}(T^\diamond; P_w, P_e)$ where P_w is the discrete partition. 68
p	Puncture of Σ corresponding to $\partial\bar{\Sigma}$. 62
ϕ	Identification of neighborhood U_p of p with $S^1 \times (0, \infty)$. 62
U_p	Neighborhood of puncture p of Σ . 62
P	A partition of the e punctures of a decorated source S^\triangleright . 64
$[P]$	Multi-sets of Reeb chords associated to partition P of east punctures. 66
$ P $	Number of parts of partition P . 65
\vec{P}	An ordered partition of $e\infty$ punctures. 59, 66
$[\vec{P}]$	Sequence of multi-sets of Reeb chords associated to ordered partition \vec{P} of east punctures. 66
P_i	A set of $e\infty$ punctures; a part in a partition or ordered partition. 59, 64
P_e	A partition of the e punctures of a bi-decorated source T^\diamond . 68
P_w	A partition of the w punctures of a bi-decorated source T^\diamond . 68
$\pi_{\mathbb{D}}$	Projection $\Sigma \times [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$. 62
π_Σ	Projection $\Sigma \times [0, 1] \times \mathbb{R} \rightarrow \Sigma$. 62
\natural	Pregluing of holomorphic curves; also used for gluing homology classes. 73, 154
\mathbb{R}^E/\mathbb{R}	Quotient of \mathbb{R}^E by simultaneous translation; target space for evaluation maps. 67
ρ_i	A Reeb chord. 59
s	Coordinate on $[0, 1]$. 62
S^\triangleright	A decorated source. 59, 62
$E(S^\triangleright)$	Set of east punctures of S^\triangleright . 64
E	Same as $E(S^\triangleright)$. Used when S^\triangleright is clear from context. 64
S	Riemann surface underlying a decorated source S^\triangleright . 59
T^\diamond	A bi-decorated source. 67
$E(T^\diamond)$	Set of east punctures of T^\diamond . 68
E	Same as $E(T^\diamond)$. Used when T^\diamond is clear from context. 68
T	Riemann surface underlying a bi-decorated source T^\diamond . 67
$W(T^\diamond)$	Set of west punctures of T^\diamond . 68
W	Same as $W(T^\diamond)$. Used when T^\diamond is clear from context. 68
$\omega_{\mathbb{D}}$	Standard symplectic form on \mathbb{D} . 62
ω_Σ	Symplectic form on Σ (fixed). 62
t	Coordinate on \mathbb{R} . 59, 62
t_i	Height (t -coordinate) of Reeb chord ρ_i . 59
τ_R	Translation of $\Sigma \times [0, 1] \times \mathbb{R}$ by $R \in \mathbb{R}$. 66
u	A holomorphic curve in $\Sigma \times [0, 1] \times \mathbb{R}$. 63
$[u]$	Homology class of u . 63
U	A holomorphic comb. 71
(u, v)	A simple holomorphic comb. 71
v	A holomorphic curve in $Z \times \mathbb{R} \times [0, 1] \times \mathbb{R}$. 67

Type D modules

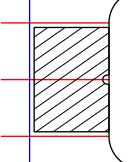
$a_{\mathbf{x}, \mathbf{y}}^B$	Coefficient of \mathbf{y} in ∂x coming from homology class B . 104
$a_{\mathbf{x}, \mathbf{y}}$	Coefficient of \mathbf{y} in ∂x ; the sum over B of $a_{\mathbf{x}, \mathbf{y}}^B$. 107
$\widehat{CFD}(Y)$	Type D invariant $\widehat{CFD}(\mathcal{H})$ associated to any bordered Heegaard diagram \mathcal{H} . 134



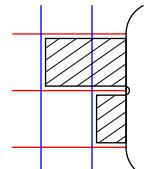
$\widehat{CFD}(Y, \mathfrak{s})$	Summand of $\widehat{CFD}(Y)$ corresponding to spin ^c -structure \mathfrak{s} . 134
$\widehat{CFD}(\mathcal{H})$	Type D invariant associated in Section 6.1 to bordered Heegaard diagram \mathcal{H} . 104
$\widehat{CFD}(\mathcal{H}, \mathfrak{s})$	Summand of $\widehat{CFD}(\mathcal{H})$ corresponding to spin ^c -structure \mathfrak{s} . 103
$\widehat{CFD}(\mathcal{H}, \mathbf{x}_0)$	Totally twisted variant of $\widehat{CFD}(\mathcal{H})$, with base generator \mathbf{x}_0 . 134
$I_D(\mathbf{x})$	Idempotent for \widehat{CFD} for generator \mathbf{x} ; $I_D(\mathbf{x}) = I([2k] \setminus o(\mathbf{x}))$. 103, 189
Invariance of \widehat{CFD}	
α^H	α -curves gotten by handlesliding α_1^a over α_1^c and performing an appropriate isotopy. 113
\mathbf{a}^H	$\alpha^H \cap \partial\bar{\Sigma}$. 113
$\alpha_i^{a,H}$	i^{th} arc in α^H ; for $i \geq 2$ a small perturbation of α_i^a intersecting α_i^a in one point; for $i = 1$, a handleslide of α_1^a over α_1^c , appropriately perturbed. 113
$\alpha_i^{c,H}$	i^{th} circle in α^H ; a small perturbation of α_i^c intersecting α_i^c in two points. 113
α'	Appropriately chosen small perturbation of α . 130
$\alpha_i^{a'}$	i^{th} arc in α' , a small deformation of α_i^a . 130
$\alpha_i^{c'}$	i^{th} circle in α' , a small deformation of α_i^c . 130
$b(B)$	Domain in $\pi_2(\mathbf{x}', \mathbf{y})$ (in $(\Sigma, \alpha^H, \beta)$) corresponding to domain $B \in \pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$, as in Lemma 6.24. 116
$\widehat{CFD}(\mathcal{H}, \mathfrak{s}; J)$	Module $\widehat{CFD}(\mathcal{H}, \mathfrak{s})$ computed with respect to the almost complex structure J . 110
Δ	A triangle (disk with three boundary punctures). 113
e_1, e_2, e_3	Edges of Δ . 113
v_{12}, v_{23}, v_{13}	Vertices of Δ ; v_{ij} is adjacent to e_i and e_j . 113
F^{J_r}	Continuation map associated to path J_r of almost complex structures. 110, 111
$F_{\alpha, \alpha^H, \beta}$	Continuation map $\widehat{CFD}(\Sigma, \alpha^H, \beta, z) \rightarrow \widehat{CFD}(\Sigma, \alpha, \beta, z)$ for a handleslide gotten by counting triangles. 113, 120
$\text{ind}(B, T^\Delta, P, P^H)$	Expected dimension of $\widehat{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$. 115
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; \vec{\rho}; J)$	Open moduli space of embedded curves with respect to t -dependent complex structure J in homology class B with asymptotics \mathbf{x}, \mathbf{y} and $\vec{\rho}$. 111
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P; J)$	Moduli space of curves for t -dependent almost complex structure J with decorated source S^\triangleright asymptotic to \mathbf{x} at $-\infty$ and \mathbf{y} at $+\infty$, with partition of $e\infty$ punctures P . 111
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$	Moduli space of holomorphic triangles with decorated source T^Δ asymptotic to \mathbf{x} at v_{31} , \mathbf{y} at v_{23} and Θ_o at v_{12} . 115
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho})$	Moduli space of embedded holomorphic triangles asymptotic to \mathbf{x} at v_{31} , \mathbf{y} at v_{23} , some Θ_o at v_{12} and $\vec{\rho}$ along e_1 and e_2 . 120
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; \vec{\rho}, \vec{\rho}^H)$	Moduli space of embedded holomorphic triangles asymptotic to partially-diagonal generator w_{pd}^- at v_{12} , $\vec{\rho}$ along e_1 and $\vec{\rho}^H$ along e_2 . 123
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; \vec{\rho}_1, \vec{\rho}_2)$	Moduli space of embedded holomorphic triangles asymptotic to \mathbf{x} at v_{31} , \mathbf{y} at v_{23} , Θ_o at v_{12} , $\vec{\rho}_1$ along e_1 and $\vec{\rho}_2$ along e_2 . 120
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$	Subspace of $\widehat{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta)$ with ordered partition of east punctures \vec{P} along e_1 and \vec{P}^H along e_2 . 115
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$	Naive compactification of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ via combs. 115
$\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \mathbf{w}_{\text{pd}}^-; T^\Delta; \vec{P}, \vec{P}^H)$	Moduli space of triangles with source T^Δ asymptotic to partially diagonal generator \mathbf{w}_{pd}^- at v_{12} . 123
$\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$	Closure of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$ in $\overline{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}, \Theta_o; T^\Delta; \vec{P}, \vec{P}^H)$. 115
$\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+)$	Moduli space of holomorphic curves in $(\Sigma, \alpha^H, \alpha)$. 122
$\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+; S^\triangleright)$	Same as $\mathcal{M}^B(\Theta_o, \mathbf{w}_{\text{pd}}^+)$; used when we want to name the source. 122
μ	Matching between partially diagonal generators. 123
$\pi_2(\mathbf{x}, \mathbf{y}, \Theta_o)$	Set of homology classes for $(\Sigma, \alpha, \alpha^H, \beta)$ connecting \mathbf{x} , \mathbf{y} , and Θ_o . 115
$\rho_\alpha(\mathbf{w}_{\text{pd}}^+, \mathbf{w}_{\text{pd}}^-)$	Sequence of α -Reeb chords associated to partially diagonal generators $\mathbf{w}_{\text{pd}}^\pm$ and matching μ . 123



$\rho_{\alpha^H}(\mathbf{w}_{\text{pd}}^+, \mathbf{w}_{\text{pd}}^-)$	Sequence of α^H -Reeb chords associated to partially diagonal generators $\mathbf{w}_{\text{pd}}^\pm$ and matching μ . 123
$*_{ij}$	Operation of attaching a homology class at corner v_{ij} of Δ . 116
Θ_o	Top-graded generator for $(\Sigma, \alpha^H, \alpha)$ with idempotent $I_D(o)$. 115
T_x	Canonical small triangle homology class in $\pi_2(\mathbf{x}, \mathbf{x}', \Theta_{o(\mathbf{x})})$. 116
T^Δ	Decorated source for maps to $\Sigma \times \Delta$. 114
w_{pd}^\pm	Positive / negative partially-diagonal generators. 122
\mathbf{x}'	Nearest generator in $\mathfrak{S}(\alpha^H, \beta)$ to $\mathbf{x} \in \mathfrak{S}(\alpha, \beta)$. 115
Type A modules	
$\partial(x)$	Alternate notation for $m_1(\mathbf{x})$. 138
$\widehat{\mathcal{CFA}}(\mathcal{H})$	Type A invariant associated in Section 7.1 to bordered Heegaard diagram \mathcal{H} . 138
$\widehat{\mathcal{CFA}}(\mathcal{H}, \mathfrak{s})$	Summand of $\widehat{\mathcal{CFA}}(\mathcal{H})$ corresponding to spin c -structure \mathfrak{s} . 137, 138
$\widehat{\mathcal{CFA}}(\mathcal{H}, \mathbf{x}_0)$	Totally twisted variant of $\widehat{\mathcal{CFA}}(\mathcal{H})$, with base generator \mathbf{x}_0 . 147
$f_n^{J_r}$	\mathcal{A}_∞ chain homotopy equivalence on $\widehat{\mathcal{CFA}}$ associated to change in almost complex structure. 143
$I_A(\mathbf{x})$	Idempotent for $\widehat{\mathcal{CFA}}$ for generator \mathbf{x} ; $I_A(\mathbf{x}) = I(o(\mathbf{x}))$. 137, 189
m_{n+1}	\mathcal{A}_∞ products on $\widehat{\mathcal{CFA}}$. 138
$\mathbf{x} \cdot a$	Alternate notation for $m_2(\mathbf{x}, a)$. 138
The pairing theorem	
$\mathcal{H}_i = (\Sigma_i, \alpha_i, \beta_i, z)$	Provincially admissible Heegaard diagrams with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$. 154
Σ, α, β	$\overline{\Sigma}_1 \cup_\partial \overline{\Sigma}_2$, $\overline{\alpha}_1 \cup_\partial \overline{\alpha}_2$ and $\beta_1 \cup \beta_2$, respectively. 154
$\partial_T(\mathbf{x}_1 \times \mathbf{x}_2)$	T -deformed differential, on $\widehat{\mathcal{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$. 157
$\widehat{\mathcal{CF}}(T; \mathcal{H}_1, \mathcal{H}_2)$	Analogue of $\widehat{\mathcal{CF}}(\mathcal{H}_1 \cup_\partial \mathcal{H}_2)$ computed using T -matched holomorphic curves. 157
S^\diamond	The pregluing $S_1^\diamond \natural S_2^\diamond$ of sources S_i^\diamond for curves in \mathcal{H}_i . 154
F_{T_1, T_2}	Chain homotopy equivalence $\widehat{\mathcal{CF}}(T_1; \mathcal{H}_1, \mathcal{H}_2) \rightarrow \widehat{\mathcal{CF}}(T_2; \mathcal{H}_1, \mathcal{H}_2)$ from counting ψ -matched curves. 161
$\text{ind}(B_1, S_1^\diamond; B_2, S_2^\diamond)$	Expected dimension of $\widetilde{\mathcal{MM}}_0^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$. 154
$\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Matched moduli space, $\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\diamond) \times_{\text{ev}_1=\text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\diamond)$. 154
$\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Quotient $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)/\mathbb{R}$. 155
$\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Compactification of $\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$. 158
$\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Embedded matched moduli space, union of matched moduli spaces with $\chi(S_1^\diamond \natural S_2^\diamond) = \chi_{\text{emb}}(B)$. 156
$\mathcal{MM}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Quotient $\widetilde{\mathcal{MM}}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)/\mathbb{R}$. 156
$\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	T -matched moduli space, the fibered product $\widetilde{\mathcal{M}}^{B_1}(\mathbf{x}_1, \mathbf{y}_1; S_1^\diamond) \times_{T \text{ ev}_1=\text{ev}_2} \widetilde{\mathcal{M}}^{B_2}(\mathbf{x}_2, \mathbf{y}_2; S_2^\diamond)$. 157
$\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Quotient $\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)/\mathbb{R}$. 157
$\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Embedded T -matched moduli space, union of T -matched moduli spaces with $\chi(S_1^\diamond \natural S_2^\diamond) = \chi_{\text{emb}}(B)$. 157
$\mathcal{MM}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Quotient $\widetilde{\mathcal{MM}}^B(T; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)/\mathbb{R}$. 157
$\mathcal{MM}^B(\geq T_0; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Union of T -matched moduli spaces for $T \geq T_0$. 171
$\mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2, S_2^\diamond)$	Moduli space of ψ -matched pairs. 161
$\mathcal{MM}^B(\psi; \mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Union of ψ -matched moduli spaces with $\chi(S_1^\diamond \natural S_2^\diamond) = \chi_{\text{emb}}(B)$. 161
$\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$	Moduli space of embedded trimmed simple ideal-matched curves, given by the union $\bigcup_{\{S_1^\diamond, S_2^\diamond \chi(S_1^\diamond \natural S_2^\diamond) = \chi_{\text{emb}}(B)\}} \mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2; S_2^\diamond)$. 169
$\mathcal{MM}_{tsic}^B(\mathbf{x}_1, \mathbf{y}_1, S_1^\diamond; \mathbf{x}_2, \mathbf{y}_2; S_2^\diamond)$	Moduli space of trimmed simple ideal-matched curves (modulo translation). 168
φ	Correspondence between the punctures of U_1 and U_2 in an ideal-matched holomorphic comb. 162



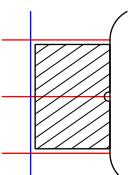
T	Dilation parameter, a positive real number. 157
$T(U)$	Time parameter space of a holomorphic comb U . 162
$\psi(t)$	Dilation function interpolating between T_1 and T_2 . Also used for grading refinement data. 161
Manifolds with torus boundary	
A	Alexander depth (filtration). 200
$\tilde{G}(\mathcal{Z})$	$G(\mathcal{Z}) \times \mathbb{Z}$; the extra \mathbb{Z} is the Alexander factor. 204
\tilde{S}_D	\tilde{G} -set for $CFD^-(\mathcal{H}, z, w)$ given by $\tilde{G}/\tilde{P}(\mathbf{x}_0)$. 204
$\tilde{P}(\mathbf{x}_0)$	$\tilde{g}(\pi_2(\mathbf{x}_0, \mathbf{x}_0)) \subset \tilde{G}$. 204
\tilde{g}	Enhanced grading on CFD in presence of knot given by $\tilde{g}(B) \cdot \tilde{P}(\mathbf{x}_0)$ for $B \in \pi_2(\mathbf{x}, \mathbf{x}_0)$. 204
\tilde{g}	Enhanced grading of domains in presence of knot, $(g(B), n_w(B) - n_z(B)) \in \tilde{G}$. 204
$\mathcal{A}(\mathbb{T})$	Algebra $\mathcal{A}(\mathbb{T}, 0)$ associated to the unique pointed matched circle for a torus in the central ($i = 0$) spin ^c structure. 195
α_1^a, α_2^a	α -arcs in a Heegaard diagram with torus boundary, labeled as in Section 11.1. 196
C^h	Horizontal complex of a \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$. 207
C^v	Vertical complex of a \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$. 207
D_I	Coefficient map for type D structure over $\mathcal{A}(\mathbb{T})$ corresponding to $I \subset \{1, 2, 3\}$; or generalized coefficient maps for $I \subset \{0, 1, 2, 3\}$ a cyclic interval. 197, 208, 211, 215
\mathfrak{S}_K	Set of generators for $\widehat{CFK}(\mathcal{H}_K)$. 219
$\mathfrak{S}(n)$	Set of generators for $\widehat{CFD}(\mathcal{H}(n))$. 219
G'	$G'(4)$, the unrefined grading group for the torus algebra. 196
G	$G(\mathbb{T})$, the refined grading group for the torus algebra. 197
$(m; a, b, c)$	Element of G' . 196
$(m; a, b)$	Element of G . 197
$(\Sigma_0, \alpha_0, \beta_0, z, w)$	Doubly-pointed Heegaard diagram for $K \subset S^3$. 217
$\mathcal{H}_K := (\Sigma, \alpha, \beta, z, w)$	Doubly-pointed Heegaard diagram for K from stabilizing $(\Sigma_0, \alpha_0, \beta_0, z, w)$. 217
α_g	New α -circle in \mathcal{H}_K , contained in the new handle. 217
β_g	New β -circle in \mathcal{H}_K meeting α_g in a single point. 217
$\mathcal{H}(n) := (\Sigma, \alpha_1^a, \alpha_2^a, \alpha_1, \dots, \alpha_{g-1}, \beta)$	Bordered Heegaard diagram for $S^3 \setminus \text{nbd}(K)$ with framing n , coming from \mathcal{H}_K . 218
$\mathcal{H}_{-1}, \mathcal{H}_0, \mathcal{H}_\infty$	Standard Heegaard diagrams for -1 -, 0 - and ∞ -framed solid tori, respectively. 199
S	Function $\mathfrak{S}(\mathcal{H}) \rightarrow \frac{1}{2}\mathbb{Z}$ characterized by Equations (11.41) and (11.44). Also used for source of holomorphic curves. 218, 219
B_0	Generator for periodic domains in $\mathcal{H}(n)$, with local multiplicities $(0, 1, 1-n, -n)$ in regions $(0, 1, 2, 3)$. 218
H^h	Horizontal complex of a \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$. 207
H^v	Vertical complex of a \mathbb{Z} -filtered, \mathbb{Z} -graded chain complex over $\mathbb{F}_2[U]$. 207
ι_0, ι_1	Indecomposable idempotents in $\mathcal{A}(\mathbb{T})$, labeled so that ι_0 corresponds to α_1^a occupied on type A side (unoccupied on type D side). 196
$CFK^-(K)$	Knot Floer complex, a filtered complex generated over $\mathbb{F}_2[U]$ by \mathfrak{S}_K . 199
$gCFK^-(K)$	Associated graded complex to $CFK^-(K)$. 200
$\widehat{CFK}(K)$	$U = 0$ specialization of $CFK^-(K)$, a filtered complex generated over \mathbb{F}_2 by \mathfrak{S}_K . 200
$\widehat{gCFK}(K)$	Associated graded complex to $\widehat{CFK}(K)$. 200
C	Shorthand for $\widehat{gCFK}(K)$. 214
$C(r)$	Summand $\widehat{gCFK}(\mathcal{H}, r)$ of $\widehat{gCFK}(K)$ in Alexander grading r . 214
$C(\leq s)$	$\bigoplus_{r \leq s} C(r)$. 214
$C(\geq s)$	$\bigoplus_{r \geq s} C(r)$. 214
∂^i	Map $C(r) \rightarrow C(r+i)$ defined by counting holomorphic disks ϕ with $n_z(\phi) = -i$ and $n_w(\phi) = 0$ if $i \leq 0$, and $n_z(\phi) = 0$ and $n_w(\phi) = i$ if $i \geq 0$. 214
∂_w	$\sum_{i \geq 0} \partial^i$. 214



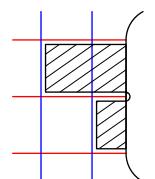
∂_z	$\sum_{i \leq 0} \partial^i$. 214
\mathfrak{S}_K	Generators for knot Floer complex. 199
$(\Sigma, \alpha, \beta, w, z)$	A doubly pointed Heegaard diagram for a knot in a 3-manifold. 199
$HFK^-(K)$	Knot Floer homology, homology of $gCFK^-(K)$. 200
$\widehat{HFK}(K)$	Homology of $\widehat{gCFK}(K)$. 201
$CFA^-(\mathcal{H}, z, w)$	Type A variant of bordered knot Floer complex. 203
$\widehat{CFA}(\mathcal{H}, z, w)$	$U = 0$ specialization of $CFA^-(\mathcal{H}, z, w)$. 203
$CFD^-(\mathcal{H}, z, w)$	Type D variant of bordered knot Floer complex. 203
$\widehat{CFD}(\mathcal{H}, z, w)$	$U = 0$ specialization of $CFD^-(\mathcal{H}, z, w)$. 203
M	Maslov grading on knot Floer complex. 200
N	Renormalized Maslov grading on knot Floer complex. $N = M - 2A$. 200
n	The framing parameter in Section 11.7, a large positive integer. 214
ϕ_k	Distinguished homology class in winding region connecting x_k to x_{k-1} with multiplicity +1 at regions 2 and 3. 221, 222
ψ_k	Distinguished homology class in winding region connecting x_{k-1} to x_k with multiplicity 0 at regions 2 and 3. 222
$\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$	Non-idempotent basic generators for $\mathcal{A}(\mathbb{T})$. 196
τ	The tau invariant from knot Floer homology. 201
U	Formal variable used in knot Floer homology. 199
$\tilde{\text{gr}}(Ux)$	$(\lambda^{-2}, -1) \cdot \tilde{\text{gr}}(x)$. 204
V_s^1	$C(\leq s + \frac{n-1}{2})$ if $s \leq -n/4$; \mathbb{F}_2 if $ s < n/4$; and $C(\geq s - \frac{n-1}{2})$ if $s \geq n/4$. 214
V^1	$\bigoplus_{s \in \mathbb{Z} + \frac{n+1}{2}} V_s^1$. 214
V_s^0	$C(s)$. 214
V^0	$\bigoplus_{s \in \mathbb{Z}} V_s^0$. 214
V	$V^0 \oplus V^1$. 214
W^i	Summand of $\widehat{CFD}(\mathcal{H})$ (for \mathcal{H} as in Section 11.7) corresponding to idempotent i . 217
$\bigoplus_{s \in \frac{1}{2}\mathbb{Z}} W_s^i$	Decomposition of W^i according to a grading function S . 218
\mathcal{W}	The winding region. 217
$\{x_i\}_{i=-\frac{n}{2}}^{i=\frac{n}{2}}$	Intersections between β_g and $\alpha_1^a \cup \alpha_2^a$, contained in winding region. 219
$\{\mathbf{x}_i\}_{i=-\frac{n}{2}}^{i=\frac{n}{2}}$	Generators of $\widehat{CFD}(\mathcal{H}(n))$ corresponding to generator $\mathbf{x} \in \mathfrak{S}_K$. 219

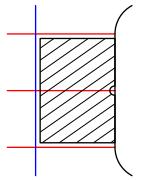
Bimodules and change of framing

$\mathcal{H} = (\overline{\Sigma}, \overline{\alpha}, \overline{\beta}, \mathbf{z})$	Arced bordered Heegaard diagram. Also used for other kinds of Heegaard diagrams. 242
∂_L, ∂_R	Left and right boundaries of $\overline{\Sigma}$, Y or \mathcal{H} . 242
${}_{\mathcal{A}(\mathcal{Z})} \widehat{CFDA}(\psi)_{\mathcal{A}(\mathcal{Z})}$	Type DA bimodule associated to mapping class ψ . 241
$m_{i,1,j}(a_1, \dots, a_i, \mathbf{x}, b_1, \dots, b_j)$	\mathcal{A}_∞ bimodule structure on \widehat{CFDA} . 244
${}_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})} \widehat{CFAA}(\mathbb{I})$	Type AA bimodule associated to the identity map of $F(\mathcal{Z})$; tensoring with this turns type D invariants into type A invariants. 242, 245
$m_{1,i,j}(\mathbf{x}; a_1, \dots, a_i, b_1, \dots, b_j)$	\mathcal{A}_∞ bimodule structure on $\widehat{CFAA}(\mathbb{I}_\mathcal{Z})$. 246
${}_{\mathcal{A}(\mathcal{Z}), \mathcal{A}(-\mathcal{Z})} \widehat{CFDD}(\mathbb{I})$	Type DD bimodule associated to the identity map of $F(\mathcal{Z})$; tensoring with this turns type A invariants into type D invariants. 242, 245
∂	Differential on $\widehat{CFDD}(\mathbb{I}_\mathcal{Z})$ given by Equation (A.4). 245
F_L, F_R	Surfaces $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$. 242
$\mathfrak{S}(\mathcal{H})$	The set of generators for the arced bordered Heegaard diagram \mathcal{H} . 244
$M_{R,S}$	Bimodule with commuting right actions of R and S . 242
${}_{R,S} M$	Bimodule with commuting left actions of R and S . 242
$o^L(\mathbf{x}), o^R(\mathbf{x})$	Set of α^L arcs and of α^R arcs occupied by \mathbf{x} , respectively. 244
ϕ_L, ϕ_R	Homeomorphisms $\phi_L: F(\mathcal{Z}_L) \rightarrow \partial_L Y$ and $\phi_R: F(\mathcal{Z}_R) \rightarrow \partial_R Y$ specified by a bordered Heegaard diagram for Y . 242
MCG_0	Strongly based mapping class group. 243



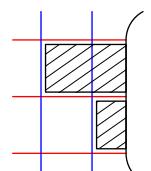
$\mathcal{Z}_L, \mathcal{Z}_R$	Left and right boundaries of \mathcal{H} , i.e., $\mathcal{Z}_L = \partial_L \mathcal{H}$ and $\mathcal{Z}_R = \partial_R \mathcal{H}$. 242
$\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$	Generators for type <i>DA</i> invariants of $\tau_\mu^{\pm 1}$, $\tau_\lambda^{\pm 1}$, with idempotents specified by Equation (A.7). 247
τ_λ	Dehn twist of the standard torus around its longitude. 247
τ_μ	Dehn twist of the standard torus around its meridian. 247



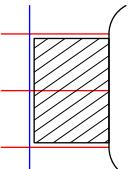


Index of Definitions

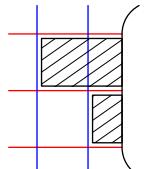
- \mathcal{A}_∞
 algebra, *see* algebra, \mathcal{A}_∞
 homomorphism, *see* homomorphism of \mathcal{A}_∞
 modules
 module, *see* module, \mathcal{A}_∞
 homotopy equivalent, 14
 tensor product, *see* tensor product, \mathcal{A}_∞
- \mathbb{R} -action
 on $\Sigma \times [0, 1] \times \mathbb{R}$, 62, 66
 on $\widetilde{\mathcal{M}}^B$, 66
 on matched pairs, 155, 157, 161, 169
- $\mathbb{R} \times \mathbb{R}$ -action, 67, 68
- λ energy, 75
 ω energy, 75
 (C-1)–(C-5), 72
 (J-1), 229
 (J1)–(J4), 62
 (M1)–(M11), 63
 (ME-1)–(ME-4), 92
 (MT-1)–(MT-9), 126
 (SB-1'), (SB-2'), 88
 (SB-1), (SB-2), (SB-3), 88
 (T-A)–(T-B), 167
 (T-B1)–(T-B7), 167
 (T-B6'), 167
 (c-1)–(c-2), 220
 (c-1)–(c-4), 121
 (i-1')–(i-2'), 172
 (i-1)–(i-6), 172
- abutting
 Reeb chords, 32
 sequence of Reeb chords, 169
- admissible
 complex structure, *see* complex structure,
 admissible
 Heegaard diagram, 56
 alternate characterization of, 56
 provincially, 56
- Alexander factor (of grading), 204
 Alexander filtration, 200, 206
 Alexander grading, *see* grading, Alexander
- algebra
 \mathcal{A}_∞ , 10
 operationally bounded, 11
 strictly unital, 10
 associated to pointed matched circle, 35
 grading on, *see* grading
 strands, *see* strands algebra
 almost complex structure, *see* complex structure
 amalgamation of chords, 169
 arced bordered Heegaard diagram, 242
 area function, 56
 arrow
 horizontal, 207
 length, 207
 vertical, 207
 arrows
 diagonal, 209
- bar resolution, 15
 basic generator, 36
 basis
 filtered, 206
 horizontally simplified, 207
 vertically simplified, 207
 becoming nodal, 84
 bordered
 3-manifold, 46
 cutting, 57
 equivalent, 46
 gluing, 57
 represented by Heegaard diagram, 47
 Heegaard diagram, *see* Heegaard diagram,
 bordered
 bottom puncture, 69
 boundary connect sum
 of bordered 3-manifolds, 52
 of bordered Heegaard diagrams, 52
 boundary degeneration, *see* degeneration,
 boundary, 211
 boundary double point, *see* degeneration,
 boundary double point
 boundary monotonicity



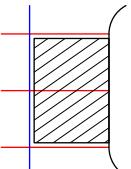
- strong, 63, 88
 for comb, 89
 for moduli space, 88
 for pair $(\mathbf{s}, \vec{\rho})$, 88
 for pair $(\mathbf{x}, \vec{\rho})$, 88
 weak, 63
- bounded
 \mathcal{A}_∞ module, 11, 138
 homomorphism of \mathcal{A}_∞ modules, 13
 operationally b. \mathcal{A}_∞ algebra, 11
 type D homomorphism, 19
 type D structure, 18
- box product, *see* tensor product, box
- Bruhat order, 31
- bubble, sphere or disk, 84
- building, *see* holomorphic building
- chord, *see* Reeb chords
- circle, matched, *see* matched circle
- coefficient maps, 197
 generalized, 211
- collision of levels, *see* end, collision of levels
- comb, *see* holomorphic comb
- compactified moduli space, *see* moduli space, compactified
- compatible pair
 $(B, \vec{\rho})$, 94
 of decorated sources, 154
 of generators, 58, 154
- complex
 \mathbb{Z} -graded, \mathbb{Z} -filtered over $\mathbb{F}_2[U]$, 206
 horizontal, 207
 reduced, 207
 vertical, 207
- complex structure
 admissible, 62
 on $\Sigma \times \Delta$, 114
 path of, 110
 on \mathbb{D} , 62
 on $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$, 67
 on Σ , 62
 on $\Sigma \times [0, 1] \times \mathbb{R}$, 62
 on $\Sigma_{\bar{e}}$, 62
 sufficiently generic, 65
 for path, 111
 sufficiently pinched, 220
- component
 generalized split, 81, 166
 ghost, 87
 join, 69
 shuffle, 69
 even, 70
 odd, 70
 split, 69
- composable
 sequence of sets of Reeb chords, 32
 sets of Reeb chords, 32
 weakly, 91
- composite of \mathcal{A}_∞ homomorphisms, 13
- concatenation, 53, 54
- consistent set of Reeb chords, 32
- convergence
 to holomorphic comb, 72
 to ideal-matched comb, 164
- curve
 generalized split, 81
 haunted, 87
 ideal-matched
 simple, 165
 trimmed simple, 167
- join, 60, 69
- shuffle, 60, 70
 even, 70, 82
 odd, 70, 82
 split, 60, 69
- cutting
 bordered 3-manifolds, 57
 bordered Heegaard diagrams, 57
- cylindrical end, 51, 62
- decorated source, *see* source, decorated
- degeneration of holomorphic curves, 60, 83, *see also* end
 becoming nodal, 84
 boundary, 87, 211
 does not occur, 87
- boundary double point, 87
 does not occur, 89
 splitting at $\pm\infty$, 83
 splitting at east ∞ , 83
- degree near p , odd, 80
- Dehn twists, 247
- destabilization, 49
- dg module, 12
- diagonal, partial, 64
- diagram, Heegaard, *see* Heegaard diagram
- differential
 on $\mathcal{A}(n, k)$, 30, 31
 on \widehat{CFA} , 138
 on \widehat{CFD} , 104
 on \boxtimes product, 21
- differential graded module, 12
- dimension, *see* expected dimension
- disk bubble, 84
- domain, 53
 of triangle, 115
 periodic, 53
 positive, 56



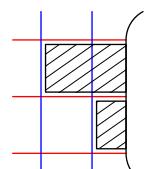
- provincial, 54
- triply-periodic, 116
- double point, boundary, *see* degeneration, boundary double point
- drop shadow, 84
- duality theorem, 242
- east infinity
 - bimodule case, 244
 - holomorphic curves at, 67
 - of $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$, 67
- east puncture, *see* puncture, east
- east-most level, 158
- embedded
 - Euler characteristic, 95
 - index, *see* index, embedded
 - moduli space, 95
- end, *see also* degeneration
 - collision of levels, 91
 - cylindrical, *see* cylindrical end
 - join curve, 91
 - shuffle curve, 91
 - two-story, 90
- energy, 63, 75
 - λ , 75
 - ω , 75
- energy filtration, 131
- equivalent bordered 3-manifolds, 46
- Euler measure, 65
- even
 - shuffle component, *see* component, shuffle, even
 - shuffle curve, *see* curve, shuffle, even
- exact triangle, surgery, 198
- expected dimension, *see also* index
 - at embedded curve, 93–97
 - of $\mathcal{M}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, 66
 - of $\widehat{\mathcal{M}}^B(\mathbf{x}, \mathbf{y}; S^\triangleright; P)$, 65
- exterior generator, 219
- Farkas's lemma, 56
- filtered
 - basis, 206
 - linearly-independent, 206
- filtration, Alexander, 200, 206
- formal dg algebra, 36
- G -graded algebra, *see* grading
- generalized
 - coefficient maps, *see* coefficient maps, generalized
 - generator, 63
 - split component, *see* component, generalized split
 - split curve, *see* curve, generalized split
- generator, 52
- compatible pair of, 58, 154
- exterior, 219
- generalized, 63
- gluing g.'s, 58
- partially-diagonal, 121
 - negative, 122
 - positive, 121
- generators
 - partially-diagonal
 - compatible, 123
 - matching of, 123
- generic, *see* complex structure, sufficiently generic
- ghost component, 87
- gluing
 - bordered 3-manifolds, 57
 - bordered Heegaard diagrams, 57
 - generators, 58
 - holomorphic curves, 73, 79
 - homology classes, 58
 - profile, 82
- grading
 - spin^c component, 39
 - Alexander
 - bordered case, 204
 - relationship to S , 219
 - differential algebra graded by a group, 25
 - differential module graded by G -set, 25
 - for pairing theorem, 192
- group
 - big, 39
 - dependence on orientation, 186
 - refined, 188
 - small, 41
- group-valued, on algebra, 25
 - homogeneous element of, 25
- Maslov, 206
 - on CFK , 200
 - z -normalized, 200
- Maslov component, 39, 179
 - multiplication by U on, 204
 - on $\mathcal{A}(\mathcal{Z})$, 39–41, 179–180
 - refined, 41–43, 188–189
 - on \widehat{CFA} , 185
 - refined, 189
 - on $\widehat{CFAA}(\mathbb{I})$ and $\widehat{CFDD}(\mathbb{I})$, 246
 - on \widehat{CFD} , 188
 - refined, 190
 - on \widehat{CFDA} , 244
 - on \widehat{CFA} , 183
 - refined, 189
 - on \widehat{CFD} , 187
 - refined, 190



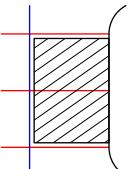
- on tensor product of group-graded vector spaces, 25
- on tensor product of set-graded modules, 26
- refinement data, 42, 188
dependence on, 42
- spin^c component, 179
- handlebody, bordered Heegaard diagrams for, 52
- handleslide, 49
- Hasse diagram, 31
- haunted curve, 87
rare, 89
- Heegaard diagram
bordered, 46
arced, 242
cutting, 57
doubly pointed, 203
for handlebody, 52
for solid torus, 198
for trefoil complement, 51
gluing, 57
represents bordered 3-manifold, 47
- capped, 65
- closed, 45
- doubly pointed for knot, 199
- nice, 149
- pointed, 45
- Heegaard moves, 49
destabilization, 49
handleslide, 49
invariance under, *see* invariance
isotopy, 49
stabilization, 49
- height, 59
- Heisenberg group, 26
- holomorphic building, 71
two-story, 60
- holomorphic comb, 60, 71
 ψ -matched, 161
height, 71
zero, 71
- homology class of, 71
- horizontal level of, 71
- ideal-matched, 162
- nodal, 71
- simple, 70
- smooth, 71
- spine of, 71
- story of, 71
- T -matched, 158
- toothless, 71
- trivial, 71
- vertical level of, 71
- holomorphic story, 71
- horizontal level of, 71
- T -matched, 158
- holomorphic triangle, 113
- homogeneous element of group-graded algebra,
see grading, group-valued
- homology class
connecting generators, 53
gluing, 58
of comb, 71
of curve, 63
positive, 56
provincial, 54
- homomorphism
of type D modules, 19
of \mathcal{A}_∞ modules, 12
composition, 13
homotopic, 14
identity, 13
null homotopic, 14
of type D modules, 17, 19
bounded, 19
homotopic, 19
homotopy of, 17
- homotopic, *see* homomorphism, homotopic
- homotopy equivalent, *see* module, homotopy equivalent
- horizontal
arrow, 207
length, 207
complex, 207
homology, 207
level of holomorphic story, 71
- horizontally simplified basis, 207
- ideal-matched holomorphic comb, 162
- idempotents
minimal, 20, 29
of $\mathcal{A}(\mathcal{Z})$, 35
of $\mathcal{A}(n, k)$, 29
- index, *see also* expected dimension
embedded, 95
is additive, 100
of matched pair, 154
- infinity
 \pm , 59
east, 59, 67, 244
of $\mathbb{R} \times \mathbb{Z} \times [0, 1] \times \mathbb{R}$, 67
of $\Sigma \times [0, 1] \times \mathbb{R}$, 59
west, 67, 244
- interleaved Reeb chords, 32
- intersection number, jittered, 97
- invariance
of $\widehat{\mathcal{CFA}}$, 142–147
graded, 185



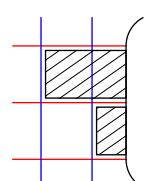
- of \widehat{CFD} , 110–134
 - graded, 188
- inversions, 29
- isotopy, *see* Heegaard moves, isotopy
- jittered intersection number, 97
- join
 - component, *see* component, join
 - curve, *see* curve, join
 - curve end, *see* end, join curve
 - of Reeb chords, 32
- knot Floer
 - complex, 199
 - homology, 200
 - pairing theorem, 204
- left boundary, 244
- length
 - of arrow, 207
- length of arrow, 207
- local multiplicity, 53
- longitude Floer complex, 202
- mapping class group, 243
 - oid, 243
- Maslov, *see* grading, Maslov
- matched circle, 34
 - pointed, *see* pointed matched circle
- matched pairs, *see* moduli space of matched pairs
- matching, 34
- minimal idempotent, *see* idempotents, minimal
- module
 - \mathcal{A}_∞ , 11
 - bounded, 11
 - strictly unital, 11
 - differential graded, 12
 - type D , 17
 - bounded, 18
 - coefficient maps, *see* coefficient maps
- moduli space, 59, 63, 64, 66
 - compactified, 73
 - is compact, 74
 - embedded, 95
 - of ψ -matched pairs, 161
 - of combs, *see* holomorphic comb
 - of curves at east ∞ , 68
 - of matched pairs, 154
 - embedded, 156
 - expected dimension, 154
 - of T -matched pairs, 157
 - reduced, 60, 66
 - rigid, 60
- monotonicity, *see* boundary monotonicity
- Morse function
- compatible with bordered 3-manifold, 47
- compatible with pointed matched circle, 34
- multiplicity of point in homology class, 39
- nice diagram, 149
- nodal
 - becoming, 84
 - holomorphic comb, *see* holomorphic comb, nodal
- nose, 55
- null homotopic, *see* homomorphism, null homotopic
- occupied α -arcs, 52, 189
- odd
 - degree near p , 80
 - shuffle component, *see* component, shuffle, odd
 - shuffle curve, *see* curve, shuffle, odd
- operationally bounded \mathcal{A}_∞ algebra, 11
- ordered partition, 59
- ordering of Reeb chords at $e\infty$, 59
- orientation reversal, 34, 57, 103, 104, 106, 186, 245
- pairing theorem
 - for change of framing, 241
 - for knot Floer, 204
 - graded, 192
 - relating \widehat{CFA} and \widehat{CFD} , 242
 - twisted coefficients, 174
- partial diagonal, 64
- partially-diagonal generator, *see* generator, partially-diagonal
- partition
 - ordered, *see* ordered partition, 66
- periodic domain, *see* domain, periodic
- pointed Heegaard diagram, *see* Heegaard diagram, pointed
- pointed matched circle, 34
 - boundary of bordered Heegaard diagram, 46
 - orientation reverse, 34
 - surface associated to, 34
- positive homology class, 56
- pregluing sources, 73
 - giving closed source, 154
- proper near p , 80
- provincial, *see* homology class, provincial
- puncture
 - bottom, 69
 - boundary of Σ is a, 62
 - east, 59, 62, 67
 - top, 69
 - west, 67
- reduced

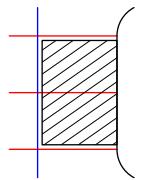


- complex, 207
 moduli space, 60, 66
- Reeb chord, 59
 Reeb chords, 32
 abutting, 32, 169
 amalgamation, 169
 composable
 sequence of sets of, 32
 sets of, 32
 interleaved, 32
 join of, 32
 nested, 32
 set of
 consistent, 32
 shuffle of, 33
 splitting of, 33
 weak shuffle of, 33
 weak splitting of, 33
 splitting of, 33
 set of, 33
- refinement data, *see* grading refinement data
 region, 53
 twin, 121
 winding, 217
 right boundary, 244
- S*-graded module, 25
 section of matching, 35
 set-graded, *see* grading
 shuffle
 component, *see* component, shuffle
 curve, *see* curve, shuffle
 curve end, *see* end, shuffle curve
 simple
 holomorphic comb, *see* holomorphic comb,
 simple
 simple ideal-matched curve, *see* curve, simple
 ideal-matched
 smeared neighborhood, 79
 smooth holomorphic comb, *see* holomorphic
 comb, smooth
 smoothing a crossing, 31
 solid torus, Heegaard diagram for, 198
 source
 bi-decorated, 67
 decorated, 59, 62
 for map to $\Sigma \times \Delta$, 114
 sphere bubble, 84
 spin^c component of grading, 39, 179
 spin^c-structure, 54
 associated to generator, 55
 associated to vector field, 54
 relative
 associated to generator, 55
- spin^c-structure
 relative, 55
 spine
 of comb, 71
 of ideal-matched curve, 169
- split
 component, *see* component, split
 curve, *see* curve, split
- splitting
 at $\pm\infty$, 83
 at east ∞ , 83
- stabilization, 49
 stable
 holomorphic curve, 66
 matched pair, 155
 T-matched story, 158
 stable holomorphic curve
 at east ∞ , 68
 story, *see* holomorphic story
 strands algebra, 30
 element associated to $a_0(\rho)$, 32
 strands diagram, 30
 stratified
 map, 80
 space, 80
 strictly unital
 \mathcal{A}_∞ algebra, 10
 \mathcal{A}_∞ module, 11
 $\widehat{\text{CFA}}$ is, 138
 homomorphism of \mathcal{A}_∞ modules, 12
 strong boundary monotonicity, *see* boundary
 monotonicity, strong
 strongly based mapping class group, 243
 sufficiently generic
 complex structure, 65
 path of almost complex structures, 111
 sufficiently pinched complex structure, 220
 surface associated to pointed matched circle, 34
 surgery exact triangle, 198
 symplectic form
 on \mathbb{D} , 62
 on Σ , 62
 on $\Sigma \times [0, 1] \times \mathbb{R}$, 62
- T*-matched
 comb, 158
 story, 158
 tautological correspondence, 212
 tensor product
 \mathcal{A}_∞ , 15
 is triangulated functor, 24
 box, 21
 is triangulated functor, 24
 of homomorphisms, 22



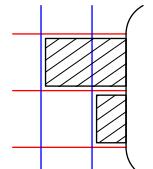
- time-parameter space, 162
- toothless comb, 71, 169
- top puncture, 69
- torus
 - algebra, 196
 - coefficient maps, *see* coefficient maps
 - refined grading group, 197
 - unrefined grading group, 196
- total east source, 163
- total west source, 163
- transversality, 64
 - for matched pair, 154
- trefoil, 51
- triangle, holomorphic, 113
- triply-periodic domain, 116
- trivial
 - component, 68
 - holomorphic comb, *see* holomorphic comb,
 - trivial
- twin regions, 121
- twisted coefficients
 - for \widehat{CFD} , 134
 - pairing theorem with, 174
- two-story end, *see* end, two-story
- type *A* invariant, 137–138
 - for solid torus, 205
- type *D*
 - homomorphism, *see* homomorphism, type *D*
 - invariant, 103–105
 - module, *see* module, type *D*
- vertical
 - arrow, 207
 - length, 207
 - complex, 207
 - homology, 207
 - level of comb, 71
- vertically simplified basis, 207
- weakly composable sets of chords, 91
- weight of algebra element, 36
- west infinity
 - bimodule case, 244
 - of $\mathbb{R} \times Z \times [0, 1] \times \mathbb{R}$, 67
- west puncture, *see* puncture, west
- west-most level, 158
- winding region, 217
- z*-normalized Maslov grading, 200



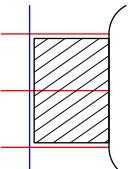


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