Notes from *Introduction to Dynamical Systems*

by Michael Brin and Garret Stuck

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Publisher's Description

This book provides a broad introduction to the subject of dynamical systems, suitable for a one- or two-semester graduate course. In the first chapter, the authors introduce over a dozen examples, and then use these examples throughout the book tomotivate and clarify the development of the theory. Topics include topological dynamics, symbolic dynamics, ergodic theory, hyperbolic dynamics, one-dimensional dynamics, complex dynamics, and measure-theoretic entropy. The authors top off hte presentation with some beautiful and remarkable applications of dynamical systesm sto such areas as number theory, data storage, and Internet search engines.

The book grew out of lecture ntoes from the graduate dynamical systems course at the University of Maryland, College Park, and reflects not only the tastes of the authors, but also to some extent the collectiv eopinion of hte Dynamics Group at the University of Maryland, which includes experts in virtually every major area of dynamical systems.

Transcription Notes

Take without proofs for an independent study course.

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0.1 Hyperbolic Dynamics

0.1.1 Hyperbolic Sets

Throughout, M is a C^1 Riemannian manifold, $U \subset M$ a non-empty open subset, and $f : U \to M$ a C^1 diffeomorphism.

Definition 1. *A compact, $f-$invariant subset $\Lambda \subset U$ is called hyperbolic if there are $\lambda \in (0,1)$, $C > 0$, and regular distributions $E_x^s, E_x^u \subset T_xM$; $x \in \Lambda$ such that for all x:*

1. $T_xM = E_x^s \oplus E_x^u$
2. $\|T_xf^n v^s\| \leq C\lambda^n \|v^s\|$ *for all* $v^s \in E_x^s$
3. $\|T_xf^{-n} v^u\| \leq C\lambda^n \|v^u\|$ *for all* $v^u \in E_x^u$
4. $(T_xf)(E_x^s) = E_{f(x)}^s$ *and* $(T_xf)(E_x^u) = E_{f(x)}^u$

The distributions E^s and E^u are called the stable and unstable distribution of $f \restriction_\Lambda$. If $\Lambda = M$, then f is called an *Anosov diffeomorphism.*

Proposition 1. *Let Λ be a hyperbolic set of f. Then the stable and unstable distributions are smooth and regular.*

Proposition 2. *Let Λ be a hyperbolic set of f with constants C and λ. Then for $\varepsilon > 0$ there is a C^1 Riemannian metric $\langle \cdot, \cdot \rangle'$ in a neighborhood of Λ called the Lyapunov or adapted metric, for which f is hyperbolic with new constants $C' = 1$ and $\lambda' = \lambda + \varepsilon$, and the unstable and unstable distributions are $\varepsilon-$orthogonal ($\langle v^s, v^u \rangle' < \varepsilon$ for unit vectors in the respective distributions).*

A fixed point of a differentiable map f is *hyperbolic* if no eigenvalue of $T_x f$ lies on the unit circle. A periodic point of period k is called *hyperbolic* if no eigenvalue of $T_x f^k$ lies on the unit circle.

0.1.2 ε-Orbits

An ε-orbit is a finite or infinite sequence $(x_n) \subset U$ satisfying $d(f(x_n), x_{n+1})\, \forall n$. These are also called *pseudo-orbits*.

Theorem 1. *Let Λ be a hyperbolic set of $f : U \to M$. Then there is an open $O \subset U$ containing Λ and there are positive ε_0, δ_0 satisfying: $\forall \varepsilon > 0\, \exists \delta\, \forall g : O \to M$ with $\mathrm{dist}_1(g, f) < \varepsilon_0$, any homeomorphism $h : X \to X$ and any continuous map $\phi : X \to O$ with $\mathrm{dist}_0(\phi \circ h, g \circ \phi) < \delta$, then there is a continuous map $\psi : X \to O$ with $\psi \circ h = g \circ \psi$ and $\mathrm{dist}_0(\phi, \psi) < \varepsilon$. Additionally, ψ is unique in the sense that $\psi' \circ h = g \circ \psi'$ & $\mathrm{dist}_0(\phi, \psi) < \delta_0$, then $\psi = \psi'$.*

Corollary 1. *Let Λ be a hyperbolic set of $f : U \to M$. Then for every $\epsilon > 0$ there is $\delta > 0$ such that if (x_k) is a (in)finite $\delta-$orbit of f and $\mathrm{dist}(x_k, \Lambda) < \delta$ then there is $x \in \Lambda_\varepsilon$ with $\mathrm{dist}((f^k(x), x_k) < \varepsilon$.*

Proof. *Choose O satisfying the conditions in 1 and δ such that $\Lambda_\delta \subset O$. If (x_k) is (semi-in)finite, add to (x_k) the preimages of some $y_0 \in \Lambda$ whose distnace to the first point in the sequence is $< \delta$, and/or the images of some $y_m \in \Lambda$ whose distance to the last point of the sequence is $< \delta$ to obtain a $\delta-$orbit lying in the $\delta-$ neighborhood of Λ. Let $X = (x_k)$ with the discrete topology, $g = f$, $h : X \to X$ the shift $x_k \mapsto x_{k+1}$ and $\phi : X \to U$ be the inclusion into the manifold. Since (x_k) is a $\delta-$orbit, $\mathrm{dist}(\phi(h(x_k)), f(\phi(x_k))) <\leq$, then theorem 1 applies and the corollary follows.*

Recall the set of nonwandering points $\mathrm{NW}(f)$ is the set of points where the iterate of any neighborhood intersects the neighborhood, and the Periodic points of f, $\mathrm{Per}(f)$. If Λ is $f-$invariant, we can speak of $\mathrm{NW}(f\restriction_\Lambda)$. In general, $\mathrm{NW}(f\restriction_\Lambda) \neq \mathrm{NW}(f) \cap \Lambda$.

Proposition 3. *If Λ is a hyperbolic set of $f : U \to M$, then $\overline{\mathrm{Per}(f\restriction_\Lambda)} = \mathrm{NW}(f\restriction_\Lambda)$.*

Corollary 2. *If $f : M \to M$ is Anosov, then $\overline{\mathrm{Per}(f)} = \mathrm{NW}(f)$.*

0.1.3 Invariant Cones

Let Λ be a hyperbolic set of $f : U \to M$. Since the distributions E^s and E^u are continuous, we can extend them to continuous distributions in a neighborhood

$U(\Lambda) \supset \Lambda$. If $x \in \Lambda$ and $v \in T_xM$, then $v = v^s + v^u$. Now assume the metric is adapted with constant λ. For $\alpha > 0$, define the (un)stable cones of size α by

$$K^s_\alpha(x) = \{v \in T_xM| \ : \|v^u\| \leq \alpha \|v^s\|\}$$

$$K^u_\alpha(x) = \{v \in T_xM : \|v^s\| \leq \alpha \|v^u\|\}$$

For a cone K, let $\mathring{K} = \mathrm{int}(K) \cup \{0\}$. Let $\Lambda_\varepsilon = d_\Lambda^{-1}([0, \varepsilon))$.

Proposition 4. *For every $\alpha > 0$ there is $\varepsilon = \varepsilon(\alpha)$ such that $f^i(\Lambda_\varepsilon) \subset U(\Lambda)$, $i = -1, 0, 1$ and for every $x \in \Lambda_\varepsilon$:*

$$T_xf(K^u_\alpha(x)) \subset \mathring{K}^u_\alpha(f(x)); \ (T_{f(x)}f^{-1})(K^s_\alpha(f(x))) \subset \mathring{K}^s_\alpha(x)$$

Proposition 5. *For every $\delta > 0$, there are $\alpha > 0$ and $\varepsilon > 0$ such that $f^i(\Lambda_\varepsilon \subset U(\Lambda)$, $i = -1, 0, 1$ and for every $x \in \Lambda_\varepsilon$:*

$$\|T_xf^{-1}(v)\| \leq (\lambda + \delta)\|v\|, \ v \in K^u_\alpha(x)$$
$$\|T_xf(v)\| \leq (\lambda + \delta)\|v\|, \ v \in K^s_\alpha(x)$$

Proposition 6. *Let Λ be a compact invariant set of $f : U \to M$. Suppose that there is a $\alpha > 0$ and for every $x \in \Lambda$ there are continuous subspaces E^s_x, E^u_x such that $E^s_x \oplus E^u_x = T_xM$ and the $\alpha-$cones $K^s_\alpha(x)$ and $K^U_\alpha(x)$ determined by the subspaces satisfy*

1. *$(T_xf)(K^u_\alpha(x)) \subset K^u_\alpha(x)$ and $(T_{f(x)}f^{-1})(K^u_\alpha(x)) \subset K^s_\alpha(x)$*
2. *$\|T_xf(v)\| < \|v\|$ for non-zero $v \in K^s_\alpha(x)$, and $\|T_xf^{-1}v\| < \|v\|$ for non-zero $v \in K^u_\alpha(x)$.*

Then Λ is a hyperbolic set of f.

Let

$$\Lambda^s_\varepsilon = \{x \in U : d_\Lambda(f^n(x)) < \varepsilon \ \forall n\}$$

$$\Lambda^u_\varepsilon = \{x \in U : d_\Lambda(f^{-n}(x)) \quad \forall n\}$$

Note that both sets are contained in Λ_ε and $f(\Lambda^s_\varepsilon) \subset \Lambda^s_\varepsilon$, and $f^{-1}(\Lambda^u_\varepsilon) \subset \Lambda^u_\varepsilon$.

Proposition 7. *Let Λ be a hyperbolic set of f with adapted metric. Then for every $\delta > 0$ there is $\varepsilon > 0$ such that the distributions E^s and E^u can be extended to Λ_ε so that*

1. *E^s is continuous on Λ^s_ε, E^u is continuous on Λ^u_ε.*
2. *$x \in \Lambda_\varepsilon \cap f(\Lambda_\varepsilon) \Rightarrow (T_xf)(E^s_x) = E^s_{f(x)}$ and $(T_xf)(E^u_x) = E^u_{f(x)}$*
3. *$\|(T_xf)(v)\| < (\lambda + \delta)\|v\|$ for every $x \in \Lambda_\varepsilon$ and $v \in E^s_x$.*
4. *$\|(T_xf^{-1})(v)\| < (\lambda + \delta)\|v\|$ for every $x \in \Lambda_\varepsilon$ and $v \in E^u_x$.*

0.1.4 Stability of Hyperbolic Sets

Proposition 8. *Let Λ be a hyperbolic set of $f : U \to M$. There is an open set $U(\Lambda) \supset \Lambda$ and $\varepsilon_0 > 0$ such that if $K \subset U(\Lambda)$ is a compact invariant subset of a diffeomorphism $g : U \to M$ with $\text{dist}_1(g, f) < \varepsilon_0$, then K is a hyperbolic set of g.*

Let $\text{Diff}^1(M)$ be the space of C^1 diffeomorphisms of M with the C^1 topology.

Corollary 3. *The set of Anosov diffeomorphisms of a given compact manifold is open in $\text{Diff}^1(M)$.*

Proposition 9. *Let Λ be a hyperbolic set of $f : U \to M$. For every open set $V \subset U$ containing Λ and every $\varepsilon > 0$, there is $\delta > 0$ such that $\forall g : V \to M$ with $\text{dist}_1(g, f) < \delta$, there is a hyperbolic set $K \subset V$ of g and a homeomorphism $\chi : K \to \Lambda$ such that χ cojugates f to g and $\text{dist}_0(\chi, \text{Id}) < \varepsilon$.*

A C^1 diffeomorphism f of a C^1 manifold is called *structurally stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $g \in \text{Diff}^1(M)$ and $\text{dist}_1(g, f) < \delta$, then there is a homeomorphsim $h : M \to M$ conjugated f and g and $\text{dist}_0(h, \text{Id}) < \varepsilon$.

Corollary 4. *Anosov diffeomorphisms are structurally stable.*

0.1.5 Stable and Unstable Manifolds

For $\delta > 0$, let B_δ be the ball of radius δ at 0.

Proposition 10 (Hadamard-Perron)**.** *Let $f_n : B_\delta to \mathbb{R}^m$ be a sequence of C^1 diffeomorphisms onto their images such that $\forall n\ f_n(0) = 0$. Suppose that for each n there is a splitting $\mathbb{R}^m = E_n^s \oplus E_n^u$ and $\lambda \in (0, 1)$ such that*

1. *$T_0 f_n(E_n^s) = E_{n+1}^s$ and $T_0 f_n(E_n^u) = E_{n+1}^u$*
2. *$\|T_0 f_n v^s\| < \lambda \|v^s\|$ for all $v^s \in E_n^s$*
3. *$\|T_0 f_n v^u\| > \lambda \|v^u\|$ for all $v^u \in E_n^u$*
4. *The angles between E_n^u and E_n^s are uniformly bounded away from 0*
5. *$(T f_n)$ are an equicontinuous family of fuctions $T f_n : B_\delta \to \text{GL}_m(\mathbb{R})$.*

THEN there are $\varepsilon > 0$ and a sequence $\phi = (\phi_n)$ of uniformly Lipschitz continuous maps $\phi_n : B_\varepsilon^s = E_n^s \cap B_\varepsilon \to E_n^u$ such that

1. *$\text{graph}(\phi_n) \cap B_\varepsilon = W_\varepsilon^s(n)$, where the latter set is defined as $\{x \in B_\varepsilon : \|f_{n+k-1} \circ ... \circ f_{n+1} \circ f_n(x)\| \to 0$ as $k \to \infty\}$*
2. *$f_n(\text{graph}(\phi_n)) \subset \text{graph}(\phi_{n+1})$*
3. *$x \in \text{graph}(\phi_n) \Rightarrow \|f_n(x)\| \leq \lambda \|x\| \Rightarrow f_n^k(x) \to 0$ exponentially as $k \to \infty$*

4. *for* $x \in B_\varepsilon \backslash \mathrm{graph}(\phi_n)$,

$$\|P^u_{n+1} f_n(x) - \phi_{n+1}\left(P^s_{n+1} f_n(x)\right)\| > \lambda^{-1} \|P^u_n x - \phi_n\left(P^s_n x\right)\|$$

Where P^s_n (P^u_n) *denotes the projection onto* E^s_n (E^u_n) *parallel to the other subspace*

5. ϕ_n *is differentiable at 0,* $T_0\phi_n 0 = 0 \Rightarrow$ *the tangent plane to* $\mathrm{graph}(\phi_n)$ *is* E^s_n.

6. ϕ *depends continuously on* f *in the topologies by the following distance functions:*

$$d_0(\phi, \psi) = \sup_{x,n} 2^{-n} |\phi_n(x) - \phi_n(x)|$$

$$d(f, g) = \sup_n \mathrm{dist}_1(f_n, g_n)$$

Let $\Phi(L, \varepsilon)$ be the space of sequences $\phi = (\phi_n)$ where $\phi_n : B^s_\varepsilon \to E^u_n$ is Lipschitz-continuous map with Lipschitz constant L and $\phi_n(0) = 0$, with a metric $d(\phi, \psi) = \sup_{n,x} |\phi_n(x) - \psi_n(x)|$, which is complete.

We now define an operator $F : \Phi(L, \varepsilon) \to \Phi(L, \varepsilon)$ called the *graph transform*. Let $\phi \in \Phi(L, \varepsilon)$. The next lemma will show that $f^{-1}_n(\mathrm{graph}(\phi_{n+1}))$ projected onto E^s_n covers $E^s_\varepsilon(n)$ and $f^{-1}_n(\mathrm{graph}(\phi_{n+1}))$ contains the graph of a continuous function $\psi_n : B^s_\varepsilon \to E^u_\varepsilon(n)$ with Lipschitz constant L. Take $F(\phi)_n = \psi_n$.

Lemma 1. *For any* $L > 0$*, there exists* $\varepsilon > 0$ *such that the graph transform* F *is a well-defined operator on* $\Phi(L, \varepsilon)$.

Lemma 2. *There are* $\varepsilon > 0$ *and* $L > 0$ *such that* F *is a contracting operator.*

Theorem 2. *Let* $f : M \to M$ *be a* C^1 *diffeomorphism of a differentiable manifold and* Λ *a hyperbolic set of* f *with constant* λ *and adapted metric.*

Then there are $\varepsilon > 0$, $\delta > 0$ *such that for every* $x^s \in \Lambda^s_\delta$ *and every* $x^u \in \Lambda^u_\delta$*:*

1. *the sets*

$$W^s_\varepsilon(x^s) = \{y \in M : \mathrm{dist}(f^n(x^s), f^n(y)) < \varepsilon \ \forall n\}$$

$$W^u_\varepsilon(x^u) = \{y \in M : \mathrm{dist}(f^{-n}(x^u), f^{-n}(y)) < \varepsilon \ \forall n\}$$

called the local stabale manifold of x^s *and the local unstable manifold of* x^u*, are* C^1 *embedded disks,*

2. $T_{y^s} W^s_\varepsilon(x^s) = E^s_{y^s}$ *for all* $y^s \in W^s_\varepsilon(x^s)$ *and similarly for the unstable manifolds and subspaces,*

3. $f(W^s_\varepsilon(x^s)) \subset W^s_\varepsilon(f(x^s))$ *and* $f^{-1}(W^u_\varepsilon(f(x^u))) \subset W^u_\varepsilon(x^u)$

4. *if* $y^s, z^s \in W^s_\varepsilon(x^s)$*, then* $d^s(f(y^s), f(z^s)) < \lambda d^s(y^s, z^s)$*, where* d^s *is the distance along* $W^s_\varepsilon(x^s)$*, and a similar result for the local unstable manifold using the inverse map*

5. *if* $0 < \mathrm{dist}(x^s, y) < \varepsilon$ *and* $\exp^{-1}_{x^s}(y)$ *lies in the* $\delta-$*cone* $K^u_\delta(x^s)$, *then* $\mathrm{dist}(f(x^s), f(y)) > \ lambda^{-1}\mathrm{dist}(x^s, y)$ *and if* $0 < \mathrm{dist}(x^u, y) < \varepsilon$ *and* $\exp^{-1}_{x^u}(y)$ *lies in the* $\delta-$*cone* $K^s_\delta(x^u)$, *then* $\mathrm{dist}(f(x^u), f(y)) < \lambda\mathrm{dist}(x^s, y)$

6. *if* $y^s \in W^s_\varepsilon(x^s)$, *then* $W^s_\alpha(y^s) \subset W^s_\varepsilon(x^s)$ *for some* $\alpha > 0$, *and if* $y^u \in W^u_\varepsilon(x^u)$, *then* $W^u_\beta(y^u) \subset W^s_\varepsilon(x^u)$ *for some* $\beta > 0$.

Let Λ be a hyerbolic set of $f : U \to M$ and $x \in \Lambda$. The *(global) stable and unstable manifolds* of x are defined by

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0, n \to \infty\}$$

$$W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0, n \to \infty\}$$

Proposition 11. *There is* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and every* $x \in \Lambda$,

$$W^s(x) = \bigcup_{n>0} f^{-n}(W^s_\varepsilon(f^n(x)))$$

$$W^u(x) = \bigcup_{n>0} f^{n}(W^u_\varepsilon(f^{-n}(x)))$$

Corollary 5. *The global stable and unstable manifolds are embedded* C^1 *submanifolds of* M *homeomorphic to unit balls in corresponding dimensions.*

0.1.6 Inclination Lemma

Recall the definition of two submanifolds to intersect transversely.

Denote by B^i_ε the open ball of radius ε centered at 0 in $\mathbb{R}^i$. For $v \in \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^l$ denote by $v^u \in \mathbb{R}^k$ and $v^s \in \mathbb{R}^l$ the components of $v = v^u + v^s$, and $\pi^u : \mathbb{R}^m \to \mathbb{R}^k$ the projection. For $\delta > 0$ let $K^u_\delta = \{v \in \mathbb{R}^m : \|v^s\| \leq \delta\|v^u\|\}$ and the stable cone $K^s_\delta = \{v \in \mathbb{R}^m : \|v^s\| \leq \delta\|v^u\|\}$

Lemma 3. *Let* $\lambda \in (0, 1), \varepsilon > 0, \delta \in (0, 0.1)$. *Suppose* $f : B^k_\varepsilon \times B^l_\varepsilon \to \mathbb{R}^m$ *and* $\phi : B^k_\varepsilon \to B^l_\varepsilon$ *are* C^1 *amps such that:*

1. *0 is a hyperbolic fixed point of* f
2. $W^u_\varepsilon(0) = B^k_\varepsilon \times \{0\}$ *and* $W^s_\varepsilon = \{0\} \times B^l_\varepsilon$
3. $\|T_xf(v)\| \geq \lambda^{-1}\|v\|$ *for every* $v \in K^u_\delta$ *whenever both* $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$
4. $\|T_xf(V)\| \leq \lambda\|v\|$ *for every* $v \in K^s_\delta$ *whenever both* $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$
5. $T_xf(K^u_\delta) \subset K^u_\delta$ *whenever* $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$
6. $T_xf^{-1}(K^s_\delta) \subset K^s_\delta$ *whenever* $x, f^{-1}(x) \in B^k_\varepsilon \times B^l_\varepsilon$
7. $T_{(y,\phi(y))}\mathrm{graph}(\phi) \subset K^u_\delta$ *for every* $y \in B^k_\varepsilon$

Then for every n there is a subset $D_n \subset B^k_\varepsilon$ diffeomorphic to B^k such that the image I_n under f^n of the graph of the restriction $\phi \restriction_{D_n}$ has the following properties: $\pi^u(I_n) \supset B^k_{\varepsilon/2}$ and $T_x I_n \subset K^u_{\delta\lambda^{2n}}$ for each $x \in I_n$.

The meaning of the lemma is that the tangent planes to the image of the grap of ϕ under f^n are exponentially (in n) close to the "horizontal" space $\mathbb{R}^k$, and the image spreads over B^k_ε in the horizontal direction.

The next theorem, sometimes called the Lambda Lemma, implies that if f is C^r with $r \geq 1$, and D is any C^1-disk that intersects transversely the stable manifold $W^s(x)$ of a hyperbolic fixed point of x, then the forwards images of D converge in the C^r topology to the unstable manifold $W^u(x)$. The proof only covers C^1 convergence. Let B^u_R be the ball of radius R centered at x in $W^u(x)$ in the induced metric.

Theorem 3 (Inclination Lemma)**.** *Let x be a hyperbolic fixed point of a diffeomorphism $f : U \to M$, $\dim(W^u(x)) = k$ and $\dim(W^s(x)) = l$. Let $y \in W^s(x)$ and suppose that $D \ni y$ is a C^1 submanifold of dimension k intersecting $W^s(x)$ transversely at y.*

Then for every $R > 0$ and $\beta > 0$ there are n_0 and for each $n \geq n_0$, a subset $\tilde{D} = \tilde{D}(R, \beta, n)$, $y \in \tilde{D} \subset D$, diffeomorphic to an open $k-$disk and such that the C^1 distance between $f^n(\tilde{D})$ and B^u_R is less than β.

0.1.7 Horseshoes and Transverse Homoclinic Points

Let $\mathbb{R}^> = \mathbb{R}^k \times \mathbb{R}^l$. We will refer to $\mathbb{R}^k$ and $\mathbb{R}^l$ as the unstable and stable subspaces, respectively, and denote by π^u and π^s the projections to these spaces. For $v \in \mathbb{R}^m$ denoted by $v^u = \pi^u(v) \in \mathbb{R}^k$ and $v^s = \pi^s(v) \in \mathbb{R}^l$. For $\alpha \in (0,1)$, call the sets $K^u_\alpha = \{v \in \mathbb{R}^m : |v^s| \leq \alpha |v^u|\}$ and $K^s_\alpha = \{v \in \mathbb{R}^m : |v^u| \leq \alpha |v^s|\}$ the unstable and stable cones, respectively. Let $R^u = \{x \in \mathbb{R}^k : |x| \leq 1\}$, $R^s = \{x \in \mathbb{R}^l : |x| \leq 1\}$, and $R = R^u \times R^s$. For $z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^l$, the sets $F^s(z) = \{x\} \times R^s$ and $F^u(z) = R^u \times \{y\}$ will be called the stable and unstable fibers, respectively. We say that a C^1 map $f : R \to \mathbb{R}^m$ has a *horseshoe* if there are $\lambda, \alpha \in (0,1)$ such that:

1. f is one-to-one on R
2. $f(R) \cap R$ has at least two components $\Delta_0, \dots, \Delta_{q-1}$
3. if $z \in R$ and $f(z) \in \Delta_i$, $0 \leq i < q$, then the sets $G^u_i(z) = f(F^u(z)) \cap \Delta_i$ and $G^s_i(z) = f^{-1}(F^s(f(z)) \cap \Delta_i)$ are connected, and the restriction of π^u to $G^u_i(z)$ and of π^s to $G^s_i(z)$ are bijective
4. if $z, f(z) \in R$, then the derivative $T_z f$ preserves the unstable cones K^u_α and $\lambda |T_z f(v)| \geq |v|$ for every $v \in K^u_\alpha$, and the inverse $T_{f(z)} f^{-1}$ preserves the stable cones K^s_α and $\lambda |T_{f(z)} f^{-1}(v)| \geq |v|$.

The intersection $\Lambda = \bigcap_{n>0} f^n(R)$ is called a *horseshoe*.

Theorem 4. *The horseshoe $\Lambda = \bigcap_{n>0} f^n(R)$ is a hyperbolic set of f. If $f(R) \cap R$ has q components, then the restriction of f to Λ is topologically conjugate tot he full two-sided shift σ in the space of Σ_q of bi-infinite sequences in the alphabet $\{0, 1, \dots, q-1\}$*

Corollary 6. *If a diffeomorphism has a horseshoe, then the topological entropy of f is positive.*

Let p be a hyperbolic fixed periodic point of a diffeomorphism $f : U \to M$. A point q is called *homoclinic* (for p) if $q \neq p$ and $q \in W^s(p) \cap W^u(p)$; it is called *transverse homoclinic* (for p) if in addition $W^s(p)$ and $W^u(p)$ intersect transversely at q.

Theorem 5. *Let p be a hyperbolic periodic point of a diffeomorphism $f : U \to M$, and let q be a transverse homoclinic point of p. Then for every $\varepsilon > 0$ the union of $\varepsilon-$neighborhoods of the orbits of p and q contains a horseshoe of f.*

0.1.8 Local Product Structure and Locally Maximal Hyperbolic Sets

A hyperbolic set Λ of $f : U \to M$ is called *locally maximal* if there is an open set V such that $\Lambda \subset V \subset U$ and $\Lambda = \bigcap_{n>0} f^n(V)$. Since every closed invariant subset of a hyperbolic set is also a hyperbolic set, the geometric structure of a hyperbolic set may be very complicated and difficult to describe. However, due to their special properties, locally maximal hyperbolic sets allow a geometric characterization.

Since $E^s_x \cap E^u_x = \{0\}$, the local stable and unstable manifolds of x intersect at x transversely. By continuity, this transversality extends to a neighborhood of the diagonal in $\Lambda \times \Lambda$.

Proposition 12. *Let Λ be a hyperbolic set of f. For every samll enough $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is transverse and consists of exactly one point $[x, y]$, which depends continuously on x and y. Furthermore, there is $C_p = C_p(\delta) > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $d^s(x, [x, y]) \leq C_p d(x, y)$ and $d^u(x, [x, y]) \leq C_p d(x, y)$, where d^s and d^u are distances along the stable and unstable manifolds, respectively.*

Let $\varepsilon > 0, k, l \in \mathbb{N}$, let $B^k_\varepsilon \subset \mathbb{R}^k$, and $B^l_\varepsilon \subset \mathbb{R}^l$ be $\varepsilon-$balls.

Lemma 4. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\phi : B^k_\varepsilon \to \mathbb{R}^l$ and $\psi : B^l_\varepsilon \to \mathbb{R}^k$ are differentiable maps and $|\phi(x)|, \|T\phi(x)\|, |\psi(y)|, \|T\phi(y)\| < \delta$ for all $x \in B^k_\varepsilon$ and $y \in B^l_\varepsilon$, then the intersection $\mathrm{graph}(phi) \cap \mathrm{graph}(psi) \subset \mathbb{R}^{k+l}$ is transverse and consists of exactly one point, which depend continuously on ϕ and ψ in the C^1 topology.*

The following property of hyperbolic sets plays a major role in their geometric description and is equivalent to local maximality. A hyperbolic set Λ has a *local product structure* if there area (small enough) $\varepsilon > 0$ and $\delta > 0$ such that

1. $\forall x, y \in \Lambda$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of at most one point, belonging to Λ

2. $\forall x, y \in \Lambda$ with $d(x, y) < \delta$, the intersection consists of exactly one point of Λ, denoted by $[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$, and the intersection is transverse.

If a hyperbolic set Λ has a local product structure, then for every $x \in \Lambda$ there is a neighborhood $U(x)$ such that

$$U(x) \cap \Lambda = \{[y, z] : y \in U(x) \cap W^s_\varepsilon(x), z \in U(x) \cap W^u_\varepsilon(x)\}$$

Proposition 13. *A hyperbolic set Λ is locally maximal iff it has a local product structure.*

0.1.9 Anosov Diffeomorphisms

Recall that a C^1 diffeomorphism f of a connected differentaible manifold M is called *Anosov* if M is a hyperbolic set for f; it follows then that M is a locally maximal and compact.

An important class of Anosov diffeomorphisms is as follows: Let N be a simply connected nilpotent Lie group, and Γ a uniform discrete subgroup of N. The quotient $M = N/\Gamma$ is a compact *nilmanifold*. Let $\overline{f}$ be an automorphism of N that preserves Γ and whose derivative at the identity is hyperbolic. The induced diffeomorphism f of M is Anosov. Up to finite coverings, all known Anosov diffeomorphisms are topologically conjugate to automorphisms of nilmanifolds.

The families of stable and unstable manifolds of an Anosov diffeomorphism for two foliations called the *stable foliation* W^s and unstable foliation W^u These foliations are in general not C^1, or even Lipschitz, but they are Hölder continuous. In spite of lack of Lipschitz continuity, the stable and unstable foliations possess a uniqueness property similar to the uniqueness theorem for ordinary differential equations.

Proposition 14. *LEt $f : M \to M$ be an Anosov diffeomorphism. Then there are $\lambda \in (0, 1)$, $C_p > 0$, $\varepsilon > 0$, $\delta > 0$ and for each $x \in M$, a splitting $T_xM = E^s_x \oplus E^u_x$ such that:*

1. $T_xf(E^s_x) = E^s_{f(x)}$ *and* $T_xf(E^u_x) = E^u_{f(x)}$

2. $\|T_xf(v^s)\| \leq \lambda\|v^s\|$ *and* $T_xf^{-1}(v^u) \leq \lambda\|v^u\|$ *for* $v^s \in E^s_x, v^u \in E^u_x$.

3. $W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0$ *as* $n \to \infty\}$ *and* $d^s(f(x), f(y)) \leq \lambda d^s(x, y)$ *for every* $y \in W^s(x)$

4. $W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0$ *as* $n \to \infty\}$ *and* $d^u(f^{-1}(x), f^{-1}(y)) \leq \lambda d^u(x, y)$ *for every* $y \in W^u(x)$

5. $f(W^s(x)) = W^s(f(x))$ *and* $f(W^u(x)) = W^u(f(x))$

6. $T_xW^s(x) = E^s_x$ *and* $T_xW^u(x) = E^u_x$
7. *if* $d(x,y) < \delta$, *then the intersection* $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ *is exactly one point* $[x,y]$, *which depends continuously on* x *and* y, *and* $d^s([x,y],x) \leq C_p d(x,y)$; $d^u([x,y],y) \leq C_p d(x,y)$.

A diffeomorphism is structurally stable if $\forall \varepsilon > 0$ there is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that $\forall g \in \mathcal{U}$ there is a homeomorphism h conjugating f and g and $\mathrm{dist}_0(h, \mathrm{Id}) < \varepsilon$.

Proposition 15. 1. *Anosov diffemorphisms forma n open (possibly empty) subset in the* C^1 *topology.*

2. *Anosov diffeomorphisms are structurally stable.*
3. *The set of periodic points of an Anosov diffeomorphism is dense in the set of non-wandering points.*

Theorem 6. *Let* $f : M \to M$ *be an Anosov diffeomorphism. Then TFAE:*

1. $\mathrm{NW}(f) = M$
2. *Every unstable manifold is dense in* M
3. *every stable manifold is dense in* M
4. f *is topologically transitive*
5. f *is topologically mixing*

0.1.10 Axiom A and Structural Stability

A diffeomorphism satisfies Smale's *Axiom A* if the set $\mathrm{NW}(f)$ is hyperbolic and $\overline{\mathrm{Per}(f)} = \mathrm{NW}(f)$.

For a hyperbolic periodic point p of f, denote by $W^s(O(p))$ and $W^u(O(p))$ the unions of the stable and unstable manifolds of p and its images, respectively. If p and q are hyperbolic periodic points, we write $p \leq q$ when $W^s(O(p))$ and $W^u(O(p))$ have a point of transverse intersection. $\leq$ is reflexive and transitive. If $p \leq q$ and $q \leq p$, we write $p\ q$ and say that p and q are *heteroclinically related.* This is an equivalence relation.

Theorem 7 (Smale's Spectral Decomposition Theorem)**.** *If* f *satisfies Axiom A, then there is a unique representation of* $\mathrm{NW}(f)$,

$$\mathrm{NW}(f) = \Lambda_1 \cup \cdots cup \Lambda_k$$

as a partition of closed $f-$*invariant subsets (called basic sets) such that:*

1. *each* Λ_i *is a locally maximal hyperbolic set of* f

2. *f is topologically transitive on each λ_i*

3. *each Λ_i is a disjoint union of closed sets $\Lambda_i^j, i \leq j \leq m_i$, with f cycically permuting the set Λ_i^j and f^{m_i} is topologically mixing on each Λ_i^j.*

The basic sets are precisely the closures of the equivalence classes of . For two basic sets, we write $\Lambda_i \leq \Lambda_j$ if there are periodic points $q \in \Lambda_j$ and $p \in \Lambda_i$ such that $p \leq q$.

Let f satisfy Axiom A. f satisfies the *strong transversality condition* if $W^s(x)$ intersects $W^u(y)$ transversely (at all point of intersection) for all $x, y \in \mathrm{NW}(f)$.

Theorem 8. *A C^1 diffeomorphism is structurally stable iff it satisfies Axiom A and the strong transversality condition.*