1 Linear Sympletic Geometry

1.1 Symplectic Vector Spaces

Definition 1 *A **Symplectic Vector Space** is a pair (V, ω) of a finite-dimensional vector space V and a non-degenerate skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. A **Linear Symplectomorphism** is a linear map preserving the symplectic form. The set of linear symplectomorphisms is denoted by $\mathrm{Sp}(V, \omega)$.*

Definition 2 *The symplectic complement of a subspace W is the subspace $W^\omega = \{v \in V | \omega(v, w) = 0 \ \forall w \in W\}$. Subspaces are called*

isotropic	$W \subset W^\omega$
coisotropic	$W^\omega \subset W$
symplectic	$W \cap W^\omega = \{0\}$
Lagrangian	$W = W^\omega$

Lemma 1 $\mathrm{dim} W + \mathrm{dim} W^\omega = \mathrm{dim} V$*;* $W^{\omega\omega} = W$

Corollary 1 *ω is nondegenerate iff the associated volume element is nonzero: $\omega \wedge \ldots \wedge \omega = \omega^n$*

Lemma 2 *Any isotropic subspace is contained in a Lagrangian subspace. Additionally, any basis of a Lagrangian subspace can be extendded to a sympletic basis of (V, ω).*

Lemma 3 *Let (V, ω) be a symplectic vector space of $W \subset V$ a coisotropic subspace. Then:*

1. *$V' = W/W^\omega$ carries a natural symplectic structure ω' induced by ω.*
2. *If $\Lambda \subset V$ is a Lagrangian subspace then $\Lambda' = ((\Lambda \cap W) + W^\omega)/W^\omega$ is a Lagrangian subspace of V'*

1.2 The Symplectic Linear Group

Lemma 4 $\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)$

Lemma 5 *$\lambda \in \sigma(\Psi) \iff \lambda^{-1} \in \sigma(\Psi)$, and their multiplicities are identical (Here Ψ is a linear symplectomorphism, and $\sigma(\Psi)$ its spectrum). Moreover, distinct eigenvectors z_1, z_2 have the property that $\omega(z_1, z_2) = 0$.*

Lemma 6 *Every real symmetric positive-definite symplectic matrix, taken to any positive real power, is again a symplectic matrix*

Proposition 1 *$\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$ and the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible.*

Proposition 2 *The fundamental group of $\mathrm{U}(n)$ is isomorphic to the integers. The determinant map $\det : \mathrm{U}(n) \to \mathrm{S}^1$ induces an isomorphism of fundamental groups.*

1.3 The Maslov Index

Theorem 1 *There is a unique function $\mu : \Omega \mathrm{Sp}(2n) \to \mathbb{Z}$ satisfying the following:*

1. *(homotopy) Two loops have the same Maslov index $\iff$ they are homotopic*
2. *(product) For any two loops $\Psi_t, \Phi_t : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$, $\mu(\Psi_t \Phi_t) = \mu(\Psi_t) + \mu(\Phi_t)$*
3. *(direct sum) Identifying $\mathrm{Sp}(2a) \bigoplus \mathrm{Sp}(2b) \subset \mathrm{Sp}(2a+2b)$, then $\mu(\Psi \bigoplus \Phi) = \mu(\Psi) + \mu(\Phi)$*
4. *(normalization) The loop $\theta_t = e^{2\pi i t} \in \mathrm{U}(1)$ has Maslov index 1.*

The Maslov index can also be considered the intersection number of a loop with a certain submanifold. Decompose a symplectic matrix into a 2x2 block matrix form, then take the upper right matrix and set its determinant equal to zero. This forms a codimension one submanifold. Then take the Maslov index to be the intersection number of the loop with this submanifold.

1.4 Lagrangian Subspaces

Let $\mathcal{L}(V, \omega)$ be the set of Lagrangian subspaces of (V, ω)

Lemma 7 *Let X and Y be real $n \times n$ matrices and define $\Lambda \subset \mathbb{R}^{2n}$ by $\Lambda = \mathrm{range}(Z)$; $Z = (XY)^T$. Then Λ is a Lagrangian subspace $\iff$ Z is of full rank and $X^T Y = Y^T X$*

The matrix that satisfies the above is a **Lagrangian Frame**. $\mathcal{L}(n)$ is a manifold of dimension $n(n+1)/2$.

Lemma 8

1. *Any symplectic transformation of a Lagrangian subspace is again Lagrangian*
2. *There is a symplectic transform between any two Lagrangian subspaces.*
3. *There is a natural isomorphism $\mathcal{L}(n) \approx \mathcal{U}(n)/\mathcal{O}(n)$.*

Theorem 2 *There is a unique function $\mu : \Omega\mathcal{L}(n) \to \mathbb{Z}$ satisfying the following:*

1. *(homotopy) Two loops have the same Maslov index $\iff$ they are homotopic*
2. *(product) For a loop $\Lambda_t \in \Omega\mathcal{L}(n)$ and a loop $\Psi_t \in \Omega\mathrm{Sp}(2n)$, then $\mu(\Psi_t \Lambda_t) = \mu(\Lambda_t) + 2\mu(\Psi_t)$*
3. *(direct sum) Identifying $\mathcal{L}(a) \bigoplus \mathcal{L}(b) \subset \mathcal{L}(a+b)$, then the Maslov index of a direct sum of two loops is the sum of their Maslov indices.*

4. *(normalization) The loop* $\Lambda_t = e^{2\pi it}\mathbb{R} \subset \mathbb{C}$ *has Maslov index 1.*

Similarly to the Maslov indices of symplectomorphism loops, we can view the Maslov index of Lagrangian subspaces as the intersection number of the loop with a submanifold of $\mathcal{L}(n)$. The desired submanifold is the set of planes $\bigcup_{c\in\mathbb{R}}\{\mathrm{Re}(z) = c\}$ in complex symplectic basis.

1.5 The Affine Non-Squeezing Theorem

Definition 3 *An* ***Affine Symplectomorphism*** *is a map that is a symplectomorphism followed by a translation. A symplectic cylinder* $Z^{2n}(R)$ *is* $B^2(R) \times \mathbb{R}^{2n-2}$

Theorem 3 *Let* ψ *be an affine symplectormophism, and that* $\psi(B^{2n}(r)) \subset Z^{2n}(R)$*. Then* $r \leq R$*.*

Theorem 4 *Let* Ψ *be a nonsingular matrix with the non-squeezing property. Then* Ψ *is symplectic or anti-symplectic* ($\Psi^*\omega = -\omega$)*.*

Definition 4 *The* ***Linear Symplectic Width*** *of a subset* $A \subset \mathbb{R}^{2n}$ *is the area of the largest symplectic ball that fits inside the subset:*

$$w_L(A) = \sup\{\pi r^2 | \psi(B^{2n}(r)) \subset A\}$$

For any affine symplectic transform ψ*. It has the following properties:*

1. *(monotonicity) if* $\psi(A) \subset B$ *then* $w_L(A) \leq w_L(B)$*.*
2. *(conformality)* $w_L(\lambda A) = \lambda^2 w_L(A)$
3. *(nontriviality)* $w_L(B^{2n}(r)) = w_L(Z^{2n}(r)) = \pi r^2$

Theorem 5 *Let* $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be a linear map. Then TFAE:*

1. Ψ *preserves the linear symplectic width of ellipsoids centered at* 0*.*
2. Ψ *is either symplectic or anti-symplectic, i.e.* $\Psi^*\omega = \pm\omega$

Lemma 9 *Let* (V, ω) *be a symplectic vector space with an inner product* g*. Then there is a basis of* V *which is* g*-orthogonal and* ω*-standard, and can be chosen so that* $g(u_j, u_j) = g(v_j, v_j)$

Lemma 10 *Given an ellipsoid*

$$E = \{W \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} w_i w_j \leq 1\}$$

There is a symplectic linear tranformation $\Psi in \mathrm{Sp}(2n)$ *such that* Ψ *turns the matrix* a_{ij} *diagonal.*

Remark 1 *For $n = 1$, the previous lemma tells us that every ellipse in $\mathbb{R}^2$ can be mapped into a circle by an area-preserving transformation.*

Definition 5 *The **Symplectic Spectrum** of an ellipsoid to be the increasing n-tuple $(r_1, \ldots, r_n)$ such that the ellipsoid is 'diagonalized' by a linear transformation into a ellipsoid with diagonal matrix $\mathrm{diag}(r_1, \ldots, r_n)$. Symplectic spectra have the following properties:*

1. *Two ellipsoids are linearly symplectomorphic $\iff$ they have the same spectrum*
2. *An ellipsoid with its spectrum of the form $(r, \ldots, r)$ are symplectic balls*
3. *The volume of a symplectic ellipsoid is $\mathrm{Vol}(E) = \pi^n \Pi_i r_i^2$*

Theorem 6 *Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0. Then $w_L(E) = \sup_{B \subset E} w_L(B) = \inf_{E \subset Z} w_L(Z)$, where B are symplectic balls and Z are symplectic cylinders.*

1.6 Complex Structures

Definition 6 *A **Complex Structure** is an automorphism $J : V \to V$ such that $-J^2$ is the identity. We can 'complexify' the vector space by "complex' scalar multiplication: $\mathbb{C} \times V \to V : (s + it, v) \mapsto sv + tJv$, which means V has even dimension over the reals. The set of complex structures is denoted by $\mathcal{J}(V)$*

Proposition 3 *Every almost complex structure is a linear transform away from the standard complex structure:*

$$J = \begin{bmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{bmatrix}$$

Definition 7 *If (V, ω) is symplectic vector space, a complex structure $J \in \mathcal{J}(V)$ is said to be compatible with ω if $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$. This defines an inner product on V, defined by $g_J(v, w) = \omega(w, Jw)$. This inner product makes J skew-adjoint, i.e. $g_J(v, Jw) = -g_J(Jv, w)$. The space of compatible complex structures is denotes $\mathcal{J}(V, \omega)$*

Proposition 4

1. *$\mathcal{J}(V, \omega)$ is homeomorphic to the space $\mathcal{P}$ of symmetric positive definite symplectic matrices.*
2. *There is a continuous map $r : met(V) \to \mathcal{J}(V, \omega)$ such that $r(g_J) = J$ & $r(\Phi^* g) = \Phi^* r(g)$, $\forall J \in \mathcal{J}(V, \omega), g \in met(V), \Phi \in Sp(V, \omega)$.*
3. *$\mathcal{J}(V, \omega)$ is contractible.*

Definition 8 *A complex structure $J \in \mathcal{J}(V, \omega)$ is called ω-tame if $\omega(v, Jv) > 0$ for all nonzero v. The space of ω-tame complex structures is denotes by $\mathcal{J}_T(V, \omega)$. There is an associated inner product for each ω-tame complex structure given by $g_J(v, w) = \frac{1}{2}(w(v, Jw) + \omega(w, Jv))$.*

Proposition 5 *The space $\mathcal{J}_T(V, \omega)$ is contractible.*

1.7 Symplectic Vector Bundles

Definition 9 *A* ***Symplectic Vector Bundle*** *over a manifold is a real vector bundle* $E \xrightarrow{\pi} M$ *equipped with a smoothly symplectic form* $\omega \in \Gamma(E \bigotimes E^*)$.

Definition 10 *Symplectic vector bundles are isomorphic* $\iff$ *their underlying complex vector bundles are isomorphic. Symplectic vector bundles with compatible almost complex structure and a metric are called a* ***Hermitian Structure****.*

Proposition 6 *Let* $E \to M$ *be a* $2n$*-dimensional vector bundle.*

1. *Every symplectic form has a compatible almost complex structure. The space* $\mathcal{J}(E, \omega)$ *is contractible.*

2. *The space of symplectic forms compatible with a given almost complex structure is contractible.*

Definition 11 *A* ***Unitary Trivialization*** *is a smooth map of a symplectic vector bundle, an almost complex structure, and a metric into Euclidean space, transforming each structure into its standard structure. A* ***Unitary Trivialization along a curve*** *is a trivialization of the pull-back bundle along a curve.*

Lemma 11 *If a curve has unitary trivializations at its endpoints, then it can be extended to a unitary transformation along the entire curve*

Proposition 7 *A Hermitian vector bundle* $E \to \Sigma$ *over a compact Riemann surface* Σ *with non-empty boundary* $\partial\Sigma$ *admits a unitary trivialization.*

1.8 First Chern Classes

Theorem 7 *There is a unique function called the* ***First Chern Number****, that assigns an integer* $c_1(E)$ *to every symplectic vector bundle* E *over a compact oriented Riemann surface* Σ *without boundary and satisfies the following axioms:*

1. *(naturality) Two isomorphic vector bundles have the same Chern number*

2. *(functoriality) Any smooth map* $\phi : \Sigma' \to \Sigma$ *of oriented Riemann surfaces and any symplectic vector bundle* $E \to \Sigma$*, then* $c_1(\phi^* E) = \deg(\phi) c_1(E)$

3. *(additivity) For any two symplectic vector bundles* $E_1 \to \Sigma$ *and* $E_2 \to \Sigma$*,* $c_1(E_1 \bigoplus E_2) = c_1(E_1) + c_1(E_2)$.

4. *(normalization) The Chern number of the tangent bundle is* $c_1(T\Sigma) = 2 - 2g = \chi(\Sigma)$

Remark 2

1. *The first Chern number vanishes* $\iff$ *the bundle is trivial; so the first Chern number is an indicator of if the bundle can be symplectically trivialized.*

2. *Usually the Chern number is defined for complex vector bundles, which is fine for our definition.*