1 Covering Spaces

1.1 The Definition of Riemann Surfaces

Definition 1 *Let X be 2-d manifold. A complex chart on X is a homeomorphism $\phi : U \to V$ of an open subset U of X onto an open subset $V \subset \mathbb{C}$. Two chart ϕ_1, ϕ_2 are said to be holomorphically compatible if the overlap map*

$$\phi_2 \circ \phi_1^{-1} : \phi(U_1 \cap U_2) \to \phi(U_1 \circ U_2)$$

is biholomorphic. A complex atlas is a collection of mutually holomorphically compatible charts whose domains cover X.

Remark 1 *Open subdomains of complex charts naturally induce a holomorphically compatible chart by restriction. Additionally, holomorphic compatibility is an equivalence relation.*

Definition 2 *A complex structure on a two-dimensional manifold is an equivalence class of holomorphically compatible atlases. A Riemann Surface is a pair of a connected 2-d manifold and a complex structure on the manifold.*

Definition 3 *Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f : Y \to \mathbb{C}$ is called holomorphic is for every chart ψ, the composition $f \circ \psi^{-1} : U \cap V \to \mathbb{C}$ is holomorphic. The set of holomorphic functions on Y will be denoted by $\mathcal{O}(Y)$.*

Remark 2

1. *The sum and product of holomorphic functions are again holomorphic, and constant functions are holomorphic. Thus $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.*
2. *One only needs check the holomorphicity of a covering set of charts for Y, not every single chart.*
3. *The 'coordinate charts' ψ is trivially holomorphic. One usually uses the letter z instead of ψ.*

Theorem 1 (Riemann's Removable Singularities Theorem) *Let U be an open subset of a Riemann surface and $a \subset U$. Suppose $f \in \mathcal{O}(U \backslash \{a\})$ is bounded in some neighborhood of a. Then f can be uniquely extended to a function $\overline{f} \in \mathcal{O}(U)$*

Definition 4 *Suppose X and Y are Riemann surfaces. A cotninuous mapping $f : X \to Y$ is called holomorphic if every coordinate representation of the function is holomorphic as a map from $\mathbb{C}$ to $\mathbb{C}$.*

A mapping is biholomorphic if it is bijctive, holomorphic, and its inverse is holomorphic. Two surfaces are isomorphic if there is a biholomorphic mapping between them.

Remark 3 *1. When the target space is the complex plane, holomorphic mappings are clearly the same as holomorphic functions.*

2. Composition of holomorphic mappings are again holomorphic.

3. A holomorphic mapping induces a ring homomorphism:

$$f^* : \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V)); \;\; f^*(\psi) = \psi \circ f$$

Theorem 2 (Identity Theorem) *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ with limit point $a \in X$. Then f_1, f_2 are identically equal.*

Theorem 3 *Let $Y \subset_{op} X$ be an open subset of a Riemann surface X. A meromorphic function on Y is a holomorphic function $f : Y' \to \mathbb{C}$, Y' an open subset with the following:*

1. *$Y \backslash Y'$ consists of only isolated points.*
2. *For every point $p \in Y \backslash Y'$,*

$$\lim_{x \to p} |f(x)| = \infty$$

The points of $Y \backslash Y'$ are called the poles of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

Theorem 4 *Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f, define $f(p) = \infty$. Then $f : X \to \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f : X \to \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞, or $f^{-1}(\infty)$ is a set of isolated points and thus $f : X \backslash f^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.*

1.2 Elementary Properties of Holomorphic Mappings

Theorem 5 (Local Behavior of Holomorphic Mappings) *Suppose X and Y are Riemann surfaces and $f : X \to Y$ a holomorphic mapping. Suppose $a \in X$ and $b = f(a)$. Then there exists an integer $k \geq 1$ and charts $\phi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:*

1. *$a \in U; \phi(a) = 0; \;\; b \in U'; \psi(b) = 0$*
2. *$f(U) \subset U'$*
3. *The map $F = \psi \circ f \circ \phi^{-1} : V \to V'$ is given by $F(z) = z^k$*

Remark 4 *The number k is theorem 5 can be characterized in the following way. For every neighborhood U_0 of a there exist neighborhoods $U \subset U_0$ of a and W of $b = f(a)$ such that the set $f^{-1}(y) \cap U$ contains k elements for every points $y \in W, y \neq b$. One calls k the multiplicity of f as a.*

Corollary 1 *Let X and Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic mapping. Then f is open; taking open sets to open sets.*

Corollary 2 *Let X and Y be Riemann surfaces, and let $f : X \to Y$ be an injective holomorphic mapping. Then f is a biholomorphic mapping of X onto $f(X)$.*

Corollary 3 (Maximum Principle) *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a non-constant holomorphic function. Then the absolute value of f does not attain its maximum.*

Theorem 6 *Suppose X and Y are Riemann surfaces. Suppose X is compact and $f : X \to Y$ is a non-constant holomorphic mapping. Then Y is compact and f is surjective.*

Corollary 4 *Every holomorphic function on a compact Riemann surface is constant.*

Corollary 5 *Every meromorphic function f on $\mathbb{P}^1$ is a rational function.*

Theorem 7 (Liouville's Theorem) *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

1.3 Branched and Unbranched Coverings

Definition 5 *Suppose X and Y are topological spaces and $p : Y \to X$ is a continous map. For $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. If $y \in p^{-1}(x)$, we say y lies over x. If $p : Y \to X$ and $q : Z \to X$ are continuous maps, then a map $f : Y \to Z$ is called fiber-preserving if $p = q \circ f$. This means that ny points $Y \in Y$ lying over the point $x \in X$ is mapped to a point which also lies over x.*

A subset A of a topological space is called discrete if the subspace topology on A is discrete. A mapping $p : Y \to X$ between topological spaces X and Y is said to be discrete if every fiber is a discrete subset of Y.

Theorem 8 *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. Then p is open and discrete.*

If $p : YtoX$ is a non-constant holomorphic map, then we will say Y is a domain over X.

A holomorphic (meromorphic) function f may also be considered as a multivalued holomorphic function on X (??? this doesn't make sense).

Definition 6 *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. A point $y \in Y$ is called a branch point or ramification point of p, if there is no neighborhood V of y such that $p \restriction_V$ is injective. The map p is called an unbranched holomorphic map if it has no branch points.*

Theorem 9 *Suppose X and Y are Riemann surfaces. A non-constant holomorphic map $p : Y \to X$ has no branch points iff p is a local homeomorphism, i.e. every point $y \in Y$ has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.*