Hyperbolic Dynamics

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1 Hyperbolic Sets

Throughout, M is a C^1 Riemannian manifold, $U \subset M$ a non-empty open subset, and $f : U \to M$ a C^1 diffeomorphism.

Definition 1. *A compact, $f-$invariant subset $\Lambda \subset U$ is called hyperbolic if there are $\lambda \in (0, 1)$, $C > 0$, and regular distributions $E^s_x, E^u_x \subset T_x M$; $x \in \Lambda$ such that for all x:*

1. $T_x M = E^s_x \oplus E^u_x$
2. $\|T_x f^n v^s\| \leq C\lambda^n \|v^s\|$ *for all* $v^s \in E^s_x$
3. $\|T_x f^{-n} v^u\| \leq C\lambda^n \|v^u\|$ *for all* $v^u \in E^u_x$
4. $(T_x f)(E^s_x) = E^s_{f(x)}$ *and* $(T_x f)(E^u_x) = E^u_{f(x)}$

The distributions E^s and E^u are called the stable and unstable distribution of $f \restriction_\Lambda$. If $\Lambda = M$, then f is called an *Anosov diffeomorphism.*

Proposition 1. *Let Λ be a hyperbolic set of f. Then the stable and unstable distributions are smooth and regular.*

Proposition 2. *Let Λ be a hyperbolic set of f with constants C and λ. Then for $\varepsilon > 0$ there is a C^1 Riemannian metric $\langle \cdot, \cdot \rangle'$ in a neighborhood of Λ called the Lyapunov or adapted metric, for which f is hyperbolic with new constants $C' = 1$ and $\lambda' = \lambda + \varepsilon$, and the unstable and unstable distributions are $\varepsilon-$orthogonal ($\langle v^s, v^u \rangle' < \varepsilon$ for unit vectors in the respective distributions).*

A fixed point of a differentiable map f is *hyperbolic* if no eigenvalue of $T_x f$ lies on the unit circle. A periodic point of period k is called *hyperbolic* if no eigenvalue of $T_x f^k$ lies on the unit circle.

2 ε-Orbits

An ε-orbit is a finite or infinite sequence $(x_n) \subset U$ satisfying $d(f(x_n), x_{n+1}) \, \forall n$. These are also called *pseudo-orbits.*

Theorem 1. *Let Λ be a hyperbolic set of $f : U \to M$. Then there is an open $O \subset U$ containing Λ and there are positive ε_0, δ_0 satisfying: $\forall \varepsilon > 0\, \exists \delta\, \forall g : O \to M$ with $\operatorname{dist}_1(g, f) < \varepsilon_0$, any homeomorphism $h : X \to X$ and any continuous map $\phi : X \to O$ with $\operatorname{dist}_0(\phi \circ h, g \circ \phi) < \delta$, then there is a continuous map $\psi : X \to O$ with $\psi \circ h = g \circ \psi$ and $\operatorname{dist}_0(\phi, \psi) < \varepsilon$. Additionally, ψ is unique in the sense that $\psi' \circ h = g \circ \psi'$ & $\operatorname{dist}_0(\phi, \psi) < \delta_0$, then $\psi = \psi'$.*

Corollary 1. *Let Λ be a hyperbolic set of $f : U \to M$. Then for every $\epsilon > 0$ there is $\delta > 0$ such that if (x_k) is a (in)finite $\delta-$orbit of f and $\operatorname{dist}(x_k, \Lambda) < \delta$ then there is $x \in \Lambda_\varepsilon$ with $\operatorname{dist}((f^k(x), x_k) < \varepsilon$.*

Proof. *Choose O satisfying the conditions in 1 and δ such that $\Lambda_\delta \subset O$. If (x_k) is (semi-in)finite, add to (x_k) the preimages of some $y_0 \in \Lambda$ whose distnace to the first point in the sequence is $< \delta$, and/or the images of some $y_m \in \Lambda$ whose distance to the last point of the sequence is $< \delta$ to obtain a $\delta-$orbit lying in the $\delta-$ neighborhood of Λ. Let $X = (x_k)$ with the discrete topology, $g = f$, $h : X \to X$ the shift $x_k \mapsto x_{k+1}$ and $\phi : X \to U$ be the inclusion into the manifold. Since (x_k) is a $\delta-$orbit, $\operatorname{dist}(\phi(h(x_k)), f(\phi(x_k))) <\leq$, then theorem 1 applies and the corollary follows.*

Recall the set of nonwandering points $\mathrm{NW}(f)$ is the set of points where the iterate of any neighborhood intersects the neighborhood, and the Periodic points of f, $\mathrm{Per}(f)$. If Λ is $f-$invariant, we can speak of $\mathrm{NW}(f \restriction_\Lambda)$. In general, $\mathrm{NW}(f \restriction_\Lambda) \neq \mathrm{NW}(f) \cap \Lambda$.

Proposition 3. *If Λ is a hyperbolic set of $f : U \to M$, then $\overline{\mathrm{Per}(f \restriction_\Lambda)} = \mathrm{NW}(f \restriction_\Lambda)$.*

Corollary 2. *If $f : M \to M$ is Anosov, then $\overline{\mathrm{Per}(f)} = \mathrm{NW}(f)$.*

3 Invariant Cones

Let Λ be a hyperbolic set of $f : U \to M$. Since the distributions E^s and E^u are continuous, we can extend them to continuous distributions in a neighborhood $U(\Lambda) \supset \Lambda$. If $x \in \Lambda$ and $v \in T_xM$, then $v = v^s + v^u$. Now assume the metric is adapted with constant λ. For $\alpha > 0$, define the (un)stable cones of size α by

$$K^s_\alpha(x) = \{v \in T_xM| \ : \|v^u\| \leq \alpha \|v^s\|\}$$

$$K^u_\alpha(x) = \{v \in T_xM : \|v^s\| \leq \alpha \|v^u\|\}$$

For a cone K, let $\mathring{K} = \operatorname{int}(K) \cup \{0\}$. Let $\Lambda_\varepsilon = d_\Lambda^{-1}([0, \varepsilon))$.

Proposition 4. *For every $\alpha > 0$ there is $\varepsilon = \varepsilon(\alpha)$ such that $f^i(\Lambda_\varepsilon) \subset U(\Lambda)$, $i = -1, 0, 1$ and for every $x \in \Lambda_\varepsilon$:*

$$T_xf(K^u_\alpha(x)) \subset \mathring{K}^u_\alpha(f(x)); \ (T_{f(x)}f^{-1})(K^s_\alpha(f(x))) \subset \mathring{K}^s_\alpha(x)$$

Proposition 5. *For every $\delta > 0$, there are $\alpha > 0$ and $\varepsilon > 0$ such that $f^i(\Lambda_\varepsilon \subset U(\Lambda)$, $i = -1, 0, 1$ and for every $x \in \Lambda_\varepsilon$:*

$$\|T_x f^{-1}(v)\| \leq (\lambda + \delta)\|v\|, \; v \in K^u_\alpha(x)$$
$$\|T_x f(v)\| \leq (\lambda + \delta)\|v\|, \; v \in K^s_\alpha(x)$$

Proposition 6. *Let Λ be a compact invariant set of $f : U \to M$. Suppose that there is a $\alpha > 0$ and for every $x \in \Lambda$ there are continuous subspaces E^s_x, E^u_x such that $E^s_x \oplus E^u_x = T_x M$ and the $\alpha-$cones $K^s_\alpha(x)$ and $K^U_\alpha(x)$ determined by the subspaces satisfy*

1. *$(T_x f)(K^u_\alpha(x)) \subset K^u_\alpha(x)$ and $(T_{f(x)} f^{-1})(K^u_\alpha(x)) \subset K^s_\alpha(x)$*
2. *$\|T_x f(v)\| < \|v\|$ for non-zero $v \in K^s_\alpha(x)$, and $\|T_x f^{-1} v\| < \|v\|$ for non-zero $v \in K^u_\alpha(x)$.*

Then Λ is a hyperbolic set of f.

Let
$$\Lambda^s_\varepsilon = \{x \in U : d_\Lambda(f^n(x)) < \varepsilon \;\forall n\}$$

$$\Lambda^u_\varepsilon = \{x \in U : d_\Lambda(f^{-n}(x)) \quad \forall n\}$$
Note that both sets are contained in Λ_ε and $f(\Lambda^s_\varepsilon) \subset \Lambda^s_\varepsilon$, and $f^{-1}(\Lambda^u_\varepsilon) \subset \Lambda^u_\varepsilon$.

Proposition 7. *Let Λ be a hyperbolic set of f with adapted metric. Then for every $\delta > 0$ there is $\varepsilon > 0$ such that the distributions E^s and E^u can be extended to Λ_ε so that*

1. *E^s is continuous on Λ^s_ε, E^u is continuous on Λ^u_ε.*
2. *$x \in \Lambda_\varepsilon \cap f(\Lambda_\varepsilon) \Rightarrow (T_x f)(E^s_x) = E^s_{f(x)}$ and $(T_x f)(E^u_x) = E^u_{f(x)}$*
3. *$\|(T_x f)(v)\| < (\lambda + \delta)\|v\|$ for every $x \in \Lambda_\varepsilon$ and $v \in E^s_x$.*
4. *$\|(T_x f^{-1})(v)\| < (\lambda + \delta)\|v\|$ for every $x \in \Lambda_\varepsilon$ and $v \in E^u_x$.*

4 Stability of Hyperbolic Sets

Proposition 8. *Let Λ be a hyperbolic set of $f : U \to M$. There is an open set $U(\Lambda) \supset \Lambda$ and $\varepsilon_0 > 0$ such that if $K \subset U(\Lambda)$ is a compact invariant subset of a diffeomorphism $g : U \to M$ with $\mathrm{dist}_1(g, f) < \varepsilon_0$, then K is a hyperbolic set of g.*

Let $\mathrm{Diff}^1(M)$ be the space of C^1 diffeomorphisms of M with the C^1 topology.

Corollary 3. *The set of Anosov diffeomorphisms of a given compact manifold is open in $\mathrm{Diff}^1(M)$.*

Proposition 9. *Let Λ be a hyperbolic set of $f : U \to M$. For every open set $V \subset U$ containing Λ and every $\varepsilon > 0$, there is $\delta > 0$ such that $\forall g : V \to M$ with $\operatorname{dist}_1(g, f) < \delta$, there is a hyperbolic set $K \subset V$ of g and a homeomorphism $\chi : K \to \Lambda$ such that χ cojugates f to g and $\operatorname{dist}_0(\chi, \mathrm{Id}) < \varepsilon$.*

A C^1 diffeomorphism f of a C^1 manifold is called *structurally stable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $g \in \mathrm{Diff}^1(M)$ and $\operatorname{dist}_1(g, f) < \delta$, then there is a homeomorphsim $h : M \to M$ conjugated f and g and $\operatorname{dist}_0(h, \mathrm{Id}) < \varepsilon$.

Corollary 4. *Anosov diffeomorphisms are structurally stable.*

5 Stable and Unstable Manifolds

For $\delta > 0$, let B_δ be the ball of radius δ at 0.

Proposition 10 (Hadamard-Perron)**.** *Let $f_n : B_\delta to \mathbb{R}^m$ be a sequence of C^1 diffeomorphisms onto their images such that $\forall n$ $f_n(0) = 0$. Suppose that for each n there is a splitting $\mathbb{R}^m = E_n^s \oplus E_n^u$ and $\lambda \in (0, 1)$ such that*

1. *$T_0 f_n(E_n^s) = E_{n+1}^s$ and $T_0 f_n(E_n^u) = E_{n+1}^u$*
2. *$\|T_0 f_n v^s\| < \lambda \|v^s\|$ for all $v^s \in E_n^s$*
3. *$\|T_0 f_n v^u\| > \lambda \|v^u\|$ for all $v^u \in E_n^u$*
4. *The angles between E_n^u and E_n^s are uniformly bounded away from 0*
5. *$(T f_n)$ are an equicontinuous family of fuctions $T f_n : B_\delta \to \mathrm{GL}_m(\mathbb{R})$.*

THEN there are $\varepsilon > 0$ and a sequence $\phi = (\phi_n)$ of uniformly Lipschitz continuous maps $\phi_n : B_\varepsilon^s = E_n^s \cap B_\varepsilon \to E_n^u$ such that

1. *$\operatorname{graph}(\phi_n) \cap B_\varepsilon = W_\varepsilon^s(n)$, where the latter set is defined as $\{x \in B_\varepsilon : \|f_{n+k-1} \circ ... \circ f_{n+1} \circ f_n(x)\| \to 0$ as $k \to \infty\}$*
2. *$f_n(\operatorname{graph}(\phi_n)) \subset \operatorname{graph}(\phi_{n+1})$*
3. *$x \in \operatorname{graph}(\phi_n) \Rightarrow \|f_n(x)\| \le \lambda \|x\| \Rightarrow f_n^k(x) \to 0$ exponentially as $k \to \infty$*
4. *for $x \in B_\varepsilon \backslash \operatorname{graph}(\phi_n)$,*

   $$\|P_{n+1}^u f_n(x) - \phi_{n+1}\left(P_{n+1}^s f_n(x)\right)\| > \lambda^{-1} \|P_n^u x - \phi_n\left(P_n^s x\right)\|$$

   Where P_n^s (P_n^u) denotes the projection onto E_n^s (E_n^u) parallel to the other subspace
5. *ϕ_n is differentiable at 0, $T_0 \phi_n 0 = 0 \Rightarrow$ the tangent plane to $\operatorname{graph}(\phi_n)$ is E_n^s.*

6. *ϕ depends continuously on f in the topologies by the following distance functions:*

$$d_0(\phi, \psi) = \sup_{x,n} 2^{-n} |\phi_n(x) - \phi_n(x)|$$

$$d(f, g) = \sup_n \mathrm{dist}_1(f_n, g_n)$$

Let $\Phi(L, \varepsilon)$ be the space of sequences $\phi = (\phi_n)$ where $\phi_n : B^s_\varepsilon \to E^u_n$ is Lipschitz-continuous map with Lipschitz constant L and $\phi_n(0) = 0$, with a metric $d(\phi, \psi) = \sup_{n,x} |\phi_n(x) - \psi_n(x)|$, which is complete.

We now define an operator $F : \Phi(L, \varepsilon) \to \Phi(L, \varepsilon)$ called the *graph transform*. Let $\phi \in \Phi(L, \varepsilon)$. The next lemma will show that $f_n^{-1}(\mathrm{graph}(\phi_{n+1}))$ projected onto E^s_n covers $E^s_\varepsilon(n)$ and $f_n^{-1}(\mathrm{graph}(\phi_{n+1}))$ contains the graph of a continuous function $\psi_n : B^s_\varepsilon \to E^u_\varepsilon(n)$ with Lipschitz constant L. Take $F(\phi)_n = \psi_n$.

Lemma 1. *For any $L > 0$, there exists $\varepsilon > 0$ such that the graph transform F is a well-defined operator on $\Phi(L, \varepsilon)$.*

Lemma 2. *There are $\varepsilon > 0$ and $L > 0$ such that F is a contracting operator.*

Theorem 2. *Let $f : M \to M$ be a C^1 diffeomorphism of a differentiable manifold and Λ a hyperbolic set of f with constant λ and adapted metric.*

Then there are $\varepsilon > 0$, $\delta > 0$ such that for every $x^s \in \Lambda^s_\delta$ and every $x^u \in \Lambda^u_\delta$:

1. *the sets*

$$W^s_\varepsilon(x^s) = \{y \in M : \mathrm{dist}(f^n(x^s), f^n(y)) < \varepsilon \ \forall n\}$$
$$W^u_\varepsilon(x^u) = \{y \in M : \mathrm{dist}(f^{-n}(x^u), f^{-n}(y)) < \varepsilon \ \forall n\}$$

   called the local stabale manifold of x^s and the local unstable manifold of x^u, are C^1 embedded disks,

2. *$T_{y^s} W^s_\varepsilon(x^s) = E^s_{y^s}$ for all $y^s \in W^s_\varepsilon(x^s)$ and similarly for the unstable manifolds and subspaces,*

3. *$f(W^s_\varepsilon(x^s)) \subset W^s_\varepsilon(f(x^s))$ and $f^{-1}(W^u_\varepsilon(f(x^u))) \subset W^u_\varepsilon(x^u)$*

4. *if $y^s, z^s \in W^s_\varepsilon(x^s)$, then $d^s(f(y^s), f(z^s)) < \lambda d^s(y^s, z^s)$, where d^s is the distance along $W^s_\varepsilon(x^s)$, and a similar result for the local unstable manifold using the inverse map*

5. *if $0 < \mathrm{dist}(x^s, y) < \varepsilon$ and $\exp^{-1}_{x^s}(y)$ lies in the $\delta-$cone $K^u_\delta(x^s)$, then $\mathrm{dist}(f(x^s), f(y)) > \ lambda^{-1}\mathrm{dist}(x^s, y)$ and if $0 < \mathrm{dist}(x^u, y) < \varepsilon$ and $\exp^{-1}_{x^u}(y)$ lies in the $\delta-$cone $K^s_\delta(x^u)$, then $\mathrm{dist}(f(x^u), f(y)) < \lambda \mathrm{dist}(x^s, y)$*

6. *if $y^s \in W^s_\varepsilon(x^s)$, then $W^s_\alpha(y^s) \subset W^s_\varepsilon(x^s)$ for some $\alpha > 0$, and if $y^u \in W^u_\varepsilon(x^u)$, then $W^u_\beta(y^u) \subset W^s_\varepsilon(x^u)$ for some $\beta > 0$.*

Let Λ be a hyerbolic set of $f : U \to M$ and $x \in \Lambda$. The *(global) stable and unstable manifolds* of x are defined by

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0, n \to \infty\}$$

$$W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0, n \to \infty\}$$

Proposition 11. *There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and every $x \in \Lambda$,*

$$W^s(x) = \bigcup_{n>0} f^{-n}(W^s_\varepsilon(f^n(x)))$$

$$W^u(x) = \bigcup_{n>0} f^n(W^u_\varepsilon(f^{-n}(x)))$$

Corollary 5. *The global stable and unstable manifolds are embedded C^1 submanifolds of M homeomorphic to unit balls in corresponding dimensions.*

6 Inclination Lemma

Recall the definition of two submanifolds to intersect transversely.

Denote by B^i_ε the open ball of radius ε centered at 0 in $\mathbb{R}^i$. For $v \in \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^l$ denote by $v^u \in \mathbb{R}^k$ and $v^s \in \mathbb{R}^l$ the components of $v = v^u + v^s$, and $\pi^u : \mathbb{R}^m \to \mathbb{R}^k$ the projection. For $\delta > 0$ let $K^u_\delta = \{v \in \mathbb{R}^m : \|v^s\| \leq \delta\|v^u\|\}$ and the stable cone $K^s_\delta = \{v \in \mathbb{R}^m : \|v^s\| \leq \delta\|v^u\|\}$

Lemma 3. *Let $\lambda \in (0,1), \varepsilon > 0, \delta \in (0, 0.1)$. Suppose $f : B^k_\varepsilon \times B^l_\varepsilon \to \mathbb{R}^m$ and $\phi : B^k_\varepsilon \to B^l_\varepsilon$ are C^1 amps such that:*

1. *0 is a hyperbolic fixed point of f*
2. *$W^u_\varepsilon(0) = B^k_\varepsilon \times \{0\}$ and $W^s_\varepsilon = \{0\} \times B^l_\varepsilon$*
3. *$\|T_x f(v)\| \geq \lambda^{-1}\|v\|$ for every $v \in K^u_\delta$ whenever both $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$*
4. *$\|T_x f(V)\| \leq \lambda\|v\|$ for every $v \in K^s_\delta$ whenever both $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$*
5. *$T_x f(K^u_\delta) \subset K^u_\delta$ whenever $x, f(x) \in B^k_\varepsilon \times B^l_\varepsilon$*
6. *$T_x f^{-1}(K^s_\delta) \subset K^s_\delta$ whenever $x, f^{-1}(x) \in B^k_\varepsilon \times B^l_\varepsilon$*
7. *$T_{(y,\phi(y))}\mathrm{graph}(\phi) \subset K^u_\delta$ for every $y \in B^k_\varepsilon$*

Then for every n there is a subset $D_n \subset B^k_\varepsilon$ diffeomorphic to B^k such that the image I_n under f^n of the graph of the restriction $\phi \restriction_{D_n}$ has the following properties: $\pi^u(I_n) \supset B^k_{\varepsilon/2}$ and $T_x I_n \subset K^u_{\delta\lambda^{2n}}$ for each $x \in I_n$.

The meaning of the lemma is that the tangent planes to the image of the grap of ϕ under f^n are exponentially (in n) close to the "horizontal" space $\mathbb{R}^k$, and the image spreads over B^k_ε in the horizontal direction.

The next theorem, sometimes called the Lambda Lemma, implies that if f is C^r with $r \geq 1$, and D is any C^1-disk that intersects transversely the stable manifold $W^s(x)$ of a hyperbolic fixed point of x, then the forwards images of D converge in the C^r topology to the unstable manifold $W^u(x)$. The proof only covers C^1 convergence. Let B^u_R be the ball of radius R centered at x in $W^u(x)$ in the induced metric.

Theorem 3 (Inclination Lemma)**.** *Let x be a hyperbolic fixed point of a diffeomorphism $f : U \to M$, $\dim(W^u(x)) = k$ and $\dim(W^s(x)) = l$. Let $y \in W^s(x)$ and suppose that $D \ni y$ is a C^1 submanifold of dimension k intersecting $W^s(x)$ transversely at y.*

Then for every $R > 0$ and $\beta > 0$ there are n_0 and for each $n \geq n_0$, a subset $\tilde{D} = \tilde{D}(R, \beta, n)$, $y \in \tilde{D} \subset D$, diffeomorphic to an open $k-$disk and such that the C^1 distance between $f^n(\tilde{D})$ and B^u_R is less than β.

7 Horseshoes and Transverse Homoclinic Points

Let $\mathbb{R}^{>} = \mathbb{R}^k \times \mathbb{R}^l$. We will refer to $\mathbb{R}^k$ and $\mathbb{R}^l$ as the unstable and stable subspaces, respectively, and denote by π^u and π^s the projections to these spaces. For $v \in \mathbb{R}^m$ denoted by $v^u = \pi^u(v) \in \mathbb{R}^k$ and $v^s = \pi^s(v) \in \mathbb{R}^l$. For $\alpha \in (0, 1)$, call the sets $K^u_\alpha = \{v \in \mathbb{R}^m : |v^s| \leq \alpha |v^u|\}$ and $K^s_\alpha = \{v \in \mathbb{R}^m : |v^u| \leq \alpha |v^s|\}$ the unstable and stable cones, respectively. Let $R^u = \{x \in \mathbb{R}^k : |x| \leq 1\}$, $R^s = \{x \in \mathbb{R}^l : |x| \leq 1\}$, and $R = R^u \times R^s$. For $z = (x, y) \in \mathbb{R}^k \times \mathbb{R}^l$, the sets $F^s(z) = \{x\} \times R^s$ and $F^u(z) = R^u \times \{y\}$ will be called the stable and unstable fibers, respectively. We say that a C^1 map $f : R \to \mathbb{R}^m$ has a *horseshoe* if there are $\lambda, \alpha \in (0, 1)$ such that:

1. f is one-to-one on R
2. $f(R) \cap R$ has at least two components $\Delta_0, \ldots, \Delta_{q-1}$
3. if $z \in R$ and $f(z) \in \Delta_i$, $0 \leq i < q$, then the sets $G^u_i(z) = f(F^u(z)) \cap \Delta_i$ and $G^s_i(z) = f^{-1}(F^s(f(z)) \cap \Delta_i)$ are connected, and the restriction of π^u to $G^u_i(z)$ and of π^s to $G^s_i(z)$ are bijective
4. if $z, f(z) \in R$, then the derivative $T_z f$ preserves the unstable cones K^u_α and $\lambda |T_z f(v)| \geq |v|$ for every $v \in K^u_\alpha$, and the inverse $T_{f(z)} f^{-1}$ preserves the stable cones K^s_α and $\lambda |T_{f(z)} f^{-1}(v)| \geq |v|$.

The intersection $\Lambda = \bigcap_{n>0} f^n(R)$ is called a *horseshoe.*

Theorem 4. *The horseshoe $\Lambda = \bigcap_{n>0} f^n(R)$ is a hyperbolic set of f. If $f(R) \cap R$ has q components, then the restriction of f to Λ is topologically conjugate tot he full two-sided shift σ in the space of Σ_q of bi-infinite sequences in the alphabet* $\{0, 1, \ldots, q-1\}$

Corollary 6. *If a diffeomorphism has a horseshoe, then the topological entropy of f is positive.*

Let p be a hyperbolic fixed periodic point of a diffeomorphism $f : U \to M$. A point q is called *homoclinic* (for p) if $q \neq p$ and $q \in W^s(p) \cap W^u(p)$; it is called *transverse homoclinic* (for p) if in addition $W^s(p)$ and $W^u(p)$ intersect transversely at q.

Theorem 5. *Let p be a hyperbolic periodic point of a diffeomorphism $f : U \to M$, and let q be a transverse homoclinic point of p. Then for every $\varepsilon > 0$ the union of $\varepsilon-$neighborhoods of the orbits of p and q contains a horseshoe of f.*

8 Local Product Structure and Locally Maximal Hyperbolic Sets

A hyperbolic set Λ of $f : U \to M$ is called *locally maximal* if there is an open set V such that $\Lambda \subset V \subset U$ and $\Lambda = \bigcap_{n>0} f^n(V)$. Since every closed invariant subset of a hyperbolic set is also a hyperbolic set, the geometric structure of a hyperbolic set may be very complicated and difficult to describe. However, due to their special properties, locally maximal hyperbolic sets allow a geometric characterization.

Since $E^s_x \cap E^u_x = \{0\}$, the local stable and unstable manifolds of x intersect at x transversely. By continuity, this transversality extends to a neighborhood of the diagonal in $\Lambda \times \Lambda$.

Proposition 12. *Let Λ be a hyperbolic set of f. For every samll enough $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is transverse and consists of exactly one point $[x, y]$, which depends continuously on x and y. Furthermore, there is $C_p = C_p(\delta) > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $d^s(x, [x, y]) \le C_p d(x, y)$ and $d^u(x, [x, y]) \le C_p d(x, y)$, where d^s and d^u are distances along the stable and unstable manifolds, respectively.*

Let $\varepsilon > 0, k, l \in \mathbb{N}$, let $B^k_\varepsilon \subset \mathbb{R}^k$, and $B^l_\varepsilon \subset \mathbb{R}^l$ be $\varepsilon-$balls.

Lemma 4. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\phi : B^k_\varepsilon \to \mathbb{R}^l$ and $\psi : B^l_\varepsilon \to \mathbb{R}^k$ are differentiable maps and $|\phi(x)|, \|T\phi(x)\|, |\psi(y)|, \|T\phi(y)\| < \delta$ for all $x \in B^k_\varepsilon$ and $y \in B^l_\varepsilon$, then the intersection* $\mathrm{graph}(phi) \cap \mathrm{graph}(psi) \subset \mathbb{R}^{k+l}$ *is transverse and consists of exactly one point, which depend continuously on ϕ and ψ in the C^1 topology.*

The following property of hyperbolic sets plays a major role in their geometric description and is equivalent to local maximality. A hyperbolic set Λ has a *local product structure* if there area (small enough) $\varepsilon > 0$ and $\delta > 0$ such that

1. $\forall x, y \in \Lambda$, the intersection $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of at most one point, belonging to Λ

2. $\forall x, y \in \Lambda$ with $d(x, y) < \delta$, the intersection consists of exactly one point of Λ, denoted by $[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$, and the intersection is transverse.

If a hyperbolic set Λ has a local product structure, then for every $x \in \Lambda$ there is a neighborhood $U(x)$ such that

$$U(x) \cap \Lambda = \{[y, z] : y \in U(x) \cap W^s_\varepsilon(x), z \in U(x) \cap W^u_\varepsilon(x)\}$$

Proposition 13. *A hyperbolic set Λ is locally maximal iff it has a local product structure.*

9 Anosov Diffeomorphisms

Recall that a C^1 diffeomorphism f of a connected differentaible manifold M is called *Anosov* if M is a hyperbolic set for f; it follows then that M is a locally maximal and compact.

An important class of Anosov diffeomorphisms is as follows: Let N be a simply connected nilpotent Lie group, and Γ a uniform discrete subgroup of N. The quotient $M = N/\Gamma$ is a compact *nilmanifold*. Let $\overline{f}$ be an automorphism of N that preserves Γ and whose derivative at the identity is hyperbolic. The induced diffeomorphism f of M is Anosov. Up to finite coverings, all known Anosov diffeomorphisms are topologically conjugate to automorphisms of nilmanifolds.

The families of stable and unstable manifolds of an Anosov diffeomorphism for two foliations called the *stable foliation* W^s and unstable foliation W^u These foliations are in general not C^1, or even Lipschitz, but they are Hölder continuous. In spite of lack of Lipschitz continuity, the stable and unstable foliations possess a uniqueness property similar to the uniqueness theorem for ordinary differential equations.

Proposition 14. *LEt $f : M \to M$ be an Anosov diffeomorphism. Then there are $\lambda \in (0,1)$, $C_p > 0$, $\varepsilon > 0$, $\delta > 0$ and for each $x \in M$, a splitting $T_xM = E^s_x \oplus E^u_x$ such that:*

1. $T_xf(E^s_x) = E^s_{f(x)}$ *and* $T_xf(E^u_x) = E^u_{f(x)}$
2. $\|T_xf(v^s)\| \leq \lambda\|v^s\|$ *and* $T_xf^{-1}(v^u) \leq \lambda\|v^u\|$ *for* $v^s \in E^s_x, v^u \in E^u_x$.
3. $W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0$ *as* $n \to \infty\}$ *and* $d^s(f(x), f(y)) \leq \lambda d^s(x,y)$ *for every* $y \in W^s(x)$
4. $W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0$ *as* $n \to \infty\}$ *and* $d^u(f^{-1}(x), f^{-1}(y)) \leq \lambda d^u(x,y)$ *for every* $y \in W^u(x)$
5. $f(W^s(x)) = W^s(f(x))$ *and* $f(W^u(x)) = W^u(f(x))$
6. $T_xW^s(x) = E^s_x$ *and* $T_xW^u(x) = E^u_x$
7. *if* $d(x,y) < \delta$, *then the intersection* $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ *is exactly one point* $[x,y]$, *which depends continuously on* x *and* y, *and* $d^s([x,y],x) \leq C_p d(x,y)$; $d^u([x,y],y) \leq C_p d(x,y)$.

A diffeomorphism is structurally stable if $\forall\varepsilon > 0$ there is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that $\forall g \in \mathcal{U}$ there is a homeomorphism h conjugating f and g and $\mathrm{dist}_0(h, \mathrm{Id}) < \varepsilon$.

Proposition 15.

1. *Anosov diffemorphisms forma n open (possibly empty) subset in the C^1 topology.*
2. *Anosov diffeomorphisms are structurally stable.*

3. *The set of periodic points of an Anosov diffeomorphism is dense in the set of non-wandering points.*

Theorem 6. *Let $f : M \to M$ be an Anosov diffeomorphism. Then TFAE:*

1. $\mathrm{NW}(f) = M$
2. *Every unstable manifold is dense in M*
3. *every stable manifold is dense in M*
4. *f is topologically transitive*
5. *f is topologically mixing*

10 Axiom A and Structural Stability

A diffeomorphism satisfies Smale's *Axiom A* if the set $\mathrm{NW}(f)$ is hyperbolic and $\overline{\mathrm{Per}(f)} = \mathrm{NW}(f)$.

For a hyperbolic periodic point p of f, denote by $W^s(O(p))$ and $W^u(O(p))$ the unions of the stable and unstable manifolds of p and its images, respectively. If p and q are hyperbolic periodic points, we write $p \leq q$ when $W^s(O(p))$ and $W^u(O(p))$ have a point of transverse intersection. $\leq$ is reflexive and transitive. If $p \leq q$ and $q \leq p$, we write $p\ q$ and say that p and q are *heteroclinically related*. This is an equivalence relation.

Theorem 7 (Smale's Spectral Decomposition Theorem)**.** *If f satisfies Axiom A, then there is a unique representation of* $\mathrm{NW}(f)$*,*

$$\mathrm{NW}(f) = \Lambda_1 \cup \cdots cup \Lambda_k$$

as a partition of closed $f-$invariant subsets (called basic sets) such that:

1. *each Λ_i is a locally maximal hyperbolic set of f*
2. *f is topologically transitive on each λ_i*
3. *each Λ_i is a disjoint union of closed sets $\Lambda_i^j, i \leq j \leq m_i$, with f cycically permuting the set Λ_i^j and f^{m_i} is topologically mixing on each Λ_i^j.*

The basic sets are precisely the closures of the equivalence classes of . For two basic sets, we write $\Lambda_i \leq \Lambda_j$ if there are periodic points $q \in \Lambda_j$ and $p \in \Lambda_i$ such that $p \leq q$.

Let f satisfy Axiom A. f satisfies the *strong transversality condition* if $W^s(x)$ intersects $W^u(y)$ transversely (at all point of intersection) for all $x, y \in \mathrm{NW}(f)$.

Theorem 8. *A C^1 diffeomorphism is structurally stable iff it satisfies Axiom A and the strong transversality condition.*