Notes from *Curvature and Homology*

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Publisher's Description

This systematic and self-contained treatment examines the topology of differentiable manifolds, curvature and homology of Riemannian manifolds, compact Lie groups, complex manifolds, and curvature and homology of Kaehler manifolds. It generalizes the theory of Riemann surfaces to that of Riemannian manifolds. Includes four helpful appendixes

Transcription Notes

Copied without proofs for independent learning. Some less abstract and more computational sections were skipped.

Contents

0.1 Topology of Differentiable Manifolds

0.1.1 Complexes

Definition 0.1.1. *A closure finite abstract complex K is a countable collection of object $\{S_i^p\}$ called simplexes satisfying the following properties:*

1. *To each simplex S_i^p there is associated an integer $p \geq 0$ called its dimension;*

2. *To the simplexes S_i^p and S_j^{p-1} is associated an integer denoted by $\left[S_i^p : S_j^{p-1}\right]$ called their incidence number;*

3. *There are only a finite number of simplexes S_j^{p-1} such that $\left[S_i^p : S_j^{p-1}\right] \neq 0$;*

4. *For every pait of simplexes* S_i^{p+1}, S_j^{p-1} *whose dimensions differ by two*

$$\sum_k \left[S_i^{p+1} : S_k^p\right] \left[S_k^p : S_j^{p-1}\right] = 0$$

We associate with K *an integer* $\dim K$ *called its dimension which is the max dimension of its simplexes.*

Definition 0.1.2. *An algebraic structure is imposed on* K *as follows: the p-simplexes are taken as free generators of an abelian group. A finite sum*

$$C_p = \sum_i g_i S_i^p; \;\; g_i \in G$$

where G *is an abelian group group is called a p-dimensional chain or a p-chain. Two p-chains may be addded, with their sum being the sum of their coefficients of each simplex. This way, p chains form an abelian group denoted by* $C_p(K, G)$.

Definition 0.1.3. *Let* Λ *be a ring with unity 1. A* Λ*-module is an abelian group* A *together with a map* $(\lambda, a) \to \lambda a$ *of* $\Lambda \times A \to A$ *satisfying*

1. $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2$
2. $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a$
3. $(\lambda_1 \lambda_2)a = \lambda_1 (\lambda_2 a)$
4. $1a = a$

Definition 0.1.4. *Let* A *be a right* Λ*-module and* B *a left* Λ*-module. Let* $F_{A\times B}$ *the free abelian group having as a basis the set* $A \times B$ *of pairs* (a, b) *and let* Γ *be the subgroup of* $F_{A\times B}$ *the subgroup of* $F_{A\times B}$ *generated by elements of the form*

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$

$$(a\lambda, b) - (a, \lambda b)$$

The quotient group $F_{A\times B}/\Gamma$ *is called the tensor product of* A *and* B *and it is an abelian group.*

Definition 0.1.5. *The boundary map* $\partial : C_p(K, G) \to C_{p-1}(K, G)$ *is defined by the formula*

$$\partial C_p = \sum_i g_i \partial S_i^p = \sum_i \sum_j g_i \left[S_i^p : S_j^{p-1}\right] S_j^{p-1}$$

where since $\left[S_i^p : S_j^{p-1}\right]$ *is an integer, its multiplication against* g_i *is considered as a multiple of* g_i *in the* $\mathbb{Z}$*-module of* G*. As a linear function, the boundary map is a group homomorphism.*

Definition 0.1.6. *The kernel of ∂ is denoted by $Z_p(G, K)$, and its elements are called p-cycles. Since $\partial^2 = 0$, the set of p-cycles contains the image of ∂ on $C_{p-1}(K, G)$, denoted by $B_p(K, G)$ whose elements are called boundaries. The quotient group*

$$H_p(K, G) = Z_p(K, G)/B_p(K, G)$$

is called the p-th homology group of K with coefficient group G. the elements of $H_p(K, G)$ are called homology classes.

Definition 0.1.7. *Let $C_p(K) = C_p(K, \mathbb{Z})$, elements of which we will call integral p-chains of K. A linear function f^p defined on $C_p(K)$ with values in a commutative topological group G:*

$$f^p : C_p(K) \to G$$

is called a p-dimensional cochain or a p-cochain. We define groups dual to the homology groups by using function addition as the group operation on p-cochains.

Definition 0.1.8. *The operator ∂^* dual to ∂ called hte coboundary operator is defined by*

$$\left(\partial^* f\right)\left(C_{p+1}\right) = f^p\left(\partial C_{p+1}\right)$$

It is a linear, square-free map.

Definition 0.1.9. *The kernel of ∂^* is denoted by $Z^p(K, G)$ and its elements are called p-cocycles. The image of $C^{p-1}(K, G)$ under ∂^* is denoted by $B^p(K, G)$ and its elements are called coboundaries. The quotient group*

$$H^p(K, G) = Z^p(K, G)/B^p(K, G)$$

is called the p-th cohomology group of K wit coefficient group G. Its elements are called cohomology classes.

0.1.2 Singular Homology

Definition 0.1.10. *A geometric realization K_E of an abstract complex K we mean a complex whose simplexes are geometric simplexes; i.e., points, lines, triangles, tetrahedrons in Euclidean space $\mathbb{R}^n$ of sufficiently high dimension, in such a way that distinct abstract simplexes correspond to disjoint geometric simplexes. The union of all the simplexes in K_E, written $|K_E|$ is called a polyhedron and the abstract complex is said to be a covering of $|K_E|$.*

Definition 0.1.11. *Two complexes are isomorphic if there is a bijection between the two preserving incidences.*

Proposition 0.1.1. *Isomorphic complexes induce a homeomorphism between their geometric realizations. The homology groups of isomorphic complexes are isomorphic.*

Definition 0.1.12. *If the group of coefficient G form a ring F, the homology groups become modules over F. The rank of $H_p(K,F)$ as a module over F is called the p-th betti number $b_p(K)$. If F has characteristic zero, $H_p(K)$ is a vector space. The expression $\sum_p (-1)^p b_p(K)$ is called the Euler-Poincaré characteristic of K.*

Definition 0.1.13. *A p-simplex $[\phi : S^p]$ on a differentiable manifold M is a geometric simplex and a differentiable map $\phi : S^p \to M$. A singular p-chain s^p on M is a formal sum of p-simplexes with coefficients in a group G.*
The support of s^p is $\phi(S^p)$, and a chain is locally finite if each compact set in M meets only a finite number of supports with $g_i \neq 0$.
The faces of a p-simplex $s^p = [\phi, S^p]$ are the simplexes $\phi\left(S_i^{p-1}\right)$. A boundary operator ∂ is defined by putting

$$\partial s^p = \sum_i (-1)^i s_i^{p-1}$$

and $\partial s^0 = 0$. Cycles and boundaries are defined with respect to this boundary map, giving rise to the p^{th} singular homology space of M, ddenoted SH_p.

0.1.3 Stokes' Theorem

Definition 0.1.14. *Let ϕ be a singular p-simplex and α a p-form on the differentiable manifold M. Since ϕ is continuous, the intersection of the supports of ϕ and α is compact. Define the integral of α over $s^p = [\phi, S^p]$ by*

$$\int_{s^p} \alpha = \int_{S^p} \phi^* \alpha$$

For a chain $C_p = \sum_i g_i s_i^p$, extend the integral by linearity over the region of integration:

$$\int_{C_p} = \sum_i g_i \int_{s_i^p} \alpha$$

Proposition 0.1.2. *Consider the functional L_α defined by*

$$L_\alpha (C_p) = \int_{C_p} \alpha$$

L_α being linear makes it a cochain, and Stokes' theorem makes L_α a cocycle if α is closed and a coboundary if α is exact.

0.1.4 De Rham Cohomology

Theorem 0.1.1.

$$D^p(M) \approx H^p(M)$$

where $D^p(M)$ is the space of p-forms on M and $H^p(M) = H^p(M,\mathbb{R})$.

Theorem 0.1.2. *$b_p(M)$ is the number of linearly independdent closed differential forms modulo exact forms of degree p.*

0.1.5 Periods

Skipped.

0.1.6 Decomposition theorem for compact Riemann surfaces

Theorem 0.1.3. *For a compact Riemann surface S,*

$$D^1(S) \approx H^1(S)$$

Definition 0.1.15. *The linear map*

$$* : D^1(S) \to D^1(S); \;\; *(pdx + qdy) = -qdx + pdy$$

which has the following properties

1. $*^2 = -1$
2. $(\alpha, \beta) = \alpha \wedge *\beta$ *is an inner product.*

Define as well:

1. $(*d)f = *(df)$ *for a function* f
2. $(*d)\alpha = -d(*\alpha)$ *for a 1-form* α
3. $\Delta = d * d$

So that for a function f,

$$\Delta f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy$$

A function is called harmonic if Δf vanishes on S. A 1-form α is called a harmonic form if it is locally the exterior derivative of a harmonic function.

Proposition 0.1.3. *α is harmonic if and only if $d\alpha = 0$ and $d * \alpha = 0$.*

Theorem 0.1.4. *The first betti number of a compact Riemann surface is equal to the number of linearly independent harmonic 1-forms on the surface.*

0.1.7 The Star Isomorphism

Definition 0.1.16. *We define the Laplacian as an operator*

$$\Delta : C^\infty(M) \to C^\infty(M)$$

by

$$-\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} g^{ij} D_j f \right)$$

And Laplace's equation is $\Delta f = 0$.

Definition 0.1.17. *We define the isomorphism* $* : \Lambda^p(M) \to \Lambda^{n-p}(M)$ *by*

$$\alpha = a_{(i_1 \ldots i_p)} du^i \wedge \ldots \wedge du^{i_p} \Rightarrow *\alpha = a^*_{(j_1 \ldots j_{n-p})} du^{j_i} \wedge \cdots \wedge du^{j_{n-p}}$$

where

$$a^*_{j_1 \ldots j_{n-p}} = \epsilon_{(i_1 \ldots i_p) j_1 \ldots j_{n-p}} a^{(i_1 \ldots i_p)}$$

where $\epsilon_{i_1 \ldots i_p j_1 \ldots j_{n-p}}$ *is the Levi-Civitas symbol, scaled to* $\sqrt{g}$. $*\alpha$ *is called the adjoint of* α.

Proposition 0.1.4. $*$ *has the following properties:*

1. $* * \alpha = (-1)^{pn+p}\alpha$
2. $\alpha \wedge *\beta$ *is an inner product on* $\Lambda^p(M)$.

Definition 0.1.18. *We define (global) scalar product of* α *and* β *as the number*

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta$$

Proposition 0.1.5. $*$ *is an isometry on* $\Lambda^p(M)$ *with the scalar product above.*

0.1.8 Harmonic Forms

Definition 0.1.19. *The co-differential of a form* α *is defined as*

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha$$

Proposition 0.1.6. *The co-differential has the following properties*

1. $\delta\delta\alpha = 0$
2. $*\delta\alpha = (-1)^p d * \alpha$; $*d\alpha = (-1)^{p+1}\delta * \alpha$

Definition 0.1.20. *A form is co-closed if its co-differential is zero, and if* $\alpha = \delta\beta$ *means that* α *is co-closed. A form is closed in the sense of Hodge if it is closed and co-closed, and closed in the sense of Kodaira if its Laplacian vanishes.*

0.1.9 Orthogonality Relations

Definition 0.1.21. *Two linear operators* A, A' *are said to be dual if* $(A\alpha, \beta) = (\alpha, A'\beta)$. d *and* δ *are dual.*

Proposition 0.1.7. *The following are the orthogonality relations:*

1. *A form is closed if and only if and only if it is orthogonal to all co-exact forms of the appropriate degree.*
2. *A form is co-closed if and only if it is orthogonal to all exact forms.*
3. *On a compact Riemannian manifold, the the definitions of harmonic in the senses of Kodaira and Hodge are equivalent.*
4. *A harmonic function on a compact harmonic manifold is necessarily a constant.*

0.1.10 Decomposition Theorem for Compact Riemannian Manifolds

Theorem 0.1.5 (Hodge-de Rham)**.** *A regular form α of degree p may be uniquely decomposed into the sum*

$$\alpha = \alpha_d + \alpha_\delta + \alpha_H$$

where α_d is exact, α_δ is co-exact, and α_H is harmonic.

0.1.11 Fundamental Theorem

The following are the *existence theorems of De Rham.*

Theorem 0.1.6 (R1)**.** *Let $\left\{\Gamma_p^i\right\};\ 1 \leq ileqb_p(M)$ be a base for the (rational) p-cycles of a compact differentiable manifold M and ω_p^i be $b_p(M)$ arbitrary real constants. Then there exists a regular closed p-form α on M having the prescribed periods on the cycles; i.e.,*

$$\int_{\Gamma_p^i} \alpha = \omega_p^i$$

Theorem 0.1.7 (R2)**.** *A closed form having zero periods is exact. In addition, there exists a unique harmonic form α having arbitrarily prescribed periods on $b_p(M)$ independent p-cycles of a compact and orientable Riemannian manifold.*

Theorem 0.1.8. *Let M be a compact and orientable Riemannian manifold. Then the number of linearly inddependent real harmonic forms of degree p is equal to the p^{th} betti number of M.*

0.1.12 Explicit Expressions for d, δ, and Δ

Skipped.

0.2 Curvature and Homology of Riemannian Manifolds

0.2.1 Some Contributions of S. Bochner

Proposition 0.2.1. *If $\tilde{M}$ is a covering manifold of M which is also compact, then*

$$b_p(M) \leq b_p(\tilde{M})$$

0.2.2 Curvature and Betti Numbers

Lemma 0.2.1. *For a regular 1-form α on a compact and orientable Riemannian manifold M*

$$\int_M \delta\alpha * 1 = \int_M \delta\alpha \wedge *1 = 0$$

Definition 0.2.1. *Define the operator Q by*

$$(Q\alpha)_i = R_i^j \alpha_j$$

where R_i^j is the Ricci curvature tensor.

Theorem 0.2.1. *The first betti number of a compact and orientable Riemannian manifold of positive definite Ricci curvature is zero.*

Definition 0.2.2. *A parallel vector field is one for which*

$$A(t) = \alpha^i(t) \frac{\partial}{\partial u^i}$$

is parallel along any parametrized curve $u(t)$.

Theorem 0.2.2. *In a compact and orientable Riemannian manifold a harmonic vector field for which the quadratic form $\langle Q\alpha, \alpha \rangle$ is positive semi-definite is necessarily a parallel vector field.*

Theorem 0.2.3. *In a coordinate neighborhood of a compact and orientable Riemannian manifold with the local coordinates $u^1, \ldots, u^n$, α is a harmonic one form if and only if*

$$R_i^j \alpha_j = g^{ik} D_k D_j \alpha_i = 0$$