1 Hamiltonian and Lagrangian Systems

1.1 Symplectic Geometry

Definition 1 *Let M be a manifold and $\omega \in \Omega^2(M)$. Then define the isomorphism $\flat : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$; $\;X \mapsto X^\flat = i_X\omega$, and the map $\sharp$ be its inverse.*

Theorem 1 (Darboux) *Suppose ω is a nondegenerate two-form on a $2n$-manifold. Then $d\omega = 0$ iff there is a chart (U, ϕ) around each point m such that $\phi(m) = 0$ and $\omega|_U$ is canonical.*

Definition 2 *A **sympletic form** on a manifold M is a nondegenerate, closed two-form ω on M. A **Symplectic Manifold** is a manifold equipped with a symplectic form. The associated volume form is $\Omega_\omega = [(-1)^{[n/2]}/n!]\omega^n$. The charts in which the symplectic form takes the canonical form are called **symplectic charts**, and the coordinate functions are called **canonical coordinates**.*

Definition 3 *If (M, ω) and (N, ρ) are symplectic manifolds, a C^∞ map between them that preserves the symplectic structure is called a **canonical transformation**.*

Proposition 1 *A canonical transformation has determinant 1 and is a local diffeomorphism.*

Theorem 2 *Let $M = T^*Q$, with $\tau_Q^* : M \to Q$ and $T\tau_Q^* : TQ \to TM$. Let $\alpha_q \in M$ and $\omega_{\alpha_q} \in T_{\alpha_q}M$. Then let $\theta_{alpha_q} : T_{\alpha_q}M \to \mathbb{R} : \omega_{\alpha_q} \mapsto \langle \alpha_q, T\tau_Q^*(\omega_{\alpha_q})\rangle$, and $\theta_0 : \alpha_q \mapsto \theta_{\alpha_q}$. Then $\omega_0 = -d\theta_0$ is symplectic and the forms ω_0 and θ_0 are called the **canonical forms**.*

The canonical forms, given in the usual coordinates of a cotangent bundle, take the form:

$$\theta_0 = \sum p_i dq^i$$

$$\omega = \sum dq^i \wedge dp_i$$

The canonical one-form can be thought of as a 'formal adjoint' to the projection operator:

$$\langle \theta(\alpha_q), w_{\alpha_q} \rangle = \langle T\tau_Q^* w_{\alpha_q}, \alpha_q \rangle$$

1.2 Hamiltonain Vector Fields and Poisson Brackets

Definition 4 *On a symplectic manifold, given a function $H : M \to \mathbb{R}$, the **Hamiltonian Vector Field** associate to the function is a the vector field X_H satisfying $\omega(X_H, Y) = \langle dH, Y\rangle$, or that $i_{X_H}\omega = dH$.*

Proposition 2 *H is constant along the flow of X_H.*

Proposition 3 *Along a Hamiltonian flow, the symplectic form is conserved.*

Definition 5 *A vector field X is **locally Hamiltonian** if for every point, there is a neighborhood U of m such that $X|_U$ is Hamiltonian*

Proposition 4 *TFAE:*

1. *X is locally Hamiltonian*
2. *$\mathcal{L}_X\omega = 0$*
3. *The flow of X consists of canonical transformations*

Remark 1 *Locally Hamiltonian vector fields for a Lie subalgebra of $\mathfrak{X}(M)$. Globally Hamiltonian vector fields are locally Hamiltonian, but the other way around requires $H^1(M) = 0$.*

Definition 6 *Let $\alpha, \beta \in \mathfrak{X}^*(M)$. Then the Poisson Bracket of alpha and β is the one-form $-[\alpha^\sharp, \beta^\sharp]^\flat$*

Definition 7 *Let M be a sympletic manifold and $f, g : M \to \mathbb{R}$, then the Poisson bracket of f and g is $\{f, g\} = -i_{X_f} i_{X_g}\omega$.*

Proposition 5

$$\{f, g\} = -\mathcal{L}_{X_f} g = \mathcal{L}_{X_g} f$$

Which mean the Poisson bracket is a derivation over f and g individually.

Corollary 1

1. *$\frac{d}{dt}(f \circ F_t^{X_H}) = \{f \circ F_t^{X_H}, H\}$*
2. *$d\{f, g\} = \{df, dg\}$*

Definition 8 *The **Lagrange Bracket** of two vector fields is the function $[[X, Y]] = \omega(X, Y)$ and the Lagrange bracket of a chart is a matrix formed from the Lagrange bracket of each coordinate vector.*

Theorem 3 *Let (u, φ) be a chart on a symplectic manifold. Then*

1. *$\omega|_U = \sum [[u^i, u^j]] du^i \wedge du^j$*
2. *In a symplectic chart, the matrix ω_{ij} takes the off-diagonal block matrix form of a almost-complex structure.*
3. *If $f(q, p) = (Q, P)$, then $[[Q, P]] = \sum \left(\frac{\partial q^i}{\partial Q} \frac{\partial p^i}{\partial P} - \frac{\partial q^i}{\partial P} \frac{\partial p^i}{\partial Q} \right)$*
4. *$[[q, p]] \circ f^{-1} = [[Q, P]]$*

Theorem 4 *If X is a locally Hamiltonian vector field, and the pushforward of the canonical coordinates by the flow is denoted (Q_t, P_t), then $[[Q_t, P_t]] \circ F_t^X = [[q, p]]$*

1.3 Integral Invariants, Energy Surfaces, and Stability

Definition 9 *An invariant form for a vector field is one whose Lie derivative is zero.*

Proposition 6 *Let X be a vector field and α, β invariant forms of it. Then*

1. $i_X\alpha$ *is invariant*
2. $d\alpha$ *is invariant*
3. $\mathcal{L}_X\gamma$ *is closed* $\iff$ $d\gamma$ *is invariant*
4. $\alpha \wedge \beta$ *is invariant*

Definition 10 α *is relatively invariant* $\iff$ $\mathcal{L}_X\alpha$ *is closed.*

Definition 11 $\mathcal{A}_X$ *is the algebra of all invariant forms of X, $\mathcal{R}_X$ the relatively invariant forms of X, $\mathcal{C}$ the closed forms of $\Omega(M)$ and $\mathcal{E}$ the exact forms.*

Theorem 5 *The following sequences are exact:*

1. $0 \to \mathcal{A}_X \overset{i}{\to} \Omega(M) \overset{\mathcal{L}_X}{\to} \Omega(M) \overset{\pi}{\to} \Omega(M)/Im(\mathcal{L}_X) \to 0$
2. $0 \to \mathcal{C} \overset{i}{\to} \mathcal{R}_X \overset{d}{\to} \mathcal{A}_X \overset{\pi}{\to} \mathcal{A}_X/(\mathcal{E} \cap \mathcal{A}_X) \to 0$

Let Σ_e be a connected component of $H^{-1}(e)$, where e is a regular value of H.

Theorem 6 *There is a volume element μ_e invariant on Σ_e invariant under $X|\Sigma_e$*

Definition 12 *$V \subset M$ is a submanifold is an invariant manifold of a vector field if the vector field is tangent to V at every point.*

Definition 13 *Let $f_k : M \to \mathbb{R}$ be constants of motion for a Hamiltonian system X_H, and let $\vec{F} = (f_1, \ldots, f_n) : M \to \mathbb{R}^k$, and c a regular value of $\vec{F}$, and let $\Sigma_c = \vec{F}^{-1}(c)$. Then Σ_c is an invariant manifold of X_H of codimension n and there is an invariance volume μ_c defined on Σ_c.*

1.4 Lagrangian Systems

Definition 14 *Let f be any map between vector bundles E, F over the same base space. Then the **Fiber Derivative** of the function f is the function $\boldsymbol{F}f : E \to L(E, F);\ e \mapsto Df(e)$.*

Proposition 7 *Let $L : TQ \to \mathbb{R}$. Then $\boldsymbol{F}L : TQ \to T^*Q$ is smooth and fiber-preserving.*

Definition 15 *Let ω_0 be the canonical symplectic form on T^*Q and let $L : TQ \to \mathbb{R}$. Then the **Lagrange two-form** is $\omega_L = (\boldsymbol{F}L)^*\omega_0$*

Definition 16 *Let Q be a manifold and L a function on the tangent bundle. Then L is a regular Lagrangian if every point is a regular point of $\boldsymbol{F}L$*

Definition 17 *Given $L : TQ \to \mathbb{R}$, define the action $A : TQ \to \mathbb{R}$ by $A(v) = \langle \boldsymbol{F}L(v), v \rangle$ and the energy $E = A - L$. A Lagrangian vector field for L is a vector field X_L s.t. $i_{X_L}\omega_L = dE$.*

Theorem 7 *Let X_L be a Lagrangian vector field for L, then in a chart, the integral curves $(u(t), v(t))$ satisfy Lagrange's Equations:*

$$\frac{d}{dt}u(t) = v(t)$$

$$\frac{d}{dt}\left(\langle D_2L(u(t), v(t)), w \rangle\right) = \langle D_1L(u(t), v(t)), w \rangle$$

$\forall w \in TQ$.

Theorem 8 *Let L and $\tilde{L}$ be regular Lagrangians, and $X_L, X_{\tilde{L}}$ be their respective vector fields. Then TFAE:*

1. *$L = \tilde{L} + \alpha + C$, $d\alpha = 0$*
2. *$X_L = X_{\tilde{L}}$ & $\omega_L = \omega_{\tilde{L}}$*

The set of closed one-forms on Q form the 'gauge group' of Lagrangians, i.e. Lagrangians can be transformed without changing the dynamics.

1.5 The Legendre Transformation

Definition 18 *L is a hyperregular Lagrangian if $\boldsymbol{F}L : TQ \to T^*Q$ is a diffeomorphism.*

Theorem 9 *Let L be a hyperregular Lagrangian on Q and let $H = E \circ (\boldsymbol{F}L)^{-1} : T^*Q \to \mathbb{R}$, where E is the energy of L. Then $\boldsymbol{F}L$ conjugates the flow X_L to X_H.*

Theorem 10 $\boldsymbol{F}H = (\boldsymbol{F}L)^{-1}$

Corollary 2 *Hyperregular Hamiltonians and Lagrangians correspond bijectively by their fiber derivatives.*

1.6 Variational Principles in Mechanics

Definition 19 *The path space between two points is defined as $\Omega(q_1, q_2, [a, b]) = \{c : [a, b] \to Q |\ c$ is a C^2 curve,$c(a) = q_1$; $c(b) = q_2\}$*

Proposition 8 *The tangent space of the path space is $T_c\Omega(q_1, q_2, [a, b]) = \{v : [a, b] \to TQ | \pi_Q(v) = c, v(a) = 0, v(b) = 0\}$*

Theorem 11 *A function satisfies the Euler-Lagrange equations iff the resulting curve is a critical point of the action functional.*

Theorem 12 *(Euler-Lagrange-Jacobi-Maupertuis Principle of Least Action)*

Let $c_0(t)$ *be a base integral curve of* X_L, $q_1 = c_0(a); q_2 = c_0(b)$, *and* e *be the energy of* $c_0(t)$ *and be a regular value of a* e. *Let* A *be the accumulated (integrate) action along a path. Then* $dA(c) = 0$, *and the converse holds.*