Notes from *Lectures on Riemann Surfaces*

by Otto Forster

taken by Samuel T. Wallace

Publisher's Description

This book grew out of lectures on Riemann surfaces given by Otto Forster at the universities of Munich, Regensburg, and Mnster. It provides a concise modern introduction to this rewarding subject, as well as presenting methods used in the study of complex manifolds in the special case of complex dimension one.

Transcription Notes

Taken verbatim without proofs for an independent study project.

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0.1 Covering Spaces

0.1.1 The Definition of Riemann Surfaces

Definition 1. *Let X be 2-d manifold. A complex chart on X is a homeomorphism $\phi : U \to V$ of an open subset U of X onto an open subset $V \subset \mathbb{C}$. Two chart ϕ_1, ϕ_2 are said to be holomorphically compatible if the overlap map*

$$\phi_2 \circ \phi_1^{-1} : \phi(U_1 \cap U_2) \to \phi(U_1 \circ U_2)$$

is biholomorphic. A complex atlas is a collection of mutually holomorphically compatible charts whose domains cover X.

Remark 1. *Open subdomains of complex charts naturally induce a holomorphically compatible chart by restriction. Additionally, holomorphic compatibility is an equivalence relation.*

Definition 2. *A complex structure on a two-dimensional manifold is an equivalence class of holomorphically compatible atlases. A Riemann Surface is a pair of a connected 2-d manifold and a complex structure on the manifold.*

Definition 3. *Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f : Y \to \mathbb{C}$ is called holomorphic is for every chart ψ, the composition $f \circ \psi^{-1} : U \cap V \to \mathbb{C}$ is holomorphic. The set of holomorphic functions on Y will be denoted by $\mathcal{O}(Y)$.*

Remark 2.
1. *The sum and product of holomorphic functions are again holomorphic, and constant functions are holomorphic. Thus $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.*
2. *One only needs check the holomorphicity of a covering set of charts for Y, not every single chart.*

3. *The 'coordinate charts' ψ is trivially holomorphic. One usually uses the letter z instead of ψ.*

Theorem 1 (Riemann's Removable Singularities Theorem)**.** *Let U be an open subset of a Riemann surface and $a \subset U$. Suppose $f \in \mathcal{O}(U \backslash \{a\})$ is bounded in some neighborhood of a. Then f can be uniquely extended to a function $\overline{f} \in \mathcal{O}(U)$*

Definition 4. *Suppose X and Y are Riemann surfaces. A cotninuous mapping $f : X \to Y$ is called holomorphic if every coordinate representation of the function is holomorphic as a map from $\mathbb{C}$ to $\mathbb{C}$.*

A mapping is biholomorphic if it is bijctive, holomorphic, and its inverse is holomorphic. Two surfaces are isomorphic if there is a biholomorphic mapping between them.

Remark 3. 1. *When the target space is the complex plane, holomorphic mappings are clearly the same as holomorphic functions.*

2. *Composition of holomorphic mappings are again holomorphic.*

3. *A holomorphic mapping induces a ring homomorphism:*

$$f^* : \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V)); \;\; f^*(\psi) = \psi \circ f$$

Theorem 2 (Identity Theorem)**.** *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ with limit point $a \in X$. Then f_1, f_2 are identically equal.*

Theorem 3. *Let $Y \subset_{op} X$ be an open subset of a Riemann surface X. A meromorphic function on Y is a holomorphic function $f : Y' \to \mathbb{C}$, Y' an open subset with the following:*

1. *$Y \backslash Y'$ consists of only isolated points.*

2. *For every point $p \in Y \backslash Y'$,*

$$\lim_{x \to p} |f(x)| = \infty$$

The points of $Y \backslash Y'$ are called the poles of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

Theorem 4. *Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f, define $f(p) = \infty$. Then $f : X \to \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f : X \to \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞, or $f^{-1}(\infty)$ is a set of isolated points and thus $f : X \backslash f^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.*

0.1.2 Elementary Properties of Holomorphic Mappings

Theorem 5 (Local Behavior of Holomorphic Mappings)**.** *Suppose X and Y are Riemann surfaces and $f : X \to Y$ a holomorphic mapping. Suppose $a \in X$ and $b = f(a)$. Then there exists an integer $k \geq 1$ and charts $\phi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:*

1. $a \in U; \phi(a) = 0; \quad b \in U'; \psi(b) = 0$
2. $f(U) \subset U'$
3. *The map $F = \psi \circ f \circ \phi^{-1} : V \to V'$ is given by $F(z) = z^k$*

Remark 4. *The number k is theorem 5 can be characterized in the following way. For every neighborhood U_0 of a there exist neighborhoods $U \subset U_0$ of a and W of $b = f(a)$ such that the set $f^{-1}(y) \cap U$ contains k elements for every points $y \in W, y \neq b$. One calls k the multiplicity of f as a.*

Corollary 1. *Let X and Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic mapping. Then f is open; taking open sets to open sets.*

Corollary 2. *Let X and Y be Riemann surfaces, and let $f : X \to Y$ be an injective holomorphic mapping. Then f is a biholomorphic mapping of X onto $f(X)$.*

Corollary 3 (Maximum Principle)**.** *Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is a non-constant holomorphic function. Then the absolute value of f does not attain its maximum.*

Theorem 6. *Suppose X and Y are Riemann surfaces. Suppose X is compact and $f : X \to Y$ is a non-constant holomorphic mapping. Then Y is compact and f is surjective.*

Corollary 4. *Every holomorphic function on a compact Riemann surface is constant.*

Corollary 5. *Every meromorphic function f on $\mathbb{P}^1$ is a rational function.*

Theorem 7 (Liouville's Theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

0.1.3 Branched and Unbranched Coverings

Definition 5. *Suppose X and Y are topological spaces and $p : Y \to X$ is a continous map. For $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. If $y \in p^{-1}(x)$, we say y lies over x. If $p : Y \to X$ and $q : Z \to X$ are continuous maps, then a map $f : Y \to Z$ is called fiber-preserving if $p = q \circ f$. This means that ny points $Y \in Y$ lying over the point $x \in X$ is mapped to a point which also lies over x.*

A subset A of a topological space is called discrete if the subspace topology on A is discrete. A mapping $p : Y \to X$ between topological spaces X and Y is said to be discrete if every fiber is a discrete subset of Y.

Theorem 8. *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. Then p is open and discrete.*

If $p : YtoX$ is a non-constant holomorphic map, then we will say Y is a domain over X.

A holomorphic (meromorphic) function f may also be considered as a multivalued holomorphic function on X (??? this doesn't make sense).

Definition 6. *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. A point $y \in Y$ is called a branch point or ramification point of p, if there is no neighborhood V of y such that $p \restriction_V$ is injective. The map p is called an unbranched holomorphic map if it has no branch points.*

Theorem 9. *Suppose X and Y are Riemann surfaces. A non-constant holomorphic map $p : Y \to X$ has no branch points iff p is a local homeomorphism, i.e. every point $y \in Y$ has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.*