Notes from *Functional Analysis, Sobolev Spaces, and PDEs*

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Publisher's Description

Uniquely, this book presents a coherent, concise and unified way of combining elements from two distinct worlds, functional analysis (FA) and partial differential equations (PDEs), and is intended for students who have a good background in real analysis. This text presents a smooth transition from FA to PDEs by analyzing in great detail the simple case of one-dimensional PDEs (i.e., ODEs), a more manageable approach for the beginner. Although there are many books on functional analysis and many on PDEs, this is the first to cover both of these closely connected topics. Moreover, the wealth of exercises and additional material presented, leads the reader to the frontier of research. This book has its roots in a celebrated course taught by the author for many years and is a completely revised, updated, and expanded English edition of the important Analyse Fonctionnelle (1983). Since the French book was first published, it has been translated into Spanish, Italian, Japanese, Korean, Romanian, Greek and Chinese. The English version is a welcome addition to this list. The first part of the text deals with abstract results in FA and operator theory. The second part is concerned with the study of spaces of functions (of one or more real variables) having specific differentiability properties, e.g., the celebrated Sobolev spaces, which lie at the heart of the modern theory of PDEs. The Sobolev spaces occur in a wide range of questions, both in pure and applied mathematics, appearing in linear and nonlinear PDEs which arise, for example, in differential geometry, harmonic analysis, engineering, mechanics, physics etc. and belong in the toolbox of any graduate student studying analysis.

Transcription Notes

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0.1 The Hahn-Banach Theorems.

0.1.1 The Analytic Form of the Hahn-Banach Theorem: Extensions of Linear Functionals

Let E be a vector space over $\mathbb{R}$. We recall that a *functional* is a function defined n E, or a subspace of E, *with values in* $\mathbb{R}$.

Theorem 1 (Helly, Hahn-Banach analytic form). *Let $p : E \to \mathbb{R}$ be a function satisfying*

1. $p(\lambda x) = \lambda p(x)$

2. $p(x+y) \leq p(x) + p(y)$

Let $G \subset E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that

$$g(x) \leq p(x) \ \forall x \in G$$

Then there exists a linear functional $f : E \to \mathbb{R}$ such that $f \restriction_G = g$.

0.1.2 The Geometric Forms of the Hahn-Banach Theorem: Separation of Convex Sets

Definition 1. *An affine hyperplane is a subset $H \subset E$ of the form*

$$H = \{x \in E : f(x) = \alpha\}$$

where f is a linear functional that does not vanish identically and $\alpha \in \mathbb{R}$. We write $H = [f = \alpha]$ and say that $f = \alpha$ is the equation of H.

Proposition 1. *The hyperplane $H = [f = \alpha]$ is closed if and only if f is continuous.*

Definition 2. *Let A and B be two subsets of E. We say the hyperplane $H = [f = \alpha]$ separates A and B if*

$$f(a) \leq \alpha \ \forall a \in A \text{ and } f(b) \geq \alpha \ \forall b \in B$$

We say that H strictily separates A and B if there exists an $\epsilon > 0$ such that

$$f(a) \leq \alpha - \epsilon \ \forall a \in A \text{ and } f(b) \geq \alpha + \epsilon \ \forall b \in B$$

A subset $A \subset E$ is convex if

$$tx + (1-t)y \in A \ \forall x, y \in A \ \forall t \in [0,1]$$

Theorem 2 (Hahn-Banach, first geometric form)**.** *Let $A \subset E$ and $B \subset E$ be two nonempty convex disjoint subsets, one of which is open. Then there exists a closed hyperplane separating them.*

Theorem 3 (Hahn-Banach,second geometric form)**.** *Let $A \subset E$ and $B \subset E$ be two empty convex disjoint subsets. If A is closed and B is compact, then there exists a closed hyperplan separating A and B.*

Corollary 1. *Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^*$ not identically zero such that $f(F) = 0$*

0.1.3 The Bidual E^{**}. Orthogonality Relations

Let E be a normed vector space and let E^* be the dual space with norm

$$\|f\|_{E^*} = \sup_{\|x\| \leq 1} |\langle f, x \rangle|$$

The bidual E^{**} is the dual of E^* with norm

$$\|\xi\|_{E^{**}} = \sup_{\|f\|\leq 1} |\langle xi, f\rangle|$$

There is a canonical injection $J : E \to E^{**}$ defined as

$$\langle Jx, f\rangle_{E^{**},E^*} = \langle f, x\rangle_{E^*,E}$$

which is an *isometry*. J may not be surjective, but if it is, we say E is reflexive.

Definition 3. *If $M \subset E$ is a linear subspace, let*

$$M^{\perp} = \{f \in E^* : \langle f, x\rangle = 0 \ \forall x \in M\}$$

If $N \subset E^*$ is a linear subspace we set

$$N^{\perp} = \{x \in E : \langle f, x\rangle = 0 \ \forall f \in N\}$$

Proposition 2. *Let $M \subset E$ be a linear subspace. Then*

$$\left(M^{\perp}\right)^{\perp} = \overline{M}$$

Let $N \subset E^*$ be a linear subspace. Then

$$\overline{N} \subset \left(N^{\perp}\right)^{\perp}$$

0.1.4 A Quick Introduction to the Theory of Conjugate Convex Functions

Definition 4. *Let E be a set, and $\phi : E \to (-\infty, +\infty]$ a function. Let*

$$D(\phi) = \{x \in E : \phi(x) < +\infty\}$$

be the domain of ϕ. We define the epigraph of ϕ

$$\text{epi}\phi = \{[x, \lambda] \in E \times \mathbb{R}; \phi(x) \leq \lambda\}$$

If E is a topological space, we say ϕ is lower semicontinuous if $\lambda \in \mathbb{R}$ the set

$$[\phi \leq \lambda] = \{x \in E : \phi(x) \leq \lambda\}$$

is closed.

Proposition 3. *If ϕ is lower-semicontinuous, then*

1. epiϕ *is closed in $E \times \mathbb{R}$ and conversely,*
2. *for every $x \in E$ and $\epsilon > 0$ there is a neighborhood V of x such that*

   $$\phi(y) \geq \phi(x) - \epsilon \ \forall y \in V$$

   and conversely.

3. *If ϕ_1 and ϕ_2 are lower semicontinuous, then so is $\phi_1 + \phi_2$*

4. *If $(\phi_i)_{i\in I}$ is a family of lsc functions then so is*

$$\phi(x) = \sup_{i\in I} \phi_i(x)$$

   called the superior envelope.

5. *If E is compact and ϕ is lsc, then $\inf_E \phi$ is achieved.*

Definition 5. *A function $\phi : E \to (-\infty, +\infty]$ is convex if*

$$\phi\left(tx + (1-t)y\right) \le t\phi(x) + (1-t)\phi(y) \;\; \forall x, y \in E, \;\; \forall t \in (0,1)$$

Proposition 4. *If ϕ is a convex function, then*

1. $\mathrm{epi}\phi$ *is a convex set in $E \times \mathbb{R}$ and conversely*

2. *$\forall \lambda \in \mathbb{R}$ the set $[\phi \le \lambda]$ is convex, but not the converse*

3. *a sum of convex functions is again convex*

4. *the superior envelope of a family of convex functions is again convex.*

Let E be a normed vector space.

Definition 6. *Let $\phi : E \to (-\infty, +\infty]$ be a function with nonempty domain. We define the conjugate function $\phi^* : E^* \to (-\infty, +\infty]$ by*

$$\phi^*(f) = \sup_{x\in E} \{\langle f, x\rangle - \phi(x)\}$$

Proposition 5. *Assume that $\phi : E \to (-\infty, +\infty]$ is convex lsc with nonempty domain. Then ϕ^* has nonempty domain and is bounded below by an affine continuous function.*

Definition 7. *Instead of defining ϕ^{**} on E^{**}, we can define it on E by*

$$\phi^{**}(x) = \sup_{f\in E^*} \{\langle f, x\rangle - \phi^*(f)\}$$

Theorem 4 (Fenchel-Moreau)**.** *Let $\phi : E \to (-\infty, +\infty]$ is convex lsc with nonempty domain. Then $\phi^{**} = \phi$.*

Theorem 5 (Fenchel-Rockafeller)**.** *Let ϕ, ψ be two convex functions. Assume there is some $x_0 \in D(\phi) \cap D(\psi)$ such that ϕ is continuous at x_0. Then*

$$\inf_{x\in E} \{\phi(x) + \psi(x)\} = \sup_{f\in E^*} \{-\phi^*(-f) - \psi^*(f)\}$$

$$= \max_{f\in E^*} \{-\phi^*(-f) - \psi^*(f)\} = -\min_{f\in E^*} \{\phi^*(-f) + \psi^*(f)\}$$

0.2 The Uniform Boundedness Principle and the Closed Graph Theorem

0.2.1 The Baire Category Theorem

Theorem 6 (Baire). *Let X be a complete metric space and $(X_n)_{n\geq q}$ be a sequence of closed subsets in X. If*

$$\text{Int}X_n = \emptyset$$

Then

$$\text{Int}\left(\bigcup_n X_n\right) = \emptyset$$

0.2.2 The Uniform Boundedness Principle

Definition 8. *Let E and F be two normed vector spaces. Let $\mathcal{L}(E,F)$ be the space of continuous (bounded) linear operators equipped with the norm*

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\|x\|\leq 1} \|Tx\|$$

And we write $\mathcal{L}(E) = \mathcal{L}(E,E)$.

Theorem 7 (Banach-Steinhaus, uniform boundedness principle). *Let E and F be two Banach spaces and let $(T_i)_{i\in I}$ be a family of continuous linear operator from E into F. If*

$$\forall x \in E \ \sup_{i\in I} \|T_i x\| \leq \infty$$

Then

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)}$$

Corollary 2. *Let E and F be two Banch spaces. Let (T_n) be a sequence of continuous linear operators from E into F such that $\forall x \in E$ $T_n x$ converges (to a limit we call Tx). Then*

1. $\sup_n \|T_n\|_{\mathcal{L}(E,F)} < \infty$
2. $T \in \mathcal{L}(E,F)$
3. $\|T\|_{\mathcal{L}(E,F)} \leq \text{liminf}_n \|T_n\|_{\mathcal{L}(E,F)}$

Corollary 3. *Let G be a Banach space and let B be a subset of G. If*

$$\forall f \in G^* \ f(B) \text{ is bounded in } \mathbb{R}$$

Then B is bounded.

Corollary 4. *Let G be a Banach space and let B^* be a subset of G^*. If*

$$\forall x \in G \ \langle B^*, x\rangle \text{ is bounded in } \mathbb{R}$$

Then B^* is bounded.

0.2.3 The Open Mapping Theorem and the Closed Graph Theorem

Theorem 8 (Open Mapping Theorem)**.** *Let E and F be two Banach spaes and let T be a continuous linear operator from E into F that is surjective. Then there exists $\delta > 0$ such that*

$$T\left(B_E(0,1)\right) \supset B_F(0,\delta)$$

Which says T is an open mapping.

Corollary 5. *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is bijective. Then T^{-1} is also continuous.*

Corollary 6. *Let E be a vector space with two norms $\|\cdot\|_1, \|\cdot\|_2$ with both make E into a Banach Space, and that there is a constant $C \geq 0$ such that*

$$\|x\|_2 \leq C\|x\|_1$$

Then the two norms are equivalent.

Theorem 9 (Closed Graph Theorem)**.** *Let E and F be two Banach spaces. Let T be a linear operator from E to F. If the graph of T, $G(T)$, is closed in $E \times F$, then T is continuous.*

0.2.4 Complementary Subspaces. Right and Left Invertibility of Linear Operators

Theorem 10. *Let E be a Banach space. Assume that G and L are two closed linear suspaces such that $G + L$ is closed. Then there exists a constant $C \geq 0$ such that $z \in G+L \Rightarrow z = x+y$ with $C\|z\| \geq \|x\|, x \in G$ and $C\|z\| \geq \|y\|, y \in L$.*

Definition 9. *Let $G \subset E$ be a closed subspace of a Banach space E. A subspace $L \subset E$ is said to be a topological complement or simply a complement of G if*

1. *L is closed*
2. *$G \cap L = 0$ and $G + L = E$*

We also say G and L are complementary subspaces of E. If this holds, then every z can be decomposed into components in G and L, for which the projection operators are continuous.

Definition 10. *Let $T \in \mathcal{L}(E,F)$. A right inverse is an operator $S \in \mathcal{L}(F,E)$ such that $T \circ S = I_F$. A left inverse is an operator $S \in \mathcal{L}(F,E)$ such that $S \circ T = I_E$.*

Theorem 11. *Let $T \in \mathcal{L}(E,F)$ be surjective. The following are equivalent:*

1. *T admits a right inverse.*

2. *$N(T) = T^{-1}(0)$ admits a complements in E.*

Theorem 12. *Let $T \in \mathcal{L}(E, F)$ be injective. The following are equivalent:*

1. *T admits a left inverse.*
2. *$R(T) = T(E)$ is closed and admits a complement in F.*

0.2.5 Orthogonality Revisited

Proposition 6. *Let G and L be two closed subspaces in E. Then*

$$G \cap L = \left(G^{\perp} + L^{\perp}\right)^{\perp}$$

$$G^{\perp} \cap L^{\perp} = (G + L)^{\perp}$$

Corollary 7.

$$(G \cap L)^{\perp} \supset \overline{G^{\perp} + L^{\perp}}$$

$$\left(G^{\perp} \cap L^{\perp}\right)^{\perp} = \overline{G + L}$$

Theorem 13. *Let G and L be two closed subpsaces in a Banach spaces E. The following are equivalent:*

1. *$G + L$ is closed in E*
2. *$G^{\perp} + L^{\perp}$ is closed in E^**
3. *$G + L = \left(G^{\perp} + L^{\perp}\right)^{\perp}$*
4. *$G^{\perp} + L^{\perp} = (G \cap L)^{\perp}$*

0.2.6 An Introduction to Unbounded Linear Operators. Definition of the Adjoint

Definition 11. *Let E and F be two Banach spaces. An unbounded linear operator from E into F is a linear map $A : D(A) \subset E \to F$ where $D(A)$ is a linear subspace called the domain of A.*

A is bounded (or continuous) if $D(A) = E$ and there is a $c \geq 0$ such that

$$\|Au\| \leq c\|u\|$$

The norm of a bounded operator is defined as

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}$$

Some additional definitions are as follows:

1. *$G(A) = \{(u, Au) : u \in D(A)\} \subset E \times F$, the Graph of A*

2. $R(A) = \{Au : u \in D(A)\} \subset F$, *the range of* A

3. $N(A) = \{u \in D(A) : Au = 0\} \subset E$, *the kernel of* A.

An operator A *is closed if* $G(A)$ *is closed in* $E \times F$.

Definition 12. *Let* $A : D(A) \subset E \to F$ *be an unbounded linear operator that is densely defined* ($D(A)$ *is dense in* E)*. We introduce a new operator* $A^* : D(A^*) \subset F^* \to E^*$ *as follows. First we define*

$$D(A^*) = \{v \in F^* : \exists c \geq 0 \ |\langle v, Au\rangle| \leq c\|u\| \ \forall u \in D(A)\}$$

Now we go about defining A^*v*. Given* $v \in D(A^*)$*, we define* $g(u) = \langle v, Au\rangle$*. Use Hahn-Banach to extend* g *to a bounded functional* $f \in E^*$*, which is unique if* $D(A)$*is dense in* E*. Let* $A^*v = f$*. In brief,*

$$\langle v, Au\rangle_{F^*,F} = \langle A^*v, u\rangle_{E^*,E}$$

Proposition 7. *Let* $A : D(A) \subset E \to F$ *be a densely defined unbounded linear operator. Then* A^* *is closed.*

Corollary 8. *Let* $A : D(A) \subset E \to F$ *be an unbounded linear operator that is densely defined and closed. Then*

1. $N(A) = R(A^*)^\perp$

2. $N(A^*) = R(A)^\perp$

3. $N(A)^\perp \supset \overline{R(A^*)}$

4. $N(A^*)^\perp = \overline{R(A)}$

0.2.7 A Characterization of Operators with Closed Range. A Characterization of Surjective Operators

Theorem 14. *Let* $A : D(A) \subset E \to F$ *be an unbounded linear operator that is densely defined and closed. The following are equivalent:*

1. $R(A)$ *is closed*

2. $R(A^*)$ *is closed*

3. $R(A) = N(A^*)^\perp$

4. $R(A^*) = N(A)^\perp$

Theorem 15. *Let* $A : D(A) \subset E \to F$ *be a linear operator that is densely defined and closed. The following are equivalent:*

1. A *is surjective*

2. *There is a constant C such that*

$$\|v\| \leq C\|A^*v\|$$

3. $N(A^*) = \{0\}$ *and* $R(A^*)$ *is closed.*

Theorem 16. *Let $A : D(A) \subset F$ ($\to E^*$?) be an unbounded linear operator that is densely defined and closed. The following are equivalent:*

1. A^* *is surjective*

2. *there is a constant C such that*

$$\|u\| \leq C\|Au\|$$

3. $N(A) = 0$ *and* $R(A)$ *is closed.*