Notes from *Curvature and Homology*

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Contents

0.1 Topology of Differentiable Manifolds

0.1.1 Complexes

Definition 1. *A closure finite abstract complex K is a countable collection of object $\{S_i^p\}$ called simplexes satisfying the following properties:*

1. *To each simplex S_i^p there is associated an integer $p \geq 0$ called its dimension;*
2. *To the simplexes S_i^p and S_j^{p-1} is associated an integer denoted by $\left[S_i^p : S_j^{p-1}\right]$ called their incidence number;*
3. *There are only a finite number of simplexes S_j^{p-1} such that $\left[S_i^p : S_j^{p-1}\right] \neq 0$;*
4. *For every pait of simplexes S_i^{p+1}, S_j^{p-1} whose dimensions differ by two*

$$\sum_k \left[S_i^{p+1} : S_k^p\right] \left[S_k^p : S_j^{p-1}\right] = 0$$

We associate with K an integer $\dim K$ called its dimension which is the max dimension of its simplexes.

Definition 2. *An algebraic structure is imposed on K as follows: the p-simplexes are taken as free generators of an abelian group. A finite sum*

$$C_p = \sum_i g_i S_i^p; \;\; g_i \in G$$

where G is an abelian group group is called a p-dimensional chain or a p-chain. Two p-chains may be addded, with their sum being the sum of their coefficients of each simplex. This way, p chains form an abelian group denoted by $C_p(K, G)$.

Definition 3. *Let Λ be a ring with unity 1. A Λ-module is an abelian group A together with a map $(\lambda, a) \to \lambda a$ of $\Lambda \times A \to A$ satisfying*

1. $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2$
2. $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a$
3. $(\lambda_1 \lambda_2)a = \lambda_1 (\lambda_2 a)$
4. $1a = a$

Definition 4. *Let A be a right Λ-module and B a left Λ-module. Let $F_{A\times B}$ the free abelian group having as a basis the set $A \times B$ of pairs (a, b) and let Γ be the subgroup of $F_{A\times B}$ the subgroup of $F_{A\times B}$ generated by elements of the form*

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$
$$(a\lambda, b) - (a, \lambda b)$$

The quotient group $F_{A\times B}/\Gamma$ is called the tensor product of A and B and it is an abelian group.

Definition 5. *The boundary map $\partial : C_p(K,G) \to C_{p-1}(K,G)$ is defined by the formula*
$$\partial C_p = \sum_i g_i \partial S_i^p = \sum_i \sum_j g_i \left[S_i^p : S_j^{p-1} \right] S_j^{p-1}$$
where since $\left[S_i^p : S_j^{p-1} \right]$ is an integer, its multiplication against g_i is considered as a multiple of g_i in the $\mathbb{Z}$-module of G. As a linear function, the boundary map is a group homomorphism.

Definition 6. *The kernel of ∂ is denoted by $Z_p(G,K)$, and its elements are called p-cycles. Since $\partial^2 = 0$, the set of p-cycles contains the image of ∂ on $C_{p-1}(K,G)$, denoted by $B_p(K,G)$ whose elements are called boundaries. The quotient group*
$$H_p(K,G) = Z_p(K,G)/B_p(K,G)$$
is called the p-th homology group of K with coefficient group G. the elements of $H_p(K,G)$ are called homology classes.

Definition 7. *Let $C_p(K) = C_p(K,\mathbb{Z})$, elements of which we will call integral p-chains of K. A linear function f^p defined on $C_p(K)$ with values in a commutative topological group G:*
$$f^p : C_p(K) \to G$$
is called a p-dimensional cochain or a p-cochain. We define groups dual to the homology groups by using function addition as the group operation on p-cochains.

Definition 8. *The operator ∂^* dual to ∂ called hte coboundary operator is defined by*
$$(\partial^* f)(C_{p+1}) = f^p(\partial C_{p+1})$$
It is a linear, square-free map.

Definition 9. *The kernel of ∂^* is denoted by $Z^p(K,G)$ and its elements are called p-cocycles. The image of $C^{p-1}(K,G)$ under ∂^* is denoted by $B^p(K,G)$ and its elements are called coboundaries. The quotient group*
$$H^p(K,G) = Z^p(K,G)/B^p(K,G)$$
is called the p-th cohomology group of K wit coefficient group G. Its elements are called cohomology classes.

0.1.2 Singular Homology

Definition 10. *A geometric realization K_E of an abstract complex K we mean a complex whose simplexes are geometric simplexes; i.e., points, lines, triangles, tetrahedrons in Euclidean space $\mathbb{R}^n$ of sufficiently high dimension, in such a way that distinct abstract simplexes correspond to disjoint geometric simplexes. The union of all the simplexes in K_E, written $|K_E|$ is called a polyhedron and the abstract complex is said to be a covering of $|K_E|$.*

Definition 11. *Two complexes are isomorphic if there is a bijection between the two preserving incidences.*

Proposition 1. *Isomorphic complexes induce a homeomorphism between their geometric realizations. The homology groups of isomorphic complexes are isomorphic.*

Definition 12. *If the group of coefficient G form a ring F, the homology groups become modules over F. The rank of $H_p(K, F)$ as a module over F is called the p-th betti number $b_p(K)$. If F has characteristic zero, $H_p(K)$ is a vector space. The expression $\sum_p (-1)^p b_p(K)$ is called the Euler-Poincaré characteristic of K.*

Definition 13. *A p-simplex $[\phi : S^p]$ on a differentiable manifold M is a geometric simplex and a differentiable map $\phi : S^p \to M$. A singular p-chain s^p on M is a formal sum of p-simplexes with coefficients in a group G.*
The support of s^p is $\phi\,(S^p)$, and a chain is locally finite if each compact set in M meets only a finite number of supports with $g_i \neq 0$.
The faces of a p-simplex $s^p =$