Summary of Curvatures of Left Invariant Metrics on Lie Groups

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This is a summary of the paper "Curvatures of Left Invariant metrics on Lie Groups" by John Milnor available here.

1 Sectional Curvature

Let G be an n-dimensional Lie group, and $\mathfrak{g}$ its associated Lie algebra. Choosing some basis $e_1, \dots, e_n$ for $\mathfrak{g}$, there is obviously only one metric making this basis orthonormal. In fact, we can choose exactly one metric making the inner product $\langle e_i, e_j \rangle$ is the $i-j$-th component of a specific matrix. So there are a $\frac{1}{2}n(n+1)$ dimensional manifold of left-invariant metrics on G. The sectional curvature of a metric is defined to be

$$\kappa(x, y) = \langle R_{xy}(x), y \rangle \tag{1}$$

For orthonormal vectors x, y. This is Gaussian curvature of the surface swept out by the vectors x, y.

The structure constant of a Lie group are the numbers α_{ijk} such that

$$[e_i, e_j] = \sum_k \alpha_{ijk} e_k \tag{2}$$

The next fact is not practically useful, but theoretically interesting.

Lemma 1 *The sectional curvature is given in terms of structure constants by*

$$\kappa(e_i, e_j) = \sum_k \frac{1}{2}\alpha_{ijk}(-\alpha_{ijk} + \alpha_{jki} + \alpha_{kij}) - \\ \frac{1}{4}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})(\alpha_{ijk} + \alpha_{jki} + \alpha_{kij}) - \alpha_{kii}\alpha_{kjj}$$

The next fact is slightly more useful.

Lemma 2 *If* $\mathrm{ad}(u)$ *is skew-adjoint, then* $\kappa(u, v) \geq 0$ *when* $u \perp [v, \mathfrak{g}]$

There is an important corollary:

Corollary 1 *If u belongs to the center of $\mathfrak{g}$ (i.e. $[v, \mathfrak{g}] = 0$), then for any left-invariant metric and any vector v, $\kappa(u, v) \geq 0$.*

Lemma 3 *A left-invariant metric on a connected Lie group is also right-invariant iff $\mathrm{ad}(x)$ is skew-adjoint for all $x \in \mathfrak{g}$. A Lie group admits a bi-invariant metric iff it is isomorphic to a Cartesian product of a compact group and a commutative group.*

Corollary 2 *Every compact Lie group admits a left-invariant and a bi-invariant metric so that $K \geq 0$ for all sectional curvatures.*

Theorem 1 *A Lie Group with left-invariant metric is flat iff the associated Lie algebra splits as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$ where $\mathfrak{b}$ is a commutative subalgebra, $\mathfrak{u}$ is a commutative ideal, and if $\mathrm{ad}(b)$ is skew-adjoint for every $b \in \mathfrak{b}$.*

The necessary and sufficient conditions for a left-invariant metric to have negative sectional curvature is that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbb{R}x$ and that $\mathrm{ad}(x) \restriction_{[\mathfrak{g},\mathfrak{g}]}$ has eigenvalues with positive real part. $K \leq 0$ groups have been classified in the following statements.

Theorem 2 *If a connected Lie group G has a left-invariant metric with $K \leq 0$, then it is solvable. If a left-invariant Haar measure is also right-invariant (unimodular), then the $K = 0$.*

2 Ricci Curvature

Another curvature is given by the Ricci curvature, defined by

$$r(x) = \sum_i \kappa(x, e_i) = \sum_i \langle R_{xe_i}(x), e_i \rangle \tag{3}$$

For a unit vector u, $r(u)$ is the Ricci curvature of the direction. It is equal to $(n-1)$ times the average of the sectional curvature of all tangent planes containing u. It will become more convenient to work with the *Ricci transformation*, defined by

$$\widehat{r}(x) = \sum_i R_{e_i x}(e_i) \tag{4}$$

Which gives the relation

$$r(x) = \langle \widehat{r}(x), x \rangle \tag{5}$$

The eigenvalues of $\widehat{r}$ are called the principal Ricci curvatures. Now back to left-invariant metrics.

Lemma 4 *If $\mathrm{ad}(u)$ is skew-ajoint, then $r(u) \geq 0$, where there is only equality if $u \perp [\mathfrak{g}, \mathfrak{g}]$.*

Theorem 3 *A connected Lie group admits a left-invariant metric with all Ricci curvatures strictly positive iff it is compact with finite funamental group.*

Lemma 5 *If $u \perp [\mathfrak{g}, \mathfrak{g}]$, then $r(u) \leq 0$ with equality iff $\mathrm{ad}(u)$ is skew-adjoint.*

Definition 1 *A Lie algebra is **nilpotent** if some term in the series*

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supset \dots \tag{6}$$

is zero.

Theorem 4 *Suppose $\mathfrak{g}$ is nilpotent but not commutative. Then for any left-invariant metric there is a direction of strictly negative Ricci curvature and one of strictly positive Ricci curvature.*

Theorem 5 *If the Lie algebra of G contains linearly independent vectors x, y, z so that $[x, y] = z$, then there is a left-invariant metric so that $r(x) < 0$ and $r(z) > 0$.*

3 Scalar Curvature

Definition 2 *Choose an orthonormal basis e_i for the tangent space, then*

$$\rho = \sum_i r(e_i) \tag{7}$$

is the scalar curvature. It is $n(n-1)$ times the average of all sectional curvatures at a point.

Theorem 6 *If G is solvable, then every left-invariant metric on G is either flat, or has strictly negative curvature.*

Corollary 3 *If G is solvable and unimodular, then every left-invariant metric on G is either flat, or has both positive and negative sectional curvatures.*

Theorem 7 *If $\mathfrak{g}$ is noncommutative, then G has a left-invariant metric of strictly negative curvature.*

Theorem 8 (Wallach) *If the universal covering of G is not homeomorphic to Euclidean space, then G admits a left-invariant metric of strictly positive scalar curvature.*