1 Hamiltonian Systems With Symmetry

1.1 The Momentum Mapping

Definition 1 *Let (P, ω) be a connected symplectic manifold and $\Phi : G \times P \to P$ a symplectic action of the Lie group G on P. Then a **Momentum Mapping** for the action is a map $J : P \to \mathfrak{g}^*$ provided that $dJ(\xi) = i_{\xi_P}\omega$, where $\xi_P(x) = \frac{d}{dt}\Phi(\exp(\xi t)x)$.*

Theorem 1 *Let Φ be a symplectic action of a Lie group with momentum mapping J. Suppose a function $H : P \to \mathbb{R}$ is an invariant of the action, i.e. $\mathcal{L}_{\xi_P} H = 0\, \forall \xi \in \mathfrak{g}$. Then J is invariant for the flow of H, i.e. $\mathcal{L}_{X_H} J = 0$.*

Proposition 1 *Let (Φ, J) be a symplectic action and a momentum mapping. Define for $g \in G$ & $\xi \in \mathfrak{g}$: $\psi_{g,\xi} : P \to \mathbb{R} :\ x \mapsto J(\xi)(\Phi_g(x)) - J(\mathrm{Ad}_{g^{-1}}\xi)(x)$*

*Then $\psi_{g,\xi}$ is constant on P. Let $\sigma : G \to \mathfrak{g}^*$ be defined by $\sigma(g)\cdot\xi = \psi_{g,\xi}$, the **co-adjoint cocycle** associated to J. It satisfies the **cocycle identity** $\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h)$.*

Proposition 2 *Let G be a Lie group and $\mathfrak{g}$ its Lie Algebra. A **(co-adjoint) cocycle** is a map $\sigma : G \to \mathfrak{g}^*$ that satisfies the cocycle identity: $\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h)$.*

*A cocycle Δ is a **coboundary** if there is a $\Delta(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu$. The cocycles form a vector space and coboundaries form a subspace, so the quotient space of cocycles over coboundaries, $[\sigma]$, is the **cohomology** of G.*

Proposition 3 *Let Φ be a symplectic action of G and P, and two momentum mappings J_1 and J_2. Then $[\sigma_1] = [\sigma_2]$. So every symplectic group action there is a well-defined cohomology class.*

Definition 2 *A momentum mapping is Ad^*-**equivariant** when $J(\Phi_g(x)) = \mathrm{Ad}^*_{g^{-1}}J(x)$*

Proposition 4 *Let J be a momentum mapping for the symplectic action Φ with cocycle σ. Then:*

1. *The map $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*;\ (g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g)$*
2. *J is equivariant with respect to the action in 1.*

Theorem 2 *Let Φ be symplectic action of a Lie group with momentum mapping J with cocycle σ and define $\Sigma : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R};\ \ \Sigma(\xi, \eta) = d\widehat{\sigma}_\eta(e)\cdot\xi$ Where $\widehat{\sigma}_\eta : G \to \mathbb{R} :\ g \mapsto \sigma(g)\cdot\eta$ then:*

1. *Σ is a skew symmetric bilinear form on $\mathfrak{g}$ and satisfies Jacobi's identity*
2. *$J([\xi, \eta]) - \{J(\xi), J(\eta)\} = \Sigma(\xi, \eta)$*

Since $\Sigma(\xi, \eta)$ is constant, we have that $X_{\{J(\xi),J(\eta)\}} = X_{J([\xi,\eta])}$

Proposition 5 *If J is an Ad^*-equivariant momentum mapping, then $\{J(\xi), J(\eta)\} = J([\xi, \eta])$.*

Σ satisfying the Jacobi identity means that Σ defines a **two-cocycle** on $\mathfrak{g}$. A two-cocycle is called **exact** if there is a $\mu \in \mathfrak{g}^*$ such that $\Sigma(\xi, \eta) = \langle \mu, [\xi, \eta] \rangle$. So requiring any two-cocycle to be exact is a limitation on the cohomology condition on $\mathfrak{g}$. If Σ is exact, then $J(\xi) - \mu(\xi)$ is again a momentum mapping.

Theorem 3 *Let Φ be a symplectic action on P, where the symplectic form is exact, i.e. $\omega = -d\theta$, and the group action leaves θ invariant. Then θ forms an Ad^*-equivariant momentum map by $J(x) \cdot \xi = (i_{\xi_P}\theta)(x)$.*

Every symplectic action Φ can be lifted to an action on the tangent bundle by adjointness to the tangent map. i.e. $\langle T^*\Phi w, v \rangle = \langle w, T\Phi v \rangle$. We will write $T^*\Phi$ as Φ^{T^*} to avoid confusion.

Corollary 1 *The canonical momentum mapping for a canonical symplectic structure is given by $J : T^*Q \to \mathfrak{g}^*$; $J(\xi)(\alpha_q) = \langle \alpha_q, \xi_Q(q)$ for each one-form (momentum vector) α_q.*

Corollary 2 *Let G act on Q by the map Φ (not necessarily symplectically) and let Φ^T denote the pushforward on the tangent bundle. Now let L be a regular Lagrangian, and let $\theta_L = (FL)^*\theta_0$, and L is invariant under the action of Φ. Then*

1. *$(\Phi_g^T)^*\theta_L = \theta_L$*
2. *The momentum for this action is $J(\xi)(v_q) = \langle FL(v_q), xi_Q(q) \rangle$ and is Ad^*-equivariant*
3. *The momentum of 2. is a conserved quantity of the Lagrange Equations.*

1.2 Reduction of Phase Spaces with Symmetry

Definition 3 *The **Isotropy Group** of a group action on an element is the set of mappings which hold that element fixed.*

Theorem 4 *Let (P, ω) is a (weak, i.e. degenerate) symplectic manifold on which the Lie group G acts symplectically and let $J : P \to \mathfrak{g}^*$ be an Ad^*-equivariant momentum mapping for the action.*

Let $\mu \in \mathfrak{g}^*$ be a regular value of J, and that G_μ, acting coadjointly on $\mathfrak{g}^*$, is the isotropy group acts freely and transitively on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ has a unique weakly symplectic form ω_μ with the property $\pi_\mu^*\omega_\mu = i_\mu^*\omega$

Lemma 1 *For $p \in J^{-1}(\mu)$,*

1. *$T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_p(J^{-1}(\mu))$*
2. *$v \in T_p(J^{-1}(\mu)), w \in T_p(G \cdot p) \Rightarrow \omega(v, w) = 0$, i.e. $T_p(J^{-1}(\mu))$ is the ω-orthogonal complement of $T_p(G \cdot p)$*

Remark 1 *If μ is a regular value of J, the action of G_μ is locally free, even if not globally free and proper. A sufficient condition for later work is that μ is **weakly regular**, $J^{-1}(\mu)$ is submanifold with $T_pJ^{-1}(\mu) = \mathrm{ker} T_pJ$.*

Theorem 5 *Let G act on Q and cotangently on T^*Q and let $J(\xi)(\alpha_q) = \langle \alpha_q, \xi_Q(q)\rangle$ be the canonical momentum mapping, and let the conditions on a regular value μ of J from theorem 4 theorem hold.*

Additionally assume there is a G_μ-equivariant one-form α_μ on Q with values in $J^{-1}(\mu)$. Now let $\Omega_\mu = \omega_0 + (\tau_Q^*)^* d\alpha_\mu$ be a symplectic form on T^*Q and let T^*Q_μ be given the corresponding induced symplectic form (where $Q_\mu = Q/G_\mu$).

Then there exists a symplectic embedding $\phi_\mu : (T^*Q)_\mu \to T^*Q_\mu$ onto a subbundle over Q_μ. The map is a diffeomorphism onto $T^*Q_\mu \iff \mathfrak{g} = \mathfrak{g}_\mu$.

Theorem 6 *Under the assumptions of theorem 4, let $H : P \to \mathbb{R}$ be invariant under the action of G. Then the flow $F_t^{X_H}$ leaves $J^{-1}(\mu)$ invariant and commutes with the action of G_μ on $J^{-1}(\mu)$, so there is a flow H_t on P_μ satisfying $\pi_\mu \circ F_t^{X_H} = H_t \circ \pi_\mu$. This flow is a Hamiltonian flow on P_μ satisfying $H_\mu \circ \pi_\mu = H \circ i_\mu$. H_μ is called the **Reduced Hamiltonian**.*

If we know the flow H_t on the reduced system P_μ, then we can find the flow of $F_t^{X_H}$ on $J^{-1}(\mu)$ by the following: Let $p_0 \in J^{-1}(\mu)$ and let $c(t)$ and $[c(t)]$ be the integral curves of X_H and X_{H_μ} with $c(0) = p_0$. Pick $d(t) \in [c(t)]$ so that $c(t) = \Phi_{g(t)}(d(t))$, and we try to find $g(t)$. It can be found by solving $\xi_P(d(t)) = X_H(d(t)) - d'(t)$ and then solving for $\xi(t) \in \mathfrak{g}$ in $g'(t) = TL_{g(t)}\xi(t)$.

Definition 4 *Under the conditions of theorem 4 and theorem 6, a point is called a **relative equilibrium** if $\pi_\mu \in P_\mu$ is a fixed point for the reduced Hamiltonian system over $\mu \in \mathfrak{g}^*$. A point is relatively periodic if it is a periodic point of the reduced action.*

Proposition 6 *Under the conditions of theorems 4 and 6, let $p \in J^{-1}(\mu)$. Let Φ be a symplectic group action on P and let $F_t^{X_H}$ be a Hamiltonian flow of X_H.*

1. *TFAE:*
   - *(a) $p \in P$ is a relative equilibrium*
   - *(b) There is a one-parameter subgroup $g(t)$ of G such that $\forall t \in \mathbb{R}$, $F_t^{X_H}(p) = \Phi(g(t), p)$*
2. *TFAE:*
   - *(a) $p \in P$ is a relative periodic point*
   - *(b) There exists $g \in G$ and $\tau > 0$ such that $F_{t+\tau}^{X_H}(p) = \Phi(g, F_t^{X_H}(p))$ for all $t \in \mathbb{R}$*

Proposition 7 (Souriau-Smale-Robbin) *Let the conditions of theorems 6 and 4 hold. Then $p \in J^{-1}(\mu)$ is a relative equilibrium $\iff$ p is a relative equilibrium of $H \times J : P \times \mathfrak{g}^* \to \mathbb{R} \times \mathfrak{g}^*$.*

Lemma 2 (Lagrange Multiplier Theorem) *Let $T : \boldsymbol{E} \to \mathbb{R}$ and $A : \boldsymbol{E} \to \boldsymbol{F}$ be linear maps, A is surjective and $\boldsymbol{E}, \boldsymbol{F}$ are finite-dimensional vector spaces. Then T is surjective on $\ker A \iff T \times A : \boldsymbol{E} \to \mathbb{R} \times \boldsymbol{F}$ is surjective.*

Definition 5 *Let (P, ω) be a symplectic manifold and G a Lie group acting symplectically on P and leaving a Hamiltonian H invariant. Assume that the hypotheses of theorems 6 and 4 hold. A relative equilibrium $p \in P$ is* ***relatively stable*** *is π_μ is stable for the induced dynamical system X_{H_μ} on P_μ where $\pi_\mu(p)$.*

Theorem 7 *Let the conditions of theorems 4 and 6 hold. Suppose that the Hessian $(\mathrm{Hess} H_\mu)(\pi_\mu(p))$ is postive (or negative) definite. Then p is relatively stable.*

Definition 6 *Let (P, ω) be a sympletic manifold. A map is* ***antisymplectic*** *if $\mu^* \omega = -\omega$. A Hamiltonian system is called* ***reversible*** *if there is an antisymplectic involution such that $H \circ \mu = H$.*

Proposition 8 *Let H be reversible and let $c(t)$ be an integral curve of X_H. Then $\mu \circ c(-t)$ is also an integral curve of X_H. So $F_{-t}^{X_H}(x) = \mu F_t^{X_H}(\mu(x))$.*

1.3 Hamiltonian Systems on Lie Groups and the Rigid Body

Let G be a (finite-dimensional) Lie group. Then the tangent bundle is trivial. There are two isomorphisms on the tangent bundle: $\lambda(v) = (g, TL_g^{-1}(v))$; $\rho(v) = (g, TR_g^{-1}(v))$

λ is sometimes called the **body coordinates** and ρ the **space coordinates**. The transition is given by:

$$(\rho \circ \lambda^{-1})(g, \xi) = (g, \mathrm{Ad}_g \xi) \tag{1}$$

Now we will establish the relationship between time derivatives in space and body coordinates. Let $x(t)$ be a curve in G and let $v_0(t)$ be a curve such that $v_0(t) \in T_{x(t)}G$. Let $\xi(t)$ be $x(t)$ in body coordinates, i.e. $\xi(t) = \lambda(x(t)) = TL_{x(t)^{-1}} v_0(t)$, so that $\tilde{\xi}(t) = \mathrm{Ad}_{x(t)}(\xi(t))$. Then

$$\dot{\tilde{\xi}}(t) = \mathrm{Ad}_{x(t)} \dot{\xi}(t) + [\rho(\dot{x}), \tilde{\xi}(t)] = \tilde{\dot{\xi}}(t) + [v_s(t), \tilde{\xi}(t)] \tag{2}$$

Where $v_s(t)$ is the velocity in space coordinates.

Now we look at the analogous situation on the cotangent bundle. Here we have two isomorphism, $\overline{\lambda}$ and $\overline{\rho}$. They are defined adjointly:

$$\overline{\lambda}(\alpha) = (g, \alpha \circ TL_g) \tag{3}$$

$$\overline{\rho}(\alpha) = (g, \alpha \circ TR_g) \tag{4}$$

And the conversion between them:

$$(\overline{\rho} \circ \overline{\lambda}^{-1})(g, \mu) = (g, \mathrm{Ad}^*_{g^{-1}}(\mu)) \tag{5}$$

And the time derivatives are related by:

$$\dot{\tilde{\mu}} = \tilde{\dot{\mu}} - \langle \mathrm{ad}^*(v_b(t)), \tilde{\mu}\rangle \tag{6}$$

Now onto Hamiltonian systems on T^*G and TG, but with the canonical forms in body coordinates, i.e. $\theta_B = \overline{\lambda}_*\theta_0$ and $\omega_B = \overline{\lambda}_*\omega_0$.

Proposition 9 *Let $(g, \mu) \in G \times \mathfrak{g}^*$ and $(v, \rho), (w, \sigma) \in T_{(g,\mu)}(G \times \mathfrak{g}^*)$. Then*

1. $\langle \theta_B(g, \mu), (v, \rho)\rangle = \mu(TL_{g^{-1}}v)$
2. $\omega_B(g, \mu)((v, \rho), (w, \sigma)) = \sigma(TL_{g^{-1}}v) - \rho(TL_{g^{-1}}w) + \mu([TL_{g^{-1}}v, TL_{g^{-1}}w])$

Now a Riemannian metric pulls the natural canonical structure on the cotangent bundle to one on the tangent bundle. This one-form Θ has the action $\langle \Theta(v), w_v\rangle = \langle T\pi(w), v\rangle$, which induces a symplectic form $\Omega = -d\Theta$. Now we look for Θ and Ω in body coordinates: $\Theta = \lambda_*\Theta$; $\Omega_B = \lambda_*\Omega$.

Proposition 10 *Let $(g, \xi) \in G \times \mathfrak{g}^*$ and $(v, \zeta), (w, \eta) \in T_{(g,\xi)}(G \times \mathfrak{g})$. Then*

1. $\langle \Theta(g, \xi), (v, \zeta)\rangle = \langle TL_{g^{-1}}(v), \xi\rangle$
2. $\Omega(g, \xi)((v, \zeta), (w, \eta)) = \langle \eta, TL_{g^{-1}}(v)\rangle - \langle \zeta, TL_{g^{-1}}(w)\rangle + \langle \xi, [TL_{g^{-1}}(v), TL_{g^{-1}}w]\rangle$

Let $\Lambda : G \times G \to G$ be the action of G on itself by left translations.

Theorem 8 (Euler Conservation Laws) *1. The Ad^*-equivariant momentum mapping $\overline{J}$ of the action Λ^{T^*} on T^*G is given by $\overline{J} : T^*G \to \mathfrak{g}^*$; $\overline{J}(\alpha_g)(\xi) = \alpha_g(TR_g(\xi))$. This is a momentum mapping for left-invariant H.*

2. *If G has a left-invariant metric $\langle , \rangle$, then the Ad^*-equivariant momentum mapping J of the action Λ^T on TG is given by $J : TG \to \mathfrak{g}^*$; $J(v_g)(\xi) = \langle v_g, TR_g(\xi)\rangle$. This is a momentum mapping for left-invariant Lagrangians, in particular, the kinetic energy $K = \frac{1}{2}\langle v, v\rangle$.*

3. *The action Λ^{T^*} in body coordinates is given by $\Lambda_B^{T^*} : G \times G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$; $(g, (h, \mu)) \mapsto (gh, \mu)$. The momentum mapping of this action $\overline{J}_B : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is given by $\overline{J}_B(\xi) = \overline{J}(\xi) \circ \overline{\lambda}^{-1}$. If a Hamiltonian is left invariant, then this is a momentum mapping for it.*

4. *The action Λ^T on a group G with left-invariant metric $\langle , \rangle$. Then a left-invariant Lagrangian has an invariant momentum mapping $J_B(\xi) = J(\xi) \circ \lambda^{-1}$.*

5. *Let $\langle , \rangle$ be a left-invariant metric. The action Λ^T in space coordinates is given by*

$$\Lambda_S^T : G \times G \times \mathfrak{g} \to G \times \mathfrak{g};\ (g, (h, \xi)) \mapsto (gh, \mathrm{Ad}_g(\xi)) \tag{7}$$

The Ad^*-equivariant momentum mapping of this action is $J_S(\xi) = J(\xi) \circ \rho^{-1}$. Every left-invariant Lagrangian has this as a momentum mapping.

If $\langle x, y\rangle = \langle A(y), x\rangle$ for some $A : \mathfrak{g} \to \mathfrak{g}^*$, then $J_S(\xi) = (\mathrm{Ad}^*_{g^{-1}} \circ A \circ \mathrm{Ad}_{g^{-1}})(\xi)$. The Euler's conservation laws then becomes a conservation of a vector quantity

$$L_{\xi,g} = (Ad^*_{g^{-1}}) \circ A \circ \mathrm{Ad}_{g^{-1}})\xi \tag{8}$$

The plane $\mathcal{I}_{\xi,g} = \{\eta \in \mathfrak{g} | \mathcal{L}_\xi(\eta) = 0\}$ is called the **invariable plane** for the initial condition $\xi \in \mathfrak{g}$.

Theorem 9 *In reference to theorem 8 part 5, let $E = L = K = \frac{1}{2}\langle TR_g\xi, TR_g\xi\rangle = \frac{1}{2}\langle \mathrm{Ad}_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\xi\rangle$. Let $w(t)$ be an integral curve of X_L in space coordinates. Let $E_0 = \frac{1}{2}\langle w(0), w(0)\rangle$, and let $S(t)$ be the image of the inertial ellipsoid $\langle \xi, \xi\rangle = eE_0$ after t seconds, and in space coordinates, that is*

$$S(t) = \{\xi \in \mathfrak{g} | \langle Ad_{x(t)^{-1}}\xi, Ad_{x(t)^{-1}}\xi\rangle = 2E_0\} \tag{9}$$

Then letting $\mathcal{I}_{w(t),x(t)}$ denote the invariable plane,

1. *$\mathcal{I}_{w(t),x(t)}$ is tangent to $S(t)$ at $w(t)$*
2. *$\mathcal{I}_{w(t),x(t)}$ is independent of t*

On T^*G, consider a left-invariant Hamiltonian and let $H_B = H \circ \overline{\lambda}^{-1}$ be its expression in body coordinates. Clearly $\overline{\lambda}_* X_H = X_{H_B}$ so that

$$X_{H_B} : G \times \mathfrak{g}^* \to TG \times (\mathfrak{g}^* \times \mathfrak{g}^*); \tag{10}$$

$$X_{H_B}(g, \mu) = (\overline{X}(g, \mu), \mu, \overline{Y}(g, \mu) \tag{11}$$

So that

$$\overline{X} : G \times \mathfrak{g}^* \to TG;\ \overline{Y} : G \times \mathfrak{g}^* \to \mathfrak{g}^* \tag{12}$$

So that for any μ, $\overline{X}(\cdot, \mu)$ is a left-invariant vector field on G, and $\overline{Y}$ is independent of g. $\overline{Y}$ is called the **Euler Vector field** or the **Euler equations in cotangent formulation**. This, and the flow of $\overline{Y}$ is summarized in the following proposition.

Proposition 11

1. *Let $X \in \mathfrak{X}(T^*G)$ be left invariant and let $X_B = \overline{\lambda}_* X$ be its expression in body coordinates; then $X_B(g, \mu) = (\overline{X}(g, \mu), \mu, \overline{Y}(\mu))$ where $\overline{Y} : \mathfrak{g}^* \to \mathfrak{g}^*$ and $\overline{X}^\mu : g \mapsto \overline{X}(g, \mu)$ are a family of left-invariant vector fields on G depending smoothly on $\mu \in \mathfrak{g}^*$. The flow of $\overline{Y}$, denoted by $\overline{H}_t$, is given by*

$$\overline{H}_t(\nu) = F^X_t(\nu) \circ TL_{x(t)} \tag{13}$$

Where $x(t) = \pi(F_t(\nu))$. $\overline{Y}$ is called the ***Cotangent Euler Vector Field****. In particular, this holds for X_H and X_{H_B}.*

2. *Assume G has a left-invariant metric $\langle,\rangle$. Let $X \in \mathfrak{X}(TG)$ be left invariant, and let $X_B = \lambda_* X$ be its expression in body coordinates. Then $X_B(g, \xi) = (X^\xi(g), \xi, Y(\xi))$, where $Y : \mathfrak{g} \to \mathfrak{g}$ and X^ξ are a family of left-invariant vector fields on G. The flow of Y, denoted by H_t, is given by $H_t(\xi) = TL_{x(t)^{-1}}(F^X_t(\xi))$. We call Y the* ***Tangent Euler Vector Field****. This applies specifically when $X = X_L$ is left-invariant and $X_B = X_{L_B} = \lambda_* X_L$.*

Theorem 10 *1. Let $X \in \mathfrak{X}(T^*G)$ be a left-invariant vector field with flow F_t^X. Let $\overline{Y} : \mathfrak{g}^* \to \mathfrak{g}^*$ be the corresponding cotangent Euler vector field with flow $\overline{H}_t$. Then*

$$\langle \overline{Y}(\mu), \eta \rangle = (\langle d\overline{J}(\eta), \mu \rangle)(X(\mu)) + \langle \mu, [\dot{x}(0), \eta] \rangle \tag{14}$$

Where $x(t) = \pi(F_v(\mu))$. In particular, if $X = X_H$ is left-invariant, then the first term drops out.

2. Let G be a Lie group with left-invariant metric $\langle , \rangle$ and $X \in \mathfrak{X}(TG)$ a left-invariant vector field with flow F_t^X. Let $Y : G \to G$ be the correpsonding tangent Euler vector field with flow H_t. Then

$$\langle Y(\xi), \eta \rangle = \langle [\xi, \eta], \eta \rangle + (\langle dJ(\eta), \xi \rangle)(X(\xi)). \tag{15}$$

In particular, if $X = X_L$, then the second term drops out.

Theorem 11 *1. Let $H : T^*G \to \mathbb{R}$ be a left-invariant Hamiltonian, and let $G \cdot \mu = \{\mathrm{Ad}^*_{g^{-1}}\mu | g \in G\}$ be the image of a covector under the adjoint representation of the group. Then $(G \cdot \mu, \omega_\mu$ is a symplectic manifold with*

$$\omega_\mu(\mathrm{Ad}^*_{g^{-1}}\mu)\left(\xi(\mathrm{Ad}^*_{g^{-1}}\mu), \eta(\mathrm{Ad}^*_{g^{-1}}\mu)\right) = -\left(\mathrm{Ad}^*_{g^{-1}}\mu\right)([\xi, \eta])] \tag{16}$$

$\overline{Y} \restriction G \cdot \mu$ is a Hamiltonian vector field with Hamiltonian $H_\mu : G \cdot \mu \to \mathbb{R}$ given by

$$H_\mu\left(\mathrm{Ad}^*_{g^{-1}}\mu\right) = H\left(TR_{g^{-1}}(\mu)\right) = H\left(\mathrm{Ad}^*_{g^{-1}}\right) \tag{17}$$

2. Let G have a bi-invariant metric $(,)$. Let $G \cdot \xi = \{\mathrm{Ad}_g \xi | g \in G\}$. Then $(G \cdot \xi, \omega_\xi)$ is a symplectic manifold with

$$\omega_\xi\left(\mathrm{Ad}_g \xi\right)(\eta(\mathrm{Ad}_g \xi), \zeta(\mathrm{Ad}_g \xi)) = -\left([\eta, \zeta], \mathrm{Ad}_g \xi\right) \tag{18}$$

And $Y \restriction G \cdot \xi$ is a Hamiltonian vector field with $H_\xi : G \cdot \xi \to \mathbb{R}$ given by

$$H_\xi\left(\mathrm{Ad}_g \xi\right) = E\left(\mathrm{Ad}_g \xi\right) \tag{19}$$

Theorem 12 (Arnold) *Let Y be the tangent Euler field and $Y(\xi) = 0$. Let Q be a bilinear form defined by*

$$Q(\eta, \zeta) = \langle A^{-1}(\mathrm{ad}\eta)^* A\xi, A^{-1}(\mathrm{ad}\zeta)^* A\xi \rangle + \langle \xi, A^{-1}(\mathrm{ad}\zeta)^*(\mathrm{ad}\eta)^* A\xi \rangle \tag{20}$$

If Q is positive or negative definite, then ξ is a stable equilibrium point of $Y \restriction G{\cdot}\xi$

Now we consider Hamiltonians and Lagrangians that are not left-invariant. Consider the energy function $E = K + V \circ \pi$, for V not left-invariant. Let $F_t(\eta)$ be the flow of H. Define

$$H_t(\eta) = TL_{x(t)^{-1}} F_t(\eta) \tag{21}$$

This won't give a flow on $\mathfrak{g}$ because F_t is not left-invariant. We will, however, have a time-dependent vector field by setting

$$Y_t(H_t(\xi)) = \frac{d}{dt} H_t(\xi) \tag{22}$$

Proposition 12

$$Y_t(H_t(\xi)) = Y(H_t(\xi)) - TL_{x(t)^{-1}}\mathrm{grad}V(x(t)) \tag{23}$$

Where Y is the Euler Vector field for G.

Proposition 13 *Let $\langle , \rangle$ be a left-invariant metric and $K(v) = \frac{1}{2}\langle v, v\rangle$. Let V be smooth and bounded below. Then the flow of $E = K + V \circ \pi$ is complete.*

The Euler Equations become simpler if the Lie algebra $\mathfrak{g}$ carries a nondegenerate symmetric bilinear form $(,)$ that is invariant under the adjoint maps:

$$(\mathrm{Ad}(g)\xi, \mathrm{Ad}(g)\eta) = (\xi, \eta) \tag{24}$$

Then this is a pseudo-Riemannian metric that is both right- and left-invariant.

Suppose that $(,)$ is a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Then $\langle \xi, \eta\rangle = (I\xi, \eta)$, where $I : \mathfrak{g} \to \mathfrak{g}$ is linear and symmetric with respect to $(,)$. The Euler equations then read:

$$(IY(\xi), \eta) = (I\xi, [\xi, \eta]) = (I\xi, \mathrm{ad}(\xi)\eta) \tag{25}$$

Lemma 3 *Suppose that $(,)$ is invariant under $\mathrm{Ad}(g)$ for all g. Then for each $\xi \in \mathfrak{g}$, $\mathfrak{ad}(\xi)$ is skew-symmetric with respect to $(,)$.*

So that if $(,)$ is invariant, we have that $Y(\xi) = I^{-1}[I\xi, \xi]$.

Proposition 14 *Suppose that $\langle , \rangle$ is invariant under all the actions $\mathrm{Ad}(\exp(t\eta))$ $\forall t \in \mathbb{R}$. Then the function μ_η defined by:*

$$\mu_\eta(\xi) = \langle \eta, \xi\rangle \tag{26}$$

is a constant for the motion for H_t. In fact, $\mu_\eta(Y(\xi)) = 0$ $\forall \xi \in \mathfrak{g}$.

Corollary 3 *Suppose that $\langle , \rangle$ is invariant under adjoint actions of G. Then the corresponding Euler vector field vanishes identically, and the geodesic flow is given by the exponential map:*

$$\lambda \circ F_t \circ \lambda^{-1}(g, \xi) = ((\exp(t\xi)g, \xi) \tag{27}$$

1.4 The Topology of Simple Mechanical Systems

Stephen Smale set out a topological program for studying Hamiltonian systems with symmetry, which goes as follows. Let H be a Hamiltonian on a symplectic manifold (P, ω) and let G be a Lie group acting on P, leaving H invariant and having a momentum mapping $J : P \to \mathfrak{g}^*$. Then we can form the **Energy Momentum Mapping**:

$$H \times J : P \to \mathbb{R} \times \mathfrak{g}^*, \ (H \times J)(p) = (H(p), J(p)) \tag{28}$$

So that the sets

$$I_c = (H \times J)^{-1}(c) \tag{29}$$

Are invariant under the flow of X_H. To understand the topological features of X_H we should figure out:

1. the topology of I_c for all c
2. the bifurcation set $\Sigma_{H\times J}$ of $H \times J$
3. the flow of X_H on each I_c
4. How the set I_c 'fit together' as μ is varied to understand the level set $H^{-1}(e)$

Now we will define the **Bifurcation Set**. A smooth map $f : M \to N$ is locally trivial at a point y_0 in its range if there is a neighborhood U of y_0 such that $\forall y \in U$ $f^{-1}(y)$ is a smooth submanifold of M and there is a smooth map $h : f^{-1}(U) \to f^{-1}(y_0)$ such that $f \times h$ is a diffeomorphism from $f^{-1}(U)$ to $U \times f^{-1}(y_0)$. The bifurcation set of f is

$$\Sigma_f = \{y_0 \in N | f \text{ fails to be locally trivial at } y_0\} \tag{30}$$

Now let $\sigma(f)$ be the set of critical points of f, and Σ'_f be the set of critical values of f. Then we have the following result:

Proposition 15

$$\Sigma'_f \subset \Sigma_f \tag{31}$$

If f is proper (takes compact sets to compact sets), then $\Sigma'_f = \Sigma_f$. Most of the time, f does not have compact level sets. However, these are the 'interesting' ones, because other systems arise by breaking the symmetry of symmetric systems. Now we'll get into the real meat of Smale's program.

Definition 7 *A* ***Simple Mechanical System with Symmetry*** *is* (M, K, V, G), *where:*

1. *M is a Riemannian manifold with metric $\gamma = \langle , \rangle$; M is called the* ***configuration space*** *and T^*M with its canonical symplectic structure is called the* ***phase space*** *of the system*
2. *$K : T^*M \to \mathbb{R}$ is the kinetic energy of the system defined by $K(\alpha) = \frac{1}{2}\langle \alpha, \alpha \rangle$, with the usual lift of the metric to the cotangent bundle*
3. *$V : M \to \mathbb{R}$ is the* ***potential energy*** *of the system;*
4. *G is a connected Lie Group acting on M which preserves the metric and the function V. G is called the* ***symmetry group*** *of the system.*
5. *$H : T^*M \to \mathbb{R}$ is defined by $H = K + V \circ \pi$ is the* ***Hamiltonian*** *of the system.*

For most values (h, μ), the sets $I_{h,\mu} = (H \times J)^{-1}(h, \mu)$ are, for non-bifurcation values, manifolds; we will call them **invariant manifolds**, even if they are not truly manifolds. The isotropy group G_μ acts on $J^{-1}(\mu)$, but since H is invariant as well, G_μ acts on $H^{1-}(h)$ invariantly. So G_μ acts on $I_{h,\mu} = H^{-1}(h) \cap J^{-1}(\mu)$.

Then we can symplectically reduce $I_{h,\mu}$ by the group action to $\widehat{I}_{h,\mu}$, which is a submanifold of $J^{-1}(\mu)/G_\mu$. Here the Hamiltonian vector field reduces $(\pi_\mu)_*(X_H \restriction_{J^{-1}(\mu)}= X_{H_\mu}$ and can be projected down by $\pi_{h,\mu} : I_{h,\mu} \to \widehat{I}_{h\mu}$. So then it is clear that $\widehat{I}_{h,\mu}$ is the energy surface of the reduced Hamiltonian $H_\mu^{-1}(h)$. We will put in some legwork with bifurcation points first before examining the topology of the sets $\widehat{I}_{h,\mu}$.

Proposition 16

$$\sigma(H \times J) = \sigma(J) \cup \left(\bigcup_{\mu \in \mathfrak{g}^* \setminus J(\sigma(J))} \sigma(H \restriction_{J^{-1}(\mu)}) \right) \tag{32}$$

In words, $\alpha \in T^*M$ is a critical point of $H \times J$ iff $T_\alpha J$ is not surjective or if α is a critical point of $H \restriction_{J^{-1}(\mu)}$ for some μ.

Lemma 4 *1. Let $\Lambda = \{x \in M | J_x = J \restriction_{T_x M}$ is not surjective$\}$. Then $\Lambda = \{x \in M : \Xi_x : \mathfrak{g} \to T_x M,\ \xi \mapsto \xi_M(x)$ is not injective$\} = \{x \in M | \mathrm{dim} G_x \geq 1\}$ where $G_x = \{g \in G | \Phi(g,x) = x\}$*

2. Λ is closed and G-invariant.

This proposition tells us that $\Lambda \supset \pi_M(\sigma(J))$, so that J has only regular values if Λ is excluded. This means we can deal with $M \backslash \Lambda$ and Λ separately since Λ can be figured out nicely.

From before, we can reduce a phase space if we can find a one-form $\alpha_\mu \in T^*M$ with values in $J^{-1}(\mu)$. We can do this explicitly, then examine the Hamiltonian induced on T_μ^M.

For $\mu \in \mathfrak{g}^*$ let $\alpha_\mu \in \Omega^1(M \Lambda)$ satisfy the following conditions:

1. $\alpha_\mu(x) \in J_x^{-1}(\mu) = J^{-1}(\mu) \cap T_x^* M$
2. $K(\alpha_\mu(x)) \inf_{\alpha \in J_x^{-1}(\mu)}$ where K is the kinetic energy.

The existence and uniqueness follows from existence and uniqueness of elements of minimal norm in certain sets of Hilbert spaces.

Proposition 17 *1. $\alpha_\mu \in Omega^1(M \backslash \Lambda)$, that is, a smooth one-form that is the unique critical point of K.*

2. $\alpha_\mu \perp \mathrm{ker} J_x = J_x(0)$ with respect to the kinetic energy.

3. α_μ is G_μ-equivariant.

We have a symplectic embedding $\phi_\mu : P_\mu \to T^* M_\mu$ given by 5. We know that H induces H_μ on P_μ where $H = H_\mu \circ \pi_\mu$ and X_H induces X_{H_μ} on P_μ. In T^*M_μ we have $\widehat{H}_\mu = H_\mu \circ \phi_\mu^{-1}$ induced on $\phi_\mu(P_\mu)$. Points in T^*M_μ are

G_μ orbits of covectors α_x vanishing on $T_x(G_\mu \cdot x)$ (cf. lemma 1), so we can suggestively write them as $G_\mu \cdot \alpha_x$. Then we have that

$$\begin{aligned}
\widehat{H}_\mu(G_\mu \cdot \alpha_x) &= H(\alpha_x + \alpha_\mu(x)) = K(\alpha_x + \alpha_\mu(x)) + V(x) \\
&= K(\alpha_x) + 2\langle \alpha_x, \alpha_\mu(x)\rangle + K(\alpha_\mu(x)) + V(x) \\
&= K(\alpha_x) + K(\alpha_\mu(x)) + V(x)
\end{aligned}$$

So if we set $V_\mu(x) = K(\alpha_\mu(x)) + V(x)$ then the projection of $\widehat{H}_\mu$ is the projection of $K + V_\mu$. Then we can form a simple mechanical system on $\phi_\mu(P_\mu)$ by restricting $K + V_\mu \circ \pi_\mu$.

Theorem 13 *The reduction of a simple mechanical system on T^*M with $\Lambda = \emptyset$ is a simple mechanical system on T^*M_μ, where $M_\mu = M/G_\mu$. The kinetic energy of the reduced system is $\widehat{K}$ induced from the G_μ-invariant kinetic energy K on T^*M and the potential $\widehat{V}_\mu$ induced from the function defined by*

$$V_\mu(x) = V(x) + K(\alpha_\mu(x)) = H(\alpha_\mu(x)) \tag{33}$$

*Called the **effective** or **amended potential**. Note that the symplectic structure on T^*M may not be the canonical structure.*

Since $\Lambda = \emptyset$ we are assuming μ is a regular value of J and the actions of G_μ are free and proper, or else we work with local statements. If $\Lambda \neq \emptyset$, then we work with $M \backslash \Lambda$ and determine the behavior of Λ separately.

In T^*M_μ the subbundle $\phi(M_\mu)$ over M_μ, equilibrium points are points in the zero section that are critical points of $\widehat{V}_\mu$. These critical points are one-to-one with critical orbits of V_μ. It's also worth knowing that a critical point of $\widehat{V}_\mu$ is nondegenerate, then the indices of $\widehat{V}_\mu$ and V_μ along the corresponding nondegenerate critical manifold are equal.

Critical points of V_μ are one-to-one with relative equilibria, which are critical points of $H \times J$ on $J^{-1}(\mu)$, or equivalently to critical points of $H \restriction_{J^{-1}(\mu)}$. In summary:

Corollary 4 *1. $V_\mu \circ \Phi_g = V_\mu$ and $\sigma(V_\mu)$ is G_μ invariant.*

2. $\sigma(H \restriction J^{-1}(\mu)) = \alpha_\mu(\sigma(V_\mu))$ and is G_μ and X_H invariant.

Theorem 14 *Let μ be a regular value of J. Then*

$$\Sigma'_{H \times J \restriction_{T^*(M\backslash\Lambda)}} = \{(h, \mu) | h \in V_\mu(\sigma(V_\mu))\} \tag{34}$$

To continue we will introduce some definitions related to vector bundles.

Definition 8 *The **Unit Disk Bundle** of a vector bundle E with an inner product $\langle , \rangle_E$ is the set*

$$D_1(E) = \{v \in E | \|v\| \leq 1\} \tag{35}$$

and the ***Unit Sphere Bundle*** *is the set*

$$S_1(E) = \{v \in E | \|v\| = 1\} \tag{36}$$

And if the base space has no boundary, then $\partial D_1(E) = S_1(E)$. If not, then for each x in the boundary, identify $\pi^{-1}(x) \cap D_1(E)$ with the point x, forming the space $\alpha(E)$, the ***Reduced Disk Bundle*** *of E. Doing the same with $S_1(E)$ we get the* ***Reduced Sphere Bundle*** *$\beta(E)$ of E. These can be given smooth manifold structures and $\partial\alpha(E) = \beta(E)$.*

For trivial bundles of M the sphere and disk bundles have explicit forms.

1. If the base space is boundaryless, then $\alpha_k(M) \approx M \times D^k$; $\beta_k(M) \approx M \times S^{k-1}$, where α_k and β_k are the disk and sphere bundles of the trivial real bundles of rank k.

2. If the base space is boundaryless, then $\beta_1(M)$ is the 'double' of M. The double of M is the gluing of a second copy of M with reverse orientation to its boundary, so the result is oriented and boundaryless.

3. $\beta_k(D^m) \approx S^{k+m-1}$

4. For boundaryless manifolds, $\alpha_k(M_1 \times M_2) \approx M_1 \times \alpha_k(M_2)$, $\beta_k(M_1 \times M_2) \approx M_1 \times \beta_k(M_2)$.

For a simple mechanical system with symmetry (M, K, V, G) with $\Lambda = \emptyset$, let

$$M_{h,\mu} = V_\mu^{-1}\left((-\infty, h]\right) \tag{37}$$

If h is a regular value for V_μ, then $M_{h,\mu}$ is a smooth manifold with boundary and $\partial M_{h,\mu} = V_\mu^{-1}(h)$. Let

$$E_{h,\mu} = \{\alpha \in T^*M | J(\alpha) = \mu, H(\alpha) \leq h\} = (H \restriction_{J^{-1}(\mu)})^{-1}((-\infty, h]) \tag{38}$$

If h is a regular value of $V_\mu = H \circ \alpha_\mu$, then it is also a regular value of $H \restriction_{J^{-1}(\mu)}$ by the second part of corollary 4. So then $E_{h,\mu}$ is a submanifold (with boundary) of T^*M. We have that

$$\partial E_{h,\mu} = I_{h,\mu} \tag{39}$$

Theorem 15 *Given a mechanical system with symmetry (M, K, V, G) with $\Lambda = \emptyset$ and h a regular value of V_μ, the following are true:*

1. *(a)* $E_{h,\mu} = \{\alpha_x \in J^{-1}(\mu) | K(\alpha_x) - K(\alpha_\mu(x)) \leq h - V_\mu(x)\}$

   (b) $\partial E_{h,\mu} = I_{h,\mu}$

   (c) $\pi_M(E_{h,\mu}) \subset M_{h,\mu}$

2. *If $F = J^{-1}(0)$, then $\pi_M \restriction_F : F \to M$ is a vector subbundle of T^*M. Let $F \restriction_{M_{h,\mu}}$ its restriction to $M_{h,\mu} \subset M$. Then if $(h, \mu) \notin \Sigma'_{H \times J}$, $E_{h,\mu}$ is diffeomorphic to $\alpha(F \restriction_{M_{h,\mu}})$.*
   More precisely, there is a G_μ-invariant diffeomorphism of manifolds with boundary $\phi_{h,\mu} : \alpha(F \restriction_{M_{h,\mu}}) \to E_{h,\mu}$.

3. *The induced diffeomorphism on the boundaries*

$$\partial\phi_{h,\mu} : \beta(F \restriction_{M_{h,\mu}}) \to I_{h,\mu} \tag{40}$$

is G_μ equivariant.

4. *If C is a nondegnerate critical submanifold of $V_\mu \restriction_{M_{h,\mu}}$ of index λ, then $\alpha_\mu(C)$ is a nondegenerate critical submanifold of $H \restriction_{E_{h,\mu}}$ of the same index.*

All the objects defined so far have been G_μ-equivariant, so that symplectic reduction is possible. Since we can reduce dimension, it would be beneficial to have a "topological" theorem similar to the above for the reduced manifolds $\widehat{M_{h,\mu}} = M_{h,\mu}/G_\mu$, and so on.

We will denote the passing of object to quotients by a hat: $\widehat{H} : \widehat{E}_{h,\mu} \to \mathbb{R}$, $\widehat{\alpha_\mu} : \widehat{M} \to \widehat{T^*M}$, and $\widehat{V}_\mu = \widehat{H} \circ \widehat{\alpha}_\mu$. The relation $\partial\widehat{E}_{h,\mu} = \widehat{I}_{h,\mu}$ still holds.

Theorem 16 (Reduced Invariant Manifold Theorem of Smale) *Assume G_μ acts freely and properly on $M_{h,\mu}$ and $E_{h,\mu}$. Then $\widehat{M}_{h,\mu}$ and $\widehat{E}_{h,\mu}$ are manifolds. Further assume that $\widehat{V}_\mu : \widehat{M}_{h,\mu} \to \mathbb{R}$ has nondegenerate critical points.*

1. *If $\widehat{x} \in \widehat{M}_{h,\mu}$ in a nondegenrate critical point of $\widehat{V}_\mu$, then $\widehat{\alpha}_\mu(\widehat{x})$ will be a nondegenerate critical point of $\widehat{H} \restriction_{\widehat{E}_{h,\mu}}$ of the same index. This index is the same index of V_μ on the nondegenerate critical manifold $\pi_\mu^{-1}(\widehat{x}) \subset M_{h,\mu}$ where $\pi_\mu : M_{h,\mu} \to \widehat{M}_{h,\mu}$ is the projection.*

2. *If the vector bundle $J^{-1}(0) \restriction_{M_{h,\mu}}$ is trivial, then*

$$I_{h,\mu} = \beta_S(M_{h,\mu});\ \ \widehat{I}_{h,\mu} = \beta_S(\widehat{M}_{h,\mu}) \tag{41}$$

Theorem 17 *Let (M, K, V, G) be a simple mechanical system with symmetry and assume that $\alpha_{x_0} \in J^{-1}(\mu)$ for a regular value of J. Denote by the $H = K + V \circ \pi_M$ the Hamiltonian, by $E = K + V \circ \pi_M$ the energy function, and by $L = K - V \circ \pi_M$ the Legendre transformation of H, with $\gamma^\flat$ the Legendre transform. Then TFAE:*

1. *α_{x_0} is a relative equilibrium*
2. *$\exists \xi \in \mathfrak{g}$ satisfying $(J \circ \gamma^\flat \circ \xi_M)(x_0) = \mu$ satisfying $X_L(\xi_M(x_0)) = \xi_T M(\xi_M(x_0))$*
3. *$\exists \xi \in \mathfrak{g}$ satisfying $(J \circ \gamma^\flat \circ \xi_M)(x_0) = \mu$ such that x_0 is critical point of $L \circ \xi_M$*
4. *$\exists \xi \in \mathfrak{g}$ such that $(J \circ \gamma^\flat \circ \xi_M)$ such that x_0 is a critical point of $V - K \circ \gamma^\flat \circ \xi_M$*
5. *x_0 is a critical point of the amended potential V_μ and $\alpha_{x_0} = \alpha_\mu(x_0)$*

Here are some remarks about the α, β constructions.

Proposition 18 *Let M be a manifold without boundary and $\pi : E \to M$ a vector bundle with metric $\langle , \rangle_E$. Let $f : M \to \mathbb{R}$ and c a regular value for f. Define $g : E \to \mathbb{R}$ by $g(v) = \langle v, v\rangle_E + (f \circ \pi)(v)$. Then:*

1. *c is a regular value for $g \iff c$ is a regular value for f; $g^{-1}((-\infty, c])$ is a smooth manifold with boundary $g^{-1}(c)$*
2. *$g^{-1}((-\infty, c])$ is homeomorphic to $\alpha(E \restriction_{f^{-1}((-\infty,c])})$, which is in fact a diffeomorphism*
3. *If $\pi_1 : E_1 \to M$ is another Riemannian vector bundle over M and $f_1 : M \to \mathbb{R}$ is another smooth map such that c is a regular value for both f and f_1, $f_1^{-1}((-\infty, c]) = f^{-1}((-\infty, c])$, and $E_1 \restriction_{f_1^{-1}((-\infty,c])}$ is vector bundle isomorphic to $E \restriction_{f^{-1}((-\infty,c])}$, then $\alpha(E_1 \restriction_{f_1^{-1}((-\infty,c])})$ is diffeomorphic to $\alpha(E \restriction_{f^{-1}((-\infty,c])})$.*

Proposition 19 *Let $M, \pi : E \to M, f, c$ be as in Proposition 1. Assume there is a map $h : E \to \mathbb{R}$ satisfying:*

1. *For each $x \in M$, $h \restriction_{E_x} : E_x \to \mathbb{R}$ and has a unique nondegenerate minimum at the origin of the fiber*
2. *$f(x) = h(0_x)$*

Then c is a regular value for h and $\alpha(E \restriction_{f^{-1}((-\infty,c])})$ is diffeomorphic to $h^{-1}((-\infty, c])$.