Summary of Jacobi Metric and Morse Theory of Dynamical Systems

Samuel Wallace

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This is a summary of notes from the paper "Jacobi Metric and Morse Theory of Dynamical Systems" by A. A. Izquierdo et al., available here.

1 The Maupertius-Jacobi Principle

1.0.1 Geodesics

Geodesics on a Riemannian manifold are extremals of the functionals:

$$S_0[\gamma] = \int_{t_1}^{t_2} \frac{1}{2}\|\dot{\gamma}(t)\|^2 dt; \quad L[\gamma] = \int_{t_1}^{t_2} \|\dot{\gamma}(t)\| dt \tag{1}$$

And the solution satisfy the Geodesic Equation:

$$\delta S_0 = 0 \Rightarrow \nabla_{\dot{\gamma}}\dot{\gamma} = 0 \tag{2}$$

To see if a path is a minimum we look at the second-variation formula:

$$\delta^2 S_0 = \int_{s_1}^{s_2} \langle \Delta V, V \rangle ds; \quad \Delta V = -\nabla_{\gamma'}\nabla_{\gamma'} V - R(\gamma', V)\gamma' \tag{3}$$

Where V is an infinitesimal variation of a path from the initial to final points, so that $V(s_1) = V(s_2) = 0$.

We can form a 'mechanical system' action by

$$S[\gamma] = \int_{t_1}^{t_2} \left(\frac{1}{2}\|\dot{\gamma}(t)\|^2 - U(\gamma(t)) \right) dt \tag{4}$$

Giving Euler-Lagrange Equations:

$$\frac{D\dot{\gamma}}{dt} = \nabla_{\dot{\gamma}}\dot{\gamma} = -\text{grad}U \tag{5}$$

And the second variation formula is

$$\delta^2 S = \int_{t_1}^{t_2} \left(\langle \Delta V, V \rangle - H(U)(V, V) \right) = \int_{t_1}^{t_2} \langle \Delta V - \nabla_V \text{grad}U, V \rangle dt \tag{6}$$

1.1   The Jacobi Metric

The definition of the Jacobi metric is

$$h(X, Y) = 2(E - U(x))g(X, Y)$$

Where g is the 'kinetic energy' metric and E is a constant. To remain Riemannian, this metric only holds in the region $U(x) < E$.

Theorem 1 (Jacobi) *The extremal trajectories of 4 are the geodesics of the Jacobi metric.*

2   Geometric and Dynamical Stability

Theorem 2 *Let $\gamma(t)$ be an extremal of the S functional 4, and S_0^J the corresponding geodesic extremal of the associated Jacobi metric. Then*

$$\delta^2 S_0^J[\gamma] = \delta^2 S[\gamma] + \int_{t_1}^{t_2} 2\langle \dot{\gamma}, \frac{DV}{dt} \rangle \langle F, V \rangle dt \tag{7}$$

Where $F = grad\, ln(2(E - U(x)))$.

Theorem 3 *Let γ be a extremal of the $S[\gamma]$ functional and let $L^J[\gamma]$ be the length functional of the Jacobi metric associated with $S[\gamma]$. Then*

$$\delta^2 L^J[\gamma] = \delta^2 S[\gamma] - \int_{t_1}^{t_2} \frac{dt}{2(E - U(x))}[\langle \nabla_{\dot{\gamma}} \dot{\gamma}, V \rangle - \langle \dot{\gamma} \nabla_{\dot{\gamma}} V \rangle]^2 \tag{8}$$

3   Jacobi Fields and Morse Series

Jacobi Fields are easy to find outside of direct computation from the second variation formula:

Proposition 1 *If $\gamma = \gamma(t, a)$ is a one-parameter family of geodesics, then $\frac{d\gamma}{da}$ is a Jacobi field.*

Morse Theory relates the topology of a manifold to the extremals of its functions. A summary is as follows:

Let ΩM be the loop space of M with a fixed base point. Then some topological features of ΩM are encoded into the Poincaré series:

$$\sum_k b_k t^k; \quad b_k = \dim H_k(\Omega M, \mathbb{R}) \tag{9}$$

For any functional S on ΩM, we can define a Morse Series

$$\mathcal{M}(\mathcal{S}) = \sum_{N_c} P_t(N_c) t^{\mu(N_c)} \tag{10}$$

Where N_c are critical submanifolds of ΩM and $\mu(N_c)$ is the dimension of the submanifold, called the Morse Index. An important result is the following:

Theorem 4 (Morse Index Theorem) *The Morse index of a critical path γ_c is equal to the number of conjugate points to the base point crossed by γ_c counted with multiplicity.*

The critical result of this section is that $\mathcal{M}(S) = \mathcal{M}(L^J)$, so the Morse structure of the two Riemannian metrics is the same.

4   Conclusion

The paper is finished with an example, the Garnier System. The Jacobi fields and the Morse series is computer for the system, and the Morse structure of the system is related to a topological space.