Notes from *Topological Vector Spaces, Distributions and Kernels*

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Publisher's Description

This text for upper-level undergraduates and graduate students focuses on key notions and results in functional analysis. Extending beyond the boundaries of Hilbert and Banach space theory, it explores aspects of analysis relevant to the solution of partial differential equations.

The three-part treatment begins with topological vector spaces and spaces of functions, progressing to duality and spaces of distribution, and concluding with tensor products and kernels. The archetypes of linear partial differential equations (Laplace's , the wave, and the heat equations) and the traditional problem (Dirichlet's and Cauchy's) are the volume's main focus. Most of the basic classical results appear here. There are 390 exercises, several of which contain detailed information that will enable readers to reconstruct the proofs of some important results.

A Note From the Transcriber

I have owned this book for several months but have never put dedicated study time into it. Now is the time. I have just entered graduate school, and plan to devote some time for studying. I have previous exposure to some topics in functional analysis. I know enough Banach space theory to prove the uniqueness of fixed points of contraction mappings, and enough Hilbert space theory to understand the spectral theorem in some form. I have also seen distributions before in some depth as a way to derive fundamental solutions and Green's functions for linear constant-coefficient PDEs. I think it's time for a dedicated advancement of these topics.

0.1 Filters. Topological Spaces. Continuous Mappings

Skipped, as nothing outside of a standard point-set topology class is covered.

0.2 Vector Spaces. Linear Mappings

Skipped, as nothing outside of a standard linear algebra class is covered.

0.3 Topological Vector Spaces. Definition

Let E be a vector space over the complex numbers. Let

$$A : E \times E \to E;\ \ (x, y) \mapsto x + y$$

$$M : \mathbb{C} \times E \to E;\ \ (\lambda, x) \mapsto \lambda x$$

be the basic vector operations on E . A topology $\mathcal{T}$ in E is said to be *compatible with the linear structure* of E if the two maps are continuous in the product topology on each domain. We call E a *topological vector space* if it has a topology compatible with the linear structure.

Note that the topology is "translation-invariant," i.e. neighborhoods uniformly shifted are still neighborhoods. Thus is it only necessary to study the topology near the origin.

Theorem 0.3.1. *A filter $\mathcal{F}$ on a vector space E is the filter of neighborhoods of the origin compatible with the linear structure of E if and only if the following hold:*

1. *The origin belongs to every $U \in \mathcal{F}$*
2. *For every $U \in \mathcal{F}$ there is $V \in \mathcal{F}$ such that $V + V \subset U$*
3. *For every $U \in \mathcal{F}$ and every $\lambda \in \mathbb{C}$ nonzero, we have that $\lambda U \in \mathcal{F}$*
4. *Every $U \in \mathcal{F}$ is absorbing*
5. *Every U contains some $V \in \mathcal{F}$ which is balanced*

Definition 0.3.1. *A subset A of a vector space E is said to be absorbing if for every $x \in E$ there exists $c_x > 0$ such that for all $\lambda \in \mathbb{C}$, $|\lambda| \leq c_x$ $\lambda x \in A$.*

Definition 0.3.2. *A subset A of a vector space E is said to be absorbing if for every $x \in A$ and every $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ then $\lambda x \in A$.*

Proposition 0.3.1. *There is a basis of neighborhoods of zero in a TVS E consisting of only closed sets.*

Corollary 0.3.1. *There is a basis of neighborhoods of 0 in E consisting of closed balanced sets.*

Proposition 0.3.2. *In a TVS E , if a vector subspace M is open, then it is the entire space*

0.4 Hausdorff Topological Vector Spaces. Quotient Topological Vector Spaces. Continuous Linear Mappings

Throughout we denote by E a TVS over the field of complex numbers.

0.4.1 Hausdorff Topological Vector Spaces

A topological space X is said to be *Hausdorff* if, given any two distinct points x and y , there is a neighborhood U of x and a neighborhood V of y that do not intersect.

Theorem 0.4.1. *A filter on a Hausdorff topological space converges to at most one point.*

Corollary 0.4.1. *Every one-point set in a topological space is closed.*

Proposition 0.4.1. *E is Hausdorff iff for every point $x \neq 0$ there is a neighborhood U of 0 such that $x \notin U$.*

Proposition 0.4.2. *The intersection of all neighborhoods of the origin is a vector space of E , which is the closure of the origin.*

Corollary 0.4.2. *For E to be Hausdorff, it is necessary and sufficient that the singleton set containing the origin be closed in E .*

Proposition 0.4.3. *Let f, g be two continuous mappings of a topological space X into E . The set A on which f and g are equal in value is closed in X .*

Proposition 0.4.4. *Let X, f, g be as in the previous. If f and g are equal on a dense subset Y of X , then they are equal everywhere on X .*

0.4.2 Quotient Topological Vector Spaces

Let E be a TVS and M a subspace of E . Then the map $\phi : E \to E/M$ can be made a quotient map in the topological sense by saying a set $\dot{U}$ of E/M if there is an open set $U \subset E$ such that $\phi(U) = \dot{U}$. In other words, we impose the finest topology such that ϕ is an open map (in the topological space). Clearly the quotient topology is compatible with the linear structure on E/M .

Proposition 0.4.5. *Let E be a TVS and M a vector subspace of E . The following two properties are equivalent:*

1. *M is closed*
2. *E/M is Hausdorff*

Corollary 0.4.3. *The TVS $E/\overline{\{0\}}$ is Hausdorff.*

0.4.3 Continuous Linear Mappings

Let E , F be two TVS, and $f : E \to F$ a linear map. If F is Hausdorff and f is continuous, then $\mathrm{Ker} f$ is is closed. Note that these are not necessary conditions, there are other ways of having a closed kernel.

0.5 Cauchy Filters. Complete Subsets. Completion

The definition of a *Cauchy sequence* in a TVS E is simple.

Definition 0.5.1. *Let S be a sequence in E, then S is a Cauchy sequence if for every neighborhood U of the origin in E, there is an integer N such that $n, m \geq N \Rightarrow x_m - x_n \in U$.*

Definition 0.5.2. *$A \subset E$ is said to be complete if every Cauchy sequence in A converges.*

Proposition 0.5.1. *In a Haussdorff TVS E, any complete subset is closed.*

Proposition 0.5.2. *In a complete TVS E, any closed subset is complete.*

Definition 0.5.3. *Let $A \subset E$. A mapping $f : A \to F$ is said to be uniformly continuous if for every neighborhood, V, of zero in F, there is a neighborhood of $U \subset E$ of zero such that for $x_1, x_2 \in A$, $x_1 - x_2 \in U \Rightarrow f(x_1) - f(x_2) \in V$.*

Proposition 0.5.3. *Every continuous linear map of a linear subspace A of TVS E into a TVS F is uniformly continuous.*

Proposition 0.5.4. *Let f be a uniformly continuous map of $A \subset E$ into F. The image under f of a Cauchy sequence is again a Cauchy sequence.*

Theorem 0.5.1. *Let E, F be two Hausdorff TVS, A a dense subset of E, and f a uniformly continuous mapping of A into F.*

If F is complete, there is a unique continuous mapping $\tilde{f}$ of E into F which extends f, i.e. they agree on points of A. Moreover, $\tilde{f}$ is uniformly continuous, and $\tilde{f}$ if A is a linear subspace and if f is linear.

Theorem 0.5.2. *Let E be a TVS. If E is Hausdorff, there exists a complete Hausdorff TVS $\widehat{E}$ and a mapping i of E into $\widehat{E}$ with the following properties:*

1. *The mapping i is an isomorphism of E into $\widehat{E}$.*
2. *The image of E under i is dense in $\widehat{E}$.*
3. *To every complete Hausdorff TVS F and to every continuous linear map $f : E \to F$ there is a continuous linear map $\widehat{f} : \widehat{E} \to F$ such that $f = \widehat{f} \circ i$.*

Furthermore, these maps are unique up to isomorphism.

0.6 Compact Sets

A topological space X is said to be *compact* if X is Hausdorff and if every open cover contains a finite subcovering.

Proposition 0.6.1. *A closed subset of a compact space is compact.*

Proposition 0.6.2. *Let f be a continuous mapping of a compact set X into a Hausdorff topological space Y . Then $f(X)$ is a compact subset of Y .*

Proposition 0.6.3. *Let f be a 1-1 continuous mapping of a compact space X onto a compact space Y . Then f is a homeomorphism.*

Let E be a TVS.

Definition 0.6.1. *$x \in E$ is an accumulation point of a sequence if it belongs to the closure of the set of points of the sequence.*

Proposition 0.6.4. *If a sequence converges to x , then x is an accumulation point of the sequence.*

Proposition 0.6.5. *If a Cauchy sequence in E has an accumulation point, then it it converges to that point.*

Proposition 0.6.6. *Let K be a Hausdorff topological space. The following are equivalent:*

1. *K is compact*
2. *every sequence on K has at least one accumulation point*

Corollary 0.6.1. *A compact subset K of a Hausdorff topological space E is closed.*

Corollary 0.6.2. *In compact topological spaces, every sequence has an accumulation point.*

Definition 0.6.2. *$A \subset X$ is said to be relatively compact (or precompact) if the closure of A is compact.*

0.7 Locally Convex Spaces. Seminorms

A subset K of a vector space E is convex if, whenever K contains two points x and y , K also contains the straight line joining them. Now let S be any subset of E . We define the *convex hull* of S to be the set of all finite linear combinations with nonnegative coefficients that sum to one. A set is convex if it is equal to its own convex hull.

Intersections of convex subsets, but unions may not be. Sums of convex sets are convex, and the image of convex sets under linear maps are convex.

Proposition 0.7.1. *Let E be a TVS. The closure and interior of convex sets are convex.*

Definition 0.7.1. *A subset T of a TVS E is called a barrel if T has the following four properties:*

1. *T is absorbing*
2. *T is balanced*
3. *T is closed*
4. *T is convex*

We can construct barrels in the following way: let U be a neighborhood of 0 in E . Then we define $T(U)$ the closure of the convex hull of the set

$$\bigcup_{||\lambda|\leq 1|} \lambda U$$

And $T(U)$ is a barrel.

Definition 0.7.2. *A TVS E is said to be a locally convex space if there is a neighborhoods in E consisting of convex sets.*

Proposition 0.7.2. *In a locally convex space E , there is a basis of neighborhoods of zero consisting of barrels.*

Definition 0.7.3. *A nonnegative function $p : E \to \mathbb{R}$ on a vector space E is called a seminorm if it satisfies the following conditions:*

1. *p is subadditive: $p(x + y) \leq p(x) + p(y)$*
2. *p is positively homogeneous of degree 1: $p(\lambda x) = |\lambda| p(x)$*
3. *$p(0) = 0$*

A seminorm on a vector space E is called a norm if $p(x) = 0 \Rightarrow x = 0$.

Definition 0.7.4. *A vector space E over the field of complex numbers, provided with a Hermitian nonnegative form, is called a complex pre-Hilbert space.*

Definition 0.7.5. *Let E be a vector space, and p a seminorm on E . The sets*

$$U_p = \{x \in E : p(x) \leq 1\} \; \dot{U}_p = \{x \in E : p(x) < 1\}$$

are called the closed and open unit semiball of p .

Proposition 0.7.3. *Let E be a topological vector space, and p a seminorm on E . Then the following are equivalent:*

1. *the open unit semiball of p is an open set*
2. *p is continuous at the origin*
3. *p is continuous at every point*

Proposition 0.7.4. *If p is a continuous seminorm on a TVS E , its closed unit semiball is a barrel.*

Proposition 0.7.5. *The E be a TVS, and T a barrel in E . There exists a unique seminorm p on E such that T is the closed unit semiball of p . The seminorm p is continuous if and only if T is a neighborhood of* 0 .

Corollary 0.7.1. *Let E be a locally convex space. The closed unit semiballs of the continuous seminorms on E form a basis of neighborhoods of the origin.*

Definition 0.7.6. *A family $\mathcal{P}$ of continuous seminorms on a locally convex space E will be called a basis of continuous seminorms on E if for any continuous seminorm p on E there is a continuous seminorm q belonging to $\mathcal{P}$ and a constant $C > 0$ such that $p(x) \leq Cq(x)$.*

Proposition 0.7.6. *Let $\mathcal{P}$ be a basis of continuous seminorms on the locally convex space E . Then the sets λU_p where U_p is the closed unit semiball of $p \in \mathcal{P}$ and λ is a positive number, form a basis of neighborhoods of 0. Conversely, given a family, $\mathcal{B}$ of neighborhoods of zero and consisting of barrels and such that the sets λU, $U \in \mathcal{B}$ form a basis of neighborhoods of 0 in E , then the seminorms whose closed unit semiballs are the barrels belonging to $\mathcal{B}$ form a basis of continuous seminorms in E .*

We shall even say that a basis of continuous seminorms on a locally convex space E *defines* the topology of E . We shall also use the expression *"a family of seminorms on E defining the topology of E"* in which the family under consideration need not be a basis of continuous seminorms. The meaning it is the following: first: every seminorm p_α is continuous; second, the family obtained by forming the supremum of finite numbers of seminorms p_α is a basis of of continuous seminorms on E . This family consists of the seminorms

$$x \mapsto p_{(B)}(x) = \sup_{\alpha \in B} p_\alpha(x)$$

where B ranges over all the finite subsets of the set of indices A of the family $\{p_\alpha\}$. Forming the supremum of a finite number of seminorms is the equivalent of forming the intersection of their closed unit semiballs and taking the "gauge" of this intersection (a seminorm p is the *gauge* of a set U if U is the closed unit semiball of p).

Proposition 0.7.7. *Let E, F be two locally convex spaces. A linear map $f : E \to F$ is continuous if and only if to every continuous seminorm q on F there is a continuous seminorm p on E such that*

$$q\left(f\left(x\right)\right) \leq p(x)$$

Corollary 0.7.2. *A linear form f on a locally convex space, E , is continuous if and only if there is a continuous seminorm p on E such that $|f(x)| \leq p(x)$.*

Proposition 0.7.8. *Let E be a locally convex space, and M a linear subspace of E . Let ϕ be the canonical mapping of $E \to E/M$. Then the following are true:*

1. *the topology of the quotient* TVS *is locally convex*
2. *if* $\mathcal{P}$ *is a basis of continuous seminorms on* E *, let us dentoe by* $\dot{\mathcal{P}}$ *the family of seminorms on* E/M *consisting of the seminorms*

$$\dot{x} \mapsto \dot{p}(\dot{x}) = \inf_{\phi(x)=\dot{x}} p(x)$$

Then $\dot{\mathcal{P}}$ *is a basis of continuous seminorms on* E/M*.*

We call the *kernel* of a seminorm p on E the set of vectors with vanishing seminorm. This is a subspace of E and we denote is by $\mathrm{Ker}p$. It is closed, and in a locally convex space E , the closure of the origin is

$$\bigcap_p \mathrm{Ker}p$$

Proposition 0.7.9. *In a locally convex space* E *, the closure of the singleton set containing zero is the intersection of the closed linear subspaces* $\mathrm{Ker}p$ *, where* p *varies over a basis of continuous seminorms on* E *.*

Proposition 0.7.10. *Let* E *be a locally convex Hausdorff TVS, and* K *a precompact subset of* E *. Then the convex hull* $\Gamma(K)$ *of* K *is precompact.*

Corollary 0.7.3. *If* E *is complete the closed convex hull of a compact subset of* E *is compact.*

0.8 Metrizable Topological Vector Spaces

A TVS E is said to be *metrizable* if it is Hausdorff and if there is a *countable* basis of neighborhoods of zero in E . The motivation for the name metrizable lies in the following fact (which we shall not prove in such a general form):

The topology of a TVS E can be defined by a metric if and only if E is Hausdorff and has a countable basis of neighborhoods of 0.

Note that a norm on a vector space defines a metric on E , by the rule $d(x, y) = \|x - y\|$, but not all metrizable spaces can be defined by a norm.

Proposition 0.8.1. *Let* E *be a locally convex metrizable TVS, and* $\{p_1, p_2, \ldots\}$ *a nondecreasing countable basis of seminorms on* E *. Let* a_n *be a sequence positive of numbers whose sum converges. Then the function*

$$(x, y) \mapsto \sum_{j=1}^{\infty} a_j p_j(x - y) / [1 + p_j(x - y)]$$

is a translation invariant metric on E *defining the topology of* E *.*

Proposition 0.8.2. *A subset K of a metrizable space E is complete if and only if every Cauchy sequence in K converges to a point of K .*

Proposition 0.8.3. *A complete metrizable TVS E is a Baire space, i.e., has the equivalent properties:*

1. *The union of a countable family of closed sets with empty interior again has empty interior*
2. *The intersection of any countable family of open dense subsets is again dense.*

Remark 0.8.1. *There exist complete TVS which are not Baire spaces, for example the LF spaces we will study later.*

Remark 0.8.2. *Note that completeness and metrizability and not necessary conditions for being a Baire space: there are nonmetrizable Baire spaces and incomplete Baire spaces.*

Proposition 0.8.4. *In a metrizable TVS E , a set K is compact if and only if every sequence has a limit point in K .*

Definition 0.8.1. *A mapping f toa topological space E into a topological space F is said to be sequentially continuous if for every convergent sequence x_n , the sequence $f(x_n)$ also converges to $f(x)$.*

Proposition 0.8.5. *A mapping f of a metrizable TVS into a TVS F is continuous if and only if it is sequentially continuous.*

0.9 Finite Dimensional Hausdorff Topological Vector spaces. Linear Subspaces with Finite Codimension. Hyperplanes

Theorem 0.9.1. *Let E be a finite-dimensional Hausdorff TVS.*

1. *E is isomorphic, as a TVS, to $\mathbb{C}^d$ where $d = \dim E$.*
2. *Every linear functional on E is continuous.*
3. *Every linear map of E into any TVS F is continuous.*

Corollary 0.9.1. *Every finite-dimensional Hausdorff TVS is complete.*

Corollary 0.9.2. *Every finite dimensional linear subspace of a Hausdorff TVS is closed.*

A space is said to be *locally compact* if it has a basis of compact neighborhoods.

Theorem 0.9.2. *A locally compact TVS is finite dimensional.*

Definition 0.9.1. *A linear subspace of codimension one is called a hyperplane.*

Proposition 0.9.1. *A hyperplane H is a vector space E is a maximal proper linear subspace or E .*

Proposition 0.9.2. *A hyperplane H in a TVS E is either everywhere dense or closed.*

Proposition 0.9.3. *Let E be a TVS and M a closed linear subspace of E of finite codimension. Then there is a homomorphism p of E onto M such that $p^2 = p$. Also, $E = M \oplus \ker p$. The homomorphism is called a projection. It is both continuous and open.*

0.10 Fréchet Spaces. Examples

A *Fréchet space* (or *F-space*) is a TVS with the following three properties:

1. it is metrizable ($\Rightarrow$ metrizable)
2. it is complete ($\Rightarrow$ Baire)
3. it is *locally convex* (and carries a metric based on seminorms as presented in previous propositions)

Proposition 0.10.1. *1. Any closed subspace of an F-space is an F-space*

2. Any product of two F-spaces is an F-space

3. The quotient of an F-space module a closed subspace is an F-space.

0.10.1 Example I. The space of $\mathcal{C}^k$ Functions in an open subset Ω of $\mathbb{R}^n$.

The set of functions under consideration is $\mathcal{C}^k(\Omega)$, where Ω is open and $k \geq 0, k = \infty$. We impose the seminorms

$$|f|_{m,K} = \sup_{|p| \leq m} \left(\sup_{x \in K} |(\partial/\partial x)^p f(x)| \right)$$

Where $0 < m < k$ and $K \subset \Omega$ is compact. Clearly the seminorms are defined for every function since we deal with continuously differentiable functions of order $\leq k$. We give $\mathcal{C}^k$ the topology induced by these seminorms, called the *topology of uniform convergence on compact sets.*

This space is a Freèchet space as it is indeed metrizable, complete, and locally convex.

0.10.2 Example II. The Space of Holomorphic Functions in an Open subset Ω of $\mathbb{C}^n$

This is clearly a subspace of $\mathcal{C}^\infty(\mathbb{R}^{2n})$, and with the fact that uniform convergence of a holomorphic function is again holomorphic, we are therefore led to the conclusion that $H(\Omega)$ is a closed subspace. Thus $H(\Omega)$ is an F-space.

0.10.3 Example III. The Space of Formal Power Series in n Indeterminates

Skipped. I don't like it.

0.10.4 Example IV. The space $\mathscr{S}$ of $\mathcal{C}^\infty$ functions Rapidly Decreasing at Infinity