# Notes from *Functional Analysis, Sobolev Spaces, and PDEs*

by Haim Brezis

taken by Samuel T. Wallace

## Publisher's Description

Uniquely, this book presents a coherent, concise and unified way of combining elements from two distinct worlds, functional analysis (FA) and partial differential equations (PDEs), and is intended for students who have a good background in real analysis. This text presents a smooth transition from FA to PDEs by analyzing in great detail the simple case of one-dimensional PDEs (i.e., ODEs), a more manageable approach for the beginner.

Although there are many books on functional analysis and many on PDEs, this is the first to cover both of these closely connected topics. Moreover, the wealth of exercises and additional material presented, leads the reader to the frontier of research. This book has its roots in a celebrated course taught by the author for many years and is a completely revised, updated, and expanded English edition of the important Analyse Fonctionnelle (1983). Since the French book was first published, it has been translated into Spanish, Italian, Japanese, Korean, Romanian, Greek and Chinese. The English version is a welcome addition to this list. The first part of the text deals with abstract results in FA and operator theory.

The second part is concerned with the study of spaces of functions (of one or more real variables) having specific differentiability properties, e.g., the celebrated Sobolev spaces, which lie at the heart of the modern theory of PDEs. The Sobolev spaces occur in a wide range of questions, both in pure and applied mathematics, appearing in linear and nonlinear PDEs which arise, for example, in differential geometry, harmonic analysis, engineering, mechanics, physics etc. and belong in the toolbox of any graduate student studying analysis.

## A Note From the Transcriber

These notes were taken for self-study in Spring 2020. I was already familiar with some aspects of FA, through a mathematical methods course which essentially was a "FA for Scientists' class where distributions and the basic theory of operators on Banach and Hilbert spaces were covered, finishing with the spectral theorem.

The reader is assumed to have a semester in point-set topology, some exposure to PDEs, and familiarity with abstract vector spaces. These notes were taken without proof for brevity, and like all my notes, is meant as a 'theorem cheat sheet,' rather than a comprehensive self-teaching course.

# Contents

## 0.1 The Hahn-Banach Theorems.

### 0.1.1 The Analytic Form of the Hahn-Banach Theorem: Extensions of Linear Functionals

Let $E$ be a vector space over $\mathbb{R}$. We recall that a *functional* is a function defined n $E$, or a subspace of $E$, *with values in* $\mathbb{R}$.

**Theorem 0.1.1** (Helly, Hahn-Banach analytic form)**.** *Let $p : E \to \mathbb{R}$ be a function satisfying*

1. $p(\lambda x) = \lambda p(x)$
2. $p(x + y) \leq p(x) + p(y)$

*Let $G \subset E$ be a linear subspace and let $g : G \to \mathbb{R}$ be a linear functional such that*

$$g(x) \leq p(x) \ \forall x \in G$$

*Then there exists a linear functional $f : E \to \mathbb{R}$ such that $f \restriction_G = g$.*

### 0.1.2 The Geometric Forms of the Hahn-Banach Theorem: Separation of Convex Sets

**Definition 0.1.1.** *An affine hyperplane is a subset $H \subset E$ of the form*

$$H = \{x \in E : f(x) = \alpha\}$$

*where $f$ is a linear functional that does not vanish identically and $\alpha \in \mathbb{R}$. We write $H = [f = \alpha]$ and say that $f = \alpha$ is the equation of $H$.*

**Proposition 0.1.1.** *The hyperplane $H = [f = \alpha]$ is closed if and only if $f$ is continuous.*

**Definition 0.1.2.** *Let $A$ and $B$ be two subsets of $E$. We say the hyperplane $H = [f = \alpha]$ separates $A$ and $B$ if*

$$f(a) \leq \alpha \ \forall a \in A \text{ and } f(b) \geq \alpha \ \forall b \in B$$

*We say that $H$ strictily separates $A$ and $B$ if there exists an $\epsilon > 0$ such that*

$$f(a) \leq \alpha - \epsilon \ \forall a \in A \text{ and } f(b) \geq \alpha + \epsilon \ \forall b \in B$$

*A subset $A \subset E$ is convex if*

$$tx + (1 - t)y \in A \ \forall x, y \in A \ \forall t \in [0, 1]$$

**Theorem 0.1.2** (Hahn-Banach, first geometric form)**.** *Let $A \subset E$ and $B \subset E$ be two nonempty convex disjoint subsets, one of which is open. Then there exists a closed hyperplane separating them.*

**Theorem 0.1.3** (Hahn-Banach,second geometric form)**.** *Let $A \subset E$ and $B \subset E$ be two empty convex disjoint subsets. If $A$ is closed and $B$ is compact, then there exists a closed hyperplan separating $A$ and $B$.*

**Corollary 0.1.1.** *Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^*$ not identically zero such that $f(F) = 0$*

### 0.1.3 The Bidual $E^{**}$. Orthogonality Relations

Let $E$ be a normed vector space and let $E^*$ be the dual space with norm

$$\|f\|_{E^*} = \sup_{\|x\| \leq 1} |\langle f, x \rangle|$$

The bidual $E^{**}$ is the dual of $E^*$ with norm

$$\|\xi\|_{E^{**}} = \sup_{\|f\| \leq 1} |\langle xi, f \rangle|$$

There is a canonical injection $J : E \to E^{**}$ defined as

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E}$$

which is an *isometry*. $J$ may not be surjective, but if it is, we say $E$ is reflexive.

**Definition 0.1.3.** *If $M \subset E$ is a linear subspace, let*

$$M^\perp = \{f \in E^* : \langle f, x \rangle = 0 \ \forall x \in M\}$$

*If $N \subset E^*$ is a linear subspace we set*

$$N^\perp = \{x \in E : \langle f, x \rangle = 0 \ \forall f \in N\}$$

**Proposition 0.1.2.** *Let $M \subset E$ be a linear subspace. Then*

$$\left(M^\perp\right)^\perp = \overline{M}$$

*Let $N \subset E^*$ be a linear subspace. Then*

$$\overline{N} \subset \left(N^\perp\right)^\perp$$

### 0.1.4 A Quick Introduction to the Theory of Conjugate Convex Functions

**Definition 0.1.4.** *Let $E$ be a set, and $\phi : E \to (-\infty, +\infty]$ a function. Let*

$$D(\phi) = \{x \in E : \phi(x) < +\infty\}$$

*be the domain of $\phi$. We define the epigraph of $\phi$*

$$\text{epi}\phi = \{[x, \lambda] \in E \times \mathbb{R}; \phi(x) \leq \lambda\}$$

*If $E$ is a topological space, we say $\phi$ is lower semicontinuous if $\lambda \in \mathbb{R}$ the set*

$$[\phi \leq \lambda] = \{x \in E : \phi(x) \leq \lambda\}$$

*is closed.*

**Proposition 0.1.3.** *If $\phi$ is lower-semicontinuous, then*

1. epi$\phi$ *is closed in $E \times \mathbb{R}$ and conversely,*
2. *for every $x \in E$ and $\epsilon > 0$ there is a neighborhood $V$ of $x$ such that*
$$\phi(y) \geq \phi(x) - \epsilon \ \forall y \in V$$
*and conversely.*
3. *If $\phi_1$ and $\phi_2$ are lower semicontinuous, then so is $\phi_1 + \phi_2$*
4. *If $(\phi_i)_{i \in I}$ is a family of lsc functions then so is*
$$\phi(x) = \sup_{i \in I} \phi_i(x)$$
*called the superior envelope.*
5. *If $E$ is compact and $\phi$ is lsc, then $\inf_E \phi$ is achieved.*

**Definition 0.1.5.** *A function $\phi : E \to (-\infty, +\infty]$ is convex if*
$$\phi\left(tx + (1-t)y\right) \leq t\phi(x) + (1-t)\phi(y) \ \forall x, y \in E, \ \forall t \in (0,1)$$

**Proposition 0.1.4.** *If $\phi$ is a convex function, then*

1. epi$\phi$ *is a convex set in $E \times \mathbb{R}$ and conversely*
2. *$\forall \lambda \in \mathbb{R}$ the set $[\phi \leq \lambda]$ is convex, but not the converse*
3. *a sum of convex functions is again convex*
4. *the superior envelope of a family of convex functions is again convex.*

Let $E$ be a normed vector space.

**Definition 0.1.6.** *Let $\phi : E \to (-\infty, +\infty]$ be a function with nonempty domain. We define the conjugate function $\phi^* : E^* \to (-\infty, +\infty]$ by*
$$\phi^*(f) = \sup_{x \in E} \{\langle f, x\rangle - \phi(x)\}$$

**Proposition 0.1.5.** *Assume that $\phi : E \to (-\infty, +\infty]$ is convex lsc with nonempty domain. Then $\phi^*$ has nonempty domain and is bounded below by an affine continuous function.*

**Definition 0.1.7.** *Instead of defining $\phi^{**}$ on $E^{**}$, we can define it on $E$ by*
$$\phi^{**}(x) = \sup_{f \in E^*} \{\langle f, x\rangle - \phi^*(f)\}$$

**Theorem 0.1.4** (Fenchel-Moreau)**.** *Let $\phi : E \to (-\infty, +\infty]$ is convex lsc with nonempty domain. Then $\phi^{**} = \phi$.*

**Theorem 0.1.5** (Fenchel-Rockafeller)**.** *Let $\phi, \psi$ be two convex functions. Assume there is some $x_0 \in D(\phi) \cap D(\psi)$ such that $\phi$ is continuous at $x_0$. Then*
$$\inf_{x \in E} \{\phi(x) + \psi(x)\} = \sup_{f \in E^*} \{-\phi^*(-f) - \psi^*(f)\}$$
$$= \max_{f \in E^*} \{-\phi^*(-f) - \psi^*(f)\} = -\min_{f \in E^*} \{\phi^*(-f) + \psi^*(f)\}$$

## 0.2 The Uniform Boundedness Principle and the Closed Graph Theorem

### 0.2.1 The Baire Category Theorem

**Theorem 0.2.1** (Baire). *Let $X$ be a complete metric space and $(X_n)_{n\geq q}$ be a sequence of closed subsets in $X$. If*

$$\text{Int}X_n = \emptyset$$

*Then*

$$\text{Int}\left(\bigcup_n X_n\right) = \emptyset$$

### 0.2.2 The Uniform Boundedness Principle

**Definition 0.2.1.** *Let $E$ and $F$ be two normed vector spaces. Let $\mathcal{L}(E,F)$ be the space of continuous (bounded) linear operators equipped with the norm*

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\|x\|\leq 1} \|Tx\|$$

*And we write $\mathcal{L}(E) = \mathcal{L}(E,E)$.*

**Theorem 0.2.2** (Banach-Steinhaus, uniform boundedness principle). *Let $E$ and $F$ be two Banach spaces and let $(T_i)_{i\in I}$ be a family of continuous linear operator from $E$ into $F$. If*

$$\forall x \in E \ \sup_{i\in I} \|T_i x\| \leq \infty$$

*Then*

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)}$$

**Corollary 0.2.1.** *Let $E$ and $F$ be two Banch spaces. Let $(T_n)$ be a sequence of continuous linear operators from $E$ into $F$ such that $\forall x \in E$ $T_n x$ converges (to a limit we call $Tx$). Then*

1. $\sup_n \|T_n\|_{\mathcal{L}(E,F)} < \infty$
2. $T \in \mathcal{L}(E,F)$
3. $\|T\|_{\mathcal{L}(E,F)} \leq \text{liminf}_n \|T_n\|_{\mathcal{L}(E,F)}$

**Corollary 0.2.2.** *Let $G$ be a Banach space and let $B$ be a subset of $G$. If*

$$\forall f \in G^* \ f(B) \textit{ is bounded in } \mathbb{R}$$

*Then $B$ is bounded.*

**Corollary 0.2.3.** *Let $G$ be a Banach space and let $B^*$ be a subset of $G^*$. If*

$$\forall x \in G \ \langle B^*, x\rangle \textit{ is bounded in } \mathbb{R}$$

*Then $B^*$ is bounded.*

### 0.2.3 The Open Mapping Theorem and the Closed Graph Theorem

**Theorem 0.2.3** (Open Mapping Theorem)**.** *Let $E$ and $F$ be two Banach spaes and let $T$ be a continuous linear operator from $E$ into $F$ that is surjective. Then there exists $\delta > 0$ such that*

$$T\left(B_E(0,1)\right) \supset B_F(0,\delta)$$

*Which says $T$ is an open mapping.*

**Corollary 0.2.4.** *Let $E$ and $F$ be two Banach spaces and let $T$ be a continuous linear operator from $E$ into $F$ that is bijective. Then $T^{-1}$ is also continuous.*

**Corollary 0.2.5.** *Let $E$ be a vector space with two norms $\|\cdot\|_1, \|\cdot\|_2$ with both make $E$ into a Banach Space, and that there is a constant $C \geq 0$ such that*

$$\|x\|_2 \leq C\|x\|_1$$

*Then the two norms are equivalent.*

**Theorem 0.2.4** (Closed Graph Theorem)**.** *Let $E$ and $F$ be two Banach spaces. Let $T$ be a linear operator from $E$ to $F$. If the graph of $T$, $G(T)$, is closed in $E \times F$, then $T$ is continuous.*

### 0.2.4 Complementary Subspaces. Right and Left Invertibility of Linear Operators

**Theorem 0.2.5.** *Let $E$ be a Banach space. Assume that $G$ and $L$ are two closed linear suspaces such that $G + L$ is closed. Then there exists a constant $C \geq 0$ such that $z \in G + L \Rightarrow z = x + y$ with $C\|z\| \geq \|x\|, x \in G$ and $C\|z\| \geq \|y\|, y \in L$.*

**Definition 0.2.2.** *Let $G \subset E$ be a closed subspace of a Banach space $E$. A subspace $L \subset E$ is said to be a topological complement or simply a complement of $G$ if*

1. *$L$ is closed*
2. *$G \cap L = 0$ and $G + L = E$*

*We also say $G$ and $L$ are complementary subspaces of $E$. If this holds, then every $z$ can be decomposed into components in $G$ and $L$, for which the projection operators are continuous.*

**Definition 0.2.3.** *Let $T \in \mathcal{L}(E, F)$. A right inverse is an operator $S \in \mathcal{L}(F, E)$ such that $T \circ S = I_F$. A left inverse is an operator $S \in \mathcal{L}(F, E)$ such that $S \circ T = I_E$.*

**Theorem 0.2.6.** *Let $T \in \mathcal{L}(E, F)$ be surjective. The following are equivalent:*

1. *$T$ admits a right inverse.*
2. *$N(T) = T^{-1}(0)$ admits a complements in $E$.*

**Theorem 0.2.7.** *Let $T \in \mathcal{L}(E, F)$ be injective. The following are equivalent:*

1. *$T$ admits a left inverse.*
2. *$R(T) = T(E)$ is closed and admits a complement in $F$.*

### 0.2.5 Orthogonality Revisited

**Proposition 0.2.1.** *Let $G$ and $L$ be two closed subspaces in $E$. Then*

$$G \cap L = \left(G^{\perp} + L^{\perp}\right)^{\perp}$$

$$G^{\perp} \cap L^{\perp} = (G + L)^{\perp}$$

**Corollary 0.2.6.**

$$(G \cap L)^{\perp} \supset \overline{G^{\perp} + L^{\perp}}$$

$$\left(G^{\perp} \cap L^{\perp}\right)^{\perp} = \overline{G + L}$$

**Theorem 0.2.8.** *Let $G$ and $L$ be two closed subpsaces in a Banach spaces $E$. The following are equivalent:*

1. *$G + L$ is closed in $E$*
2. *$G^{\perp} + L^{\perp}$ is closed in $E^*$*
3. *$G + L = \left(G^{\perp} + L^{\perp}\right)^{\perp}$*
4. *$G^{\perp} + L^{\perp} = (G \cap L)^{\perp}$*

### 0.2.6 An Introduction to Unbounded Linear Operators. Definition of the Adjoint

**Definition 0.2.4.** *Let $E$ and $F$ be two Banach spaces. An unbounded linear operator from $E$ into $F$ is a linear map $A : D(A) \subset E \to F$ where $D(A)$ is a linear subspace called the domain of $A$.*

*$A$ is bounded (or continuous) if $D(A) = E$ and there is a $c \geq 0$ such that*

$$\|Au\| \leq c\|u\|$$

*The norm of a bounded operator is defined as*

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}$$

*Some additional definitions are as follows:*

1. *$G(A) = \{(u, Au) : u \in D(A)\} \subset E \times F$, the Graph of $A$*
2. *$R(A) = \{Au : u \in D(A)\} \subset F$, the range of $A$*
3. *$N(A) = \{u \in D(A) : Au = 0\} \subset E$, the kernel of $A$.*

*An operator $A$ is closed if $G(A)$ is closed in $E \times F$.*

**Definition 0.2.5.** *Let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined ($D(A)$ is dense in $E$). We introduce a new operator $A^* : D(A^*) \subset F^* \to E^*$ as follows. First we define*

$$D(A^*) = \{v \in F^* : \exists c \geq 0 \ |\langle v, Au\rangle| \leq c\|u\| \ \forall u \in D(A)\}$$

*Now we go about defining $A^*v$. Given $v \in D(A^*)$, we define $g(u) = \langle v, Au\rangle$. Use Hahn-Banach to extend $g$ to a bounded functional $f \in E^*$, which is unique if $D(A)$is dense in $E$. Let $A^*v = f$. In brief,*

$$\langle v, Au\rangle_{F^*,F} = \langle A^*v, u\rangle_{E^*,E}$$

**Proposition 0.2.2.** *Let $A : D(A) \subset E \to F$ be a densely defined unbounded linear operator. Then $A^*$ is closed.*

**Corollary 0.2.7.** *Let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined and closed. Then*

1. *$N(A) = R(A^*)^\perp$*
2. *$N(A^*) = R(A)^\perp$*
3. *$N(A)^\perp \supset \overline{R(A^*)}$*
4. *$N(A^*)^\perp = \overline{R(A)}$*

### 0.2.7 A Characterization of Operators with Closed Range. A Characterization of Surjective Operators

**Theorem 0.2.9.** *Let $A : D(A) \subset E \to F$ be an unbounded linear operator that is densely defined and closed. The following are equivalent:*

1. *$R(A)$ is closed*
2. *$R(A^*)$ is closed*
3. *$R(A) = N(A^*)^\perp$*
4. *$R(A^*) = N(A)^\perp$*

**Theorem 0.2.10.** *Let $A : D(A) \subset E \to F$ be a linear operator that is densely defined and closed. The following are equivalent:*

1. *$A$ is surjective*

2. *There is a constant $C$ such that*

$$\|v\| \leq C\|A^* v\|$$

3. $N(A^*) = \{0\}$ *and* $R(A^*)$ *is closed.*

**Theorem 0.2.11.** *Let $A : D(A) \subset F$ ($\rightarrow E^*$?) be an unbounded linear operator that is densely defined and closed. The following are equivalent:*

1. $A^*$ *is surjective*

2. *there is a constant $C$ such that*

$$\|u\| \leq C\|Au\|$$

3. $N(A) = 0$ *and* $R(A)$ *is closed.*

## 0.3 Weak Topologies. Reflexive Spaces. Separable Spaces. Uniform Convexity

### 0.3.1 The Coarsest Topology for Which a Collection of Maps Becomes Continuous

**Definition 0.3.1.** *The weak topology $\sigma(E, E^*)$ on $E$ is the coarsest topology such that the collection*

$$\{\phi_f : E \rightarrow \mathbb{R};\ x \mapsto \langle f, x\rangle : f \in E^*\}$$

*of functions contains continuous functions.*

**Proposition 0.3.1.** *The weak topology $\sigma(E, E^*)$ is Hausdorff.*

**Proposition 0.3.2.** *Let $x_0 \in E$ and let $\epsilon > 0$ and let $F = \{f_1, f_2, \ldots, f_k\}$ be a finite set in $E^*$. Let*

$$V = V(f_1, \ldots f_k, \epsilon) = \{x \in E; |\langle f_i, x - x_0\rangle| < \epsilon\}$$

*Then $V$ is a neighborhood of $x_9$ for the topology $\sigma(E, E^*)$. Neighborhood of this form make up a basis for all the neighborhoods around $x_0$.*

**Proposition 0.3.3.** *Let $(x_n)$ be a sequence in $E$. Then*

1. $x_n \rightarrow x$ *weakly* $\iff \forall f \in E^* \langle f, x_n\rangle \rightarrow \langle f, x\rangle$

2. $x_n \rightarrow x$ *weakly then* $x_n \rightarrow x$ *strongly.*

3. $x_n \rightarrow x$ *weakly, then* $\|x_n\|$ *is bounded and* $\|x\| \leq \liminf \|x_n\|$

4. *If* $x_n \rightarrow x$ *weakly and if* $f_n \rightarrow f$ *strongly in* $E^*$*, then* $\langle f_n, x_n\rangle \rightarrow \langle f, x\rangle$

**Proposition 0.3.4.** *When $E$ is **finite-dimensional**, the weak topology and strong topology coincide.*

### 0.3.2 Weak Topology Convex Sets, and Linear Operators

**Theorem 0.3.1.** *Let $C$ be a convex subset of $E$. Then $C$ is closed in the weak topology if and only iff it is closed in the strong topology.*

**Corollary 0.3.1** (Mazur). *Assume $(x_n)$ converges **weakly** to $x$. Then there is a sequence $(y_n)$ made up of convex combinations of the $x_n$'s that converge **strongly** to $x$.*

**Corollary 0.3.2.** *Assume that $\phi : E \to (-\infty, +\infty]$ is convex and lsc in the strong topology. Then $\phi$ is lsc in the weak topology.*

**Theorem 0.3.2.** *Let $E$ and $F$ be two Banach spaces and let $T$ be a linear operator from $E$ into $F$. Assume that $T$ is continuous in the strong topologies. Then $T$ is continuous on $E$ with the weak topology to $F$ with its weak topology.*

### 0.3.3 The Weak* Topology $\sigma(E^*, E)$

**Definition 0.3.2.** *The weak* topology, $\sigma(E^*, E)$ is the coarsest topology on $E^8$ associated to the evaluation maps*

$$\{\phi_x : E^* \to \mathbb{R} : x \in E\}$$

*such that all the evaluation maps are continuous.*

**Proposition 0.3.5.** *The weak* topology is Hausdorff.*

**Proposition 0.3.6.** *Let $f_0 \in E^*$; given a **finite** set $\{x_1, \dots, x_k\}$ and an $\epsilon > 0$, let*

$$V = V(x_1, \dots, x_k, \epsilon) = \{f \in E^* : |\langle f - f_0, x_i\rangle|\}$$

*Then $V$ is a neighborhood of $f_0$ for the weak* topology; neighborhoods of this form become a **basis of neighborhoods** for $f_0$.*

**Definition 0.3.3.** *If a sequence $(f_n)$ converges to $f$ in the weak* topology we will write*

$$f_n \xrightarrow{*} f$$

**Proposition 0.3.7.** *Let $(f_n)$ be a sequence in $E^*$.*

1. $f_n \xrightarrow{*} f$ *if and only iff* $\langle f_n, x\rangle \to \langle f, x\rangle$
2. $f_n \to f$ *strongly* $\Rightarrow f_n \to f$ *in the weak topology on* $E^* \Rightarrow f_n \xrightarrow{*} f$
3. $f_n \xrightarrow{*} f$ *then* $\|f_n\|$ *is bounded and* $\|f\| \leq \operatorname{liminf}\|f_n\|$
4. $f_n \xrightarrow{*} f$ *and if* $x_n \to x$ *strongly, then* $\langle f_n, x_n\rangle \to \langle f, x\rangle$

**Proposition 0.3.8.** *Let $\phi : E^* \to \mathbb{R}$ be a linear functional that is continuous in the weak* topology. Then there is some $x_0 \in E$ such that*

$$\phi(f) = \langle f, x_0\rangle$$

**Lemma 0.3.1.** *Let $X$ be a vector space and let $\phi, \phi_1, \ldots, \phi_k$ be linear functionals on $X$ such that*

$$\phi_i(v) = 0 \ \forall i \Rightarrow \phi(v) = 0$$

*Then there are constants $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\phi = \sum_i \lambda_i \phi_i$*

**Corollary 0.3.3.** *Assume that $H$ is a hyperplane in $E^*$ that is closed in the weak* topology. Then $H$ has the form*

$$H = \{f \in E^* : \langle f, x_0 \rangle = \alpha\}$$

*for some $x_0 \in E$ and an $\alpha \in \mathbb{R}$.*

**Theorem 0.3.3** (Banach-Alaoglu-Bourbaki)**.** *The closed unit ball*

$$B_{E^*} = \{f \in E^* : \|f\| \leq 1\}$$

*is compact in the weak* topology on $E^*$*

### 0.3.4 Reflexive Spaces

**Definition 0.3.4.** *Let $E$ be a Banach space and let $J : E \to E^{**}$ be the canonical injetion from $E$ to $E^{**}$. The space $E$ is reflexive if $J$ is surjective.*

*When $E$ is reflexive, $E^{**}$ is usually identified with $E$.*

**Theorem 0.3.4.** *Let $E$ be a Banach space. Then $E$ is reflexive if and only if*

$$B_E = \{x \in E : \|x\| \leq 1\}$$

*is compact in the weak topology.*

**Theorem 0.3.5.** *Assume $E$ is a reflexive Banach space and let $(x_n)$ be a bounded sequence in $E$. Then there is a subsequence $x_{n_k}$ that converges in the weak topology.*

**Theorem 0.3.6** (Eberlein-Smulian)**.** *Assume $E$ is a Banach space such that every bounded sequence in $E$ admits a weakly convergent subsequence. Then $E$ is reflexive.*

**Proposition 0.3.9.** *Assume $E$ is a reflexive Banach space and let $M \subset E$ be a closed subspace of $E$. The $M$ is reflexive.*

**Corollary 0.3.4.** *A Banach space $E$ is reflexive if and only if its dual space $E^*$ is reflexive.*

**Corollary 0.3.5.** *Let $E$ be a reflexive Banach space. Let $K \subset E$ be a bounded, closed, convex subset of $E$. Then $K$ is compact in the weak topology.*

**Corollary 0.3.6.** *Let $E$ be a reflexive Banach space and let $A \subset E$ be a nonempty closed, convex subset of $E$. Let $\phi : A \to (-\infty, +\infty]$ be a convex lsc function with nonempty domain and*

$$\lim_{\|x\| \to \infty} \phi(x) = +\infty$$

*Then $\phi$ achieves its minimum on $A$.*

**Theorem 0.3.7.** *Let $E$ and $F$ be two reflexive Banach spaces. Let $A : D(A) \subset E \to F$ be a linear operator that is densely defined and closed. Then $D(A^*)$ is dense in $F^*$. Thus $A^{**}$ is well-defined ( $A^{**} : D(A^{**}) \subset E^{**} \to F^{**}$) and it may be viewed as an unbounded operator from $E$ to $F$ (by identifying $E^{**} = E$ and respectively). Then*

$$A^{**} = A$$

### 0.3.5 Separable Spaces

**Definition 0.3.5.** *A metric space is separable if there exists a subset $D \subset E$ that is countable and dense.*

**Proposition 0.3.10.** *Let $E$ be a separable metric space and let $F \subset E$ be any subset. Then $F$ is also separable.*

**Theorem 0.3.8.** *Let $E$ be a Banach space such that $E^*$ is separable. Then $E$ is separable.*

**Corollary 0.3.7.** *Let $E$ be a Banach space. Then $E$ is reflexive and separable if and only if $E^*$ is reflexive and separable.*

**Theorem 0.3.9.** *Let $E$ be a separable Banach space. Then $B_{E^*}$ is metrizable in the weak* topology. Conversely, if $B_{E^*}$ is metrizable in the weak* topology, then $E$ is separable.*

**Theorem 0.3.10.** *Let $E$ be a Banach space such that $E^*$ is separable. Then $B_E$ is metrizable in the weak topology. Conversely, if $B_E$ is metrizable in the weak topology, then $E^*$ is separable.*

**Corollary 0.3.8.** *Let $E$ be a separable Banach space and let $(f_n)$ be a bounded sequence in $E^*$. Then there is a subsequence $(f_{n_k})$ that converges in the weak* topology.*

### 0.3.6 Uniformly Convex Spaces

**Definition 0.3.6.** *A Banach space is uniformly convex if*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ x, y \in B_E \& \|x - y\| > \epsilon \Rightarrow \|\frac{x+y}{2}\| < 1 - \delta$$

**Theorem 0.3.11** (Milman-Pettis)**.** *Every uniformly convex Banach space is reflexive.*

**Proposition 0.3.11.** *Assume that $E$ is a uniformly convex Banach space. Let $(x_n)$ be a sequence such that $x_n \to x$ weakly and*

$$\text{limsup} \|x_n\| \leq \|x\|$$

*Then $x_n \to x$ strongly.*

## 0.4 $L^p$ Spaces

Let $(\Omega, \mathcal{M}, \mu)$ denote a *measure space*, i.e. $\Omega$ is a set such that

1. $\mathcal{M}$ is a $\sigma$-algebra in $\Omega$, i.e. $\mathcal{M}$ is a collection of subset of $\Omega$ such that
   (a) $\emptyset \in \mathcal{M}$
   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
   (c) $\cup_n A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}$
2. $\mu$ is a *measure*, i.e., $\mu : \mathcal{M} \to [0, \infty]$ satisfies
   (a) $\mu(\emptyset) = 0$
   (b) $\mu(\cup_n A_n) = \sum_n \mu(A_n)$; members of $\mathcal{M}$ are called the *measurable sets*. Sometimes $|A|$ is written for $\mu(A)$.
3. $\Omega$ is $\sigma$-finite; i.e., there is a countable family $(\Omega_n)$ in $\mathcal{M}$ that cover $\Omega$ and the measure of each element is finite.

The sets $E : \mu(E) = 0$ are called *null sets.* We say a property holds almost everywhere (a.e.) if it holds except on a null set. We will not review the notion of measurable, integrable functions. We will write $\int f = \int_\Omega f$ and use the notation

$$\|f\|_{L^1} = \|f\|_1 = \int |f|$$

and identify funtions that agree almost everywhere.

### 0.4.1 Some Results about Integration That Everyone Must Know

**Theorem 0.4.1** (monotone convergence theorem, Beppo Levi). *Let $(f_n)$ be a sequence of functions in $L^1$ that satisfy*

1. $f_1 \le f_2 \le \cdots \le f_n \le f_{n+1} \le \cdots$ *a.e. on* $\Omega$
2. $\sup_n \int f_n < \infty$

*Then $f_n(x)$ converges a.e. on $\Omega$ to a finite limit (denoted by $f(x)$), the function is in $L^1$ and $\|f_n - f\|_1 \to 0$.*

**Theorem 0.4.2** (dominated convergence theorem, Lebesgue). *Let $(f_n)$ be a sequence of functions in $L^1$ that satisfy*

1. $f_n(x) \to f(x)$ *a.e. on* $\Omega$
2. *there is a function $g \in L^1$ such that for all $n$, $|f_n(x) \le g(x)|$ a.e. on $\Omega$*

*Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.*

**Lemma 0.4.1** (Fatou's Lemma). *Let $(f_n)$ be a sequence of functions in $L^1$ that satisfy*

1. $f_n \geq 0$
2. $\sup_n \int f_n < \infty$

*For almost all* $x \in \Omega$ *we set* $f(x) = \text{liminf}_n f_n(x) \leq +\infty$. *Then* $f \in L^1$ *and*

$$\int f \leq \text{liminf}_n \int f_n$$

**Definition 0.4.1.** *We denote by* $C_c\left(\mathbb{R}^N\right)$ *the space ofall continuous functionson* $\mathbb{R}^N$ *with compact support.*

**Theorem 0.4.3** (density)**.** *The space* $C_c\left(\mathbb{R}^N\right)$ *is dense in* $L^1\left(\mathbb{R}^N\right)$, *which is to say*

$$\forall f \in L^1 \ \forall \epsilon > 0 \ \exists f_1 \in C_c\left(\mathbb{R}^N\right): \ \|f - f_1\|_1 \leq \epsilon$$

**Theorem 0.4.4** (Tonelli)**.** *Let* $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be a measurable function satifying*

1. $\int_{\Omega_1} |F(x,y)| d\mu_2(y) < \infty$ *for a.e.* $x \in \Omega_1$
2. $\int_{\Omega_1} \left[\int_{\Omega_2} |F(x,y)| d\mu_2(y)\right] d\mu_1(x) < \infty$

*Then* $F \in L^1(\Omega_1 \times \Omega_2)$

**Theorem 0.4.5** (Fubini)**.** *Assume that* $F \in L^1(\Omega_1 \times \Omega_2)$. *Then a.e. on* $\Omega_1$,

### 0.4.2 Definition and Elementary Properties of $L^p$ Spaces

**Definition 0.4.2.** *Let* $p \in \mathbb{R}$ *with* $1 < p < \infty$; *we set*

$$L^p(\Omega) = \left\{f : \Omega \to \mathbb{R} : f \textit{ is measurable and } |f|^p \in L^1(\Omega)\right\}$$

*with*

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu(x)\right]^{1/p}$$

**Definition 0.4.3.** *Let*

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : f \textit{ is measurable and bounded.}\}$$

**Definition 0.4.4.** *Let* $1 \leq p \leq \infty$, *we denote by* $p'$ *the conjugate exponent:*

$$\frac{1}{p} + \frac{1}{p'} = 1$$

**Theorem 0.4.6** (Hölder's inequality)**.** *Assume that* $f \in L^p$ *and* $g \in L^{p'}$ *with* $1 \leq p \leq \infty$. *Then* $fg \in L^1$ *and*

$$\int |fg| \leq \|f\|_p \|g\|_{p'}$$

**Theorem 0.4.7.** *$L^p$ is a vector space and $\|\cdot\|_p$ is a norm for any $p$, $1 \leq p \leq \infty$.*

**Theorem 0.4.8** (Fischer-Riesz)**.** *$L^p$ is a Banach space for any $p$, $1 \leq p \leq \infty$.*

**Theorem 0.4.9.** *Let $(f_n)$ be a sequence in $L^p$ and let $f \in L^p$ be such that $\|f_n - f\|_p \to 0$. Then there is a subsequence $(f_{n_k})$ and a function $h \in L^p$ such that*

1. *$f_{n_k}(x) \to f(x)$ a.e. on $\Omega$*
2. *$|f_{n_k}(x) \leq h(x)|$ a.e. on $\Omega$*

### 0.4.3 Reflexivity, Separability, Dual of $L^p$