Notes from *Introduction to Symplectic Topology*

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Publisher's Description

Over the last number of years powerful new methods in analysis and topology have led to the development of the modern global theory of symplectic topology, including several striking and important results. The first edition of Introduction to Symplectic Topology was published in 1995. The book was the first comprehensive introduction to the subject and became a key text in the area. A significantly revised second edition was published in 1998 introducing new sections and updates on the fast-developing area. This new third edition includes updates and new material to bring the book right up-to-date.

A Note From the Transcriber

These notes were taken as part of an independent study course at the University of Florida in Fall 2019. This book was suggested to me as I learned about symplectic geometry and made some comments on the properties of symplectic structures. I have not touched these notes in a while and will probably not return to them soon.

These notes assume familiarity with a wide range of math: modern differential geometry (vector bundles, complex geometry), functional analysis, and algebraic topology (homology and cohomology). It is still possible to get something out of the book without all these prerequisites, since the book covers a wide range of topics. If you are invested in the topic, though, you should think about familiarizing yourself with all the prereqs.

Contents

0.1 Linear Sympletic Geometry

0.1.1 Symplectic Vector Spaces

Definition 0.1.1. *A* ***Symplectic Vector Space*** *is a pair (V, ω) of a finite-dimensional vector space V and a non-degenerate skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. A* ***Linear Symplectomorphism*** *is a linear map preserving the symplectic form. The set of linear symplectomorphisms is denoted by* $\mathrm{Sp}(V, \omega)$.

Definition 0.1.2. *The symplectic complement of a subspace W is the subspace $W^\omega = \{v \in V | \omega(v, w) = 0 \ \forall w \in W\}$. Subspaces are called*

isotropic	$W \subset W^\omega$
coisotropic	$W^\omega \subset W$
symplectic	$W \cap W^\omega = \{0\}$
Lagrangian	$W = W^\omega$

Lemma 0.1.1. $\dim W + \dim W^\omega = \dim V$*;* $W^{\omega\omega} = W$

Corollary 0.1.1. *ω is nondegenerate iff the associated volume element is nonzero:* $\omega \wedge \ldots \wedge \omega = \omega^n$

Lemma 0.1.2. *Any isotropic subspace is contained in a Lagrangian subspace. Additionally, any basis of a Lagrangian subspace can be extendded to a sympletic basis of (V, ω).*

Lemma 0.1.3. *Let (V, ω) be a symplectic vector space of $W \subset V$ a coisotropic subspace. Then:*

1. *$V' = W/W^\omega$ carries a natural symplectic structure ω' induced by ω.*
2. *If $\Lambda \subset V$ is a Lagrangian subspace then $\Lambda' = ((\Lambda \cap W) + W^\omega)/W^\omega$ is a Lagrangian subspace of V'*

0.1.2 The Symplectic Linear Group

Lemma 0.1.4. $\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)$

Lemma 0.1.5. $\lambda \in \sigma(\Psi) \iff \lambda^{-1} \in \sigma(\Psi)$*, and their multiplicities are identical (Here Ψ is a linear symplectomorphism, and $\sigma(\Psi)$ its spectrum). Moreover, distinct eigenvectors z_1, z_2 have the property that $\omega(z_1, z_2) = 0$.*

Lemma 0.1.6. *Every real symmetric positive-definite symplectic matrix, taken to any positive real power, is again a symplectic matrix*

Proposition 0.1.1. $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proposition 0.1.2. *The fundamental group of* $\mathrm{U}(n)$ *is isomorphic to the integers. The determinant map* $\det : \mathrm{U}(n) \to \mathrm{S}^1$ *induces an isomorphism of fundamental groups.*

0.1.3 The Maslov Index

Theorem 0.1.1. *There is a unique function $\mu : \Omega\mathrm{Sp}(2n) \to \mathbb{Z}$ satisfying the following:*

1. *(homotopy) Two loops have the same Maslov index $\iff$ they are homotopic*
2. *(product) For any two loops $\Psi_t, \Phi_t : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$, $\mu(\Psi_t \Phi_t) = \mu(\Psi_t) + \mu(\Phi_t)$*
3. *(direct sum) Identifying $\mathrm{Sp}(2a) \bigoplus \mathrm{Sp}(2b) \subset \mathrm{Sp}(2a+2b)$, then $\mu(\Psi \bigoplus \Phi) = \mu(\Psi) + \mu(\Phi)$*
4. *(normalization) The loop $\theta_t = e^{2\pi i t} \in \mathrm{U}(1)$ has Maslov index 1.*

The Maslov index can also be considered the intersection number of a loop with a certain submanifold. Decompose a symplectic matrix into a 2x2 block matrix form, then take the upper right matrix and set its determinant equal to zero. This forms a codimension one submanifold. Then take the Maslov index to be the intersection number of the loop with this submanifold.

0.1.4 Lagrangian Subspaces

Let $\mathcal{L}(V,\omega)$ be the set of Lagrangian subspaces of (V,ω)

Lemma 0.1.7. *Let X and Y be real $n \times n$ matrices and define $\Lambda \subset \mathbb{R}^{2n}$ by $\Lambda = \text{range}(Z)$; $Z = (XY)^T$. Then Λ is a Lagrangian subspace $\iff$ Z is of full rank and $X^T Y = Y^T X$*

The matrix that satisfies the above is a **Lagrangian Frame**. $\mathcal{L}(n)$ is a manifold of dimension $n(n+1)/2$.

Lemma 0.1.8.

1. *Any symplectic transformation of a Lagrangian subspace is again Lagrangian*
2. *There is a symplectic transform between any two Lagrangian subspaces.*
3. *There is a natural isomorphism $\mathcal{L}(n) \approx \mathcal{U}(n)/\mathcal{O}(n)$.*

Theorem 0.1.2. *There is a unique function $\mu : \Omega\mathcal{L}(n) \to \mathbb{Z}$ satisfying the following:*

1. *(homotopy) Two loops have the same Maslov index $\iff$ they are homotopic*
2. *(product) For a loop $\Lambda_t \in \Omega\mathcal{L}(n)$ and a loop $\Psi_t \in \Omega\text{Sp}(2n)$, then $\mu(\Psi_t\Lambda_t) = \mu(\Lambda_t) + 2\mu(\Psi_t)$*
3. *(direct sum) Identifying $\mathcal{L}(a) \bigoplus \mathcal{L}(b) \subset \mathcal{L}(a+b)$, then the Maslov index of a direct sum of two loops is the sum of their Maslov indices.*
4. *(normalization) The loop $\Lambda_t = e^{2\pi it}\mathbb{R} \subset \mathbb{C}$ has Maslov index 1.*

Similarly to the Maslov indices of symplectomorphism loops, we can view the Maslov index of Lagrangian subspaces as the intersection number of the loop with a submanifold of $\mathcal{L}(n)$. The desired submanifold is the set of planes $\bigcup_{c\in\mathbb{R}}\{\text{Re}(z) = c\}$ in complex symplectic basis.

0.1.5 The Affine Non-Squeezing Theorem

Definition 0.1.3. *An **Affine Symplectomorphism** is a map that is a symplectomorphism followed by a translation. A symplectic cylinder $Z^{2n}(R)$ is $B^2(R) \times \mathbb{R}^{2n-2}$*

Theorem 0.1.3. *Let ψ be an affine symplectormophism, and that $\psi(B^{2n}(r)) \subset Z^{2n}(R)$. Then $r \leq R$.*

Theorem 0.1.4. *Let Ψ be a nonsingular matrix with the non-squeezing property. Then Ψ is symplectic or anti-symplectic ($\Psi^*\omega = -\omega$).*

Definition 0.1.4. *The **Linear Symplectic Width** of a subset $A \subset \mathbb{R}^{2n}$ is the area of the largest symplectic ball that fits inside the subset:*

$$w_L(A) = \sup\{\pi r^2 | \psi(B^{2n}(r)) \subset A\}$$

For any affine symplectic transform ψ. It has the following properties:

1. *(monotonicity) if $\psi(A) \subset B$ then $w_L(A) \leq w_L(B)$.*
2. *(conformality) $w_L(\lambda A) = \lambda^2 w_L(A)$*
3. *(nontriviality) $w_L(B^{2n}(r)) = w_L(Z^{2n}(r)) = \pi r^2$*

Theorem 0.1.5. *Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map. Then TFAE:*

1. *Ψ preserves the linear symplectic width of ellipsoids centered at 0.*
2. *Ψ is either symplectic or anti-symplectic, i.e. $\Psi^*\omega = \pm\omega$*

Lemma 0.1.9. *Let (V, ω) be a symplectic vector space with an inner product g. Then there is a basis of V which is g-orthogonal and ω-standard, and can be chosen so that $g(u_j, u_j) = g(v_j, v_j)$*

Lemma 0.1.10. *Given an ellipsoid*

$$E = \{W \in \mathbb{R}^{2n} | \sum_{i,j=1}^{2n} a_{ij} w_i w_j \leq 1\}$$

There is a symplectic linear tranformation $\Psi in \mathrm{Sp}(2n)$ such that Ψ turns the matrix a_{ij} diagonal.

Remark 0.1.1. *For $n = 1$, the previous lemma tells us that every ellipse in $\mathbb{R}^2$ can be mapped into a circle by an area-preserving transformation.*

Definition 0.1.5. *The **Symplectic Spectrum** of an ellipsoid to be the increasing n-tuple $(r_1, \ldots, r_n)$ such that the ellipsoid is 'diagonalized' by a linear transformation into a ellipsoid with diagonal matrix $\mathrm{diag}(r_1, \ldots, r_n)$. Symplectic spectra have the following properties:*

1. *Two ellipsoids are linearly symplectomorphic $\iff$ they have the same spectrum*
2. *An ellipsoid with its spectrum of the form $(r, \ldots, r)$ are symplectic balls*
3. *The volume of a symplectic ellipsoid is $\mathrm{Vol}(E) = \pi^n \Pi_i r_i^2$*

Theorem 0.1.6. *Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0. Then $w_L(E) = \sup_{B \subset E} w_L(B) = \inf_{E \subset Z} w_L(Z)$, where B are symplectic balls and Z are symplectic cylinders.*

0.1.6 Complex Structures

Definition 0.1.6. *A* ***Complex Structure*** *is an automorphism $J : V \to V$ such that $-J^2$ is the identity. We can 'complexify' the vector space by "complex' scalar multiplication: $\mathbb{C} \times V \to V : (s + it, v) \mapsto sv + tJv$, which means V has even dimension over the reals. The set of complex structures is denoted by $\mathcal{J}(V)$*

Proposition 0.1.3. *Every almost complex structure is a linear transform away from the standard complex structure:*

$$J = \begin{bmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{bmatrix}$$

Definition 0.1.7. *If (V, ω) is symplectic vector space, a complex structure $J \in \mathcal{J}(V)$ is said to be compatible with ω if $\omega(Jv, Jw) = \omega(v, w)$ and $\omega(v, Jv) > 0$. This defines an inner product on V, defined by $g_J(v, w) = \omega(w, Jw)$. This inner product makes J skew-adjoint, i.e. $g_J(v, Jw) = -g_J(Jv, w)$. The space of compatible complex structures is denotes $\mathcal{J}(V, \omega)$*

Proposition 0.1.4.

1. *$\mathcal{J}(V, \omega)$ is homeomorphic to the space $\mathcal{P}$ of symmetric positive definite symplectic matrices.*
2. *There is a continuous map $r : met(V) \to \mathcal{J}(V, \omega)$ such that $r(g_J) = J$ & $r(\Phi^* g) = \Phi^* r(g)$, $\forall J \in \mathcal{J}(V, \omega), g \in met(V), \Phi \in Sp(V, \omega)$.*
3. *$\mathcal{J}(V, \omega)$ is contractible.*

Definition 0.1.8. *A complex structure $J \in \mathcal{J}(V, \omega)$ is called ω-tame if $\omega(v, Jv) > 0$ for all nonzero v. The space of ω-tame complex structures is denotes by $\mathcal{J}_T(V, \omega)$. There is an associated inner product for each ω-tame complex structure given by $g_J(v, w) = \frac{1}{2}(w(v, Jw) + \omega(w, Jv))$.*

Proposition 0.1.5. *The space $\mathcal{J}_T(V, \omega)$ is contractible.*

0.1.7 Symplectic Vector Bundles

Definition 0.1.9. *A* ***Symplectic Vector Bundle*** *over a manifold is a real vector bundle $E \xrightarrow{\pi} M$ equipped with a smoothly symplectic form $\omega \in \Gamma(E \bigotimes E^*)$.*

Definition 0.1.10. *Symplectic vector bundles are isomorphic $\iff$ their underlying complex vector bundles are isomorphic. Symplectic vector bundles with compatible almost complex structure and a metric are called a* ***Hermitian Structure****.*

Proposition 0.1.6. *Let $E \to M$ be a $2n$-dimensional vector bundle.*

1. *Every symplectic form has a compatible almost complex structure. The space $\mathcal{J}(E, \omega)$ is contractible.*
2. *The space of symplectic forms compatible with a given almost complex structure is contractible.*

Definition 0.1.11. *A **Unitary Trivialization** is a smooth map of a symplectic vector bundle, an almost complex structure, and a metric into Euclidean space, transforming each structure into its standard structure. A **Unitary Trivialization along a curve** is a trivialization of the pull-back bundle along a curve.*

Lemma 0.1.11. *If a curve has unitary trivializations at its endpoints, then it can be extended to a unitary transformation along the entire curve*

Proposition 0.1.7. *A Hermitian vector bundle $E \to \Sigma$ over a compact Riemann surface Σ with non-empty boundary $\partial\Sigma$ admits a unitary trivialization.*

0.1.8 First Chern Classes

Theorem 0.1.7. *There is a unique function called the **First Chern Number**, that assigns an integer $c_1(E)$ to every symplectic vector bundle E over a compact oriented Riemann surface Σ without boundary and satisfies the following axioms:*

1. *(naturality) Two isomorphic vector bundles have the same Chern number*
2. *(functoriality) Any smooth map $\phi : \Sigma' \to \Sigma$ of oriented Riemann surfaces and any symplectic vector bundle $E \to \Sigma$, then $c_1(\phi^* E) = \deg(\phi) c_1(E)$*
3. *(additivity) For any two symplectic vector bundles $E_1 \to \Sigma$ and $E_2 \to \Sigma$, $c_1(E_1 \bigoplus E_2) = c_1(E_1) + c_1(E_2)$.*
4. *(normalization) The Chern number of the tangent bundle is $c_1(T\Sigma) = 2 - 2g = \chi(\Sigma)$*

Remark 0.1.2.

1. *The first Chern number vanishes $\iff$ the bundle is trivial; so the first Chern number is an indicator of if the bundle can be symplectically trivialized.*
2. *Usually the Chern number is defined for complex vector bundles, which is fine for our definition.*

0.2 Symplectic Manifolds

0.2.1 Basic Concepts

Throughout we will assume M to be a smooth manifold without boundary. Most of the time M will be compact.

Definition 0.2.1. *A **Symplectic Structure** is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. The manifold is necessarily even-dimensional and orientable.*

Definition 0.2.2. *A **Symplectomorphsim** is a diffeomorphism that preserves the symplectic form. The set of Symplectomorphisms is denoted by* $\mathrm{Symp}(M, \omega)$ *or* $\mathrm{Symp}(M)$.

Definition 0.2.3. *A vector field $X \in \mathfrak{X}(M)$ is called **symplectic** or **Locally Hamiltonian** is $i_X\omega$ is closed. The set of locally Hamiltonian vector fields is denoted by $\mathfrak{X}(M, \omega)$.*

Proposition 0.2.1. *Let M be a closed manifold. If $t \mapsto \psi_t \in \mathrm{Diff}(M)$ a smooth homotopy, generating smooth vector fields $X_t \circ \phi_t = \frac{d}{dt}\psi_t$, then*

$$\psi_t \in \mathrm{Symp}(M, \omega) \iff X_t \in \mathfrak{X}(M, \omega) \tag{1}$$

In addition, if $X, Y \in \mathfrak{X}(M, \omega)$ then $[X, Y] \in \mathfrak{X}(M, \omega)$ and

$$i_{[X,Y]}\omega = dH; \quad H = \omega(X, Y) \tag{2}$$

0.2.2 Hamiltonian Flows

Definition 0.2.4. *For any smooth function $H : M \to \mathbb{R}$ the vector field $X_H : M \to TM$ determined by $i_{X_H}\omega = dH$ is called the **Hamiltonian Vector Field** associated to the **Hamiltonian Function** H. The flow associated with this vector field is called the **Hamiltonian Flow** associated to H.*

Definition 0.2.5. *The **Poisson Bracket** of two functions F, G is the new function*

$$\{F, H\} = \omega(X_F, X_H) = dF(X_H) \tag{3}$$

Proposition 0.2.2. *Let (M, ω) be a symplectic manifold.*

1. *Hamiltonian flows are symplectomorphisms, and are tangent to the level surfaces of their Hamiltonian function.*

2. *For every Hamiltonian function H and every symplectomorphism ψ, $X_{H\circ\psi} = \phi^* X_H$*

3. $[X_F, X_G] = X_{\{F,G\}}$

Thus Hamiltonian vector fields form a Lie subalgebra of the symplectic vector fields. The map $H \mapsto X_H$ is a surjective Lie Algebra homomorphism from the Lie algebra of smooth functions to Hamiltonian vector fields. The kernel of this homomorphism is constant functions.

Since $\mathcal{L}_{X_H}H = 0$, every level set of H is an invariant submanifold of the Hamiltonian vector field. Conversely, let $S \subset M$ be a compact orientable hypersurface (codimension 1) of a symplectic manifold. An exercise (not in these notes) showed that this is a coisotropic submanifold. Hence the vector space

$$L_q = T_qS^\omega = \{v \in T_qM | \omega(v, w) = 0 \ \forall w \in T_qS\} \tag{4}$$

is a 1-dimensional subspace of T_qS for every $q \in S$ and hence defines a real line bundle L over S. It integrates to give the **Characteristic Foliation**. The leaves of this foliation are the integral curves of any Hamiltonian vector field which for which S is a regular level surface of the associated Hamiltonian function.

0.2.3 Hamiltonian Isotopies

Consider a smooth map $t \mapsto \psi_t \in \mathrm{Symp}(M, \omega)$ with $\psi_0 = \mathrm{id}_M$. This generates a smooth vector field

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \tag{5}$$

Because ψ_t is symplectic, the X_t are locally Hamiltonian. If they are all globally Hamiltonian, then we have that

$$X_t = X_{H_t} \tag{6}$$

H_t are **time-dependent Hamiltonians** and ψ_t is a **Hamiltonian Isotopy**. If there is a Hamiltonian Isotopy ending with $\psi \in \mathrm{Symp}(M, \omega)$, then ψ is called **Hamiltonian**. The space of Hamiltonian symplectomorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

$\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$, and it Lie algebra is the space of all Hamiltonian vector fields. This makes it an infinite dimensional Lie group, markedly different from the Riemannian case.

0.2.4 Isotopies and Darboux's Theorem

Lemma 0.2.1. *Let M be a $2n$-dimensional manifold and $Q \subset M$ a compact submanifold. Suppose ω_0, ω_1 are closed degenerate 2-forms such that at each $q \in Q$, $(\omega_0)_q = (\omega_1)_q$. Then there are open neighborhoods $\mathcal{N}_0, \mathcal{N}_1$ of Q and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}_1$ such that*

$$\psi \restriction_Q = \mathrm{id}; \;\; \psi^*\omega_1 = \omega_0 \tag{7}$$

Theorem 0.2.1. *Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

Theorem 0.2.2 (Moser Stability Theorem for Symplectic Structures)**.** *Let M be a closed manifold and suppose ω_t is a smooth family of cohomologous (i.e. all lying in the same cohomology class) symplectic forms on M. Then there is a family of diffeomorphisms ψ_t satisfying*

$$\psi_0 = \mathrm{id}; \;\; \psi_t^*\omega_t = \omega_0 \tag{8}$$

Definition 0.2.6.

1. *An isotopy preserving a symplectic structure is called a* ***Symplectic Isotopy****.*
2. *Two symplectic forms ω_0, ω_1 on M are* ***isotopic*** *if they can be joined by a smooth family ω_t of cohomologous symplectic forms on M.*
3. *Two isotopic symplectic forms are* ***strongly isotopic*** *is there is an isotopy ψ_t of M such that $\psi_1^*\omega_1 = \omega_0$*

Theorem 0.2.3 (Symplectic Isotopy Extension Theorem)**.** *Let (M, ω) be a compact symplectic manifold and let $Q \subset M$ be a compact subset. Let $\phi_t : U \to M$ be a symplectic isotopy of an open neighborhood U of Q and assume $H^2(M, Q, \mathbb{R}) = 0$.*

Then there exists a neighborhood $\mathcal{N} \subset U$ of Q and a symplectic isotopy ψ_t such that

$$\psi_t \restriction_{\mathcal{N}} = \phi_t \restriction_{\mathcal{N}} \tag{9}$$

0.2.5 Submanifolds of Symplectic Manifolds

Definition 0.2.7. *A submanifold $Q \subset M$ is called **symplectic** (resp. **isotropic, coisotropic, Lagrangian**) is for every $q \in Q$, the symplectic vector space (T_qM, ω_q) is symplectic (resp. isotropic, coisotropic, Lagrangian).*

Proposition 0.2.3. *The graph $\Gamma_\sigma \subset T^*L$ of a 1-form σ on L is Lagrangian $\iff$ σ is closed.*

Proposition 0.2.4. *Let ψ be a diffeomorphism of a symplectic manifold (M, ω). Then ψ is a symplectomorphism $\iff$ its graph*

$$\mathrm{graph}(\psi) = \{(q, \psi(q))\} \subset M \times M \tag{10}$$

is a Lagrangian submanifold of $(M \times M, (-\omega) \times \omega)$

Theorem 0.2.4 (Symplectic Neighborhood Theorem)**.** *For $j = 0, 1$, let (M_j, ω_j) be symplectic manifolds with compact symplectic submanifolds Q_j. Suppose there is an isomorphism $\Phi : \nu_{Q_0} \to \nu_{Q_1}$ of the symplectic normal bundles which covers a symplectomorphism $\psi : (\mathcal{N}(Q_0), \omega_0) \to (\mathcal{N}(Q_1), \omega_1)$ such that $d\psi$ induces the map Φ on $\nu_{Q_0} = (TQ_0)^\omega$.*

Theorem 0.2.5. *Let (M, ω) be a symplectic manifold*