Notes from *Spectral Methods: Algorithms, Analysis, and Applications*

by Jie Shen, Tao Tang, Li-Lian Wang

taken by Samuel T. Wallace

Publisher's Description

Along with finite differences and finite elements, spectral methods are one of the three main methodologies for solving partial differential equations on computers. This book provides a detailed presentation of basic spectral algorithms, as well as a systematical presentation of basic convergence theory and error analysis for spectral methods. Readers of this book will be exposed to a unified framework for designing and analyzing spectral algorithms for a variety of problems, including in particular high-order differential equations and problems in unbounded domains. The book contains a large number of figures which are designed to illustrate various concepts stressed in the book. A set of basic matlab codes has been made available online to help the readers to develop their own spectral codes for their specific applications.

A Note From the Transcriber

These notes were taken over summer 2020 as part of self-study preparing for PhD in applied math and numerical analysis. I am reading this without much interpolation theory knowledge, and some prior exposure to spectral methods through Trefethen's book *Spectral Methods in MATLAB* (sorry, no notes for that book). This book comes from a suggestion by a professor as my grad school, and it looked temptingly challenging.

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0.1 Introduction

0.1.1 Weighted Residual Methods

Consider the following general problem:

$$\partial_t u(x,t) - \mathcal{L}u(x,t) = \mathcal{N}(u)(x,t), \quad t > 0, x \in \Omega \tag{1}$$

Where $\mathcal{L}$ is a leading spatial derivative operator, and $\mathcal{N}$ is a lower-order linear or non-linear operator involving only spatial derivatives. Here, Ω denotes a bounded domain of $\mathbb{R}^d$, $d = 1, 2,$ or 3. This equation is to be supplemented with an initial condition and suitable boundary conditions.

We shall only consider the WRM for the spatial discretization, and assume that the time derivative is discretized with a suitable time-stepping scheme. Among various time-stepping methods, semi-implicit schemes or linearly-implicit schemes, in which the principal linear operators are treated *implicitly* to reduce the associated stability constraint, while the non-linear equations are treated explicitly to avoid the expensive process of solving nonlinear equations at each time step, are most frequently used int eh context of spectral methods.

Let τ be the step size, and $u^k(\cdot)$ be an approximation of $u(\cdot, k\tau)$. As an example, we consider the Crank-Nicolson leap-frog scheme for the equation:

$$\frac{u^{n+1} - u^{n-1}}{2\tau} - \mathcal{L}\left(\frac{u^{n+1} + u^{n-1}}{2}\right) = \mathcal{N}\left(u^n\right) \;\; n \geq 1 \tag{2}$$

We can rewrite this as

$$\mathbf{L}u(x) := \alpha u(x) - \mathcal{L}u(x) = f(x), \;\; x \in \Omega \tag{3}$$

where $u = \frac{u^{n+1} + u^{n+1}}{2}, \alpha = \tau^{-1}$ and $f = \alpha^{n-1} + \mathcal{N}\left(u^n\right)$. Hence, at each time step, we need to solve a steady-state problem of the form