Notes from *Principles of Algebraic Geometry*

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Publisher's Description

A comprehensive, self-contained treatment presenting general results of the theory. Establishes a geometric intuition and a working facility with specific geometric practices. Emphasizes applications through the study of interesting examples and the development of computational tools. Coverage ranges from analytic to geometric. Treats basic techniques and results of complex manifold theory, focusing on results applicable to projective varieties, and includes discussion of the theory of Riemann surfaces and algebraic curves, algebraic surfaces and the quadric line complex as well as special topics in complex manifolds.

Transcription Notes

Copied without proofs for independent study.

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0.1 Foundational Material

0.1.1 Rudiments of Several Complex Variables

Cauchy's Formula and Applications

Theorem 1. *For Δ a disc in $\mathbb{C}$, $f \in C^\infty(\overline{\Delta}), z \in \Delta$,*

$$f(z) = \frac{-1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)dw}{w-z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{\partial f(w)}{d\overline{w}} \frac{dw \wedge d\overline{w}}{w-z}$$

Lemma 1. *Given $g \in C^\infty(\overline{\Delta})$, the function*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{g(w)}{w-z} dw \wedge d\overline{w}$$

is defined, belongs to $C^\infty(U)$; $U \subset \Delta$ and satisfies

$$\frac{\partial f}{\partial \overline{z}} = g$$

Several Variables

Definition 1. *The total differential of a function f on $\mathbb{C}^n$ is defined as*

$$df = \sum \left(\frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i \right)$$

We call the first set of summands ∂f and the second set $\overline{\partial} f$. A function is holomorphic if $\overline{\partial} f = 0$ (which is the same as being holomorphic in each variable separately).

Many results of one-variable complex analysis extend to multiple variable; for example, a function is analytic in each variable iff it is holomorphic, two functions holomorphic on a connected open set, equal on an open subset, are identical, and the absolute value of a holomorphic function has no maximum in an open subset.There are some differences, however.

Theorem 2. *Any holomorphic function defined in a neighborhood of $U \backslash V$ extends to a holomorphic function on U.*

Definition 2. *A Weierstrass polynomial in w is a polynomial of the form*

$$w^d + a_1(z)w^{d-1} + \cdots + a_d(z);\ \ a_i(0) = 0$$

Theorem 3. *If f is holomorphic around the origin in $\mathbb{C}^n$ and is not identically zero on the w-axis, then in some neighborhood of the origin, f can be written uniquely as $f = g \cdot h$ where g is a Weierstrass polynomial and $h(0) \neq 0$.*

This theorem says that the zero locus (set of zeros) of a function f, $\mathcal{Z}(f)$ is the zero locus of a Weierstrass polynomial

$$g(z, w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z)$$

The roots $b_i(z)$ of the polynomial $g(z, \cdot)$ are, away from those values for which $g(z, \cdot)$ has a root with nonunity multiplicity, locally single valued holomorphic functions of z. Since the discriminant of $g(z, \cdot)$ is an analytic function of z,

The zero locus of an analytic function $f(z_1, \ldots, z_{n-1}, w)$ not vanishing identically on the w-axis projects locally onto the hyperplane ($w = 0$) as a finite-sheeted cover branched over the zero locus of an analytic function.

Theorem 4. *Suppose $f(z, w)$ is holomorphic in a disc $\Delta \subset \mathbb{C}^n$ and $g(z, w)$ is holomorphic in $\overline{\Delta} \backslash \mathcal{Z}(f)$ and is bounded. Then g extends to a holomorphic function on Δ.*

Now we recall some basic algebra. Let R be an integral domain (a ring in which the zero product property holds). An element u is a unit if there exists $v \in R$ such that $u \cdot v = 1$; u is irreducible if $u = v \cdot w \Rightarrow u$ is a unit or v is a unit. R is a Unique Factorization Domain (UFD) if every element can be written as a product of irreducible elements unique up to multiplication by units. The main facts we will use are:

1. R is a UFD $\Rightarrow R[t]$ is a UFD (Gauss' lemma)

2. R is a UFD, $u, v \in R[t]$ are relatively prime, then there are relatively prime polynomials α, β and γ such that $\alpha u + \beta v = \gamma$, γ is called the resultant of u and v.

Let $\mathcal{O}_{n,z}$ be the ring of holomorphic functions defined in some neighborhood of $z \in \mathbb{C}^n$; write $\mathcal{O}_n$ for $\mathcal{O}_{n,0}$. $\mathcal{O}_n$ is an integral domain by the identity theorem, and moreover is a local ring whose maximal ideal m is $\{f : f(0) = 0\}$. $f \in \mathcal{O}_n$ is a unit iff $f(0) \neq 0$. Now we begin with the results.

Proposition 1. *$\mathcal{O}_n$ is a UFD.*

Proposition 2. *If f and g are relatively prime in $\mathcal{O}_n$, then for $\|z\| < \epsilon$, f and g are relatively prime in $\mathcal{O}_{n,z}$.*

Theorem 5. *Let $g(z,w) \in \mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree k in w. Then for any $f \in \mathcal{O}_n$, we can write $f = g \cdot h + r$ with $r(z,w)$ a polynomial of degree strictly less than k in w.*

Corollary 1. *If $f(z,w) \in \mathcal{O}_n$ is irreducible and $h \in \mathcal{O}_n$ vanishes on $\mathcal{Z}(f)$, then f divides h in $\mathcal{O}_n$.*

Analytic Varieties

The main purpose of the results given so far is describe the properties of analytic varieties in $\mathbb{C}^n$.

Definition 3. *We say a subset V of an open set $U \subset \mathbb{C}^n$ is an analytic variety in U if, for any $p \in U$ there exists a neighborhood U' of p in U such that $V \cap U'$ is the common zero locus of a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U'. V is called an analytic hypersurface if V is locally the zero locus of a single nonzero holomorphic function f.*

An analytic variety $V \subset U \subset \mathcal{C}^n$ is said to be irreducible if V cannot be written as the union of proper analytic subvarieties; it is irreducible at a point p if the variety is irreducible in any neighborhood of the point.

Note first that if $f \in \mathcal{O}_n$ is irreducible in the ring $\mathcal{O}_n$, then the analytic hypersurface V equal to the zero locus of f is irreducible at 0. Additionally:

1. Suppose V is an analytic hypersurface that is $\mathcal{Z}(f)$ in some neighborhood of 0. Since $\mathcal{O}_n$ is a UFD, we can write $f = f_1 \cdots f_n$ with f_i irreducible in $\mathcal{O}_n$; if we set $V_i = \mathcal{Z}(f_i)$, then $V = V_1 \cup \cdots \cup V_k$. Thus for $p \in V$, *V can be expressed uniquely in some neighborhood of p as the union of a finite number of analytic hypersurfaces irreducible at p.*

2. Let $W \subset U \subset \mathbb{C}^n$ be an analytic variety given in a neighborhood Δ of $0 \in W$ as the zero locus of two functions $f, g \in \mathcal{O}_n$. If W contains no analytic hypersurface through 0, then f and g are necessarily relatively prime in $\mathcal{O}_n$; if W does not contain the line $z' = 0$, the by taking linear combinations we may assume that $\mathcal{Z}(f)$ or $\mathcal{Z}(g)$ contains $z' = 0$, and hence that f and g are Weierstrass polynomials in z_n. Let

$$\gamma = \alpha f + \beta g \neq 0 \in \mathcal{O}_{n-1}$$

be the resultant of f and g. We claim that the image under the projection map $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is just $\mathcal{Z}(\gamma)$. To see this, write

$$\alpha = hg + r$$

with $\deg(r) < \deg(g)$. Then $\gamma = rf + (\beta + hf)g$. Now if for some $z \in \mathbb{C}^{n-1}, \gamma$ vanishes at z but f and g have no common zeros of along the line $\pi^{-1}(z)$, if follows that r vanishes at all the zeroes of g in $\pi^{-1}(z)$; since $\deg(r)\deg(g)$, this means that r, and hence $\beta + hf$, vanish identically on $\pi^{-1}(z)$. Thus r and $\beta + hf$ both are zero on the inverse image of any components of the zero locus of γ other than $\pi(W)$; but r and $\beta + hf$ are relatively prime and so have no common components. We see then that $\pi(W)$ *is an analytic hypersurface in a neighborhood of the origin in* $\mathbb{C}^{n-1}$, and that *projection of* W *onto a suitably chosen* $(n-2)$ *plane expresses* W *locally as a finite sheeted branche dcover of a neighborhood of the origin in* $\mathbb{C}^{n-2}$

3. Let $V \subset U \subset \mathbb{C}^n$ be an analytic variety irreducible at $0 \in V$ such that for small neighborhoods Δ of 0 in $\mathbb{C}^n$, $\pi(V \cap \Delta)$ contains a neighborhood of 0 in $\mathbb{C}^{n-1}$. Let $V = \mathcal{Z}(f_1) \cap \cdots \cap \mathcal{Z}(f_k)$ near 0. Then the f_i must all have a common factor in $\mathcal{O}_n$, since otherwise V would be contained in the comon locus of two relatively prime functions, and by 2, $\pi(V \cap \Delta)$ would be a proper analytic subvariety of $\mathbb{C}^{n-1}$. If we let $g(z)$ be the greatest common divisor of all the f_i's, then we can write

$$V = \mathcal{Z}(g) \cup (\mathcal{Z}(f_1/g) \cap \cdots \cap \mathcal{Z}(f_k/g))$$

Since V is irreducible at 0 and since the loci $\mathcal{Z}(f_i/g)$ cannot contain $\mathcal{Z}(g)$, we must have $V = \mathcal{Z}(g)$ near 0.

The previous 3 points, with our basic description of an analytic hypersurface, give us a picture of an analytic hypersurface, gives us a picture of the local behavior of those analytic varieties cut out locally by one or two holomorphic function. In fact, the same picture is in almost all respects valid for general analytic varieties, but to prove this requires some relatively sophisticated techniques from the theory of several complex variables. Since the primary focus of the book is on the codimension 1 case, we will simply state here without proof the analogous results for general analytic varieties:

1. If $V \subset U \subset \mathbb{C}^n$ is any analytic variety and $p \in V$, then in some neighborhood of p, V can be uniquely written as disjoint union of analytic varieties irreducible at p.

2. Any analytic variety can be expressed locally by a projection map as a finite-sheeted cover of a poydisc Δ branched over an analytic hypersurface of Δ.

3. If $V \subset \mathcal{C}^n$ does not contain the line $\mathcal{Z}(z_1) \cap \cdots \cap \mathcal{Z}(z_{n-1})$ then the image of a neighborhood of 0 in V under the projection map $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is an analytic subvariety in a neighborhood of 0.

0.1.2 Complex Manifolds

Complex Manifolds

Definition 4. *A complex manifold is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic for all α, β.*

A function on an open set $U \subset M$ is holomorphic if, its representation in coordinates is holomorphic. A set of functions $\{z_1, \ldots, z_n\}$ is a holomorphic coordinate system if their coordinate representation is biholomorphic. A map between complex manifolds is holomorphic if its representation in holomorphic coordinate system is holomorphic.

[Examples...]

Submanifolds and Subvarieties

Theorem 6. *Let U, V be open sets in $\mathbb{C}^n$ with $0 \in U$ and $f : U \to V$ a holomorphic map with with Jacobian $\mathcal{J}(f) = [\frac{\partial f_i}{\partial z_j}]$ nonsingular at 0. Then f is injective in a neighborhood of 0, and f^{-1} is holomorphic at $f(0)$.*

Theorem 7. *Given $f_1, \ldots, f_k \in \mathcal{O}_n$ with*

$$\det\left(\frac{\partial f_i}{\partial z_j}(0)\right) \neq 0$$

there exist functions $w_1, \ldots, w_k \in \mathcal{O}_{n-k}$ such that in a neighborhood of 0 in $\mathbb{C}^n$,

$$f_1(z) = \ldots = f_k(z) = 0 \iff z_i = w_i(z_{k+1}, \ldots, z_n)$$

Definition 5. *A complex submanifold S of a complex manifold M is a subset $S \subset M$ given locally either as the zeros of a collection $f_1, \ldots, f_k$ of holomorphic functions with rank $\mathcal{J}(f) = k$, or as the image of an open set $U \subset \mathbb{C}^{n-k}$ under a map $f : U \to M$ with rank $\mathcal{J}(f) = n - k$.*

Definition 6. *An Analytic Subvariety V of a complex manifold M is a subset given locally as the zeros of a finite collection of holomorphic functions. A point $p \in V$ is called a smooth point of V if V is a submanifold in some sufficiently small neighborhood of p. The set of smooth point of V is denoted V^*. A point $p \in V \backslash V^*$ is called a singular point, the set of singular points is denoted by V_s. V is called smooth or nonsingular if V is a submanifold of M.*

In particular, if p is a point of an analytic hypersurface $V \subset M$ given in terms of local coordinates z by the function f at p, we define the multiplicity of f at p, $\mathrm{mult}_p(V)$, to be the order of vanishing of f at p, that is, the greatest integer m such that all partial derivatives

$$\frac{\partial^k f}{\partial z_{i_1} \cdots \partial z_{i_k}}(p) = 0$$

Definition 7. *A commonly used word in algebraic geometry is generic. When dealing with a family of objects parametrized by a complex manifold a property of the family being 'generic' means the set of objects not having this property is contained in a submanifold of strictly smaller dimension.*

Proposition 3. *V_s is contained in an analytic subvariety of M not equal to V.*

Proposition 4. *An analytic variety V is irreducible iff V^* is connected.*

We take the *dimension* of an irreducible analytic variety V to be the dimension of the complex manifold V^*; a general analytic variety is of dimension k if all of its components are.

A note: if $V \subset M$ is an analytic subvariety of a complex manifold M, then we may define the *tangent cone* $T_p(V) \subset T'_p(M)$ to V at any point $p \in V$ as follows: if $V = \mathcal{Z}(f)$ is an analytic hypersurface, and in terms of holomorphic coordinates $z_1, \ldots, z_n$ on M centered around p, we write

$$f(z_1, \ldots, z_n) = f_m(z_1, \ldots, z_n) + f_{m+1}(z_1, \ldots, z_n) + \cdots$$

with f_k a homogeneous degree k polynomial in all its variables, then the tangent cone to V at p is taken to be the subvariety of $T'_p(M)$ defined by

$$\left\{ \sum \alpha_i \frac{\partial}{\partial z_i} : f_m(\alpha_1, \ldots, \alpha_n) = 0 \right\}$$

The *multiplicity* of a subvariety V of dimension k in M at a point p, denoted by $\mathrm{mult}_p(V)$, is taken to be the number of sheets in the projection, in a small coordinate polydisc on M around p, of V onto a generic k-dimensional polydisc; note that p isa smooth point of V iff $\mathrm{mult}_p(V) = 1$. In general, if $W \subset M$ is an irreducible subvariety, we define the *multiplicity* $\mathrm{mult}_W(V)$ *of V along W* to be simply the multiplicity of V at a generic point of W.

De Rham and Dolbeault Cohomology

Let M be a differentiable manifold. Let $A^P(M, \mathbb{R})$ denote the space of differential forms of degree p on M, and $Z^p(M, \mathbb{R})$ the subspace of closed p-forms. Since $d^2 = 0$, $d(A^{p-1}(M, \mathbb{R})) \subset Z^p(M, \mathbb{R})$; the quotient groups

$$H^P_{DR}(M, \mathbb{R}) = \frac{Z^p(M, \mathbb{R})}{dA^{p-1}(M, \mathbb{R})}$$

of closed forms modulo exact forms are called the *de Rham cohomology groups of M.*

In the same way, we can let $A^p(M)$ and $Z^p(M)$ denote the spaces of complex-valued p-forms and closed complex-valued p-forms on M, respectively, and let

$$H^P_{DR}(M) = \frac{Z^p(M)}{dA^{p-1}(M)}$$

be the corresponding quotient; clearly

$$H^p_{DR}(M) = H^p_{DR}(M, \mathbb{R}) \otimes \mathbb{C}$$

Now let M be a complex manifold. By linear algebra, the decomposition

$$T^*_{\mathbb{C},z}(M) = T^*_z(M) \bigoplus \overline{T}^*_z(M)$$

of the cotangent space to M at each point z gives a decomposition

$$\wedge^n T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=n} (\wedge^p T^*_z(M) \otimes \wedge^q \overline{T}^*_z(M))$$

Correspondingly, we can write

$$A^n(M) = \bigoplus_{p+q=n} A^{p,q}(M)$$

And we have projection maps

$$\pi^{(p,q)} : A^*(M) \to A^{p,q}(M)$$

that correspond to the direct sum decomposition. We define the operators

$$\overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$$

$$\partial : A^{p,q}(M) \to A^{p+1,q}(M)$$

by

$$\overline{\partial} = \pi^{(p,q+1)} \circ d, \partial = \pi^{(p+1,q)} \circ d$$

So that $d = \partial + \overline{\partial}$. A form ϕ is holomorphic if $\overline{\partial}\phi = 0$. Note that since the decomposition of holomorphic and antiholomorphic forms is preserved under holomorphic maps, $\overline{\partial} \circ f^* = f^* \circ \overline{\partial}$. Note that $\overline{\partial}^2 = 0$, we can analogously define

$$H^{p,q}_{\overline{\partial}}(M) = \frac{Z^{p,q}_{\overline{\partial}}(M)}{\overline{\partial} A^{p,q-1}(M)}$$

Note that holomorphic maps induce maps of $\overline{\partial}$-cohomology groups, and the Poincarè lemma guarantees that de Rham groups are locally trivial.

Lemma 2. *For $\Delta = \Delta(r)$ a polycylinder in $\mathbb{C}^n$,*

$$H^{p,q}_{\overline{\partial}}(\Delta) = 0$$

Calculus on Complex Manifolds

Let M be a complex manifold of dimension n. A *Hermitian metric* on M is given by a positive definite Hermitian inner product

$$(,) : T_z(M) \otimes \overline{T_z(M)} \to \mathbb{C}$$

on the holomorphic tangent bundle at z for each $z \in M$, with smooth dependence on z. We can write a hermitian metric in terms of its basis as an element of $\left(T_z(M) \otimes \overline{T_z(M)}\right)^* = T_z^*(M) \otimes T_z^*(M)$, $dz_i \otimes d\overline{z}_j$:

$$ds^2 = \sum h_{ij} dz_i \otimes d\overline{z}_j$$

A *coframe* for a hermitian metric is an n-tuple of holomorphic one-forms ϕ_i such that

$$ds^2 = \sum \phi_i \otimes \overline{\phi_i}$$

so that the ϕ_i are an orthonormal basis for the cotangent space. Clearly coframes exist locally.

The real and imaginary part of a hermitian inner product on a complex vector space give an ordinary inner product and an alternating two-form, respectively, on the underlying real vector space. Since we have an natural $\mathbb{R}$-linear isomorphism $R_{\mathbb{R},z}(M) \to T_z(M)$ we see that

$$\mathrm{Re} ds^2 : T_{\mathbb{R},z} \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

is a *Riemannian metric* on M, called the induced Riemannian metric of the hermitian metric. we also see that the two-form

$$\mathrm{Im} ds^2 : T_{\mathbb{R},p}(M) \otimes T_{\mathbb{R},p}(M) \to \mathbb{R}$$

is a real differential two-form on M; $\omega = -\frac{1}{2}\mathrm{Im} ds^2$ is called the *associated (1,1) form* of the metric.

If a coframe ϕ_i has the associated decomposition $\phi_i = \alpha_i + \sqrt{-1}\beta_i$, then the associated metric and two-form take the form:

$$\mathrm{Re} ds^2 = \sum (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i)$$

$$\omega = \frac{\sqrt{-1}}{2} \sum \phi_i \wedge \overline{\phi_i}$$

Theorem 8. *If S is a d-dimensional submanifold*

$$\mathrm{vol}(S) = \frac{1}{d!} \int_S \omega^d$$

Proposition 5. *V^* has finite volume in bounded regions.*

Theorem 9. *For M a complex manifold, $V \subset M$ an analytic subvariety of dimension k, and ϕ a differential form of degree $2k-1$ with compact support in M,*

$$\int_V d\phi = 0$$

Theorem 10. *If M, N are complex manifolds, $f : M \to N$ a holomorphic map, and $V \subset M$ an analytic variety such that $f \restriction_V$ is proper, then $f(V)$ is an analytic subvariety of N.*

0.1.3 Sheaves and Cohomology

Origins: The Mittag-Leffler Problem

Let S be a Riemann surface, not necessarily compact, with $p \in S$ and a local coordinate z centered at p. A *principal part* is the polar part $\sum a_k z^{-k}$ of a Laurent series. If $\mathcal{O}_p$ is the local ring of holomorphic functions around p, $\mathcal{M}_p$ the field of meromorphic functions around p, a principal part is an element of the quotient group $\mathcal{M}_p \backslash \mathcal{O}_p$. The *Mittag Leffler* question is, given a discrete set $\{p_n\}$ and a principal part at p_n for every n, does there exist a function holomorphic away from the $\{p_n\}$ that has the prescribed principal parts at each p_n? The question is cleary a global one. There are two approaches (Cech and Dolbeault) both of which lead to cohomology theories.

Sheaves

Given X a topological space, a *sheaf* $\mathcal{F}$ on X associates to each open set $U \subset X$ a group $\mathcal{F}(U)$, called the sections over U, and to each pair $U \subset V$ a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, called the restriction map, satisfying:

1. For any triple $U \subset V \subset W$ of open sets,

   $$r_{W,U} = r_{V,U} \circ r_{W,V}$$

   So that we can write $\sigma \restriction_U = r_{V,U}(\sigma)$ without loss of information.

2. For any pair of open sets $U, V \subset M$ and sections $\sigma \in \mathcal{F}(U), \tau \in \mathcal{F}(V)$ such that

   $$\sigma \restriction_{U \cap V} = \tau \restriction_{U \cap V}$$

   there exists a section $\rho \in \mathcal{F}(U \cup V)$ with

   $$\rho \restriction_U = \sigma; \;\; \rho \restriction_V = \tau$$

3. If $\sigma \in \mathcal{F}(U \cup V)$ and

   $$\sigma \restriction_U = \sigma \restriction_V = 0$$

   then $\sigma = 0$.

The most commonly used sheaves we will use are listed below:

1. On any C^∞ manifold M, we define sheaves $C^\infty, C^*, \mathcal{A}^p, \mathcal{Z}^p, \mathbb{Z}, \mathcal{Q}, \mathcal{R}, \mathcal{C}$ by
   (a) $C^\infty(U)$, the smooth functions on U,
   (b) $C^*(U)$, the smooth nonzero functions on U under multiplication
   (c) $\mathcal{A}^p(U)$, smooth p-forms on U
   (d) $\mathbb{Z}(U), \mathbb{Q}(U), \mathbb{C}(U)$, the locally constant sheaves with value of the respective field
2. If M is a complex manifold, $V \subset M$, an analytic subvariety of M, and $E \to M$ a holomorphic vector bundle, we define the sheaves $\mathcal{O}, \mathcal{O}^*, \Omega^*, \mathcal{A}^{p,q}, \mathcal{Z}^{p,q}_{\overline{\partial}}, \mathcal{J}_V, \mathcal{O}(E), \mathcal{A}^{p,q}(E)$ by
   (a) $\mathcal{O}(U)$, the holomorphic functions on U
   (b) $\mathcal{O}^*(U)$, the multiplicative group of nonzero holomorphic functions on U
   (c) $\Omega^p(U)$, the holomorphic p forms on U
   (d) $\mathcal{A}^{p,q}(U)$, the closed forms of type (p,q)
   (e) $\mathcal{Z}^{p,q}_{\overline{\partial}}(U)$, $\overline{\partial}$-closed forms on U,
   (f) $\mathcal{J}_V(U)$, holomorphic functions vanishing on $V \cap U$
   (g) $\mathcal{O}(E)(U)$, the holomorphic sections of E over U,
   (h) $\mathcal{A}^{p,q}(E)(U)$, E-valued (p,q)-forms over U.
3. If M is a complex manifold, a *meromorphic function* on an open set $U \subset M$ is given locally as the quotient of two holomorphic functions. A meromorphic function is *not*, strictly speaking, a function even if we consider ∞ a value, as the function is undefined still at points were the numerator and denominator both vanish. The sheaf of meromorphic function on M is denoted $\mathcal{M}$; the multiplicative sheaf of meromorphic functions not identically zero is denoted $\mathcal{M}^*$.

A *map of sheaves* $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ on M is given by a collection of homomorphisms $\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \subset M}$ such that for $U \subset V \subset M$, α_U and α_V commute with the restriction maps. The kernel of the map $\alpha : \mathcal{F} \to \mathcal{G}$ is just the sheaf given by the kernel of each homomorphsm; it does indeed define a sheaf. The *cokernel* is harder to define; it may not even define a sheaf.
Instead of taking a direct cokernel of each homomorphism, we define $\mathrm{Coker}(\alpha)$ as follows: the section over U is given by an open cover $\{U_\alpha\}$ of U together with section $\sigma_\alpha \in \mathcal{G}(U_\alpha)$ such that for all α, β,

$$\sigma_\alpha \restriction_{U_\alpha \cap U_\beta} - \sigma_\beta \restriction_{U_\alpha \cap U_\beta} \in \alpha_{U_\alpha \cap U_\beta} \left(\mathcal{F}(U_\alpha \cap U_\beta)\right)$$

and we identify two such collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(U'_\alpha, \sigma'_\alpha)\}$ if for all $p \in U$ and $p \in U_\alpha, U'_\beta$ there exists V with $p \in V \subset \left(U_\alpha \cap U'_\beta\right)$ such that $\sigma'_\alpha \restriction_V - \sigma'_\beta \restriction_V \in \alpha_V\left(\mathcal{F}(V)\right)$. We say that a sequence of sheaf maps

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

is *exact* if $\mathcal{E} = \operatorname{Ker}(\beta)$ and $\mathcal{G} = \operatorname{Coker}(\alpha)$; in this case, we also call $\mathcal{E}$ a *subsheaf* of $\mathcal{F}$ and $\mathcal{G}$ the *quotient sheaf* of $\mathcal{F}$ by $\mathcal{G}$, written $\mathcal{F} \backslash \mathcal{G}$. More generally, we say a sequence

$$\cdots \to \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \xrightarrow{\alpha_{n+1}} \mathcal{F}_{n+2} \to \cdots$$

is exact if $\alpha_{n+1} \circ \alpha_n = 0$ and

$$0 \to \operatorname{Ker}(\alpha_n) \to \mathcal{F}_n \to \operatorname{Ker}(\alpha_{n+1}) \to 0$$

is exact for each n. Note that by our definition of Coker, this does not imply that

$$0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0$$

is exact for all U; it does imply that this sequence is exact at the first two stages for U, and that for any section $\sigma \in \mathcal{G}(U)$ and at any point $p \in U$ there exists a neighborhood V of p in U such that $\sigma \restriction_V$ is in the image of β_V.

Cohomology of Sheaves

Let $\mathcal{F}$ be a sheaf on M, and $\underline{U} = \{U_\alpha\}$ a locally finite open cover. We define

$$C^0(\underline{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$$

$$C^1(\underline{U}, \mathcal{F}) = \prod_{\alpha_1 \neq \alpha_2} \mathcal{F}(U_{\alpha_1} \cap U_{\alpha_2})$$

$$\vdots$$

$$C^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_1 \neq \alpha_2 \neq \cdots \neq \alpha_p} \mathcal{F}\left(U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}\right)$$

An element $\sigma \in C^p(\underline{U}, \mathcal{F})$ is called a *p-cochain* of $\mathcal{F}$. We define a *coboundary operator*

$$\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_1,\ldots,i_{p+1}} = \sum_j (-1)^j \sigma_{i_1,\ldots,\hat{i_j},\ldots,i_{p+1}} \restriction_{U_{i_1} \cap \cdots \cap U_{i_p}}$$

A p-cochain is called a *cocycle* if $\delta\sigma = 0$. Note that any cocycle σ must satisfy the skew symmetry condition; i.e. switching two indices gives the negative of the coboundary. σ is called a *coboundary* if $\sigma = \delta\tau$ for some $\tau \in C^{p-1}(\underline{U}, \mathcal{F})$. We define the usual cohomology classes:

$$H^p(\underline{U}, \mathcal{F}) = \frac{Z^p(\underline{U}, \mathcal{F})}{\delta C^{p-1}(\underline{U}, \mathcal{F})}$$

Now for a refinement $\underline{U}'$ of $\underline{U}$, there is a map

$$\rho : C^p(\underline{U}, \mathcal{F}) \to C^p\left(\underline{U}', \mathcal{F}\right)$$

given by taking the 'smaller elements of the refinement' and restricting the section to these elements. Explicitly, given a map $\phi : \underline{U}' \to \underline{U}$ putting 'smaller elements in the bigger one', so that

$$(\rho\sigma)_{i_1 \cdots i_p} = \sigma_{\phi i_0 \cdots \phi i_p} \restriction_{U_{i_1} \cap \cdots \cap U_{i_p}}$$

Since $\delta \circ \rho = \rho \circ \delta$, ρ induces a homomorphism of $H^p\left(\underline{U}, \mathcal{F}\right) \to H^p\left(\underline{U}', \mathcal{F}\right)$, which is independent of ϕ. We define the p-th Čech cohomology group of $\mathcal{F}$ on M to be the direct limit of the $H^p\left(\underline{U}, \mathcal{F}\right)$ as $\underline{U}$ becomes finer:

$$H^p\left(M, \mathcal{F}\right) = \xrightarrow[U]{\lim} H^p\left(\underline{U}, \mathcal{F}\right)$$

If there is possibility of confusion, these groups may also be denoted by Ȟ.

Theorem 11. *If the covering $\underline{U}$ is acyclic for the sheaf $\mathcal{F}$, in the sense that*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0, \;\; q > 0,$$

then $H^*\left(\underline{U}, \mathcal{F}\right) \approx H^*(M, \mathcal{F})$.

Proposition 6. *For an exact sequence $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ of sheaves on M, the induced sequence*

$$\begin{gathered} 0 \to H^0\left(M, \mathcal{E}\right) \to H^0\left(M, \mathcal{F}\right) \to H^0\left(M, \mathcal{G}\right) \\ \to H^1\left(M, \mathcal{E}\right) \to H^1\left(M, \mathcal{F}\right) \to H^1\left(M, \mathcal{G}\right) \to \cdots \\ \vdots \\ \to H^p\left(M, \mathcal{E}\right) \to H^p\left(M, \mathcal{F}\right) \to H^p\left(M, \mathcal{G}\right) \to \cdots \end{gathered}$$

is exact.

There are a couple observations to make about certain sheaves:

1. $H^P\left(M, \mathcal{A}^{r,s}\right) = 0$ for strictly positive p.
2. For K a simplicial complex with underlying topological space M,

   $$H^*\left(K, \mathbb{Z}\right) \approx H^*\left(M, \mathbb{Z}\right)$$

   which is to say, the simplicial cohomology is isomorphic to the Čech cohomology.

The de Rham Theorem

Let M be a real C^∞ manifold. We say that a singular p-chain σ on M, given as a formal linear combination $\sum a_i f_i$ of maps $\Delta \xrightarrow{f_i} M$ of the standard p-simplex $\Delta \subset \mathbb{R}^p$ to M, is *piecewise smooth* if the maps f_i extend to C^∞ maps of a neighborhood of Δ to M. Let $C_p^{ps}(M,\mathbb{Z})$ denote the space of piece-wise smooth integral p-chains. Clearly the boundary of a piecewise smooth chain is again piecewise smooth, so $C_*^{ps}(M,\mathbb{Z})$ forms a subcomplex of $C_*(M,\mathbb{Z})$ so that we can set

$$Z_p^{ps}(M,\mathbb{Z}) = \mathrm{Ker}\partial$$

$$H_p^{ps}(M,\mathbb{Z}) = \frac{Z_p^{ps}(M,\mathbb{Z})}{\partial C_{p+1}^{ps}(M,\mathbb{Z})}$$

By a big result in differential topology, the inclusion map $C_*^{ps}(M,\mathbb{Z}) \to C_*(M,\mathbb{Z})$ induces an isomorphism

$$H_p^{ps}(M,\mathbb{Z}) \approx H_p(M,\mathbb{Z})$$

Now let $\phi \in A^p(M)$ be a C^∞ p-form and $\sigma = \sum a_i f_i$ a piecewise smooth p-chain; set

$$\langle \phi, \sigma \rangle = \int_\sigma \phi = \sum a_i \int_{f_i(\Delta)} \phi$$

Now Stokes' theorem says that the addition of a coboundary or an exact form will not change this value, so we have an isomorphism

$$H_{DR}^*(M) \to H_{\mathrm{sing}}^*(M,\mathbb{R})$$

The Dolbeault Theorem

Theorem 12. *For M a complex manifold,*

$$H^q(M,\Omega^p) \approx H_{\overline{\partial}}^{p,q}(M)$$

Now for some computations:

1. If M is any n-dimensional complex manifold, then

$$H^q(M,\mathcal{O}) \approx H_{\overline{\partial}}^{0,q}(M) = 0$$

2. By te $\overline{\partial}$-Poincaré lemma,

$$H^q(\mathbb{C}^n,\mathbb{O}) = 0$$

3. $H^0(\mathbb{P}^1,\mathcal{O}) \approx \mathbb{C}$
4. Let $M = \mathbb{C}^2 \setminus \{0\}$. Then $\dim H^1(M,\mathcal{O}) = \infty$.

0.1.4 Topology of Manifolds

Intersection of Cycles

Definition 8. *Let M be an oriented n-manifold, A and B two peicewise smooth cycles on M of dimension k and $n-k$, respectively, and $p \in A \cap B$ a point of transverse intersection of A and B. Let $v_1, \ldots, v_k \in T_p(A)$ be an oriented basis for T_pA, $w_1, \ldots, w_{n-k}$ an oriented basis for T_pB, we define the intersection index $\iota_p(A \cdot B)$ of A with B at p to be the $+1$ if $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$ is an oriented basis for T_pM, and -1 otherwise. then define the intersection number $\#(A \cdot B)$ to be*

$$\#(A \cdot B) = \sum_{p \in A \cap B} \iota_p(A \cdot B)$$

Proposition 7. *$\#(A \cdot B)$ depends only on the homology classes of A and B.*

Poincaré Duality

Theorem 13. *If M is a compact, oriented n-manifold, the intersection pairing:*

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$$

is unimodular; i.e., any linear functional $H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}$ is expressible as intersection with some class $\alpha \in H_k(M, \mathbb{Z})$ and any class $\alpha \in H_k(M, \mathbb{Z})$ having intersection number 0 with all classes in $H_{n-k}(M, \mathbb{Z})$ is a torsion class.

Proposition 8. *The Künneth Formula formula states that*

$$H_*(M \times N, \mathbb{Q}) \approx H_*(M, \mathbb{Q}) \otimes H_*(N, \mathbb{Q})$$

which says that the intersection of cycles in homology is Poincaré dual to wedge product in cohomology.

Intersection of Analytic Cycles

Suppose M is a compact complex manifold of dimension n, $V \subset M$ a possibly singular analytic subvariety of dimension k. As we have seen,

$$\int_V d\phi = 0$$

holds for any $(2k-1)$ form ϕ on M. We may thus define a linear functional on $H^{2k}_{DR}(M)$ by

$$[\phi] \mapsto \int_V \phi$$

We may also define fundamental class of V by means of the intersection pairing. For any homology class $\alpha \in H_{2n-2k}(M, \mathbb{Z})$ we may find a cycle A representing α and intersecting V transversely in smooth points. In fact, the intersection number

$$\#(V \cdot A)$$

depends only on the cohomology class of α.

Proposition 9. *The intersection number of two analytic subvarieties meeting transversely is always positive.*

Let V and W be two analytic varieties of dimension k and $n-k$ in the polycylinder Δ of radius 1 in $\mathbb{C}^n$ having the origin as their only point of intersection. Consider in the product $\Delta \times \Delta$ of the polycylinder of radius $\frac{1}{2}$ with itself the two varieties

$$\tilde{V} = \pi_1^{-1}(V) = \{(z, w) : z \in V\}$$

and

$$\tilde{W} = \{(z, w) : z - w \in W\}$$

For each ϵ, the varieties $\tilde{V}$ and $\tilde{W}$ meet the fiber $\pi_2^{-1}(\epsilon) = \Delta' \times \{\epsilon\}$ in the analytic variety V and the analytic $W + \epsilon$ (W translated by ϵ) respectively; moreover, $\pi_2^{-1}(\epsilon)$ will meet the intersection $\tilde{V} \cap \tilde{W}$ transvresely at a point (p, ϵ) exactly when V and $W + \epsilon$ meet tranversely at p. The intersection $\tilde{V} \cap \tilde{W} \subset \Delta' \times \Delta'$ is an analytic variety of dimension n, and so the projection $\pi_2 : \tilde{V} \cap \tilde{W} \to \Delta'$ expresses $\tilde{V} \cap \tilde{W}$ as a branched μ sheeted cover of Δ'. We are led to the following definition:

Definition 9. *For $\epsilon \in \Delta'$ lying outside an analytic subvariety of Δ', the varieties V and $W + \epsilon$ will meet transversely in μ points in Δ'. μ is called the intersection mutliplicity of V and W of at 0 and is written $\mu = m_0(V \cdot W)$.*

Proposition 10.

$$\#(V \cdot W) = \sum_{p \in V \cap W} m_p(V \cdot W)$$

Theorem 14. *The topological intersection number $\#(V \cdot W)$ of two analytic subvarieties of complementary dimension meeting in a finite set of point ona compact complex manifold is given by*

$$\#(V \cdot W) = \sum_{p \in V \cap W} m_p(V \cdot W)$$

The intersection multiplicity satisfies

$$m_p(V \cdot W) \geq 1$$

with equality holding if and only if V and W meet transversely at p.

Corollary 2. *If M is any complex submanifold of projective space $\mathbb{P}^n$, $V \subset M$ an analytic subvariety, then the fundamnetal class of V is nonzero in the homology of M.*

0.1.5 Vector Bundles, Connections, and Curvature

Complex and Holomorphic Vector Bundles

Definition 10. *Let M be a differentiable manifold. A C^∞ compex vector bundle on M consists of a family $\{E_x\}_{x \in M}$ of complex vector spaces parametrized by M, together with a C^∞ manifold structure on $E = \cup_{x \in M} E_x$ such that*

1. *The projection map $\pi : E \to M$ taking E_x to x is C^∞*
2. *For every $x_0 \in M$, there exists an open set U in M containing x_0 and a diffeomorphism*
$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$$
taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$; ϕ_U is called a trivialization of E over U.

The dimension of the fibers E_x of E is called the rank of E; in particular, a vector bundle of rank 1 is called a line bundle.

Definition 11. *For any pair of trivialization ϕ_U and ϕ_V the map*

$$g_{UV} : U \cap V \to \mathrm{GL}(k, \mathbb{C})$$

given by
$$g_{UV}(x) = \left(\phi_U \circ \phi_V^{-1}\right) \restriction_{\{x\} \times \mathbb{C}^k}$$
is called a transition function for E relative to the trivializations ϕ_U, ϕ_V

All of these constructions can be brought to the complex category by replacing smooth with holomorphic, and differentiable with complex.

Proposition 11. *There is a natural exterior derivative $\overline{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$ from E valued (p, q)-forms to E-valued $(p, q + 1)$ forms, given in coordinates by*

$$\overline{\partial}\sigma = \sum \overline{\partial}\omega_i \otimes e_i$$

The holomorphicity of transition functions shows that $\overline{\partial}\sigma$ does not depend on the frame chosen.

Metrics, Connections, and Curvature

Definition 12. *Let $E \to M$ be a complex vector bundle. A hermitian metric on E is a hermitian inner product on each fiber E_x of E, varying smoothly with $x \in M$. A frame ζ is called unitary if $\zeta_1, \ldots, \zeta_2$ is an orthonormal basis for E_x for each x; unitary frames exist locally by the Graham-Schmidt process for a local basis. A holomorphic vector bundle with a hermitian metric is called a hermitian vector bundle.*

Definition 13. *A connection D on a complex vector bundle $E \to M$ is a linear map*
$$D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$
satisfying Leibnitz' rule

$$D(f \cdot \zeta) = df \otimes \zeta + f \cdot D(\zeta)$$

Definition 14. *Let D be a connection on E, and $\{e_1, \ldots, e_n\}$ be a local frame. We can decompose De_i as*

$$De_i = \sum_j \theta_{ij} e_j$$

where θ_{ij} is called the connection matrix of the connection, giving a general decomposition of section $\sigma = \sum \sigma e_i$

$$D\sigma = \sum_j \left(d\sigma + \sum_i \_i\theta_{ij} \right) e_j$$

And the connection matrix transforms as:

$$\theta_{e'} = dg \cdot g^{-1} + g \cdot \theta_e \cdot g^{-1}$$

Where θ_e is the connection matrix in the original frame, $\theta_{e'}$ is the connection matrix an the new frame, and g is the vector bundle cocycle transforming the original frame into the new frame.

Definition 15. *Writing $D = D' + D''$ with $D' : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ and $D'' : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$, then D is compatible with the complex structure if $D'' = \overline{\partial}$. If E is hermitian, D is said to be compatible with the metric if*

$$d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta)$$

Lemma 3. *If E is a hermitian vector bundle, there is a unique connection D on E that is compatible with both the metric and complex structure.*

Lemma 4. *Let $E \to M$ be a hermitian vector bundle and $F \subset E$ a holomorphic subbundle. Then F is itself a hermitian bundle with metric connection D_F induced by the metric connection of E.*

Lemma 5.

$$D_{E \otimes E'} = D_E \otimes 1 + 1 \otimes D_{E'}$$

Definition 16. *For a connection D on a complex vector bundle E, the operator D^2 is linear over $\mathcal{A}^0$, which means it takes the form*

$$D^2 e_i = \sum \Theta_{ij} \otimes e_j$$

Θ_e is called the curvature matrix of D in the frame e. It transforms, when the frame transforms by the cocycle g,

$$\Theta_{e'} = g \cdot \Theta_e \cdot g^{-1}$$

Proposition 12.

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e$$

Proposition 13. *If the connection D is compatible with both the metric and complex structure, then the curvature matrix is a hermitian matrix of $(1, 1)$-forms.*

Lemma 6. *There exists a unique matrix ψ_{ij} of 1-forms such that $\psi + \overline{\psi} = 0$ and*

$$d\phi_i = \sum_j \psi_{ij} \wedge \phi_j + \tau_i$$

where τ_i is of type $(2,0)$.

0.1.6 Harmonic Theory on Compact Complex Manifolds

The Hodge Theorem

Throughout, M will be a connected, compact, complex manifold of complex dimension n. We will choose a hermitian metric ds^2 with associate (1,1) form

$$\omega = \frac{\sqrt{-1}}{2} \sum_j \phi_j \wedge \overline{\phi_j}$$

in terms of a unitary cofram $\{\phi_1, \ldots, \phi_n\}$. The metric ds^2 induces a hermitian metric on all tensor bundles $T^{*(p,q)}(M)$; the inner product in $T^{*(p,q)}(M)$ is by taking the basis$\left\{\phi_I(z) \wedge \overline{\phi_J}(z)\right\}_{\#I=p, \#J=q}$ to be orthogonal and of length 2^{p+q}. Let $C_n = (-1)^{n(n-1)/2}\left(\sqrt{-1}/2\right)$ and

$$\Phi = \frac{\omega^n}{n!} = C_n \phi_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi_1} \wedge \cdots \wedge \overline{\phi_n}$$

be the volume form on M associated to the metric. The global inner product

$$(\psi, \eta) = \int_M (\psi(z), \eta(z))\, \Phi(z)$$

makes $A^{p,q}(M)$ into a pre-Hilbert space. We ask: *Given a $\overline{\partial}$-closed form $\psi \in Z^{p,q}_{\overline{\partial}}(M)$, among all the forms $\left\{\psi + \overline{\partial}\eta\right\}$ representing the Dolbeault cohomology class $[\psi] \in H^{p,q}_{\overline{\partial}}(M)$ of ϕ, can we fine one of smallest norm?* To answer, we pretend for a moment that $A^{p,q}(M)$ is complete and $\overline{\partial}$ is bounded and define the adjoint operator

$$\overline{\partial}^* : A^{p,q}(M) \to A^{p,q-1}(M)$$

by requiring that

$$\left(\overline{\partial}^* \psi, \eta\right) = \left(\psi, \overline{\partial}\eta\right)$$

for all $\eta \in A^{p,q-1}(M)$.

Lemma 7. *A $\overline{\partial}$-closed form $\psi \in Z^{p,q}_{\overline{\partial}}(M)$ is of minimal norm in $\overline{\partial}A^{p,q-1}(M)$ if and only if $\overline{\partial}^* \psi = 0$.*

So at least formally, the Dolbeault cohomology group $H^{p,q}_{\overline{\partial}}(M)$ is represented by the solutions of two first-order equations

$$\overline{\partial}\psi = 0 \;\; \overline{\partial}^* \psi = 0$$

These two may be replaced by the single second order equation

$$\Delta_{\overline{\partial}} = \left(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right)\psi = 0$$

As a positive definite operator, the vanishing of $\Delta_{\overline{\partial}}\psi$ implies the vanishing of the two components. The operator

$$\Delta_{\overline{\partial}} A^{p,q}(M) \to A^{p,q}(M)$$

is called the $\overline{\partial}$-*Laplacian*, or simple the *Laplacian* (written Δ) if no ambiguity is likely. Differential forms satisfying the Laplace Equation

$$\Delta\psi = 0$$

are called *Harmonic forms*; the space of harmonic forms of type (p, q) is denoted by $\mathscr{H}^{p,q}(M)$ and called the *harmonic space*. What the previous argument suggests is an isomorphism

$$\mathscr{H}^{p,q}(M) \approx H^{p,q}_{\overline{\partial}}(M)$$

Definition 17. *The star or duality operator is the map*

$$* : A^{p,q}(M) \to A^{n-p,n-q}(M)$$

by requiring

$$(\psi(z), \eta(z))\,\Phi(z) = \psi(z) \wedge *\eta(z)$$

which satisfies the following:

1. $**\,\eta = (-1)^{p+q}\eta$
2. $\overline{\partial}^* = -*\overline{\partial}*$

Theorem 15. 1. $\dim \mathscr{H}^{p,q}(M) < \infty$

2. *The orthogonal projection*

$$\mathscr{H} : A^{p,q}(M) \to \mathscr{H}^{p,q}(M)$$

is well-defined, and there exists a unique operator, the Green's operator,

$$G : A^{p,q}(M) \to A^{p,q}(M)$$

with $G(\mathscr{H}^{p,q}(M)) = 0$, $\overline{\partial}G = G\overline{\partial}$, $\overline{\partial}^*G = G\overline{\partial}^*$ *and*

$$I = \mathscr{H} + \Delta G$$

This implies the orthogonal direct-sum relationship

$$A^{p,q}(M) = \mathscr{H}^{p,q}(M) \bigoplus \overline{\partial}A^{p,q-1}(M) \bigoplus \overline{\partial}^* A^{p,q+1}(M)$$

Applications of the Hodge Theorem

Theorem 16.

$$\dim H^q(M, \Omega^p) < \infty$$

Theorem 17. *1.* $H^n(M, \Omega^n) \to \mathbb{C}$

2. the pairing

$$H^q(M, \Omega^p) \otimes H^{n-q}\left(M, \Omega^{n-p} \to H^n(M, \Omega^n)\right)$$

is nondegenerate.

0.1.7 Kähler Manifolds

0.1.8 The Kähler Condition

We look for the condition and equivalent conditions for the $\overline{\partial}$-Laplacian to be equal to the d-Laplacian,

$$\overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial} = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d = d\delta + \delta d$$

Where δ is the codifferential of the associated metric of the hermitian metric on M.

Definition 18. *A metric ds^2 on M is Kähler if its associated $(1,1)$-form*

$$\omega = \frac{\sqrt{-1}}{2}\sum \phi \wedge \overline{\phi_i}$$

is d-closed (i.e. ω is symplectic).

Definition 19. *A metric ds^2 on M osculates to order k to the Euclidean metric on $\mathbb{C}^n$ if the metric locally takes the form*

$$ds^2 = \sum (\delta_{ij} + g_{ij})\, dz_i \otimes d\overline{z_j}$$

where g_{ij} vanishes up to order k at z_0; this is usually written as

$$ds^2 = \sum (\delta_{ij} + [k])\, dz_i \otimes d\overline{z_j}$$

Lemma 8. *ds^2 is Kähler if and only if it osculates to order 2 to the Euclidean metric everywhere.*

Definition 20. *A manifold is called Kähler if it admits a Kähler metric.*

Proposition 14. *For M a compact Kähler manifold,*

1. *The even numbers $b_{2q}(M)$ are positive*
2. *There is an injection of holomorphic q-forms $H^0(M, \Omega^q)$ into the cohomology $H^q_{DR}(M)$*
3. *The fundamental class η_V of any analytic subvariety $V \subset M$ is non-zero.*

The Hodge Identities and the Hodge Decomposition

Definition 21. *The following are differential operator:*

1. $d : A^r(M) \to A^{r+1}(M)$
2. $\partial : A^{p,q}(M) \to A^{p+1,q}(M)$
3. $\overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$
4. $d^c = \frac{\sqrt{-1}}{4\pi}(\partial - \overline{\partial}) : A^r(M) \to A^{r+1}(M)$

Note that these satisfy the following:

1. $d = \partial + \overline{\partial}$
2. $d^c = \partial - \overline{\partial}$
3. $dd^c + d^c d = 0$
4. $\partial\overline{\partial} + \overline{\partial}\partial = 0$
5. $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}$

We additionally define the following:

$$\Pi^{p,q} : A^*(M) \to A^{p,q}(M)$$

$$\Pi^r = \bigoplus_{p+q=r} \Pi^{p,q}$$

We also define

$$L : A^{p,q}(M) \to A^{p+1,q+1}(M);\ \ L\eta = \eta \wedge \omega$$

where ω is the associated $(1,1)$-form of the Hermitian metric ds^2.

Theorem 18. *The following are identities on Kähler manifolds:*

1. $[\Lambda, d] = -4\pi d^{c*}$
2. $[L, \Delta_d] = 0$

Theorem 19 (Hodge Decomposition)**.** *For a compact Kähler manifold M, the complex cohomology satisfies*

1. $H^r(M, \mathbb{C}) \approx \bigoplus_{p+q=r} H^{p,q}(M)$
2. $H^{p,q}(M) = \overline{H^{q,p}(M)}$

In addition, the odd Betti numbers $b_{2q+1}(M)$ are even.

The Lefschetz Decomposition