

Notes from *Curvature and
Homology*

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0.1 Topology of Differentiable Manifolds

0.1.1 Complexes

Definition 1. A closure finite abstract complex K is a countable collection of object $\{S_i^p\}$ called simplexes satisfying the following properties:

1. To each simplex S_i^p there is associated an integer $p \geq 0$ called its dimension;
2. To the simplexes S_i^p and S_j^{p-1} is associated an integer denoted by $[S_i^p : S_j^{p-1}]$ called their incidence number;
3. There are only a finite number of simplexes S_j^{p-1} such that $[S_i^p : S_j^{p-1}] \neq 0$;
4. For every pair of simplexes S_i^{p+1}, S_j^{p-1} whose dimensions differ by two

$$\sum_k [S_i^{p+1} : S_k^p] [S_k^p : S_j^{p-1}] = 0$$

We associate with K an integer $\dim K$ called its dimension which is the maximum dimension of its simplexes.

Definition 2. An algebraic structure is imposed on K as follows: the p -simplexes are taken as free generators of an abelian group. A finite sum

$$C_p = \sum_i g_i S_i^p; \quad g_i \in G$$

where G is an abelian group is called a p -dimensional chain or a p -chain. Two p -chains may be added, with their sum being the sum of their coefficients of each simplex. This way, p chains form an abelian group denoted by $C_p(K, G)$.

Definition 3. Let Λ be a ring with unity 1. A Λ -module is an abelian group A together with a map $(\lambda, a) \rightarrow \lambda a$ of $\Lambda \times A \rightarrow A$ satisfying

1. $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2$
2. $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a$
3. $(\lambda_1 \lambda_2)a = \lambda_1 (\lambda_2 a)$
4. $1a = a$

Definition 4. Let A be a right Λ -module and B a left Λ -module. Let $F_{A \times B}$ the free abelian group having as a basis the set $A \times B$ of pairs (a, b) and let Γ be the subgroup of $F_{A \times B}$ the subgroup of $F_{A \times B}$ generated by elements of the form

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$

$$(a\lambda, b) - (a, \lambda b)$$

The quotient group $F_{A \times B}/\Gamma$ is called the tensor product of A and B and it is an abelian group.

Definition 5. The boundary map $\partial : C_p(K, G) \rightarrow C_{p-1}(K, G)$ is defined by the formula

$$\partial C_p = \sum_i g_i \partial S_i^p = \sum_i \sum_j g_i [S_i^p : S_j^{p-1}] S_j^{p-1}$$

where since $[S_i^p : S_j^{p-1}]$ is an integer, its multiplication against g_i is considered as a multiple of g_i in the \mathbb{Z} -module of G . As a linear function, the boundary map is a group homomorphism.

Definition 6. The kernel of ∂ is denoted by $Z_p(G, K)$, and its elements are called p -cycles. Since $\partial^2 = 0$, the set of p -cycles contains the image of ∂ on $C_{p-1}(K, G)$, denoted by $B_p(K, G)$ whose elements are called boundaries. The quotient group

$$H_p(K, G) = Z_p(K, G)/B_p(K, G)$$

is called the p -th homology group of K with coefficient group G . the elements of $H_p(K, G)$ are called homology classes.

Definition 7. Let $C_p(K) = C_p(K, \mathbb{Z})$, elements of which we will call integral p -chains of K . A linear function f^p defined on $C_p(K)$ with values in a commutative topological group G :

$$f^p : C_p(K) \rightarrow G$$

is called a p -dimensional cochain or a p -cochain. We define groups dual to the homology groups by using function addition as the group operation on p -cochains.

Definition 8. The operator ∂^* dual to ∂ called the coboundary operator is defined by

$$(\partial^* f)(C_{p+1}) = f^p(\partial C_{p+1})$$

It is a linear, square-free map.

Definition 9. The kernel of ∂^* is denoted by $Z^p(K, G)$ and its elements are called p -cocycles. The image of $C^{p-1}(K, G)$ under ∂^* is denoted by $B^p(K, G)$ and its elements are called coboundaries. The quotient group

$$H^p(K, G) = Z^p(K, G)/B^p(K, G)$$

is called the p -th cohomology group of K with coefficient group G . Its elements are called cohomology classes.