1 Symplectic Manifolds

1.1 Basic Concepts

Throughout we will assume M to be a smooth manifold without boundary. Most of the time M will be compact.

Definition 1 *A* ***Symplectic Structure*** *is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. The manifold is necessarily even-dimensional and orientable.*

Definition 2 *A* ***Symplectomorphsim*** *is a diffeomorphism that preserves the symplectic form. The set of Symplectomorphisms is denoted by $\mathrm{Symp}(M, \omega)$ or $\mathrm{Symp}(M)$.*

Definition 3 *A vector field $X \in \mathfrak{X}(M)$ is called* ***symplectic*** *or* ***Locally Hamiltonian*** *is $i_X\omega$ is closed. The set of locally Hamiltonian vector fields is denoted by $\mathfrak{X}(M, \omega)$.*

Proposition 1 *Let M be a closed manifold. If $t \mapsto \psi_t \in \mathrm{Diff}(M)$ a smooth homotopy, generating smooth vector fields $X_t \circ \phi_t = \frac{d}{dt}\psi_t$, then*

$$\psi_t \in \mathrm{Symp}(M, \omega) \iff X_t \in \mathfrak{X}(M, \omega) \tag{1}$$

In addition, if $X, Y \in \mathfrak{X}(M, \omega)$ then $[X, Y] \in \mathfrak{X}(M, \omega)$ and

$$i_{[X,Y]}\omega = dH; \quad H = \omega(X, Y) \tag{2}$$

1.2 Hamiltonian Flows

Definition 4 *For any smooth function $H : M \to \mathbb{R}$ the vector field $X_H : M \to TM$ determined by $i_{X_H}\omega = dH$ is called the* ***Hamiltonian Vector Field*** *associated to the* ***Hamiltonian Function*** *H. The flow associated with this vector field is called the* ***Hamiltonian Flow*** *associated to H.*

Definition 5 *The* ***Poisson Bracket*** *of two functions F, G is the new function*

$$\{F, H\} = \omega(X_F, X_H) = dF(X_H) \tag{3}$$

Proposition 2 *Let (M, ω) be a symplectic manifold.*

1. *Hamiltonian flows are symplectomorphisms, and are tangent to the level surfaces of their Hamiltonian function.*
2. *For every Hamiltonian function H and every symplectomorphism ψ, $X_{H\circ\psi} = \phi^* X_H$*
3. *$[X_F, X_G] = X_{\{F,G\}}$*

Thus Hamiltonian vector fields form a Lie subalgebra of the symplectic vector fields. The map $H \mapsto X_H$ is a surjective Lie Algebra homomorphism from the Lie algebra of smooth functions to Hamiltonian vector fields. The kernel of this homomorphism is constant functions.

Since $\mathcal{L}_{X_H} H = 0$, every level set of H is an invariant submanifold of the Hamiltonian vector field. Conversely, let $S \subset M$ be a compact orientable hypersurface (codimension 1) of a symplectic manifold. An exercise (not in these notes) showed that this is a coisotropic submanifold. Hence the vector space

$$L_q = T_q S^\omega = \{v \in T_q M | \omega(v, w) = 0 \ \forall w \in T_q S\} \tag{4}$$

is a 1-dimensional subspace of $T_q S$ for every $q \in S$ and hence defines a real line bundle L over S. It integrates to give the **Characteristic Foliation**. The leaves of this foliation are the integral curves of any Hamiltonian vector field which for which S is a regular level surface of the associated Hamiltonian function.

1.3 Hamiltonian Isotopies

Consider a smooth map $t \mapsto \psi_t \in \mathrm{Symp}(M, \omega)$ with $\psi_0 = \mathrm{id}_M$. This generates a smooth vector field

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \tag{5}$$

Because ψ_t is symplectic, the X_t are locally Hamiltonian. If they are all globally Hamiltonian, then we have that

$$X_t = X_{H_t} \tag{6}$$

H_t are **time-dependent Hamiltonians** and ψ_t is a **Hamiltonian Isotopy**. If there is a Hamiltonian Isotopy ending with $\psi \in \mathrm{Symp}(M, \omega)$, then ψ is called **Hamiltonian**. The space of Hamiltonian symplectomorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

$\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$, and it Lie algebra is the space of all Hamiltonian vector fields. This makes it an infinite dimensional Lie group, markedly different from the Riemannian case.

1.4 Isotopies and Darboux's Theorem

Lemma 1 *Let M be a $2n$-dimensional manifold and $Q \subset M$ a compact submanifold. Suppose ω_0, ω_1 are closed degenerate 2-forms such that at each $q \in Q$, $(\omega_0)_q = (\omega_1)_q$. Then there are open neighborhoods $\mathcal{N}_0, \mathcal{N}_1$ of Q and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}_1$ such that*

$$\psi \restriction_Q = \mathrm{id}; \ \psi^* \omega_1 = \omega_0 \tag{7}$$

Theorem 1 *Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

Theorem 2 (Moser Stability Theorem for Symplectic Structures) *Let M be a closed manifold and suppose ω_t is a smooth family of cohomologous (i.e. all lying in the same cohomology class) symplectic forms on M. Then there is a family of diffeomorphisms ψ_t satisfying*

$$\psi_0 = \mathrm{id}; \quad \psi_t^* \omega_t = \omega_0 \tag{8}$$

Definition 6 *1. An isotopy preserving a symplectic structure is called a **Symplectic Isotopy**.*

*2. Two symplectic forms ω_0, ω_1 on M are **isotopic** if they can be joined by a smooth family ω_t of cohomologous symplectic forms on M.*

*3. Two isotopic symplectic forms are **strongly isotopic** is there is an isotopy ψ_t of M such that $\psi_1^* \omega_1 = \omega_0$*

Theorem 3 (Symplectic Isotopy Extension Theorem) *Let (M, ω) be a compact symplectic manifold and let $Q \subset M$ be a compact subset. Let $\phi_t : U \to M$ be a symplectic isotopy of an open neighborhood U of Q and assume $H^2(M, Q, \mathbb{R}) = 0$.*

Then there exists a neighborhood $\mathcal{N} \subset U$ of Q and a symplectic isotopy ψ_t such that

$$\psi_t \restriction_{\mathcal{N}} = \phi_t \restriction_{\mathcal{N}} \tag{9}$$

1.5 Submanifolds of Symplectic Manifolds

Definition 7 *A submanifold $Q \subset M$ is called **symplectic** (resp. **isotropic, coisotropic, Lagrangian**) is for every $q \in Q$, the symplectic vector space $(T_q M, \omega_q)$ is symplectic (resp. isotropic, coisotropic, Lagrangian).*

Proposition 3 *The graph $\Gamma_\sigma \subset T^* L$ of a 1-form σ on L is Lagrangian $\iff$ σ is closed.*

Proposition 4 *Let ψ be a diffeomorphism of a symplectic manifold (M, ω). Then ψ is a symplectomorphism $\iff$ its graph*

$$\mathrm{graph}(\psi) = \{(q, \psi(q))\} \subset M \times M \tag{10}$$

is a Lagrangian submanifold of $(M \times M, (-\omega) \times \omega)$

Theorem 4 (Symplectic Neighborhood Theorem) *For $j = 0, 1$, let (M_j, ω_j) be symplectic manifolds with compact symplectic submanifolds Q_j. Suppose there is an isomorphism $\Phi : \nu_{Q_0} \to \nu_{Q_1}$ of the symplectic normal bundles which covers a symplectomorphism $\psi : (\mathcal{N}(Q_0), \omega_0) \to (\mathcal{N}(Q_1), \omega_1)$ such that $d\psi$ induces the map Φ on $\nu_{Q_0} = (TQ_0)^\omega$.*

Theorem 5 *Let (M, ω) be a symplectic manifold*