1 Hamilton-Jacobi Theory and Mathematical Physics

1.1 Time-Dependent Systems

Definition 1 *Let ω be an exterior two-form on M. Then*

$$R_\omega = \{v \in TM : \omega(v, \cdot) = 0\}$$

*is called the **characteristic bundle** of ω. A **Characteristic Vector Field** is a vector field X such that $i_X\omega = 0$.*

Proposition 1 *Let ω be a two-form on M of constant rank. Then R_ω is a subbundle of TM. If ω is closed, then R_ω is integrable as well.*

Theorem 1 (Darboux) *Let M be a $(2n+k)-$manifold and ω a closed two-form of constant rank $2n$. For each point, there is a neighborhood of that point such that ω takes the local form*

$$\omega \restriction_U = \sum dx^i \wedge dy^i$$

Definition 2 *A **contact manifold** is a pair (M, ω) consisting of an odd-dimensional manifold M and a closed two-form ω of maximal rank on M. An **exact contact manifold** (M, θ) consists of a $(2n+1)-$dimensional manifold M and a one-form θ on M such that $\theta \wedge (d\theta)^n$ is a volume on M.*

Note that the characteristic bundle R_ω of a contact form ω has one-dimensional fibers, so it is sometimes called the *characteristic line bundle*.

Theorem 2 *Let (M, ω) be a contact manifold. Then for each point there is a neighborhood of that point in which*

$$\omega \restriction_U = dq^i \wedge dp_i$$

Similarly, if (M, θ) is an exact contact manifold, there a chart of a neighborhood of every point such that

$$\theta \restriction_U = dt + p_i dq^i$$

Proposition 2 *Let θ be a nowhere zero oneform on a $(2n+1)-$manifold M and let $R_\theta = \{v \in TM : \theta(v) = 0\}$ be the characteristic line bundle. Then (M, θ) is an exact contact manifold ifff $d\theta$ is nondegenerate on the fibers of R_θ.*

Proposition 3 *Let (P, ω, H) be a Hamiltonian system and Σ_e a regular energy surface. Then $(\Sigma_e, i^*\omega)$ is a contact manifold, where $i : \Sigma \to P$ is the inclusion. Moreover, $X_H \restriction_{\Sigma_e}$ is a characteristic vector field of $i^*\omega$ generating the characteristic line bundle of $i^*\omega$.*

Proposition 4 *Let (P, ω) be a symplectic manifold, $\mathbb{R} \times P$ the product manifold. Let $\pi_2 : \mathbb{R} \times P \to P$ the projection onto P, and let $\tilde{\omega} = \pi_2^* \omega$. Then $(\mathbb{R} \times P, \tilde{\omega})$ is a contact manifold.*

The characteristic line bundle of $\tilde{\omega}$ if generated by the vector field $\underline{t}$ on $\mathbb{R} \times P$ is given by

$$\underline{t}(s, p) = ((s, 1), 0)$$

If $\omega = d\theta$ and $\tilde{\theta} = dt + \pi_2 \theta$ where $t : \mathbb{R} \times P \to \mathbb{R}$ the projection on the first factor, then $\tilde{\omega} = d\tilde{\theta}$ and $(\mathbb{R} \times P, \tilde{\theta})$ is an exact contact manifold.

For a time dependent vector field $X : \mathbb{R} \times M \to TM$, we can define $\tilde{X} : \mathbb{R} \times M \to T(\mathbb{R} \times M) \approx T\mathbb{R} \times TM$ by $\tilde{X}(t, m) = ((t, 1), (X(t, m))$ so that $\tilde{X} \in \mathfrak{X}(\mathbb{R} \times M)$ and that $\tilde{X} = \underline{t} + X$. We call $\tilde{X}$ the *suspension* of X, and its flow takes the form $F_{t,s} : \mathbb{R} \times M \to \mathbb{R} \times M$.

Definition 3 *Let (P, ω) be a symplectic manifold and $H : \mathbb{R} \times P \to \mathbb{R}$ be smooth and for each $t \in \mathbb{R}$ define $H_t : P \to \mathbb{R}$; $p \mapsto H(t, p)$. Then let $X_H(t, p) = X_{H_t}(p)$ and define the suspension $\tilde{X_H}$ as above.*

Proposition 5

$$\mathcal{L}_{\tilde{X}_H} H = \frac{\partial H}{\partial t}$$

Theorem 3 *Let (P, ω) be a symplectic manifold and $H : \mathbb{R} \times P \to \mathbb{R}$ be smooth. Let $\tilde{\omega}$ be as above, and let*

$$\omega_H = \tilde{\omega} + dH \wedge dt$$

Then

1. *$(\mathbb{R} \times P)$ is a contact manifold*

2. *$\tilde{X}_H$ generates the line bundle of ω_H; in fact, $\tilde{X}_H$ is the unique vector field satisfying*

   $$i_{\tilde{X}_H} \omega_H = 0 \text{ and } i_{\tilde{X}_H} dt = 1$$

   Moreover, if F is the flow of X_H, then $F^* \omega = \tilde{\omega} - dH \wedge dt$.

3. *if $\omega = -d\theta$ and $\theta_H = \pi_2^* \theta - H dt$, then $\omega_H = -d\theta_H$; if $H + (\theta \circ \pi_2)(X_H)$ is nowhere zero, then $(\mathbb{R} \times P, \theta_H)$ is an exact contact manifold.*

Theorem 4 *Let (P, ω) be a symplectic manifold, H a Hamitonian function and ω_H be its associated contact form. Then:*

1. *$\omega_H, \omega_H^2, \ldots, \omega_H^n$ are invariant forms of $\tilde{X}_H$.*

2. *$dt \wedge \omega_H^n = dt \wedge \tilde{\omega}^n$ is an invariant volume element for $\tilde{X}_H$.*

1.2 Canonical Transformations and Hamilton-Jacobi Theory

Proposition 6 *Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds, $P_1 \times P_2$ the product with projection maps π_i, and*

$$\Omega = \pi_1^* \omega_1 - \pi_2^* \omega_2$$

Then:

1. *Ω is a symplectic form on $P_1 \times P_2$*
2. *a map $f : P_1 \to P_2$ is symplectic iff $i_f^* \Omega = 0$, where $i_f : \Gamma_f \to P_1 \times P_2$ is the inclusion and Γ_f is the graph of f.*

Definition 4 *Suppose we define a local form Θ such that $\Omega = -d\Theta$ ($\Theta = \pi_1^* \theta_1 - \pi_2^* \theta_2$ works, but is not the only choice). Thus $i_f^* d\Theta = d i_f^* \Theta = 0$, that is, $i_f^* \Theta$ is closed is equivalent to f being symplectic. Locally, $i_f^* \Theta = -dS$ for a function $S : \Gamma_f \to \mathbb{R}$.*

Theorem 5 *Let $P = T^*Q$ with the canonical symplectic structure. Let X_H be a given Hamiltonian vector field on P, and let $S : Q \to \mathbb{R}$. Then TFAE:*

1. *A curve $c(t)$ satisfying*

   $$c'(t) = T\pi_Q^* X_H \left(dS(c(t)) \right)$$

   has the property that the curve $t \mapsto dS(c(t))$ is an integral curve of X_H
2. *S satisfies the Hamilton-Jacobi Equation:*

   $$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E$$

Definition 5 *Let $(P_i, \omega_i), i = 1, 2$ be symplectic manifolds and $(\mathbb{R} \times P_i, \tilde{\omega}_i)$ the corresponding contact manifolds. A smooth mapping $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ is called a canonical transformation if the following hold:*

- *C1 F is a diffeomorphism*
- *C2 F preserves time, that is $F^* t = t$*
- *C3 There is function $K_F : \mathbb{R} \times P_1$ such that $F^* \tilde{\omega}_2 = \omega_{K_f}$, where $\omega_{K_f} = \tilde{\omega}_1 + dK_F \wedge dt$*

Proposition 7 *The set of all canonical transformations on $(\mathbb{R} \times P, \tilde{\omega})$ forms a group under composition.*

Definition 6 *Let $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ be a smooth mapping satisfying (C1). Then F is said to have property (S) iff $F_t : P \to P$ is symplectic for each $t \in \mathbb{R}$.*

Proposition 8 *A mapping $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ has property (S) iff there is a one form α on $\mathbb{R} \times P$ such that $F^* \tilde{\omega_2} = \tilde{\omega_1} + \alpha \wedge dt$.*

Proposition 9 *(C3) $\Rightarrow$ (S). Take $\alpha = dK_F$. In the case where the symplectic forms ω_i are exact, $\omega_i = -d\theta_i$, (C3) is clearly equivalent to :*

(C4) There is a K_F such that $F^*\tilde{\theta_2} - \theta_{K_F}$ is closed, where, as usual,

$$\tilde{\theta_i} dt + \pi_2^* \theta_i$$

and

$$\theta_{K_F} = \tilde{\theta_1} - K_F dt$$

Proposition 10 *Suppose $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ satisfies (C2). Then (C3) is equivalent to the following:*

C5 For all $H \in \mathfrak{F}(\mathbb{R} \times P_2)$ there is a $K \in \mathfrak{F}(\mathbb{R} \times P_1)$ such that

$$F^* \omega_H = \omega_K$$

Proposition 11 *Let $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ satisfy (C1) and (C2). Then (C3) is equivalent to each of the following.*

C6 (S) holds and, for all $H \in \mathfrak{F}(\mathbb{R} \times P_2)$, there is a $K \in \mathfrak{F}(\mathbb{R} \times P_1)$ such that $F^* \tilde{X}_H = \tilde{X}_K$.

C7 (S) holds, and there is a function $K_F \in \mathcal{F}(\mathbb{R} \times P_1)$ such that $F^* \underline{t} = X_{K_F}$.

Theorem 6 (Jacobi) *If $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ satisfies (C1) and (C2), then (C3) is equivalent to the following:*

C8 There is a function $K_F \in \mathfrak{F}(\mathbb{R} \times P_1)$ such that for all $H \in \mathfrak{F}(\mathbb{R} \times P_2)$, $F^* \tilde{X}_H = \tilde{X}_K$, where $K = H \circ F + K_F$.

Definition 7 *Let F be canonical and locally write $\omega_1 = -d\theta_1$, $\omega_2 = -d\theta_2$, and so on as in (C4). Then if we locally write*

$$F^* \tilde{\theta_2} - \theta_{K_F} = dW$$

for $W : \mathbb{R} \times P_1 \to \mathbb{R}$, we call W a generating function for F.

Proposition 12 *If F is canonical and has generating function W, then*

$$K_F = \partial W / \partial t = \dot{F}$$

and thus for a Hamiltonian function H on $\mathbb{R} \times P_2$,

$$F^* \tilde{X}_H = \tilde{X}_K$$

where

$$K = H \circ F + (\partial W / \partial t) - \dot{F}$$

Definition 8 *Let $F : \mathbb{R} \times P_1 \to \mathbb{R} \times P_2$ be a canonical transformation and $H \in \mathfrak{F}(\mathbb{R} \times P_2)$. We say that F **transforms** H to **equilibrium** if $K = H \circ F + K_F =$ constant.*

Definition 9 *Let (P, ω) be a symplectic manifold $H \in \mathfrak{P}$ a Hamiltonian, and $f_1 (= H), f_2, \ldots, f_k$ constants of the motion (i.e. $\{f_i, H\} = 0$ for each i). The set is said to be in involution if $\{f_i, f_j\} = 0$. The set of f_i are said to be independdent if the set of critical points of $F = f_1 \times \ldots \times f_k$ has measure zero in P. A set of constants of the motion is called **integrable** if k is half the dimension of P.*

Theorem 7 *Let (P, ω) be a symplectic manifold, $H \in \mathfrak{F}(P)$ a Hamiltonian, and f_i an independent, integrable system of constants of motion. Denote by $F = f_1 \times \ldots \times f_k : P \to \mathbb{R}^n$ and let $U \subset \mathbb{R}^n$ be an open set such that $F^{-1}(U) \cap \sigma(F) = \emptyset$.*

1. *If $F \restriction F^{-1}(U) : F^{-1}(U) \to U$ is a proper map, then each of $X_{f_i} \restriction F^{-1}(U)$ is complete, $U \subset \mathbb{R}^n\ \Sigma(F)$ and the fibers of the locally trivial fibration $F \restriction F^{-1}(U)$ are a disjoint union of manifolds diffeomorphic with the torus $\mathbb{T}^n$.*

2. *If $F \restriction F^{-1}(U) : F^{-1}(U) \to U$ is not proper, but we assume $X_{f_i} \restriction F^{-1}(U)$ is complete and $U \subset \mathbb{R}^n\ \Sigma(F)$, then each fiber of $F \restriction F^{-1}(U)$ is a disjoint union of manifolds diffeomorphic to the cylinders $\mathbb{R}^k \times \mathbb{T}^{n-k}$.*

Definition 10 *Let $\vec{v} \in \mathbb{R}^n$ be a fixed vector and consider the flow $F_t : \mathbb{R}^n \to \mathbb{R}^n$ by $F_t(\vec{w}) = \vec{w} + t\vec{v}$. Denote the canonical projection $\pi : \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{T}^{n-k}$ and let $\phi_t : \mathbb{R}^k \times \mathbb{T}^{n-k} \to \mathbb{R}^k \times \mathbb{T}^{n-k}$ be the unique flow satisfying $\pi \circ F_t = \phi_t \circ \pi$. ϕ_t is called a **translation-type flow**.*

When $k = 0$, the flow is called *quasi-periodic*. Then the numbers $v_i = \vec{v} \cdot \vec{e}_i$ are called the *frequencies of the flow* and they determine completely its character, as will be seen in the next proposition.

Proposition 13 *Each orbit of ϕ_t is dense in $\mathbb{T}^n$ if and only if $\{v_i\}$ are linearly independent over $\mathbb{Z}$.*

Theorem 8 *If I_c^0 denotes a connected component of $I_c = F^{-1}(c)$ and $\phi_t = \phi_t^1$ denotes the flow of $X_H = X_{f_1}$, then $\phi_t \restriction I_c^0$ is smoothly conjugate to a translation type flow on $\mathbb{R}^k \times \mathbb{T}^{n-k}$.*

Definition 11 *A Hamiltonian $H \in \mathfrak{F}(P)$ on a symplectic manifold (P, ω) **admits action angle coordinates** (I, ϕ) in some open set $U \subset P$ if:*

1. *there exists a symplectic diffeomorphism $\psi : U \to B^n \times \mathbb{T}^n$*

2. *$H \circ \psi^{-1} \in \mathfrak{F}(B^n \times \mathbb{T}^n)$ admits "action-angle coordinates" in $B^n \times \mathbb{T}^n$, that is, the Hamiltonian vector field $\psi_* X_H$ has the form*

$$\psi_* X_H = -\sum \frac{\partial (H \circ \psi^{-1})}{\partial I} \frac{\partial}{\partial \phi}$$

We will now show a quick way to construct action-angle coordinates based on argument from Arnold. Suppose the following: Suppose we work in an open subset of a symplectic manifold (P, ω) with a given Hamiltonian function H and n independent integrals of motion in involution $f_1, \dots, f_n$. Let Σ_F be the bifurcation set of $F = f_1 \times \dots \times f_n$, and $U \subset \mathbb{R}^n \backslash \Sigma_F$, and that $F^{-1}(U)$ is diffeomorphic to $U \times \mathbb{T}^n$.

We shall construct the symplectic diffeomorphism $\psi : F^{-1}(U) \to B^n \times \mathbb{T}^n$. Locally, the symplectic form is exact ($\omega = -d\theta;\ \theta = \sum p_i dq^i$), and the preimage of a state specified by its integrals of motion, $I_c = F^{-1}(c) \approx \mathbb{T}^n$. Denote by $\gamma_i(c)$ the single loops in each S^1 factor of $\mathbb{T}^n$, then define $\lambda : U \to \mathbb{R}^n$ by

$$\lambda_i(c) = \oint_{\gamma_i(c)} i_c^*(\theta)$$

Where $i_c : I_c \to P$ is the inclusion. Assume λ is a diffeomorphism onto its image. We can shrink U until $\lambda(U) \subset B^n$. This gives us the B^n half of $\psi : F^{-1}(U) \to \mathbb{T}^n$.

Now we look for a map Γ such that $(\lambda \circ F) \times \Gamma : F^{-1}(U) \to B^n \times \mathbb{T}^n$ is a diffeomorphism; i.e. look for the 'angle coordinates.' The first step is to show $i_c^*(\theta)$ is closed. We first note that because the f_i are in independent integrals in involution, the vector fields X_{f_i} form a basis for the tangent space at every point of U. So all we need to show is that

$$di_c^*(\theta)(X_{f_i}, X_{f_j} = 0$$

But this is clear since

$$di_c^*(\theta)(X_{f_i}, X_{f_j} = -i_c^*(\omega)(X_{f_i}, X_{f_j}) = \{f_i, f_j\} \circ i_c$$

Since the matrix df_i/dp_j has nonzero determinant, we can solve the equation $F(\vec{q}, \vec{p}) - \lambda^{-1}(\vec{I}) = 0$ can be solved for $\vec{p}$. We now define

$$S(\vec{q}, \vec{I}) = \int_{(\vec{q_0}, \vec{p_0})}^{(\vec{q}, \vec{p})} i^*_{\lambda^{-1}(\vec{I})}(\theta)$$

Where the integral is taken over any path lying in the torus $I_{\lambda^{-1}(\vec{I}}$. Define the map $\Gamma : F^{-1}(U) \to \mathbb{T}^n$ by

$$\Gamma_i(\vec{q}, \vec{p}) = \left. \frac{\partial S}{\partial I_i} \right|_{\vec{I} = (\lambda \circ F)(\vec{q}, \vec{p}}$$

The Γ_i are multi-valued functions, as we want for angular variables. The variation of Γ_i on each fundamental cycle of the torus is given by

$$\oint_{\gamma_k(\lambda^{-1}(\vec{I}} d(\Gamma_i \circ i_{\lambda^{-1}(\vec{I}} = \oint_{\gamma_k(\lambda^{-1}(\vec{I})} d\left(\frac{\partial S}{\partial I^i} \circ i_{\lambda^{-1}(\vec{I})}\right)$$
$$= \frac{\partial}{\partial I^i} \int_{\gamma_k(\lambda^{-1}(\vec{I}))} dS = \frac{\partial}{\partial I^i} \int_{\gamma_k(\lambda^{-1}(\vec{I})} i^*_{\lambda^{-1}(\vec{I})}(\theta) = \frac{\partial I^k}{\partial I_j}$$

Note that S is a generating function of the map $\psi : (\vec{q}, \vec{p}) \to (\vec{I}, \varphi)$.

1.3 Lagrangian Submanifolds