Notes from *Geometry of Infinite-Dimensional Groups*

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Contents

0.1 Preliminaries

0.1.1 Lie Groups and Lie Algebras

Lie Groups and an Infinite-Dimensional Setting

Definition 1. *A nonempty collection G of transformations of some set is called a group if along with every two transformations $g, h \in G$ belonging to the collection, the composition $g \circ h$ and the inverse transformation g^{-1} belong to the same collection G.*

Definition 2. *A Lie Group is a smooth manifold G with a group structure such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps.*

Definition 3. *Let V, W be Frèchet spaces, i.e. complete locally convex Hausdorff metrizable vector spaces, and let U be an open subset of V. A map $f : U \subset V \to W$ is said to be differentiable at a point $u \in U$ in a direction $v \in V$ if the limit*

$$Df(u; v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} \tag{1}$$

exists. The function f is said to be continuously differentiable on U if the limit exists for all $u \in U$ and all $v \in V$, and if the function $Df : U \times V \to W$ is continuous as a function on $U \times V$. Similarly, we can build a function $D^2 f : U \times V \times V \to W$, and so on. A function f is called smooth or C^∞ if all its derivatives exist and are continuous.

Definition 4. *A Frèchet manifold is a Hausdorff space with a coordinate atlas taking value in a Frèchet space such that all transition functions are smooth maps.*

Remark 1. *Now one can start defining vector fields, tangent sapces, differential forms, principal bundles, and the like on a Freèchet manifold exactly like finite-dimensional manifolds.*

For example, for a manifold M, a tangent vector at some point $m \in M$ is defined as an equivalence class of smooth parametrized curves $f : \mathbb{R} \to M$ such that $f(0) = m$. The set of all such equivalence classes is the tangent space of $T_m M$ at m. The union of tangent spaces $T_m M$ for all $m \in M$ can be give the structure of a Freèchet manifold TM, the tangent bundle of M. Now a smooth vector field is smooth map $v : M \to TM$, and one defines in a similar vien the directional derivative of a function and the Lie bracket of two vector fields.

Sinc the dual of a Frèchet space need not be Frèchet, we define 1-forms directly, as smooth maps $\alpha : TM \to \mathbb{R}$ such that the restriction for any $m \in M$, $\alpha \upharpoonright_{T_m M}$ is a linear map. Higher rank tensor bundles follow similar constructions. The directional derivative of functions generalizes through an axiomatic description to the exterior derivative.

0.1.2 The Lie Algebra of A Lie Group

Definition 5. *Let G be a Lie Group with identity elements e. The tangent space to the group G at e is the vector space component to the Lie algebra of G, $\mathfrak{g}$. The group multiplication on G endows its Lie algebra $\mathfrak{g}$ with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ to be descibed as follows.*

Notice that $\mathfrak{g}$ can be identifies with the space of left-invariant vector fields on G. Thus there is a map $X \in \mathfrak{g} \mapsto \tilde{X} \in \mathfrak{X}(M)$. We define the Lie bracket on the Lie algebra as the usual Lie bracket on the corresponding left-invariant vector field (The 'usual Lie bracket' is the vector field $[X, Y]$ such that $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$). The combination of vector space and Lie bracket gives the whole Lie algebra.

Remark 2. *The definition of Lie algebra given above imbues the Lie bracket with the following properties:*

1. *It is antisymmetric in its arguments, i.e. $[X, Y] = -[Y, X]$*

2. *It satisfies the Jacobi Identity*

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X]$$

0.1.3 The Exponential Map

Definition 6. *The exponential map from a Lie algebra to the corresponding Lie group $\exp : \mathfrak{g} \to G$ is defined as follows: $X \in \mathfrak{g}$ thought of as a left-invariant vector field, so there is an associated flow, $\phi_{X,t} : G \to G$. Then we define $\exp(X) = \phi_{X,1}(e)$.*

Remark 3. *The definition of the exponential map relies on the existence and uniqueness of first-order ordinary differential equations. In genral, solutions of differential equations in Frèchet spaces may not be unique. However, in the case of the exponential map, the particular differential equation has a unique solution.*

Remark 4. *The exponential map exists for all finite-dimensional Lie groups and for Lie groups modeled on Banach spaces, but there may exist infinite-dimensional Lie groups that do nto admit and exponential map. Moreover, even when the exponential map does exist, it may not be as well-behaved as well as one would like.*

0.1.4 Abstract Lie Algebras

The properties outlined about the Lie bracket can be taken as axioms for soe sort of 'abstract' Lie algebra.

Definition 7. *An (abstract) Lie algebra is a real or complex vector space $\mathfrak{g}$ together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (The Lie bracket) that is antisymmetric and satisfies the jacobi identity.*

Definition 8. *A map $\rho : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a Lie algebra homomorphsim if it satisfies $rho([X, Y]) = [\rho(X), \rho(Y)]$. A linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ is called a derivation if*

$$\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]$$

Notably, the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$; $Y \mapsto [X, Y]$ is a derivation by the Jacobi identity. A derivation that can be expressed as an adjoint action is called an inner derivation; otherwise, it is an outer derivation.

Definition 9. *A subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $X \in \mathfrak{g} \Rightarrow [X, \mathfrak{h}] \subset \mathfrak{h}$. The importance of ideals comes from the fact that the quotient $\mathfrak{g}/\mathfrak{h}$ is again a Lie algebra.*

Definition 10. *A Lie algebra is simple (respectively, semisimple) if it does not contain any nontrivial ideals (nontrivial abelian ideals).*

The group analogue of an ideal is a normal subgroup.

0.1.5   Adjoint and Coadjoint Orbits

The Adjoint Representation

A *representation* of a Lie group G on a vector space is a linear action ϕ of the group G on V that is smooth. If V is real, (V, ϕ) is called a real representation, and if V is complex, it is a complex representation. Often V is a Frèchet space or a Hilbert space. If the action of G on V is contained in the group of unitary transformations on V, then the representation is said to be unitary.   Every Lie group has two distinguished representations: the adjoint and coadjoint representations.

Definition 11. *The map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ defines a representation of the group G on the space $\mathfrak{g}$ and is called the group adjoint representation The orbits of the group action are called the adjoint orbits of G. The differential of $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at e defines a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, called the adjoint representation of the Lie algebra $\mathfrak{g}$.*

0.1.6   The Coadjoint Representation

The dual object to the adjoint representation representation of a Lie group G on its Lie algebra $\mathfrak{g}$ is called the coadjoint representation of G on $\mathfrak{g}^*$, the dual space to $\mathfrak{g}$.

Definition 12. *The coadjoint representation Ad^* of the group G on the space $\mathfrak{g}^*$ is the dual of the adjoint representation. Let $\langle \cdot, \cdot \rangle$ denote the pairing of elements from $\mathfrak{g}$ and $\mathfrak{g}^*$; i.e., $\langle \xi, X \rangle = \xi(X)$. Then the coadjoint action of the group G*

on the dual space $\mathfrak{g}^*$ *is given by the operators* $\mathrm{Ad}^*_G : \mathfrak{g}^* \to \mathfrak{g}^*$ *for any* $g \in G$ *that are defined the relation*

$$\langle \mathrm{Ad}^*_g(\xi), X \rangle = \langle \xi, \mathrm{Ad}^*_{g^{-1}}(X) \rangle$$

The orbits of the group action on $\mathfrak{g}^*$ are called the coadjoint orbits of G.

The differential $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ of the group representation Ad^* at the identity e is called the coadjoint representation of the Lie algebra $\mathfrak{g}$. It is given by the relation

$$\langle \mathrm{ad}^*_Z(\xi), X \rangle = -\langle \xi, \mathrm{ad}_Z(X) \rangle$$

Remark 5. *The dual space of a Frèchet space is not necessarily again a Frèchet space. In this case, instaed of considering the full dual to an infinite-dimensional Lie algebra* $\mathfrak{g}$*, we will usually confine ourselves to considering only "smooth duals," the functionals from certain G-invariant Frèchet subspace* $\mathfrak{g}^*_s \subset \mathfrak{g}^*$*. Natural smooth duals will be different according to the type of the infinited dimensional groups considered, but they all have a (weak) nondegenerate pairing with the corresponding Lie algebra* $\mathfrak{g}$ *in the following sense: for every nonzero* $X \in \mathfrak{g}$*, ther is a* $\xi \in \mathfrak{g}^*_s$ *such that* $\langle \xi, g \rangle \neq 0$*, and the other way around. This ensures that the coadjoint orbit is uniquely determined. The smooth dual space and the adjoint action is called the regular (or smooth) part of the coadjoint representation of G, and often we will omit the subscript s.*

0.1.7   Central Extension

In this section we collect several basic facts about central extensions of Lie groups and Lie algebras. A central extension of a Lie group as a new bigger Lie group $\tilde{G}$ over the initial group G in such a way that the fiber over the identity of G lies in the center of $\tilde{G}$.

Lie Algebra Central Extensions

Definition 13. *A central extension of a Lie algebra* $\mathfrak{g}$ *by a vector space* $\mathfrak{n}$ *is a Lie algebra* $\tilde{\mathfrak{g}}$ *whose underlying vector space* $Tilde\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{n}$ *is equipped with the following Lie bracket:*

$$[(X, u), (Y, v)] = ([X, Y], \omega(X, Y))$$

for some continuous map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{n}$*. To make this a bonafide Lie algebra, we require* ω *satisfy the cocycle identity*

$$\omega([X, Y], Z) + \omega([Z, X], Y) + \omega([Y, Z], X) = 0 \in \mathfrak{n}$$

A 2-cocycle ω *on* $\mathfrak{g}$ *is called a 2-coboundary if there exists a linear map* $\alpha : \mathfrak{g} \to \mathfrak{n}$ *such that* $\omega(X, Y) = \alpha([X, Y])$*. The central extension defined by such a 2-coboundary becomes he trivial extension by the zero cocycle after a change of basis* $(X, u) \mapsto (X, u - \alpha(X))$*.*

Hence in describing different central extensions we are interested in the 2-cocyles modulo the 2-coboundaries; i.e. $\omega \; \omega' \iff \omega - \omega'$ *is a 2-coboundary. The equivalence classes are called the second cohomology* $H^2(\mathfrak{g}; \mathfrak{n})$.

Remark 6. *A central extension of a Lie algebra* $\mathfrak{g}$ *by an abelian Lie algebra* $\mathfrak{n}$ *can be defined by the exact sequence*

$$\{0\} \to \mathfrak{n} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to \{0\}$$

Proposition 1. *There is a one-to-one correspondence between the equivalence classes of central extensions of* $\mathfrak{g}$ *by* $\mathfrak{n}$ *and the elements of* $H^2(\mathfrak{g}; \mathfrak{n})$.

Definition 14. *A central extension* $Tilde\mathfrak{g}$ *of* $\mathfrak{g}$ *is called universal if for any other central extension* $\tilde{\mathfrak{g}}'$, *there is a unique morphism* $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$ *of the central extensions.*

Remark 7. *A sufficient condition for a Lie algebra* $\mathfrak{g}$ *to have a universal central extension is that* $\mathfrak{g}$ *be perfect; i.e. that it coincide with its own derrived algebra* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. *Any finite-dimensional semisimple Lie algebra is perfect. The universal central extension of a semisimple Lie algebra* $\mathfrak{g}$ *coincide with itself: such algebras do not admit nontrivial central extensions.*

No abelian Lie algebra is perfect. Nevertheless, abelian Lie algebras can have universal central extensions.

Central Extensions of Lie Groups

Central extensions of Lie groups can be defined similarly to those of Lie algebras. However, unlike the case of Lie algebras, not all group extensions can be described explicitly by cocycles. So we have the following alternative definition:

Definition 15. *A central extension* $\tilde{G}$ *of a Lie group* G *by an abelian Lie Group* H *is an exact sequence of Lie groups*

$$\{e\} \to H \to \tilde{G} \to Gto\{e\}$$

such that the image of H *lies in the center of* $\tilde{G}$. *Morphisms are done how you would think.*

If the central extension $\tilde{G}$ is topologically a direct product of G and H, $\tilde{G} = G \times H$ (or equivalently, there is a smooth section in the principal H-bundle $\tilde{G} \to G$), one can define the multiplication in $\tilde{G}$ as follows:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, \gamma(g_1, g_2) h_1 h_2)$$

for a smooth map $\gamma : G \times G \to H$ (note the similarity to the Lie algebra definition). The associativity of this multiplication corresponds to the *group cocycle identity* on the map γ.

Definition 16. *Let G and H be Lie groups and suppose H is abelian. A smooth map $\gamma : G \times G \to H$ that satisfies*

$$\gamma(g_1 g_2, g_3)\gamma(g_1, g_2) = \gamma(g_1, g_2 g_3)\gamma(g_2, g_3)$$

is called a smooth group 2-cocycle on G with values in H.

     A smooth 2-cocycle on G with values in H is called a 2-coboundary is there exists a smooth map $\lambda : G \to H$ such that $\gamma(g_1, g_2) = \lambda(g_1)\lambda(g_2)\lambda(g_1 g_2)^{-1}$. We are also interested in the non-trivial extensions, which are those generated by 2-cocycles which are not 2-coboundaries.

Proposition 2. *There is a one-to-one correspondence between the set of central extensions of G by H that admit a smooth section and the elements in the second cohomology group $H^2(G, H)$.*

It is important to note that there are central extension of Lie groups that do not admit a smooth section, and hence cannot be defined by smooth cocycles.

     A central extension of a Lie group G always defines a central extension of the corresponding Lie algebra. The converse need not be true: the existence of a Lie group for a given Lie algebra is not guaranteed in infinite dimensions. Instead, one says that a central extension $\tilde{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ lifts to the group level if there are corresponding group central extensions with the appropriate Lie algebras. If the group central extension $\tilde{G}$ by H is defined by a group 2-cocycle γ, one can recover the 2-cocycle defining the central extension of the group cocycle by differentiation.

Proposition 3. *Let H be an abelian Lie group with Lie algebra $\mathfrak{h}$, and let γ be an H valued 2-cocycle on G defining a central extension $\tilde{G}$. Then the $\mathfrak{h}$-valued cocycle ω defining the corresponding central extension $\tilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ is given by*

$$\omega(X, Y) = \frac{d^2}{dt ds} \upharpoonright_{(s,t)=(0,0)} \gamma(g_t, h_s) \frac{d^2}{dt ds} \upharpoonright_{(s,t)=(0,0)} \gamma(h_s, g_t)$$

Where $g_0' = X, \ h_0' = Y$.

0.1.8   The Euler Equations for Lie Groups

The Euler equations form a class of dynamical systems closely related to Lie groups and to the geometry of their coadjoint orbits. To describe them, we start with generalities on Poisson structures and Hamiltonian systems, before bridging them to Lie groups. Note that we will use $\mathfrak{g}^*$ will be the *smooth* dual of $\mathfrak{g}$.

Poisson Structures on Manifolds

Definition 17. *A Poisson Structure on a manifold M is a bilinear operation on functions*

$$\{,\} : C^\infty(M) \to C^\infty(M)$$

satisfying the following properties:

1. *antisymmetry*

$$\{f, g\} = -\{g, f\}$$

2. *the Jacobi identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

3. *the Leibniz rule:*

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

The first two properties make a Poisson structure a Lie algebra with $\{,\}$ as the bracket on the vector space $C^\infty(M)$. The third propety identifies functions with vector fields, since every derivation on a manifold is a vector field.

Definition 18. *Let $H : M \to \mathbb{R}$ be any smooth function on a Poisson manifold M. Such a function defines a vector field ξ_H on M by $\mathcal{L}_{\xi_H} g = \{H, g\}$ for any smooth function g. The vector field ξ_H is called the Hamiltonian field corresponding to the Hamiltonian function H wiht respect to the Poisson bracket.*

We call a function $F : M \to \mathbb{R}$ a Casimir function on a Poisson manifold M if it generates a zero vector field; i.e., $\forall g \; F, g = 0$.

Remark 8. *Let M be a manifold with a Poisson structure $\{,\}$ and we fix some point $m \in M$. All Hamiltonian vector fields on M evaluated at the point m spans a subspace of the tangent space $T_m M$. Thus, the Poisson structure defines a distribution of suc subspaces on the manifold by varying the points m. This distribution is integrable, since the commutator of two Hamiltonian vector fields is again Hamiltonian. Therefore, we have a foliation of the Poisson manifold M.*

The leaves of this foliation are called symplectic leaves. This means two points are on the same symplectic leaf if there is a Hamiltonian flow taking on to the other.

Definition 19. *A pair (N, ω) consisting of a closed manifold N and a 2-form ω on N is called a symplectic manifold if ω is closed and nondegenerate. (Weak nondegeneracy in infinite dimension if $\forall X \exists Y \; \omega(X, Y) \neq 0$). The 2-form ω is called the symplectic form on the manifold N.*

Remark 9. *Locally, a Poisson manifold near any point p splits into the product of a symplectic space and Poisson manifold whose rank at p is zero. The symplectic space is aneighborhood of the symplectic leaf passing through p while the Poisson manifold of zero rank represents the transverse Poisson structure at p.*

Below we will see that this splitting works in many (but not all!) infinite-dimensional examples: Poisson structures can have infinite-dimensional symplectic leaves and finite-dimensional Poisson transversals.

Hamiltonian Equations on the Dual of a Lie Algebra

Let G be a Lie group (finite- or infinite-dimensional) with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ denote the (smooth part of) its dual.

Definition 20. *The natural Lie-Poisson structure $\{,\}_{LP}$ on the dual Lie algebra $\mathfrak{g}^*$,*

$$\{,\}_{LP} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$$

is defined via

$$\{f, g\}_{LP}(m) = \langle [df_m, dg_m], m \rangle$$

Proposition 4. *The Hamiltonian equation corresponding to a function H and the natural Lie-Poisson structure $\{,\}_{LP}$ on $\mathfrak{g}^*$ is given by*

$$\frac{d}{dt} m(t) = -\mathrm{ad}^*_{dH_{m(t)}} m(t)$$

This equation is called the Euler-Poisson Equation on $\mathfrak{g}^*$.

Corollary 1. *The symplectic leaves of $\{,\}_{LP}$ on $\mathfrak{g}^*$ are the coadjoint orbits of G. In particular, all (finite-dimensional) coadjoint orbits have even dimension.*

Corollary 2. *Let $A : \mathfrak{g} \to \mathfrak{g}^*$ be an invertible self-adjoint operator. For the quadratic Hamiltonian function $H : \mathfrak{g}^* \to \mathbb{R}$ defined by $H(m) = \frac{1}{2}\langle m, A^{-1}m \rangle$ the corresponding Hamiltonian equation is*

$$\frac{d}{dt} m(t) = -\mathrm{ad}^*_{A^{-1}m(t)} m(t)$$

Definition 21. *An invertible self-adjoint operator $A : \mathfrak{g} \to \mathfrak{g}^*$ defining the quadratic Hamiltonian H is called an inertia operator on $\mathfrak{g}$.*

A Riemannian Approach to the Euler Equations

Let $(,)$ be some left-invariant metric on the group G. The geodesic flow with respect to this metric is a dynamical system on the tangent bundle TG of the group G. We can pull back this system to the Lie algebra $\mathfrak{g}$ by left translation. Hence we get a dynamical system

$$\frac{d}{dt} v(t) = B(v(t))$$

on the Lie algebra $\mathfrak{g}$, where $B : \mathfrak{g} \to \mathfrak{g}$ is a (nonlinear) operator.

Definition 22. *The dynamical system on the Lie algebra $\mathfrak{g}$ describing the evolution of the velocity vector of a geodesic in a left-invariant metric on the Lie group G is called Euler (or Euler-Arnold) equation corresponding to the metric on G.*

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Observe that the metric $(,)_e$ at the identity defines a non-degenerate bilinear form on $\mathfrak{g}$, and therefore an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ by $A(v) = (v, \cdot)$. This identification allows us to write the Euler equations on the dual space $\mathfrak{g}^*$. We can identify $m = A(v)$ and write a Hamiltonian function $H(m) = \frac{1}{2}\langle m, A^{-1}m \rangle$.

Theorem 1. *For the left-invariant emtric on a group generated by an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, the Euler (or geodesic) equation assumes the form*

$$\frac{d}{dt}m(t) = -\mathrm{ad}^*_{A^{-1}m(t)}m(t)$$

Remark 10. *The underlying reason for the Riemannian reformulation is the fact taht an geodesic problem in Riemannian geometry can be described in terms of symplectic geometry. Geodesics on M are extremals of a quadratic Lagrangian on TM. They can also be described by the Hamiltonian flow on T^*M for the quadratic Hamiltonian function given by the Legendre Transform of the Lagrangian.*

Poisson Pairs and Bi-Hamiltonian Structures

A *first integral* (or a conservation law) for a vector field ξ on a manifold M is a function on M invariant under the flow of this field. In this section we will show that if the vector field ξ is a Hamiltonian vector field with respect to two different Poisson structures that are compatible in a certan sense, there is a way of constructing first integrals of such a field.

Definition 23. *Two Poisson structures $\{,\}_0$ and $\{,\}_1$ on a manifold M are said to be compatible or form a Poisson pair if for every $\lambda \in \mathbb{R}$ the linear combination $\{,\}_0 + \lambda\{,\}_1$ is again a Poisson bracket on M.*

A dynamical system $\frac{d}{dt}m = \xi(m)$ on M is called bi-Hamiltonian if the vector field ξ is Hamiltonian with respect to both $\{,\}_0$ and $\{,\}_1$.

The main example of a manifold admitting a Poisson pair is $\mathfrak{g}^*$. One is the usual Lie-Poisson bracket $\{,\}_{LP}$ and the other is a 'frozen' Lie-Poisson bracket.

Definition 24. *The constant Poisson bracket on $\mathfrak{g}^*$ associated to a point $m_0 \in \mathfrak{g}^*$ is the bracket $\{,\}_0$ defined by*

$$\{f, g\}_0(m) = \langle [df_m, dg_m], m_0 \rangle$$

The Poisson bracket $\{,\}_0$ depends on the *freezing point m_0*. The two brackets $\{,\}_{LP}$ and $\{,\}_0$ coincide at m_0. The symplectic leaves of the constant bracket $\{,\}_0$ are given by translations of the stangent space $T_{m_0}\mathcal{O}_{m_0}$ to the coadjoint orbit $\mathcal{O}_{m_0}$ thorugh the point m_0.

Lemma 1. *The Poisson brackets $\{,\}_{LP}$ and $\{,\}_0$ are compatible for every point $m_0 \in \mathfrak{g}^*$.*

Remark 11. *Explicitly, the Hamiltonian equation on $\mathfrak{g}^*$ with the Hamiltonian function F and computed with respect to the frozen bracket is*

$$\frac{dm}{dt} = -\mathrm{ad}^*_{dF_m} m_0$$

Going back to the general situation, let $\{,\}_0$ and $\{,\}_1$ be a Poisson pair on a manifold M. One can generate a bi-Hamiltonian dynamical systems by producing a sequence of Hamiltonians in involution, according to the follow *Lenard-Magri scheme.* Consider $\{,\}_\lambda = \{,\}_0 + \lambda\{,\}_1$ for any λ. Let h_λ be a *Casimir function* on M for this bracket (a Casimir function generates the zero vector field). Furthermore, suppose h_λ can be expanded in a power series of λ, so that we can write

$$h_\lambda = \sum_i \lambda^i h_i$$

where each coefficient h_i is a smooth function on M. We can generate Hamiltonian vector fields ξ_i on M by using the new Poisson bracket.

Theorem 2. *The function h_i are Hamiltonians of a hierarchy of bi-Hamiltonian systems. In other words,*

$$\{h_i, f\}_1 = \mathcal{L}_{\xi_i} f = -\{h_{i+1}, f\}_1$$

And the other functions are constants of motion:

$$\{h_i, h_j\}_k = 0$$

For $k = 0, 1$.

Remark 12. *The fact that $\{h_i, h_j\}_k = 0$ means the functions h_j are first integrals of the vector fields ξ_i. So if the function h_j are independent, the previous theorem gives an infinite list of first integrals of motion for ξ_i.*

Integrable Systems and the Liouville-Arnold Theorem

Definition 25. *A Hamiltonian system on a symplectic $2n$-dimensional manifold M is calle (completely) integrable if it has n integrals in involution that are functionally independent almost everywhere on on M. The Hamiltonian function is one of the first integrals.*

Theorem 3. *For a compact manifold M, connected components of noncritical common level set M_c of the first n integrals are n-dimensional tori, while the Hamiltonaian system defines a (quasi)-periodic motionon each of them. In a neighborhood of such a component in M there are coordinates $(\phi_1, \ldots, \phi_n, I_1, \ldots, I_n)$ where ϕ_i are angular coordinates (cyclic) along the tori and I_i are the first integrals, such that the dynamical system assume the form $\dot{\phi}_i = \Omega_i(I_1, \ldots, I_n)$, and the symplectic form is $\omega = \sum_i dI_i \wedge d\phi_i$.*

0.1.9   Symplectic Reduction

Hamiltonian Group Actions

Definition 26. *Consider a finite-dimensional symplectic manifold (M, ω). Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and suppose that the exponential map exists. If the group G acts smoothly on M, each element X of the Lie algebra $\mathfrak{g}$ defines a vector field ξ_X on the manifold M as an infinitesimal action of the group:*

$$\xi_X(m) = \frac{d}{dt}\exp(tX)m \restriction_{t=0}$$

The action of the group G on the manifold M is called symplectic if it leaves the symplectic form ω invariant.

Definition 27. *The action of a Lie Group G is called Hamiltonian if for every $X \in \mathfrak{g}$ there exists a globally-defined Hamiltonian function H_X that can be chosen in such a way that the map $\mathfrak{g} \to C^\infty(M)$, associating to X the corresponding Hamiltonian H_X, is a Lie algbra homomorphism:*

$$H_{[X,Y]} = \{H_X, H_Y\}$$

Definition 28. *Assume the action of G on M is Hamiltonian. Then the moment map is the map $\Phi : M \to \mathfrak{g}^*$ defined by*

$$H_X(m) = \langle \Phi(m), X \rangle$$

where $\langle, \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$

Symplectic Quotients

Theorem 4. *Suppose λ is a regular value of the moment map and suppose that $\Phi^{-1}(\lambda)/G_\lambda$ is a manifold (satisfied if G_λ is compact and acts freely on $\Phi^{-1}(\lambda)$). Then there exists a unique symplectic struture ω_λ on the reduced space $\Phi^{-1}(\lambda)/G_\lambda$ such that*

$$\iota^*\omega = \pi^*\omega_\lambda$$

Where ι is the embedding $\Phi^{-1}(\lambda) \to M$ and π is the projection $\Phi^{-1}(\lambda) \to \Phi^{-1}(\lambda)/G_\lambda$. The resulting manifold is called one of the following symplectic reduction, Hamiltonian, Marsden-Weinstein reduction or the sympectic quotient.

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