Simulation
Lecture 18 - Output Analysis for Steady-State Simulations

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Steady-State Simulations

- What do we mean by steady-state simulations?
- System starts and no definite end time for simulation
- Formal Definition:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_{\{X(s)=k\}}(s)ds = p_k > 0$$

- What is not a steady-state simulation?
- Consider a queueing system where
  - interarrivals are exponential with rate 6
  - services are exponential with rate 4

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Performance Measures for a Steady-State Simulations

- Average System Size

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Q(s)ds = \mu.$$

- Average Cycle Time

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} C_i = \mu.$$

- If we run a single replication until infinity we will get the real performance measure!!!

Performance Measures for a Steady-State Simulations

- We cannot run the system until infinity
- When should we terminate the simulation?
  - It may not have converged
- How should we start our simulation
  - Bias is introduced by the initial conditions

Initialization Bias

- Remedy to get rid of initialization bias:
  - Run the system until some time t_w
  - Reset the statistics collected
  - Calculate your statistics between $[t_w, t]$

$$X_t = \int_{t_w}^{t_w+t} X(s)ds.$$

- The time interval $[0, t_w]$ is called the warm up period
- How to set the warm up length?

Initialization Bias – Warm-up Period

- We can use moving averages
- Suppose you have a data sequence (time series)

$$Y_1, Y_2, Y_3, \cdots, Y_n, \cdots$$

- $Y_i(m) = \frac{\sum_{k=i}^{i+m} Y_k}{m}$
- Plot $Y_i(m)$ and see when it starts to fluctuate

Calculating the Confidence Interval – Truncated Replications

1. Perform n replications to run from time 0 to $t_w + t$.
2. For each replication i calculate $X_t^i = \int_{t_w}^{t_w+t} X(s)ds$.
3. Find
$$\bar{X} = \frac{\sum_{i=1}^{n} X_t^i}{n} \text{ and } s^2 = \frac{\sum_{i=1}^{n}(X_t^i - \bar{X})^2}{n-1}.$$
4. Construct the $1-\alpha \times 100\%$ confidence interval as
$$(\bar{X} - \frac{t_{n-1,1-\alpha/2}s}{\sqrt{n}}, \bar{X} + \frac{t_{n-1,1-\alpha/2}s}{\sqrt{n}})$$

- Too much data is discarded for warm-up!!!

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Batch-Means Method

- The idea is run a long replication and divide it into n parts
  1. Perform a single replication to run from time 0 to $t_w + nt$.
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Truncated Replications vs Batch Means

- The warm-up period is discarded *n* times in truncated replications
- The warm up period is discared only once
- The observations for truncated replications are independent
- The observations for batch means are mildly dependent (How can we reduce this dependency)
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