# Measuring the Fine Structure Constant via Compton Cross Section (Draft)*

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As a first-order test of quantum electrodynamics (QED), we measured the Compton cross section of photon-electron scattering by sending gamma radiation from a cesium-137 source through absorbers made of aluminum, carbon, copper, and lead, respectively, and detecting unscattered photons on the other side. By approximating the valence electrons in the cylinder as free electrons and taking into account of the photoelectric effect of high Z materials, the Compton cross section can be computed from the unscattered-photon count. Comparing this cross section with the Klein-Nishina formula, a first-order QED prediction, the inverse fine structure constant was extracted and found to be $1/\alpha = 141.7 \pm 1.0$, which is close to the well-known value of 137. This indirectly implies that QED passes our first-order test.

**Keywords:** Quantum Electrodynamics, Cross Section, Precision Measurement, Compton Effect, Photoelectric Effect

## I. INTRODUCTION

I Quantum electrodynamics (QED) remains the most successful theory of nature to date. It describes electrons, photons, and their interactions using the language of quantum fields [1]. The fine structure constant $\alpha$ is a fundamental parameter of QED that describes the strength of the coupling between the electron field and the photon gauge field. In terms of fundamental physical constants in SI unit, the fine structure constant is a unitless number given by

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}. \tag{1}$$

The fact that $\alpha$ is unitless and that its inverse is so close to an integer (it was believed to be exactly 137 at one point) historically led many physicists to believe that there must be some first-principle derivation of the number 137 [2]. Although it has lost some of its "magic" as more careful measurements [3] revealed that it is in fact not exactly the inverse of 137 but

$$\frac{1}{\alpha} = 137.035999046(27), \tag{2}$$

this number is still of significant importance for the following reason: due to the small value of $\alpha$, the expression of the cross section of any QED process can be Taylor expanded in powers of $\alpha$, and the first few terms often constitute a very good approximation.

Each of these Taylor expansion term can be represented by a corresponding Feynman diagram [1]. In practice, usually a few Feynman diagrams are drawn and are used to calculate cross section (using the known value of $\alpha$). This then becomes a prediction that can be verified experimentally as a test of the validity of the theory itself.

In this experiment, we made such a test of QED but with a slightly different point of view: we began by being agnostic of the value of $\alpha$ and assumed the validity of QED. We measured the cross section of Compton scattering and fit it to the expression predicted by QED. The fitted parameter is the value of $\alpha$ that is needed for QED to be correct. We finally compared this value with the famous $\frac{1}{137}$. This is logically equivalent to the usual test but operationally, our analysis treats $\alpha$ as the primary object of investigation, as opposed to cross section, primarily for educational purpose and a historical perspective of the subject.

Concretely, we follow the lab manual prepared by Bailey and Paul [4]. Consider the lowest order Feynman diagram for the scattering of a photon off an electron, as shown in Fig. 1. In this picture, the photon is absorbed

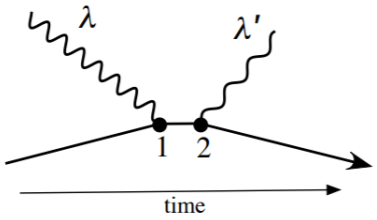


FIG. 1. Lowest order Feynman diagram for Compton scattering. $\lambda$ is the initial wavelength of the photon while $\lambda'$ is the final wavelength. Image taken from [4].

and re-emitted by the electron. One can think of this as the photon bouncing off the electron, with its final frequency altered due to energy-momentum transfer to the electron. This is known as the Compton effect [4].

Technically, the Compton effect only describes asymptotically free electrons. Here, we make the approximation that valence electrons of the atoms of a material are free electrons because their binding energy ( $\sim$ 1keV even for lead [5]) are much smaller than typical energy (6617 keV) of gamma radiation of cesium-137 [4]. Note that this approximation is better for low Z elements such as carbon than for high Z elements such as lead, but we will use the approximation for both cases.

We put a cesium-137 sample near an "absorber," which is a cylinder made of a pure element. Since the polarization of the gamma photons and the spins of the electrons in the material are unpolarized, we can average over dif-

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ferent spins. The Compton cross section of the Feynman diagram in Fig. 1, when averaged over spins, is given exactly by the Klein-Nishina formula (integrated over solid angle) [4],

$$\sigma = 2\pi r_e^2 \left\{ \frac{1+\gamma}{\gamma^2} \left[ \frac{2(1+\gamma)}{1+2\gamma} - \frac{1}{\gamma} \ln(1+2\gamma) \right] + \frac{1}{2\gamma} \ln(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2} \right\}, \tag{3}$$

where $\gamma$ is the lab-frame photon energy in unit of electron mass, which for the case of cesium-137 is given by (uncertainties negligible compared to other measurements)

$$\gamma = \frac{E_{\gamma,\text{lab}}}{m_e c^2} = \frac{0.6617 \text{ MeV}}{0.511 \text{ MeV}} = 1.295, \tag{4}$$

and $r_e$ is the classical electron radius

$$r_e = \frac{\hbar c}{m_e c^2} \alpha. \tag{5}$$

The Klein-Nishina formula gives the Compton cross section of a single photon interacting with a single electron. When a stream of photons pass through an absorber of width $x$, made of elements of atomic number $Z$, and with atomic number density $n$, one can use Eq. (3) to show [4] that the number of photons $I(x)$, over a fixed period of time $\Delta t$, that do not get scattered and pass straight through the absorber (and into a detector) is given by the equation

$$I(x) = I(0) e^{-n\sigma_{\text{tot}} x}, \tag{6}$$

where $\sigma_{\text{tot}}$ is the total cross section of various processes:

$$\sigma_{\text{tot}} = \sigma_{\text{pe}} + \sigma_{\text{pair}} + \sigma_{\text{elastic}} + Z\sigma. \tag{7}$$

Here, $\sigma_{\text{pe}}$ is the cross section of the photoelectric effect [4] given by

$$\sigma_{\text{pe}} = pZ^{4.2}, \tag{8}$$

where $p$ is an undetermined coefficient that we will measure. Note that due to the high power of this power law, the photoelectric cross section is much smaller for low Z elements than for high Z elements. This property will be crucial for us to experimentally extract the value of $p$.

$\sigma_{\text{pair}}$ is the cross section of pair production of electron-positron pairs. According to the famous formula of $E = mc^2$ and the conservation of energy, it can be easily shown that the energy in this experiment is insufficient to create electron-positron pairs, and $\sigma_{\text{pair}}$ is effectively zero [4].

$\sigma_{\text{elastic}}$ is the cross section of elastic scattering. This means that the initial photon and final photon have the same energy and frequency. This is distinct from the Compton scattering where the initial and final frequency are different, as depicted by Fig. 1. It can be assumed that this cross section is small in this experiment [4].

In effect, we only need to consider the total cross section

$$\sigma_{\text{tot}} = pZ^{4.2} + Z\sigma, \tag{9}$$

and for low Z element, even simpler,

$$\sigma_{\text{tot}} = Z\sigma. \tag{10}$$

The strategy of the experiment is to first measure the photon count $I(x)$ as a function of absorber width $x$ for a few low Z material (as well as measure their number density $n$). We then combine Eq. (6) and Eq. (10) to get data points of $\sigma$ as a function of $Z$. At this point, we can already calculate the $\alpha$'s that corresponds to these $\sigma$'s. But to take into account of the photoelectric effect, we fit the $(Z, \sigma_{\text{tot}})$ data points to Eq. (9) (where $\sigma$ is taken to be a constant, averaged over the low-Z cross section), which allows us to extract the photoelectric coefficient $p$. Armed with this number, we can get a better estimate of $\sigma$ and use Eq. (3) and Eq. (5) to extract the fine structure constant $\alpha$. Finally, we compare $1/\alpha$ with the known value published in the literature (Eq. (2)) as a test of the Klein-Nishina formula (Eq. (3)).

## II. APPARATUS

The experimental setup is shown in Fig. 2. To protect the whole experiment from external photons, we start with a rectangular box made of lead. We placed 5 $\mu$Ci of cesium-137 on one end of the box. Next to it, we put a cylindrical absorber made of a pure element. We used aluminum, carbon, copper, and lead, respectively. For each element, we made a few measurements with absorbers of various width $x$, which is labelled in Fig. 2. We also measured every cylinder's mass $m$ (using a digital balance) and diameter $d$ (using a caliper), for calculation of its atomic density $n$.

Following the path of a gamma photon, after passing the absorber it hits a lead collimator, which narrows the gamma radiation in the direction to the right on the diagram. Then a scintillator gives off light when struck by gamma photons and a photomultiplier tube further amplifies the signal. Finally, a SPECTECH USC30 spectrometer [6] is in placed to count the number of photons in the signal (not shown in Fig. 2). This detector is connected to a computer that has a USX software, which we used to analyze data.

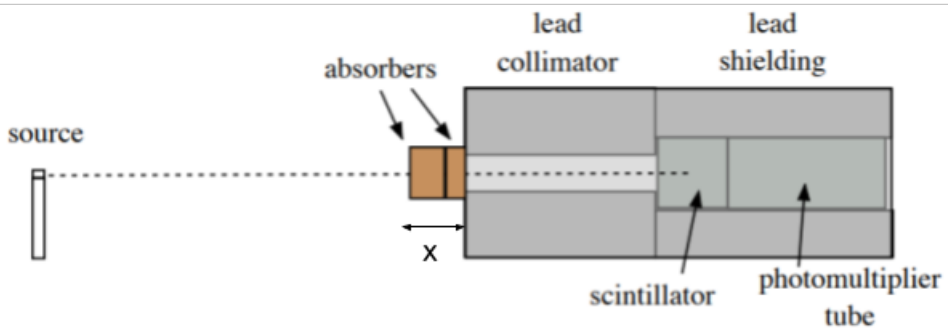


FIG. 2. Experimental setup. The whole apparatus is shielded inside a lead box (not shown). Image modified from [4].

## III. TECHNIQUE

### A. Time Dependence Convention

Since the photon count $I$ is directly proportional to the time elpased $\Delta t$, we made the convention of always using $\Delta t = 600$ s, so that in Eq. (6), dependence of $\Delta t$ cancel out. The USX software has a "real time" function that allows us to preset the amount of time data is collected.

### B. Noise Measurement

We first made a measurement of the photon count without radiation source and without absorber. This gives us a distribution of noise photon count as a function of energy (discretized by the detector to "channels"), as shown in FIG. 3. For all other measurements, we took

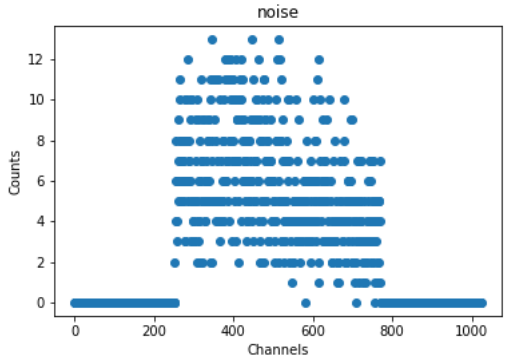


FIG. 3. The distribution of noise photon count as a function of energy over a 600 s time period.

into account of the noise by first subtracting the noise distribution (and taking absolute values to ensure $I(x) > 0$).

### C. Photon Count Measurements

We then made a series of measurements of the spectrum with radiation source and different absorbers. For each measurement, we get a (discrete) distribution of photon count $y(E)$ as a function of energy (or channel) $E$, just like the plot we got in FIG. 3. We now describe the general algorithm we used to convert each measurement to a photon count $I$.

#### 1. *Calculating I*

The first step is to perform the aforementioned noise subtraction. We then restrict the data to a channel subdomain that actually contains signal (note that most of the channels have zero or almost zero photon count, as in FIG. 2; including these data would make it hard to fit to a Gaussian).

After pre-processing the data, we fit the distribution of $y(E)$ to a shifted Gaussian

$$y(E) = a \exp\left(-\frac{(E-b)^2}{2c^2}\right) + d. \tag{11}$$

Ideally, while fitting, we should take into account of the counting error, which is $\sqrt{N}$ for any particular count of $N$ data points. However, in this case, we found that including such error made the least-square fit algorithm impossible to find a good fit to a Gaussian. So we decided to ignore the counting error. All is not lost, however, as we can still get some reasonable uncertainty $\Delta a, \Delta b, \Delta c, \Delta d$ of the parameters of fitted $a, b, c, d$ from their covariance matrix. A typical measurement with its fitted $y(E)$ is shown in FIG. 4.

Since the photon count $I$ is just the area under the distribution curve $y(E)$, we numerically integrate the fitted Gaussian

$$I = \int_{E_0}^{E_f} y(E)dE \tag{12}$$

$$= \int_{E_0}^{E_f} \left[a \exp\left(-\frac{(E-b)^2}{2c^2}\right) + d\right] dE, \tag{13}$$

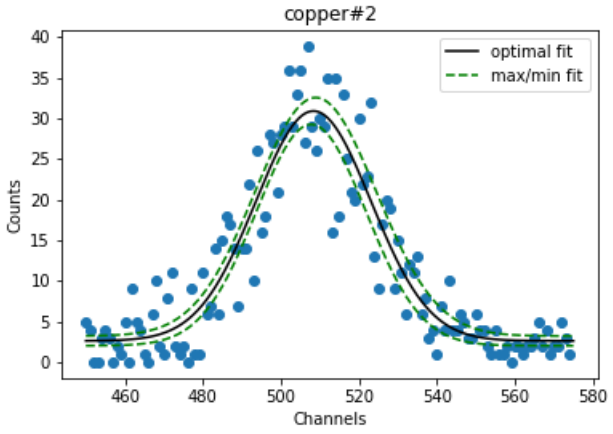


FIG. 4. The spectrum measurement for a copper absorber. The black curve is the least-square fitted Gaussian distribution $y(E)$, while the two green dotted curves are two alternative fits within one standard deviation for the fitted parameter with the maximum and minimum area underneath.

where the range $(E_0, E_f)$ is some domain of the channels where there are signal. Technically, we need to divide the above expression by $\frac{1}{E_f - E_0}\frac{1}{\Delta t}$. But since all of the data are integrated over the same energy range and were exposed for the same period of time $\Delta t$, these factors cancel in Eq. (6). So we shall ignore them to avoid increasing uncertainty.

#### 2. *Estimating* $\Delta I$

To find the uncertainty of $I$, we considered the $y(E)$ curves with the largest and smallest possible area that is still permitted by one standard deviation of the parameters $a, b, c, d$, as shown in FIG. 4. Concretely, we define [7]:

$$I_{\max} \equiv \int_{E_0}^{E_f} \left[(a+\Delta a)\exp\left(-\frac{(E-b)^2}{2(c+\Delta c)^2}\right) + (d+\Delta d)\right] dE$$

$$I_{\min} \equiv \int_{E_0}^{E_f} \left[(a-\Delta a)\exp\left(-\frac{(E-b)^2}{2(c-\Delta c)^2}\right) + (d-\Delta d)\right] dE.$$

Then we have an asymmetric error on $I$ [8]

$$I^{+(I_{\max}-I)}_{-(I-I_{\min})}. \tag{14}$$

However, in practice the $I$ is always very close to the middle of $I_{\max}$ and $I_{\min}$, as can be seen from FIG. 4. So we average the higher and lower error

$$\Delta I \equiv \frac{(I_{\max}-I)+(I-I_{\min})}{2} = \frac{I_{\max}-I_{\min}}{2}. \tag{15}$$

#### 3. *Measuring* $I(0)$

We repeated the above protocol for a variety of absorber material and width. We also made an additional measurement with radiation source but no absorber. This gives us $I(0)$, since having no absorber is equivalent to having an absorber with vanishing width $x = 0$.

### D. Atomic Number Density Measurements

For each cylindrical absorber, we measured its mass $m$ and diameter $d$. The volume is clearly given by

$$V = \pi\left(\frac{d}{2}\right)^2 x. \tag{16}$$

We can then find the atomic number density $n$ by

$$n = \frac{N}{V} = \left(\frac{m}{M}N_A\right)\frac{1}{V}, \tag{17}$$

where in the first equality, $N$ is the total number of atoms in the absorber, and $n$ is by definition the number of atoms divided by volume. In the second equality, $M$ is the molar mass of the element, which can be looked up in a periodic table, and $N_A$ is the Avogadro's constant. The second equality holds because the ratio between the mass and the molar mass tells us the number of mol present in the absorber. Since each mol corresponds to an Avogadro's constant worth of particles, multiplying by $N_A$ gives us $N$.

### E. Extracting Total Cross Section

Once we have computed $I(x)$ for a few values of $x$ for the same element, as well as having calculated $I(0)$ and $n$, we can fit them to Eq. (6) and extract $\sigma_{\text{tot}}$. An example of the plot of $I(x)/I(0)$ as a function of $x$ is shown in FIG. 5, as well as its least-square fit to a decaying exponential function.

### F. Extracting Compton Cross Section and $\alpha$

For the three low Z elements we used, namely aluminum, carbon, and copper, we computed the Compton cross section $\sigma$ by dividing $\sigma_{\text{tot}}$ by the atomic number $Z$, according to Eq. (10). As for lead, which has a high $Z = 82$, we fitted the collection of points $(Z, \sigma_{\text{tot}})$ to Eq. (9), where $\sigma$ is taken to be a constant, averaged over the low-Z cross section. We then extracted the photoelectric coefficient $p$. A plot of this fit is shown in FIG. 6.

Finally, with the value of $p$ in hand, we computed the Compton cross section $\sigma$ of lead using Eq. (9). Substituting into Eq. (3) and Eq. (5), we can calculate the fine structure constant $\alpha$.

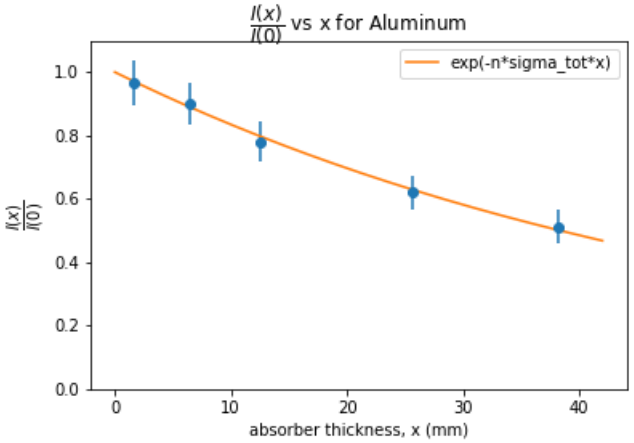


FIG. 5. The ratio of $I(x)/I(0)$ as a function of $x$ for Aluminum, fitted to a decaying exponential Eq. (6).

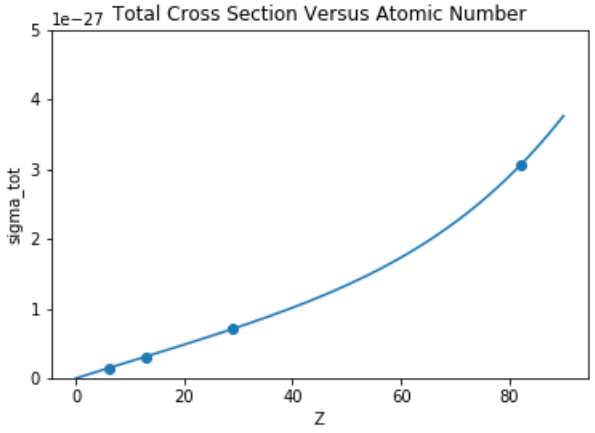


FIG. 6. The total cross section $\sigma_{\text{tot}}$ as a function of atomic number $Z$, including the photoelectric effect, as in Eq. (9).

## IV. RESULTS

The inverse fine structure constant, computed from measurements using various elements, is shown in TABLE I. The final average value of $141.7 \pm 1.0$ is close to, but distinct from, the well-known value of 137. Our experimental value is within 3% of the well-known value. So we conclude that QED has passed our test. The small discrepancy comes from various factors. First, we would not expect the answer to be exactly 137 since the Klein-Nishina formula is only for the first-order Feynman diagram. The true value of $\alpha$ corresponds to a calculation that comes from infinitely many Feynman diagram. Secondly, as explained in section I, We also made the approximation that the valence electrons are free electrons. We also discarded the elastic scattering cross section as well as the pair production cross section. Finally, as explained in Bailey's lab manual [4], there are also corrections that this experiment cannot take into account, such as forward scattering. In this experiment, we assumed all the photons that pass straight through the absorber must be unscattered. This is only an approximation, for photons can be scattered in the forward direction, in which case our cross section is being overestimated.

TABLE I. The results for the inverse fine structure constants using measurements of various elements.

| Element | $1/\alpha$ |
|---|---|
| Aluminum | $144.2 \pm 1.8$ |
| Carbon | $140.4 \pm 1.2$ |
| Copper | $140.5 \pm 0.6$ |
| Lead | $141.7 \pm 3.3$ |
| Average | $141.7 \pm 1.0$ |

## ACKNOWLEDGMENTS

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[6] Spectrum Techniques, UCS30 – universal computer spectrometer, `http://www.spectrumtechniques.com/products/instruments/ucs30-spectrometer/`.
[7] Based on experimental results, $\Delta b \approx 0$ for all cases.
[8] Due to the nature of the Gaussian, the two errors are in general different.