Dec 20.

Rereading Paper

I continue to read section 5 of the paper. I am completely convinced now that the formula of the action is correct.

Equation of Motion

I now verify the equation of motion, which is the equation I will solve for the rest of the project.

Claim 1. *The static equation of motion is*

$$\nabla^2 x^a = \frac{1}{4}\frac{\partial}{\partial(x^a)^*}\left|\frac{dW}{d\vec{x}}\right|^2 + 2\pi i\sum_j (\vec{c}_j)_a \partial_z \delta(z - z_j)\int_{y_j}^{\pm\infty} dy\delta(y - y') \tag{1}$$

Proof. The action is

$$S = \frac{g^2}{16\pi^2 L}\int dt\int dz\int dy\left(|\partial_0 x^a|^2 - |\partial_i x^a|^2 - \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2 - 4\pi\sum_j (\vec{c}_j)_a\partial_z\delta(z - z_j)\int_{y_j}^{\pm\infty} dy'\delta(y - y')Im(x^a(z, y))\right) \tag{2}$$

from which we can extract the Lagrangian density (we can discard the overall coefficient since it cancels out in the EL equation),

$$\mathscr{L} = |\partial_0 x^a|^2 - |\partial_i x^a|^2 - \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2 - 4\pi\sum_j (\vec{c}_j)_a\partial_z\delta(z - z_j)\int_{y_j}^{\pm\infty} dy'\delta(y - y')Im(x^a(z, y))$$

For static equation of motion,

$$\frac{\partial\mathscr{L}}{\partial(\partial_i x^{a*})} = \frac{\partial}{\partial(\partial_i x^{a*})}\left(-\partial_i x^b\partial_i x^{b*}\right) = -\partial_i x^b\delta_a^b = -\partial_i x^a$$

Hence,

$$\partial_i\frac{\partial\mathscr{L}}{\partial(\partial_i x^{a*})} = -\partial_i\partial_i x^a = -\nabla^2 x^a$$

On the other hand,

$$\frac{\partial\mathscr{L}}{\partial x^{a*}} = -\frac{1}{4}\frac{\partial}{\partial(x^a)^*}\left|\frac{dW}{dx^a}\right|^2 - 4\pi\sum_j (\vec{c}_j)_a\partial_z\delta(z - z_j)\int_{y_j}^{\pm\infty} dy'\delta(y - y')\frac{\partial}{\partial(x^a)^*}\frac{x^a - x^{a*}}{2i}$$

$$\frac{\partial\mathscr{L}}{\partial x^{a*}} = -\frac{1}{4}\frac{\partial}{\partial(x^a)^*}\left|\frac{dW}{dx^a}\right|^2 - \frac{2\pi}{i}\sum_j (\vec{c}_j)_a\partial_z\delta(z - z_j)\int_{y_j}^{\pm\infty} dy'\delta(y - y')(-1)$$

$$\frac{\partial \mathscr{L}}{\partial x^{a*}} = -\frac{1}{4}\frac{\partial}{\partial (x^a)^*}\left|\frac{dW}{dx^a}\right|^2 - 2\pi i \sum_j (\vec{c}_j)_a \partial_z \delta(z - z_j) \int_{y_j}^{\pm\infty} dy' \delta(y - y')$$

By The Euler-Lagrange equation,

$$\partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i x^{a*})} = \frac{\partial \mathscr{L}}{\partial x^{a*}}$$

Hence, we have,

$$-\nabla^2 x^a = -\frac{1}{4}\frac{\partial}{\partial (x^a)^*}\left|\frac{dW}{dx^a}\right|^2 - 2\pi i \sum_j (\vec{c}_j)_a \partial_z \delta(z - z_j) \int_{y_j}^{\pm\infty} dy' \delta(y - y')$$

$$\nabla^2 x^a = \frac{1}{4}\frac{\partial}{\partial (x^a)^*}\left|\frac{dW}{d\vec{x}}\right|^2 + 2\pi i \sum_j (\vec{c}_j)_a \partial_z \delta(z - z_j) \int_{y_j}^{\pm\infty} dy' \delta(y - y')$$

□

Derivative of Dirac Delta

Reading the next part of the paper, which was written by Andrew, there is a clever argument for the numerical implementation of derivative of 1 dimensional Dirac Delta, but it is exactly the same equation as what one would get by naive analogy with positive/negative infinite slope.

Let $g(x) = \partial_x \delta(x - x_i)$.

$$g(x_k) = \begin{cases} \frac{1}{h^2}, & k = i-1 \\ -\frac{1}{h^2}, & k = i \end{cases} = \frac{1}{h^2}(\delta_{k,i-1} - \delta_{k,i}) \tag{3}$$