Dec 19.

Start of Project

I finally begin my senior thesis project. I technically started last semester, but all of my time was spent on writing and publishing the summer work. My rule for myself: I have to write a diary entry for this project everyday, even if I did nothing. This way, I can stay engaged with the research and can clearly tell if I did not touch research for too long.

Goal of Project

The goal is to compute the confining string tension dependence on N-ality (p) and rank of gauge group (N).

GitHub

I created a public GitHub repository called Confining-String. I will copy some code from the summer project. But the overall structure will be distinct as I want to do things cleaner this time.

Rereading the Paper

As a refresher, my first step is to reread the paper and re-derive the Lagrangian and equation of motion I am trying to solve. I downloaded the paper from JHEP and started rereading section 5.

Source Part of Action

Consider the general case of n quarks, each with charge $\vec{c}_i$, with position $\vec{r}_i = (y_i, z_i)$, where $i = 1, \ldots, n$.

Claim 1. *The source part of the action is given by*

$$S_{source} = -\frac{g^2}{4\pi L}\int dt \int dz \int dy \sum_i (\vec{c}_i)_a \partial_z \delta(z - z_i) \int_{y_i}^{\pm\infty} dy' \delta(y - y')\sigma^a(z, y) \tag{1}$$

Proof. The action of external particle is the spacetime integral of the delta function of its worldline. Since the quarks are stationary, and the energy of a charged particle (just like in classical EM) is the charge times the voltage:

$$S_{source} = -\int dt \sum_i (\vec{c}_i)_a A_0^a(\vec{r}_i)$$

where a labels the $N-1$ fields.

Using the identity (where we assume the potential is zero at infinity),

$$A_0^a(\vec{r_i}) = -\int_{y_i}^{\pm\infty} dy \partial_y A_0^a(z_i, y)$$

and the definition of the field strength tensor,

$$\partial_y A_0 = F_{y0} \implies -\partial_y A_0 = F_{0y}$$

we have

$$A_0^a(\vec{r_i}) = \int_{y_i}^{\pm\infty} dy F_{0y}^a(z_i, y)$$

Next, use the definition of the dual photon

$$\frac{g^2}{4\pi L}\epsilon_{\mu\nu\lambda}\partial^\lambda \sigma^a = F_{\mu\nu}^a$$

This gives

$$F_{0y}^a = \frac{g^2}{4\pi L}\epsilon_{0yz}\partial^z \sigma^a$$

Take the convention that $\epsilon_{0yz} = 1$. Recall that bringing the spatial component down gives a negative sign. We have

$$F_{0y}^a = -\frac{g^2}{4\pi L}\partial_z \sigma^a$$

So the potential can be written as

$$A_0^a(\vec{r_i}) = -\frac{g^2}{4\pi L}\int_{y_i}^{\pm\infty} dy \partial_z \sigma^a$$

Subbing this back into the action,

$$S_{source} = \frac{g^2}{4\pi L}\int dt \sum_i (\vec{c_i})_a \int_{y_i}^{\pm\infty} dy \partial_z \sigma^a(z, y)|_{z=z_i}$$

We can rewrite the fact that $z = z_i$ by introducing an integral and a delta function

$$S_{source} = \frac{g^2}{4\pi L}\int dt \int dz \sum_i (\vec{c_i})_a \int_{y_i}^{\pm\infty} dy \partial_z \sigma^a(z, y)\delta(z - z_i)$$

Change the variable y to y':

$$S_{source} = \frac{g^2}{4\pi L}\int dt \int dz \sum_i (\vec{c_i})_a \int_{y_i}^{\pm\infty} dy' \partial_z \sigma^a(z, y')\delta(z - z_i)$$

Without changing anything, introduce an integral over y as well as a delta function that converts it back to y':

$$S_{source} = \frac{g^2}{4\pi L}\int dt \int dz \int dy \sum_i (\vec{c_i})_a \int_{y_i}^{\pm\infty} dy' \partial_z \sigma^a(z, y')\delta(z - z_i)\delta(y - y')$$

By the property of the delta function, we can drop the prime on the y' in the argument of dual photon

$$S_{source} = \frac{g^2}{4\pi L}\int dt \int dz \int dy \sum_i (\vec{c}_i)_a \int_{y_i}^{\pm\infty} dy' \partial_z \sigma^a(z,y)\delta(z-z_i)\delta(y-y')$$

Integrating by parts for z,

$$S_{source} = -\frac{g^2}{4\pi L}\int dt \int dz \int dy \sum_i (\vec{c}_i)_a \partial_z \delta(z-z_i) \int_{y_i}^{\pm\infty} dy' \delta(y-y')\sigma^a(z,y)$$

We have successfully written the source action in terms of spacetime integral. □

Full Action

Claim 2. *The kinetic and potential part of the action for the complex field is given by*

$$S_{kin,pot} = \frac{g^2}{16\pi^2 L}\int dt \int dz \int dy \left(|\partial_0 x^a|^2 - |\partial_i x^a|^2 - \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2\right) \tag{2}$$

such that the total action is

$$\begin{aligned} S = \frac{g^2}{16\pi^2 L}\int dt \int dz \int dy \{ &|\partial_0 x^a|^2 - |\partial_i x^a|^2 - \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2 \\ &-4\pi \sum_i (\vec{c}_i)_a \partial_z \delta(z-z_i) \int_{y_i}^{\pm\infty} dy' \delta(y-y') Im(x^a(z,y))\} \end{aligned} \tag{3}$$

Proof. The kintetic part of the action is just the usual absolute square of the complex field, under the usual Lorentz metrics. The overall constant in front of the kinetic and potential action, as well as the potential term, will both be justified later by consistency checks.

When we add the source part and kinetic and potential part of the action, it is clear that the coefficient differs by 4π, which is where the relative factor comes from. Also, in going from the real field to the complex field, if we only want to talk about the dual photon, which is the imaginary part, we have to refer to it in terms of the complex field, $x^a = \phi^a + i\sigma^a$, via

$$Im(x^a) = \frac{x^a - x^{a*}}{2i} = \sigma^a$$

Finally, the last step that remains is to check that the coefficient and the source terms are consistent with known special case. We will do two checks: the action has to agree with the energy of a one dimensional BPS soliton when we ignore one direction and set the charge to zero; if we set $\phi^a = 0$ and set the superpotential $W = 0$, and specialize to two quarks, we should get back the classical field action from Erich's note.

Test 1: If we set the charge to zero and forget the z direction, the action becomes

$$S = \frac{g^2}{16\pi^2 L}\int dt \int dy \left(|\partial_0 x^a|^2 - |\partial_i x^a|^2 - \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2\right)$$

Recall that action = time integral of (kinetic energy - potential energy). Therefore, we can extract kinetic energy + potential energy by changing the signs in front of terms with no time dependence and discarding the time integral to get

$$E = \frac{g^2}{16\pi^2 L}\int dy \left(|\partial_0 x^a|^2 + |\partial_i x^a|^2 + \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2 \right)$$

Taking the static solution (soliton), the energy is

$$E = \frac{g^2}{16\pi^2 L}\int_{-\infty}^{\infty} dy \left(\left|\frac{dx^a}{dy}\right|^2 + \frac{1}{4}\left|\frac{dW}{dx^a}\right|^2 \right)$$

Up to an overall factor, this is exactly the expression of the energy of a BPS soliton in equation (18.3) in "Mirror Symmetry" book.

Test 2: Set $W = 0$ and $\phi^a = 0$.

$$S = \frac{g^2}{16\pi^2 L}\int dt \int dz \int dy \left((\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 - 4\pi \sum_i (\vec{c}_i)_a \partial_z \delta(z - z_i) \int_{y_i}^{\pm\infty} dy'\delta(y - y')\sigma^a(z, y) \right)$$

Specialize to the case of two quarks (quark-antiquark pair) lying on the y-axis and a distance R apart, with opposite charges $\pm\vec{c}$.

$$S = \frac{g^2}{16\pi^2 L}\int dt \left(\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] + 4\pi \int dz \int dy \sum_i (\vec{c}_i)_a \delta(z - z_i) \int_{y_i}^{\pm\infty} dy'\delta(y - y')\partial_z\sigma^a(z, y) \right)$$

where we revert the integration by part of z. Since $z_1 = z_2 = 0$ for both quarks are on the y-axis,

$$S = \frac{g^2}{16\pi^2 L}\int dt \left(\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] + 4\pi \int dy \sum_i (\vec{c}_i)_a \int_{y_i}^{\pm\infty} dy'\delta(y - y')\partial_z\sigma^a(z, y)|_{z=0} \right)$$

Equating y and y' and taking out the delta function.

$$S = \frac{g^2}{16\pi^2 L}\int dt \left(\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] + 4\pi \sum_i (\vec{c}_i)_a \int_{y_i}^{\pm\infty} dy\partial_z\sigma^a(z, y)|_{z=0} \right)$$

Now let the negative quark be at the origin $y_1 = 0$, and the positive quark be at $y_2 = R$.

$$S = \frac{g^2}{16\pi^2 L}\int dt \left(\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] - 4\pi(\vec{c})_a \int_0^{\infty} dy\partial_z\sigma^a(z, y)|_{z=0} + 4\pi(\vec{c})_a \int_R^{\infty} dy\partial_z\sigma^a(z, y)|_{z=0} \right)$$

Interchanging lower and upper limit and combining integrals,

$$S = \frac{g^2}{16\pi^2 L}\int dt \left(\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] - 4\pi(\vec{c})_a \int_0^{\infty} dy\partial_z\sigma^a(z, y)|_{z=0} - 4\pi(\vec{c})_a \int_{\infty}^{R} dy\partial_z\sigma^a(z, y)|_{z=0} \right)$$

$$S = \frac{g^2}{16\pi^2 L}\int dt \left[\int dz \int dy \left[(\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2 \right] - 4\pi(\vec{c})_a \int_0^{R} dy\partial_z\sigma^a(z, y)|_{z=0} \right] \tag{4}$$

But this is exactly the same as the formula for action of quark anti-quark on page 18 of Erich's note. □