4 Duality: "electric" and "magnetic" formulation of (2+1)d EM

Take the gauge action and reduce to 3d, forgetting about A_3 (below we have $i, j = 1, 2$, the sum is over the Cartan indices only, and F_{0i}, F_{ij} do not have commutator terms in them, all the signs related to the metric have been taken into account, and repeated indices are summed over):

$$S_g = \sum_{a=1}^{N-1} \int dt \left[\int d^2x \left(\frac{L}{2g^2}(F_{0i}^a)^2 - \frac{L}{4g^2}(F_{ij}^a)^2 \right) - \frac{1}{\sqrt{2}}(\nu^B)_a (A_0^a(0,0) - A_0^a(R,0)) \right] , \tag{4.1}$$

where the extra term added represents two oppositely charged fundamental static sources at $x = 0$ and $x = R$ ($y = 0$) in $\mathbb{R}^2$, whose charges under the Cartan subalgebra $U(1)$'s are given by $n \in \mathbb{Z}$ times $\boldsymbol{\nu}^B$, some chosen weight of the fundamental representation. The normalization is consistent with our normalization of generators. The description of the $U(1)^{N-1}$ electrodynamics using the gauge potentials A_0^a, A_i^a is called the "electric description." If you ignore the multiplicity of $U(1)$'s, this is EM as you know it, except in (2+1)d.

It is instructive to solve for the field A_0^a of a single static charge. The equation of motion for A_0^a (putting $A_i = 0$), taking only the charge at the origin

$$\partial_i \partial_i A_0^a(x) = -\frac{g^2}{\sqrt{2}L}(\boldsymbol{\nu}^A)_a \delta^{(2)}(0) \ . \tag{4.2}$$

Now integrate the above over a disk shaped region S in $\mathbb{R}^2$ which contains the origin (but not the other charge(s)) to find

$$\int_S dxdy \ \partial_i F_{0i}^a = \frac{g^2}{\sqrt{2}L}(\boldsymbol{\nu}^A)_a \tag{4.3}$$

($F_{0i} = -\partial_i A_0$, since we took $A_i = 0$) and integrate by parts to find the 2d version of Gauss' law

$$\oint_{C=\partial S} dl \ n^i F_{0i}^a = \frac{g^2}{\sqrt{2}L}(\boldsymbol{\nu}^A)_a, \tag{4.4}$$

where n^i is a unit outward normal to the boundary $C = \partial S$ (imagine S a disk and C a circle) and the integral runs counterclockwise along C.

We now want to develop a dual, "magnetic description" of (2+1)d electrodynamics, not in terms of a vector potential. Let us now introduce the following "dual photon" field σ^a

$$F_{0i}^a \equiv \frac{g^2}{2\pi\sqrt{2}L}\epsilon_{ij}\partial_j \sigma^a, \ \epsilon_{12} = 1. \tag{4.5}$$

In terms of σ^a (4.4) reads

$$\oint_{C=\partial S} dl\ n^i \epsilon_{ij} \partial_j \sigma^a = 2\pi (\boldsymbol{\nu}^A)_a, \tag{4.6}$$

Now the integrand has[5] $dln^1 \partial_2 \sigma - dln^2 \partial_1 \sigma = -dx^2 \partial_2 \sigma - dx^1 \partial_1 \sigma$, hence (4.6) becomes

$$\oint_{C=\partial S} d\vec{l} \cdot \vec{\nabla} \sigma^a = -2\pi (\boldsymbol{\nu}^A)_a, \tag{4.7}$$

or, the circulation of $\boldsymbol{\sigma}$ along C is $\mathbb{Z}$ times $\boldsymbol{\nu}^A$, for any A. Notice that for the case of $N = 2$, i.e. a single dual photon field, the periodicity is $\sqrt{2}\pi$, because the weights of the fundamental for $SU(2)$ are $\pm 1/\sqrt{2}$.

MORAL: in order to allow for static charge sources, our "dual photon" field is allowed to have nonzero circulation (it is convenient, but is not precise in this case, to think of σ as the phase of some complex field). In our normalization, the circulation takes values in the weight lattice (the one spanned by the vectors $2\pi \boldsymbol{\nu}^A$). The circulation of σ around static charges is called "monodromy". It is convenient, for many purposes, to think of the dual photon field $\sigma^a(t, x, y)$ as a map from $\mathbb{R}^{1,2}$ (3d spacetime) into a $N-1$-dimensional torus (a field is just that, a map from spacetime to the field space!), the weight lattice of the $SU(N)$ gauge group. This torus is defined by the identifications $\sigma^a \equiv \sigma^a + 2\pi(\nu^A)^a$, for $A = 1, ..., N-1$ (like a normal rectangular torus can be thought as $(x, y) \equiv (x + 2\pi, y) \equiv (x, y + 2\pi)$). Notice that we can exclude ν^N, since $\nu^1 + \nu^2 + ... + \nu^{N-1} + \nu^N = 0$, so the $N-1$ periods define the torus.

In addition to using the dual photon (4.5) to describe the electric field, we shall also describe the magnetic field, F_{ij} (which really only has one component, F_{12}), introducing

$$F^a_{0i} \equiv \frac{g^2}{2\pi\sqrt{2}L} \epsilon_{ij} \partial_j \sigma^a, \ \ \epsilon_{12} = 1 \tag{4.8}$$

$$F^a_{12} \equiv \frac{g^2}{2\pi\sqrt{2}L} \partial_0 \sigma^a \tag{4.9}$$

Notice that the above two relations combine into one if one uses 3d Lorentz invariant notation: $F^a_{\mu\nu} = \frac{g^2}{2\pi\sqrt{2}L} \epsilon_{\mu\nu\lambda} \partial^\lambda \sigma^a$, with $\epsilon_{012} = -1$ (and metric $(+,-,-)$).

[5] We take ϕ to be the angle between the tangent to C and the positive x axis; then $n^1 = -\sin\phi$ and $n^2 = \cos\phi$, hence $dln^1 = -dy \equiv -dx^2$ and $dln^2 = dx \equiv dx^1$.

Skip! *The duality in canonical quantiation:* In order to argue for signs and overall coefficients and short-circuit a bit, let us imagine that we replace the above classical relation by operators.

$$\frac{g^2}{L}\hat{\Pi}_i^a \equiv \frac{g^2}{2\pi\sqrt{2}L}\epsilon_{ij}\partial_j\hat{\sigma}^a \to \hat{\Pi}_i^a \equiv \frac{1}{2\pi\sqrt{2}}\epsilon_{ij}\partial_j\hat{\sigma}^a \qquad \hat{F}_{12}^a \equiv c'\hat{\Pi}_\sigma^a \ , \tag{4.10}$$

where the constant c' is to be determined. Let us postulate the usual relations for σ, $[\hat{\Pi}_\sigma^a(x), \hat{\sigma}^b(y)] = -i\delta^{(2)}(x-y)\delta^{ab}$. The gauge commutation relation gives

$$[\hat{\Pi}_i^a(x), \hat{F}_{12}^b(y)] = -i\delta^{ab}(\delta_{i2}\partial_1 - \delta_{i1}\partial_2)\delta^{(2)}(x-y) = i\delta^{ab}\epsilon_{ij}\partial_j\delta^{(2)}(x-y). \tag{4.11}$$

Now, try with the $\sigma's$ using the map (4.10):

$$[\frac{1}{2\pi\sqrt{2}}\epsilon_{ij}\partial_j\hat{\sigma}^a(x), c'\hat{\Pi}_\sigma^b(y)] = \frac{c'}{2\pi\sqrt{2}}\epsilon_{ij}[\partial_j\hat{\sigma}^a(x), \hat{\Pi}_\sigma^b(y)] = i\delta^{ab}\frac{c'}{2\pi\sqrt{2}}\epsilon_{ij}\partial_j\delta^{(2)}(x-y) \ . \tag{4.12}$$

Thus, $c' = 2\pi\sqrt{2}$ and our duality map is, in the operator formalism:

$$\frac{g^2}{L}\hat{\Pi}_i^a \equiv \frac{g^2}{\sqrt{2}L}\epsilon_{ij}\partial_j\hat{\sigma}^a \to \hat{\Pi}_i^a \equiv \frac{1}{2\pi\sqrt{2}}\,\epsilon_{ij}\partial_j\hat{\sigma}^a \qquad \hat{F}_{12}^a \equiv 2\pi\sqrt{2}\,\hat{\Pi}_\sigma^a \ . \tag{4.13}$$

Notice that the duality map, in terms of operators, relates gauge invariant (abelian) operators and interchanges coordinates and momenta.

Read! Continuing with the classical theory, the dual photon σ is a free (2+1)d massless scalar field, dimensionless, and it should retain Lorentz invariance, so its action should be

$$S_\sigma = \int dtd^2x\ A'\frac{g^2}{L}\left((\partial_0\sigma^a)^2 - (\partial_i\sigma^a)^2\right)$$

where the overall coefficient was written on dimensional grounds and we need to fix A'. The best way, without doing a duality in the path integral, is to consider the electrostatic energy of a pair of charges. Let's flesh it out. Two equal and opposite static charges' action (the source term in (4.1)) can be written as

$$\begin{aligned} S_{source} &= -\int dt\frac{1}{\sqrt{2}}(\nu^B)_a(A_0^a(0,0) - A^a(R,0)) = \frac{1}{\sqrt{2}}(\nu^B)_a\int dt\int_0^R dx\ \partial_x A_0^a(x,0) \\ &= -\frac{1}{\sqrt{2}}(\nu^B)_a\int dt\int_0^R dx\ \partial_y F_{0x}^a(x,0). \end{aligned} \tag{4.14}$$

This can be rewritten, using the duality relation as

$$S_{source} = -\frac{g^2}{4\pi L}(\nu^B)_a \int dt \int\limits_0^R dx \partial_y \sigma^a(x,y)|_{y=0}. \tag{4.15}$$

Let us now extremize $S_\sigma + S_{source}$ w.r.t. $\sigma(x,y)$. The time independent EOM is

$$2A'(\partial_x^2 + \partial_y^2)\sigma(x,y)^a = \frac{1}{4\pi}(\nu^B)_a \partial_{y'}\delta(y-y')|_{y'=0} \int\limits_0^R dx' \delta(x-x') \ .$$

Now $\int\limits_0^R dx'\delta(x-x') = 1$ if $0 < x < R$ and zero otherwise. For simplicity, let us take the limit $R \to \infty$ (and $0 \to -\infty$) (or look away from the endpoints, physically speaking). Then, the x-integral of the delta function is just unity. In this limit one does not expect x-dependence, so we have

$$\partial_y^2 \sigma^a(y) = \frac{1}{8\pi A'}(\nu^B)_a \partial_y \delta(y) \ .$$

Dropping one of the derivatives on each side, we find $\partial_y \sigma^a(y) = \frac{1}{8\pi A'}(\nu^B)_a \delta(y)$, or $\sigma^a(0^+) - \sigma^a(0^-) = \frac{n}{8\pi A'}(\nu^B)^a$. Remember that the monodromy of σ is $2\pi(\nu^A)_a$ (in the limit where the charges are at infinity the entire monodromy is at the $y = 0$ plane only), we conclude that $A' = \frac{1}{16\pi^2}$. In summary, our action for σ with electric sources, corresponding to two fundamental charges a distance R apart, now reads

$$S_{\sigma+source} = \frac{g^2}{16\pi^2 L} \int dt \left[\int d^2x \ \left((\partial_0 \sigma^a)^2 - (\partial_i \sigma^a)^2 \right) - 4\pi(\nu^B)_a \int\limits_0^R dx \partial_y \sigma^a(x,y)|_{y=0} \right] \ .$$

Let's play a bit more with this. Back to the 2d static charges a distance R apart. With the normalization fixed, the equation of motion reads:

$$(\partial_x^2 + \partial_y^2)\sigma^a(x,y) = 2\pi(\nu^B)_a \ \partial_y \delta(y) \int\limits_0^R dx' \delta(x-x').$$

Also specialize to $SU(2)$, where periodicity of the single dual photon σ is $\sqrt{2}\pi$, hence obtaining (taking $\boldsymbol{\nu}$ to be the positive weight of the fundamental of $SU(2)$),

$$(\partial_x^2 + \partial_y^2)\sigma(x,y) = \sqrt{2}\pi \ \partial_y \delta(y) \int\limits_0^R dx' \delta(x-x').$$

Claim: solution is $\sigma(x,y) = \frac{1}{\sqrt{2}}(\theta_0 - \theta_R)$, where θ_0 is the polar angle measured from the origin and θ_R—the one measured from the point $(R, 0)$. The convention assumed is that θ_0 changes from $0+\epsilon$ along the positive x direction, just above $y=0$ to π in the negative x-direction, to $2\pi - \epsilon$ in the positive x-direction just below the $y=0$ axis (and similar for θ_R).

Notice that the polar angle obeys Laplace's equation away from the origin. In mathematica, this convention for the polar angle can be defined using the built-in function *ArcTan* like

$$myat[x_,y_] := If[y >= 0, ArcTan[x,y], 2Pi + ArcTan[x,y]]$$

which defines the convention for θ and the next line plots it

$$Plot3D[myat[x,y] - myat[x-2,y], x, -3, 3, y, -3, 3, AxesLabel- > x, y]$$

Compute:

1. The leading R-dependence (as R becomes large, larger than any "size of the point charge") of the electrostatic energy of two charges in SU(2), using the electric picture and the dual photon picture. Verify that they agree, i.e. we have not made any mistake in coefficients! Plot the electric field vector in the electric picture and plot the $\nabla\sigma$ vector in the magnetic picture and compare.

2. After this, repeat for charges in SU(N) given by an arbitrary weight of the fundamental $\boldsymbol{\nu}^A$, defined below.

3. Finally, by imposing monodromies "± 1" around the two charges, numerically solve using relaxation method. Picture the resulting configuration in a manner of your choice.

4.1 Lie algebraic notation, repeated

Skip! We denote the fundamental $SU(N)$ Cartan subalgebra generators by H^a, $a = 1, ..., N-1$. An explicit form is $H^a = \frac{1}{\sqrt{2}}\text{diag}[\lambda^{a1}, ..., \lambda^{aN}]$, where

$$H^a = \frac{1}{\sqrt{2}}\text{diag}[\lambda^{a1}, ..., \lambda^{aN}] = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{a(a+1)}}\text{diag}[\underbrace{1,1,...,1}_{a\,\text{times}}, -a, \underbrace{0,0,...0}_{N-1-a\,\text{times}}] \,. \quad (4.16)$$

Read! Above, we defined the symbols:

$$\lambda^{aA} \equiv \frac{1}{\sqrt{a(a+1)}}(\theta^{aA} - a\delta_{a+1,A}),\ a = 1, ..., N-1,\ A = 1, ..., N,\ \theta^{aA} \equiv \begin{cases} 1, a \geq A \\ 0, a < A \end{cases}$$

The utility in introducing λ^{aA} in (4.16) is to note that the weights of the fundamental representation $\boldsymbol{\nu}^A$ can be expressed in this $N-1$-dimensional basis (we denote its a-th component by $(\boldsymbol{\nu}^A)_a$) as:

$$(\boldsymbol{\nu}^A)_a \equiv \lambda^{aA}, \quad \boldsymbol{\nu}^A \cdot \boldsymbol{\nu}^B \equiv \sum_{a=1}^{N-1} \lambda^{aA}\lambda^{aB} = \delta^{AB} - \frac{1}{N}, \; \sum_{A=1}^{N} \lambda^{aA}\lambda^{bA} = \delta^{ab} \;, \qquad (4.17)$$

where we also noted the properties of the λ^{aA} implying that $\operatorname{tr} H^a H^b = \frac{1}{2}\delta^{ab}$ (like the complete set of generators T^a used above).

Compute:

4. Check the relations of (4.17) from the definition of λ. Also check **all** *numbered relations that follow till the end of this Section. For us, the importance of the weights of the fundamental representation $\boldsymbol{\nu}^A$ are in that they represent the charges of probe (heavy) quarks under the $N-1$ gluons in the Cartan subalgebra of $SU(N)$. These words should suffice for now.*

Furthermore, the fundamental weights $\boldsymbol{\omega}^a$ and simple roots $\boldsymbol{\alpha}^a$ of $SU(N)$ ($a = 1, \ldots, N-1$) are, respectively (bold face vectors denote vectors in the Cartan subalgebra) are

$$\begin{aligned} \boldsymbol{\omega}^a &\equiv \sum_{A=1}^{a} \boldsymbol{\nu}^A, \; a = 1, \ldots N-1, \\ \boldsymbol{\alpha}^a &\equiv \boldsymbol{\nu}^a - \boldsymbol{\nu}^{a+1}, \; a = 1, \ldots N-1. \end{aligned} \qquad (4.18)$$

The fundamental weights span a lattice in the $N-1$ dimensonal space called the "weight lattice" (what we mean is the lattice of points spanned by linear combinations $\sum\limits_{a=1}^{N-1} n_a 2\pi \boldsymbol{\omega}^a$ for all integer choices of n_a). The simple roots $\boldsymbol{\alpha}^a$, on the other hand, span the so-called "root lattice". The root lattice is coarser than the weight lattice, as you will momentarily observe.

Compute:

5. In order to get a feel for the weight and root lattices, draw the vectors and associated lattice points on the plane for $SU(3)$, i.e. $N-1=2$. Observe which one is coarser than the other and try to infer the symmetries of the lattice.

We also define the positive roots $\boldsymbol{\beta}^{AB}$, $A < B$:

$$\boldsymbol{\beta}^{AB} \equiv \boldsymbol{\nu}^A - \boldsymbol{\nu}^B \;, \; A, B = 1, \ldots N. \qquad (4.19)$$

The simple roots are a subset, $\boldsymbol{\alpha}^a = \boldsymbol{\beta}^{a\, a+1}$ and the affine root is $\boldsymbol{\alpha}^0 = -\sum\limits_{k=1}^{N-1} \boldsymbol{\alpha}^k$. We shall need several relations that follow from the definitions (4.16, 4.17, 4.18):

$$\boldsymbol{\omega}^a \cdot \boldsymbol{\omega}^b = \min(a,b) - \frac{ab}{N} \tag{4.20}$$

$$\boldsymbol{\omega}^b \cdot \boldsymbol{H} = \sum_{a=1}^{N-1} (\boldsymbol{\omega}^b)_a H^a = \frac{1}{\sqrt{2}} \mathrm{diag}[\underbrace{1-\frac{b}{N}, 1-\frac{b}{N}, ..., 1-\frac{b}{N}}_{b\,\text{times}}, \underbrace{-\frac{b}{N}, -\frac{b}{N}, ... -\frac{b}{N}}_{N-b\,\text{times}}]$$

$$\boldsymbol{\omega}^b \cdot \boldsymbol{\beta}^{AB} = \sum_{k=1}^{b} \delta^{kA} - \delta^{kB} = \begin{cases} 0, & b < A \\ 1, & A \le b < B \\ 0, & b \ge B \end{cases} . \tag{4.21}$$

Skip! We prefer to keep the normalization of roots and weights as above, so that roots have length 2, and include $1/\sqrt{2}$ in the definition of the generators. The reason for this somewhat painful convention is that the topological terms have been worked out in detail and certainty for a $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$ normalization of generators and reworking them is too much to ask for.