DERIVED EQUIVALENCE OF VARIETIES

(with particular emphasis on K3 Surfaces)

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CONTENTS

Introduction

Let X be a smooth projective variety over a field k. The notion of a coherent sheaf on X was introduced in algebraic geometry by J-P. Serre in his foundational FAC paper [**28**], at a time when the importance of cohomological methods was recognized and the foundations of modern (scheme-theoretic) algebraic geometry were laid down. Subsequently the properties of the k-linear abelian category **Coh**(X) of coherent sheaves on X were investigated, of which we shall only mention the dévissage lemma by A. Grothendieck ([**14**], III, théorème 3.1.2) and the Krull-Schmidt theorem for **Coh**(X) by M. Atiyah ([**1**], theorem 2).

While it is of course not unexpected that **Coh**(X) contains quite some amount of information about X, essentially by definition all cohomological information, one can in fact recover X from **Coh**(X) up to isomorphism by a theorem of P. Gabriel ([**13**], chapitre IV). Thus the invariant **Coh**(X) of X is in a sense too rigid to be interesting. As any abelian category, **Coh**(X) has a bounded derived category $\mathrm{D}^b(\mathbf{Coh}(X))$, which is customarily abbreviated as D(X). Whereas the technical convenience of the derived category (in, say, formulating Grothendieck duality) was acknowledged from the outset, only the pioneering work of S. Mukai [**21**] started what is the subject matter of this thesis: *the study of X in terms of* $\mathrm{D}(X)$. Considering Gabriel's theorem it is only natural to ask whether one can recover X from $\mathrm{D}(X)$. This is false, for Mukai [**21**] (theorem 2.2) proved that if X is an abelian variety with dual $\hat{X}$ (in general, $\hat{X}$ is not isomorphic to X), then the functor

$$\Phi_P = \mathrm{R}p_*(P \otimes^{\mathrm{L}} q^*(-)) : \mathrm{D}(X) \to \mathrm{D}(\hat{X})$$

is an equivalence. (Here p and q denote the projections from $X \times \hat{X}$ to X and $\hat{X}$ respectively, and P is the Poincaré bundle on $X \times \hat{X}$.) The functor Φ_P is nowadays called a Fourier-Mukai transform with kernel P, and has also proven to be useful in the study of moduli spaces of stable sheaves. (We should specify here that we view $\mathrm{D}(X)$ as a triangulated category. It was proven by P. Balmer [**2**], corollary 8.6, that one can reconstruct $\mathrm{D}(X)$ from X if $\mathrm{D}(X)$ is viewed as a *monoidal* triangulated category.) Thus arose the notion of derived equivalence of varieties, as an equivalence of the associated bounded derived categories of coherent sheaves.

The interest in derived equivalent varieties (over $k = \mathbf{C}$) was renewed by M. Kontsevich's homological mirror symmetry conjecture [**19**], for in that conjecture derived equivalence is related to the non-uniqueness of the mirror variety. Shortly afterwards, D. Orlov [**22**] (theorem 2.2) proved that any equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ is a Fourier-Mukai transform; together with A. Bondal ([**5**], theorem 2.5) he also obtained a reconstruction theorem for X with ample or anti-ample canonical sheaf. These papers have inspired (and continue to inspire) much of the subsequent development.

Summary. In the first section we discuss some technical results concerning the derived category of coherent sheaves, which are frequently used in the following sections. The second section discusses a few derived invariants (such as dimension) as well as some recent work towards the conjectured derived invariance of the Hodge polynomial. The third section contains a proof of the Bondal-Orlov reconstruction, while the fourth summarizes some general results about lattices that are used in section 5. In section 5 we address the counting of Fourier-Mukai partners of K3 surfaces, for which we discuss the proof of a counting formula due to [**15**]. The sixth (and last) section is devoted to the largely unexplored topic of Fourier-Mukai partners of Hilbert schemes of points.

While this paper is largely expositional, several proofs are our own, some of which are essentially our solutions to exercises in [**16**]. We do not know of a published proof of theorem 1.3, and we have arrived at theorem 2.7 (in an attempt to prove theorem 2.1 over an arbitrary field) independently of [**23**]. D. Huybrecht's book [**16**] and R. Rouquier's exposé [**27**] were very useful to us, both in learning the subject as well as a reference.

Notations and conventions. Let $\mathscr{A}$ be an abelian category and $\mathrm{D}^b(\mathscr{A})$ be its bounded derived category. Then there is a canonical equivalence between $\mathscr{A}$ and the subcategory of $\mathrm{D}^b(\mathscr{A})$ consisting of complexes with cohomology concentrated in degree 0; we view this as an identification. For objects X and Y in $\mathscr{A}$ we define $\mathrm{Ext}^n_{\mathscr{A}}(X,Y) = \mathrm{Hom}_{\mathrm{D}^b(\mathscr{A})}(X, Y[n])$. (If $\mathscr{A}$ has enough injectives, then this agrees with the usual definition of Ext.) The *projective dimension* (resp *injective dimension*) of X is defined by $\mathrm{dimproj}_{\mathscr{A}}(X) = \sup\{n \mid \mathrm{Ext}^n_{\mathscr{A}}(X,-) \neq 0\}$ (resp. $\mathrm{diminj}_{\mathscr{A}}(X) = \sup\{n \mid \mathrm{Ext}^n_{\mathscr{A}}(-,X) \neq 0\}$). The *cohomological dimension* of $\mathscr{A}$ is $\mathrm{dimcoh}(\mathscr{A}) = \sup\{n \mid \mathrm{Ext}^n_{\mathscr{A}}(-,-) \neq 0\}$. In particular, the cohomological dimension of a ring A is $\mathrm{dimcoh}(A) = \mathrm{dimcoh}(A-\mathbf{Mod})$. The Grothendieck (K-theory) group of $\mathscr{A}$ will be denoted by $\mathrm{K}_0(\mathscr{A})$. A functor of triangulated categories is *exact* if it commutes with the shift functor [1] and preserves distinguished triangles.

In all that follows we work in the category of schemes over a field k; thus $X \times Y$ abbreviates $X \times_k Y$. All varieties are assumed to be projective and irreducible. For X a scheme and $x \in X$ a point we denote by $\mathscr{O}_{X,x}$ the local ring of X at x, and by $\kappa(x)$ its residue field. The diagonal map $X \to X \times X$ is denoted by Δ. The category of coherent sheaves on X is $\mathbf{Coh}(X)$, while $\mathrm{D}(X) = \mathrm{D}^b(\mathbf{Coh}(X))$ denotes its bounded derived category. The derived tensor product will be denoted simply by $\otimes$, and for an object E of $\mathrm{D}(X)$ we let $E^\vee = \mathrm{R}\mathscr{H}\mathrm{om}(E, \mathscr{O}_X)$. If E and F are objects of $\mathrm{D}(X)$ and $\mathrm{D}(Y)$, respectively, then $E \boxtimes F$ is the object $p^*(E) \otimes q^*(F)$ of $\mathrm{D}(X \times Y)$ (where p and q are the projections from $X \times Y$ to X and Y, respectively). For a coherent sheaf $\mathscr{F}$ on X we use the notation $\mathscr{F}^n$ for $\bigotimes_{i=1}^n \mathscr{F}$. If x is closed then $\mathscr{O}_x$ designates the structure sheaf of the closed subscheme $\{x\}$ of X (endowed with the induced reduced subscheme structure); $\mathscr{O}_x$ is a skyscraper sheaf at x with stalk $\kappa(x)$. Over $k = \mathbf{C}$ we do not distinguish notationally between a smooth variety X and its associated analytic space $X(\mathbf{C})$.

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1. The Categories $\mathbf{Coh}(X)$ and $\mathrm{D}(X)$.

1.1. **Preliminaries.** Let X be a projective variety over a field k. The k-linear abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X is a Noetherian category with a tensor product $\otimes_{\mathscr{O}_X}$, as well as an inner hom $\mathscr{H}\mathrm{om}$. It is an intermediate between the category of quasi-coherent sheaves and the category of vector bundles. While it is geometrically more natural than the former category, it is much better behaved than the latter category (which is, among other things, not abelian).

Since $\mathbf{Coh}(X)$ is an abelian category, we have an associated bounded derived category

$$\mathrm{D}(X) = \mathrm{D}^b(\mathbf{Coh}(X)).$$

To study the category $\mathrm{D}(X)$ it would be convenient if $\mathbf{Coh}(X)$ had enough injectives, which is unfortunately not the case. For that reason we also need to consider the inclusion $\mathbf{Coh}(X) \to \mathbf{QCoh}(X)$ into the abelian category of quasi-coherent sheaves on X, which is a Grothendieck abelian category and therefore has enough injectives. As the inclusion functor is exact, it induces a functor $\mathrm{D}^b(\mathbf{Coh}(X)) \to \mathrm{D}^b(\mathbf{QCoh}(X))$. The latter induces an equivalence of categories of $\mathrm{D}(X)$ with the triangulated subcategory $\mathrm{D}_c(X)$ of $\mathrm{D}^b(\mathbf{QCoh}(X))$ consisting of bounded complexes with coherent cohomology sheaves (see [**16**], proposition 3.5). The Hom sets of $\mathrm{D}(X)$ are in general hard to describe, but one can obtain some information by considering various spectral sequences, for instance by successive use of the (Grothendieck) spectral sequences

$$\mathrm{E}_2^{p,q} = \mathrm{Hom}_{\mathrm{D}(X)}(E, \mathrm{H}^q(F)[p]) \Rightarrow \mathrm{Hom}_{\mathrm{D}(X)}(E, F[p+q]) \tag{1.1}$$

$$\mathrm{E}_2^{p,q} = \mathrm{Hom}_{\mathrm{D}(X)}(\mathrm{H}^{-q}(E)[-p], F) \Rightarrow \mathrm{Hom}_{\mathrm{D}(X)}(E, F[p+q]) \tag{1.2}$$

holding for $E, F \in \mathrm{D}(X)$. (See for instance [**16**], remark 3.7).

If $f : X \to Y$ is a proper morphism of projective varieties over k, then the push-forward $f_* : \mathbf{QCoh}(X) \to \mathbf{QCoh}(Y)$ is left-exact and since the category $\mathbf{QCoh}(X)$ has enough injectives, the derived functor

$$\mathrm{R}f_* : \mathrm{D}^+(\mathbf{QCoh}(X)) \to \mathrm{D}^+(\mathbf{QCoh}(Y))$$

exists. To define a functor $\mathrm{D}^b(\mathbf{Coh}(X)) \to \mathrm{D}^b(\mathbf{Coh}(Y))$ we will use the following fundamental theorem regarding the higher direct images of coherent sheaves.

THEOREM 1.1. *Let $f : X \to Y$ be a proper morphism of schemes, assume Y to be locally Noetherian, and let $\mathcal{F}$ be a coherent sheaf on X.*
(i) The higher direct image sheaves $\mathrm{R}^n f_*(\mathcal{F})$ ($n \geqslant 0$) are coherent sheaves on Y.
(ii) For every $y \in Y$ we have $\mathrm{R}^n f_*(\mathcal{F})_y = 0$ for all $n > \dim f^{-1}(y)$.

Part (i) is an application of the dévissage lemma ([**14**], III, théorème 3.2.1), while part (ii) is a corollary of the theorem on formal functions ([**14**], III, corollaire 4.2.2) as well as Grothendieck's vanishing theorem for sheaf cohomology. By using theorem 1.1 and applying the equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}_c(X)$ we can define the functor $\mathrm{R}f_* : \mathrm{D}(X) \to \mathrm{D}(Y)$. Unless f is flat, we will have no occasion to use the left derived functor of f^* (if f is flat, then f^* is exact and directly induces a functor of derived categories).

COROLLARY 1.1. *The category* $\mathrm{D}(X)$ *is Hom-finite.*

Proof. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X consider the local-to-global spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}xt_X^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F}, \mathcal{G}).$$

By applying theorem 1.1 (i) to the structural morphism $f : X \to \operatorname{Spec} k$ we obtain that all $\mathrm{E}_2^{p,q}$ are finite-dimensional, and the local-to-global spectral sequence shows that the spaces $\mathrm{Ext}_X^n(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{F}, \mathcal{G}[n])$ are also finite-dimensional. Finally, for $E, F \in \mathrm{D}(X)$ the spectral sequence (1.1) implies that the Hom sets $\mathrm{Hom}_{\mathrm{D}(X)}(\mathrm{H}^{-q}(E), F)$ are finite-dimensional, while the spectral sequence (1.2) gives the finite-dimensionality of $\mathrm{Hom}_{\mathrm{D}(X)}(E, F)$. ■

1.2. **Serre duality and Smoothness.** We will mostly be concerned with the case where X is smooth over k. Then the sheaf of differentials $\Omega_X^1 = \Omega_{X/k}^1$ is a locally free sheaf of rank $\dim X$ and the canonical sheaf

$$\omega_X = \bigwedge^{\dim X} \Omega_X^1$$

is an invertible sheaf on X. It will play a special role in all that follows, and the following cases are of particular interest: ω_X is trivial; ω_X is ample (general type case); ω_X is antiample, i.e. ω_X^{-1} ample (Fano case). As we shall see, many important questions have been resolved in the second and third case.

Example 1.1. (i) If X is a Calabi-Yau variety (for instance a K3 surface, see section 5) or a group variety, then ω_X is trivial (in the latter case even Ω_X^1 is trivial).
(ii) Let X be a smooth hypersurface of degree d in $\mathbf{P}^n$, $n \geqslant 2$. Then

$$\omega_X = \mathcal{O}_X(d - n - 1)$$

and hence ω_X is ample if and only if $d > n + 1$, antiample if and only if $d < n + 1$, and trivial if and only if $d = n + 1$. Thus the canonical sheaf of a hypersurface is usually ample; however, many common examples (quadrics such as $\mathrm{Gr}(2, 4) \subset \mathbf{P}^5$) are Fano varieties. In fact, Grassmannians are always Fano varieties.

The importance of ω_X is also reflected in Serre's duality theorem, which is crucial for the study of derived categories of coherent sheaves.

THEOREM 1.2 (Serre). *Let* X *be a smooth projective variety* k*, and denote by* S *the functor* $- \otimes \omega_X[\dim X] : \mathrm{D}(X) \to \mathrm{D}(X)$*. Then we have a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{D}(X)}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(X)}(F, S(E))^*.$$

(The existence of such a duality is one of the main motivations for the homological mirror symmetry conjecture, see [**19**].) This can be deduced from the classical formulation of Serre duality (i.e. the special case where $E = \mathcal{O}_X$ and F is a coherent sheaf on X) using purely formal arguments, see for instance proposition 5.1.1 in [**6**]. We note in passing that the existence of such a functorial isomorphism is often expressed by saying that S is a *Serre functor* for $\mathrm{D}(X)$; one can show that any two Serre functors are functorially isomorphic. It is also worthwhile to note that if X has trivial canonical sheaf then S is the shift functor $[\dim X]$, and the Serre duality theorem is formulated entirely at the level of arbitrary triangulated categories. This observation leads to the notion of Calabi-Yau categories.

We finish this section by discussing the role of the smoothness hypothesis, showing that it enters the categories $\mathbf{Coh}(X)$ and $\mathrm{D}(X)$ as a finiteness condition. In accordance with established terminology (see for instance [**6**]), we shall call $\mathrm{D}(X)$ *Ext-finite* whenever

$$\sum_{n\in\mathbf{Z}} \dim_k \mathrm{Hom}_{\mathrm{D}(X)}(E, F[n]) < \infty$$

for all objects E and F of $\mathrm{D}(X)$. Our discussion will rely on two lemmas from commutative algebra, the first of which we also state for future reference.

LEMMA 1.1. *(i) Let A be a Noetherian local ring with residue field κ, and M be a finite-type A-module M. If* $\mathrm{Hom}_A(M,\kappa) = 0$ *then $M = 0$. If* $\mathrm{Ext}^1_A(M,\kappa) = 0$ *then M is free. (ii) Let $\mathcal{F}$ be a coherent sheaf on X. A closed point $x \in X$ belongs to the support of $\mathcal{F}$ if and only if* $\mathrm{Hom}_X(\mathcal{F}, \mathcal{O}_x) \neq 0$. *The sheaf $\mathcal{F}$ is locally free if and only if* $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{O}_x) = 0$ *for all $x \in X$.*

(Here $\mathcal{O}_x$ is the skyscraper sheaf at x with stalk $\kappa(x)$.)

Proof. (i) By the tensor-hom adjunction $\mathrm{Hom}_\kappa(M \otimes_A \kappa, \kappa) \simeq \mathrm{Hom}_A(M,\kappa) = 0$ and hence $M \otimes_A \kappa = 0$. Nakayama's lemma implies $M = 0$. Let

$$0 \to K \to A^n \to M \to 0$$

be a short exact sequence of A-modules with $n = \dim_\kappa M \otimes_A \kappa$. Then

$$0 \to \mathrm{Hom}_A(M,\kappa) \to \mathrm{Hom}_A(A^n,\kappa) \to \mathrm{Hom}_A(K,\kappa) \to \mathrm{Ext}^1_A(M,\kappa) = 0$$

and the first map is an injective map of κ-vector spaces of dimension n. Hence $\mathrm{Hom}_A(K,\kappa) = 0$ and $K = 0$.

(ii) Since $\mathrm{Hom}_X(\mathcal{F}, \mathcal{O}_x) \simeq \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \kappa(x))$ this follows from (i). By the local-to-global spectral sequence $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{O}_x) = \mathrm{Ext}^1(\mathcal{F}_x, \kappa(x))$, and (i) shows that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all x. ■

LEMMA 1.2. *Let A be a Noetherian local ring with residue field κ. Then*

$$\operatorname{dimcoh}(A) = \operatorname{diminj}_A(\kappa) = \sup\{n \mid \mathrm{Ext}^n_A(\kappa,\kappa) \neq 0\},$$

and A is regular if and only if $\operatorname{dimcoh}(A) < \infty$.

Proof. The second equality follows from theorem 4.4.9 in [**30**], whereas for the first equality we need to prove that for any finite type A-module M we have $\operatorname{dimproj}_A(M) \leqslant \operatorname{diminj}_A(\kappa)$. If $F \to M \to 0$ is the minimal free resolution of M, then by definition

$$\operatorname{Ext}^m_A(M, \kappa) = \mathrm{H}^m(\operatorname{Hom}_A(F, \kappa)) = \operatorname{Hom}_A(F_m, \kappa) = 0$$

for $m > \operatorname{diminj}_A(\kappa)$. By lemma 1.1 (i) we deduce $F_m = 0$ for $m > \operatorname{diminj}_A(\kappa)$, and therefore $\operatorname{dimproj}_A(M) \leqslant \operatorname{diminj}_A(\kappa)$. The remaining statement is a well-known theorem of Serre ([**14**], IV, théorème 17.3.1). ■

THEOREM 1.3. *Let X be a projective variety over a perfect field k. Then the following are equivalent:*
(i) X is smooth over k;
(ii) the cohomological dimension of $\mathbf{Coh}(X)$ *is finite;*
(iii) $\mathrm{D}(X)$ *is Ext-finite.*
If either condition is satisfied, then $\operatorname{dimcoh}(\mathbf{Coh}(X)) = \dim X$.

Proof. We take for granted the well-known fact that smoothness over a field is equivalent to geometric regularity, which over a perfect field is equivalent to regularity (see for instance [**14**], IV, corollaire 17.15.2). That (i) implies (ii) follows immediately from Serre duality: if $n > \dim X$, then the vanishing

$$\operatorname{Ext}^n_X(\mathcal{F}, \mathcal{G}) = \operatorname{Ext}^{\dim X - n}_X(\mathcal{G}, \mathcal{F} \otimes \omega_X) = 0$$

for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X proves $\operatorname{dimcoh}(\mathbf{Coh}(X)) \leqslant \dim X$. To show that (ii) implies (iii) note that (by assumption) we have

$$\sum_{n \in \mathbf{Z}} \dim_k \operatorname{Hom}_{\mathrm{D}(X)}(E, F[n]) < \infty$$

for all objects E and F of $\mathrm{D}(X)$ with cohomology sheaves concentrated in a single degree. This implies the general result by using the spectral sequences (1.1), (1.2). At last we prove that (iii) implies (i). Let $x \in X$ be a closed singular point, and $\mathcal{O}_x$ its structure sheaf. For any q the sheaf $\mathcal{E}xt^q(\mathcal{O}_x, \mathcal{O}_x)$ is a skyscraper sheaf at x with stalk $\operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\kappa(x), \kappa(x))$, and the local-to-global spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}xt^q(\mathcal{O}_x, \mathcal{O}_x)) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{O}_x, \mathcal{O}_x)$$

collapses to give an isomorphism $\operatorname{Ext}^n_X(\mathcal{O}_x, \mathcal{O}_x) = \operatorname{Ext}^n_{\mathcal{O}_{X,x}}(\kappa(x), \kappa(x))$ for all n. By lemma 1.2 it follows that

$$\sup\{n \mid \operatorname{Ext}^n_X(\mathcal{O}_x, \mathcal{O}_x) \neq 0\}$$

is the cohomological dimension of $\mathcal{O}_{X,x}$, which is infinite. As for the remaining statement, note that $\operatorname{Ext}^n_X(\mathcal{O}_X, \omega_X) = \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)^* \neq 0$ by Serre duality. ■

Remark 1.1. (i) If X is a projective variety over k, then the *cohomological dimension* of X is the cohomological dimension

$$\sup\{n \mid \mathrm{R}^n\mathrm{H}^0(X,-) \neq 0\}$$

of the global sections functor $\mathrm{H}^0 : \mathbf{Coh}(X) \to k-\mathbf{Vect}$. As a theorem of Lichtenbaum asserts that the cohomological dimension of X is equal to $\dim X$, it follows that the cohomological dimension of X coincides with dimcoh($\mathbf{Coh}(X)$) provided X is smooth. If X is not smooth, then we conclude from above that the cohomological dimension of X is distinct from dimcoh($\mathbf{Coh}(X)$), as the cohomological dimension is always bounded from above by $\dim X$ by a theorem of Grothendieck.
(ii) Most of what we said above breaks down for affine varieties, and for that reason we restrict ourselves to projective varieties. For if X is affine n-space over k the global sections $\mathrm{H}^0(X, \mathcal{O}_X) = k[T_1, \ldots, T_n]$ is of course not a finite dimensional k-vector space (thus $\mathrm{D}(X)$ is not Hom-finite), the cohomological dimension of X is 0 ([**14**], III, théorème 1.3.1), and

$$\mathrm{dimcoh}(\mathbf{Coh}(X)) = \mathrm{dimcoh}(k[T_1, \ldots, T_n]) = n$$

by Hilbert's syzygy theorem. (Notice that if A is a Noetherian ring, then the cohomological dimension of the category of A-modules coincides with the cohomological dimension of the subcategory consisting of finite type A-modules.)

2. **Derived Equivalence and Derived Invariants of Varieties.**

Let X be a smooth projective variety over a field k. A projective variety Y over k is said to be *derived equivalent* to X if there is an equivalence of k-linear triangulated categories $\mathrm{D}(X) \simeq \mathrm{D}(Y)$. As we mentioned in the introduction, this notion is weaker than isomorphism. By a *derived invariant* we shall mean a property of varieties that is shared by derived equivalent varieties. In this section we discuss a few derived invariants, as well as some conjectures regarding them.

THEOREM 2.1. *Over a perfect field smoothness is a derived invariant.*

Proof. This follows immediately from theorem 1.3. ■

We do not know a proof of this theorem over an arbitrary ground field. In all that follows, we restrict ourselves to derived equivalences between smooth varieties.

LEMMA 2.1. *Let $F : \mathrm{D}(X) \to \mathrm{D}(Y)$ be an equivalence of categories. Then there is a functorial isomorphism*

$$S \circ F \xrightarrow{\sim} S \circ F.$$

(We do not distinguish notationally between the Serre functors on X and Y.) This follows immediately from Yoneda's lemma, and will be used a few times in the sequel.

2.1. **Dimension.** Dimension is also a derived invariant ([**16**], proposition 4.1).

THEOREM 2.2. *If X and Y are derived equivalent, then* $\dim X = \dim Y$ *and the order of ω_X equals to the order of ω_Y. In particular, ω_Y is trivial whenever ω_X is trivial.*

(The order of ω_X is the order of ω_X in the Picard group of X.)

Proof. Let $x \in X$ be a closed point, and $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ be an equivalence. Then $\mathcal{O}_x \simeq S(\mathcal{O}_x)[-\dim X]$, and thus

$$F(\mathcal{O}_x) \simeq F(\mathcal{O}_x) \otimes \omega_Y[\dim Y - \dim X]$$

by lemma 2.1. Let m be maximal (resp. minimal) with $\mathrm{H}^m(F(\mathcal{O}_x)) \neq 0$. By taking H^m of the above equation and using that the exact functor $- \otimes \omega_Y$ commutes with cohomology we obtain $\dim X = \dim Y$. Denoting the latter by n, it is evident that for $k \geq 1$ the sheaf ω_X^k is trivial if and only if the functor $S^k \circ [-kn]$ is functorially isomorphic to the identity. ∎

2.2. **K-Theory.** The K-theory (Grothendieck group) of triangulated categories can be defined in a similar way as for abelian categories, by imposing the relation $[A] = [A'] + [A'']$ for every distinguished triangle

$$A' \to A \to A'' \to A'[1].$$

The distinguished triangle $A \to 0 \to A[1] \to A[1]$ (whose existence is guaranteed by the axioms of a triangulated category) shows that $[A[1]] = -[A]$, while $A \to A \oplus B \to B \to A[1]$ gives $[A \oplus B] = [A] + [B]$. The canonical functor $\mathbf{Coh}(X) \to \mathrm{D}(X)$ induces a morphism of abelian groups $\mathrm{K}_0(X) = \mathrm{K}_0(\mathbf{Coh}(X)) \to \mathrm{K}_0(\mathrm{D}(X))$.

THEOREM 2.3. *The canonical map* $\mathrm{K}_0(X) \to \mathrm{K}_0(\mathrm{D}(X))$ *is an isomorphism.*

Proof. We show that the Euler characteristic $\chi : \mathrm{K}_0(\mathrm{D}(X)) \to \mathrm{K}_0(X)$ given by

$$\chi([E]) = \sum (-1)^i [\mathrm{H}^i(E)]$$

is an inverse. To see that the composition $\mathrm{K}_0(\mathrm{D}(X)) \to \mathrm{K}_0(X) \to \mathrm{K}_0(\mathrm{D}(X))$ is the identity, let E in $\mathrm{D}(X)$ be fixed and $n = \min\{m \mid \mathrm{H}^m(E) \neq 0\}$. Then the standard distinguished triangle

$$\mathrm{H}^n(E)[-n] \to E \to E' \to \mathrm{H}^n(E)[-n][1]$$

with $\mathrm{H}^m(E') = 0$ for $m \leqslant n$, $\mathrm{H}^m(E) = \mathrm{H}^m(E')$ for $m > n$, gives

$$[E] = [\mathrm{H}^n(E)[-n]] + [E'] = (-1)^n[\mathrm{H}^n(E)] + [E']$$

and the desired equality $[E] = \sum(-1)^i[\mathrm{H}^i(E)]$ follows by iterating. ■

This makes it clear that K-theory $\mathrm{K}(X)$ is a derived invariant of X; it is also evident that the proof works for any abelian category instead of $\mathbf{Coh}(X)$.

2.3. **Fourier-Mukai transforms.** A fundamental theorem of D. Orlov [**22**] describes equivalences of derived categories of coherent sheaves much more explicitly than one might expect. Let X and Y be smooth projective varieties over k, and let

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

be the projections. The morphisms p and q are obtained from the structural morphism of X and Y via base change, and are therefore proper and flat. Let us define a functor

$$\Phi : \mathrm{D}(X \times Y) \to \mathrm{Hom}(\mathrm{D}(X), \mathrm{D}(Y))$$

by $\Phi_K = \mathrm{R}q_*(K \otimes p^*(-))$. The functor Φ_K is called the *Fourier-Mukai transform with kernel K*; as a composite of exact functors Φ_K is itself exact. (In the terminology of [**18**], Φ_K is referred to as an *integral functor*.)

THEOREM 2.4 (Mukai). *Let X, Y, and Z be smooth projective varieties over k; let K be an object of $\mathrm{D}(X \times Y)$ and L an object of $\mathrm{D}(Y \times Z)$.*
(i) Φ_K has left adjoint and right adjoints

$$\Phi_{K^\vee \otimes q^* \omega_Y[\dim Y]} \quad \text{and} \quad \Phi_{K^\vee \otimes p^* \omega_X[\dim X]},$$

respectively.
(ii) Let

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "s"'] \arrow[d, "r"'] \arrow[dr, "t"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

be the projections. Then

$$\Phi_L \circ \Phi_K \simeq \Phi_{\mathrm{R}r_*(s^*(K) \otimes t^*(L))} : \mathrm{D}(X) \to \mathrm{D}(Z).$$

In part (i), which is a consequence of Grothendieck-Verdier duality ([**16**], proposition 5.9), we identify $X \times Y$ with $Y \times X$, whereas part (ii) follows from the projection formula ([**16**], proposition 5.10).

Example 2.1. The shift functor $[n] : \mathrm{D}(X) \to \mathrm{D}(X)$ is functorially isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[n]$, where $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_X$ is the structure sheaf of the diagonal. For

$$\begin{aligned}\mathrm{R}q_*(\mathcal{O}_\Delta[n] \otimes p^*(-)) &\xrightarrow{\sim} \mathrm{R}q_*(\mathrm{R}\Delta_*(\Delta^*\mathcal{O}_X[n] \otimes p^*(-)))\\ &\xrightarrow{\sim} \mathrm{R}(q \circ \Delta)_* \circ (p \circ \Delta)^* \circ [n] \xrightarrow{\sim} [n]\end{aligned}$$

by the projection formula and Grothendieck's theorem for derived functors of composites.

Example 2.2. Let $f : X \to Y$ a proper morphism, and denote by $\Gamma : X \to X \times Y$ the graph morphism; it satisfies $f = q \circ \Gamma$ and $p \circ \Gamma = \mathrm{id}_X$. If $K = \Gamma_* \mathcal{O}_X$ is the structure sheaf of the graph of f, then $\Phi_K \xrightarrow{\sim} \mathrm{R}f_*$ for

$$\begin{aligned}\Phi_K = \mathrm{R}q_*(\Gamma_*\mathcal{O}_X \otimes p^*(-)) &\xrightarrow{\sim} \mathrm{R}q_*(\mathrm{R}\Gamma_*(\mathcal{O}_X \otimes \Gamma^* \circ p^*(-)))\\ &\xrightarrow{\sim} \mathrm{R}q_* \circ \mathrm{R}\Gamma_* \xrightarrow{\sim} \mathrm{R}(q_* \circ \Gamma_*) = \mathrm{R}f_*\end{aligned}$$

again by the projection formula and Grothendieck's theorem for derived functors of composites.

In particular, a Fourier-Mukai transform need not be fully faithful.

THEOREM 2.5 (Orlov). *For any fully faithful exact functor $F : \mathrm{D}(X) \to \mathrm{D}(Y)$ there is an object K in $\mathrm{D}(X \times Y)$ and a functorial isomorphism*

$$F \xrightarrow{\sim} \Phi_K.$$

The object K is determined up to isomorphism by F.

Remark 2.1. (i) It was a folklore conjecture that every exact functor $\mathrm{D}(X) \to \mathrm{D}(Y)$ is isomorphic to one of the form Φ_K for some K (in other words, that Φ is essentially surjective); this turned out to be wrong. We refer to [**26**], theorem 1.4, for an example of an exact functor $\mathrm{D}(X) \to \mathrm{D}(\mathbf{P}^4)$ (X a smooth quadric in $\mathbf{P}^4$) which is not a Fourier-Mukai transform.
(ii) Orlov's theorem can be used to prove Gabriel's theorem (X is determined by $\mathbf{Coh}(X)$), see corollary 5.24 in [**16**].

It is natural to attempt to classify those Fourier-Mukai transforms which are equivalences; this problem can be split into two parts: (i) for a given X find the set of isomorphism classes of smooth projetive varieties derived equivalent to X; (ii) classify the group of autoequivalences of $\mathrm{D}(X)$. We shall address only part (i), and refer to (for instance) [**16**] for a discussion of (ii).

The proof of theorem 2.5 is long and difficult; originally, in [**22**] (theorem 2.2), it was assumed that F admits a right adjoint. It turns out that this condition is automatically satisfied. To show that F has a right adjoint, it suffices to show that for any object E in $\mathrm{D}(Y)$ the cohomological functor

$$H = \mathrm{Hom}_{\mathrm{D}(X)}(F(-), E)$$

is representable, which follows from the following representability theorem (see [**6**], theorem 1.1) as well as theorem 1.3.

THEOREM 2.6 (Bondal, Van den Bergh). *Let* $H : \mathrm{D}(X)^{op} \to k - \mathbf{Vect}$ *be a cohomological functor with* $\sum_{n \in \mathbf{Z}} \dim_k H(E[i]) < \infty$ *for all* E *in* $\mathrm{D}(X)$. *Then* H *is representable.*

Let us give an application of theorem 2.5; the following theorem is due to [**23**], lemma 2.12.

LEMMA 2.2. *Let* K *be an object of* $\mathrm{D}(X \times Y)$ *and choose the notations as in theorem 2.4 (ii) (with* $Y = Z$*). Then* Φ_K *is an equivalence if and only if*

$$\mathrm{R}r_*(s^*(K) \otimes t^*(K^\vee \otimes q^*\omega_Y[\dim Y])) \simeq \mathcal{O}_{\Delta(X)},$$

$$\mathrm{R}r_*(s^*(K^\vee \otimes q^*\omega_Y[\dim Y]) \otimes t^*(K) \simeq \mathcal{O}_{\Delta(Y)}.$$

(We view $K^\vee$ as an object of $\mathrm{D}(Y \times X)$; in the second equality we have exchanged X and Y in the notation of the first equality.)

Proof. If Φ_K is an equivalence, then any quasi-inverse to Φ_K is isomorphic to the right and left adjoints of Φ_K. By combining parts (i) and (ii) of theorem 2.4 and using that the identity of $\mathrm{D}(X)$ is $\Phi_{\mathcal{O}_{\Delta(X)}}$ ($n = 0$ in example 2.1) we obtain the first isomorphism by the uniqueness statement of theorem 2.5. The second isomorphism follows similarly. On the other hand, if these isomorphisms hold then we see by theorem 2.4, (ii), that $\Phi_{K^\vee \otimes q^*\omega_Y[\dim Y])}$ is a quasi-inverse to Φ_K. ■

THEOREM 2.7. *Let* X *and* Y *be smooth projective varieties over* k*, and* L *be a field extension of* k. *If* X *and* Y *are derived equivalent, then* X_L *and* Y_L *are also derived equivalent.*

(Here we abbreviate $X \times_k \operatorname{Spec} L$ by X_L.)

Proof. Let $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ be an equivalence and K a kernel for F, $F \xrightarrow{\sim} \Phi_K$ (theorem 2.5). We choose our notations as in the following commutative diagram, in which all maps are (induced by) projections.

$$\begin{tikzcd}
 & & X_L \times Y_L & & \\
X_L \arrow[d, "x"'] & X_L \times_L Y_L \arrow[l, "p_L"'] \arrow[d, "f"'] & X_L \times_L Y_L \times_L X_L \arrow[l, "s_L"'] \arrow[u, "r_L"] \arrow[r, "t_L"] \arrow[d, "g"] & Y_L \times_L X_L \arrow[r, "q_L"] \arrow[d, "f"] & Y_L \arrow[d, "y"] \\
X & X \times_k Y \arrow[l, "p"'] & X \times_k Y \times_k X \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "r"] & Y \times_k X \arrow[r, "q"] & Y \\
 & & X \times_k X & &
\end{tikzcd}$$

Consider the object $K_L = f^*(K)$ of $\mathrm{D}(X_L \times_L Y_L)$. To prove that

$$\Phi_{K_L} : \mathrm{D}(X_L) \to \mathrm{D}(Y_L)$$

is an equivalence, it suffices (by the above lemma) to show

$$\mathrm{R}(r_L)_*(s_L^*(K_L) \otimes t_L^* K_L^\vee \otimes q_L^* \omega_{Y_L}[\dim Y]) \simeq \mathscr{O}_{\Delta(X_L)} \quad \text{in} \quad \mathrm{D}(X_L \times_L X_L).$$

(The second isomorphism required by the lemma is proven analogously.) To see this we note that since $\omega_{Y_L} = y^* \omega_Y$ the left hand side is isomorphic to

$$\mathrm{R}(r_L)_*((f \circ s_L)^*(K) \otimes \mathrm{R}\mathscr{H}\mathrm{om}((t \circ g)^*(K), (q \circ t \circ g)^*(\omega_Y)))[\dim Y],$$

which (since $\mathrm{R}\mathscr{H}\mathrm{om}$ commutes with pullback) is in turn isomorphic to

$$(\mathrm{R}(r_L)_* \circ g^*)(s^*(K) \otimes t^* \mathrm{R}\mathscr{H}\mathrm{om}(K, q^* \omega_Y))[\dim Y].$$

By flat base change $\mathrm{R}(r_L)_* \circ g^* \xrightarrow{\sim} (x \times x)^* \circ \mathrm{R}r_*$, which then gives ($\Phi_K$ is an equivalence) the desired isomorphism. ■

Remark 2.2. Similarly one proves that if X and Y are derived equivalent, X' and Y' are derived equivalent, then $X \times X'$ and $Y \times Y'$ are also derived equivalent.

Orlov's theorem explains why derived equivalent varieties are often called *Fourier-Mukai partners*. We shall follow this terminology and denote by $\mathrm{FM}(X)$ the set of isomorphism classes of Fourier-Mukai partners of X. The natural finiteness conjecture regarding $\mathrm{FM}(X)$ has not yet been resolved, but it has been established (over $k = \mathbf{C}$) in dimension $\leqslant 2$ by T. Bridgeland and A. Maciocia ([**7**], corollary 1.2), and Y. Kawamata ([**18**], theorem 1.6) using the Enriques-Kodaira classification.

CONJECTURE 2.1. $\#\mathrm{FM}(X) < \infty$.

This conjecture is also known to be true for abelian varieties ([**23**], corollary 2.8). A natural question is whether $\#\mathrm{FM}(X) = 1$, i.e. whether $\mathrm{D}(X)$ determines X. This is false ([**21**], theorem 2.2).

THEOREM 2.8 (Mukai). *Let k be algebraically closed, X an abelian variety over k with dual $\hat{X}$. If P is the Poincaré bundle on $X \times \hat{X}$, then the Fourier-Mukai transform $\Phi_P : \mathrm{D}(X) \to \mathrm{D}(\hat{X})$ is an equivalence of categories.*

However, for a certain class of varieties one has a reconstruction theorem ([**5**], theorem 2.5). We refer to example 1.1 for a few explicit members of this class.

THEOREM 2.9 (Bondal, Orlov). *Let X be a smooth projective variety over a field k with ω_X ample or anti-ample. If Y is a smooth projective variety over k that is derived equivalent to Y, then X is isomorphic to Y.*

Thus $\#\mathrm{FM}(X) = 1$ whenever ω_X is ample or anti-ample. We will explain a proof of this theorem in section 3; note that Y is not assumed to have ample or anti-ample canonical sheaf. (This accounts for the asymmetry of X and Y in the proof.) Furthermore, the proof only uses that the equivalence commutes with the shift functor. The main idea of the proof is to give a categorical characterization of the skyscraper sheaves $\mathcal{O}_x$ and line bundles.

There is also a reconstruction theorem for certain surfaces ([**7**], proposition 6.1 and proposition 6.2).

THEOREM 2.10 (Bridgeland, Maciocia). *If X is a Enriques or a bielliptic surface, then* $\#\mathrm{FM}(X) = 1$.

2.4. **The bigraded algebra** $\mathrm{HA}(X)$**.** Let $\Delta : X \to X \times X$ be the diagonal map, and consider the bigraded k-algebra

$$\mathrm{HA}(X) = \bigoplus_{i,j \geqslant 0} \mathrm{HA}_{i,j}(X),$$

where $\mathrm{HA}_{i,j}(X) = \mathrm{Ext}^i_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^j) = \mathrm{Hom}_{\mathrm{D}(X\times X)}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^j[i])$, and where the product $\mathrm{HA}_{i,j} \times \mathrm{HA}_{k,l} \to \mathrm{HA}_{i+k,j+l}$ is the isomorphism

$$\begin{aligned} &\mathrm{Hom}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^j[i]) \times \mathrm{Hom}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^l[k]) \\ \xrightarrow{\sim}\ &\mathrm{Hom}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X^j[i]) \times \mathrm{Hom}(\Delta_*\omega_X^j[i], \Delta_*\omega_X^{j+l}[i+k]) \end{aligned}$$

(induced by the equivalence $- \otimes \Delta_*\omega_X^j[i]$) followed by composition in $\mathrm{D}(X \times X)$. We have also used the isomorphisms

$$\Delta_*(\omega_X^j \otimes \omega_X^l) \xrightarrow{\sim} \Delta_*(\omega_X^j \otimes \Delta^*\Delta_*\omega_X^l) \xrightarrow{\sim} \Delta_*\omega_X^j \otimes \Delta_*\omega_X^l$$

coming from the projection formula.

THEOREM 2.11. *The bigraded k-algebra* $\mathrm{HA}(X)$ *is a derived invariant of* X.

Proof. We merely sketch the main ideas of the proof, which can be adapted from the proof of proposition 6.1 in [**16**]. Let F be an equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$

and (by theorem 2.5) K a kernel for F, $F \xrightarrow{\sim} \Phi_K$. Let $n = \dim X = \dim Y$ (by theorem 2.2) and consider the object

$$L = K^\vee \otimes p^*\omega_X[n]$$

of $\mathrm{D}(X \times Y)$, where $K^\vee = \mathrm{R}\mathscr{H}\mathrm{om}(K, \mathscr{O}_{X\times Y})$ is the derived dual of K. Then $L \boxtimes K$ may be regarded as an object of $\mathrm{D}((X \times X) \times (Y \times Y))$, and one shows that the Fourier-Mukai transforms $\mathrm{D}(Y) \to \mathrm{D}(Y)$ with kernel $\Phi_{L\boxtimes K}(\omega_X^k)$, respectively ω_Y^k, are both isomorphic to $S_Y^k[kn]$. Hence $\Phi_{L\boxtimes K}(\omega_X^k)$ and ω_Y^k are isomorphic by uniqueness of kernel (theorem 2.5). The equivalence $\Phi_{L\boxtimes K}$ thus provides isomorphisms $\mathrm{HA}_{i,j}(X) \xrightarrow{\sim} \mathrm{HA}_{i,j}(Y)$. ■

In particular, the canonical algebra

$$\mathrm{R}(X) = \bigoplus_{j\geqslant 0} \mathrm{HA}_{0,j}(X)$$

(Δ_* gives an isomorphism $\mathrm{Hom}_X(\mathscr{O}_X, \omega_X^i) \xrightarrow{\sim} \mathrm{Hom}_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\omega_X^i)$ since Δ is a closed immersion) and the Hochschild homology algebra

$$\mathrm{HH}(X) = \bigoplus_{i\geqslant 0} \mathrm{HA}_{i,1}(X)$$

are derived invariants.

2.5. **Hodge numbers and birational equivalence.** Let $k = \mathbf{C}$ and X a smooth projective variety of dimension n over $\mathbf{C}$. The *Hodge numbers* of X are

$$h^{p,q} = h^{p,q}(X) = \dim_{\mathbf{C}} \mathrm{H}^p(X, \Omega_X^q) \quad (0 \leqslant p, q \leqslant n).$$

They satisfy the relations $h^{p,q} = h^{q,p}$ (Hodge symmetry) and $h^{p,q} = h^{n-p,n-q}$ (Serre duality). (For the last equality notice that

$$\mathrm{Hom}(\mathscr{O}_X, \Omega_X^q[p]) \simeq \mathrm{Hom}(\Omega^q, \Omega^n[n-p]) \simeq \mathrm{Hom}(\mathscr{O}_X, \Omega^{n-q}[n-p])$$

since $\Omega^{n-q} \simeq (\Omega^q)^\vee \otimes \Omega^n$.) Let us assemble these numbers into the Hodge diamond

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & \iddots & & \ddots & \\ h^{n,0} & & & & h^{0,n} \\ & \ddots & & \iddots & \\ & & h^{0,0} & & \end{array}$$

as well as the Hodge polynomial

$$h = \sum_{p,q=0}^{n} h^{p,q} S^p T^q \in \mathbf{Z}[S, T].$$

The Hodge symmetry gives a vertical symmetry of the Hodge diamond and asserts that h is a symmetric polynomial, $h(S, T) = h(T, S)$. Serre duality, on the other hand, gives a radial symmetry of the Hodge diamond as well as $h(S, T) = h(1/S, 1/T)S^n T^n$. The sum $\sum_{p+q=m} h^{p,q}$ of the entries of the m-th row is the Betti number b_m, and thus $h(S, S)$ is just the Poincaré polynomial.

Example 2.3. (i) The Hodge polynomial of a cubic surface X in $\mathbf{P}^3$ can be computed as $1 + 7ST + S^2T^2$.
(ii) The Hodge polynomial of a K3 surface X is $1 + S^2 + T^2 + 20ST + S^2T^2$ (see section 5).

CONJECTURE 2.2. *The Hodge polynomial is a derived invariant of X.*

The derived invariance of the Poincaré polynomial is also merely conjectural; however, the derived invariance of the topological Euler characteristic is known ([**16**], exercise 5.38).

LEMMA 2.3. *For every $m \geqslant 0$ the sum $\sum_{p+q=m} h^{p,n-q}$ is a derived invariant.*

(Notice that this is the sum of the entries of the m-th column of the Hodge diamond.)

Proof. Consider the local to global spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X \times X, \mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X)) \Rightarrow \mathrm{Ext}^{p+q}_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X).$$

Since $\mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X)$ is supported on the diagonal, we have

$$\mathrm{H}^p(X \times X, \mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X)) \simeq \mathrm{H}^p(X, \Delta^* \mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X)).$$

As ω_X is locally free, we also have isomorphisms

$$\Delta^* \mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \omega_X) \simeq \Delta^* \mathscr{E}xt^q_{X \times X}(\Delta_* \mathscr{O}_X, \Delta_* \mathscr{O}_X) \otimes \omega_X \simeq \bigwedge^q T_X \otimes \omega_X \simeq \Omega_X^{n-q}.$$

This yields a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \Omega_X^{n-q}) \Rightarrow \mathrm{HA}_{p+q,1}.$$

Swan [**29**] (corollary 2.6) shows that the latter spectral sequence degenerates to give an isomorphism

$$\mathrm{HA}_{m,1}(X) = \bigoplus_{p+q=m} \mathrm{H}^p(X, \Omega_X^{n-q}).$$

The conclusion now follows from theorem 2.11. ∎

This result can be used to verify conjecture 2.2 in low dimensions.

THEOREM 2.12 (Popa, Schnell). *The conjecture holds if X has dimension $\leqslant 3$.*

Proof. We merely explain the main ideas of the proof for $\dim X = 3$, which is due to [**25**]. The Hodge diamond

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

satisfies $h^{0,0} = 1$. Hodge symmetry and Serre duality show that the diamond is determined by the Hodge numbers

$$h^{1,0}, h^{1,1}, h^{2,0}, h^{2,1}, h^{3,0}.$$

According to the above lemma, the sums

$$h^{3,0}, h^{2,0} + h^{3,1}, h^{1,0} + h^{2,1} + h^{3,2}, 1 + h^{1,1} + h^{2,2} + 1$$

are derived invariants. Thus it is clear that $h^{1,1}$ and $h^{2,0}$ are derived invariants, and it remains to show that $h^{1,0}$ and $h^{2,1}$ are derived invariants. Now the main theorem of [**25**] is that derived equivalent varieties have isogenous Picard variety Pic^0. (Conjecturally the Picard varieties are even derived equivalent.) But the dimension of Pic^0 is $h^{1,0}$, which is thus invariant, and the invariance of $h^{1,0}+h^{2,1}+h^{3,2}$ yields the invariance of $h^{2,1}$. ■

Remark 2.3. The proof in dimensions 1 and 2 is of course analogous, but one can readily check that one does not need to rely on the invariance of $h^{1,0}$.

We finish this section by indicating (conjectural) relations between derived equivalence, birational equivalence, K-equivalence, as well as the derived invariance of the Hodge polynomial. Since there are few invariants which are both derived and birational invariants (such as dimension, the canonical algebra, $h^{1,0}$), one can expect a close relation.

Recall that smooth projective varieties X and Y are called *K-equivalent* if there is a smooth projective variety Z and birational morphisms

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

such that $f^*\omega_X = g^*\omega_Y$ in the Picard group of Z. (If X and Y have trivial canonical sheaf, then it is obvious that X and Y are K-equivalent if and only if they are birationally equivalent.) It is known that K-equivalent curves or surfaces are isomorphic ([**16**], corollary 12.8).

The following conjecture is due to [**18**].

CONJECTURE 2.3 (Kawamata). *If X and Y are K-equivalent, then they are derived equivalent.*

For Calabi-Yau varieties this is a conjecture of Bondal and Orlov; it is known to be true in dimension $\leqslant 3$ ([**9**], theorem 1.1.) The converse of that conjecture is wrong, however ([**4**]).

One reason why K-equivalence is a fundamental relationship is the next theorem of Kontsevich, proven using the change of variables formula in motivic integration

THEOREM 2.13 (Kontsevich). *If X and Y are K-equivalent, then $[X] = [Y]$ in the ring $\hat{\mathcal{M}}_{\mathbf{C}}$.*

Here $\hat{\mathcal{M}}_{\mathbf{C}}$ is the completion (with respect to a certain filtration) of the localization

$$\mathcal{M}_{\mathbf{C}} = K_0(\mathrm{Var}_0)[\mathbf{L}^{-1}]$$

of the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbf{C}})$ at $\mathbf{L} = [\mathbf{A}^1_{\mathbf{C}}]$. The class of X in $\hat{\mathcal{M}}_{\mathbf{C}}$ entails information about the Hodge theory of X, since the Hodge polynomial induces a ring morphism from $\hat{\mathcal{M}}_{\mathbf{C}}$; in particular, $[X] = [Y]$ implies that X and Y have the same Hodge polynomial. Thus the following theorem of Kawamata ([**18**], theorem 2.3) shows the derived invariance of the Hodge polynomial for varieties of general type.

THEOREM 2.14. *If smooth projective varieties of general type are derived equivalent, then they are K-equivalent.*

3. **Decomposition of Derived Categories and Reconstruction.**

3.1. **Semi-orthogonal decompositions.** For a geometrically very simple variety such as the projective line $\mathbf{P}^1$ one can describe the objects of $\mathrm{D}(\mathbf{P}^1)$ rather explicitly. Namely, every object is a direct sum of shifts of coherent sheaves (see for instance [**16**], corollary 3.15) and every coherent sheaf on $\mathbf{P}^1$ is a direct sum of line bundles (which are the twisting sheaves $\mathcal{O}(m)$) and degree 1 skyscraper sheaves.

For more complicated varieties X one cannot expect such a simple description of $\mathrm{D}(X)$, but there are notions that facilitate a description of the structure of $\mathrm{D}(X)$.

A full triangulated subcategory D of $\mathrm{D}(X)$ is said to be *admissible* if the inclusion functor $D \to \mathrm{D}(X)$ admits a right adjoint. We denote by $D^{\perp}$ the full subcategory of $\mathrm{D}(X)$ consisting of all objects E such that $\mathrm{Hom}(-, E)$ vanishes on D. For an object E of $\mathrm{D}(X)$ we let $\langle E \rangle$ be the smallest full triangulated subcategory of $\mathrm{D}(X)$ containing E. We call E *exceptional* if $\mathrm{Hom}(E, E) = k$ and $\mathrm{Hom}(E, E[n]) = 0$ for $n \neq 0$; then $\langle E \rangle$ is admissible ([**16**], lemma 1.58).

Example 3.1. (i) The objects $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ in $\mathrm{D}(\mathbf{P}^n)$ are exceptional.
(ii) If ω_X is trivial and $\dim X > 0$, then $\mathrm{Hom}(E, E) \simeq \mathrm{Hom}(E, E[\dim X])$ shows that $\mathrm{D}(X)$ has no exceptional objects.

A *semi-orthogonal decomposition* of $\mathrm{D}(X)$ is a sequence $D_1, \ldots, D_n$ of full admissible triangulated subcategories of $\mathrm{D}(X)$ such that $D_j \subset D_i^{\perp}$ for all $i > j$ and $\mathrm{D}(X)$ is generated by $D_1, \ldots, D_n$. (In the sense that if D is the smallest full triangulated subcategory of $\mathrm{D}(X)$ containing $D_1, \ldots, D_n$, then the inclusion $D \to \mathrm{D}(X)$ is an equivalence.

A *full exceptional sequence* in $\mathrm{D}(X)$ is a sequence $E_1, \ldots, E_n$ of exceptional objects such that $\langle E_1 \rangle, \ldots, \langle E_n \rangle$ is a semi-orthogonal decomposition of $\mathrm{D}(X)$.

THEOREM 3.1 (Beilinson). *The sequence* $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ *is a full expectional sequence in* $\mathrm{D}(\mathbf{P}^n)$

(The proof is an application of the Beilinson spectral sequence, and we refer to [**16**], corollary 8.29.)

The *Euler form* of $E, F \in \mathrm{D}(X)$ is defined by

$$\chi(E, F) = \sum_{n \in \mathbf{Z}} (-1)^n \dim_k \mathrm{Hom}(E, F[n]).$$

(If $\mathcal{F}$ is a coherent sheaf on X, then $\chi(\mathcal{O}_X, \mathcal{F})$ is the Euler characteristic of $\mathcal{F}$.) Since $\mathrm{D}(X)$ is Ext-finite (theorem 1.3), this expression is well-defined. Clearly χ is preserved by any equivalence. In general χ is not symmetric or anti-symmetric, but if X has trivial canonical sheaf then Serre duality implies the relation

$$\chi(E, F) = (-1)^{\dim X} \chi(F, E).$$

As χ is biadditive with respect to distinguished triangles, it induces a bilinear form

$$\chi : \mathrm{K}_0(X) \times \mathrm{K}_0(X) \to \mathbf{Z}.$$

(Here we identify $\mathrm{K}_0(X)$ with $\mathrm{K}_0(\mathrm{D}(X))$ by the canonical isomorphism of theorem 2.3.) The existence of a finite generating set of $\mathrm{D}(X)$ yields a finiteness condition for $\mathrm{K}_0(X)$ ([**3**], proposition 2.3.6).

PROPOSITION 3.1. *Let $E_1, \dots, E_n$ be a full exceptional sequence in* $\mathrm{D}(X)$. *Then the classes $[E_1], \dots, [E_n]$ form a* $\mathbf{Z}$*-module basis of* $K_0(X)$. *Furthermore*

$$\chi([E_i],[E_i]) = 1 \quad \text{and} \quad \chi([E_j],[E_k]) = 0$$

for all $1 \leqslant i \leqslant n$ *and* $1 \leqslant k < j \leqslant n$

(In other words, the matrix of χ with respect to this basis is upper triangular with ones on the diagonal.)

Proof. The definition of a generating class of objects of $\mathrm{D}(X)$ and the remarks before theorem 2.3 make it clear that $[E_1], \dots, [E_n]$ is a generating set for $\mathrm{K}_0(X)$. The relations $\chi([E_i],[E_i]) = 1$ and $\chi([E_j],[E_k]) = 0$ follow immediately from the definition of a full exceptional sequence. To see that $[E_1], \dots, [E_n]$ are $\mathbf{Z}$-linearly independent, let $\sum_{i=1}^n \lambda_i [E_i] = 0$ with $\lambda_i \in \mathbf{Z}$. Taking $\chi(-,[E_1]), \dots, \chi(-,[E_n])$ of $\sum_{i=1}^n \lambda_i [E_i]$ we obtain $\lambda_1 = \cdots = \lambda_n = 0$. ■

In particular the length n of a full exceptional sequence in $\mathrm{D}(X)$ is uniquely determined by X as $n = \dim_{\mathbf{Q}} \mathrm{K}_0(X) \otimes \mathbf{Q} = \dim_{\mathbf{Q}} \mathrm{A}(X) \otimes \mathbf{Q}$.

COROLLARY 3.1. $\mathrm{K}_0(\mathbf{P}^n)$ *is a free* $\mathbf{Z}$*-module of rank* $n+1$.

Proof. Combine theorem 3.1 with proposition 3.1. ■

3.2. **A spanning class for** $\mathrm{D}(X)$**.** The objects $\mathcal{O}_x$ $(x \in X)$ are mutually orthogonal in $\mathrm{D}(X)$.

LEMMA 3.1. *(i)* $\mathrm{Hom}(E, \mathcal{O}_x[n]) = 0$ *for all closed points* $x \in X$ *and* $n \in \mathbf{Z}$ *implies* $E = 0$.
(ii) For $x \neq y$ *we have* $\mathrm{Hom}(\mathcal{O}_x, \mathcal{O}_y[i]) = 0$ *for* $i \in \mathbf{Z}$.
(iii) If $\mathcal{O}_x[i] \simeq \mathcal{O}_y[j]$, *then* $i = j$ *and* $x = y$.

(In common parlance the set of all $\mathcal{O}_x$ forms a *spanning class* for $\mathrm{D}(X)$.)

Proof. (i) If $E \neq 0$ let n be maximal with $\mathrm{H}^n(E) \neq 0$ and x be a point in the support of $\mathrm{H}^n(E)$. The spectral sequence

$$E_2^{p,q} = \mathrm{Hom}(\mathrm{H}^{-q}(E), \mathcal{O}_x[p]) \Rightarrow \mathrm{Hom}(E, \mathcal{O}_x[p+q])$$

has $E_2^{0,-n} \neq 0$ and by the choice of n we obtain $\mathrm{Hom}(E, \mathcal{O}_x[n]) \neq 0$. As for (ii), we have $\mathrm{Hom}_{\mathrm{D}(X)}(\mathcal{O}_x, \mathcal{O}_y[i]) = \mathrm{Ext}_X^i(\mathcal{O}_x, \mathcal{O}_y)$. The sheaf $\mathcal{E}xt^q(\mathcal{O}_x, \mathcal{O}_y)$ vanishes identically (its support is empty), and hence the local-to-global spectral sequence yields $\mathrm{Ext}_X^i(\mathcal{O}_x, \mathcal{O}_y) = \mathrm{H}^0(X, \mathcal{E}xt^i(\mathcal{O}_x, \mathcal{O}_y)) = 0$. (iii) Compare the cohomology sheaves and their supports. ■

3.3. **Proof of the Bondal-Orlov reconstruction theorem.** For any invertible sheaf L_0 one defines a graded k-algebra

$$\mathrm{R}(X, L_0) = \bigoplus_{m \geqslant 0} \mathrm{Hom}(L_0, L_0 \otimes \omega_X^m).$$

(The product is defined as for the algebra $\mathrm{HA}(X)$ of section 2.4, but by using the equivalence $S^m \circ [-m \dim X]$.) The algebra $\mathrm{R}(X, \mathcal{O}_X)$ is just the canonical algebra $\mathrm{R}(X)$.

LEMMA 3.2. *Let X be a smooth projective variety with ample canonical sheaf. Then there is a canonical isomorphism $X \xrightarrow{\sim} \mathrm{Proj}\, \mathrm{R}(X, \mathcal{O}_X)$ of schemes over k.*

The previous lemma follows by applying proposition. 4.6.3 in [**14**], II, to the structural morphism $f : X \to \mathrm{Spec}\, k$. Recall that a family $(\mathcal{L}_i)_{i \in I}$ of invertible sheaves on X is an *ample family* if the set of supports

$$\{X_f \mid f \in \mathrm{H}^0(X, \mathcal{L}_i^n),\ n \in \mathbf{N},\ i \in I\}$$

is a basis for the topology on X. Clearly for any invertible sheaf $\mathcal{L}$ on X the family $(\mathcal{L}^n)_{n \in \mathbf{N}}$ is ample if and only if $\mathcal{L}$ is.

LEMMA 3.3. *Any smooth variety admits an ample family of invertible sheaves.*

An object P of $\mathrm{D}(X)$ is a *point object of codimension s* if $S(P) \cong P[s]$, $\mathrm{Hom}(P, P[i]) = 0$ for $i < 0$ and the ring $k(P) = \mathrm{Hom}(P, P)$ is a field. (The last condition implies that P is indecomposable.) By using the same arguments as in the proof of theorem 2.2 we see that a point object in $\mathrm{D}(X)$ always has codimension $\dim X$. It is also clear that an equivalence $\mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ sends point objects in $\mathrm{D}(X)$ to point objects in $\mathrm{D}(Y)$. If $x \in X$ is a closed point, then $P = \mathcal{O}_x[n]$ is a point object with

$$k(P) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\kappa(x), \kappa(x)) = \kappa(x).$$

LEMMA 3.4. *If X has ample or antiample canonical sheaf, then an object P of $\mathrm{D}(X)$ is a point object if and only if it is of the form $\mathcal{O}_x[m]$ for some $m \in \mathbf{Z}$ and $x \in X$.*

The idea of the proof is that since

$$\mathrm{H}^i(P) \otimes \omega_X \simeq \mathrm{H}^i(P)$$

and ω_X is ample, it follows (by, say, considering Hilbert polynomials) that the cohomology sheaves of P consist of a single point. For more details we refer to [**5**]. The assumption on ω_X is necessary.

Example 3.2. If $\omega_X = \mathcal{O}_X$ then $\mathcal{O}_X$ is a point object in $\mathrm{D}(X)$.

An object L of $\mathrm{D}(X)$ is an *invertible object* if for every point object P of $\mathrm{D}(X)$ there is $m \in \mathbf{Z}$ such that $\mathrm{Hom}(L, P[m]) = k(P)$ and $\mathrm{Hom}(L, P[i]) = 0$ for $i \neq m$.

LEMMA 3.5. *Let X be a smooth projective variety such that all point objects in $\mathrm{D}(X)$ are of the form $\mathcal{O}_x[m]$. Then an object L of $\mathrm{D}(X)$ is invertible if and only if there is $m \in \mathbf{Z}$ such that $L \simeq \mathcal{L}[m]$ for an invertible sheaf $\mathcal{L}$ on X.*

Proof. Let L be a invertible object in $\mathrm{D}(X)$, let n be maximal with $\mathrm{H}^n(L) \neq 0$, and x a closed point in the support of $\mathrm{H}^n(L)$. The spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Hom}(\mathrm{H}^q(L), \mathcal{O}_x[p]) \Rightarrow \mathrm{Hom}(L, \mathcal{O}_x[p-q])$$

and the definition of an invertible object shows that $\mathrm{Hom}(\mathrm{H}^n(L), \mathcal{O}_x) = \kappa(x)$ and $\mathrm{Ext}^1_X(\mathrm{H}^n(L), \mathcal{O}_x) = 0$. The second condition implies that $\mathrm{H}^n(L)$ is locally free (lemma 1.1), while the first one shows that the rank is one. Hence

$$\mathrm{Hom}(\mathrm{H}^n(L), \mathcal{O}_x[i]) = 0$$

for $i > 0$, and the above spectral sequence implies that the cohomology of L is concentrated in degree n, thus $L \cong \mathrm{H}^n(L)[n]$. ■

Proof of theorem 2.9. Let $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ be an equivalence with inverse G. Let $P_{\mathrm{D}(X)}$ be the set of isomorphism classes of point objects in $\mathrm{D}(X)$, and denote

$$P_X = \{\mathcal{O}_x[m] \mid x \in X \text{ closed},\ m \in \mathbf{Z}\}.$$

The canonical map $P_X \to P_{\mathrm{D}(X)}$ is bijective by lemma 3.4. The functor G induces a map $P_Y \to P_{\mathrm{D}(X)}$ which is injective by lemma 3.1. To see that it is also surjective, let P be an element of $P_{\mathrm{D}(X)}$ that is not contained in the image of the previous map. Since that image is closed under translation we obtain for all y and j

$$0 = \mathrm{Hom}_{\mathrm{D}(X)}(P, G(\mathcal{O}_y[j])) = \mathrm{Hom}_{\mathrm{D}(Y)}(F(P), \mathcal{O}_y[j])$$

by Lemma 3.1 (b) and hence $F(P) = 0$ by Lemma 3.1 (a). Now $P = 0$ as F is an equivalence and $k(P) = 0$ cannot be a field.

Let $L_{\mathrm{D}(X)}$ be the set of isomorphism classes of invertible objects in $\mathrm{D}(X)$, and

$$L_X = \{L[m] \mid L \text{ invertible sheaf on X},\ m \in \mathbf{Z}\};$$

by lemma 3.5 the natural map $L_X \to L_{\mathrm{D}(X)}$ is a bijection. By the previous step both X and Y satisfy the assumptions of Lemma 3.5, hence we also obtain a bijection $L_{\mathrm{D}(X)} \to L_Y$.

Let L_0 be a fixed invertible sheaf on X. After composing F with a translation we may assume that $F(L_0)$ is isomorphic to an invertible sheaf on Y, and we adjust F so that $F(L_0)$ *is* an invertible sheaf on Y (and not merely isomorphic to one).

We know from above that

$$p_{\mathrm{D}(X)} = \{P \in P_{\mathrm{D}(X)} \mid \mathrm{Hom}(L_0, P) = k(P)\}$$

is in bijection with $p_X = \{\mathcal{O}_x \mid x \in X\}$ and p_Y, and thus we obtain a bijection between the (closed) points of X and Y. (In the following we do not distinguish notationally between $\mathcal{O}_x$ and x.) Let l_X be the subset of L_X consisting of isomorphism classes of invertible sheaves on X, it is taken to l_Y by F and in natural bijection with

$$l_{\mathrm{D}(X)} = \{L \in L_{\mathrm{D}(X)} \mid \mathrm{Hom}_{\mathrm{D}(X)}(L, P) = k(P) \text{ for any } P \in p_{\mathrm{D}(X)}\}.$$

For $\alpha \in \mathrm{Hom}_{\mathrm{D}(X)}(L_1, L_2)$ with $L_1, L_2 \in l_{\mathrm{D}(X)}$ and $P \in p_{\mathrm{D}(X)}$ let α_P^* be the induced map $\alpha_P^* : \mathrm{Hom}_{\mathrm{D}(X)}(L_2, P) \to \mathrm{Hom}_{\mathrm{D}(X)}(L_1, P)$. If U_α denotes the set of $P \in p_D$ with $\alpha_P^* \neq 0$, then by lemma 3.3 the set

$$\{U_\alpha \mid L_1, L_2 \in l_{\mathrm{D}(X)},\ \alpha \in \mathrm{Hom}_{\mathrm{D}(X)}(L_1, L_2)\}$$

forms a basis for the topology on X. Let $n = \dim X = \dim Y$ by theorem 2.2, and let $L_m = S^m \circ [-nm](L_0) = L_0 \otimes \omega_X^m$ for $m \in \mathbf{Z}$, and after adjusting F we may assume $F(L_m) = F(L_0) \otimes \omega_Y^m$. As ω_X is ample (or anti-ample), the set

$$\{U_\alpha \mid \alpha \in \mathrm{Hom}_{\mathrm{D}(X)}(L_i, L_j),\ i, j \in \mathbf{Z}\}$$

with $\mathrm{Hom}_{\mathrm{D}(X)}(L_i, L_j) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(X)}(\omega_X^i, \omega_X^j)$ is a basis of the Zariski topology on X. Hence

$$\{U_\alpha \mid \alpha \in \mathrm{Hom}_{\mathrm{D}(Y)}(F(L_i), F(L_i)),\ i, j \in \mathbf{Z}\}$$

with $\mathrm{Hom}_{\mathrm{D}(Y)}(F(L_i), F(L_j)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(Y)}(\omega_Y^i, \omega_Y^j)$ is a basis for the topology on Y, and ω_Y is also ample.

We have isomorphisms of graded k-algebras

$$\begin{tikzcd} R(X, L_0) \arrow[r, "\sim"] & R(Y, F(L_0)) \\ R(X, \mathcal{O}_X) \arrow[u, "\wr"] & R(Y, \mathcal{O}_Y) \arrow[u, "\wr"'] \end{tikzcd}$$

where the left one is induced by the equivalence $L_0 \otimes - : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(X)$, the middle one by $F : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ and the right one by $F(L_0) \otimes - : \mathrm{D}(Y) \xrightarrow{\sim} \mathrm{D}(Y)$. The ring structure is preserved as equivalences commute with the Serre functors (lemma 2.1). Hence we obtain isomorphisms of schemes over k

$$X \xrightarrow{\sim} \mathrm{Proj}(R(X)) \xrightarrow{\sim} \mathrm{Proj}(R(Y)) \xrightarrow{\sim} Y,$$

where we use lemma 3.2. ∎

As a first application of the Bondal-Orlov reconstruction theorem, we indicate that a curve X is always determined by $\mathrm{D}(X)$.

Example 3.3. Let k be algebraically closed and X a curve of genus g. Then the Riemann-Roch theorem shows that ω_X is ample if and only if

$$\deg \omega_X = 2g - 2 > 0.$$

Thus ω_X is ample if and only if $g \geqslant 2$, trivial if and only if $g = 1$, and ω_X is antiample if and only if $g = 0$. By the Bondal-Orlov theorem it follows that X is determined by $\mathrm{D}(X)$ for $g \geqslant 2$ and $g = 0$, while for the case $g = 1$ (elliptic curves) one can use the Torelli theorem (see [**16**], corollary 5.46).

4. **Lattices and Hodge Structures.**

4.1. **Generalities on lattices.** Let A be a ring. We consider pairs consisting of a free finite type A-module Λ and a symmetric A-bilinear form

$$(\cdot,\cdot) : \Lambda \times \Lambda \to A.$$

For these there are obvious notions of morphism, embedding, and isomorphism (isometry). For $a \in A$ there is an induced symmetric bilinear form

$$a(\cdot,\cdot) : \Lambda \times \Lambda \to A$$

on Λ, which will be denoted by $\Lambda(a)$. If $(\cdot,\cdot)' : \Lambda' \times \Lambda' \to A$ is a symmetric bilinear form on Λ', then we obtain a symmetric bilinear form on $\Lambda \oplus \Lambda'$; we abbreviate $\oplus_{j=1}^{n} \Lambda$ by $n\Lambda$. The group of automorphisms of $(\cdot,\cdot)$ will be denoted $O(\Lambda)$. Let

$$\Lambda \to \Lambda^* = \mathrm{Hom}_A(\Lambda, A)$$

be the canonical map induced by $(\cdot,\cdot)$ and $\mathrm{Rad}(\Lambda)$ (resp. D_Λ) be its kernel (resp. cokernel). We call Λ *nondegenerate* if $\mathrm{Rad}(\Lambda) = 0$, and *unimodular* if it is nondegenerate and $D_\Lambda = 0$. (Over a field these notions are of course equivalent.) If B is an A-algebra, then $(\cdot,\cdot) : \Lambda \times \Lambda \to A$ induces a symmetric B-bilinear form

$$\Lambda \otimes_A B \times \Lambda \otimes_A B \to B,$$

and it is obvious that an isometry $\Lambda \xrightarrow{\sim} \Lambda'$ induces an isometry $\Lambda \otimes_A B \xrightarrow{\sim} \Lambda' \otimes_A B$. If B is flat then we obtain an exact sequence

$$0 \to \mathrm{Rad}(\Lambda) \otimes_A B \to \Lambda \otimes_A B \to \Lambda^* \otimes_A B \to D_\Lambda \otimes_A B \to 0,$$

and under the canonical isomorphism $\Lambda^* \otimes_A B \xrightarrow{\sim} (\Lambda \otimes_A B)^*$ the map

$$\Lambda \otimes_A B \to \Lambda^* \otimes_A B$$

becomes the map $\Lambda \otimes_A B \to (\Lambda \otimes_A B)^*$ induced by the bilinear form on $\Lambda \otimes_A B$. This shows that

$$\mathrm{Rad}(\Lambda) \otimes_A B \simeq \mathrm{Rad}(\Lambda \otimes_A B) \quad \text{and} \quad D_{\Lambda \otimes_A B} \simeq D_\Lambda \otimes_A B,$$

and hence $\Lambda \otimes_A B$ is nondegenerate (resp. unimodular) whenever Λ is so.

We recall some classical results regarding the classification of nondegenerate symmetric bilinear forms over fields. (Part (iii) is essentially the Hasse-Minkowski theorem.)

THEOREM 4.1. *(i) Nondegenerate symmetric bilinear forms over* **R** *are isometric if and only if they have the same signature.*
(ii) Nondegenerate symmetric bilinear forms over **C** *are isometric if and only if they have the same rank.*
(iii) Nondegenerate symmetric bilinear forms over an algebraic number field k are isometric if and only if their extensions to every completion of k are isometric.

By a *lattice* we shall mean a nondegenerate symmetric bilinear form over **Z**. The *signature* of a lattice Λ is the signature of $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. The **Z**-module D_Λ is a finite abelian group (for $D_\Lambda \otimes_{\mathbf{Z}} \mathbf{Q} = 0$), the *discriminant group* Λ. The minimal number of generators of D_Λ is customarily denoted by $\ell(\Lambda)$, obviously

$$\ell(\Lambda) \leqslant \operatorname{rank} \Lambda.$$

By inspection the determinant of any Gram matrix $G(e)$, $G(e)_{ij} = (e_i, e_j)$ (where $e = (e_1, \ldots, e_n)$ is a **Z**-module basis of Λ) is independent of e; it is called the *discriminant* $\mathrm{disc}(\Lambda)$ of Λ. We have $\# D_\Lambda = |\mathrm{disc}(\Lambda)|$.

Example 4.1. Let $E/\mathbf{Q}$ be an algebraic number field. Then its ring of integers $\mathcal{O}_E$ is a nonsingular lattice of rank $[E : \mathbf{Q}]$, the symmetric bilinear form being given by the composition of the product $\mathcal{O}_E \times \mathcal{O}_E \to \mathcal{O}_E$ with the field trace $\mathrm{Tr}_{E/\mathbf{Q}} : \mathcal{O}_E \to \mathbf{Z}$. According to algebraic number theory (Minkowski's bound) the discriminant of the lattice $\mathcal{O}_E$ is never ± 1 (unless $E = \mathbf{Q}$), and hence this lattice is unimodular if and only if $E = \mathbf{Q}$. Let p be an odd prime, ζ a primitive p-th root of unity, and $E = \mathbf{Q}(\zeta)$ the p-th cyclotomic field. Then the Gram matrix of $\mathcal{O}_E = \mathbf{Z}[\zeta]$ with respect to the basis $(1, \zeta, \ldots, \zeta^{p-2})$ is

$$\begin{pmatrix} p-1 & -1 & \ldots & -1 & -1 \\ -1 & -1 & \iddots & -1 & p-1 \\ \vdots & \iddots & \iddots & p-1 & -1 \\ -1 & \iddots & \iddots & \iddots & \vdots \\ -1 & p-1 & -1 & \ldots & -1 \end{pmatrix}$$

from which one can readily compute that

$$\operatorname{disc}(\mathbb{G}_E) = (-1)^{p-1/2} p^{p-2}, \quad D_{\mathbb{G}_E} \simeq (\mathbf{Z}/p\mathbf{Z})^{p-2}, \quad \ell(\mathbb{G}_E) = p-2.$$

(One can obtain $D_{\mathbb{G}_E}$ from the Smith normal form of the Gram matrix.)

Example 4.2. We denote by E_8 the unique even unimodular lattice of signature $(8,0)$. In a certain basis its Gram matrix takes the form

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Let U the unique even unimodular lattice of signature $(1,1)$. In a certain basis its Gram matrix takes the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The lattices E_8 and U play a peculiar role in the classification of even unimodular lattices ([**11**], proposition 1.3.1).

THEOREM 4.2 (Milnor). *Even unimodular lattices of signature (p,q) exist if and only if $p \equiv q \mod 8$. Indefinite even unimodular lattices are isometric if and only if they have the same signature.*

Thus an even unimodular indefinite lattice of signature (p,q) with $p \equiv q \mod 8$ is isomorphic to

$$qU \oplus \frac{p-q}{8} E_8(-1) \text{ for } q \leqslant p$$
$$pU \oplus \frac{q-p}{8} E_8(-1) \text{ for } p \leqslant q.$$

4.2. **Primitive embeddings and genus of a lattice.** An embedding of lattices $\Lambda \to \Lambda'$ is *primitive* if its cokernel is a free group. We denote by

$$P(\Lambda, \Lambda')$$

the set of primitive embeddings $\Lambda \to \Lambda'$. If G is a subgroup of $O(\Lambda)$, then two elements j and k of $P(\Lambda, \Lambda')$ are said to be *G-equivalent*, if they fit into a commutative diagram

$$\begin{tikzcd} \Lambda' \arrow[r] & \Lambda' \\ \Lambda \arrow[u, "j"] \arrow[r] & \Lambda \arrow[u, "k"'] \end{tikzcd}$$

where the top map is an element of $O(\Lambda')$ and the bottom map an element of G. The classes of G-equivalent primitive embeddings form the set $P(\Lambda, \Lambda')^G$.

Lattices Λ and Λ' *are in the same genus* if $\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p \simeq \Lambda' \otimes_{\mathbf{Z}} \mathbf{Z}_p$ for all primes p and $\Lambda \otimes_{\mathbf{Z}} \mathbf{R} \simeq \Lambda' \otimes_{\mathbf{Z}} \mathbf{R}$. (Here $\mathbf{Z}_p$ denotes the ring of p-adic integers.) By the Hasse-Minkowski theorem this implies in particular that $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \Lambda' \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $\mathscr{G}(\Lambda)$ be the set of isomorphism classes of lattices that are in the same genus as Λ; it is known that $\mathscr{G}(\Lambda)$ is always finite ([**11**], 1.3). By requiring that the isomorphisms should be of > 0 determinant, we obtain the variant $\tilde{\mathscr{G}}(\Lambda)$ of $\mathscr{G}(\Lambda)$. Clearly $\mathscr{G}(\Lambda)$ is in natural bijection with the orbit space $\tilde{\mathscr{G}}(\Lambda)/(\mathbf{Z}/2\mathbf{Z})$, where we identify $\mathbf{Z}/2\mathbf{Z} \simeq \mathrm{GL}_2(\mathbf{Z})/\mathrm{SL}_2(\mathbf{Z})$.

The lattice Λ is called *even* if (x, x) is even for all $x \in \Lambda$. In this case we have an associated *discriminant quadratic form* $q_\Lambda : D_\Lambda \to \mathbf{Q}/2\mathbf{Z}$ defined by

$$q_\Lambda(x \mod \Lambda) = (x, x) \mod 2\mathbf{Z},$$

as well as a natural group morphism $O(\Lambda) \to O(D_\Lambda)$.

THEOREM 4.3 (Nikulin). *Let Λ be an indefinite even lattice with rank at least $2 + \ell(\Lambda)$. Then the genus $\mathscr{G}(\Lambda)$ consists of Λ alone, and the canonical map $O(\Lambda) \to O(D_\Lambda)$ is surjective.*

THEOREM 4.4 (Nikulin). *If Λ is even, then an even lattice Λ' is in the same genus as Λ if and only if $\Lambda \otimes_{\mathbf{Z}} \mathbf{R} \simeq \Lambda' \otimes_{\mathbf{Z}} \mathbf{R}$ and $D_\Lambda \simeq D_{\Lambda'}$, where the isomorphism is required to preserve the quadratic forms.*

The first theorem is proposition 1.4.2 and proposition 1.4.7 in [**11**], while the second is proposition 1.3.3 in [**11**].

4.3. **A counting formula for primitive embeddings.** To give a counting formula for the Fourier-Mukai number of a K3 surfaces, we will rely on a purely lattice-theoretic counting formula due to [**15**]. Let Λ be an even unimodular indefinite lattice, T an even lattice admitting a primitive embedding $j_0 \in P(T, \Lambda)$, which we fix. Fix a subgroup G of $O(T)$, a lattice S isomorphic to $j_0(T)^\perp \subset \Lambda$ and an isomorphism $S \xrightarrow{\sim} j_0(T)^\perp$. Then we have a disjoint union

$$P(T, \Lambda)^G = \bigcup_{R \in \mathscr{G}(S)} P_R(T, \Lambda)^G,$$

where $P_R(T,\Lambda)^G = \{[j] \in P^G(T,\Lambda) \mid j(T)^\perp \simeq R\}$. For every $R \in \mathcal{G}(S)$ we have an isomorphism $D_T \to D_R$; we let $O(R)$ act on $O(D_R)$ by the canonical map, and G by $G \to O(T) \to O(D_T) \to O(D_R)$.

THEOREM 4.5 (Hosono-Lian-Oguiso-Yau). *We have*

$$\#P_R^G(T,\Lambda) = \#O(R)\backslash O(D_R)/G,$$

and hence

$$\#P^G(T,\Lambda) = \sum_{R\in\mathcal{G}(S)} \#O(R)\backslash O(D_R)/G.$$

4.4. **Hodge structures.** A (pure) *Hodge structure* of weight n on a $\mathbf{Z}$-module H is a decomposition

$$\mathrm{H} \otimes_{\mathbf{Z}} \mathbf{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$

of $\mathrm{H}\otimes_{\mathbf{Z}}\mathbf{C}$ into $\mathbf{C}$-subspaces $\mathrm{H}^{p,q}$ with $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$ (Hodge symmetry). (Here $\overline{\mathrm{H}^{p,q}}$ denotes the image of $\mathrm{H}^{p,q}$ under the endomorphism $\mathrm{id}_{\mathrm{H}} \otimes c$ of $\mathrm{H} \otimes_{\mathbf{Z}} \mathbf{C}$ induced by complex conjugation $c : \mathbf{C} \to \mathbf{C}$.)

Example 4.3. Let X be a compact Kähler manifold. The Hodge spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p,q}(X) \implies \mathrm{H}^{p+q}(X;\mathbf{C})$$

degenerates to give a weight n Hodge structure

$$\mathrm{H}^n(X;\mathbf{Z}) \otimes \mathbf{C} \simeq \mathrm{H}^n(X;\mathbf{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$

on $\mathrm{H}^n(X;\mathbf{Z})$, where we have used the universal coefficient theorem.

This example will be of use in the next section, since embedding a smooth projective variety X into some projective space induces a structure of compact Kähler manifold on its analytification.

If lattices Λ and Λ' are endowed with Hodge structures of the same weight, then a *Hodge isometry* $\Lambda \to \Lambda'$ is an isometry whose complexification takes $\Lambda^{p,q}$ to $\Lambda'^{p,q}$. The subgroup of $O(\Lambda)$ consisting of Hodge isometries will be denoted $HO(\Lambda)$.

5. Fourier-Mukai Numbers of K3 Surfaces.

5.1. **Generalities on K3 surfaces.** In this section we restrict ourselves to the ground field $k = \mathbf{C}$. By a *K3 surface* we shall mean a smooth projective surface X over $\mathbf{C}$ with $\omega_X = \mathcal{O}_X$ and $h^1(X,\mathcal{O}_X) = 0$. Theorems 1.3, 2.2 and the invariance of Hodge numbers imply that a projective variety derived equivalent to a K3 surface is also a K3 surface. The analytification of a K3 surface X (which we denote, by abuse of notation, also by X) is a simply connected orientable compact complex 2-manifold. Hence by Poincaré duality and the Hurewicz theorem

$$\mathrm{H}^1(X;\mathbf{Z}) \simeq \mathrm{Hom}(\pi_1(X),\mathbf{Z}) = 0, \quad \mathrm{H}^3(X;\mathbf{Z}) \simeq \mathrm{H}_1(X;\mathbf{Z}) \simeq \pi_1(X)^{ab} = 0.$$

If p is a prime number, then by Poincaré duality and the universal coefficient theorem

$$\mathrm{H}_3(X;\mathbf{Z}/p\mathbf{Z}) \simeq \mathrm{H}_1(X;\mathbf{Z}/p\mathbf{Z}) \simeq \mathrm{H}_1(X;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} = 0.$$

Whence $\mathrm{Tor}_1^{\mathbf{Z}}(\mathrm{H}_2(X;\mathbf{Z}),\mathbf{Z}/p\mathbf{Z}) = 0$ by the universal coefficient theorem and the finite-type $\mathbf{Z}$-module $\mathrm{H}_2(X;\mathbf{Z}) = \mathrm{H}^2(X;\mathbf{Z})$ is torsion-free. Serre duality gives

$$h^2(X,\mathcal{O}_X) = h^0(X,\omega_X) = h^0(X,\mathcal{O}_X) = 1,$$

in particular $\chi(\mathcal{O}_X) = 2$. Since $c_1(T_X) = -c_1(\Omega^1_X) = -c_1(\omega_X) = 0$, the Hirzebruch-Riemann-Roch theorem gives

$$\chi(\mathcal{O}_X) = \frac{1}{12}\int_X (c_1(T_X)^2 + c_2(T_X)) = \frac{1}{12}\int_X c_2(T_X)$$

and the topological Euler characteristic of X is therefore 24, giving that the rank of $\mathrm{H}^2(X;\mathbf{Z})$ is 22.

Poincaré duality implies that the intersection pairing

$$\langle \cdot \cup \cdot, [X]\rangle : \mathrm{H}^2(X;\mathbf{Z}) \times \mathrm{H}^2(X;\mathbf{Z}) \to \mathbf{Z}$$

makes $\mathrm{H}^2(X;\mathbf{Z})$ into a unimodular lattice. It is known that Wu's formula (relating the Wu classes to the Stiefel-Whitney classes) shows that this lattice is even. As X is a Kähler manifold, it follows by example 4.3 that the lattice $\mathrm{H}^2(X;\mathbf{Z})$ also has a natural Hodge structure, which in fact determines X.

THEOREM 5.1 (Classical Torelli). *Two K3 surfaces X and Y are isomorphic if and only if there is a Hodge isometry between* $\mathrm{H}^2(X;\mathbf{Z})$ *and* $\mathrm{H}^2(Y;\mathbf{Z})$.

The Hodge diamond of any K3 surface X is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

for $h^{2,0} = h^0(X,\mathcal{O}_X) = 1$ and $22 = 2h^{2,0} + h^{1,1}$. The Hodge index theorem says that the signature of the intersection pairing is given by

$$(2h^{2,0}+1, h^{1,1}-1) = (3,19).$$

Thus by Theorem 4.2 the lattice $\mathrm{H}^2(X;\mathbf{Z})$ is isomorphic to the *K3 lattice*

$$\Lambda_{K3} = 3U \oplus 2E_8(-1).$$

According to the Lefschetz theorem on $(1,1)$-classes the Chern class map $c_1 : \mathrm{Pic}(X) \to \mathrm{H}^2(X;\mathbf{Z})$ induces an isomorphism of $\mathrm{Pic}(X)$ with $\mathrm{H}^2(X;\mathbf{Z}) \cap \mathrm{H}^{1,1}(X)$, whence the Picard rank $\rho(X) = \mathrm{rank}\,\mathrm{Pic}(X)$ of X is at most $h^{1,1} = 20$. (The theorem on $(1,1)$-classes can be proved by considering the cohomology long exact sequence associated to the exponential sequence.) The *Néron-Severi lattice* $\mathrm{NS}(X)$ of X is the primitive sublattice $\mathrm{H}^2(X;\mathbf{Z}) \cap \mathrm{H}^{1,1}(X)$ of $\mathrm{H}^2(X;\mathbf{Z})$. It has signature $(1, \rho(X)-1)$, and the *transcendental lattice* $\mathrm{T}(X) = \mathrm{NS}(X)^{\perp}$ has signature $(2, 20-\rho(X))$. The lattice $\mathrm{T}(X)$ is the minimal primitive sublattice of $\mathrm{H}^2(X;\mathbf{Z})$ whose complexification contains $\mathrm{H}^{2,0}$; it is a Hodge substructure with the same $\mathrm{H}^{2,0}$ and $\mathrm{H}^{0,2}$.

Example 5.1. Let X be a smooth quadric in $\mathbf{P}^3$. Then its ideal sheaf is isomorphic to $\mathcal{O}(-4)$, and the cohomology long exact sequence associated to

$$0 \to \mathcal{O}(-4) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_X \to 0$$

gives

$$0 = \mathrm{H}^1(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) \to \mathrm{H}^1(\mathbf{P}^3, \mathcal{O}_X) \to \mathrm{H}^2(\mathbf{P}^3, \mathcal{O}(-4)) = 0$$

and hence $\mathrm{H}^1(X, \mathcal{O}_X) \simeq \mathrm{H}^1(\mathbf{P}^3, \mathcal{O}_X) = 0$. By the adjunction formula

$$\omega_X \simeq \omega_{\mathbf{P}^3} \otimes \mathcal{O}(4)|X = \mathcal{O}(-4) \otimes \mathcal{O}(4)|X = \mathcal{O}_X.$$

Thus X is a K3 surface. If in particular X is the Fermat quartic $X_0^4 + \cdots + X_3^4 = 0$, then it is known ([**17**], chapter 3, section 2.6) that

$$\rho(X) = 20, \quad \mathrm{disc}\,\mathrm{NS}(X) = -64,$$
$$\mathrm{T}(X) \simeq \mathbf{Z}(8) \oplus \mathbf{Z}(8), \quad \mathrm{NS}(X) \simeq 2E_8(-1) \oplus U \oplus \mathbf{Z}(-8) \oplus \mathbf{Z}(-8).$$

The lattice $\mathrm{T}(X)$ determines X up to derived equivalence by Orlov's derived Torelli theorem [**22**] (theorem 3.3).

THEOREM 5.2 (Derived Torelli). *Two K3 surfaces X and Y are derived equivalent if and only if there is a Hodge isometry between* $\mathrm{T}(X)$ *and* $\mathrm{T}(Y)$.

5.2. **A counting formula for the Fourier-Mukai partners of a K3 surface.** Define the *period domain*

$$\Omega = \{[\mu] \in \mathbf{P}(\Lambda_{K3} \otimes \mathbf{C}) \mid (\mu, \mu) = 0,\ (\mu, \bar{\mu}) > 0\}.$$

A *marked K3 surface* is a pair (X, τ_X) consisting of a K3 surface X and an isomorphism of lattices $\tau_X : \mathrm{H}^2(X;\mathbf{Z}) \xrightarrow{\sim} \Lambda_{K3}$. Then the image of the line $\mathrm{H}^{2,0}$ under the complexification of τ_X gives a point in Ω, and according to Todorov's theorem on the surjectivity of the period map every point of Ω is obtained from some marked K3 surface in this way.

THEOREM 5.3 (Hosono-Lian-Oguiso-Yau). *Let X be a K3 surface. Then there is a natural map $P(\mathrm{T}(X), \mathrm{H}^2(X;\mathbf{Z})) \to \mathrm{FM}(X)$ inducing a bijection*

$$P^{HO(\mathrm{T}(X))}(\mathrm{T}(X), \mathrm{H}^2(X;\mathbf{Z})) \xrightarrow{\sim} \mathrm{FM}(X).$$

Combining this with the counting formula of theorem 4.5, we obtain a counting formula for $\mathrm{FM}(X)$.

Proof. Fix a marking $\tau_X : \mathrm{H}^2(X;\mathbf{Z}) \xrightarrow{\sim} \Lambda_{K3}$, and let $j : \mathrm{T}(X) \to \Lambda_{K3}$ be a primitive embedding. Since $\mathrm{H}^{2,0}$ is contained in $\mathrm{T}(X) \otimes \mathbf{C}$, j induces an element of Ω. By surjectivity of the period map we obtain a marked K3 surface Y and a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(Y;\mathbf{Z}) \arrow[r, "\sim"] & \Lambda_{K3} \\
\mathrm{T}(Y) \arrow[u] \arrow[r] & \mathrm{T}(X) \arrow[u, "j"']
\end{tikzcd}$$

since (by minimality of the transcendental lattice) the image of

$$\mathrm{T}(Y) \to \mathrm{H}^2(Y;\mathbf{Z}) \to \Lambda_{K3}$$

is contained in the image of j. The bottom map, which is defined by commutativity, is a Hodge isometry and thus Y is a Fourier-Mukai partner of X by theorem 5.2. If Y' is another marked K3 surface with the same commutative diagram, then it follows that the composition $\tau_Y^{-1} \circ \tau_{Y'} : \mathrm{H}^2(Y';\mathbf{Z}) \to \mathrm{H}^2(Y;\mathbf{Z})$ is a Hodge isometry, and thus Y and Y' are isomorphic by the Torelli theorem 5.1. Thus we obtain a well-defined map

$$P(\mathrm{T}(X), \mathrm{H}^2(X;\mathbf{Z})) \to \mathrm{FM}(X)$$

by taking j to (the class of) Y. We claim that this map is surjective: let Y be a Fourier-Mukai partner of X. Then by theorem 5.2 we have a Hodge isometry $\mathrm{T}(X) \xrightarrow{\sim} \mathrm{T}(Y)$ and the composite

$$\mathrm{T}(X) \xrightarrow{\sim} \mathrm{T}(Y) \to \mathrm{H}^2(Y;\mathbf{Z}) \xrightarrow{\sim} \Lambda_{K3}$$

(where the last map is a choice of marking) is a primitive embedding that maps to Y under our map. It is clear that $HO(\mathrm{T}(X))$-equivalent primitive embeddings induce the same Fourier-Mukai partner of X, and we get an induced surjection

$$P^{HO(\mathrm{T}(X))}(\mathrm{T}(X), \mathrm{H}^2(X;\mathbf{Z})) \to \mathrm{FM}(X).$$

It remains to show that this map is also injective. Let j and j' be (representatives of) classes in $P^{HO(T(X))}(\mathrm{T}(X), \Lambda_{K3})$, and denote by Y and Y' their images in $\mathrm{FM}(X)$. If $Y = Y'$ then we obtain a Hodge isometry $\mathrm{H}^2(Y;\mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^2(Y';\mathbf{Z})$ by the Torelli theorem, and we have a commutative diagram

$$\begin{tikzcd}
\Lambda_{K3} & \mathrm{H}^2(Y;\mathbf{Z}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{H}^2(Y';\mathbf{Z}) \arrow[r, "\sim"] & \Lambda_{K3} \\
\mathrm{T}(X) \arrow[u, "j"] & \mathrm{T}(Y) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & \mathrm{T}(Y') \arrow[r, "\sim"] \arrow[u] & \mathrm{T}(X) \arrow[u, "j'"']
\end{tikzcd}$$

where the the middle vertical maps are the inclusions, and the horizontal maps are (restrictions of) choices of markings for Y and Y'. The top composite is an isometry, whereas the bottom composite is a Hodge isometry. ■

COROLLARY 5.1. *If X has Picard rank $\rho(X) \geqslant 12$, then* $\#\mathrm{FM}(X) = 1$.

Proof. We have the estimate

$$\ell(\mathrm{NS}(X)) = \ell(\mathrm{T}(X)) \leqslant \operatorname{rank} \mathrm{T}(X) \leqslant 10 \leqslant \rho(X) - 2,$$

hence $\rho(X) \geqslant \ell(\mathrm{NS}(X)) + 2$ and we conclude by applying theorem 4.3, theorem 4.5, and theorem 5.3. ■

5.3. **K3 surfaces and real quadratic fields of class number one.** The next lemma is based on the well-known connection between class numbers of quadratic fields and integral quadratic forms.

LEMMA 5.1. *Let p be a prime $\equiv 1 \mod 4$, and Λ a even lattice of signature* $(1,1)$ *with* $\operatorname{disc}\Lambda = -p$. *Then there is a bijection between the ideal class group of the real quadratic field $\mathbf{Q}(\sqrt{p})$ and the set $\tilde{G}(\Lambda)$. Furthermore there is only one member of $\tilde{G}(\Lambda)$ that is fixed under action of the cyclic group* $\mathbf{Z}/2\mathbf{Z} \cong \mathrm{GL}_2(\mathbf{Z})/\,\mathrm{SL}_2(\mathbf{Z})$.

For this we refer to theorem 3.2 in [**15**] and the references contained therein; for the formulation of the above lemma we have also used that the ideal class group of $\mathbf{Q}(\sqrt{p})$ is isomorphic to the narrow ideal class group, see theorem 6.19 and theorem 9.3 in [**10**]. In this reference one can also find a proof that the class number $h(p)$ of $\mathbf{Q}(\sqrt{p})$ is odd, which also follows from

THEOREM 5.4. *Let p be a prime $\equiv 1 \mod 4$, X a K3 surface with $\rho(X) = 2$ and* $\operatorname{disc}\mathrm{NS}(X) = -p$. *Then the number of Fourier-Mukai partners of X is*

$$\#\mathrm{FM}(X) = \frac{h(p)+1}{2}.$$

Proof. For $\Lambda = \mathrm{NS}(X)$ we count the genus $\mathcal{G}(\Lambda)$ of Λ in two different ways. We have $\mathcal{G}(S) = \tilde{\mathcal{G}}(\Lambda)/(\mathbf{Z}/2\mathbf{Z})$ and hence

$$\#\mathcal{G}(\Lambda) = \frac{\#\tilde{\mathcal{G}}(\Lambda) + 1}{2} = \frac{h(p) + 1}{2},$$

by Burnside's lemma and Lemma 5.1. On the other hand, the group D_Λ has p elements and thus $D_\Lambda \simeq \mathbf{Z}/p\mathbf{Z}$, $O(D_\Lambda) = \{\pm 1\} \subset HO(\mathrm{T}(X))$, and the counting formula implies that $\#\mathrm{FM}(X) = \sum^{\#\mathcal{G}(S)} 1 = \#\mathcal{G}(S)$. ■

Unfortunately we do not know of an (explicit) example of a K3 surface satisfying the hypothesis of the above theorem. As pointed out in [**15**], this theorem may be used to reformulate an old conjecture regarding real quadratic fields of class number one.

CONJECTURE 5.1 (Gauss). *There is an infinite number of primes p such that the class number $h(p)$ of $\mathbf{Q}(\sqrt{p})$ is* 1.

The theorem implies that Gauss' conjecture admits a geometric reformulation in terms of K3 surfaces, which might shed some light on it.

CONJECTURE 5.2. *There is an infinite number of K3 surfaces X with $\rho(X) = 2$, $\#\mathrm{FM}(X) = 1$, such that the numbers $|\operatorname{disc}\mathrm{NS}(X)|$ are distinct primes.*

6. **Derived Categories of Hilbert Schemes of Points.**

6.1. **Generalities on Hilbert Schemes.** Let k be a field, X be a projective scheme over k endowed with a very ample invertible sheaf $\mathcal{O}_X(1)$. The *Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ on X (with respect to $\mathcal{O}_X(1)$) is the unique polynomial $p_{\mathcal{F}} \in \mathbf{Q}[T]$ such that

$$p_{\mathcal{F}}(n) = \chi(\mathcal{F}(n)) \text{ for all } n \in \mathbf{Z},$$

where $\mathcal{F}(n) = \mathcal{F}(n) \otimes \mathcal{O}_X(1)^{\otimes n}$. It follows from Serre's cohomological characterization of ample sheaves that for large n we have $\chi(\mathcal{F}(n)) = \dim_k \mathrm{H}^0(X, \mathcal{F}(n))$; furthermore it is known that $\deg p_{\mathcal{F}} = \dim \operatorname{Supp}(\mathcal{F})$ ([**14**], III, théorème 2.5.3).

Example 6.1. A hypersurface of degree d in $\mathbf{P}^n$ has Hilbert polynomial

$$\binom{T+n}{n} - \binom{T+n-d}{n}.$$

Now let S denote a Noetherian scheme, $\mathbf{Sch}/S$ the category of *locally Noetherian* schemes over S. Let X be a projective scheme over S endowed with a very ample invertible sheaf $\mathcal{O}_X(1)$. For every $s \in S$ the fibre $X_s = X \times_S \operatorname{Spec} \kappa(s)$ is a projective scheme over the residue field $\operatorname{Spec} \kappa(s)$ with an induced very ample sheaf

$\mathcal{O}_{X_s}(1)$. For a fixed polynomial P define a functor

$$\mathcal{H}\mathrm{ilb}^P_{X/S} : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$$

by letting $\mathcal{H}\mathrm{ilb}^P_{X/S}(T)$ be the set of closed subschemes $Z \subset X \times_S T$ that are flat over T such that for every $t \in T$ the fibre Z_t has Hilbert polynomial P.

THEOREM 6.1 (Grothendieck). *The functor* $\mathcal{H}\mathrm{ilb}^P_{X/S}$ *is representable by a a projective scheme* $\mathrm{Hilb}^P_{X/S}$ *over* S.

The scheme $\mathrm{Hilb}^P_{X/S}$ is the *Hilbert scheme* of X over S with respect to P.

6.2. **Hilbert schemes of points.** For the remaining part we take $\mathbf{C}$ as our ground field, and put $\mathrm{Hilb}^P_X = \mathrm{Hilb}^P_{X/\mathrm{Spec}(\mathbf{C})}$; let $n \geqslant 0$. The *Hilbert scheme of n points of X* is Hilb^n_X, where n denotes the constant polynomial with value n. We will consider the case where X has dimension $\dim X \leqslant 2$. Then Hilb^n_X is a smooth projective variety of dimension $n \dim X$. The n-th symmetric power Sym^n_X of X is the quotient variety X^n/S_n (where X^n is the n-fold product of X with itself). It is a Gorenstein variety, and there is a canonical *Hilbert-Chow morphism*

$$\tau : \mathrm{Hilb}^n_X \to \mathrm{Sym}^n_X$$

which is a crepant resolution of singularities (in the sense that the pullback of the canonical sheaf on Sym^n_X under τ is the canonical sheaf on Hilb^n_X). It is semi-small in the sense that $\dim \mathrm{Hilb}^n_X \times_{\mathrm{Sym}^n_X} \mathrm{Hilb}^n_X \leqslant \dim \mathrm{Hilb}^n_X$. In case $\dim X = 1$, τ is an isomorphism.

The very definition of Hilb^n_X gives a universal closed subscheme Σ of $X \times \mathrm{Hilb}^n_X$, whence a tautological functor

$$\Phi_{\mathcal{O}_\Sigma} : \mathrm{D}(X) \to \mathrm{D}(\mathrm{Hilb}^n_X).$$

For $n = 1$ Σ is the diagonal, so this is just the identity functor. Since $\dim \mathrm{Hilb}^n_X > \dim X$ for $n \geqslant 2$ this is never an equivalence for higher n, but it is fully faithful in the case where X is an Enriques surface ([**20**], theorem 1.2). One of the few known general structural results about $\mathrm{D}(\mathrm{Hilb}^n_X)$ is that if X is a surface that admits a full exceptional sequence, then Hilb^n_X also admits a full exceptional sequence ([**20**], proposition 1.3). The proof of this result relies on the following theorem, which describes $\mathrm{D}(\mathrm{Hilb}^n_X)$ as an S_n-equivariant derived category. (The formalism of equivariant derived categories is described elsewhere, for instance in [**24**].)

THEOREM 6.2 (Bridgeland, King, Reid). *Let X be a smooth projective surface. There is a canonical equivalence of categories*

$$\mathrm{D}(\mathrm{Hilb}^n_X) \xrightarrow{\sim} \mathrm{D}^{S_n}(X^n).$$

(This is a special case of theorem 1.1 in [**8**]. The assumptions are satisfied because the Hilbert-Chow morphism is semi-small; we have also used that Hilb_X^n is isomorphic to $S_n - \mathrm{Hilb}(X^n)$.) The next application of theorem 6.2 is well-known.

COROLLARY 6.1. *If X and Y are derived equivalent surfaces, then the Hilbert schemes of points $\mathrm{Hilb}^n(X)$ and $\mathrm{Hilb}^n(Y)$ are also derived equivalent.*

Proof. If $F = \Phi_K : \mathrm{D}(X) \xrightarrow{\sim} \mathrm{D}(Y)$ is an equivalence, then the equivalence $\Phi_{K\boxtimes\cdots\boxtimes K} : \mathrm{D}(X^n) \to \mathrm{D}(Y^n)$ induces an equivalence $\mathrm{D}^{S_n}(X^n) \xrightarrow{\sim} \mathrm{D}^{S_n}(Y^n)$. ■

Assuming the general finiteness conjecture 2.1, the sets $\mathrm{FM}(\mathrm{Hilb}_X^n)$ are finite. The generating series

$$\mu_X = \sum_{n=0}^{\infty} \#\mathrm{FM}(\mathrm{Hilb}_X^n) T^n \in \mathbf{Z}[[T]]$$

is also a derived invariant by the last corollary. Of course μ_X is always a invertible power series, and never a polynomial. From previous experience with other invariants one might expect a closed-form expression of μ_X in terms of other invariants of $X = \mathrm{Hilb}_X^1$. The only example we can compute so far is the case of a genus 0 curve.

Example 6.2. Let $X = \mathbf{P}^1$. Then $\mathrm{Hilb}_X^n = \mathbf{P}^n$ and we find

$$\mu_X = \sum_{n=0}^{\infty} T^n = \frac{1}{1-T}$$

by the Bondal-Orlov theorem.

If X is a curve of genus $g \geqslant 2$, then the canonical sheaf of Hilb_X^n is ample for $1 \leqslant n \leqslant g-2$, and hence $\#\mathrm{FM}(\mathrm{Hilb}_X^n) = 1$ for $1 \leqslant n \leqslant g-2$. Since Hilb_X^g is birationally equivalent to the Jacobian of X, it is reasonable to expect that $\#\mathrm{FM}(\mathrm{Hilb}_X^g) = \#\mathrm{FM}(\mathrm{Jac}(X))$. For $n > 2g-2$ the variety Hilb_X^n is a projective bundle over $\mathrm{Jac}(X)$; this also points to a close relationship between $\#\mathrm{FM}(\mathrm{Hilb}_X^n)$ and $\#\mathrm{FM}(\mathrm{Jac}(X))$.

The last theorem in this thesis gives a reconstruction theorem.

THEOREM 6.3 (Fakhruddin). *Let X and Y smooth projective curves of genus $g \geqslant 3$. If there is $n \geqslant 1$ such that Hilb_X^n is isomorphic to Hilb_Y^n, then X and Y are isomorphic.*

COROLLARY 6.2. *Let X and Y be curves of genus $g \geqslant 3$. If there is $1 \leqslant n \leqslant g-2$ such that Hilb_X^n and Hilb_Y^n are derived equivalent, then X and Y are isomorphic.*

Proof. By the Bondal-Orlov theorem Hilb_X^n and Hilb_Y^n are isomorphic, so this follows from Fakhruddin's theorem. ■

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