Proof of a Theorem of Macaulay

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THEOREM (Macaulay). *Let $(f_0, \dots, f_n)$ be a regular sequence of homogeneous polynomials in $\mathbf{C}[T_0, \dots, T_n]$, and put $A = \mathbf{C}[T_0, \dots, T_n]/(f_0, \dots, f_n)$, and $d_p = \deg f_p$, as well as $\sigma = \sum_{p=0}^{n}(d_p - 1)$. Then*
(i) $A_d = 0$ for $d > \sigma$,
(ii) $A_\sigma = \mathrm{Soc}(A)$ is 1-dimensional,
(iii) the multiplication pairing $A_d \times A_{\sigma-d} \to A_\sigma$ is perfect for $0 \leqslant d \leqslant \sigma$.

PROOF. (i) *Vanishing*. The short exact sequences induced by the injectivity of

$$f_p : \mathbf{C}[T_0, \dots, T_n]/(f_1, \dots, f_{p-1})[-d_p] \to \mathbf{C}[T_0, \dots, T_n]/(f_1, \dots, f_{p-1})$$

allow one to compute the Hilbert–Poincaré series of A as

$$\sum_{d=0}^{\infty} \dim_{\mathbf{C}}(A_d) T^d = \frac{\prod_{p=0}^{n}(1 - T^{d_p})}{(1 - T)^{n+1}} = \prod_{p=0}^{n} \sum_{q=0}^{d_p - 1} T^q = T^\sigma + \text{lower order terms},$$

showing (i) and that A_σ is 1-dimensional.
(ii) *Socle*. From (i) it follows that A_σ is contained in the socle

$$\mathrm{Soc}(A) = \{a \in A \mid at_i = 0,\ 0 \leqslant i \leqslant n\}.$$

To see that $A_\sigma = \mathrm{Soc}(A)$, we show that $\mathrm{Soc}(A)$ is 1-dimensional by computing

$$T = \mathrm{Tor}_{n+1}^{\mathbf{C}[T_0, \dots, T_n]}(A, \mathbf{C})$$

in two different ways. By computing it through the Koszul resolution of the module $\mathbf{C} = \mathbf{C}[T_0, \dots, T_n]/(T_0, \dots, T_n)$, we see that T can be identified with the kernel of the map $A \to A^{\oplus(n+1)}$ given by $a \mapsto (at_0, -at_1, \dots, (-1)^n at_n)$, i.e. $T \simeq \mathrm{Soc}(A)$. On the other hand, using the Koszul resolution of A, the map obtained by tensoring $\mathbf{C}[T_0, \dots, T_n] \to \mathbf{C}[T_0, \dots, T_n]^{\oplus(n+1)}$, $g \mapsto (gf_0, -gf_1, \dots, (-1)^n gf_n)$ with $\mathbf{C}$ vanishes identically, which yields $T \simeq \mathbf{C}[T_0, \dots, T_n] \otimes_{\mathbf{C}[T_0, \dots, T_n]} \mathbf{C} \simeq \mathbf{C}$.
(iii) *Perfect pairing*. Let $d < \sigma$, and $a \in A_d$ such that the multiplication map

$$a : A_{\sigma-d} \to A_\sigma$$

vanishes identically. Let m be (the class of) a monomial of degree $\sigma - d - 1$. Then mt_i is a monomial of degree $\sigma - d$, and hence $amt_i = 0$, i.e. $am \in \mathrm{Soc}(A) \cap A_{\sigma-1} = 0$. Proceeding in this way, we get that $at_i = 0$, i.e. $a \in \mathrm{Soc}(A) \cap A_d = 0$. Thus the map

$$A_d \to \mathrm{Hom}_{\mathbf{C}}(A_{\sigma-d}, A_\sigma)$$

is injective for all $0 \leqslant d \leqslant \sigma$. In particular, $A_{\sigma-d} \to \mathrm{Hom}_{\mathbf{C}}(A_d, A_\sigma)$ is also injective, and so $\dim_{\mathbf{C}} A_d \leqslant \dim_{\mathbf{C}} A_{\sigma-d} \leqslant \dim_{\mathbf{C}} A_d$, which shows that the map $A_d \to \mathrm{Hom}_{\mathbf{C}}(A_{\sigma-d}, A_\sigma)$ is in fact an isomorphism. ∎

Remark. (i) By Euler's formula one can write $f_i = \sum T_j \partial f_i / \partial T_j$. A general theorem of Tate [(1)] allows one to deduce from this representation that the socle $\mathrm{Soc}(A)$ is generated by the Jacobian determinant $\det(\partial f_i / \partial T_j)$.
(ii) Of course, one cannot simply drop the regularity assumption. Consider the (non-regular) sequence (XY, Y^2) in $\mathbf{C}[X, Y]$. Here $\sigma = 2$, $A_d = \mathbf{C}x^d$ for $d > 2$, $A_2 = \mathbf{C}x^2$, and the multiplication map $y : A_1 \to A_2$ is the zero map.
(iii) A more sophisticated but 'geometric' proof (using the cohomology of line bundles on $\mathbf{P}^n$, Serre duality, Koszul resolutions) of Macaulay's theorem can be found in the book 'Period Mappings and Period Domains'.

Example. Let $f \in \mathbf{C}[T_0, \ldots, T_n]$ be a homogenous polynomial of degree d, and assume that the hypersurface $X \subset \mathbf{P}^n$ defined by f is smooth. Then the partial derivatives $f_p = \partial f / \partial T_p$ form a regular sequence [(2)], $\sigma = (n+1)(d-2)$, and

$$A = \mathbf{C}[T_0, \ldots, T_n]/(\partial f/\partial T_0, \ldots, \partial f/\partial T_n)$$

is called the *Jacobian ring* of f. Griffiths proved that the graded pieces of the Jacobian ring encode the primitive cohomology of X.

(1) For a proof (and statement) we refer to Theorem A.3 in B. Mazur, L. Roberts, Local Euler Characteristics, Invent. Math. 9, 201-234 (1970).

(2) By Hilbert's Nullstellensatz it follows that the radical of $(\partial f/\partial T_0, \ldots, \partial f/\partial T_n)$ is $(T_0, \ldots, T_n)$. in particular, the Jacobian ring has exactly one prime ideal, and hence that it is a finite $\mathbf{C}$-algebra. This shows that $\partial f/\partial T_0, \ldots, \partial f/\partial T_n$ is a system of parameters in $\mathbf{C}[T_0, \ldots, T_n]$, which is therefore a regular sequence (using that $\mathbf{C}[T_0, \ldots, T_n]$ is Cohen-Macaulay).