The Quot Scheme $\mathrm{Quot}^{l}(\mathscr{E})$

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Oxford Geometry and Analysis Seminar

16 March 2021

§1. QUOT SCHEMES — DEFINITIONS

Let $\mathscr{E}$ be a coherent sheaf over a projective scheme S over $\mathbf{C}$.

* The Quot scheme $\mathrm{Quot}_{\mathrm{S}}(\mathscr{E})$ is the moduli space of coherent sheaf quotients of $\mathscr{E}$: a point q of $\mathrm{Quot}_{\mathrm{S}}(\mathscr{E})$ corresponds to an exact sequence

$$0 \to \mathscr{S}_q \to \mathscr{E} \to \mathscr{Q}_q \to 0 \quad \text{on} \quad \mathrm{S},$$

  up to equivalence. If P is a polynomial, then $\mathrm{Quot}_{\mathrm{S}}^{\mathrm{P}}(\mathscr{E})$ stands for all q with $\mathscr{Q}_q$ of Hilbert polynomial P, in particular $\dim \mathscr{Q}_q = \deg \mathrm{P}$.

* **Theorem** (Grothendieck, 1961). *The Quot scheme* $\mathrm{Quot}_{\mathrm{S}}^{\mathrm{P}}(\mathscr{E})$ *exists. It is a projective scheme, and the tangent space of* $\mathrm{Quot}_{\mathrm{S}}^{\mathrm{P}}(\mathscr{E})$ *at* q *can be identified with* $\mathrm{Hom}(\mathscr{S}_q, \mathscr{Q}_q)$.

* In a sense, Quot schemes are the most fundamental moduli spaces — many other moduli spaces (of curves, vector bundles, ...) can be constructed by taking GIT quotients of subschemes of Quot schemes (Mumford, 1962; ...).

§1. QUOT SCHEMES — EXAMPLES

Let $\mathscr{E}$ be a coherent sheaf over a projective scheme S over $\mathbf{C}$.

* Let S be a point, and P the constant polynomial l. Then $\mathrm{Quot}_S^P(\mathscr{E})$ is the Grassmannian $\mathrm{Gr}(l, \mathrm{H}^0(\mathscr{E}))$ of l-dimensional quotient spaces of $\mathrm{H}^0(\mathscr{E})$.
* Let S be a smooth curve of genus g, $\mathscr{E} = \mathscr{O}^{\oplus n}$, $\mathrm{P(T)} = r\deg(\mathscr{O}(1))\mathrm{T} + d + r(1-g)$. Then for every point q of $\mathrm{Quot}_S^P(\mathscr{E})$ the sheaf $\mathscr{Q}_q$ on S is of rank r and degree d. If $\mathscr{Q}_q$ is locally free, then q corresponds to a map

$$\mathrm{S} \to \mathrm{Gr}(r, n).$$

* Let $\mathrm{S} = \mathbf{P}^3$, $\mathscr{E} = \mathscr{O}$, $\mathrm{P(T)} = d\mathrm{T} + 1 - g$. Every smooth curve

$$\mathrm{C} \subset \mathbf{P}^3 \quad \text{of genus } g \text{ and degree } d$$

defines a point of $\mathrm{Quot}_S^P(\mathscr{E})$. For certain g and d $\mathrm{Quot}_S^P(\mathscr{E})$ can be very singular (Mumford, 1962). In a sense, Quot schemes of this form can be as singular as possible (Vakil, 2006).

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — GENERALITIES

Let $\mathscr{E}$ be locally free of rank r on a smooth projective variety S, $\mathrm{P} = l$.

* For every point q of $\mathrm{Quot}^l_{\mathrm{S}}(\mathscr{E})$ the support of $\mathscr{Q}_q$ is finite, and $h^0(\mathscr{Q}_q) = l$.
  The Hilbert scheme of points $\mathrm{S}^{[l]} = \mathrm{Quot}^l_{\mathrm{S}}(\mathscr{O})$ can be reducible (Iarrobino, 1972) and very singular (Jelisiejew, 2020) if $\dim \mathrm{S} \gg 0$.
* If S is a curve, then $\mathrm{S}^{[l]}$ is the l-th symmetric product of S.
  If S is a surface, then $\mathrm{S}^{[l]}$ is smooth and irreducible of dimension $2l$, in fact a canonical resolution of singularities of $\mathrm{S}^{(l)} = \mathrm{S}^l/\mathfrak{S}_l$ (Fogarty, 1968). The geometry of $\mathrm{S}^{[l]}$ is very rich (Beauville, 1983; Göttsche, 1990; Grojnowski, 1996; Nakajima, 1997; ...)
* What about $\mathrm{Quot}^l_{\mathrm{S}}(\mathscr{E})$ when $\mathscr{E}$ is of rank $\geqslant 2$?

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — GENERALITIES

In all that follows, $\mathscr{E}$ is locally free of rank r on a smooth projective surface S.

- A point q of $\mathrm{Quot}^l(\mathscr{E})$ corresponds to

$$0 \to \mathscr{S}_q \to \mathscr{E} \to \mathscr{Q}_q \to 0 \quad \text{on} \quad \mathrm{S},$$

  with $\mathscr{S}_q$ torsion-free and $\mathscr{Q}_q$ torsion.
  Exact sequences of this form occur in the context of compactifying moduli of vector bundles.

- $\mathrm{Quot}^1(\mathscr{E}) \simeq \mathbf{P}(\mathscr{E})$, and $\mathrm{Quot}^l(\mathscr{E})$ is isomorphic to $\mathrm{Sym}^l\,\mathbf{P}(\mathscr{E})$ over the locus of all q such that thesupport of $\mathscr{Q}_q$ consists of l distinct points. $\mathrm{Quot}^l(\mathscr{E})$ is irreducible (Rego, 1982; Gieseker-Li, 1996; Ellingsrud-Lehn, 1999).

- How does $\mathrm{Quot}^l(\mathscr{E})$ depend on $\mathscr{E}$?

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — SINGULARITIES

- If q is a point of $\mathrm{Quot}^l(\mathscr{E})$, then

$$\dim \mathrm{Hom}(\mathscr{S}_q, \mathscr{Q}_q) = rl + \dim \mathrm{Hom}(\mathscr{Q}_q, \mathscr{Q}_q).$$

  In particular, q is smooth if and only if $\dim \mathrm{Hom}(\mathscr{Q}_q, \mathscr{Q}_q) = l$; this is also equivalent to $\mathscr{Q}_q$ benig If $r, l \geqslant 2$, then $\mathrm{Quot}^l(\mathscr{E})$ is singular. E.g.:

- What is the nature of the singularities of $\mathrm{Quot}^l(\mathscr{E})$? Is $\mathrm{Quot}^l(\mathscr{E})$ a variety (reduced)? Is $\mathrm{Quot}^l(\mathscr{E})$ normal? Cohen-Macaulay? Are the singularities rational? How can they be resolved?

- Locally, $\mathrm{Quot}^l_S(\mathscr{E})$ looks like $\mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$.

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — SINGULARITIES

The Quot scheme $\mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$ has an ADHM description.

* Let V be a vector space of dimension l, and $\mathrm{C}(\mathfrak{gl}(\mathrm{V}))$ the commuting scheme, i.e. the subscheme of $\mathfrak{gl}(\mathrm{V}) \times \mathfrak{gl}(\mathrm{V})$ cut out by

$$[x, y] = 0.$$

* The group $\mathrm{GL}(\mathrm{V})$ acts on $\mathrm{C}(\mathfrak{gl}(\mathrm{V})) \times \mathrm{V}^{\times r}$. This action is free on the open subset U of all (x, y, v) such that there is no proper subspace of V which is invariant for x, y and contains v.
* There is an isomorphism

$$\mathrm{U}/\mathrm{GL}(\mathrm{V}) \xrightarrow{\sim} \mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$$

which takes (x, y, v) to $\mathscr{O}^{\oplus r} \to \mathrm{V}$, $f \mapsto \sum_{i=1}^r f_i(x, y) v_i$.

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — SINGULARITIES

- **Folklore Conjecture** (... , Artin, Hochster, ...). *The commuting scheme* $\mathrm{C}(\mathfrak{gl}(\mathrm{V}))$ *is normal and Cohen-Macaulay.*

  Known to be true for small $l = \dim \mathrm{V}$.

  Theorem (Ginzburg, 2012). *The normalisation of* $\mathrm{C}(\mathfrak{gl}(\mathrm{V}))$ *is Cohen-Macaulay.*

- Since the action of $\mathrm{GL}(\mathrm{V})$ is free, the quotient

$$\mathrm{U} \to \mathrm{U}/\mathrm{GL}(\mathrm{V}) \simeq \mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$$

  is a principal $\mathrm{GL}(\mathrm{V})$-bundle.

- Hence $\mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$ is as singular as U. The above conjecture implies that $\mathrm{Quot}^l_{\mathbf{A}^2}(\mathscr{O}^{\oplus r})$ is normal and Cohen-Macaulay.

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — TAUTOLOGICAL SHEAVES

* Consider the universal quotient $\mathscr{Q}$ on $\mathrm{S} \times \mathrm{Quot}^l(\mathscr{E})$, and let

$$\mathrm{S} \xleftarrow{\pi_1} \mathrm{S} \times \mathrm{Quot}^l(\mathscr{E}) \xrightarrow{\pi_2} \mathrm{Quot}^l(\mathscr{E})$$

  denote the projections.

* If $\mathscr{F}$ is a locally free sheaf on S, then the associated tautological sheaf

$$\mathscr{F}^{[l]} = \pi_{2*}(\mathscr{Q} \otimes \pi_1^*\mathscr{F}) \quad \text{on} \quad \mathrm{Quot}^l(\mathscr{E})$$

  is locally free, with fibre $\mathrm{H}^0(\mathscr{Q}_q \otimes \mathscr{F})$ over a point q.

* A section s of $\mathscr{E} \otimes \mathscr{F}$ induces a section $s^{[l]}$ of $\mathscr{F}^{[l]}$ whose value $s^{[l]}(q)$ at a point q is

$$\mathscr{O} \xrightarrow{s} \mathscr{E} \otimes \mathscr{F} \to \mathscr{Q}_q \otimes \mathscr{F}.$$

§2. THE QUOT SCHEME $\mathrm{Quot}^l(\mathscr{E})$ — THE $l = 2$ CASE

- We have $\mathrm{C}(\mathfrak{gl}_2) \simeq \mathrm{C}(\mathfrak{sl}_2) \times \mathbf{A}^2$, and

$$\mathrm{C}(\mathfrak{sl}_2) \subset \mathfrak{sl}_2 \times \mathfrak{sl}_2 \text{ is cut out by } x_1y_2 - x_2y_1 = x_1y_3 - x_3y_1 = x_2y_3 - x_3y_2 = 0,$$

  Thus $\mathrm{C}(\mathfrak{sl}_2)$ is the determinantal variety $\{\mathrm{rank} \leqslant 1\} \subset \mathrm{M}(3,2)$.

- Let $p : \mathbf{P}(\mathscr{E}) \to \mathrm{S}$ be the projection. We have a resolution of singularities

$$\phi : \mathrm{Hilb}^2 \mathbf{P}(\mathscr{E}) \to \mathrm{Quot}^2(\mathscr{E})$$

  by taking Z to the quotient $\mathscr{E} = p_*\mathscr{O}(1) \to p_*\mathscr{O}(1)|_{\mathrm{Z}}$. Observe $\phi^*\mathscr{F}^{[2]} = p^*\mathscr{F}(1)^{[2]}$.

- Since $\mathrm{Quot}^2(\mathscr{E})$ has rational singularities, $\mathrm{R}\phi_*\mathscr{O} = \mathscr{O}$, we get

$$\mathrm{H}^k(\mathscr{F}^{[2]}) \simeq \bigoplus_{i+j=k} \mathrm{H}^i(\mathscr{E} \otimes \mathscr{F}) \otimes \mathrm{H}^j(\mathscr{O}_{\mathbf{P}(\mathscr{E})}), \quad \mathrm{H}^n(\mathscr{O}_{\mathrm{Quot}^2(\mathscr{E})}) \simeq \mathrm{H}^n(\mathscr{O}_{\mathrm{Hilb}^2 \mathbf{P}(\mathscr{E})}).$$

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$

* Consider a short exact sequence of locally free sheaves on S

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0.$$

* The map $\mathscr{E} \to \mathscr{E}''$ allows us to obtain a quotient of $\mathscr{E}$ from one of $\mathscr{E}''$, which gives an embedding

$$\mathrm{Quot}^l(\mathscr{E}'') \to \mathrm{Quot}^l(\mathscr{E}).$$

* Regarding $\mathscr{E}' \to \mathscr{E}$ as a section of $\mathscr{H}\mathrm{om}(\mathscr{E}', \mathscr{E}) \simeq \mathscr{E}'^{\vee} \otimes \mathscr{E}$, we obtain a section s of $\mathscr{E}'^{\vee[l]}$. Let q be a point of $\mathrm{Quot}^l(\mathscr{E})$. Then

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}' \arrow[r] \arrow[dr, "s(q)"'] & \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}'' \arrow[r] \arrow[dl, dashed] & 0 \\
& & \mathscr{Q}_q & &
\end{tikzcd}$$

shows $\mathrm{Z}(s) = \mathrm{Quot}^l(\mathscr{E}'')$.

§3. $\mathrm{Quot}^l(\mathcal{E})$ AND QUOTIENTS OF $\mathcal{E}$ — FUNDAMENTAL CLASSES

- The relation $\mathrm{Z}(s) = \mathrm{Quot}^l(\mathcal{E}'')$ suggests a relation between the fundamental classes of $\mathrm{Quot}^l(\mathcal{E}'')$ and $\mathrm{Quot}^l(\mathcal{E})$.
- Let $\iota : \mathrm{Quot}^l(\mathcal{E}'') \to \mathrm{Quot}^l(\mathcal{E})$ be the inclusion.

  Proposition. *We have*

$$\iota_*[\mathrm{Quot}^l(\mathcal{E}'')] = e(\mathcal{E}'^{\vee[l]}) \cap [\mathrm{Quot}^l(\mathcal{E})] \quad in \quad \mathrm{A}_{l(r''+1)}(\mathrm{Quot}^l(\mathcal{E})).$$

- Assume that s is regular, i.e. that the Koszul complex of s is exact. Fulton-MacPherson then construct a Gysin map

$$\iota^* : \mathrm{A}_*(\mathrm{Quot}^l(\mathcal{E})) \to \mathrm{A}_*(\mathrm{Quot}^l(\mathcal{E}''))$$

  such that

$$\iota_* \circ \iota^* = e(\mathcal{E}'^{\vee[l]}) \cap (-) \quad \text{and} \quad \iota^*[\mathrm{Quot}^l(\mathcal{E})] = [\mathrm{Quot}^l(\mathcal{E}'')].$$

* Observe that the codimension of $Z(s) = \mathrm{Quot}^l(\mathcal{E}'')$ in $\mathrm{Quot}^l(\mathcal{E})$ is

$$l(r+1) - l(r''+1) = l(r - r'') = lr'.$$

* If $\mathrm{Quot}^l(\mathcal{E})$ were Cohen-Macaulay, then

$$\mathrm{codim}\, Z(s) = \mathrm{rank}\, \mathcal{E}'^{\vee[l]}$$

  would imply that s is regular.

* Consider a point q of $Z(s) = \mathrm{Quot}^l(\mathcal{E}'')$. Then $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ in a neighbourhood of $\mathrm{Supp}(q)$ in S, with q of the form $(0, *) : \mathcal{E}' \oplus \mathcal{E}'' \to \mathcal{Q}_q$. There is a neighbourhood of q in $\mathrm{Quot}^l(\mathcal{E})$ isomorphic to

$$\mathbf{A}^{r'l} \times \mathrm{Quot}^l(\mathcal{E}'').$$

  The section s corresponds to the pullback of the tautological section of $\mathcal{O}^{\oplus r'l}$.

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — VIRTUAL FUNDAMENTAL CLASSES

* The scheme $\mathrm{Quot}^l(\mathscr{E})$ carries a virtual fundamental class in the sense of Behrend-Fantechi (1997)

$$[\mathrm{Quot}^l(\mathscr{E})]^{\mathrm{vir}} \in \mathrm{A}_{rl}(\mathrm{Quot}^l(\mathscr{E})),$$

  where $rl = \dim \mathrm{Ext}^0(\mathscr{S}_q, \mathscr{Q}_q) - \dim \mathrm{Ext}^1(\mathscr{S}_q, \mathscr{Q}_q)$ (Marian-Oprea-Pandharipande, 2015).

* Grothendieck's deformation-obstruction theory has
  deformations $\mathrm{Hom}(\mathscr{S}_q, \mathscr{Q}_q)$, obstructions $\mathrm{Ext}^1(\mathscr{S}_q, \mathscr{Q}_q)$, higher obstructions $\mathrm{Ext}^{\geqslant 2}(\mathscr{S}_q, \mathscr{Q}_q)$.
  It is thus governed by the complex $\mathrm{T}^{\mathrm{vir}} = \mathrm{R}\mathscr{H}om_{\pi_2}(\mathscr{S}, \mathscr{Q})$.

* For every point q we have

$$\mathrm{Ext}^2(\mathscr{S}_q, \mathscr{Q}_q) \simeq \mathrm{Hom}(\mathscr{Q}_q, \mathscr{S}_q \otimes \omega_{\mathrm{X}})^{\vee} = 0$$

  since $\mathscr{Q}_q$ is torsion and $\mathscr{S}_q$ torsion-free.

§3. $\mathrm{Quot}^l(\mathcal{E})$ AND QUOTIENTS OF $\mathcal{E}$ — VIRTUAL FUNDAMENTAL CLASSES

* The virtual fundamental class is given by Siebert's formula

$$[\mathrm{Quot}^l(\mathcal{E})]^{\mathrm{vir}} = \left\{ c(\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathcal{E})})^{-1} \cap c_{\mathrm{F}}(\mathrm{Quot}^l(\mathcal{E})) \right\}_{rl}.$$

  Here $c_{\mathrm{F}}(\mathrm{Quot}^l(\mathcal{E}))$ is the Fulton class of $\mathrm{Quot}^l(\mathcal{E})$. (If $\mathrm{X} \subset \mathrm{A}$ with A smooth, then $c_{\mathrm{F}}(\mathrm{X}) = c(\mathrm{T_A}|_{\mathrm{X}}) \cap [\mathrm{X}]$ is independent of A.)

* **Theorem** ($\mathcal{E} = \mathcal{O}^{\oplus r}$: Oprea-Pandharipande, 2019). *(i) If ω_{S} is trivial, then $[\mathrm{Quot}^l_{\mathrm{S}}(\mathcal{E})]^{\mathrm{vir}} = 0$. (ii) If ω_{S} has a section whose zero locus is a smooth irreducible curve C, then $[\mathrm{Quot}^l_{\mathrm{S}}(\mathcal{E})]^{\mathrm{vir}}$ is the pushforward of $(-1)^l[\mathrm{Quot}^l_{\mathrm{C}}(\mathcal{E}|_{\mathrm{C}})]$ with respect to the embedding $\mathrm{Quot}^l_{\mathrm{C}}(\mathcal{E}|_{\mathrm{C}}) \to \mathrm{Quot}^l_{\mathrm{S}}(\mathcal{E})$.*

* **Theorem**. *If we have an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, then*

$$\iota_*[\mathrm{Quot}^l(\mathcal{E}'')]^{\mathrm{vir}} = e(\mathcal{E}'^{\vee[l]}) \cap [\mathrm{Quot}^l(\mathcal{E})]^{\mathrm{vir}}.$$

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — VIRTUAL FUNDAMENTAL CLASSES

$\ast$ Compatibility of the obstruction theories:

$$\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E}'')} \to \mathrm{L}\iota^*\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E})} \to \iota^*\mathscr{E}'^{\vee[l]} \to \mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E}'')}[1].$$

$\ast$ Compatibility of the Fulton classes:

$$\iota^* c_{\mathrm{F}}(\mathrm{Quot}^l(\mathscr{E})) = c(\iota^*\mathscr{E}'^{\vee[l]}) \cap c_{\mathrm{F}}(\mathrm{Quot}^l(\mathscr{E}'')).$$

$\ast$ Siebert's formula,

$$\begin{aligned}
\iota^*[\mathrm{Quot}^l(\mathscr{E})]^{\mathrm{vir}} &= \left\{\iota^* c(\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E})})^{-1} \cap \iota^* c_{\mathrm{F}}(\mathrm{Quot}^l(\mathscr{E}))\right\}_{r''l} \\
&= \left\{c(\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E}'')})^{-1} c(\iota^*\mathscr{E}'^{\vee[l]})^{-1} c(\iota^*\mathscr{E}'^{\vee[l]}) c_{\mathrm{F}}(\mathrm{Quot}^l(\mathscr{E}''))\right\}_{r''l} \\
&= \left\{c(\mathrm{T}^{\mathrm{vir}}_{\mathrm{Quot}^l(\mathscr{E}'')})^{-1} \cap c_{\mathrm{F}}(\mathrm{Quot}^l(\mathscr{E}''))\right\}_{r''l} = [\mathrm{Quot}^l(\mathscr{E}'')]^{\mathrm{vir}}.
\end{aligned}$$

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — TAUTOLOGICAL INTEGRALS

- The formation of tautological sheaves is compatible with the embedding $\iota : \mathrm{Quot}^l(\mathscr{E}'') \hookrightarrow \mathrm{Quot}^l(\mathscr{E})$.
- We then have

$$\int_{[\mathrm{Quot}^l(\mathscr{E})]^{\mathrm{vir}}} e(\mathscr{E}'^{\vee[l]})(-) = \int_{\iota_*[\mathrm{Quot}^l(\mathscr{E}'')]^{\mathrm{vir}}} (-) = \int_{[\mathrm{Quot}^l(\mathscr{E}'')]^{\mathrm{vir}}} \iota^*(-) = \int_{[\mathrm{Quot}^l(\mathscr{E}'')]^{\mathrm{vir}}} (-)$$

  where $(-)$ denotes any polynomial expression in Chern classes of tautological sheaves. Of course, the same equation holds with $[\mathrm{Quot}^l(\mathscr{E})]$ in lieu of $[\mathrm{Quot}^l(\mathscr{E})]^{\mathrm{vir}}$.
- Consider $\mathscr{E} = \mathscr{O}^{\oplus r}$, and the exact sequence

$$0 \to \mathscr{O} \to \mathscr{O}^{\oplus r} \to \mathscr{O}^{\oplus(r-1)} \to 0.$$

  The sheaf $\mathscr{O}^{[l]}$ is precisely the pushforward of the universal sheaf, and the above equality says the insertion $e(\mathscr{O}^{[l]})$ lowers the rank of the defining sheaf by 1.

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — TAUTOLOGICAL INTEGRALS

- The Chern classes of tautological sheaves play an important role in the intersection theory of Hilbert schemes of points $S^{[l]} = \mathrm{Quot}^l(\mathscr{O})$ (Lehn, 1997; ...), e.g.

$$c_1(\mathscr{O}^{[l]}) = -\frac{1}{2}[\partial S^{[l]}].$$

- Pushing forward the universal exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}^{\oplus r} \to \mathscr{Q} \to 0 \quad \text{on} \quad S \times \mathrm{Quot}^l(\mathscr{O}^{\oplus r})$$

  to $\mathrm{Quot}^l(\mathscr{O}^{\oplus r})$ defines a tautological section of $(\mathscr{O}^{\oplus r})^{[l]}$: the value at a point q is the element of $H^0(\mathscr{Q}_q)^{\oplus r}$ given by $\mathscr{O}^{\oplus r} \to \mathscr{Q}_q$. Hence

$$e(\mathscr{O}^{[l]})^r = 0.$$

- In the case $l = 1$ we have $\mathrm{Quot}^1(\mathscr{O}^{\oplus r}) \simeq \mathbf{P}(\mathscr{O}^{\oplus r})$, $\mathscr{O}^{[1]} \simeq \mathscr{O}(1)$.

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — TAUTOLOGICAL INTEGRALS

- We have
$$\int_{\mathrm{Quot}^l(\mathscr{O}^{\oplus r})} e(\mathscr{O}^{[l]})^{r-1}(-) = \int_{\mathrm{S}^{[l]}}(-)$$
for any expression (-) in Chern classes of tautological sheaves. Integrals of the above kind are rather difficult to compute explicitly (e.g. 'Lehn's conjecture' on Segre classes $s_{2l}(\mathscr{L}^{[l]})$: Marian-Oprea-Pandharipande, 2017; Voisin, 2017).
- If $\mathscr{L}_1$ and $\mathscr{L}_2$ are invertible sheaves on S, then
$$\sum_{l=0}^{\infty} q^l \int_{\mathrm{Quot}^l(\mathscr{O}^{\oplus r})} e(\mathscr{O}^{[l]})^{r-1} e(\mathscr{L}_1^{[l]}) e(\mathscr{L}_2^{[l]}) = (1+q)^{\int_{\mathrm{S}} c_1(\mathscr{L}_1)c_1(\mathscr{L}_2)}.$$
- If we take $\mathscr{L}_1$ to be ample, and $\mathscr{L}_2 = \mathscr{L}_1^{\vee}$, then we obtain in particular $e(\mathscr{O}^{[l]})^{r-1} \neq 0$.

§3. $\mathrm{Quot}^l(\mathscr{E})$ AND QUOTIENTS OF $\mathscr{E}$ — TAUTOLOGICAL INTEGRALS

* Arbesfeld-Johnson-Lim-Oprea-Pandharipande (2020) prove that for any r_1, r_2 and $\mathscr{L}$

$$(-1)^{r_1 l} \int_{[\mathrm{Quot}^l(\mathscr{O}^{\oplus r_1})]^{\mathrm{vir}}} s(\mathscr{L}^{[l]})^{r_2} = (-1)^{r_2 l} \int_{[\mathrm{Quot}^l(\mathscr{O}^{\oplus r_2})]^{\mathrm{vir}}} s(\mathscr{L}^{[l]})^{r_1}$$

* We can reformulate this as an equality

$$\int_{[\mathrm{Quot}^l(\mathscr{O}^{\oplus r_2})]^{\mathrm{vir}}} e(\mathscr{O}^{[l]})^{r_2-r_1} s(\mathscr{L}^{[l]})^{r_2} = (-1)^{(r_2-r_1)l} \int_{[\mathrm{Quot}^l(\mathscr{O}^{\oplus r_2})]^{\mathrm{vir}}} s(\mathscr{L}^{[l]})^{r_1}$$

  over the same Quot scheme.

* The Euler class $e(\mathscr{O}^{[l]})$ should play an important role in the intersection theory of $\mathrm{Quot}^l(\mathscr{O}^{\oplus r})$.