Homological Mirror Symmetry for the Quartic Surface

[*after* P. SEIDEL]

S. STARK

Plan

§ 1. **Mirror Symmetry**

§ 2. **Homological Mirror Symmetry**

§ 3. **Seidel's HMS for the Quartic Surface**

§ 1. **Mirror Symmetry**

CALABI-YAU MANIFOLDS

* In String theory spacetime is described by a product $\mathbf{R}^{1,3} \times X$ of Minkowski space $\mathbf{R}^{1,3}$ with a Calabi-Yau 3-fold X (Candelas, Horowitz, Strominger, Witten; 1985).

§ 1. Mirror Symmetry

CALABI-YAU MANIFOLDS

* In String theory spacetime is described by a product $\mathbf{R}^{1,3} \times X$ of Minkowski space $\mathbf{R}^{1,3}$ with a Calabi-Yau 3-fold X (Candelas, Horowitz, Strominger, Witten; 1985).
* A *Calabi-Yau manifold* is a connected compact Kähler manifold X with trivial canonical bundle and $h^{p,0} = 0$ $(0 < p < \dim X)$.
  Calabi-Yau curves are the same as *elliptic curves*,
  Calabi-Yau surfaces are called *K3 surfaces*.

§ 1. Mirror Symmetry

CALABI-YAU MANIFOLDS

* In String theory spacetime is described by a product $\mathbf{R}^{1,3} \times X$ of Minkowski space $\mathbf{R}^{1,3}$ with a Calabi-Yau 3-fold X (Candelas, Horowitz, Strominger, Witten; 1985).
* A *Calabi-Yau manifold* is a connected compact Kähler manifold X with trivial canonical bundle and $h^{p,0} = 0$ $(0 < p < \dim X)$.
  Calabi-Yau curves are the same as *elliptic curves*,
  Calabi-Yau surfaces are called *K3 surfaces*.
* Example: Hypersurfaces $X \subset \mathbf{P}^{d+1}$ of degree $d + 2$ ($d = 2$: quartic surface, $d = 3$: quintic 3-fold).

§ 1. **Mirror Symmetry**

* To every Calabi-Yau 3-fold X there should be an associated physical theory, which has two ‘models’: an A-model $A(X)$, and a B-model $B(X)$.

* To every Calabi-Yau 3-fold X there should be an associated physical theory, which has two 'models': an A-model $A(X)$, and a B-model $B(X)$.
* $A(X)$ is described by the symplectic geometry of X, while $B(X)$ is described by the complex geometry of X.

§ 1. Mirror Symmetry

* To every Calabi-Yau 3-fold X there should be an associated physical theory, which has two 'models': an A-model $A(X)$, and a B-model $B(X)$.
* $A(X)$ is described by the symplectic geometry of X, while $B(X)$ is described by the complex geometry of X.
* There exist pairs $(X, \check{X})$ of Calabi-Yau 3-folds yielding the same physical theory, but with A and B-models exchanged:

$$A(X) \simeq B(\check{X}) \quad \text{and} \quad B(X) \simeq A(\check{X}) \quad \text{(mirror symmetry)}.$$

Mirror symmetry holds *in families*.

§ 1. Mirror Symmetry

* Mirror pairs in particular satisfy

$$h^{1,1}(X) = h^{2,1}(\check{X}), \quad h^{2,1}(X) = h^{1,1}(\check{X}),$$

$$\chi(X) = -\chi(\check{X}),$$

but there is much more to Mirror symmetry.

§ 1. Mirror Symmetry

IMPLICATIONS OF MIRROR SYMMETRY

* Mirror pairs in particular satisfy

$$h^{1,1}(X) = h^{2,1}(\check{X}), \quad h^{2,1}(X) = h^{1,1}(\check{X}),$$

$$\chi(X) = -\chi(\check{X}),$$

but there is much more to Mirror symmetry.

* Greene and Plesser (1991) obtained the mirror family $\check{X}_\psi$ to a quintic 3-fold X by an orbifolding construction.

§ 1. Mirror Symmetry

IMPLICATIONS OF MIRROR SYMMETRY

* Mirror pairs in particular satisfy

$$h^{1,1}(X) = h^{2,1}(\check{X}), \quad h^{2,1}(X) = h^{1,1}(\check{X}),$$

$$\chi(X) = -\chi(\check{X}),$$

but there is much more to Mirror symmetry.

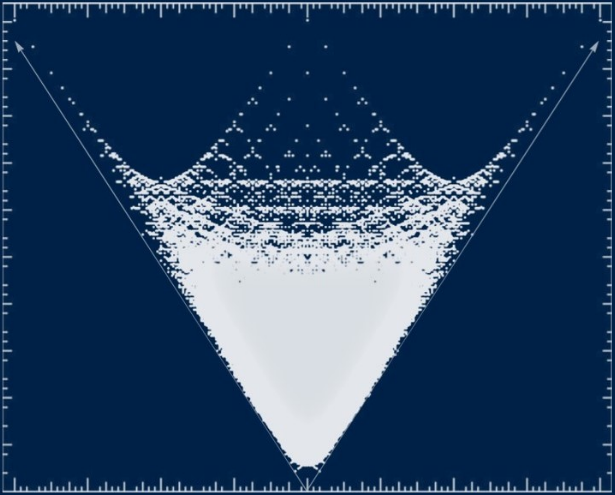
* Greene and Plesser (1991) obtained the mirror family $\check{X}_\psi$ to a quintic 3-fold X by an orbifolding construction.
* **Miracle** (Candelas, de la Ossa, Green, Parkes; 1991). For a general quintic 3-fold X mirror symmetry $A(X) \simeq B(\check{X}_\psi)$ predicts the equality

$$\begin{aligned} & 5 + n_1 q + (2^3 n_2 + n_1) q^2 + (3^3 n_3 + n_1) q^3 + \dots \\ = & \ 5 + 2875q + 4876875q^2 + 8564575000q^3 + \dots, \end{aligned}$$

where n_d is the (virtual) number of curves of degree d on X.

§ 1. Mirror Symmetry

HODGE PLOT



The 'Hodge plot' of Candelas-Constantin-Skarke, which displays $\chi = 2(h^{1,1} - h^{2,1})$ on the horizontal axis and $h^{1,1} + h^{2,1}$ on the vertical axis.

§ 2. **Homological Mirror Symmetry**

KONTSEVICH'S CATEGORIFICATION OF MIRROR SYMMETRY

* How can mirror symmetry be formulated and explained mathematically? Kontsevich proposed an answer (ICM 1994). '*Our conjecture, if it is true, will unveil the mystery of Mirror Symmetry*': $A(X)$ and $B(X)$ are described by (triangulated) *categories*, and mirror symmetry is an equivalence between these categories (homological mirror symmetry, HMS).

§ 2. Homological Mirror Symmetry

KONTSEVICH'S CATEGORIFICATION OF MIRROR SYMMETRY

* How can mirror symmetry be formulated and explained mathematically? Kontsevich proposed an answer (ICM 1994). '*Our conjecture, if it is true, will unveil the mystery of Mirror Symmetry*': $A(X)$ and $B(X)$ are described by (triangulated) *categories*, and mirror symmetry is an equivalence between these categories (homological mirror symmetry, HMS).
* The category corresponding to $A(X)$ is $D^{\pi}\mathcal{F}(X)$, the split-closed derived category of the Fukaya category $\mathcal{F}(X)$), while $D^b\mathrm{Coh}(X)$ (bounded derived category of the category $\mathrm{Coh}(X)$ coherent sheaves) corresponds to $B(X)$.

§ 2. Homological Mirror Symmetry

KONTSEVICH'S CATEGORIFICATION OF MIRROR SYMMETRY

* How can mirror symmetry be formulated and explained mathematically? Kontsevich proposed an answer (ICM 1994). '*Our conjecture, if it is true, will unveil the mystery of Mirror Symmetry*': $A(X)$ and $B(X)$ are described by (triangulated) *categories*, and mirror symmetry is an equivalence between these categories (homological mirror symmetry, HMS).
* The category corresponding to $A(X)$ is $D^\pi\mathcal{F}(X)$, the split-closed derived category of the Fukaya category $\mathcal{F}(X)$), while $D^b\mathrm{Coh}(X)$ (bounded derived category of the category $\mathrm{Coh}(X)$ coherent sheaves) corresponds to $B(X)$.
* Kontsevich's arguments in favour of HMS: one should be able to recover numerical predictions by taking Hochschild cohomology; Witten's intuition; both categories satisfy the duality $\mathrm{Hom}(C, D) \simeq \mathrm{Hom}(D, C[\dim X])^\vee$.

§ 2. **Homological Mirror Symmetry**

'DEFINITION' OF $D^b\mathrm{Coh}(X)$

* The category $\mathrm{Coh}(X)$ of coherent sheaves on X includes not only vector bundles, but also structure sheaves of subschemes of X. It is an *abelian category*, and hence has a bounded derived category $D^b\mathrm{Coh}(X)$.

§ 2. Homological Mirror Symmetry

'DEFINITION' OF $D^b\mathrm{Coh}(X)$

* The category $\mathrm{Coh}(X)$ of coherent sheaves on X includes not only vector bundles, but also structure sheaves of subschemes of X. It is an *abelian category*, and hence has a bounded derived category $D^b\mathrm{Coh}(X)$.
* The category $D^b\mathrm{Coh}(X)$ is obtained from the category $\mathrm{Kom}^b\mathrm{Coh}(X)$ of bounded complexes of coherent sheaves by inverting quasi-isomorphisms. These categories have the same objects, but different Hom sets; for coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on X we have $\mathrm{Hom}(\mathcal{F}[0], \mathcal{G}[n]) \simeq \mathrm{Ext}^n_X(\mathcal{F}, \mathcal{G})$, and composition is given by the Yoneda product.

§ 2. Homological Mirror Symmetry

'DEFINITION' OF $D^b\mathrm{Coh}(X)$

* The category $\mathrm{Coh}(X)$ of coherent sheaves on X includes not only vector bundles, but also structure sheaves of subschemes of X. It is an *abelian category*, and hence has a bounded derived category $D^b\mathrm{Coh}(X)$.
* The category $D^b\mathrm{Coh}(X)$ is obtained from the category $\mathrm{Kom}^b\mathrm{Coh}(X)$ of bounded complexes of coherent sheaves by inverting quasi-isomorphisms. These categories have the same objects, but different Hom sets; for coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on X we have $\mathrm{Hom}(\mathcal{F}[0], \mathcal{G}[n]) \simeq \mathrm{Ext}^n_X(\mathcal{F}, \mathcal{G})$, and composition is given by the Yoneda product.
* It is a *triangulated category*: it has a shift functor [1] and comes with a class of distinguished triangles $K' \to K \to K'' \to K'[1]$, satisfying some axioms.

§ 2. **Homological Mirror Symmetry**

'DEFINITION' OF $\mathcal{F}(X)$

* The objects of the Fukaya category $\mathcal{F}(X)$ are (rational) Lagrangian submanifolds L of X carrying additional data (grading, spin structure,...). Denote by $\Lambda = \Lambda_{\mathbf{Q}}$ (Novikov field) the algebraic closure of the fraction field $\mathbf{C}((q))$ of $\mathbf{C}[[q]]$; elements of Λ are of the form $\sum_{n\in\mathbf{Z}} a_n q^{n/d}$ (Puiseux series) for some $d \geqslant 1$, where $a_n \in \mathbf{C}$ is 0 for $n \ll 0$.

§ 2. Homological Mirror Symmetry

'DEFINITION' OF $\mathscr{F}(X)$

- The objects of the Fukaya category $\mathscr{F}(X)$ are (rational) Lagrangian submanifolds L of X carrying additional data (grading, spin structure,...). Denote by $\Lambda = \Lambda_{\mathbf{Q}}$ (Novikov field) the algebraic closure of the fraction field $\mathbf{C}((q))$ of $\mathbf{C}[[q]]$; elements of Λ are of the form $\sum_{n \in \mathbf{Z}} a_n q^{n/d}$ (Puiseux series) for some $d \geqslant 1$, where $a_n \in \mathbf{C}$ is 0 for $n \ll 0$.
- At least for L_0, L_1 transverse $\mathrm{Hom}(L_0, L_1)$ is the free Λ-module generated by the $x \in L_0 \cap L_1$, $\mathbf{Z}$-graded by Maslov index. (If L_0, L_1 are not transverse one 'perturbs'.) It carries a Λ-linear map μ^1 defined by counts of pseudoholomorphic 2-gons u in X with boundary in L_0, L_1, $[u] = \beta \in \pi_2(X, L_0 \cup L_1)$:

$$\mu^1(x) = \sum_{y,\beta} \#\mathcal{M}(x, y, \beta) y q^{\omega(\beta)}.$$

We pretend that $\omega(\beta) = \int u^* \omega \in \frac{1}{d}\mathbf{Z}$ for some $d = d(L_0, L_1)$.

§ 2. Homological Mirror Symmetry

'DEFINITION' OF $\mathcal{F}(X)$

* More generally, for any $d \geqslant 1$ we have Λ-linear maps

$$\mu^d : \mathrm{Hom}(L_{d-1}, L_d) \otimes \cdots \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_d)$$

of degree $2 - d$ defined by weighted counts of pseudoholomorphic $(d+1)$-gons in X with boundary in $L_0, \ldots, L_d$.

* The maps μ^d satisfy the 'A_∞-associativity equations': $(\mu^1)^2 = 0$, μ^2 is a chain map, μ^3 is the obstruction to μ^2 (which we view as the 'composition' of $\mathscr{F}(X)$) being associative, etc. The cohomology of $\mathrm{Hom}(L_0, L_1)$ is the Floer cohomology $\mathrm{HF}(L_0, L_1)$, and the map induced by μ^2 on cohomology is the Donaldson product.

§ 2. Homological Mirror Symmetry

- The maps μ^d satisfy the 'A_∞-associativity equations': $(\mu^1)^2 = 0$, μ^2 is a chain map, μ^3 is the obstruction to μ^2 (which we view as the 'composition' of $\mathscr{F}(X)$) being associative, etc. The cohomology of $\mathrm{Hom}(L_0, L_1)$ is the Floer cohomology $\mathrm{HF}(L_0, L_1)$, and the map induced by μ^2 on cohomology is the Donaldson product.
- So $\mathscr{A} = \mathscr{F}(X)$ is an A_∞-*category*. It need not be a category in the ordinary sense. However, its cohomology category $\mathrm{H}^*(\mathscr{A})$ (same objects as $\mathscr{A}$, morphisms given by cohomology groups) is one, and so is $\mathrm{H}^0(\mathscr{A})$ (morphisms: H^0). A *quasi-equivalence* between A_∞-categories is an A_∞-functor such that the induced functor between cohomology categories is an equivalence.

§ 2. **Homological Mirror Symmetry**

TWISTED COMPLEXES

* For HMS we need to form a triangulated category out of $\mathscr{F}(X)$ (or indeed any A_∞-category $\mathscr{A}$).

§ 2. Homological Mirror Symmetry

TWISTED COMPLEXES

* For HMS we need to form a triangulated category out of $\mathscr{F}(X)$ (or indeed any A_∞-category $\mathscr{A}$).
* We first form an auxiliary A_∞-category $\Sigma\mathscr{A}$, whose objects are finite formal sums $K = \bigoplus_{i \in I} K_i[\sigma_i]$, with morphisms 'matrices',

$$\mathrm{Hom}(K, L) = \bigoplus_{i \in I, j \in J} \mathrm{Hom}(K_i, L_j)[\tau_j - \sigma_i]$$

and composition by matrix multiplication. This has μ^d's induced by the ones of $\mathscr{A}$. A *twisted complex* is then an object K of $\Sigma\mathscr{A}$ with $\delta \in \mathrm{Hom}^1(K, K)$ satisfying $\mu^1(\delta) + \mu^2(\delta, \delta) + \cdots = 0$. This gives an A_∞ category $\mathrm{Tw}\,\mathscr{A}$ (same Hom sets, but δ's inserted into μ^d's).

§ 2. Homological Mirror Symmetry

TWISTED COMPLEXES

* For HMS we need to form a triangulated category out of $\mathscr{F}(X)$ (or indeed any A_∞-category $\mathscr{A}$).
* We first form an auxiliary A_∞-category $\Sigma\mathscr{A}$, whose objects are finite formal sums $K = \bigoplus_{i \in I} K_i[\sigma_i]$, with morphisms 'matrices',

$$\mathrm{Hom}(K, L) = \bigoplus_{i \in I, j \in J} \mathrm{Hom}(K_i, L_j)[\tau_j - \sigma_i]$$

  and composition by matrix multiplication. This has μ^d's induced by the ones of $\mathscr{A}$. A *twisted complex* is then an object K of $\Sigma\mathscr{A}$ with $\delta \in \mathrm{Hom}^1(K, K)$ satisfying $\mu^1(\delta) + \mu^2(\delta, \delta) + \cdots = 0$. This gives an A_∞ category $\mathrm{Tw}\,\mathscr{A}$ (same Hom sets, but δ's inserted into μ^d's).
* We then define $D^b(\mathscr{A}) = \mathrm{H}^0(\mathrm{Tw}\,\mathscr{A})$, which is a *triangulated category* (distinguished triangles via mapping cones, as usual).

§ 2. Homological Mirror Symmetry

SUMMARY OF CATEGORIFICATION

	$A(X)$	Type	$B(X)$	Type
I	$\mathscr{F}(X)$	A_∞	$\mathrm{Coh}(X)$	abelian
$\downarrow$				
II	$\mathrm{Tw}\,\mathscr{F}(X)$	A_∞	$\mathrm{Kom}^b\mathrm{Coh}(X)$	abelian
$\downarrow$	H^0		localisation	
III	$D^b\mathscr{F}(X)$	triang.	$D^b\mathrm{Coh}(X)$	triang.
	$D^\pi\mathscr{F}(X)$	triang.	$D^b\mathrm{Coh}(X)$	triang.

$*$ As $D^b\mathrm{Coh}(X)$ is idempotent complete (every idempotent splits), we use the idempotent completion $D^\pi\mathscr{F}(X)$ of $D^b\mathscr{F}(X)$ (*split-closed Fukaya category*).

§ 2. **Homological Mirror Symmetry**

SUMMARY OF CATEGORIFICATION

	$A(X)$	Type	$B(X)$	Type
I	$\mathcal{F}(X)$	A_∞	$\mathrm{Coh}(X)$	abelian
$\downarrow$				
II	$\mathrm{Tw}\,\mathcal{F}(X)$	A_∞	$\mathrm{Kom}^b\mathrm{Coh}(X)$	abelian
$\downarrow$	H^0		localisation	
III	$D^b\mathcal{F}(X)$	triang.	$D^b\mathrm{Coh}(X)$	triang.
	$D^\pi\mathcal{F}(X)$	triang.	$D^b\mathrm{Coh}(X)$	triang.

* As $D^b\mathrm{Coh}(X)$ is idempotent complete (every idempotent splits), we use the idempotent completion $D^\pi\mathcal{F}(X)$ of $D^b\mathcal{F}(X)$ (*split-closed Fukaya category*).
* A set S of objects in an idempotent complete triangulated category $\mathcal{C}$ *split generates* $\mathcal{C}$ if any object in $\mathcal{C}$ is obtained from objects of S by shifts, cones of morphisms between S, and splitting off direct summands.

§ 3. Seidel's HMS for the Quartic Surface

STATEMENT OF SEIDEL'S THEOREM

* Let X_0 be any quartic surface in $\mathbf{P}^3_{\mathbf{C}}$, viewed as a symplectic manifold (any two such are symplectomorphic).

§ 3. Seidel's HMS for the Quartic Surface

STATEMENT OF SEIDEL'S THEOREM

* Let X_0 be any quartic surface in $\mathbf{P}^3_{\mathbf{C}}$, viewed as a symplectic manifold (any two such are symplectomorphic).
* Consider the quartic surface Y_q^* in $\mathbf{P}^3_{\Lambda}$ cut out by the equation

$$y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0,$$

with its natural action of the subgroup Γ_{16} of PSL_4 given by 4×4 diagonal matrices with determinant 4 and entries 4-th roots of unity. Let $Z_q^* \to Y_q^* / \Gamma_{16}$ be the crepant resolution.

§ 3. Seidel's HMS for the Quartic Surface

STATEMENT OF SEIDEL'S THEOREM

- Let X_0 be any quartic surface in $\mathbf{P}^3_{\mathbf{C}}$, viewed as a symplectic manifold (any two such are symplectomorphic).
- Consider the quartic surface Y_q^* in $\mathbf{P}^3_\Lambda$ cut out by the equation

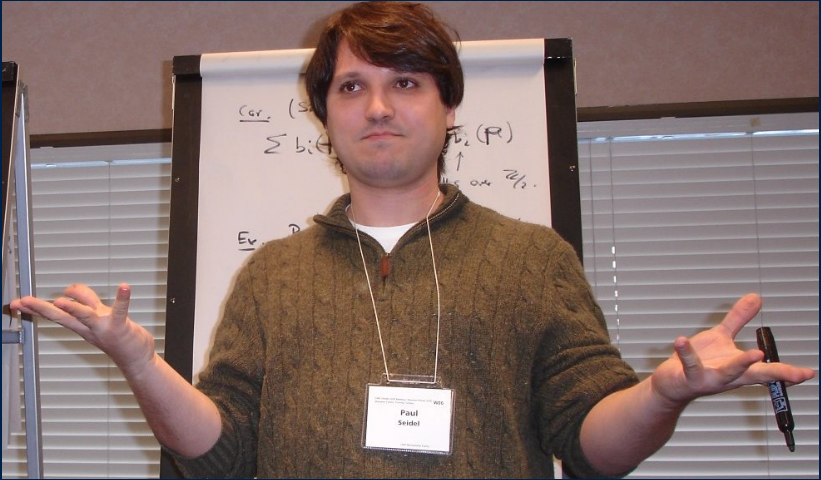
$$y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0,$$

  with its natural action of the subgroup Γ_{16} of PSL_4 given by 4×4 diagonal matrices with determinant 4 and entries 4-th roots of unity. Let $Z_q^* \to Y_q^* / \Gamma_{16}$ be the crepant resolution.
- **Theorem** (Seidel; 2003). *There exist a local automorphism ψ of* $\mathbf{C}[[q]]$ *with $\psi(q)'(0) \neq 0$, and an equivalence of Λ-linear triangulated categories*

$$D^\pi \mathscr{F}(X_0) \simeq \hat{\psi} D^b \mathrm{Coh}(Z_q^*).$$

§ 3. Seidel’s HMS for the Quartic Surface

PICTURE OF P. SEIDEL



§ 3. Seidel's HMS for the Quartic Surface

REMARKS ON THE FORMULATION OF SEIDEL'S THEOREM

* **Theorem** (Seidel; 2003). *There exist a local automorphism* ψ *of* $\mathbf{C}[[q]]$ *with* $\psi(q)'(0) \neq 0$*, and an equivalence of* Λ*-linear triangulated categories*

$$D^{\pi}\mathscr{F}(X_{\mathbf{0}}) \simeq \hat{\psi}\, D^{b}\mathrm{Coh}(Z_q^{*}).$$

§ 3. Seidel's HMS for the Quartic Surface

REMARKS ON THE FORMULATION OF SEIDEL'S THEOREM

* **Theorem** (Seidel; 2003). *There exist a local automorphism ψ of $\mathbf{C}[[q]]$ with $\psi(q)'(0) \neq 0$, and an equivalence of Λ-linear triangulated categories*

$$D^\pi \mathscr{F}(X_0) \simeq \hat{\psi} D^b \mathrm{Coh}(Z_q^*).$$

* We have chosen a lift of $\psi : \mathbf{C}[[q]] \to \mathbf{C}[[q]]$ to an automorphism $\hat{\psi}$ of Λ, and $\hat{\psi} D^b \mathrm{Coh}(Z_q^*)$ denotes the category obtained from $D^b \mathrm{Coh}(Z_q^*)$ by restriction of scalars with respect to $\hat{\psi}$. The map ψ is not known explicitly, but it could be the 'mirror map'.

§ 3. Seidel's HMS for the Quartic Surface

REMARKS ON THE FORMULATION OF SEIDEL'S THEOREM

* **Theorem** (Seidel; 2003). *There exist a local automorphism ψ of $\mathbf{C}[[q]]$ with $\psi(q)'(0) \neq 0$, and an equivalence of Λ-linear triangulated categories*

$$D^{\pi}\mathscr{F}(X_0) \simeq \hat{\psi}^{*} D^b \mathrm{Coh}(Z_q^*).$$

* We have chosen a lift of $\psi : \mathbf{C}[[q]] \to \mathbf{C}[[q]]$ to an automorphism $\hat{\psi}$ of Λ, and $\hat{\psi}^{*} D^b\mathrm{Coh}(Z_q^*)$ denotes the category obtained from $D^b\mathrm{Coh}(Z_q^*)$ by restriction of scalars with respect to $\hat{\psi}$. The map ψ is not known explicitly, but it could be the 'mirror map'.
* For elliptic curves HMS was proven by Polishchuk and Zaslow (1998), setting up a very explicit correspondence which relies on Atiyah's classification of vector bundles on elliptic curves.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF I

* Find split generators for $D^{\pi}\mathscr{F}(X_0)$. Consider the pencil $(X_z)_{z\in\mathbf{P}^1}$ generated by the sections $\sigma_0 = x_0^4 + \cdots + x_3^4$ and $\sigma_\infty = x_0 \cdots x_3$ of the line bundle $\mathcal{O}(4)$ on $\mathbf{P}^3_{\mathbf{C}}$. Picard-Lefschetz theory yields 64 Lagrangian spheres in $X_0 - X_{0,\infty}$ (16 for each of the four singular fibre), where $X_{0,\infty} = X_0 \cap X_\infty$. Considering the monodromy along a large circle at 0 and using the correspondence between Dehn twists and algebraic twists shows that the 64 spheres are split generators.

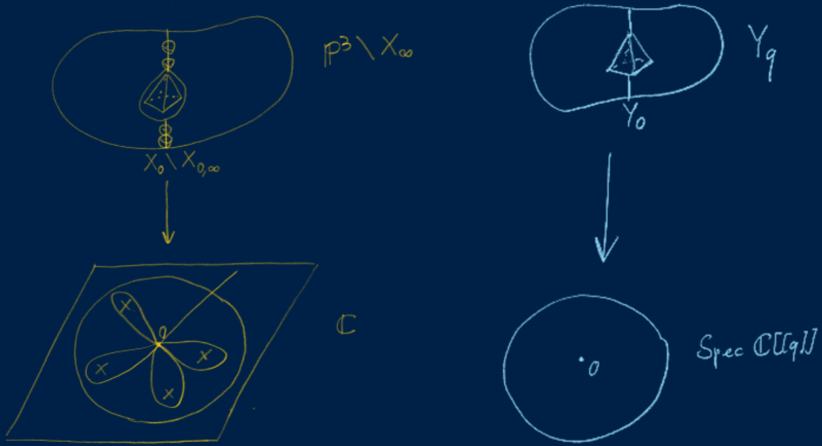
§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF I

- Find split generators for $D^\pi\mathcal{F}(X_0)$. Consider the pencil $(X_z)_{z\in\mathbf{P}^1}$ generated by the sections $\sigma_0 = x_0^4 + \cdots + x_3^4$ and $\sigma_\infty = x_0 \cdots x_3$ of the line bundle $\mathcal{O}(4)$ on $\mathbf{P}^3_{\mathbf{C}}$. Picard-Lefschetz theory yields 64 Lagrangian spheres in $X_0 - X_{0,\infty}$ (16 for each of the four singular fibre), where $X_{0,\infty} = X_0 \cap X_\infty$. Considering the monodromy along a large circle at 0 and using the correspondence between Dehn twists and algebraic twists shows that the 64 spheres are split generators.
- Find split generators for $D^b\mathrm{Coh}(Z_q^*)$. One uses $D^b\mathrm{Coh}(Z_q^*) \simeq D^b\mathrm{Coh}_{\Gamma_{16}}(Y_q^*)$, which has 64 split generators induced by Beilinson's four split generators $\Omega^1(1), \Omega^2(2), \Omega^3(3), \Omega^4(4)$ for $D^b\mathrm{Coh}(\mathbf{P}^3)$, there are 16 ways to twist each $\Omega^i(i)$ by a character of Γ_{16}.

§ 3. Seidel's HMS for the Quartic Surface

ACCOMPANYING PICTURE



§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF II

* Both sets of generators come in 1-parameter families.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF II

* Both sets of generators come in 1-parameter families.
* There are variants of the Fukaya category: A_∞-categories $\mathcal{F}(X_0 - X_\infty)$ and $\mathcal{F}(X_0, X_\infty)$. Both have objects exact Lagrangians in $X_0 - X_\infty$, but they defined over different ground rings: $\mathbf{C}$ resp. $\mathbf{C}[[q]]$. In fact, $\mathcal{F}(X_0 - X_\infty)$ is obtained from $\mathcal{F}(X_0, X_\infty)$ by taking $\otimes_{\mathbf{C}[[q]]}\mathbf{C}$, while taking $\otimes_{\mathbf{C}[[q]]}\Lambda$ gives a full A_∞-subcategory of $\mathcal{F}(X)$. The 64 Lagrangian spheres induce an A_∞-subcategory $\mathcal{F}_{64,q}$ of $\mathcal{F}(X_0, X_\infty)$.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF II

* Both sets of generators come in 1-parameter families.
* There are variants of the Fukaya category: A_∞-categories $\mathcal{F}(X_0 - X_\infty)$ and $\mathcal{F}(X_0, X_\infty)$. Both have objects exact Lagrangians in $X_0 - X_\infty$, but they defined over different ground rings: $\mathbf{C}$ resp. $\mathbf{C}[[q]]$. In fact, $\mathcal{F}(X_0 - X_\infty)$ is obtained from $\mathcal{F}(X_0, X_\infty)$ by taking $\otimes_{\mathbf{C}[[q]]}\mathbf{C}$, while taking $\otimes_{\mathbf{C}[[q]]}\Lambda$ gives a full A_∞-subcategory of $\mathcal{F}(X)$. The 64 Lagrangian spheres induce an A_∞-subcategory $\mathcal{F}_{64,q}$ of $\mathcal{F}(X_0, X_\infty)$.
* On the B-side we need to introduce a dg enhancement (underlying A_∞-category $\mathcal{S}(Z_q^*)$ with $\mu^d = 0$ for $d \geqslant 3$) of $D^b\mathrm{Coh}(Z_q^*)$. We then have an A_∞-category $\mathcal{S}_{64,q}$ induced by the 64 generators of $D^b\mathrm{Coh}(Z_q^*)$.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF III

* Both $\mathcal{F}_{64,q}$ and $\mathcal{S}_{64,q}$ are 1-parameter A_∞-*deformations* of some A_∞-algebra $\mathbb{Q}_{64}$, i.e. the structural maps of each family are of the form $\mu_q^d = \mu^d + q\mu_{q,1}^d + q^2\mu_{q,2}^d + \cdots$, where μ^d is the structural map of $\mathbb{Q}_{64}$ and each $\mu_{q,n}^d$ is a linear map $\mathbb{Q}_{64}^{\otimes d} \to \mathbb{Q}_{64}$.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF III

* Both $\mathcal{F}_{64,q}$ and $\mathcal{S}_{64,q}$ are 1-parameter A_∞-*deformations* of some A_∞-algebra $\mathbb{Q}_{64}$, i.e. the structural maps of each family are of the form $\mu_q^d = \mu^d + q\mu_{q,1}^d + q^2\mu_{q,2}^d + \cdots$, where μ^d is the structural map of $\mathbb{Q}_{64}$ and each $\mu_{q,n}^d$ is a linear map $\mathbb{Q}_{64}^{\otimes d} \to \mathbb{Q}_{64}$.
* The algebra $\mathbb{Q}_{64}$ has $\mu^1 = 0$, and every 1-parameter deformation with $\mu_q^1 = 0$ defines a class in the (truncated) second Hochschild cohomology of $\mathbb{Q}_{64}$.

§ 3. Seidel's HMS for the Quartic Surface

STRATEGY OF THE PROOF III

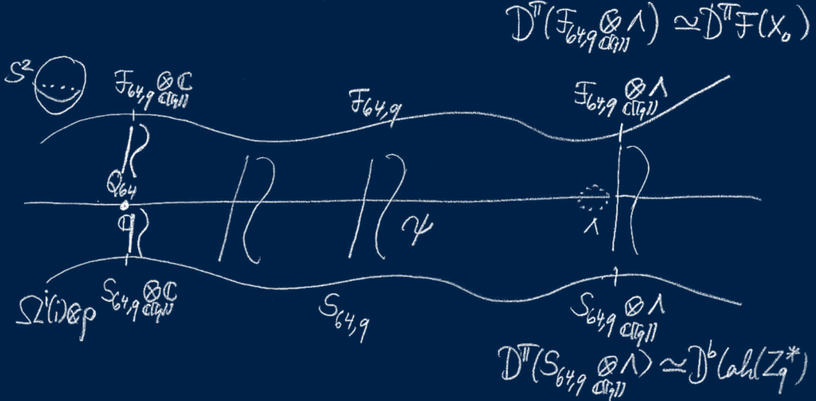
* Both $\mathcal{F}_{64,q}$ and $\mathcal{S}_{64,q}$ are 1-parameter A_∞-*deformations* of some A_∞-algebra $\mathbb{Q}_{64}$, i.e. the structural maps of each family are of the form $\mu_q^d = \mu^d + q\mu_{q,1}^d + q^2\mu_{q,2}^d + \cdots$, where μ^d is the structural map of $\mathbb{Q}_{64}$ and each $\mu_{q,n}^d$ is a linear map $\mathbb{Q}_{64}^{\otimes d} \to \mathbb{Q}_{64}$.
* The algebra $\mathbb{Q}_{64}$ has $\mu^1 = 0$, and every 1-parameter deformation with $\mu_q^1 = 0$ defines a class in the (truncated) second Hochschild cohomology of $\mathbb{Q}_{64}$.
* At least the invariant part of the second Hochschild cohomology of $\mathbb{Q}_{64}$ is 1-dimensional, which implies that any two (equivariant) 1-parameter deformation with $\mu_q^1 = 0$ and nontrivial class are equivalent after reparametrisation by some $\psi : \mathbf{C}[[q]] \to \mathbf{C}[[q]]$. Seidel proves that $\mathcal{F}_{64,q}$ and $\mathcal{S}_{64,q}$ have nontrivial class, hence

$$D^\pi\mathcal{F}(X_0) \simeq D^\pi(\mathcal{F}_{64,q} \otimes_{\mathbf{C}[[q]]} \Lambda) \simeq D^\pi(\psi\mathcal{S}_{64,q} \otimes_{\mathbf{C}[[q]]} \Lambda).$$

$$\simeq \hat{\psi} D^\pi(\mathcal{S}_{64,q} \otimes_{\mathbf{C}[[q]]} \Lambda) \simeq \hat{\psi} D^b\mathrm{Coh}(Z_q^*).$$

§ 3. Seidel’s HMS for the Quartic Surface

PICTORIAL DESCRIPTION OF THE PROOF



References

* Auroux, Introductions to everything.
* Fukaya, Morse Homotopy, A_∞-Categories, and Floer Homologies.
* Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry.
* Kontsevich, Homological Algebra of Mirror Symmetry.
* Seidel, Fukaya Categories and Picard-Lefschetz Theory.
* Seidel, Fukaya Categories and Deformations.
* Seidel, Homological Mirror Symmetry for the Quartic Surface.
* Sheridan, Lectures on Homological Mirror Symmetry.
* Rouquier, Catégories dérivées et géométrie birationnelle.