Hochschild Homology and the HKR Theorem

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Introduction.

The principal subject matter of this bachelor thesis is a homology theory for associative algebras known as Hochschild homology. Historically, Hochschild homology first arose in the form of its dual, Hochschild cohomology, which is intimately connected to algebraic deformation theory (see Gerstenhaber [**4**] for an account of this). However, this thesis is mainly devoted to the connection between Hochschild homology and the exterior algebra of differential forms, which is established by proving the Hochschild-Kostant-Rosenberg (HKR) theorem (first proven in [**7**]). It is emphasized that this thesis is of an expository nature; thus there are no new theorems proven here.

In section 1 (based on chapter 1 of Loday [**8**] and chapter 9 of Weibel [**11**]) Hochschild homology of a commutative k-algebra A is defined using the Hochschild complex, and the connection to the derived functor $\mathrm{Tor}_*^{A\otimes A}$ is pointed out. Section 2 (based on chapter 9 of Matsumara [**9**] and chapter 1 of Loday [**8**]) proceeds with the definition and certain properties of the module of Kähler differentials $\Omega^1_{A|k}$, which is identified with the first Hochschild homology module $\mathrm{HH}_1(A)$ of A. Section 3 (based on chapter 3 of Loday [**8**], chapter 9 of Matsumara [**9**] and §19, §20, §22 in Grothendieck [**6**]) discusses two different definitions of the geometric notion of a smooth algebra and their relation to one another. Section 4 introduces the Shuffle product structure in Hochschild homology and contains the proof of the HKR theorem (following [**8**]). The last section (which places emphasis on concepts and not on the proofs themselves) is based mainly on Quillen [**10**] and André [**1**], and introduces the André-Quillen homology theory, points out a HKR theorem for formal smoothness (giving necessary and sufficient conditions in terms of André-Quillen homology), which is then used to prove Serre's theorem on regular local rings. The appendix is intended to collect (for the reader's convenience) results and terminology that are often used (explicit or implicit). On the reader's part it is assumed a knowledge of methods of commutative algebra and homological algebra, which can be found e.g. in Matsumara [**9**], and Weibel [**11**]. The language of category theory is also frequently used.

It remains to settle certain conventions regarding notation, apart from those which are traditional. Throughout, homological indexing is used, the ground ring k is taken to be commutative, and A denotes a commutative k-algebra. (In general, all rings and algebras considered are commutative, apart from the obvious exceptions.) Unless otherwise stated, all tensor products are taken over k (thus $\otimes$ denotes $\otimes_k$); occasionally the notation $A^{\otimes n} = \bigotimes_{i=1}^{n} A$ is employed (where it is understood that $A^{\otimes 0} = k$). The k-algebra homomorphism $A \otimes A \to A$ given by multiplication plays a prominent role, and will be denoted by μ.

1. **Hochschild homology.**

1.1. **Hochschild homology defined.** Let k be a commutative ring, and A be a commutative k-algebra. If M is an (A, A)-bimodule (which we can view also as an $A \otimes A$-module) then we define the k-modules $C_n(A, M) = M \otimes A^{\otimes n}$ for $n \geqslant 0$ and $C_n(A, M) = 0$ for $n < 0$ (viewing M also as a k-module by restriction of scalars, see section A.1); it is immediate that for every $n \geqslant 1$ there are unique k-module morphisms $d_n^i : C_n(A, M) \to C_{n-1}(A, M)$ $(0 \leqslant i \leqslant n)$ satisfying the conditions

$$\begin{aligned} d_n^0(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n, \\ d_n^i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \quad (0 < i < n), \\ d_n^n(m \otimes a_1 \otimes \cdots \otimes a_n) &= a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

These maps satisfy the identity $d_{n-1}^i d_n^j = d_{n-1}^{j-1} d_n^i$ $(0 \leqslant i < j \leqslant n)$ which are easily checked on generators. If we define maps $b_n : C_n(A, M) \to C_{n-1}(A, M)$ by $b_n = \sum_{j=0}^{n} (-1)^j d_n^j$ then we get an associated complex, the *Hochschild complex* $C_*(A, M)$

$$\cdots \xrightarrow{b} C_n(A, M) \xrightarrow{b} C_{n-1}(A, M) \xrightarrow{b} \cdots \xrightarrow{b} C_1(A, M) \xrightarrow{b} C_0(A, M) \xrightarrow{b} 0 \xrightarrow{b} \cdots,$$

since the square of b is

$$b^2 = \sum_{0 \leqslant i < j \leqslant n} (-1)^{i+j} d_{n-1}^{j-1} d_n^i + \sum_{0 \leqslant j \leqslant i \leqslant n-1} (-1)^{i+j} d_{n-1}^i d_n^j = 0,$$

where we have applied the above identity to the first term. Note that $C_n(A, M)$ has a unique A-module structure such that $a(m \otimes a_1 \otimes \cdots \otimes a_n) = am \otimes a_1 \otimes \cdots \otimes a_n$, which makes the maps d_n^i and hence the differential b into A-module maps; thus $C_*(A, M)$ is a complex of A-modules.

REMARK 1.1. It is striking that the differential of the Hochschild complex is of the same form as the differential of the singular chain complex of a topological space (in the latter case the maps d_i are induced by the face maps). This reflects the fact that these are simplical modules; a systematic treatment of this relationship can be found for instance in chapter 8 of Weibel [**11**].

The *Hochschild homology* $H_*(A, M)$ of A with coefficients in M is defined to be the homology of the Hochschild complex $C_*(A, M)$; $H_*(A, M)$ is also used to denote the direct sum $\bigoplus_{n=0}^{\infty} H_n(A, M)$. The notations $C_*(A) = C_*(A, A)$ and $\mathrm{HH}_*(A) = H_*(A, A)$ will be used, since coefficients in A are of most interest to us. Notice that the zeroth Hochschild homology module of A with coefficients in A is obviously $\mathrm{HH}_0(A) = A$ as the relevant boundary maps vanish identically. (Here it is essential that A is commutative; more generally $H_0(A, M)$ is the module of coinvariants of M by A.) The first Hochschild homology $\mathrm{HH}_1(A)$ will be computed in section 2.3.

1.2. **The bar resolution.** We now establish the connection between Hochschild homology and the derived functor Tor. Let $\mu : A \otimes A \to A$ be the product k-algebra homomorphism of A, $\mu(x \otimes y) = xy$, and consider A as an $A \otimes A$-module by restriction of scalars. The *bar* (or *standard*) *complex* of A is the complex $C_*(A, A \otimes A)$ of $A \otimes A$-modules.

LEMMA 1.1. *The bar complex gives a resolution (with augmentation μ) of the $A \otimes A$-module A.*

PROOF. It is clear that μ is the cokernel of the boundary map $A^{\otimes 3} \to A^{\otimes 2}$; we have a contracting homotopy $s : A^{\otimes n} \to A^{\otimes(n+1)}$ given by $s(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$. ■

PROPOSITION 1.1. *If A is a flat k-algebra, then for every (A, A)-bimodule M we have an isomorphism $H_*(A, M) \cong \mathrm{Tor}_*^{A\otimes A}(M, A)$.*

PROOF. As A is flat as a k-module, $A^{\otimes n}$ is also a flat k-module, and $C_n(A, A \otimes A) = (A \otimes A) \otimes A^{\otimes n}$ is a flat $A \otimes A$-module (it is isomorphic to the extension of scalars of $A^{\otimes n}$). Thus the bar resolution is a flat resolution of the $A \otimes A$-module A; as M is naturally an $A \otimes A$-module, we have an isomorphism of complexes $M \otimes_{A\otimes A} C_*(A, A \otimes A) \cong C_*(A, M)$ (taking $m \otimes (a_0 \otimes \cdots \otimes a_{n+1}$ to $a_{n+1} m a_0 \otimes (a_1 \otimes \cdots \otimes a_n)$). The homology of the right complex is by definition Hochschild homology $H_*(A, M)$, whereas the homology of the left complex is $\mathrm{Tor}_*^{A\otimes A}(M, A)$ as Tor can be computed using flat resolutions. ■

REMARK 1.2. One can drop the flatness assumption in the above proposition, provided one replaces $\mathrm{Tor}_*^{A\otimes A}(M, A)$ by the relative Tor $\mathrm{Tor}_*^{A\otimes A/k}(M, A)$ for the structure morphism $k \to A \otimes A$ of $A \otimes A$ (see Lemma 9.1.3 in Weibel [**11**]).

2. Kähler differentials and their relations to Hochschild homology

2.1. **The functor $\mathrm{Der}_k(A, -)$.** If M is an A-module, a *k-derivation* $\delta : A \to M$ of A with values in M is a k-module morphism satisfying the equation $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in A$. The set $\mathrm{Der}_k(A, M)$ of k-derivations of A with values in M has a natural (pointwise defined) A-module structure, and every A-module morphism $f : M \to M'$

induces an A-module morphism $f_* : \mathrm{Der}_k(A, M) \to \mathrm{Der}_k(A, M')$ by $f_*(\delta) = f \circ \delta$; we thus get a functor $\mathrm{Der}_k(A, -) : A-\mathbf{Mod} \to A-\mathbf{Mod}$, which happens to be representable.

EXAMPLE 2.1. Let M be a C^∞-manifold, let $x \in M$ be a point and consider $\mathbf{R}$ as a $C^\infty(M)$-module by the ring morphism $C^\infty(M) \to \mathbf{R}$ given by evaluation at x. Then we have (by definition) $T_x M = \mathrm{Der}_{\mathbf{R}}(C^\infty(M), \mathbf{R})$, the tangent space of M at x. (The $\mathbf{R}$-module structure on $\mathbf{R}$ induced by $\mathbf{R} \to C^\infty(M)$ is obviously the usual one.) The case is similar for affine algebraic varieties.

2.2. $\mathrm{Der}_k(A, -)$ **is representable.** A module of *Kähler differentials* for A over k is a pair $(\Omega^1_{A|k}, d)$ consisting of an A-module $\Omega^1_{A|k}$ and a k-derivation $d : A \to \Omega^1_{A|k}$, universal in the sense that for every k-derivation $\delta : A \to M$ into a A-module M there is a unique A-module map $f : \Omega^1_{A|k} \to M$ which makes the diagram

$$\begin{tikzcd} A \arrow[rr, "\delta"] \arrow[dr, "d"'] & & M \\ & \Omega^1_{A|k} \arrow[ur, "f"'] & \end{tikzcd}$$

commutative. The usual argument establishes that the pair $(\Omega^1_{A|k}, d)$ is determined uniquely up to unique isomorphism. Also, if S is subset of A that generates A as a k-algebra, then it is easily seen that the image $d(S)$ of S under d generates $\Omega^1_{A|k}$ as an A-module; in particular $\mathrm{Im}\, d$ generates $\Omega^1_{A|k}$ as an A-module.

If $(\Omega^1_{A|k}, d)$ is a module of Kähler differentials for A over k, then it is clear that the maps

$$d^* : \mathrm{Hom}_A(\Omega^1_{A|k}, M) \to \mathrm{Der}_k(A, M), \quad d^*(f) = f \circ d$$

give a functorial isomorphism $\mathrm{Hom}_A(\Omega^1_{A|k}, -) \to \mathrm{Der}_k(A, -)$; in other words, $\Omega^1_{A|k}$ represents the functor $\mathrm{Der}_k(A, -)$.

PROPOSITION 2.1. $(\Omega^1_{A|k}, d)$ *exists for every k-algebra A.*

PROOF. Consider the k-algebra morphism $\mu : A \otimes A \to A$ given by multiplication. Let $\mathfrak{J}$ denote the kernel of μ, and put $\Omega^1_{A|k} = \mathfrak{J}/\mathfrak{J}^2$. The morphism $A \to A \otimes A$ by $a \mapsto a \otimes 1$ allows us to view $A \otimes A$ as an A-module, thus inducing an A-module structure on $\Omega^1_{A|k}$. We define the universal derivation $d : A \to \Omega^1_{A|k}$ by $d(x) = (x \otimes 1 - 1 \otimes x) + \mathfrak{J}^2$. Let $\delta \in \mathrm{Der}_k(A, M)$; to see that there is at most one A-module map $f : \Omega^1_{A|k} \to M$ with $f \circ d = \delta$ we note that the image of d generates $\Omega^1_{A|k}$. This follows from the fact that $\mathfrak{J}$ is generated as an A-module by the elements of the form $x \otimes 1 - 1 \otimes x$ for $x \in A$: for $\sum x_i \otimes y_i \in \mathfrak{J}$ we have

$$\sum x_i \otimes y_i = \sum (x_i \otimes y_i - x_i y_i \otimes 1) = -\sum x_i (y_i \otimes 1 - 1 \otimes y_i).$$

The latter also implies that $\mathfrak{J}^2$ is generated by the elements $(x \otimes 1 - 1 \otimes x)(y \otimes 1 - 1 \otimes y)$ for $x, y \in A$. For the existence of f, the unique A-module morphism $\mathfrak{J} \to M$ with $x \otimes y \mapsto -x\delta(y)$ vanishes on generators of $\mathfrak{J}^2$, thus induces an A-module map $f : \Omega^1_{A|k} \to M$ which clearly satisfies $f \circ d = \delta$. ∎

Let $\Omega^*_{A|k} = \bigwedge_A \Omega^1_{A|k}$ denote the exterior A-algebra of the A-module $\Omega^1_{A|k}$; the elements of $\Omega^n_{A|k}$ are called the *exterior differential n-forms* of A over k.

REMARK 2.1. There is a unique k-antiderivation $d : \Omega^*_{A|k} \to \Omega^*_{A|k}$ of degree 1 with $d^2 = 0$ extending the universal derivation $d : A \to \Omega^1_{A|k}$ (d is explicitly given by $d(ydx_1 \wedge \cdots \wedge dx_n) = dy \wedge dx_1 \wedge \cdots \wedge dx_n$, see Bourbaki [**3**], chapitre 10, §2.10). The complex $(\Omega^*_{A|k}, d)$ is the *de Rham complex* of A over k.

2.3. **Computation of** $\mathrm{HH}_1(A)$**.** We now establish the connection between the first Hochschild homology and the Kähler differentials, which holds without imposing further conditions on A or k.

PROPOSITION 2.2. *There is a canonical A-module isomorphism* $\Omega^1_{A|k} \to \mathrm{HH}_1(A)$ *mapping xdy to* $[x \otimes y]$.

PROOF. The boundary map $b : A \otimes A \to A$ vanishes identically, therefore every element of $A \otimes A$ is a 1-cycle. The k-module map $A \to \mathrm{HH}_1(A)$ by $x \mapsto [1 \otimes x]$ is a k-derivation, since $[x \otimes y] - [1 \otimes xy] + [y \otimes x] = 0$ as $x \otimes y - 1 \otimes xy + y \otimes x = b(1 \otimes x \otimes y)$ is a boundary. By definition of $\Omega^1_{A|k}$ we have an induced A-module map $\Omega^1_{A|k} \to \mathrm{HH}_1(A)$ mapping xdy to $[x \otimes y]$. The kernel of the k-module map $A \otimes A \to \Omega^1_{A|k}$ with $x \otimes y \mapsto xdy$ contains the 1-boundaries, thus we have an induced map $\mathrm{HH}_1(A) \to \Omega^1_{A|k}$ sending $[x \otimes y]$ to xdy, which is an inverse of our map $\Omega^1_{A|k} \to \mathrm{HH}_1(A)$. ∎

2.4. **Functoriality of** Ω^1 **and the fundamental exact sequences.** The A-module $\Omega^1_{A|k}$ depends functorially on $k \to A$ in the following sense: for every commutative square

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}$$

of ring homomorphisms there is a unique A-module map $\Omega^1_{A|k} \to \Omega^1_{A'|k'}$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "d"'] & A' \arrow[d, "d"] \\ \Omega^1_{A|k} \arrow[r] & \Omega^1_{A'|k'} \end{tikzcd}$$

commutes, for the composite $A \to A' \to \Omega^1_{A'|k'}$ is a k-derivation. (This property is a functorial property in the following sense. If one has another commutative square, adjacent to the original one, then the composite of the induced maps on Ω^1 is equal to the map induced by the commutative square that is obtained by merging the two squares.)

The following exact sequences for Ω^1 are often useful; for the proof (which is not difficult at any rate) we refer to Theorems 25.1 and 25.2 of Matsumara [**9**].

PROPOSITION 2.3. *Every sequence of ring homomorphisms $k \to A \to B$ induces an exact sequence of B-modules*

$$\Omega^1_{A|k} \otimes_A B \to \Omega^1_{B|k} \to \Omega^1_{B|A} \to 0,$$

which is called the first fundamental exact sequence. If in addition $A \to B$ is surjective and I denotes its kernel, then we have an exact sequence of B-modules

$$I/I^2 \to \Omega^1_{A|k} \otimes_A B \to \Omega^1_{B|k} \to 0,$$

which is called the second fundamental exact sequence.

(Note that by Example 3.1 we have $\Omega^1_{B|A}$=0 in the second sequence.) The maps occurring in these sequences are described by using the functorial property of $\Omega^1_{A|k}$ mentioned above. For instance the map $\Omega^1_{A|k} \otimes_A B \to \Omega^1_{B|k}$ is obtained by taking $k = k'$, $B = A'$ and applying extension of scalars (see section A.1) to the A-module map $\Omega^1_{A|k} \to \Omega^1_{B|k}$ obtained by functoriality of Ω^1 (using also that $\Omega^1_{B|k}$ is a B-module).

As an application of the second fundamental exact sequence, we compute a presentation of $\Omega^1_{A|k}$ for any k-algebra A (using a presentation of A).

EXAMPLE 2.2. First we show that $\Omega^1_{k[(x_i)_{i\in I}]|k} \cong \bigoplus_{i\in I} k[(x_i)_{i\in I}]dx_i$. As the indeterminates x_i ($i \in I$) generate $k[(x_i)_{i\in I}]$ as a k-algebra, the elements dx_i ($i \in I$) generate $\Omega^1_{k[(x_i)_{i\in I}]|k}$ as a $k[(x_i)_{i\in I}]$-module. These are also $k[(x_i)_{i\in I}]$-linearly independent: let $\sum f_i dx_i = 0$ for some $f_i \in k[(x_i)_{i\in I}]$ (zero for almost all $i \in I$); for $j \in I$ the partial derivative $\partial/\partial x_j \in \mathrm{Der}_k(k[(x_i)_{i\in I}], k[(x_i)_{i\in I}])$ induces a $k[(x_i)_{i\in I}]$-module map $f : \Omega^1_{k[(x_i)_{i\in I}]|k} \to k[(x_i)_{i\in I}]$ with $f(dx_i) = \partial x_i/\partial x_j = \delta_{ij}$, and thus we have $f_j = f(\sum f_i dx_i) = 0$.

Now let A be any k-algebra, and I a subset of A which generates A as a k-algebra. The inclusion map $I \to A$ induces a surjective k-algebra morphism $k[(x_i)_{i\in I}] \to A$; let $(f_j)_{j\in J}$ be a set of generators for the kernel of this map. Since we have $\Omega^1_{k[(x_i)_{i\in I}]|k} \cong \bigoplus_{i\in I} k[(x_i)_{i\in I}]dx_i$, we see by the second fundamental sequence that $\Omega^1_{A|k}$ is the quotient of $\bigoplus_{i\in I} k[(x_i)_{i\in I}]dx_i$ by the submodule generated by the elements $df_j = \sum(\partial f_j/\partial x_m)dx_m$ ($j \in J$).

3. **Smooth algebras.**

The notion of a smooth algebra has of course a geometric origin; several different definitions of this notion can be found in the literature. Our definition will be the one of Loday [**8**], section 3.4, which is closely related to the notion of a (local) complete intersection. It has the advantage that in the statement of the HKR theorem no (finiteness) conditions on the ground ring k have to be imposed. (In Weibel's statement of the HKR theorem, formal smoothness is used and k is taken to be a field, but it is mentioned that the proof can be modified to hold if k is merely a Noetherian ring. It follows from Theorem 3.1 that Weibel's statement is less general than the one of Loday.)

3.1. **Smooth algebras defined.** A k-algebra A is said to be *smooth* if it is flat and if for every maximal ideal $\mathfrak{m}$ of A the kernel of the surjective local homomorphism $\mu_{\mathfrak{m}} : (A \otimes A)_{\mu^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ induced by μ is generated by a regular sequence in $(A \otimes A)_{\mu^{-1}(\mathfrak{m})}$. (The flatness assumption is very convenient, in particular because it allows us to use the Tor definition of Hochschild homology.)

3.2. **Formal smoothness.** Another notion of smoothness, which is due to Grothendieck ([**6**], Définition 19.3.1), is based on a certain infinitesimal lifting property and is particularly useful for studying deformations. A k-algebra is said to be *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if for every k-algebra B and ideal I of B with $I^2 = 0$ the map $\mathrm{Hom}_{k-\mathbf{Alg}}(A, B) \to \mathrm{Hom}_{k-\mathbf{Alg}}(A, B/I)$ induced by $B \to B/I$ is surjective (resp. injective, resp. bijective). Note that A is formally unramified iff $\Omega^1_{A|k} = 0$ (in other words, iff $\mathrm{Der}_k(A, -) = 0$). Formal smoothness also yields splitting of the fundamental exact sequences of Proposition 2.3: if $A \to B$ is a formally smooth A-algebra (resp. $k \to B$ a formally smooth k-algebra) then the sequence obtained from the first (resp. second) exact sequence by inserting $0 \to$ at the left is a split exact sequence.

EXAMPLE 3.1. The polynomial k-algebra $k[(x_i)_{i\in I}]$ is of course formally smooth. If A is a k-algebra such that the structural morphism $k \to A$ is surjective, then it is clear that A is formally unramified.

EXAMPLE 3.2. If S is a multiplicative subset of k, then the k-algebra $\phi : k \to S^{-1}k$ given by localization is formally étale. To see this, let $\eta : k \to B$ be a k-algebra and I an ideal of B with $I^2 = 0$. If $f : S^{-1}k \to B/I$ is a k-algebra homomorphism, then for every $s \in S$ the element $f(s/1) = [\eta(s)] \in B/I$ is invertible, and as $I^2 = 0$ it follows that $\eta(s)$ is a unit in B. Thus η admits a unique factorization $k \to S^{-1}k \to B$ and $S^{-1}k \to B$ is the unique preimage of f under $\mathrm{Hom}_{k-\mathbf{Alg}}(S^{-1}k, B) \to \mathrm{Hom}_{k-\mathbf{Alg}}(S^{-1}k, B/I)$.

The following result will be used in the proof of the HKR theorem.

LEMMA 3.1. *If S is a multiplicative subset of A, then we have a canonical isomorphism $\Omega^1_{S^{-1}A|k} \cong S^{-1}\Omega^1_{A|k}$ of $S^{-1}A$-modules, in particular a canonical isomorphism $S^{-1}\bigwedge_A \Omega^1_{A|k} \cong \bigwedge_{S^{-1}A} \Omega^1_{S^{-1}A|k}$ of graded $S^{-1}A$-algebras.*

PROOF. The first fundamental exact sequence for $k \to A \to S^{-1}A$ reads

$$0 \to \Omega^1_{A|k} \otimes_A S^{-1}A \to \Omega^1_{S^{-1}A|k} \to 0$$

since $S^{-1}A$ is a formally étale A-algebra by Example 3.2. By the results of section A.1 and section A.2 we obtain canonical isomorphisms

$$(\bigwedge\nolimits_A \Omega^1_{A|k}) \otimes_A S^{-1}A \cong \bigwedge\nolimits_{S^{-1}A}(\Omega^1_{A|k} \otimes_A S^{-1}A) \cong \bigwedge\nolimits_{S^{-1}A} \Omega^1_{S^{-1}A|k}.$$

■

3.3. **The connection between smoothness and formal smoothness.** (The finiteness conditions for algebras occurring in this section are summarized in section A.3.) The following result connects our definitions of the notion of a smooth algebra; for the rather long and technical proof (which uses methods of André-Quillen homology, see section 5) we refer to Appendix E of Loday [**8**] (which also proves the equivalence with several other definitions).

THEOREM 3.1. *Let k be a Noetherian ring, and A be a k-algebra which essentially of finite type and which satisfies* $\mathrm{Tor}^k_n(A, A) = 0$ *for $n \geqslant 1$. Then A is smooth iff A is formally smooth.*

EXAMPLE 3.3. Let k be a perfect field, and A a finite type k-algebra. Then it follows by the above theorem that A is smooth iff A is formally smooth; by combining Théorème 22.5.8, Corollaire 22.6.6, Proposition 22.6.7 in Grothendieck [**6**] it follows that A is a formally smooth k-algebra iff A is a regular ring (see section 5.3 for the latter terminology). In particular, if V is an affine algebraic variety (over $\mathbf{C}$, say), then V is nonsingular iff the ring of regular functions on V is a smooth $\mathbf{C}$-algebra. Thus is the connection to classical algebraic geometry; in Grothendieck's treatment of modern algebraic geometry, the notion of a smooth scheme $X \to S$ over S is defined using the notion of formal smoothness. The definition is such that a morphism of affine schemes Spec $B \to$ Spec A is smooth iff the corresponding A-algebra $A \to B$ is formally smooth and of finite presentation.

4. **The HKR theorem.**

4.1. **The product in Hochschild homology.** A (p, q)-*shuffle* is a permutation $\sigma \in S_{p+q}$ satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. There is a unique left action of the symmetric group S_n on $C_n(A) = A^{\otimes(n+1)}$ such that $\sigma(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$. The *shuffle product* of A is the unique family of A-linear maps $\nabla = \nabla_{p,q} : C_p(A) \otimes_A C_q(A) \to C_{p+q}(A)$ such that

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_p)\nabla(a_0^{'} \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) = \sum_{\sigma} \mathrm{sgn}(\sigma)\sigma(a_0 a_0^{'} \otimes a_1 \otimes \cdots \otimes a_{p+q}),$$

where the sum on the right hand side is taken over all (p,q)-shuffles $\sigma \in S_{p+q}$. The shuffle product ∇ makes $C_*(A)$ into a graded-commutative DG-algebra over A (this is Proposition 4.2.2 in Loday [**8**]), thus passes to homology and gives $\mathrm{HH}_*(A)$ the structure of a graded commutative A-algebra. The canonical A-module isomorphism $\Omega^1_{A|k} \to \mathrm{HH}_1(A)$ of Proposition 2.2 therefore extends to a unique homomorphism of graded A-algebras $\varepsilon : \Omega^*_{A|k} \to \mathrm{HH}_*(A)$, which is called the *antisymmetrization map*. (Explicitly, one has the expression $\varepsilon(a_0 d_1 \wedge \cdots \wedge da_n) = [a_0 \otimes a_1]\nabla \cdots \nabla[1 \otimes a_n] = [\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma(a_0 \otimes \cdots \otimes a_n)]$.) If A is flat, then one has an isomorphism of graded A-algebras $\mathrm{HH}_*(A) \cong \mathrm{Tor}_*^{A\otimes A}(A,A)$, where the product of $\mathrm{Tor}_*^{A\otimes A}(A,A)$ is the canonical one (which is described in detail in chapitre XIV of André [**1**]).

REMARK 4.1. The shuffle map probably first occurred in algebraic topology (in the context of the Eilenberg-Zilber theorem), where it gives a homology inverse to the Alexander-Whitney map.

4.2. **Proof of the HKR theorem.** We first prove two results that will be needed in the proof of the HKR theorem.

LEMMA 4.1. *Let Λ be a ring and I an ideal of Λ. Then we have an isomorphism of Λ-modules $I/I^2 \to \mathrm{Tor}_1^{\Lambda}(\Lambda/I, \Lambda/I)$.*

PROOF. The short exact sequence of Λ-modules

$$0 \to I \to \Lambda \to \Lambda/I \to 0$$

induces a Tor long exact sequence, whose last part is

$$0 \to \mathrm{Tor}_1^{\Lambda}(\Lambda/I, \Lambda/I) \to \Lambda/I \otimes_\Lambda I \to \Lambda/I \otimes_\Lambda \Lambda \to \Lambda/I \otimes_\Lambda \Lambda/I \to 0$$

as $\mathrm{Tor}_1^{\Lambda}(\Lambda/I, \Lambda) = 0$. The connecting homomorphism $\mathrm{Tor}_1^{\Lambda}(\Lambda/I, \Lambda/I) \to \Lambda/I \otimes_\Lambda I$ thus induces an isomorphism onto its image, which is the kernel of the map $\Lambda/I \otimes_\Lambda I \to \Lambda/I \otimes_\Lambda \Lambda$. But under the usual identifications $\Lambda/I \otimes_\Lambda I \cong I/I^2$ and $\Lambda/I \otimes_\Lambda \Lambda \cong \Lambda/I$ this kernel identifies with the kernel of the trivial homomorphism $I/I^2 \to \Lambda/I$. ■

REMARK 4.2. In particular, if $\Lambda = A \otimes A$ and $I = \mathfrak{J}$ (using the notation introduced in the proof of Proposition 2.1), then the short exact sequence

$$0 \to \mathfrak{J} \to A \otimes A \to (A \otimes A)/\mathfrak{J} \to 0$$

is isomorphic (the isomorphism being induced by $\mu : A \otimes A \to A$) to the short exact sequence

$$0 \to \mathfrak{J} \to A \otimes A \xrightarrow{\mu} A \to 0,$$

and we thus have an isomorphism $\mathfrak{J}/\mathfrak{J}^2 \to \mathrm{Tor}_1^{A\otimes A}(A,A)$. If A is a flat k-algebra, then one has a commutative diagram

$$\begin{tikzcd} \Omega^1_{A|k} \arrow[r] \arrow[d] & \mathfrak{J}/\mathfrak{J}^2 \arrow[d] \\ \mathrm{HH}_1(A) \arrow[r] & \mathrm{Tor}_1^{A\otimes A}(A,A) \end{tikzcd}$$

of isomorphisms (which are due to Proposition 1.1, the proof of Proposition 2.1, Proposition 2.2, Lemma 4.1), which shows that the antisymmetrization $\Omega^*_{A|k} \to \mathrm{HH}_*(A)$ becomes the map $\Omega^*_{A|k} \to \mathrm{Tor}_*^{A\otimes A}(A,A)$ under $\mathrm{HH}_*(A) \cong \mathrm{Tor}_*^{A\otimes A}(A,A)$.

LEMMA 4.2. *Let Λ be a ring and $\Lambda \to \Lambda'$ be a flat Λ-algebra. Then for every Λ-modules M and N we have an isomorphism*

$$\mathrm{Tor}_*^{\Lambda}(M,N) \otimes_\Lambda \Lambda' \to \mathrm{Tor}_*^{\Lambda'}(M \otimes_\Lambda \Lambda', N \otimes_\Lambda \Lambda')$$

of graded Λ-modules.

PROOF. Let $P \to M$ be a flat resolution of M; as Λ' is a flat Λ-module, $P \otimes_\Lambda \Lambda' \to M \otimes_\Lambda \Lambda'$ is a flat resolution of the extension of scalars $M \otimes_\Lambda \Lambda'$ of M. As the functor $- \otimes_\Lambda \Lambda'$ is exact, it follows that

$$\begin{aligned} \mathrm{Tor}_*^{\Lambda}(M,N) \otimes_\Lambda \Lambda' &\cong H_*(P \otimes_\Lambda N) \otimes_\Lambda \Lambda' \cong H_*(P \otimes_\Lambda N \otimes_\Lambda \Lambda') \\ &\cong H_*((P \otimes_\Lambda \Lambda') \otimes_{\Lambda'} (N \otimes_\Lambda \Lambda')) \cong \mathrm{Tor}_*^{\Lambda'}(M \otimes_\Lambda \Lambda', N \otimes_\Lambda \Lambda'). \end{aligned}$$

■

For the proof of the following result (which uses that the Koszul resolution associated to a regular sequence generating I gives a free resolution of the Λ-module Λ/I) we refer to Proposition 3.4.7 of Loday [**8**].

LEMMA 4.3. *Let Λ be a ring and I an ideal of Λ. If I is generated by a regular sequence in Λ, then the map $\bigwedge_{\Lambda/I} I/I^2 \to \mathrm{Tor}_*^{\Lambda}(\Lambda/I, \Lambda/I)$ induced by the isomorphism $I/I^2 \to \mathrm{Tor}_1^{\Lambda}(\Lambda/I,\Lambda/I)$ above is an isomorphism of graded Λ/I-algebras.*

We are now in position to prove the HKR theorem.

THEOREM 4.1 (Hochschild-Kostant-Rosenberg). *If A is a smooth k-Algebra then the antisymmetrization map $\varepsilon : \Omega^*_{A|k} \to \mathrm{HH}_*(A)$ is an isomorphism of graded A-algebras.*

PROOF. According to the local to global principle of commutative algebra, it is enough to prove that for fixed n the $A_\mathfrak{m}$-module map $\varepsilon_\mathfrak{m} : (\Omega^n_{A|k})_\mathfrak{m} \to \mathrm{HH}_n(A)_\mathfrak{m} = \mathrm{Tor}_n^{A\otimes A}(A,A)_\mathfrak{m}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of A. One has a commutative diagram of maps

$$\begin{tikzcd}
(\Omega^n_{A|k})_{\mathfrak{m}} \arrow[r] \arrow[d] & \Omega^n_{A_{\mathfrak{m}}|k} \arrow[d] \\
\operatorname{Tor}_n^{A\otimes A}(A,A)_{\mathfrak{m}} \arrow[r] & \operatorname{Tor}_n^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_{\mathfrak{m}},A_{\mathfrak{m}})
\end{tikzcd}$$

where the top map is the isomorphism of Lemma 3.1; the right map is the isomorphism of Lemma 4.3 for $\Lambda = (A \otimes A)_{\mu^{-1}(\mathfrak{m})}$, I the kernel of $\mu_{\mathfrak{m}}$ (which is generated by a regular sequence, as A is smooth), and where we identify $A_{\mathfrak{m}} \cong \Lambda/I$ by the isomorphism induced by $\mu_{\mathfrak{m}}$. The bottom map is the $(A \otimes A)_{\mu^{-1}(\mathfrak{m})}$-module isomorphism obtained by Lemma 4.2 for the flat $A \otimes A$-algebra $A \otimes A \to (A \otimes A)_{\mu^{-1}(\mathfrak{m})}$ given by localization, and the $(A\otimes A)_{\mu^{-1}(\mathfrak{m})}$-module isomorphisms $M \otimes_{A\otimes A} (A \otimes A)_{\mu^{-1}(\mathfrak{m})} \cong M_{\mu^{-1}(\mathfrak{m})} \cong M_{\mathfrak{m}}$ (holding for any A-module M) of section A.1 and A.2. ■

REMARK 4.3. In the original paper of Hochschild-Kostant-Rosenberg [**7**], the assumptions of the theorem (Theorem 5.2 in the paper) were much more restrictive; namely, k was taken to be a perfect field and A to be a finite type k-algebra which is also an integral domain. It should be mentioned that also the notion the of smoothness was a different one: in their treatment, smoothness meant being a regular ring. It follows from Example 3.3 that the statement of the HKR theorem presented here (the one of Loday [**8**], Theorem 3.4.4) is indeed a generalization of the original statement. However, the main ideas of the proof are mostly the same.

REMARK 4.4. A partial converse to the HKR theorem has been proven by L. L. Avramov and S. Iyengar [**2**] (Theorem 5.3). Namely, if k is a Noetherian ring and A a flat k-algebra, which is essentially of finite type and such that the Hochschild homology algebra $\mathrm{HH}_*(A)$ is a finite type A-algebra, then A is smooth k-algebra.

5. **André-Quillen homology.**

In this last section, a HKR type theorem for formal smoothness is formulated; it gives necessary and sufficient conditions in terms of the André-Quillen homology theory. This theory is the proper tool for dealing with smooth algebras, and is defined via the cotangent complex. To introduce the latter, there are mainly two approaches (each having its advantages): by an explicit free resolution (see chapitre III of André [**1**] or section 8.8 of Weibel [**11**]) or using methods of homotopical algebra. Here the latter approach is used, which is conceptually more satisfying, but introducing all relevant concepts of homotopical algebra would take more space than I am willing to give. For this and the more technical homotopy-theoretic details the reader is referred to the article by Goerss [**5**].

5.1. **Preliminaries.** By Δ we denote the category of finite nonempty ordered sets $[n] = \{0, \dots, n\}$ $(n \geqslant 0)$ with morphisms the order-preserving functions. For $n \geqslant 1$ and

$0 \leqslant i \leqslant n$ the morphism $\varepsilon_n^i : [n-1] \to [n]$ is the unique injection missing $i \in [n]$; we have $\varepsilon_n^j \circ \varepsilon_{n-1}^i = \varepsilon_n^i \circ \varepsilon_{n-1}^{j-1}$ for $0 \leqslant i < j \leqslant n$. If $\mathscr{C}$ is a category, then the category $s\mathscr{C}$ of *simplical objects* in $\mathscr{C}$ is the category of functors $\Delta^{\mathrm{op}} \to \mathscr{C}$. The diagonal functor $\mathscr{C} \to s\mathscr{C}$ gives an embedding, and we identify an object of $\mathscr{C}$ with its image in $s\mathscr{C}$. Any functor $\mathscr{C} \to \mathscr{D}$ induces a functor $s\mathscr{C} \to s\mathscr{D}$; an adjunction of functors between $\mathscr{C}$ and $\mathscr{D}$ induces an adjunction of the corresponding functors between $s\mathscr{C}$ and $s\mathscr{D}$.

If X is a simplical object in $A-\mathbf{Mod}$, then we get an *associated chain complex* $C(X)$ with $C(X)_n = X[n]$ and differential $d_n = \sum_{j=0}^n (-1)^j X(\varepsilon_n^j)$ (the proof is the same as in section 1.1). One also forms the *normalized chain complex* $N(X)$, which is the subcomplex of $C(X)$ with $N(X)_n = \bigcap_{j=0}^{n-1} \operatorname{Ker} X(d_n^j)$. We then obtain functors C and N from $s(A-\mathbf{Mod})$ to the category of complexes of A-modules, and it is a fact that the components of the functorial morphism $N \to C$ given by inclusion are chain homotopy equivalences. The Dold-Kan theorem asserts that N gives an equivalence of categories between $s(A-\mathbf{Mod})$ and the category of complexes of A-modules concentrated in non-negative degrees.

An object X of a category $\mathscr{C}$ is an *abelian group object* if for every object Y in $\mathscr{C}$ the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ has the structure of an abelian group, such that $\mathrm{Hom}_{\mathscr{C}}(-, X)$ becomes a functor to the category of abelian groups. These objects give a subcategory $\mathscr{C}_{\mathrm{ab}}$ of $\mathscr{C}$, and if the inclusion functor $\mathscr{C}_{\mathrm{ab}} \to \mathscr{C}$ admits a left adjoint, then it is natural (keeping in mind the case $\mathscr{C} = \mathbf{Grp}$) to call the left adjoint the *abelianization* of $\mathscr{C}$.

5.2. **The cotangent complex.** In the following, we consider the comma category $k-\mathbf{Alg} \downarrow A$ of k-algebras over a fixed k-algebra A. If M is an A-module, then the direct sum $A \oplus M$ becomes a ring with $(a,m)(a',m') = (aa', am' + a'm)$ and a k-algebra over A by using the morphism $A \to A \oplus M$ by $a \mapsto (a, 0)$ and the projection $A \oplus M \to A$. This gives a functor $A \oplus - : A - \mathbf{Mod} \to k - \mathbf{Alg} \downarrow A$, and the functorial isomorphism $\mathrm{Hom}_{k-\mathbf{Alg}\downarrow A}(B, A \oplus M) \to \mathrm{Der}_k(B, M)$ induced by the projection $A \oplus M \to M$ shows that $A \oplus M \in (k - \mathbf{Alg} \downarrow A)_{\mathrm{ab}}$; in fact $A \oplus -$ gives an equivalence of categories $A - \mathbf{Mod} \cong (k - \mathbf{Alg} \downarrow A)_{\mathrm{ab}}$. The functor $A \otimes_{-} \Omega^1_{-|k} : k - \mathbf{Alg} \downarrow A \to A - \mathbf{Mod}$ (defined on morphisms by using the functoriality of Ω^1, see section 2.4) is the left adjoint of $A \oplus -$, which follows from the functorial isomorphisms

$$\begin{aligned}\mathrm{Hom}_{k-\mathbf{Alg}\downarrow A}(B, A \oplus M) &\cong \mathrm{Der}_k(B, M) \cong \mathrm{Hom}_B(\Omega^1_{B|k}, M)\\ &\cong \mathrm{Hom}_B(\Omega^1_{B|k}, \mathrm{Hom}_A(A, M)) \cong \mathrm{Hom}_A(A \otimes_B \Omega^1_{B|k}, M).\end{aligned}$$

We thus have an induced adjunction of functors between $s(k-\mathbf{Alg} \downarrow A)$ and $s(A-\mathbf{Mod})$; as the functor $A \oplus -$ preserves weak equivalences and fibrations (the model structures on these categories are described in detail in ...), we obtain a Quillen adjunction and we may form the total left derived functor $\mathbf{L}(A \otimes_{-} \Omega^1_{-|k}) : \mathrm{Ho}(s(k - \mathbf{Alg} \downarrow A)) \to$

$\mathrm{Ho}(s(A - \mathbf{Mod}))$. The image of $A \to A$ under $\mathbf{L}(A \otimes_{-} \Omega^1_{-|k})$ is said to be the *cotangent complex* of A over k and is denoted $\mathbf{L}_{A|k}$. As the normalization functor preserves the model structures, the cotangent complex $\mathbf{L}_{A|k}$ may also be viewed as an element of the homotopy category of complexes of A-modules (i.e. the derived category of A-modules).

5.3. **HKR type theorems for formal smoothness.** The *André-Quillen homology* $D_*(A|k, M)$ of A over k with values in the A-module M is the homology of the complex $\mathbf{L}_{A|k} \otimes_A M$. It has many desirable formal properties; for instance $D_*(A|k, -)$ is a homological functor (in fact the functor $\mathbf{L}_{A|k} \otimes_A -$, defined on the derived category of A-modules, turns out to be an exact functor). It also gives a long exact sequence (called the Jacobi-Zariski exact sequence) extending the first fundamental sequence of Proposition 2.3, which arises from a certain distinguished triangle.

The following is Theorem 6.13 in Quillen [**10**]; it appears also (in slightly modified form) in André [**1**], chapter XIV, Théorème 22. It seems to be due to Quillen.

THEOREM 5.1. *Let Λ be a ring and I an ideal of Λ. Then the following are equivalent: (i) I/I^2 is a flat Λ/I-module and the canonical morphism of graded Λ/I-algebras $\bigwedge_{\Lambda/I} I/I^2 \to \mathrm{Tor}^{\Lambda}_*(\Lambda/I, \Lambda/I)$ is an isomorphism; (ii) $D_n(\Lambda/I|\Lambda, -) = 0$ for $n \geqslant 2$; (iii) I/I^2 is a flat Λ/I-module and $\mathbf{L}_{\Lambda/I|\Lambda} \cong I/I^2[1]$ in the derived category of Λ/I-modules.*

Using $\Lambda = A \otimes A$, $I = \mathfrak{J}$ and taking into account the following result, the above theorem may be viewed as a HKR type theorem for formal smoothness.

PROPOSITION 5.1. *A is a formally smooth k-algebra iff $D_1(A|k, A) = 0$ and $\Omega^1_{A|k}$ is a projective A-module.*

5.4. **Applications to regular local rings.** In this last section we outline a proof of Serre's theorem on local rings, using André-Quillen homology and Theorem 5.1. A minimal amount of general homological dimension theory is introduced and used without proof.

Let Λ be a ring. If M is an Λ-module, then the *projective dimension* of M is by definition $\mathrm{dp}_\Lambda(M) = \inf\{n \in \mathbf{N} \mid \mathrm{Ext}^{n+1}_\Lambda(M, -) = 0\}$; the infimum being taken in $\mathbf{Z} \cup \{-\infty, +\infty\}$. (Here and in the following it is assumed that $0 \in \mathbf{N}$.) The *homological dimension* of Λ is the supremum $\mathrm{dh}(\Lambda) = \sup\{\mathrm{dp}_\Lambda(M) \mid M \in \Lambda - \mathbf{Mod}\}$ taken in $\mathbf{Z} \cup \{-\infty, +\infty\}$.

LEMMA 5.1. *If $n \in \mathbf{N}$ then we have $\mathrm{dh}(\Lambda) \leqslant n$ iff $\mathrm{dp}_\Lambda(M) \leqslant n$ for every finitely-generated Λ-module.*

This result is Proposition 17.2.8 in Grothendieck [**6**]; it is due to M. Auslander.

LEMMA 5.2. *If Λ is a Noetherian ring, M a finitely generated Λ-module and $n \in \mathbf{N}$, then we have $\mathrm{dp}_\Lambda(M) \leqslant n$ iff $\mathrm{Tor}^\Lambda_m(-, M) = 0$ for all $m > n$.*

This is Proposition 2 of §8.1 of chapitre 10 in Bourbaki [**3**].

If $(\Lambda, \mathfrak{m}, \kappa)$ is a Noetherian local ring, then it follows from the fundamental theorem of (classical) dimension theory (for instance Matsumara [**9**], Theorem 13.4) that its Krull dimension satisfies $\dim \Lambda \leqslant \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) < \infty$; Λ is said to be a *regular local ring* if equality holds. A *regular ring* is a Noetherian ring Λ such that for every $\mathfrak{p} \in \operatorname{Spec} \Lambda$ the localization $\Lambda_\mathfrak{p}$ is a regular local ring.

The following result is Proposition 4.4.11 in Weibel [**11**].

LEMMA 5.3. *Let $(\Lambda, \mathfrak{m}, \kappa)$ be a Noetherian local ring and $n \in \mathbf{N}$. If M a finitely-generated Λ-module, then we have* $\mathrm{dp}_\Lambda(M) \leqslant n$ *iff* $\mathrm{Tor}^\Lambda_{n+1}(M, \kappa) = 0$.

The following two results are more specific and establish the connection to André-Quillen homology; the first one is a characterization of regular local rings in terms of André-Quillen homology, and appears as Proposition 26 of chapitre VI in André [**1**].

PROPOSITION 5.2. *Let $(\Lambda, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then the following are equivalent: (i) Λ is regular local ring; (ii) $D_2(\kappa|\Lambda, \kappa) = 0$; (iii) $D_n(\kappa|\Lambda, -) = 0$ for $n \neq 1$.*

The following is (a special case of) Proposition 2 of chapitre XVII in André [**1**].

PROPOSITION 5.3. *Let $(\Lambda, \mathfrak{m}, \kappa)$ be a Noetherian local ring. If we have $\mathrm{Tor}^\Lambda_m(\kappa, \kappa) = 0$ for some even integer $m \in \mathbf{N}$ and the functor $D_2(\kappa|\Lambda, -)$ is left exact, then we have $D_2(\kappa|\Lambda, -) = 0$.*

The following proof of Serre's theorem is a modification of the proofs of Théorème 11 and Théorème 12 of chapitre XVII in André [**1**].

THEOREM 5.2 (Serre). *Let $(\Lambda, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then Λ is a regular local ring iff* $\mathrm{dh}(\Lambda) < \infty$*, and if either condition is satisfied, we have* $\mathrm{dh}(\Lambda) = \dim \Lambda$.

PROOF. Suppose that $\mathrm{dh}(\Lambda) < \infty$. Then there exists $n \in \mathbf{N}$ with $\mathrm{dp}_\Lambda(\kappa) \leqslant n$, hence by Lemma 5.2 we have $\mathrm{Tor}^\Lambda_m(-, \kappa) = 0$ for $m > n$. As κ is a field, the additive functor $D_2(\kappa|\Lambda, -)$ is exact (for every short exact sequence in $\kappa - \mathbf{Mod}$ is split exact), and by Proposition 5.3 we obtain $D_2(\kappa|\Lambda, \kappa) = 0$. Hence Λ is a regular local ring by Proposition 5.2.

For the remaining implication, let Λ be a regular local ring. As κ is a field, the κ-module $\mathfrak{m}/\mathfrak{m}^2$ is free; by Proposition 5.2 we may apply Theorem 5.1, which thus gives an isomorphism $\mathrm{Tor}^\Lambda_n(\kappa, \kappa) \cong \bigwedge^n_\kappa(\mathfrak{m}/\mathfrak{m}^2)$ for every $n \in \mathbf{N}$. Therefore we have

$$\mathrm{dp}_\Lambda(\kappa) = \sup\{n \in \mathbf{N} \mid \bigwedge\nolimits^n_\kappa(\mathfrak{m}/\mathfrak{m}^2) \neq 0\} = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2,$$

where the first equality is due to Lemma 5.3. By Lemma 5.2 we have $\mathrm{Tor}^{\Lambda}_{\dim_\kappa \mathfrak{m}/\mathfrak{m}^2+1}(-,\kappa) = 0$, and from Lemma 5.3 it follows that we have $\mathrm{dp}_\Lambda(M) \leqslant \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ for every finitely-generated Λ-module M. Hence $\mathrm{dh}(\Lambda) \leqslant \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim \Lambda$ by Lemma 5.1, and $\mathrm{dp}_\Lambda(\kappa) = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$ shows that we have equality. ∎

REMARK 5.1. In André [**1**] and Grothendieck [**6**], this theorem is attributed to both J.-P. Serre and D. Hilbert, perhaps because Hilbert's syzygy theorem can be deduced rather easily from this theorem. Namely, if k is a field, then as $k[x_1,\dots,x_n]$ is regular ring of Krull dimension n, Serre's theorem gives the second equality in

$$\begin{aligned}\mathrm{dh}(k[x_1,\dots,x_n]) &= \sup\{\mathrm{dh}(k[x_1,\dots,x_n]_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Spec} k[x_1,\dots,x_n]\}\\ &= \sup\{\dim k[x_1,\dots,x_n]_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Spec} k[x_1,\dots,x_n]\} = n,\end{aligned}$$

which is Hilbert's syzygy theorem.

A. **Appendix**

Here we gather some results that are often used or somewhat specific.

A.1. **Restriction and extension of scalars.** Let $\phi : A \to B$ be a homomorphism of rings. If N is a B-module, then ϕ induces a natural A-module structure on N, namely $(a,n) \mapsto \phi(a)n$ for $(a,n) \in A \times N$; this gives the restriction of scalars functor $\phi_* : B - \mathbf{Mod} \to A - \mathbf{Mod}$. (The notation $\phi_*(N)$ for the A-module obtained from N by restriction of scalars is used in the Bourbaki books; however, it is often convenient to denote $\phi_*(N)$ and N by the same symbol N, the A-module structure being understood.)

The ring B thus has a natural (A,B)-bimodule structure; if M is an A-module, then the A-module $M \otimes_A \phi_*(B)$ is naturally a B-module (such that $b'(m \otimes b) = m \otimes b'b$. This gives the extension of scalars functor $\phi^* : A - \mathbf{Mod} \to B - \mathbf{Mod}$. The tensor-hom adjunction for B

$$\mathrm{Hom}_B(M \otimes_A B, N) \to \mathrm{Hom}_A(M, \mathrm{Hom}_B(B,N))$$

induces an adjunction

$$\mathrm{Hom}_B(\phi^*(M), N) \to \mathrm{Hom}_A(M, \phi_*(N));$$

we thus have an adjoint pair (ϕ^*, ϕ_*).

If M is an A-module, then the extension $\bigwedge_B(M \otimes_A B) \to \bigwedge_A(M) \otimes_A B$ of the B-module map $M \otimes_A B \to \bigwedge_A(M) \otimes_A B$ (obtained by applying the extension of scalars functor to the canonical A-module map $M \to \bigwedge_A(M)$) is an isomorphism of graded B-algebras (see Bourbaki [**3**], chapitre 3, Proposition 8 of §7.5).

A.2. **Restriction of scalars and localization.** Let S be a multiplicative subset of A, and consider the canonical ring homomorphism $\phi : A \to S^{-1}A$. As we have another adjoint pair (S^{-1}, ϕ_*), we obtain a canonical functorial isomorphism $S^{-1} = \phi^* = - \otimes_A$

$S^{-1}A$. If $\eta : A \to B$ is a homomorphism of rings, denote by $\eta' : S^{-1}A \to \eta(S)^{-1}B$ the ring homomorphism induced by the composite $A \to B \to \eta(S)^{-1}B$. Then we have a functorial isomorphism $S^{-1}\eta_* \to \eta'_*\eta(S)^{-1}$ (whose components are given by $n/s \mapsto n/\eta(s)$) of functors $B-\mathbf{Mod} \to S^{-1}A-\mathbf{Mod}$.

A.3. **Finiteness conditions for algebras.** Let A be a k-algebra. Then A is said to be *of finite type* if for some $n \geqslant 0$ there is a surjective k-algebra homomorphism $k[x_1, \ldots, x_n] \to A$; A is *of finite presentation* if for some $n \geqslant 0$ there is a surjective k-algebra homomorphism $k[x_1, \ldots, x_n] \to A$ whose kernel is a finitely generated ideal; A is *of essentially finite type* if there is a finite-type k-algebra A' and a multiplicative subset S of A' such that $S^{-1}A'$ is isomorphic to A as a k-algebra. It is clear that if A is of finite type then it is also of essentially finite type; if k is a Noetherian ring, then A is of finite presentation iff it is of finite type, and if A is of essentially finite type then A is a Noetherian ring by Hilbert's basis theorem.

A.4. **The de Rham complex.** If M is a smooth manifold, then $\Omega^1(X) = \Gamma(T^*M)$ is the $C^\infty(M)$-module of smooth sections of the cotangent bundle (differential 1-forms). Put $\Omega^*(X) = \bigwedge_{C^\infty(M)} \Omega^1(X)$, the elements of $\Omega^k(X)$ are called differential k-forms (for $k > n$ we have $\Omega^k(X) = 0$). There is a unique homomorphism $d : \Omega^*(M) \to \Omega^*(M)$ of degree 1 such that $\Omega^*(M)$ becomes a DG-algebra over $\mathbf{R}$ and for $f \in \Omega^0(M) = C^\infty(M)$ df is the differential of f. Also d commutes with pullback, and the fact that it is a functorial morphism $\Omega^* \to \Omega^*[1]$ determines it up to a scalars (R. Palais). The cohomology of $\Omega^*(M)$ is the *de Rham cohomology* of M. (We have a contravariant functor Ω^* from the category of differentiable manifolds to Kom($\mathbf{R}-\mathbf{Mod}$).) Homotopy invariance: if f and g are smoothly homotopic, then f^* and g^* (pullback of forms) are chain homotopic. Thus the Poincaré lemma: if X is contractible, then its de Rham cohomology vanishes in degrees at least 1.

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