The Spectral Sequence of a Filtered Complex.

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In all that follows, cohomological indexing is used throughout. Our definitions and general considerations hold for objects in any abelian category $\mathcal{C}$; at least locally (by Freyd-Mitchell) and for the sake of concreteness the reader may think of $\mathcal{C}$ as being a category of modules over a ring.

Introduction

Consider a short exact sequence of complexes of objects of $\mathcal{C}$

$$0 \to K' \to K \to K'' \to 0.$$

The associated long exact sequence in cohomology

$$\cdots \to \mathrm{H}^n(K') \to \mathrm{H}^n(K) \to \mathrm{H}^n(K'') \to \mathrm{H}^{n+1}(K') \to \cdots$$

is the fundamental relationship between the cohomology objects $\mathrm{H}(K)$ and $\mathrm{H}(K')$, $\mathrm{H}(K'')$. It is natural to ask whether there is an analogue of the long exact sequence in cohomology associated to any descending chain of subcomplexes ('filtration')

$$F(K) = (F^p(K))_{p \in \mathbf{Z}}$$

of K. (The above short exact sequence is of course nothing but the special case where $F^p(K) = K$ for $p \leqslant 0$, $F^1(K) = K'$, $F^p(K) = 0$ for $p \geqslant 2$.) This analogue is the 'spectral sequence' of the filtered complex K. It essentially relates the cohomology $\mathrm{H}(K)$ of K to the cohomologies

$$\mathrm{H}(F^p(K)/F^{p+1}(K)) \quad (p \in \mathbf{Z})$$

by successive approximations.

1. Preliminaries on Filtrations.

1.1. **Filtered objects and their associated graded objects.** A *filtered object* of $\mathcal{C}$ is an object A with a *filtration* of A, i.e. a family $F(A) = (F^p(A))_{p \in \mathbf{Z}}$ of subobjects of A satisfying

$$\cdots \supset F^p(A) \supset F^{p+1}(A) \supset \cdots .$$

The *associated graded object* $\mathrm{Gr}(A) = (\mathrm{Gr}^p(A))_{p\in\mathbf{Z}}$ of A is defined by

$$\mathrm{Gr}^p(A) = F^p(A)/F^{p+1}(A).$$

By definition we have a family of short exact sequences

$$0 \to F^{p+1}(A) \to F^p(A) \to \mathrm{Gr}^p(A) \to 0 \quad (p \in \mathbf{Z}). \tag{1.1}$$

A morphism $f : A \to B$ of filtered objects is a morphism in C which preserves the filtration in the sense that $f(F^p(A)) \subset F^p(B)$ for all $p \in \mathbf{Z}$. We have an induced morphism of associated graded objects

$$\mathrm{Gr}(f) : \mathrm{Gr}(A) \to \mathrm{Gr}(B) \tag{1.2}$$

turning Gr into a functor.

1.2. **Euler characteristics and Serre subcategories.** Let Λ be an abelian group. A Λ-valued *Euler characteristic* χ on C associates to any object A of C an element $\chi(A) \in \Lambda$ such that the equation

$$\chi(A) = \chi(A') + \chi(A'')$$

holds for any exact sequence $0 \to A' \to A \to A'' \to 0$ in C.

Example 1.1. Let k be a field, and X a proper scheme over k. A $\mathbf{Z}$-valued Euler characteristic on the category $\mathbf{Coh}(X)$ of coherent $\mathcal{O}_X$-modules is given by

$$\chi = \sum_i (-1)^i \dim_k \mathrm{H}^i(X, -);$$

in particular, $\dim_k$ is a $\mathbf{Z}$-valued Euler characteristic on finite-dimensional k-vector spaces. If X is projective and $\mathcal{O}_X(1)$ a very ample sheaf on X, then the Hilbert polynomial (with respect to $\mathcal{O}_X(1)$) defines a $\mathbf{Q}[T]$-valued Euler characteristic on coherent sheaves.

By a *Serre subcategory* of C we mean a nonempty full subcategory S such that for any short exact sequence $0 \to A' \to A \to A'' \to 0$ of objects of C one has that A is in S if and only if both A' and A'' are in S. (This is equivalent to requiring that for any exact sequence $A' \to A \to A''$ with A' and A'' in S we also have A in S, which is how Serre defined it.)

Example 1.2. If $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module (this condition is satisfied if X is a locally Noetherian scheme or a complex manifold), then $\mathbf{Coh}(X)$ is a Serre subcategory of the category of $\mathcal{O}_X$-modules. On the other hand, the category of locally free $\mathcal{O}_X$-modules is usually not a Serre subcategory.

1.3. **Finite Filtrations.** A filtered object A is *finitely filtered* if $F^p(A) = A$ for p sufficiently small and $F^p(A) = 0$ for p sufficiently large. For the sake of future reference, let us note the following properties which follow immediately from the short exact sequences (1.1). (To prove (iv) one applies the snake lemma to the diagram obtained from (1.1), (1.2).)

LEMMA 1.1. *Let A and B be a finitely filtered object.*
(i) We have $\mathrm{Gr}^p(A) = 0$ *for all but finitely many $p \in \mathbf{Z}$, and an isomorphism*

$$A \simeq \bigoplus_{p \in \mathbf{Z}} \mathrm{Gr}^p(A)$$

if the exact sequences (1.1) split.
(ii) If χ is an Euler characteristic, then we have an equality

$$\chi(A) = \sum_{p \in \mathbf{Z}} \chi(\mathrm{Gr}^p(A)).$$

(iii) If S is a Serre subcategory and $\mathrm{Gr}(A)$ *consists of objects of S, then A is an object of S.*
(iv) If $f : A \to B$ is a morphism, then f is an isomorphism if $\mathrm{Gr}(f)$ *is one.*

2. The Spectral Sequence of a Filtered Complex.

2.1. **The Category of Spectral Sequences.** A *spectral sequence E* consists of
(i) a family $(E_r)_{r \geqslant 1}$ of bigraded objects $E_r = (E_r^{p,q})_{p,q \in \mathbf{Z}}$ of C, each of which carries a differential $d_r : E_r \to E_r$ of bidegree $(r, -r+1)$;
(ii) a family of isomorphisms $E_{r+1}^{p,q} \simeq \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d^{p-r,q+r-1})$;
(iii) a family $(E^n)_{n \in \mathbf{Z}}$ of finitely filtered objects with isomorphisms

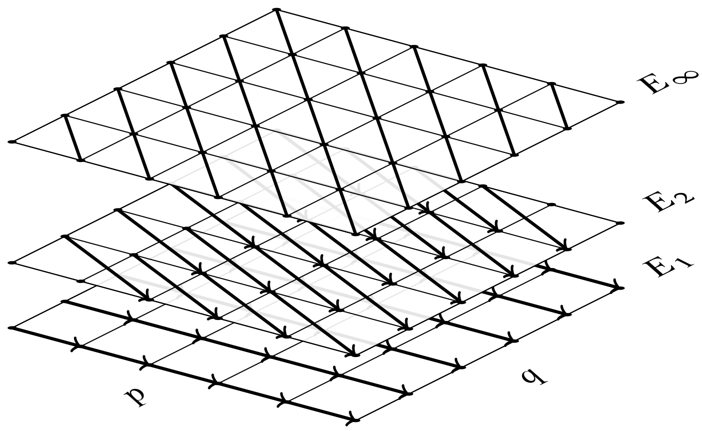
$$E_\infty^{p,q} \simeq \mathrm{Gr}^p(E^{p+q}),$$

where the bigraded object $E_\infty = (E_\infty^{p,q})_{(p,q) \in \mathbf{Z}^2}$ is defined as follows. We assume that for every pair $(p, q) \in \mathbf{Z}^2$ there exists some r such that

$$d_r^{p,q} = d_r^{p-r,q+r-1} = 0.$$

Then we have $E_r^{p,q} \simeq E_{r+1}^{p,q} \simeq \cdots$ by the isomorphisms (ii); this object is the one denoted by $E_\infty^{p,q}$ in (iii).

For a visual representation of a spectral sequence E, let us identify each E_r $(1 \leqslant r \leqslant \infty)$ with a planar lattice $\mathbf{Z}^2$ (depending on r), with $E_r^{p,q}$ corresponding to (p, q). We may then view (E_r) as an infinite stack of planes with the plane E_∞ being on top. The bigraded object E_∞ does not carry a differential, but it contains the graded objects $\mathrm{Gr}(E^n)$ on the diagonal lines $p + q = n$.



It is customary to denote the spectral sequence E by

$$E_1^{p,q} \Rightarrow E^{p+q} \quad \text{or} \quad E_2^{p,q} \Rightarrow E^{p+q},$$

and the spectral sequence E is said to *converge* to $(E^n)_{n\in\mathbf{Z}}$, and $(E^n)_{n\in\mathbf{Z}}$ (without the filtrations) is said to be the *limit term* (or *abutment*) of E. The bigraded object E_r is sometimes called the r-*th page* of E. The differentials of E are usually not specified explicitly as they may be very hard to determine; the isomorphisms (ii), (iii) are often regarded as identifications. In most applications one has knowledge of E_1 or E_2 and wants to obtain information about the limit term $(E^n)_{n\in\mathbf{Z}}$.

Remark 2.1. Our definition is the one used in Grothendieck's landmark paper [**6**]. The definition of spectral sequences in [**4**] is identical to our definition, apart from the finiteness condition on the filtration of E^n; they merely require the equalities $E^n = \bigcup_{p\in\mathbf{Z}} F^p(E^n)$ and $\bigcap_{p\in\mathbf{Z}} F^p(E^n) = 0$. However, if there is some r such that for any $n \in \mathbf{Z}$ the set

$$\{(p,q) \in \mathbf{Z}^2 \mid p+q = n,\ E_r^{p,q} \neq 0\} \tag{2.1}$$

is finite, then the filtration of each E^n is actually finite. (This condition is often satisfied in practice, for instance by 'first quadrant' spectral sequences.) For an even more general definition we refer to [**8**].

By a *morphism* of spectral sequences $f : E \to E'$ is meant a family $(f_r)_{r\geqslant 1}$ of morphisms $f_r : E_r \to E'_r$ of bigraded objects compatible with the differentials and the isomorphisms (ii), with a family $(f^n)_{n\in\mathbf{Z}}$ of morphisms $f^n : E^n \to E'^n$ of filtered objects compatible with the isomorphisms (iii). (The family $(f_r)_{r\geqslant 1}$ induces a morphism $f_\infty : E_\infty \to E_\infty$.)

For the sake of completeness we mention the following *comparison theorem*, which essentially follows from lemma 1.1 (iv).

THEOREM 2.1. *Let $f : E \to E'$ be a morphism of spectral sequences such that f_r is an isomorphism for some r. Then f_s is an isomorphism for all $r \leqslant s \leqslant \infty$, and the morphisms $f^n : E^n \to E'^n$ ($n \in \mathbf{Z}$) are isomorphisms as well.*

Of its many application we only mention that it is used in the proof of the GAGA theorem, more precisely the equivalence of categories $\mathbf{Coh}(X) \simeq \mathbf{Coh}(X^{an})$ (see [**9**], exposé 12, théorème 4.4), and the projective case of Grothendieck's theorem on algebraic de Rham cohomology ([**7**], theorem 1'), which also relies on GAGA.

2.2. **The Spectral Sequence of a Filtered Complex.** Let K be a filtered complex. The filtration $F(K)$ of K induces a filtration $F(K^n)$ on each K^n by $F^p(K^n) = F^p(K)^n$. By imposing a finiteness condition we obtain the spectral sequence mentioned in the introduction; its limit term consists of the cohomology objects $\mathrm{H}(K) = (\mathrm{H}^n(K))_{n\in\mathbf{Z}}$ of K endowed with the following filtration: we let $F^p\mathrm{H}^n(K)$ be the image of the canonical map $\mathrm{H}^n(F^p(K)) \to \mathrm{H}^n(K)$ induced by the inclusion $F^p(K) \to K$.

THEOREM 2.2. *Assume that each K^n is a finitely filtered object. There is a spectral sequence $E = E(K)$ functorial in K with*

$$E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p(K)) \Rightarrow \mathrm{H}^{p+q}(K). \tag{2.2}$$

The construction of (2.2) is somewhat laborious and may be found in [**4**]. In any case, the spectral sequence (2.2) satisfies the finiteness condition (2.1), and the construction shows that the first differential

$$d_1^{p,q} : \mathrm{H}^{p+q}(\mathrm{Gr}^p(K)) \to \mathrm{H}^{p+q+1}(\mathrm{Gr}^{p+1}(K))$$

may be identified with the boundary morphism of the cohomology long exact sequence of

$$0 \to \mathrm{Gr}^{p+1}(K) \to F^p(K)/F^{p+2}(K) \to \mathrm{Gr}^p(K) \to 0.$$

The following special case is worth pointing out. Consider a double complex $D = (D^{p,q})_{(p,q)\in\mathbf{Z}^2}$ such that for any n we have $D^{p,q} = 0$ for all but a finite number of (p,q) with $p+q = n$, and let K be the total complex of D, i.e.

$$K^n = \bigoplus_{p+q=n} D^{p,q}$$

and $d = d_I + d_{II}$, where d_I is the horizontal (bidegree $(1,0)$) differential of D and d_{II} is the vertical (bidegree $(0,1)$) one. Then K has two associated filtrations: the horizontal filtration $F_I(K)$ given by

$$F_I^p(K)^n = \bigoplus_{i+j=n, i\geqslant p} D^{i,j},$$

and the vertical filtration $F_{II}(K)$ given by

$$F_{II}^p(K)^n = \bigoplus_{i+j=n, j\geqslant p} D^{i,j}.$$

The finiteness condition on D guarantees that these filtrations satisfy the assumptions of theorem 2.2, and we obtain two spectral sequences

$$'E_1^{p,q} = \mathrm{H}_{II}^q(D^{p,\cdot}) \Rightarrow \mathrm{H}^{p+q}(K), \tag{2.3}$$

$$"E_1^{p,q} = \mathrm{H}_I^q(D^{\cdot,p}) \Rightarrow \mathrm{H}^{p+q}(K). \tag{2.4}$$

The first differential of $'E$ (resp. $"E$) is the map induced by d_I (resp. d_{II}). We may thus use the notation $'E_2^{pq} = \mathrm{H}_I^p(\mathrm{H}_{II}^q(D))$ (resp. $"E_2^{pq} = \mathrm{H}_{II}^p(\mathrm{H}_I^q(D)))$ for the second page.

2.3. **Degeneration.** The spectral sequence E *degenerates* at the r-th page if $d_s = 0$ for all $s \geqslant r$. This implies $E_r \simeq E_\infty$ and occurs often in applications. In fact one usually encounters an even stronger (for $r \geqslant 2$) form of degeneration: E *collapses* at the r-th page if there is p_0 (resp. q_0) such that $E_r^{p,q} = 0$ whenever $p \neq p_0$ (resp. $q \neq q_0$). In this case E^n we have isomorphisms $E^n \simeq E_r^{p_0, n-p_0}$ (resp. $E^n \simeq E_r^{n-q_0, q_0}$).

Example 2.1. Let K be a complex and K' a subcomplex of K. This defines a filtration $F(K)$ of K with $F^p(K) = K$ for $p \leqslant 0$, $F^1(K) = K'$ and $F^p(K) = 0$ for $p \geqslant 2$. Then $\mathrm{Gr}^0(K) = K/K'$, $\mathrm{Gr}^1(K) = K'$, and $\mathrm{Gr}^p(K) = 0$ for $p \neq 0, 1$. Hence $E_1^{0,q} = \mathrm{H}^q(K/K')$, $E_1^{1,q} = \mathrm{H}^{q+1}(K')$, $E_1^{p,q} = 0$ for $p \neq 0, 1$, and the spectral sequence (2.2) associated to this filtration degenerates at E_2. The first differential $d_1^{0,q} : E_1^{0,q} \to E_1^{1,q}$ has

$$\mathrm{Ker}(d_1^{0,q}) = E_2^{0,q} = E_\infty^{0,q} = \mathrm{Coker}(\mathrm{H}^q(K') \to \mathrm{H}^q(K))$$
$$\mathrm{Coker}(d_1^{0,q}) = E_2^{1,q} = E_\infty^{1,q} = \mathrm{Im}(\mathrm{H}^{q+1}(K') \to \mathrm{H}^{q+1}(K))$$

We thus obtain the long exact sequence in cohomology

$$\cdots \to \mathrm{H}^q(K') \to \mathrm{H}^q(K) \to \mathrm{H}^q(K/K') \to \mathrm{H}^{q+1}(K') \to \cdots,$$

which we have shown to be nothing but a a degenerate case of the spectral sequence (2.2).

3. **Examples and Applications of Spectral Sequences.**

3.1. **The Atiyah-Hirzebruch Spectral Sequence.** Let X be a finite CW complex, and h an extraordinary cohomology theory. The Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X; h^q(*)) \Rightarrow h^{p+q}(X) \tag{3.1}$$

relates the singular cohomology of X with coefficients in $h(*)$ to $h(X)$. As X is a finite CW complex, there is some n such that $\mathrm{H}^p(X; A) = 0$ for all $p > n$ and abelian groups A; this shows that this spectral sequence satisfies (2.1). Notice that if h is an ordinary cohomology theory (i.e. $h^q(*) = 0$ for $q \neq 0$), then the spectral sequence collapses to give isomorphisms $\mathrm{H}^n(X; h^0(*)) \simeq h^n(X)$.

For a concrete example take h to be complex K-theory K. The complex K-theory of a point is $\mathbf{Z}[q, q^{-1}]$, where q has degree 2. If $X = \mathbf{P}^n(\mathbf{C})$ then we have $E_2^{p,q} = 0$ if p or q is odd. Thus the Atiyah-Hirzebruch spectral sequence (3.1) of $\mathbf{P}^n(\mathbf{C})$ degenerates at the E_2-page, and we obtain an isomorphism $K^0(\mathbf{P}^n(\mathbf{C})) \simeq \mathbf{Z}^{n+1}$ by using part (i) of lemma 1.1 (the integral cohomology groups of $\mathbf{P}^n(\mathbf{C})$ are free $\mathbf{Z}$-modules).

In fact, for complex K-theory it is not uncommon that the spectral sequence (3.1) of X degenerates at the E_2-page; it occurs whenever the integral cohomology of X is concentrated in even degrees of if it is torsion-free. This was used by Adams and Atiyah [**2**] to give a very elegant K-theoretic proof of the following *Hopf invariant one theorem*.

THEOREM 3.1 (Adams). *Let $X = X_f$ be the finite CW complex obtained by attaching a $2n$-ball to a sphere S^n by a continuous map $f : S^{2n+1} \to S^n$. The $\cup$-square*

$$\mathrm{H}^n(X; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^{2n}(X; \mathbf{Z}/2\mathbf{Z})$$

is zero for $n \neq 1, 2, 4, 8$.

Remark 3.1. If S^{n-1} is an H-space, then the 'Hopf construction' yields a continuous map $f : S^{2n+1} \to S^n$ such that the $\cup$-square of X_f is nonzero.

In essence, one assumes $n > 1$ and that the $\cup$-square is nonzero. This is clearly impossible if n is odd, and so one may write $n = 2m$. The spectral sequence (3.1) for X degenerates at the E_2-page, which allows one to identify the integral cohomology ring $\mathrm{H}^*(X; \mathbf{Z})$ with $\mathrm{Gr}(K^0(X))$. Now $K^0(X)$ is a filtered *ring* and this identification respects the ring structures. On this ring there are certain functorial endomorphisms ψ^k ('Adams operations') which for $X = X_f$ satisfy relations that are strong enough to yield the divisibility condition $2^m | 3^m - 1$. (This implies $m = 0, 1, 2, 4$.) We refer to [**2**] for details.

3.2. **The Frölicher Spectral Sequence.** Let M be a complex manifold. The de Rham complex $A(M) = \Omega(M) \otimes \mathbf{C}$ of M may be regarded as the total complex of the Dolbeault complex $(A^{p,q}(M))$ of M, since we have decompositions $A^n(M) = \bigoplus_{p+q=n} A^{p,q}(M)$ and $d = \partial + \bar{\partial}$. The first spectral sequence (2.3) associated to this double complex

$$E_1^{p,q} = \mathrm{H}^{p,q}_{\bar{\partial}}(M) \Rightarrow \mathrm{H}^{p+q}_{dR}(M) \tag{3.2}$$

is called the *Frölicher spectral sequence* (also known as the *Hodge-to-de Rham spectral sequence*). By a theorem of Dolbeault $\mathrm{H}^{p,q}_{\bar\partial}(M)$ can be identified with the sheaf cohomology group $\mathrm{H}^p(M, \Omega^q)$. Note that $\dim_{\mathbf{C}} E^{p,q}_{r+1} \leqslant \dim_{\mathbf{C}} E^{p,q}_r$ with equality if and only if $d^{p,q}_r = d^{p-r,q+r-1}_r = 0$. Combining this with part (ii) of lemma 1.1 yields the *Frölicher relations*

$$\sum_{p+q=n} h^{p,q}(M) \geqslant b_n(M), \quad \sum_{p,q}(-1)^{p+q} h^{p,q}(M) = \chi(M).$$

The first inequality is an equality if and only if (3.2) degenerates at the E_1-page.

THEOREM 3.2. *Let M be compact surface or a compact Kähler manifold. Then the Frölicher spectral sequence of M degenerates at E_1.*

For compact Kähler manifolds this follows by Hodge theory; for compact surfaces it is immediate from the fact (due to Kodaira) that holomorphic 1-forms on surfaces are closed. Thus the higher differentials of (3.2) may be regarded as obstructions to M being a manifold of Kähler type. Let us give an example (taken from [**5**], chapter 3) of a complex 3-fold M whose Hodge-to-de Rham spectral sequence does not degenerate at E_1.

Example 3.1 (Iwasawa). Let G be the complex Lie group of upper-triangular complex 3×3-matrices with ones on the diagonal, and let $a, b, c : G \to \mathbf{C}$ be the projections. The subgroup Γ of elements of G with entries in $\mathbf{Z}[i]$ acts properly discontinuously on G, and so $M = G/\Gamma$ is a complex 3-fold. The holomorphic 1-forms $\omega_1 = da$, $\omega_2 = dc$, $\omega_3 = -c\,da + db$ are right-invariant 1-forms on G and hence induce 1-forms on M. We have $d\omega_3 = \omega_1 \wedge \omega_2$, which implies that the differential $d = \partial : E^{1,0}_1 \to E^{2,0}_1$ is nonzero.

It used to be a folklore conjecture that the spectral sequence (3.2) degenerates at E_2 for any compact complex manifold M; this turned out to be false.

THEOREM 3.3 ([**3**]). *For every n there is a compact complex manifold whose Hodge-to de Rham spectral sequence has $d_n \neq 0$.*

3.3. **The Grothendieck Spectral Sequence.** The right derived functors $(R^n F)_{n \geqslant 0}$ of a left exact functor F may be viewed as a 'universal' cohomology theory associated to $F = R^0 F$. (More precisely, they form a universal ∂-functor in the sense of [**6**].) If G is another left exact functor, then $G \circ F$ of is also left exact, and we may wonder how $(R^n G)$ and $(R^n F)$ are related to $(R^n(G \circ F))$.

THEOREM 3.4 (Grothendieck [**6**]). *Let C, C', C'' be abelian categories, and assume that C and C' have enough injectives. Let $F : C \to C'$ and $G : C' \to C''$ be left exact functors such that F maps injective objects to G-acylic objects. Then* *for every object A of C there is a first-quadrant spectral sequence*

$$E_2^{p,q} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G\circ F)(A)$$

which is functorial in A.

The proof consists in taking an injective resolution J of A and comparing the two spectral sequences (2.3) and (2.4) associated to the double complex $G(I)$, where I is a 'Cartan-Eilenberg resolution' of $F(J)$. The first spectral sequence has ${}'E_2^{p,q} = R^pG(R^qF(A))$, while the second one collapses to give isomorphisms ${}''E^n \simeq {}''E_2^{n,0} = R^n(G\circ F)(A)$. But the limit terms of (2.3) and (2.4) are the same.

Remark 3.2. If one defines the right derived functors at the level of derived categories, then this spectral sequence may be replaced by the 'chain rule' $R(G\circ F) \simeq RG\circ RF$ (see [**4**] for a technical treatment or [**10**] for a more informal exposition).

Example 3.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of sheaves of $\mathcal{O}_X$-modules has enough injectives, and we obtain the following special cases (see [**1**]).

(i) The Grothendieck spectral sequence of $\Gamma\circ\mathcal{H}om_X(\mathcal{F},-) = \mathrm{Hom}_X(\mathcal{F},-)$ is the *local-to-global spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{E}xt_X^q(\mathcal{F},\mathcal{G})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{F},\mathcal{G}).$$

(ii) The Grothendieck spectral sequence of $g_*\circ f_* = (g\circ f)_*$ (where f and g are morphisms of ringed spaces) is the *Leray spectral sequence*

$$E_2^{p,q} = R^pg_*(R^qf_*(\mathcal{F})) \Rightarrow R^{p+q}(g\circ f)_*(\mathcal{F}). \tag{3.3}$$

(iii) The Grothendieck spectral sequence of $\check{\mathrm{H}}^0(\mathcal{U},-)\circ j = \Gamma$ (where j is the inclusion of presheaves of $\mathcal{O}_X$-modules into sheaves of $\mathcal{O}_X$-modules) is the *Čech-to-derived functor spectral sequence*

$$E_2^{p,q} = \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{H}^q(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X,\mathcal{F}).$$

The presheaf $\mathcal{H}^q(\mathcal{F}) = R^qj(\mathcal{F})$ has $\mathrm{H}^0(U,\mathcal{H}^q(\mathcal{F})) = \mathrm{H}^q(U,\mathcal{F})$. If X is a seperated scheme with affine open cover $\mathcal{U}$, and $\mathcal{F}$ is quasi-coherent, then $E_2^{p,q} = 0$ for $q > 0$ (by a theorem of Serre affine schemes have no higher cohomology), hence the spectral sequence collapses to yield functorial isomorphisms $\check{\mathrm{H}}^n(\mathcal{U},\mathcal{F}) \simeq \mathrm{H}^n(X,\mathcal{F})$.

(iv) Let A be a ring, and B an A-algebra. If M is a B-module, then $\mathrm{Hom}_B(M,-)\circ \mathrm{Hom}_A(B,-) \simeq \mathrm{Hom}_A(M,-)$, and we get a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_B^p(M, \mathrm{Ext}_A^q(B,N)) \Rightarrow \mathrm{Ext}_A^{p+q}(M,N).$$

This immediately implies the inequality $\operatorname{dimproj}_A(M) \leqslant \operatorname{dimproj}_B(M)+\operatorname{dimproj}_A(B)$ of projective dimensions. (If r is the right-hand side, then $E_2^{p,q} = 0$ for all p, q with $p + q > r$, and hence $\operatorname{Ext}^n_A(M, N) = 0$ for $n > r$.)

We mention somewhat more involved arguments in which the Grothendieck spectral sequence facilitates reductions.

Coherence of Higher Direct Images. The importance of the following result cannot be overestimated; it first appeared in [**8**].

THEOREM 3.5 (Grothendieck). *Let $f : X \to Y$ be a proper morphism of schemes, and assume Y to be locally Noetherian. If $\mathscr{F}$ is a coherent sheaf on X, then $R^p f_*(\mathscr{F})$ is a coherent sheaf on Y for all $p \geqslant 0$.*

We assume the projective case, which is due to Serre[1], and show that the category $\mathscr{K}$ of coherent sheaves $\mathscr{F}$ on X satisfying the conclusion of the theorem fulfills the assumptions of the following *dévissage lemma* (see [**8**] for a proof).

LEMMA 3.1. *Let X be a Noetherian scheme, and $\mathscr{K}$ a full subcategory of the category* $\operatorname{Coh}(X)$ *of coherent sheaves on X. Assume that $\mathscr{K}$ contains* 0*, is closed under coherent direct factors, and that for any short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ in* $\operatorname{Coh}(X)$ *we have that if any two of $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ are in $\mathscr{K}$, then the third is also in $\mathscr{K}$. If for any closed irreducible subset $Y = \overline{\{y\}}$ of X there is some $\mathscr{F}$ in $\mathscr{K}$ with $\mathscr{F}_y \neq 0$, then $\mathscr{K} = \operatorname{Coh}(X)$.*

To prove Grothendieck's theorem, one can of course assume that Y is Noetherian, which implies that X is also Noetherian. We only check the last assumption of the dévissage lemma, as the others follow easily from the long exact sequence of higher direct images. We may assume that X is irreducible with generic point x. By Chow's lemma there is an irreducible scheme X' and a surjective projective morphism $g : X' \to X$ such that $f \circ g$ is projective. By applying the projective case to g we obtain an integer n such that $\mathscr{F} = g_*(\mathscr{O}_{X'}(n))$ is coherent, $R^q g_*(\mathscr{O}_{X'}(n)) = 0$ for $q > 0$, and $g^*(\mathscr{F}) \to \mathscr{O}_{X'}(n)$ is surjective. The latter implies $\mathscr{F}_x \neq 0$, and the isomorphisms $R^p f_*(\mathscr{F}) \simeq R^p(f \circ g)_*(\mathscr{O}_{X'}(n))$ coming from the collapse of the Leray spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathscr{O}_{X'}(n)) \Rightarrow R^{p+q}(f \circ g)_*(\mathscr{O}_{X'}(n))$$

show that $\mathscr{F}$ indeed lies in $\mathscr{K}$.

Remark 3.3. (i) The above theorem in particular shows that if X is a proper scheme over a Noetherian ring A and $\mathscr{G}$ a coherent sheaf on X, then the A-modules

[1]It can easily be reduced to Serre's computation of the cohomology of line bundles on projective space

$\mathrm{H}^n(X, \mathscr{G}) = \mathrm{Ext}^n_X(\mathscr{O}_X, \mathscr{G})$ are of finite type. Using lemma 1.1 (iii), example 1.2, and the local-to-global spectral sequence we see that $\mathrm{Ext}^n_X(\mathscr{F}, \mathscr{G})$ is also an A-module of finite type.

(ii) The main part of the GAGA theorem is the existence of a canonical isomorphism $(R^p f_* \mathscr{F})^{an} \to R^p f_*^{an}(\mathscr{F})$ for a proper morphism f; the reduction to the projective case is essentially identical to the above argument (see [**9**], exposé 12, théorème 4.2).

Serre Duality. This cohomological duality theorem was proven by Serre in the analytic case; for more details and a complete proof we refer to [**1**].

THEOREM 3.6 (Serre). *Let X be an equidimensional smooth projective scheme over a field k, and $d = \dim X$. For every p we have a canonical isomorphism*

$$\mathrm{Ext}^p_X(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Ext}^{d-p}_X(\mathscr{G}, \mathscr{F} \otimes \omega_X)^\vee$$

functorial in the coherent sheaves $\mathscr{F}$ and $\mathscr{G}$.

One can reduce to the case where $\mathscr{F} = \mathscr{O}_X$ (essentially because one can $\mathscr{F}$ by locally free sheaves of finite rank). In this case the Yoneda pairing

$$\mathrm{Ext}^p_X(\mathscr{O}_X, \mathscr{G}) \times \mathrm{Ext}^{d-p}_X(\mathscr{G}, \omega_X) \to \mathrm{Ext}^d_X(\mathscr{O}_X, \omega_X) \xrightarrow{\sim} k$$

yields a morphism of ∂-functors $\mathrm{Ext}^{d-p}_X(\mathscr{G}, \omega_X) \to \mathrm{H}^p(X, \mathscr{G})^\vee$. For $X = \mathbf{P}^n$ one shows 'explictly' that this is an isomorphism for line bundles and deduces the general case by using the fact that any $\mathscr{G}$ is a quotient of a direct sum of line bundles.

For general X one can reduce to the case $X = \mathbf{P}^n$ as follows. Pick a closed immersion of X into some projective space $\mathbf{P}^n$, and identify coherent sheaves on X with coherent sheaves on $\mathbf{P}^n$ supported on X. By taking stalks and using local algebra one sees that $\mathscr{E}xt^q_{\mathbf{P}^n}(\mathscr{O}_X, \omega_{\mathbf{P}^n})$ vanishes for $q \neq n-d$; on the other hand, $\mathscr{E}xt^{n-d}_{\mathbf{P}^n}(\mathscr{O}_X, \omega_{\mathbf{P}^n})$ can be identified with ω_X (use a Koszul resolution of $\mathscr{O}_X$ and the adjunction formula). The Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathscr{G}, \mathscr{E}xt^q_{\mathbf{P}^n}(\mathscr{O}_X, \omega_{\mathbf{P}^n})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{P}^n}(\mathscr{G}, \omega_{\mathbf{P}^n})$$

thus collapses to give isomorphisms $\mathrm{Ext}^{d-p}_X(\mathscr{G}, \omega_X) \xrightarrow{\sim} \mathrm{Ext}^{n-p}_{\mathbf{P}^n}(\mathscr{G}, \omega_{\mathbf{P}^n})$, which are compatible with the Serre duality morphism in the sense that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^{d-p}_X(\mathscr{G}, \omega_X) \arrow[r] \arrow[d, "\wr"'] & \mathrm{H}^p(X, \mathscr{G})^\vee \arrow[d, "\wr"] \\
\mathrm{Ext}^{n-p}_{\mathbf{P}^n}(\mathscr{G}, \omega_{\mathbf{P}^n}) \arrow[r] & \mathrm{H}^p(\mathbf{P}^n, \mathscr{G})^\vee
\end{tikzcd}$$

is commutative. Thus Serre duality for $\mathbf{P}^n$ implies Serre duality for X.

Cohomological Dimension. The *cohomological dimension* of an abelian category C is the supremum $\sup\{n \mid \mathrm{Ext}^n_C(-,-) \neq 0\}$. The cohomological dimension of a ring A is the cohomological dimension of the category of A-modules.

Remark 3.4. The cohomological dimension of a Noetherian ring A coincides with the cohomological dimension of the category of A-modules of finite type. If A is in addition local, then the cohomological dimension of A is given by the supremum $\sup\{n \mid \mathrm{Ext}^n_A(\kappa,\kappa) \neq 0\}$.

Recall that a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue field κ satisfies $\dim(A) \leqslant \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$, and that A is *regular* if equality holds. A stunning theorem of Serre shows that regularity is actually a finiteness condition for cohomological dimension.

THEOREM 3.7 (Serre). *Let A be a Noetherian local ring. Then A is regular if and only if its cohomological dimension is finite, in which case it coincides with the Krull dimension.*

Standard localisation arguments show that Serre's theorem holds more generally for any Noetherian ring of finite Krull dimension. It is perhaps instructive to formulate an analogue of Serre's theorem for projective schemes.

THEOREM 3.8. *Let k be a field, and X be an equidimensional projective scheme over k of Krull dimension d. Then X is smooth if and only if the category of coherent sheaves on X has finite cohomological dimension, in which case it equals d.*

If X is smooth, Serre duality implies the vanishing $\mathrm{Ext}^p_X(\mathscr{F},\mathscr{G}) = 0$ for $p > d$ and $\mathrm{Ext}^d_X(\mathcal{O}_X,\omega_X) \neq 0$. On the other hand, if X is not smooth, let $x \in X$ be a closed singular point with structure sheaf $\mathcal{O}_x$. The local-to-global spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{E}xt^q(\mathcal{O}_x,\mathcal{O}_x)) \Rightarrow \mathrm{Ext}^n_X(\mathcal{O}_x,\mathcal{O}_x)$$

collapses to give isomorphisms $\mathrm{Ext}^n_{\mathcal{O}_{X,x}}(\kappa(x),\kappa(x)) \simeq \mathrm{Ext}^n_X(\mathcal{O}_x,\mathcal{O}_x)$. Serre's theorem and remark 3.4 imply that X must be of infinite cohomological dimension.

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'Mais je suis débarrassé de mon horreur de la suite spectrale.' — A. G., letter to J-P. S. (18. 2. 1955).