The Malgrange-Ehrenpreis Theorem.

Let $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$. A polynomial $P = \sum_{|\alpha| \leqslant N} a_\alpha X^\alpha \in \mathbf{C}[X_1, \ldots, X_n]$ induces a constant coefficient partial differential operator $P(\partial) = \sum_{|\alpha| \leqslant N} a_\alpha \partial^\alpha$. It extends to $\mathcal{D}'$ by $\langle P(\partial)T, \phi \rangle = \langle T, P(-\partial)\phi \rangle$ for $T \in \mathcal{D}'$, $\phi \in \mathcal{D}$. A *fundamental solution* for the partial differential operator $P(\partial)$ is a distribution $E \in \mathcal{D}'$ with $P(\partial)E = \delta_0$. Notice that such an E gives an *explicit* smooth solution of the partial differential equation $P(\partial)u = f$ $(f \in \mathcal{D})$, namely $u = E * f \in C^\infty$. (We have $P(\partial)u = P(\partial)E * f = \delta_0 * f = f$.)

Example. The Laplacian $-\Delta$ on $\mathbf{R}^3$ has a fundamental solution E given by $E(x) = 1/4\pi \,\|x\| \in L^1_{\mathrm{loc}}(\mathbf{R}^3)$. For if $B_\varepsilon = B_\varepsilon(0)$, then we have for all $\phi \in \mathcal{D}(\mathbf{R}^3)$

$$\begin{aligned}
\langle -\Delta E, \phi \rangle &= \lim_{\varepsilon \downarrow 0} \int_{\mathbf{R}^3 - B_\varepsilon} E(-\Delta\phi)dx \\
&= \lim_{\varepsilon \downarrow 0} \left\{ \int_{\mathbf{R}^3 - B_\varepsilon} \phi(-\Delta E)dx + \int_{\partial B_\varepsilon} \left(-E\frac{\partial\phi}{\partial n} + \phi\frac{\partial E}{\partial n} \right) d\sigma \right\} \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{4\pi\varepsilon^2} \int_{\partial B_\varepsilon} \phi d\sigma \\
&= \phi(0),
\end{aligned}$$

by Green's identity. Here we have used that $-\Delta E = 0$ on $\mathbf{R}^3 - \{0\}$, as well as

$$\left| \int_{\partial B_\varepsilon} E \frac{\partial\varphi}{\partial n} d\sigma \right| \leqslant \frac{C}{4\pi\varepsilon} \int_{\partial B_\varepsilon} d\sigma = C\varepsilon \to 0 \quad (\varepsilon \downarrow 0).$$

Thus we obtain a smooth solution of the Poisson equation $-\Delta u = f$, which is the well-known

$$u(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{f(y)}{|x - y|} dy.$$

(This happens to be the only solution of the Poisson equation with $u(x) \to 0$ for $|x| \to \infty$, as the difference of any two such solutions would be a bounded harmonic function on $\mathbf{R}^3$.)

For differential operators with variable coefficients a fundamental solution need not exist.

Example. For $f \in \mathcal{D}(\mathbf{R})$ the operator $f d/dx$ on $\mathbf{R}$ need not have a fundamental solution. (By definition we must have

$$\left\langle E, -\frac{d}{dx}(f\phi) \right\rangle = \left\langle f\frac{d}{dx}E, \phi \right\rangle = \phi(0).$$

Now pick f and ϕ with disjoint supports, as well as $\phi(0) \neq 0$.)

On the other hand, there is the following fundamental existence theorem for constant coefficient operators.

Theorem (Malgrange-Ehrenpreis). *If $P \neq 0$, then $P(\partial)$ has a fundamental solution E.*

The proof of this result relies on two lemmas.

Lemma 1. *If $P : \mathbf{C}^n \to \mathbf{C}$ is a polynomial function, then there is a constant C such that for all entire functions $f : \mathbf{C}^n \to \mathbf{C}$ and $z \in \mathbf{C}^n$ we have*

$$|f(z)| \leqslant C \int_{T^n} |(fP)(z+w)| dm(w),$$

where m is the Haar measure on $T^n \subset \mathbf{C}^n$.

We omit the proof of this lemma, as it properly belongs to complex analysis.

Lemma 2. *Define a seminorm $\|\cdot\|$ on $\mathcal{D}$ by*

$$\|\psi\| = \int_{T^n} \int_{\mathbf{R}^n} |\hat{\psi}(t+w)| dt\, dm(w).$$

Then for any sequence $(\psi_k)_{k=1}^{\infty}$ in $\mathcal{D}$ we have that $\psi_k \to 0$ implies $\|\psi_k\| \to 0$.

Proof. We have $\hat{\psi}(t+w) = (\chi_{-w}\psi)\hat{}\,(t)$, where $\chi_w(x) = \exp(i\langle x, w\rangle)$. If $\psi_k \to 0$ in $\mathcal{D}$, then there is a compact set $K \subset \mathbf{R}^n$ with $\operatorname{supp} \psi_k \subset K$. As the χ_w $(w \in T^n)$ are uniformly bounded on K, we obtain for every α

$$\|\partial^\alpha(\chi_{-w}\psi_k)\|_\infty = \|\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\chi_{-w}\partial^\beta \psi_k\|_\infty \leqslant C(K,\alpha) \max_{\beta \leqslant \alpha} \|\partial^\beta \psi_k\|_\infty \to 0 \quad (k \to \infty)$$

independently of $w \in T^n$. Thus for every $\varepsilon > 0$ there is a k_0 with $\|(1-\Delta)^n(\chi_{-w}\psi_k)\|_2 < \varepsilon$ for all $k > k_0$, $w \in T^n$. By the Plancherel theorem this is equivalent to

$$\sqrt{\int_{\mathbf{R}^n} |(1+|t|^2)^n \hat{\psi}_k(t+w)|^2 dt} < \varepsilon.$$

(We see by induction on n that $(1+|t|^2)^n\hat{\psi}_k(t+w)$ is the Fourier transform of $(1-\Delta)^n(\chi_{-w}\psi_k)$.) But by the Cauchy-Schwartz inequality we have

$$\begin{aligned} \|\psi_k\| &= \int_{T^n}\int_{\mathbf{R}^n} |(\chi_{-w}\psi)\hat{}\,(t)| dt\, dm(w) \\ &\leqslant \int_{T^n} \sqrt{\int_{\mathbf{R}^n} \frac{dt}{(1+|t|^2)^{2n}}} \sqrt{\int_{\mathbf{R}^n} |(1+|t|^2)^n \hat{\psi}_k(t+w)|^2 dt\, dm(w)} \\ &< C\varepsilon \end{aligned}$$

whenever $k > k_0$. ∎

Proof of the Malgrange-Ehrenpreis theorem. For $\phi \in \mathcal{D}$ we have $(P(-\partial)\phi)\hat{}\,(\xi) = P(-i\xi)\hat{\phi}(\xi)$, and $\hat{\phi}$ is entire. If $P(-\partial)\phi = P(-\partial)\psi$ with $\psi \in \mathcal{D}$, then $\hat{\phi} = \hat{\psi}$ by the identity theorem (using that $P \neq 0$). By Fourier inversion we have $\phi = \psi$, and therefore we have a well-defined linear functional $l : P(-\partial)\mathcal{D} \to \mathbf{C}$ by $\langle l, P(-\partial)\phi\rangle = \phi(0)$. Applying Lemma 1 gives

$$|\hat{\phi}(t)| \leqslant C \int_{T^n} |(P(-\partial)\phi)\hat{}\,(t+w)| dm(w)$$

for all $t \in \mathbf{R}^n$. Now

$$|\phi(0)| \leqslant \int_{\mathbf{R}^n} |\hat{\phi}(t)| dt \leqslant C \| P(-\partial)\phi \|$$

by the Fourier inversion theorem and the above estimate. By the Hahn-Banach theorem l extends to $E : \mathcal{D} \to \mathbf{C}$ with $|E(\phi)| \leqslant C\|P(-\partial)\phi\|$. Now E is sequentially continuous by Lemma 2 and the continuity of $P(-\partial) : \mathcal{D} \to \mathcal{D}$, and $\langle P(\partial)E, \phi\rangle = \langle E, P(-\partial)\phi\rangle = \phi(0)$ for all $\phi \in \mathcal{D}$. The proof is complete. ∎

Since the existence of the convolution $E * f$ requires some decay of E or f (it is convenient to impose more decay on E rather than on f), it is natural to ask whether it is possible to find a *tempered* fundamental solution E. This is indeed the case; we merely indicate how one can deduce this result from the following result, the proof of which goes way beyond the scope of these notes. (It is impossible, however, to have a *compactly supported* fundamental solution, as in that case $\hat{E}$ would be an entire function.)

Theorem (Hörmander). *If $P \neq 0$ is a polynomial, then the map $\mathcal{S} \to \mathcal{S}$ by $f \mapsto Pf$ is a homeomorphism onto its image.*

This theorem resolves the following division problem, which then implies (by taking fourier transforms) the existence of tempered fundamental solutions.

Corollary. *For every $T \in \mathcal{S}'$ there is $S \in \mathcal{S}'$ with $PS = T$.*

PROOF. The functional $l : P\mathcal{S} \to \mathbf{C}$ by $\langle l, P\phi\rangle = \langle T, \phi\rangle$ is continuous, for $P\phi_k \to 0$ implies $\phi_k \to 0$ by Hörmander's theorem, and thus $\langle l, \phi\rangle \to 0$. By the Hahn-Banach theorem there is $S \in \mathcal{S}'$ with $S|P\mathcal{S} = l$. Then $\langle PS, \phi\rangle = \langle S, P\phi\rangle = \langle T, \phi\rangle$ for every $\phi \in \mathcal{S}$. ∎

Example. If $f(x) = 1/P(ix) \in L^1_{\text{loc}}(\mathbf{R}^n)$, then we get a tempered fundamental solution by $E = \check{f} \in \mathcal{S}'(\mathbf{R}^n)$ (fourier transform in the sense of distributions). This holds for instance if $x \mapsto P(ix)$ has no zeros on $\mathbf{R}^n$; the size of the zero set of this polynomial is an obstruction to $f \in L^1_{\text{loc}}(\mathbf{R}^n)$.