ETH zürich

Quot schemes of surfaces

(University of Cambridge, Algebraic geometry seminar, 9/3/2022)

S projective scheme, $\mathcal{E}$ coherent sheaf on S

Quot scheme $\mathrm{Quot}_S(\mathcal{E})$ is the moduli space of quotients of $\mathcal{E}$:

$q \in \mathrm{Quot}_S(\mathcal{E}) \leftrightarrow \mathcal{E} \twoheadrightarrow Q_q$ on S

$\mathrm{Quot}_S^\ell(\mathcal{E}) \subset \mathrm{Quot}_S(\mathcal{E})$ given by $\dim Q_q = 0$, $h^0(Q_q) = \ell$

Notation: $S_q = \mathrm{Ker}(\mathcal{E} \twoheadrightarrow Q_q)$

§1. General properties (Grothendieck)

(I) $\mathcal{E} \twoheadrightarrow \mathcal{E}''$ induces $\mathrm{Quot}_S^\ell(\mathcal{E}'') \hookrightarrow \mathrm{Quot}_S^\ell(\mathcal{E})$

(II) $\mathcal{L} \in \mathrm{Pic}(S)$ —— $\mathrm{Quot}_S^\ell(\mathcal{E}) \xrightarrow{\sim} \mathrm{Quot}_S^\ell(\mathcal{E} \otimes \mathcal{L})$

(III) $\mathrm{Aut}(\mathcal{E})$ acts on $\mathrm{Quot}_S^\ell(\mathcal{E})$

Deformation-obstruction theory: $T_q^i = \mathrm{Ext}^{i-1}(S_q, Q_q)$

Tautological sheaves: locally free $\mathcal{F}$ on S induces

$\mathcal{F}^{[\ell]}$ on $\mathrm{Quot}_S^\ell(\mathcal{E})$ with $\mathcal{F}^{[\ell]}(q) = H^0(\mathcal{F} \otimes Q_q)$, compatible with (I)–(III)

§2. Surface case

S smooth surface, $\mathcal{E}$ locally free of rank r

S irreducible $\Rightarrow$ $\mathrm{Quot}_S^\ell(\mathcal{E})$ irreducible (Rego, Gieseker-Li)

Dimension: $\ell(r+1)$, in fact $\mathrm{Quot}_S^\ell(\mathcal{E}) \dashleftarrow\dashrightarrow \mathbb{P}(\mathcal{E})^{(\ell)}$

Tangent dim: $\dim T_q \mathrm{Quot}_S^\ell(\mathcal{E}) = \ell r + h^0(\mathrm{End}\, Q_q)$

Virtual dim: $\dim T_q^1 - \dim T_q^2 = \ell r$, since $\mathrm{Ext}^2(S_q, Q_q) = 0$.

Get virtual class $[\mathrm{Quot}_S^\ell(\mathcal{E})]^{vir} \in A_{\ell r}(\mathrm{Quot}_S^\ell(\mathcal{E}))$ (obstruction theory 2-term)

<u>Examples</u>. (i) $S^{[\ell]} = \mathrm{Quot}_S^\ell(\mathcal{O})$ smooth, $S^{[\ell]} \to S^{(\ell)}$ resolution of singularities, well-studied (Fogarty, Beauville, Ellingsrud-Strømme, Göttsche, Nakajima, Grojnowski, Lehn, ...)

$\det(\mathcal{O}^{[\ell]})^{\otimes 2} = \mathcal{O}(-\partial S^{[\ell]})$, $[S^{[\ell]}]^{vir} = (-1)^\ell e(\omega_S^{[\ell]}) \cap [S^{[\ell]}]$

(ii) $\mathbb{P}(\mathcal{E}) = \mathrm{Quot}_S^1(\mathcal{E})$, $\mathcal{O}^{[1]} = \mathcal{O}(1)$, $[\mathbb{P}(\mathcal{E})]^{vir} = -e(p^*\omega_S) \cap [\mathbb{P}(\mathcal{E})]$

(iii) $\mathrm{Quot}_{\mathcal{E}}^2(S)$ singular for $r \geq 2$: $Q_q = \mathcal{O}_Z$ $(Z \in S^{[2]})$, $Q_q = \mathcal{O}_s^{\oplus 2}$ $(s \in S)$

§3. Results

(I) gives relations in intersection theory

$(*)\quad 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, $\mathcal{E}''$ locally free

$\mathcal{E}' \to \mathcal{E}$ induces $\sigma \in H^0(\mathcal{E}'^{\vee[\ell]})$ such that $\sigma(q) = \mathcal{E}' \to \mathcal{E} \to Q_q$

Hence $Z(\sigma) = \mathrm{Quot}^{\ell}_S(\mathcal{E}'') \overset{\iota}{\subset} \mathrm{Quot}^{\ell}_S(\mathcal{E})$

<u>Theorem</u>. σ regular;

(i) $\iota_*[\mathrm{Quot}^{\ell}_S(\mathcal{E}'')] = e(\mathcal{E}'^{\vee[\ell]}) \cap [\mathrm{Quot}^{\ell}_S(\mathcal{E})]$;

(ii) $-[\ \text{———}\ ]^{vir} = \text{———}\ [\ \text{———}\ ]^{vir}$;

(iii) $-[\ \text{———}\ ]^{red} = \text{———}\ [\ \text{———}\ ]^{red}$ (if S K3).

<u>Proof idea</u>: if $(*)$ split, write $\phi_q = (\phi'_q, \phi''_q) : \mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'' \to Q_q$.

Let $\mathcal{U} = \{q \mid \phi''_q \text{ surjective}\}$. Then $\mathcal{U} \to \mathrm{Quot}^{\ell}_S(\mathcal{E}'')$, $q \mapsto \phi''_q$ is the vector bundle associated to $\mathcal{E}'^{\vee[\ell]}$, $\sigma|_{\mathcal{U}}$ its universal section

(i) Get ι^* with $\iota_*\iota^* = e(\mathcal{E}'^{\vee[\ell]}) \cap (-)$, $\iota^*[\mathrm{Quot}^{\ell}_S(\mathcal{E})] = [\mathrm{Quot}^{\ell}_S(\mathcal{E}'')]$ (Fulton-MacPherson)

(ii) Obtain $\iota^*[\mathrm{Quot}^{\ell}_S(\mathcal{E})]^{vir} = [\mathrm{Quot}^{\ell}_S(\mathcal{E}'')]^{vir}$ from

$[\mathrm{Quot}^{\ell}_S(\mathcal{E})]^{vir} = \{c(T^{vir}_{\mathrm{Quot}^{\ell}_S(\mathcal{E})})^{-1} \cap c_F(\mathrm{Quot}^{\ell}_S(\mathcal{E}))\}_{e_r}$ (Siebert),

$$\begin{tikzcd} & \mathcal{E}'^{\vee[\ell]} \arrow[dl, "{[1]}"'] & \\ T^{vir}_{\mathrm{Quot}^{\ell}_S(\mathcal{E}'')} \arrow[r] & L\iota^* T^{vir}_{\mathrm{Quot}^{\ell}_S(\mathcal{E})} \arrow[u] \end{tikzcd}$$

and $\iota^* c_F(\mathrm{Quot}^{\ell}_S(\mathcal{E})) = c(\mathcal{E}'^{\vee[\ell]}) \cap c_F(\mathrm{Quot}^{\ell}_S(\mathcal{E}''))$

(iii) $T^{red} = T^{vir} - \mathcal{O}$

<u>Example</u> $\displaystyle\int_{\mathrm{Quot}^{\ell}_S(\mathcal{O}^{\oplus r})} e(\mathcal{O}^{[\ell]})^{r-1}(-) = \int_{S^{[\ell]}} (-)$, $e(\mathcal{O}^{[\ell]})^r = 0$.

(Use $0 \to \mathcal{O} \to \mathcal{O}^{\oplus r} \to \mathcal{O}^{\oplus(r-1)} \to 0$)

<u>Theorem</u> If $C \subset S$ smooth irreducible canonical curve, then $\qquad$ ($\mathcal{E} = \mathcal{O}^{\oplus r}$: Oprea-Pandharipande)

$\iota_*[\mathrm{Quot}^{\ell}_C(\mathcal{E}|_C)] = (-1)^{\ell}[\mathrm{Quot}^{\ell}_S(\mathcal{E})]^{vir}$

smooth!

<u>Application</u>. Any virtual integral of Chern classes of $\mathcal{F}_1^{[\ell]}, \dots, \mathcal{F}_m^{[\ell]}, T^{vir}$ over $\mathrm{Quot}^{\ell}_S(\mathcal{E})$ is given by a universal polynomial in $c_1(\mathcal{F}_i \otimes \mathcal{E})c_1(S)$, $c_1(\mathcal{E})c_1(S)$, $c_1(S)^2$

$$\int_{[\mathrm{Quot}^{\ell}_S(\mathcal{E})]^{vir}} (-) \overset{(\mathrm{II})}{=} \int_{[\mathrm{Quot}^{\ell}_S(\mathcal{E}\otimes\mathcal{L})]^{vir}} (-) \overset{\mathrm{Thm.}}{=} \int_{[\mathrm{Quot}^{\ell}_S(\mathcal{O}^{\oplus n})]^{vir}} e(K^{\vee[\ell]})(-) = \int_{\substack{[GP],\ S^{[\ell]} \\ (\mathrm{III})}} (-) \underset{[EGL]}{=} \dots$$

$0 \to K \to \mathcal{O}^{\oplus N} \to \mathcal{E}\otimes\mathcal{L} \to 0$