Naive Set Theory Refresher

Lecture 2, No Associated Chapter

John Connor

Definition: Set

What is a **set**?

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What is a **set**? A set is a collection of elements.

But what's a collection and what is an element?

Set: Definition by Example

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Set Relations: Membership

If an element x is an element of a set A then we denote this as

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Note: The emptyset is a subset of *every* set!

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This can also be stated using subsets:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

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$$|\mathcal{P}(A)| = 2^{|A|}$$

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Set Operations: Difference (Relative Complement)

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Definition: Tuple

Like a set, a tuple is a collection of elements, but unlike a set a tuple

1. is ordered: $(0, 1, 2, 3) \neq (1, 2, 3, 0)$.
2. can have "duplicate" elements: $(0, 0, 1, 2, 3) \neq (0, 1, 2, 3)$.

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Note: A tuple with exactly two elements is called a **pair**.

Tuple Operations: Projection

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$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$

Beyond Sets: Multisets and Lists

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