$$f(x) = \sum_{1}^{n} C_n P_n(x)$$

$$f(x) = C_1 P_1(x) + C_2 P_2(x) \cdots + C_n P_n(x)$$

$$P_n f(x) = C_1 P_1(x) P_n(x) + C_2 P_2(x) P_n(x) \cdots + C_n P_n(x) P_n(x)$$

$\Rightarrow$ por relación de completez:

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0 & , m \neq n \\ \frac{2}{2n+1} & , m = n \end{cases}$$

$$\Rightarrow \int_{-1}^{1} P_n f(x)\,dx = \int_{-1}^{1} C_1 P_1(x) P_n(x)\,dx + \int_{-1}^{1} C_2 P_2(x) P_n(x)\,dx \cdots \int_{-1}^{1} C_n P_n(x) P_n(x)\,dx$$

$$\int_{-1}^{1} P_n f(x)\,dx = \int_{-1}^{1} C_n P_n(x) P_n(x)\,dx = C_n \frac{2}{2n+1}$$

$$\Rightarrow C_n = \frac{2n+1}{2} \int_{-1}^{1} P_n f(x)\,dx$$