Parcial 2. Integración: 1 y 3

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Integración:

1

Método del trapecio simple:

$$I = \int_a^b f(x)\,dx \quad \text{donde}$$

$$I \approx \int_a^b p_1(x)\,dx \quad \text{con} \quad p_1(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

Luego:

$$\int_a^b p_1(x) = \int_a^b \frac{x-b}{a-b} f(a)\,dx + \int_a^b \frac{x-a}{b-a} f(b)\,dx$$

$$= \frac{f(a)}{a-b}\int_a^b (x-b)\,dx + \frac{f(b)}{b-a}\int_a^b (x-a)\,dx$$

$$= \frac{f(a)}{a-b}\left(\frac{x^2}{2} - xb\right)\Big|_a^b + \frac{f(b)}{b-a}\left(\frac{x^2}{2} - ax\right)\Big|_a^b$$

$$= \frac{f(a)}{a-b}\left(\frac{b^2}{2} - b^2 - \frac{a^2}{2} + ab\right) + \frac{f(b)}{b-a}\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right)$$

$$= \frac{f(a)}{a-b}\left(\frac{-b^2 - a^2 + 2ab}{2}\right) + \frac{f(b)}{b-a}\left(\frac{a^2 + b^2 - 2ab}{2}\right)$$

$$= \frac{f(a)}{b-a}\left(\frac{(b-a)^2}{2}\right) + \frac{f(b)}{b-a}\left(\frac{(b-a)^2}{2}\right)$$

$$= f(a)\left(\frac{b-a}{2}\right) + f(b)\left(\frac{b-a}{2}\right) = \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)$$

$$= f(a)\left(\frac{b-a}{2}\right)+f(b)\left(\frac{b-a}{2}\right)=\left(\frac{b-a}{2}\right)\left(f(a)+f(b)\right)$$

3

$$I=\int_a^b f(x)\,dx$$

$$f(x)\approx p_2(x)=\frac{(x-b)(x-x_m)}{(a-b)(a-x_m)}f(a)+\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)}f(x_m)+\frac{(x-a)(x-x_m)}{(b-a)(b-x_m)}f(b) \quad y$$

$I\approx\int_a^b p_2(x)\,dx$, luego:

$$\int_a^b p_2(x)\,dx=\int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)}f(a)\,dx+\int_a^b\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)}f(x_m)\,dx+\int_a^b\frac{(x-a)(x-x_m)}{(b-a)(b-x_m)}f(b)\,dx$$

$$=\frac{f(a)}{(a-b)(a-x_m)}\underbrace{\int_a^b(x-b)(x-x_m)\,dx}_{\alpha}+\frac{f(x_m)}{(x_m-a)(x_m-b)}\underbrace{\int_a^b(x-a)(x-b)\,dx}_{\beta}+\frac{f(b)}{(b-a)(b-x_m)}\underbrace{\int_a^b(x-a)(x-x_m)\,dx}_{\gamma}$$

donde para α tenemos:

$$\int_a^b(x-b)(x-x_m)\,dx=\int_a^b(x^2-x_mx-bx+bx_m)\,dx=\int_a^b\left[x^2-x(x_m+b)+bx_m\right]dx$$

$$=\left(\frac{x^3}{3}-(x_m+b)\frac{x^2}{2}+bx_mx\right)\Big|_a^b$$

$$=\frac{b^3}{3}-(x_m+b)\frac{b^2}{2}+b^2x_m-\frac{a^3}{3}+(x_m+b)\frac{a^2}{2}-bax_m$$

$$=-\frac{b^3}{6}-\frac{a^3}{3}+\frac{b^2x_m}{2}+(x_m+b)\frac{a^2}{2}-bax_m=\frac{b-a}{6}$$

para β tenemos:

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$$\beta=\int_a^b (x-a)(x-b)\,dx=\int_a^b\left[x^2-x(b+a)+ab\right]dx$$

$$=\left(\frac{x^3}{3}-\frac{x^2}{2}(b+a)+xab\right)\Big|_a^b$$

$$=\frac{b^3}{2}-\frac{b^2}{2}(b+a)+b^2a-\frac{a^3}{3}+\frac{a^2}{2}(b+a)-a^2b$$

$$=\frac{a^3}{6}+\frac{b^2a}{2}-\frac{a^2b}{2}=\frac{2b-2a}{3}$$

para γ tenemos,

$$\gamma=\int_a^b (x-a)(x-x_m)\,dx=\int_a^b\left[x^2-x(x_m+a)+ax_m\right]dx$$

$$=\left(\frac{x^3}{3}-\frac{x^2}{2}(x_m+a)+ax_mx\right)\Big|_a^b$$

$$=\frac{b^3}{3}-\frac{b^2}{2}(x_m+a)+abx_m-\frac{a^3}{3}+\frac{a^2}{2}(x_m+a)-a^2x_m$$

$$=\frac{b-a}{6}$$

$$\int_a^b p_2(x)\,dx=\left(\frac{b-a}{6}\right)f(a)+\frac{2(b-a)}{3}f(x_m)+\left(\frac{b-a}{6}\right)f(b)$$

$$=\left(\frac{b-a}{3}\right)\left[\frac{1}{2}f(a)+2f(x_m)+\frac{1}{2}f(b)\right]$$

$$=\left(\frac{b-a}{6}\right)\left[f(a)+4f(x_m)+f(b)\right]=\frac{h}{3}\left[f(a)+4f(x_m)+f(b)\right]$$