of Y. Either way, $f(f(\pi)) = \pi$, and f is a well-defined, sign-reversing involution, as desired. ■

In summary, we have shown combinatorially that for all values of n, there are almost as many even derangements as odd derangements of n elements. Or to put it another way, when randomly choosing a derangement with at least five elements, the *odds* of having an even derangement are nearly *even.*

Acknowledgment. We are indebted to Don Rawlings for bringing this problem to our attention and we thank Magnhild Lien, Will Murray, and the referees for many helpful ideas.

REFERENCES

1. R. A. Brualdi, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, New Jersey, 1999.
2. C. A. McCarthy and A. T. Benjamin, Determinants of the tournaments, this MAGAZINE, **69** (1996), 133–135.
3. C. D. Olds, Odd and even derangements, Solution E907, *Amer. Math. Monthly*, **57** (1950), 687–688.

Volumes of Generalized Unit Balls

XIANFU WANG
UBC Okanagan
3333 University Way, Kelowna
B.C., Canada, V1V 1V7
shawn.wang@ubc.ca

Diamonds, cylinders, squares, stars, and balls. These geometric figures are familiar to undergraduate students, but what could they possibly have in common? One answer is: They are generalized balls. The standard Euclidean ball can be distorted into a variety of strange-shaped balls by linear and nonlinear transformations. The purpose of this note is to give a unified formula for computing the volumes of generalized unit balls in n-dimensional spaces.

A generalized unit ball in $\mathbb{R}^n$ is described by the set

$$\mathbb{B}_{p_1 p_2 \cdots p_n} = \{\mathbf{x} = (x_1, \ldots, x_n) : |x_1|^{p_1} + \cdots + |x_n|^{p_n} \leq 1\}, \tag{1}$$

where $p_1 > 0, p_2 > 0, \ldots, p_n > 0$.

When the numbers $p_1, \ldots, p_n$ are all greater than or equal to 1, the unit ball $\mathbb{B}_{p_1 \cdots p_n}$ is convex. Since $|x|^p$ is not concave on $[-1, 1]$ for $0 < p < 1$, $\mathbb{B}_{p_1 \cdots p_n}$ is not necessarily convex anymore when $n > 1$. When $p_1 = p_2 = \cdots = p_n = p \geq 1$, we obtain the usual l_p ball. The l_2 ball is denoted by $\mathbb{B}$. By choosing different numbers p_i, we can alter the appearance of the generalized balls greatly, as shown in FIGURE 1 with examples in $\mathbb{R}^3$.

Motivated by an article by Folland [**5**], I derived a unified formula for calculating the volume of these balls. Although the volume formulas for the standard Euclidean ball $\mathbb{B}$ and simplex have been known for a long time [**4**, pp. 208, 220], the unified formula is (relatively) new. It is surprising that no matter how strange the balls look, the volume of any ball can be computed by a single formula, as follows:

THEOREM. *Assume $p_1, \ldots, p_n > 0$. The volume of the unit ball $\mathbb{B}_{p_1 p_2 \cdots p_n}$ in $\mathbb{R}^n$ is equal to*

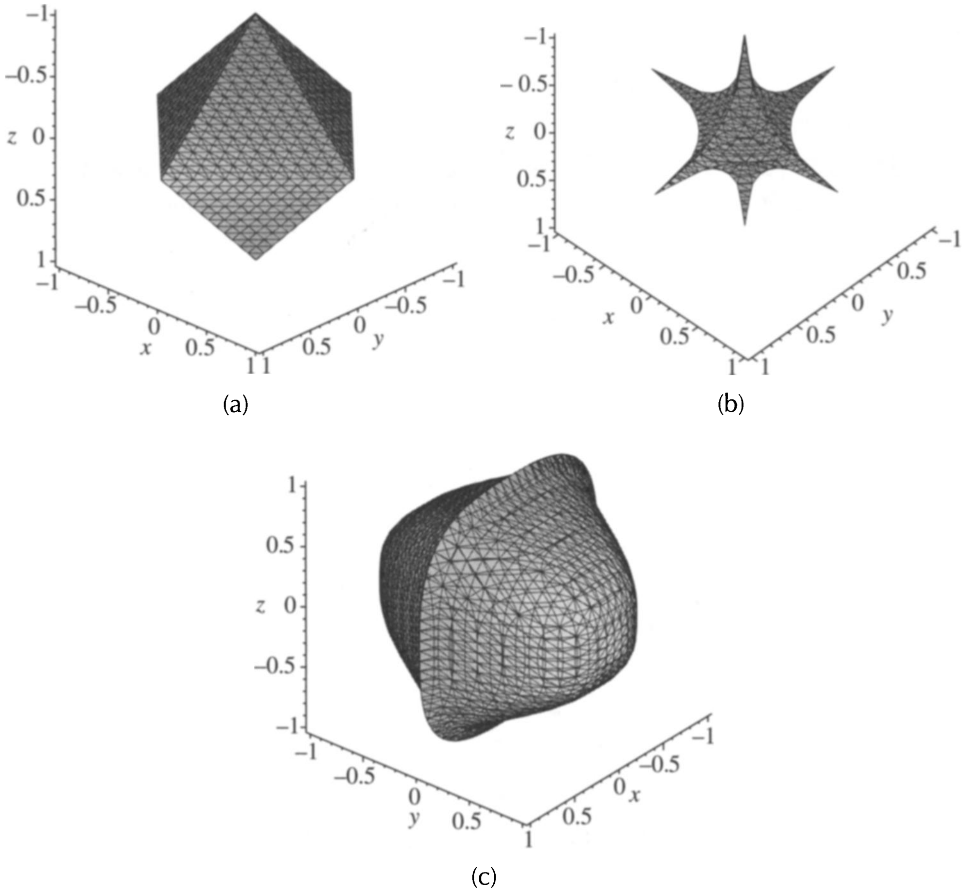


Figure 1 (a) $|x_1| + |x_2| + |x_3| \leq 1$; (b) $|x_1|^{1/2} + |x_2|^{1/2} + |x_3|^{1/2} \leq 1$; (c) $|x_1|^3 + |x_2|^{1/2} + |x_3|^3 \leq 1$

$$2^n \frac{\Gamma(1 + 1/p_1) \cdots \Gamma(1 + 1/p_n)}{\Gamma(1/p_1 + 1/p_2 + \cdots + 1/p_n + 1)}. \tag{2}$$

The volume of the positive orthant part, where all x-values are positive, may be obtained by removing the factor of 2^n from the formula.

The formula involves the *gamma function*, which we review for readers who may be unfamiliar with it. For $0 < t < \infty$, we define

$$\Gamma(t) := \int_0^\infty s^{t-1} e^{-s}\, ds.$$

The integral converges for $t > 0$. The following facts will be needed: For $u > 0$ and $v > 0$, we have

$$\Gamma(u + 1) = u\Gamma(u), \tag{3}$$

and

$$\int_0^1 s^{u-1}(1-s)^{v-1}\,ds = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{4}$$

Although the integral in $\Gamma(t)$ becomes infinite for $t \leq 0$, (3) provides an analytic continuation formula to define $\Gamma(t)$ for $t < 0$. The function Γ has discontinuities only at $t = 0, -1, -2, \ldots$. More details can be found in Folland [**6**, pp. 344–346].

Proof.

Step 1. We begin with the fact that

$$V(\mathbb{B}_{p_1\cdots p_n}) = \int_{\mathbb{B}_{p_1\cdots p_n}} 1\,d\mathbf{x}$$

and apply a change of variables that deforms the generalized ball into $\mathbb{B}$, the standard ball: Let $y_1 = x_1^{p_1/2}, \ldots, y_n = x_n^{p_n/2}$. For the function

$$\phi(\mathbf{y}) := (y_1^{2/p_1}, \ldots, y_n^{2/p_n}),$$

the Jacobian determinant is

$$J\phi(\mathbf{y}) = \frac{2}{p_1}\cdots\frac{2}{p_n} y_1^{\frac{2}{p_1}-1}\cdots y_n^{\frac{2}{p_n}-1}.$$

Readers may consult Folland [**6**, p. 432] for a detailed proof of the change of variables formula, which is our next ingredient. We use it to obtain

$$\int_{\mathbb{B}_{p_1\cdots p_n}} 1\,d\mathbf{x} = \int_{\mathbb{B}} \left|J\phi(y)\right|d\mathbf{y} = \frac{2^n}{p_1\cdots p_n}\int_{\mathbb{B}} |y_1|^{2/p_1-1}\ldots|y_n|^{2/p_n-1}\,d\mathbf{y}.$$

Step 2. Assume $\alpha_1, \ldots, \alpha_n > -1$. We claim:

$$\int_{\mathbb{B}} |x_1|^{\alpha_1}\ldots|x_n|^{\alpha_n}\,d\mathbf{x} = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\cdots+\beta_n+1)}, \tag{5}$$

where $\beta_i := (\alpha_i + 1)/2$ for $i = 1, \ldots, n$.

To verify this claim, we develop a recursion formula. Let $I(\alpha_1, \ldots, \alpha_n)$ denote the integral in (5). We then evaluate this as an iterated integral starting with x_1 as outermost variable.

$$I(\alpha_1, \ldots, \alpha_n) = \int_{-1}^1 |x_1|^{\alpha_1}\int_{x_2^2+\cdots+x_n^2\leq 1-x_1^2} |x_2|^{\alpha_2}\cdots|x_n|^{\alpha_n}dx_2\ldots dx_n\,dx_1$$

The inner integration takes place over a ball of radius $r = \sqrt{1-x_1^2}$. Changing variables again, we set $(x_2, \ldots, x_n) = r(y_2, \ldots, y_n)$ to get

$$\begin{aligned}\int_{x_2^2+\cdots+x_n^2\leq r^2} &|x_2|^{\alpha_2}\cdots|x_n|^{\alpha_n}dx_2\cdots dx_n \\ &= \int_{y_2^2+\cdots+y_n^2\leq 1} r^{(n-1)+\alpha_2+\cdots+\alpha_n}|y_2|^{\alpha_2}\cdots|y_n|^{\alpha_n}dy_2\cdots dy_n.\end{aligned}$$

This gives $I(\alpha_1, \ldots, \alpha_n) =$

$$= \int_{-1}^{1} |x_1|^{\alpha_1} (1 - x_1^2)^{(n-1)/2 + (\alpha_2 + \cdots + \alpha_n)/2} \int_{y_2^2 + \cdots + y_n^2 \le 1} |y_2|^{\alpha_2} \cdots |y_n|^{\alpha_n} dy_2 \ldots dy_n \, dx_1$$

$$= 2 \int_0^1 x_1^{\alpha_1} (1 - x_1^2)^{(n-1)/2 + (\alpha_2 + \cdots + \alpha_n)/2} dx_1 \cdot \int_{y_2^2 + \cdots + y_n^2 \le 1} |y_2|^{\alpha_2} \ldots |y_n|^{\alpha_n} dy_2 \ldots dy_n$$

$$= \int_0^1 (x_1^2)^{(\alpha_1 - 1)/2} (1 - x_1^2)^{(\alpha_2 + \cdots + \alpha_n + n + 1)/2 - 1} d(x_1^2) \int_{y_2^2 + \cdots + y_n^2 \le 1} |y_2|^{\alpha_2} \cdots |y_n|^{\alpha_n} dy_2 \ldots dy_n.$$

Hence by (4),

$$I(\alpha_1, \ldots, \alpha_n) = \frac{\Gamma((\alpha_1 + 1)/2)\Gamma((\alpha_2 + \cdots + \alpha_n + n + 1)/2)}{\Gamma((\alpha_1 + \cdots + \alpha_n + n + 2)/2)} \cdot I(\alpha_2, \ldots, \alpha_n).$$

This provides a recursion formula connecting $I(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $I(\alpha_2, \ldots, \alpha_n)$. Applying the recursion formula $(n - 1)$ times, after cancellation, we obtain

$$I(\alpha_1, \ldots, \alpha_n) = \frac{\Gamma(\frac{\alpha_1+1}{2}) \cdots \Gamma(\frac{\alpha_{n-1}+1}{2}) \frac{\alpha_n+1}{2} \Gamma(\frac{\alpha_n+1}{2})}{\Gamma(\frac{\alpha_1 + \cdots + \alpha_n + n}{2} + 1)} \cdot I(\alpha_n). \tag{6}$$

But

$$I(\alpha_n) = \int_{x^2 \le 1} |x|^{\alpha_n} dx = 2 \int_0^1 x^{\alpha_n} dx = \frac{2}{\alpha_n + 1}.$$

Putting this into (6) yields (5).

Step 3. When $\alpha_i = 2/p_i - 1$ for $i = 1, \ldots, n$, (5) gives

$$I(2/p_1 - 1, \ldots, 2/p_n - 1) = \frac{\Gamma(1/p_1) \cdots \Gamma(1/p_n)}{\Gamma(1/p_1 + \cdots + 1/p_n + 1)}.$$

Hence

$$\begin{aligned} V(\mathbb{B}_{p_1 \ldots p_n}) &= 2^n \frac{1}{p_1} \cdots \frac{1}{p_n} I(2/p_1 - 1, \ldots, 2/p_n - 1) \\ &= 2^n \frac{\Gamma(1 + 1/p_1) \cdots \Gamma(1 + 1/p_n)}{\Gamma(1/p_1 + \cdots + 1/p_n + 1)}. \end{aligned}$$

The volume of positive orthant part follows from there being 2^n orthants in $\mathbb{R}^n$. ■

In (1), you might argue that p_i cannot be infinite, but, my dear readers, we can consider a limiting case. Let us write

$$x^\infty = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

We proceed to single out a few special cases (calculus students' delights):

1. Some $p_i = +\infty$: as Γ is continuous on $(0, +\infty)$, we have $V(\mathbb{B}_{p_1 \ldots p_n}) =$

$$2 \cdot 2^{n-1} \frac{\Gamma(1 + 1/p_1) \cdots \Gamma(1 + 1/p_{i-1})\Gamma(1 + 1/p_{i+1}) \cdots \Gamma(1 + 1/p_n)}{\Gamma(1/p_1 + \cdots + 1/p_{i-1} + 1/p_i + \cdots + 1/p_n + 1)}.$$

In particular, when $p_1 = p_2 = \cdots = p_n = +\infty$, the volume of the ball is 2^n, and the shape is an n-dimensional hypercube (excluding the portions of its boundary where two or more x_is are simultaneously 1). When $p_1 = p_2 = 2$, $p_3 = \infty$, the generalized ball is a circular cylinder in $\mathbb{R}^3$.

2. When $p_1 = p_2 = \cdots = p_n = p > 0$, we have $V(\mathbb{B}_{p\ldots p}) =$

$$2^n \frac{(\Gamma(1+1/p))^n}{\Gamma(n/p+1)} = \frac{(2/p)^n(\Gamma(1/p))^n}{(n/p)\Gamma(n/p)}.$$

Recall that $\Gamma(1) = 1$, $\Gamma(1/2) = \pi^{1/2}$, and $\Gamma(n) = (n-1)!$. For $p = 2$, the generalized ball is the standard Euclidean ball with volume $2\pi^{n/2}/(n\Gamma(n/2))$. For $p = 1$, the generalized ball

$$\{(x_1, \ldots, x_n) : |x_1| + \cdots + |x_n| \le 1\},$$

is an n-dimensional diamond, and has volume $2^n/n!$. For $p = 1/2$, the generalized ball has volume $2^{2n}/(2n)!$, and its shape is an n-dimensional star. These are two of the balls shown in FIGURE 1.

Surprisingly, for $0 < p < \infty$ we find the n-dimensional ball has smaller volume when n becomes larger, and that

$$\lim_{n\to\infty} V(B_{\underbrace{p,p,\ldots,p}_{n \text{ terms}}}) = \lim_{n\to\infty} \frac{(2/p)^n(\Gamma(1/p))^n}{(n/p)\Gamma(n/p)} = 0.$$

Here we use Stirling's formula: $\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}$, where $\sim$ means that the ratio of the quantity on the left and right approaches 1 as $x \to \infty$ [**6**, p. 353].

3. For the ellipsoid $\{(x_1, \ldots, x_n) : |x_1|^{p_1}/a_1^{p_1} + \cdots + |x_n|^{p_n}/a_n^{p_n} \le 1\}$, with $a_i > 0$, a simple linear transformation reduces it to the form in (1) and the theorem yields its volume as

$$a_1 \cdots a_n \cdot 2^n \frac{\Gamma(1+1/p_1)\cdots\Gamma(1+1/p_n)}{\Gamma(1/p_1 + \cdots + 1/p_n + 1)}.$$

FIGURE 2 shows two ellipsoids in $\mathbb{R}^3$ to give readers an idea of their appearance.

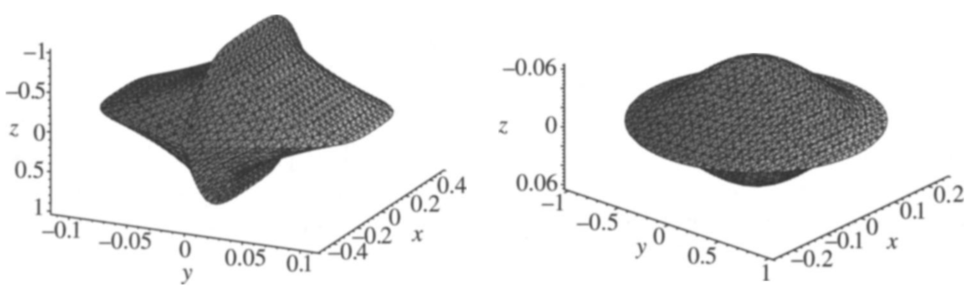


Figure 2 (a) $16|x_1|^4 + 3|x_2|^{1/2} + |x_3| \le 1$; (b) $16|x_1|^2 + |x_2|^2 + 4|x_3|^{1/2} \le 1$

Remark After I obtained this result, Dr. J. M. Borwein, at Simon Fraser University, informed me that the 19th-century French mathematician Dirichlet had obtained a similar result using a different method [**3**, pp. 153–159]. An *induction-free* proof to the volume formula of the l_p ball, via the *Laplace transform*, has been given by Borwein and Bailey in [**2**, pp. 195–197]. Similarly, one can derive an induction-free proof to the volume formula of generalized balls (2) using the Laplace transform. Finally, we remark that more properties on the gamma function and volume of Euclidean balls can be found in Stromberg [**7**, pp. 394–395].

Acknowledgment. This work was supported by NSERC and by the Grant In Aid of Okanagan University College.

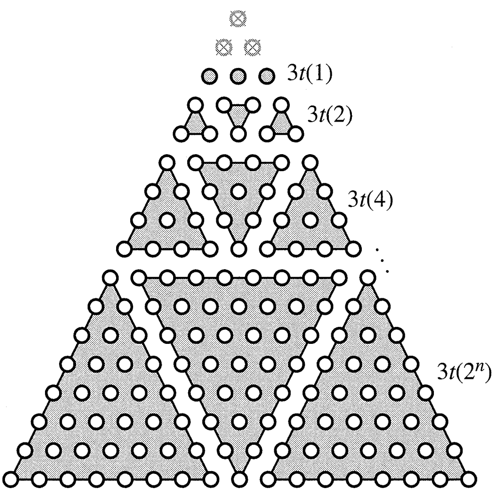
REFERENCES

1. J. A. Baker, Integration over spheres and the divergence theorem for balls, *Amer. Math. Monthly* **104** (1997), 36–47.
2. J. M. Borwein, D. H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A.K. Peters, 2003.
3. J. Edwards, *A Treatise on the Integral Calculus*, Vol. II, Chelsea Publishing Company, New York, 1922.
4. W. Fleming, *Functions of Several Variables*, Springer-Verlag, New York, 1977.
5. G. B. Folland, How to integrate a polynomial over a sphere, *Amer. Math. Monthly* **108** (2001), 446–448.
6. ———, *Advanced Calculus*, Prentice Hall, Upper Saddle River, NJ, 2002.
7. K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth International Mathematics Series, 1981.

Proof Without Words: A Triangular Sum

$$t(n) = 1 + 2 + \cdots + n \to \sum_{k=0}^{n} t(2^k) = \frac{1}{3}t(2^{n+1}+1) - 1$$

$t(2^{n+1}+1) - 3$:



$$3\sum_{k=0}^{n} t(2^k) = t(2^{n+1}+1) - 3$$

Exercises: (a) $\displaystyle\sum_{k=1}^{n} t(2^k - 1) = \frac{1}{3}t(2^{n+1} - 2)$

(b) $\displaystyle\sum_{k=0}^{n} t(3\cdot 2^k - 1) = \frac{1}{3}\left[t(3\cdot 2^{n+1} - 2) - 1\right]$

—ROGER B. NELSEN
LEWIS & CLARK COLLEGE
PORTLAND OR 97219