Vector space, dual space

Let $\mathbb{V}$ be a complex vector space, with $\psi \in \mathbb{V}$

Its dual space $\mathbb{V}^*$, is a set of all linear maps Φ of $\mathbb{V} \to \mathbb{C}$.

$$\Phi : \mathbb{V} \to \mathbb{C}, \quad \psi \to \Phi(\psi) \quad \text{with} \quad \Phi(\psi + \phi) = \Phi(\psi) + \Phi(\phi) \quad \forall\, \psi, \phi \in \mathbb{V}$$

$$\Phi(a\psi) = a\Phi(\psi) \quad \forall\, a \in \mathbb{C}$$

If $\mathbb{V}$ is equipped with scalar product, $\langle , \rangle\ \mathbb{V} \otimes \mathbb{V} \to \mathbb{C}$,

$(\psi, \phi) \to \langle \psi, \phi \rangle$

Canonical identification $\underline{J}$, between elements of $\mathbb{V}^*$ & $\mathbb{V}$.

$$J : \mathbb{V} \to \mathbb{V}^*, \quad \psi \to J(\psi) = \Psi \quad \text{s.t} \quad \Psi(\psi') = \langle \psi \underset{\substack{\uparrow \\ \mathbb{C}}}{,} \psi' \rangle \ \forall\, \psi' \in \mathbb{V}$$

All properties of J comes directly from this complex scalar product.

e.g. $\langle a\psi, \psi' \rangle = \bar{a} \langle \psi, \psi' \rangle$

In QM $\mathbb{V}$ = Hilbert space

vectors: $|\psi\rangle := \psi$ dual vectors: $\langle\psi| := J(|\psi\rangle)$

$\underline{\langle\psi|\psi\rangle} : \mathbb{C}$

Vectors (kets)

Basis kets: $|e_\sigma\rangle \in \mathbb{V}$

Label componenets of kets

$$|\phi_\alpha\rangle = |e_\sigma\rangle A^\sigma{}_\alpha$$

Dual vectors (bras)

Basis bras: $\langle e^\sigma| \in \mathbb{V}^*$

Label components of bras $\bar{A}^\sigma_\alpha = A^{\dagger\alpha}{}_\sigma$

$$\langle\phi^\alpha| = \underline{\bar{A}^\sigma}\langle e^\sigma| = \underline{A^{\dagger\alpha}{}_\sigma}\langle e^\sigma|$$

Overlaps

$\langle e^\sigma|e_\delta\rangle = \delta^\sigma{}_\delta$

$$\langle e^\sigma|\underline{\phi\rangle} = \underline{\langle e^\sigma|e_{\sigma'}\rangle A^{\sigma'}{}_\alpha} = \delta^\sigma{}_{\sigma'} A^{\sigma'}{}_\alpha = A^\sigma{}_\alpha$$

Kets; direct product basis: $|\psi_{\vec{\sigma}}\rangle = |\varphi_{\sigma_1\sigma_2\cdots\sigma_L}\rangle$

$= \underbrace{|\varphi_{\sigma_L}\rangle}_{\mathbb{V}_L} \otimes \cdots \otimes \underbrace{|\varphi_{\sigma_2}\rangle}_{\mathbb{V}_2} \otimes \underbrace{|\varphi_{\sigma_1}\rangle}_{\mathbb{V}_1} \in \mathbb{V}^L$

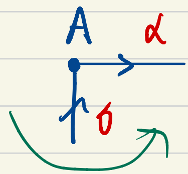
Linear comb: $|\phi_\beta\rangle = |\varphi_{\sigma_1\sigma_2\cdots\sigma_L}\rangle A^{\sigma_1\cdots\sigma_L}{}_\beta =: |\varphi_{\vec{\sigma}}\rangle A^{\vec{\sigma}}{}_\beta \in \mathbb{V}^L$

$\Rightarrow |\vec{\sigma}\rangle := |\sigma_1, \sigma_2, \ldots \sigma_L\rangle = |\sigma_L\rangle \otimes \cdots \otimes |\sigma_2\rangle \otimes |\sigma_1\rangle$

$\langle |$  $| \rangle$

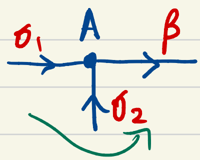
$$|\alpha\rangle = |\sigma\rangle A^{\sigma}{}_{\alpha}$$

$$\langle\sigma|\alpha\rangle = A^{\sigma}{}_{\alpha}$$

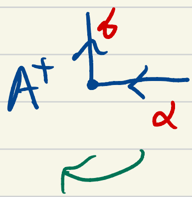


$$|\beta\rangle = |\sigma_2\rangle \otimes |\sigma_1\rangle \, A^{\sigma_1\sigma_2}{}_{\beta}$$

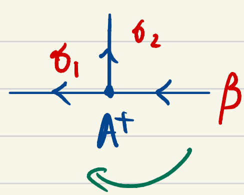
$$\left(\langle\sigma_1| \otimes \langle\sigma_2|\right)|\beta\rangle = A^{\sigma_1\sigma_2}{}_{\beta}$$



$$\langle\alpha| = A^{\dagger\alpha}{}_{\sigma} \langle\sigma|$$

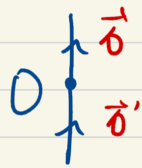


$$A^{\dagger\beta}{}_{\sigma_2\sigma_1}$$



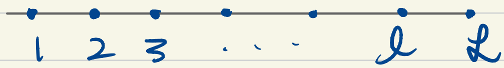
Operators: $\hat{O} = |\vec{\sigma}'\rangle O^{\vec{\sigma}'}{}_{\vec{\sigma}} \langle\vec{\sigma}|$,

$$O^{\vec{\sigma}'}{}_{\vec{\sigma}} = \langle\vec{\sigma}'|\hat{O}|\vec{\sigma}\rangle$$



3. Iterative diagonalization

spin-s chain, with Hamiltonian $H^{\mathcal{L}} = J \sum_{\ell=1}^{\mathcal{L}-1} \bar{S}_\ell \cdot \bar{S}_{\ell+1} + \sum_{\ell=1}^{\mathcal{L}} \bar{S}_\ell \cdot \vec{h}_\ell$



local state space for site: $|\sigma_\ell\rangle$, $\sigma_\ell = 1, \cdots, d = 2s+1$

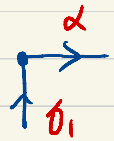
Want to find eigenstate of $H^{\mathcal{L}}$:

$$H^{\mathcal{L}} |E^{\mathcal{L}}_\alpha\rangle = E^{\mathcal{L}}_\alpha |E^{\mathcal{L}}_\alpha\rangle, \quad |E^{\mathcal{L}}_\alpha\rangle \in \mathcal{H}^{\mathcal{L}}$$

Diagonalize Hamiltonian iteratively, adding one site at at a time

$\mathcal{L}=1$: first site, diagonalize H^1 in $\mathcal{H}^1$. Eigenstates have the form

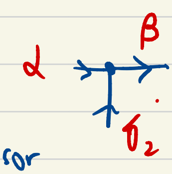
$$|\alpha\rangle \equiv |E^1_\alpha\rangle = |\sigma_1\rangle A^{\sigma_1}{}_\alpha \quad (\alpha = 1 \cdots d_1)$$



$\mathcal{L}=2$: Add 2nd site, diagonalize H^2 in H^2 in $\mathcal{H}^2$.

$$|\beta\rangle \equiv |E^2_\beta\rangle = \underset{d}{|\sigma_2\rangle} \otimes \underset{d}{|\alpha\rangle} B^{\alpha\sigma_2}{}_\beta$$

↖ coeff. tensor

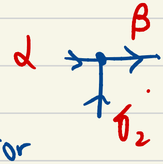


$\beta = 1, \cdots d^2$

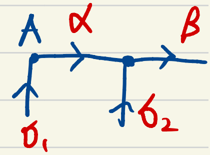
$\mathcal{L}=2$: Add 2nd site, diagonalize H^2 in H^2 in $\mathcal{H}^2$.

$$|\beta\rangle \equiv |E^2_\beta\rangle = \underset{d}{\underset{\uparrow}{|\sigma_2\rangle}} \otimes \underset{d}{\underset{\uparrow}{|\alpha\rangle}} B^{\alpha\sigma_2}{}_\beta$$

$\uparrow$ coeff. tensor



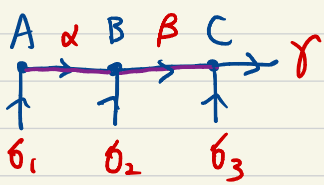
$$= |\sigma_2\rangle \otimes |\sigma_1\rangle A^{\sigma_1}{}_\alpha B^{\alpha\sigma_2}{}_\beta$$



$\mathcal{L}=3$; 3rd site

$$|\gamma\rangle = |\sigma_3\rangle \otimes |\beta\rangle C^{\beta\sigma_3}{}_\gamma \quad (\gamma = 1 \cdots d^3)$$

$$= |\sigma_3\rangle \otimes |\sigma_2\rangle \otimes |\sigma_1\rangle A^{\sigma_1}{}_{\alpha} \underbrace{\alpha B^{\alpha\sigma_2}}_{} {}_{\beta} \underbrace{\beta\, C^{\beta\sigma_3}}_{}{}_\gamma$$



⋮

add site $\mathcal{L}$

$\mathcal{H}^{\otimes \mathcal{L}}$

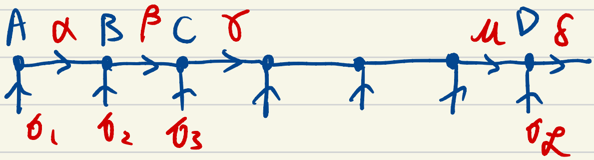
$$|E^{\mathcal{L}}_{\delta}\rangle = |\delta\rangle = |\sigma_{\mathcal{L}}\rangle \otimes \cdots \otimes |\sigma_3\rangle \otimes |\sigma_2\rangle \otimes |\sigma_1\rangle \underline{A^{\sigma_1}{}_{\alpha} B^{\alpha\sigma_2}{}_{\beta} C^{\beta\sigma_3}{}_{\gamma}}$$

$$\underline{\cdots D^{\mu\sigma_{\mathcal{L}}}{}_{\delta}}$$

$\delta = 1, \cdots d^{\mathcal{L}}$

(MPS)

$$=: \underline{C^{\vec{\sigma}}_{\delta}} \rightarrow \text{wave function}$$



σ_ℓ = physical indices, $\alpha, \beta, \gamma, \cdots$ = bond indices

Comments

1. Iterative diag. generates eigenstates whose wavefunctions are tensor that can be represented as matrix products.

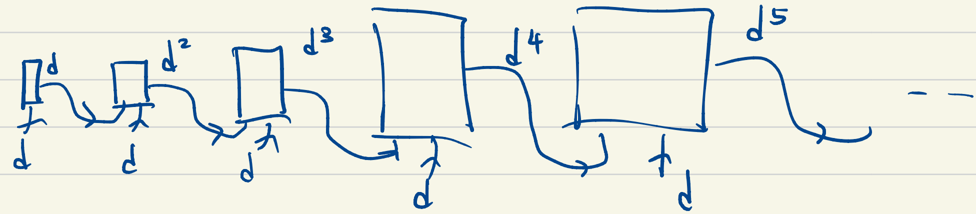
   MPS

   Matrix size $\uparrow$

for given σ_1: $A^{\sigma_1}{}_{\underline{\alpha}}$ has dimension $1 \times d$ (vector)

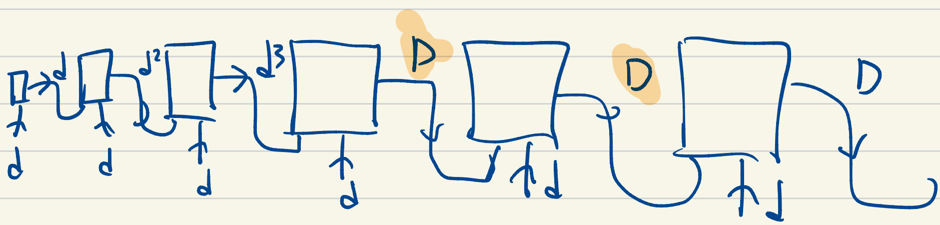
for given σ_2: $B^{\alpha\sigma_2}{}_{\beta}$ has dimension $d \times d^2$ (rectangular matrix)

σ_3: $C^{\beta\sigma_3}{}_{\gamma}$ has dimesion $d^2 \times d^3$ (larger ..



Hilbert space very large

Truncation scheme: $\alpha, \beta, \gamma \cdots \leq D$ for all virutal bonds



2. # of parameters to encode state:

$$N_{MPS} \leq L \cdot D^2 \cdot d$$