Notes on game theory

Sam Wycherley

These are a collection of notes covering topics in game theory. They are drawn from a number of sources and I claim no originality to any of the content. They are not particularly well-organized, alas.[1] They are already quite long but still work-in-progress.

Some housekeeping: I usually write "wlog" for "without loss of generality" and "iff" for "if and only if", and if I write "s.t." then I mean "such that".

Contents

[1] Also, there's a small amount of measure theory in some of the definitions and proofs in here. The only time it really matters for proofs is for Aumann's agreement theorem etc.

1 The structure of games

1.1 Extensive form games

Formally defining extensive form games is cumbersome but worth doing. The formal definition is a pretty ugly object, because it has so many components.

We first require some basic graph theory:

Definition 1 (Graph-theoretic preliminaries)**.**

(a) *Graph.* A *graph* (V, E) is a pair consisting of a set of *vertices* (or *nodes*) V and a set of *edges* E. An *edge* is a pair of vertices (u, v), with $u, v \in V$. We write this edge more compactly as uv. We say (V, E) is *directed* if edges are ordered pairs of vertices (the first entry is the *source* and the second is the *destination*), and *undirected* if edges are unordered pairs. For any directed graph (V, E), the corresponding undirected graph is obtained by treating each edge pair as unordered.

We say that two vertices u, v are *adjacent* if $uv \in E$. We say that edges $e_1 = u_1u_2$ and $e_2 = v_1v_2$ are *adjacent* if $u_2 = v_1$ or $u_1 = v_2$. In an undirected graph, two edges are adjacent if they share a common vertex. In a directed graph, two edges are adjacent if the destination of one of the edges is the source of the other.

We may sometimes write $uv \in G$ to denote that the edge uv belongs to $G = (V, E)$, i.e. $uv \in E$. We write $G - uv$ for the graph obtained from G by removing the edge uv, and $G + uv$ for the graph obtained from G by adding the edge uv.

(b) *Trails and paths.* In a graph (V, E), a *trail* $(v_1v_2, \ldots, v_{n-1}v_n)$ is a sequence of distinct adjacent edges such that $v_iv_{i+1} \in E$ for each $i = 1, \ldots, n-1$. If, moreover, each vertex that is visited by the trail is visited precisely once, then the trail is called a *path*.

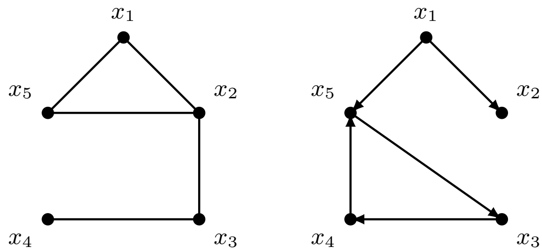
(c) *Cycles.* A *cycle* is a trail $(v_1v_2, \ldots, v_{n-1}v_n)$ such that $v_1 = v_n$. We say that a graph is *acyclic* if it contains no cycles.

(d) *Connectedness.* An undirected graph is *connected* if for any pair of vertices $u, v \in V$, there exists a path from u to v. A directed graph is connected if the corresponding undirected graph is connected.

(e) *Trees.* A graph (V, E) is a *tree* if it is connected and acyclic. A *rooted tree* (V, E, v) is a tree equipped with a *root*, which is some node $v \in V$. An *arborescence* is a directed rooted tree in which all edges are directed away from the root. That is, there is precisely one path between the root and any other vertex.

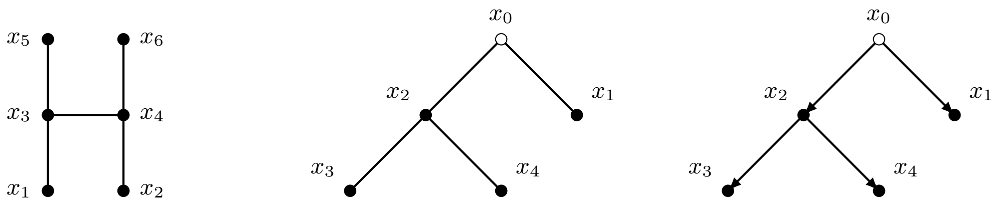
Example 1.

(a) An undirected graph (left) and a directed graph (right) on five vertices.



Both graphs are connected. The left graph contains one cycle, (x_1x_2, x_2x_5, x_5x_1), and there is a path from any node to any other. The right graph also contains a cycle: (x_5x_3, x_3x_4, x_4x_5). There is a path from x_1 to any other node in the right graph, but for all other nodes, this is not true (e.g. there is no path to x_2 from any node but x_1).

(b) An undirected tree (left), a rooted tree (centre) and an arborescence (right).



Both the rooted tree and the arborescence have root x_0. We will always denote the root by a hollow node in our diagrams. Note that in the arborescence, any node can be reached via a unique path from x_0, and there is no other node from which all the other nodes can be reached.

Definition 2 (Extensive form game)**.** An *extensive form game* Γ is a tuple

$$\Gamma = \left(\mathcal{I}, T, P, \Phi, \mathcal{A}, (u_i)_{i\in\mathcal{I}}, \eta\right),$$

where:

(i) *Players.* $\mathcal{I}$ is a finite index set of n *players*, $i = 1, \ldots, n$, and possibly including nature, conventionally indexed as player 0 or player N.[2]

(ii) *Game tree.* The *game tree* $T = (X, E, x_0)$ is a (potentially infinite) arborescence with root node x_0, which we refer to as the *initial node*. The set of nodes, X, is partitioned as $X = X_\tau \cup X_d$, where X_τ is the set of *terminal nodes*, the nodes that are not the source of any edge in T. The set X_d is the set of *decision nodes*. Any decision node is the source of some edge in T.

It is convenient to define a partial order on X. We write $x > x'$ if there is a path from x to x' in T, and we say that x *precedes* x', or x is a *predecessor* of

[2] Nature randomizes in a given way over a set of 'states of nature'.

x'. Conversely, we say that x' *succeeds* x or x' is a *successor* of x. If $x > x'$ and there is no node x'' such that $x > x'' > x$, then we say that x is the *immediate predecessor* of x', and x' is an *immediate successor* of x. Clearly, every node in the game tree except the initial node has a unique immediate predecessor, and any node that has no successors is a terminal node.

(iii) *Player allocation function.* The *player allocation function* $P : X_d \to \mathcal{I}$ is a function assigning each decision node x to a player $P(x) \in \mathcal{I}$. We define the set of decision nodes of player i as $X_i := P^{-1}(i)$. At a decision node, a player makes a choice. Recall we potentially include nature as a player in $\mathcal{I}$. We refer to decision nodes of nature as *chance nodes*. At a chance node, nature plays randomly according to a given probability distribution.

(iv) *Information sets.* $\Phi = (\Phi_i)_{i\in\mathcal{I}}$ is the set of information sets. For each player $i \in \mathcal{I}$, each X_i is partitioned into *information sets* $\phi_i \in \Phi_i$. The information sets ϕ_i of player i are disjoint nonempty subsets of X_i and $X_i = \bigcup_{\phi_i\in\Phi_i} \phi_i$. For each information set $\phi_i \in \Phi_i$, we assume every path in T intersects ϕ_i at most once, and every node in ϕ_i is the source of the same number of edges. At an information set ϕ_i, player i cannot distinguish between the nodes in ϕ_i. That is, if $\phi_i = \{x, x'\}$ then player i can determine that they are at information set ϕ_i, but cannot determine whether they are at node x as opposed to x'.

(v) *Actions.* $\mathcal{A} = (A_i)_{i\in\mathcal{I}}$ is the set of action assignment functions $A_i : \Phi_i \to 2^E$. At each information set ϕ_i of i, $A_i(\phi_i)$ is a partition of the set of edges in T that originate at decision nodes in ϕ_i. An *action* $a \in A_i(\phi_i)$ is a set of edges such that for each decision node $x \in \phi_i$, a contains precisely one edge that has x as its source. We assume $\left|A_i(\phi_i)\right| \geq 2$, or else there is no choice to be made by i at ϕ_i.

(vi) *History.* A *history* is any path from the initial node to a decision node or terminal node. A history that terminates at a terminal node is a called a *terminal history* or an *outcome path*. An *outcome* is the unique terminal node associated with an outcome path.

(vii) *Payoff function.* Payoffs for each player i are defined over each terminal node by the von Neumann-Morgenstern expected utility function $u_i : X_\tau \to \mathbb{R}$.

(viii) *Randomization over states of nature.* At each chance node, η assigns a probability distribution over the set of outgoing edges. We assume the probabilities assigned over all edges is positive.

Definition 3 (Pure strategy)**.** In an extensive form game Γ, a pure *strategy* s_i for player i is a mapping from the set of information sets Φ_i into the set of possible choices $\bigcup_{\phi_i\in\Phi_i} A_i(\phi_i)$, assigning to each information set ϕ_i an action $s_i(\phi_i) \in A_i(\phi_i)$. The set of pure strategies is denoted $S_i = \prod_{\phi_i\in\Phi_i} A_i(\phi_i)$.

Intuitively, a strategy is a *complete contingent plan of action.*

We actually only defined the extensive form game here for the case of finite action spaces. We might instead have infinitely many actions available at a decision node. Often one sees cones used to denote a continuum of possible actions at a decision node.

1.2 Normal form games

Definition 4 (Normal form game)**.** A *normal form game*, also called a *strategic form game*, is a tuple $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, where:

(i) $\mathcal{I}$ is a nonempty finite set of players, of cardinality n.

(ii) S_i is a nonempty set of pure strategies of player $i \in \mathcal{I}$. We denote the typical strategy of player i by $s_i \in S_i$.

(iii) $u_i : S_i \times S_{-i} \to \mathbb{R}$ is a von Neumann-Morgenstern expected utility (payoff) function, where $S_{-i} = \prod_{j\neq i} S_j$, the Cartesian product of the strategy sets of the j other players with typical element $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_{-i}$.

Let $S = \prod_{i\in\mathcal{I}} S_i$ denote the set of all strategy profiles with typical element $s = (s_1, \ldots, s_n)$ and let $u(s) = (u_1(s), \ldots, u_n(s))$ be the payoff profile of all players given strategy profile s.

In many cases, we can represent a normal form two-player game by means of a *payoff matrix*, which encodes all the relevant information defining the normal form game. This can be extended beyond two players. Note that the payoff matrix for a normal form game exists only if there is a finite strategy set for each player and a finite number of players.

1.3 Representational equivalence

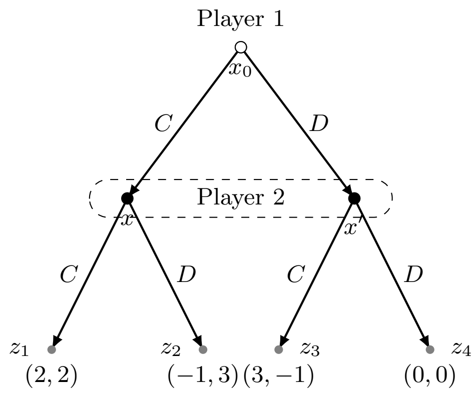
Any extensive form game can be represented in payoff matrix form.

Example 2 (Prisoner's Dilemma)**.**

(a) *Classic Prisoner's Dilemma.* The classic *Prisoner's Dilemma* has the following motivation: suppose there are two prisoners (Player 1 and Player 2) who can either confess ("defect" D) or stay silent ("cooperate" C). If Player i confesses and the other stays quiet, then Player i is released (receiving immunity for providing the State with evidence) and the other player faces a full prison sentence. If both stay silent, then both face a short jail sentence (given the State's lack of evidence to convict on serious charges). If both confess then they both face lengthy but reduced sentences. The set of players is $indexset = \{1, 2\}$ and the strategy set for each player i is $S_i = \{C_i, D_i\}$. A payoff matrix for this game is:

	C_2	D_2
C_1	$3,3$	$0,4$
D_1	$4,0$	$1,1$

Equivalently, we can represent the prisoner's dilemma in extensive form:



The dotted rectangle denotes Player 2's information set.

(b) *Repeated Prisoner's Dilemma.* Suppose the Prisoner's Dilemma game is played twice, and at the second round, each player observes and remembers the actions of both players in the first round. Suppose the payoffs in each round are as above except that the second round is discounted at some discount rate $\delta \in (0, 1)$. Again, the set of players is $\mathcal{I} = \{1, 2\}$. A strategy for player i now consists of a round 1 action and a round 2 action for each round 1 action profile that could be observed. For example, Player i could have as a strategy $CCDCD$ ("C in round 1, C in round 2 if (C, C) observed in round 1, D in round 2 if (C, D) observed in round 1, C in round 2 if (D, C) observed in round 1, D in round 2 if (D, D) observed in round 1"). For Player 1, this is a "tit-for-tat" strategy. If both players played tit-for-tat strategies (i.e. Player 1 plays $CCDCD$ and Player 2 plays $CCCDD$) then the payoff profile would be $(2 + 2\delta, 2 + 2\delta)$.

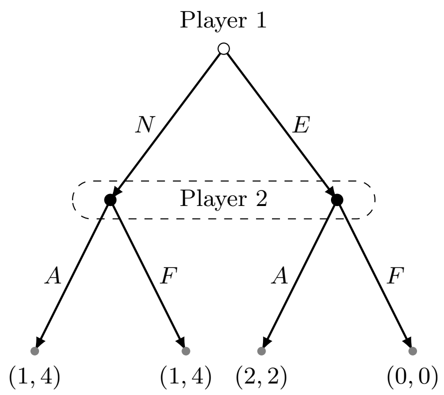
The number of strategies available to each player is $2^5 = 32$. We could, rather tediously, write out a 32×32 payoff matrix for this game. As the number of times the game is repeated increases, the number of strategies will blow up very quickly.

Note that the extensive form representation of a normal form game need not be unique. Consider the following example.

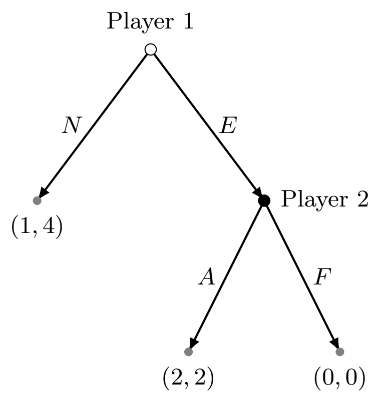
Example 3 (Market entry game I: non-unique extensive form representation)**.** Suppose Player 1 is an potential market entrant and Player 2 is an incumbent monopolist ($\mathcal{I} = \{1, 2\}$). Player 1 chooses between entering the market (E) and not entering (N), so has pure strategy set $S_1 = \{N, E\}$, Simultaneously, Player 2 chooses between acquiescing to the entrant (A) or fighting to try to deter the entrant (F), so Player 2 has strategy set $S_2 = \{A, F\}$. There are thus four potential strategy profiles: $S = \{(E, A), (E, F), (N, A), (N, F)\}$. Whether Player 2 acquiesces or fights does not matter for payoffs if Player 1 chooses not to enter. Suppose we have payoff matrix,

	A	F
N	$1,4$	$1,4$
E	$2,2$	$0,0$

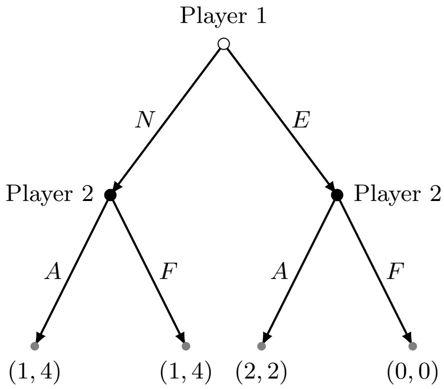
There are two equivalent ways to represent this in extensive form. We can represent this as (a) a *simultaneous move game*,



or (b) as a *chain store game*,



These are equivalent because the payoff if Player 1 plays N do not depend on Player 2's choice. Note the strategy set is the same in all the above representations (normal, simultaneous-move extensive and chain store extensive). Indeed, were we to instead have the *sequential move game*,



this would *not* be equivalent to the normal form game, for the strategy set of Player 2, $S_2 = \{(A, A), (F, A), (A, F), (F, F)\}$ is no longer that of the normal form game.

The extensive form representation of a game typically conveys more information about the game than the normal form representation. For dynamic games, such as repeated games or games with multiple stages, the extensive form representation is usually easier to interpret.[3]

1.4 Mixed strategies

Definition 5 (Randomized strategies)**.** For any topological space A, let $\mathcal{B}$ denote the Borel σ-algebra on A. Let $\Delta(A)$ be the set of all probability measures on $\mathcal{A}$:

$$\Delta(A) := \left\{\sigma : \mathcal{B} \to \mathbb{R} \mid \sigma \text{ is a measure and } \sigma(A) = 1\right\}.$$

In general, we will assume sets are endowed with their natural topology. If A is countable, this is the discrete topology, so $\mathcal{B} = 2^A$, the power set. If A is a connected subset of $\mathbb{R}^k$ (such as an interval in the real line, or a box in $\mathbb{R}^2$), then the natural topology is induced by the Euclidean metric and the Borel σ-algebra is obvious.

Recall that for σ to be a measure, we need that $\sigma \geq 0$, $\sigma(\varnothing) = 0$, and for any disjoint countable collection $\{E_n\}$, $E_n \subseteq A$, we have $\sigma\left(\bigcup_n E_n\right) = \sum_n \sigma(E_n)$ (countable additivity).

If A is finite with $|A| = k$, we can more straightforwardly define $\Delta(A)$ as a simplex in $\mathbb{R}^k$. Then each $\sigma \in \Delta(A)$ is a probability vector, and we can interpret the value of the ith entry in σ as the probability that the ith action in A is played.

Now consider a game Γ with finite set of players $\mathcal{I} = \{1, \ldots, n\}$ and where each player $i \in \mathcal{I}$ has strategy set S_i consisting of pure strategies s_i.

[3] These differences in information and the fact that the same normal form game can have non-unique extensive form representations can pose a problem. For some solution concepts, making seemingly irrelevant changes to the extensive form representation of a normal form game can lead to differing solutions. See e.g. Example 39.

(a) *Mixed strategy.* A *mixed strategy* is a probability measure $\sigma_i \in \Delta(S_i)$.

If S_i is countable, it is more convenient to focus on the probability measure of the singletons in the σ-algebra we associate with S_i. That is, abusing notation, we define $\sigma_i(s_i) = \sigma_i(\{s_i\})$, so we have $\sigma_i : S_i \to [0,1]$ satisfying $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. This is of course a probability mass function $\sigma_i \in \Delta(S_i)$.

A mixed strategy profile is a profile $(\sigma_1, \ldots, \sigma_n) \in \Delta(S_1) \times \cdots \times \Delta(S_n)$.

We define the space of mixed strategies of opponents by $\Delta_{-i}(S_{-i}) = \prod_{j \neq i} \Delta(S_j)$.

(b) *Mixed strategy game.* Given a pure strategy game $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$, a mixed strategy game is (in general) the game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$.

(c) *Correlated strategy.* In general, a *correlated strategy* is a strategy profile in $\Delta(S_1 \times \cdots \times S_n) = \Delta(S)$.

This is a generalization of a mixed strategy profile, for it allows correlation between players' strategies. Indeed, $\Delta(S_1) \times \cdots \times \Delta(S_n) \subseteq \Delta(S)$.

(d) *Behavioural strategy.* In general, a *behavioural strategy* $\pi_i : \Phi_i \to \bigcup_{\phi_i \in \Phi_i} \Delta(A_i(\phi_i))$ of player i is a function assigning to each information set $\phi_i \in \Phi_i$ a probability measure over the action set at that information set, $\pi_i(\phi_i) \in \Delta(A_i(\phi_i))$.

A mixed strategy σ_i and a behavioural strategy π_i are called (outcome) *equivalent* if for every (mixed or behavioural) strategy profile σ_{-i} and any node x in the extensive form game Γ, it follows that

$$P_{\sigma_i, \sigma_{-i}}(x) = P_{\pi_i, \sigma_{-i}}(x),$$

where $P(x)$ denotes the probability that node x is reached.

We can equivalently define a behavioural strategy of player i in terms of the set of histories $\mathcal{H}_i$ that terminate at an information set of player i. For each $h_i \in \mathcal{H}_i$, let $\phi_i(h_i)$ denote the information set at which h_i terminates. Then a behavioural strategy is a function $\pi_i : \mathcal{H}_i \to \bigcup_{\phi_i \in \Phi_i} Delta(A_i(\phi_i))$ that assigns to each history $h_i \in \mathcal{H}_i$ a probability measure $\pi_i(h_i) \in \Delta(A_i(\phi_i(h_i)))$, with the condition that if $\phi_i(h_i) = \phi_i(h_i')$ then $\pi_i(h_i) = \pi_i(h_i')$. We will stick to the definition given in (d) unless otherwise stated (when it comes to repeated games we switch to the history version).

Note that any pure strategy is a degenerate mixed strategy (that is, a mixed strategy with all probability mass placed on a single pure strategy.) I abuse notation sometimes and write the pure strategy profile where I mean a degenerate mixed strategy profile.

It is common in treatments of mixed strategies to restrict exposition to mixed strategies over finite strategy sets. I have included the more general definition of a mixed, correlated and behavioural strategy above, defined in terms of a probability measure over a Borel σ-algebra. This immediately allows for mixed strategies over continuous strategy sets, where our simplified definition for countable sets is of little use.

Conversely, any mixed strategy over S_i is a pure strategy in $\Delta(S_i)$. Indeed, mixed strategy extensions of pure strategy games are equivalent to a certain class of pure strategy games with continuous strategy spaces.

For any correlated strategy profile $\sigma \in \Delta(S)$, the expected payoff to player i is $\int_S u_i \, \mathrm{d}\sigma$ and for any mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_n) \in \prod_{i \in \mathcal{I}} \Delta(S_i)$, the expected payoff is

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \int_S u_i(s) \, \mathrm{d}\sigma \\ &= \int_{S_i \times S_{-i}} u_i(s_i, s_{-i}) \, \mathrm{d}(\sigma_i \times \sigma_{-i}) \\ &= \int_{S_1 \times \cdots \times S_n} u_i(s_i, s_{-i}) \, \mathrm{d}(\sigma_1 \times \cdots \times \sigma_n). \end{aligned}$$

Specializing to the countable case, the expected payoff to player i of correlated strategy profile $\sigma = (\sigma_1, \dots, \sigma_n) \in \Delta(S)$ is $u_i(s) = \sum_{s \in S} u_i \sigma(s)$, and for any mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_n) \in \prod_{i \in \mathcal{I}} \Delta(S_i)$, the expected payoff is

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{s \in S} u_i(s) \sigma(s) \\ &= \sum_{(s_i, s_{-i}) \in S_i \times S_{-i}} u_i(s_i, s_{-i}) \sigma_i(s_i) \sigma_{-i}(s_{-i}) \\ &= \sum_{(s_1, \dots, s_n) \in S_1 \times \cdots \times S_n} u_i(s_i, s_{-i}) \prod_{i \in \mathcal{I}} \sigma_i(s_i). \end{aligned}$$

Lemma 1. *For any mixed or behavioural strategy profile σ_{-i}, if mixed strategy σ_i is equivalent to behavioural strategy π_i then*

$$u_i(\sigma_i, \sigma_{-i}) = u_i(\pi_i, \sigma_{-i}).$$

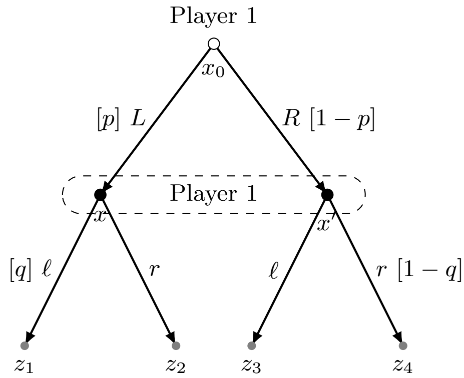
Proof. Follows immediately from the definition of equivalence and the expected payoff. □

Definition 6 (Perfect recall)**.** A game Γ is a game of *perfect recall* if every player in every stage of the game recalls all their previous information and every previous action they have taken.

Given a decision node x, let $\mathcal{E}_i(x)$ denote the the chronologically-ordered list of information sets encountered by player i on the path from initial node x_0 to x and which action i has taken at each such information sets. We call $\mathcal{E}_i(x)$ the *experience* of player i at x.

Example 4 (Games without perfect recall)**.**

(a) Player 1 cannot remember their first move:



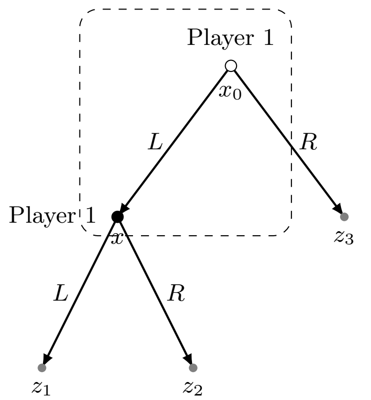
We have $\mathcal{E}_i(x) = \{x_0, L\} \neq \mathcal{E}_i(x') = \{x_0, R\}$, yet $x, x' \in \phi_1$.

The strategy set for Player 1 is $S_1 = \{L\ell, Lr, R\ell, Rr\}$. Now consider the mixed strategy $\sigma_1 = (1/2, 0, 0, 1/2)$ and consider an outcome-equivalent behavioural strategy σ_1^b s.t. Player 1 randomizes $(p, 1-p)$ over $\{L, R\}$ at node x_0 and $(q, 1-q)$ over $\{\ell, r\}$ at information set ϕ_1. We have

$$\begin{aligned}
P_{\sigma_1^b}(z_1) &= pq = \frac{1}{2} = P_{\sigma_1}(z_1), \\
P_{\sigma_1^b}(z_2) &= p(1-q) = 0 = P_{\sigma_1}(z_2), \\
P_{\sigma_1^b}(z_3) &= (1-p)q = 0 = P_{\sigma_1}(z_3), \\
P_{\sigma_1^b}(z_4) &= (1-p)(1-q) = \frac{1}{2} = P_{\sigma_1}(z_4).
\end{aligned}$$

We have $q = \frac{1}{2p}$ and hence $p(1-q) = p(1-1/2p) = p - \frac{1}{2} = 0$ so $p = \frac{1}{2}$, $q = 1$. Yet this implies $(1-p)(1-q) = 0 \neq \frac{1}{2}$, yielding a contradiction. Hence there is no behavioural strategy σ_1^b that is equivalent to σ_1.

(b) Suppose Player 1 forgets they make a first move:



We have $\mathcal{E}_i(x_0) = \varnothing \neq \mathcal{E}_i(x) = \{x_0, L\}$, but $x_0, x \in \phi_1$. The strategy set is $S_1 = \{L, R\}$. Now, any behavioural strategy assigning positive probability to both L and R can attain outcomes z_1, z_2, z_3, each with positive probability. However, there is no mixed strategy that can attain z_2 with positive probability.

In games of perfect recall, to every mixed strategy there exists an outcome-equivalent behavioural strategy. Kuhn (1953) first proved this result for finite strategy sets, but it extends to (countably, uncountably) infinite strategy sets also.

Theorem 1 (Kuhn, 1953)**.** *In every game of perfect recall, every mixed strategy has an equivalent behavioural strategy.*

Proof. We prove this for finite strategy sets, per Kuhn (1953). Aumann (1961) proved that the theorem also holds for infinite strategy sets.

We require that for any mixed strategy σ_i, there exists a behavioural strategy π_i s.t. for any (mixed or behavioural) strategy profile σ_{-i},

$$P_{\sigma_i,\sigma_{-i}}(x) = P_{\pi_i,\sigma_{-i}}(x)$$

for all nodes x. Now, for any x that lies at an information set irrelevant for σ_i (that is, not reached with positive probability given strategy σ_i for some σ_{-i}) then both sides are zero. Hence consider only those information sets ϕ_i that are relevant for σ_i. For each node x, let $R_i(\phi_i) = \{s_i \in S_i \mid \phi_i \text{ is on the path of } (s_i, s_{-i}) \text{ for some } s_{-i} \in S_{-i}\}$ be the set of relevant information sets for ϕ_i. Now for each relevant ϕ_i for σ_i, define the probability of playing move $a \in \phi_i$ under π_i by

$$\pi_i(a \mid \phi_i) = \frac{\sum_{s_i \in R_i(\phi_i): s_i(\phi_i)=a} \sigma_i(s_i)}{\sum_{s_i \in R_i(\phi_i)} \sigma_i(s_i)}.$$

Let $\phi_i^1, \ldots, \phi_i^{\bar{k}}$ denote player i's information sets preceding ϕ_i. Under perfect recall, reaching ϕ_i requires that i takes the appropriate action a^k at each ϕ_i^k:

$$R_i(\phi_i) = \{s_i \mid s_i(\phi_i^k) = a^k \text{ for all } k = 1, \ldots, \bar{k}\}.$$

Now, conditional on reaching ϕ_i, the distribution of continuation play is

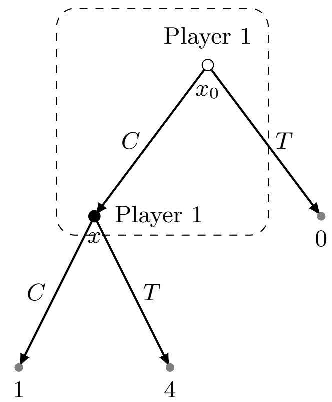
$$\pi_i(a \mid \phi_i) = \frac{\sum_{s_i \mid s_i(\phi_i^k)=a^k \text{ for all } k=1,\ldots,\bar{k} \text{ and } s_i(\phi_i)=x} \sigma_i(s_i)}{\sum_{s_i \mid s_i(\phi_i^k)=a^k \text{ for all } k=1,\ldots,\bar{k}} \sigma_i(s_i)},$$

which is precisely the probability of playing action a under σ_i conditional on reaching information set ϕ_i.

Since this holds for all ϕ_i, it follows that $P_{\sigma_i,\sigma_{-i}}(x) = P_{\pi_i,\sigma_{-i}}(x)$ for all nodes x. □

Unless otherwise specified, we assume all games are games of perfect recall. Without perfect recall, one has to be very careful in interpreting information sets, strategies etc. The optimal strategy ceases to be obvious, because different principles of optimality can conflict, as the following example shows.

Example 5 (Paradox of the absent-minded driver; Piccione and Rubinstein, 1995)**.** A weary traveller is sat in a pub planning their journey home. If they take the first exit on the way home, they will end up in a dangerous area (payoff 0). Ideally, they would to take the second exit, which leads them to their home (payoff 4). If they miss the second exit, they cannot turn around and must continue to the end of the road, where they can stay the night at a hotel (payoff 1). The driver is very absentminded, so when arriving at an exit, cannot tell if it is the first one or how many they have already passed. The driver knows this fact when planning the trip. The game can be represented as follows, where action T is taking the exit and C is continuing:



Suppose first that the driver cannot randomize. Then when planning the trip, they should plan to continue to the end of the road, receiving payoff 1 for if they instead choose to turn off when encountering an exit, they would turn off at the first exit and receive payoff 0. Yet on the road, when encountering an exit, the driver reasons there is probability $\frac{1}{2}$ that the exit is the second exit given their strategy. If they choose to turn off, their expected payoff is 2, and so it is optimal to take the exit. Hence the optimal strategy is time inconsistent.

There is a paradox here: on the one hand, the optimality of the ex ante optimal strategy should not require verification at execution if tastes or information have not changed; on the other, maximizing the driver's expected payoff given their beliefs at each stage leads them to deviate from the ex ante optimal strategy when at any exit. If pursuing this, the driver will take the first exit, and thus receive payoff 0.

The paradox persists if we allow the driver to randomize. The optimal behavioural strategy involves staying on the road with probability $\frac{2}{3}$ whenever encountering an exit. Now suppose p is the probability the driver does not exit, and α is the probability the driver assigns to being at the first exit. Then the expected payoff is $\alpha(p^2 + 4(1-p)p) + (1-\alpha)(p + 4(1-p))$, and so the optimal choice of p is $\max\left\{0, \frac{(7\alpha-3)}{6\alpha}\right\}$, which only gives $p = \frac{2}{3}$ if $\alpha = 1$, i.e. if the driver is certain they have not passed the first exit, which is implausible given the driver has no capacity to remember whether they have done so.

1.5 Knowledge

"Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know." –
Donald Rumsfeld, 2004, on failing to find Iraq's nonexistent WMD.

Reasoning about games requires developing some epistemic foundations. For us to model how players reason, we need to carefully set out what they know about the structure of the game, about what they know about each other, and so on. How much information agents have will alter their behaviour. For example, a monopolist is able to practice first-degree price discrimination if she knows her customers' individual demand schedules, whereas if she only knows the aggregate demand schedule, she must set a price uniformly.

"Higher order" knowledge also often matters. For example, suppose Firm 1 and Firm 2 are both profit-maximizing duopolists. Firm 1 can produce at a marginal cost known to both firms but Firm 2 has two possible types – either its costs are high, and so it produces a relatively smaller level of output, or its costs are low, in which case it produces a higher level of output. If Firm 1 does not know Firm 2's type, then it has to set its output based only on its estimate of Firm 2's output, based on its own beliefs about Firm 2's type. Suppose instead that Firm 1 learns Firm 2's output before setting its own. If Firm 2 does not know that Firm 1 knows its own output, then this puts Firm 1 in a much better position – whatever Firm 2's output, Firm 1 can tailor its own output to maximize its own profits. However, if Firm 2 knows that Firm 1 knows its output, then Firm 2 is in a much better position, because it becomes a Stackelberg leader – it can set higher output itself knowing that Firm 1 must optimally choose a lower output in response.

1.5.1 The standard model of knowledge

The standard model of knowledge does not claim to accurately describe knowledge as we would understand it in everyday use, but since we are not philosophers, this is unimportant – the model works well enough to capture interesting insights.

Definition 7 (States, information and knowledge)**.**

(a) *States and events.* Let $(\Omega, \mathcal{F}, p)$ be a probability space. The elements $\omega \in \Omega$ are called *states* and the sets $E \in \mathcal{F}$ are called *events.*

(b) *Information.* An *information function* for Ω is a function $h : \Omega \to \mathcal{F}$, associating to each state $\omega \in \Omega$ a nonempty set $h(\omega) \in \mathcal{F}$.

We call any partition $\mathcal{P}$ of Ω such that each member of $\mathcal{P}$ lies in $\mathcal{F}$ an *information partition.*

We say that h is *partitional* if $\{h(\omega) \mid \omega \in \Omega\}$ is an information partition and $\omega \in h(\omega)$ for each $\omega \in \Omega$. We say h corresponds to information partition $\mathcal{P}$ if $P = \{h(\omega) \mid \omega \in \Omega\}$.

Equivalently, h is partitional if

(I1) $\omega \in h(\omega)$ for all $\omega \in \Omega$ and

(I2) if $\omega' \in h(\omega)$ implies $h(\omega') = h(\omega)$.

(c) *Knowledge.* A *knowledge operator* for an agent is a function $K : \mathcal{F} \to \mathcal{F}$ defined by

$$K(E) = \{\omega \in \Omega \mid h(\omega) \subseteq E\}.$$

We say that K is the knowledge operator *induced* by information function h or information partition $\mathcal{P}$.

(I1) has the interpretation that an agent in state ω cannot rule out being in state ω. Under (I2), if ω' were also deemed possible by the agent in state ω, then it must be that the set of states the agent would consider possible in states ω and ω' are the same.

The knowledge operator captures under which states an agent knows a given event. We say that the agent *knows* event E in state ω if $\omega \in K(E)$.

Example 6. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and that we have partition $\{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ associated with the partitional information function h. We have e.g. $h(\omega_1) = h(\omega_2) = \{\omega_1, \omega_2\}$ and $h(\omega_3) = \{\omega_3\}$. The corresponding knowledge operator K has $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$, $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$, and $K(\{\omega_1\}) = \varnothing$.

Suppose instead that $\Omega = \{\omega_1, \omega_2\}$, that $h(\omega_1) = \{\omega_1\}$ but $h(\omega_2) = \{\omega_1, \omega_2\}$. Then h is not partitional. We would have $K(\{\omega_1\}) = \{\omega_1\}$, $K(\{\omega_2\}) = \varnothing$ and $K(\{\omega_1, \omega_2\}) = \{\omega_1, \omega_2\}$.

Knowledge operators derived from partitional information functions satisfy the following axioms:

Axioms.

(**AWA**) *Awareness.* $K(\Omega) = \Omega$.

(**OMN**) *Omniscience.* $K(E \cap F) = K(E) \cap K(F)$ for all $E, F \in \mathcal{F}$.

(**KNO**) *Knowledge.* $K(E) \subseteq E$ for all $E \in \mathcal{F}$.

(**TRA**) *Transparency.* $K(K(E)) = K(E)$ for all $E \in \mathcal{F}$.

(**WIS**) *Wisdom.* $\neg K(E) = K(\neg K(E))$ for all $E \in \mathcal{F}$.

This axiomatic characterization was introduced by Milgrom (1981). All of them are somewhat contentious – see Samuelson (2004) who discusses this in some detail.

The axiom of awareness (**AWA**), one of the more innocuous axioms, states that the agent always knows she is in some state, or equivalently knows the set of possible states.

The axiom of omniscience (**OMN**) is so-named because despite its appeal – knowing E and F implies knowing E and knowing F – it carries quite severe implications. Note that under the axiom, $E \subseteq F$ implies that $K(E) \subseteq K(F)$. Now $E \subseteq F$ carries the interpretation that E implies F, and so if the agent knows E, she also knows F. It follows that if we were to, say, explain the rules of chess to the agent, she would know the optimal strategy to win chess.

The next two axioms are less contentious. The axiom of knowledge (**KNO**) states that the agent knows E only if E happens. The axiom of transparency (**TRA**) states that if the agent knows E then she also knows that she knows E. The axiom implies that she also knows that she knows that she knows E and so on. Indeed, $K(E)$ implies $K^n(E)$ for all $n \in \mathbb{N}$.

Finally, the axiom of wisdom (**WIS**) implies that if the agent does not know an event then she knows she does not know it. This rules out that the agent can be unaware of any possibilities.

Proposition 1 (Bacharach, 1985)**.** *A knowledge operator K satisfies* (**AWA**)*,* (**OMN**)*,* (**KNO**)*,* (**TRA**) *and* (**WIS**) *iff it is induced by a partitional information function.*

Proof. First suppose K is induced by a partitional information function h. Since $h(\omega) \subseteq \Omega$ for all $\omega \in \Omega$, and thus $K(\Omega) = \Omega$, (**AWA**) holds. Suppose $E, F \in \mathcal{F}$. If $h(\omega) \subseteq E \cap F$, then $h(\omega) \subseteq E$ and $h(\omega) \subseteq F$. Conversely, if $h(\omega) \subseteq E$ and $h(\omega) \subseteq F$, then $h(\omega) \subseteq E \cap F$. Thus $\omega \in K(E \cap F)$ iff $\omega \in K(E) \cap K(F)$, so (**OMN**) holds. Third, if $h(\omega) \subseteq E$ then $\omega \in E$ by (I1), and so $K(E) \subseteq E$, i.e. (**KNO**) holds. Fourth, by (I2), for any $\omega' \in h(\omega)$, we have that $h(\omega') \subseteq E$ also, and so $h(\omega) \subseteq K(E)$. Thus if $\omega \in K(E)$ then $\omega \in K(K(E))$, so (**TRA**) holds. Finally, $\neg K(E) = \{\omega \in \Omega \mid h(\omega) \subseteq E\}^c$. If $h(\omega) \cap E = \varnothing$, then $h(\omega) \subseteq E^c \subseteq \neg K(E)$, and so $\omega \in K(\neg K(E))$. If $h(\omega) \cap E \neq \varnothing$ but $h(\omega) \not\subseteq E$, then $K(E) \cap h(\omega) = \varnothing$ by (I1)-(I2), and hence $h(\omega) \subset \neg K(E)$, so $\omega \in K(\neg K(E))$. It follows that (**WIS**) holds.

See Bacharach (1985) for proof of the converse, though a caution that his notation is messy. □

1.5.2 Common knowledge

Definition 8 (Common knowledge)**.** Consider a finite set $\mathcal{I}$ of n agents with partitional information functions $h_1, \dots, h_n$ and corresponding knowledge operators $K_1, \dots, K_n$.

(a) *Mutual knowledge.* We say an event $E \in \mathcal{F}$ is *mutual knowledge* in state $\omega \in \Omega$ if $\omega \in \bigcap_{i=1}^{n} K_i(E)$, that is, if E is known to all agents in state ω. If E is mutual knowledge, we write $K^1(E)$.

Recursively, let $K^k(E) := K^1(K^{k-1}(E))$. Hence $K^2(E)$ denotes that mutual knowledge of E is mutual knowledge, and so on

(b) *Common knowledge.* We say an event $E \in \mathcal{F}$ is *common knowledge* in state $\omega \in \Omega$ if $\omega \in \bigcap_{k=1}^{\infty} K^k(E)$. That is, if E is known to all agents, all agents know E is known to all agents, and so on.

(c) *Self-evident events.* We say an event $E \in \mathcal{F}$ is *self-evident* if for all $\omega \in E$ and all $i \in \mathcal{I}$, we have $h_i(\omega) \subseteq E$.

If Ω is finite, an equivalent definition of common knowledge can be stated in terms of self-evident events.

Lemma 2. *The following statements are equivalent for any $E \in \mathcal{F}$:*

(i) *$K_i(E) = E$ for all $i \in \mathcal{I}$;*

(ii) *E is a self-evident event;*

(iii) *for all $i \in \mathcal{I}$, E is a union of members of the partition of Ω induced by h_i.*

Proof. E is self-evident iff $E \subseteq K_i(E)$ for all $i \in \mathcal{I}$, and by (**KNO**), $K_i(E) \subseteq E$. Hence (i) and (ii) are equivalent. Now, if E is self-evident then by definition, $\omega \in E$ implies $h(\omega) \subseteq E$, so $E = \bigcup_{\omega \in E} h_i(\omega)$ for each $i \in \mathcal{I}$. Thus (ii) implies (iii). If E is a union of the members of the partition induced by h_i, then $h_i(\omega) \subseteq E$ iff $\omega \in E$, and so (iii) implies (i). □

Proposition 2. *Suppose Ω is a finite set. An event $E \in \mathcal{F}$ is common knowledge in a state $\omega \in \Omega$ iff there exists a self-evident event $F \subseteq E$ such that $\omega \in F$.*

Proof. Suppose E is common knowledge in state ω. By (**TRA**), $E \supseteq K^1(E) \supseteq K^2(E) \supseteq \cdots$, and $\omega \in E$, $\omega \in K^k(E)$ for all $k \in \mathbb{N}$. Given Ω is finite, there exists a set $F = K^k(E)$ for some k such that $K_i(F) = F$ for all $i \in \mathcal{I}$. Now $F \subseteq E$ and $\omega \in F$, and F is self-evident since $F \subseteq K_i(F)$ for all i.

Conversely, suppose $E \in \mathcal{F}$ is such that there exists some self-evident event $F \subseteq E$ with $\omega \in F$. Then $K_i(F) = F$ for all $i \in \mathcal{I}$, so $K^1(F) = F$ and thus $K^1(F)$ is self-evident. Iterating, it follows that $K^k(F) = F$ is self-evident for all $k \in \mathbb{N}$. Since $F \subseteq E$, by (**OMN**), we have that $F \subseteq K^k(E)$ for any k, and $\omega \in F$. Thus $\omega \in K^k(E)$ for all k, so $\omega \in \bigcup_{k=1}^{\infty} K^k(E)$, and hence E is common knowledge in state ω. □

Common knowledge plays an important role in classifying games:

Definition 9 (Games and information)**.**

(a) *Perfect and imperfect information.* An extensive form game Γ is a game of *perfect information* if all information sets are singletons; that is, if all past moves are common knowledge. Otherwise, Γ is a game of *imperfect information.*

(b) *Structure.* The *structure* of a game G (Γ) consists of all elements listed in the normal form tuple G or in the extensive form tuple Γ respectively.

(c) *Complete and incomplete information.* A (normal or extensive form) game Γ is a game of *complete information* if the structure of the game is common knowledge. Otherwise, Γ is a game of *incomplete information.*

1.5.3 Aumann's agreement theorem

The formalization of common knowledge has spawned several famous, possibly surprising theorems. Famously, *Aumann's agreement theorem* says that no two agents with a common prior, whose posterior beliefs are common knowledge, will ever disagree about an event.

Definition 10. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are information partitions of Ω, define the *join* $\mathcal{P}_1 \vee \mathcal{P}_2$ to be the coarsest common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, and define the *meet* $\mathcal{P}_1 \wedge \mathcal{P}_1$ to be the finest common coarsening of $\mathcal{P}_1$ and $\mathcal{P}_2$.[4]

Let $(\Omega, \mathcal{F}, p)$ be a probability space, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be information partitions, for agents 1 and 2 respectively, such that $\mathcal{P}_1 \vee \mathcal{P}_2$ consists of only non-null events, i.e. $p(E) > 0$ for all $E \in \mathcal{P}_1 \vee \mathcal{P}_2$. We interpret p as the *common prior* of agents 1 and 2.

Corollary 1. *An event E is common knowledge in state $\omega \in \Omega$ if the member $F \in \mathcal{P}_1 \wedge \mathcal{P}_2$ for which $\omega \in F$ is s.t. $F \subseteq E$.*

Proof. F is a self-evident event, and the 'if' direction in Proposition 2 holds generally (note that part of the proof does not rely on finiteness of Ω). $\square$

Fixing an event $E \in \mathcal{F}$, we denote the posterior probability for agent i by

$$q_i(\omega) = \frac{p(E \cap h_i(\omega))}{p(h_i(\omega))}.$$

Theorem 2 (Aumann, 1976)**.** *Let $\omega \in \Omega$. If it is common knowledge in state ω that $q_1(\omega) = \bar{q}_1$ and $q_2(\omega) = \bar{q}_2$ for numbers $\bar{q}_1$ and $\bar{q}_2$, then $\bar{q}_1 = \bar{q}_2$.*

Proof. Let $P \in \mathcal{P}_1 \wedge \mathcal{P}_2$ be such that $\omega \in P$. We can write $P = \bigcup_j P^j$ for a disjoint collection $\{P^j\} \subseteq \mathcal{P}_1$. Since $q_1 = \bar{q}_1$ on P, we have that $\bar{q}_1 = \frac{p(E \cap P^j)}{p(P^j)}$ for all j, giving $p(E \cap P^j) = \bar{q}_1 p(P^j)$. By countable additivity, $p(E \cap P) = \bar{q}_1 p(P)$.

Repeating the argument, we can write $P = \bigcup_k P^k$ for a disjoint collection $\{P^k\} \subseteq \mathcal{P}_2$ and follow the same steps to obtain $p(E \cap P) = \bar{q}_2 p(P)$. Hence $\bar{q}_1 = \bar{q}_2$. $\square$

Note the theorem generalizes beyond two agents (Bacharach, 1985; Rubinstein & Wolinsky, 1990, and Samet, 1990). The conclusion is that if agents share the same priors, they cannot "agree to disagree". Of course, people disagree a lot in real life, and we could put this down to differences in individuals' subjective priors. Yet Aumann's

[4] A *refinement* $\mathcal{P}'$ of a partition $\mathcal{P}$ is a partition such that every element of $\mathcal{P}'$ is the subset of some element of $\mathcal{P}$. We say that $\mathcal{P}'$ is a *coarsening* of $\mathcal{P}$ if $\mathcal{P}$ is a refinement of $\mathcal{P}'$. We say that $\mathcal{P}$ is a *common refinement* (*common coarsening*) of $\mathcal{P}_1$ and $\mathcal{P}_2$ if it is a refinement (coarsening) of both $\mathcal{P}_1$ and $\mathcal{P}_2$. A partition $\mathcal{P}$ is the *coarsest common refinement* of $\mathcal{P}_1$ and $\mathcal{P}_2$ if $\mathcal{P}$ is a the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ such that any other common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ is a refinement of $\mathcal{P}$. Likwise, $\mathcal{P}$ is the *finest common coarsening* of $\mathcal{P}_1$ and $\mathcal{P}_2$ if it is a common coarsening such that it is the refinement of any other common coarsening of $\mathcal{P}_1$ and $\mathcal{P}_2$.

agreement poses a challenge to Harsanyi's (1968) argument that differences in subjective priors ought to only come about from differences in information. If this were true, then two reasonable people with the same information who have common knowledge of each others posterior beliefs should never disagree – but in practice, disagreements among expert colleagues with access to the same evidence is very common. Of course, it could be that reasonable people nevertheless ascribe errors to the calculation of each others' posterior distributions, arising due to systematic biases (see e.g. the behavioural literature).

1.5.4 No trade?

The agreement theorem undergirds a group of results that are known as the no-trade theorems – Kreps (1977), Milgrom & Stokey (1982), Tirole (1982) and Rubinstein & Wolinsky (1990). These effectively rule out that rational risk-averse traders with common knowledge of rationality can engage in speculative trades unless they have different priors. We focus on the Milgrom-Stokey no-trade theorem, and follow the simple version presented in Levin's notes.

Suppose there are two agents. Let $(\Omega, \mathcal{F}, p)$ be a probability space with Ω finite, let X be a set of trading outcomes, and assume each agent's information function is partitional.

We define a *contingent contract* to be a measurable function $a : \Omega \to X$. We let A be the space of contingent contracts. Each agent i has a utility function $u_i : X \times \Omega \to \mathbb{R}$, and the agent's utility from a contract a is the random variable $U_i(a)(\omega) = u_i(a(\omega), \omega)$. We denote i's expectation of $U_i(a)$ given her information H_i by $\mathbb{E}[U_i(a) \mid H_i]$.

Proposition 3. *Let ϕ be a random variable on $(\Omega, \mathcal{F}, p)$. If p is the common prior of i and j, and the posterior distributions of both agents are common knowledge, then it cannot be common knowledge that i's expectation of ϕ is strictly greater than j's expectation of ϕ.*

Proof. Fix $\omega \in \Omega$. Let $E(t) = \{\phi \leq t\}$ for any $t \in \mathbb{R}$. Define $q_i(\omega)(E) = \frac{p(E \cap h_i(\omega))}{p(h_i(\omega))}$ for each $E \in \mathcal{F}$ and $i = 1, 2$. Suppose these posterior distributions are common knowledge. Then by Aumann's agreement theorem (Theorem 2), $q_1(\omega) = q_2(\omega)$ everywhere. Now, $\mathbb{E}[\phi \mid h_i(\omega)] = \int_\Omega \phi \, \mathrm{d}q_i(\omega)$ for each i,[5] and thus $\mathbb{E}[\phi \mid h_1(\omega)] = \mathbb{E}[\phi \mid h_2(\omega)]$. □

Call a contingent contract b *ex ante efficient* if there does not exist a contract $a \in A$ such that $\mathbb{E}[U_i(a)] > \mathbb{E}[U_i(b)]$ for both agents i.

Theorem 3 (Milgrom-Stokey, 1982)**.** *If a contingent contract b is ex ante efficient, then it cannot be common knowledge that every agent prefers some contract a to b.*

Proof. Let $E = \{\omega \in \Omega \mid \mathbb{E}[U_i(a) \mid h_i(\omega)] > \mathbb{E}[U_i(b) \mid h_i(\omega)] \text{ for all } i\}$. The theorem states there is no state ω in which E is common knowledge. Suppose otherwise. By

[5] Note this integral is taken with respect to the measure $q_i(\omega)$, *not* with respect to ω.

Proposition 2 there is a self-evident set $F \subseteq E$ with $\omega \in F$. By definition, for all $\omega' \in F$, $h_i(\omega') \subseteq F$, and thus for all $\omega' \in F$ and all i, we have $\mathbb{E}[U_i(a) - U_i(b) \mid h_i(\omega')] > 0$.

Since h_i is partitional and Ω is finite, we can write F as a disjoint union $F = \bigcup_{k=1}^{n} h_i(\omega_k)$ for some set of states $\{\omega_k\}_{k=1}^{n} \subseteq F$. Thus $\mathbb{E}[U_i(a) - U_i(b) \mid F] > 0$ for all i. That is, a strictly Pareto dominates b on F (c.f. Definition 11). Now consider the contract c defined so that $c(\omega') = a(\omega')$ for each $\omega' \in F$ and $c(\omega') = b(\omega')$ for each $\omega' \notin F$. Then c yields the same payoff as b outside F and strictly better payoff than b on F, so b cannot be *ex ante* efficient, yielding a contradiction. $\square$

The Milgrom-Stokey no-trade theorem rules out the possibility of speculative trades under conditions that are (or were) standard in models of financial markets – namely that traders are rational Bayesian agents, traders have common priors, it is common knowledge when a trade takes place that it is feasible and mutually acceptable to both parties, and that markets are in an efficient equilibrium. An interesting implication of the theorem is that receipt of private information by a trader is not helpful, because they cannot find a counterparty to agree on a favourable trade – the only motive for trading in the Milgrom-Stokey setting is speculative, and a trader only places a speculative bet if they have private information, but then the counterparty can infer that the trader has better information and so will refuse to meet the bet.

In practice, the predictions of the no-trade theorem obviously do not hold – trade volumes in financial markets are very high. There are several ways we might accommodate this fact. The first is to suppose that some traders are noise traders, that is, non-rational traders who trade anyway. These traders create trade possibilities via two channels – first, rational traders can make gains from the losses of the noise traders, and secondly, rational traders may be willing to place trades between themselves because there is uncertainty over whether their counterparty is rational or a noise trader (thus a rational trader with private information can profit by trading with a rational trader without this information, for example). We could instead relax more fundamental assumptions, such as rationality more generally. For example, as with Aumann's agreement theorem, the no-trade theorem fails if agents have systematic biases. Thirdly, we can weaken the assumption of common priors. Feinberg (2000) shows that under heterogeneous priors, there always exists some purely speculative trade that all agents would be willing to make.

Example 7 (Harrison & Kreps, 1978)**.** Harrison & Kreps (1978) present a nice example (and a formal model) to ilustrate how speculative trades become possible when agents have heterogeneous priors. Suppose there are two types of risk-neutral investor, $i = 1, 2$, and both have a common discount factor $\delta = \frac{3}{4}$. Investors can purchase a stock that pays a dividend, and in every period t, the dividend is either $d_t = 0$ or $d_t = 1$. This is the state, and so the state space is $D = \{0, 1\}$. The dividend process is perceived by both types to follow a stationary Markov process, but the types disagree on transition probabilities, with type i believing the transition matrix is Q_i, with

$$Q_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad \text{and} \qquad Q_2 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

First, consider the value $v_i(d)$ to each type i of investor of buying a unit of the stock when the current state is d and holding it forever. We have that

$$\begin{aligned} p_1(0) &= \frac{4}{3} = 1.33, \quad & p_1(1) &= \frac{11}{9} = 1.22, \\ p_2(0) &= \frac{16}{11} = 1.45, \quad & p_2(1) &= \frac{21}{11} = 1.91. \end{aligned}$$

The second type of investor always values holding the stock more than the first type, regardless of the current state. Nevertheless, there are opportunities for trade. In state 1, type 2 investors are optimistic that they will receive dividends in the future, since they assess that $d_{t+1} = 1$ with probability $\frac{3}{4}$. Type 1 investors are pessimistic about future dividends in state 1, but are unable to short-sell the stock. In state 0, however, type 1 investors are optimistic about a transition to state 1 relative to type 2 investors. Type 1 investors thus have an opportunity to trade by purchasing stock in state 0 and selling it to type 2 investors in state 1, realizing capital gains. We can thus expect that in equilibrium, the stock changes hands between investors: when there is a transition to state 1, type 1 investors sell the stock to type 2 investors, and when there is a transition to state 0, type 2 investors sell to type 1 investors.

Harrison & Kreps (1978) use the notion of *consistent equilibrium*, which imply the price $p_t(d_t)$ of the stock at time t should satisfy

$$p_t(d_t) = \max_k \delta \sum_{d_{t+1} \in D} [d_{t+1} + p_t(d_{t+1})] Q_k(d_t, d_{t+1}).$$

That is, the price of the stock in state d is the maximum discounted expected return across investors from buying and holding the asset for a single period. Note the price of the stock will be stationary, since the dividend process is stationary, and thus $p_t(d) = p_{t+k}(d)$ for any $k \in \mathbb{Z}$ and any $d \in D$. We have

$$\begin{aligned} p(0) &= \max \left\{ \frac{3}{4} \left[\frac{1}{2} p(0) + \frac{1}{2}(1 + p(1)) \right], \ \frac{3}{4} \left[\frac{2}{3} p(0) + \frac{1}{3}(1 + p(1)) \right] \right\}, \\ p(1) &= \max \left\{ \frac{3}{4} \left[\frac{2}{3} p(0) + \frac{1}{3}(1 + p(1)) \right], \ \frac{3}{4} \left[\frac{1}{4} p(0) + \frac{3}{4}(1 + p(1)) \right] \right\}. \end{aligned}$$

Solving this system gives $p(0) = \frac{24}{13} = 1.85$ and $p(1) = \frac{27}{13} = 2.04$. The price in both states is greater than the valuation "on fundamentals" of both types of investor! The type 2 investor is willing to pay in excess of their "fundamental" valuation in state 1 because they know they can sell the stock to type 1 investors in state 0 at a higher price than her valuation in that state. The type 1 investor, meanwhile, is willing to pay a high price in state 0 knowing she can sell it for a much higher price to type 2 investors in state 1.

2 Games of complete information

First, a bit about solution concepts. Solution concepts serve several purposes:

- *Descriptive.* A solution concept to a game can aim to predict strategies that players will play in practice. A large empirical literature looks at behaviour of players in experimental settings (e.g. Güth et al (1982) study experimental evidence of behaviour in the ultimatum game, Forsythe et al. (1994) likewise analyse behaviour in practice for the dictator game) and non-experimental empirical settings (e.g. the empirical IO literature)

- *Normative.* A solution concept may aim to prescribe which strategies a rational player would play. They can thus provide a guide to action.

- *Theoretical.* Given certain assumptions about players' behaviour, a solution concept can aim to predict behaviour under those assumptions (without making a broader claim that those assumptions are accurate in practice)

The desirability of a solution concept lies in several properties:

- A solution concept should be reasonable in the sense that the assumptions about behaviour on which it relies are reasonable assumptions about agents (descriptively or normatively). While it might not always seem like it, we are trying to build a theory of how people actually interact strategically in the real world here.[6]

- A solution concept should apply to a sufficiently broad class of games, or else it seems ad hoc.

- A solution concept ideally gives a clear-cut – and thus falsifiable – prediction about behaviour. This doesn't need to be a unique prediction necessarily, but some sufficiently small set to be useful. In practice, multiplicity of equilibria is the norm, and typically we might want to make equilibrium selection arguments or rely on refinements to narrow down the range of "reasonable" equilibria.

2.1 Efficiency

Definition 11.

(a) *Pareto dominance.* In the game $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$, a strategy profile s *Pareto dominates* a strategy profile s' if

$$u_i(s) \geq u_i(s') \quad \text{for all } i \in \mathcal{I}, u_i(s) \quad > u_i(s') \quad \text{for some } i \in \mathcal{I}.$$

(b) *Pareto efficiency.* A strategy s is *Pareto efficient* if there is no strategy profile s' s.t. s' Pareto dominates s.

(c) *Strong efficiency.* A strategy s is *strongly efficient* if it is the solution to

$$\max_{s \in S} \sum_{i \in \mathcal{I}} u_i(s),$$

that is, if it maximizes total payoff.

[6] And often successfully so! See https://twitter.com/ben_golub/status/1611917935399796743

The definition of strong efficiency is taken from Jackson & Wolinsky (1996), who use it in the context of network formation games. In general, we cannot expect our solution concepts to give us Pareto efficient – let alone strongly efficient – solutions. One exception is games with transferable utility (i.e. where players can costlessly transfer portions of their payoffs to other players) – here, appropriate solution concepts will typically give us strongly efficient solutions.

2.2 Zero sum and matrix games

Definition 12 (Zero sum game)**.** A *zero sum game* is a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$ s.t.

$$\sum_{i\in\mathcal{I}} u_i(s) = 0 \qquad \text{for all } s \in S = \prod_{i\in\mathcal{I}} S_i.$$

The zero sum games constitute a large class of games. Zero sum games are games of pure competition – comparing any two strategies, any increase in the payoff of some player is matched by a decrease in the aggregate payoff of the other players.

Corollary 2. *If $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$ is a zero sum game, the corresponding mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ is a zero sum game. That is,*

$$\sum_{i\in\mathcal{I}} u_i(\sigma) = 0 \qquad \textit{for all } \sigma \in \prod_{i\in\mathcal{I}} \Delta(S_i).$$

Proof. By linearity of the Lebesgue integral,

$$\sum_{i\in\mathcal{I}} u_i(s) = \sum_{i\in\mathcal{I}} \int_S u_i(s)\,\mathrm{d}\sigma = \int_S \left[\sum_{i\in\mathcal{I}} u_i(s)\right] \mathrm{d}\sigma = 0.$$

□

Any outcome of a zero sum game is clearly Pareto optimal.

Finite two-person zero sum games are representable as *matrix games.*

Definition 13 (Matrix game)**.** A *matrix game* is a real $m \times n$ matrix A, where m is the number of actions for Player 1 and n is the number of actions for Player 2. A mixed strategy of Player 1 is an m-dimensional probability vector p, and the set of mixed strategies of Player 1 is

$$\Delta^m := \left\{ p \in \mathbb{R}_+^m \mid \sum_{i=1}^m p_i = 1 \right\}.$$

A mixed strategy of Player 2 is an n-dimensional probability vector q, and the set of mixed strategies of Player 2 is

$$\Delta^n := \left\{ q \in \mathbb{R}_+^n \mid \sum_{i=1}^n q_i = 1 \right\}.$$

We call a strategy p or q of a matrix game a *pure strategy* if there is an entry of p or q with value 1. We denote the vector with ith entry 1 and all other entries 0 by e^i.

2.3 Maxmin and minmax

Definition 14.

(a) *Maximin.* Given a game $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$, the pure strategy *maximin payoff* for player i is given by

$$w_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

A pure strategy $\alpha^i = (\alpha^i_i, \alpha^i_{-i}) \in S$ is a *maximin solution* for player i if

$$(\alpha^i_i, \alpha^i_{-i}) = \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

That is, the maximin payoff is the payoff attained at the maximin solution, $w_i = u_i(\alpha^i_i, \alpha^i_{-i})$.

Given the mixed game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$, the mixed strategy maximin payoff for player i is given by

$$w^m_i = \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} u_i(\sigma_i, \sigma_{-i}).$$

(b) *Minimax.* Given a game $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$, the pure strategy *minimax payoff* for player i is given by

$$v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

A pure strategy $\gamma^i = (\gamma^i_i, \gamma^i_{-i}) \in S$ is a *minimax solution* for player i if

$$(\gamma^i_i, \gamma^i_{-i}) = \arg\min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

That is, the minimax payoff is the payoff attained at the minimax solution, $v_i = u_i(\gamma^i_i, \gamma^i_{-i})$.

Given the mixed game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$, the mixed strategy minimax payoff for player i is given by

$$v^m_i = \min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}).$$

The maximin payoff for i is the largest payoff player i can guarantee themselves in the absence of any knowledge about their opponent's strategy. The minimax payoff for i, by contrast, is the least payoff that opponents $-i$ can enforce on player i.

Lemma 3. *Consider the mixed game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$, and let w^m_i and v^m_i respectively denote the mixed strategy maximin and minimax payoffs for player i. Then*

$$v^m_i \geq w^m_i.$$

Proof. For any σ_i, σ'_{-i},

$$u_i(\sigma_i, \sigma'_{-i}) \geq \min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} u_i(\sigma_i, \sigma_{-i}).$$

Hence

$$\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma'_{-i}) \geq \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} u_i(\sigma_i, \sigma_{-i}),$$

and so

$$\min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma'_{-i}) \geq \max_{\sigma_i \in \Delta(S_i)} \min_{\sigma_{-i} \in \Delta_{-i}(S_{-i})} u_i(\sigma_i, \sigma_{-i})$$

□

In finite-strategy zero sum games, the mixed strategy maximin and minimax solutions for each player are equivalent. Indeed, mixed strategy maximin is a natural solution concept in this setting:

Theorem 4 (Minimax theorem; von Neumann, 1928)**.** *In any finite two player zero sum game* $G = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$,

$$v_1^m = w_1^m = v_2^m = w_2^m.$$

Proof. Recall we can write any finite two player zero sum game as an $m \times n$ matrix game A. First we claim that $w_1^m = v_2^m$. Let p_1, q_1 be the choices of $p \in \Delta^m$ and $q \in \Delta^n$ that solve $w_1^m = \max_{p \in \Delta^m} \min_{q \in \Delta^n} p'Aq$ and let p_2, q_2 be the choices of $p \in \Delta^m$ and $q \in \Delta^n$ that solve $v_2^m = \min_{q \in \Delta^n} \max_{p \in \Delta^m} p'Aq$. Now, we have

$$w_1^m = p_1'Aq_1 \leq p_2'Aq_1 \leq p_2'Aq_2 = v_2^m,$$

so $w_1^m \leq v_2^m$.

Suppose $w_1^m > v_2^m$. We prove this yields a contradiction using the lemma of the alternative for matrices, Lemma 22. Let B be an arbitrary $m \times n$ matrix game. Let $w_1(p) = \min_{q \in \Delta^n} p'Bq$, let $w_1(B) = \max_{p \in \Delta^m} w_1(p)$, let $v_2(q) = \max_{p \in \Delta^m} p'Bq$, and let $v_2(B) = \min_{q \in \Delta^n} v_2(q)$.

Recall from the lemma that precisely one of the following must hold:

(a) There exist $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ s.t. $(y, z) \geq 0$, $(y, z) \neq 0$ and $By + z = 0$;

(b) There is an $x \in \mathbb{R}^m$ s.t. $x > 0$ and $x'B > 0$.

Suppose (a) holds, so there exists $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, at least one of which nonzero, such that $(y, z) \geq 0$ and $By + z = 0$. If $y = 0$ then $z = 0$, yielding a contradiction, so $y \neq 0$ and $\sum_{k=1}^n y_k > 0$. Define $q \in \Delta^n$ so that $q_j = \frac{y_j}{\sum_{k=1}^n y_k}$ for each $j = 1, \ldots, n$. It follows that $Bq = -\frac{z}{\sum_{k=1}^n y_k} \leq 0$. Thus $v_2(q) \leq 0$, so $v_2(B) \leq 0$.

Suppose (b) holds. Then there exists $x \in \mathbb{R}^m$ such that $x > 0$ and $x'B > 0$. Define $p \in \Delta^m$ so that $p = \frac{x}{\sum_{k=1}^m x_k}$. Then $w_1(p) > 0$ and so $w_1(B) > 0$. It follows that we cannot have $w_1(B) \leq 0 < v_2(B)$, since at least one of (a) and (b) must hold.

Now define B so that each ijth entry of B is $B_{ij} = A_{ij} - w_1(A)$. Then $w_1(B) = v_1(A) - v_1(A) = 0$ and $v_2(B) = v_2(A) - v_1(A) > 0$. Hence $w_1(B) \leq 0 < v_2(B)$, yielding a contradiction.

Hence we conclude that $w_1^m = v_2^m$. An identical argument shows that $v_1^m = w_2^m$. The equality $v_1^m = w_1^m = v_2^m = w_2^m$ now follows from Lemma 3. $\square$

We call $v := v_1^m = v_2^m = w_1^m = w_2^m$ the *value of the game* G. Von Neumann's minimax theorem as stated above is a special case of the following, more general version:

Theorem 5 (Minimax theorem)**.** *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be nonempty, convex, compact sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function s.t.*

(i) *f is concave in its first argument, that is, for each $y \in Y$, $g(x) = f(x, y)$ is concave, and*

(ii) *f is convex in its second argument, that is, for each $x \in X$, $h(y) = f(x, y)$ is convex,*

then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Note there are a number of generalizations – Sion's minimax theorem and Parthasarthy's theorem, to name the two most important examples. More generally, the value of a game

The value of the game need not exist once we allow for infinite strategy spaces.

Example 8 (Game without a value, Sion & Wolfe, 1957)**.** Suppose Player 1 chooses a number $x \in [0, 1]$ and Player 2 chooses a number $y \in [0, 1]$. Player 1 receives payoff

$$\pi(x, y) = \begin{cases} -1 & \text{if } x < y < x + \frac{1}{2}, \\ 0 & \text{if } x = y \text{ or } y = x + \frac{1}{2}, \\ 1 & \text{otherwise,} \end{cases}$$

and Player 2 receives payoff $-\pi(x, y)$.

This is a kind of continuous *Colonel Blotto game*: imagine that Player 1 assigns a fraction x of their forces to attack one mountain pass and $1 - x$ to attack the other. Player 2 assigns a fraction y to defend the first mountain pass and $1 - y$ to defend the second, at which a second permanent garrison of $\frac{1}{2}$ is also located. A player receives payment 1 from their opponent at each pass if their forces are larger than their opponent.

If the value of this game exists, then it is the value

$$\sup_f \inf_g \iint \pi \, \mathrm{d}f \, \mathrm{d}g = \inf_g \sup_f \iint \pi \, \mathrm{d}f \, \mathrm{d}g.$$

However, it can be shown that $\sup_f \inf_g \iint \pi \, \mathrm{d}f \, \mathrm{d}g = \frac{1}{3} \neq \frac{3}{7} = \inf_g \sup_f \iint \pi \, \mathrm{d}f \, \mathrm{d}g$.

Example 9 (Matching Pennies)**.** Matching Pennies is a zero sum game in which each of two players simultaneously announce heads (H) or tails (T). Player 1 receives a payment from Player 2 if the two announcements match, and Player 2 receives a payment from Player 1 if the two announcements differ.

	H_2	T_2
H_1	$1,-1$	$-1,1$
T_1	$-1,1$	$1,-1$

Consider the mixed strategies $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$. Consider the maxmin strategy for Player 1. Player 1's expected payoff is

$$u_1(p, q) = p[q - (1-q)] + (1-p)[(1-q) - q] = (1-2p)(1-2q).$$

Given $p \in [0,1]$, Player 2's problem (in order to minimize Player 1's payoff) is

$$\begin{aligned}
\min_{q\in[0,1]} u_1(p,q) &= \begin{cases} 1-2p & \text{if } p \le \frac{1}{2}, \\ 2p-1 & \text{if } p > \frac{1}{2}, \end{cases} \\
&= \min\{u_1(p,0), u_1(p,1)\} \\
&= \min\{1-2p, 2p-1\} \\
&= -|1-2p| \, .
\end{aligned}$$

Player 1's mixed maxmin strategy thus solves

$$w_1^m = \max_{p\in[0,1]} \min_{q\in[0,1]} u_1((p,1-p),(q,1-q)) = \max_{p\in[0,1]} -|1-2p| = 0,$$

for $(p,q) = (\frac{1}{2}, \frac{1}{2})$. The game is symmetric, so Player 2's maxmin payoff is also $w_2^m = 0$.

Likewise, given $q \in [0,1]$, Player 1's problem is

$$\begin{aligned}
\max_{p\in[0,1]} u_1(p,q) &= \begin{cases} 1-2q & \text{if } q \le \frac{1}{2}, \\ 2q-1 & \text{if } q > \frac{1}{2}, \end{cases} \\
&= \max\{u_1(0,q), u_1(1,q)\} \\
&= \max\{1-2q, 2q-1\} \\
&= |1-2q| \, .
\end{aligned}$$

Player 2's minmax strategy solves

$$w_1^m = \min_{q\in[0,1]} \max_{p\in[0,1]} u_1((p,1-p),(q,1-q)) = \min_{q\in[0,1]} |1-2q| = 0,$$

for $(p,q) = (\frac{1}{2}, \frac{1}{2})$. The game is symmetric, so we also have $w_2^m = 0$.

We see that here, $v_i^m = w_i^m$. This is of course as predicted by von Neumann's minmax theorem.

2.4 Strict dominance

Strict dominance captures the idea that rational (i.e. expected payoff-maximizing) players will never play strategies that perform uniformly worse than some alternative strategy.

Definition 15 (Strict dominance)**.**

(a) *Strict dominance in pure strategies.* In a pure strategy game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a pure strategy $s_i \in S_i$ for player i is said to *strictly dominate* a strategy $s_i' \in S_i$ if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

We say that a strategy s_i' is *strictly dominated* if there exists some strategy s_i that strictly dominates s_i'.

We say that a strategy $s_i \in S_i$ is *strictly dominant* if s_i strictly dominates every $s_i' \in S_i - \{s_i\}$.

(b) *Strict dominance in mixed strategies.* In a mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, a mixed strategy $\sigma_i \in \Delta(S_i)$ is said to *strictly dominate* a strategy $\sigma_i' \in \Delta(S_i)$ if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}) \qquad \text{for all } \sigma_{-i} \in \Delta_{-i}(S_{-i}).$$

We say that σ_i' is *strictly dominated* if there exists some strategy σ_i that strictly dominates σ_i'.

Equivalently, by linearity of expectations, σ_i strictly dominates σ_i' if

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

(c) *Strict dominance of a pure strategy by a mixed strategy.* In a mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, a mixed strategy $\sigma_i \in \Delta(S_i)$ is said to *strictly dominate* pure strategy s_i if

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

(d) *Strictly dominant strategy equilibrium.* A strategy profile $s^* \in S$ ($\sigma^* \in \prod_{i\in\mathcal{I}} \Delta(S_i)$) is a *strictly dominant strategy equilibrium* if, for each player i, $s_i^* \in S_i$ ($\sigma_i^* \in \Delta(S_i)$) is a strictly dominant strategy.

Proposition 4. *In a game G, if s^* (σ^*) is a strictly dominant strategy equilibrium, then it is the unique strictly dominant strategy equilibrium.*

Proof. Wlog, consider a pure strategy game. Suppose s^* and s' are strictly dominant strategy equilibria and that $s^* \neq s'$. Then for some i, $s_i^* \neq s_i'$. Since s_i^* and s_i' are both strictly dominant strategies for i, we have

$$u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{and} \quad u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i},$$

a clear contradiction. □

A strictly dominant strategy equilibrium requires only that each player is rational. Beliefs about other players are irrelevant. However, it applies only to a very small class of games: those in which every player has a strictly dominant strategy. In general, a strictly dominant strategy equilibrium need not exist. The Prisoner's Dilemma (Example 2(a)) has a strictly dominant strategy equilibrium (D, D), for example, but the market entry game in Example 3 does not.

Definition 16 (Iterated strict dominance)**.**

(a) *Level-k rationality.* An assumption of *level-*1 *rationality* is that all players are rational (i.e. expected payoff-maximizing). We say that players $\mathcal{I}$ are *level-*2 *rational* if they are all rational (level-1) and know that all other players are rational. Iteratively, we say players are *level-k rational* if they know that all players are level-$(k-1)$ rational.

(b) *Iterated strict dominance.* Let $\mathcal{D}_i^0 = S_i$ for all i, and define

$$\mathcal{D}_i^k = \left\{ s_i \in \mathcal{D}_i^{k-1} \mid \nexists \sigma_i \in \Delta(\mathcal{D}_i^{k-1}) \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{D}_{-i}^{k-1} \right\},$$

where $\mathcal{D}_{-i}^{k-1} = \prod_{j \neq i} \mathcal{D}_j^{k-1}$. Clearly, $\mathcal{D}_i^k \subseteq \mathcal{D}_i^{k-1}$ for all $k \geq 1$.

We call $\mathcal{D}_i^k$ the set of pure strategies that survive k rounds of iterated strict dominance for player i. We might also refer to this as the level-k iterated weak dominance set.

We call $\mathcal{D}_i = \bigcap_{k=0}^{\infty} \mathcal{D}_i^k$ the *set of pure strategies that survive iterated strict dominance* for player i, and $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ the *set of pure strategy profiles that survive iterated strict dominance.*

The set of mixed strategies that survive iterated strict dominance is defined analogously.

We call the set $\mathcal{D}$ the iterated strict dominance solution.

Iterated strict dominance is often known by the more verbose but more descriptive name "iterated deletion of strictly dominated strategies". Shorter names are preferable so we don't call it that.

If the structure of the game and rationality are common knowledge, then no player i will choose a strategy that is not contained in $\mathcal{D}_i$, i.e. that does not survive iterated strict dominance. The predictive power of iterated strict dominance is limited – indeed, in some games $\mathcal{D}_i = S_i$ – though not as poor as that of strictly dominant strategy equilibrium.

If the iterated strict dominance solution isolates a unique strategy profile, then the order in which strategies are deleted in iterated strict dominance does not matter, and we have a sharp prediction of equilibrium play. Even if $\mathcal{D}_i$ is not a singleton for all players i, Iterated strict dominance may allow us to considerably reduce the size of the strategy space that we need to consider.

Example 10. Consider the game

	A_2	B_2	C_2	D_2
A_1	$0,7$	$2,5$	$7,0$	$0,1$
B_1	$5,2$	$3,3$	$5,2$	$0,1$
C_1	$7,0$	$2,5$	$0,7$	$0,1$
D_1	$0,0$	$0,-2$	$0,0$	$10,-1$

The strategy D_2 is strictly dominated by mixed strategy $(\frac{1}{2}, 0, \frac{1}{2}, 0)$, and no other strategies of either player are strictly dominated. Hence $\mathcal{D}_1^1 = \{A_1, B_1, C_1, D_1\}$ and $\mathcal{D}_1^2 = \{A_2, B_2, C_2\}$. In the second round, D_1 is strictly dominated by B_1. No other strategies of either player are strictly dominated. Hence $\mathcal{D}_1 = \mathcal{D}_1^2 = \{A_1, B_1, C_1\}$ and $\mathcal{D}_2 = \mathcal{D}_2^2 = \{A_2, B_2, C_2\}$.

2.5 Correlated rationalizability

The notion of *correlated rationalizability* is closely related to iterated strict dominance. We define a player i's *belief* μ_{-i} about opponents' play to be a subjective probability distribution over S_{-i}. That is, $\mu_{-i} \in \Delta(S_{-i})$. Intuitively, μ_{-i} gives the probabilities i attaches to the (possibly correlated) strategy profile of i's opponents.

Definition 17 (Best response)**.** A strategy $s_i \in S_i$ ($\sigma_i \in \Delta(S_i)$) is a *best response*, or *best reply*, to $s_{-i} \in S_{-i}$ ($\sigma_{-i} \in \Delta(S_{-i})$) if

$$\begin{aligned} u_i(s_i, s_{-i}) &\geq u_i(s_i', s_{-i}) \qquad && \text{for all } s_i \in S_i \\ (u_i(\sigma_i, \sigma_{-i}) &\geq u_i(\sigma_i', \sigma_{-i}) && \text{for all } \sigma_i' \in \Delta(S_i).) \end{aligned}$$

We write $B_i(s_{-i})$ ($B_i(\sigma_{-i})$) for the set of i's best responses to s_{-i} (σ_{-i}).

A subset $B_1 \times \cdots \times B_n \subseteq S = S_1 \times \cdots \times S_n$ is called a *best response set* if, for all i and all $s_i \in B_i$, there exists a $\sigma_{-i} \in \Delta(B_{-i})$ s.t. s_i is a best response to σ_{-i}.

Call a subset $B_1^I \times \cdots \times B_n^I \subseteq S$ a *best response set to independent strategies* if, for all i and all $s_i \in B_i^I$, there exists a $\sigma_{-i} \in \Delta_{-i}(B_{-i}^I)$ s.t. s_i is a best response to σ_{-i}.

To check that a mixed strategy σ_i is a best response to some opponents' strategy profile σ_{-i}, we need only compare it to pure strategies s_i:

Lemma 4. *Consider a finite mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$. A mixed strategy $\sigma_i \in \Delta(S_i)$ is a best response to $\sigma_{-i} \in \Delta(S_{-i})$ iff $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$ for all $s_i \in S_i$.*

Proof. Suppose $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$ for all $s_i \in S_i$. Consider any $\sigma_i' \in \Delta_i(S_i)$. Then

$$u(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} u(s_i, \sigma_{-i}) \sigma_i(s_i) \geq \sum_{s_i \in S_i} u(s_i, \sigma_{-i}) \sigma_i'(s_i) = u_i(\sigma_i', \sigma_{-i}),$$

so $\sigma_i \in B_i(\sigma_{-i})$. The converse is immediate by definition. $\square$

Since player i's belief μ_{-i} is a correlated strategy profile, we use $B_i(\mu_{-i})$ to denote i's best response to belief μ_{-i}.

Definition 18 (Rationalizability)**.**

(a) *Correlated rationalizability.* The *set of correlated rationalizable strategies* R is defined as
$$R = R_1 \times \cdots \times R_n = \bigcup_{\alpha} B_1^{\alpha} \times \cdots \times B_n^{\alpha},$$
where each $B^{\alpha} = B_1^{\alpha} \times \cdots \times B_n^{\alpha}$ is a best response set and the union runs through the index set $\{\alpha : B^{\alpha} \text{ is a best response set}\}$.

(b) *Independent rationalizability.* The *set of independent rationalizable strategies* R^I is
$$R^I = \bigcup_{\alpha} B_1^{I,\alpha} \times \cdots \times B_n^{I,\alpha},$$
where each $B^{I,\alpha} = B_1^{I,\alpha} \times \cdots \times B_n^{I,\alpha}$ is a best response set to independent strategies and the union runs through index set $\{\alpha : B^{I,\alpha} \text{ is a best response set to independent strategies}\}$.

Proposition 5. *R is the maximal best response set.*

Proof. Suppose $s_i \in R_i$. Then $s_i \in B_i^{\alpha}$ for some α. Thus s_i is a best response to some $\sigma_{-i} \in \Delta(B_{-i}^{\alpha})$. Since this holds for all i, R is a best response set, and since R contains all best response sets, it is thus maximal. □

Lemma 5 (Myerson, 1990)**.** *In any finite game, a strategy s_i of player i is a best response to belief μ_{-i} iff it is not strictly dominated (by a mixed strategy).*

Proof. See Myerson (1990) for a rigorous proof – this is more a sketch. Fix $s_i \in S_i$ and consider the linear program,

$$\begin{aligned} &\min_{\sigma_{-i},\delta} \delta \\ \text{s.t.} \quad & \sigma_{-i}(s_{-i}) \geq 0 \text{ for all } s_{-i} \in S_{-i}, \\ & \sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq 1, \\ & -\sum_{s_{-i}} \sigma_{-i}(s_{-i}) \geq -1, \\ \delta + \sum_{s_{-i}} \sigma_{-i}(s_{-i})&[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})] \geq 0 \text{ for all } s_i' \in S_i. \end{aligned}$$

Note that s_i is a best response to some belief μ_{-i} over opponents' play iff the solution to this LP is less than or equal to 0. Now consider the program,

$$\begin{aligned}
&\max_{\eta,\epsilon,\sigma_i} \epsilon \\
&\text{s.t.} \quad \epsilon \in \mathbb{R}_+, \\
&\sigma_i \in \mathbb{R}_+^{|S_i|}, \\
&\eta \in \mathbb{R}_+^{|S_{-i}|}, \\
&\sum_{s_i'} \sigma_i(s_i') \geq 1, \\
\eta(s_{-i}) + \epsilon + &\sum_{s_i'} \sigma_i(s_i')[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})] = 0 \text{ for all } s_i' \in S_i.
\end{aligned}$$

Note that s_i is strictly dominated iff the solution to this program is strictly greater than 0, so s_i is not strictly dominated iff the solution to this program is less than or equal to 0.

By the duality theorem, Theorem 55, both programs have the same solution if feasible (and by Proposition 59, if one is feasible then both are). This completes the proof. □

Proposition 6. *In any finite game, the set of strategies surviving iterated strict dominance and the set of correlated rationalizable strategies coincide, i.e. $\mathcal{D}_i = R_i$ for each player i.*

Proof. If $s_i \in R_i$, then s_i is the best response to some belief about opponents' play $\mu_{-i} \in \Delta(R_{-i})$. Since $R_{-i} \subseteq S_{-i}$, the lemma implies s_i is not strictly dominated. Hence $R_i \subseteq \mathcal{D}_i^1$. Applying the argument iteratively gives $R_i \subseteq \mathcal{D}_i^k$ for all $k \in \mathbb{N}$ and hence $R_i \subseteq \mathcal{D}_i$. Since this holds for all i, $R \subseteq \mathcal{D}$.

Conversely, by definition, no strategy profile in $\mathcal{D}$ is strictly dominated in the reduced game in which the strategy set is $\mathcal{D}$. Hence any $s_i \in \mathcal{D}_i$ is the best response to some beliefs μ_{-i} over $\mathcal{D}_{-i}$ and hence over S_{-i}. It follows that $\mathcal{D}$ is a best response set. Thus $\mathcal{D} \subseteq R$.

Since $\mathcal{D} \subseteq R$ and $R \subseteq \mathcal{D}$, we have $R = \mathcal{D}$. □

Corollary 3. *A strategy profile σ is a profile of correlated rationalizable strategies, i.e. $\sigma \in R$, iff for each i, σ_i is a best response to some belief μ_{-i} consistent with common knowledge of rationality and the structure of the game.*

Proof. Follows from the coincidence of R and $\mathcal{D}$ and the fact that no player i will play $s_i \notin \mathcal{D}_i$ if rationality and the structure of the game are common knowledge, while every $s_i \in \mathcal{D}_i$ is the best response to some $\sigma_{-i} \in \Delta(\mathcal{D}_{-i})$ or else it is strictly dominated. □

Bernheim (1984) and Pierce (1984) use the concept of independent rationalizability rather than correlated rationalizability. The set of strategies surviving iterated strict dominance coincides with the set of independent rationalizable strategies for two player finite games, but for $n \geq 3$, independent rationalizability is a refinement of iterated strict dominance, i.e. $R^I \subseteq \mathcal{D}$ but in general $R^I \neq \mathcal{D}$.

2.6 Weak dominance

Definition 19 (Weak dominance)**.**

(a) *Weak dominance.* In a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a pure strategy $s_i \in S_i$ weakly dominates $s_i' \in S_i$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}, \ \text{ with strict inequality for some } s_{-i}.$$

We call s_i' a *weakly dominated strategy* if there is some s_i that weakly dominates s_i'.

We call s_i a *weakly dominant strategy* if it weakly dominates all $s_i' \neq s_i$.

In a mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, a mixed strategy $\sigma_i \in \Delta(S_i)$ is said to *weakly dominate* a strategy $\sigma_i' \in \Delta(S_i)$ if

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}) \qquad \text{for all } s_{-i} \in S_{-i}.$$

(b) *Weakly dominant strategy equilibrium.* A strategy profile $s^* \in S$ ($\sigma^* \in \prod_{i\in\mathcal{I}} \Delta(S_i)$) is a *weakly dominant strategy equilibrium* if, for each player i, s_i^* (σ_i^*) is a weakly dominant strategy.

(c) *Iterated weak dominance.* Let $\mathcal{W}_i^0 = S_i$ for all i, and define

$$\mathcal{W}_i^k = \Big\{ s_i \in \mathcal{W}_i^{k-1} \mid \nexists \sigma_i \in \Delta(\mathcal{W}_i^{k-1}) \text{ s.t. } u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}, \\ \text{with strict inequality for some } s_{-i} \Big\}$$

We call $\mathcal{W}_i^k$ the set of pure strategies that survive k rounds of iterated weak dominance for player i. The set of strategies that *survive iterated weak dominance* for player i is

$$\mathcal{W}_i = \bigcap_{k=0}^{\infty} \mathcal{W}_i^k.$$

Note that weakly dominant strategy equilibrium need not be unique.

Iterated weak dominance (and indeed weakly dominant strategy equilibrium) is a bit more difficult to justify than iterated strict dominance. Strategies that are weakly but not strictly dominated are rationalizable – any such strategy is a best response to some belief about opponents' play. This said, such strategies are not robust, in the sense that if a player has any doubt about their beliefs, or believes their opponents can 'tremble' (make a mistake) with positive probability, then they are better off choosing a strategy that weakly dominates a weakly dominated strategy. Indeed, there is never any loss to choosing a strategy that weakly dominates a weakly dominated strategy, only potential gain. Luce & Raiffa (1957) thus have as an axiom of decision theory that no player will never choose a weakly dominated strategy. We could push this further to argue that players will never play a strategy profile that does not survive iterated weak dominance (see section 4.6).

Example 11. Sometimes the process of iterated strict dominance or iterated weak dominance is conceived of differently. Rather than eliminating all (strictly/weakly) dominated strategies at each step, players take turns to remove a single strategy from their strategy set. This conception of iterated dominance has a few pitfalls. First, it only works if players' strategy sets are finite. Second, under the assumption that the strategy sets are finite, for iterated weak dominance the set of strategies we arrive at can differ depending on the order in which we delete strategies (for iterated strict dominance, we will end up at the same set). Thus this version of the process is conceptually flawed.

Consider the game

	A_2	B_2	C_2
A_1	$1,2$	$2,3$	$0,3$
B_1	$2,2$	$2,1$	$3,2$
C_1	$2,1$	$0,0$	$1,0$

B_1 is a weakly dominant strategy for Player 1. However, if Player 1 is sure that Player 2 will play A_2, then C_1 is reasonable.

We have $\mathcal{W}_1^1 = \{B_1\}$ and $\mathcal{W}_2^1 = \{A_2, C_2\}$. No further deletions are possible.

The order in which we delete strategies matters here. For example, suppose we proceed in rounds where, rather than finding level-k iterated weak dominance sets, we alternate deleting strategies between players.

First delete A_1. Then we have $\{B_1, C_1\}$, and Player 2 should delete C_2, leaving $\{A_2, B_2\}$. Now Player 1 deletes C_1 leaving $\{B_1\}$, and Player 2 deletes B_2, leaving $\{A_2\}$. We are left with strategy profile (B_1, A_2).

Now suppose instead we first delete C_1, leaving $\{A_1, B_1\}$. Then Player 2 can delete A_2 or B_2. If Player 2 deletes A_2, this leaves $\{B_2, C_2\}$; Player 1 deletes A_1 leaving $\{B_1\}$, and Player 2 deletes B_2, leaving us with strategy profile (B_1, C_2). If Player 2 instead deletes B_2, this leaves $\{A_2, C_2\}$; Player 1 deletes C_1, leaving $\{B_1\}$, leaving us with strategy profiles (B_1, A_2) and (B_1, C_2).

Example 12 (Beauty contest)**.** Suppose there are n players and each player's strategy is a guess $x_i \in [0, 1]$. A player i wins if x_i is closer to 2/3 of the average guess than any other player. Ties are broken at random and a player receives payoff 1 if winning, 0 otherwise. No guess is strictly dominated. However, any guess in $(2/3, 1]$ is weakly dominated by 200/3, and $\mathcal{W}_i^1 = [0, 2/3]$ survives the first round of iterated weak dominance. Iterating, we see that $\mathcal{W}_i^k = [0, (2/3)^k]$. It follows that $\mathcal{W}_i = \bigcap_{k=0}^{\infty} \mathcal{W}_i^k = \{0\}$. The unique weakly dominant strategy equilibrium (which is also the unique Nash equilibrium) thus has every player guessing 0.

In experimental studies, players guess significantly higher numbers than 0. Nagel (1995) takes this as evidence for bounded rationality.

2.7 Nash equilibrium

Nash equilibrium captures the notion that, given their opponents' strategies, a rational player should play a strategy that does at least as well against those opponents' strategies as any other available strategy.

Definition 20 (Nash equilibrium)**.**

(a) *Best response correspondence.* The *best response correspondence* $B_i : S_{-i} \rightrightarrows S_i$ for player i is defined by

$$B_i(s_{-i}) = \{s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\},$$

for each $s_{-i} \in S_{-i}$. The *best response correspondence* $B : S \rightrightarrows S$ is defined by

$$B(s) = \{s' \in S \mid s_i' \in B_i(s_{-i}) \text{ for all } i \in \mathcal{I}\},$$

for each $s \in S$. The extension to mixed strategies is trivial.

(b) *Nash equilibrium.* A strategy profile $s^* \in S$ is a (pure strategy) *Nash equilibrium* if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i.$$

A strategy profile $\sigma^* \in \prod_{i \in \mathcal{I}} \Delta(S_i)$ is a (mixed strategy) *Nash equilibrium* if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \text{for all } s_i \in S_i.$$

(c) *Strict Nash equilibrium.* A strategy profile s^* is a *strict Nash equilibrium* if

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i - \{s_i^*\}.$$

All strict Nash equilibria are Nash equilibria. Whenever we refer to Nash equilibrium, we refer to the definition in (b).

A Nash equilibrium requires that every player plays a best response to their opponents' strategies. Indeed, a Nash equilibrium is a fixed point of the best response correspondence:

Proposition 7. *A strategy profile $s^* \in S$ is a (pure strategy) Nash equilibrium iff*

$$s^* \in B(s^*).$$

Likewise, a mixed strategy profile $\sigma^* \in \prod_{i \in \mathcal{I}} \Delta(S_i)$ is a (mixed strategy) Nash equilibrium iff

$$\sigma^* \in B(\sigma^*).$$

Proof. If $s^* \in B(s^*)$ then $s_i^* \in B_i(s_{-i}^*)$. Hence $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$. Since this holds across all $i \in \mathcal{I}$, it follows that s^* is a Nash equilibrium. Conversely, suppose s^* is a Nash equilibrium. Then $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$. Thus $s_i^* \in B_i(s_{-i}^*)$. Since this holds for all i, $s^* \in B(s^*)$.

The extension to mixed strategies follows immediately, since a mixed strategy game is a special case of a pure strategy game. □

Nash equilibrium is famously the 'workhorse' solution concept in game theory. The (mixed strategy) Nash existence result makes Nash equilibrium particularly appealing – allowing mixed strategies, any finite player game has at least one Nash equilibrium. On close reflection, however, the concept is not quite as intuitive as might be assumed. A Nash equilibrium is a point of mutual best response. But in general, to play the Nash equilibrium solution, all players must not only be rational but also have correct beliefs about the play of their opponents. In general, multiple Nash equilibria may exist, and it is not obvious which Nash equilibrium will be played, or even if a Nash equilibrium will be played at all.

How then do we get to Nash equilibrium? Here are several suggestions, though none are applicable to all settings:

- *Nash equilibrium as the outcome of introspection.* If players are rational and reason about the behaviour of the other players, then we might expect Nash equilibrium to be played. This usually requires that we have a point prediction of rational play – for example, if there is a unique iterated strict dominance solution and rationality is common knowledge, or if we have some reasonable equilibrium selection criterion that isolates a unique "reasonable" Nash equilibrium from a set of multiple ones.

- *Nash equilibrium as the outcome of a learning process.* Nash equilibrium can be thought of as the result of a learning process. There is some empirical evidence that in repeated games, agents learn to play a Nash equilibrium over time (Smith, 1990; McCabe et al., 1991; Prasnikar & Roth, 1991). Kalai & Lehrer (1993) developing a rational learning model in an infinitely repeated game, provide theoretical motivation for why Nash equilibrium play should emerge from learning. The obvious drawback is that a learning process requires repeated interaction. It does not help explain why we might expect Nash equilibrium in a one-shot game.

- *Nash equilibrium as the outcome of an evolutionary process.* An evolutionary interpretation of Nash equilibrium has large populations of types of agents playing pure strategies against each other. These types reproduce at different rates depending on their evolutionary fitness. This process will converge a Nash equilibrium. It is particularly intuitive in biological settings – for example, in explaining the mix of predators and prey in an ecosystem, as in the Hawk-Dove game. This only applies in matrix games (i.e. in the usual interpretation, games of two players with symmetric strategy sets and symmetric payoffs.) However, evolutionary processes are myopic, and thus do not provide a compelling description of strategic behaviour in repeated games. If players value outcomes in future periods and realize that their current actions affect their opponents' future play, then we should not expect an evolutionary process to predict their behaviour well.

- *Nash equilibrium as a self-enforcing agreement.* One interpretation of Nash equilibrium is contractual. Suppose players get together and reach an agreement that specifies the strategy profile that players should play. If this strategy profile is a Nash equilibrium, then players have no incentive to break the agreement, and so the agreement is self-enforcing. We discuss this in more detail when we get to correlated equilibrium (section 2.12)

Example 13. Consider the game,

	a_2	b_2	c_2
a_1	$4, 3$	$-1, -1$	$0, 0$
b_1	$-1, -1$	$-2, -2$	$-1, -1$
c_1	$0, 0$	$-1, -1$	$5, 2$

For Player 1, b_1 is strictly dominated by a_1 (or c_1). For Player 2, b_2 is strictly dominated by a_2 (or c_2). Player 1's best response to each of Player 2's pure strategies is coloured blue and Player 2's best response to each of Player 1's pure strategies is coloured red. Note that while, for compactness of representation, we have coloured the payoff of the best response by each player, the Nash equilibrium is a *strategy profile*, not a payoff profile. We see that there are two pure strategy Nash equilibria in this game: (a_1, a_2) and (c_1, c_2).

There is also a mixed strategy Nash equilibrium. Suppose Player 1 plays a_1 with probability p, b_1 with probability 0 (since b_1 is strictly dominated we can rule it out of any Nash equilibrium) and c_1 with probability $1 - p$. That is, $\sigma_1 = (p, 0, 1 - p)$. Likewise, suppose Player 2 plays a_2 with probability q, b_2 with probability 0 and c_2 with probability $1 - q$. That is, $\sigma_2 = (q, 0, 1 - q)$.

Player 1's expected payoff is

$$u_1(\sigma_1, \sigma_2) = p(4q) + (1 - p)(5(1 - q)).$$

In Nash equilibrium, Player 2 randomizes s.t. Player 1 does not have a profitable deviation from randomizing. This requires that

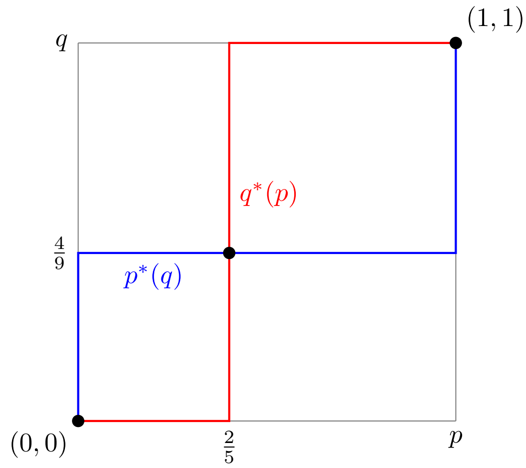
$$4q = 5(1 - q),$$

i.e. $q = \frac{5}{9}$. If $q > \frac{5}{9}$, then $u_1(a_1, \sigma_2) > u_1(\sigma_1, \sigma_2)$ for $p < 1$. If $q < \frac{5}{9}$ then $u_1(c_1, \sigma_2) > u_1(\sigma_1, \sigma_2)$ for $p > 0$.

Likewise, Player 1 randomizes so that

$$3p = 2(1 - p),$$

i.e. $p = \frac{2}{5}$. Hence we have mixed strategy Nash equilibrium $((\frac{2}{5}, 0, \frac{3}{5}), (\frac{5}{9}, 0, \frac{4}{9}))$. Graphically, we can plot the 'best response probability' $p^*(q)$ for Player 1 and $q^*(p)$ for Player 2 associated to the best response to σ_2 and σ_1 respectively. This is of course not the same as the best response itself, which is a (set of) strategy profile(s) over all the available pure strategies. However, given how we defined σ_1, σ_2, it does fully characterize these best responses.



To see that correct beliefs are, in general, necessary for Nash equilibrium play, suppose Player 1 believes that Player 2 will play a_2 (i.e. Player 1's belief over her opponent's play is $\mu_{-1} = (1, 0, 0)$). Player 1's best response to her belief is to play a_1. Now suppose Player 2 believes that Player 1 will play c_1 (i.e. $\mu_{-2} = (0, 0, 1)$). Player 2's best response to his belief is to play c_2. The result is the profile (a_1, c_2), and both players receive a payoff of 0. This is not a Nash equilibrium since both players have profitable deviations (c_1 and a_2 respectively).

Proposition 8. *In any finite game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, if a unique pure strategy profile survives iterated strict dominance then it is the unique pure strategy Nash equilibrium.*

Proof. Let (s_i^*, s_{-i}^*) be the unique strategy profile surviving iterated strict dominance. Suppose $u_i(s_i, s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*)$ for some $s_i \neq s_i^*$.

Define a strict order $>$ on S_i by $s_i > s_i'$ iff $u_i(s_i, s_{-i}^*) > u_i(s_i', s_{-i}^*)$.

Lemma 6. *$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$ s.t. $s_i \neq s_i^*$.*

Proof. Since S_i is finite, there is some k_1-level iterated strict dominance set $\mathcal{D}_i^{k_1}$ s.t. $s_i \in \mathcal{D}_i^{k_1}$ and $s_i \notin \mathcal{D}_i^{k_1+1}$. Thus we have that there is some $s_i' \in \mathcal{D}_i^{k_1}$ that strictly dominates s_i. Since $s_{-i}^* \in \mathcal{D}_{-i}^j$ for all $j \in \mathbb{N}$, it follows that

$$u_i(s_i', s_{-i}^*) > u_i(s_i, s_{-i}^*).$$

If $u_i(s_i', s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*)$, then applying the same argument as above gives us some k_2 s.t. $s_i' \in \mathcal{D}_i^{k_2}$ but $s_i' \notin \mathcal{D}_i^{k_2+1}$. We have some $s_i^{(2)}$ that strictly dominates s_i', implying $u_i(s_i^{(2)}, s_{-i}^*) > u_i(s_i', s_{-i}^*)$. Since there are only finitely many members of S_i, reapplying this argument finitely many times, we can always find a chain $s_i^* > s_i^{(n)} > \cdots > s_i' > s_i$. The lemma follows. $\square$

Since i is arbitrary, it follows from the lemma that (s_i^*, s_{-i}^*) is a pure strategy Nash equilibrium.

Now suppose there is a second pure strategy Nash equilibrium $(s'_i, s'_{-i}) \neq (s^*_i, s^*_{-i})$. Then $u_i(s'_i, s'_{-i}) \geq u_i(s_i, s'_{-i})$ for all $s_i \in S_i$. Since $s'_i \notin \mathcal{D}_i$, there is some k-level iterated strict dominance set s.t. $s'_i \in \mathcal{D}^k_i$ but $s'_i \notin \mathcal{D}^{k+1}_i$. If $s'_{-i} \in \mathcal{D}^{k+1}_{-i}$, then we must have some $s''_i \in \mathcal{D}^k_i$ s.t. s''_i strictly dominates s'_i, implying $u_i(s''_i, s'_{-i}) > u_i(s'_i, s'_{-i})$. Given (s'_i, s'_{-i}) is a Nash equilibrium, this would yield a contradiction.

if $s'_{-i} \notin \mathcal{D}^{k+1}_{-i}$, then there is some $j \neq i$ s.t. s'_j is strictly dominated in $\mathcal{D}^k_j$ by some $s''_j \in \mathcal{D}^k_j$, implying $u_j(s''_j, s'_{-j}) > u_j(s'_j, s'_{-j})$. Since (s'_j, s'_{-j}) is Nash, this again yields a contradiction. □

Proposition 9. *In a finite mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, a strategy profile σ^* is a mixed strategy Nash equilibrium only if each player i chooses $\sigma^*_i(s_i) > 0$ for some $s_i \in S_i$ only if s_i is a best response to σ^*_{-i}.*

Proof. Given a strategy σ_i,

$$u_i(\sigma_i, \sigma^*_{-i}) = \int_{S_i} u_i(s_i, \sigma^*_{-i})\, \mathrm{d}\sigma_i = \sum_{s_i\in S_i} u_i(s_i, \sigma^*_{-i})\sigma_i(s_i).$$

Suppose σ^* is s.t. for each i, $\sigma^*_i(s_i) > 0$ for $s_i \in S_i$ only if s_i is a best response to σ^*_{-i}. We mean to show this is a sufficient condition for σ^* to be a Nash equilibrium. Let

$$B^s_i(\sigma_{-i}) = \left\{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \text{ for all } s'_i \in S_i\right\},$$

the set of pure strategies that are best responses to σ_{-i}. Clearly, $u_i(s_i, \sigma^*_{-i}) = u_i(s'_i, \sigma^*_{-i}) =: \bar{u}_i(\sigma^*_{-i})$ for all $s_i, s'_i \in B^s_i(\sigma^*_{-i})$, for if $u_i(s_i, \sigma^*_{-i}) > u_i(s'_i, \sigma^*_{-i})$ then $s'_i \notin B^s_i(\sigma^*_{-i})$, and the converse if $u_i(s'_i, \sigma^*_{-i}) > u_i(s_i, \sigma^*_{-i})$. Now, by the hypothesis, σ^*_i assigns $\sigma^*_i(s_i) > 0$ only to $s_i \in B^s_i(\sigma^*_{-i})$ and $\sigma^*_i(s_i) = 0$ otherwise. Hence

$$u_i(\sigma^*_i, \sigma^*_{-i}) = \sum_{s_i\in B^s_i(\sigma^*_{-i})} u_i(s_i, \sigma^*_{-i})\sigma^*_i(s_i) = \sum_{s_i\in B^s_i(\sigma^*_{-i})} \bar{u}_i(\sigma^*_{-i})\sigma^*_i(s_i) = \bar{u}_i(\sigma^*_{-i}).$$

Thus $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i})$ for all $\sigma_i \in \Delta(S_i)$. Since this holds for all i, it follows that σ^* is a Nash equilibrium.

Conversely, suppose σ^* is a Nash equilibrium. If s'_i is not a best response to σ^*_{-i} then there is some strategy s''_i s.t. $u_i(s''_i, \sigma^*_{-i}) > u_i(s'_i, \sigma^*_{-i})$. If $\sigma^*_i(s_i) > 0$, then the strategy

$$\sigma'_i(s_i) = \begin{cases} \sigma^*_i(s_i) & \text{if } s_i \neq s'_i, s''_i, \\ 0 & \text{if } s_i = s'_i, \\ \sigma^*(s'_i) + \sigma^*(s''_i) & \text{if } s_i = s''_i, \end{cases}$$

yields expected payoff

$$\begin{aligned} u_i(\sigma'_i, \sigma^*_{-i}) &= \sum_{s_i\neq s'_i, s''_i} u_i(s_i)\sigma^*_i(s_i) + u_i(s''_i, \sigma^*_{-i})\sigma^*_i(s''_i) + u_i(s''_i, \sigma^*_{-i})\sigma^*_i(s'_i) \\ &> \sum_{s_i\neq s'_i, s''_i} u_i(s_i)\sigma^*_i(s_i) + u_i(s''_i, \sigma^*_{-i})\sigma^*_i(s''_i) + u_i(s'_i, \sigma^*_{-i})\sigma^*_i(s'_i) \\ &= u_i(\sigma^*_i, \sigma^*_{-i}). \end{aligned}$$

Since σ^* is a Nash equilibrium, this yields a contradiction. $\square$

Definition 21 (Never-best response)**.** A pure strategy $s_i \in S_i$ of i is a *never-best response* if for all profiles $\sigma_{-i} \in \Delta_{-i}(S_{-i})$, there exists a $\sigma_i \in \Delta(S_i)$ s.t.

$$u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}).$$

A strictly dominated strategy is of course a never-best response. An obvious corollary to Proposition 9 is:

Corollary 4. *Suppose s_i is a never-best response. Then if σ^* is a mixed strategy Nash equilibrium $\sigma_i^*(s_i) = 0$.*

In general, some intuitively undesirable solutions can be Nash equilibria. For example:

Proposition 10. *A Nash equilibrium can be weakly dominated.*

Proof. Proof follows from the following example. $\square$

Example 14 (Weakly dominated Nash equilibrium)**.** Consider the game with payoff matrix:

$$\begin{array}{ccc} & a_2 & b_2 \\ a_1 & 1,1 & 0,1 \\ b_1 & 1,0 & 2,2 \end{array}$$

Now, (a_1, a_2) is a Nash equilibrium since $u_1(a_1, a_2) \geq u_1(b_1, a_2)$ and $u_2(a_1, a_2) \geq u_2(a_1, b_2)$. However, b_1 is a weakly dominant strategy for Player 1 and b_2 is a weakly dominant strategy for Player 2, so (a_1, a_2) is weakly dominated.

Such a weakly dominated equilibrium is fragile in the sense that if either player's beliefs involve any degree of uncertainty over the actions of the other player, then that player would optimally choose the weakly dominant action. This is not the only kind of case where a Nash equilibrium might seem unreasonable. To isolate the more reasonable equilibria, we typically need some kind of refinement – in this case, the appropriate refinement is trembling hand perfect equilibrium.

2.7.1 Nash equilibrium inefficiency

Nash equilibria are not generally efficient. We show this by means of a counterexample.

Proposition 11. *Nash equilibria need not be Pareto efficient.*

Proof. See the following example. $\square$

Example 2 (continued)**.** In the Prisoner's Dilemma, we have payoff matrix

	C_2	D_2
C_1	$3, 3$	$0, 4$
D_1	$4, 0$	$1, 1$

The best responses are highlighted in blue and red for Player 1 and Player 2 respectively. We see that the (unique) Nash equilibrium is (D_1, D_2) [also, a strict Nash equilibrium and a strictly dominant strategy equilibrium]. This is Pareto inefficient since

$$u_1(C_1, C_2) = u_2(C_1, C_2) = 3 > 1 = u_1(D_1, D_2) = u_2(D_1, D_2).$$

2.7.2 Existence of Nash equilibrium

In finite games of pure strategies, Nash equilibrium need not exist. However, Nash (1950,1951) proves the existence of mixed strategy Nash equilibrium in games of finite players. Debreu (1952), Glicksberg (1952) and Fan (1952) prove a more general existence result.

Example 9 (continued)**.** Recall the payoff matrix for Matching Pennies is

	H_2	T_2
H_1	$1, -1$	$-1, 1$
T_1	$-1, 1$	$1, -1$

The best responses for Player 1 is highlighted in blue and for Player 2 in red. We see that there is no pure strategy Nash equilibrium.

The proof of the existence results for the Debreu-Glicksberg-Fan and Nash existence results rely on Kakutani's fixed point theorem (Theorem 6). For a discussion of correspondences and upper hemicontinuity, see the Mathematical Appendix, section 7.1.

Theorem 6 (Kakutani's fixed point theorem)**.** *Let $K \subset \mathbb{R}^n$ be a nonempty, compact, convex set and suppose $F : K \rightrightarrows K$ is a correspondence satisfying*

(i) *$F(x)$ is nonempty valued;*

(ii) *$F(x)$ is convex valued;*

(iii) *$F(x)$ is upper hemicontinuous.*

Then F has a fixed point.

We discuss fixed point theorems, including this one, in more detail in section 7.2.

Theorem 7 (Debreu-Glicksberg-Fan, 1952)**.** *Let $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$ be a game of $n \in \mathbb{N}$ players s.t. for each player $i \in \mathcal{I}$, $S_i \subset \mathbb{R}^{k_i}$ is nonempty, convex and compact. If, for each $i \in \mathcal{I}$, u_i is continuous in s and quasiconcave in s_i, then the game has a pure strategy Nash equilibrium.*

Proof. Since each S_i is convex and compact, it follows that $S = \prod_{i \in \mathcal{I}} S_i$ is convex and compact. Furthermore, since u_i is continuous in S_i and S_i is compact, to each s_{-i} there is some $s_i \in S_i$ s.t. $s_i = \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i})$, by the extreme value theorem. Hence the best response correspondence $B_i(s_{-i})$ is nonempty for all $s_{-i} \in S_{-i}$.

Recall that a function f defined on a convex set X is quasiconcave if $f(\lambda x+(1-\lambda)y) \geq \min\{f(x), f(y)\}$ for all $\lambda \in [0,1]$ and all $x, y \in X$.

Since u_i is quasiconcave in s_i, if $s_i, s'_i \in B_i(s_{-i})$ then $u_i(\lambda s_i+(1-\lambda)s'_i) \geq \min\{u_i(s_i), u_i(s'_i)\}$ for all $\lambda \in [0,1]$. Hence any convex combination of s_i, s'_i achieves payoff against s_{-i} that is at least as great as that of s_i or s'_i. Now, $u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$, for if $u_i(s_i, s_{-i}) > u_i(s'_i s_{-i})$ then $s'_i \notin B_i(s_{-i})$ and likewise if $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$ then $s_i \notin B_i(s_{-i})$. Applying the same argument to the convex combination $\lambda s_i + (1-\lambda)s'_i$, we have $u_i(\lambda s_i + (1-\lambda)s'_i, s_{-i}) = u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$ and hence $\lambda s_i + (1-\lambda)s'_i \in B_i(s_{-i})$ for all $\lambda \in [0,1]$. It follows that $B_i(s_{-i})$ is convex for all $s_{-i} \in S_{-i}$. Since this holds for all i, we have that $B(s)$ is convex for all $s \in S$, i.e. B is convex valued.

Since u_i is continuous, u^{-1} maps closed sets into closed sets. Define $\bar{u}_i(s_{-i}) := u_i(s_i, s_{-i})$ for any $s_i \in B_i(s_{-i})$. By definition of the best response correspondence, $B_i(s_{-i}) = u^{-1}(\bar{u}_i(s_{-i}))$, and the latter is a single point and thus closed. Hence $B_i(s_{-i})$ is closed for all $s_{-i} \in S_{-i}$ and all i. Since $B_i(s_{-i}) \subseteq S_i$, a compact set, it follows that $B_i(s_{-i})$ is therefore also compact. Hence B is compact valued.

This allows us to use the sequence definition of upper hemicontinuity. For any $\{s^n\}$ in S s.t. $s^n \to s$ and $\{\hat{s}^n\}$ s.t. $\hat{s}^n \in B(s^n)$ for all n and $\hat{s}^n \to \hat{s}$, we mean to prove that $\hat{s} \in B(s)$. Suppose otherwise. Then for some player i, $\hat{s}_i \notin B_i(s_{-i})$ and so there exists s'_i s.t.

$$u_i(\hat{s}_i, s_{-i}) < u_i(s'_i, s_{-i}).$$

Yet by continuity of u_i, this implies that for some $N \in \mathbb{N}$,

$$u_i(\hat{s}_i^n, s_{-i}^n) < u_i(s'_i, s_{-i}^n)$$

for all $n \geq N$, and hence $\hat{s}_i^n \notin B_i(s_{-i}^n)$ for some n, yielding a contradiction. Hence B_i is upper hemicontinuous for all i and so B is upper hemicontinuous. We could have instead concluded that B is upper hemicontinuous by using Berge's theorem of the maximum (Theorem 38) to conclude that B has a closed graph, and then applying the closed graph theorem (Theorem 37)

Applying Kakutani's fixed point theorem, we have that B has a fixed point, i.e. there exists some $s^* \in S$ s.t. $s^* \in B(s^*)$. □

Nash's existence theorem is just a special case of this:

Theorem 8 (Nash's existence theorem, 1950)**.** *Any game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$ with a finite number of players and such that S_i is finite and nonempty for all $i \in \mathcal{I}$ has a mixed strategy Nash equilibrium.*

Proof. Clearly, $\Delta(S_i)$ is a nonempty, compact and convex subset of $\mathbb{R}^{|S_i|}$.

By linearity of the Lebesgue integral, u_i is continuous.

For any $\sigma_i, \sigma_i' \in \Delta(S_i)$ and any $\lambda \in [0,1]$, we have

$$\begin{aligned}
u_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) &= \sum_{s \in S} u_i(s)[\lambda\sigma_i + (1-\lambda)\sigma_i'(s_i)]\sigma_{-i}(s_{-i}) \\
&= \sum_{s \in S} u_i(s)[\lambda\sigma_i(s_i) + (1-\lambda)\sigma_i'(s_i)]\sigma_{-i}(s_{-i}) \\
&= \lambda \sum_{s \in S} u_i(s)\sigma_i(s_i)\sigma_{-i}(s_{-i}) + (1-\lambda)\sum_{s \in S} u_i(s)\sigma_i'(s_i)\sigma_{-i}(s_{-i}) \\
&\geq \min\left\{\sum_{s \in S} u_i(s)\sigma_i(s_i)\sigma_{-i}(s_{-i}), \sum_{s \in S} u_i(s)\sigma_i'(s_i)\sigma_{-i}(s_{-i})\right\} \\
&= \min\{u_i(\sigma_i, \sigma_{-i}), u_i(\sigma_i', \sigma_{-i})\}.
\end{aligned}$$

Hence u_i is quasiconcave in σ_i.

It follows that the hypotheses of the Debreu-Glicksberg-Fan theorem are satisfied. $\square$

2.7.3 Upper hemicontinuity and Nash equilibria

We can make use of the fact that best response correspondences are upper hemicontinuous (under the appropriate assumptions on payoffs and strategy spaces) to find Nash equilibria in limits of games.

Definition 22. Let Λ be a parameter space, which we assume to be a compact metric space. Consider a finite set of players $\mathcal{I}$ and let $S = \prod_{i \in \mathcal{I}} S_i$ be the set of strategy profiles. Suppose each player $i \in \mathcal{I}$ has a payoff function $u_i : S \times \Lambda \to \mathbb{R}$ that is continuous in both strategy profiles S and parameters Λ.

1. *Family of games.* For each $\lambda \in \Lambda$, define the game $G(\lambda) := (\mathcal{I}, (S_i, u_i(\cdot, \lambda))_{i \in \mathcal{I}})$. We call $\mathcal{G}_\Lambda = \{G(\lambda) \mid \lambda \in \Lambda\}$ a *family of games* parameterized by Λ.

2. *Nash equilibrium correspondence.* Given a family of games $\mathcal{G}_\Lambda$, the *Nash equilibrium correspondence* $\mathrm{NE} : \Lambda \rightrightarrows S$ is defined as $\mathrm{NE}(\lambda) = \{s \in S \mid s \text{ is a Nash equilibrium of } G(\lambda)\}$ for all $\lambda \in \Lambda$.

We assume the value of λ is common knowledge in game $G(\lambda)$.

Proposition 12. *Under the assumptions of Definition 22, the Nash equilibrium correspondence of a family of games $\mathcal{G}$ has a closed graph.*

Proof. Consider any sequence $\{(s^n, \lambda^n)\}$ such that $(s^n, \lambda^n) \to (s, \lambda)$ with $s^n \in \mathrm{NE}(\lambda^n)$ for each $n \in \mathbb{N}$. Suppose that $s \notin \mathrm{NE}(\lambda)$. Then, for some player $i \in \mathcal{I}$, $u_i(s_i', s_{-i}, \lambda) > u_i(s_i, s_{-i}, \lambda)$ for some $s_i' \in S_i$. Since u_i is continuous and $(s^n, \lambda^n) \to (s, \lambda)$, it follows that $u_i(s_i', s_{-i}^n, \lambda^n) > u_i(s_i^n, s_{-i}^n, \lambda^n)$ for sufficiently large n, but then $s^n \notin \mathrm{NE}(\lambda^n)$, yielding a contradiction. $\square$

By the closed graph theorem (Theorem 37), the Nash equilibrium correspondence is thus upper hemicontinuous. This implies that if we take a sequence of parameters $\{\lambda_n\}$ such that $\lambda_n \to \lambda$, then $\lim_{n\to\infty} \mathrm{NE}(\lambda_n) \subseteq \mathrm{NE}(\lambda)$. Thus we can find Nash equilibria of games by looking at approximations of those games and taking limits. Note however that the set inclusion relation only goes in one direction: because the Nash equilibrium correspondence is not necessarily lower hemicontinuous, there may be Nash equilibria of the limit game that are not limits of Nash equilibria of the approximations.

Since any mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ can be thought of as effectively a pure strategy game with strategy sets $\Delta(S_i)$ for each i, Proposition 12 clearly holds for mixed strategy games.

2.8 (Trembling hand) perfect equilibrium

> **In retrospect the earlier use of the word "perfect" was premature. Therefore a perfect equilibrium point in the old sense will be called "subgame perfect". The new definition of perfectness has the property that a perfect equilibrium point is always subgame perfect but a subgame perfect equilibrium point may not be perfect.**

"Let's all aspire to the bravery of Selten (1975), renaming the original "perfect equilibrium" to make way for the new perfect equilibrium, confident that the concept has now reached perfection." – Shengwu Li (@ShengwuLi), Twitter, 31 January 2022

A serious drawback of Nash equilibrium in general is multiplicity, and so we may consider selection criteria and refinements to rule out certain Nash equilibria that are less plausible under certain criteria.

Trembling hand perfect equilibrium (or just, *perfect equilibrium*), introduced by Selten (1975), is a refinement of Nash equilibrium that excludes Nash equilibria that are fragile to noise in players' beliefs.

Example 14 (continued)**.** Recall that the game in Example 14 has payoff matrix

	a_2	b_2
a_1	$1,1$	$0,1$
b_1	$1,0$	$2,2$

This game has two pure strategy Nash equilibria, (a_1, a_2) and (b_1, b_2). Recall that b_1 and b_2 are weakly dominant strategies for Players 1 and 2 respectively. It follows that (a_1, a_2) can only be sustained if each player i has sure belief that the other player j will play a_j. The equilibrium is thus not robust – any very small doubt on the part of either player will ensure that the (rational) player plays b_i.

Definition 23 (Trembling hand perfect equilibrium)**.** Consider a mixed strategy normal form game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ with n players.

(a) *Totally mixed strategy.* A mixed strategy $\sigma_i \in \Delta(S_i)$ is called *totallly mixed* or *completely mixed* if $\sigma_i(s_i) > 0$ for all $s_i \in S_i$.

(b) *Perturbed game.* A *perturbation* for a player i is a function $\epsilon_i : S_i \to (0,1)$ s.t.

$$\sum_{s_i \in S_i} \epsilon_i(s_i) < 1.$$

A *perturbation* $\epsilon : S \to (0,1)^n$ is a function defined by $\epsilon(s) = (\epsilon_i(s_i)_{i\in\mathcal{I}})$ where ϵ_i is a perturbation for player i.

An *ϵ-perturbed game* G_ϵ of G^m is a game $G^m_\epsilon = (\mathcal{I}, (\Delta_{\epsilon_i}(S_i), u_i)_{i\in\mathcal{I}})$ with

$$\Delta_{\epsilon_i}(S_i) = \{\sigma_i \in \Delta(S_i) \mid \sigma_i(s_i) \geq \epsilon_i(s_i) \text{ for all } s_i \in S_i\}$$

for each $i \in \mathcal{I}$, where ϵ_i is a perturbation. That is, $\Delta_{\epsilon_i}(S_i)$ is the set of totally mixed strategies of player i bounded below by the perturbation ϵ_i.

(c) *Trembling hand perfect equilibrium* (normal form). A mixed strategy profile σ^* of $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ is a (*trembling hand*) *perfect equilibrium* if

  (i) there exists a sequence of perturbations $\{\epsilon^k\}$ s.t. $\epsilon^k_i(s_i) \to 0$ for all $s_i \in S_i$ and all $i \in \mathcal{I}$, and

  (ii) for each $k \in \mathbb{N}$, there exists a (totally mixed) Nash equilibrium σ^k of the ϵ^k-perturbed game $G^m_{\epsilon^k}$ s.t. $\sigma^k \to \sigma^*$.[7]

(d) *Trembling hand perfection in extensive form games.* The *agent-normal form* of an extensive form game Γ with information sets $(\Phi_i)_{i\in\mathcal{I}}$ is the corresponding normal form game $G = \Big((\Phi_i)_{i\in\mathcal{I}}, \big(A_i(\phi_i), u_i\big)_{(\Phi_i)_{i\in\mathcal{I}}}\Big)$, that is, assigning to each information set ϕ_i of player i a new 'player' with payoff function u_i and strategy set $A_i(\phi_i)$.

A (*trembling hand*) *perfect equilibrium* for Γ is a perfect equilibrium of the corresponding agent-normal form game.

Note that a perfect equilibrium σ^* need not itself be totally mixed. We also do not need to limit the definition to mixed strategy games – it makes sense to talk about perfect equilibria of pure strategy games but we must consider its mixed strategy counterpart when it comes to modelling trembles.

Note also that we can equivalently define perfect equilibrium in terms of best responses:

Proposition 13. *A mixed strategy profile σ^* of $G^m(\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ is a perfect equilibrium iff there exists a sequence of totally mixed strategy profiles $\{\sigma^k\}$ s.t.*

(i) *$\sigma^k \to \sigma^*$, and*

(ii) *for all $k \in \mathbb{N}$ and for each $i \in \mathcal{I}$, $\sigma^*_i \in B_i(\sigma^k_{-i})$, that is,*

$$u_i(\sigma^*_i, \sigma^k_{-i}) \geq u_i(s_i, \sigma^k_{-i}) \quad \textit{for all } s_i \in S_i.$$

[7]We say that a Nash equilibrium of an ϵ-perturbed game is *ϵ-perfect.*

Proof. The proof is quite longwinded. See Selten (1975), pp. 49-51. □

Perfect equilibrium excludes weakly dominated Nash equilibria:

Proposition 14. *Consider any mixed strategy finite player game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, where S_i is finite for all $i \in \mathcal{I}$.*

(i) *Any perfect equilibrium σ^* of G^m is a Nash equilibrium in weakly undominated strategies.*

(ii) *If G^m is a two player game, if σ^* is a Nash equilibrium and if σ^* is not weakly dominated, then σ^* is trembling hand perfect.*

Proof. (i) Assume G^m has a perfect equilibrium σ^*. Then there exists a sequence of perturbations $\{\epsilon^k\}$ satisfying Definition 23(c)(i) and for each k, there exists a totally mixed Nash equilibrium σ^k of the ϵ^k-perturbed game $G^m_{\epsilon^k}$ s.t. $\sigma^k \to \sigma^*$. First we show σ^* is a Nash equilibrium. Suppose otherwise. Then for some player $i \in \mathcal{I}$, there exists a strategy $s_i' \in S_i$ s.t. $u_i(s_i', \sigma^*_{-i}) > u_i(\sigma^*_i, \sigma^*_{-i})$. It follows that there exists some $K \in \mathbb{N}$,

$$u_i(s_i', \sigma^k_{-i}) > u_i(\sigma^k_i, \sigma^k_{-i})$$

for all $k \geq K$, since $\sigma^k \to \sigma^*$. Hence for some perturbed game σ^k is not a Nash equilibrium of the perturbed game $G^m_{\epsilon^k}$, yielding a contradiction.

Next we show that if s_i' is a weakly dominated strategy, then $\sigma^*_i(s_i') = 0$. Suppose s_i' is weakly dominated, necessarily by some strategy $\hat{\sigma}_i$ with $\hat{\sigma}_i(s_i') = 0$. Then there is some profile s'_{-i} s.t. $u_i(s_i', s'_{-i}) < u_i(\hat{\sigma}_i, s'_{-i})$. Now, for any totally mixed strategy profile σ_{-i},

$$\begin{aligned}
u_i(s_i', \sigma_{-i}) &= \sum_{s_{-i}\in S_{-i}} u_i(s_i', s_{-i})\sigma_{-i}(s_{-i}) \\
&= u_i(s_i', s'_{-i})\sigma_{-i}(s'_{-i}) + \sum_{s_{-i}\neq s'_{-i}} u_i(s_i', s_{-i})\sigma_{-i}(s_{-i}) \\
&< u_i(\hat{\sigma}_i, s'_{-i})\sigma_{-i}(s'_{-i}) + \sum_{s_{-i}\neq s'_{-i}} u_i(\hat{\sigma}_i, s_{-i})\sigma_{-i}(s_{-i}) \\
&= u_i(\hat{\sigma}_i, \sigma_{-i}).
\end{aligned}$$

Suppose $\sigma^*_i(s_i') > 0$ and define $\sigma_i' = \sigma^*_i(s_i) + \sigma^*_i(s_i')\hat{\sigma}_i(s_i)$ for all $s_i \in S_i$. Then for any k, and any totally mixed profile σ^k_{-i}

$$\begin{aligned}
u_i(\sigma^*_i, \sigma^k_{-i}) &= \sum_{s_i\in S_i} u_i(s_i, \sigma^k_{-i})\sigma^*_i(s_i) \\
&= u_i(s_i', \sigma^k_{-i})\sigma^*_i(s_i') + \sum_{s_i\neq s_i'} u_i(s_i, \sigma^k_{-i})\sigma^*_i(s_i) \\
&< u_i(\hat{\sigma}_i, \sigma^k_{-i})\sigma^*_i(s_i') + \sum_{s_i\neq s_i'} u_i(s_i, \sigma^k_{-i})\sigma^*_i(s_i) \\
&= u_i(\sigma_i', \sigma^k_{-i}).
\end{aligned}$$

Hence if σ_i^* plays a weakly dominated strategy with positive probabulity, it is not a best response to any totally mixed strategy σ_i^k, and thus σ^* cannot be perfect.

(ii) Proof omitted.

$\square$

Proposition 15. *In a mixed strategy finite player game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$, if a player i has a strictly dominant strategy s_i then in any perfect equilibrium σ^* of G^m, $\sigma_i^*(s_i) = 1$.*

Proof. Suppose otherwise for some i. Let s_i' be a strictly dominant strategy. Since σ^* is a perfect equilibrium, there is some sequence of totally mixed profiles $\{\sigma^k\}$ s.t. $\sigma^k \to \sigma^*$ and $u_i(\sigma_i^*, \sigma_{-i}^k) \geq u_i(s_i, \sigma_{-i}^k)$ for all k and all $s_i \in S_i$, yet $\sigma_i^*(s_i') < 1$. But, since s_i' is strictly dominant, it is the unique best response, i.e.

$$u_i(s_i', \sigma_{-i}^k) > u_i(\sigma_i, \sigma_{-i}^k)$$

for any σ_i s.t. $\sigma_i(s_i) \neq 1$. This yields a contradiction. $\square$

Lemma 7. *Any finite player ϵ-perturbed game $G_\epsilon^m = (\mathcal{I}, (\Delta_{\epsilon_i}(S_i), u_i)_{i\in\mathcal{I}})$, where S_i is finite, has a (totally) mixed strategy Nash equilibrium.*

Proof. $\Delta_{\epsilon_i}(S_i)$ is convex and compact, since $\Delta_{\epsilon_i}(S_i) = \Delta(S_i) \cap [\epsilon_i(s_1), 1] \times \cdots \times [\epsilon_i(s_{k_i}), 1]$, where $k_i = |S_i|$. Since this is the intersection of two convex sets, it is convex, and since it is the finite intersection of two compact (and thus closed) sets, it is a closed subset of a compact set in $\mathbb{R}^{k_i}$, and thus compact.

By definition of ϵ_i, $\sum_{s_i\in S_i} \epsilon_i(s_i) < 1$. Let $\delta_i = 1 - \sum_{s_i\in S_i} \epsilon_i(s_i)$. Then $\delta_i > 0$. Define $\sigma_i(s_i) = \epsilon_i(s_i) + \frac{\delta_i}{k_i}$ for all $s_i \in S_i$. Then $\sum_{s_i\in S_i} \sigma_i(s_i) = 1$, so $\sigma_i \in \Delta(S_i)$. Furthermore, $\sigma_i(s_i) \geq \epsilon_i(s_i)$ for all $s_i \in S_i$, and so $\sigma_i \in \Delta_{\epsilon_i}(S_i)$. Hence $\Delta_{\epsilon_i}(S_i)$ is nonempty. Since these properties hold for all i, it follows that $\Delta_\epsilon(S) := \prod_{i\in\mathcal{I}} \Delta_{\epsilon_i}(S_i)$ is compact, convex and nonempty.

Now u_i is continuous in σ, by linearity, and quasiconcave in σ_i, since for any $\sigma_i, \sigma_i' \in \Delta_{\epsilon_i}(S_i)$ and any $\lambda \in [0,1]$,

$$\begin{aligned}
u_i(\lambda\sigma_i + [1-\lambda]\sigma_i', \sigma_{-i}) &= \sum_{s\in S} u_i(s)[\lambda\sigma_i + (1-\lambda)\sigma_i'](s_i)\sigma_{-i}(s_{-i}) \\
&= \sum_{s\in S} u_i(s)[\lambda\sigma_i(s_i) + (1-\lambda)\sigma_i'(s_i)]\sigma_{-i}(s_{-i}) \\
&= \lambda \sum_{s\in S} u_i(s)\sigma_i(s_i)\sigma_{-i}(s_{-i}) + (1-\lambda)\sum_{s\in S} u_i(s)\sigma_i'(s_i)\sigma_{-i}(s_{-i}) \\
&\geq \min\left\{\sum_{s\in S} u_i(s)\sigma_i(s_i)\sigma_{-i}(s_{-i}), \sum_{s\in S} u_i(s)\sigma_i'(s_i)\sigma_{-i}(s_{-i})\right\} \\
&= \min\{u_i(\sigma_i, \sigma_{-i}), u_i(\sigma_i', \sigma_{-i})\}.
\end{aligned}$$

The hypotheses of the Debreu-Glicksberg-Fan theorem (Theorem 7) thus hold, and hence there exists a (necessarily totally mixed) Nash equilibrium of the ϵ-perturbed game. $\square$

Theorem 9 (Existence of perfect equilibria)**.** *Any finite-player mixed strategy game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$ for which each S_i is finite has at least one perfect equilibrium.*

Proof. Given perturbations ϵ_i for player i, call the function $\epsilon : S \to (0,1)^n$ defined by $\epsilon(s) = (\epsilon_i(s_i)_{i\in\mathcal{I}})$ a perturbation.

By the lemma, for any sequence of perturbations $\{\epsilon^k\}$ there exists a sequence $\{\sigma^k\}$ s.t. each σ^k is a Nash equilibrium of the ϵ^k-perturbed game. Suppose $\epsilon^k \to 0$. Since $\prod_{i\in\mathcal{I}} \Delta(S_i)$ is a compact set, the sequence $\{\sigma^k\}$ must have some accumulation point σ^*, and hence there exists some subsequence $\{\sigma^{k_j}\}$ s.t. $\sigma^{k_j} \to \sigma^*$. Taking this subsequence, and the corresponding subsequence $\{\epsilon^{k_j}\}$ of $\{\epsilon^k\}$, there exists a sequence of perturbations $\{\epsilon^{k_j}\}$ s.t. $\epsilon^{k_j} \to 0$ and a sequence $\{\sigma^{k_j}\}$ s.t. σ^{k_j} is a Nash equilibrium of the ϵ^{k_j}-perturbed game $G^m_{\epsilon^k}$ and $\sigma^{k_j} \to \sigma^*$. Hence σ^* is a perfect equilibrium. This completes the proof. $\square$

Given a family of games $\mathcal{G}_\Lambda$ parameterized by Λ, Proposition 12 stated that the Nash equilibrium correspondence on $\mathcal{G}_\Lambda$ has a closed graph. This is quite useful for finding Nash equilibria via approximations of a game. Define the *perfect equilibrium correspondence* $\mathrm{PE} : \Lambda \rightrightarrows \prod_{i\in\mathcal{I}} \Delta(S_i)$ by $\mathrm{PE}(\lambda) = \{\sigma \in \prod_{i\in\mathcal{I}} \Delta(S_i) \mid \sigma \text{ is a perfect equilibrium of } G^m(\lambda)\}$. Unlike the Nash equilibrium correspondence, the perfect equilibrium correspondence does not have a closed graph, as the following counterexample shows.

Example 15. Consider the following family of games $G(n)$ parameterized by $\mathbb{N}$:

	L_2	R_2
U_1	$1, 1$	$0, 0$
D_1	$0, 0$	$1/n, 1/n$

Best responses ar highlighted in blue (Player 1) and red (Player 2). For each $n \in \mathbb{N}$, we claim that (D_1, R_2) is a perfect equilibrium. To see this, define

$$\sigma_1^\epsilon(U_1) = \epsilon, \quad \sigma_1^\epsilon(D_1) = 1-\epsilon,$$
$$\sigma_2^\epsilon(L_2) = \epsilon, \quad \sigma_2^\epsilon(R_2) = 1-\epsilon.$$

For any $\epsilon < \frac{1}{n+1}$, we have

$$u_1(D_1, \sigma_2^\epsilon) = \frac{1-\epsilon}{n} > \epsilon = u_1(U_1, \sigma_2^\epsilon)$$

and so D_1 is a best response for Player 1. A symmetric calculation shows R_1 is a best response for Player 2 to σ_1^ϵ. Taking $\epsilon \to 0$ gives $(\sigma_1^\epsilon, \sigma_2^\epsilon) \to (D_1, R_2)$, and thus (D_1, R_2) is a perfect equilibrium of $G(n)$.

Now consider the limit game $G(\infty) = \lim_{n\to\infty} G(n)$. This game is:

	L_2	R_2
U_1	$1, 1$	$0, 0$
D_1	$0, 0$	$0, 0$

Now (D_1, R_2) is a weakly dominated Nash equilibrium, so by Proposition 14(i), (D_1, R_2) cannot be perfect. Hence we have found a sequence of games where the limit of the corresponding perfect equilibria is not a perfect equilibrium of the limit game.

2.9 Proper equilibrium

Perfect equilibrium is not quite as perfect as Selten had so confidently hoped. Consider the following example.

Example 16 (Myerson, 1978)**.** Consider the two player game with payoff matrix

$$\begin{array}{cccc} & a_2 & b_2 & c_2 \\ a_1 & 1,1 & 0,0 & -9,-9 \\ b_1 & 0,0 & 0,0 & -7,-7 \\ c_1 & -9,-9 & -7,-7 & -7,-7 \end{array}$$

Again, best responses are highlighted in blue (Player 1) and red (Player 2). Clearly, there are three pure strategy Nash equilibria: (a_1, a_2), (b_1, b_2) and (c_1, c_2). It can be shown there are no non-degenerate mixed strategy Nash equilibria.

(c_1, c_2) is weakly dominated and thus not a perfect equilibrium. However, both (a_1, a_2) and (b_1, b_2) are trembling hand perfect. To see that (b_1, b_2) is a perfect equilibrium, define

$$\begin{aligned} \sigma_1^\epsilon(a_1) &= \epsilon, & \sigma_1^\epsilon(b_1) &= 1 - 2\epsilon, & \sigma_1^\epsilon(c_1) &= \epsilon, \\ \sigma_2^\epsilon(a_2) &= \epsilon, & \sigma_2^\epsilon(b_2) &= 1 - 2\epsilon, & \sigma_2^\epsilon(c_2) &= \epsilon, \end{aligned}$$

Then for any $\epsilon < \frac{1}{2}$,

$$u_1(b_1, \sigma_2^\epsilon) = -7\epsilon = u_1(c_1, \sigma_2^\epsilon) > -9\epsilon = u_1(a_1, \sigma_2^\epsilon),$$

and hence b_1 is a best response to σ_2^ϵ. A symmetric calculation for Player 2 shows that b_2 is a best response to any σ_1^ϵ. Taking $\epsilon \to 0$, we have $(\sigma_1^\epsilon, \sigma_2^\epsilon) \to (b_1, b_2)$. Hence (b_1, b_2) is a perfect equilibrium.

Likewise, defining

$$\begin{aligned} \sigma_1^\epsilon(a_1) &= 1 - 2\epsilon, & \sigma_1^\epsilon(b_1) &= \epsilon, & \sigma_1^\epsilon(c_1) &= \epsilon, \\ \sigma_2^\epsilon(a_2) &= 1 - 2\epsilon, & \sigma_2^\epsilon(b_2) &= \epsilon, & \sigma_2^\epsilon(c_2) &= \epsilon, \end{aligned}$$

we have, for any $\epsilon < \frac{1}{4}$, that

$$\begin{aligned} u_1(a_1, \sigma_2^\epsilon) &= 1 - 11\epsilon \\ &> -7\epsilon = u_1(b_1, \sigma_2^\epsilon) \\ &> -9 + 4\epsilon = u_1(c_1, \sigma_2^\epsilon), \end{aligned}$$

so a_1 is a best response to σ_2^ϵ. Again, a similar calculation shows a_2 is a best response to σ_1^ϵ for all $\epsilon < \frac{1}{4}$. As $\epsilon \to 0$, $(\sigma_1^\epsilon, \sigma_2^\epsilon) \to (a_1, a_2)$, and thus (a_1, a_2) is a perfect equilibrium.

Yet there is a clear sense in which the equilibrium (b_1, b_2) seems less compelling than (a_1, a_2). In particular, we had to assume that costly trembles (c_i) happened with the same probability as less costly trembles (a_i).

Myerson thus (1978) proposes a further refinement of perfect equilibrium: *proper equilibrium*. Proper equilibrium imposes a restriction on the relative likelihood of trembles. Namely, given a strategy profile σ, consider ordering a player i's pure strategies $s_i \in S_i$ by their payoff to i. In an ϵ-proper equilibrium, we assume i plays the second-highest payoff strategy with at most ϵ times the probability of the highest payoff strategy, i plays the third-highest payoff strategy with at most ϵ times the probability of the second-highest payoff strategy, and so forth. Thus proper equilibrium captures the notion that a player will tremble to play a higher-payoff strategy relatively more often than they tremble to play a lower-payoff strategy.

Definition 24 (Proper equilibrium)**.** Consider a finite mixed strategy normal form game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$ with n players.

(a) *ϵ-proper equilibrium.* A totally mixed strategy profile σ is an *ϵ-proper equilibrium* if, for all $i \in \mathcal{I}$, if

$$u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i})$$

then

$$\sigma_i(s_i) \leq \epsilon \cdot \sigma_i(s_i').$$

That is, a totally mixed strategy profile σ is an ϵ-proper equilibrium if every player places greater probability weight on their better responses than their worse responses, by a factor of $1/\epsilon$.

(b) *Proper equilibrium.* A strategy profile σ^* is a *proper equilibrium* if

  (i) there exists a sequence $\{\epsilon^k\}$ with $\epsilon^k \geq 0$ and $\epsilon^k \to 0$, and

  (ii) there exists a sequence of profiles $\{\sigma^k\}$ s.t. each σ^k is an ϵ^k-proper equilibrium and $\sigma^k \to \sigma^*$.

Proposition 16. *Any proper equilibrium σ of a finite mixed strategy game G^m is a perfect equilibrium.*

Proof. Note that σ is a proper equilibrium if it is the limit of ϵ-proper equilibrium. Now, any ϵ-proper equilibrium is a Nash equilibrium of the $(\epsilon, \dots, \epsilon)$-perturbed game G^m_ϵ. Thus σ is the limit of ϵ-perfect equilibria as $\epsilon \to 0$, and so σ is perfect. $\square$

Theorem 10. *Every finite player mixed strategy normal form game $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i \in \mathcal{I}})$ such that S_i is finite for each $i \in \mathcal{I}$ has a proper equilibrium.*

Proof. The proof is almost identical to Theorem 9. $\square$

Example 16 (continued)**.** Returning to Myerson's (1978) example, we show that (a_1, a_2) is the only proper equilibrium. By Theorem 10, a proper equilibrium must exist. Suppose $\epsilon \in (0, 1)$, and let (σ_1, σ_2) be an ϵ-proper equilibrium. Since b_1 dominates c_1 and σ_2 is totally mixed, $u_1(b_1, \sigma_2) > u_1(c_1, \sigma_2)$, and thus $\sigma_1(c_1) \leq \epsilon \cdot \sigma_1(b_1)$. But then $u_2(c_2, \sigma_1) < u_2(a_2, \sigma_1)$, and so $\sigma_2(c_2) \leq \epsilon \cdot \sigma_2(a_2)$. It follows that $u_1(b_1, \sigma_2) < u_1(a_1, \sigma_2)$ and so

$\sigma_1(b_1) \leq \epsilon \cdot \sigma_1(a_1) \leq \epsilon$, and $\sigma_1(c_1) \leq \epsilon \cdot \sigma_1(b_1) \leq \epsilon^2$. By a similar argument, $\sigma_2(b_2) \leq \epsilon \cdot \sigma_2(a_2) \leq \epsilon$ and $\sigma_2(c_2) \leq \epsilon\sigma_2(b_2) \leq \epsilon^2$. These imply that $\sigma_1(a_1) \geq 1 - \epsilon - \epsilon^2$ and $\sigma_2(a_2) \geq 1 - \epsilon - \epsilon^2$. Taking $\epsilon \to 0$, we have that $(\sigma_1, \sigma_2) \to (a_1, a_2)$, so the only proper equilibrium is (a_1, a_2).

2.10 Focal points

In coordination games, Schelling (1960) introduced a particular refinement of Nash equilibrium relying on information external to the description of the game – in certain games, there may be a particular strategy that can act as a default in the absence of communication between players. Such a strategy is a *focal point* or *Schelling point*.

Example 17 (Schelling, 1960)**.** Consider a coordination game motivated as follows: two strangers are to meet in New York, and they cannot communicate with each other. If they choose the same location, they receive a positive payoff, and zero otherwise. Experimentally, Schelling (1960) finds that the most common strategy is to choose Grand Central Terminal, a central New York landmark, rather than some less obvious location. Yet there is nothing in the formal structure of the game that favours this particular location over any other.

Like many of Schelling's insights, this is not really something that can be easily formalized.

2.11 Payoff dominance and risk dominance

In games with coordination, equilibrium selection concepts refine Nash equilibrium – when there are multiple equilibria, determining which equilibrium will be played requires specifying additional criteria. The classic Harsanyi-Selten approach involves specifying desirable properties of equilibria to determine which agents may arrive at.

Definition 25 (Harsanyi & Selten, 1988)**.** Consider a finite player game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$.

(a) *Payoff dominance.* A Nash equilibrium s^* is said to *payoff dominate* a Nash equilibrium s' if
$$u_i(s^*) > u_i(s') \qquad \text{for all } i \in \mathcal{I}.$$
A Nash equilibrium s^* is said to be a *payoff dominant equilibrium* of G if it payoff dominates all other Nash equilibrium s' of G.

(b) *Bilateral risk dominance.* In a two-player game, the *Nash product* at a Nash equilibrium s^* relative to an alternative equilibrium s' is given by the product of deviation losses of both players, that is
$$(u_1(s_1^*, s_2^*) - u_1(s_1', s_2^*))(u_2(s_1^*, s_2^*) - u_2(s_1^*, s_2')).$$
A Nash equilibrium s^* is said to (bilaterally) *risk dominate* a Nash equilibrium s' if s^* has strictly greater Nash product than s', that is, if
$$(u_1(s_1^*, s_2^*)-u_1(s_1', s_2^*))(u_2(s_1^*, s_2^*)-u_2(s_1^*, s_2')) > (u_1(s_1', s_2')-u_1(s_1^*, s_2'))(u_2(s_1', s_2')-u_2(s_1', s_2^*)).$$

If a Nash equilibrium s^* of G risk dominates all other Nash equilibria of G then it is called a *risk dominant equilibrium*.

Example 18. Consider the coordination game:

	a_2	b_2
a_1	$6, 6$	$3, 4$
b_1	$4, 3$	$4, 4$

There are two pure strategy Nash equilibria, (a_1, a_2) and (b_1, b_2) [there is also a mixed strategy Nash equilibrium]. Since $u_i(a_1, a_2) = 6 > 3 = u_i(b_1, b_2)$ for $i = 1, 2$, we have that (a_1, a_2) payoff dominates (b_1, b_2).

However, we have Nash product relation (a_1, a_2) vs (b_1, b_2) of

$$\begin{aligned}(u_1(a_1,a_2) - u_1(b_1,a_2))(u_2(a_1,a_2) - u_2(a_1,b_2)) &= (6-4)(6-4) = 4\\ &> 1 = (4-3)(4-3)\\ &= (u_1(b_1,b_2) - u_1(a_1,b_2))(u_2(b_1,b_2) - u_2(b_1,a_2)).\end{aligned}$$

Hence (a_1, a_2) risk dominates (b_1, b_2).

2.12 Correlated equilibrium

Nash equilibrium can be envisaged as a kind of self-enforcing agreement, in the sense that in a Nash equilibrium, no player can do any better by unilaterally deviating and doing something else. This is not to say that a Nash equilibrium is necessarily a credible agreement – some Nash equilibria only exist due to threats that a player would be better off reneging on.[8] Yet Nash equilibrium is somewhat less general than the notion of a self-enforcing agreement, as the following examples show. In some situations, Nash equilibrium is the inappropriate solution concept because it is inconsistent with certain kinds of communication.

Example 19 (Battle-of-the-sexes)**.** The now rather dated motivation for this game, introduced by Luce and Raiffa (1957), imagines a couple (man and woman) hoping to meet for the evening. They have a choice between a prize fight F (preferred by the man) and a ballet B (preferred by the woman). They cannot communicate, and only receive a positive payoff if they choose the same event. Assigning the woman as Player 1, the payoff matrix is

	b_2	f_2
b_1	$3, 2$	$0, 0$
f_1	$0, 0$	$2, 3$

The pure strategy Nash equilibria are (b_1, b_2) and (f_1, f_2). There is also a mixed Nash equilibrium. Suppose Player 1 plays b_1 with probability p. Then Player 2 is indifferent

[8] In the literature, you sometimes see it argued that Nash equilibria are *not* self-enforcing, in the sense that they are not *strategically stable*, e.g. Kohlberg & Mertens (1986) say this. However, we are not using "self-enforcing" in their sense here.

over b_2 and f_2 iff $2p = 3(1-p) \Rightarrow p = \frac{3}{5}$. Likewise, if Player 2 plays b_2 with probability q, then Player 1 is indifferent over b_1 and f_1 iff $3q = 2(1-q) \Rightarrow q = \frac{2}{5}$. Hence we have mixed strategy Nash equilibrium $((\frac{3}{5}, \frac{2}{5}), (\frac{2}{5}, \frac{3}{5}))$.

Now to see that Nash equilibrium can be interpreted as a self-enforcing agreement, we might imagine that players jointly agree to play (b_1, b_2). The profile (b_1, b_2) is self-enforcing, in the sense that neither player, if rational, gains from deviating. The same argument of course applies to (f_1, f_2) and to the mixed strategy Nash equilibrium.

Yet there are self enforcing agreements that are not Nash equilibria. Suppose the couple agree to flip a fair coin. If it lands on Heads, they will both attend the ballet, i.e. play the Nash equilibrium (b_1, b_2), and if it lands on Tails, they both attend the prize fight, i.e. play the Nash equilibrium (f_1, f_2). This achieves an expected payoff of $\frac{5}{2}$ for each player, which is not a Nash equilibrium payoff – it is greater than the expected payoff in the mixed strategy Nash equilibrium, in which (b_1, f_2) and (f_1, b_2) are played with positive probability. Furthermore, this is a self-enforcing agreement. If the coin lands on Heads, neither player i can profitably deviate by playing f_i. We see then that Nash equilibrium might not allow sufficient room to communicate.

Example 20. The coin flip story is applicable in the battle-of-the-sexes, but it is a primitive way to communicate prior to play. More generally, we might imagine that there is a mediator who can perform randomizations and who tells the players what to play depending on the outcome.

Consider the following game:

	L	R
U	$5,1$	$0,0$
D	$4,4$	$1,5$

There are three Nash equilibria: pure strategy Nash equilibria (U, L) and (D, R) and a mixed strategy Nash equilibrium $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

Suppose players find a mediator who chooses $\omega \in \{1, 2, 3\}$ uniformly at random, so each with probability $\frac{1}{3}$. Players do not observe ω directly. The mediator proposes the following:

(i) if $\omega = 1$, tell Row to play U and Column to play L;

(ii) if $\omega = 2$, tell Row to play D and Column to play L;

(iii) if $\omega = 3$, tell Row to play D and Column to play R.

It is a perfect Bayesian equilibrium for players to follow the mediator's advice here.[9]

The notion of *correlated equilibrium* fully captures the notion of a self-enforcing agreement.

[9] See section 4.4 for definition and discussion of perfect Bayesian equilibria. If Row hears U then Row believes Column will play L. Row's best response to this belief is to play U. If Row hears D, then Row believes Column will play L with probability $\frac{1}{2}$ and R with probability $\frac{1}{2}$, and so D is a best response for Row. Column's strategy can be checked similarly.

Definition 26 (Correlated equilibrium)**.**

(a) *Correlating mechanism.* A *correlating mechanism* is a tuple $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$ where

(i) Ω is a finite set of states of the world;

(ii) for each player i in finite player set $\mathcal{I}$, $\mathcal{H}_i$ is a partition of Ω, with the function $h_i : \Omega \to \mathcal{H}_i$ assigning to each ω the element $h_i(\omega) = (H_i \in \mathcal{H}_i \mid \omega \in H_i)$, and

(iii) p is a probability distribution on Ω.[10]

(b) *Correlated strategy.* A *correlated strategy* for i is a function $\sigma_i : \Omega \to S_i$ that is measurable with respect to information partition $\mathcal{H}_i$, i.e. if $h_i(\omega) = h_i(\omega')$ then $\sigma_i(\omega) = \sigma_i(\omega')$.[11]

(c) *Correlated equilibrium.* A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is an (objective) *correlated equilibrium* relative to the correlating mechanism $(\Omega, \{\mathcal{H}_i\}, p)$ if, for every i and every correlated strategy σ_i,

$$\sum_{\omega\in\Omega} u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega))p(\omega) \geq \sum_{\omega\in\Omega} u_i(\sigma_i(\omega), \sigma_{-i}^*(\omega))p(\omega).$$

Unpacking the definition, correlated equilibrium requires that σ_i^* maximizes i's *ex ante* payoff. That is, the strategy can be considered a contingent plan to be implemented after learning the partition element ω. Note that this is equivalent to σ_i maximizing i's *interim* payoff for each $H_i \in \mathcal{H}_i$ that is reached with positive probability: that is, for all $i \in \mathcal{I}$, for all $\omega \in \Omega$, and for all $s_i' \in S_i$,

$$\sum_{\omega'\in h_i(\omega)} u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega'))p(\omega' \mid h_i(\omega)) \geq \sum_{\omega'\in h_i(\omega)} u_i(s_i', \sigma_{-i}^*(\omega'))p(\omega' \mid h_i(\omega)),$$

where $p(\omega' \mid h_i(\omega))$ is the conditional probability of ω' given that the true state lies in $h_i(\omega)$. By Bayes' rule, this is

$$p(\omega' \mid h_i(\omega)) = \frac{\mathbb{P}\{h_i(\omega) \mid \omega'\}p(\omega')}{\sum_{\omega''\in h_i(\omega)} \mathbb{P}\{h_i(\omega) \mid \omega''\}p(\omega'')} = \frac{p(\omega')}{p(h_i(\omega))}.$$

The definition is unwieldy, in that naively searching for all the correlated equilibria would require checking many arbitrary correlating mechanisms. Fortunately, the problem can be reduced to checking only correlating mechanisms in a particular class.

Definition 27 (Direct mechanism)**.** A *direct mechanism* is a correlating mechanism $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$ s.t. $\Omega = S$, $h_i(s) = \{s' \in S : s_i' = s_i\}$, and where p is some probability distribution over pure strategy profiles.

[10] More generally (i.e. with a continuum of states), we can define a correlating mechanism as a tuple $((\Omega, \mathcal{F}, p), \{H_i\}_{i\in\mathcal{I}})$, where $(\Omega, \mathcal{F}, p)$ is a probability space, with $\mathcal{F}$ being a σ-algebra, and $\{\mathcal{H}_i\}$ is defined as before.

[11] Note this differs from our definition of a correlated strategy *profile*, which is simply a point in $\Delta(S)$.

For any correlated equilibrium relative to some correlating mechanism, there is an outcome-equivalent correlating mechanism relative to some direct mechanism:

Theorem 11 (Revelation principle)**.** *Suppose σ^* is a correlated equilibrium relative to correlating mechanism $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$. Define $q(s) = \mathbb{P}\{\sigma^*(\omega) = s\}$. Then the strategy profile $\tilde{\sigma}$ with $\tilde{\sigma}_i(s) = s_i$ for all $s_i \in S_i$ and all $i \in \mathcal{I}$ is a correlated equilibrium relative to the direct mechanism $(S, \{\tilde{\mathcal{H}}_i\}_{i\in\mathcal{I}}, q)$.*

Proof. Suppose s_i is recommended to i with positive probability, i.e. $p(s_i, s_{-i}) > 0$ for some s_{-i}. We require that under the direct mechanism, i does not benefit from choosing some $s_i' \neq s_i$ when s_i is suggested. If s_i is recommended, then i's expected payoff from playing s_i' is

$$\sum_{s_{-i}\in S_{-i}} u_i(s_i', s_{-i})q(s_{-i} \mid s_i).$$

If there is only one information set $H_i \in \mathcal{H}_i$ s.t. $\sigma_i^*(H_i) = s_i$, then conditioning on s_i is equivalent to conditioning on H_i so the proposition holds trivially. More generally, substituting for q gives expected payoff to playing s_i' of

$$\frac{1}{\mathbb{P}\{\sigma_i^*(\omega) = s_i\}} \sum_{\omega|\sigma_i^*(\omega)=s_i} u_i(s_i', \sigma_{-i}^*(\omega))p(\omega).$$

Rearranging,

$$\frac{1}{\mathbb{P}\{\sigma_i^*(\omega) = s_i\}} \sum_{H_i|\sigma_i^*(H_i)=s_i} \mathbb{P}\{H_i\} \left[\sum_{\omega\in H_i} u_i(s_i', \sigma_{-i}^*(\omega))p(\omega \mid H_i)\right].$$

Given $(\Omega, \{\mathcal{H}_i, p, \sigma^*)$ is a correlated equilibrium, we have that each bracketed term for which $\mathbb{P}\{H_i\} > 0$ is maximized at $\sigma_i(H_i) = s_i$. Hence s_i is optimal given recommendation s_i, under the direct mechanism. □

Hence to find all correlated equilibria, we need only consider the class of direct mechanisms. Note this direct mechanism is effectively the mechanism we considered in the example, where a mediator recommends a strategy to each player.

The probability distribution over profiles is what matters for a correlated equilibrium. We call the probability distribution q over strategy profiles s a *correlated equilibrium distribution* if it is the distribution generated by some correlated equilibrium.

Proposition 17. *The distribution $q \in \Delta(S)$ is a correlated equilibrium distribution iff for all $i \in \mathcal{I}$, for all $s_i \in S_i$ with $q(s_i) > 0$, and for all $s_i' \in S_i$,*

$$\sum_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})q(s_{-i} \mid s_i) \geq \sum_{s_{-i}\in S_{-i}} u_i(s_i', s_{-i})q(s_{-i} \mid s_i).$$

Proof. Suppose q satisfies the inequality. Then the profile σ^* with $\sigma_i^*(s) = s_i$ is a correlated equilibrium given direct mechanism $(S, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, q)$, for the inequality states precisely that s_i is the best response for i on being recommended s_i.

Conversely, suppose q is a correlated equilibrium distribution. Then q corresponds to some correlated equilibrium σ^* relative to some direct mechanism $(S, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, q)$. Hence for all i and all recommendations s_i, the inequality must hold by optimality of the recommendation s_i. □

We have the corollary:

Corollary 5. *The distribution $q \in \Delta(S)$ is a correlated equilibrium distribution iff for all $i \in \mathcal{I}$, for all $s_i \in S_i$ with $q(s_i) > 0$, and for all $s_i' \in S_i$,*

$$\sum_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})q(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} u_i(s_i', s_{-i})q(s_i, s_{-i}).$$

Proof. Let $q(s_i)$ denote the marginal probability of s_i under q. Then we have $q(s_{-i} \mid s_i) = \frac{q(s_i,s_{-i})}{q(s_i)}$. Substituting into the inequality in Proposition 17 and multiplying through by $q(s_i)$ yields the result. □

Proposition 18. *Every Nash equilibrium is a correlated equilibrium.*

Proof. We provide a proof for finite player games with finite strategy sets.

We require that for all i and all s_i with $q(s_i) > 0$, that

$$\sum_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})q(s_{-i} \mid s_i) \geq \sum_{s_{-i}\in S_{-i}} u_i(s_i', s_{-i})q(s_{-i} \mid s_i)$$

for all $s_i' \in S_i$, under the distribution q induced by the Nash equilibrium. In case of a pure strategy Nash equilibrium s^*, we have

$$q(s_{-i} \mid s_i^*) = \begin{cases} 1 & \text{if } s_{-i} = s_{-i}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the inequality reduces to

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*),$$

which is precisely the definition of pure strategy Nash equilibrium.

In case of a mixed strategy Nash equilibrium, since players mix independently, we have that $q(s_{-i} \mid s_i^*) = \sigma_{-i}^*(s_{-i})$ for any s_i^* in the support of σ_i^*. Hence, for all i, s_i^* in the support of σ_i^*, and $s_i \in S_i$, we have that

$$\sum_{s_{-i}\in S_{-i}} u_i(s_i^*, s_{-i})\sigma_{-i}^*(s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})\sigma_{-i}^*(s_{-i}),$$

which is precisely the definition of mixed strategy Nash equilibrium. □

Corollary 6. *In any finite game, there exists a correlated equilibrium.*

Proof. Follows immediately from the Nash existence theorem and the fact that Nash equilibria are correlated equilibria. See Hart and Schmeidler (1989) for a direct existence proof. □

Proposition 19. *The sets of correlated equilibrium distributions and payoffs are convex.*

Proof. For any distribution $p \in \Delta(S)$, define $S_i^p := \{s_i \in S_i : p(s_i) > 0\}$. Note that if $p \in \Delta(S)$, then for any i and any $s_i \in S_i^p$, we have that $p(s_{-i} \mid s_i) = \frac{p(s_i, s_{-i})}{p(s_i)}$, where $p(s_i)$ is the marginal probability of s_i induced by the distribution p.

Suppose q and q' are correlated equilibrium distributions. For any pair $\lambda, \in (0,1)$, let $q''(s_i, s_{-i}) = \lambda q(s_i, s_{-i}) + (1-\lambda)q'(s_i, s_{-i})$, so

$$q''(s_{-i} \mid s_i) = \frac{q''(s_i, s_{-i})}{q''(s_i)} = \frac{\lambda q(s_i, s_{-i}) + (1 - \lambda q'(s_i, s_{-i}))}{q''(s_i)}.$$

By Proposition 17, it is sufficient to show that for all $i \in \mathcal{I}$, for all $s_i' \in S_i$ and for all $s_i \in S_i^{q''} = S_i^q \cup S_i^{q'}$,

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) q''(s_{-i} \mid s_i) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) q''(s_{-i} \mid s_i)$$

iff

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \frac{\lambda q(s_i, s_{-i}) + (1 - \lambda q'(s_i, s_{-i}))}{q''(s_i)} \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \frac{\lambda q(s_i, s_{-i}) + (1-\lambda) q'(s_i, s_{-i})}{q''(s_i)}.$$

Multiplying both sides through by $q''(s_i)$, we have

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})[\lambda q(s_i, s_{-i}) + (1-\lambda) q'(s_i, s_{-i})] \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i})[\lambda q(s_i, s_{-i}) + (1-\lambda) q'(s_i, s_{-i})],$$

which is equivalent to

$$\begin{aligned}\lambda \sum_{s_{-i}} u_i(s_i, s_{-i}) q(s_i, s_{-i}) + (1-\lambda) \sum_{s_{-i}} u_i(s_i, s_{-i}) q'(s_i, s_{-i}) \geq &\lambda \sum_{s_{-i}} u_i(s_i, s_{-i}) q(s_i, s_{-i}) \\ &+ (1-\lambda) \sum_{s_{-i}} u_i(s_i', s_{-i}) q'(s_i, s_{-i}).\end{aligned}$$

By Corollary 5, this holds iff

$$\begin{aligned}\lambda \sum_{s_{-i}} u_i(s_i, s_{-i}) q(s_{-i} \mid s_i) + (1-\lambda) \sum_{s_{-i}} u_i(s_i, s_{-i}) q'(s_{-i} \mid s_i) \geq &\lambda \sum_{s_{-i}} u_i(s_i, s_{-i}) q(s_{-i} \mid s_i) \\ &+ (1-\lambda) \sum_{s_{-i}} u_i(s_i', s_{-i}) q'(s_{-i} \mid s_i).\end{aligned}$$

This inequality holds, since q and q' are correlated equilibrium distributions. Hence q'' is a correlated equilibrium distribution. Since $\lambda \in (0,1)$ was arbitrary, it follows that any convex combination of correlated equilibrium distributions is a correlated equilibrium distribution, and thus the set of correlated equilibrium distributions is convex. The statement wrt payoffs is a straightforward corollary. $\square$

In fact, the sets of correlated equilibrium distributions and payoffs are *convex polytopes.*

Definition 28. *Public correlating mechanism.* A correlating mechanism $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$ is a *public correlating mechanism* if for all $i \in \mathcal{I}$, $\mathcal{H}_i = \mathcal{H}$ for some partition $\mathcal{H}$. That is, every player has the same partition wrt states of the world. Since if $\mathcal{H}$ contains some non-singleton set H, we can replace the subset $H \subseteq \Omega$ with some state ω' s.t. $p(\omega') = \sum_{\omega\in H} p(\omega)$. Hence, wlog, we can represent a public correlating mechanism by (Ω, p).

We call σ^* a *public correlated equilibrium* if it is a correlated equilibrium relative to a public correlating mechanism.

Proposition 20. *The set of public correlated equilibrium payoffs is the convex hull of the set of Nash equilibrium payoffs.*

Since Proposition 19 shows the set of correlated equilibrium payoffs is convex, it is often convenient to look at payoffs graphically.

Definition 29 (Feasibility and individual rationality)**.**

(a) *Feasible payoffs.* In a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a payoff profile $v = (v_1, \dots, v_n)$ is *feasible* if there is some probability distribution $p \in \Delta(S)$ s.t.

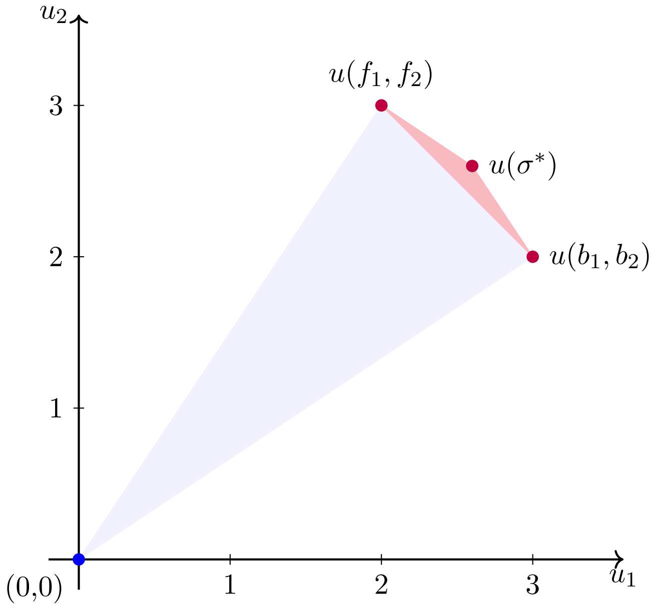
$$v_i = \sum_{s\in S} u_i(s)p(s) \qquad \text{for all } i \in \mathcal{I}.$$

(b) *Individually rational payoffs.* In a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a payoff profile $v = (v_1, \dots, v_n)$ is *individually rational* if for each $i \in \mathcal{I}$,

$$v_i \geq \min_{\sigma_{-i}\in\Delta_{-i}(S_{-i})} \max_{\sigma_i\in\Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \underline{v}_i.$$

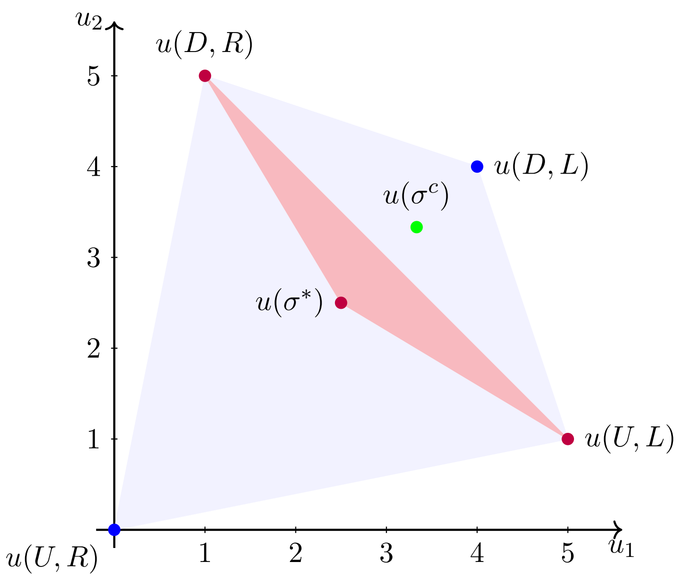
A payoff profile v is *strictly individually rational* if this holds with strict inequality for all i.

Example 19 (continued)**.** We show, for the battle-of-the-sexes game, the set of feasible payoffs (blue region), the set of Nash equilibrium payoffs (red points), and correlated equilibrium payoffs (red region).



σ^* denotes the mixed strategy Nash equilibrium profile $\sigma^* = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$. The convex hull of Nash equilibria is denoted in red. Any point in the convex hull of Nash equilibria is the payoff profile of some public correlated equilibrium.

Example 20 (continued)**.** When the correlating mechanism is not public, it is possible to achieve correlated equilibrium payoffs that lie outside the convex hull of the set of Nash equilibrium payoffs.



The set of feasible payoffs is shaded blue and the convex hull of the set of Nash equilibrium payoffs is shaded red (σ^* is the mixed strategy Nash equilibrium $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$).

The correlated equilibrium we previously considered, where (U, L), (D, L) and (D, R) were each played with probability $\frac{1}{3}$, is denoted σ^c. We see that $u(\sigma^c)$ lies outside the convex hull.

We motivated the correlated equilibrium σ^c by means of a mediator telling to each player privately which strategy to play. This is not a public correlating mechanism, since players have private information – if told to play D, for example, Row cannot infer whether the mediator told Column to play L or R. Were the mediator to announce which strategy she recommends to each player publicly, then σ^c could not be sustained. For example, if the mediator publicly announced that players should play (D, L), then Row would have a profitable deviation U and Column would have a profitable deviation R.

In the (objective) correlating mechanisms $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$ previously introduced, we assume a *common prior* p over set of states Ω. Equivalently, the players share the same probability distribution over equilibrium play (by Theorem 11). Subjective correlated equilibrium weakens this assumption.

Definition 30 (Subjective correlated equilibrium)**.**

(a) *Subjective correlating mechanism.* A *subjective correlating mechanism* is a tuple $(\Omega, \{\mathcal{H}_i, p_i\}_{i\in\mathcal{I}})$ where Ω and $\mathcal{H}_i$ are defined as in Definition 26 and for each player $i \in \mathcal{I}$, p_i is a probability distribution over Ω.

(b) *Subjective correlated equilibrium.* A profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ of functions $\sigma_i^* : \Omega \to S_i$ is a *subjective correlated equilibrium* relative to subjective correlating mechanism $(\Omega, \{\mathcal{H}_i, p_i\}_{i\in\mathcal{I}})$ if for every i and every correlated strategy σ_i,

$$\sum_{\omega\in\Omega} u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega))p_i(\omega) \geq \sum_{\omega\in\Omega} u_i(\sigma_i(\omega), \sigma_{-i}^*(\omega))p_i(\omega).$$

Example 20 (continued)**.** Subjective correlated equilibrium does not require that players are correct about the objective probability distribution over states (or about play). We can obtain (D, L) (with payoff $(4, 4)$) in subjective correlated equilibrium play. Consider the direct mechanism $p_1 = p_2 = \frac{1}{3}(U, L) + \frac{1}{3}(D, L) + \frac{1}{3}(D, R)$. This is a subjective correlated equilibrium, and since players need not be correct about beliefs, we can have (D, L) played with probability 1.

Example 19 (continued)**.** Moreover, a subjective correlated equilibrium can achieve *ex ante* payoffs that are not even feasible (of course, such payoffs can never be achieved ex post). In the battle-of-the-sexes game, suppose $\Omega = \{\omega', \omega''\}$, $p_1(\omega') = 1$ and $p_2(\omega'') = 1$. Then each player i finds it optimal to play

$$\sigma_i(\omega) = \begin{cases} b_i & \text{if } \omega = \omega', \\ f_i & \text{if } \omega = \omega''. \end{cases}$$

Given each players priors, we have *ex ante* expected payoffs,

$$\begin{aligned} u_1(\omega) = \sum_{\omega \in \Omega} u_1(\sigma(\omega))p_1(\omega) &= u_1(\sigma(\omega'))p_1(\omega') = 4, \\ u_2(\omega) = \sum_{\omega \in \Omega} u_2(\sigma(\omega))p_2(\omega) &= u_2(\sigma(\omega''))p_2(\omega'') = 4. \end{aligned}$$

So the ex ante payoff is $(4, 4)$, yet this is not feasible!

2.13 Epistemic foundations of equilibrium

Nash equilibrium and its cousins are often treated as something of a black box. Certainly, much thought about what conditions are sufficient for equilibrium to be played has only been developed long after the initial solution concepts were developed. In this section, we try to clarify what rational agents need to know in order to play equilibrium.

Consider a normal form game $G = (\mathcal{I}, (S_i, u_i)_{i \in \mathcal{I}})$ of n players, and let $(\Omega, \mathcal{F})$ be a set of states Ω equipped with a σ-algebra $\mathcal{F}$. We assume each player i has a partitional information function $h_i : \Omega \to \mathcal{F}$. Each state $\omega \in \Omega$ specifies, for each player $i \in \mathcal{I}$, i's knowledge $h_i(\omega) \in \mathcal{F}$, i's pure strategy $s_i(\omega) \in S_i$, and i's belief $\mu_i(\omega) \in \Delta(S_{-i})$ about opponents' play. We assume that if $\omega, \omega' \in h_i(\omega)$ then $\mu_i(\omega) = \mu_i(\omega')$. We use $s = (s_1, \ldots, s_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ to denote the strategy and belief profiles respectively.

Proposition 21. *Let G be a finite game. Suppose that in state $\omega \in \Omega$, for each player $i \in \mathcal{I}$*

(i) $h_i(\omega) \subseteq \{\omega' \in \Omega \mid s_{-i}(\omega') = s_{-i}(\omega)\}$ *(player i knows the actions of other players);*

(ii) *the support of $\mu_i(\omega)$ lies in $\{s_{-i}(\omega') \in S_{-i} \mid \omega' \in h_i(\omega)\}$ (player i has a belief consistent with such knowledge);*

(iii) *$s_i(\omega)$ is a best response to $\mu_i(\omega)$ (player i is rational).*

Then $s(\omega)$ is a Nash equilibrium of G.

Proof. By (i) and (ii), each player i's beliefs assign probability one to the profile $s_{-i}(\omega)$, and by (iii), $s_i(\omega)$ is a best response to $\mu_i(\omega) = s_{-i}(\omega)$. Hence $s(\omega)$ must be a Nash equilibrium. $\square$

That each player knows the strategy of their opponents is a very strong assumption. Aumann & Brandenberger (1995) ask how much we can relax this assumption. In the case of two-player games, we can replace the assumption that players know the actions of other players with the weaker assumption that players mutually know each others' beliefs and know they are both rational:

Proposition 22. *Let G be a two-player finite game. Suppose that in state $\omega \in \Omega$, for each player $i \in \mathcal{I}$,*

(i) $h_i(\omega) \subseteq \{\omega' \in \Omega \mid \mu_{-i}(\omega') = \mu_{-i}(\omega)\}$ *(player i knows the belief of her opponent);*

(ii) *for any $\omega' \in h_i(\omega)$, an action $s_{-i}(\omega')$ is in the support of $\mu_i(\omega')$ only if $s_{-i}(\omega')$ is a best response for player $-i$ to $\mu_{-i}(\omega')$ (player i knows her opponent is rational and i's beliefs are consistent with her knowledge).*

Then the strategy profile $\sigma = (\mu_1(\omega), \mu_2(\omega))$ is a Nash equilibrium of G.

Proof. Let $s_{-i} \in S_{-i}$ lie in the support of $\mu_i(\omega)$. Then there is some state $\omega' \in h_i(\omega)$ such that $s_{-i}(\omega') = s_{-i}$, and so s_{-i} is a best response to $\mu_{-i}(\omega')$ by (ii). Now by (i), $\mu_{-i}(\omega) = \mu_{-i}(\omega')$. □

The assumptions here are quite weak – we don't need beliefs to be derived from a common prior, we don't need that beliefs or rationality are common knowledge, and we don't actually need that the game is common knowledge (only mutual knowledge).

Unfortunately, this result does not extend to games with more than two players. In general, we need that beliefs are derived from a common prior, that rationality is mutual knowledge, and that beliefs are common knowledge.

Proposition 23. *Let G be a finite game of n players, and let $\mu(\omega) = (\mu_1(\omega), \ldots, \mu_n(\omega))$. Suppose players have a common prior p on Ω that assigns positive probability to the structure of the game being mutually known. Suppose that in state $\omega \in \Omega$, the structure of G is mutual knowledge, that players' rationality is mutual knowledge, and that the belief profile $\mu(\omega)$ is common knowledge. Then $\mu(\omega)$ is a Nash equilibrium of G.*

Proof. See Aumann & Brandenberger (1995) for the proof. □

Correlated equilibrium also has an epistemic foundation:

Proposition 24 (Aumann, 1987)**.** *Let G be a finite game and let $(\Omega, \{\mathcal{H}_i\}_{i\in\mathcal{I}}, p)$ be a correlating mechanism. For each player i, let $\mathcal{F}_i$ be the σ-algebra generated by $\mathcal{H}_i$. Let σ be a correlated strategy profile. For every player $i \in \mathcal{I}$, suppose that*

(i) *player i is rational;*

(ii) *player i's belief μ_i is derived from the common prior p and $p(h_i(\omega)) > 0$ for all $\omega \in \Omega$, and*

(iii) *player i's strategy $\sigma_i : \Omega \to S_i$ is measurable wrt $\mathcal{F}_i$.*

Then σ is a correlated equilibrium relative to the correlating mechanism $(\Omega, \{\mathcal{H}_i\}, p)$.

Proof. This is immediate from Definition 26 □

2.14 Learning equilibrium

We mentioned that one hypothesis for how Nash equilibrium (or other related notions) might arise is as the outcome of a learning process. There is a large literature that attempts to model such learning processes, and it is now big in computer science because of the machine learning craze.

2.14.1 Fictitious play

Brown (1951) introduced *fictitious play*, a learning rule in which each player assumes their opponents play stationary mixed strategies and choose a best response based on the empirical frequency of opponents' play. Suppose there are two players, $i = 1, 2$. Suppose the players play a finite game $G = (\{1, 2\}, \{S_i, u_i\}_{i=1,2})$ at times $t = 0, 1, 2, \ldots$. For each t, let $\eta_i^t : S_{-i} \to \mathbb{N}$ be such that $\eta_i^t(s_{-i})$ gives the number of times at time t that player i has observed s_{-i} in the past. We take players' actions in period 0 as exogenously given. Since each player assumes their opponent follows a stationary mixed strategy, each player's beliefs are some distribution μ_i^t on $\Delta(S_{-i})$. Players update their beliefs using Bayesian updating. Since the distribution of $\eta_i^t(s_{-i})$ is multinomial, a typical assumption is that μ_i^0 is a Dirichlet distribution with $\mu_i^0(\sigma_{-i}) = \frac{1}{B}\prod_{s_{-i}\in S_{-i}} \sigma_{-i}(s_{-i})^{\eta_i^0(s_{-i})}$, where B is a normalizing constant.[12] In period t, the player expects play $\hat{\mu}_i^t(s_{-i}) = \mathbb{E}_{\mu_i^t}\sigma_{-i}(s_{-i})$. Because the prior is Dirichlet, Bayesian updating implies that $\hat{\mu}_i^t(s_{-i}) = \frac{\eta_i^t(s_{-i})}{\sum_{s'_{-i}\in S_{-i}} \eta_i^t(s'_{-i})}$. Note that each player's assumption that the other is playing a stationary mixed strategy is wrong, because both are instead following fictitious play. Thus even though players update their forecasts correctly, their priors are unreasonable. At each time t, player i chooses a best response to her forecast μ_i^t, that is, she chooses a strategy $s_i^t \in \arg\max_{s_i \in S_i} u_i(s_i, \mu_i^t)$. Under the assumption that her opponent plays a stationary mixed strategy – and thus is *not* updating their own play based on her actions – this myopic choice would be optimal.

To investigate convergence, we should also track the number of times a player has played a given strategy in the preceding periods. For each t, let $\alpha_i^t : S_i \to \mathbb{N}$ be such that $\alpha_i^t(s_i)$ gives the number of times at time t that player i has previously played s_i.

Definition 31 (Convergence of fictitious play)**.**

(a) *Convergence.* We say that the sequence of pure strategy profiles $\{s^t\}$ *converges* to a pure strategy profile s if there exists a time T such that $s^t = s$ for all $t \geq T$.

(b) *Convergence in a time-average sense.* We say that a sequence of pure strategy profiles $\{s^t\}$ *converges in a time-average sense* to a mixed strategy profile σ if, for each $i \in \mathcal{I}$, $\sigma_i(s_i) = \lim_{t\to\infty} \frac{1}{t}\alpha_i^t(s_i)$ for all $s_i \in S_i$.

Proposition 25. *Given a finite game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, if a sequence of fictitious play $\{s^t\}$ converges in a time-average sense to a mixed strategy profile σ, then σ is a Nash equilibrium of $G^m = (\mathcal{I}, (\Delta(S_i), u_i)_{i\in\mathcal{I}})$.*

Proof. Suppose $s^t \to \sigma$ in a time-average sense but σ is not a Nash equilibrium of G^m. Then for some player i, there exists a pair of strategies $s_i, s_i' \in S_i$ such that $\sigma_i(s_i) > 0$ and $u_i(s_i', sigma_{-i}) > u_i(s_i, \sigma_{-i})$. Let $N = |S_{-i}|$. Choose $\epsilon > 0$ s.t. $\epsilon < \frac{1}{2}(u_i(s_i', \sigma_{-i}) - u_i(s_i, \sigma_{-i}))$ and T sufficiently large that $\left|\hat{\mu}_i^t(s_{-i}) - \sigma_{-i}(s_{-i})\right| < \frac{\epsilon}{2N}$ for all

[12] The Dirichlet distribution is the conjugate prior distribution of the multinomial distribution.

$t \geq T$. Now, for any $t \geq T$,

$$\begin{aligned} u_i(s_i, \hat{\mu}_i^t) &= \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \hat{\mu}_i^t(s_{-i}) \\ &\leq \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma_{-i}(s_{-i}) + \epsilon \\ &< \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \sigma_{-i}(s_{-i}) - \epsilon \\ &\leq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \hat{\mu}_{-i}^t(s_{-i}) = u_i(s_i', \hat{\mu}_i^t). \end{aligned}$$

Hence s_i is never a best response for $t \geq T$, and so $\lim_{t \to \infty} \frac{1}{t} \alpha_i^t(s_i) = 0$, yet $\sigma_i(s_i) > 0$, yielding a contradiction. □

This is good news for fictitious play – if fictitious play converges, then agents learn a Nash equilibrium. Even better, we know that in some two-person games, convergence under fictitious play is guaranteed. For example, Robinson (1951) shows that in two-person finite zero-sum games, fictitious play converges in the time-average sense, and Miyasawa (1951) proves the same result for 2×2 games (two-person, two-strategy).

The bad news is that in general, we often do not get convergence even in the time-average sense. Even worse, even when there is convergence in the time-average, it is often very counterintuitive, as the following examples illustrate.

Example 21.

(a) *Rock-Paper-Scissors.* Shapley (1964) shows that fictitious play can fail to converge in the following game of rock-paper-scissors:

	L	C	R
T	$0,0$	$1,0$	$0,1$
M	$0,1$	$0,0$	$1,0$
B	$1,0$	$0,1$	$0,0$

The unique Nash equilibrium in this game has both players play mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Suppose in period 0, (T, M) is played. Then in period 1, Player 1 expects M and Player 2 expects T, so (T, R) is played. (T, R) continues to be played until Player 1 switches to M, after which (M, R) is played, until Player 2 switches to L. Then (M, L) is played until Player 1 switches to B, and (B, L) is played until Player 2 switches to M, and then (B, M) is played until Player 1 switches to T, and we are back where we started. The number of periods at which each strategy pair is played before switching grows exponentially over time.

(b) *Matching pennies.* Consider matching pennies:

	H_2	T_2
H_1	$1,-1$	$-1,1$
T_1	$-1,1$	$1,-1$

Suppose players begin at (H_1, H_2). The sequence of play is then as follows:

t	$\eta_1 t$	η_2^t	s^t
0	(0,0)	(0,0)	(H_1, T_2)
1	(0,1)	(1,0)	(T_1, T_2)
2	(0,2)	(1,1)	(T_1, H_2)
3	(1,2)	(1,2)	(T_1, H_2)
4	(2,2)	(1,3)	(T_1, H_2)
5	(3,2)	(1,4)	(H_1, H_2)
6	(4,2)	(2,4)	(H_1, H_2)
7	(5,2)	(3,4)	(H_1, H_2)
8	(6,2)	(4,4)	(H_1, T_2)
9	(6,3)	(5,4)	(H_1, T_2)
10	(6,4)	(6,4)	(H_1, T_2)

Play converges in the time average sense to $((1/2, 1/2), (1/2, 1/2))$, yet neither player actually plays a mixed strategy – play here is deterministic except when $\eta_i^t = (k, k)$ for some k (in this case, either strategy is a best response). Because players wrongly believe each other to be playing stationary mixed strategies, a rational player who realizes that their opponent is following fictitious play can easily take advantage of this fact by predicting what their opponent will play next given their opponent's learning rule and past play. For example, at $t = 5$, if Player 2 knows that Player 1 is playing fictitious play, she knows Player 1 will play H_1 next, and so she is better off playing T_2.

There are of course many other models of learning. Borgers & Sarin (1997) and Erev & Roth (1998) consider models of reinforcement learning, for example. Reinforcement learning is now very big in machine learning.

2.14.2 Self-confirming equilibrium

If learning is a foundation for (some) Nash equilibria, then it is worth asking which equilibria we can expect to be the product of some learning process. Fudenberg & Levine (1993) suggest *self-confirming equilibrium* as an answer to this question.

Definition 32 (Self-confirming equilibrium)**.** Consider an extensive form game Γ. For each mixed strategy σ_i of player i, let $\pi_i(\phi_i \mid \sigma_i)$ denote the behavioural strategy induced by σ_i at information set ϕ_i. Let Π_i denote the set of player i's behavioural strategies. For each player i, let μ_i denote player i's belief over $\Pi_{-i} = \prod_{j \neq i} \Pi_j$. Let Φ_{-i} denote the set of information sets not belonging to i, and let $\Phi(s_i, \sigma_{-i})$ denote the set of information sets that can be reached with non-zero probability if player i plays pure strategy s_i and i's opponents play strategy profile σ_{-i}.

(a) *Nash equilibrium.* A mixed strategy profile σ^* is a *Nash equilibrium* of Γ if for every player $i \in \mathcal{I}$ and each pure strategy $s_i \in S_i$ in the support of σ_i,

(i) $u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i)$ for all $s_i' \in S_i$ (s_i is a best response to i's beliefs), and

(ii) for all information sets $\phi_j \in \Phi_{-i}$, we have $\mu_i(\{\pi_{-i} \mid \pi_j(\phi_j) = \pi_j(\phi_j \mid \sigma_j^*)\}) = 1$ (i's beliefs are correct.)

(b) *Self-confirming equilibrium.* A mixed strategy profile σ^* is a *self-confirming equilibrium* of Γ if for every player $i \in \mathcal{I}$ and each pure strategy $s_i \in S_i$ in the support of σ_i,

(i) $u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i)$ for all $s_i' \in S_i$ (s_i is a best response to i's beliefs), and

(ii) for all information sets $\phi_j \in \Phi(s_i, \sigma_{-i}^*)$, we have $\mu_i(\{\pi_{-i} \mid \pi_j(\phi_j) = \pi_j(\phi_j \mid \sigma_j^*)\}) = 1$ (i's beliefs are empirically correct.)

Unlike Nash equilibrium, in a self-confirming equilibrium, beliefs only need to be correct for histories that can be reached. In the definition, we assume players perfectly observe the actions of their opponents. In general, self-confirming equilibrium can be defined relaxing this assumption.

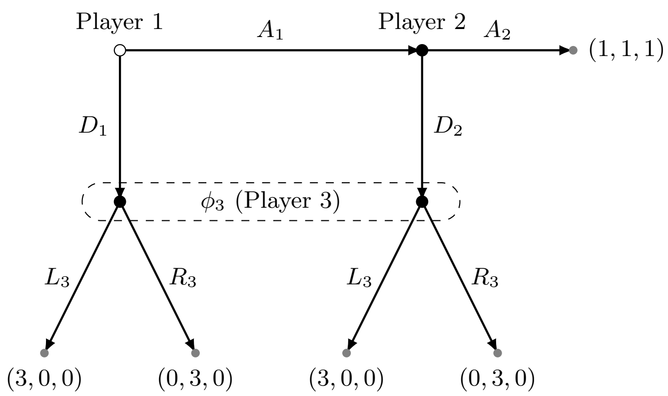
Example 9 (continued)**.** Once again, consider the game of matching pennies:

	H_2	T_2
H_1	$1, -1$	$-1, 1$
T_1	$-1, 1$	$1, -1$

The mixed strategy profile $\sigma^* = ((1/2, 1/2), (1/2, 1/2))$ supported by beliefs $\mu^* = ((1/2, 1/2), (1/2, 1/2))$ is a self-confirming equilibrium. Any other self-confirming equilibrium must involve one of the players i playing a pure strategy s_i, but then j must believe that i will play this strategy since this is the only belief j can hold that is empirically correct. Now j's best response to this belief and s_i are not mutual best responses, and hence this cannot be a self-confirming equilibrium.

A self-confirming equilibrium does not have to be a Nash equilibrium:

Example 22 (Fudenberg & Kreps, 1993)**.** Consider the following three-player game:



Suppose μ_1 is such that Player 1 is certain that Player 3 will play R_3 and Player 2 will play A_2 and μ_2 is such that Player 2 is certain that Player 3 will play L_3. Then it is optimal for Player 1 to play A_1 and for Player 2 to play A_2. Since ϕ_3 is never reached, these beliefs can support a self-confirming equilibrium in which the equilibrium path involves (A_1, A_2).

Yet there is no Nash equilibrium in which (A_1, A_2) is played. Consider any strategy $\sigma_3 = (p, 1-p)$ of Player 3. If $p \geq \frac{1}{3}$, then Player 2's best response is A_2, and Player 1's best response is thus D_1. If $p \geq \frac{1}{3}$, then Player 2's best response is D_2. In neither case is (A_1, A_2) a pair of best responses for Players 1 and 2.

3 Games of incomplete information

3.1 Bayesian games

Definition 33 (Bayesian game)**.** A *Bayesian game* or (*game of incomplete information*) is a tuple $G_\Theta = (\mathcal{I}, (S_i, \Theta_i, u_i)_{i\in\mathcal{I}}, p)$, where $\mathcal{I}$ is a set of players, S_i is a set of strategies for player i, Θ_i is a set of types θ_i for player i, $u_i : S \times \Theta \to \mathbb{R}$ is an von Neumann-Morgenstern expected payoff function for i (where $S = \prod_i S_i$ and $\Theta = \prod_i \Theta_i$), and p is a joint probability distribution over Θ.

Note that the payoff to a player i depends not only on own type θ_i but also on the type profile θ_{-i} of i's opponents. If u_i is independent of θ_{-i} (i.e. $u_i(s, \theta_i, \theta_{-i}) = u_i(s, \theta_i, \theta'_{-i})$ for all $s \in S$, $\theta_i \in \Theta_i$, and $\theta_{-i}, \theta'_{-i} \in \Theta_{-i}$, then we say that the game has *private values.*

Note also the assumption of a common prior p. This can be relaxed, although one does not gain much by relaxing this assumption. A more elaborate version of a Bayesian game is a tuple $(\mathcal{I}, \Omega, p, (S_i, \Theta_i, u_i, \tau_i)_{i\in\mathcal{I}})$ where Ω is a set of states, p is a common prior over Ω and the functions $\tau_i : \Omega \to \Theta_i$ map states of the world into types for each i. In this case, the assumption of a common prior is arguably wlog, since we can define the state space Ω appropriately to 'shift' differences in prior into the state space.

As Harsanyi (1968) notes, any game of incomplete information is equivalent to a game of complete but imperfect information with an additional player *Nature* that randomly chooses a type for each player according to the probabability distribution p. In Harsanyi's formulation, the complete but imperfect equivalent game proceeds in three stages:

1. *Ex ante stage.* Players know only the probabilities with which Nature assigns those elements of the game that are not common knowledge (i.e. in Bayesian games, types).

2. *Interim stage.* Players learn their private information (in Bayesian games, their own type θ_i) and, on the basis of this information, choose strategies simultaneously.

3. *Ex post stage.* Given realized structure of the game (types) and strategies played, payoffs are realized.

Note that in Bayesian games, player i learns only θ_i and not θ_{-i}. We assume the set of type profiles Θ is common knowledge.

In the interim stage, players' posterior beliefs p_i are generated by *Bayesian updating*, i.e.

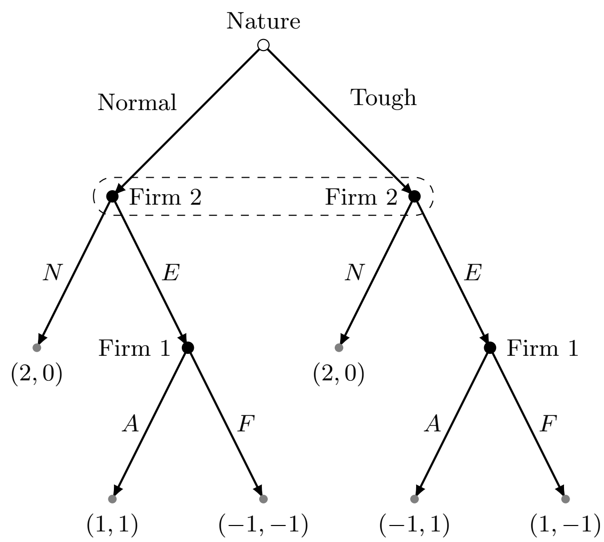
$$p_i(\theta_{-i} \mid \theta_i) = \frac{p(\theta_i, \theta_{-i})}{p(\theta_i)} = \frac{p(\theta_i, \theta_{-i})}{\int_{\Theta_{-i}} p(\theta_i, \mathrm{d}\theta_{-i})}.$$

This requirement is called *consistency* of beliefs.

Example 23 (Market entry III)**.** Suppose Firm 1 is an incumbent monopolist and Firm 2 is a potential entrant. Firm 2 can choose whether to enter (E) or not (N), and if choosing to enter, Firm 1 can choose whether to fight (F) or accommodate (A). With probability $1 - p$, the incumbent is "tough" and with probability p, the incumbent is "normal". If tough, Firm 1's payoff from fighting is greater than from accommodating, and the converse is true if normal:

	"Normal"			"Tough"	
	N	E		N	E
A	$2, 0$	$1, 1$	A	$2, 0$	$-1, 1$
F	$2, 0$	$-1, -1$	F	$2, 0$	$1, -1$

As a game of complete but imperfect information, we can envisage the game as follows:



Definition 34 (Bayesian equilibrium)**.**

(a) *Bayesian strategy.* A *Bayesian pure strategy* is a function $\sigma_i : \Theta_i \to S_i$. A *Bayesian mixed strategy* is a function $\sigma_i : \Theta_i \to \Delta(S_i)$. Let S_{Θ_i} denote the set of Bayesian pure strategies of player i and let $S_\Theta = \prod_{i \in \mathcal{I}} S_{\Theta_i}$ denote the set of Bayesian strategy profiles.

(b) *Expected payoffs.* Given a Bayesian strategy profile $\sigma \in S_\Theta$, the *ex ante expected payoff* $u_i(\sigma)$ of player i is defined by

$$u_i(\sigma) = \mathbb{E}_\theta u_i(\sigma, \theta) = \int_\Theta u_i(\sigma, \theta) p(\mathrm{d}\theta),$$

and, further given type θ_i in the support of p, the *interim expected payoff* $u_i(\sigma \mid \theta_i)$ is defined by

$$u_i(\sigma \mid \theta_i) = \mathbb{E}_\theta[u_i(\sigma, \theta) \mid \theta_i] = \int_{\Theta_{-i}} u_i(\sigma, \theta_i, \theta_{-i}) p(\mathrm{d}\theta_{-i} \mid \theta_i).$$

(c) *Bayesian equilibrium.* A (pure strategy) *Bayesian equilibrium* of an n-player Bayesian game G_Θ is a strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \in S_\Theta$ s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \qquad \text{for all } \sigma_i \in S_\Theta \text{ and } i \in \mathcal{I}.$$

Bayesian equilibrium is just the extension of Nash equilibrium to Bayesian games, and so existence is just a corollary of Nash's existence theorem (Theorem 8):

Corollary 7. *Any finite Bayesian game G_Θ has a Bayesian equilibrium.*

Proof. Immediate from Theorem 8. □

The definition we give above requires that the strategies in Bayesian equilibrium are *ex ante* optimal. This is equivalent to requiring that the strategies in Bayesian equilibrium are interim optimal:

Proposition 26. *A Bayesian strategy profile $\sigma^* \in S_\Theta$ is a (pure strategy) Bayesian equilibrium iff*

$$u_i(\sigma_i^*, \sigma_{-i}^* \mid \theta_i) \geq u_i(s_i, \sigma_{-i}^* \mid \theta_i) \qquad \textit{for all } s_i \in S_i, ,\ \theta_i \textit{ s.t. } p(\theta_i) > 0, \textit{ and } i \in \mathcal{I}.$$

Proof. Assume wlog that Θ is the support of p (otherwise we can simply redefine the type space.) Then

$$\begin{aligned}
u_i(\sigma_i^*, \sigma_{-i}^*) &= \int_\Theta u_i(\sigma_i^*, \sigma_{-i}^*, \theta) p(\mathrm{d}\theta) \\
&= \int_{\Theta_i \times \Theta_{-i}} u_i(\sigma_i^*, \sigma_{-i}^*, \theta_i, \theta_{-i}) p(\mathrm{d}\theta_i, \mathrm{d}\theta_{-i}) \\
&= \int_{\Theta_i} \left(\int_{\Theta_{-i}} u_i(\sigma_i^*, \sigma_{-i}^*, \theta_i, \theta_{-i}) p(\mathrm{d}\theta_{-i} \mid \theta_i) \right) p(\mathrm{d}\theta_i) \\
&= \int_{\Theta_i} u_i(\sigma_i^*, \sigma_{-i}^* \mid \theta_i) p(\mathrm{d}\theta_i).
\end{aligned}$$

□

More explicitly, if Θ is finite, σ^* is a Bayesian equilibrium if

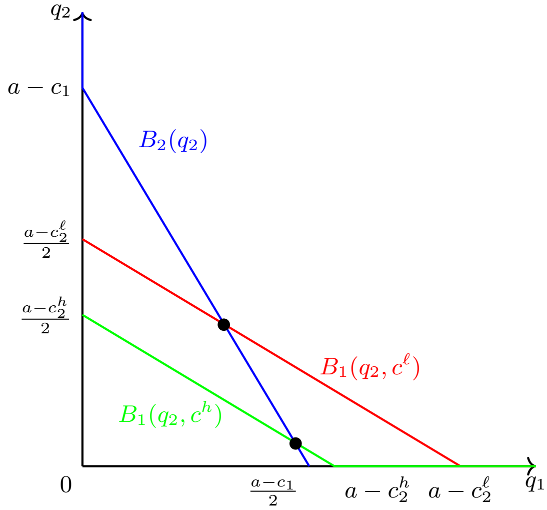
$$\sum_{\theta\in\Theta} u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*(\theta_{-i}), \theta)p(\theta) \geq \sum_{\theta\in\Theta} u_i(s_i, \sigma_{-i}^*(\theta_{-i}), \theta)p(\theta)$$

for all $i \in \mathcal{I}$ and $s_i \in S_i$, or equivalently,

$$\sum_{\theta_{-i}\in\Theta_{-i}} u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})p(\theta_{-i} \mid \theta_i) \geq \sum_{\theta_{-i}\in\Theta_{-i}} u_i(s_i, \sigma_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})p(\theta_{-i} \mid \theta_i)$$

for all $i \in \mathcal{I}$, $s_i \in S_i$ and θ_i in the support of p.

Example 24 (Cournot duopoly with uncertain cost)**.** Consider a Cournot duopoly with linear inverse demand function $P(Q) = \max\{a - Q, 0\}$ with $Q = q_1 + q_2$. Firm 1 has marginal cost c_1, known to both firms, and Firm 2 has marginal cost $c_2 \in \{c_2^h, c_2^\ell\}$, with $c^h > c^\ell$. Firm 1 does not know Firm 2's marginal cost, and believes it is c^h with probability p. We have type spaces $\Theta_1 = \{1\}$ and $\Theta_2 = \{h, \ell\}$.



A Bayesian equilibrium in this game is a strategy profile $(q_1^*, (q_2^*(h), q_2(\ell)))$ s.t.

$$q_2^*(\theta_2) = \arg\max_{q_2\in\mathbb{R}_+} u_2(q_1^*, q_2, \theta_2) = \arg\max_{q_2\in\mathbb{R}_+} (a - q_2 - q_1^* - c_2^{\theta_2})q_2,$$

for each $\theta_2 = h, \ell$, and

$$\begin{aligned} q_1^* &= \arg\max_{q_1\in\mathbb{R}_+} \mathbb{E}_{\theta_2} u_1(q_1, q_2^*(\theta_2)) \\ &= \arg\max_{q_1\in\mathbb{R}_+} p(a - q_2^*(h) - q_1 - c_1)q_1 + (1-p)(a - q_2^*(\ell) - q_1 - c_1)q_1. \end{aligned}$$

We have best response correspondences

$$\begin{aligned} q_2^*(q_1, \theta_2) &= \max\left\{\frac{a - q_1 - c_2^{\theta_2}}{2}, 0\right\}, \\ q_1^*(q_2^*) &= \max\left\{\frac{a - c_1 - pq_2(h) - (1-p)q_2(\ell)}{2}, 0\right\}. \end{aligned}$$

Assuming an interior solution, we obtain

$$\begin{aligned}
q_2^*(h) &= \frac{a - 2c^h + c_1}{3} + \frac{(1-p)(c_2^h - c_2^\ell)}{6}, \\
q_2^*(\ell) &= \frac{a - 2c_2^\ell + c_1}{3} - \frac{p(c_2^h - c_2^\ell)}{6}, \\
q_1^* &= \frac{a - 2c_1 + pc_2^h + (1-p)c_2^\ell}{3}.
\end{aligned}$$

We see that $q_2^*(\ell) > q_2^*(h) > q_2^{h*} = \frac{a-2c_2^h+c_1}{3}$. Even in the high cost case, Firm 2 benefits from Firm 1's lack of information and so produces more than if its type were known to Firm 1.

3.2 Weak dominance

The notion of weak dominance extends to Bayesian games.

Definition 35 (Weak dominance)**.**

(a) *Weak dominance.* In the Bayesian game $G_\Theta = (\mathcal{I}, (S_i, \Theta_i, u_i)_{i\in\mathcal{I}}, p)$, a Bayesian strategy σ_i *weakly dominates* σ_i' if, for all $s_{-i} \in S_{-i}$ and all $\theta \in \Theta$,

$$u_i(\sigma_i(\theta), s_{-i}, \theta_i) \geq u_i(\sigma_i'(\theta_i), s_{-i}, \theta), \qquad \text{with strict inequality for some } s_{-i} \in S_{-i}.$$

(b) *Weakly dominant equilibrium.* A Bayesian strategy profile σ^* is a *weakly dominant equilibrium* of the Bayesian game $G_\Theta = (\mathcal{I}, (S_i, \Theta_i, u_i)_{i\in\mathcal{I}}, p)$ if, for all $i \in \mathcal{I}$, $\theta \in \Theta$, $s_{-i} \in S_{-i}$ and any $\sigma_i' \in S_{\Theta_i}$,

$$u_i(\sigma_i^*(\theta_i), s_{-i}, \theta) \geq u_i(\sigma_i'(\theta_i), s_{-i}, \theta), \quad \text{with strict inequality for some } s_{-i} \in S_{-i}.$$

That is, σ^* is a weakly dominant equilibrium if, for all $i \in \mathcal{I}$, σ_i^* is a weakly dominant strategy.

Proposition 27. *In any Bayesian game $G_\Theta = (\mathcal{I}, (S_i, \Theta_i, u_i)_{i\in\mathcal{I}}, p)$, any weakly dominant equilibrium is a Bayesian equilibrium.*

Proof. Assume G_Θ has a weakly dominant equilibrium σ^*. Fix any $\theta_i \in \Theta_i$ in the support of $p$and any $s_i \in S_i$. By definition of weakly dominant equilibrium, we have

$$u_i(\sigma_i^*(\theta_i), s_{-i}, \theta_i, \theta_{-i}) \geq u_i(s_i, s_{-i}, \theta_i, \theta_{-i})$$

for all $\theta_{-i} \in \Theta_{-i}$ and $s_{-i} \in S_{-i}$. Hence

$$u_i(\sigma_i^*(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \geq u_i(s_i, \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

for all $\theta_{-i} \in \Theta_{-i}$ and $\sigma_{-i} \in S_{\Theta_{-i}}$. It follows that

$$\int_{\Theta_{-i}} u_i(\sigma_i^*(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\mathrm{d}\theta_{-i} \mid \theta_i) \geq \int_{\Theta_{-i}} u_i(s_i, \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) p(\mathrm{d}\theta_{-i} \mid \theta_i),$$

for all $\sigma_{-i} \in S_{\Theta_{-i}}$, which is to say,

$$u_i(\sigma_i^*, \sigma_{-i} \mid \theta_i) \geq u_i(s_i, \sigma_{-i} \mid \theta_i)$$

for all σ_{-i}, and in particular,

$$u_i(\sigma_i^*, \sigma_{-i}^* \mid \theta_i) \geq u_i(s_i, \sigma_{-i}^* \mid \theta_i).$$

Since this holds for all $s_i \in S_i$ all θ_i in the support of p, and all $i \in \mathcal{I}$, it follows that σ^* is a Bayesian equilibrium. □

3.3 Auction theory

Auction theory is a particular application of the theory of Bayesian games, with much influence on the design of real-world auctions.[13]

3.3.1 Independent private value model

The basic auction environment consists of:

- a set of bidders $\mathcal{I} = \{1, \ldots, n\}$, with $n \geq 2$;
- one object to be sold;
- to each bidder i, a signal $S_i \sim F(\cdot)$, where F is assumed to be continuous with density f, taking support on the interval $[\underline{v}, \bar{v}]$. The signals $S_1, \ldots, S_n$ are assumed to be independent.
- to each bidder i, a *valuation* $v_i(s_i) = s_i$ for the object.

This is Levin's treatment, in anticipation of the common value model presented later. However, at this stage, we will simply denote the realization of S_i by v_i.

Specifying a set of auction rules gives rise to an auction among the bidders. If a bidder i is awarded the object, i's payoff is given by

$$u_i(b_i, b_{-i}, v_i, v_{-i}) = v_i - t(b_i, b_{-i}),$$

where $t : \mathbb{R}_+^n \to \mathbb{R}_+$ is a payment rule. If the bidder is not awarded the object, the bidder's payoff is 0. Note i's payoff function is independent of v_{-i}.

Bidder i's information v_i is *independent* of the information of any other bidder j. Furthermore, bidder i's information v_i is *private* in the sense that it does not affect the valuation of any other player j. This model is thus known as the *independent private value* (IPV) model.

A strategy for bidder i in this context is a *bid* $b_i : [0, \bar{v}] \in \mathbb{R}_+$. The bidder who is awarded the object is said to be the *winning bidder*.

[13] See e.g. numerous spectrum auctions, or the role of Klemperer's product-mix auction in the Bank of England's liquidity insurance provision during the 2008 financial crisis.

3.3.2 Vickrey (second price) auctions

A *Vickrey auction* or *second price auction* is an auction in which bidders submit sealed bids $b_1, \ldots, b_n$, the bidder with the highest bid is awarded the object, and the winning bidder pays the amount bidded by the second highest bidder. If there is more than one highest bidder, the object is randomly assigned to one of the highest bidders by a lottery placing equal probability weight on each of the highest bidders.

Proposition 28. *In a second price auction, each player i's weakly dominant strategy is to bid i's value $b_i(v_i) = v_i$. Thus the strategy profile $b^* = (b_1^*, \ldots, b_n^*)$ with $b_i^*(v_i) = v_i$ for each $i = 1, \ldots, n$ is a weakly dominant equilibrium.*

Proof. Suppose i's value is v_i and i bids $b_i > v_i$. Let $\hat{b}_{-i} = \max_{j \neq i} b_j$, i.e. $\hat{b}_{-i}$ is the maximum bid of the other bidders. Then either

(i) $\hat{b}_{-i} > b_i, v_i$. Bidder i does not win the object and so receives payoff 0. If i bids v_i then she also does not win the object and so also receives payoff 0.

(ii) $b_i, v_i > \hat{b}_{-i}$. Bidder i is awarded the object and pays $\hat{b}_{-i}$. If i bids v_i, this would be unchanged.

(iii) $b_i > \hat{b}_{-i} > v_i$. Bidder i wins the object and pays $\hat{b}_{-i}$. Since $\hat{b}_{-i} > v_i$, she receives payoff $v_i - \hat{b}_{-i} < 0$, and the rhs is the payoff she would receive if bidding v_i.

Hence bidding v_i weakly dominates bidding $b_i > v_i$. The case for $b_i < v_i$ is similar. Hence $b_i(v_i) = v_i$ is a weakly dominant strategy. Since this holds for all i, the weakly dominant equilibrium statement follows straightforwardly. □

Since each bidder bids their truthful value, the seller's realized revenue is the second highest value. Let $S^{(k)}$ denote the kth highest of the n draws $S_1, \ldots, S_n$ from distribution F. Then the seller's expected revenue is $\mathbb{E}S^{(2)}$.

The truthful equilibrium just described is the unique symmetric Bayesian equilibrium for the second price auction. However, there exist asymmetric equilibria involving some bidders playing weakly dominated strategies. For example, suppose some player i bids $b_i(v_i) = \bar{v}$ and all other players bid $b_j(v_j) = 0$. Then i is awarded the object and pays 0, so receives payoff v_i. For any $b_i \in (0, \bar{v})$, he also wins the object and pays 0. For any other player j, if they bid $b_j \in (0, \bar{v})$, they do not win the object so receive payoff 0, as they would if bidding 0. If they bid $\bar{v}$, then they receive a nonpositive expected payoff.

Vickrey auctions are not used frequently in practice. However, *English auctions* (also known as *open ascending auctions*) are common. A way of modelling such an auction is to assume the price rises continuously from 0, and bidders can each push a button to 'drop out' from bidding, until only one bidder is left (or no bidders and the final set of multiple bidders drop out at the same time, in which case the object is assigned randomly within this set). Under this model of an English auction, the set of Bayesian equilibria coincide with that of the Vickrey auction, and the winning bidder pays the second highest bid.

3.3.3 Sealed bid (first price) auctions

In a *sealed bid* or *first price* auction, bidders submit sealed bids $b_1, \ldots, b_n$, the bidder i submitting the highest bid is awarded the object and pays b_i. Again, if multiple bidders share the highest bid, the object is allocated randomly to one of these bidders.

Under these rules, bidders will not wish to bid their truthful values. To solve for symmetric equilibrium bidding strategies, we consider two approaches to finding *necessary conditions.*

First, the first order conditions approach. This involves finding a continuous, strictly increasing and differentiable bidding strategy (function). In fact, it can be proven that any symmetric equilibrium necessarily entails that bidders all play a continuous, strictly increasing strategy.

Fix bidder i and suppose all bidders $j \neq i$ use an identical strategy $b_j = b(v_j)$ that is continuous, differentiable and strictly increasing. Then i's interim expected payoff is

$$u_i(b_i, b_{-i} \mid v_i) = (v_i - b_i)\mathbb{P}\{b(v_j) \leq b_i \text{ for all } j \neq i\}.$$

Bidder i thus solves

$$\max_{b_i \geq 0} (v_i - b_i) F^{n-1}(b^{-1}(b_i)).$$

We have first order condition,

$$(v_i - b_i)(n-1)F^{n-2}(b^{-1}(b_i))f(b^{-1}(b_i))\frac{1}{b'(b^{-1}(b_i))} - F^{n-1}(b^{-1}(b_i)) = 0.$$

In a symmetric equilibrium, $b_i = b(v_i)$ for all i, so this becomes differential equation,

$$b'(v_i) = (v_i - b(v_i))(n-1)\frac{f(v_i)}{F(v_i)}.$$

Using the boundary condition that $b(\underline{v}) = \underline{v}$, we can solve the differential equation to yield:

$$b(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}}{F^{n-1}(v)}.$$

This is increasing and differentiable.

A second approach is the envelope theorem approach. Suppose $b(v)$ is a symmetric equilibrium, and b is increasing and differentiable. Then i's equilibrium interim expected payoff is

$$\begin{aligned} u_i(b \mid v_i) &= (v_i - b(v_i))F^{n-1}(v_i) \\ &= \max_{b_i} (v_i - b_i)F^{n-1}(b^{-1}(b_i)), \end{aligned}$$

where the second line follows from the fact that i must be playing a best response in equilibrium. Applying the envelope theorem,

$$\left.\frac{\mathrm{d}u(b|v)}{\mathrm{d}v}\right|_{v=v_i} = F^{n-1}(b^{-1}(b(v_i))) = F^{n-1}(v_i).$$

By the fundamental theorem of calculus,

$$u_i(b \mid v_i) = u_i(b \mid \underline{v}) + \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}$$
$$= \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v},$$

where the second line follows since $b(v)$ is strictly increasing, so any bidder with signal $\underline{v}$ fails to win the auction almost surely, and thus $u_i(b \mid \underline{v})) = 0$.

Substituting for the two definitions of $u_i(b \mid v)$, we have

$$(v - b(v))F^{n-1}(v) = \int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v},$$

which rearranges to

$$b(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}}{F^{n-1}(v)}.$$

These two approaches yield a necessary condition. Next we show sufficiency.[14]

Definition 36 (Single crossing conditions)**.**

(a) *Single crossing condition.* A function $f : X \to [-\infty, \infty]$ where $X \subseteq [-\infty, \infty]$ is said to satisfy the *single crossing condition* if, for all $t > t'$ if $f(t') > 0$ then $f(t) > 0$ and if $f(t') \geq 0$ then $f(t) \geq 0$.

The function f is said to satisfy the *strict single crossing condition* if for all $t > t'$, if $f(t') \geq 0$ then $f(t) > 0$.

(b) *Single crossing differences condition.* The function $g : Y \to \mathbb{R}$ with $Y \subseteq \mathbb{R}^2$ is said to satisfy the *single crossing differences condition* if for any $x' > x$, the function $f(t) = g(x', t) - g(x, t)$ satisfies the single crossing condition.

Likewise, g is said to satisfy the *strict single crossing differences condition* if for any $x' > x$, the function $f(t) = g(x', t) - g(x, t)$ satisfies the strict single crossing condition.

(c) *Smooth single crossing differences condition.* A function $g : Y \to \mathbb{R}$ with $Y \subseteq \mathbb{R}^2$ is said to satisfy the *smooth single crossing differences condition* if if it satisfies the single crossing differences condition and additionally satisfies the property that for all $x \in \mathbb{R}$, if $g_1(x, t) = 0$ then for all $\delta > 0$, we have $g_1(x, t + \delta) \geq 0$ and $g_1(x, t - \delta) \leq 0$.

Theorem 12 (Monotone selection theorem; Milgrom, 2004)**.** *The function $g : \mathbb{R} \times [0, 1] \to \mathbb{R}$ satisfies the strict single crossing differences condition iff for every finite set $X \subset \mathbb{R}$, every optimal selection $x^*(t, X) \in \arg\max_{x \in X} g(x, t)$ is nondecreasing in t.*

[14] This section closely follows Milgrom (2004) *Putting auction theory to work*, Ch. 4. In particular, Theorems 4.2 and 4.6.

Proof. First, suppose g satisfies the strict single crossing differences condition. Let $x \in \arg\max_{x\in X} g(x,t)$, let $t_0 < t_1$, and let $x_0 = x^*(t_0, X)$ and $x_1 = x^*(t_1, X)$. Define $f(t) = g(x_0, t) - g(x_1, t)$. By optimality, $f(t_0) = g(x_0, t_0) - g(x_1, t_0) \geq 0$ and $f(t_1) = g(x_0, t_1) - g(x_1, t_1) \leq 0$. If $x_0 > x_1$, then the strict single crossing differences property would entail that since $t_1 > t_0$ and $f(t_0) \geq 0$, that $f(t_1) > 0$, which would yield a contradiction. Hence $x_0 \leq x_1$, so $x^*(t, X)$ is nondecreasing in t.

Conversely, suppose g does not satisfy the strict single crossing differences condition. Then there is some $t_0 < t_1$ and $x_0 > x_1$ s.t. $f(t_0) = g(x_0, t_0) - g(x_1, t_0) \geq 0$ and $f(t_1) = g(x_0, t_1) - g(x_1, t_1) \leq 0$. Since the statement of the theorem must hold for all finite sets X, take $X = \{x_0, x_1\}$ and let $x^*(t_0, X) = x_0 > x_1 = x^*(t_1, X)$. Then the optimal selection is decreasing. □

Note that any nondecreasing function $\bar{x}$ is differentiable almost everywhere, and so can be discontinuous only at a set of jumps of measure zero. Hence we can decompose $\bar{x}$ into $\bar{x} = \bar{x}_J + \bar{x}_C$ where $\bar{x}_J$ is a jump function and $\bar{x}_C$ is a continuous function. The jump function $\bar{x}_J$ is defined by

$$\bar{x}_J(t) = \sum_{t\in J, s\leq t} \lambda_-(s) + \sum_{t\in J, s<t} \lambda_+(s),$$

where J is a set of jump points and $\lambda_-(s)$ and $\lambda_+(s)$ are the sizes of the left- and right-hand jumps at s. Denote the derivative of $\bar{x}$ at t by $\bar{x}'(t)$ if it exists at t, and otherwise set $\bar{x}'(t) = 0$. The following sufficiency condition requires the regularity condition that $\bar{x}_C$ is absolutely continuous, that is, for all $t, \hat{t}$, $\bar{x}_C(t) - \bar{x}_C(\hat{t}) = \int_{\hat{t}}^{t} \bar{x}'(s)\,\mathrm{d}s$.

Theorem 13 (Sufficiency theorem; Milgrom, 2004)**.** *Suppose $g(x,t)$ is continuously differentiable and satisfies the smooth single crossing differences condition. Let $\bar{x} : [0,1] \to \mathbb{R}$ have range X and suppose $\bar{x} = \bar{x}_J + \bar{x}_C$ where $\bar{x}_J$ is a jump function and $\bar{x}_C$ is absolutely continuous. If*

(i) *$\bar{x}(t)$ is nondecreasing, and*

(ii) *the envelope formula holds, i.e. $g(\bar{x}(t), t) - g(\bar{x}(t), 0) = \int_0^t g_2(\bar{x}(s), s)\,\mathrm{d}s$,*

then $\bar{x}(t)$ is a selection from $X^*(t) = \arg\max_{x\in X} g(x,t)$.

Proof. Since $\bar{x}$ is nondecreasing, for all t we have $\lim_{\hat{t}\to t^+} \bar{x}(t) =: \bar{x}_+(t) \geq \bar{x}_-(t) := \lim_{t\to t^-} \bar{x}(t)$. Consider any $s \in J$, recalling J is the set of jump points. By (ii), $g(\bar{x}(t), t)$ is continuous, and thus $g(\bar{x}(s), s) = g(\bar{x}_+(s), s) = g(\bar{x}_-(s), s)$. By the single crossing property, for all $t > s$ we have $g(\bar{x}_-(s), t) \leq g(\bar{x}(s), t) \leq g(\bar{x}_+(s), t)$ and for all $t < s$, we have $g(\bar{x}_-(s), t) \geq g(\bar{x}(s), t) \geq g(\bar{x}_+(s), t)$.

If $s \notin J$, then $\bar{x}$ is continuous at s. Hence, by (ii), $\frac{\mathrm{d}}{\mathrm{d}s} g(\bar{x}(s), s) = g_2(\bar{x}(s), s)$. By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}s} g(\bar{x}(s), s) = g_2(\bar{x}(s), s) + g_1(\bar{x}(s), s)\bar{x}'(s)$, and thus either $g_1(\bar{x}(s), s) = 0$ or $\bar{x}'(s) = 0$. In the former case, the smooth single crossing property entails that for all $t > s$, we have $g_1(\bar{x}(s), t) \geq 0$ and for all $t < s$, we have $g_1(\bar{x}(s), t) \leq 0$. Since $\bar{x}'(s) \geq 0$, it follows that for $t > s$, we have $g_1(\bar{x}(s), t)\bar{x}'(s) \geq g_1(\bar{x}(s), s)\bar{x}'(s)$, and the reverse inequality if $t < s$.

Hence if $t > \hat{t}$,

$$\begin{aligned}
g(\bar{x}(t),t) - g(\bar{x}(\hat{t}),t) &= \int_{\hat{t}}^{t} g_1(\bar{x}(s),t)\bar{x}'(s)\,\mathrm{d}s + \sum_{s\in J,\hat{t}<s<t} \big[g(\bar{x}_+(s),t) - g(\bar{x}_-(s),t)\big] \\
&\quad + [g(\bar{x}(t),t) - g(\bar{x}_-(t),t)] + [g(\bar{x}_+(\hat{t}),t) - g(\bar{x}(\hat{t}),t)] \\
&\geq \int_{\hat{t}}^{t} g_1(\bar{x}(s),s)\bar{x}'(s)\,\mathrm{d}s + \sum_{s\in J,\hat{t}<s<t} \big[g(\bar{x}_+(s),s) - g(\bar{x}_-(s),s)\big] \\
&\quad + [g(\bar{x}(t),t) - g(\bar{x}_-(t),t)] + [g(\bar{x}_+(\hat{t}),t) - g(\bar{x}(\hat{t}),t)] = 0.
\end{aligned}$$

The inequality holds for each term of the integrand and summand. Likewise, if $t < \hat{t}$,

$$\begin{aligned}
g(\bar{x}(t),t) - g(\bar{x}(\hat{t}),t) &= \int_{\hat{t}}^{t} g_1(\bar{x}(s),t)\bar{x}'(s)\,\mathrm{d}s + \sum_{s\in J,t<s<\hat{t}} \big[g(\bar{x}_+(s),t) - g(\bar{x}_-(s),t)\big] \\
&\quad + [g(\bar{x}(t),t) - g(\bar{x}_-(t),t)] + [g(\bar{x}_+(\hat{t}),t) - g(\bar{x}(\hat{t}),t)] \\
&\leq \int_{\hat{t}}^{t} g_1(\bar{x}(s),s)\bar{x}'(s)\,\mathrm{d}s + \sum_{s\in J,t<s<\hat{t}} \big[g(\bar{x}_+(s),s) - g(\bar{x}_-(s),s)\big] \\
&\quad + [g(\bar{x}(t),t) - g(\bar{x}_-(t),t)] + [g(\bar{x}_+(\hat{t}),t) - g(\bar{x}(\hat{t}),t)] = 0.
\end{aligned}$$

Hence $g(\bar{x}(\hat{t}),t) \leq g(\bar{x}(t),t)$ for all $t, \hat{t}$. Thus $\bar{x}(t) \in X^*(t)$. □

These theorems provide the machinery to prove that our candidate symmetric equilibrium is indeed an equilibrium:

Theorem 14. *In the first price auction, the strategy given by*

$$b(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}}{F^{n-1}(v)}$$

is the unique symmetric (Bayesian) equilibrium strategy.

Proof. By construction, b satisfies the envelope formula. Furthermore, for any symmetric opponents' strategy $b_{-i} = b$, $\bar{u}_i(b_i \mid v_i) = u_i(b_i, b \mid v_i)$ satisfies the single crossing differences condition since for any $b_i' > b_i$, we have that $h(v_i) = u_i(b_i' \mid v_i) - u_i(b_i \mid v_i) = v_i[F^{n-1}(b^{-1}(b_i')) - F^{n-1}(b^{-1}(b_i))] + b_i F^{n-1}(b^{-1}(b_i)) - b_i' F^{n-1}(b^{-1}(b_i'))$ is increasing in v_i.

Furthermore, $\frac{\partial u_i(b_i|v_i)}{\partial b_i} = (v_i - b_i)(n-1)F^{n-2}(b^{-1}(b_i))f(b^{-1}(b_i))\frac{1}{b'(b^{-1}(b_i))} - F^{n-1}(b^{-1}(b_i))$. Since this is increasing in v_i, if $\frac{\partial u_i(b_i|v_i)}{\partial b_i} = 0$ then for any $\delta > 0$, $\frac{\partial u_i(b_i|v_i+\delta)}{\partial b_i} \geq 0$ and $\frac{\partial u_i(b_i|v_i-\delta)}{\partial b_i} \leq 0$, so u_i furthermore satisfies the smooth single crossing property.

Since we established $b(v)$ is nondecreasing (indeed, increasing), both hypotheses of Theorem 13 are satisfied, so it follows that $b(v)$ is a selection from $B_i^*(b) = \arg\max_{b_i \geq 0} u_i(b_i, b \mid v)$. Hence b is an equilibrium strategy. Since b was constructed as the reformulation of the envelope condition, it follows that it is the unique symmetric Bayesian equilibrium strategy. □

Some comments. First, note that in the symmetric Bayesian equilibrium, bids are almost everywhere strictly less than valuation v. This is called *bid shading*, and the level of shading is $\frac{\int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}}{F^{n-1}(v)} = \int_{\underline{v}}^{v} \left(\frac{F(\tilde{v})}{F^{n-1}(v)}\right)^{n-1} \mathrm{d}\tilde{v}$. Note that since $\frac{F(\tilde{v})}{F^{n-1}(v)} \in (0,1)$, we have that $\left(\frac{F(\tilde{v})}{F^{n-1}(v)}\right)^{n-1}$ is decreasing in n. Hence the level of shading on bids is decreasing in n, and as $n \to \infty$, $b(v) \to v$.

The revenue to the seller from the first price auction is the expected winning bid, $\mathbb{E}[b(S^{(1)})]$. Define $G(v) = F^{n-1}(v)$, the probability that for $n-1$ draws, all will lie below v. Let $S_{n-1}^{(1)}$ be the first order statistic of the sample of $n-1$ draws from F, so $G(v) = \mathbb{P}\{S_{n-1}^{(1)} \leq v\}$. We have

$$b(v) = v - \frac{\int_{\underline{v}}^{v} G(\tilde{v})\,\mathrm{d}\tilde{v}}{G(v)} = \frac{1}{G(v)} \int_{\underline{v}}^{v} \tilde{s}\,\mathrm{d}G(\tilde{s}) = \mathbb{E}[S_{n-1}^{(1)} \mid S_{n-1}^{(1)} \leq v],$$

i.e. if a bidder has value v, she sets her own bid equal to the expectation of the highest of the $n-1$ other bidders' values, conditional on all those values being less than her own. Thus

$$\mathbb{E}[b(S^{(1)})] = \mathbb{E}[S_{n-1}^{(1)} \mid S_{n-1}^{(1)} \leq S^{(1)}] = \mathbb{E}[S^{(2)}],$$

establishing the following:

Proposition 29. *The first and second price auctions yield the same revenue in expectation.*

3.3.4 Revenue equivalence

Proposition 29 is a special case of the *revenue equivalence theorem*, due to Vickrey (1961), Myerson (1981), Riley & Samuelson (1981), and Harris & Raviv (1981).

Definition 37 (Expected payment and revenue)**.** An auction rule specifies, for every $i \in \mathcal{I}$,

$$x_i : B_1 \times \cdots \times B_n \to [0,1],$$
$$t_i : B_1 \times \cdots \times B_n \to \mathbb{R},$$

where $x_i(\cdot)$ is the probability that i will be awarded the object and $t_i(\cdot)$ is i's required payment as a function of bids $(b_1, \ldots, b_n)$.

(a) *Interim expected payment.* The *interim expected payment* of bidder i with valuation v_i is defined as

$$m_i(b_i, b_{-i} \mid v_i) := \mathbb{E}_{v_{-i}}[t_i(b_i(v_i), b_{-i}(v_{-i}))].$$

(b) *Ex ante expected payment.* The *ex ante expected payment* of bidder i is defined as

$$m_i(b_i, b_{-i}) := \mathbb{E}_{v_i}[m_i(b_i, b_{-i} \mid v_i)].$$

(c) *Ex ante expected revenue.* Given a bid profile $b = (b_1, \dots, b_n)$, the *ex ante expected revenue* to the seller is

$$r(b) = \sum_{i=1}^{n} m_i(b_i, b_{-i}).$$

Theorem 15 (Revenue equivalence theorem)**.** *Suppose n risk-neutral bidders have values $v_1, \dots, v_n$, distributed iid with cdf $F(\cdot)$. Then all auction mechanisms that*

(i) *in equilibrium, always award the object to the bidder with the highest value, and*

(ii) *give any bidder with a valuation $\underline{v}$ zero profits*

generate the same revenue in expectation. Moreover, the ex ante *expected revenue to the seller is $\mathbb{E}[S^{(2)}]$, the expectation of the second highest value.*

Proof. Consider the general class of auctions where bidders submit bids $b_1, \dots, b_n$ with $b_i \in B_i$. Recall that an auction rule specifies, for every $i \in \mathcal{I}$, functions x_i and t_i defined as in Definition 37.

Given an auction rule, bidder i's expected payoff as a function of her value and bid is

$$u_i(b_i, v_i) = v_i \mathbb{E}_{b_{-i}}[x_i(b_i, b_{-i})] - \mathbb{E}_{b_{-i}}[t_i(b_i, b_{-i})].$$

Let (b_i, b_{-i}) denote an equilibrium of the auction game (recalling b_i, b_{-i} are functions). Bidder i's equilibrium *interim expected payoff* is

$$u_i(v_i) = u_i(v_i, b(v_i)) = v_i F^{n-1}(v_i) - \mathbb{E}_{v_{-i}}[t_i(b_i(v_i), b_{-i}(v_{-i}))],$$

where by (i), we have written $\mathbb{E}_{v_{-i}}[x_i(b(v_i), b(v_{-i}))] = F^{n-1}(v_i)$.

Since b is an equilibrium, $b(v_i)$ must maximize i's payoff given v_i and opponents' strategies $b_{-i}(\cdot)$, the envelope theorem implies

$$\frac{\mathrm{d}}{\mathrm{d}v} u_i(v)\bigg|_{v=v_i} = \mathbb{E}_{b_{-i}}[x_i(b_i(v_i), b_{-i}(v_{-i}))] = F^{n-1}(v_i).$$

Using (ii) to write $u_i(\underline{v}) = 0$, we have

$$u_i(v_i) = u_i(\underline{v}) + \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v} = \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v}.$$

Combining, we have that bidder i's interim expected payment is

$$m_i(b_i, b_{-i} \mid v_i) = \mathbb{E}_{v_{-i}}[t_i(b_i(v_i), b_{-i}(v_{-i}))] = v_i F^{n-1}(v_i) - \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\,\mathrm{d}\tilde{v} = \int_{\underline{v}}^{v_i} \tilde{v}\,\mathrm{d}F^{n-1}(\tilde{v}),$$

where the final equality follows by integration by parts. Since $x_i(\cdot)$ does not enter this expression, i's interim expected payment is the same under any auction rule satisfying (i) and (ii). Indeed, i's expected payment given v_i is

$$m_i(b_i, b_{-i} \mid v_i)\mathbb{E}[S^{(1)}_{n-1} \mid S^{(1)}_{n-1} < v_i] = \mathbb{E}[S^{(2)} \mid S^{(1)} = v_i].$$

Hence sellers's *ex ante* expected revenue is

$$r(b) = n\mathbb{E}_{v_i}[m_i(b_i, b_{-i} \mid v_i)] = \mathbb{E}_{v_i}[\mathbb{E}[S^{(2)} \mid S^{(1)} = v_i]] = \mathbb{E}[S^{(2)}].$$

Hence for all auctions satisfying (i) and (ii), the *ex ante* expected revenue of the seller is $\mathbb{E}[S^{(2)}]$. $\square$

We call auctions that satisfy conditions (i) and (ii) of Theorem 15 *standard auctions.* Standard auctions encompass not only the first and second price auctions but also such auctions as the third price auction (where the winning bidder pays the third highest price) and the all-pay auction (where all bidders pay their bids and the highest bidder wins the auction). Note some implicit consequences of conditions (i) and (ii):

- *Restrictions on asymmetric equilibria.* Condition (i) can be violated by asymmetric equilibria. As we discussed, in the second price auction, there is an equilibrium where some player (not necessarily the player with highest valuation) bids $\bar{v}$ and all other players bid 0. Then the (expected) revenue to the seller is 0 since the winning bidder pays the second highest price, which is 0. Clearly, a first price auction would yield higher expected revenue here. However, the class of symmetric equilibria satisfy (i).

- *Risk aversion.* If bidders are risk averse, the theorem does not hold. The symmetric equilibrium in the second price auction is not affected by risk aversion, since each bidder's payment if they win is not increasing in their own bid. In the first price auction, however, the incentive is to shade their own bid less. Hence the first price auction yields higher expected revenue.

- *Budget constraints.* Sufficiently restrictive budget constraints can cause violations of condition (i), e.g. if i has the highest value v_i but is restricted to bidding $b_i \in [0, \bar{b}_i]$ for some $\bar{b}_i < b_i$, then it is possible that some other bidder with valuation $\bar{b}_i < v_j < v_i$ but $B_j \supset [0, v_j]$ will outbid i in equilibrium and receive the object.

Example 25 (All-pay auction)**.** As in the first and second price auctions, consider bidders $1, \ldots, n$ with values $v_1, \ldots, v_n$ drawn iid from a distribution taking support on $[\underline{v}, \bar{v}]$ with cdf F. In the *all-pay auction*, the rules are as follows. Bidders submit bids $b_1, \ldots, b_n$ and the bidder with the highest bid is awarded the object. However, bidders must pay their bids regardless of whether they win the auction. While these rules seem strange in traditional auction contexts, they are reasonable in modelling lobbying activity for example.

Suppose the auction has a symmetric equilibrium with an increasing strategy $b(v)$ used by all players. Then bidder i's interim expected payoff given v_i is

$$u(b \mid v_i) = v_i F^{n-1}(v_i) - b(v_i) = \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})\, \mathrm{d}\tilde{v}.$$

Thus

$$b(v) = vF^{n-1}(v) - \int_{\underline{v}}^{v} F^{n-1}(\tilde{v})\, \mathrm{d}\tilde{v} = \int_{\underline{v}}^{v_i} \tilde{v}\, \mathrm{d}F^{n-1}(\tilde{v}),$$

which is i's interim expected payment, $m_i(b \mid v_i) = b(v_i)$. This is as we derived for the first price auction. Hence the seller's ex ante expected revenue is

$$r(b) = \mathbb{E}[S^{(2)}].$$

Two other notable examples that satisfy the revenue equivalence conditions when we assume a symmetric equilibrium with an increasing strategy:

- *English auction.* The *English auction*, or *oral ascending auction*, starts with a price of zero. The price increases continuously in time, and bidders can drop out at any point. Once a bidder has dropped out, they cannot reenter. When all but one bidder has dropped out, the remaining bidder is awarded the item and pays the price at which the second-to-last bidder dropped out.[15] The analysis of this auction mirrors that of the second price auction.

- *Dutch auction.* The *Dutch auction*, or *oral descending auction*, starts with a very high price. The price decreases continuously in time. At any point, a bidder can stop the auction and win the object, paying the current price (with a lottery if multiple bidders stop the auction simultaneously).

3.3.5 Common value auctions

The common value model generalizes the independent private value model to allow for (a) the possibility that learning bidder j's information causes bidder i to reassess her own valuation, and (b) that the information of i and j is correlated.

These are reasonable assumptions in many settings. Suppose bidders are bidding on a natural resource such as a timber tract or oil concession. The cost of harvesting and processing timber, or extracting the oil, may be independent across bidders, as in the independent private value model. However, bidders are also unlikely to be sure of the exact amount of retrievable timber in the tract or the volume or quality of the oil deposit, since such assessments of quantity or quality will typically be based on limited sampling (i.e. bidders may estimate quantity and quality by sampling limited areas of the tract/concession.) Hence if i samples a particular area and estimates on a low value, but j samples another area and arrives at a higher value, then i may revise their estimate upward given knowledge of j's valuation. If sampled areas overlap, i and j's values are unlikely to be independent.

The general model consists of bidders $1, \ldots, n$ and signals $S_1, \ldots, S_n$ with joint density $f(\cdot)$ with support $[\underline{s}, \bar{s}]$.

We use s to denote the vector $s = (s_1, \ldots, s_n)$ of valuations. Let

$$s \wedge s' := (\min(s_1, s_1'), \ldots, \min(s_n, s_n'))$$

denote the *join* of s and s', and let

$$s \vee s' := (\max(s_1, s_1'), \ldots, \max(s_n, s_n'))$$

[15] As with previous auctions we considered, if all final bidders drop out simultaneously then there is a lottery to decide who wins.

denote the *meet* of s and s'.

Definition 38.

(i) *Exchangeability.* We say that signals are *exchangeable* if:

$$\text{if } s' \text{ is a permutation of } s \text{ then } f(s) = f(s')$$

(ii) *Affiliation.* We say that signals are *affiliated* if $f(s \wedge s')f(s \vee s') \geq f(s)f(s')$. That is, if for all distinct bidders i, j, $s_j \mid s_i$ has the *monotone likelihood ratio property*:

$$\frac{f(s_j \mid s_i)}{f(s_j \mid s_i')} \geq \frac{f(s_j' \mid s_i')}{f(s_j' \mid s_i)} \quad \text{for all } s_j' > s_j \text{ and } s_i' > s_i.$$

We assume signals are (i) exchangeable and (ii) affiliated.

Finally, we have valuation $v(s_i, s_{-i})$ to each bidder i.

Example 26.

(a) *Independent private value model.* As previously considered, this is a special case with $v(s_i, s_{-i}) = s_i$ and assuming $S_1, \ldots, S_n$ are independent.

(b) *Pure common value model with conditionally independent signals.* All bidders have the same value given by a random variable V. The signals $S_1, \ldots, S_n$ are each correlated with V but independent conditional on V. For example, we could have $S_i = V + \epsilon_i$ where $\epsilon_1, \ldots, \epsilon_n$ are independent. Then $v(s_i, s_{-i}) = \mathbb{E}[V \mid s_1, \ldots, s_n]$.

(c) *Linear model with independent signals.* Each bidder has valuation $v_i(s_i, s_{-i}) = s_i + \beta \sum_{j \neq i} s_j$, and $S_1, \ldots, S_n$. This is a commonly used model, but is difficult to motivate. It has the feature that bidders' information is independent so the revenue equivalence theorem holds. But since valuations are interdependent, the winner's curse also applies.

Consider the second price auction, and again consider a symmetric increasing equilibrium strategy $b(s)$. Let s^i denote the highest signal of the bidders $j \neq i$.

Bidder i wins if she bids $b_i \geq b(s^i)$, and if winning, pays $b(s^i)$. Bidder i thus faces problem,

$$\max_{b_i} \int_{\underline{s}}^{\bar{s}} [\mathbb{E}_{S_{-i}}[v(s_i, S_{-i}) \mid s_i, S^i = s^i] - b(s^i)] \mathbf{1}_{\{b(s^i) \leq b_i\}} f(s^i \mid s_i) \, \mathrm{d}s^i,$$

or equivalently,

$$\max_{b_i} \int_{\underline{s}}^{b^{-1}(b_i)} [\mathbb{E}_{S_{-i}}[v(s_i, S_{-i}) \mid s_i, S^i = s^i] - b(s^i)] \, \mathrm{d}F(s^i \mid s_i).$$

We have first order condition,

$$-\frac{1}{b'(b^{-1}(b_i))}\left[\mathbb{E}_{S_{-i}}[v(s_i, S_{-i}) \mid s_i, S^i = b^{-1}(b_i)] - b(b^{-1}(b_i))\right] f(b^{-1}(b_i) \mid s_i) = 0,$$

which simplifies to

$$b_i = b(s_i) = \mathbb{E}_{S_{-i}}\left[v(s_i, S_{-i}) \mid s_i, \max_{j\neq i} s_j = s_i)\right].$$

Hence in equilibrium, bidder i bids her expected value conditional on her own value and conditional on the values of all her opponents being less than her own value.

What if i were to bid $b_i = \mathbb{E}_{S_{-i}}\left[v(s_i, S_{-i}) \mid s_i\right]$, that is, the unconditional interim expected value? If i wins the auction, then the fact she won is bad news – it means her opponents' valuations were lower than her own. Hence *ex post*, $v(s_i, s_{-i}) < \mathbb{E}_{S_{-i}}\left[v(s_i, S_{-i}) \mid s_i\right]$. In the second price auction, this is not so severe for i than in a first price auction, since i pays the second highest price. However, following this strategy is to succumb to the *winner's curse* – the winner was too optimistic about the valuation. In the equilibrium strategy, each bidder i accounts for winner's curse effects.

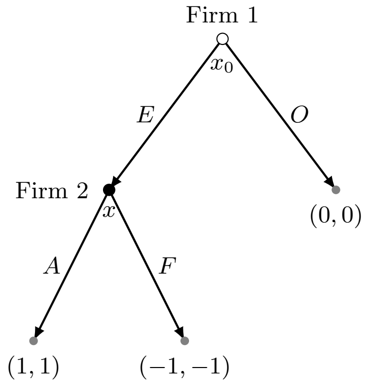
The revenue equivalence theorem fails in the common value model. Milgrom & Weber (1982) prove that in the symmetric equilibrium, revenue is increasing in information (the *linkage principle*). Hence in first price auctions, the expected revenue will be lower than in second price auctions, since the winner's payment is based only on the winning bidder's own signal, whereas in the second price auction, it is based on the winner's signal and the highest signal of her opponents.

4 Sequential games

We have touched on sequential games in some examples in the previous sections. We now discuss these in more detail and the solution concepts appropriate in these settings. Two particularly notable applications – signalling games and noncooperative approaches to bargaining – are presented at the end of this section.

4.1 Sequential rationality

Example 27 (Market entry game II)**.** Consider the following game of perfect information. Suppose Firm 1 is a potential entrant to a market and Firm 2 is an incumbent monopolist. Firm 1 chooses whether to enter (E) or stay out (O) and Firm 2 can choose to fight (F) or accommodate (A) if Firm 1 enters. This is a sequential game in that Firm 2 plays their strategy after Firm 1. The extensive form representation of this game is:



Alternatively, the normal form payoff matrix is:

	A	F
E	$1,1$	$-1,-1$
O	$0,0$	$0,0$

Clearly, there are two pure strategy Nash equilibria, (E, A) and (O, F). Furthermore, there is a mixed strategy Nash equilibrium $((0,1), (\frac{1}{2}, \frac{1}{2}))$ – that is, Firm 1 chooses not to enter with probability 1 and Firm 2 mixes $(\frac{1}{2}, \frac{1}{2})$ over accommodating and fighting.

However, the equilibrium in which Firm 2 plays F is unreasonable in the sense that once Firm 1 enters, Firm 2 is better off deviating from F to A. Provided Firm 2 is rational, its threat to play F is therefore *not credible*. Since Firm 2 cannot credibly commit to playing F if Firm 1 enters, and Firm 1 knows this, Firm 1 should enter.

Sequential rationality rules out these 'unreasonable' strategies.

Definition 39 (Sequential rationality)**.** A strategy s_i (σ_i) for player i is called *sequentially rational* if at every information set at which i is to move, s_i (σ_i) maximizes i's expected payoff conditional on some belief of i over paths that lead to that information set.

This includes at those information sets precluded by i's own strategy. The belief justifying some action a_i^* at an information set not reached must assign positive probability to paths that do not occur.

As in Example 27, Nash equilibrium is *ex ante* rational, but it is not necessarily optimal after each move by opponents.

Definition 40.

(a) *On equilibrium path.* Given an equilibrium, we say that outcomes of the game lie *on the equilibrium path* if they occur with positive probability when players implement the equilibrium strategy.

(b) *Off equilibrium path.* Given an equilibrium, we say that outcomes of the game lie *off the equilibrium path* if they occur with zero probability when players implement the equilibrium strategy.

4.1.1 Backward induction

In finite extensive form games of perfect information, we can use *backward induction* to determine a sequence of optimal actions. Suppose sequential rationality is common knowledge. Then players anticipate that other players will not take actions off the equilibrium path that are not optimal at the information set at which that action is taken. The backward induction algorithm works backward from terminal nodes to find an optimal set of actions.

Definition 41 (Backward induction)**.** The *backward induction algorithm* is as follows:

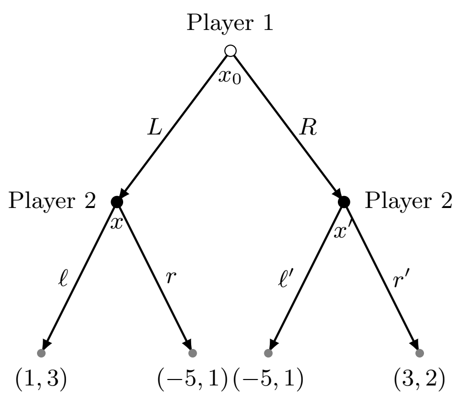
(Step 1) Start at each node that is an immediate predecessor of a terminal node. Find the optimal action for the player who moves at this node, and modify this node to a terminal node with payoffs given by the optimal action.

(Step k) Given the terminal nodes derived in the $(k-1)$th step, find each of these terminal nodes, find the optimal action for the player who moves at the predecessor of this node.

Continue until the initial node is reached. The list of optimal actions derived from the algorithm is called a *backward induction solution.*

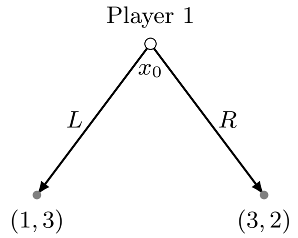
There is a unique backward induction solution if the payoffs across actions at each decision node (information set) differ. Otherwise, one needs to repeat the backward induction algorithm alternating the actions taken to derive all the backward induction solutions.

Example 28. Consider the extensive form game,



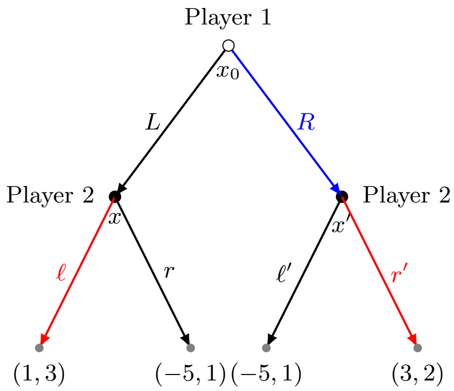
To implement the backward induction algorithm. We start at x and x' since these are the immediate predecessors to the terminal nodes. At x, Player 2's optimal action is ℓ, which yields a payoff of 3 instead of 1. At x', Player 2's optimal action is r', yielding payoff 2 instead of 1.

Replacing x and x' by terminal nodes whose payoffs are given by the optimal actions of Player 2 at x and x', we have



At x_0, Player 1's optimal action in this game is R, yielding payoff 3 instead of 1.

The backward induction solution is therefore $(R, (\ell, r'))$:



If Player 1 believes Player 2 is sequentially rational, he plays R, believing Player 2 will play ℓ at x and r' at x'.

Backward induction involves reasoning about opponents' future behaviour and beliefs, but not past choices – past choices are taken for granted.

Proposition 30. *Any backward induction solution of an extensive form game of perfect information is sequentially rational.*

Proof. In a game of perfect information, every decision node is an information set. From the first step of backward induction, the backward induction solution has player(s) at the nodes immediately preceding the terminal nodes taking actions that maximize the expected payoff. Given $k-1$ stages, at the kth stage, the backward induction solution has player(s) playing the optimal solution at the nodes immediately preceding the terminal nodes obtained from the $(k-1)$th stage given subsequent play that would follow from each of those nodes under the backward induction solution, provided that node is reached. Hence these actions maximize the player's expected payoff under the backward induction strategy profile. Proof follows by induction. $\square$

Proposition 31. *Any backward induction solution of an extensive form game of perfect information is a Nash equilibrium.*

Proof. Any backward induction solution s^* results in an equilibrium path. At every node x on the equilibrium path, the player $\iota(x)$ plays the strategy maximizing $\iota(x)$'s payoff given that x is reached. Hence no player has a profitable deviation at any node on the equilibrium path. Thus s^* is a Nash equilibrium. □

Indeed, the backward induction solution is a refinement of Nash equilibrium.

Theorem 16 (Zermelo)**.** *Every (finite) extensive form game* Γ *of perfect information has a backward induction solution.*

Furthermore, if Γ *has no ties – that is, if* $u_i(z) \neq u_i(z')$ *for all* $z, z' \in X_\tau$ *and all* $i \in \mathcal{I}$*) – then it has a unique backward induction solution.*

Proof. Recall $\Gamma = (\mathcal{I}, (N, \succ, x_0), (X_i)_{i\in\mathcal{I}}, X_\tau, (A_i(x))_{x\in X_i, i\in\mathcal{I}}, (\Phi_i)_{i\in\mathcal{I}}, \iota, (u_i)_{i\in\mathcal{I}})$, with these objects defined as in Definition 2. In this case, Φ_i is the collection of all singleton subsets of X_i.

Suppose there is no backward induction solution. Then for some $i \in \mathcal{I}$, there is some decision node $x \in X_i$ at which the backward induction algorithm prescribes no action $a_i \in A_i(x)$. However, every x precedes some terminal node z, and since $(N, \succ, x_0)$ is a finite arborescence, the path from x_0 to z is unique and includes x. By structure of the backward induction algorithm, the optimal action at the node immediately proceeding z will be determined in the first step. There are finitely many – say, k – nodes between x and z. Furthermore, at any node x' on the path between x and z, there is some (possibly non-unique) optimal strategy at each of these nodes given subsequent play, for there are finitely many strategies. Iterating from the first step, we see that node x will be reached by the algorithm at the $(k-1)$th step. Hence the algorithm will choose some action $a \in A_i(x)$ at the $(k-1)$th step given the actions chosen at steps $1, \dots, k-2$ for all nodes succeeding x on the path from x_0 to z. Since this holds for all nodes x, there is a backward induction solution.

Now suppose Γ has no ties. Then at every node x considered at the first step, there is a unique optimal action, since the set of actions at that node is finite and there are no $a, a' \in A_i(x)$ s.t. the payoff to player $\iota(x)$ from a is the same as the payoff from a'. Hence we can construct a strict order $a_1 > a_2 > \cdots > a_M$ with $M = |A_i(x)|$ and $a > a'$ implies a has strictly higher payoff than a': we thus have a unique action $a_1 \in A_i(x)$ that is optimal for $\iota(x)$. At every subsequent stage, the payoffs at each of the new terminal nodes is a subset of the payoffs of X_τ, so again we have no ties. Repeating the argument at each stage shows there is a unique backward induction solution. □

Suppose Γ has infinite nodes. If the arborescence of Γ has an infinite path, then backward induction cannot be applied. However, if the arborescence does not have an infinite path, then we can apply backward induction, but a backward induction solution need not exist.

Example 29 (Ultimatum game)**.** In the (two-player) ultimatum game, Player 1 proposes an offer $(x_1, v - x_1)$ dividing a surplus of total value $v > 0$ between Player 1 and Player 2. Given Player 1's offer, Player 2 can choose to accept the offer, in which case payoffs are $(x_1, v - zx_2)$, or reject the offer, in which case payoffs are $(0, 0)$. We assume Player 1 cannot choose $x_1 > v$.

This is a situation where backward induction is possible in the presence of an (uncountably) infinite number of terminal nodes.

The *set of feasible agreements* is $Z = \{(z, v - z) \mid z \in [0, v]\}$. The (pure) strategy set for Player 1 is $S_1 = Z$, and for Player 2, the strategy set is

$$S_2 = \{s_2 : Z \to \{\text{Accept}, \text{Reject}\}\}.$$

There are infinitely many Nash equilibria in the ultimatum game. Indeed, any strategy profile $(s_1^*, s_2^*(s_1))$ s.t., for any $z \in [0, v]$,

$$s_1^* = (z, v - z),$$
$$s_2^*(x_1, v - x_1) = \begin{cases} \text{Accept} & \text{if } x_1 \leq z, \\ \text{Reject} & \text{otherwise}, \end{cases}$$

is a Nash equilibrium. For suppose Player 1 has alternative strategy $s_1' = (z', v - z')$ with some $z' \neq z$. If $z' < z$, then Player 2 under s_2^* would accept and Player 1 receives payoff z', yet Player 2 would also accept if Player 1 played s_1^*, in which case Player 1's payoff is $z > z'$. Hence $u_1(s_1^*, s_2^*(s_1)) \geq u_1(s_1', s_2^*(s_1))$. If Player 1 chooses $z' > z$, then Player 2 rejects and so Player 1 receives payoff $u_1(s_1', s_2^*(s_1)) = 0 < z = u_1(s_1^*, s_2^*(s_1))$. Turning to Player 2, under s_2^* Player 2 receives payoff $v - z$, which is clearly maximal given s_1^*.

The *equilibrium outcome path* in these equilibria is the offer $(z, v - z)$ followed by acceptance.

However, note that if $z < v$, then it is not sequentially rational for Player 2 to reject any offer $(x_1, v - x_1)$ with $z < x_1 < v$: given such an offer, Player 2 would, if accepting, obtain a payoff $v - x_1 > 0$, and 0 if rejecting. In the Nash equilibrium, $x_1 > z$ is off the equilibrium path and so any response to $x_1 > z$ (including rejection) is *ex ante* optimal for Player 2. Since in the Nash equilibrium, Player 2 would reject in this off-equilibrium-path case, s_1^* is optimal for Player 1.

The only sequentially rational Nash equilibrium is

$$s_1^* = (v, 0),$$
$$s_2^*(x_1, v - x_1) = \text{Accept for all } x_1 \in Z.$$

This is the unique backward induction solution. Player 2 is indifferent between accepting and rejecting $(v, 0)$. In general, indifferences can imply multiple backward induction solutions (per Theorem 16, ties are a necessary condition for multiple backward induction solutions.) However, in the ultimatum game, there is no backward induction solution with Player 2 rejecting $(v, 0)$: any other backward induction solution for Player 2 would entail strategy

$$s_2(x_1, v - x_1) = \text{Accept for all } x_1 \text{ s.t. } v - x_1 > 0.$$

Then Player 1 solves

$$\max_{x_1 \in [0,v)} x_1,$$

which is not well defined since for any $x_1 \in [0, v)$, there is an $x_1' \in (x_1, v)$ yielding higher payoff to Player 1.

If the set of possible agreements were a discrete set,

$$Z = \{(z, v - z) \mid 0 \leq z \leq v, \;\; z = k\epsilon \text{for } k \in \mathbb{N}\},$$

for some fixed $\epsilon > 0$, then other backward solutions may exist. For example, the strategy profile $(s_1^*, s_2^*(s_1))$ defined by

$$s_1^* = (v - \epsilon, \epsilon),$$
$$s_2^*(s_1) = \begin{cases} \text{Accept} & \text{if } x_1 \leq v - \epsilon, \\ \text{Reject} & \text{otherwise,} \end{cases}$$

is a backward induction solution.

Example 30 (Stackelberg competition)**.** Suppose there are two firms, $i = 1, 2$ which each produce $q_i \in [0, \infty)$ of a single good. Suppose the inverse demand function is

$$p(q_1, q_2) = \max\{a - b(q_1 + q_2), 0\},$$

for some $a, b > 0$., and that both firms have the same (linear) technology and so have the same marginal cost, $c < a$. We refer to Firm 1 as the *Stackelberg leader* and Firm 2 as the *follower*. The leader, Firm 1, chooses q_1 first. Firm 2 then chooses q_2, knowing q_1, and we assume Firm 2 is sequentially rational.

The strategy set for Firm 1 is $S_1 = [0, \infty)$ and the strategy set for Firm 2 is $S_2 = \{q_2 : S_1 \to [0, \infty)\}$.

Again, this is a case where there are infinite terminal nodes, but we can nevertheless apply backward induction. Fix any $q_1 \in \left[0, \frac{a-c}{b}\right]$. Firm 2's profit is

$$\pi_2(q_1, q_2) = (a - b(q_1 + q_2) - c)q_2.$$

We have first order condition,

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = a - bq_1 - c - 2bq_2 = 0,$$

which yields solution

$$q_2^*(q_1) = \frac{a - bq_1 - c}{2b}.$$

Given the leader knows the follower is sequentially rational, the leader solves

$$\begin{aligned}
\max_{q_1} \pi(q_1, q_2^*(q_1)) &= \max_{q_1} (a - b(q_1 + q_2^*(q_1)) - c)q_1 \\
&= \max_{q_1} \left(a - \frac{a + bq_1 - c}{2} - c\right) q_1.
\end{aligned}$$

We have first order condition,

$$\frac{\partial \pi_1(q_1, q_2^*(q_2))}{\partial q_1} = \frac{1}{2}[a - 2bq_1 - c] = 0,$$

and so Firm 1's optimal strategy is $q_1^* = \frac{a-c}{2b}$.

The backward induction solution of the Stackelberg game is therefore

$$q_1^* = \frac{a-c}{2b}, \qquad q_2^*(q_1) = \max\left\{\frac{a - bq_1 - c}{2b}, 0\right\}.$$

The backward induction outcome path is therefore $q_1^* = \frac{a-c}{2b}$, $q_2^* = \frac{a-c}{4b}$.

The equilibrium payoffs are

$$\pi_1^* = \frac{(a-c)^2}{8b},$$
$$\pi_2^* = \frac{(a-c)^2}{16b}.$$

Note the Cournot equilibrium payoff is $\pi_i^c = \frac{(a-c)^2}{9b}$. Thus we see that $\pi_1^* > \pi_1^c$ and $\pi_2^* < \pi_2^c$. Hence Firm 1 has a *first mover advantage.*

Note that while this is the unique backward induction solution, there are infinite (non-sequentially rational) Nash equilibria.

4.2 Subgame perfection

Note that while backward induction can be applied in (finite) extensive form games of perfect information, it cannot necessarily be applied in the presence of imperfect information, for a player's optimal decision depends on which node they are at in their information set, and they cannot distinguish between nodes within an information set. *Subgame perfect equilibrium* is a generalization of the backward induction solution,applicable in games of imperfect information.

Definition 42 (Subgame perfect equilibrium)**.**

(a) *Subgame.* Given an extensive form game

$$\Gamma = (\mathcal{I}, T, P, \Phi, \mathcal{A}, (u_i)_{i\in\mathcal{I}}, \eta),$$

a subgame

$$\Gamma' = (\mathcal{I}, T', P', \Phi', \mathcal{A}', (u_i')_{i\in\mathcal{I}}, \eta')$$

of Γ is a part of Γ s.t.

(i) There is a unique decision node x^* in Γ' s.t. the immediate predecessor of x^* is not in Γ', and furthermore, $x^* \in \phi = \{x^*\}$ for some information set ϕ in Γ.

(ii) A node x is in the subgame Γ' iff either $x = x^*$ or x is a successor of x^*.

(iii) If $x \in \phi_i$ is in Γ' then every $x' \in \phi_i$ is in Γ'.

If Γ' is a subgame of Γ, we write $\Gamma' \subseteq \Gamma$. Note Γ is itself a subgame of Γ. If $\Gamma' \subseteq \Gamma$ and $\Gamma' \neq \Gamma$, then we call Γ' a *proper subgame* of Γ, and write $\Gamma' \subset \Gamma$.

(b) *Restricted strategies.* For any (pure) strategy s_i of player i in the game Γ, the *restriction* of s_i to a subgame $\Gamma' \subset \Gamma$ is a mapping $s_{i|\Gamma'}$ s.t. $s_{i|\Gamma'}(\phi_i) = s_i(\phi_i)$ for every $\phi_i \in \Phi'_i$.

If $n = |\mathcal{I}|$, a restriction of a (pure) strategy profile s to Γ' is a profile $s_{|\Gamma'} = (s_{1|\Gamma'}, \ldots, s_{n|\Gamma'})$.

Restrictions of mixed strategies are defined analogously.

(c) *Subgame perfect equilibrium.* A (pure) strategy profile s^* is a *subgame perfect equilibrium* of Γ if $s^*_{|\Gamma'}$ is a Nash equilibrium of Γ', for every subgame $\Gamma' \subseteq \Gamma$.

Subgame perfect equilibrium in mixed strategies is defined analogously.

Proposition 32. *In any game Γ, any $\Gamma' \subseteq \Gamma$ is an extensive form game in its own right.*

Proof. Let S be any set-valued object in the definition of Γ and let S' be the corresponding set-valued object in the definition of Γ'. Likewise, let f be any function-valued object in the definition of Γ, and let f' be the corresponding object in Γ'. Since Γ' is a part of Γ, $S' \subseteq S$ and f' is a restriction of f to a subdomain of f. Hence all the objects of Γ' are well-defined. By (i) in the definition, x^* is a root node, and since the precedence relation is preserved, $(N', \succ, x^*)$ is thus a finite arborescence. It follows that Γ' satisfies the definition of an extensive form game. $\square$

Subgame perfect equilibria are sometimes called subgame perfect Nash equilibria, since quite trivially:

Proposition 33. *If s^* is subgame perfect equilibrium of Γ, then s^* is a Nash equilibrium of Γ.*

Proof. By definition, $s^*_{|\Gamma'}$ is a Nash equilibrium of Γ' for every $\Gamma' \subseteq \Gamma$. Since $\Gamma \subseteq \Gamma$, $s^* = s^*_{|\Gamma}$ is a Nash equilibrium of Γ. $\square$

Proposition 34. *Any subgame perfect equilibrium s^* of Γ is sequentially rational.*

Proof. Fix any subgame $\Gamma' \subseteq \Gamma$. Since $s^*_{|\Gamma'}$ is a Nash equilibrium of Γ', we have that for every $i \in \mathcal{I}$ and at every information set $\phi_i \in \Phi'_i$ of Γ', $s^*_{i|\Gamma'}(\phi_i) \in B_i(\phi_i)$, where $B_i(\cdot)$ is the best response correspondence. By definition, any strategy in the best response correspondence maximizes i's payoff given belief that other players play $s^*_{-i|\Gamma}$. Since this holds for all subgames, it follows that under s^*, for all i, at every information set at which i moves, s^*_i maximizes i's expected payoff (given the continuation play prescribed by s^* if that information set is reached). Hence s^* is sequentially rational. $\square$

Proposition 35. *A strategy profile s^* is a backward induction solution of a finite extensive form game of perfect information iff s^* is a subgame perfect equilibrium.*

Proof. Let Γ be a finite game of perfect information. Then every node x in Γ corresponds to a distinct information set $\{x\}$. Hence any part of Γ consisting of any decision node x and all its successors is a subgame of Γ. Consider any backward induction solution s^* of Γ. Since any subgame $\Gamma' \subseteq \Gamma$ is a game, the backward induction solution also identifies a backward induction solution s' for Γ', and it is easy to see from the structure of Γ that s' and $s^*_{|\Gamma'}$ must coincide. By Proposition 31, $s^*_{|\Gamma'}$ is a Nash equilibrium. Since this holds for all $\Gamma' \subseteq \Gamma$, s^* is a subgame perfect equilibrium.

Conversely, suppose s^* is a subgame perfect equilibrium.

Define the *level* of a subgame of Γ as follows:

Definition 43 (Level)**.** In a finite extensive form Γ, call a subgame $\Gamma' \subseteq \Gamma$ a *level* 0 subgame if Γ' contains no proper subgame. Recursively, call Γ' a *level k* subgame if

(i) There is a proper subgame Γ'' of Γ' s.t. Γ'' is level $k-1$, and

(ii) If Γ'' is a proper subgame of Γ', then Γ'' is of level at most $k-1$.

Since Γ is finite, it is of some level K. Let $\mathcal{G}^k$ denote the set of all level k subgames of Γ, for $k = 0, 1, \ldots, K$. First, consider $\mathcal{G}^0$. Since Γ is a game of perfect information, any $\Gamma' \in \mathcal{G}$ consists of a single decision node, say x. Since s^* is a subgame perfect equilibrium, $s^*_{|\Gamma'}$ is a Nash equilibrium of Γ' and so presribes that $\iota(x)$ play the action at x that yields the highest payoff. Since this is the optimal action for $\iota(x)$ at x, there is some backward induction solution s^b s.t. the restriction $s^b_{|\Gamma'}$ has $\iota(x)$ playing the same action, and so $s^*_{|\Gamma'} = s^b_{|\Gamma'}$. Since Γ' is an arbitrary member of $\mathcal{G}^0$ and the members of $\mathcal{G}^0$ are independent, it follows that there exists some backward induction solution s^b s.t. $s^b_{|\Gamma'} = s^*_{|\Gamma'}$ for all $\Gamma' \in \mathcal{G}^0$.

Fix any $k \geq 1$ and suppose we have that there exists some backward induction solution s^b s.t. $s^b_{|\Gamma'} = s^*_{|\Gamma'}$ for all $\Gamma' \in \bigcup_{j=0}^{k-1} \mathcal{G}^j$. Then consider any subgame $\Gamma' \in \mathcal{G}^k$. Γ' has some root node, say x, and all 'downstream' play is fixed by s^*. Hence $s^*_{|\Gamma'}$ has $\iota(x)$ playing the optimal action at x given the actions fixed by s^* at each of the successor nodes of x. This optimal action is equivalent to the optimal action of the modified game at which each decision node that is an immediate successor to x is replaced by a terminal node with payoffs fixed by the continuation play prescribed by s^*. Since by assumption, $s^b_{|\Gamma''} = s^*_{|\Gamma''}$ for all proper subgames Γ'' of Γ', the same action at x as under s^* is also an optimal choice for the backward induction algorithm. Hence there is a backward induction solution s^b s.t. $s^b_{|\Gamma'} = s^*_{|\Gamma'}$. Proof follows by induction. □

Corollary 8. *If Γ is a finite extensive form game of perfect information, then Γ has a subgame perfect equilibrium in pure strategies. Furthermore, if Γ has no ties, then Γ has a unique subgame perfect equilibrium (in pure strategies).*

Proof. Let Γ be a finite extensive form game of perfect information. By Theorem 16, Γ has a backward induction solution, which is unique if Γ has no ties. Proof follows by Proposition 35. □

Proposition 36. *Any finite extensive form game has a (mixed strategy) subgame perfect equilibrium.*

Proof. Let Γ be any finite extensive form game. Since any extensive form game is equivalent to a normal form game, by Nash's existence theorem (Theorem 8), every $\Gamma' \subseteq \Gamma$ has a mixed strategy Nash equilibrium. We need to show there is a mixed strategy Nash equilibrium σ^* of Γ s.t. $\sigma^*_{|\Gamma'}$ is a Nash equilibrium of Γ' for all $\Gamma' \subseteq \Gamma$.

Define the level of a subgame as in Definition 43. Since Γ is finite, Γ is level K for some integer K. Let $\mathcal{G}^k$ denote the set of level k subgames of Γ.

We can apply a form of backward induction here. By the Nash existence theorem, each $\Gamma' \in \mathcal{G}^0$ has a Nash equilibrium. Since Γ' has no proper subgame, this is also a subgame perfect equilibrium. Very obviously:

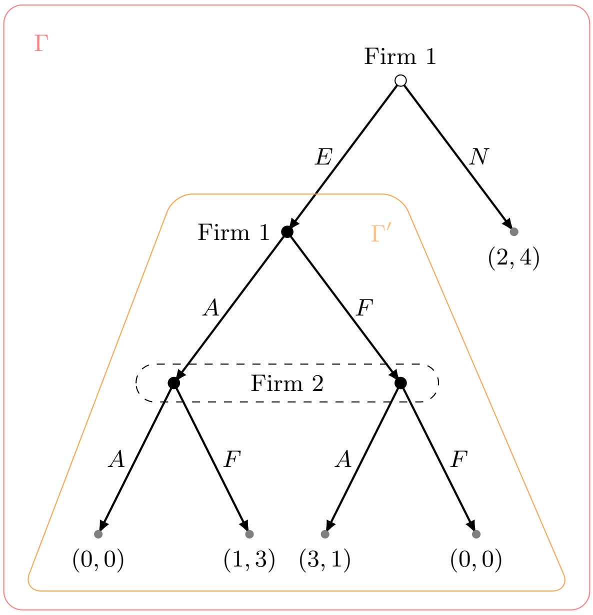
Lemma 8. *σ^* is a subgame perfect equilibrium of Γ iff $\sigma^*_{|\Gamma'}$ is a subgame perfect equilibrium of Γ' for all subgames $\Gamma' \subseteq \Gamma$. The converse is equally trivial.*

Proof. The profile σ^* is a subgame perfect equilibrium if $\sigma^*_{|\Gamma'}$ is a Nash equilibrium for every subgame Γ'. Any subgame Γ'' of Γ' is also a subgame of Γ, and hence $\sigma^*_{|\Gamma''}$ is a Nash equilibrium of Γ'' for all subgames $\Gamma'' \subseteq \Gamma'$. Thus $\sigma^*_{|\Gamma'}$ is a subgame perfect equilibrium of Γ'. □

Now fix k and suppose every $\Gamma'' \in \mathcal{G}^{k-1}$ has a subgame perfect equilibrium $\sigma^*_{|\Gamma''}$. Fix some $\Gamma' \in \mathcal{G}^k$. For each $\Gamma'' \in \mathcal{G}^{k-1}$ s.t. $\Gamma'' \subseteq \Gamma'$, replace the root node x'' of Γ'' by a terminal node with payoffs fixed by the subgame perfect equilibrium $\sigma^*_{|\Gamma''}$. By Nash's existence theorem, this modified game has a mixed strategy Nash equilibrium. Hence Γ' has a mixed strategy Nash equilibrium $\sigma^*_{|\Gamma'}$, s.t. $\left(\sigma^*_{|\Gamma'}\right)_{|\Gamma''} = \sigma^*_{|\Gamma''}$ for all $\Gamma'' \subseteq \Gamma'$. Thus $\sigma^*_{|\Gamma'}$ is a subgame perfect equilibrium of Γ'. By induction, we can therefore construct a mixed strategy subgame perfect equilibrium σ^* of Γ. □

Example 31 (Market entry game IV)**.** Consider the following market entry game. Firm 1 is a potential entrant, who first decides whether to enter (E) or not enter (N). Conditional on entering, Firm 1 chooses whether to fight (F) or accommodate (A). The incumbent monopolist, Firm 2, simultaneously chooses whether to fight (F) or accommodate (A), not knowing Firm 1's action.

The set of pure strategies for Firm 1 are $\{EA, EF, NA, NF\}$ and the set of pure strategies for Firm 2 are $\{A, F\}$. The game has one proper subgame Γ', circled in orange, and the entire game Γ is circled in pale red.



Consider subgame Γ'. There are two pure strategy Nash equilibria in this subgame, (A, F) and (F, A). Furthermore, there is a mixed strategy Nash equilibrium $((\frac{1}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}))$.

Now considering Γ, we can identify those Nash equilibria whose restrictions are Nash equilibria in Γ'. First, if (A, F) is played in Γ', then Firm 1 receives greater expected payoff from N than from E, so we have a subgame perfect equilibrium (NA, F). Second, if (F, A) is played in Γ', then Firm 1 receives greater expected payoff by playing E instead of N, so we have a subgame perfect equilibrium (EF, A). Finally, if the mixed strategy equilibrium is played in Γ', then the expected payoff to Firm 1 from E is $\frac{3}{4}$, versus 2 from N, so we have a subgame perfect equilibrium $((0, 0, \frac{1}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}))$.

4.3 One-shot deviation principle

The *one-shot deviation principle* is a particularly helpful dynamic programming result for finding subgame perfect equilibria in dynamic games (both sequential and repeated games). First, note that given the strategies of other players $-i$, we have a maximization problem for player i. We can thus consider a single person decision tree. Recall that given a decision tree, a path $y = (z_0, \ldots, z_n)$ is an ordered collection of nodes in the tree s.t. for each $z_j \in y$, z_{j+1} is an immediate successor to z_j. Given a node $x \in y$, define the *subpath* y_x to be the path with initial node x and all subsequent nodes identical to those successor nodes of x in y. We say that two paths y and y' *diverge at* x if they share the same nodes up to x but have distinct subpaths thereafter.

Let $\pi(y)$ be the *return* of path y, that is, the payoff to player i at the terminal node reached by y. We assume the following.

Assumption. Let π be a return function, mapping paths to payoffs. Assume

(**A1**) *Consistency.* If $\pi(y) \geq \pi(y')$ and if y and y' diverge at x, then $\pi(y_x) \geq \pi(y'_x)$.

(**A2**) *Continuity.* Fix any path y. For every $\epsilon > 0$, there exists an integer N s.t. if $n \geq N$ and if another path y' shares the first n nodes of y, then $\left|\pi(y) - \pi(y')\right| < \epsilon$.

Any finite game tree with payoffs at terminal nodes automatically satisfies both consistency and continuity assumptions. So do infinite problems with discounted additively separable payoffs: this covers infinitely repeated games of perfect information. Interpreting nodes as information sets and payoffs as expected payoffs, we can extend this to stochastic decision problems such as games of imperfect information.

Definition 44 (One-shot deviations)**.**

(a) Given a node x any strategy profile σ induces a probability distribution P over all paths y with initial node x. Define $\pi(x, \sigma) = \mathbb{E}_y[\pi(y) \mid \sigma]$.

(b) *Optimal strategy.* A strategy σ is *optimal* if there is no strategy σ' and node x s.t. $\pi(\sigma', x) > \pi(\sigma, x)$.

(c) *One-shot deviation.* Given a strategy σ and a node x, let σ_a denote the strategy obtained by substituting the deterministic choice $a \in A(x)$ in place of what was prescribed under σ, and leaving σ otherwise unchanged. We call σ_a a *one-shot deviation* from σ at x.

(d) *Unimprovable strategy.* We call a strategy σ *unimprovable* if there is no one-shot deviation σ_a from σ at x s.t.

$$\pi(\sigma_a, x) > \pi(\sigma, x).$$

Theorem 17 (One-shot deviation principle)**.** *Under assumptions* (**A1**) *and* (**A2**)*, any unimprovable strategy must be optimal.*

Proof. Suppose σ is an unimprovable but non-optimal strategy. Then there exists a strategy σ' and a node x_0 s.t.

$$\pi(\sigma', x_0) > \pi(\sigma, x_0).$$

Since any best stochastic strategy is payoff-equivalent to some pure strategy, this is equivalent to assuming there is some path y with initial node x_0 s.t.

$$\pi(y) \geq \pi(\sigma, x_0) + 2\epsilon$$

for some $\epsilon > 0$. Under (**A2**), there is some integer N s.t. if any path y' starting from x_0 shares the first $N + 1$ nodes with y, then

$$\pi(y') \geq \pi(y) - \epsilon.$$

Combining inequalities, we have that

$$\pi(y') \geq \pi(\sigma, x_0) + \epsilon.$$

Let $x_0, \ldots, x_N$ denote the first N ordered nodes of y. From the inequality, it follows that finitely many one-shot deviations at the nodes x_j, $j = 0, \ldots, N$ are sufficient to generate a payoff improvement from σ.

Define a family of N different strategies α_j for $j = 1, \ldots, N-1$ s.t. α_j chooses x_{k+1} at node x_k for every $k \in \{0, \ldots, j\}$, and coincides with σ elsewhere. Then by the argument of the previous paragraph,

$$\pi(\alpha_{N-1}, x_0) > \pi(\sigma, x_0).$$

Note α_{N-2} coincides with σ at node x_{N-1} and all subsequent nodes, while α_{N-1} is a one-shot deviation from σ at node x_{N-1}. Since σ is unimprovable by assumption,

$$\pi(\alpha_{N-2}, x_{N-1}) = \pi(\sigma, x_{N-1}) > \pi(\alpha_{N-1}, x_{N-1}).$$

Applying assumption (**A1**), we have that

$$\pi(\alpha_{N-2}, x_0) \geq \pi(\alpha_{N-1}, x_0),$$

since α_{N-2} and α_{N-1} share the same nodes $x_0, \ldots, x_{N-2}$ along all paths they generate. But since we also had $\pi(\alpha_{N-1}, x_0) > \pi(\sigma, x_0)$, we have

$$\pi(\alpha_{N-2}, x_0) > \pi(\sigma, x_0).$$

Reapplying this argument iteratively for α_{N-3}, then α_{N-4}, and so on, we reach

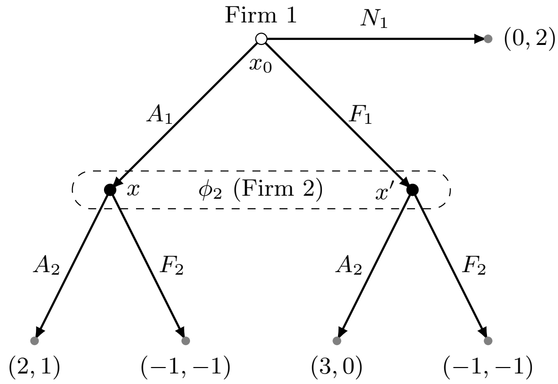
$$\pi(\alpha_0, x_0) > \pi(\sigma, x_0).$$

But α_0 is a one-shot deviation from σ: in particular, $\alpha_0 = \sigma_{x_1}$. Hence this inequality contradicts the unimprovability of σ. □

4.4 Perfect Bayesian equilibrium

There are settings where subgame perfection lacks 'bite', most notably where there is no proper subgame: in this case, any Nash equilibrium is a subgame perfect equilibrium, yet we can see that in some situations, this will result in unreasonable subgame perfect equilibria.

Example 32 (Market entry game V)**.** The following market entry game is very similar to Example 31. Firm 1 is a potential entrant, who decides either to not enter (N_1), to enter and fight (F_1) or to enter and accommodate (A_1). Firm 2 is an incumbent monopolist who simultaneously chooses to either fight (F_2) or accommodate (A_2). The game in extensive form representation is as follows:



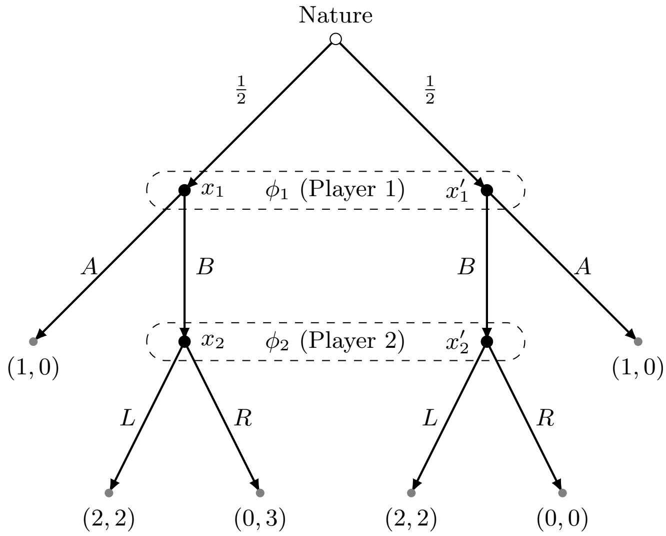
Unlike in Example 31, there is now only one subgame – the whole game itself. Hence any Nash equilibrium is a subgame perfect equilibrium. There are two pure strategy Nash equilibria – (F_1, A_2) and (N_1, F_2). Furthermore, there is a mixed strategy Nash equilibrium. Suppose Player 2's strategy σ_2 mixes between A_2 with probability p and F_2 with probability $1 - p$. Then Player 1's payoffs are

$$\begin{aligned} u_1(A_1, \sigma_2) &= 3p - 1, \\ u_1(F_1, \sigma_2) &= 4p - 1, \\ u_1(N_1, \sigma_2) &= 0. \end{aligned}$$

Hence $u_1(N_1, \sigma_2) \geq u_1(F_1, \sigma_2) > u_1(A_1, \sigma_2)$ provided $p \leq \frac{1}{4}$, so we have a family of mixed strategy Nash equilibria $\{\sigma^* = ((0, 0, 1), (p, 1 - p)) : 0 \leq p \leq \frac{1}{4}\}$ where mixing is over $\{A_1, F_1, N_1\}$ and $\{A_2, F_2\}$ in that order respectively.

While all of these Nash equilibria are trivially also subgame perfect, the mixed strategy equilibria and the pure strategy equilibrium (N_1, F_2) are not sequentially rational, since at both nodes x and x', Player 2 will obtain a higher payoff by playing A_2 for certain than by playing F_2 with any positive probability. Subgame perfection does not capture this, because for these equilibria, the information set ϕ_2 neither lies on the equilibrium path nor forms a part of a proper subgame off the equilibrium path.

Example 33. A second problem is that subgame perfect equilibrium can allow actions that are only possible if players have unreasonable beliefs. This arises in the following game:



This game has only one subgame. It has pure strategy subgame perfect equilibria (B, L) and (A, R). The latter is intuitively unreasonable. R is optimal for Player 2 if he believes he is at x_2 with probability at least $\frac{2}{3}$ given he is at $\phi_2 = \{x_2, x_2'\}$. Yet this would be an unreasonable belief: if Player 1 chooses B, then Player 2 should place equal probability on being at x_2 and x_2'.

Perfect Bayesian equilibrium addresses some of these issues by generalizing backward induction-like reasoning to games of imperfect information. In particular, perfect Bayesian equilibrium imposes sequential rationality at every information set.
Recall that a behavioural strategy σ_i for player i is a mapping $\sigma_i : \phi_i \mapsto \Delta(A_i(\phi_i))$.

Definition 45 (Perfect Bayesian equilibrium)**.**

(a) *System of beliefs.* A *system of beliefs* $\mu = (\mu_i)_{i \in \mathcal{I}}$ is a profile of belief functions μ_i that each assign to each information set $\phi_i \in \Phi_i$ a probability distribution over the nodes of ϕ_i. We write $\mu_i(x \mid \phi_i)$ for i's belief that i is at node $x \in \phi_i$ given that i is at information set ϕ_i.

Given a strategy profile σ and a system of beliefs μ, player i's *continuation payoff* at information set ϕ_i is

$$u_i(\sigma \mid \phi_i, \mu) = \sum_{x \in \phi_i} u_i(\sigma \mid x)\mu_i(x),$$

where $u_i(\sigma \mid x)$ is the expected payoff at node x under the actions prescribed by strategy profile σ for all successor information sets of x.

(b) *Assessment.* An *assessment* (σ, μ) consists of a strategy profile σ and a system of beliefs μ.

(c) *Perfect Bayesian equilibrium.* An assessment (σ^*, μ^*) is a (weak) *perfect Bayesian equilibrium* if

(i) For all $i \in \mathcal{I}$, σ_i^* is sequentially rational, that is, for all information sets $\phi_i \in \Phi_i$,

$$u_i(\sigma_i^*, \sigma_{-i}^* \mid \phi_i, \mu_i) \geq u_i(\sigma_i, \sigma_i^* \mid \phi_i, \mu_i)$$

for all strategies σ_i of i.

(ii) Beliefs μ_i are updated by Bayes' rule whenever it applies – that is, given σ^*, for all $i \in \mathcal{I}$, all $\phi_i \in \Phi_i$ s.t. $\mathbb{P}\{\phi_i \mid \sigma^*\} > 0$, and all $x \in \phi_i$,

$$\mu_i(x) = \frac{\mathbb{P}\{x \mid \sigma^*\}}{\mathbb{P}\{\phi_i \mid \sigma^*\}} = \frac{\mathbb{P}\{x \mid \sigma^*\}}{\sum_{x' \in \phi_i} \mathbb{P}\{x' \mid \sigma^*\}}.$$ [16]

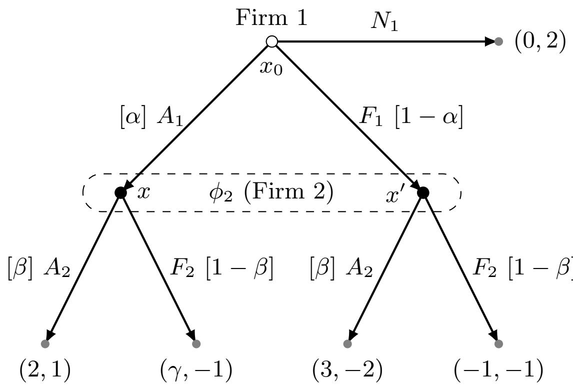
(d) *Strong perfect Bayesian equilibrium.* A profile of assessments (σ, μ) is a *strong perfect Bayesian equilibrium* of a game Γ if $(\sigma_{|\Gamma'}, \mu_{|\Gamma'})$ is a perfect Bayesian equilibrium in every subgame $\Gamma' \subseteq \Gamma$.

Perfect Bayesian equilibrium is now quite a dated solution concept. Sequential equilibrium, discussed in the next section, is preferred.

Example 32 (continued)**.** A system of beliefs μ must assign to each information set a probability distribution over that set. We have information sets $\{x_0\}$ and $\phi_2 = \{x, x'\}$. Hence μ assigns probability $\mu_1(x_0) = 1$ and probabilities $\mu_2(x), \mu_2(x')$ s.t. $\mu_2(x) + \mu_2(x') = 1$. If Firm 2 maximizes its expected payoff at ϕ_2 given belief μ_2. We see that A_2 is the optimal action for any belief Firm 2 may have. Hence (F_1, A_2) is the only perfect Bayesian equilibrium.

Example 33 (continued)**.** Suppose $\sigma_1(B) > 0$. Then Bayesian updating implies $\mu_2(x_2) = \mu_2(x_2') = \frac{1}{2}$. Under these beliefs, Player 2's optimal action at ϕ_2 is L. Player 1's optimal strategy at ϕ_1 is then B. The belief profile consistent with Bayes' rule given strategies (B, L) is $\mu = ((\mu_1(x_1), \mu_1(x_1')), (\mu_2(x_2), \mu_2(x_2'))) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. Hence the assessment $((B, L), \mu)$ is a perfect Bayesian equilibrium.

Example 34. Slightly modifying the payoffs in Example 32 to create a less computationally trivial example, consider the following game:



[16] To see how this follows from Bayes' theorem, note that $\mu_i(x) = \mathbb{P}\{x \mid \phi_i, \sigma^*\} = \frac{\mathbb{P}\{\phi_i \mid x, \sigma^*\}\mathbb{P}\{x \mid \sigma^*\}}{\mathbb{P}\{\phi_i \mid \sigma^*\}}$, and $\mathbb{P}\{\phi_i \mid x, \sigma^*\} = 1$.

Assume that $\gamma > 0$, and let μ be a system of beliefs assigning $\mu_1(x_0) = 1$ and $\mu_2(x) + \mu_2(x') = 1$. At ϕ_2, Firm 2's expected payoffs given these beliefs are

$$\begin{aligned} u_2(s_1, A_2 \mid \phi_2, \mu) &= \mu_2(x) - 2(1 - \mu_2(x)) = 3\mu_2(x) - 2, \\ u_2(s_1, F_2 \mid \phi_2, \mu) &= -1, \end{aligned}$$

where s_1 is the strategy of Firm 1 (irrelevant for the calculation of expected payoffs here since continuation play from ϕ_2 does not involve Firm 1 taking decisions.)

Given belief system μ, we see that Firm 2's optimal action is A_2 if $3\mu_2(x) - 2 \geq -1$ $\Rightarrow \mu_2(x) \geq \frac{1}{3}$ and Firm 2's optimal action is F_2 if $\mu_2(x) \leq \frac{1}{3}$, noting that if $\mu_2(x) = \frac{1}{3}$, then any arbitrary randomization over $\{A_2, F_2\}$ is optimal.

First, suppose $\mu_2(x) < \frac{1}{3}$ and $\sigma^* = (A_1, F_2)$. Then applying Bayes' rule, we have

$$\mu_2(x) = \frac{\mathbb{P}\{x \mid \sigma^*\}}{\mathbb{P}\{x \mid \phi_2\}} = \frac{1}{1} = 1 \not< \frac{1}{3},$$

so $\mu_2(x) < \frac{1}{3}$ is not consistent with Bayes' rule.

Second, suppose $\mu_2(x) > \frac{1}{3}$ and $\sigma^* = (F_1, A_2)$. Then applying Bayes' rule, we have

$$\mu_2(x) = \frac{\mathbb{P}\{x \mid \sigma^*\}}{\mathbb{P}\{x \mid \phi_2\}} = \frac{0}{1} = 0 \not> \frac{1}{3},$$

and thus $\mu_2(x) > \frac{1}{3}$ is not consistent with Bayes' rule.

This leaves $\mu_2(x) = \frac{1}{3}$. Suppose Firm 1's strategy σ_1 mixes over $\{A_1, F_1\}$, playing A_1 with probability $\alpha \in [0, 1]$ and F_1 with complementary probability. Suppose Firm 2's strategy σ_2 mixes over $\{A_2, F_2\}$, playing A_2 with probability $\beta \in [0, 1]$ and F_2 with complementary probability. Then

$$\begin{aligned} u_1(A_1, \sigma_2 \mid \{x_0\}, \mu) &= 2\beta + \gamma(1 - \beta) &&= (2 - \gamma)\beta + \gamma, \\ u_1(F_1, \sigma_2 \mid \{x_0\}, \mu) &= 3\beta - (1 - \beta) &&= 4\beta - 1. \end{aligned}$$

Hence Firm 1 mixes iff $4\beta - 1 = (2 - \gamma)\beta + \gamma$, that is, iff $\beta = \frac{1+\gamma}{2+\gamma}$.

Now, Firm 1 randomizes between A_1 and F_1 only since N_1 is strictly dominated by A_1 given $\gamma > 0$. It is optimal for Firm 2 to randomize over A_2 and F_2 only if $\alpha = \frac{1}{3}$. Finally, we check that $\mu_2(x) = \frac{1}{3}$, is consistent with Bayes' rule:

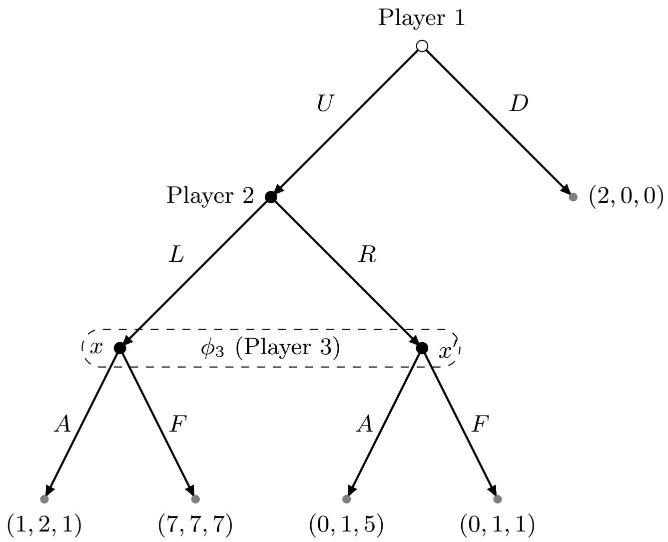
$$\mu_2(x) = \frac{\mathbb{P}\{x \mid \sigma\}}{\mathbb{P}\{\phi_2 \mid \sigma\}} = \frac{\alpha}{1} = \alpha = \frac{1}{3}.$$

Hence we have verified that the unique perfect Bayesian equilibrium is

$$\left((\sigma_1, \sigma_2) = \left(\left(\frac{1}{3}, \frac{2}{3}, 0 \right), \left(\frac{1+\gamma}{2+\gamma}, \frac{1}{2+\gamma} \right) \right), \mu = \left(1, \left(\frac{1}{3}, \frac{2}{3} \right) \right) \right).$$

A defect of (weak) perfect Bayesian equilibrium is that a perfect Bayesian equilibrium need not be subgame perfect. This can result in intuitively unreasonable perfect Bayesian equilibria. A straightforward refinement addressing this – strong perfect Bayesian equilibrium – imposes subgame perfection by requiring that the assessment induces a perfect Bayesian equilibrium in every subgame, thus restricting some beliefs off the equilibrium path.

Example 35. Consider the following game:



Suppose $\mu_3(x') = 1$. Then Player 3's optimal action at ϕ_3 is A, since this yields expected payoff 3 to Player 3, versus 2 if playing F. Now, for Player 2, L is a strictly dominant strategy. Thus if Player 1 chooses U, his expected payoff is $u_1(U, L, A) = 0 < u_1(D, L, A) = 2$, so D is optimal. Clearly, $\mu_3(x') = 1$ is unreasonable, since Player 2's strictly dominant strategy is L, so were ϕ_3 to be reached, Player 3 ought to reason that he is at x for certain. But since the information set ϕ_3 is never reached on the equilibrium path, this doesn't matter – so the assessment (σ, μ) where $\sigma = (D, L, A)$ and $\mu = (1, 1, (0, 1)))$ is a perfect Bayesian equilibrium.

Strong perfect Bayesian equilibrium does not suffer this defect, since any strong perfect Bayesian equilibrium induces a perfect Bayesian equilibrium in all subgames. Considering the proper subgame here, we have that ϕ_3 is reached with positive probability. Given L is strictly dominant for Player 2, Player 3 must reason that $\mu_3(x) = 1$. Given this belief, the optimal action at ϕ_3 is F. Player 1's optimal action given continuation play is then U. We therefore have a strong perfect Bayesian equilibrium involving strategy profile (U, L, F) and belief system $(1, 1, (1, 0))$. This is of course also a perfect Bayesian equilibrium.

4.5 Sequential equilibrium

Even strong perfect Bayesian equilibrium allows for equilibria that can seem quite unreasonable. In Example 33, for example, there is only one subgame, so problems with unreasonable off-the-equilibrium-path beliefs are left unchecked:

Example 33 (continued)**.** We previously showed (B, L) is a perfect Bayesian equilibrium strategy profile. However, note that there is another. Suppose $\sigma_1(B) = 0$. Then Bayes' rule does not apply, so μ_2 can be arbitrary. If $\mu_2(x_2) > \frac{2}{3}$, then it is sequentially rational for Player 2 to play R. Given Player 2 plays R at ϕ_2, it is optimal for Player 1 to play A at ϕ_1. Hence we also have a family of perfect Bayesian equilibria

$$\left\{ \left(\sigma^* = (A, R), \mu = \left(\left(\frac{1}{2}, \frac{1}{2} \right), (\alpha, 1 - \alpha) \right) \right) : \alpha \in \left(\frac{2}{3}, 1 \right] \right\}.$$

Since there is only one subgame, these are also strong perfect Bayesian equilibria.

Yet as we had originally noted, Player 2's beliefs here are unreasonable.

Kreps and Wilson (1982) introduce *sequential equilibrium*, which imposes strong restrictions on the beliefs that players can hold off the equilibrium path – stronger than strong perfect Bayesian equilibrium, which only restricts some beliefs in subgames off the equilibrium path, but slightly weaker than perfect equilibrium. The main innovation in sequential equilibrium is that it imposes that perfect Bayesian equilibrium holds even if players tremble. Sequential equilibrium has displaced perfect Bayesian equilibrium in the literature as the go-to solution concept in games of incomplete information.

Definition 46.

(a) *Consistency.* An assessment (σ, μ) is called *consistent* if there exists some sequence of assessments (σ^n, μ^n) s.t. σ^n is totally mixed, μ^n is derived from σ^n by Bayes' rule, and $(\sigma, \mu) = \lim_{n\to\infty}(\sigma^n, \mu^n)$.

(b) *Sequential rationality.*[17] An assessment (σ, μ) is *sequentially rational* if, for every player $i \in \mathcal{I}$ and every information set $\phi_i \in \Phi_i$,

$$u_i(\sigma_i, \sigma_{-i} \mid \phi_i, \mu_i) \geq u_i(\sigma_i', \sigma_{-i} \mid \phi_i, \mu_i)$$

for all strategies σ_i' of i.

(c) *Sequential equilibrium.* An assessment (σ^*, μ^*) is a *sequential equilibrium* if

(i) (σ^*, μ^*) is sequentially rational, and

(ii) (σ^*, μ^*) is consistent.

[17] We defined sequential rationality in the context of perfect Bayesian equilibrium, but restate it here for convenience.

To relate this to perfect Bayesian equilibrium, a sequential equilibrium is a PBE with an additional consistency requirement on off-path beliefs.

The requirement that beliefs are consistent is pretty strong. Namely, consistency requires that players have common beliefs following a deviation from equilibrium behaviour. This is sometimes criticised for being too strong – if something goes wrong, then should we expect that different players have the same conjectures about what happened?

Example 33 (continued)**.** The unique sequential equilibrium in Example 33 is

$$(\sigma^*, \mu^*) = \Big((B, L), \big((1/2, 1/2), (1/2, 1/2)\big)\Big) .$$

To see this, note that for any strategy with $\sigma_1^n(A), \sigma_1^n(B) > 0$, we must have that $\mu_2^n(x) = \frac{1}{2}$. Yet then $\lim_{n\to\infty} \mu_2^n(x) = \frac{1}{2}$, so $\mu_2(x) = \mu_2(x') = \frac{1}{2}$ is the only possible beliefs of Player 2 in any consistent assessment. Choosing some sequence $\{\sigma_1^n, \sigma_2^n\}$ s.t. $\sigma_1^n(B) \to 1$ and $\sigma_2^n(L) \to 1$, we see that (σ^*, μ^*) is a consistent assessment and therefore a sequential equilibrium. Since none of the other perfect Bayesian equilibria involved $\mu_2 = (1/2, 1/2)$, they cannot be sequential equilibria.

As with the Nash equilibrium correspondence, we can ask whether in a sequence of games, limits of sequential equilibria are sequential equilibria of the limit game. For the family of extensive form games $\mathcal{G}_\Lambda = \{\Gamma(\lambda) \mid \lambda \in \Lambda\}$ parameterized by Λ, define the *sequential equilibrium correspondence* $\mathrm{SE} : \Lambda \rightrightarrows \prod_{i=1}^n \Delta(S_i)$ by $\mathrm{SE}(\lambda) = \{\sigma \mid \sigma \text{ is a sequential equilibrium of } \Gamma\}$. As with the Nash equilibrium correspondence, we assume the payoff function of each player is continuous in both strategy profiles and parameters.

Proposition 37. *Let $\mathcal{G}_\Lambda$ be a family of extensive form games parameterized by Λ. The sequential equilibrium correspondence of $\mathcal{G}_\Lambda$ has a closed graph.*

Proof. Let $\{\Gamma^k\}$ be a sequence of games in $\mathcal{G}_\Lambda$ s.t. $\Gamma^k \to \Gamma$, let $\{u^k\}$ be the corresponding sequence of payoff functions with $u^k \to u$, and let $\{(\sigma^k, \mu^k)\}$ be a corresponding sequence of sequential equilibria of the games Γ^k s.t. $(\sigma^k, \mu^k) \to (\sigma, \mu)$. Since the expected payoffs conditional on reaching any information set are continuous in the payoff functions and beliefs, we have that (σ, μ) is sequentially rational.

Now, since each (σ^k, μ^k) is consistent, there exists a sequence of totally mixed strategies $\{\sigma^{m,k}\}$ s.t. $\sigma^{m,k} \to \sigma^k$, and corresponding induced beliefs $\{\mu^{m,k}\}$ s.t. $\mu^{m,k} \to \mu^k$. For each k, there is m_k sufficiently large that $\sigma^{m,k}$ lies in a $(1/k)$-ball about σ^k and $\mu^{m,k}$ lies in a $(1/k)$-ball about μ^k for all $m \geq m_k$. Now since $\sigma^k \to \sigma$ and $\mu^k \to \mu$, it follows that $\sigma^{m_k,k} \to \sigma$ and $\mu^{m_k,k} \to \mu$, and thus (σ, μ) is consistent. □

It turns out that sequential equilibria nest the perfect equilibria (recall that for extensive form games, a perfect equilibrium is a perfect equilibrium of the corresponding agent-normal form game):

Theorem 18. *Consider a finite extensive form game Γ. For every perfect equilibrium σ of Γ, there exists a system of beliefs μ such that (σ, μ) is a sequential equilibrium of Γ.*

Proof. Suppose σ is a perfect equilibrium of Γ. Then there is some sequence of totally mixed strategies $\{\sigma^m\}$ in the corresponding agent-normal form game s.t. $\sigma^m \to \sigma$, and σ^m_ϕ is a best response to $\sigma^m_{-\phi}$ for every information set ϕ and each $m \in \mathbb{N}$. For each m, the induced system of beliefs μ^m is well-defined, and there exists some subsequence of $\{\mu^{m_k}\} \subseteq \{\mu^m\}$ such that $\mu^{m_k} \to \mu$ for some system of beliefs μ. Thus (σ, μ) is consistent, and since σ_ϕ is a best response given $\sigma^{m_k}_{-\phi}, \mu^{m_k}(\cdot \mid \phi)$ for each m_k, we have that (σ, μ) is sequentially rational. □

Theorem 19. *Every finite extensive form game Γ has a sequential equilibrium.*

Proof. By Theorem 18, for every perfect equilibrium σ of Γ, there exists beliefs μ s.t. (σ, μ) is a sequential equilibrium. Now by Theorem 9, there is some perfect equilibrium σ of Γ. □

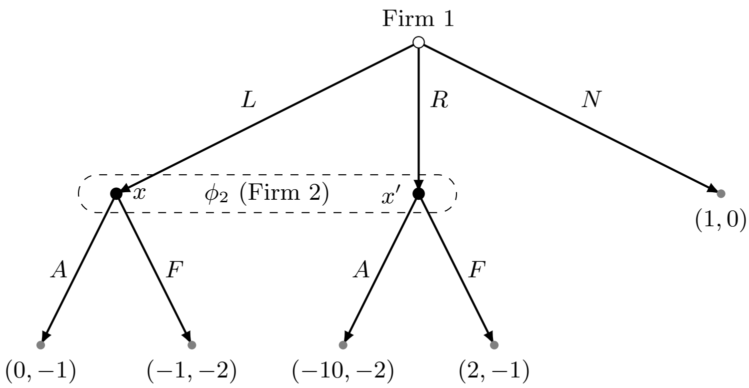
Sequential equilibrium may still fail to eliminate certain kinds of unreasonable equilibrium. Techniques such as forward induction (see sections 4.6 and 4.8.2) can be used to further refine the set of equilibria.

4.6 Forward induction

Backward induction captures the notion that whatever a player chooses, players making subsequent decisions will behave rationally. *Forward induction* extends this to previous decisions – it captures the notion that players assume that even if confronted with an unexpected event, their opponents chose rationally in the past and will continue to choose rationally in subsequent play. If a player finds herself off the equilibrium path, she assumes this is the result of opponents having maximized their utility, as long as such an assumption is reasonable.

Example 36.

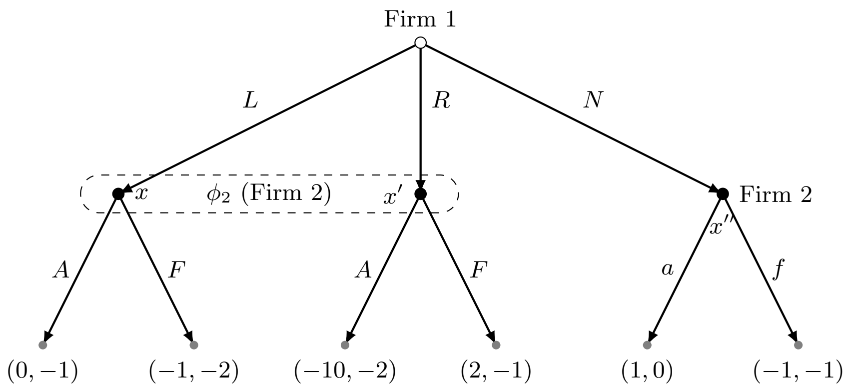
(a) Here is yet another iteration of the market entry game:



There are two sets of pure strategy sequential equilibria: $((R, F), \mu_2(x') = 1)$ and $((N, A), \mu_2(x) > 1/2)$. We can apply forward induction by a dominance argument.

If Firm 2 finds itself at ϕ_2, then Firm 1 must have played L or R, yet L is strictly dominated by N. Since Firm 2 assumes Firm 1 must acted rationally, Firm 2 deduces that Firm 1 played R, so $\mu_2(x) = 0$ and $\mu_2(x') = 1$. The family of equilibria involving strategy profile (N, A) thus fail forward induction. Given μ_2, Firm 2's optimal strategy is F, and thus $((R, F), \mu(x') = 1)$ is consistent with forward induction.

(b) Suppose that if Firm 1 chooses N, Firm 2 faces a decision between a and f:



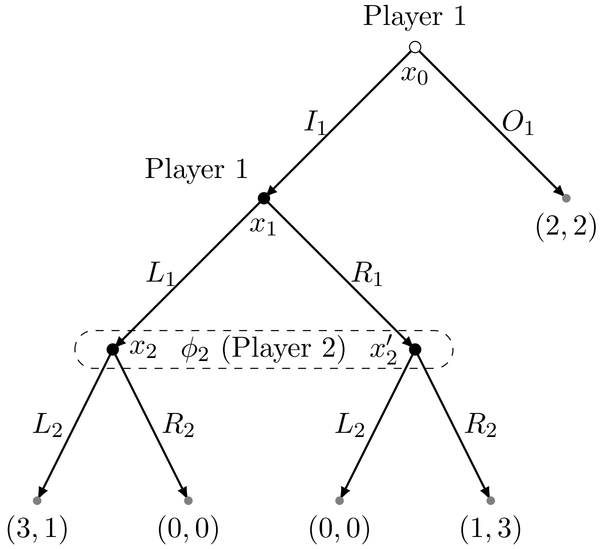
Again, we have two sets of pure strategy equilibria: $((R, Fa) : \mu_2(x') = 1)$ and $((N, Aa), \mu_2(x) > 1/2)$. In this case, N does not strictly dominate L. However, we can apply forward induction by appeal to equilibrium payoffs. Consider the family of equilibria involving the strategy profile (N, Aa). Firm 1 achieves a payoff of 1 in equilibrium. If Firm 1 deviates to play L, then the maximum payoff Firm 1 can achieve is 0. Thus if Firm 1 is rational, it will not deviate to play L since it obtains a strictly worse payoff, regardless of Firm 2's action. On the other hand, if Firm 1 deviates to play R, the maximum payoff Firm 1 can achieve is 2. Thus, if Firm 2 finds itself at ϕ_2, it can reason that Firm 1 chose R in the belief that Firm 2 would choose F. Thus the only belief consistent with forward induction is $\mu_2(x) = 0$, $\mu_2(x') = 1$. This eliminates the family of equilibria involving (N, Aa). Firm 2's optimal action at ϕ_2 given these beliefs is F. Hence the equilibrium $((R, Fa), \mu_2(x') = 1)$ is consistent with forward induction.

Like sequential rationality (at least, for a long time), forward induction reasoning in its entirety is somewhat abstract and not rigorously defined. Capturing the notion formally is difficult and there have been many attempts. Iterated weak dominance (see section 2.6) partially captures both sequential rationality and forward induction reasoning.

In games of perfect information, iterated weak dominance implies backward induction. At a penultimate node, any action that is not optimal is weakly dominated, so does not survive iterated weak dominance. Considering sets of immediate predecessor nodes in turn, we see that the set of strategy profiles that survive iterated weak dominance are precisely the set of backward induction solutions. In games of imperfect information, however, iterated weak dominance lacks the bite of, say, sequential equilibrium.

Now, iterated weak dominance also partially captures forward induction reasoning. To see this, consider the following extensive form version of battle-of-the-sexes:

Example 37. Suppose two players are playing battle-of-the-sexes. Player 1 first has the option of whether to leave before the game is played, in which case the payoff is $(2, 2)$:



This game has a subgame perfect equilibrium $((O_1, R_1), R_2)$. Yet a forward induction argument implies that this equilibrium is unreasonable. For Player 1, I_1 is only optimal if she expects Player 2 to play L_2, in which case she plays L_1. Hence if information set ϕ_2 is reached, Player 2 can infer that Player 1 must have played L_1, and so his best response is L_2.

In reduced normal form, the game has the representation:

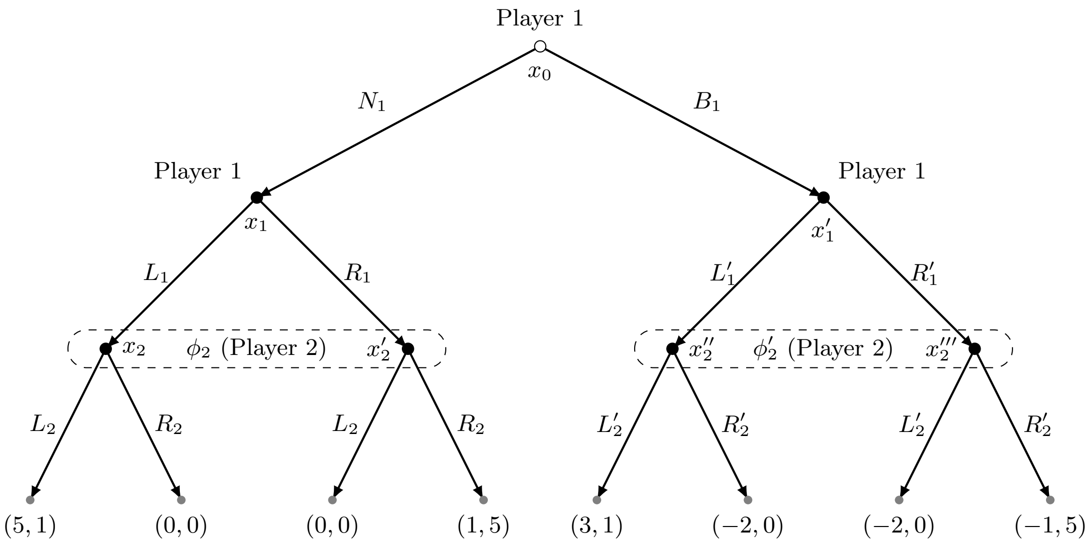
	L_2	R_2
O_1	$2, 2$	$2, 2$
I_1L_1	$3, 1$	$0, 0$
I_1R_1	$0, 0$	$1, 3$

Consider applying iterated weak dominance to this game. I_1R_1 is strictly dominated by O_1, and thus we eliminate it. Mo other strategies of Player 1 are weakly dominated at this stage. Having eliminated I_1R_1, we see that R_2 is weakly dominated by L_2, so we eliminate R_2. Finally, of the remaining strategies, O_1 is strictly dominated, leaving only (I_1L_1, L_2). Iterated weak dominance in the reduced normal form game thus successfully captures our forward induction argument here.

There are many other concepts that formalize forward induction reasoning in a partial way. Stable equilibria, which we are about to discuss, capture a lot of the force of forward induction. In signalling games, the (possibly iterated) intuitive criterion and divine equilibrium also capture forward induction reasoning – see section 4.8.2.

Before we move on to stable equilibrium, it is worth mentioning a striking example of forward induction reasoning due to Ben-Porath & Dekel (1992):

Example 38 (Burning money)**.** Consider the following game:



Two players are playing battle-of-the-sexes. Before the game, Player 1 has the choice to "burn" two units of utility. The subgame in which Player 1 has not burned utility (played N_1) is on the left, and the subgame in which Player 1 has burned two units of utility (played B_1) is on the right. There are four pure-strategy subgame perfect equilibria (which are also all sequential equilibria for some consistent system of beliefs): $s^1 = ((N_1, L_1, R_1'), (L_2, R_2'))$, $s^2 = ((N_1, L_1, L_1'), (L_2, L_2'))$, $s^3 = ((N_1, R_1, R_1'), (R_2, R_2'))$ and $s^4 = ((B_1, R_1, L_1'), (R_2, L_2'))$.

Let $\succ_1$ be a ranking of these equilibria by how favourable the outcome is for Player 1. Clearly, $s^1 \sim_1 s^2 \succ_1 s^4 \succ_1 s^3$. At first glance, Player 1 has an equilibrium selection problem. If she plays N_1, then there is nothing in the notion of subgame perfection or sequential rationality to fix Player 2's beliefs at the information set ϕ_2. If $\mu_2(x_2 \mid \phi_2) \geq \frac{5}{6}$, then L_2 is optimal and if $\mu_2(x_2 \mid \phi_2) \leq \frac{1}{6}$ then R_2 is optimal for Player 2.

A forward induction argument eliminates all but one of these equilibria, however, as we see if we apply iterated weak dominance. Note that any strategy of Player 1 that involves (B_1, R_1') is strictly dominated – Player 1 will only burn if she intends to play L_1'. Having eliminated these strategies, R_2' is weakly dominated in the right subgame by L_2' for Player 2. Thus Player 1 can guarantee herself a payoff of 3 by burning. Having eliminated these strategies, we see that any strategy involving (B_1, L_1') strictly dominates any strategy involving (N_1, R_1), and that (N_1, L_1, L_1') strictly dominates (B_1, L_1, L_1'). The only subgame perfect equilibrium surviving weak iterated dominance is thus $s^2 = ((N_1, L_1, L_1'), (L_2, L_2'))$.

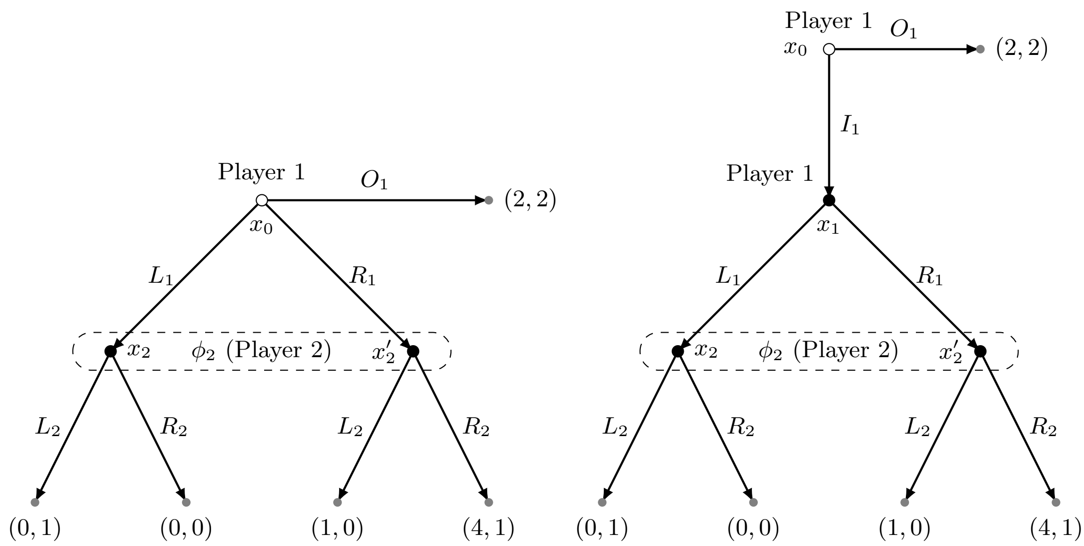
In the language of forward induction, at ϕ_2, Player 2 can conclude that ϕ_2 is only reached because Player 1 has played (N_1, L_1), since if Player 1 instead played (N_1, R_1), she would have been better off burning and playing (B_1, L_1') to achieve a payoff of 3. Thus Player 2 concludes that his optimal action at ϕ_2 is L_2.

The interesting conclusion is that the *option* to burn cash allows Player 1 to select their most preferred equilibrium, even though Player 1 never exercises the option.

4.7 Stable equilibria

One principle we might hope an equilibrium concept satisfies is that irrelevant ("strategically neutral") changes to the game tree do not affect the equilibrium outcome. Kohlberg & Mertens (1986) point out that sequential equilibrium fails this principle.

Example 39. Consider the following two games:



These games are identical, except that in the right game, Player 1's actions have been split in a way that seems irrelevant to play. Rather than one information set $\{x_0\}$, Player 1 now has two information sets $\{x_0\}$ and $\{x_1\}$, and chooses whether to play I_1 or O_1 at the first, and L_1 or R_1 at the second.

Intuitively, we would expect equilibrium outcomes to be the same in both games. Now, (A_1, L_2) can be supported as a sequential equilibrium strategy profile in the left game for beliefs $\mu_1 = (1, 0)$ over (L_2, R_2) and $\mu_2 = (1, 0, 0)$ over (A_1, L_1, R_1). To see $((O_1, L_2), \mu)$ is consistent, consider a sequence $\epsilon_n \to 0$ and the totally mixed strategy profiles $\sigma^n = ((1 - \epsilon_n, 2\epsilon_n/3, \epsilon_n/3), (1 - \epsilon_n, \epsilon_n))$. Then the limit of the induced beliefs is $\lim_{n\to\infty} \mu^n = \mu$, and $\sigma^n \to \sigma$.

Yet there is no sequential equilibrium strategy profile involving O_1 in the right game. Indeed, the unique sequential equilibrium strategy profile in the right game is $((I_1, R_1), R_2)$. To see this, note L_1 is strictly dominated by R_1 for Player 1 (in both games). In the right game, therefore, R_1 is optimal at x_1 for any beliefs of Player 1, and thus in a consistent system of beliefs μ, $\mu_2(x_2') = 1$, so Player 2's optimal action at ϕ_2 is R_2.

Indeed, there are good reasons to think that in the left game, (O_1, L_2) is not a credible equilibrium strategy profile, on a forward induction argument: if ϕ_2 is reached, then because R_1 dominates L_1 for Player 1, Player 2 must conclude he is at x_2', so $\mu_2(x_2') = 1$. Player 2's best response given this belief is R_2. We see then that the reason that sequential equilibrium gives us different answers in the left and right games is because it does not capture forward induction arguments. Whereas the left game requires a forward induction argument to eliminate (O_1, L_2), the structure of the right game is such that the forward induction argument is unnecessary.

Kohlberg & Mertens (1986) set out to develop an set-valued equilibrium concept that captures what they term *strategic stability.* A strategy profile σ is strategically stable if no player $i \in \mathcal{I}$ ever has an incentive to deviate from σ_i. Put another way, a strategy profile is strategically stable if it is *credible*, in the sense that at no point in a game, for any history, can any rational player make inferences that would lead them to prefer to do otherwise than σ. What does this require?

- *Sequential rationality.* As we discussed in e.g. Example 27, certain Nash equilibria are not credible because they rely on threats that, if it comes to it, a player would prefer not to follow through with. Sequential rationality rules this out. However, it is clearly not sufficient for strategic stability, as Example 39 illustrates.

- *Invariance to extensive form representation.* The addition or removal of irrelevant elements of the game tree should not affect the equilibrium outcome. Kohlberg and Mertens conceptualize this as follows. Recall that any extensive form game has a "reduced" normal form representation (section 1.3). Kohlberg and Mertens' argue that the normal form representation of the game is all that should matter. This is a controversial take. Selten (1975) argues that a lot of information is potentially lost in the normal form representation of an extensive form game, and Kreps & Wilson (1982) argue analyses of normal form representations, in ignoring the role of beliefs off-equilibrium-path, lack the power of extensive form analyses.

- *Weak dominance.* Kohlberg and Mertens argue that players never have a good reason to play a weakly dominated strategy, and strategic stability thus requires that no weakly dominated strategy is played in equilibrium. This could be extended to rule out equilibria that do not survive iterated weak dominance. But they don't quite go this far when they define sets of stable equilibria, because then such sets are not guaranteed to exist. Thus they just require that strategic stability rules out that weakly dominated strategies are played in equilibrium.

- *Forward induction.* A credible Nash equilibrium should survive forward induction reasoning.

Definition 47 (Strategically stable equilibrium)**.** Let G be a game. Let $\mathcal{E}$ be the set of all sets E of Nash equilibria of G with the property that for every $\delta > 0$, there exists an

ϵ-perturbation G_ϵ of G such that there is a Nash equilibrium σ^ϵ of G_ϵ within distance δ of some equilibrium $\sigma \in E$.[18]

The *strategically stable set* S of Nash equilibria is the set

$$S = \bigcap_{E \in \mathcal{E}} E.$$

In the definition G can be a normal form or extensive form game.

Kohlberg and Mertens show that:

- For any game, a strategically stable set exists.
- There exists a stable set that lies in a connected component of the set of Nash equilibria (and thus equilibrium outcomes are the same throughout this set; only off-equilibrium-path actions differ).
- Stable sets are invariant to irrelevant changes in the structure of the game tree.
- Any stable set contains a proper equilibrium (and thus a perfect equilibrium). However, if the game is extensive form, then the stable set is only guaranteed to contain a perfect equilibrium of the reduced normal form, not of the corresponding agent-normal form, and thus a stable set of an extensive form game does not necessarily contain a sequential equilibrium (Gul provides a counterexample.)
- No weakly dominated strategy is played in any strategy profile in any stable set.
- Any stable set of a game also contains the stable set of the game obtained by deleting a weakly dominated strategy (and thus contains the stable set of the game corresponding to those strategies that survive iterated weak dominance).
- Any stable set of the game also contains the stable set of the game obtained by deleting a strategy s_i of some player i that is never a best response to any of the strategy profiles s_{-i} of opponents that lie in the set. This captures forward induction.

Note that Kohlberg & Mertens' interpretation of strategic stability doesn't quite capture credibility (they need to sacrifice this to ensure that e.g. stable sets always exist), though strategically stable sets do contain all the credible equilibria.

4.8 Signalling games

An important class of dynamic games with incomplete information are *signalling games*, introduced by Spence (1974). There are two periods and two players – a sender and a receiver. The structure of a signalling game proceeds as follows:

- Stage 0: Nature chooses a type $\theta \in \Theta$ of Player 1 from a distribution p with support Θ.

[18] That is, the Hausdorff distance between σ^ϵ and E is less than δ.

- Stage 1: Player 1 (the *sender*) observes θ and chooses a *message* $m \in M$ (or *signal*).
- Stage 2: Player 2 (the *receiver*) observes m and chooses action $a \in A$.
- Payoffs: at the end of stage 2, payoffs $u_1(m, a, \theta)$ and $u_2(m, a, \theta)$ are realized.
- Strategies: A pure strategy of the sender is a mapping $s_1 : \Theta \to M$, and a pure strategy of the receiver is a mapping $s_2 : M \to A$.

Signalling games arise in many economic settings. Some notable examples:

Example 40.

(a) *Job market signalling.* This is Spence's original example. Player 1 is a prospective worker and Player 2 is a perfectly competitive labour market. The prospective worker's type θ is her ability, and she can choose to acquire a level of education e, the cost of which is decreasing in ability. The firm observes the worker's education level and makes a wage offer based on her expected ability conditional on her education level. The worker hopes to signal her ability via her education level.

(b) *Initial public offerings.* Player 1 is a founder of a private company and Player 2 is a set of potential investors. The founder's type θ is the future profitability of the company, and she chooses what fraction of the company to float publicly and the price at which these shares will be offered. The potential investors choose whether to accept or reject the founder's offer in response. The founder hopes to signal high future expected profitability via offer price and the size of the stake to be sold.

(c) *Monetary policy.* Player 1 is a central bank, and Player 2 is the set of firms. The central bank's type is its policy preferences over unemployment and inflation. In the first period, the central bank sets an inflation level $m \in M$. Firms form expectations a about inflation in the second period given inflation in the first period. The central bank wishes to signal its policy preferences so that period 2 inflation is low.

(d) *Pretrial negotiations.* Player 1 is a defendant in a civil case, and Player 2 is the plaintiff. The defendant's type is the strength of his defence. The defendant makes a settlement offer $m \in M$, and the plaintiff responds by accepting or rejecting. If the plaintiff rejects, the parties go to trial. The defendant hopes to signal he has a strong case.

Definition 48 (Sequential equilibrium in signalling games)**.** In a (finite) signalling game, a sequential equilibrium is an assessment (s, μ) s.t.

(i) Player 1's strategy $s_1(\theta)$ is optimal given Player 2's strategy $s_2(m)$, that is, $s_1(\theta)$ solves

$$\max_{m \in M} u_1(m, s_2(m), \theta) \qquad \text{for all } \theta \in \Theta.$$

(ii) Player 2's beliefs are compatible with Bayes' rule, that is, if $\mathbb{P}\{s_1(\theta) = m\} > 0$ then

$$\mu_2(\theta \mid m) = \frac{\mathbb{P}\{s_1(\theta) = m\}p(\theta)}{\sum_{\theta' \in \Theta} \mathbb{P}\{s_1(\theta') = m\}p(\theta')}.$$

(iii) Player 2's strategy is optimal given μ_2 and m, that is, $s_2(m)$ solves

$$\max_{a \in A} \sum_{\theta \in \Theta} u_2(m, a, \theta)\mu_2(\theta \mid m) \qquad \text{for all } m \in M.$$

The definition can be easily extended if the type space or action spaces are infinite.[19]

It is helpful to partition the sequential equilibria in signalling games into three categories:

(a) *Separating equilibria.* Player 1 will play different actions in equilibrium depending on her type, and so Player 2 learns Player 1's type perfectly.

(b) *Pooling equilibrium.* Player 1 will play the same action in equilibrium regardless of her type, so no information is transmitted to Player 2.

(c) *Semi-separating equilibrium.* Some actions are chosen by several types of Player 1, and others are chosen by a single type. Player 2 thus learns Player 1's type imperfectly.

4.8.1 Job market signalling

Spence's original model concerned job market signalling. Player 1 is a *worker* whose ability (i.e. productivity) is given by $\theta \in \{\theta_L, \theta_H\}$, where $\theta_H > \theta_L > 0$. The worker knows her own ability, and the labour market assigns probability λ that she has productivity θ_H. The worker chooses an education level $e \in E$. To acquire e costs $c(e, \theta)$.

Assumption. $\frac{\partial c(e,\theta)}{\partial e} > 0$ and $\frac{\partial c(e,\theta)}{\partial e \partial \theta} < 0$.

This assumption captures the idea that (i) on the margin, it costs more to acquire more education since this takes more time and money, and (ii) higher ability find education less costly on the margin – i.e. they learn more efficiently and so can complete education with less effort or time.

[19] In particular, let $f_{s_1(\theta)}$ be the pdf of the distribution over M induced by $s_1(\theta)$ and let p be the pdf of the distribution from which θ is drawn. (ii) becomes: If $f_{s_1(\theta)}(m) > 0$ then

$$\mu_2(\theta \mid m) = \frac{f_{s_1(\theta)}(m)p(\theta)}{\int_\Theta f_{s_1(\theta')}(m)p(\theta')\,\mathrm{d}\theta'},$$

where μ_2 now represents a density. (iii) becomes: $s_2(m)$ solves

$$\max_{a \in A} \int_\Theta u_2(m, a, \theta)\mu_2(\theta \mid m)\,\mathrm{d}\theta.$$

Once the worker chooses her education level, the perfectly competitive firm make wage offers. For simplicity, assume the reservation wage of workers is 0, regardless of ability.

Player 2 is a competitive labour market – for example, a set of two or more identical firms in Bertrand competition. Let $\mu(e)$ denote the market's belief that the worker with education level e is high productivity (type θ_H). Since each firm in the market is perfectly competitive, it offers a wage equal to expected productivity,

$$w(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L.$$

This generates a wage schedule $w : E \to \mathbb{R}_+$.

Consider the problem facing the worker given the market's beliefs $\mu(e)$ and the wage schedule. The worker with ability θ chooses e to solve

$$\max_e \; w(e) - c(e, \theta).$$

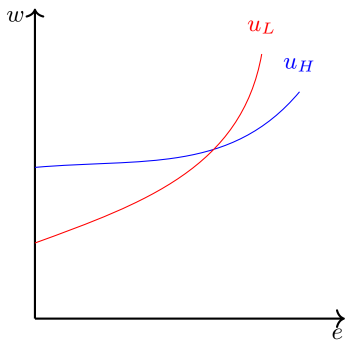
Consider the worker's indifference curves in (e, w)-space. Implicitly differentiating $u_1(w, e, \theta) = \bar{u}$ gives

$$\left.\frac{\mathrm{d}w}{\mathrm{d}e}\right|_{u=\bar{u}} = \frac{\partial c(e, \theta)}{\partial e} > 0,$$

so the indifference curves are upward-sloping. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left.\frac{\mathrm{d}w}{\mathrm{d}e}\right|_{u=\bar{u}} = \frac{\partial c(e, \theta)}{\partial e \partial \theta} < 0.$$

Thus indifference curves are flatter for higher ability workers. This implies that the indifference curves have the *Mirrlees-Spence single crossing property*, which is closely related to the smooth single crossing differences condition discussed in Definition 36 (see Milgrom & Shannon, 1994).



An example where $u_H(e, w, \theta_H) = u_L(e, w, \theta_L) = \bar{u}$. The two indifference curves cross only once. Note this implies that a worker of type θ_H will choose a weakly higher education level than a worker of type θ_L. More formally, we can apply Topkis' theorem:

Definition 49 (Supermodularity)**.** A function $f : X \times \Theta \to \mathbb{R}$ is called *supermodular* in (x, θ), or equivalently has *increasing differences* in θ, if for all $x, x' \in X$ s.t. $x' \geq x$ and all $\theta, \theta' \in \Theta$ s.t. $\theta' \geq \theta$, we have that

$$f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta),$$

i.e. $f(x', \theta) - f(x, \theta)$ is nondecreasing in θ.

If f is a utility function, this property is simply that the incremental gain to choosing a higher value of x is weakly greater if θ is higher. If f is twice continuously differentiable, this is equivalent to f having a nonnegative cross-partial derivative:

Lemma 9. *A twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (x, θ) iff $\frac{\partial f(x,\theta)}{\partial x \partial \theta} \geq 0$ for all (x, θ).*

Theorem 20 (Topkis' theorem)**.** *If $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (x, θ), then $x^*(\theta) = \arg\max_x f(x, \theta)$ is nondecreasing.*

Proof. Suppose $\theta' \geq \theta$, $x \in x^*(\theta)$ and $x' \in x^*(\theta')$. Since $x \in x^*(\theta)$, we have $f(x, \theta) - f(\min\{x, x'\}, \theta) \geq 0$. There are two cases:

(i) $x \geq x'$. Then this becomes $f(x, \theta) - f(x', \theta) \geq 0$. Since $x = \max\{x, x'\}$, we have $f(\max\{x, x'\}, \theta) - f(x', \theta) \geq 0$.

(ii) $x' \geq x$. Then $x' = \max\{x, x'\}$, so $f(\max\{x', x\}, \theta) - f(x', \theta) = f(x', \theta) - f(x', \theta) = 0$.

From the two cases, we establish that $f(\max\{x, x'\}, \theta) - f(x', \theta) \geq 0$. By supermodularity, we thus have $f(\max\{x, x'\}, \theta') - f(x', \theta) \geq 0$, implying $\max\{x, x'\} \in x^*(\theta')$. Since $x' \in x^*(\theta')$, we must have $f(x', \theta') - f(\max\{x, x'\}, \theta') \geq 0$, equivalent to $f(\max\{x, x'\}, \theta') - f(x', \theta') \leq 0$. By supermodularity, $f(\max\{x, x'\}, \theta) - f(x', \theta) \leq 0$. Again, there are two cases:

(i) $x \geq x'$. Then this becomes $f(x, \theta) - f(x', \theta) \leq 0$ and $x' = \min\{x, x'\}$ so $f(x, \theta) - f(\min\{x, x'\}, \theta) \leq 0$.

(ii) $x' \geq x$. Then $x = \min\{x, x'\}$, so $f(x, \theta) - f(\min\{x, x'\}, \theta) = 0$.

From the two cases, we have that $f(x, \theta) - f(\min\{x, x'\}, \theta) \leq 0$. Thus $\min\{x, x'\} \in x^*(\theta)$. $\square$

Consider the worker's utility function $u_1(e, w(e), \theta) = w(e) - c(e, \theta)$. Note this is a function in two arguments, (e, θ). Because $\frac{\partial c(e,\theta)}{\partial e \partial \theta} < 0$ and $\frac{\partial w(e)}{\partial e \partial \theta} = 0$, we have that $\frac{\partial u_1}{\partial e \partial \theta} > 0$ and thus u_1 is supermodular in (e, θ). Applying Topkis' theorem, we immediately have that the optimal education level for a type-θ_H worker is weakly greater than for a type-θ_L worker.

On the equilibrium path, the wage schedule $w(e)$ (or equivalently, the market's belief $\mu(e)$) is pinned down by the worker's choice. For those levels of education e that are not chosen in equilibrium, however, $w(e)$ can be anything in the interval (θ_L, θ_H), since perfect Bayesian equilibrium does not restrict beliefs off the equilibrium path. This allows for many possible equilibria.

Example 41 (Equilibria in the Spence job market signalling game)**.**

(a) *Separating equilibria.* First, consider separating equilibria, i.e. equilibria for which $e(\theta_L) \neq e(\theta_H)$.

Lemma 10. *In a separating equilibrium, $w(e(\theta)) = \theta$ for $\theta \in \{\theta_L, \theta_H\}$.*

Proof. Since $e(\theta_L) \neq e(\theta_H)$, if μ is to be consistent with Bayes' rule, we must have

$$\mu(e(\theta_L)) = \frac{\mathbb{P}\{e(\theta_H) = \theta_L\}\lambda}{\mathbb{P}\{e(\theta_H) = \theta_L\}\lambda + \mathbb{P}\{e(\theta_L) = \theta_L\}(1-\lambda)} = \frac{0}{1-\lambda} = 0,$$
$$\mu(e(\theta_H)) = \frac{\mathbb{P}\{e(\theta_H) = \theta_L\}\lambda}{\mathbb{P}\{e(\theta_H) = \theta_H\}\lambda + \mathbb{P}\{e(\theta_L) = \theta_H\}(1-\lambda)} = \frac{\lambda}{\lambda} = 1.$$

Given these beliefs, we have wages $w(e(\theta_H)) = \theta_H$ and $w(e(\theta_L)) = \theta_L$. □

Lemma 11. *In a separating equilibrium, $e(\theta_L) = 0$.*

Proof. Suppose otherwise. By the previous lemma, $w(e(\theta_L)) = \theta_L$. Suppose a type-$\theta_L$ worker instead chooses $e = 0$. Now, $\mu(0) \geq 0$ so $w(0) \geq \theta_L$, and $c(0, \theta_L) < c(e(\theta_L), \theta_L)$. Hence $u_1(0, w(\cdot), \theta) > u_1(e(\theta_L), w(\cdot), \theta)$, so for any $e(\theta_L) > 0$, $e = 0$ is a profitable deviation. □

Finally, we need to find $e(\theta_H)$ s.t. high ability workers prefer to select $e(\theta_H)$ over $e = 0$ and earn the higher wage $w = \theta_H$, while lower ability workers prefer to select $e(\theta_L) = 0$.

Lemma 12. *For any $e(\theta_H)$ that satisfies*

$$\theta_H - c(e(\theta_H), \theta_H) \geq \theta_L - c(0, \theta_H),$$
$$\theta_L - c(0, \theta_L) \geq \theta_H - c(e(\theta_H), \theta_L),$$

there is a separating equilibrium in which type-θ_H workers earn θ_H.

Proof. If the two conditions hold, then clearly no type of worker benefits from choosing the signal of the other type. The only other thing to check is that no type of worker benefits by choosing some $e \notin \{e(\theta_L), e(\theta_H)\}$. This requires setting $\mu(e)$ sufficiently low for all $e \notin \{e(\theta_L), e(\theta_H)\}$, so that

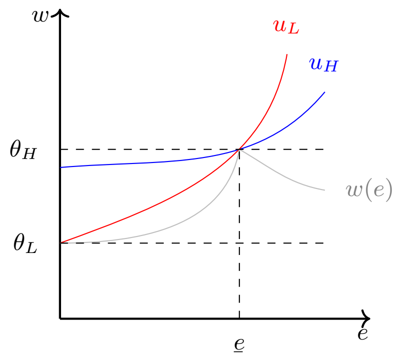
$$w(e) - c(e, \theta_H) \leq \theta_H - c(e(\theta_H), \theta_H),$$
$$w(e) - c(e, \theta_L) \leq \theta_L - c(0, \theta_L).$$

This can be assured trivially by setting $\mu(e) = 0$ for all $e < e(\theta_H)$. Then $w(e) = \theta_L$ for all $e < \theta_H$. □

We can find bounds on the range of values of $e(\theta_H)$ that are possible in a separating equilibrium. By the single crossing property, if the second condition in Lemma 12 is satisfied with equality then the first condition holds with strict inequality. Thus define $\underline{e}$ so that

$$\theta_L - c(0, \theta_L) = \theta_H - c(\underline{e}, \theta_H).$$

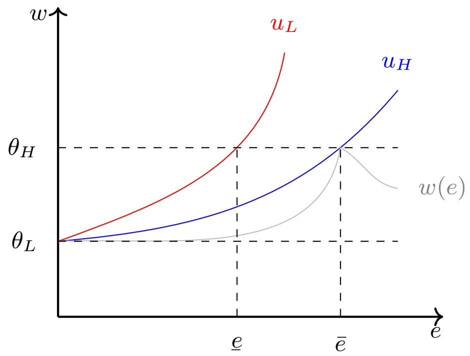
We have a separating equilibrium with $e(\theta_H) = \underline{e}$:



Likewise, the single crossing property implies that if the first condition in Lemma 12 holds with equality, then the second condition holds with strict inequality. Thus define $\bar{e}$ so that

$$\theta_H - c(\bar{e}, \theta_H) = \theta_L - c(0, \theta_H).$$

We have a separating equilibrium with $e(\theta_H) = \bar{e}$:



Clearly, $\underline{e}$ is the minimum value of $e(\theta_H)$ that can be supported in a separating equilibrium, and $\bar{e}$ is the maximum. Any value in $[\underline{e}, \bar{e}]$ is a value of $e(\theta_H)$ in some separating equilibrium.

The single crossing property is key to separating equilibrium – that allows high ability workers to acquire education that is relatively low cost for them but prohibitively costly for lower ability workers. The differential costs allow separation.

(b) *Pooling equilibria.* In a pooling equilibrium, every worker chooses the same education level $e^P = e(\theta_L) = e(\theta_H)$. Since the market's beliefs in equilibrium must be consistent with Bayes' rule, we must have $\mu(e^P) = \lambda$. Therefore the equilibrium wage is

$$w(e^P) = \lambda\theta_H + (1-\lambda)\theta_L =: \mathbb{E}\theta.$$

Define $\hat{e}$ to satisfy

$$\mathbb{E}\theta - c(\hat{e}, \theta_L) = \theta_L = c(0, \theta_L),$$

i.e. $\hat{e}$ is s.t. a lower-ability worker is indifferent between acquiring $\hat{e}$ and no education at all.

Proposition 38. *For any $e^P \in [0, \hat{e}]$, there is a pooling equilibrium in which all types of worker choose e^P with probability* 1.

Proof. Fix $e^P \in [0, \hat{e}]$. Suppose $\mu(e^P) = \lambda$ and $\mu(e) = 0$ for all $e \neq e^P$, with corresponding wages $w(e) = \theta_L$. Then $w(e) < w^{(}e^P)$. Hence no worker benefits by deviating to $e > e^P$, and by definition of $\hat{e}$, type-θ_L workers prefer e^P to any $e < e^P$. By the single crossing property, type-θ_H workers must also prefer e^P to any $e < e^P$. □

An interesting consequence of Proposition 38 is that there can be inefficiency even in a pooling equilibrium. If $e^P > 0$, then all workers are needlessly acquiring some costly education.

(c) *Hybrid equilibria.* In a hybrid equilibrium, one or both types of worker randomize. For example, suppose $e(\theta_L) = 0$ but that type θ_H chooses $e(\theta_H) = 0$ with probability q and $e(\theta_H) = \bar{e}$ for some $\bar{e} > 0$ with probability $1 - q$. The market knows on observing education level $\bar{e}$ that the worker is type θ_H, so $\mu(\bar{e}) = 1$ and the market pays wage $w(\bar{e}) = \theta_H$. By Bayes' rule, the market's belief that a worker with education level 0 is type θ_H is $\mu(0) = \frac{\lambda q}{\lambda q + 1 - \lambda}$, and so the market pays wage $w(0) = \frac{\lambda q}{\lambda q + 1 - \lambda}\theta_H + \frac{1-\lambda}{\lambda q + 1 - \lambda}\theta_L$. Now we need that type θ_H is indifferent between education levels 0 and $\bar{e}$, or else θ_H will not randomize in equilibrium. Thus we require that $\bar{e}$ solves $w(0) = w(\bar{e}) - c(\bar{e}, \theta_H)$. Finally, let $\mu(e) = 0$ for all $e \notin \{0, \bar{e}\}$. Then this constitutes a hybrid equilibrium.

All three types of equilibrium exist in the Spence model, because sequential equilibrium imposes no restrictions on off-equilibrium-path beliefs in this setting.

In the standard job market signalling model we have discussed, education has no effect on productivity. In the separating equilibrium, education reveals information, and so correlates with wages. In the pooling equilibrium, education reveals no information, and yet workers may still end up acquiring some positive level of education.

Any equilibrium in which some type of worker acquires any positive level of education is inefficient. If firms in the labour market were able to observe types directly, agents would not need to burden the cost of acquiring an education. Again, this reflects the (unrealistic) assumption that education does not improve productivity.

4.8.2 Forward induction in signalling games

"Despite the name we have given it, the Intuitive Criterion is not completely intuitive" – In-Koo Cho and David M. Kreps, 1987.

While there are generally infinitely many equilibria in the signalling game, some of these equilibria are arguably less appealing than others. Cho & Kreps (1987) introduce the *intuitive criterion*, a criterion for refinement of equilibrium in signalling games to eliminate these arguably less appealing equilibria. This is an application of forward induction reasoning.

Definition 50 (Intuitive criterion)**.**

(a) *Best responses.* Let $\mathrm{BR}(T, m)$ denote the set of best responses of Player 2 if Player 1 has chosen m and Player 2's beliefs have support in $T \subseteq \Theta$. That is,

$$\mathrm{BR}(T, m) = \bigcup_{\mu \in \Delta(T)} \arg\max_{a \in A} \sum_{\theta \in T} u_2(m, a, \theta)\mu(\theta).$$

(b) *Intuitive criterion.* A sequential equilibrium (or PBE) (s^*, μ) of a signalling game G is said to *fail the intuitive criterion* if there exists $m \in M$, $\theta' \in \Theta$ and $J \subseteq \Theta$ s.t.

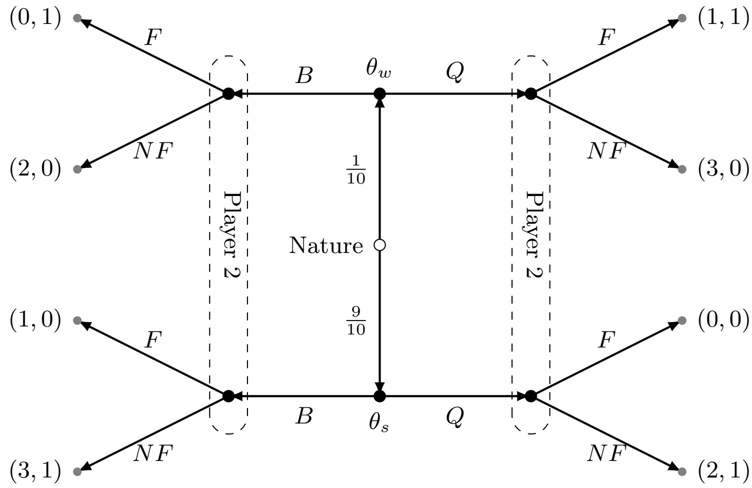
$$u_1(s^*, \theta) > \max_{a_2 \in \mathrm{BR}(\Theta, a_1)} u_1(m, a, \theta) \qquad \text{for all } \theta \in J, \text{ and}$$
$$u_1(s^*, \theta) < \min_{a \in \mathrm{BR}(\Theta - J, m)} u_1(m, a, \theta').$$

The first condition implies types in J would not play m since they would do strictly worse if they do so, even if they can persuade Player 2 that they are of a particular type. The second condition implies type θ' performs strictly better by playing m than the equilibrium provided that type θ' player can convince Player 2 that her type is not in J.

The key idea of the intuitive criterion is that if a certain action is dominated in equilibrium for some subset J of types, then Player 2 should reason that if Player 1 takes that action, then Player 1 cannot be of a type in J. We can see this is a forward induction argument – Player 2 is assuming Player 1 has acted rationally, and making inferences about Player 1's type on that basis.

Example 42 (Beer-quiche game; Cho & Kreps, 1987)**.** In the beer-quiche game Player 1 is wimpy (θ_w) with probability $\frac{1}{10}$ and surly (θ_s) with probability $\frac{9}{10}$. Player 2 is a bully who would prefer to fight (F) the wimpy type and prefers to not fight (NF) the surly type. Player 1 orders breakfast – either beer B or quiche Q – and Player 2 then decides whether to fight.[20] The game, with payoffs, is depicted below.

[20] This story seems very typical of Kreps' style.



In the beer-quiche game, there are no separating equilibria:

- Suppose $\sigma_1(\theta_w) = Q$ and $\sigma_1(\theta_s) = B$. The only consistent beliefs of Player 2 are then
$$\mu_2(\theta_w \mid Q) = 1, \qquad \mu_2(\theta_s \mid Q) = 0,$$
$$\mu_2(\theta_w \mid B) = 0, \qquad \mu_2(\theta_s \mid B) = 1.$$
Player 2's best responses are thus $\sigma_2(B) = NF$ and $\sigma_2(Q) = F$. But then $u_1(B, \sigma_2(B), \theta_w) = 2 > 1 = u_1(Q, \sigma_2(Q), \theta_w)$, so this cannot be an equilibrium.

- Suppose $\sigma_1(\theta_w) = B$ and $\sigma_1(\theta_s) = Q$. The only consistent beliefs of Player 2 are now
$$\mu_2(\theta^w \mid Q) = 0, \qquad \mu_2(\theta^s \mid Q) = 1,$$
$$\mu_2(\theta_w \mid B) = 1, \qquad \mu_2(\theta_s \mid B) = 0.$$
Player 2's best responses are thus $\sigma_2(B) = F$ and $\sigma_2(Q) = NF$. But then $u_1(Q, \sigma_2(Q), \theta_w) = 3 > 0 = u_1(B, \sigma_2(B), \theta_w)$, so this cannot be an equilibrium.

However, there are two classes of sequential equilibrium, both of which are pooling equilibria:

- *Pooling on quiche.* Suppose $\sigma_1(\theta_w) = \sigma_1(\theta_s) = Q$. Then any consistent beliefs of Player 2 must involve
$$\mu_2(\theta_w \mid Q) = \frac{1}{10} \qquad \text{and} \qquad \mu_2(\theta_s \mid Q) = \frac{9}{10}.$$
Thus Player 2 has a best response $\sigma_1(Q) = NF$, since this has expected payoff $u_2(NF, Q \mid \mu_2) = \frac{9}{10} > \frac{1}{10} = u_2(F, Q \mid \mu_2)$.

  However, off-equilibrium-path beliefs are not pinned down by consistency here. We have $u_2(NF, B \mid \mu_2) = \mu_2(\theta_s \mid B)$ and $u_2(F, B \mid \mu_2) = \mu_2(\theta_1 \mid B)$. Hence if

$\mu_2(\theta_w \mid B) \geq \mu_2(\theta_s \mid B)$, i.e. $\mu_2(\theta_w \mid B) \geq \frac{1}{2}$, then Player 2's best response is $\sigma_2(B) = F$. In this case,

$$u_1(Q, \sigma_2(Q), \theta_w) = 3 > 0 = u_1(B, \sigma_2(B), \theta_w),$$
$$u_1(Q, \sigma_2(Q), \theta_s) = 2 > 1 = u_1(B, \sigma_2(B), \theta_s),$$

and thus if $\sigma = (Q, (F, NF))$, where $\sigma_2 = (\sigma_2(B), \sigma_2(F))$, and $\mu_2 = ((p, 1 - p), (1/10, 9/10))$ with $p \in [1/2, 1]$, then (σ, μ_2) is a sequential equilibrium.

- *Pooling on beer.* Suppose $\sigma_1(\theta_w) = \sigma_1(\theta_s) = B$. Now any consistent beliefs of Player 2 must involve

$$\mu_2(\theta_w \mid B) = \frac{1}{10} \qquad \text{and} \qquad \mu_2(\theta_s \mid B) = \frac{9}{10}.$$

  Player 2's best response is then $\sigma_1(B) = F$, since this has expected payoff $u_2(NF, B \mid \mu_2) = \frac{9}{10} > \frac{1}{10} = u_2(F, B \mid \mu_2)$.

  Again, off-equilibrium-path beliefs are not pinned down by consistency here, and we have $u_2(NF, Q \mid \mu_2) = \mu_2(\theta_s \mid Q)$ and $u_2(F, Q \mid \mu_2) = \mu_2(\theta_1 \mid Q)$. So if $\mu_2(\theta_w \mid Q) \geq \mu_2(\theta_s \mid Q)$, i.e. $\mu_2(\theta_w \mid Q) \geq \frac{1}{2}$, then Player 2's best response is $\sigma_2(Q) = F$. Thus,

$$u_1(B, \sigma_2(Q), \theta_w) = 2 > 1 = u_1(Q, \sigma_2(Q), \theta_w),$$
$$u_1(B, \sigma_2(Q), \theta_s) = 3 > 1 = u_1(Q, \sigma_2(Q), \theta_s),$$

  and so if $\sigma = (B, (NF, F))$ and $\mu_2 = ((1/10, 9/10), (p, 1 - p))$ with $p \in [1/2, 1]$, then (σ, μ_2) is a sequential equilibrium.

The pooling equilibrium on quiche seems unreasonable on the forward induction argument captured by the intuitive criterion. It is never a best response for type θ_w to choose B: θ_w cannot conceivably want to deviate to Q, since he receives payoff 3 from the quiche equilibrium. Type θ_s can conceivably choose B, however, if he believes Player 2 will not fight if he does so. Hence Player 2, on observing B, should conclude that Player 1's type is θ_s. Thus the pooling equilibrium on quiche fails the intuitive criterion.

Proposition 39. *In the job market signalling game with two types, the only equilibrium outcome surviving the intuitive criterion is the separating equilibrium in which $e(\theta_L) = 0$ and $e(\theta_H) = \underline{e}$.*

Proof. First, we claim any pooling equilibrium fails the intuitive criterion. Consider a pooling equilibrium with equilibrium education level e^P. We require an education level e satisfying

$$\mathbb{E}\theta - c(e^P, \theta_L) > \theta_H - c(e, \theta_L),$$
$$\mathbb{E}\theta - c(e^P, \theta_H) < \theta_H - c(e, \theta_H).$$

Fix $\bar{u} = u_1(e^P, w = \mathbb{E}\theta, \theta_L)$, i.e. $\bar{u}$ is the utility level a type-θ_L attains in the pooling equilibrium. Let $\hat{e}$ be s.t. $u_1(\hat{e}, w = \theta_H, \theta_L) = \bar{u}$, i.e. so that a type-$\theta_L$ player is indifferent between the equilibrium and $(\hat{e}, w = \theta_H)$. Since $c(\hat{e}, \theta_H) < c(\hat{e}, \theta_L)$, a type-$\theta_H$ player will strictly prefer $(\hat{e}, w = \theta_H)$ to the equilibrium. Hence choosing some $e = \hat{e} + \epsilon$ for some sufficiently small $\epsilon > 0$ achieves our two conditions.

Next, we claim all separating equilibria with $e(\theta_H) > \underline{e}$ fail the intuitive criterion. Consider any $e' \in (\underline{e}, e(\theta_H))$. We have $\theta_L - c(0, \theta_L) < \theta_H - c(e', \theta_L)$ (c.f. Lemma 12 and the single-crossing property) but $\theta_H - c(e', \theta_H) > \theta_H - c(e(\theta_H), \theta_H)$, so the equilibrium fails the intuitive criterion.

Thus, the only equilibrium surviving is the separating equilibrium with $e(\theta_H) = \underline{e}$. □

In the Spence model with n types $\theta_1 < \cdots < \theta_n$, the *Riley outcome* is the separating equilibrium in which each type θ_k chooses the best education level for themselves assuming they will be paid according to their type, subject to the constraint that types $\theta_j < \theta_k$ do not prefer to acquire that level of education and pretend to be type θ_k. This is the most efficient of the separating equilibria.

The intuitive criterion is very effective in the Spence model with two types – it selects the Riley equilibrium. Once we enlarge the number of possible types of worker, however, the intuitive criterion does not isolate the Riley outcome.

The intuitive criterion can be applied iteratively (imaginatively, the *iterated intuitive criterion*). Suppose we have a proposed equilibrium (s, μ). One can apply the intuitive criterion to eliminate type-message pairs (θ, m) (messages m that type θ cannot conceivably ever want to send). Having done so, one can then eliminate actions for Player 2 that are not best responses to some belief about the type-message pairs that have not yet been eliminated. We can then carry out subsequent rounds of eliminating type-message pairs and then actions of Player 2, until we arrive at a set of type-message pairs and actions that survive this iterative procedure.

Banks & Sobel (1987) introduce an alternative to the intuitive criterion: *divinity.*

Definition 51 (Divinity)**.** Consider a signalling game.

(a) *Mixed best responses.* Let $\mathrm{MBR}(T, m)$ denote the set of mixed strategy best responses of Player 2 if Player 1 has chosen m and Player 2's beliefs have support in $T \subseteq \Theta$. That is,

$$\mathrm{MBR}(T, m) = \bigcup_{\mu \in \Delta(T)} \operatorname*{arg\,max}_{\sigma_2 \in \Delta(A)} \sum_{\theta \in T} u_2(m, \sigma_2, \theta)\mu(\theta).$$

(b) *Divinity criteria.* Let $T(m) \subseteq \Theta$ denote the set of types θ that might have sent message m. Fix a sequential equilibrium s^*. For each type $\theta \in \Theta$ and message $m \in M$, define

$$D_\theta(m, \theta) = \left\{ \sigma_2 \in \mathrm{MBR}(T(m), m) \mid u_1(s^*, \theta) < \sum_{a \in A} u_1(m, a, \theta)\sigma_2(a) \right\},$$

$$D^0_\theta(m, \theta) = \left\{ \sigma_2 \in \mathrm{MBR}(T(m), m) \mid u_1(s^*, \theta) = \sum_{a \in A} u_1(m, a, \theta)\sigma_2(a) \right\}.$$

Define the following criteria:

(D1) Type-message pair (θ, m) is said to *fail criterion (D1)* if there exists a type θ' such that $D_\theta(m) \cup D^0_\theta(m) \subseteq D_{\theta'}(m)$.

In this case, we say that type θ' is *infinitely more likely* to choose the out-of-equilibrium message m than type θ.

(D2) Type-message pair (θ, m) is said to *fail criterion (D2)* if $D_\theta(m) \cup D^0_\theta(m) \subseteq \bigcup_{\theta' \neq \theta} D_{\theta'}(m)$.

(c) *Universal divinity.* We say a sequential equilibrium (s^*, μ) is a *universally divine equilibrium* if it survives iterated application of (D2).

Banks & Sobel (1987) also define a weaker notion of *divine equilibrium*, but it is difficult to map into the above framework. See their paper and Cho & Kreps (1987) for more details.

4.8.3 Cheap talk

Sometimes, effective costly signals may not be available to players, but players may be able to transmit costless signals. These signals are *cheap talk* – signals that have no direct effect on payoffs.

The structure of a cheap talk model is very similar to the signalling model:

- Stage 0: Nature chooses a type $\theta \in \Theta$ of Player 1 from a distribution p with support Θ.
- Stage 1: Player 1 (the *sender*) observes θ and chooses a *message* $m \in M$.
- Stage 2: Player 2 (the *receiver*) observes m and chooses $a \in A$.
- Payoffs: at the end of stage 2, payoffs $u_1(a, \theta)$ and $u_2(a, \theta)$ are realized.
- Strategies: A pure strategy of the sender is a mapping $s_1 : \Theta \to M$, and a pure strategy of the receiver is a mapping $s_2 : M \to A_2$.

As before, the solution concept of interest here is perfect Bayesian equilibrium.

Whether cheap talk can be effective depends on the setting. Some settings where it does not work:

Example 43.

(a) *Different types of senders have same preferences over actions.* Suppose that in the job market game, instead of acquiring costly education, the worker simply announces their type, possibly not truthfully. Suppose we have payoffs:

	a_L	a_M	a_H
$\theta = L$	$1, 1$	$2, 0$	$3, -2$
$\theta = H$	$1, -2$	$2, 0$	$3, 1$

There is no separating perfect Bayesian equilibrium in this game. Suppose otherwise, i.e. that $s_1(\theta = L) = m$ and $s_1(\theta = H) = m'$ with $m \neq m'$. Bayes' rule implies $\mu_2(L \mid m) = 1$ and thus $a(m) = a_L$ and $\mu_2(H \mid m) = 0$ so $a(m) = a_H$. Yet then Player 1, if type L, has a profitable deviation by playing m', since $u_1(a(m), L) = 1 < 3 = u_1(a(m'), L)$. Thus this is not a perfect Bayesian equilibrium.

(b) *Different types of senders have completely opposed preferences over actions.* Consider a game where Player 1 wants to accept a job if he is of type A and not if he is of type B. Player 2, conversely, wants to hire (H) Player 1 if Player 1 is of type B and does not want to hire (N) if Player 1 is of type A. We have payoffs:

	H	N
$\theta = A$	$2, -2$	$0, 0$
$\theta = B$	$-2, 2$	$0, 0$

There is no separating perfect Bayesian equilibrium in this game. Suppose $s_1(A) = m$ and $s_1(B) = m'$ with $m \neq m'$. By Bayes' rule, we have that $\mu_2(A \mid m) = 1$ so the optimal action is $a(m) = N$, and $\mu_2(A \mid m') = 0$ so the optimal action is $a(m') = H$. But both types of Player 1 gain from switching messages, since

$$\begin{aligned} u_1(a(m), A) &= 0 < 2 = u_1(a(m'), A), \qquad \text{and} \\ u_1(a(m), B) &= 0 > -2 = u_1(a(m'), A). \end{aligned}$$

Hence this cannot be a perfect Bayesian equilibrium.

However, cheap talk can be effective in coordination games:

Example 44. Consider a version of the meeting game in Example 17 where players go to either Grand Central Station (G) or the Empire State Building (E). Suppose Player 1 is already at one location and cannot move from there – this is Player 1's type θ. Player 1 can text Player 2 his location. Player 2 reads the text and then chooses which location to go to. The payoffs are:

	G	E
G	$1, 1$	$0, 0$
E	$0, 0$	$1, 1$

This game has a separating equilibrium:

$$\begin{aligned} s_1(G) &= m, & s_1(E) &= m', \\ \mu_2(\theta = G \mid m) &= 1, & \mu_2(\theta = G \mid m') &= 0, \\ a_2(m) &= G & a_2(m') &= E. \end{aligned}$$

Note that there is also a *babbling equilibrium*, where Player 1 announces the same message regardless of his type and Player 2 does not change her beliefs based on the message, or where Player 1 randomizes the message independent of type.

4.9 Noncooperative theory of bargaining

The typical noncooperative approach to modelling bargaining envisages bargaining as a dynamic game, in which players make offers sequentially until an agreement is reached. The cooperative approach to bargaining is discussed in a later section.

A *bilateral bargaining* model involves two players dividing a surplus of size $v > 0$. In each period, one of the players is assigned to the position of *proposer*, and the other is assigned the position of *responder*. The proposer proposes an *offer* – a division of the surplus between the players – and the responder can accept or reject the offer. If accepting, the offer is implemented and payoffs are obtained. If rejecting, then (i) if the period is not a final period, then the period is ended and the next period begins; (ii) if the period is the final period, payoffs corresponding to failure to reach an agreement are realized. There is of course only a final period in finite-period bargaining games. Payoffs for each player i are typically discounted according to some discount factor $\delta_i \in (0, 1)$, capturing impatience – players would rather reach an agreement sooner rather than later.

Wlog, we can normalize the surplus v to be divided among the players to 1. The *set of feasible agreements* is

$$Z = \{(z, 1 - z) \mid z \in [0, 1]\},$$

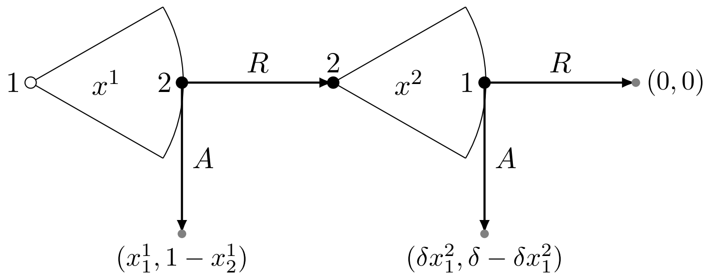
where z is Player 1's share and $1 - z$ is Player 2's share.

4.9.1 Finite period alternating bargaining: the Ståhl model

Ståhl (1972) analyses a finite period *alternating bargaining model.* This assigns one player (say, Player 1) as the proposer in odd periods and the other player (Player 2) in even periods. We can think of this as an *offer-counteroffer* situation – one player makes an offer, if the other player rejects they propose a counteroffer, and the process of haggling continues until an agreement is reached (or not). In period 1, Player 1 offers $x^1 = (x_1^1, 1 - x_1^1) \in Z$. For any offer $x^1 \in Z$, Player 2 accepts or rejects. If Player 2 rejects, then in period 2, Player 2 proposes an offer $x^2 = (x_1^2, 1 - x_1^2) \in Z$, which Player 1 accepts or rejects. In general, if Player 1 rejects, then in period 3, Player 1 makes an offer, and so on.

The simplest case is one period, in which case this is simply the ultimatum game. The two periods case is slightly more interesting:

Example 45 (Two-period alternating bargaining)**.** Suppose there are two periods. If no offer is accepted by the end of the second period, the payoff to both players is 0. Assume payoffs are discounted at a common discount rate $\delta \in (0, 1)$ so if the offer $(x_1^1, 1 - x_1^1)$ is accepted in period 1, payoff profile is $(x_1^1, 1 - x_1^1)$ and if the offer $(x_1^2, 1 - x_1^2)$ is accepted in period 2. the payoff profile is $(\delta x_1^2, \delta - \delta x_1^2)$. We can represent the game in extensive form as:



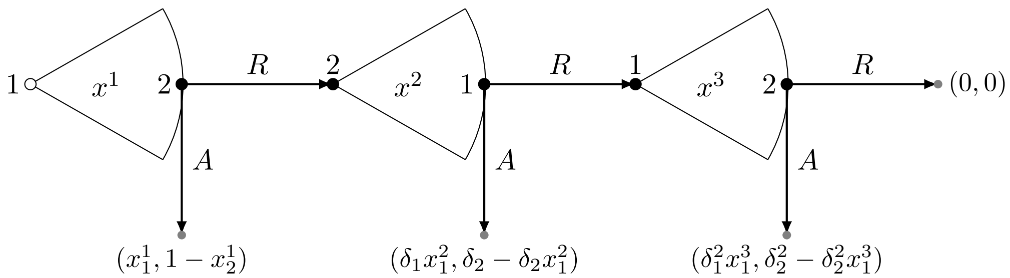
As in the ultimatum game, there are infinite Nash equilibria. However, there is a unique backward induction solution (and so a unique subgame perfect equilibrium):

$$\begin{aligned}
s_1^*(\varnothing) &= (1-\delta, \delta), \\
s_1^*(x^1, R, x^2) &= \text{Accept for all } x^2 \in Z, \\
s_2^*(x^1) &= \begin{cases} \text{Accept} & \text{if } x_1^1 \leq 1-\delta, \\ \text{Reject} & \text{if } x_1^1 > 1-\delta, \end{cases} \\
s_2^*(x^1, R) &= (0, 1).
\end{aligned}$$

The situation in the second period is simply that of the ultimatum game. As previously discussed, Player 2's optimal strategy in this case is to offer $(0, 1)$, and Player 1's optimal strategy is to accept. The payoff profile if the second round is reached and these strategies were implemented would be $(0, \delta)$. In period 1, given an offer $(x_1^1, 1-x_1^1)$, Player 2 knows that if he rejects, he can obtain payoff δ. Hence it is optimal for him to reject any offer $x_1^1, 1 - x_1^1$ s.t. $1 - x_1^1 < \delta \Rightarrow x_1^1 > 1 - \delta$. Conversely, it is optimal for him to accept any offer s.t. $1 - x_1^1 \geq \delta \Rightarrow x_1^1 \leq \delta$. Given this, Player 1 can maximize her payoff by offering $(1 - \delta, \delta)$ in the first round.

The equilibrium outcome path is thus that Player 1 offers $(1-\delta, \delta)$ in the first period and Player 2 accepts.

The more general setting involves $T = 2k + 1$ periods for some $k \in \mathbb{N}$, and unlike the example, we do not assume a common discount factor. Player 1 proposes in odd periods and Player 2 proposes in even periods.



Ståhl model with $T = 3$

We focus on the subgame perfect equilibrium.

The situation in the final period $T = 2k+1$ is again equivalent to the ultimatum game, so Player 1 offers $x^{2k+1} = (1,0)$ and Player 2 accepts any offer in Z.

In the penultimate period $t = 2k$, Player 1 knows she be assured of payoff δ_1 if she rejects, so she finds it optimal to accept any offer $x^{2k} = (x_1^{2k}, 1 - x_1^{2k})$ s.t. $x_1^{2k} \geq \delta_1$ and reject any offer s.t. $x_1^{2k} < \delta_1$. Hence Player 2's optimal offer is $x^{2k} = (\delta_1, 1 - \delta_1)$.

In period $t = 2k-1$, Player 2 knows he can be assured of payoff $\delta_2(1-\delta_1)$, so his optimal strategy is to accept any offer x^{2k-1} s.t. $1 - x_1^{2k-1} \geq \delta_2(1-\delta_1) \Rightarrow x_1^1 \leq 1 - \delta_2(1-\delta_1)$ and reject any offer x^{2k-1} s.t. $1 - x_1^{2k-1} < \delta_2(1-\delta_1) \Rightarrow x_1^1 > 1 - \delta_2(1-\delta_1)$. Knowing this, Player 1's optimal strategy is to offer $x^{2k-1} = (1 - \delta_2 + \delta_2\delta_1, \delta_2 - \delta_2\delta_1)$.

Recursively, we have that in every odd period $t = 2k - 2\ell + 1$, Player 1 offers

$$x^{2(k-\ell)+1} = \frac{1 - \delta_2 + \delta_1^\ell \delta_2^{\ell+1}(1-\delta_1)}{1 - \delta_1\delta_2},$$

and Player 2 will accept this offer. In particular, in period 1, Player 1 offers,

$$x_1^1 = \frac{1 - \delta_2 + \delta_1^k \delta_2^{k+1}(1-\delta_1)}{1 - \delta_1\delta_2}.$$

Hence the backward induction outcome path is that Player 1 offers x^1 as above and Player 2 will accept.

Note that as k increases, x_1^1 decreases. Indeed, as $k \to \infty$, $x_1^1 \to \frac{1-\delta_2}{1-\delta_2\delta_1}$, so $x^1 \to \left(\frac{1-\delta_2}{1-\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$. If $\delta_1 = \delta_2 = \delta$, this simplifies to $x^1 \to \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$. If additionally players become very patient, so $\delta \to 1$, we have $\lim_{\delta\to 1} \lim_{k\to\infty} x^1 = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Example 46 (Ståhl model extension: costly proposing, no discounting)**.** Suppose that players do not discount, but every time it is a player i's turn to propose, they must pay a cost $c \in (0,1)$. The game proceeds for T periods and if no agreement is reached in any period, the payoffs are $(0,0)$. As before, Player 1 proposes in odd rounds and Player 2 proposes in even rounds. For any given T, this game has a unique subgame perfect equilibrium, which differs depending on whether T is odd or even.

First, suppose T is odd. Then in the final round, Player 1 proposes. Player 1's optimal proposal is clearly $(1,0)$, which Player 2 will accept if sequentially rational given the alternative is that Player 2 receives 0. The payoffs in this period would be $(1-c, 0)$. In period $T-1$, Player 1 can obtain $1-c$ if rejecting Player 2's offer, and therefore Player 2's optimal offer is $(1-c, c)$, and accepting is optimal for Player 1. Player 2's payoff is $c - c = 0$. Iterating, we have that in every odd period, Player 1 proposes $(1,0)$ and Player 2 accepts, and in every even period, Player 2 offers $(1-c, c)$ and Player 1 accepts. The equilibrium path has an agreement in the first period, with equilibrium payoffs $(1-c, 0)$.

If T is even, then in the final round, it is Player 2 who proposes. The argument is identical to the case where T is odd, except with the role of Players 1 and 2 reversed: we will have that Player 1 offers $(c, 1-c)$ in every odd period, and Player 2 accepts, while in every even period, Player 2 offers $(0,1)$ and Player 1 accepts. On the equilibrium path, an agreement is reached in the first period and the equilibrium payoffs are $(0, 1-c)$. Unlike the standard Ståhl model, in this version of the game there is a *last-mover advantage*.

Now suppose $T \to \infty$. Then there is no unique subgame perfect equilibrium. Let Γ^T denote the game of T periods and let π_i^T denote the subgame perfect equilibrium payoff of player i in Γ^T. Consider the sequence $\{\pi_i^T\}$. For player 1,

$$\pi_1^T = \begin{cases} 1-c & \text{if } T \text{ is odd,} \\ 0 & \text{if } T \text{ is even.} \end{cases}$$

Hence the subsequence $\{\pi_1^{2k+1}\}_{k\in\mathbb{N}}$ has $\pi_1^{2k+1} = 1-c$ for all k, so $\lim_{k\to\infty} \pi_1^{2k+1} = 1-c$. Conversely, the subsequence $\{\pi_1^{2k}\}_{k\in\mathbb{N}}$ has $\pi_1^{2k} = 0$ for all k so $\lim_{k\to\infty} \pi_1^{2k} = 0$. We therefore have that

$$\limsup_{T\to\infty} \pi_1^T = 1-c > 0 = \liminf_{T\to\infty} \pi_1^T,$$

so the set of subgame perfect equilibrium payoffs for Player 1 in the limit as $T \to \infty$ is not a singleton. The same can be derived for Player 2.

4.9.2 Infinite period alternating bargaining: the Rubinstein model

Rubinstein (1982) analyses an infinite period alternating bargaining model. The structure of the model is identical to Ståhl's: two players bargain to divide a surplus of normalized value 1, with Player 1 proposing in odd periods and Player 2 proposing in even periods. We let Player 1 have discount rate δ_1 and Player 2 have discount rate δ_2.

Proposition 40. *There is a unique subgame perfect equilibrium in the Rubinstein sequential bargaining game.*[21] *Furthermore, the subgame perfect equilibrium is as follows: Whenever Player 1 proposes, she offers a division $(x, 1-x)$ with $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Player 2 accepts any division giving her at least $1-x$. Whenever Player 2 proposes, she offers a division $(y, 1-y)$ with $y = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$. Player 1 accepts any division giving her at least y. Thus on the equilibrium path, bargaining ends in the first round with division $(x, 1-x)$.*

Proof. First, we confirm the proposed strategies constitute a subgame perfect equilibrium. By the one-shot deviation principle (Theorem 17), it is sufficient to check one-shot deviations. Note $y = \delta_1 x$. First consider any period where Player 1 makes an offer. Player 1 has no profitable one-shot deviation: suppose she offers $(x', 1-x')$; if $x' < x$, Player 2 accepts and Player 1 receives a strictly lower payoff $x' < x$; if $x' > x$, Player 2 rejects and Player 1 receives, in present value terms, $\delta_1 y = \delta_1^2 x < x$. Likewise, if Player 2 rejects Player 1's offer, Player 2 receives, in present value terms $\delta_2(1-y) = \delta_2(1-\delta_1 x) = \delta_2 - \delta_1\delta_2 x = 1-x$ [to see this final equality, note it rearranges to $x = \frac{1-\delta_2}{1-\delta_1\delta_2}$ which is definitional.] The argument for periods where Player 2 makes an offer is similar.

Next, we show the equilibrium is unique. Let Π_1 be the set of subgame perfect equilibrium payoffs for Player 1. Let $\underline{v}_1 = \inf \Pi_1$ and $\bar{v}_1 = \sup \Pi_1$. Consider any period

[21] Technically, the SPE is unique up to the decision to accept or reject Pareto-inefficient offers.

where Player 2 makes an offer. Player 1 will accept any offer greater than $\delta_1\bar{v}_1$ and reject any offer less than $\delta_1\underline{v}_1$. Hence Player 2 can secure at least $1-\delta_1\bar{v}_1$ by proposing division $(\delta_1\bar{v}_1, 1-\delta_1\bar{v}_1)$, and can secure at most $1-\delta_1\underline{v}_1$ by proposing $(\delta_1\underline{v}_1, 1-\delta_1\underline{v}_1)$.

Now consider a period where Player 1 makes an offer. For Player 2 to accept Player 1's offer, Player 1 must offer at least $\delta_2(1-\delta_1\bar{v}_1)$, and hence

$$\bar{v}_1 \leq 1-\delta_2(1-\delta_1\bar{v}_1),$$

implying

$$\bar{v}_1 \leq \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

Likewise, Player 2 will certainly accept if offered more than $\delta_2(1-\delta_1\underline{v}_1)$. Hence

$$\underline{v}_1 \geq 1-\delta_2(1-\delta_1\underline{v}_1),$$

implying

$$\underline{v}_1 \geq \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

Combining inequalities, we have

$$\bar{v}_1 \leq \frac{1-\delta_2}{1-\delta_1\delta_2} \leq \underline{v}_1.$$

Since by definition, $\underline{v}_1 \leq \bar{v}_1$, it follows that in any subgame perfect equilibrium, Player 1 receives $v_1 = \underline{v}_1 = \bar{v}_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Making a similar argument for Player 2 completes the proof. □

As with the Ståhl model, relatively more patient players in the Rubinstein model receive higher payoffs – Player 1's payoff is increasing in δ_1 and decreasing in δ_2, whereas Player 2's payoff is increasing in δ_2 and decreasing in δ_1. If there is a common discount factor $\delta = \delta_1 = \delta_2$, then Player 1's offer in the first period is $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$. This illustrates Player 1's *first-mover advantage* – since the players are impatient, Player 2 is willing to accept a smaller slice of the pie now than a slightly larger slice in the next period – and likewise, Player 1 would be willing to accept a slightly smaller slice were we to reach a period where Player 2 was proposing.

Example 47 (Rubinstein extension: outside options)**.** Suppose now that when considering a proposal, the responding player i has an option to either continue to the next period or to end the game immediately in which case Player 1 and Player 2 receive payoffs $c_1, c_2 \in [0,1]$ respectively, with $c_1 + c_2 \leq 1$. We assume a common discount rate δ. The game is otherwise identical to the standard Rubinstein model

Proposition 41. *In the modified Rubinstein bargaining game with outside options and a common discount rate δ, there is a unique subgame perfect equilibrium. Furthermore,* *the subgame perfect equilibrium is as follows: Whenever Player* 1 *proposes, she offers a division* $(x, 1-x)$ *with*

$$x = \begin{cases} \frac{1}{1+\delta} & \textit{if } c_i \le \frac{\delta}{1+\delta} \textit{ for all } i = 1, 2, \\ 1 - \delta(1 - c_1) & \textit{if } c_1 > \frac{\delta}{1+\delta} \textit{ and } c_2 \le \delta(1 - c_1), \\ 1 - c_2 & \textit{otherwise,} \end{cases}$$

in every odd period. Player 2 *accepts any division giving her at least* $1 - x$*. Whenever Player* 2 *proposes, she offers a division* $(y, 1-y)$ *with*

$$y = \begin{cases} \frac{\delta}{1+\delta} & \textit{if } c_i \le \frac{\delta}{1+\delta} \textit{ for all } i = 1, 2, \\ \delta(1 - c_2) & \textit{if } c_2 > \frac{\delta}{1+\delta} \textit{ and } c_1 \le \delta(1 - c_1), \\ c_1 & \textit{otherwise,} \end{cases}$$

and Player 1 *accepts any division giving her at least* y*.*

Proof. First we verify the proposed strategies constitute a subgame perfect equilibrium. Consider any period where Player 1 proposes. There are three cases to check:

(i) $c_1, c_2 \le \frac{\delta}{1+\delta}$. Suppose Player 1 offers $(x', 1-x')$. If $x' < x$, then Player 2 accepts and Player 1's payoff will clearly be lower. If $x' > x$, then Player 2 rejects. Player 1 accepts $y = \delta x$ in the next period which in present value terms is worth $\delta y = \delta^2 x < x$. Hence Player 1 has no profitable deviation. Suppose Player 2 rejects $(x, 1-x)$. If she opts to end the game, she receives $c_2 \le \frac{\delta}{1+\delta} = 1 - x$, so this is not a profitable deviation. If she advances the game to the next period, she receives, in present value terms, a payoff of $\delta(1-y) = \delta(1-\delta x) = \delta - \delta^2 x = 1 - x$ [to verify the final equality holds, note that rearranging it yields $x = \frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta}$]. This is no greater than accepting, so is not a profitable deviation.

(ii) $c_1 > \frac{\delta}{1+\delta}$, $c_2 \le \delta(1 - c_1)$. Note that $c_2 \le \delta(1 - c_1)$ implies $c_2 < \frac{\delta}{1+\delta}$. Suppose Player 1 offers $(x', 1-x')$. Again, if $x' < x$ the Player 2 accepts and Player 1 receives strictly lower payoff. If $x' > x$, Player 2 rejects and continues to the next period, so Player 1 receives, in present value terms, $\delta^2 x < x$. Hence Player 1 has no profitable deviation. Suppose Player 2 rejects $(x, 1-x)$. If Player 2 ends the game, she receives $c_2 \le \delta(1-c_1) = 1-x$, so this is not a profitable deviation. If Player 2 continues the game, she receives in present value terms $\delta(1-c_1)$, which is precisely what she receives if accepting the offer, so this is not a profitable deviation.

(iii) $c_1 > \frac{\delta}{1+\delta}$, $c_2 > \delta(1 - c_1)$. Suppose Player 1 offers $(x', 1-x')$. As before, if $x' < x$ then Player 2 accepts and Player 1 is strictly worse off compared to the division $(x, 1-x)$. If $x' > x$, then Player 2 rejects and ends the game to receive outside payoff c_2. In this case, Player 1 receives $c_1 \le 1 - c_2$, so this is not a profitable deviation. Suppose Player 2 rejects $(x, 1-x)$. Since Player 2 would receive c_2 if accepting, ending the game immediately is not a profitable deviation. If advancing to the next period, Player 2 receives, in net present value terms, $\delta(1-c_1) < c_2$ or $\delta(1-\delta(1-c_2)) < c_2$, where the inequality follows from $c_2 > \frac{\delta}{1+\delta}$. Hence this is not a profitable deviation for Player 2.

The situation in even periods is symmetric.

Next, we show the equilibrium is unique. Let Π_i be the set of subgame perfect equilibrium payoffs for Player i, and define $\underline{v}_i = \inf \Pi_i$ and $\bar{v}_i = \sup \Pi_i$. Fixing $i = 1, 2$ consider any period where i is proposing. Player $j \neq i$ will certainly accept any offer that gives her at least $\max\{\delta \bar{v}_j, c_j\}$, and hence player i can obtain at least $\underline{v}_i \geq 1 - \max\{\delta \bar{v}_j, c_j\}$. Likewise, for player j to accept, she must receive at least $\max\{\delta \underline{v}_j, c_j\}$, so player i receives at most $\bar{v}_i \leq 1 - \max\{\delta \underline{v}_j, c_j\}$. Hence

$$\bar{v}_i - \underline{v}_i \leq \max\{\delta \bar{v}_j, c_j\} - \max\{\delta \underline{v}_j, c_j\}.$$

If $c_j \geq \delta \bar{v}_j \geq \delta \underline{v}_j$, then we have $\bar{v}_i - \underline{v}_i \leq 0$, and since $\bar{v}_i \geq \underline{v}_i$, it follows that $\bar{v}_i = \underline{v}_i$. If $c_j \leq \delta \underline{v}_j \leq \delta \bar{v}_j$, then we have $\bar{v}_i - \underline{v}_i \leq \delta(\underline{v}_j - \bar{v}_j) \leq 0$ Since $\bar{v}_i - \underline{v}_i \geq 0$, it follows that $\bar{v}_i = \underline{v}_i$. Finally, if $\delta \underline{v}_j \leq c_j \leq \delta \bar{v}_j$, then we have $\bar{v}_i - \underline{v}_i \leq c_j - \delta \bar{v}_j \leq 0$, and since $\bar{v}_i - \underline{v}_i \geq 0$, it again follows that $\bar{v}_i = \underline{v}_i$. Hence Π_i contains only one value for both i. This completes the proof. $\square$

5 Repeated games

A repeated game is a (typically normal form) game that is played repeatedly among the same set of players, with discounted payoffs aggregated over time. For the moment, we consider only repeated games with *perfect monitoring.*

Definition 52.

(a) *Stage game.* In the context of a repeated game, a *stage game* G is a normal form game $G = (\mathcal{I}, (A_i, u_i)_{i \in \mathcal{I}})$.

(b) *Perfect monitoring.* We say there is *perfect monitoring* if in every period t, every player perfectly recall the full history of actions by all players in all periods $0 \leq t' < t$.

(c) *History.* In a repeated game, a *history* at time t is a profile $h^t = (a^0, a^1, \ldots, a^{t-1})$, where each $a^\tau \in A_1 \times \cdots \times A_n$ is the profile of actions played in period τ. The set of histories at time t is defined as

$$\mathcal{H}^t = \{(a^0, \ldots, a^{t-1}) \mid a^\tau \in A_1 \times \cdots \times A_n \text{ for all } \tau = 0, \ldots, t-1\}.$$

(d) *Finitely repeated game.* A $(T+1)$-period repeated game $G^T(\delta)$ corresponding to stage game $G = (\mathcal{I}, (A_i, u_i)_{i \in \mathcal{I}})$ is a game $(\mathcal{I}, \mathcal{H}, (\Sigma_i, U_i)_{i \in \mathcal{I}})$, where

  (i) $\mathcal{H} = \bigcup_{t=0}^{T+1}$ is the set of all histories of the game. We call $\mathcal{H}^{T+1}$ the set of *terminal histories* of the game;

(ii) $\Sigma_i = \{\sigma_i \mid \sigma_i : \mathcal{H} \to \Delta(A_i)\}$, that is, Σ_i is the set of all mappings from the set of histories into i's stage game mixed strategy set. A strategy $\sigma_i \in \Sigma_i$ of the repeated game specifies for each t and each history $h^t \in \mathcal{H}^t$ a strategy $\sigma^t(h^t)$ in the tth stage game G;

(iii) $U_i(\sigma) = \mathbb{E}_\sigma\left[\frac{1}{T}\sum_{t=0}^T \delta^t u_i(\sigma^t(h^t))\right]$ is the payoff function, the expected value of the discounted sum of stage-game payoffs under strategy σ.

(e) *Infinitely repeated game.* An infinitely repeated game $G^\infty(\delta) = (\mathcal{I}, (\Sigma_i, U_i)_{i\in\mathcal{I}})$ with stage game $G = (\mathcal{I}, (A_i, u_i)_{i\in\mathcal{I}})$ is a game $(\mathcal{I}, \mathcal{H}, (\Sigma_i, U_i)_{i\in\mathcal{I}})$, where

(i) $\mathcal{H} = \bigcup_{t=0}^\infty \mathcal{H}^t$ is the set of all histories of the game;

(ii) $\Sigma_i = \{\sigma_i : \mathcal{H} \to \Delta(A_i)\}$ is the set of all mappings from the set of histories into i's stage game mixed strategy set. Again, a strategy $\sigma_i \in \Sigma_i$ of the repeated game specifies for each t and each history $h^t \in \mathcal{H}^t$ a strategy $\sigma^t(h^t)$ in the tth stage game G;

(iii) $U_i(\sigma) = \mathbb{E}_\sigma\left[(1-\delta)\sum_{t=0}^\infty \delta^t u_i(\sigma^t(h^t))\right]$ is the payoff function.

(f) *Continuation payoffs.* In a finitely repeated game, we define the *(normalized) continuation payoff* for player i given history h^t at time t by

$$U_i(\sigma \mid h^t) = \mathbb{E}_\sigma\left[\frac{1}{T-t}\sum_{\tau=t}^{T} \delta^{\tau-1} u_i(\sigma^\tau(h^\tau)) \mid h^t\right].$$

In an infinitely repeated game, we define the *(normalized) continuation payoff* for player i given history h^t at time t by

$$U_i(\sigma \mid h^t) = \mathbb{E}_\sigma\left[(1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-1} u_i(\sigma^\tau(h^\tau)) \mid h^t\right].$$

Note we defined repeated games so that the stage game is always a mixed strategy game. We assume that A_i is finite throughout. Note that we have defined the strategies σ_i of the repeated game to be behavioural strategies. By Theorem 1, this is wlog – these strategies are each equivalent to some mixed strategy and vice versa.

We have also defined repeated games so that the payoff function in the repeated game is in *average payoff* form. This normalization allows us to directly compare $U_i(\sigma)$ with the payoffs of the stage game. We could equivalently have defined the payoff function in *total payoff* form, which would be TU_i in the finitely repeated case and $U_i/(1-\delta)$ in the infinitely repeated case.

In the definitions of repeated games, we assumed a common discount rate δ. This can be relaxed by replacing δ with a vector of individual discount rates $(\delta_i)_{i\in\mathcal{I}}$.

We assume the existence of a public randomization device $([0,1], \mathcal{B}, p)$, which produces signals ω^t according to the same distribution p in every period. We assume the public randomization device is common knowledge, and so p is a common prior. It is conventional not to include past signals in the specification of the history, though technically it should be included. Why the need for such a device? This

As with extensive form sequential games, there are generally many Nash equilibria in both finitely and infinitely repeated games, many of which are not sequentially rational. Indeed, in infinitely repeated games we can show there are infinitely many Nash equilibria. Subgame perfection is usually the suitable refinement, though even then, the number of subgame perfect equilibria is potentially very large.

5.1 Finitely repeated games

Finitely repeated games benefit from being solvable by backward induction.

First, note that even in finitely repeated games, there may be many subgame perfect equilibria.

Example 48. Consider the following stage game G:

	A_2	B_2	C_2
A_1	$4,4$	$0,0$	$0,5$
B_1	$0,0$	$1,1$	$0,0$
C_2	$5,0$	$0,0$	$3,3$

This game has three Nash equilibria – two pure strategy equilibria (B_1, B_2) and (C_1, C_2) and a mixed strategy equilibrium $((0, \frac{3}{4}, \frac{1}{4}), (0, \frac{3}{4}, \frac{1}{4}))$.[22]

Now suppose the stage game is repeated twice and that both players discount the future at discount rate δ. Player i's payoff given strategy profile σ is therefore

$$U_i(\sigma) = u_i(\sigma_1(1), \sigma_2(1)) + \delta u_i(\sigma_1(2), \sigma_2(2)).$$

There are a large number of subgame perfect equilibria here. First, any σ^* s.t. $\sigma^*(t)$ is one of the three Nash equilibria for each t is a subgame perfect equilibrium in this game. This gives us 9 different subgame perfect equilibria. Furthermore, by making the choice of which Nash equilibrium to play in the second period contingent on first period play, we can construct a subgame perfect equilibrium where play in the first period differs from Nash play. Consider the following strategy profile:

1. Play (A, A) in the first period;
2. If first period play is (A, A), play (C, C) in the second period. Else play (B, B).

To confirm this is a subgame perfect equilibrium, consider one-shot deviations. In the second period, (C, C) and (B, B) are Nash equilibria so there is no gain to deviating in

[22] A_1 and A_2 are never-best responses so we can exclude mixing over them. Suppose under σ_2, Player 2 plays B_2 with probability p and C_2 with probability $1 - p$. Then for Player 1 to mix, we need $u_1(B_1, \sigma_2) = p = 3(1 - p) = u_1(C_1, \sigma_2)$, implying $p = \frac{3}{4}$. The game is symmetric so Player 1 mixes with the same probability in equilibrium.

either case. In period 1, if player i follows the strategy they receive $4 + 3\delta$. The best deviation in period 1 is C, which yields 5, but this is followed in period 2 by (B, B), and thus the payoff to deviating is $5 + \delta$. We thus have that the strategy above is a subgame perfect equilibrium provided that $4 + 3\delta \geq 5 + \delta$, i.e. $\delta \geq \frac{1}{2}$.

Example 49 (Repeated prisoners' dilemma)**.** In the previous example, there were multiple Nash equilibria in the stage game. If there is only a single Nash equilibrium in the stage game, then there is a unique subgame perfect equilibrium in the finitely repeated game, which involves that Nash equilibrium being played in every period.

Suppose the stage game G is the prisoner's dilemma:

	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$0, 0$

Consider the repeated game $G^T(\delta)$, i.e. the prisoner's dilemma repeated T times with common discount rate δ. We proceed by backward induction. In the final period T, a Nash equilibrium must be played since both players choose optimal strategies, and the only Nash equilibrium is (D_1, D_2). In period $T-1$, both players know that in period T, (D_1, D_2) will be played. Hence D_i is the strictly dominant strategy for each player i in period $T-1$ and so (D_1, D_2) is played in equilibrium. Recursively, we have that D_i is the strictly dominant strategy for i in every period $t = 0, 1, \ldots, T$ and thus (D_1, D_2) is played in each period t in equilibrium. This is therefore the unique subgame perfect equilibrium.

Since this holds for any $T \in \mathbb{N}$, we have that as $T \to \infty$, the unique subgame perfect equilibrium is (D_1, D_2) every period. This is a subgame perfect equilibrium of the limit game $G^\infty(\delta)$, but importantly, it is not unique.[23]

As we might expect, this result holds generally:

Proposition 42. *Suppose G is a stage game possessing a unique Nash equilibrium α^*. Then $\sigma^* = (\alpha^*)_{t=0}^T$ is the unique subgame perfect equilibrium of the $(T+1)$-period repeated game $G^T(\delta)$.*

Proof. The proof is almost identical to the example immediately above, applying backward induction. Since α^* is a unique Nash equilibrium, it must be played in the Tth subgame under any subgame perfect equilibrium. Suppose σ^* is played in all periods $k+1, \ldots, T$. Applying the backward induction algorithm in the kth game, we can substitute $u_i(\alpha_i, \alpha_{-i}) + \sum_{j=1}^T \delta^j u_i(\sigma^*)$ for $u_i(\alpha_i, \alpha_{-i})$. Since α^* is the unique Nash equilibrium,

$$u_i(\alpha_i^*, \alpha_{-i}^*) \geq u_i(s_i, \alpha_{-i}^*)$$

for all $s_i \in S_i$ and all i, and so

$$u_i(\alpha_i^*, \alpha_{-i}^*) + \sum_{j=1}^T \delta^j u_i(\alpha^*) \geq u_i(s_i, \alpha_{-i}^*) + \sum_{j=1}^T \delta^j u_i(\alpha^*),$$

[23] This comes down to the fact that the subgame perfect equilibrium correspondence is not lower hemicontinuous.

for all $s_i \in S_i$ and all i so α^* is optimal in period k. Furthermore, no other strategy profile α is optimal in period k since α^* is the unique Nash equilibrium of the stage game. Proof follows by induction. $\square$

5.2 Infinitely repeated games

In an infinitely repeated game, backward induction is not applicable. The one-shot deviation principle is thus key to finding subgame perfect equilibria here.

Note that in an infinitely repeated game, δ has two possible interpretations:

- *Time preference.* As in a finitely repeated game, δ can represent time preference, measuring how patient players are. If there is an interest rate r, for example, then the discount rate is $\delta = \frac{1}{1+\delta}$.
- *Uncertainty over end date.* An alternative interpretation is that δ is the probability that the interaction will be repeated in the subsequent period. Here, the game ends almost surely in finite time, but the end date is stochastic. $1 - \delta$ could represent the probability that an agent dies before the next period or that a firm becomes exogenously bankrupt before the next period, for example.

Example 49 (continued)**.** Continuing the repeated prisoners' dilemma example, suppose now that the game is repeated for infinite periods. Recall we have stage game G with payoff matrix:

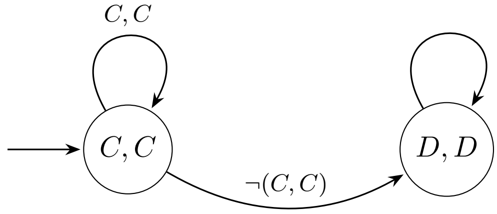
	C_2	D_2
C_1	$1, 1$	$-1, 2$
D_1	$2, -1$	$0, 0$

As in the finite case, playing (D_1, D_2) in every period constitutes a subgame perfect equilibrium. However, if $\delta \geq \frac{1}{2}$, there is a subgame perfect equilibrium in which (C_1, C_2) is played every period.

Consider the following symmetric strategies, known as *grim trigger* strategies, since they involve playing the minimax strategies if any player deviates:

1. Play C_i in the first period and every subsequent period provided no player j ever plays D_j;
2. In each period, if any player j has ever played D_j, play D_i.

It can be convenient to visualize these strategies by means of an automaton:



Call this strategy s.

To see this is a subgame perfect equilibrium, consider any period t and suppose that D_j has already been played by some player. Then there is one one-shot deviation for player i (play C_i) which yields payoff

$$\pi(s_{C_i}, t) = -1 > 0 = \pi(s, t),$$

so the only one-shot deviation is not profitable. Now suppose neither player j has played D_j. Then the one shot deviation for player i is to play D_i, which gives return $\pi(s_{D_i}, t) = 2(1-\delta)$, against $\pi(s, t) = (1-\delta)\sum_{k=0}^{\infty} \delta^t = 1$. Rearranging, we thus have that $\pi(s, t) \geq \pi(s_{D_i}, t)$ provided $\delta \geq \frac{1}{2}$.

5.3 The folk theorems

There are many folk theorems, most of which say something like the following:

> If players are sufficiently patient in an infinitely repeated game, then any feasible, strictly individually rational payoff profile in the stage game can be supported as an equilibrium average payoff profile in the repeated game.

Not all of them say this – some apply to finitely repeated games, some apply to infinitely repeated games without discounting, and some restrict to feasible payoffs that are strictly greater than some Nash equilibrium of the stage game.

We defined feasible and individually rational payoff profiles in Definition 29. Since it is very important here, we will restate it:

Definition 29 (Feasibility and individual rationality)**.**

(a) *Feasible payoffs.* In a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a payoff profile $v = (v_1, \ldots, v_n)$ is *feasible* if there is some probability distribution $p \in \Delta(S)$ s.t.

$$v_i = \sum_{s\in S} u_i(s)p(s) \qquad \text{for all } i \in \mathcal{I}.$$

(b) *Individually rational payoffs.* In a game $G = (\mathcal{I}, (S_i, u_i)_{i\in\mathcal{I}})$, a payoff profile $v = (v_1, \ldots, v_n)$ is *individually rational* if for each $i \in \mathcal{I}$,

$$v_i \geq \min_{\sigma_{-i}\in\Delta_{-i}(S_{-i})} \max_{\sigma_i\in\Delta(S_i)} u_i(\sigma_i, \sigma_{-i}) = \underline{v}_i.$$

A payoff profile v is *strictly individually rational* if this holds with strict inequality for all i.

The game G in the definition refers to a stage game in the current context, and v to an average payoff profile. We will use $\underline{v} = (\underline{v}_1, \ldots, \underline{v}_n)$ to denote the profile of minimax payoffs throughout this section.

The interpretation of the minimax payoff $\underline{v}_i$ of a player i is that it is i's reservation utility. This is only true under the assumption of perfect monitoring:

Proposition 43. *Consider a repeated game of perfect monitoring with stage game G and set of players $\mathcal{I}$. For any player $i \in \mathcal{I}$, if α^* is a Nash equilibrium α^* of the stage game G then $u_i(\alpha^*) \geq \underline{v}_i$, and if σ^* is a Nash equilibrium of the repeated game then $U_i(\sigma^*) \geq \underline{v}_i$.*

Proof. First consider α^* and fix player $i \in \mathcal{I}$. Let $\hat{\alpha} \in \prod_{i=1}^n \Delta(S_i)$ be the strategy profile that solves $\min_{\alpha_{-i}} \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})$. By definition, $\hat{\alpha}_i$ is a best response to $\hat{\alpha}_{-i}$. Now,

$$u_i(\alpha_i^*, \alpha_{-i}^*) \geq u_i(\hat{\alpha}_i, \alpha_{-i}^*) \geq u_i(\hat{\alpha}_i, \hat{\alpha}_{-i}) = \underline{v}_i.$$

Next consider σ^*. Construct a strategy $\hat{a}_i$ for player i as follows. For each history $h^t \in \mathcal{H}$, suppose player i myopically chooses $\hat{a}_i(h^t) \in \arg\max_{a_i \in A_i} u_i(a_i, \sigma_{-i}^*(h^t))$.[24] Since the information available at t is identical across players,[25] i's opponents can randomize only independently, and so $\underline{v}_i$ is the minimum payoff that i's opponents can enforce on i in any period, and thus $u_i(\hat{a}_i(h^t), \sigma_{-i}(h^t)) \geq \underline{v}_i$ for all histories $h^t \in \mathcal{H}_t$ and every period t. Finally, since σ_i^* is a best response to σ_{-i}^*, we have $U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\hat{a}_i, \sigma_{-i}^*) \geq \underline{v}_i$. □

One problem in dealing with the folk theorems is that the set of feasible payoffs is not necessarily convex if δ is sufficiently small. For δ is sufficiently close to 1, Sorin (1986) and Fudenberg & Maskin (1991) show that the set of feasible payoffs is convex. They show this by showing that any convex combination of pure strategy payoff profiles in the stage game can be supported as average payoffs by some time-varying deterministic strategy.

To avoid the issue of dealing with time-varying deterministic strategies, it is convenient to instead assume the existence of a public randomization device. A *public randomization device* is a probability space $([0, 1], \mathcal{B}, p)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0, 1]$. We assume the public randomization device is common knowledge and that p is a common prior. At the start of each period t, a signal $\omega^t \in \Omega$ is observed by all players, and players can condition strategies on this signal. This allows us to "convexify" the set of feasible payoff profiles. To see this, suppose that v and v' are two pure strategy payoff profiles in the stage game, and let $F(x) = p(\omega \leq x)$. Now any convex combination $\lambda v + (1 - \lambda)v'$ can be supported as an average payoff profile in the repeated game if a strategy profile supporting v is played whenever $\omega \leq F^{-1}(\lambda)$ is observed and a strategy profile supporting v' is played otherwise.

A *history* in the repeated game is now a profile $h^t = ((a^0, \omega^0), (a^1, \omega^1), \ldots, (a^{t-1}, \omega^{t-1}), \omega^t)$, where each $a^\tau \in A_1 \times \cdots \times A_n$ is the profile of actions played in period τ and $\omega^\tau \in [0, 1]$ is the signal observed in period τ. We now define the set of histories at time t as

$$\mathcal{H}^t = \Bigg\{ ((a^0, \omega^0), \ldots, (a^{t-1}, \omega^{t-1}), \omega^t) \; |a^\tau \in \prod_{i=1}^n A_i \text{ for all } \tau = 0, \ldots, t-1$$
$$\text{and } \omega^\tau \in [0, 1] \text{ for all } \tau = 0, \ldots, t \Bigg\}.$$

[24] This is not necessarily optimal (for example, consider the grim trigger strategy in Example 49).

[25] This is important because the minimax payoff assumes independent strategies for the opponents, not correlated strategies.

The definition of the repeated game is otherwise unchanged. For sufficiently low discount factor δ, the set of feasible payoffs will differ between the repeated game without a public randomization and the repeated game with the device. However, Fudenberg & Maskin (1991) show that the assumption there exists a public randomization device is innocuous for δ sufficiently close to 1.

5.3.1 In discounted infinitely repeated games

The most commonly studied class of folk theorems are for infinitely repeated games with $\delta \in (0, 1)$. One of the simpler folk theorems (at least in terms of proof) is down to Friedman:

Theorem 21 (Friedman, 1971)**.** *Consider a finite stage game G. If e is a payoff profile of a Nash equilibrium of G, and if v is a feasible payoff profile such that $v_i > e_i$ for every player $i \in \mathcal{I}$, then there exists a $\underline{\delta} < 1$ such that for any $\delta \in [\underline{\delta}, 1)$, there is a subgame perfect equilibrium σ^* of the infinitely repeated game $G^\infty(\delta)$ with payoff profile v.*

Proof. In general, the proof requires detailing public randomizations. To simplify matters, suppose there is some action profile a s.t. $u_i(a) = v_i$ for all players $i \in \mathcal{I}$. Let α^* be the Nash equilibrium of the stage game that yields payoff profile e. Consider the following grim trigger strategy σ_i. for each player $i \in \mathcal{I}$:

(I) Play a_i in the first period and every subsequent period providered no player j ever plays some action $a'_j \neq a_j$.

(II) In each period, if any player j has ever played some action $a'_j \neq a_j$, play α^*_i.

We claim that σ_i is a best response to σ_{-i} for sufficiently large $\delta < 1$. Let σ'_i be a one-shot deviation from σ_i in some period t, and let h^t be the history in which σ has been played in all periods prior to t. Then i's continuation payoff is at most $U_i(\sigma'_i, \sigma_{-i} \mid h^t) = (1-\delta)\max_{a_i \in A_i} u_i(a_i, \sigma_{-i}) + \delta e_i$, whereas her continuation payoff from sticking to σ_i is $U_i(\sigma_i, \sigma_{-i} \mid h^t) = v_i$. Defining $\underline{\delta} = \frac{\max_{a \in A_i - v_i} u_i(a, \sigma_{-i})}{\max_{a_i \in A_i} u_i(a_i, \sigma_{-i}) - e_i} < 1$ and applying the one-shot deviation principle, we have that σ is optimal for each player on the equilibrium path, and thus is a Nash equilibrium. Off-the-equilibrium path, the Nash equilibrium α^* is played in every stage game, and thus it is never optimal for any player to deviate. Thus σ is a subgame perfect equilibrium.

In the case that there is no action profile a generating payoffs v_i for every player i, we need to randomize action profiles on the equilibrium path via the public randomization device such that v_i is the expected payoff to i in each period. The strategies and the application of the one-shot deviation principle are otherwise unchanged. $\square$

Fudenberg & Maskin (1986) prove the following folk theorem, which has become standard:

Theorem 22 (Folk theorem)**.** *Suppose the set of feasible payoff profiles of stage game G has dimension $|\mathcal{I}|$. Then for any feasible and strictly individually rational payoff profile*

v of G, there exists a $\underline{\delta} < 1$ such that for any $\delta \in [\underline{\delta}, 1)$, there is a subgame perfect equilibrium σ^* of the infinitely repeated game $G^{\infty}(\delta)$ such that the payoff profile of σ^* is v.

Proof. Let V denote the set of feasible and strictly individually rational payoff profiles and let $n = |\mathcal{I}|$. For simplicity, suppose there is some action profile $\bar{a}$ such that $u_i(a) = \underline{v}_i$ for all $i \in \mathcal{I}$. For any $v \in V$, let σ be a strategy profile such that $\pi_i(\sigma) = v_i$ for every player $i \in \mathcal{I}$. Let v' be another payoff profile in the interior of V, such that $v_i > v_i'$ for each player i. Since v' is in the interior of V and V has dimension $n = |\mathcal{I}|$, there is some $\epsilon > 0$ such that

$$(v_1' + \epsilon, \ldots, v_{j-1}' + \epsilon, v_j', v_{j+1}' + \epsilon, \ldots, v_n' + \epsilon) \in V$$

for each player j. Let $l^j = (l_1^j, \ldots, l_n^j)$ denote a strategy profile that realizes these payoffs. Normalize j's minmax payoff to $\underline{v}_j = 0$, let $m^j = (m_1^j, \ldots, m_n^j)$ be the strategy profile that minmaxes player j and let $w_i^j = u_i(m^j)$, i's payoff when minmaxing j.

Again for simplicity, assume that for each $i \in \mathcal{I}$, there exists a pure action profile $a(i)$ so that $u_j(a) = l_j^i$ for all $j \in \mathcal{I}$. This is not necessary to prove the theorem, it just removes the need to spend a long time on the details of public randomizations. Choose an integer T with $\max_a u_i(a) + T\underline{v}_i < \min_a u_i(a) + Tv_i'$.

Consider the following multiple phase strategy for player i:

(I) Play σ_i in each period t (generating stage payoffs v) provided σ is played in period $t-1$. If a single player j deviates from (I) then go to (II$_j$).

(II$_j$) play m_i^j for T periods, then go to (III$_j$).

(III$_j$) play l_i^j thereafter.

If a player k deviates in phase (II$_j$) or (III$_j$), go to (II$_k$). If more than one player deviates in any phase, remain in that phase.

Suppose a player i deviates in phase (I) and conforms thereafter. She receives $\bar{v}_i$ at most, zero for the subsequent T periods and v_i' thereafter, i.e. her gain from deviating here is at most

$$\bar{v}_i + \frac{\delta^{T+1}}{1-\delta} v_i'.$$

Not deviating instead yields $\frac{v_i}{1-\delta}$. Hence the gain from deviating is

$$\bar{v}_i - \frac{1-\delta^{T+1}}{1-\delta} v_i'$$

at most. Note $\frac{1-\delta^{T+1}}{1-\delta} \to T+1$ as $\delta \to 1$. By the condition that $\frac{\bar{v}_i}{v_i'} < T_i + 1$, the gain from deviating will therefore be negative for any $\delta > \underline{\delta}$ for some lower bound $\underline{\delta} < 1$.

If i deviates while being punished himself in phase (II$_i$), then i obtains 0 at most in this period, and lengthens punishment, postponing positive payoff v_i' (and hence receiving a strictly lower discounted payoff). If i deviates when $j \neq i$ is being punished in phase (II_j), then he receives at most

$$\bar{v}_i + \frac{\delta^{T+1}}{1-\delta} v_i',$$

as for deviation in phase (I). However, by not deviating, he receives at least

$$w_i^j \frac{1-\delta^{T'}}{1-\delta} + \frac{\delta^{T'+1}}{1-\delta}(v_i' + \epsilon),$$

where $1 < T' < T$. The gain to deviating is at most

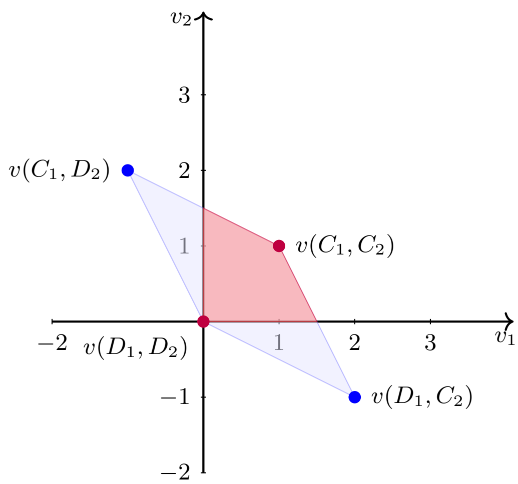
$$\bar{v}_i + \frac{1-\delta^{T'+1}}{1-\delta}(v_i' - w_i^j) - \frac{\delta^{T'+1}}{1-\delta}\epsilon - \delta^{T'} w_i^j.$$

As $\delta \to 1$, the second term stays finite but the third converges to $-\infty$ and so the gain to deviating stays negative. Hence there is some $\underline{\delta}_i < 1$ such that i will not deviate for any $\delta > \underline{\delta}_i$. The proof ruling out deviations in phases (III_i) and (III_j) are very similar to that of phase (I).

If instead, the minimax strategies are mixed, we need to alter the proof so that in phase (II_j), player i is indifferent between all of the possible length-T realizations of sequences of actions that are prescribed by the strategy profile for minimaxing j. This can be achieved by tailoring the reward ϵ to each possible such sequence, making i indifferent by promising a greater future payoff in phase (III_j) for those sequences that yield him a lower payoff in phase (II_j). See Fudenberg & Maskin (1986) for details. □

The proof hinges on a punishment scheme whereby a player i's opponents are rewarded for minimaxing player i if he deviates: in phase (III_i), they all receive strictly greater payoffs than they would if they had returned to (I).

Example 49 (continued)**.** In the infinitely repeated prisoners' dilemma, the set of payoff profiles can be depicted as follows:



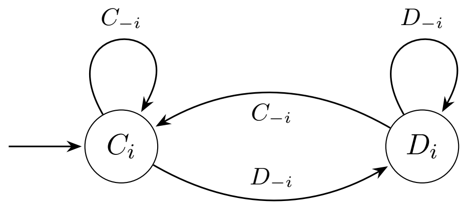
The set of feasible payoff profiles is shaded blue and the set of feasible and individually rational payoff profiles is shaded red. By the folk theorem, any payoff profile in the red set can be sustained as a profile of average payoffs for some subgame perfect equilibrium of the infinitely repeated prisoners' dilemma provided players are sufficiently patient.

We previously considered a subgame perfect equilibrium consisting of grim trigger strategies s.t. on the equilibrium path, (C_1, C_2) was played in every period. Consider instead the following tit-for-tat strategy σ_i for each player i:

1. Play C_i in the first period. In any subsequent period, play C_i if player $-i$ played C_{-i} in the previous period.

2. Play D_i if player $-i$ played D_{-i} in the previous period.

As an automaton, we can visualize this strategy as follows:



The nodes in this case are player i's own actions, rather than the action profiles of both players. These tit-for-tat strategies do not constitute a Nash equilibrium: for any $\delta < 1$, we have $U_i(\sigma_i, \sigma_{-i} \mid C_i, C_{-i}) = 1 - \delta < (1-\delta)(2-\delta) + \delta^2 = U_i(D_i, \sigma_{-i} \mid C_i, C_{-i})$, so a deviation is profitable. Now suppose that instead, for each player i, σ_i calls for i to play D_i for 3 periods, and only return to playing C_i if player $-i$ plays C_i for 3 consecutive periods. Then this constitutes a subgame perfect equilibrium for sufficiently high δ. Now we have that $U_i(\sigma_i, \sigma_{-i} \mid C_i, C_{-i}) = 1 - \delta \geq (1-\delta)(2 - \delta - \delta^2 - \delta^3) + \delta^4 = U_i(D_i, \sigma_{-i} \mid C_i, C_{-i})$ for all $\delta \geq \underline{\delta}$ where $\underline{\delta}$ solves $0 = 1 - \underline{\delta} - \underline{\delta}^2 - \underline{\delta}^3 + (1 - \underline{\delta})\underline{\delta}^4$, which gives us $\underline{\delta} \approx 0.780$. The other states are straightforward to check.

6 Cooperative game theory

6.1 Cooperative games

In non-cooperative game theory, solutions can always be envisaged as a self-enforcing agreement, in the sense that no rational player will choose to deviate. In cooperative game theory, solutions can be envisaged as an agreement which is enforced externally, such as via a system of legal institutions that enforce contracts. The interesting question is which groups will come together to form an agreement and what their payoffs will be – a more high-level approach, since how these coalitions come about is not important.

Definition 53 (Cooperative game)**.** A *cooperative game* or *coalitional game* is a tuple $(N, (A_S)_{S \in 2^N}, (u_i)_{i \in N})$, where:

(i) N is a nonempty finite set of players. Each $S \subseteq N$ is a *coalition*, and we call N the *grand coalition.*

(ii) A_S is a nonempty set of actions available to coalition S.

(iii) An *outcome* $(\mathcal{P}, (a_S)_{S\in\mathcal{P}})$ consists of a partition $\mathcal{P}$ of N and a list of actions $a_S \in A_S$ for each coalition $S \in \mathcal{P}$. Let X denote the set of possible outcomes.

(iv) $u_i : X \to \mathbb{R}$ is a payoff function for player i, mapping outcomes to payoffs.

Some examples:

Example 50.

(a) *Marriage market.* Suppose D is a group of doctors and H is a group of hospital vacancies. Doctors have preferences over which vacancy they would like to fill, if any, and each hospital hiring team (which is different for each vacancy) has preferences over which doctors they wish to hire. A *matching* is a partition of $D \cup H$ into doctor-vacancy pairs and single doctors/vacancies. Each doctor and hiring team cares only about their own match.

(b) *Control of a resource.* Suppose there are three shepherds who have access to a well, which they need to supply water to their sheep. Unfortunately, a large rock is blocking access to the well, and it is too large for any one of the shepherds to move. However, if any group of two or more of the shepherds work together, they can move the rock and claim control of the well, deciding how to divide access to the water within the group. Each shepherd only cares about what share of the water they receive, and they prefer more water to less. An action for any coalition S is an allocation of water among the members of the coalition.

Definition 54. A cooperative game is *cohesive* if for every outcome $(\mathcal{P}, (a_S)_{S\in\mathcal{P}})$, there exists some outcome $(\{N\}, a_N)$ generated by the grand coalition such that $u_i(\{N\}, a_N) \geq u_i(\mathcal{P}, (a_S)_{S\in\mathcal{P}})$ for every player $i \in N$.

In other words, a cooperative game is cohesive if for any outcome, the grand coalition can always produce an outcome that every player weakly prefers to it.

6.2 Cooperative theory of bargaining

Bargaining nicely illustrates the difference between cooperative and noncooperative approaches. In the noncooperative approach, we specified the structure of bargaining, and this could take many forms. The structure of bargaining heavily influenced equilibrium outcomes. In the ultimatum game, the proposer gets away with awarding herself the entire surplus. Where there is multiperiod bargaining and alternating proposers, outcomes depended on whether the bargaining process continued indefinitely or for finitely many rounds, and on the patience of the players. The cooperative approach to bargaining abstracts away from this.

Definition 55 (Bargaining problem)**.** An n-player bargaining problem is a pair $\mathcal{B} = (U, d)$, consisting of

(i) a *feasible set* $U \subseteq \mathbb{R}^n$, also called a *utility possibility set*, and

(ii) a *disagreement point* $d \in U$.

We call any point $u \in U$ an *agreement*.

We assume the set U is nonempty, convex and compact and that d lies in the interior of U. This implies there is some $u \in U$ s.t. $u \gg d$ (i.e. $u_i > d_i$ for all i), and thus the game is cohesive.

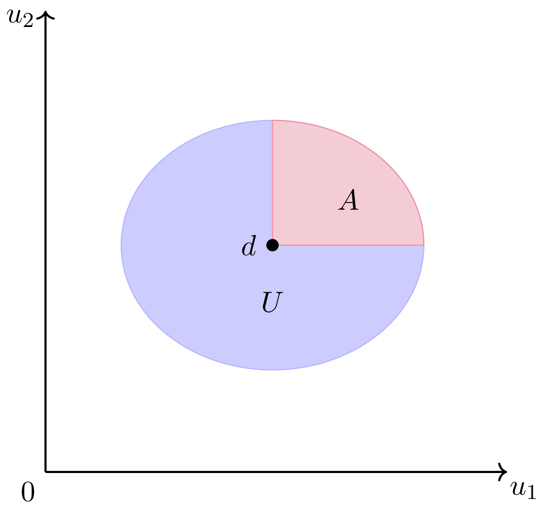
The disagreement point is interpretable as the utility agents would receive if bargaining fails, so it is equivalent to a reservation utility.

Hereafter, we consider bargaining problems with two agents, though everything can be generalized to $n \geq 2$ agents.

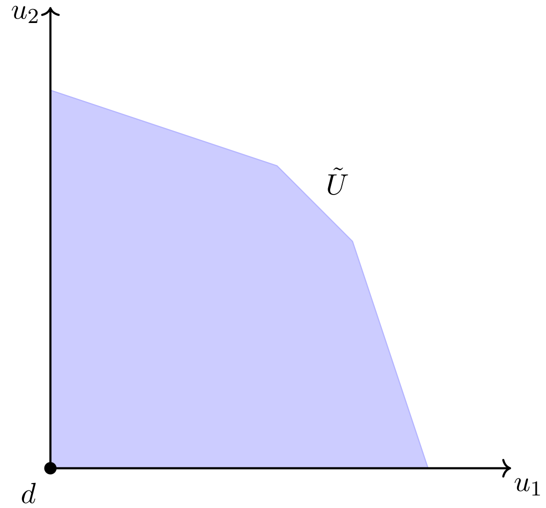
Definition 56 (Set of possible agreements)**.** In a bargaining problem $\mathcal{B} = (U, d)$, the *set of possible agreements* A is the set of individually rational agreements, that is,

$$A = \{u \in U : u \gg d\}.$$

Example 51 (A two-agent bargaining problem)**.** The feasible set U is given in blue, and the set of possible agreements $A \subseteq U$ is given in red.



Intuitively, if agents are rational then no bargaining process can result in an agreement outside the set of possible agreements, since then some agent would receive a higher payoff from the disagreement point. Thus we can restrict attention to the set of possible agreements and renormalize the disagreement point to the origin:



We can always translate the feasible set U so that the disagreement point is at the origin, i.e. by considering $\tilde{U} = \{(u_1 - d_1, u_2 - d_2) : (u_1, u_2) \in U\}$. For the solution concepts we consider, there is no loss of generality to such translations.

Definition 57 (Bargaining solution)**.** A *bargaining solution* f is a function mapping any bargaining problem $\mathcal{B} = (U, d)$ to a feasible outcome $f(\mathcal{B}) \in U$.

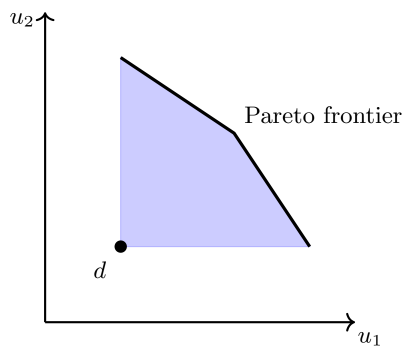
A bargaining solution is chosen to satisfy a set of desirable axioms.

6.2.1 Nash bargaining solution

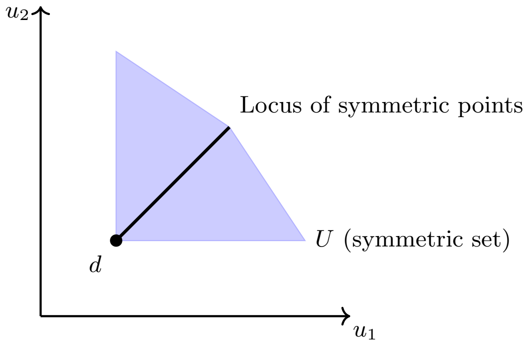
Introduced by Nash (1950), the Nash bargaining solution satisfies the following four axioms:

Axioms.

(**WPO**) *Weak Pareto optimality.* If $u^* = f(U, d)$ then there is no point $u' = (u_1', u_2') \in U$ such that $u_i' > u_i^*$ for each $i = 1, 2$.

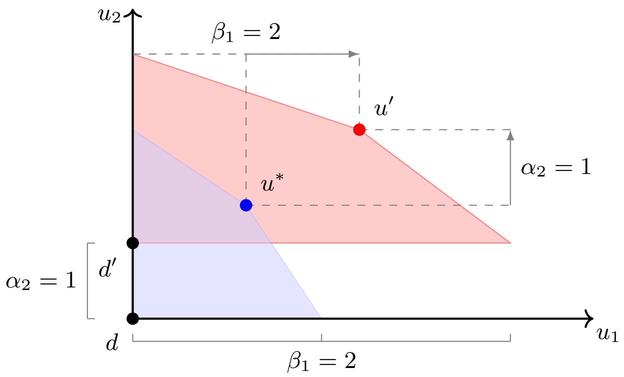


(SYM) *Symmetry.* If (U, d) is a symmetric problem [i.e. $d_1 = d_2$ and if $(u, u') \in U$ then $(u', u) \in U$] then $u^* = f(U, d)$ is such that $u_1^* = u_2^*$.

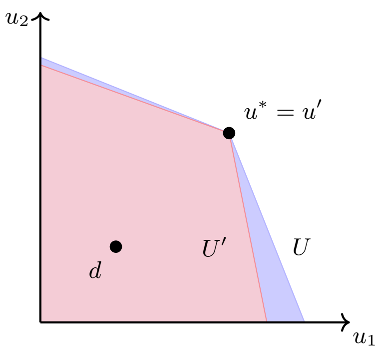


(IER) *Invariance to equivalent payoff representations.* For any $U' = \{\beta' u + \alpha : u \in U\}$ where $\alpha, \beta \in \mathbb{R}^2$ with $\beta \gg 0$ and $d' = \beta' d + \alpha$,

$$u^* = f(U, d) \quad \text{iff} \quad \beta' u^* + \alpha = f(U', d').$$



(IIA) *Independence of irrelevant alternatives.* If $U' \subseteq U$ and $f(U, d) \in U'$ then $f(U', d) = f(U, d)$.



Of these axioms, the independence of irrelevant alternatives (A4) is the most controversial. For example, an asymmetric change in the feasible set intuitively might change the relative bargaining power of the two players and thus the outcome, yet under the independence of irrelevant alternatives axiom it does not (as in the above figure). Kalai & Smorodinsky (1975) make this criticism.

Definition 58 (Nash bargaining solution)**.** Given a bargaining problem $\mathcal{B} = (U, d)$, the *Nash bargaining solution* $f^N(\mathcal{B})$ is the point u^* that solves

$$\max_{u_1 \geq d_1, u_2 \geq d_2} (u_1 - d_1)(u_2 - d_2) \quad \text{s.t.} \quad (u_1, u_2) \in U, u_1 \geq d_1, u_2 \geq d_2.$$

Theorem 23 (Nash, 1950)**.** *Every bargaining problem $\mathcal{B} = (U, d)$, where U is a compact convex set containing d, has a unique Nash bargaining solution $f^N(\mathcal{B})$, and the Nash bargaining solution is the unique bargaining solution satisfying axioms* (**WPO**)*,* (**SYM**)*,* (**IER**) *and* (**IIA**)*.*

Proof. First, we prove f^N is well-defined and unique. Fix any bargaining problem (U, d). First, since $g(u_1, u_2) = (u_1 - d_1)(u_2 - d_2)$ is a continuous function on a compact set $V = \{U \in u : u \geq d\}$, g attains a maximum on V by the extreme value theorem, so a Nash solution exists. Suppose there exist points $u, u' \neq u'$ in U s.t. u and u' are both Nash bargaining solutions. Then

$$(u_1 - d_1)(u_2 - d_2) = (u_1' - d_1)(u_2' - d_2),$$

and we must thus have that if$u_1 - u_1' < 0$ then $u_2 - u_2' > 0$, or if $u_1 - u_1' > 0$ then $u_2 - u_2' < 0$. Since $u \neq u'$, one of these two sets of inequalities must hold. Because U is convex, $u'' = \frac{1}{2}(u + u')$ lies in U. Now, let $x = 2(u_1'' - d_1)(u_2'' - d_2) - (u_1 - d_1)(u_2 - d_2) - (u_1' - d_1)(u_2' - d_2)$. Then

$$\begin{aligned} x &= \frac{1}{2}(u_1 + u_1')(u_2 + u_2') - u_1 u_2 - u_1' u_2' \\ &= \frac{1}{2}[u_1' u_2 + u_1 u_2' - u_1 u_2 - u_1' u_2'] \\ &= \frac{1}{2}(u_1 - u_1')(u_2' - u_2) > 0. \end{aligned}$$

Hence u'' has a strictly greater Nash product than u and u', so u and u' do not maximize the Nash product, yielding a contradiction. Thus the Nash bargaining solution is unique.

Now to prove f^N satisfies the four axioms:

(**WPO**) Let $g(u_1, u_2) = (u_1 - d_1)(u_2 - d_2)$. On the set $\{u \in U : u \gg d\}$, this is strictly increasing in u_1 and u_2. Consider any non-Pareto optimal point $u \gg d$ in U. There is some u' where $u_i' \geq u_i$ with strict inequality for at least one $i = 1, 2$. We have $g(u') > g(u)$, and hence u does not maximize g. Thus f^N is Pareto optimal.

(**SYM**) Suppose $u = f^N(U,d)$ is asymmetric given a symmetric problem (U,d). Let $\bar{d} = d_1 = d_2$. Then either $u_1 > u_2$ or $u_2 > u_1$. Define $u' = (u'_1, u'_2)$ with $u'_1 = u'_2 = \frac{1}{2}(u_1 + u_2)$. Since U is convex, $u' \in U$. We have

$$\begin{aligned}(u_1 - \bar{d})(u_2 - \bar{d}) &= u_1 u_2 - \bar{d}u_1 - \bar{d}u_2 + \bar{d}^2 \\ &< \frac{1}{4}(u_1^2 + 2u_1u_2 + u_2^2) - u_1\bar{d} + u_2\bar{d} + \bar{d}^2 \\ &= (u'_1 - \bar{d})(u'_2 - \bar{d}),\end{aligned}$$

where the inequality follows since

$$u_1^2 + 2u_1u_2 + u_2^2 - 4u_1u_2 = u_1^2 + u_2^2 - 2u_1u_2 = (u_1 - u_2)^2 > 0.$$

(**IER**) Fix a bargaining problem (U,d). For any $\alpha, \beta \in \mathbb{R}^2$ s.t. $\beta \gg 0$ consider the tranformed problem (U', d') with $U' = \{(\beta_1 u_1 + \alpha_1, \beta_2 u_2 + \alpha_2) : (u_1, u_2) \in U\}$ and $d' = (\beta_1 d_1 + \alpha_1, \beta_2 d_2 + \alpha_2)$, The Nash solution for (U', d') solves

$$\begin{aligned}&\max_{u' \in U : u' \gg d'} (u'_1 - d'_1)(u'_2 - d'_2) \\ &= \max_{u \in U : u \gg d} (\beta_1 u_1 + \alpha_1 - \beta_1 d_1 - \alpha_1)(\beta_2 u_2 + \alpha_2 - \beta_2 d_2 - \alpha_2) \\ &= \max_{u \in U : u \gg d} \beta_1\beta_2(u_1 - d_1)(u_2 - d_2) \\ &= \max_{u \in U : u \gg d} (u_1 - d_1)(u_2 - d_2),\end{aligned}$$

where the final line follows because $\beta_1\beta_2 > 0$.

(**IIA**) Suppose u^* is the Nash solution for bargaining problem (U,d). Then

$$(u_1^* - d_1)(u_2^* - d_2) \geq (u_1 - d_1)(u_2 - d_2) \qquad \text{for all } u \in U.$$

This inequality thus holds for all $u \in U' \subseteq U$, so if $u^* \in U'$ it must maximize the Nash product within U'.

Finally, suppose f is a bargaining solution satisfying (**WPO**), (**SYM**), (**IER**) and (**IIA**). Consider any bargaining problem (U,d) and let $u^* = f^N(U,d)$. Define

$$U' = \{\beta' u + \alpha \mid u \in U, \beta' u + \alpha = (1/2, 1/2)' \text{ and } \beta' d + \alpha = (0,0)'\}.$$

That is, U' results from mapping U u^* to $(1/2, 1/2)$ and d to $(0,0)$. Since f and f^N satisfy (**A1.3**), $f^N(U', 0) = (1/2, 1/2)$ and so we need only show $f(U', 0) = (1/2, 1/2)$.

First, note there is no $u \gg 0$ in U' with $u_1 + u_2 > 1$. Suppose otherwise, and define $t(\lambda) = [1 - \lambda](1/2, 1/2) + \lambda(u_1, u_2)$ for $\lambda \in (0,1)$. By convexity of U', $t(\lambda) \in U'$ for all $\lambda \in (0,1)$. We have Nash product

$$t_1(\lambda)t_2(\lambda) = (1-\lambda)^2/4 + \lambda(1-\lambda)[u_1 + u_2]/2 + \lambda^2 u_1 u_2 > (1-\lambda)^2/4 + \lambda(1-\lambda) + \epsilon$$

where $\epsilon \in (0, u_1 u_2]$. This exceeds 1/4, the Nash product of $f^N(U', 0)$ for any $\lambda \in (0,1)$ s.t.

$$(1-\lambda)^2 + 2\lambda(1-\lambda) + \epsilon > 1,$$

i.e. for $\lambda \in (0, \sqrt{\epsilon})$. But this implies $f^N(U', 0)$ does not maximize the Nash product on U, yielding a contradiction. Thus U' is bounded.

Since U' is bounded and convex, we can find a rectangle $U'' \supset U'$ s.t. U'' is symmetric about the line $\{u \mid u_1 = u_2\}$ and $(1/2, 1/2)$ lies on the boundary of U''.[26] By the axioms (**WPO**) and (**SYM**), $f(U'', 0) = (1/2, 1/2)$. Since $U' \subseteq U''$, by axiom (**IIA**), we must have that $f(U', 0) = (1/2, 1/2)$. Thus $f(U', 0) = f^N(U', 0)$. By axiom (**IER**), it follows that $f(U, d) = f^N(U, d)$. $\square$

The Nash bargaining solution and theorem easily extends for all finite n-player bargaining problems, not just the 2 player setting here. It is straightforward to generalize the Nash axioms to n players, and the Nash bargaining solution f^N is then the point u^* solving

$$\max_{u \in U \cap \{\tilde{u} \mid \tilde{u} \gg d\}} \prod_{i=1}^{n} (u_i - d_i).$$

Example 52 (Risk aversion and the Nash bargaining solution)**.** Consider the constant relative risk aversion preferences $u(x) = x^\rho$ where $\rho \in (0, 1]$.

The Nash bargaining solution generally favours less risk averse players. For example, suppose player 1 is risk neutral ($\rho = 1$) and player 2 is strictly risk averse ($\rho < 1$). That is, $u_1(x_1) = x_1$ and $u_2(x_2) = x_2^\rho$. As usual, there is a surplus of 1 to be divided between the two players. Let the disagreement point be $d = (0, 0)$. The Nash bargaining solution f^N solves

$$\max_{x_1} \ u_1(x_1) u_2(1 - x_1) = \max_{x_1} x_1 (1 - x_1)^\rho,$$

or equivalently,

$$\max_{x_1} \ \log x_1 + \rho \log(1 - x_1).$$

We have first order condition,

$$\frac{1}{x_1} - \frac{\rho}{1 - x_1} = 0,$$

which yields solution $x_1^* = \frac{1}{1+\rho}$, so the risk neutral player secures a larger share of the surplus.

We previously discussed non-cooperative models of bargaining at length – models in which a lot of care was taken to explicitly describe a bargaining process. A natural question to ask is whether these models and the Nash bargaining solution coincide. Binmore, Rubinstein & Wolinsky (1985) show that the Nash bargaining solution can be obtained as the limit of the subgame perfect equilibrium of the Rubinstein alternating bargaining model as the rate of time preference δ tends to 1.

[26]The rectangle has an edge passing through $(1/2, 1/2)$ of slope -1.

Proposition 44. *In the Rubinstein alternating bargaining model, suppose $\delta := \delta_1 = \delta_2$, and let $x^*(\delta)$ and $y^*(\delta)$ denote the unique subgame perfect equilibrium offers of players 1 and 2 respectively. Let $(U, 0)$ be the bargaining problem corresponding to this game. Then $\lim_{\delta\to 1} x^*(\delta) = \lim_{\delta\to 1} y^*(\delta) = f^N(U, 0)$.*

Proof. The bargaining problem $(U, 0)$ is symmetric, and therefore the Nash bargaining solution is $(1/2, 1/2)$ by the symmetry and Pareto axioms.

Write $x^*(\delta) = (x(\delta), 1 - x(\delta))$ and $y^*(\delta) = (y(\delta), 1 - y(\delta))$. By Proposition 40 where $x(\delta) = \frac{1-\delta}{1-\delta^2}$ and $y(\delta) = \frac{\delta(1-\delta)}{1-\delta^2}$. Applying l'Hôpital's rule, we have that $\lim_{\delta\to 1} x(\delta) = \frac{1}{2} = \lim_{\delta\to 1} y(\delta)$, and so $\lim_{\delta\to 1} x^*(\delta) = (1/2, 1/2) = \lim_{\delta\to 1} y^*(\delta)$. □

6.2.2 Raiffa-Kalai-Smorodinsky bargaining solution

We could take issue with the independence of irrelevant alternatives axiom, because an increase in one's feasible set could arguably improve one's bargaining position, even if the Nash solution does not change. This leads Kalai & Smorodinsky (1975) to suggest an alternative to the Nash bargaining solution, which was first introduced by Raiffa (1953). The Raiffa-Kalai-Smorodinsky solution drops the independence of irrelevant alternatives axiom in favour of *individual monotonicity.*

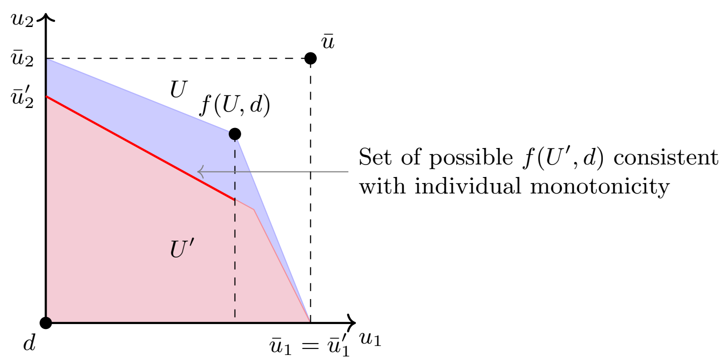
Given a bargaining problem $\mathcal{B} = (U, d)$, define the *utopia point* of $\mathcal{B}$ by

$$\begin{aligned}\bar{u}(U, d) &:= (\max\{u_1 : u \in U, u_1 \geq d_1\}, \max\{u_2 : u \in U, u_2 \geq d_2\})\\ &= (\bar{u}_1, \bar{u}_2).\end{aligned}$$

Now consider the following axiom:

Axiom.

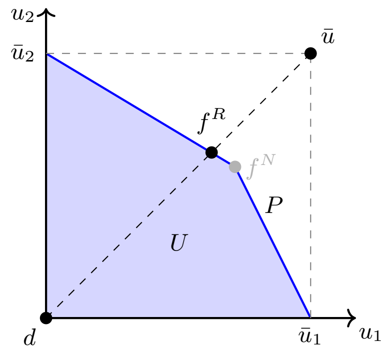
(IM) *Individual monotonicity.* If $U' \subseteq U$, $\bar{u}_i(U', d) = \bar{u}_i(U, d)$ for some $i \in \{1, 2\}$, then $f_j(U', d) \leq f_j(U, d)$ for $j \neq i$.



Definition 59 (Raiffa-Kalai-Smorodinsky solution)**.** The *Raiffa-Kalai-Smorodinsky solution* $f^R(U, d)$ is given by the intersection of the straight line between d and $\bar{u}$ with the

weak Pareto optimal boundary of U. That is, if $P \subseteq U$ is the Pareto frontier of U, then $f^R(U,d)$ is the point $u^* \in P$ s.t.

$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{\bar{u}_1 - d_1}{\bar{u}_2 - d_2}.$$



An example of the Raiffa-Kalai-Smorodinsky bargaining solution f^R. In this bargaining problem, f^R prescribes a different allocation to the Nash bargaining solution f^N.

Theorem 24 (Kalai & Smorodinsky, 1975)**.** *The Raiffa-Kalai-Smorodinsky bargaining solution is the unique bargaining solution satisfying axioms* (**WPO**)*,* (**SYM**)*,* (**IER**) *and* (**IM**)*.*

Proof. □

6.3 Transferable utility games

We say that utility is *transferable* if a player can losslessly transfer part of her utility to any other player. In the context of a cooperative game, this implies that the payoff for a coalition can be summarized by a single number, and this payoff can then be distributed in some way to coalition members. In a characteristic function game, there are no externalities – players only care about the actions of the coalition to which they belong, and not to the actions of other coalitions.

Definition 60 (TU games)**.** A *characteristic function game* or *TU-game* is a pair (N,v) where $N = \{1,\ldots,n\}$ is a set of n players and $v : 2^N \to \mathbb{R}$ is a function assigning to each subset $S \subseteq N$ a real number $v(S)$ such that $v(\varnothing) = 0$.

A subset $S \subseteq N$ is called a *coalition* and the set N is called a *grand coalition.* The function v is called the *characteristic function* and the number $v(S)$ is called the *worth* of S.

An *allocation* is a vector $x \in \mathbb{R}^n$.

We use $\mathcal{G}^N$ to denote the set of all TU-games with set of players N.

Given a coalition S and an allocation $x = (x_1,\ldots,x_n)$, we write $x(S) := \sum_{i\in S} x_i$ for the total payoff of coalition S under allocation x. To avoid needlessly complicating notation, we write $v(i)$ for $v(\{i\})$, the worth of a player i on her own, and we write $S \cup i$ for $S \cup \{i\}$.

The set of actions available to a coalition S in a game (N, v) is $A_S = \{x \in \mathbb{R}^n \mid x(N) \leq v(S)\}$. That is, coalitions' actions are allocations. The coalition generates worth $v(S)$, and thus S can only implement an allocation x if $x(N) = \sum_{i \in N} x_i \leq v(S)$.

In much of the literature, the actions of a coalition S are instead taken to be vectors $(x_i)_{i \in S}$ such that $\sum_{i \in S} x_i = v(S)$. Conceptually, the difference in approach is that we allow S to implement allocations that give something to players who are not in S. This is without consequence in the games we consider, because players only care about their own payoffs. When we consider coalitional deviations from the grand coalition, the players in coalition S will always most prefer some allocation that splits $v(S)$ only among themselves.

Definition 61 (Imputation)**.** Given a TU-game (N, v), a vector $x \in \mathbb{R}^n$ is called an *imputation* if

(i) x is *individually rational*, that is, if $x_i \geq v(i)$ for all $i \in N$, and

(ii) x is *efficient*, that is, $x(N) = v(N)$.

We denote the set of imputations of (N, v) by $I(v)$.

An imputation $x \in I(v)$ is an allocation of the worth of the grand coalition N such that each player is given at least as great a payoff as she could earn on her own.

Because $v : 2^N \to \mathbb{R}$ already encodes N (for v is defined on the power set of N), we often call v a game instead of (N, v). We detail some properties of v:

Definition 62 (Properties of the characteristic function)**.** Consider a characteristic function $v : 2^N \to \mathbb{R}$.

(a) *Convexity.* We call a game v *convex* (or *supermodular*) if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \qquad \text{for all } S, T \subseteq N.$$

(b) *Superadditivity.* We call a game v *superadditive* if

$$v(S \cup T) \geq v(S) + v(T) \qquad \text{for all } S, T \subseteq N \text{ s.t. } S \cap T = \varnothing,$$

that is, for all disjoint coalitions S, T.

If $v(S \cup T) = v(S) + v(T)$ for all disjoint coalitions S, T, then v is called *additive*.

(c) *Monotonicity.* We call a game v *monotone* if $S \subseteq T$ implies $v(S) \leq v(T)$.

(d) *Essential game.* We call a game v is *essential* if $v(N) \geq \sum_{i=1}^{n} v(i)$.

An essential game is one in which the worth of the grand coalition is weakly greater than the sum of the payoffs that individual players can earn on their own. If a game is not essential, then clearly there is no imputation of the game, since any efficient allocation x must involve $x_i < v(i)$ for some i.

Proposition 45. *Let (N, v) be a TU-game.*

(i) *If v is convex then it is superadditive.*

(ii) *If v is superadditive, it is essential.*

(iii) *If v is superadditive or monotone, it is cohesive.*

(iv) *If v is nonnegative and superadditive, then it is monotone.*

(v) *v is convex iff for every player $i \in N$,*

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$$

for all coalitions $S \subseteq T \subseteq N - \{i\}$.

Proof. (i) If v is convex and S, T are disjoint coalitions, then $S \cap T = \varnothing$, and thus by convexity, $v(S \cup T) = v(S \cup T) + v(\varnothing) \geq v(S) + v(T)$.

(ii) Let $S_1 = N - \{1\}$ and let $S_k = S_{k-1} - \{k\}$ for $k = 2, \ldots, n-1$. By superadditivity, we have $v(N) \geq v(S_1) + v(1)$ and for each $k = 1, \ldots, n-1$, we have $v(S_k) \geq v(S_{k+1}) - v(k)$. Combining gives $v(N) \geq \sum_{i=1}^{n} v(i)$.

(iii) Monotonicity immediately implies cohesiveness. Suppose v is superadditive. Applying the definition repeatedly, we see that for any partition $\mathcal{P}$ of N, we have $v(N) \geq \sum_{S \in \mathcal{P}} v(S)$. Hence for any allocation y such that $y(S) = v(S)$ for all $S \in \mathcal{P}$, there is some imputation x such that $x_i \geq y_i$ for all players i. Thus v is cohesive.

(iv) Suppose v is nonnegative and superadditive. Suppose $S \subseteq T$. Then $v(T) = v(S \cup (T - S)) \geq v(S) + v(T - S)$, and $v(T - S) \geq 0$, so $v(T) \geq v(S)$.

(v) Suppose v is convex. Then $v((S \cup i) \cup T) + v((S \cup i) \cap T) \geq v(S \cup i) + v(T)$, and since $S \subseteq T$ and $i \notin S \cup T$, $v((S \cup i) \cap T) = v(S)$ and $v((S \cup i) \cup T) = v(T \cup i)$. Hence $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$. Conversely, suppose $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$ for all $S \subseteq T \subseteq N - \{i\}$, for all players i. Fix $S_0 \subseteq T_0 \subseteq N$ and any $R = \{i_1, \ldots, i_k\} \subseteq N - T_0$. Repeatedly applying the inequality gives

$$v(S_0 \cup \{i_1, \ldots, i_j\}) - v(S_0 \cup \{i_1, \ldots, i_{j-1}\}) \leq v(T_0 \cup \{i_1, \ldots, i_j\}) - v(T_0 \cup \{i_1, \ldots, i_{j-1}\})$$

for $j = 1, \ldots, k$. Combining the inequalities gives $v(S_0 \cup R) - v(S_0) \leq v(T_0 \cup R) - v(T_0)$ for any $R \subseteq N - T_0$. Taking arbitrary S, T and setting $S_0 = S \cap T$, $T_0 = T$ and $R = S - T$ yields the convex inequality, and thus v is convex.

□

In essential games, the set of imputations is a convex set that we can easily derive:

Proposition 46. *Consider any essential game (N, v). For each $i \in N$, define $f^i = (f^i_1, \ldots, f^i_n)$ by*

$$f^i_k = \begin{cases} v(k) & \text{if } k \neq i, \\ v(N) - \sum_{j \in N - \{i\}} v(j) & \text{if } k{=}i. \end{cases}$$

Then the set of imputations $I(v)$ is given by

$$I(v) = \mathrm{co}(f^1, \ldots, f^n),$$

that is, the convex hull of the points $f^1, \ldots, f^n$.

Proof. Consider any f^i. For $k \neq i$, k's payoff is $f^i_k = v(k)$ and i's payoff is $v(N) - \sum_{j \neq i} v(j) \geq v(i)$, where the inequality holds since the game is essential. Hence f^i is individually rational. Furthermore, $f^i(N) = \sum_{k \neq i} v(k) + v(N) - \sum_{k \neq i} v(k) = v(N)$, so f^i is efficient. It follows that f^i is an imputation.

Next, consider any convex combination $x = \lambda_1 f^1 + \lambda_2 f^2 + \cdots + \lambda_n f^n$ (with $\sum_{i \in N} \lambda_i = 1$, $\lambda_i \geq 0$.) For each $k \in N$, since $f^i_k \geq v(k)$ for all $i \in N$, we have that $x_k \geq v(k)$. Thus x is individually rational. Furthermore, $x(N) = \sum_{i=1}^n \lambda_i f^i(N) = \sum_{i=1}^n \lambda_i v(N) = v(N) \sum_{i=1}^n \lambda_i = v(N)$, so x is efficient. Thus x is an imputation. Since $x \in \mathrm{co}(f^1, \ldots, f^n)$ iff x is a convex combination of $f^1, \ldots, f^n$, it follows that $\mathrm{co}(f^1, \ldots, f^n) \subseteq I(v)$.

If $v(N) = \sum_k v(k)$, then trivially, the only imputation is x defined by $x_k = v(k)$ for all k, and we have $f^1 = \cdots = f^n = x$. If $v(N) > \sum_k v(k)$, then $\{f^1, \ldots, f^n\}$ is linearly independent and spans $\mathbb{R}^n$. Consider any $x \in I(v)$. We must have $x_k \geq v(k)$ and $\sum_i x_i = v(N)$. Let F be an $n \times n$ matrix with ith column f^i, and let x be a column vector and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a column vector. Since the columns of F form a basis for $\mathbb{R}^n$, the system of equations $F\lambda = x$ has a unique solution $\bar{\lambda}$, and so $x = \sum_i \bar{\lambda}_i f^i$. If $\sum_i \bar{\lambda}_i \neq 1$, then $x(N) = \sum_{i=1}^n \bar{\lambda}_i f^i(N) = v(N) \sum_{i=1}^n \bar{\lambda}_i \neq v(N)$, which contradicts $x \in I(v)$. Likewise, if $\bar{\lambda}_i < 0$ for some i, then for player i,

$$\begin{aligned} x_i &= -|\lambda_i| \left[v(N) - \sum_{j \neq i} v(j) \right] + \sum_{j \neq i} \bar{\lambda}_j v(i) \\ &= -|\lambda_i| \left[v(N) - \sum_{j \neq i} v(j) \right] + (1 + |\lambda_i|) v(i) \\ &< v(i), \end{aligned}$$

where the inequality follows from $v(N) - \sum_{j \neq i} v(j) > v(i)$. Thus x is not individually rational, yielding a contradiction. It follows that $I(v) \subseteq \mathrm{co}(f^1, \ldots, f^n)$. This completes the proof. □

6.4 The core and stable sets

The core and stable set are two set-valued solution concepts. In cohesive games, the grand coalition always forms, and thus a natural question to ask is which actions the grand coalition chooses. These solution concepts lack much interpretation in non-cohesive games.

6.4.1 The core

Definition 63 (Domination)**.** Let (N, v) be a game, let $y, z \in I(v)$ and let $S \subseteq N$ be nonempty. Then y *dominates* z in coalition S if

(i) $y_i > z_i$ for all $i \in S$, and

(ii) $y(S) \leq v(S)$.

For $y, z \in I(v)$, we say that y *dominates* z if there exists a (nonempty) coalition $S \subseteq N$ s.t. y dominates z in S.

For each coalition S, we define the *set of imputations dominated in* S by

$$D(S) := \{z \in I(v) \mid \text{there exists } y \in I(v) \text{ s.t. } y \text{ dominates } z \text{ in } S\}.$$

We say that an imputation $x \in I(v)$ is *undominated* if $x \in I(v) - \bigcup_{S \subseteq N : S \neq \varnothing} D(S)$, that is, if x is not dominated in any coalition S.

Clearly $D(\{i\}) = \varnothing$ for any singleton $\{i\}$, since for any $y, z \in I(v)$, $y_i \geq v(i)$ and $z_i \geq v(i)$ by individual rationality. For y to dominate z in $\{i\}$, we have $y_i \leq v(i)$ so $y_i = v(i)$ and $y_i > z_i \geq v(i)$, which would be a contradiction.

Likewise, $D(N) = \varnothing$. Any $y, z \in I(v)$ has $y(N) = z(N) = v(N)$. Hence if $y_i > z_i$ for some $i \in N$ then there must be some $j \in N$ s.t. $y_i < z_i$, and so y cannot dominate z in N.

Domination gives rise to a set-valued solution concept, the *domination core*, which is simply the set of all undominated imputations. This relates closely to the *core*, the set of imputations that cannot be improved on by any coalition.

Definition 64 (Core)**.**

(a) *Domination core.* In a game (N, v), the *domination core* or D-core is the set

$$DC(v) := I(v) - \bigcup_{S \subseteq N : S \neq \varnothing} D(S).$$

(b) *Core.* In a game (N, v), the *core* is the set

$$C(v) := \{x \in I(v) \mid x(S) \geq v(S) \text{ for all nonempty } S \subseteq N\}.$$

This definition of the core is specialized to TU-games. More generally, we say that a coalition S *blocks* an action a_N of the grand coalition if there is some action a_S of S that all members of S prefer to a_N. The core is the set of actions of the grand coalition that are not blocked by any coalition. We say an action in the core is *stable*, and *unstable* otherwise. In these terms, in a TU-game (N, v), a coalition S blocks the allocation $x \in I(v)$ iff $x(S) < v(S)$.

In general, the core may be empty. If the core is empty, we might want to find an approximation:

Definition 65 (ϵ-core)**.** In a game (N, v), the *ϵ-core* is the set $\{x \in I(v) \mid x(S) \geq v(S) - \epsilon$ for all nonempty $S \subseteq N\}$.

The ϵ-core has the interpretation that it is the set of stable allocations if any coalition has to pay a cost ϵ in order to block an allocation.

Proposition 47. *For any TU-game (N, v), $C(v) \subseteq DC(v)$.*

Proof. Consider any $x \in I(v)$ s.t. $x \notin DC(v)$. Then there is a $y \in I(v)$ and a coalition $S \neq \varnothing$ s.t. y dominates x in S. Thus $v(S) \geq y(S) > x(S)$, implying $x \notin C(v)$. □

Example 53 (Game with empty core)**.** Consider the game (N, v) where $N = \{1, 2, 3\}$ and $v(S) = 1$ if $|S| \geq 2$ and $v(S) = 0$ otherwise.

For any imputation $x \in I(v)$, we have $x \geq 0$ and $x_1 + x_2 + x_3 = 1$. Suppose $x_i < \frac{1}{3}$. Since $\sum_j x_j = 1$, we cannot have that $x_j \geq \frac{2}{3}$ for both $j \neq i$. Fix $j \neq i$ with $x_j < \frac{2}{3}$ and let $S = \{i, j\}$. Consider an imputation y with $y_i = \frac{1}{3}$ and $y_j = \frac{2}{3}$. Then y dominates x in S. Thus, for x to be in the core, we need $x_i \geq \frac{1}{3}$ for all i, and thus the only imputation that can lie in the core is $x = (1/3, 1/3, 1/3)$. But this is dominated in $\{1, 2\}$ by $x = (1/2, 1/2, 0)$. Hence the core is empty.

Under a fairly mild condition (implied by superadditivity), the core and domination-core coincide:

Theorem 25. *Consider any TU-game (N, v). If*

$$v(N) \geq v(S) + \sum_{i \notin S} v(i) \quad \textit{for all } S \in 2^N - \{\varnothing\},$$

then $C(v) = DC(v)$.

Proof. Since $C(v) \subseteq DC(v)$ by Proposition 47, we need only show $DC(v) \subseteq C(v)$ if the condition holds.

Lemma 13. *Suppose the condition of Theorem 25 holds. Then if $x \in I(v)$ and $x(S) < v(S)$ for some nonempty coalition S, then there is a $y \in I(v)$ s.t. y dominates x in S.*

Proof. If $i \in S$, define $y_i := x_i + |S|^{-1}(v(S) - x(S))$. If $i \notin S$, define $y_i := v(i) + \Big(v(N) - v(S) - \sum_{j \notin S} v(j)\Big)|N - S|^{-1}$. Then $y \in I(v)$, where $y_i \geq v(i)$ follows from the condition of the theorem. Clearly, y dominates x in S since $y_i \geq x_i$ for all $i \in S$. □

Now, suppose $x \in DC(v)$. Then there is no $y \in I(v)$ s.t. y dominates x. From the lemma, it follows that $x(S) \geq v(S)$ for all $S \in 2^N - \{\varnothing\}$. Hence $x \in C(v)$. We thus prove that $DC(v) \subseteq C(v)$. □

Note that the condition of this theorem is implied by superadditivity.

Proposition 48. *If (N, v) is a game with a nonempty core, then $v(N) \geq v(S) + \sum_{i \notin S} v(i)$ for all $S \in 2^N - \{\varnothing\}$, and thus $C(v) = DC(v)$.*

Proof. If $C(v) \neq \varnothing$, then there exists some imputation x s.t. $x(S) \geq v(S)$ for all nonempty coalitions S. Fix some (nonempty) coalition S and define $T := N - S$. Since x is an imputation, $x(N) = v(N)$ and $x_i \geq v(i)$ for all $i \in N$, so $x(T) \geq \sum_{i \in T} v(i)$. Since $x(S) \geq v(S)$, it follows that

$$v(N) = x(N) = x(S) + x(T) \geq v(S) + \sum_{i \in T} v(i).$$

□

Definition 66 (Simple game)**.**

(a) *Simple game.* A game (N, v) is a *simple game* if $v(N) = 1$ and $v(S) \in \{0, 1\}$ for all $S \in 2^N - \{\varnothing\}$.

In a simple game, we call a coalition S a *winning coalition* if $v(S) = 1$ and a *losing coalition* if $v(S) = 0$. We say a winning coalition S is *minimal* if every nonempty proper subset $S' \subseteq S$ is losing.

(b) *Dictator.* In a simple game, a player i is called a *dictator* if $v(S) = 1$ iff $i \in S$.

(c) *Veto player.* In a simple game, a player i is called a *veto player* if i is a member of all winning coalitions. The *set of veto players* is given by

$$\text{veto}(v) = \bigcap \left\{ S \in 2^N \mid v(S) = 1 \right\}.$$

Simple games arise in many real-world contexts:

- *United Nations Security Council.* The Security Council has 15 members, 5 of which are permanent members who are veto players. Excluding abstentions for simplicity, the set of winning coalitions is the set of coalitions involving all 5 permanent members and at least 8 members in total.
- *Treaty or contract negotiations.* The agreement of a treaty between governments or a contract between two or more parties, requires unanimity among parties. In these settings, typically all parties to the treaty or contract are veto players.
- *Legislatures.* In most legislatures, bills typically pass if a majority of members vote in favour. Some player(s) (e.g. the Speaker, certain government members) may have veto power via agenda-setting powers, and in some legislatures, rules – such as those governing the filibuster in the US Senate (as of 2022) – may give conditional veto powers to all members.

Example 54.

(a) *Dictator game.* Fix $i \in N$. The *dictator game* δ_i is the simple game with characteristic function

$$\delta_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The set of imputations is $I(\delta_i) = \{e^i\}$, where e^i is the matrix with ith entry 1 and all other entries 0. Furthermore, $\text{veto}(\delta_i) = \{i\}$ and $C(\delta_i) = DC(\delta_i) = \{e^i\}$.

(b) *Majority game.* A n-player majority game has characteristic function

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq \lceil n/2 \rceil, \\ 0 & \text{otherwise.} \end{cases}$$

With three players, $v(S) = 1$ iff $|S| \in \{2, 3\}$. We have

$$\text{veto}(v) = \{1, 2\} \cap \{2, 3\} \cap \{1, 3\} \cap \{1, 2, 3\} = \varnothing,$$

and

$$C(v) = DC(v) = \varnothing.$$

(c) *T-unanimity game.* Let T be a nonempty coalition. The T-unanimity game is the simple game with characteristic function

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\text{veto}(u_T) = T$ and

$$C(u_T) = DC(u_T) = \text{co}\{e^i \mid i \in T\}.$$

A simple game has a nonempty core iff it has veto players:

Theorem 26. *Let (N, v) be a simple game. Then*

(i) $C(v) = \text{co}\{e^i \in \mathbb{R}^n \mid i \in \text{veto}(v)\}$*;*

(ii) *If* $\text{veto}(v) = \varnothing$ *and* $\{i \in N \mid v(i) = 1\} = \{k\}$ *then* $C(v) = \varnothing$ *and* $DC(v) = \{k\}$*. Otherwise,* $DC(v) = C(v)$*.*

Proof. Suppose $i \in \text{veto}(v)$. Let $S \in 2^N - \{\varnothing\}$. If $i \in S$, then $e^i(S) = 1 \geq v(S)$; else $e^i(S) = 0 = v(S)$. Clearly, $e^i(N) = 1 = v(N)$. Thus $e^i \in C(v)$. Since $C(v)$ is convex, it follows that $C(v) \supset \text{co}\{e^i \in \mathbb{R}^n \mid i \in \text{veto}(v)\}$.

Now, let $x \in C(v)$. To show $C(v) \subseteq \text{co}\{e^i \in \mathbb{R}^n \mid i \in \text{veto}(v)\}$, we need only prove that if $i \notin \text{veto}(v)$, then $x_i = 0$. Suppose otherwise, i.e. $x_i > 0$ for some $i \notin \text{veto}(v)$. Take S with $v(S) = 1$ and $i \notin S$ (were such an S not to exist, then i would be a veto player). Then $x(S) = x(N) - x(N - S) \leq 1 - x_i < 1$, contradicting $x \in C(v)$. This proves (i).

For (ii), if $\text{veto}(v) = \varnothing$ and $\{i \in N \mid v(i) = 1\} = \{k\}$ then $C(v) = \varnothing$ follows immediately from (i). Suppose instead that $\text{veto}(v)$ is nonempty. Then (i) implies that $C(v)$ is nonempty by (i). By Proposition 48, it follows that $C(v) = DC(v)$. ☐

Lastly, convex games are guaranteed to have a nonempty core:

Theorem 27. *If (N, v) is a convex game, then it has a nonempty core.*

Proof. We claim the allocation x defined by $x_i = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ lies in the core. For every $i_1 < i_2 < \cdots < i_j$, since v is convex we have that

$$\begin{aligned}\sum_{k=1}^{j} x_{i_k} &= \sum_{k=1}^{j} \left[v(\{i_1, \dots, i_k\}) - v(\{i_1, \dots, i_{k-1}\})\right] \\ &\geq \sum_{k=1}^{j} \left[v(\{i_1, \dots, i_{k-1}\} \cup i_k) - v(\{i_1, \dots, v_{k-1}\})\right] \\ &= v(\{i_1, \dots, i_k\}).\end{aligned}$$

□

6.4.2 Stable sets

Another solution concept based on domination is the *stable set*, due to von-Neumann and Morgenstern (1944).

Definition 67 (Stable set)**.** Let v be a game and let $A \subseteq I(v)$. The set A is called a *stable set* if

(i) If $x, y \in A$ then x does not dominate y (*internal stability*);

(ii) If $x \in I(v) - A$ then there exists a $y \in A$ that dominates x (*external stability*).

Example 55. Suppose v is a three person game, and the value of any multiperson coalition has value 1, but all single person coalition has value 0. Consider the set $A = \{x, y, z\}$ made up of three imputations: $x = (\frac{1}{2}, \frac{1}{2}, 0)$, $y = (\frac{1}{2}, 0, \frac{1}{2})$ and $z = (0, \frac{1}{2}, \frac{1}{2})$. We show this set is stable.

(i) *Internal stability.* For each vector $a, b \in A$, $a_i > b_i$ for one i only. Hence a could only dominate b in a coalition consisting of one person. But $v(i) = 0$ for all i, and so $a(i) \not\leq v(i)$. Hence a does not dominate b in any coalition, so a does not dominate b.

(ii) *External stability.* Consider any $h \in I(v)$ s.t. $h \notin A$. Consider any coalition $S = \{i, j\}$ of two people and let $\{k\}$ be the third person, excluded from S. Then $v(S) = 1$. Now there is $a \in A$ with $a_i = a_j = \frac{1}{2}$ and $a_k = 0$, so $a(S) = 1 = v(S)$. If $h_i < \frac{1}{2}$ and $h_j < \frac{1}{2}$ then we have that $a_i > h_i$ for all $i \in S$ and so a dominates h in S and thus a dominates h. Hence we need only consider the case where $h_i > \frac{1}{2}$ or $h_j > \frac{1}{2}$. Suppose wlog that $h_i > \frac{1}{2}$. Then $h_j < \frac{1}{2}$ and $h_k < \frac{1}{2}$, since $h(N) = 1$ and therefore $h(S) = h_i + h_j \leq 1$. Now consider the coalition $T = \{j, k\}$. There is a $b \in A$ s.t. $b_j = b_k = \frac{1}{2}$ and $b_i = 0$, so $b(T) = 1 = v(T)$. Since $h_j < \frac{1}{2}$, $b_j > h_j$ and since $h_k < \frac{1}{2}$, $b_k > h_k$. Thus b dominates h in T, so b dominates h.

We have thus shown that for any imputation $h \notin A$, one of the elements of A dominates h.

Hence A is stable. However, A is not a unique stable set. Indeed, if $c \in [0, \frac{1}{2})$ and $B = \{x \in I(v) \mid x_3 = c\}$ then B is stable (proof is a slight modification of the above).

In general, stable sets are non-unique. However, even if unique, a selection has to be made from the stable set – stability is a property of sets and not particular allocations. The core on the other hand is by definition a single set. Unlike stable sets, selection from the core can be plausibly motivated (e.g. the nucleolus).

The question of existence of stable sets is partially solved. First, note any essential simple game has a stable set, namely the set of imputations that assign nothing to players outside a minimal winning coalition:

Proposition 49. *Let v be a simple game and suppose S is a minimal winning coalition. Define $\Delta^S := \{x \in I(v) \mid x_i = 0$ for all $i \notin S\}$. Then Δ^S is a stable set if it is nonempty.*

Proof. Since S is a minimal winning coalition, $v(S) = 1$ and $v(S') = 0$ for any proper subset $S' \subseteq S$. Suppose some $x \in \Delta^S$ dominates some $y \in \Delta^S$. Since $x_i = y_i = 0$ for all $i \notin S$, x can only dominate y in some coalition $T \subseteq S$.If T is a proper subset of S, then $v(T) = 0$. If $x(T) > 0$ then x cannot dominate y in T, and if $x(T) = 0$ then we must have $x_i = y_i = 0$ for all $i \in T$, and again x cannot dominate y. Finally, if $T = S$, then $x(T) = y(T) = v(T) = 1$. Since $\sum_{i \in S} x_i = \sum_{i \in S} y_i$, we must have that if $x_i > y_i$ for some $i \in S$ then $x_j < y_j$ for at least one $j \in S$. Hence x does not dominate y.

We have thus established internal stability. Now we turn to external stability. Consider any $y \notin \Delta^S$. By definition, $y(S) < 1$, and so $\epsilon := 1 - y(S) > 0$. Define, x by $x_i = 0$ for all $i \notin S$ and $x_i = y_i + \frac{1}{|S|}\epsilon$ for all $i \in S$. Then $x \in \Delta^S$, $x(S) = 1 = v(S)$ and $x_i > y_i$ for all $i \in S$, so x dominates y in S. Hence we have established external stability. □

Example 56 (Stable sets in zero-one games)**.** A *zero-one game* (N, v) is a game in which $v(i) = 0$ for all $i \in N$ and $v(N) = 1$. This is not necessarily a simple game, because in games of three or more players, coalitions that are not the grand coalition but consist of more than one person can have worth not in $\{0, 1\}$.

Consider a three person zero-one game (N, v) where $N = \{1, 2, 3\}$, $v(i) = 0$ for all $i \in N$, $v(N) = 1$, and for every two-person coalition S, $v(S) = \alpha$ for some $\alpha \in [0, 1]$.

(i) If $\alpha \geq \frac{2}{3}$, then the set

$$A = \left\{(\lambda, \lambda, 1 - 2\lambda), (\lambda, 1 - 2\lambda, \lambda), (1 - 2\lambda, \lambda, \lambda) \mid \frac{\alpha}{2} \leq \lambda \leq \frac{1}{2}\right\}$$

is a stable set.

Each $x \in A$ has $x(N) = 1$. Consider any $x, y \in A$ and let λ_x and λ_y be the values of λ associated with each respectively. Since $x(i) = \lambda_x > 0 = v(i)$, x cannot dominate y in any singleton coalition. In the grand coalition, we also have that since $y(N) = x(N) = 1$ and $y_i < x_i$ for at least one $i \in N$, we must have $y_j > x_j$ for some $j \in N$, so x does not dominate y in N. Consider any two-person coalition $S = \{i, j\}$. We have several cases:

(1) $x_i = \lambda_x$ and $y_i = \lambda_y$ (note the case where $x_j = \lambda_x$ and $y_j = \lambda_y$ is also covered because we can just switch i, j). Suppose x dominates y in S. We then require $x_i > y_i$ and thus $\lambda_x > \lambda_y \geq \frac{\alpha}{2}$. If $x_j = \lambda_x$ then we would have $x(S) > \alpha = v(S)$ and so x would not dominate y in S. Hence suppose $x_j = 1 - 2\lambda_x$. If $y_j = 1 - 2\lambda_y$ then because $\lambda_x > \lambda_y$, $y_j > x_j$ and so x does not dominate y in S. Hence $y_j = \lambda_y$. Since $\lambda_x > \frac{\alpha}{2}$, $1 - 2\lambda_x < 1 - \alpha \leq \frac{1}{3}$, and $\lambda_y \geq \frac{\alpha}{2} \geq \frac{1}{3}$. Hence $y_j > x_j$, yielding a contradiction.

(2) $x_i = \lambda_x$, $x_j = 1 - 2\lambda_x$, $y_i = 1 - 2\lambda_y$, $y_j = \lambda_y$ (again, we can swap i, j labels to cover the converse case). Suppose x dominates y in S. Then $x_i = \lambda_x > 1 - 2\lambda_y = y_i$ and $x_j = 1 - 2\lambda_x > \lambda_y = y_j$. Rearranging these gives us $\lambda_x + 2\lambda_y > 1 > 2\lambda_x + \lambda_y$. Subtracting $\lambda_x + \lambda_y$ from both sides gives us $\lambda_y > \lambda_x \geq \frac{\alpha}{2}$ and so $2\lambda_x + \lambda_y > \frac{3}{2}\alpha \geq 1$, yielding a contradiction.

Thus A is internally stable. Next, we consider external stability. Consider any imputation $y \notin A$. Let $i = \arg\min_{k \in N} y_k$ and let $j = \arg\min_{k \in N - \{i\}} y_k$. Consider the two-person coalition $S = \{i, j\}$ and let $\{\ell\} = N - S$. Now, $y_i + y_j \leq \frac{2}{3}$, since if $y_i, y_j > \frac{2}{3}$ then $y_\ell < \frac{1}{3}$ and since $y_i < y_j < y_\ell$, it follows that $y_i, y_j < \frac{1}{3}$, yielding a contradiction. Hence $y(S) \leq \alpha$. Consider the $x \in A$ with $x_i = x_j = \frac{\alpha}{2}$. Then we have $x_i > y_i$, $x_j > y_j$ and so x dominates y. This completes the proof.

(ii) If $\alpha < \frac{2}{3}$ then A is not stable. We show that external stability does not hold here (A remains internally stable). Consider $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note $y \in A$ iff $\alpha = \frac{2}{3}$, and by hypothesis $\alpha < \frac{2}{3}$. For any singleton $\{i\}$, we have $y_i = \frac{1}{3} > 0 = v(i)$, so there is no $x \in A$ dominating y in $\{i\}$. For the grand coalition we have $y(N) = 1 = v(N)$ but for any $x \in A$, $x_i < \frac{1}{3}$ for some i since $y \notin A$ and $\sum_{i=1}^{3} x_i = 1$. Finally, in any two person coalition S, $y(S) = \frac{2}{3} > \alpha = v(S)$, so there is no $x \in A$ dominating y in S. It follows that y is undominated. Note if $\alpha > \frac{2}{3}$ then y would be dominated by $(\alpha/2, \alpha/2, 1 - \alpha)$.

(iii) If $\alpha \leq \frac{1}{2}$ then the core is the unique stable set.

Proposition 50. *Let (N, v) be a TU-game. Then*

(i) *The D-core of v is a subset of any stable set;*

(ii) *If A and B are stable and $A \neq B$ then $A \not\subseteq B$;*

(iii) *If the D-core of v is a stable set, then it is the unique stable set of v.*

Proof. (i) Recall the D-core is the set of all undominated imputations. Suppose there is some undominated imputation $x \in I(v)$ and some stable set A of v s.t. $x \notin A$. Since x is not dominated in any coalition, there is no $y \in A$ s.t. y dominates x, contradicting the external stability of A.

(ii) Suppose $A \subseteq B$. Then there is some $x \in B$ s.t. $x \notin A$. For A to satisfy external stability, we must have that some $y \in A$ dominates x. Yet since $x, y \in B$, internal stability of B requires that y does not dominate x, yielding a contradiction.

(iii) Suppose $DC(v)$ and A are stable and $A \neq DC(v)$. Since $DC(v)$ consists of all undominated imputations, $A \supset DC(v)$, for were there some $x \in DC(v)$ s.t. $x \notin A$, then there would be no $y \in A$ s.t. y dominates x, which would violate external stability. But by (ii), if $A \supset DC(v)$ and A is stable then either $DC(v) = A$ or $DC(v)$ is not stable, both of which contradict the hypotheses.

□

6.4.3 Balanced games

Definition 68 (Balanced games)**.** Let $N = \{1, \ldots, n\}$.

(a) *Characteristic vector.* Given a coalition $S \subseteq N$, the vector e^S defined by

$$e_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \in N - S, \end{cases}$$

is called the *characteristic vector* for S.

(b) *Balanced map.* Call a map $\lambda : 2^N - \{\varnothing\} \to \mathbb{R}_+$ a *balanced map* if

$$\sum_{S \in 2^N - \{\varnothing\}} \lambda(S) e^S = e^N.$$

(c) *Balanced coalition.* A collection B of nonempty coalitions is called a *balanced coalition* if there exists a balanced map λ s.t.

$$B = \left\{ S \in 2^N - \{\varnothing\} \mid \lambda(S) > 0 \right\}.$$

(d) *Balanced game.* A game (N, v) is called a *balanced game* if for each balanced map λ, we have

$$\sum_S \lambda(S) v(S) \leq v(N).$$

(e) *Totally balanced game.* A game (N, v) is called *totally balanced* if for every coalition $S \subseteq N$, the restriction $v_{|S}$ of v to S is a balanced game.

To give balancedness an intuitive interpretation, suppose each player is endowed with one unit of time to distribute over the various coalitions of which she is a member. The distribution is balanced if it corresponds to a balanced map λ, where we interpret $\lambda(S)$ as the length of time for which the coalition S exists. Balancedness of the map λ requires that each player spends their entire time endowment of 1 over the coalitions. Under this interpretation, a balanced game is a game in which operating the grand coalition the entire time is at least as productive as any balanced distribution of the time endowment involving smaller coalitions, where we interpret the worth of a coalition as productivity. Intuitively, in any balanced game, it is natural to form the grand coalition. In fact, the Bondareva-Shapley theorem demonstrates that the core of a game is nonempty iff the game is balanced:

Theorem 28 (Bondareva-Shapley)**.** *Let (N, v) be a TU-game. Then the following are equivalent:*

(i) *The core $C(v)$ is nonempty;*

(ii) *(N, v) is a balanced game.*

Proof. $C(v)$ is nonempty iff

$$v(N) = \min \left\{ \sum_{i=1}^{n} x_i \mid x \in \mathbb{R}^N, \;\; x(S) \geq v(S) \text{ for all } S \in 2^N - \{\varnothing\} \right\}.$$

By Theorem 55 (duality theorem), this equality holds iff

$$v(N) = \max \left\{ \sum_S \lambda(S) v(S) \mid \sum_S \lambda(S) e^S = e^N, \;\; \lambda \geq 0 \right\},$$

if we take the matrix with characteristic vectors e^S as columns for A, define $c = e^N$ and let b be the vector of coalitional worths. This holds iff $\sum_S \lambda(S) v(S) \leq v(N)$, and so (i) is equivalent to (ii). □

6.4.4 Implementing the core

The core tells us which imputations – efficient allocations of the worth of the grand coalition – we can expect might be implemented in a cohesive TU-game. For any unstable imputation – not in the core – some subset of players can form a coalition and obtain an allocation that each strictly prefers. As with Nash bargaining, we can map this into a non-cooperative setting – there must be some process by which such a blocking coalition can form over time. Perry & Reny (1994) introduce a non-cooperative dynamic game that implements any and all core allocations.

Let (N, v) be a superadditive TU-game with n players. Suppose time is continuous and that players do not discount.[27] At time t, any player i has several options:

- i can *remain quiet* (q).
- i can *make a proposal* (x, S), which consists of a coalition S and an allocation x, which has $x(N) \leq v(S)$ (i.e. $\sum_{j \in N} x_j \leq v(S)$). If i makes a proposal at t, then this proposal becomes active, replacing any previous proposal. A proposal remains active until a new proposal is made.
- i can *accept* (a) the current proposal, if one is on the table. Note if i makes a proposal that does not necessarily mean she accepts it.

[27] Perry & Reny (1994) have a nice discussion of why they think continuous time is the natural setting here.

- i can *announce she is leaving* (ℓ), in which case she leaves and then consumes.

A proposal (x, S) becomes *binding* if all members of S accept it while it is active. If a proposal (x, S) becomes binding, we say S has *formed and accepted* (x, S). Once this happens, any player in S can either leave and consume x_i, or remain at the bargaining table, in the hope of a more favourable proposal. If S has formed and accepted (x, S) and some player i in S leaves to consume x_i, then all players j in S are forced to leave and consume x_j. Any player i in S thus guarantees herself a payoff of at least x_i. If a player i never leaves to consume, she receives payoff $d_i \leq v(i)$.

A binding proposal is no longer active, and thus we can potentially have several binding proposal and an active proposal on the table together at any point in time. However, if making a proposal that includes some member of a binding proposal, we assume the proposal must include all members of that binding proposal, so that all members of the coalition in the binding proposal agree to annul it before it can be superceded. If all members of a binding proposal agree to an active proposal, then the binding proposal is annuled and removed from consideration. A player can only be associated with one binding proposal at a time.

For the game to be well-defined, we need to make a technical assumption. Assume that at every time t, for every history up to t, and for every vector of actions available to the players at t, players strategies are such that they cannot make a proposal, accept a proposal or leave (i.e. take a non-quiet action) just before or just after t. Formally, we assume there exists some $\epsilon > 0$ such that players cannot, according to their strategies, take a non-quiet action in the intervals $(t - \epsilon, t)$ and $(t, t + \epsilon)$. This ensures that for any history up to t, the players' strategies induce a unique continuation path and that payoffs are well-defined. Moreover, if players can take non-quiet actions instantaneously, then no player can convince them not to leave by placing an appropriately timed blocking proposal. Since ϵ is not bounded away from 0, players can react arbitrarily quickly.

A history h^t up to time t in this game is an n-tuple $h^t = (h_1^t, \ldots, h_n^t)$, where each h_i^t is a function $h_i^t : [0, t) \to A_i$, where A_i is the space of player i's actions, i.e. $A_i = \{q, a, \ell\} \cup P$ with $P := \{(x, S) \mid S \subseteq N,\ x_j \geq v(j) \text{ for all } j \in S,\ x(N) \leq v(S)\}$ being the set of possible proposals. Let:

- $H(t)$ denote the set of all histories up to time t with $t \geq 0$, where $H(0) = \varnothing$;
- $H = \bigcup_{t=0}^{\infty} H(t)$ denote the set of all histories;
- $p(h) \in P$ denote the current active proposal under history $h \in H(t)$;
- $\tau(h)$ denote the length of time that has elapsed since $p(h)$ was proposed under $h \in H(t)$;
- $N(h) \subseteq N$ denote the set of players who have not yet left under $h \in H(t)$;
- $A(h) \subseteq N(h)$ denote the set of players who have accepted $p(h)$ under $h \in H(t)$;
- $\Pi(h)$ denote the set of current binding proposals among the players in $N(h)$, and

- $\eta(h) = (p(h), \tau(h), N(h), A(h), \Pi(h))$ denotes the state under h.

A strategy $s_i : H \to \{q, a, \ell\} \cup P$ for a player i specifies an action $s_i(h)$ for each history $h \in H$, and we write $s(h) = (s_1(h), \ldots, s_n(h))$. We let S_i denote the set of strategies of player i. The solution concept here is stationary subgame perfect equilibrium (SSPE):

Definition 69 (Stationary subgame perfect equilibrium)**.** A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a *stationary subgame perfect equilibrium* if

(i) for every player $i \in N$ and every history $h \in H$, $u_i(s_i^*, s_{-i}^* \mid h) \geq u_i(s_i, s_{-i}^* \mid h)$ for all $s_i \in S_i$ (subgame perfection), and

(ii) for all $h, h' \in H$, if $\eta(h) = \eta(h')$ then $s^*(h) = s^*(h')$, that is, players' strategies depend only on the state (stationarity).

Proposition 51. *Every stationary subgame perfect equilibrium allocation of the game is in the core of* (N, v)*.*

Proof. Suppose towards contradiction that x is an SSPE allocation but that x is not in the core $C(v)$. Suppose s is an SSPE that supports x as an outcome. We must have $x_i \geq v(i)$ for every player i. Since $x \notin C(v)$, there is some proposal (y, S) with $y_i > x_i$ for all $i \in S$. Wlog, let $S = \{1, \ldots, k\}$.

At time t, consider any history $h \in H(t)$ s.t. $N(h) = N$, $\Pi(h) = \varnothing$, players $1, \ldots, k-1$ have accepted (y, S) under h, and $s_i(h) = q$ for all $i \in N$. Now, under s, player k will accept (y, S) before a new proposal is made. To see this, suppose, under s, a proposal (z, T) is made before player k accepts (y, S). The state then becomes $\bar{\eta} = ((z, T), 0, \varnothing, N, \varnothing)$. Since k could have accepted (y, S) and guaranteed herself a payoff $y_k > x_k$ but chose to let it be replaced by (z, T), the continuation for k must have payoff $w_k \geq y_k$. By stationarity, whenever $\eta(h) = \bar{\eta}$, the continuation payoff for k under s is $w_k \geq y_k > x_k$. But since s is an SSPE, this yields a contradiction, because player k can always ensure the state $\bar{\eta}$ is reached by proposing (z, T) herself near enough to $t = 0$. Thus under s, player k accepts (y, S) before any new proposal is made.

Now at time t, consider any history $h \in H(t)$ s.t. $N(h) = N$, $\Pi(h) = \varnothing$, players $1, \ldots, k-2$ have accepted (y, S) under h, and $s_i(h) = q$ for all $i \in N$. Now, under s, both players $k-1$ and k will accept (y, S) before a new proposal is made. To see this, note that $k-1$ can guarantee himself payoff at least $y_{k-1} > x_{k-1}$ by accepting the proposal at time t, because by the previous paragraph, this will induce player k to accept (y, S) so the proposal becomes binding. Suppose instead that (y, S) is not accepted by $k-1$ and k in the continuation of play after h. If under s, no new proposal is made in the continuation, then player $k-1$ receives $d_{k-1} < x_{k-1} < y_{k-1}$, so this cannot be optimal. If under s, a new proposal (z, T) is made, then we arrive at a contradiction by a similar argument to the previous paragraph: player $k-1$ must receive a continuation payoff of at least y_{k-1}, and by stationarity, $k-1$ can ensure this by making the proposal himself close enough to $t = 0$.

Proceeding inductively, we can conclude that at any time t and any history $h \in H(t)$ with $N(h) = N$, $\Pi(h) = \varnothing$ player 1 accepting (y, S) under h, and $s_i(h) = q$ for all players i, the continuation has every player in the coalition S accepting the proposal (y, S). But then x cannot be an SSPE outcome, because player 1 can propose (y, S) and accept it at a time sufficiently close to $t = 0$, ensuring herself a payoff $y_1 > x_1$. □

Proposition 52. *If (N, v) is a totally balanced game, then any imputation in its core can be supported as a stationary subgame perfect equilibrium.*

Proof. See Perry & Reny (1994). Repeating the proof would take several pages. □

6.4.5 Competitive equilibrium and the core

There is a close relationship between the core and general equilibrium theory. In particular, competitive equilibria lie in the core. This is easiest to see in the case of a pure-exchange economy, though the result extends to production economies provided we envisage blocking opportunities appropriately.

Definition 70 (Economy)**.**

(a) *Pure exchange economy.* A *pure exchange economy* is a tuple $\mathcal{E} = (H, (X_i, u_i, \omega_i)_{i \in H})$, where

  (i) H is a finite set of n *consumers*;

  (ii) $X_i \subseteq \mathbb{R}^k$ is a consumption set for consumer i, and k is the number of commodities;

  (iii) $u_i : X_i \to \mathbb{R}$ is a utility function for consumer h;

  (iv) $\omega_i \in X_i$ is consumer i's *endowment vector.*

(b) *Private ownership economy.* A *private ownership economy* is a tuple

$$\mathcal{P} = (H, F, (X_i, u_i, \omega_i, s_i)_{i \in H}, (Y_j)_{j \in F})$$

where

  (i) H, $X_i \subseteq \mathbb{R}^k$, $u_i : X_i \to \mathbb{R}$ and $\omega_i \in X_i$ are defined as above;

  (ii) F is a finite set of m *firms*;

  (iii) $Y_j \subseteq \mathbb{R}^k$ is a *production set* for firm j;

  (iv) $s_i = (s_{ij})_{j \in F}$ is consumer i's vector of *shareholdings.* We assume $s_{ij} \in \mathbb{R}_+$ and $\sum_{i=1}^n s_{ij} = 1$ for each firm $j \in F$.

For each consumer $i \in H$, we assume i's consumption set is $X_i = \mathbb{R}^k_+$ and i's utility function u_i is continuous. For each firm $j \in F$, we assume j's production set Y_j is nonempty and closed. We define an economy's *total endowment* to be $\bar{\omega} = \sum_{i=1}^n \omega_i$.

In a pure-exchange economy $\mathcal{E}$, an allocation is a vector $x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. An allocation x is *feasible* if $\sum_{i=1}^n x_i = \bar{\omega}$.

In a private ownership economy $\mathcal{P}$, an allocation is a vector $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in X_1 \times \cdots \times X_n \times Y_1 \times \cdots \times Y_m$, consisting of a *consumption allocation* x and a vector of *production plans* y. An allocation (x, y) is *feasible* if $\sum_{i=1}^n x_i = \bar{\omega} + \sum_{j=1}^m y_j$.

Definition 71 (Competitive equilibrium)**.**

(a) In a pure-exchange economy $\mathcal{E}$, a *competitive equilibrium* is a price vector $p \in \mathbb{R}^k$ and an allocation x^* such that

(i) for each consumer $i \in H$, x_i^* is a solution to

$$\max_{x_i \in X_i} u_i(x_i) \quad \text{subject to} \quad p \cdot x \leq p \cdot \omega_i;$$

(ii) markets clear, that is, $\sum_{i=1}^n x_i = \bar{\omega}$.

(b) In a private ownership economy $\mathcal{P}$, a *competitive equilibrium* is a price vector $p \in \mathbb{R}^k$ a consumption allocation x^*, and a vector of production plans y^* such that

(i) for each consumer $i \in H$, $x_i^* \in X_i$ is a solution to

$$\max_{x_i \in X_i} u_i(x_i) \quad \text{subject to} \quad p \cdot x \leq p \cdot \omega_i + \sum_{j=1}^m s_{ij} p \cdot y_j;$$

(ii) for each firm $j \in F$, $y_j^* \in Y_j$ is a solution to

$$\max_{y_j \in Y_j} p \cdot y_j;$$

(iii) markets clear, that is, $\sum_{i=1}^n x_i = \bar{\omega} + \sum_{j=1}^m y_j$.

This is also called *Walrasian equilibrium.* Existence conditions for Walrasian equilibrium are very well-known. A good overview is e.g. Chapter 14.4 of Kreps (2013) or Chapter 17.C in MWG (1995).

Focus for now on a pure-exchange economy $\mathcal{E}$. The pure-exchange economy has an interpretation as a cooperative game, where the set of actions A_S of each coalition $S \subseteq H$ is the set of all allocations of their total endowment $\sum_{i \in S} \omega_i$:

$$A_S = \left\{ x \in X_1 \times \cdots \times X_n \mid \sum_{i \in H} x_i \leq \sum_{i \in S} \omega_i \right\}.$$

The payoff to a consumer $i \in S$ from action x is $u_i(x_i)$. Note that this is not, in general, a TU-game.

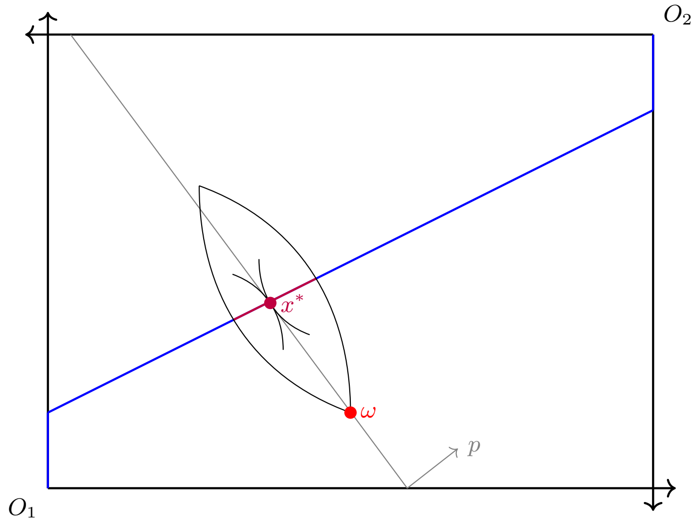
Recall that the core is the set of allocations that cannot be blocked by any coalition. In the context of a pure-exchange economy, we say that a coalition S can block a feasible allocation x if there exists an allocation x' such that $\sum_{i=1}^n x_i' \leq \sum_{i \in S} \omega_i$ and $u_i(x_i') > u_i(x_i)$ for every consumer $i \in S$. That is, S can block an allocation x if they can redistribute their own endowments among themselves in such a way that they are all better off than under x.

Theorem 29. *Any competitive equilibrium of a pure-exchange economy $\mathcal{E}$ with locally insatiable consumers is in the core.*

Proof. Fix a pure-exchange economy $\mathcal{E}$, and let (p, x^*) be a competitive equilibrium of $\mathcal{E}$, where p is a price vector and x^* is an allocation. Suppose there is some coalition S that can block x^*. Then there exists an allocation x s.t. $\sum_{i=1}^n x_i \leq \sum_{i \in S} \omega_i$ and $u_i(x_i) > u_i(x_i^*)$ for all $i \in S$. But by definition of competitive equilibrium, this implies that x_i is unaffordable for each consumer $i \in S$, since $x_i^* = \max u_i(x_i)$ subject to the consumer's budget constraint. Hence $p \cdot x_i > p \cdot \omega_i$ for all $i \in S$. Thus $p \cdot \sum_{i \in H} x_i \geq p \cdot \sum_{i \in S} x_i > p \cdot \sum_{i \in S} \omega_i$, so $\sum_{i=1}^n x_i > \sum_{i \in S} \omega_i$, yielding a contradiction. $\square$

The converse is not true – allocations in the core are not necessarily competitive equilibria, as the following Edgeworth box economy illustrates.[28]

Example 57 (The core in an Edgeworth box economy)**.** Consider the following Edgeworth box economy.



The initial endowment point is ω, and the competitive equilibrium allocation is x^*. The Pareto set is shown in blue, and the purple line, the segment of the Pareto set enclosed between the indifference curves of the two consumers at ω, is the core. This is of course identical to the contract curve. While the competitive equilibrium is a single point, the core is a continuum of points.

A similar result to Theorem 29 holds for private ownership economies, if we conceive of blocking in the right way. The problem is that it is not obvious how to allocate production to coalitions that are smaller than the grand coalition. For example:

(a) *Coalitions as peasants.* We could imagine that coalitions have no control over production at all, and so can only redistribute their total endowment. The set of

[28]However, because any allocation in the core is Pareto optimal, under the conditions of the second theorem of welfare economics, any core allocation can be supported as a quasi-equilibrium.

actions A_S of coalition S is thus the same as in the pure-exchange economy case, namely,

$$A_S = \left\{ x \in X_1 \times \cdots \times X_n \mid \sum_{i \in H} x_i \leq \sum_{i \in S} \omega_i \right\}.$$

(b) *Coalitions as revolutionaries.* Another option is that any coalition seizes the means of production and controls the productive technology of all firms, so the set of actions available to coalition S is

$$A_S = \left\{ x \in X_1 \times \cdots \times X_n \mid \sum_{i \in H} x_i \leq \sum_{i \in S} \omega_i + \sum_{j=1}^{m} y_j \text{ for some } y \in Y_1 \times \cdots \times Y_m \right\}.$$

(c) *Coalitions as private equity investors.* We could also imagine that coalitions can only employ the productive capacity of firms they fully control. For a coalition S, let $F(S)$ denote the set of firms $j \in F$ for which $\sum_{i \in S} s_{ij} = 1$. The set of actions available to S is

$$A_S = \left\{ x \in X_1 \times \cdots \times X_n \mid \sum_{i \in H} x_i \leq \sum_{i \in S} \omega_i + \sum_{j \in F(S)} y_j \text{ for some } y \in Y_1 \times \cdots \times Y_m \right\}.$$

(d) *Coalitions as activist investors.* Similarly, we could imagine coalitions can employ the productive of firms they have majority control over. For a coalition S, redefine $F(S)$ to be the set of firms $j \in F$ for which $\sum_{i \in S} s_{ij} > \frac{1}{2}$. Then A_S is as above.

(e) *Coalitions as shared owners.* We might also imagine that coalitions can use a copy of each firm scaled by their own shareholdings. For a coalition S, Let $s_{Sj} = \sum_{i \in S} s_{ij}$. Then the

$$A_S = \left\{ x \in X_1 \times \cdots \times X_n \mid \sum_{i \in H} x_i \leq \sum_{i \in S} \omega_i + \sum_{j=1}^{m} s_{Sj} y_j \text{ for some } y \in Y_1 \times \cdots \times Y_m \right\}.$$

It turns out that if $0 \in Y_j$ for each firm $j \in F$ and consumers have locally nonsatiated preferences, then (a), (c) and (e) ensure the competitive equilibrium is in the core for a private ownership economy $\mathcal{P}$. Moreover, if firms have constant returns to scale technologies, then all of (a)-(e) ensure the competitive equilibrium of $\mathcal{P}$ is in the core. See Kreps (2013), Proposition 15.9. Kreps also has a detailed discussion of how the core converges to the competitive equilibrium in N-replica economies as they grow large enough, a result due to Debreu and Scarf (1963).

6.5 Shapley value

Unlike the previous solution concepts, the Shapley value is a point-valued solution concept. It has several different characterizations. First, Shapley's own:

Definition 72 (Shapley value)**.** Given a TU game (N, v), let $\sigma : N \to N$ denote a permutation of the player set, and let $\Pi(N)$ denote the set of all permutations of the player set. For a given permutation $\sigma \in \Pi(N)$, we define the *set of predecessors of i in σ* by

$$P_\sigma(i) = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$$

for all $i \in N$, and we define the *marginal vector m^σ* of σ by

$$m_i^\sigma = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)).$$

The *Shapley value* $\Phi(v)$ of game v is then the average of the marginal vectors of the game, that is,

$$\Phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma.$$

Unpacking the definition, imagine that players enter a room in the order according to permutation σ. The ith element of the marginal vector m^σ tells us the marginal contribution of player i entering the room and joining the existing coalition of all players who already entered. The Shapley value is the average across all possible orderings with which players might arrive. Equivalently, suppose players enter the room at random, and are paid their marginal contribution to the coalition formed from the players already in the room. Then the Shapley value gives the expected payoff for each player.

We can also rewrite the Shapley value as

$$\begin{aligned}
\Phi_i(v) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \left[v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))\right] \\
&= \sum_{S : i \notin S} \frac{|S|!(n-1-|S|)!}{n!} \left[v(S \cup \{i\}) - v(S)\right] \\
&= \frac{1}{n} \sum_{S : i \notin S} \binom{n-1}{|S|}^{-1} \left[v(S \cup \{i\}) - v(S)\right]
\end{aligned}$$

for each player i. The second and third lines follow because for any given coalition S not containing i, there are $\frac{1}{n}\binom{n-1}{|S|} = \frac{|S|!(n-1-|S|)!}{n!}$ permutations σ for which the set of predecessors of i in σ is S. Probabilistically, then, we can also interpret the Shapley value as follows. Suppose we choose a number $k \in \{0, 1, \ldots, n-1\}$ uniformly at random. Given a draw k, we draw a set of size k from the set $N - \{i\}$, with each set S s.t. $|S| = k$ being drawn with probability $\binom{n-1}{k}^{-1}$. Player i then receives payoff $v(S \cup \{i\}) - v(S)$. The Shapley value gives the expected payoff for i under this procedure.

Example 58 (Glove game)**.** In a glove game, players each have either a left-hand or right-hand glove, and a coalition receives payoff 1 if it can form a left-right pair and 0 otherwise. Suppose $N = \{1, 2, 3\}$, that players 1 and 2 have right-hand gloves and player 3 has a left-hand glove. Then

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3\}, \{2,3\}, \{1,2,3\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to compute the marginal contributions for each player for each permutation:

σ	m_1^σ	m_2^σ	m_3^σ
1,2,3	0	0	1
1,3,2	0	0	1
2,1,3	0	0	1
2,3,1	0	0	1
3,1,2	1	0	0
3,2,1	0	1	0

We see that the Shapley value is $\Phi(v) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$.

6.5.1 Some axiomatic characterizations

The Shapley value can also be characterized as the unique value satisfying a certain set of attractive axioms.

Definition 73 (Value)**.** We say that a mapping $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is a *value* on $\mathcal{G}^N$, the space of TU games with player set N.

Definition 74 (Summation of games)**.** Given two TU games (N, v) and (N, w), we define the sum $(N, v + w)$ by $(v + w)(S) = v(S) + w(S)$ for each $S \in 2^N$.

The Shapley value satisfies the following four axioms:

Axioms.

(**EFF**) *Efficiency.* $\sum_{i \in N} \psi_i(v) = v(N)$ for all $v \in \mathcal{G}^N$.

(**NPP**) *Null player property.* In a game (N, v), call a player i a *null player* if $v(S \cup \{i\}) - v(S) = 0$ for all coalitions S. Under the *null player property*, for all games $v \in \mathcal{G}^N$, if i is a null player in v then $\psi_i(v) = 0$.

(**SYM**) *Symmetry.* In a game (N, v), calls players i, j *symmetric* if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions $S \subseteq N - \{i, j\}$. Under symmetry, for all $v \in \mathcal{G}^N$, if i and j are symmetric players then $\psi_i(v) = \psi_j(v)$.

(**ADD**) *Additivity.* $\psi_i(v + w) = \psi_i(v) + \psi_i(w)$ for all $v, w \in \mathcal{G}^N$ and all $i \in N$.

To prove the following theorem, it is worth noting some bases for $\mathcal{G}^N$. We denote by $1_T \in \mathcal{G}^N$ the game defined by

$$1_T(S) = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T. \end{cases}$$

We denote by $u_T \in \mathcal{G}^N$ the T-unanimity game (see Example 54(c)).

Lemma 14.

(a) *The set* $\left\{1_T \in \mathcal{G}^N \mid T \in 2^N - \{\varnothing\}\right\}$ *is a basis for* $\mathcal{G}^N$.

(b) *The set of unanimity games* $\left\{u_T \in \mathcal{G}^N \mid T \in 2^N - \{\varnothing\}\right\}$ *is a basis for* $\mathcal{G}^N$.

Proof. (a) The set $\mathcal{S} = \left\{1_T \in \mathcal{G}^N \mid T \in 2^N - \{\varnothing\}\right\}$ is clearly linearly independent. Consider any game $v \in \mathcal{G}^N$. Define $u(S) = \sum_{T \in 2^N - \{\varnothing\}} v(S) 1_T(S)$. Then we have $u = v$. Thus $\mathcal{S}$ spans $\mathcal{G}^N$.

(b) Given (a), we need only show that $\mathcal{U} = \left\{u_T \in \mathcal{G}^N \mid T \in 2^N - \{\varnothing\}\right\}$ is linearly independent, for $|\mathcal{U}| = |\mathcal{S}|$ and $\mathcal{S}$ is a basis for $\mathcal{G}^N$. Towards contradiction, assume there exist scalars c_T, some of which are nonzero, s.t. $v = \sum_{T \in 2^N - \{\varnothing\}} c_T u_T = 0$. Fix some U s.t. $c_U \neq 0$ in this sum. Then we must have $\sum_{S : S \supsetneq U} c_S = -c_U$, so at least one $S \supsetneq U$ is s.t. $c_S \neq 0$. Fix a sequence $\{S_k\}$ with $S_0 = U$, $S_{k+1} \supsetneq S_k$, and $c_{S_k} \neq 0$ for all k. Since N is finite, this sequence is of finite length K, and we can choose S_K s.t. there is no $R \supsetneq S_K$ with $c_R \neq 0$. Then $v(S_K) = c_{S_K} \neq 0$, so $v \neq 0$, yielding a contradiction.

□

Theorem 30. *The Shapley value* $\Phi : \mathcal{G}^N \to \mathbb{R}^N$ *is the unique value on* $\mathcal{G}^N$ *satisfying* (**EFF**), (**NPP**), (**SYM**) *and* (**ADD**).

Proof. First we show Φ satisfies the axioms:

(**EFF**) Fix any $v \in \mathcal{G}^N$. For any permutation σ, we have $\sum_{i=1}^n \big(v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))\big) = v(N)$. Thus,

$$\begin{aligned} \sum_{i=1}^n \Phi_i(v) &= \sum_{i=1}^n \left(\frac{1}{n!} \sum_{\sigma \in \Pi(N)} \left[v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \right] \right) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \left[\sum_{i=1}^n \big(v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \big) \right] \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} v(N) = v(N). \end{aligned}$$

(**NPP**) Suppose i is a null player in v. Then $v(S \cup \{i\}) - v(S) = 0$ for all coalitions S so $\Phi_i(v) = \frac{1}{n} \sum_{S:i \notin S} \binom{n-1}{|S|}^{-1} [v(S \cup \{i\}) - v(S)] = 0$.

(**SYM**) If i, j are symmetric players, then $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all coalitions $S \subseteq N - \{i, j\}$. Furthermore, $v(S \cup \{i, j\}) - v(S \cup \{i\}) = v(S \cup \{i, j\}) - v(S \cup \{j\})$ for any $S \subseteq N - \{i, j\}$, since $v(S \cup \{i\}) = v(S \cup \{j\})$. Thus $\Phi_i(v) = \Phi_j(v)$.

(**ADD**) Fix $v, w \in \mathcal{G}^N$. Now,

$$\begin{aligned}
\Phi_i(v+w) &= \frac{1}{n} \sum_{S:i \notin S} \binom{n-1}{|S|}^{-1} \big[(v+w)(S \cup \{i\}) - (v+w)(S)\big] \\
&= \frac{1}{n} \sum_{S:i \notin S} \binom{n-1}{|S|}^{-1} \big[v(S \cup \{i\}) - v(S) + w(S \cup \{i\}) - w(S)\big] \\
&= \frac{1}{n} \sum_{S:i \notin S} \binom{n-1}{|S|}^{-1} \big[v(S \cup \{i\}) - v(S)\big] + \frac{1}{n} \sum_{S:i \notin S} \binom{n-1}{|S|}^{-1} \big[w(S \cup \{i\}) - w(S)\big] \\
&= \Phi_i(v) + \Phi_i(w).
\end{aligned}$$

Conversely, suppose $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is a value satisfying all four axioms, and fix $v \in \mathcal{G}^N$. By Lemma 14, there are unique numbers c_T s.t. $v = \sum_{T \neq \varnothing} c_T u_T$. By (**ADD**), we must have

$$\psi(v) = \sum_{T \neq \varnothing} \psi(c_T u_T) \quad \text{and} \quad \Phi(v) = \sum_{T \neq \varnothing} \Phi(c_T u_T).$$

Thus it is sufficient to show that $\psi(cu_T) = \Phi(cu_T)$ for all nonempty coalitions T and scalars c. Fix $T \neq \varnothing$ and $c \in \mathbb{R}$. By definition of unanimity games, we have

$$cu_T(S \cup \{i\}) - cu_T(S) = 0$$

for all S and any $i \in N - T$. Thus i is a null player in cu_T. By (**NPP**), we have

$$\psi_i(cu_T) = \Phi_i(cu_T) = 0$$

for all $i \in N - T$. Suppose instead that $i, j \in T$ and $i \neq j$. Then for all $S \subseteq N - \{i, j\}$, we have $cu_T(S \cup \{i\}) = cu_T(S \cup \{j\}) = 0$, and so i and j are symmetric players in cu_T. By (**SYM**), we have

$$\psi_i(v) = \psi_j(v) \quad \text{and} \quad \Phi_i(v) = \Phi_j(v)$$

for all $i, j \in T$. Together with (**EFF**), it follows that

$$\psi_i(cu_T) = \Phi_i(cu_T)|T|^{-1}\, c$$

for all $i \in T$. Proof follows. $\square$

The Shapley value also satisfies stronger axioms than the null player property and symmetry:

Axioms.

(**DPP**) *Dummy player property.* In a game (N, v), call a player i a *dummy player* if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subseteq N - \{i\}$. Under the *dummy player property*, for all games $v \in \mathcal{G}^N$, if i is a dummy player then $\psi_i(v) = v(i)$.

(**ANO**) *Anonymity.* For a permutation $\sigma \in \Pi(N)$, define the game v^σ by $v^\sigma(S) := v(\sigma^{-1}(S))$ for all coalitions $S \in 2^N$. Define $\sigma^* : \mathbb{R}^N \to \mathbb{R}^N$ by $\sigma^*(x)_{\sigma(k)} = x_k$ for all $x \in \mathbb{R}^N$ and $k \in N$. Under *anonymity*, $\psi(v^\sigma) = \sigma^*(\psi(v))$ for all $v \in \mathcal{G}^N$ and all $\sigma \in \Pi(N)$.

In words, a dummy player contributes only their worth to every coalition of which they are a member. If a value satisfies the dummy player property than a dummy player receives only their worth. A value is anonymous if player labels are irrelevant to the payment players receive under the value – that is, relabelling the players would not change what each is paid.

Proposition 53.

(i) (**DPP**) *implies* (**NPP**)*, and*

(ii) (**ANO**) *implies* (**SYM**)*.*

Proof. (i) A null player i has worth $v(i) = 0$. Hence under (**DPP**), we have $\psi_i(v) = 0$.

(ii) Let i, j be symmetric players and suppose ψ is anonymous. Consider permutation σ defined by

$$\sigma(k) = \begin{cases} j & \text{if } k = i, \\ i & \text{if } k = j, \\ k & \text{otherwise.} \end{cases}$$

By anonymity, $\psi_i(v^\sigma) = \sigma_i^*(\psi(v)) = \psi_{\sigma(i)}(v) = \psi_j(v)$. Since i, j are symmetric, $v(S \cup \{i\}) = v(S \cup \{j\}) = v^\sigma(S \cup \{i\}) = v^\sigma(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$. Hence $\psi_i(v) = \psi_i(v^\sigma)$.

□

Proposition 54.

(i) *The Shapley value* Φ *satisfies the dummy player property, and*

(ii) *the Shapley value* Φ *is anonymous.*

Proof. (i) Let i be a dummy player. Then the result follows immediately from $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subseteq N - \{i\}$ and the definition of the Shapley value written in terms of coalitions S.

(ii) Note that for all $\rho, \sigma \in \Pi(N)$ and all $v \in \mathcal{G}^N$,

$$\rho^*(m^\sigma(v)) = m^{\rho\sigma}(v^\rho),$$

since

$$\begin{aligned} m^{\rho\sigma}(v^\rho)_{\rho\sigma(i)} &= v^\rho\left(\{\rho\sigma(1), \ldots, \rho\sigma(i)\}\right) - v^\rho\left(\{\rho\sigma(1), \ldots, \rho\sigma(i-1)\}\right) \\ &= v\left(\{\sigma(1), \ldots, \sigma(i)\}\right) - v\left(\{\sigma(1), \ldots, \sigma(i-1)\}\right) \\ &= m^\sigma(v)_{\sigma(i)} = \rho^*(m^\sigma(v))_{\rho\sigma(i)}. \end{aligned}$$

Take $v \in \mathcal{G}^N$ and $\rho \in \Pi(N)$. Note $\rho \mapsto \rho\sigma$ is a surjection on $\Pi(N)$, and ρ^* is linear. Together, all the above implies that

$$\begin{aligned} \Phi(v^\rho) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v^\rho) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\rho\sigma}(v^\rho) \\ &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \rho^*(m^\sigma(v)) = \rho^*\left(\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma\right) = \rho^*(\Phi(v)). \end{aligned}$$

□

Another axiomatic characterization of the Shapley value involves only three axioms: efficiency, symmetry and strong monotonicity.

Axiom.

(**SMO**) *Strong monotonicity.* For any $i \in N$, $\psi_i(v) \geq \psi_i(w)$ for all $v, w \in \mathcal{G}^N$ for which

$$v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S) \quad \text{for all } S \in 2^N.$$

In words, strong monotonicity is the property that if player i contributes at least as much to any coalition in game v as in game w, then i's payoff under the Shapley value is weakly greater in v than in w.

To show that the Shapley value uniquely satisfies the three axioms, we make use of the *inclusion-exclusion principle*:

Proposition 55 (Inclusion-exclusion principle)**.** *Let E be a finite set, and let f, g be functions on 2^E s.t.*

$$f(T) = \sum_{S : S \subseteq T} g(S).$$

Then

$$g(T) = \sum_{S : S \subseteq T} (-1)^{|T-S|} f(S).$$

Proof.

$$\sum_{S:S\subseteq T}(-1)^{|T-S|}f(S)=\sum_{S:S\subseteq T}\sum_{R:R\subseteq S}(-1)^{|T-S|}g(R)$$
$$=\sum_{R:R\subseteq T}g(R)\sum_{S:R\subseteq S\subseteq T}(-1)^{|T-S|}=g(T).$$

□

Theorem 31. *The Shapley value $\Phi:\mathcal{G}^N\to\mathbb{R}^N$ is the unique value on $\mathcal{G}^N$ satisfying* (**EFF**)*,* (**SYM**) *and* (**SMO**)*.*

Proof. From Theorem 30, we have that Φ satisfies (**EFF**) and (**SYM**). Fix player i and suppose v and w are s.t. $v(S\cup\{i\})-v(S)\geq w(S\cup\{i\})-w(S)$ for all $S\in 2^N$. Then

$$\Phi_i(v)=\frac{1}{n}\sum_{S:i\notin S}\binom{n-1}{|S|}^{-1}\big[v(S\cup\{i\})-v(S)\big]$$
$$\geq\frac{1}{n}\sum_{S:i\notin S}\binom{n-1}{|S|}^{-1}\big[w(S\cup\{i\})-w(S)\big]=\Phi_i(w),$$

so Φ satisfies (**SMO**).

Conversely, suppose $\psi:\mathcal{G}^N\to\mathbb{R}^N$ is a value satisfying (**EFF**), (**SYM**) and (**SMO**). Let $0\in\mathcal{G}^N$ denote the game that is identically zero. All players are symmetric in 0, so (**EFF**) and (**SYM**) imply $\psi(0)=0$.

Now consider any game v in which i is a null player. By (**SMO**), we have $\psi_i(v)\geq\psi_i(0)$ and $\psi_i(0)\geq\psi_i(v)$. Since $\psi_i(0)=0$, we have $\psi_i(v)=0$. Next, let $c\in\mathbb{R}$ and $T\in 2^N-\{\varnothing\}$. Then since any $i\in N-T$ is a null player in the T-unanimity game u_T, it follows that $\psi_i(cu_T)=0$ for all $i\in N-T$. Together with (**SYM**) and (**EFF**), this implies $\psi_i(cu_T)=c|T|^{-1}e^T$ for all $i\in T$, since T consists of symmetric players.

Next, note:

Lemma 15. *For each $v\in\mathcal{G}^N$, v has a unique representation*

$$v=\sum_{T\in 2^N-\{\varnothing\}}c_Tu_T\qquad\textit{where } c_T:=\sum_{S:S\subseteq T}(-1)^{|T-S|}v(S).$$

Proof. Fix $T\in\in^N$. Now $v(T)=\sum_{S:S\subseteq T}c_S$. Proof follows immediately from the inclusion-exclusion principle (Proposition 55) □

Thus v has form $v=\sum c_Tu_T$. Let $\alpha(v)$ denote the number of terms c_T in this sum for which $c_T\neq 0$.

We proceed by induction. If $\alpha(v)=0$ then $v=0$ so $\psi(v)=\Phi(v)=0$. If $\alpha(v)=1$, then v is some T-unanimity game u_T and so we must have $\psi(v)=c_Tu_Te^T=\Phi(v)$.

Fix $k\geq 2$ and suppose $\psi(w)=\Phi(w)$ for all w s.t. $\alpha(w)<k$. Let v be s.t. $\alpha(v)=k$. Then there exist coalitions $T_1,\ldots,T_k$ and numbers $c_{T_1},\ldots,c_{T_k}\neq 0$ s.t. $v=\sum_{s=1}^k c_{T_s}u_{T_s}$. Define $D:=\bigcap_{s=1}^k T_s$.

For $i \in N - D$, define $w^i := \sum_{s:i\in T_s} c_s u_{T_s}$. Now, because $\alpha(w^i) < k$, we have, by the induction hypothesis, that $\psi_i(w^i) = \Phi_i(w^i)$. Now, note that for all $S \in 2^N$,

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= \sum_{s=1}^{k} c_s u_{T_s}(S \cup \{i\}) - \sum_{s=1}^{k} c_s u_{T_s}(S) \\ &= \sum_{s:i\in T_s} c_s u_{T_s}(S \cup \{i\}) - \sum_{s:i\in T_s} c_s u_{T_s}(S) \\ &= w^i(S \cup \{i\}) - w^i(S). \end{aligned}$$

By (**SMO**), it follows that $\psi_i(v) = \psi_i(w^i) = \Phi_i(w^i) = \Phi_i(v)$. Hence $\psi_i(v) = \Phi_i(v)$ for all $i \in N - D$. By (**EFF**), we have

$$\sum_{i\in D} \psi_i(v) = \sum_{i\in D} \Phi_i(v).$$

Instead consider $i, j \in D$. For every $S \subseteq N - \{i, j\}$, we have

$$0 = v(S \cup \{i\}) = \sum_{s=1}^{k} c_s u_{T_s}(S \cup \{i\}) = \sum_{s=1}^{k} c_s u_{T_s}(S \cup \{j\}) = v(S \cup \{j\}).$$

Thus i and j are symmetric players, and so by (**SYM**), $\psi_i(v) = \psi_j(v)$ and $\Phi_i(v) = \Phi_j(v)$. By (**EFF**) again, it follows $\psi(v) = \Phi(v)$. □

One peculiarity about the Shapley value is that in games with three players or more, it does not necessarily assign a point in the core, even when the core is nonempty:

Example 59. Consider the game (N, v) with $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 2 & \text{if } S = \{1\}, \\ 3 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

The core of this game is $C(v) = \mathrm{co}(\{(2, 1, 0), (2, 0, 1), (3, 0, 0)\})$. Now, the marginal contributions are:

σ	m_1^σ	m_2^σ	m_3^σ
1,2,3	2	-2	3
1,3,2	2	3	-2
2,1,3	0	0	3
2,3,1	3	0	0
3,1,2	0	3	0
3,2,1	3	0	0

Thus $\Phi(v) = \left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Note $\Phi_1(v) < 2$ but $v(1) = 2$. Hence not only is $\Phi(v) \notin C(v)$, but also $\Phi(v) \notin I(v)$, the imputation set!

6.5.2 Harsanyi dividends

Another characterization of the Shapley value involves Harsanyi dividends. The dividend can be considered a measure of synergy – the total extra worth generated by players cooperating.

Definition 75 (Harsanyi dividend)**.** For any TU-game (N, v), for each coalition $T \subseteq N$, the *dividend* $\Delta_v(T)$ of T is defined recursively by

$$\begin{aligned} &\Delta_v(\varnothing) := 0, \\ &\Delta_v(T) := v(T) - \sum_{S:S\subsetneq T} \Delta_v(S) \qquad \text{if } |T| \geq 1. \end{aligned}$$

Suppose that for each coalition involving player i, the dividend of the coalition is divided equally between all players. The Shapley value is the sum of these equally divided dividends across all the coalitions involving i:

Theorem 32. *Let Φ be the Shapley value on $\mathcal{G}^N$. Then*

$$\Phi_i(v) = \sum_{T:i\in T} \frac{\Delta_v(T)}{|T|}.$$

Proof.

Lemma 16. *Let $v = \sum_{T\in 2^N-\{\varnothing\}} c_T u_T$ for T-unanimity games u_T and scalars c_T. Then $\Delta_v(T) = c_T$ for all $T \neq \varnothing$.*

Proof. We proceed by induction. Suppose $|T| = 1$, and wlog take $T = \{i\}$. Then $v(i) = \Delta_v(T)$. Now suppose $\Delta_v(T) = c_T$ holds for all $S \subsetneq T$. Then $\Delta_v(T) = v(T) - \sum_{S\subsetneq T} \Delta_v(S) = v(T) - \sum_{S\subsetneq T} c_S = c_T$, for $v(T) = \sum_{S\subseteq T} c_S$. $\square$

Since $\{u_T \in \mathcal{G}^N \mid T \in 2^N - \{\varnothing\}\}$ is a basis for $\mathcal{G}^N$ (Lemma 14), we can write $v = \sum_{T\in 2^N-\{\varnothing\}} c_T u_T$. From the proof of Theorem 30, we have that $\Phi(c_T u_T) = |T|^{-1} c_T e^T$ for all coalitions T. Hence by (**ADD**), we have $\Phi(v) = \sum_{T\neq\varnothing} c_T |T|^{-1} e^T$, and thus $\Phi_i(v) = \sum_{T:i\in T} c_T |T|^{-1}$. Proof now follows from the lemma. $\square$

6.5.3 Multilinear extensions

Another alternative characterization is as follows.

Definition 76 (Multilinear function)**.** We call a function $g : \mathbb{R}^N \to \mathbb{R}$ *multilinear* if g has form

$$g(x) = \sum_{S:S\subseteq N} c_S \left(\prod_{i\in S} x_i \right)$$

for some real numbers c_S.

Proposition 56. *Let (N, v) be a game. Then there is a unique multilinear function $f : 2^N \to \mathbb{R}$ s.t. $f(e^S) = v(S)$ for all $S \in 2^N$. Moreover,*

$$f(x) = \sum_{S \in 2^N} \left(\prod_{i \in S} x_i \prod_{i \in N-S} (1 - x_i) \right) v(S).$$

We call f the multilinear extension of v.

Proof. By definition, f has form

$$f(x) = \sum_{S \subseteq N} c_S \prod_{i \in S} x_i.$$

Now, $f(e^S) = v(S)$ implies that

$$f(e^S) = \sum_{T : T \subseteq S} c_T \qquad \text{for all } S \subseteq N.$$

This is a system of linear equations, and so has a unique solution if, when $v(S) = 0$ for all S, we have that $c_T = 0$ for all T is the only solution. Towards contradiction, assume the system

$$\sum_{T : T \subseteq S} c_T = 0 \qquad \text{for all } S \subseteq N$$

has a nonzero solution. Let S be s.t. $c_S \neq 0$ but $c_T = 0$ for all $T \subseteq S$. Then $\sum_{T : T \subseteq S} c_T = c_S \neq 0$, yielding a contradiction. Hence the system has no nonzero solutions, and thus the original system has a unique solution.

Now, let

$$f(x) = \sum_{S : S \subseteq N} \left(\sum_{T : T \subseteq S} (-1)^{|S-T|} v(T) \right) \left(\prod_{i \in S} x_i \right).$$

Taking $c_S = \sum_{T : T \subseteq S} (-1)^{|T-S|} v(T)$, we have $f(x) = \sum_{S : S \subseteq N} c_S \left(\prod_{i \in S} x_i \right)$, so f is multilinear. Furthermore, $f(e^S) = v(S)$ for all coalitions S. Now f can equivalently be written as $f(x) = \sum_{S \in 2^N} \left(\prod_{i \in S} x_i \prod_{i \in N-S} (1 - x_i) \right) v(S)$. $\square$

Note that the set of extreme points of $[0, 1]^N$ is $\text{ext}([0, 1]^N) = \{e^S \mid S \in 2^N\}$, and $f(e^S) = v(S)$. Thus we can also consider f to be the unique multilinear extension of the function $\tilde{v} : \text{ext}([0, 1]^N) \to \mathbb{R}$ defined by $\tilde{v}(S) := e^S$ for all coalitions S.

The multilinear extension f of v has several possible interpretations. Probabilistically, we can imagine that each player i independently chooses to cooperate with probability x_i. Then the probability that coalition S forms is $\prod_{i \in S} x_i \prod_{i \in N-S} (1 - x_i)$. Thus $f(x)$ gives the expected value of the worth of the coalition formed through this process. Another interpretation of $x \in [0, 1]^N$ is that $(N, i \mapsto x_i)$ is a fuzzy set (with membership function $i \mapsto x_i$). In this case, x_i is the intensity with which i is available to cooperate.

The payoff to player i under the Shapley value is the integral of $D_i f$ along the main diagonal of $[0, 1]^N$.

Theorem 33. *For any $v \in \mathcal{G}^N$ with multilinear extension f,*

$$\Phi_i(v) = \int_0^1 D_i f(t, \ldots, t)\,\mathrm{d}t \qquad \textit{for each } i \in N.$$

Proof. Note

$$\begin{aligned} D_i f(x) &= \sum_{T: i \in T} \left(\prod_{j \in T - \{i\}} x_j \prod_{j \in N - T} (1 - x_j) \right) v(T) - \sum_{S: i \notin S} \left(\prod_{j \in S} x_i \prod_{j \in N - (S \cup \{i\})} (1 - x_j) \right) v(S) \\ &= \sum_{S: i \notin S} \left(\prod_{j \in S} x_j \prod_{j \in N - (S \cup \{i\})} (1 - x_j) \right) [v(S \cup \{i\}) - v(S)]. \end{aligned}$$

We thus have $\int_0^1 D_i f(t, \ldots, t)\,\mathrm{d}t = \sum_{S: i \notin S} \left(\int_0^1 t^{|S|} (1-t)^{n-|S|-1}\,\mathrm{d}t \right) \left[v(S \cup \{i\}) - v(S) \right]$.

Lemma 17 (Beta-integral formula)**.** *Let m, n be positive integers. Then*

$$\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t = \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

Proof. Let $B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t$. Now, for any integer p we can write $(p-1)! = \Gamma(p) := \int_0^\infty t^{p-1} e^{-t}\,\mathrm{d}t$. [29] Now

$$\begin{aligned} (m-1)!(n-1)! &= \left(\int_0^\infty u^{m-1} e^{-u}\,\mathrm{d}u \right) \left(\int_0^\infty v^{n-1} e^{-v}\,\mathrm{d}v \right) \\ &= \int_0^\infty \int_0^\infty u^{m-1} v^{n-1} e^{-u-v}\,\mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

Now consider a change of variables with $u = st$ and $v = s(1-t)$. We have

$$\begin{aligned} (m-1)!(n-1)! &= \int_0^\infty \int_0^1 (st)^{m-1} [s(1-t)]^{n-1} e^{-s} s\,\mathrm{d}t\,\mathrm{d}s \\ &= \left(\int_0^\infty e^{-s} s^{m+n-1}\,\mathrm{d}s \right) \left(\int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t \right) \\ &= (m+n-1)! \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}t. \end{aligned}$$

□

From the Beta-integral formula, it follows that $\int_0^1 t^{|S|}(1-t)^{n-|S|-1}\,\mathrm{d}t = \frac{|S|!(n-|S|-1)!}{n!}$. Thus $\int_0^1 D_i f(t, \dot{,} t)\,\mathrm{d}t = \sum_{S: i \notin S} \frac{|S|!(n-|S|-1)}{n!} \left[v(S \cup \{i\}) - v(S) \right] = \Phi_i(v)$. □

[29] B is the *beta function* and Γ the *gamma function*. To see that $(p-1)! = \Gamma(p)$, note that in general, for $z \in \mathbb{C}$, $\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,\mathrm{d}t = \left[-t^z e^{-t} \right]_0^\infty + z \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t = \lim_{t \to \infty} [-t^z e^{-t}] + z \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t = z\Gamma(z)$. Now, we have $\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = \left[-e^{-t} \right]_0^\infty = 1$. By induction, we have $\Gamma(p) = (p-1)!$.

6.5.4 Potential games

Let

$$\mathcal{G} = \bigcup_{N:N\subseteq\mathbb{N},|N|<\infty} \mathcal{G}^N,$$

i.e. $\mathcal{G}$ is the family of all games (N, v) with finite player sets N. Note $(\varnothing, v) \in \mathcal{G}$.

Definition 77 (Potential)**.** A *potential* is a function $P : \mathcal{G} \to \mathbb{R}$ s.t.

$$\begin{aligned} P(\varnothing, v) &= 0, \text{ and} \\ \sum_{i\in N} D_i P(N, v) &= v(N) \qquad \text{for all } (N, v) \in \mathcal{G}, \\ \text{where} \qquad D_i P(N, v) &:= P(N, v) - P(N - \{i\}, v). \end{aligned}$$

That is, P is a potential if the empty game has potential zero and for each game (N, v), the gradient $\nabla P(N, v) := (D_i P(N, v))_{i\in N}$ of P is an efficient payoff vector for (N, v).

Theorem 34.

(i) *There is a unique potential* $P : \mathcal{G} \to \mathbb{R}$*;*

(ii) *For each* $v = \sum_{T\in 2^N-\{\varnothing\}} c_T u_T$*, we have*

$$P(N, v) = \sum_{T\in 2^N-\{\varnothing\}} c_T |T|^{-1};$$

(iii) $\nabla P(N, v) = \Phi(v)$.

Proof. (i) Note from the gradient condition that

$$P(N, v) = |N|^{-1} \left(v(N) + \sum_{i\in N} P(N - \{i\}, v) \right).$$

If $P(T, v)$ has a unique value for all $T \subseteq N$ s.t. $T = N - \{i\}$ for some $i \in N$, then it follows that $P(N, v)$ has a unique value. Now $P(\varnothing, v) = 0$. Proof follows by induction.

(ii) Define $Q : \mathcal{G} \to \mathbb{R}$ by

$$\begin{aligned} Q(\varnothing, v) &:= 0, \\ Q(N, v) &:= \sum_{T\in 2^N-\{\varnothing\}} c_T |T|^{-1} \quad \text{for all } v = \sum c_T u_T \text{ if } N \neq \varnothing. \end{aligned}$$

For each (N, v), we have

$$D_i Q(N, v) = \sum_{T\in 2^N-\{\varnothing\}} c_T |T|^{-1} - \sum_{T\in \in^{N-\{i\}}-\{\varnothing\}} c'_T |T|^{-1}$$

for all $i \in N$, where $v = \sum_{T \in 2^N - \{\varnothing\}} c_T u_T$ and where $v' = \sum_{T \in 2^{N-\{i\}} - \{\varnothing\}} c'_T u_T$ is the restriction of v to $2^{N-\{i\}}$. For each $S \subseteq N - \{i\}$, we have

$$\sum_{T \in 2^N - \{\varnothing\}} c_T u_T(S) = v(S) = v'(S) = \sum_{T \in 2^{N-\{i\}} - \{\varnothing\}} c'_T u_T(S),$$

and thus by recursion, $c_T = c'_T$ for all $T \in 2^{N-\{i\}} - \{\varnothing\}$. Together with Theorem 32, it follows that

$$D_i Q(N, v) = \sum_{T: i \in T} c_T |T|^{-1} = \Phi_i(v)$$

for all $i \in N$. Since Φ satisfies axiom (**EFF**), we have

$$\sum_{i \in N} D_i Q(N, v) = \sum_{i \in N} \Phi_i(N, v) = v(N).$$

Thus Q is a potential so by (i), $P = Q$.

(iii) Follows from $P = Q$ and $D_i Q(N, v) = \Phi_i(v)$.

□

Proposition 57. *For each* $(N, v) \in \mathcal{G}$*,*

$$P(N, v) = \sum_{S \subseteq N} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} v(S).$$

Proof. For each (N, v), define

$$Q(N, v) := \sum_{S \subseteq N} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} v(S).$$

Clearly, $Q(\varnothing, v) = 0$. To show that $Q(N, v) = P(N, v)$, we need only show $D_i Q(N, v) = \Phi_i(N, v)$ for all $i \in N$, in view of Theorem 34. We have

$$\begin{aligned}
D_i Q(N, v) &= Q(N, v) - Q(N - \{i\}, v) \\
&= \sum_{T \subseteq N} \frac{(|T| - 1)!(|N| - |T|)!}{|N|!} v(T) - \sum_{S \subseteq N - \{i\}} \frac{(|S| - 1)!(|N| - 1 - |S|)!}{(|N| - 1)!} v(S) \\
&= \sum_{S \subseteq N - \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} v(S \cup \{i\}) + \sum_{S \subseteq N - \{i\}} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} v(S) \\
&\quad - \sum_{S \subseteq N - \{i\}} \frac{(|S| - 1)!(|N| - 1 - |S|)!}{(|N| - 1)!} v(S) \\
&= \sum_{S \subseteq N - \{i\}} \frac{|S|!(|N| - 1 - |S|)!}{|N|!} \big(v(S \cup \{i\}) - v(S)\big) \\
&= \Phi_i(N, v).
\end{aligned}$$

□

Proposition 57 has the following interpretation: Suppose the probability that each coalition S forms is $\binom{|N|}{|S|}^{-1} |N|^{-1}$. Then $\frac{P(N,v)}{|N|}$ is the expected normalized worth $\mathbb{E}\left[\frac{v(S)}{|S|}\right]$.

6.5.5 Reduced games

Definition 78 (Reduced game)**.** Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. For any $U \in 2^N - \{\varnothing\}$ and any $(N, v) \in \mathcal{G}$, define the game $(N - U, v_U^\psi)$ by

$$v_U^\psi(S) := \begin{cases} 0 & \text{if } S = \varnothing, \\ v(S \cup U) - \sum_{i \in U} \psi_i(S \cup U, v) & \text{otherwise.} \end{cases}$$

We call v_U^ψ the (U, ψ)-*reduced game* of v.

Reduced games can be motivated as follows. Suppose a given solution concept is used to allocate payoffs in a game. If any subset of players consider the game arising among themselves and agree to apply the same solution concept, then their payoffs should be identical to those of the original game. Thus the reduced game provides a way to define a certain notion of *consistency* that values ought to satisfy. The standard notion of consistency in TU-games is *Hart-Mas-Colell consistency* (*HM-consistency*).

Definition 79 (HM-consistency)**.** A value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is called *HM-consistent* if it satisfies the following property: For all games (N, v) and all $U \in 2^N - \{\varnothing\}$,

$$\psi_i(N - U, v_U^\psi) = \psi_i(N, v) \quad \text{for all } i \in N - U.$$

Lemma 18. *Let $(N, v) \in \mathcal{G}$ be a TU game with potential P, and suppose $Q : 2^N \to \mathbb{R}$ is s.t.*

$$\sum_{i \in S} (Q(S) - Q(s - \{i\}))$$

for all nonempty $S \subseteq N$. Then

$$Q(S) = P(S, v) + Q(\varnothing)$$

for all $S \subseteq N$.

Proof. We proceed by induction. If $|S| = 0$ then $S = \varnothing$, $P(\varnothing, v) = 0$ and so $Q(S) = P(S, v) + Q(\varnothing)$. Now suppose $Q(S) = P(S, v) + Q(\varnothing)$ for all $S \subseteq N$ s.t. $|S| < k$, and take some $T \subseteq N$ with $|T| = k$. Then

$$\begin{aligned}
Q(T) &= |T|^{-1} \left(v(T) + \sum_{i \in T} Q(T - \{i\}) \right) \\
&= |T|^{-1} \left(v(T) + |T|\, Q(\varnothing) + \sum_{i \in T} P(T - \{i\}, v) \right) \\
&= Q(\varnothing) + |T|^{-1} \left(v(T) + \sum_{i \in T} P(T - \{i\}, v) \right) \\
&= Q(\varnothing) + P(T, v).
\end{aligned}$$

□

Proposition 58. *The Shapley value Φ is HM-consistent.*

Proof. Fix a TU-game $(N, v) \in \mathcal{G}$. Fix a nonempty coalition U and consider the reduced game v_U^Φ. The reduced game v_U^Φ satisfies

$$\begin{aligned}
v_U^\Phi(S) = v(S \cup U) - \sum_{i \in U} \Phi_i(S \cup U, v) &= \sum_{i \in S} \Phi_i(S \cup U, v) \\
&= \sum_{i \in S} \left[P(S \cup U, v) - P((S \cup U) - \{i\}, v) \right],
\end{aligned}$$

where the first equality is by definition of the reduced game, the second is by efficiency, and the third is because $\Phi(N, v) = \nabla P(N, v)$ (Theorem 34), the gradient of the potential. For each $S \in 2^{N-U}$, define $Q(S) := P(S \cup U, v)$. By Lemma 18, this implies that

$$Q(S) = P(S, v_U^\Phi) + Q(\varnothing) = P(S, v_U^\Phi) + P(U, v)$$

for all $S \in 2^{N-U}$. By definition of Q, we have $P(S \cup U, v) = P(S, v_U^\Phi) + P(U, v)$, and so

$$\begin{aligned}
\Phi_i(N - U, v_U^\Phi) &= P(N - U, v_U^\Phi) - P((N - U) - \{i\}, v_U^\Phi) \\
&= P(N, v) - P(N - \{i\}, v) = \Phi_i(N, v).
\end{aligned}$$

Thus for each $U \in 2^N - \{\varnothing\}$, we have $\Phi_i(N - U, v_U^\Phi) = \Phi_i(N, v)$ for all $i \in N - U$. Thus Φ is HM-consistent. □

6.5.6 Myerson value

The Shapley value allocates payoffs to players based on the values of all possible coalitions. Thus it treats all possible coalitions in a symmetric way. In practice, however, there may be factors that prevent certain coalitions from forming. Two people who do not know about each other or who live far apart probably cannot form a coalition, at least not without some intermediary. In many settings, cooperation takes place within structures such as social structures, networks of business relationships or international agreements, and so on.

Myerson (1977) considers cooperative games where cooperation structures are described by graphs. Fix a set of players N and consider a graph $G = (N, E)$ (c.f. Definition 1). Let $\mathcal{N}^N$ denote the set of all graphs on N.

Definition 80. Consider a TU-game (N, v) and a graph $G = (N, E)$.

(a) *Connectedness.* Given a coalition S, call a pair of players $i, j \in S$ *connected in S by G* if there is a path connecting i and j that stays within S. Call the coalition S *connected* if every pair of players in S is connected in S by G. That is, S is connected if the subgraph of G induced by S is connected.[30]

For each coalition S, let $S_{|G} := \{\{i \in S \mid i \text{ and } j \text{ are connected in } S \text{ by } G\} \mid j \in S\}$ be the partition of S into sets of players connected in S by G. Note that S is connected if $S_{|G} = \{S\}$.

(b) *Allocation rule.* An *allocation rule* $\psi : \mathcal{N}^N \to \mathbb{R}^N$ gives, for each graph $G \in \mathcal{N}^N$, an allocation $\psi_i(G)$ for each player $i \in N$, with the property that

$$\sum_{i \in S} \psi_i(G) = v(S)$$

for every coalition $S \in N_{|G}$, for every graph $G \in \mathcal{N}^N$.

(c) *Fairness.* We call an allocation rule ψ *fair* if

$$\psi_i(G) - \psi_i(G - ij) = \psi_j(G) - \psi_j(G - ij)$$

for every $ij \in G$ and every graph $G \in \mathcal{N}^N$.

(d) *Worth of disconnected coalitions.* Given G, the worth of any coalition $S \in 2^N$, possibly disconnected, is given by

$$v_{|G}(S) = \sum_{T \in S_{|G}} v(T).$$

Intuitively, fairness requires that in any bilateral relationship between players (represented by the edge $ij \in G$), both players benefit equally.

The *Myerson value* ψ of a game (N, v) under graph G is the Shapley value of the game $(N, v_{|G})$.

Theorem 35 (Myerson, 1977)**.** *For any TU-game (N, v), the unique fair allocation rule ψ is given by $\psi(G) = \Phi(v_{|G})$ for every graph $G \in \mathcal{N}^N$, where Φ is the Shapley value.*

[30] A graph $G = (V, E)$ is *connected* if there is a path connecting any pair of vertices $i, j \in E$. The subgraph of G induced by some $U \subset V$ is a graph (U, F) with $ij \in F$ iff $i, j \in U$ and $ij \in E$.

Proof. Suppose ψ and ψ' are two distinct fair allocation rules. Let $G = (N, E)$ be a network with $|E|$ minimal that $\psi(G) \neq \psi'(G)$. Now for all $i, j \in N$, $\psi_i(G - ij) = \psi'_i(G - ij)$ and $\psi_j(G - ij) = \psi'_j(G - ij)$. Since ψ and ψ' are fair, we have $\psi_i(G) - \psi'_i(G) = \psi_j(G) - \psi'_j(G)$. Now, $\sum_{i \in N}(\psi_i(G) - \psi'_i(G)) = 0$, and so $\psi_i(G) - \psi'_i(G) = 0$ for all $i \in N$, and thus $\psi = \psi'$, yielding a contradiction. Hence there is a unique fair allocation rule.

It remains to show that the Myerson value $\psi(G) = \Phi(v_{|G})$ is a fair allocation rule. First, we show it is indeed an allocation rule. Fix a coalition T. Now, $T_{|G} = \bigcup_{S \in N_{|G}}(T \cap S)_{|G}$, and thus $v_{|G} = \sum_{S \in N_{|G}} u_S$, where u_S is defined by $u_S(T) = \sum_{R \in (T \cap S)_{|G}} v(R)$ for all coalitions $T \subseteq N$. For u_S, all players not in S are dummy players. Since Φ satisfies the dummy player property (**DPP**), we have $\sum_{i \in S} \Phi_i(u_S) = u_S(N) = v(S)$ and $\sum_{i \in T} \Phi_i(u_S) = 0$ for all $T \in N_{|G}$ s.t. $T \neq S$. Now by additivity (**ADD**), $\Phi(v_{|G}) = \sum_{S \in N_{|G}} \Phi(u_S)$, and thus for any coalition $T \in N_{|G}$, we have $\sum_{i \in T} \Phi_i(v_{|G}) = \sum_{S \in N_{|G}} \sum_{i \in T} \Phi_i(u_S) = u_T(N) = v(T)$. Hence the Myerson value is an allocation rule.

Next, to show the Myerson value is fair. Define a new game $w = v_{|G} - v_{|G-ij}$. Now, i, j are interchangeable in w and $w(S \cup i) = w(S \cup j) = 0$ for all coalitions $S \not\supseteq \{i, j\}$. By symmetry (**SYM**), $\Phi_i(w) = \Phi_j(w)$. By additivity, we conclude that $\Phi_i(v_{|G}) - \Phi_i(v_{|G-ij}) = \Phi_j(v_{|G}) - \Phi_j(v_{|G-ij})$. □

7 Mathematical appendix

Here we give an overview and prove important results, many of which we relied on throughout.

7.1 Correspondences

A *correspondence* $F : X \rightrightarrows Y$ is a set-valued function from X into 2^Y.

Call a correspondence $F : X \rightrightarrows Y$ *nonempty-valued* if $F(x) \neq \varnothing$ for all $x \in X$, *convex-valued* if $F(x)$ is convex for all $x \in X$, *closed-valued* if $F(x)$ is closed for all $x \in X$, and *compact-valued* if $F(x)$ is compact for all $x \in X$.

Definition 81 (Hemicontinuity)**.** Let $F : X \rightrightarrows Y$ be a correspondence from a topological space X into a topological space Y.

(a) *Upper hemicontinuity.* F is said to be *upper hemicontinuous* at a point $x \in X$ if for any open neighbourhood V of $F(x)$, there exists an open neighbourhood U of x s.t. $F(U) \subseteq V$.

   F is called *upper hemicontinuous* if F is upper hemicontinuous at every $x \in X$.

(b) *Lower hemicontinuity.* F is said to be *lower hemicontinuous* at a point $x \in X$ if for any open set V s.t. $V \cap F(x) \neq \varnothing$ (i.e. s.t. V intersects $F(x)$), there exists a neighbourhood U of x s.t. $F(t) \cap V \neq \varnothing$ for all $t \in U$.

   F is called *lower hemicontinuous* if it is lower hemicontinuous at every $x \in X$.

(c) *Continuity.* F is said to be *continuous* at $x \in X$ if it is both upper and lower hemicontinuous at x. If F is continuous at every $x \in X$, it is said to be *continuous*.

Only upper hemicontinuity is required for Kakutani's FPT. For completeness, I include lower hemicontinuity and continuity, to show how these concepts relate. Hemicontinuity is in some sense the correspondence analogue of semicontinuity for functions.[31]

Theorem 36. *Suppose that X and Y are metric spaces, that $F : X \rightrightarrows Y$ is nonempty-valued and that Y is compact.*

(i) *If for every sequence $\{x_n\}$ in X s.t. $x_n \to x$ and for every sequence $\{y_n\}$ in Y s.t. $y_n \to y$ and $y_n \in F(x_n)$ for all n, we have $y \in F(x)$, then F is upper hemicontinuous at x.*

(ii) *If F compact valued, then F is upper hemicontinuous iff for every sequence $\{x_n\}$ in X s.t. $x_n \to x$ for some $x \in X$ and for every sequence $\{y_n\}$ in Y s.t. $y_n \to y$ and $y_n \in F(x_n)$ for all n, we have $y \in F(x)$.*

(iii) *F is lower hemicontinuous iff for all $y \in F(x)$ and every sequence $\{x_n\}$ in X s.t. $x_n \to x$ for some $x \in X$, there exists a sequence $\{y_n\}$ in Y s.t. $y_n \in F(x)$ for all n and $y_n \to y$.*

Proof. (i) Suppose otherwise, i.e. the property holds for F at x but F is not upper hemicontinuous. Then there exists an open set $V \supseteq F(x)$ such that for any open set U containing x, there is an $x' \in U$ such that $F(x') \nsubseteq V$. Taking successively smaller U, we can find a sequence $\{x_n\}$ with $x_n \to x$ and $y_n \in F(x_n)$ for each n but $y_n \notin V$.

Now since V is open, V^c is closed and each $y_n \in V^c$ for all n. Since Y is compact, $\{y_n\}$ has a convergent subsequence. We can thus wlog suppose $y_n \to y$ (since we can take the subsequence otherwise). Since $y_n \in V^c$, we have that $y := \lim_n y_n \in V^c$, but this implies $y \notin F(x)$, yielding a contradiction.

(ii) Since (i) proves the implication, we need only prove the converse. Suppose F is compact-valued and upper hemicontinuous, and fix any sequence $\{x_n\}$ s.t. $x_n \to x$ and any sequence $\{y_n\}$ s.t. $y_n \in F(x_n)$ for each n. Since Y is compact, $\{y_n\}$ contains a convergent subsequence $\{y_{n_k}\}$ i.e. $y_{n_k} \to y$ for some y. Suppose $y \notin F(x)$. Since $F(x)$ is compact, it is closed, and thus the distance between y and $F(x)$ is strictly positive. Thus there is some closed ϵ-ball B_ϵ containing $F(x)$ s.t. B_ϵ does not contain y. Moreover, B_ϵ contains $F(x_n)$ for sufficiently large n, by upper hemicontinuity, and so $F(x_{n_k})$ lies in B_ϵ for sufficiently large k. But then we must have that $y \in B_\epsilon$, yielding a contradiction.

(iii) Suppose F is lower hemicontinuous, fix some sequence $\{x_n\}$ s.t. $x_n \to x$ for some $x \in X$, and fix any $y \in F(x)$. Consider a sequence of $1/k$-balls $B_{1/k}(y)$ centred on

[31] See e.g. Rudin's RCA, Chapter 2, Definition 8 (p. 37 in the third edition).

y. Since $y \in F(x)$, $B_{1/k}(y) \cap F(x) \neq \varnothing$. Since F is lower hemicontinuous, for each k there exists a neighbourhood U_k of x s.t. $F(z) \cap B_{1/k}(y) \neq \varnothing$ for each $z \in U_k$. Since $x_n \to x$, $x_n \in U_k$ for each k and n sufficiently large, and thus we can choose a subsequence $\{x_{n_k}\}$ s.t. $x_{n_k} \in U_k$ for each k. Now $y_{n_k} \in B_{1/k}(y) \cap F(x_{n_k})$, and so $y_{n_k} \to y$.

Conversely, suppose the property holds for F but F is not lower hemicontinuous. Then there exists an open set V s.t. $F(x) \cap V \neq \varnothing$ and every neighbourhood U of x contains some point z with $F(z) \cap V = \varnothing$. Taking $\{U_n\}$ to be a sequence of $1/n$-balls centred on x, we can choose $x_n \in U_n$ for each n so that $x_n \to x$ with $F(x_n) \cap V = \varnothing$ for each n. Now any sequence $\{y_n\}$ with $y_n \in F(x_n)$ for each n is contained in the closed set V^c, and so, if convergent, converges in V^c. Hence $\{y_n\}$ cannot converge to any $y \in V \cap F(x)$, yielding a contradiction.

□

Definition 82.

(a) *Graph.* Given a correspondence $F : X \rightrightarrows Y$ or function $F : X \to Y$, the *graph* of F is the set $G(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

(b) *Closed graph property.* Let X and Y be topological vector spaces. We say that a correspondence $F : X \rightrightarrows Y$ has a *closed graph* if $G(F)$ is a closed subset of $X \times Y$.

Theorem 37 (Closed graph theorem)**.** *Suppose X and Y are topological spaces and Y is also a compact Hausdorff space. Then a correspondence $F : X \rightrightarrows Y$ has a closed graph iff F is upper hemicontinuous and closed-valued.*

Proof.

Lemma 19. *If X, Y are topological spaces with Y Hausdorff, and if $F : X \rightrightarrows Y$ is an upper hemicontinuous correspondence, and if F is compact-valued, then F has a closed graph.*

Proof. Suppose $(x, y) \notin G(F)$, i.e. $y \notin F(x)$. Since Y is Hausdorff and F is compact-valued, there exist neighbourhoods V of y and W of $F(x)$ s.t. V and W are disjoint. Define the upper inverse of F by $F^u(A) = \{x \in X \mid F(x) \subseteq A\}$. Now $U = F^u(W)$ is open, and thus $U \times V$ is a neighbourhood of (x, y) and $U \times V$ is disjoint of $G(F)$. Thus $G(F)$ is closed. □

Now since closed subsets of compact sets are compact, it follows that if F is closed-valued into a compact space then it is compact-valued. Thus if F is closed-valued and upper hemicontinuous, it has a closed graph by the lemma.

For the converse, see the proof of Theorem 17.11 in Aliprentis (2005). □

Theorem 38 (Berge's theorem of the maximum)**.** *Suppose $f : X \times Y \to \mathbb{R}$ is a continuous function, where X and Y are metric spaces with Y compact. Then*

(i) *the function* $g : X \to \mathbb{R}$ *defined by*

$$g(x) = \max_{y \in Y} f(x, y)$$

is continuous, and

(ii) *the correspondence* $F : X \rightrightarrows Y$ *defined by*

$$F(x) = \arg\max_{y \in Y} f(x, y)$$

is nonempty-valued and has a closed graph.

Proof. See Lemmas 17.29, 17.30, and the proof of Theorem 17.31 in Aliprentis (2005). □

7.2 Fixed point theorems

There are many fixed point theorems. We only state the most useful here. The proof of Theorem 7 relied on Kakutani's fixed point theorem. Nash's original proof of the existence of Nash equilibrium relied on Brouwer's fixed point theorem.

Recall a function $f : X \to X$ has a *fixed point* $x^* \in X$ if $f(x^*) = x^*$. The most elementary of the fixed point theorems is as follows:

Theorem 39. *Suppose* $f : [0, 1] \to [0, 1]$ *is a continuous function. Then* f *has a fixed point.*

Proof. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - x$. Clearly, g is continuous. If $g(0) = 0$ then $f(0) = 0$; if $g(1) = 0$ then $f(1) = 1$. In either of the above cases we are done. If neither of these two cases hold then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. By the intermediate value theorem, we have that there is some $x \in [0, 1]$ s.t. $g(x) = 0$, which implies $f(x) = x$. □

The above theorem is a special case of a more general, and much more harder to prove, result:

Theorem 40 (Brouwer's fixed point theorem)**.** *Let* $K \subseteq \mathbb{R}^n$ *be a nonempty compact convex subset, and suppose* $f : K \to K$ *is a continuous function. Then* f *has a fixed point.*

Kakutani's fixed point theorem generalizes Brouwer's fixed point theorem to correspondences.

Theorem 6 (Kakutani's fixed point theorem)**.** *Let* $K \subset \mathbb{R}^n$ *be a nonempty, compact, convex set and suppose* $F : K \rightrightarrows K$ *is a correspondence satisfying*

(i) $F(x)$ *is nonempty valued;*

(ii) $F(x)$ *is convex valued;*

(iii) $F(x)$ *is upper hemicontinuous.*

Then F has a fixed point.

Kakutani's fixed point theorem is sufficient for proving e.g. the Debreu-Glicksberg-Fan existence theorem (Theorem 7) and thus Nash's existence theorem (Theorem 8). It is also sufficient to prove the existence of most kinds of other equilibria we care about, under the right conditions on the strategy sets and so on.

This said, Kakutani's fixed point theorem has a generalizations that sometimes proves useful in game theory and economics generally:

Theorem 41 (Cellina's fixed point theorem)**.** *Let $K \subset \mathbb{R}^n$ be a nonempty, compact, convex set and consider a correspondence $F : K \rightrightarrows K$. If there exists a compact, convex set $L \subset \mathbb{R}^m$, a correspondence $G : K \rightrightarrows L$ having a closed graph, and a continuous function $f : K \times F \to K$ such that for each $x \in K$,*

$$F(x) = \{f(x, y) \mid y \in G(x)\},$$

then F has a fixed point.

Another useful fixed point theorem is Banach's fixed point theorem, often known as the contraction mapping theorem. It is particularly important in the theory of dynamic programming.

Definition 83 (Lipschitz continuity)**.** Let (X, ρ_X) and (Y, ρ_Y) be a metric space. We call a function $f : X \to Y$ *Lipschitz continuous* if there exists a $K > 0$ s.t.

$$\rho_Y(f(x), f(y)) \leq K\rho_X(x, y) \qquad \text{for all } x, y \in X.$$

We call any such value K a *Lipschitz constant* of f.

Definition 84 (Contraction mapping)**.** Let (X, ρ) be a metric space. We call an operator $T : X \to X$ a *contraction mapping* with *modulus* β if there exists a $\beta \in [0, 1)$ s.t.

$$\rho(Tx, Ty) \leq \beta\rho(x, y) \qquad \text{for all } x, y \in X.$$

That is, T is a contraction mapping if it has a Lipschitz constant $\beta < 1$.

A contraction mapping is simply a Lipschitz continuous function with a Lipschitz constant less than one.

Theorem 42 (Banach's fixed point theorem)**.** *Let X be a nonempty complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point x^*. Furthermore, for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n(x_0)$.*

Proof. Fix any point $x_0 \in X$, and let $x_n = T^n(x_0)$ for each $n \in \mathbb{N}$. Since T is a contraction mapping with modulus, say, $\beta < 1$, we have $\rho(x_n, x_{n+1}) \leq \beta\rho(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Applying this inequality repeatedly, we have $\rho(x_n, x_{n+1}) \leq \beta^n\rho(x_0, x_1)$ for each $n \in \mathbb{N}$. Since $\beta < 1$, this implies that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) \leq \epsilon$ for all $n, m \geq N$, and thus $\{x_n\}$ is a Cauchy sequence, so $x^* = \lim_{n\to\infty} x_n$ exists by completeness. By construction, we have $Tx^* = x^* = \lim_{n\to\infty} T^n(x_0)$.

Suppose x^*, y^* are two fixed points of T. Since T is a contraction mapping $\rho(x^*, y^*) = \rho(Tx^*, Ty^*) \leq \beta\rho(x^*, y^*)$ with $\beta < 1$. This holds iff $\rho(x^*, y^*) = \rho(Tx^*, Ty^*) = 0$. Hence $x^* = y^*$. $\square$

There is a generalization of Banach's fixed point theorem to correspondences – Nadler's fixed point theorem – though we lose uniqueness.

This requires us to think about distances between sets. Given a metric space X with metric ρ, we define the distance between a set $A \subseteq X$ and a point $x \in X$ by

$$\rho(x, A) = \inf_{y\in A} \rho(x, y).$$

We define the *Hausdorff distance* $\rho_H : 2^X \to [0, \infty)$ as

$$\rho_H(A, B) := \max\left\{\sup_{x\in A} \rho(x, B), \sup_{y\in B} \rho(y, A)\right\}$$

for any two sets $A, B \subseteq X$.[32]

Definition 85 (Contractiveness)**.** Let (X, ρ) be a metric space. We call a correspondence $F : X \rightrightarrows X$ *contractive* if there exists a $\beta \in (0, 1)$ s.t.

$$\rho_H(F(x), F(y)) \leq \beta\rho(x, y) \quad \text{for all } x, y \in X.$$

Theorem 43 (Nadler's fixed point theorem)**.** *Let X be a nonempty complete metric space and let $F : X \rightrightarrows X$ be a nonempty-valued, compact-valued contractive correspondence. Then F has a fixed point x^*.*

Proof. For any point $x \in X$ and any compact set $K \subseteq X$, we have that there exists $y \in K$ s.t. $y = \arg\min_{y'\in A} \rho(x, y')$ and thus $\rho(x, y) = \rho(x, K)$. We therefore have that for any two compact sets $A, B \subseteq X$, there exists a pair of points $x \in A, y \in B$ s.t. $\rho(x, y) = \rho_H(A, B)$. Denote such a pair of points $\hat{x}(A, B), \hat{y}(A, B)$.

Fix any pair of points $x_0, x_1 \in X$. For each $n \in \mathbb{N}$, define $x_{2n} = \hat{x}(F(x_{2n-2}), F(x_{2n-1}))$ and $x_{2n+1} = \hat{y}(F(x_{2n-2}), F(x_{2n-1}))$. Since F is compact-valued, such a pair of points

[32] This actually has a game-theoretic interpretation. It's just after New Year, and you've put on a few kilos (or so-called "pounds") over the holidays. Your opponent is an overly-paternalistic friend who is worried about your health. The opponent picks a point in any one of the sets A, B. You need to walk to the other set from the point they choose, but you've become pretty lazy over the holidays and want to walk the shortest distance possible to reach the other set (say you walk distance d, then your payoff is $-d$.) Your opponent thinks you need the exercise and so wants to maximize the distance you are forced to walk (their payoff is d). The Hausdorff distance is the most that your opponent can force you to walk. This is a zero-sum game, and so the Hausdorff distance is quite similar to the maximin (the sets are not necessarily closed though.) You could also interpret the opponent as being sadistic and taking pleasure in seeing you suffer. Your call.

always exists. This yields a sequence $\{x_n\}$, with $x_{n+1} \in F(x_{n-1})$ for each $n \in \mathbb{N}$. Since F is contractive, there is some positive $\beta < 1$ s.t. $\rho(x_{2n}, x_{2n+1}) = \rho_H(F(x_{2n-2}, x_{2n-1})) \leq \beta\rho(x_{2n-2}, x_{2n-1})$ for all $n \in \mathbb{N}$. Iterating this inequality gives $\rho(x_{2n}, x_{2n+1}) \leq \beta^n \rho(x_0, x_1)$ for each $n \in \mathbb{N}$. Since $\beta < 1$, it follows that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ s.t. $\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) \leq \epsilon$ for all $n, m \geq N$ and thus $\{x_n\}$ is a Cauchy sequence. By completeness of X, $\{x_n\}$ thus converges to a limit $x^* = \lim_{n\to\infty} x_n$. By construction, $x^* \in F(x^*)$. □

The Knaster-Tarski and Tarski fixed point theorems gives a nice result for monotonic functions. This is particularly useful because unlike the preceding fixed point theorems, we do not need continuity or 'pre-continuity' in any sense. The Knaster-Tarski fixed point theorem applies to monotone functions on partially ordered sets in which every chain has a supremum. Tarski's fixed point theorem applies to monotone functions on complete lattices. Tarski's fixed point theorem is of great importance in the matching literature and in the study of supermodular games.

Recall that in a partially ordered set L, the *meet* $x \wedge y$ of points $x, y \in L$ is the greatest lower bound (infinum) of x and y, and the *join* $x \vee y$ is the least upper bound (supremum) of x and y. This generalizes straightforwardly to subsets. Given a subset $E \subseteq L$, we denote the meet of E by $\bigwedge E$ and the join of E by $\bigvee E$.

Given a partially ordered set $(X, \leq)$, we say that a subset $C \subseteq X$ is a *chain* if C is totally ordered under $\leq$. That is, if for all $x, y \in C$, we have $x \leq y$ or $y \leq x$. *Zorn's lemma* states that if every chain in X has an upper bound, then X has a maximal element.

Definition 86 (Lattices)**.**

(a) *Semilattice.* A partially ordered set $(L, \geq)$ is a *meet-semilattice* if every pair of points $x, y \in L$ has a meet $x \wedge y$, and a *join-semilattice* if every pair of points has a join $x \vee y$.

(b) *Lattice.* A partially ordered set $(L, \geq)$ is a *lattice* if every pair of points $x, y \in L$ has a meet $x \wedge y$ and a join $x \vee y$. That is, $(L, \geq)$ is a lattice if it is both a meet-semilattice and a join-semilattice.

(c) *Sublattice.* A subset $M \subseteq L$ of a lattice $(L, \geq)$ is a *sublattice* if for any $x, y \in P$, we have $x \wedge y \in M$ and $x \vee y \in P$. That is, M is a sublattice of L if $(M, \geq)$ is a lattice.

    We say that a sublattice $M \subseteq L$ is *closed* if for every $N \subseteq M$, we have $\bigwedge N \in M$ and $\bigvee N \in M$.

(d) *Completeness.* We call a lattice L *complete* if every $E \subseteq L$ has a meet $\bigwedge E$ and a join $\bigvee E$.

Theorem 44 (Knaster-Tarski fixed point theorem)**.** *Let $(X, \geq)$ be a partially ordered set such that every chain in X has a supremum. Let $f : X \to X$ be a monotonically increasing function wrt $\geq$, and suppose there is some $a \in X$ such that $a \leq f(a)$. Then f has a fixed point and the set of fixed points of f has a maximum.*

Proof. Let $P = \{x \in X \mid x \leq f(x)\}$. Since $a \in P$, P is nonempty. Suppose C is a chain in P and $b = \bigvee C$. Since $c \leq b$ for all $c \in C$, we have $f(c) \leq f(b)$. Since $C \subseteq P$, $c \leq f(c)$ for all $c \in C$, and thus $f(b)$ is an upper bound for C. Since b is the least upper bound for C, we conclude $b \leq f(b)$, and thus $b \in P$. Hence the supremum of any chain in P lies in P, and so by Zorn's lemma, P has a maximal element, say x_0.

Since $x_0 \in P$, $x_0 \leq f(x_0)$, and since f is monotonically increasing, $f(x_0) \leq f(f(x_0))$. Thus $f(x_0) \in P$. Since x_0 is maximal in P, it follows that $f(x_0) = x_0$. Hence f has a fixed point. Now, any other fixed point of f lies in P, and so x_0 is the maximum of the fixed points of f. $\square$

We could make a similar statement for monotonically decreasing functions on X if every chain in X has an infinum.

Theorem 45 (Tarski's fixed point theorem)**.** *Let $(L, \geq)$ be a complete lattice and suppose $f : L \to L$ is a monotone function wrt $\geq$. Then the set of fixed points of f is a nonempty complete lattice under $\geq$.*

Proof. Wlog, suppose f is monotonically increasing. Let $P = \{x \in L \mid x \leq f(x)\}$ and let E be the set of fixed points of f. Since $(L, \geq)$ is complete, there is some minimal element x_0 of L and so $x_0 \leq f(x_0)$, so P is nonempty. By the argument of the proof of the Knaster-Tarski theorem (Theorem 44), $\bar{x} := \bigvee P$ is a fixed point of f. Since $E \subseteq P$, we have that $\bar{x} = \bigvee E$. Let $Q = \{x \in L \mid x \geq f(x)\}$. Again Q is nonempty since L has a maximal element. A similar argument shows that $\underset{\sim}{x} := \bigwedge Q$ is a fixed point of f, and since $E \subseteq Q$, $\underset{\sim}{x} = \bigwedge E$.

Now we claim $(E, \geq)$ is a complete lattice. Fix any nonempty subset $A \subseteq E$, and let $\bar{a}$ be the supremum of A in L. Let I denote the order interval $I = [\bar{a}, 1] := \{x \in L \mid \bar{a} \leq x\}$. Then I is a complete lattice. To see this, consider any subset $B \subseteq I$. Since L is a complete lattice, B has a supremum and infinum in L. Since $B \subseteq I$, $x \geq \bar{a}$ for all $x \in B$, so $\bar{a}$ is a lower bound of B. Hence $\bigwedge B \geq \bar{a}$ so $\bigwedge B \in I$, and so also $\bigvee B \in I$.

We claim that $f(I) \subseteq I$. If $x \in A$, then $x \leq \bar{a}$, and thus $f(x) \leq f(\bar{a})$. Yet since $A \subseteq F$, $x = f(x)$, and so $x \leq f(\bar{a})$. Thus $f(\bar{a})$ is an upper bound of A, so $\bar{a} \leq f(\bar{a})$. Since $z \geq \bar{a}$ for all $z \in I$, we have that $\bar{a} \leq f(\bar{a}) \leq f(z)$, and thus $f(z) \in I$ for all $z \in I$.

Now let $\hat{f}$ be the restriction of f to I, and let $\hat{E}$ be the set of fixed points of $\hat{f}$. Since I is a complete lattice, $\hat{E}$ is nonempty. From the above, $\underset{\sim}{z} := \bigwedge \hat{E}$ is a fixed point of $\hat{f}$, and therefore of f. Since $\underset{\sim}{z} \in I$, it is an upper bound of A lying in E. Moreover, it is the least upper bound of A, since $\underset{\sim}{z}$ is the infinum of $\hat{E}$, and thus for any other upper bound $b \in I$ of A such that b is a fixed point of f, we have that $\underset{\sim}{z} \leq b$. Hence A has a least upper bound in E. A similar argument shows it has a greatest lower bound in E. Thus E is a complete lattice. $\square$

Finally, there is a generalization of Tarski's fixed point theorem to correspondences, due to Zhou (1994).

Definition 87 (Ascending and descending correspondences)**.** Given partially ordered sets X and Y, we say a correspondence $F : X \rightrightarrows Y$ is *ascending* if for any $x, x' \in X$

such that $x \geq x'$, for any $y \in F(x)$ and any $y' \in F(x')$, we have that $y \vee y' \in F(x)$ and $y \wedge y' \in F(y)$.

Conversely, we say a correspondence $F : X \rightrightarrows Y$ is *descending* if for any $x, x' \in X$ such that $x \geq x'$, for any $y \in F(x)$ and any $y' \in F(x')$, we have that $y \wedge y' \in F(x)$ and $y \vee y' \in F(y)$.

To see the relation with monotonicity, consider any $x \geq x'$ and let $y \in F(x)$. If F is ascending then there is some $y' \in F(x')$ such that $y \geq y'$, whereas if F is descending, then there is some $y' \in F(x')$ such that $y \leq y'$.

Theorem 46 (Zhou's fixed point theorem)**.** *Let $(L, \geq)$ be a complete lattice and suppose $F : L \rightrightarrows L$ is an ascending or descending correspondence wrt $\geq$. If $F(x)$ is a closed sublattice for every $x \in L$, then the set of fixed points of F is a nonempty complete lattice under $\geq$.*

Proof. Wlog, suppose F is ascending. Let E denote the set of fixed points of F. Let $P = \{x \in L \mid \text{there exists } y \in F(x) \text{ s.t. } y \geq x\}$. Since $(L, \geq)$ is complete, it has some minimal element x_0 and $x_0 \leq y$ for all $y \in F(x_0)$, in view of the ascending property. Let $\bar{x} := \bigvee P$. Now, for any $x \in P$, there exists a $y_x \in F(x)$ with $x \leq y_x$. Since F is ascending, and $x \leq \bar{x}$, there exists a $z_x \in F(\bar{x})$ such that $x \leq y_x \leq z_x$. Define $z := \bigvee\{z_x \mid x \in P\}$. Then $\bar{x} \leq z$, because $\bar{x} = \bigvee E \leq \bigvee\{z_x \mid x \in P\} = z$, given $x \leq z_x$ for all $x \in P$. Since $F(\bar{x})$ is a closed sublattice, $z \in F(\bar{x})$. Since F is ascending, there is some $t \in F(z)$ s.t. $t \geq z$ and so $z \in P$. Since $\bar{x}$ is the supremum of P, $z \leq \bar{x}$. Hence $\bar{x} = z \in F(\bar{x})$, so $\bar{x}$ is a fixed point of F. Moreover, since $E \subseteq P$, $\bar{x} = \bigvee E$.

Let $Q = \{x \in L \mid \text{there exists } y \in F(x) \text{ s.t. } y \leq x\}$. Again, this is nonempty because L has some maximal element. Now a similar argument to the above shows $\underline{x} := \bigwedge Q$ is a fixed point of F and since $E \subseteq Q$, $\underline{x} = \bigwedge E$.

Finally, we claim E is a complete lattice. Fix any nonempty subset $A \subseteq E$, and let $\bar{a}$ be the supremum of A in L. For any $x \in A$, since $x \in F(x)$ and F is ascending, there exists $y_x \in F(\bar{a})$ s.t. $y_x \geq x$. Let $y = \bigwedge\{y_x \mid x \in A\}$. Then $y \geq \bar{a}$, and $y \in F(\bar{a})$ since $F(\bar{a})$ is a closed sublattice of L. Since F is ascending, there exists an $x_t \in F(t)$ s.t. $x_t \geq \bar{a}$ for all $t \geq a$. Let I denote the order interval $I = [\bar{a}, 1] := \{x \in L \mid \bar{a} \leq x\}$, and let $\hat{F}$ be defined on I by $\hat{F}(t) := F(t) \cap I$ for all $t \in I$. Then by the above, $\hat{F}(t)$ is nonempty for all $t \in I$. Since for any $t \in I$, $F(t)$ and I are both closed sublattices of L, we have that $\hat{F}(t)$ is a closed sublattice of I. Define $G(t) = I$ for all $t \in I$. Then G is an ascending correspondence. Since $\hat{F}(t) = F(t) \cap G(t)$ for all $t \in I$ and F and G are ascending, $\hat{F}$ is ascending. Let $\hat{E}$ be the set of fixed points of $\hat{F}$. By the argument of the preceding paragraphs, $\hat{E}$ is nonempty, and $\underline{z} := \bigwedge \hat{E}$ is a fixed point of $\hat{F}$, and thus of F. Since $\underline{z} \in I$, it is an upper bound of A lying in E, and indeed it is the least upper bound of A, since it is the infinum of $\hat{E}$, so A has a least upper bound in E. A similar argument shows it has a greatest lower bound in E. Thus E is a complete lattice. $\square$

7.3 Linear programming results

Linear programming lies at the heart of much game theory. Indeed, the early history of game theory is effectively a history of linear programming, as von Neumann's minmax theorem illustrates.

7.3.1 Hyperplanes, convex sets and extreme points

Definition 88 (Hyperplane)**.**

(a) *Hyperplane.* Given a point $p \in \mathbb{R}^n - \{0\}$ and a scalar $c \in \mathbb{R}$, the *hyperplane generated by p and c* is the set

$$H_{p,c} = \{x \in \mathbb{R}^n \mid p \cdot x = c\}.$$

(b) *Half-space.* Given a hyperplane $H_{p,c}$, the sets

$$\{x \in \mathbb{R}^n \mid p \cdot x \geq c\} \qquad \text{and} \qquad \{x \in \mathbb{R}^n \mid p \cdot x \leq c\}$$

are called, respectively, the (closed) *half-spaces* above and below $H_{p,c}$.

(c) *Supporting hyperplane.* Given a set $S \subseteq \mathbb{R}^n$, we say that a hyperplane H is a *supporting hyperplane* of S if

(i) S is contained in either the half-space above or the half-space below H, and

(ii) S has at least one boundary point on H.

There are several variations of the separating hyperplane theorem.

Theorem 47 (Separating hyperplane theorem I)**.** *Let $A \subseteq \mathbb{R}^n$ be a closed convex set and let $x \in \mathbb{R}^n - A$. Then there exists a $y \in \mathbb{R}^n$ with $y \cdot z > y \cdot x$ for all $z \in A$.*

Proof. If A is empty the theorem holds vacuously. Hence suppose A is nonempty.

First note the following lemma:

Lemma 20. *Let C be a (nonempty) closed convex subset of $\mathbb{R}^n$. Then C contains a unique vector of minimum norm.*

Proof. Define $\delta = \inf\{\|x\| \mid x \in C\}$. Consider any sequence $\{x_i\}$ s.t. $x_i \in C$ for all i and $\|x_i\| \to \delta$. Since C is convex, $\frac{x_i + x_j}{2} \in C$ for any i, j. Thus $\left\|x_i + x_j\right\|^2 \geq 4\delta^2$. Now,

$$\begin{aligned} \left\|x_i - x_j\right\|^2 &= \|x_i\|^2 + \left\|x_j\right\|^2 - 2x_i \cdot x_j \\ &= 2\|x_i\|^2 + 2\left\|x_j\right\|^2 - \left\|x_i + x_j\right\| \\ &\leq \left\|x_i^2\right\| + \left\|x_j\right\|^2 - 4\delta^2 \to 0. \end{aligned}$$

Thus $\{x_i\}$ is a Cauchy sequence, so $x := \lim_{i \to \infty} x_i \in C$. Hence C contains a vector of minimum norm.

For uniqueness, suppose $y \in C$ is also s.t. $\|y\| = \delta$. Then $\|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2 = 0$, so $x = y$. $\square$

By the lemma, there exists some $z' \in A$ s.t. $0 < \|x - z'\| \leq \|x - z\|$ for all $z \in A$.

Let $y = z' - x$ and fix $z \in A$. For any $\alpha \in [0, 1]$, we have that $z' + \alpha(z - z') \in C$ by the convexity of A. Hence

$$\|z' + \alpha(z - z') - x\|^2 \geq \|z' - x\|^2 .$$

It follows that

$$2\alpha(z' - x) \cdot (z - z') + \alpha^2\|z - z'\|^2 \geq 0.$$

Dividing through by 2α and taking limits as $\alpha \to 0$, we see that $(z' - x) \cdot (z - z') \geq 0$. Hence $(z' - x) \cdot z \geq (z' - x) \cdot z' = (z' - x) \cdot x + (z' - x) \cdot (z' - x) > (z' - x) \cdot x$. Since z is arbitrary, the theorem follows. □

Corollary 9. *In the context of Theorem 47, there exists a real number α s.t. $y \cdot z > \alpha$ and $y \cdot x < \alpha$ and a real number β s.t. $y \cdot z > \beta$ and $y \cdot x = \beta$.*

Proof. From the proof of the theorem, $y \cdot z \geq y \cdot z'$ for all $z \in A$, so $y \cdot z'$ is a lower bound on $y \cdot z$. Taking $\alpha = \frac{1}{2}(y \cdot z' + y \cdot x)$, we have $y \cdot z > \alpha$, and since $y \cdot z' > y \cdot x$, we have $y \cdot x < \alpha$. The assertion involving β is trivial. □

Theorem 48 (Separating hyperplane theorem II)**.** *Let A and B be two disjoint convex subsets of $\mathbb{R}^n$. Then there exists a vector $y \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$ s.t.*

$$x \cdot y \geq c \qquad \textit{and} \qquad z \cdot y \leq c$$

for all $x \in A$ and all $z \in B$.

Proof. Define

$$C = \{x - z \mid x \in A,\ \ z \in B\},$$

i.e. C is the Minkowski sum $C = A + (-B)$. Since $-B$ is convex and Minkowski sums of convex sets are convex, C is convex. The closure $\bar{C}$ of C is also convex. By Lemma 20, $\bar{C}$ contains a (unique) vector y of minimum norm. By convexity of $\bar{C}$, for any $v \in C$ we have

$$y + t(v - y) \in \bar{C}$$

for all $t \in [0, 1]$. Thus

$$\|y\|^2 \leq \|y + t(v - y)\|^2 = \|y\|^2 + 2ty \cdot (v - y) + t^2\|v - y\|^2 .$$

If $t \in (0, 1]$,

$$0 \leq 2y \cdot v - 2\|y\|^2 + t\|v - y\|^2 .$$

Taking $t \to 0$ gives $\|y\|^2 \leq y \cdot v$. Thus $(x - z) \cdot y \geq \|y\|^2$ for any $x \in A$ and $z \in B$. Since $(x - z) \cdot y \geq \inf_{x \in A, z \in B}\{(x - z) \cdot y\} \geq \|y\|^2$ and $\inf_{x \in A, z \in B}\{(x - z) \cdot y\} = \inf_{x \in A} x \cdot y - \sup_{z \in B} z \cdot y$, we have $\inf_{x \in A} x \cdot y \geq \|y\|^2 + \sup_{z \in B} z \cdot y$. Proof follows provided $y \neq 0$.

Extending the argument to the general case, suppose the interior C° of C is nonempty. Construct a sequence of nonempty compact convex sets $\{C_i\}$ as follows:

$$C_i = [-i, i]^n \cap \left\{ x \in C^\circ \mid \|x - x'\| \geq \frac{1}{i} \text{ for all } x' \in (C^\circ)^c \right\}.$$

Then $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$, each $C_i \subseteq C^\circ$, and $\bigcup_{i=1}^\infty C_i = C^\circ$. Now $0 \notin C^\circ$, and thus $0 \notin C_i$ for any i. Hence each C_i contains a nonzero vector y_i of minimum norm, by Lemma 20. By the argument of the preceding part, $x \cdot y_i \geq 0$ for all $x \in C_i$. Normalize each y_i s.t. $\|y_i\| = 1$. Letting $S = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, we have $y_i \in S$ for each S. Since S is compact, $\{y_i\}$ contains a convergent subsequence $\{y_{i_k}\}$. Let $y = \lim_{k\to\infty} y_{i_k}$ and note $y \neq 0$. Now, $x \cdot y \geq 0$ for all $x \in C^\circ$ and, by continuity, for all $x \in C$. Proof follows by the previous arguments.

If the interior of C is empty, then the affine set spanning C has dimension strictly less than n. Thus there is some hyperplane $H_{c,y}$ s.t. $C \subseteq H_{c,y}$. Then $x \cdot y \geq c$ for all $x \in C$. The remainder of the proof precedes as above. □

A related result is the supporting hyperplane theorem:

Theorem 49 (Supporting hyperplane theorem)**.** *Suppose $A \subseteq \mathbb{R}^n$ is a convex set and $x \notin A^\circ$. Then there exists a $y \in \mathbb{R}^n$ with $y \neq 0$ s.t. $y \cdot x \geq y \cdot z$ for all $z \in A$.*

Proof. If A is empty the theorem holds vacuously. Hence suppose $A \neq \varnothing$. Suppose $x \notin A$. There exists a sequence $\{x_i\}$ s.t. $x_i \to x$ and $x_i \notin \bar{A}$ for all i. By the separating hyperplane theorem (Theorem 47), for each i there exists a $y_i \in \mathbb{R}^i$ $y_i \neq 0$ and a $c_i \in \mathbb{R}$ s.t.

$$y_i \cdot x_i > c_i \geq y_i \cdot z$$

for all $z \in A$. Normalize each y_i s.t. $\|y_i\| = 1$. Let $S = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Then each $y_i \in S$ and S is compact, so $\{y_i\}$ contains a convergent subsequence $\{y_{i_k}\}$ with $\lim_{k\to\infty} y_{i_k} =: y \neq 0$ and $\lim_{k\to\infty} c_{i_k} =: c$. Then

$$y \cdot x \geq c \geq y \cdot z$$

for all $z \in A$. □

An extreme point of a convex set is a point that is not a convex combination of two distinct points in that set:

Definition 89 (Extreme point)**.** Given a convex set S in a linear space V, a vector $x \in S$ is called an *extreme point* if there are no distinct points $y, z \in S$ and no constant $\lambda \in (0, 1)$ s.t. $x = \lambda y + (1 - \lambda)z$.

We denote the set of extreme points of S by $\text{ext}(S)$.

There are several equivalent definitions:

Theorem 50. *Let S be a convex set in a linear space V. Then the following are equivalent:*

(i) $x \in \mathrm{ext}(S)$;

(ii) *for all* $y, z \in S$ *s.t.* $x = \frac{1}{2}(y+z)$, *we have that* $x = y = z$;

(iii) $S - \{x\}$ *is convex.*

Proof. Clearly (i) immediately implies (ii). Conversely, suppose x satisfies (ii) but there exists a constant $\lambda \in (0,1)$ s.t. $x = \lambda y + (1-\lambda)z$ for some distinct $y, z \in S$. If $\lambda > \frac{1}{2}$, let $\alpha = 2\lambda - 1$. Define $z' = \alpha y + (1-\alpha)z$. Then we have $x = \frac{1}{2}(y + z')$. By convexity of S, $z' \in S$, so we have a contradiction. Similarly, if $\lambda < \frac{1}{2}$, let $\alpha = 2\lambda$ and define $y' = \alpha y + z$. We have $x = \frac{1}{2}(y' + z)$. Likewise, $y' \in S$, yielding a contradiction. Hence (ii) implies (i).

Suppose x satisfies (i) but $S - \{x\}$ is not convex. Then there must be some $y, z \in S - \{x\}$ and some $\lambda \in [0,1]$ s.t. $\lambda y + (1-\lambda)z \notin S - \{x\}$. Yet by convexity of S, $\lambda y + (1-\lambda)z \in S$. Hence we must have $x = \lambda y + (1-\lambda)z$, but this contradicts (i) [noting $\lambda \in (0,1)$ since $y, z \neq x$]. Thus (i) implies (iii). Conversely, suppose $S - \{x\}$ is convex. Suppose there is some $\lambda \in (0,1)$ and distinct points $y, z \in S$ s.t. $x = \lambda y + (1-\lambda)z$. Then $y, z \neq x$, so $y, z \in S - \{x\}$, but convexity of $S - \{x\}$ would then imply $x \in S - \{x\}$, yielding a contradiction. Hence (iii) implies (i). □

Theorem 51 (Minkowski-Carathéodory)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex compact set. Then every $x \in K$ can be expressed as a convex combination of at most $n+1$ extreme points of K.*

Proof. We proceed by induction. Suppose the theorem holds for any convex compact set in $\mathbb{R}^{n-1}$. Let $K \subseteq \mathbb{R}^n$ be a convex compact set, and fix $x \in K$. Now we can choose some extreme point $y \in K$ and some boundary point $z \in K$ s.t. $x = (1-\lambda)y + \lambda z$ for some $\lambda \in [0,1]$. By the supporting hyperplane theorem, K has a supporting hyperplane H at z. Now $H \cap K \subseteq H$ is a convex compact set, and H has dimension $n-1$. Furthermore, the extreme points of $H \cap K$ are extreme points of K. By hypothesis, we can represent z as a convex combination of at most n extreme points $z_1, \ldots, z_n$, i.e. $z = \sum_{i=1}^n c_i z_i$ for some scalars $c_i \in [0,1]$ with $\sum_{i=1}^n c_i = 1$. Since x is a convex combination of z and y, it follows that x is a convex combination of at most $n+1$ extreme points of K.

To complete the proof, note that the induction hypothesis holds for $n = 1$, since any convex set in $\mathbb{R}$ is an interval $[a,b]$, which has 2 extreme points, a and b. For any $x \in [a,b]$, we have $x = (1-\lambda)a + \lambda b$ for $\lambda = \frac{x-a}{b-a}$, so any $x \in [a,b]$ is a convex combination of at most 2 extreme points. □

The Krein-Milman theorem establishes that any compact convex subset of an Euclidean space is equal to the (closed) convex hull of its extreme points.[33] To prove this, we first need a separation lemma:

Lemma 21. *Let C be a nonempty convex subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n - C^\circ$. Then there exists a vector $y \in \mathbb{R}^n - \{0\}$ s.t. $y \cdot x \leq y \cdot z$ for all $z \in C$.*

[33] The theorem actually holds more generally for Hausdorff locally convex topological vector spaces.

Proof. First, suppose $x \notin \bar{C}$. Then the lemma follows immediately from the separating hyperplane theorem, Theorem 47, taking $A = \bar{C}$.

Second, suppose $x \in \bar{C}$. Since $x \notin C^\circ$, there is a sequence $\{x_i\}$ with each $x_i \in \mathbb{R}^n - \bar{C}$ and $x = \lim_{i\to\infty} x_i$. Applying Theorem 47, for each i there exists a $y_i \in \mathbb{R}^n - \{0\}$ s.t. $y_i \cdot x_i \leq y_i \cdot z$ for all $z \in \bar{C}$. Normalize each y_i so that $\|y_i\| = 1$. Since the closed ball $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is compact, $\{y_i\}$ has some convergent subsequence $\{y_{i_k}\}$ with limit $y = \lim_{k\to\infty} y_{i_k}$. We have $y \cdot x = \lim_{k\to\infty} y_{i_k} \cdot x \leq \lim_{k\to\infty} y_{i_k} \cdot z = y \cdot z$ for all $z \in \bar{C}$. □

Theorem 52 (Krein-Milman)**.** *Let K be a nonempty compact convex set in $\mathbb{R}^n$. Then $\mathrm{ext}(K)$ is nonempty and $K = \mathrm{co}(\mathrm{ext}(K))$.*

Proof. Since K is compact and $\rho : x \mapsto \|x\|$ is continuous, it follows by the extreme value theorem that ρ attains a maximum on K. Let $x \in \arg\max_{y\in K}\|y\|$. Then $x \in \mathrm{ext}(K)$. For suppose that $x = \frac{1}{2}(x_1 + x_2)$ for some $x_1, x_2 \in K$. Now

$$\|x\| = \left\|\frac{1}{2}(x_1 + x_2)\right\| \leq \frac{1}{2}\|x_1\| + \frac{1}{2}\|x_2\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|x\|,$$

so $\|x_1\| = \|x_2\| = \left\|\frac{1}{2}(x_1 + x_2)\right\|$. By definition of the Euclidean norm, it follows that $x_1 = x_2 = x$, and thus $x \in \mathrm{ext}(K)$. Thus $\mathrm{ext}(K)$ is nonempty.

Next, we show that $K = \mathrm{co}(\mathrm{ext}(K))$ by induction. First, note that if $\dim K = 0$ then $K = \{x\}$ for some $x \in \mathbb{R}^n$, and thus $\mathrm{ext}(K) = \{x\}$ and $\mathrm{co}(\mathrm{ext}(K)) = \{x\} = K$.

Now fix an integer k and suppose that $\mathrm{co}(\mathrm{ext}(A)) = A$ for all nonempty compact convex $A \subseteq \mathbb{R}^n$ for which $\dim A < k$. Suppose $\dim K = k$. Clearly $\mathrm{co}(\mathrm{ext}(K)) \subseteq K$, so we need only prove $K \subseteq \mathrm{co}(\mathrm{ext}(K))$. Wlog, assume $0 \in K$, and let $W = \bigcap\{E \subseteq \mathbb{R}^n \mid E \text{is affine and } K \subseteq E\}$. Then $\dim W = k$. Since $\mathrm{ext}(K)$ is nonempty, there is some $x \in \mathrm{ext}(K)$. Fix $y \in K$. If $y = x$ then $y \in \mathrm{co}(ext(K))$. Hence suppose that $y \neq x$. Let L be the line through x and y. Now $L' = L \cap K$ is a line segment, with one endpoint being x and the other endpoint being some boundary point b of K. By Lemma 21, there is a linear function $f : W \to \mathbb{R}$ s.t. $f(b) = \min\{f(z) \mid z \in K\}$ and $f \neq 0$. In particular, the lemma shows there is some nonzero vector y so that $f(z) = y \cdot z$ for $z \in W$ has these properties. Now let $A := \{z \in K \mid f(z) = f(b)\}$. Then A is a compact convex subset of K. Note $\mathrm{ext}(A) \subseteq \mathrm{ext}(K)$. Since $f \neq 0$, we have that $\dim A < k$. By the induction hypothesis, $A = \mathrm{co}(\mathrm{ext}(A))$. Thus $b \in A = \mathrm{co}(\mathrm{ext}(A)) \subseteq \mathrm{co}(\mathrm{ext}(K))$. Since $x \in \mathrm{ext}(K)$ and $y \in \mathrm{co}\{x, b\}$, it follows that $y \in \mathrm{co}(\mathrm{ext}(K))$, and thus $K \subseteq \mathrm{co}(\mathrm{ext}(K))$. □

Definition 90.

(a) *Stochastic matrices.* We call an $n \times n$ real matrix D

- (i) *right-stochastic* if $d_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{j=1}^n d_{ij} = 1$ for each row i;
- (ii) *left-stochastic* if $d_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^n d_{ij} = 1$ for each column j;
- (iii) *doubly stochastic* if it is both right-stochastic and left-stochastic.

(b) *Permutation matrix.* If an $n \times n$ real matrix P is doubly stochastic and each entry $p_{ij} \in \{0, 1\}$, we say that P is a *permutation matrix.*

Theorem 53 (Birkhoff-von Neumann)**.** *Let $\mathcal{D}_{n\times n}$ denote the set of all $n \times n$ doubly stochastic matrices and let $\mathcal{P}_{n\times n}$ denote the set of all $n \times n$ permutation matrices. Then*

(a) $\text{ext}(\mathcal{D}_{n\times n}) = \mathcal{P}_{n\times n}$ *and*

(b) $\mathcal{D}_{n\times n} = \text{co}(\mathcal{P}_{n\times n})$.

Proof.

(a) First, we show $\text{ext}(\mathcal{D}_{n\times n}) \subseteq \mathcal{P}_{n\times n}$. Let P be an $n \times n$ permutation matrix s.t. $P = \frac{1}{2}(A + B)$ for some matrices $A, B \in \mathcal{D}_{n\times n}$, with ijth entry p_{ij}. We have $p_{ij} = \frac{1}{2}(a_{ij} + b_{ij})$ and $p_{ij} \in \{0, 1\}$. If $p_{ij} = 0$ then since $a_{ij}, b_{ij} \geq 0$ we must have $a_{ij} = b_{ij} = 0$. If $p_{ij} = 1$ then since $a_{ij}, b_{ij} \leq 1$, we must have $a_{ij} = b_{ij} = 1$. Thus $A = B$, and so $P \in \text{ext}(\mathcal{D}_{n\times n})$.

Next, let $D \in \mathcal{D}_{n\times n}$ be s.t. D is not a permutation matrix, with ijth entry d_{ij}. We claim D is not an extreme point. Now, D is not an extreme point if there exists an $n \times n$ matrix $C \neq 0$ s.t. (i) $c_{ij} = 0$ if $d_{ij} \in \{0, 1\}$, (ii) $\sum_{i=1}^{n} c_{ij} = 0$ for all j s.t. $d_{ij} \neq 1$ for all i, and (iii) $\sum_{j=1}^{n} c_{ij} = 0$ for all i s.t. $d_{ij} \neq 1$ for all j. If such a matrix C exists, then we have that $D + \epsilon C$ and $D - \epsilon C$ are distinct doubly stochastic matrices for sufficiently small $\epsilon > 0$, and $D = \frac{1}{2}([D + \epsilon C] + [D - \epsilon C])$.

Clearly, for any row or column of D containing a 1, the corresponding row or column of C must contain only 0s. Now, since D is not a permutation matrix, there are $k \geq 2$ rows (and columns) of D that do not contain an entry 1. In each such row, there are at least $2k$ entries d_{ij} s.t. $d_{ij} \notin \{0, 1\}$. The corresponding elements of C must be chosen to satisfy the system of $2k$ homogeneous linear equations described by (ii) and (iii). We can, wlog, assume these equations correspond to the first k rows and columns. If $\sum_{j=1}^{k} c_{ij} = 0$ for each $i \in \{1, \ldots, k-1\}$ and $\sum_{i=1}^{k} c_{ij} = 0$ for all $j \in \{1, \ldots, k\}$, then $\sum_{j=1}^{k} c_{kj} = 0$, so the last equation is redundant. Hence the system of equations has fewer than $2k$ independent equations and weakly more than $2k$ variables, so has a nonzero solution, C. Thus we have constructed a matrix C satisfying (i)-(iii).

(b) Fix any $A, B \in \mathcal{D}_{n\times n}$ and any $\lambda \in [0, 1]$. Let $D = (1-\lambda)A + \lambda B$. Since $a_{ij}, b_{ij} \geq 0$ for all i, j, we have $d_{ij} \geq 0$ for all i, j. For each row i, $\sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n}[(1-\lambda)a_{ij} + \lambda b_{ij}] = (1-\lambda)\sum_{j=1}^{n} a_{ij} + \lambda \sum_{j=1}^{n} b_{ij} = 1$, and for each column j, $\sum_{i=1}^{n} d_{ij} = (1-\lambda)\sum_{i=1}^{n} a_{ij} + \lambda \sum_{i=1}^{n} b_{ij} = 1$. Hence $D \in \mathcal{D}_{n\times n}$, so $\mathcal{D}_{n\times n}$ is convex.

Clearly, $\mathcal{D}_{n\times n}$ is bounded since $0 \leq d_{ij} \leq 1$ for each i, j for any $D \in \mathcal{D}_{n\times n}$. Now let A be an accumulation point of $\mathcal{D}_{n\times n}$. Consider any sequence $\{\epsilon^k\}$ s.t. $\epsilon^k > 0$ for all k and $\lim_{k\to\infty} \epsilon^k = 0$. Since A is an accumulation point, we can construct a sequence $\{D^k\}$ s.t. for each k, $D^k \in \mathcal{D}_{n\times n}$ and $\left\| D^k - A \right\| < \epsilon^k$. Then $A = \lim_{k\to\infty} D^k$. Since $d_{ij}^k \geq 0$ for all i, j and all k, we have that $a_{ij} = \lim_{k\to\infty} d_{ij}^k \geq 0$.

Since $\sum_{j=1}^n d_{ij}^k = 1$ for all i and all k, $\sum_{j=1}^n a_{ij} = \lim_{k\to\infty} \sum_{j=1}^n d_{ij}^k = 1$ for all i. Likewise, $\sum_{i=1}^n a_{ij} = \lim_{k\to\infty} \sum_{i=1}^n d_{ij}^k = 1$ for all j. Thus $A \in \mathcal{D}_{n\times n}$. Since $\mathcal{D}_{n\times n}$ contains all its accumulation points, it is closed. Applying the Heine-Borel theorem, we see $\mathcal{D}_{n\times n}$ is compact. Proof of (b) now follows by application of the Krein-Milman theorem.

□

7.3.2 Lemmas of the alternative

Lemmas of the alternative are lemmas that describe two systems of linear equations, precisely one of which has a solution.

Lemma 22 (Lemma of the alternative for matrices)**.** *Let A be an $m \times n$ matrix. Then exactly one of the following statements holds:*

(a) *There exist $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ s.t. $(y, z) \geq 0$, $(y, z) \neq 0$ and $Ay + z = 0$;*

(b) *There is an $x \in \mathbb{R}^m$ s.t. $x > 0$ and $x'A > 0$.*

Proof. First suppose both (a) and (b) hold. Since $Ay+z = 0$, we have that $x'(Ay+z) = x'Ay + x \cdot z = 0$. Since $x'A > 0$ and $y \geq 0$, we have $x'Ay = 0$ iff $y = 0$. Likewise, since $x > 0$ and $z \geq 0$, we have $x \cdot z = 0$ iff $z = 0$. Hence $x'(Ay + z) = 0$ iff $(y, z) = 0$, yielding a contradiction. Hence at most one of (a) and (b) can hold.

Let A_j be the jth column of A and let $e^1, \ldots, e^m \in \mathbb{R}^m$ be the standard basis vectors of $\mathbb{R}^m$. Let $Z = \text{co}(A_1, \ldots, A_n, e^1, \ldots, e^m)$. In (a), dividing $Ay + z = 0$ by $\sum_{i=1}^n y_i + \sum_{j=1}^m z_j$, we see that 0 is a convex combination of the vectors in Z.

Suppose (a) does not hold. We need only prove that (b) must then have a solution. Since (a) does not hold, $0 \notin Z$. By Theorem 47 and Corollary 9, there exists $x \in \mathbb{R}^m$ and a number $\beta \in \mathbb{R}$ such that $x \cdot p > \beta$ for all $p \in Z$ and $x \cdot 0 = \beta$. Thus $\beta = 0$ and so $x'A > 0$ and $x > 0$ given the columns of A and all e^j all lie in Z. □

The most well-known lemma of the alternative is Farkas' lemma. Proof requires the following:

Lemma 23. *Let A be an $m \times n$ matrix and let*

$$C := \{c \in \mathbb{R}^n \mid \text{ there exists an } x \in \mathbb{R}^m, x \geq 0 \text{ s.t. } x'A = c\}.$$

Then C is closed.

Proof. Suppose $\operatorname{rank} A = m$. Then $\operatorname{rank}(AA') = m$.[34] Hence AA' is invertible. Let $\{c^n\}$ be a sequence in C s.t. $\lim_{n\to\infty} c^n = c$, and let $c^n = x^{n\prime}A$ with $x^n \geq 0$ for all n. Since

[34] If an $m \times n$ matrix A has $\operatorname{rank} A = k$ then $\operatorname{rank}(AA') = k$. The kernel of AA' is $\{x \in \mathbb{R}^m \mid AA'x = 0\}$, i.e. the set of $x \in \mathbb{R}^m$ s.t. $\sum_{j=1}^m (\sum_{k=1}^m a_{ik}a_{jk})x_j = 0$ for all i. Follows that $\sum_{i=1}^m x_i \sum_{j=1}^m (\sum_{k=1}^m a_{ik}a_{jk})x_j = 0$ and thus $x'AA'x = x'(AA'x) = x \cdot 0 = 0$. Now, $0 = x'AA'x = (A'x)'(A'x) = \|A'x\|^2$. Hence x must be s.t. $A'x = 0$, so the kernel of AA' and A' are the same. By the rank-nullity theorem, we have that $\operatorname{rank}(AA') = \operatorname{rank}(A') = \operatorname{rank} A = k$.

$x^{n\prime} = x^{n\prime}(AA')(AA')^{-1}$ for all n, $x^{n\prime}A \to c$ implies that $x^{n\prime} \to c'A'(AA')^{-1} =: x$. In particular, $x \geq 0$. Since $x^{n\prime}A \to x'A$, we have $x'A = c$ and therefore $c \in C$.

Now fix $b \in C - \{0\}$ and choose $x \in \mathbb{R}^m$, $x \geq 0$ with $x'A = b$ s.t. $|S|$ is maximal, where $S := \{i \in \{1, \ldots, m\} \mid x_i > 0\}$. We mean to show that the rows of A indexed by $i \in S$ are linearly independent. Suppose otherwise. Then there exists $\mu \in \mathbb{R}^m$ s.t. $\mu_j \neq 0$ for some $j \in S$, $\mu_i = 0$ for all $i \notin S$, and $\mu' A = 0$. Then $(x - t\mu)'A = x'A = b$ for all $t \in \mathbb{R}$. Choose $\hat{t}$ s.t. $x_j - \hat{t}\mu_j \geq 0$ for all $j \in S$, with equality for some $j \in S$. Then $b = (x - \hat{t}\mu)'A \geq 0$ and $\left|\{i \in \{1, \ldots, m\} \mid x_i - t\mu_i > 0\}\right| \leq |S| - 1$, yielding a contradiction.

Now given this, we can write C as

$$C = \bigcup_B \{x'B \mid B \text{ is a } k \times n \text{ submatrix of } A,\ \operatorname{rank}(B) = k \leq \operatorname{rank}(A),\ 0 \leq x \in \mathbb{R}^k\}.$$

By the first part of the proof, each set in the union is closed. Since there are finitely many such B, we therefore have that C is closed. $\square$

The canonical Farkas' lemma is as follows:

Lemma 24 (Farkas' lemma)**.** *Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^n$. Then exactly one of the following statements holds:*

(a) *There is an $x \in \mathbb{R}^m$ s.t. $x'A = b$ and $x \geq 0$;*

(b) *There is a $y \in \mathbb{R}^n$ s.t. $Ay \geq 0$ and $b \cdot y < 0$.*

Proof. First, suppose (a) holds. Consider any $y \in \mathbb{R}^n$ s.t. $b \cdot y < 0$. If $Ay \geq 0$ then for $x \geq 0$, we have $x'Ay \geq 0$. But $x'A = b$ and so $x'Ay = b \cdot y < 0$. Hence if (a) holds, (b) cannot hold. Thus at most one of the statements can be true.

Now suppose (a) does not hold. Define

$$C := \{c \in \mathbb{R}^n \mid \text{there exists } x \geq 0 \text{ s.t. } x'A = c\}.$$

By Lemma 23, C is closed.

By Theorem 47 and its corollary, there is a $y \in \mathbb{R}^n$ and a real number α s.t. $y \cdot b < \alpha$ and $y \cdot c > \alpha$ for all $c \in C$. Given $0 \in C$, we have $\alpha < 0$ and so $y \cdot b < \alpha < 0$. We claim $Ay \geq 0$. Suppose otherwise. Then there is some i s.t. $(Ay)_i < 0$. It follows that $e^{i\prime}Ay < 0$, and so $(Me^i)'Ay \to -\infty$ as $M \to \infty$. But for every $M > 0$, $(Me^i)'A \in C$ and therefore $(Me^i)'Ay > \alpha$, yielding a contradiction. Hence if (a) does not hold, (b) must hold. $\square$

From this, we can derive the following variant of Farkas' lemma:

Lemma 25 (Variant Farkas' lemma)**.** *Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^n$. Then exactly one of the following statements holds:*

(a) *There is an $x \in \mathbb{R}^m$ s.t. $x'A \leq b$ and $x \geq 0$;*

(b) *There is a* $y \in \mathbb{R}^n$ *s.t.* $Ay \geq 0$, $b \cdot y < 0$ *and* $y \geq 0$.

Proof. First, suppose (a) holds. Fix any $y \in \mathbb{R}^n$ s.t. $b \cdot y < 0$ and suppose $Ay \geq 0$. Then $x'Ay \geq 0$ for any $x \geq 0$. But since $x'A \leq b$, we have $x'Ay = (x'A) \cdot y \leq b \cdot y < 0$, yielding a contradiction. Hence if (a) holds, (b) cannot hold, so at most one of the statements holds.

Suppose the system in (a) has no solution. Then the system

$$\begin{pmatrix} x' & \mu' \end{pmatrix} \begin{pmatrix} A \\ I \end{pmatrix} = b$$

has no solution for $x \geq 0$ and $\mu \geq 0$. By Lemma 24, it follows that system

$$\begin{pmatrix} A \\ I \end{pmatrix} y \geq 0, \qquad b \cdot y < 0$$

has a solution. Thus (b) holds. □

7.3.3 Duality theorems

Theorem 54 (Duality theorem of linear programming)**.** *Let* A *be an* $n \times p$ *matrix, let* $b \in \mathbb{R}^p$ *and let* $c \in \mathbb{R}^n$*. Suppose* $V := \{x \in \mathbb{R}^n \mid x'A \geq b, x \geq 0\}$ *and* $W := \{y \in \mathbb{R}^p \mid Ay \leq c, y \geq 0\}$ *are both nonempty sets. Then* $\min\{x \cdot c \mid x \in V\} = \max\{b \cdot y \mid y \in W\}$.

Proof. Note that if $x \in V$ and $y \in W$, then $x \cdot c \geq x'Ay \geq b \cdot y$, since $x'A \geq b$ and $Ay \leq c$.

Next, suppose $\hat{x} \in V$ and $\hat{y} \in W$, and that $\hat{x} \cdot c = \hat{y} \cdot b$. Since $x \cdot c \geq b \cdot y$, for all $x \in V$ and $y \in W$, we have $x \cdot c \geq \hat{x} \cdot c = b \cdot \hat{y}$. Hence $\hat{x} \cdot c = \min\{x \cdot c \mid x \in V\}$. By a similar argument, $b \cdot \hat{y} = \max\{b \cdot y \mid y \in W\}$.

In view of these two results, we need only show that there exists a solution to the system

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} -A & 0 & c \\ 0 & A' & -b \end{pmatrix} \leq \begin{pmatrix} -b & c & 0 \end{pmatrix}, \qquad x \geq 0, \qquad y \geq 0.$$

Suppose otherwise. By Lemma 25, there exists a vector $(z, w, t) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}$ s.t.

$$\begin{pmatrix} -A & 0 & c \\ 0 & A' & -b \end{pmatrix} \begin{pmatrix} z \\ w \\ t \end{pmatrix} \geq 0, \qquad (-b, c, 0) \cdot (z, w, t) < 0, \qquad z \geq 0, \quad w \geq 0, \quad t \geq 0.$$

That is, $Az \leq tc$, $w'A \geq tb$, and $c \cdot w < b \cdot z$.

If $t = 0$, then $w'A \geq 0 \leq Az$. Thus we would have, for any $x \in V$ and $y \in W$,

$$b \cdot z \leq x'Az \leq 0 \leq w'Ay \leq w \cdot c,$$

i.e. $b \cdot z \leq w \cdot c$, yielding a contradiction. Hence consider $t > 0$. Since $Az \leq tc$ and $z \geq 0$, $\frac{1}{t}z \in W$. Likewise, since $w'A \geq tb$ and $w \geq 0$, $\frac{1}{t}w \in V$. Given this, we must have $\frac{1}{t}w \cdot c \geq b \cdot \frac{1}{t}z$. Yet this implies $w \cdot c \geq b \cdot z$, which contradicts $c \cdot w < b \cdot z$. Hence our original system must have a solution. □

A variant duality theorem (which we use to prove the Bondareva-Shapley theorem):

Theorem 55 (Variant duality theorem)**.** *Let A be an $n \times p$ matrix, let $b \in \mathbb{R}^p$ and let $c \in \mathbb{R}^n$. Suppose the sets $V := \{x \in \mathbb{R}^n \mid x'A \geq b\}$ and $W := \{y \in \mathbb{R}^p \mid Ay = c,\ \ y \geq 0\}$ are both nonempty. Then* $\min\{x \cdot c \mid x \in V\} = \max\{b \cdot y \mid y \in W\}$.

Proof. Define $B = \begin{pmatrix} A \\ -A \end{pmatrix}$. Note we can write W as

$$W = \{y \in \mathbb{R}^p \mid By \leq (c, -c),\ \ y \geq 0\}.$$

Define

$$V' = \{(u, w) \in \mathbb{R}^n \times \mathbb{R}^n \mid (u, w)'B \geq b, (u, w) \geq 0\}.$$

By Theorem 54, we have that

$$\min\{(u, w) \cdot (c, -c) \mid (u, w) \in V'\} = \max\{b \cdot y \mid y \in W\}.$$

Now,

$$\begin{aligned} \min\{(u, w) \cdot (c, -c) \mid (u, w) \in V'\} &= \min\{(u - w) \cdot c \mid (u - w)'A \geq b,\ \ (u, w) \geq 0\} \\ &= \min\{x \cdot c \mid x'A \geq b, x \geq 0\} \\ &= \min\{x \cdot c \mid x \in V\}. \end{aligned}$$

□

We say that a program $\min\{x \cdot c \mid x \in V\}$ or $\max\{x \cdot c \mid x \in V\}$ for some set V and vector c is *infeasible* if the set V is empty. If V is nonempty, we say that the program is *feasible*. We say a program has an *optimal solution* if there is a point in V for which the minimum (resp. maximum) is attained.

Proposition 59. *In Theorem 55, if one of the two programs is infeasible, then both programs lack an optimal solution.*