When is a continuous local martingale a martingale?

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Say we have an Itô process

$$X_t = x + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s,$$

where B is a Brownian motion in $\mathbb{R}^d$. Equivalently, in the form of an SDE,

$$\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t; \qquad X_0 = x.$$

This is a *local martingale* iff it has zero drift, i.e. $\mu_t = 0$ for all t. It is also continuous (as an Itô process).

Great. But when is this thing actually a martingale?

1. *If* $\mathbb{E}_t[X_s] = X_t$ *for all* $s \geq t$*.* Sometimes we can just check the definition of a martingale holds directly.

   Example 1 (Geometric Brownian motion with no drift)**.** Consider a geometric Brownian motion with zero drift:

   $$X_t = x \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma B_t\right\}.$$

   Note in general, geometric Brownian motions have drift!

   Note that for $s > t$, we have

   $$X_s = X_t \exp\left\{-\frac{1}{2}\sigma^2 (s-t) + \sigma(B_s - B_t)\right\}.$$

   Now,

   $$\begin{aligned}\mathbb{E}_t[X_s] &= \mathbb{E}_t\left[X_t \exp\left\{-\frac{1}{2}\sigma^2(s-t) + \sigma(B_s - B_t)\right\}\right] \\ &= X_t \exp\left\{\sigma(\mathbb{E}_t[B_s] - B_t)\right\} \\ &= X_t,\end{aligned}$$

   since $\mathbb{E}_t[B_s] = B_t$.

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2. *If X is bounded.* If X is a bounded process, then it is a martingale. If it has a lower bound, it is a supermartingale and if it has an upper bound, it is a submartingale.

   This is sometimes useful, but most processes we're interested in are not bounded (though they are often bounded below, e.g. price processes).

3. *If X has finite expected quadratic variation for all t.* For an Itô process, the quadratic variation of X is given by

$$[X,X]_t = \int_0^t \sigma_s \cdot \sigma_s \, \mathrm{d}s.$$

   Sometimes, you see this denoted as $[X]_t$.[1]

   If

$$\mathbb{E}[X,X]_t = \mathbb{E}\left[\int_0^t \sigma_s \cdot \sigma_s \, \mathrm{d}s\right] < \infty,$$

   for all $t \in (0,\infty)$, then X is a martingale. Moreover, if

$$\mathbb{E}\left[\left([X,X]_t\right)^{1/2}\right] = \mathbb{E}\left[\left(\int_0^t \sigma_s \cdot \sigma_s \, \mathrm{d}s\right)^{1/2}\right] < \infty,$$

   then X is a martingale. This second condition is weaker than the first.

   Example 2 (Scaled Brownian motion)**.** Suppose $X_t = x + \int_0^t \sigma \, \mathrm{d}B_s$ for a constant $\sigma > 0$, with B being a Brownian motion in $\mathbb{R}$. Trivially,

$$\mathbb{E}[X,X]_t = \mathbb{E}\left[\int_0^t \sigma^2 \, \mathrm{d}s\right] = \int_0^t \sigma^2 \, \mathrm{d}s = t\sigma^2 < \infty$$

   for all $t \in (0,\infty)$. Thus X is a martingale

   A bit less trivially, say $X_t = x + \int_0^t \sigma_s \, \mathrm{d}s$ where σ_t is an i.i.d. random variable with $\mathbb{E}[\sigma_0^2] < \infty$. Then Fubini's theorem gives

$$\mathbb{E}[X,X]_t = \mathbb{E}\left[\int_0^t \sigma_s^2 \, \mathrm{d}s\right] = \int_0^t \mathbb{E}[\sigma_s^2] \, \mathrm{d}s = t\mathbb{E}[\sigma_0^2] < \infty,$$

   and we again have that X is a martingale.

4. *If X satisfies a nice bound.* Say we can write $X_t = f(B_t, t)$ for some $C^{2,1}$ function f. Since X has drift zero, Itô's formula tells us that $f_t(x,t) + \frac{1}{2} f_{xx}(x,t) = 0$.

[1] A related notion is *predictable quadratic variation*, denoted $\langle X, X\rangle_t$, which is the right-continuous, increasing, predictable process such that $X^2 - \langle X, X\rangle$ is a local martingale and $\langle X, X\rangle_0 = 0$. The Doob-Meyer decomposition theorem tells us the process $\langle X, X\rangle$ will be unique for any locally square-integrable martingale X. If X is a continuous local martingale, then $\langle X, X\rangle_t = [X,X]_t$.

A sufficient condition for X to be a martingale is that for every $\epsilon > 0$, we can find a constant $c > 0$ so that $\left|f(x,t)\right| \leq ce^{\epsilon x^2}$ for all $t \geq 0$.

The next two conditions hold when X takes the exponential form

$$X_t = x \exp\left\{\int_0^t \alpha_s \, \mathrm{d}s + \int_0^t \eta_s \, \mathrm{d}B_s\right\}.$$

5. *X satisfies Novikov's condition.* If X has the exponential form above, then X has zero drift iff $\alpha_t = -\frac{1}{2}\eta_t \cdot \eta_t$ almost everywhere, so

$$X_t = \exp\left\{-\frac{1}{2}\int_0^t \eta_s \cdot \eta_s \, \mathrm{d}s + \int_0^t \eta_s \, \mathrm{d}B_s\right\}.$$

   A sufficient condition for X to be a martingale is Novikov's condition:

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t \eta_s \cdot \eta_s \, \mathrm{d}s\right\}\right] < \infty \qquad \text{for all } t \in (0, \infty).$$

6. *X satisfies an easy-to-check but obscure non-integrability condition.* Suppose

$$\mathrm{d}X_t = h(X_t) X_t \, \mathrm{d}B_t$$

   for some nonzero function h with $\int_K \frac{1}{h(x)^2} \, \mathrm{d}x < \infty$ for all compact subsets K of $(0, \infty)$. This local integrability condition holds if $\int_a^b \frac{1}{h(x)^2} \, \mathrm{d}x < \infty$ for all $0 < a < b < \infty$.

   Then X is a martingale iff for all $a > 0$,

$$\int_a^\infty \frac{1}{x h(x)^2} \, \mathrm{d}x = \infty.$$

Example 3 (Geometric Brownian motion, again)**.** Suppose

$$X_t = x \exp\left\{-\frac{1}{2}\sigma^2 t + \sigma B_t\right\},$$

so X is a geometric Brownian motion without drift. Here, $h(x) = \sigma$. For any $a > 0$,

$$\int_a^\infty \frac{1}{\sigma^2 x} \, \mathrm{d}x = \frac{1}{\sigma^2} \int_a^\infty \frac{1}{x} \, \mathrm{d}x = \infty,$$

given $\int_a^\infty \frac{1}{x} \, \mathrm{d}x$ diverges to infinity for any finite $a > 0$.

There are more technical conditions that are typically harder to check. Also, in discrete time, any local martingale is a martingale.