

When is a continuous local martingale a martingale?

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Say we have an Itô process

$$X_t = x + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s,$$

where B is a Brownian motion in \mathbb{R}^d . Equivalently, in the form of an SDE,

$$dX_t = \mu_t dt + \sigma_t dB_t; \quad X_0 = x.$$

This is a *local martingale* iff it has zero drift, i.e. $\mu_t = 0$ for all t . It is also continuous (as an Itô process).

Great. But when is this thing actually a martingale?

1. If $\mathbb{E}_t[X_s] = X_t$ for all $s \geq t$. Sometimes we can just check the definition of a martingale holds directly.

Example 1 (Geometric Brownian motion with no drift). Consider a geometric Brownian motion with zero drift:

$$X_t = x \exp \left\{ -\frac{1}{2}\sigma^2 t + \sigma B_t \right\}.$$

Note in general, geometric Brownian motions have drift!

Note that for $s > t$, we have

$$X_s = X_t \exp \left\{ -\frac{1}{2}\sigma^2(s-t) + \sigma(B_s - B_t) \right\}.$$

Now,

$$\begin{aligned} \mathbb{E}_t[X_s] &= \mathbb{E}_t \left[X_t \exp \left\{ -\frac{1}{2}\sigma^2(s-t) + \sigma(B_s - B_t) \right\} \right] \\ &= X_t \exp \left\{ \sigma(\mathbb{E}_t[B_s] - B_t) \right\} \\ &= X_t, \end{aligned}$$

since $\mathbb{E}_t[B_s] = B_t$.

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2. *If X is bounded.* If X is a bounded process, then it is a martingale. If it has a lower bound, it is a supermartingale and if it has an upper bound, it is a submartingale. This is sometimes useful, but most processes we're interested in are not bounded (though they are often bounded below, e.g. price processes).
3. *If X has finite expected quadratic variation for all t .* For an Itô process, the quadratic variation of X is given by

$$[X, X]_t = \int_0^t \sigma_s \cdot \sigma_s \, ds.$$

Sometimes, you see this denoted as $[X]_t$ or even $\langle X, X \rangle_t$ (e.g. Darrell Duffie's notation in *Dynamic Asset Pricing Theory*).

If

$$\mathbb{E}[X, X]_t = \mathbb{E} \left[\int_0^t \sigma_s \cdot \sigma_s \, ds \right] < \infty,$$

for all $t \in (0, \infty)$, then X is a martingale. Moreover, if

$$\mathbb{E} \left[([X, X]_t)^{1/2} \right] = \mathbb{E} \left[\left(\int_0^t \sigma_s \cdot \sigma_s \, ds \right)^{1/2} \right] < \infty,$$

then X is a martingale. This second condition is weaker than the first.

Example 2 (Scaled Brownian motion). Suppose $X_t = x + \int_0^t \sigma \, dB_s$ for a constant $\sigma > 0$, with B being a Brownian motion in \mathbb{R} . Trivially,

$$\mathbb{E}[X, X]_t = \mathbb{E} \left[\int_0^t \sigma^2 \, ds \right] = \int_0^t \sigma^2 \, ds = t\sigma^2 < \infty$$

for all $t \in (0, \infty)$. Thus X is a martingale

A bit less trivially, say $X_t = x + \int_0^t \sigma_s \, ds$ where σ_t is an i.i.d. random variable with $\mathbb{E}[\sigma_0^2] < \infty$. Then Fubini's theorem gives

$$\mathbb{E}[X, X]_t = \mathbb{E} \left[\int_0^t \sigma_s^2 \, ds \right] = \int_0^t \mathbb{E}[\sigma_s^2] \, ds = t\mathbb{E}[\sigma_0^2] < \infty,$$

and we again have that X is a martingale.

4. *If X satisfies a nice bound.* Say we can write $X_t = f(B_t, t)$ for some $C^{2,1}$ function f . Since X has drift zero, Itô's formula tells us that $f_t(x, t) + \frac{1}{2}f_{xx}(x, t) = 0$.

A sufficient condition for X to be a martingale is that for every $\epsilon > 0$, we can find a constant $c > 0$ so that $|f(x, t)| \leq ce^{\epsilon x^2}$ for all $t \geq 0$.

The next two conditions hold when X takes the exponential form

$$X_t = x \exp \left\{ \int_0^t \alpha_s \, ds + \int_0^t \eta_s \, dB_s \right\}.$$

5. *X satisfies Novikov's condition.* If X has the exponential form above, then X has zero drift iff $\alpha_t = -\frac{1}{2}\eta_t \cdot \eta_t$ almost everywhere, so

$$X_t = \exp \left\{ -\frac{1}{2} \int_0^t \eta_s \cdot \eta_s \, ds + \int_0^t \eta_s \, dB_s \right\}.$$

A sufficient condition for X to be a martingale is Novikov's condition:

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^t \eta_s \cdot \eta_s \, ds \right\} \right] < \infty \quad \text{for all } t \in (0, \infty).$$

6. *X satisfies an easy-to-check but obscure non-integrability condition.* Suppose

$$dX_t = h(X_t)X_t \, dB_t$$

for some nonzero function h with $\int_K \frac{1}{h(x)^2} \, dx < \infty$ for all compact subsets K of $(0, \infty)$.

This local integrability condition holds if $\int_a^b \frac{1}{h(x)^2} \, dx < \infty$ for all $0 < a < b < \infty$.

Then X is a martingale iff for all $a > 0$,

$$\int_a^\infty \frac{1}{xh(x)^2} \, dx = \infty.$$

Example 3 (Geometric Brownian motion, again). Suppose

$$X_t = x \exp \left\{ -\frac{1}{2}\sigma^2 t + \sigma B_t \right\},$$

so X is a geometric Brownian motion without drift. Here, $h(x) = \sigma$. For any $a > 0$,

$$\int_a^\infty \frac{1}{\sigma^2 x} \, dx = \frac{1}{\sigma^2} \int_a^\infty \frac{1}{x} \, dx = \infty,$$

given $\int_a^\infty \frac{1}{x} \, dx$ diverges to infinity for any finite $a > 0$.

There are more technical conditions that are typically harder to check. Also, in discrete time, any local martingale is a martingale.