Linking risk neutral pricing with Feynman-Kac

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Say we have an underlying asset with price process X an Itô process given by

$$X_t = x + \int_0^t \mu(X_u, u)\,\mathrm{d}u + \int_0^t \sigma(X_u, u)\,\mathrm{d}B_u,$$

and we want to price a derivative security on that asset that pays out $g(X_T)$ at time T, where $g : \mathbb{R} \to \mathbb{R}$. We have riskless borrowing with short rate process r, which may depend on X and t, and assume there is a self-financing trading strategy replicating the derivative security payoff.

Suppose, after deflation by $Y_t = e^{-\int_0^t r(X_u,u)\,\mathrm{d}u}$, the price process X admits an equivalent martingale measure Q. Diffusion invariance and Itô's lemma together imply

$$\mathrm{d}\hat{X}_t = Y_t \sigma(X_t, t)\,\mathrm{d}B_t^Q,$$

where B^Q is a standard Brownian motion that is a martingale under Q. By Itô's lemma, this implies

$$\mathrm{d}X_t = r(X_t, t)X_t\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}B_t^Q.$$

The price process C of the derivative security satisfies

$$C_t = \mathbb{E}_t^Q\left[e^{-\int_t^T r(X_u,u)\,\mathrm{d}u} g(X_T)\right].$$

The deflated price process $\hat{C}$ given by

$$\hat{C}_t = Y_t C_t = \mathbb{E}_t^Q\left[Y_T g(X_T)\right]$$

is a Q-martingale.

We conjecture $C_t = f(X_t, t)$ where $f : (0, \infty) \times [0, \infty)$ is a $C^{2,1}$ function. This allows us to apply Itô's lemma:

$$\mathrm{d}\hat{C}_t = Y_t\left[\mathcal{D}f(X_t, t) - r(X_t, t)f(X_t, t)\right]\mathrm{d}t + Y_t f_x(X_t, t)\sigma_t\,\mathrm{d}B_t^Q,$$

where $\mathcal{D}f(x, t) = f_t(x, t) + f_x(x, t)r(x, t)X_t + \frac{1}{2}f_{xx}(x, t)\sigma(x, t)^2$. No arbitrage implies that $f(X_T, T) = g(X_T)$.

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Since $\hat{C}$ is a martingale, it has zero drift. Thus the pricing function f must solve the Cauchy problem,

$$\mathcal{D}f(x,t) - r(x,t)f(x,t) = 0 \qquad \text{for } (x,t) \in (0,\infty) \times (0,T]$$

with terminal condition $f(x,T) = g(x)$ for $x \in (0,\infty)$.

If this has a unique solution, it is the Feynman-Kac solution

$$f(x,t) = \mathbb{E}_t\left[e^{-\int_t^T r(Z_u^{x,t},u)\,\mathrm{d}u} g(Z_T^{x,t})\right]$$

where $Z^{x,t}$ solves $\mathrm{d}Z_u^{x,t} = r(Z_u^{x,t},u)Z_u^{x,t}\,\mathrm{d}u + \sigma(Z_u^{x,t},u)\,\mathrm{d}B_u$ for $u \geq t$, with initial condition $Z_t^{x,t} = x$.