# Bayesian filtering in continuous time

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In many applications, agents want to learn about some variable they cannot fully observe, which might evolve over time:

- *Asset pricing.* In models of insider trading such as Kyle (1985) and Back (1992), outside investors make inferences about the profitability of a firm from the price of its equity, with the price impact of insiders trading on private information partially obfuscated by noise traders.

  In regime-switching models, a security's cashflows depend on the current regime, which follows a Markov chain. Traders make inferences about the underlying state, and the state will change over time.

- *Labour.* Suppose a firm has employed a worker. The worker's output over time depends stochastically on the worker's productivity type. At any instant, the firm has the option to fire the worker and hire another worker with productivity type drawn randomly from a known distribution. The firm uses output to make inferences about the worker's type, informing the decision to fire or retain.

Liptser & Shiryaev (2001) developed much of the theory for optimal inference in these settings when time is continuous and their two-volume book seems to be a classic. Here we go through some of these results with examples of applications. See Liptser & Shiryaev (2001) chapters 8-10 for proofs.

## 1 Nonlinear filtering

We fix a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and a time horizon $T$. I use $\mathcal{N}$ to denote the null sets. Let $\mathcal{L}$ denote the set of adapted processes $X$ satisfying $\int_0^T |X_t| \, \mathrm{d}t < \infty$ a.s. and let $\mathcal{L}^2$ denote the set of adpated processes $X$ satisfying $\int_0^T |X_t|^2 \, \mathrm{d}t < \infty$ a.s. I use $C_T$ to denote the set of Borel-measurable functions in $\mathbb{R}^{[0,T]}$. Throughout, $B$ denotes a standard Brownian motion under $P$ with respect to $\mathbb{F}$. We assume there is no initial information, i.e. $\mathcal{F}_0 = \{\varnothing, \Omega\}$.

### 1.1 General case

Suppose we have a pair of processes $(\theta, X)$, where $\theta_t$ is unobserved and $X_t$ is observed at each $t$. The agent cares about estimating an adapted process $Z_t = f_t(\theta_t, X_t)$ where $f_t$

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is some $\mathcal{F}_t$-measurable function. Let $\mathcal{G}_t = \sigma(\{X_s \mid s \le t\} \cup \mathcal{N})$, the $\sigma$-algebra generated by the observed process up to $t$. This is the information actually available to the agent at $t$. Likewise, let $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ denote the corresponding filtration. At each time $t$, given her information, the agent optimally constructs an estimate $\hat{Z}_t$ for the value of $Z_t$. By "optimal", we mean that $\hat{Z}_t$ minimizes the mean squared error $\mathbb{E}\left[(\hat{Z}_t - Z_t)^2\right]$.

The mean-squared-error-minimizing estimate of $Z_t$ at $t$ is the posterior conditional expectation

$$\hat{Z}_t(Z) = \mathbb{E}\left[Z_t \mid \mathcal{G}_t\right].$$

To go further than this, we need more structure on the processes. First, suppose $Z$ takes the form

$$Z_t = z_0 + \int_0^t m_s \,\mathrm{d}s + M_t,$$

for some $z_0 \in \mathbb{R}$, some martingale $M$, and $m \in \mathcal{L}$.

Next, suppose $X$ is an Itô process,

$$X_t = x_0 + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s(X)\,\mathrm{d}B_s,$$

for some $x_0 \in \mathbb{R}$, $\mu \in \mathcal{L}$ and $\sigma(X) \in \mathcal{L}^2$ with $\sigma$ being uniformly bounded away from zero and each $\sigma_t$ being Borel-measurable, and where $B$ denotes a standard Brownian motion.

We also assume there are constants $K_1, K_2 \ge 0$ and a nondecreasing right-continuous function $\alpha : \mathbb{R}_+ \to [0,1]$ such that for all $t \in [0,T]$,

$$\begin{aligned} \left|B_t(x) - B_t(y)\right|^2 &\le K_1 \int_0^t (x_s - y_s)^2 \,\mathrm{d}\alpha_s + K_2(x_t - y_t)^2, \quad \text{and} \\ B_t(x)^2 &\le K_1 \int_0^t (1 + x_s^2)\,\mathrm{d}\alpha_s + K_2(1 + x_t^2), \end{aligned}$$

for all $x, y \in C_T$. The first of these is a kind of Lipschitz condition and the second is a kind of growth condition.

We take $\hat{Z}_t(Z) = \mathbb{E}[Z_t \mid \mathcal{G}_t]$ to be progressively measurable, which is without loss since $\mathbb{E}[Z_t \mid \mathcal{G}_t]$ has a measurable modification.

Finally, there is one more process involved in characterizing the evolution of $\pi$. By the Doob-Meyer decomposition, we can isolate the "martingale part" $\langle B, M\rangle_t$ of the product of the martingales $B$ and $M$, i.e. $\langle B, M\rangle$ is the adapted process satisfying

$$\mathbb{E}\left[B_t M_t - B_s M_s \mid \mathcal{F}_s\right] = \mathbb{E}\left[\langle B, M\rangle_t - \langle B, M\rangle_s\right] \qquad \text{for all } t \ge s \text{ and all } s.$$

We define the process $D$ so that $D_t = \dfrac{\mathrm{d}\langle B, M\rangle_t}{\mathrm{d}t}$. This captures the correlation of the Brownian motion $B$ with the martingale $M$ involved in the process $Z$.

Under these conditions, we have:

**Proposition 1.** *Suppose* $(Z, X)$ *satisfies the conditions above, and suppose* $\sup_t \mathbb{E}[Z_t^2] < \infty$*,* $\int_0^T \mathbb{E}[\mu_t^2]\,\mathrm{d}t < \infty$ *and* $\int_0^T \mathbb{E}[m_t^2]\,\mathrm{d}t < \infty$*.*

*Then the posterior conditional expectation $\hat{Z}_t(Z) = \mathbb{E}[Z_t \mid \mathcal{G}_t]$ is given by*

$$\hat{Z}_t(Z) = \hat{Z}_0 + \int_0^t \hat{Z}_s(m)\,\mathrm{d}s + \int_0^t \Big(\hat{Z}_s(D) + (\hat{Z}_s(Z\mu) - \hat{Z}_s(Z)\hat{Z}_s(\mu))\sigma_s^{-1}(X)\Big)\,\mathrm{d}\bar{B}_s,$$

*where $\bar{B}$ is a standard Brownian motion wrt $\mathbb{G}$ defined by*

$$\mathrm{d}\bar{B}_t = \frac{1}{\sigma_t(X)}\,\mathrm{d}X_t - \frac{\hat{Z}_t(\mu)}{\sigma_t(X)}\,\mathrm{d}t.$$

### 1.2 Markov states

We now suppose the unobservable component $\theta$ is a Markov process taking values in $E \subseteq \mathbb{Z}$, and the observable process $X$ evolves according to

$$\mathrm{d}X_t = \mu_t(\theta_t, X)\,\mathrm{d}t + \sigma_t(X)\,\mathrm{d}B_t,$$

with $\sigma$ uniformly bounded away from zero.

Assume $X_0 = x_0$ for some fixed $x_0 \in \mathbb{R}$.

As before, we assume some growth and Lipschitz conditions on the mean $\mu_t(\epsilon, x)$ and the volatility $\sigma_t(x)$. Namely, we assume there exist constants $K_1, K_2$ and a nondecreasing right-continuous function $\alpha : \mathbb{R}_+ \to [0,1]$ such that for all $t \in [0,T]$,

$$\begin{aligned} \mu_t(x)^2 &\le K_1 \int_0^t (1 + x_s^2)\,\mathrm{d}\alpha_s + K_2(1 + \epsilon_t^2 + x_t^2), \\ \sigma_t^2(x) &\le K_1 \int_0^t (1 + x_s^2)\,\mathrm{d}\alpha_s + K_2(1 + x_t^2), \qquad \text{and} \\ \big|\mu_t(\epsilon_t, x) - \mu_t(\epsilon_t, y)\big|^2 + \big|\sigma_t(x) - \sigma_t(y)\big|^2 &\le K_1 \int_0^t (x_s - y_s)^2\,\mathrm{d}\alpha_s + K_2(x_t - y_t)^2. \end{aligned}$$

for all $x, y \in C_T$ and $\epsilon \in E^{[0,T]}$.

For each $a, b \in E$, we assume the transition probability governing the transition from state $a$ to state $b$ has a continuous, bounded density $\lambda_t(a,b)$ at each time $t$. We assume that for any $t$ and any time interval $\Delta$,

$$\big|P(\theta_{t+\Delta} = b \mid \theta_t = a) - 1_{b=a} - \lambda_t(a,b)\Delta\big| \le o(\Delta),$$

where $\frac{o(\Delta)}{\Delta} \to 0$ uniformly as $\Delta \to 0$, over all $a, b, t$.

If the Markov transition probabilities are time invariant, then these densities are constant.

As before, we let $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$, with $\mathcal{G}_t = \sigma(\{X_s \mid s \le t\} \cup \mathcal{N})$ characterizing the information available from the observable process up to time $t$. The agent believes that the current state is $b$ with probability,

$$\pi_t(b) = P(\theta_t = b \mid \mathcal{G}_t).$$

**Proposition 2.** *Suppose the above conditions hold and that*

$$\mathbb{E}\left[\int_0^T \theta_t^2 \, \mathrm{d}t\right] < \infty.$$

*Then the posterior belief process* $\pi$ *satisfies* $\pi_0(b) = P(\theta_0 = b)$

$$\mathrm{d}\pi_t(b) = \sum_{a \in E} \lambda_t(a,b)\pi_t(a)\,\mathrm{d}t + \pi_t(b)\frac{\mu_t(b,X) - m_t(\pi_t, X)}{\sigma_t(X)}\,\mathrm{d}\bar{B}_t \qquad \text{for all } b \in E,$$

*where*

$$m_t(\pi_t, X) = \sum_{a \in E} \mu_t(a, X)\pi_t(a),$$

*and* $\bar{B}$ *is a standard Brownian motion with respect to* $\mathbb{G}$ *with*

$$\mathrm{d}\bar{B}_t = \frac{1}{\sigma_t(X)}\,\mathrm{d}X_t - \frac{\sum_{a \in E} \mu(a, X)\pi_t(a)}{\sigma_t(X)}\,\mathrm{d}t.$$

Here are some example special cases.

**Example 1** (Two-state Markov chain)**.** Suppose the time trend for the price of a risky asset depends on which of two possible regimes are in force, with $\theta_t \in \{0, 1\}$ denoting the current regime at time $t$. We denote the agent's belief that the state is 1 at time $t$ by $p_t \in [0, 1]$. Suppose the price process $S$ evolves according to

$$\mathrm{d}S_t = \mu(\theta_t)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t.$$

Letting $m(\alpha) = \alpha\mu(1) + (1-\alpha)\mu(0)$ and $\bar{B}_t = \frac{1}{\sigma}\left[\mu(\theta_t) - m(p_t)\right]\mathrm{d}t + \mathrm{d}B_t$, we have that

$$\mathrm{d}S_t = m(p_t)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}\bar{B}_t.$$

The agent observes only the price $S_t$ and uses this to update her belief about the current regime. If we have transition-rate matrix

$$\Pi = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix},$$

then we have

$$\mathrm{d}p_t = \left[\lambda_0(1-p_t) - \lambda_1 p_t\right]\mathrm{d}t + \frac{1}{\sigma}(\mu(1) - \mu(0))p_t(1-p_t)\,\mathrm{d}\bar{B}_t.$$

**Example 2** (Inference about a static parameter)**.** Suppose an agent wants to learn about a static parameter that can take at most countably many possible values. For example, suppose a worker with productivity parameter $\bar{\theta}$ produces output $Y_t = e^{(\bar{\theta} - \sigma^2/2)t + \sigma B_t}$ for some volatility parameter $\sigma$ (here, $\bar{\theta}$ is the expected growth rate of the worker's output). Then

$$\mathrm{d}Y_t = \bar{\theta}Y_t\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}B_t.$$

Let $Q$ be the (countable) set of possible types. The firm employing the worker wants to infer the worker's type $\bar{\theta}$. In this case, the transition-rate matrix is just the zero matrix. The firm's belief $\pi_t(q)$ that the worker's type is $\bar{\theta} = q$ evolves according to

$$\mathrm{d}\pi_t(q) = \frac{1}{\sigma}\pi_t(q)\left[q - m(\pi_t)\right]\mathrm{d}\bar{B}_t,$$

where

$$m(\pi_t) = \sum_{q' \in Q} q' \pi_t(q'),$$

and

$$\begin{aligned} \bar{B}_t &= \frac{1}{\sigma Y_t}\,\mathrm{d}Y_t - \frac{1}{\sigma} m(\pi_t)\,\mathrm{d}t \\ &= \frac{\bar{\theta} - m(\pi_t)}{\sigma}\,\mathrm{d}t + \mathrm{d}B_t. \end{aligned}$$

In the simplest case, we have only two types, say $h$ and $\ell < h$. Let $p_t = P(\bar{\theta} = h \mid \mathcal{G}_t)$. Then

$$\mathrm{d}p_t = \frac{p_t(h - m(p_t))}{\sigma}\,\mathrm{d}\bar{B}_t = \frac{(h-\ell)p_t(1-p_t)}{\sigma}\,\mathrm{d}\bar{B}_t,$$

where $m(\alpha) = (1-\alpha)\ell + \alpha h$ and $\bar{B}_t = \frac{\theta - m(p_t)}{\sigma}\,\mathrm{d}t + \mathrm{d}B_t$.

## 2 Linear filtering

Previously (e.g. in Proposition 1), we had an optimal filter that was nonlinear. Under more restrictive but often-natural assumptions, it turns out the optimal filter will be *linear*. This setting was originally studied in discrete time by Kálmán (1960) and Kálmán & Bucy (1961), and the filtering method is thus generally known as the *Kalman filter*.[1]

The setting is a special case of section 1.1. Again, we fix a probability space $(\Omega, \mathcal{F}, P)$, a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, and a finite time horizon $T$, and assume there is no initial information (i.e. $\mathcal{F}_0 = \{\varnothing, \Omega\}$).

Likewise, we again have a pair of $\mathcal{F}_t$-adapted processes $(\theta, X)$, with $\theta_t$ unobserved and $X_t$ observed at each $t$. The information available to the agent at $t$ is $\mathcal{G}_t = \sigma(\{X_s \mid s \leq t\} \cup \mathcal{N})$. This gives us a filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$.

Previously, $\theta$ was largely arbitrary and we assumed $X$ was an Itô process. Now, we assume both $\theta$ and $X$ are Itô processes, satisfying

$$\begin{aligned} \mathrm{d}\theta_t &= a_t \theta_t\,\mathrm{d}t + b_t\,\mathrm{d}B_{1t}, \\ \mathrm{d}X_t &= \alpha_t \theta_t\,\mathrm{d}t + \beta_t\,\mathrm{d}B_{2t}, \end{aligned}$$

where $B_{1t}$ and $B_{2t}$ are two mutually independent Brownian motions under $\mathbb{F}$, and $\theta_0, \xi_0$ are given. We assume $a, \alpha \in \mathcal{L}$, $b, \beta \in \mathcal{L}^2$ and $b, \beta > 0$.

[1] The Kalman filter is best known for its engineering applications – a classic example is in navigation, where you want to estimate the position of an object (say, the Apollo command module – the Apollo navigation computer was one of the earliest systems to use the filter) given noisy sensor estimates and an understanding of the linear dynamic system describing how the position of the object evolves.

These two equations define a two-dimensional linear dynamical system with Gaussian noise. We can verify from Itô's lemma that

$$\theta_t = A_t\theta_0 + A_t \int_0^t \frac{b_s}{A_s}\,\mathrm{d}B_{1s},$$

where $A_t = \exp\left\{\int_0^t a_u\,\mathrm{d}u\right\}$.[2]

The agent faces the problem of optimally constructing an estimate $\hat{\theta}_t$ for $\theta_t$ at each $t$, given her information $\mathcal{G}_t$, in the sense of minimizing mean-squared error $\mathbb{E}\left[(\hat{\theta}_t - \theta_t)^2\right]$. As noted before, the optimal estimate is the posterior conditional expectation, so

$$\hat{\theta}_t = \mathbb{E}[\theta_t \mid \mathcal{G}_t].$$

Let $\gamma_t = \mathbb{E}\left[(\hat{\theta}_t - \theta_t)^2\right]$, so $\gamma_t$ denotes the mean squared-error of filtering at $t$.

We need two more assumptions. Assume $\alpha \in \mathcal{L}^2$ and $\beta^2$ is uniformly bounded away from 0.

**Proposition 3** (Kalman filter)**.** *Suppose $(\theta, X)$ satisfy the conditions above. Then the posterior conditional expectation $\hat{\theta}_t = \mathbb{E}[\theta_t \mid \mathcal{G}_t]$ and the mean-squared error $\gamma_t = \mathbb{E}\left[(\hat{\theta}_t - \theta_t)^2\right]$ satisfy*

$$\begin{aligned}
\mathrm{d}\hat{\theta}_t &= \left(a_t - \frac{A_t^2}{B_t^2}\gamma_t\right)\theta_t\,\mathrm{d}t + \frac{A_t}{B_t^2}\gamma_t\,\mathrm{d}X_t,\\
\mathrm{d}\gamma_t &= \left(2a_t\gamma_t - \frac{A_t^2}{B_t^2}\gamma_t^2 + b_t^2\right)\mathrm{d}t,
\end{aligned}$$

*with $\hat{\theta}_0 = \mathbb{E}[\theta_0]$ and $\gamma_0 = \mathbb{E}[(\hat{\theta}_0 - \theta_0)^2] = \operatorname{var}\theta_0$. Imposing $\gamma \geq 0$, this system of equations has a unique solution.*

[2] Take $Y_t = \theta_0 + \int_0^t \frac{b_s}{A_s}\,\mathrm{d}B_{1s}$, and apply Itô's lemma to $f(A_t, Y_t) = A_t Y_t$.