

Add a Good Title Here

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Abstract—Write your abstract text here.

I. WAVEPORT BOUNDARY CONDITION

The following is not necessarily written using the proper prose as mentioned in lecture. This is just a derivation, I will re-word this later.

Starting with (3.3.34) in Ch. 9 of our textbook

$$\mathbf{E}(u, v, w) = \mathbf{E}^{inc}(u, v, w) + \mathbf{E}^{ref}(u, v, w). \quad (1)$$

Substituting in the \mathbf{e}_{10} definition in (9.3.35) yields the following

$$\mathbf{E}(u, v, w) = \hat{v}E_0 \sin \frac{\pi u}{a} e^{-j\beta w} + \hat{v}RE_0 \sin \frac{\pi u}{a} e^{j\beta w}. \quad (2)$$

where R is the reflection coefficient and a is the width of the waveguide.

This can be cleanly written in vector notation as

$$\begin{aligned} \mathbf{E}(u, v, w) &= \begin{bmatrix} 0 \\ E_0 \sin \frac{\pi u}{a} (e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}. \end{aligned} \quad (3)$$

With

$$\mathbf{E}^{inc}(u, v, w) = \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \quad (4)$$

The waveport source boundary condition as found in (9.3.37) is

$$\hat{n} \times (\nabla \times \mathbf{E}) + j\beta \hat{n} \times (\hat{n} \times \mathbf{E}) = -2j\beta \mathbf{E}^{inc}. \quad (5)$$

Substituting the above field definitions into 5 yields the following

$$\begin{aligned} \hat{n} \times \left(\nabla \times \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \right) \\ + j\beta \hat{n} \times \left(\hat{n} \times \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \right) \\ = -2j\beta \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \end{aligned} \quad (6)$$

Assuming $\hat{u}, \hat{v}, \hat{w}$ form an orthogonal, right-handed set of basis vectors with $\hat{n} = \hat{w}$ as it is normal to the waveport surface in the direction of propagation \hat{w} becomes

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} \partial_u \\ \partial_v \\ \partial_w \end{bmatrix} \times \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \right) \\ + j\beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \right) \\ = -2j\beta \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \end{aligned} \quad (7)$$

Evaluating the first set of cross products results in

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ 0 \\ \partial_u(c(u)(e^{-j\beta w} + Re^{j\beta w})) \end{bmatrix} \\ + j\beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \\ 0 \end{bmatrix} \\ = -2j\beta \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \end{aligned} \quad (8)$$

Evaluating the second set of cross products equals

$$\begin{aligned} \begin{bmatrix} 0 \\ -\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ 0 \end{bmatrix} \\ + j\beta \begin{bmatrix} 0 \\ -c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \\ = -2j\beta \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \end{aligned} \quad (9)$$

Resulting in the following expression for the \hat{v} direction

$$\begin{aligned} -\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ - j\beta c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ = -2j\beta c(u)e^{-j\beta w}. \end{aligned} \quad (10)$$

Which is equal to

$$\partial_w E_v + j\beta E_v = 2j\beta E_v^{inc}. \quad (11)$$

Substituting in the definition of beta yields

$$\partial_w E_v + j\sqrt{k^2 - \frac{\pi^2}{a^2}} E_v = 2j\sqrt{k^2 - \frac{\pi^2}{a^2}} E_v^{inc}. \quad (12)$$

Pulling out $k = \frac{\omega\sqrt{\mu_r\epsilon_r}}{c_0}$ from square roots yields

$$\begin{aligned} \partial_w E_v + j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v \\ = 2j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v^{inc}. \end{aligned} \quad (13)$$

Using the second order approximation of $\partial_w E_v \approx -j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v$ the above becomes

$$\begin{aligned} -j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v + j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v \\ = 2j\omega\frac{\sqrt{\mu_r\epsilon_r}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2 a^2}} E_v^{inc}. \end{aligned} \quad (14)$$

Canceling out like terms gives

$$E_v = E_v^{inc}. \quad (15)$$

Which feels incorrect but is true for $R = 0$.