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***Abstract*—Write your abstract text here.**

I. Waveport Boundary Condition

The following is not necessarily written using the proper prose as mentioned in lecture. This is just a derivation, I will re-word this later.

Starting with (3.3.34) in Ch. 9 of our textbook

$$\mathbf{E}(u,v,w) = \mathbf{E}^{inc}(u,v,w) + \mathbf{E}^{ref}(u,v,w). \tag{1}$$

Substituting in the $\mathbf{e}_{10}$ definition in (9.3.35) yields the following

$$\mathbf{E}(u,v,w) = \hat{v}E_0 \sin\frac{\pi u}{a}e^{-j\beta w} + \hat{v}RE_0\sin\frac{\pi u}{a}e^{j\beta w}. \tag{2}$$

where R is the reflection coefficient and a is the width of the waveguide.

This can be cleanly written in vector notation as

$$\begin{aligned}\mathbf{E}(u,v,w) &= \begin{bmatrix} 0 \\ E_0\sin\frac{\pi u}{a}(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}.\end{aligned} \tag{3}$$

With

$$\mathbf{E}^{inc}(u,v,w) = \begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}. \tag{4}$$

The waveport source boundary condition as found in (9.3.37) is

$$\hat{n}\times(\nabla\times\mathbf{E}) + j\beta\hat{n}\times(\hat{n}\times\mathbf{E}) = -2j\beta\mathbf{E}^{inc}. \tag{5}$$

Substituting the above field definitions into 5 yields the following

$$\begin{aligned}&\hat{n}\times\left(\nabla\times\begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}\right) \\ &\quad + j\beta\hat{n}\times\left(\hat{n}\times\begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}\right) \\ &\quad = -2j\beta\begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}.\end{aligned} \tag{6}$$

Assuming $\hat{u},\hat{v},\hat{w}$ form an orthogonal, right-handed set of basis vectors with $\hat{n} = \hat{w}$ as it is normal to the waveport surface in the direction of propagation 6 becomes

$$\begin{aligned}&\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\times\left(\begin{bmatrix} \partial_u \\ \partial_v \\ \partial_w \end{bmatrix}\times\begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}\right) \\ &\quad + j\beta\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\times\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} 0 \\ c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix}\right) \\ &\quad = -2j\beta\begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}.\end{aligned} \tag{7}$$

Evaluating the first set of cross products results in

$$\begin{aligned}&\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} -\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ 0 \\ \partial_u(c(u)(e^{-j\beta w} + Re^{j\beta w})) \end{bmatrix} \\ &\quad + j\beta\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\times\begin{bmatrix} -c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \\ 0 \end{bmatrix} \\ &\quad = -2j\beta\begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}.\end{aligned} \tag{8}$$

Evaluating the second set of cross products equals

$$\begin{aligned}&\begin{bmatrix} 0 \\ -\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ 0 \end{bmatrix} \\ &\quad + j\beta\begin{bmatrix} 0 \\ -c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ 0 \end{bmatrix} \\ &\quad = -2j\beta\begin{bmatrix} 0 \\ c(u)e^{-j\beta w} \\ 0 \end{bmatrix}.\end{aligned} \tag{9}$$

Resulting in the following expression for the $\hat{v}$ direction

$$\begin{aligned}&-\partial_w(c(u)(e^{-j\beta w} + Re^{j\beta w})) \\ &\quad - j\beta c(u)(e^{-j\beta w} + Re^{j\beta w}) \\ &\quad = -2j\beta c(u)e^{-j\beta w}.\end{aligned} \tag{10}$$

Which is equal to

$$\partial_w E_v + j\beta E_v = 2j\beta E_v^{inc}. \tag{11}$$

Substituting in the definition of beta yields

$$\partial_w E_v + j\sqrt{k^2 - \frac{\pi^2}{a^2}} E_v = 2j\sqrt{k^2 - \frac{\pi^2}{a^2}} E_v^{inc}. \tag{12}$$

Pulling out $k = \frac{\omega\sqrt{\mu\epsilon}}{c_0}$ from square roots yields

$$\partial_w E_v + j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v = 2j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v^{inc}. \tag{13}$$

Using the second order approximation of $\partial_w E_v \approx -j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v$ the above becomes

$$-j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v + j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v = 2j\omega\frac{\sqrt{\mu\epsilon}}{c_0}\sqrt{1 - \frac{\pi^2}{k^2a^2}} E_v^{inc}. \tag{14}$$

Canceling out like terms gives

$$E_v = E_v^{inc}. \tag{15}$$

Which feels incorrect but is true for $R = 0$.