CSE 499A : Assignment 1

Name: Samya Sunibir Das

ID: 1911563642

Topic: Basics Of Quantum Computing 1

Class Notes — Matrix & Linear Algebra

Lecture 1: Part 1

New Notation: Bra-Ket notation was invented by Paul Dirac.

The notation: $|\rangle$, $\langle|$: This notation shortened the calculations of quantum mechanics by a lot.

Summary Chapter 1: Essential Mathematical Methods Book (Riley)

1. **Trace:** Sum of the elements on the leading diagonal.

Let, $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow Tr\,A = 1 + 5 = 6$

2. **Determinate:** To calculate the determinant of a matrix, the matrix must be a square matrix.

Calculating a 3x3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$|A| = 1 \times \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35) = 0$$

3. **Transpose:** Interchanging Rows, Columns. No need for square matrix. $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

4. **Complex Conjugate:** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $A^* = \begin{bmatrix} 1^* & 2^* \\ 3^* & 5^* \end{bmatrix}$

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5. <u>Hermitian Conjugate:</u> Transpose the complex conjugate or Complex conjugate the transpose. $(A^T)^* = (A^*)^T = A^+$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \rightarrow (A^T)^* = \begin{bmatrix} 1^* & 3^* \\ 2^* & 5^* \end{bmatrix} \rightarrow A^+$$

6. <u>Cofactor:</u> Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

<u>Cofactors:</u> $C_1 = (-1)^{1+1} \cdot 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$, $C_2 = (-1)^{1+2} \cdot 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$,

$C_3 = (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

7. <u>Inverse Matrix:</u> For unitary matrix, $U = A^{-1} = (A^T)^*$

For generic case, $A^{-1} = \frac{1}{|A|} C^T$

<u>Matrix Concept:</u> • $AB \neq BA$ in general, not in the case of diagonal matrix.

Special Type of Square Matrices

Real Matrix: If the conjugate of a matrix is the same as the original matrix. $A^* = A$.

Imaginary Matrix: If the conjugate of a matrix is the same as the negative of the original matrix. $A^* = -A$.

Diagonal Matrix: Every off diagonal element is zero.

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Symmetric Matrix: Transpose of a matrix is the same of the original matrix. $A = A^T$.

Anti Symmetric Matrix: Transpose of the matrix is the same as the negative of the matrix. $A^T = -A$.

Orthogonal Matrix: Unitary matrix is a kind of Orthogonal matrix. Transpose of the matrix is the same as inverse of the matrix. $A^T = A^{-1}$.

Hermitian Matrix(***): Transpose, Conjugate of the matrix, the hermitian conjugate is the same as the original matrix.

$$(A^T)^* = A \quad , \quad A^+ = A .$$

Real world applications of Hermitian Matrix: Observables in quantum mechanics can be measured using this such as: energy, momentum etc.

Anti Hermitian: $(A^*)^T = -A$.

Unitary Matrix: (***) Hermitian conjugate is equal to the inverse.

$$(A^*)^T = A^{-1} .$$

If B is a matrix, and if B → real, (Summary)

then $\boxed{B = B^T} \Rightarrow$ Hermitian, $\boxed{B^{-1} = B^T} \Rightarrow$ Unitary

Normal Matrix: $A^+A = AA^+$. The matrix Commutes with its Hermitian conjugate.

Singular Matrix: Determinant is zero. $|A| = 0$, Cramer's Rule is an instance where these are seen.

Lecture 2: Part 2

Unitary Matrix Ex.

$$U = \begin{bmatrix} a & b \\ b^* e^{-i\phi} & a^* \cdot e^{i\phi} \end{bmatrix}$$ Generic format for 2x2

$\det(U) = 1 \cdot e^{i\phi}$. U^{-1}, U^{T} will always be same.

Unitary, Hermitian, symmetric, orthogonal matrices follow the properties of normal matrices.

Effects of Matrix operations on matrix products

Trace : $Tr(AB \ldots G) = Tr(BG \ldots BA)$

Determinant: $|AB \ldots G| = |A|\,|B| \ldots |G|$

Transpose: $(AB \ldots G)^T = G^T \ldots B^T A^T$.

Complex Conjugate: $(AB \ldots G)^* = A^* B^* \ldots G^*$

Hermitian " : $(AB \ldots G)^\dagger = G^\dagger \ldots B^\dagger A^\dagger$

Inverse : $(AB \ldots G)^{-1} = G^{-1} \ldots B^{-1} A^{-1}$

Matrix Multiplication Example

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (3\times2)+(2\times1)+(-1\times3) & (3\times-2)+(2\times1)+(-1\times2) & (3\times3)+(2\times0)+(-1\times1) \\ (0\times2)+(3\times1)+(2\times3) & (0\times-2)+(3\times1)+(2\times2) & (0\times3)+(3\times0)+(2\times1) \\ (1\times2)+(-3\times1)+(4\times3) & (1\times-2)+(-3\times1)+(4\times2) & (1\times3)+(-3\times0)+(4\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 & 8 \\ 9 & 7 & 2 \\ 11 & 3 & 7 \end{bmatrix}, \quad \text{Similarly, } BA = \begin{bmatrix} 9 & -11 & 6 \\ 3 & 5 & 1 \\ 10 & 9 & 5 \end{bmatrix}$$

Hence, $AB \neq BA$ for this instance

Let, $\psi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$ [Fourier in Matrix form]

Then, c and ϕ are in matrix multiplication form in, $\psi = c^T\phi$,

as $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_{n\times1}$, $c^T = \begin{bmatrix} c_1 \dots c_n \end{bmatrix}_{1\times n}$, $\phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}_{n\times1}$

After c^T, only then it is multipliable. The resultant is 1×1 matrix.

Inverse Matrix Example

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$$

First, calculate the conjugates and represent them as matrix, $\therefore C = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{bmatrix}$

Transposing them, $C^T = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$

$$\therefore A^{-1} = \frac{C^T}{|A|}$$

Determinant = 11, $A^{-1} = \frac{1}{11} \begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$

A^{-1} does not exist when $|A| = 0$

Cremer's Rule Example

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix}$$

$|A| = 11$, the cremer determinants are

$$\Delta_1 = \begin{vmatrix} 4 & 4 & 3 \\ 0 & -2 & -2 \\ -7 & 3 & 2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 0 & -2 \\ -3 & -7 & 2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} 2 & 4 & 4 \\ 1 & -2 & 0 \\ -3 & 3 & -7 \end{vmatrix}$$

$\Delta_1 = 22$, $\Delta_2 = -33$, $\Delta_3 = 44$, $\therefore x_1 = \frac{22}{11} = 2$, $x_2 = \frac{-33}{11} = -3$, $x_3 = \frac{44}{11} = 4$

* Eigenvectors and Eigenvalues

In matrix, $AB = \lambda B$
λ ↳ eigenvalue

B is an eigenvector of A.

Eigen values and the characteristic polynomial — from wiki

$$|A - \lambda I| = 0 \rightarrow \lambda_1, \lambda_2$$

After getting λ_1, λ_2, $AX = \lambda_1 X$ — ①

$AY = \lambda_1 Y$ — ④

After finding λ_1, λ_2, then find the components of X, Y. Then solve the equations.

* $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

Now, to find eigen vectors,

$A\varphi_1 = \lambda \varphi_1$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{l} 2x_1 + x_2 = x_1 \\ x_1 + 2x_2 = x_2 \end{array} \right] \rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{array}$

This has infinite amount of solutions, we will use one equation and normalize that.

$x_1 = -x_2,$

$\Rightarrow x_1^2 + x_2^2 = 1 \qquad [\because \varphi_1^T \varphi = 1]$

$\Rightarrow x_1^2 + (-x_1)^2 = 1$

$\Rightarrow 2x_1^2 = 1$

$\Rightarrow x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = -\frac{1}{\sqrt{2}}$

$\therefore \varphi_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ first eigenvector solved

Now, $A\varphi_2 = \lambda_2 \varphi_2, \qquad \lambda_2 = 3, \quad \varphi_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$

Similarly calculating

Normalized eigenvector $\varphi_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* Book Example — Eigenvalue, Eigenvectors

$A\varphi_1 = 2\varphi_1$ Page-63

$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ determinant = 0

Equations, $x_1 + x_2 + 3x_3 = 2x_1$

$x_1 + x_2 - 3x_3 = 2x_2$

$3x_1 - 3x_2 - 3x_3 = 2x_3$

A suitable eigen vector, $x^1 = (k \quad k \quad 0)^T$, $\lambda_1 = 2$

Applying normalization, $k^2 + k^2 + 0^2 = 1$, $\Rightarrow k = \frac{1}{\sqrt{2}}$

$\therefore x^1 = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\right)^T = \frac{1}{\sqrt{2}}(1 \quad 1 \quad 0)^T$

replacing $\lambda_2 = 3$, $\lambda_3 = -6$

$x^2 = \frac{1}{\sqrt{3}}(1 \quad -1 \quad 1)^T$, $x^3 = \frac{1}{\sqrt{6}}(1 \quad -1 \quad -2)^T$

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Lecture 2 Part 1

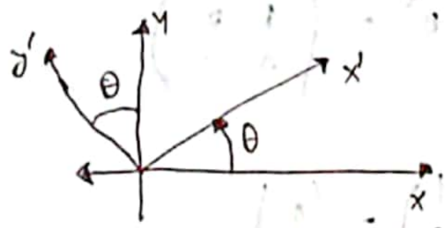
* If x,y is in 2D space

$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow$ ket notation for x, $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow$ ket notation for y

$|x\rangle + |y\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, if the same matrices are in vector form,

$\vec{x} = 1\hat{i} + 0\hat{j}$, $\vec{y} = 0\hat{i} + 1\hat{j}$, $\vec{x} + \vec{y} = 1\hat{i} + 1\hat{j}$, $\therefore$ The sum of ket notations yielded similar results to the vector sum.

* If x,y is rotated by θ.



$\hat{R}|x\rangle = |x'\rangle$, $\hat{R}|y\rangle = |y'\rangle$, $\hat{R}$ is the rotational matrix.

Basis stays the same for x, y matrix values.

let, $\hat{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\hat{R} \times |x\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$\hat{R} \times |y\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = |y'\rangle$

From above, in base form, $|x'\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

In base form, $|y'\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$$= -\sin\theta\,|x\rangle + \cos\theta\,|y\rangle$$

<u>Bra Notation</u>

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

$\therefore \langle x| = |x\rangle^{\dagger} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ <u>BRA</u>

Similarly, $\langle y| = \begin{pmatrix} 0 & 1 \end{pmatrix}$

<u>Outer Product</u>

$$|x\rangle\langle x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

<u>Inner Product</u>

$$\langle x|x\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

Self inner product is 1.

* $\langle y|x\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$, inner product of x, y is 0

Similarly, $\langle y|y\rangle$ inner product of y is 1.

* We will work with orthonormal basis.

Outer product for $|y\rangle\langle y| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

If we sum all the outer products we get Identity matrix.

$$|x\rangle\langle x| + |y\rangle\langle y| = I$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This is the completeness theorem. For 3x3, we need outer product of z.

Now from before, $(\hat{R}|x\rangle)\langle x| = (|x'\rangle)\langle x|$

$$\Rightarrow \hat{R}(|x\rangle\langle x|) = |x'\rangle\langle x| \quad \text{---} (1)$$

$$\hat{R}|y\rangle\langle y| = |y'\rangle\langle y| \quad \text{---} (2)$$

$\therefore$ (1) + (2) : $\hat{R}(|x\rangle\langle x| + |y\rangle\langle y|) = |x'\rangle\langle x| + |y'\rangle\langle y|$

$\therefore \hat{R} = |x'\rangle\langle x| + |y'\rangle\langle y|$ [$|x'\rangle$ = output, $|y'\rangle$ = input]

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longrightarrow \text{[The rotational matrix we saw before]}$$

* For a generic case,

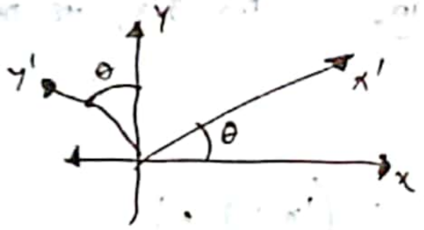
$$\hat{A}|w\rangle = 5|f\rangle$$
$$\hat{A}|g\rangle = 7|e\rangle$$

[$|w\rangle, |g\rangle$ are in orthonormal state]

$$\therefore \hat{A} = 5|f\rangle\langle w| + 7|e\rangle\langle g|$$

[Formula for generating elements of logic gates, a generic example]

Alternative Procedure



$$\hat{R} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad a_{ij} \to \text{final } (x', y')$$

i ↓ initial (x, y)

$$= \begin{bmatrix} a_{xx'} & a_{xy'} \\ a_{yx'} & a_{yy'} \end{bmatrix} = \begin{bmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle x|R|x\rangle & \langle x|R|y\rangle \\ \langle y|R|x\rangle & \langle y|R|y\rangle \end{bmatrix}$$

[Plugging values from before]

$$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

limitations of these techniques:

(1) Bases must be orthonormal base

Lecture 2 Part 2

* vectors are represented in column matrix forms.
* matrix helps shortening quantum math's as opposed to using Shrodinger's equation.
* Linear Algebra Demystified → Ch 4, Ch 5 basics for vector.

Book Notation = $\begin{bmatrix} (x, Rx) & (x, Ry) \\ (y, Ry) & (y, Ry) \end{bmatrix} \equiv \begin{bmatrix} \langle x|R|x\rangle & \langle x|R|y\rangle \\ \langle y|R|x\rangle & \langle y|R|y\rangle \end{bmatrix}$

Example 7.8

We know, $\hat{R} = |x'\rangle\langle x| + |y'\rangle\langle y|$

$\hat{R}|x\rangle = |x'\rangle \; ; \; \hat{R}|y\rangle = |y'\rangle$

Now, Hadamard operator, $\hat{H}|v_1\rangle = |v_1'\rangle = \frac{v_1+v_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}|v_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle$

$\hat{H}|v_2\rangle = |v_2'\rangle = \frac{v_1-v_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}|v_1\rangle - \frac{1}{\sqrt{2}}|v_2\rangle$

Calculating $|v_1'\rangle = \begin{vmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|v_2'\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now, hadamard eqn. $\hat{H} = |v_1'\rangle\langle v_1| + |v_2'\rangle\langle v_2|$

$\therefore \quad \hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \;\; 0) + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(0 \;\; 1)$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Alternative method, inner product:

$$\hat{R} = \begin{vmatrix} \langle x|\hat{R}|x\rangle & \langle x|\hat{R}|y\rangle \\ \langle y|\hat{R}|x\rangle & \langle y|\hat{R}|y\rangle \end{vmatrix}$$

$$\hat{H} = \begin{vmatrix} \langle v_1|v_1'\rangle & \langle v_1|v_2'\rangle \\ \langle v_2|v_1'\rangle & \langle v_2|v_2'\rangle \end{vmatrix}$$

$$\langle v_1|v_1'\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \left[|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |v_1'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right.$$

$$= \frac{1}{\sqrt{2}} (1)$$

$$\langle v_2|v_2'\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-1)$$

$$\langle v_2|v_1'\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1)$$

$$\langle v_1|v_2'\rangle = \frac{1}{\sqrt{2}}(1)$$

$$\therefore \hat{H} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

Another Procedure

Given $|v_1\rangle = -$

$|v_2\rangle = -$

$\hat{H}|v_1\rangle = \frac{1}{\sqrt{2}}|v_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle$ ——— (i)

$\hat{H}|v_2\rangle = \frac{1}{\sqrt{2}}|v_1\rangle - \frac{1}{\sqrt{2}}|v_2\rangle$ ——— (ii)

$\hat{H} = ?$

$$\hat{H} = \begin{vmatrix} \langle v_1|v_1'\rangle & \langle v_1|v_2'\rangle \\ \langle v_2|v_1'\rangle & \langle v_2|v_2'\rangle \end{vmatrix} = \begin{vmatrix} \langle v_1|Hv_1\rangle & \langle v_1|Hv_2\rangle \\ \langle v_2|Hv_1\rangle & \langle v_2|Hv_2\rangle \end{vmatrix}$$

Now, $\langle v_1| \times (i) \rightarrow \langle v_1|Hv_1\rangle = \langle v_1| \frac{1}{\sqrt{2}}|v_1\rangle + \frac{1}{\sqrt{2}}|v_2\rangle)$

$$= \frac{1}{\sqrt{2}}(\langle v_1|v_1\rangle + \langle v_1|v_2\rangle)$$

$$= \frac{1}{\sqrt{2}}(1+0) = \frac{1}{\sqrt{2}}(1)$$

Now, $\langle v_1| \times (ii) \rightarrow \frac{1}{\sqrt{2}}(1)$

Similarly, $\langle v_2| \times (i) \rightarrow \frac{1}{\sqrt{2}}(1)$

Finally, $\langle v_2| \times (ii) \rightarrow \frac{1}{\sqrt{2}}(-1)$

Yields : $\hat{H} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

Ex. 7.7

$$\hat{A}|v_1\rangle = |v_2\rangle + 4|v_3\rangle$$

$$\hat{A}|v_2\rangle = 2|v_1\rangle$$

$$\hat{A}|v_3\rangle = |v_1\rangle - |v_3\rangle$$

$$\hat{A} = \begin{vmatrix} \langle v_1|Av_1\rangle & \langle v_1|Av_2\rangle & \langle v_1|Av_3\rangle \\ \langle v_2|Av_1\rangle & \langle v_2|Av_2\rangle & \langle v_2|Av_3\rangle \\ \langle v_3|Av_3\rangle & \langle v_3|Av_2\rangle & \langle v_3|Av_3\rangle \end{vmatrix}$$

using 3rd technique

$$\hat{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & -1 \end{bmatrix}$$

Example 7.9 — Third Rule   Book Page → 159 (Linear Algebra Demystified)

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |y\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |z\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{L}\,|x\rangle = |x'\rangle$$

$$\hat{L}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$$\hat{L}\,|y\rangle = |y'\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|x'\rangle = 1\,|x\rangle + 3\,|y\rangle + 2\,|z\rangle$$

$$|y'\rangle = 1\,|x\rangle + 0\,|y\rangle + 0\,|z\rangle$$

$$|z'\rangle = 0\,|x\rangle + (-2)\,|y\rangle + 4\,|z\rangle$$

$$\therefore \quad \hat{L} = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle & \langle x|z'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle & \langle y|z'\rangle \\ \langle z|x'\rangle & \langle z|y'\rangle & \langle z|z'\rangle \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -2 \\ 2 & 0 & 4 \end{bmatrix}.$$

Schaum's Book

Example 6.1   when there is no orthonormal basis

$\hat{F}|(u_1)\rangle = c_1|u_1\rangle + c_2|u_2\rangle$ ——— (i)

$\hat{F}|(u_2)\rangle = c_3|u_1\rangle + c_4|u_2\rangle$ ——— (ii)

$$\hat{F} = \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

Given $\hat{F} = (2x+3y, 4x-5y)$

$S = \{u_1, u_2\} = \{(1,2), (2,5)\}$

$|\hat{F}(1,2)\rangle = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}2\\5\end{pmatrix}$

$\Rightarrow \begin{pmatrix}8\\-6\end{pmatrix} = \begin{pmatrix}1c_1 + 2c_2 \\ 2c_1 + 5c_2\end{pmatrix}$

$\Rightarrow c_1 + 2c_2 = 8, \quad 2c_1 + 5c_2 = -6$

$c_1 = 52, \quad c_2 = -22$

Now, $\hat{F}|u_2\rangle = |\hat{F}(2,5)\rangle = \begin{pmatrix}19\\-17\end{pmatrix}$

$= c_3|u_1\rangle + c_4|u_2\rangle$

$= c_3\begin{pmatrix}1\\2\end{pmatrix} + c_4\begin{pmatrix}2\\5\end{pmatrix}$

$$= \begin{pmatrix} 1 \cdot C_3 + 2C_4 \\ 2 \cdot C_3 + 5C_4 \end{pmatrix} = \begin{pmatrix} 19 \\ -17 \end{pmatrix}$$

$\therefore \; C_3 + 2C_4 = 19$

$2C_3 + 5C_4 = -17$

$\therefore \; C_3 = 129, \; C_4 = -55, \; \therefore \; \hat{F} = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$

Lecture 3 Part 1

Change of Basis: topic for this lecture

Revision: Inner Product: $\langle x|y\rangle$

$$= (|x\rangle^T)^* \, |y\rangle$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 0$$

Structure: $\langle$ Final State | Initial State $\rangle$

Outer Product / Tensor Product: $|x\rangle\langle y|$

$$= X \otimes Y$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Inner Product vs Outer Product without Bra-Ket

$(x^T)^* y$, $x (y^T)^*$

Linearly Independent

Let $A = 2\hat{i} + 3\hat{j} + 9\hat{k}$, i, j, k are linearly independent.

If orthonormality, orthogonality property is maintained, then linearly independent.

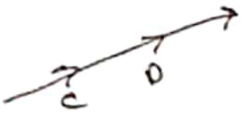
Let, $\bar{C}, \bar{D}, \bar{E}$ vectors,

$$x_1 C + x_2 D + x_3 E = 0$$

It is linearly independent, if $x_1 = 0, x_2 = 0, x_3 = 0$

Else, linearly dependent.

For vectors:



C & D are not linearly independent.

Book Examples: 5.10, 5.11 [Schaum]

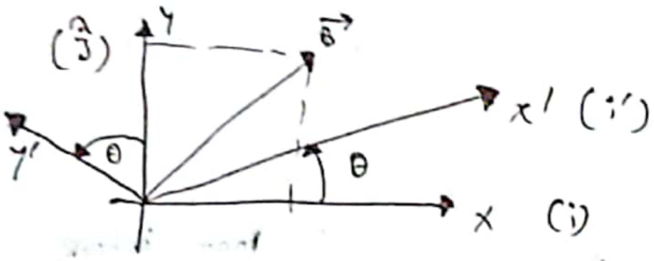
Expanding a vector in terms of Basis

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + \dots + c_n|\phi_n\rangle \longrightarrow \text{[Initial State]}$$

$$c_1 = \langle\phi_1|\psi\rangle, \quad c_2 = \langle\phi_2|\psi\rangle$$

After measurement, we get the Final State.

* Change of Basis



$\vec{B} = 2\hat{i} + 5\hat{j}$ $\qquad |x\rangle \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $|y\rangle \longrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

Now, $\vec{B} = a\hat{i}' + d\hat{j}'$ [Change of Basis]

$|B\rangle = 2|x\rangle + 5|y\rangle = \begin{pmatrix}2\\5\end{pmatrix}$

$|B\rangle = a|x'\rangle + d|y'\rangle = \begin{pmatrix}2\\5\end{pmatrix}$

$\hat{R}|x\rangle = |x'\rangle$

$\hat{R}|y\rangle = |y'\rangle$

$|x'\rangle = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

$|y'\rangle = \begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$

We can use this as basis.

To find out a, d, we know $\langle \text{Initial}|\text{Final}\rangle$

$|B\rangle = a|x'\rangle + d|y'\rangle$

$a \to \langle x'|B\rangle$, $\quad d \to \langle y'|B\rangle$

$\therefore \langle x'|B\rangle = \langle x'|(2|x\rangle + 5|y\rangle)$

$= 2\underbrace{\langle x'|x\rangle}_{\cos\theta} + 5\underbrace{\langle y'|y\rangle}_{\cos\theta}$

$= (2\cos\theta + 5\cos\theta)$ [Value for a]

$\langle y'|B\rangle = 2\langle y'|x\rangle + 5\langle y'|y\rangle$

$= 2\cdot(-\sin\theta) + 5\cdot(\cos\theta)$ [Value for d]

Lecture 3 Part 2

For rotated base,

$\vec{B} = 2\hat{i} + 5\hat{j} = B_x\hat{i} + B_y\hat{j}$

Now, $\langle x'|B\rangle = B_x\langle x'|x\rangle + B_y\langle x'|y\rangle$

This is for a.

From before,

$A\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}$

$\hat{i} \rightarrow |x'\rangle$

$\hat{j} \rightarrow |y'\rangle$

$\hat{A}\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}$ Target

For d, $d = \langle y'|B\rangle = B_x\langle y'|x\rangle + B_y\langle y'|y\rangle$

$\hat{A}\begin{pmatrix} B_x \rightarrow |x\rangle \\ B_y \rightarrow |y\rangle \end{pmatrix} = \begin{pmatrix} a \rightarrow |x'\rangle \\ d \rightarrow |y'\rangle \end{pmatrix}$

$\Rightarrow \begin{bmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{bmatrix}\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}$

Now, $\hat{A} = \begin{bmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{bmatrix}$

The vector hasn't changed. we need to represent $\vec{B}$ with $\begin{pmatrix} a \\ d \end{pmatrix}$

Shortcut: operate B_x, B_y over A to find a, d.

$$\therefore \hat{A} \begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}$$

we know, $\hat{R}|x\rangle = |x'\rangle$, $\hat{R}|y\rangle = |y'\rangle$

$$\hat{R} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{bmatrix}$$

$$(\hat{R}^T)^* = \begin{bmatrix} \langle x|x'\rangle^* & \langle y|x'\rangle^* \\ \langle x|y'\rangle^* & \langle y|y'\rangle^* \end{bmatrix} = \begin{bmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{bmatrix}$$

$$\therefore (\hat{R}^T)^* = \hat{A}$$

initial basis : $|u_1\rangle, |u_2\rangle, |u_3\rangle$

Final basis : $|v_1\rangle, |v_2\rangle, |v_3\rangle$

$$\hat{A} = \begin{bmatrix} \langle v_1|u_1\rangle & \langle v_1|u_2\rangle & \langle v_1|u_3\rangle \\ \langle v_2|u_1\rangle & \langle v_2|u_2\rangle & \langle v_2|u_3\rangle \\ \langle v_3|u_1\rangle & \langle v_3|u_2\rangle & \langle v_3|u_3\rangle \end{bmatrix}$$

Linear Algebra Demystified Book

Example 7.12

$$\hat{A} = \begin{bmatrix} \langle u_1 | v_1 \rangle & \langle u_1 | v_2 \rangle \\ \langle u_2 | v_1 \rangle & \langle u_2 | v_2 \rangle \end{bmatrix}$$

$$\hat{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \Bigg| \quad \begin{aligned} \bar{D} &= \alpha |v_1\rangle + \beta |v_2\rangle \\ \bar{D} &= p |u_1\rangle + q |u_2\rangle \end{aligned}$$

$p, q = ?$

$$\cancel{\langle u_1 | v_1 \rangle}\ \langle u_1 | v_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_1 | v_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_2 | v_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_2 | v_1 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{2}}$$

$$\therefore \hat{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \hat{A} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix} \qquad \begin{aligned} p &= \alpha + \beta \\ q &= \alpha - \beta \end{aligned}$$

Now,

1. Making the basis orthonormal
   → Gram-Schmidt Procedure

2. Diagonalizing a Matrix

Gram Schmidt

We can produce an orthonormal basis for this procedure.

6.6 Example

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$$

$\bar{w}_1 = v_1$

Normalizing the vector, $\langle v_1 | v_1 \rangle = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 6$

To find 2nd vector, $\langle \hat{w}_1, v_2 \rangle = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 3$

$$|\hat{w}_2\rangle = |v_2\rangle - \frac{\langle \bar{w}_1 | v_2 \rangle}{\langle \bar{w}_1 | \bar{w}_1 \rangle} |\bar{w}_1\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{3}{6} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

Similarly,

$$\langle \bar{w}_2 | \bar{w}_2 \rangle = \frac{1}{2}$$

$$|\hat{w}_2\rangle = \frac{1}{\sqrt{\langle \bar{w}_2 | \bar{w}_2 \rangle}} |\bar{w}_2\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

third vector,

$$|\bar{w}_3\rangle = |v_3\rangle - \frac{\langle \bar{w}_1 | \bar{v}_3\rangle}{\langle \bar{w}_1 | \bar{w}_1\rangle}\bar{w}_1 - \frac{\langle \bar{w}_2 | v_3\rangle}{\langle \bar{w}_2 | \bar{w}_2\rangle}\bar{w}_2$$

Now, $\langle \bar{w}_2 | v_3\rangle = -2$, $|\bar{w}_3\rangle = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

Normalize, $\langle \bar{w}_3 | \bar{w}_3\rangle = \cancel{0}\ 27$

last normalized basis vector

$$|w_3\rangle = \frac{1}{\sqrt{\langle \bar{w}_3 | \bar{w}_3\rangle}}\bar{w}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Example 87 : The eigen value Problem (Diagonalizing a matrix)

$$|A^{-1}| = \frac{1}{|A|}$$

Now, $0 = |B - \lambda I| = |S^{-1}AS - \lambda I|$

as $S^{-1}S = I$, $\therefore \lambda I = \lambda(S^{-1}S)I = \lambda S^{-1}IS$

now rewriting the determinant

$$S^{-1}AS - \lambda I = S^{-1}(A - \lambda I)S$$

$$\therefore 0 = |S^{-1}(A - \lambda I)S|$$

now, $|AB| = |A|\ |B|$

This gives $0 = |S^{-1}(A-\lambda I)S| = d|S^{-1}| \cdot |A-\lambda I| \cdot |S|$

$\therefore\ 0 = |S^{-1}|\ |A-\lambda I|\ |S|$

$= |A-\lambda I|$, A, B have same characteristics, same eigenvalues.

Ex. 8.8 [Diagonalizing a Matrix]

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S^{-1} X S = B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

eigenvectors, $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore\ U = [v_1\ \ v_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^{+} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = U$$

$$\therefore\ U^{+} X U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ [Diagonalized Matrix]

Theorem 3.12

$$|v\rangle = c_1|u_1\rangle + c_2|u_2\rangle + \dots c_n|u_n\rangle$$

These are linearly independent

If this is orthonormal

Fourier coeff $k_i = \dfrac{\langle u_1|v\rangle}{\langle u_1|u_1\rangle}$

Ex 4.10

$u = |1\ 1\ 0\rangle$, $v = |1\ 0\ 3\ 2\rangle$, $w = |4\ 9\ 5\rangle$

Then w, v, u are linearly independent.

because $3u + 5v - 2w = (0\ 0\ 0) = 0$