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Course Code: CSE 499A.4

Assignment # 2

Topic: Basics of Quantum Computing

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1. Basics of Bra-Ket in Quantum

We know, from Fourier Series:

$$\Psi = c_1 \Phi_1 + c_2 \Phi_2 + \dots$$

$$\langle \Psi, \Phi_1 \rangle \qquad \langle \Psi, \Phi_2 \rangle \qquad \text{[Inner Product]}$$

$$= \int_{\langle T \rangle} \Psi \, \Phi_1^* \, dt$$

This is the approach / notation followed by mathematetians. This can be altered via $\boxed{\text{Bra-Ket notation}}$, which will be different than the former approach.

Bra-Ket notation is crucial to understand the recent Journals in Quantum Computing. It is also the standard notation.

Now, same previous notation written differently,

$$\text{Wave packet} \rightarrow \psi = c_1 \underset{}{\overset{\text{Eigen}}{\phi_1}} + c_2 \phi_2 + \ldots$$

$$\langle \psi, \phi_1 \rangle \qquad \langle \psi, \phi_2 \rangle$$

$$\langle \phi_1 | \psi \rangle \quad ; \quad \boxed{\text{Here, } \langle \text{Final State} | \text{initial state} \rangle}$$

Why use this? In this case, this notation says after reducing the wave packets ψ and doing the measurements, we get Eigen Functions ϕ_n. Here, Initial State was ψ and the final state was ϕ_1. If we square this we get $|c_1|^2$ probability.

$$|\langle \phi_1 | \psi \rangle^2| = |c_1|^2$$

The ket-notation for this would be

$$\langle \varphi_1 | \psi \rangle \longrightarrow | \psi \rangle$$

Now suppose, for a notation $\langle \psi | \psi \rangle = 1$, $\langle \psi, \psi \rangle = 1$

If The ket notation here, $|\psi\rangle$ is a 1×1 matrix,

The other part must match the condition for matrix multiplication result.

So, $\underbrace{\langle \psi}_{\text{must be } 1\times2} | \underbrace{\psi \rangle}_{\text{must be } 2\times1} = 1 \Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$ [Satisfied]

So generally, if $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, $\langle \psi | = \left(|\psi\rangle \right)^{*T}$ [Adjoint]

$\langle \psi |$ = is called Bra, $|\psi\rangle$ = is called ket

Together, $\langle \psi | \psi \rangle$ is Bra-Ket.

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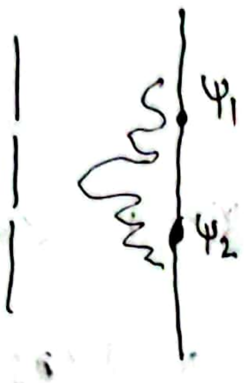
In double slit experiments,



$|\psi_1 + \psi_2|^2$

Wave mechanics is tough to calculate these ψ_1, ψ_2 entities.

But, with Bra-Ket notations, $C_1 = \langle \phi | \psi \rangle$

Using the probability amplitude, we can complete these calculations very easily. And, even more complex slit experiments can be solved by easily by using matrix mechanics, especially bra-ket notations.

In fourier transformation, we know,

$$\psi(k) = \phi(k) = \int \psi(x) \frac{e^{jkx}}{\sqrt{2\pi}} dx$$

Using Bra-Ket, $\langle k|\psi\rangle = \int \langle x|\psi\rangle \langle k|x\rangle \, dx$

$$= \int \langle k|x\rangle \langle x|\psi\rangle \, dx$$

These are treated as probability amplitudes.

<u>Projection:</u> Let, ψ be a quantity. Before measurement or calculations, ψ could be like a form of wave - packets. If we don't project it into anything, it still persists as an independent state like $|\psi\rangle$. If we project it to x, $\langle x|\psi\rangle \rightarrow$ it becomes $\psi(x)$. If we project it to k, $\langle k|\psi\rangle \rightarrow$ it becomes $\psi(k)$. It remains an independent quantity otherwise.

* Quantum Mechanics For Pedestrians Book *

- Electromagnetic Wave, $\bar{E} = \left[E_{0x}\, e^{i(kz-\omega t)}\right]\hat{x}$

Where, Intensity $I \propto |E_{0x}|^2$

In the interpretation of quantum mechanics, the amplitudes do not lead to intensities, but leads to probabilities.

For example, $\psi = c_1 \varphi_1 + c_2 \varphi_2 + \dots$

here, $|c_1|^2$ is connected to intensity.

Where the intensity is higher, the probability amplitude is higher.

The $\bar{E}$ is a plane wave, we can also rewrite it as,

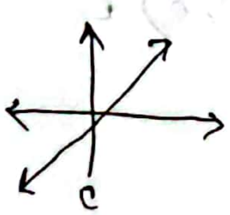
$$\bar{E} = \left[E_{0x}\, e^{i(kz-\omega t)}\right]\hat{x} + \left[E_{0y}\, e^{i(kz-\omega t)}\right]\hat{y}$$

The short form of the previous equation is,

$$\bar{E}(v,t) = \left(|E_{ox}|, |E_{oy}|, 0\right) e^{i(kz-\omega t)} \quad [\text{Plane-wave}]$$

The relative phase of the previous two equations was 0°. Now, the polarization could be 0°, 180° both.

Like



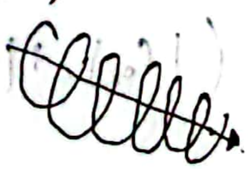
Now, if this is not the case and there is $e^{i\delta}$, where δ is relative phase.

Then, $\bar{E}(v,t) = \left(|E_{ox}|, |E_{oy}| e^{i\delta}, 0\right) e^{i(kz-\omega t)}$

Now, if $\delta = \pm \frac{\pi}{2}$, and $|E_{ox}| = |E_{oy}|$

Then the wave above is called circularly polarized wave.

For instance,



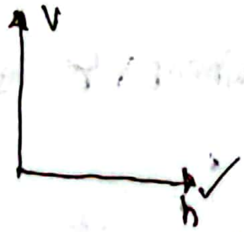
If this circular wave is circulating with respect to the right hand, (thumb Rule) then it is called Right hand Circular Polarized wave (RCP), same for left hand is (LCP).

If the phases are not equal, then it is called elliptical polarization.

We have previously seen,

$$\bar{E}(r,t) = \left(|E_{0x}|, |E_{0y}| e^{i\theta}, 0\right) e^{i(kz-\omega t)}$$

If $-i$, $+i$ then:



If polarization is only by the horizontal, (only x component)

$$E_h = \left(|A_{0x}|, 0, 0\right) e^{i(kz-\omega t)}$$

If polarization is only by the vertical, (only y component)

$$E_v = \left(0, |B_{0y}|, 0\right) e^{i(kz-\omega t)}$$

If <u>RCP</u>,

$$E_R = \left(|C_{0x}|, i|C_{0x}|, 0\right) e^{i(kz-\omega t)}$$

If <u>LCP</u>,

$$E_L = \left(|C_{0x}|, -i|C_{0x}|, 0\right) e^{i(kz-\omega t)}$$

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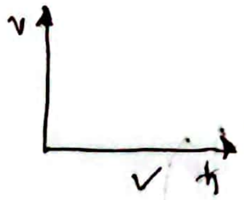
Simplifying the equations

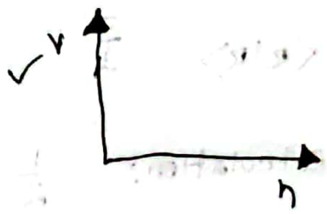
• All of the four rules above have the term $e^{i(kz - \omega t)}$ in common.

We can ommit this part.

• All of the z axis values are 0. We can ommit these too.

Now,





For horizontal polarization, we can assign a horizontal state as $|h\rangle$

For vertical polarization we can assign a vertical state as $|v\rangle$

*Ket vectors are column vectors.

$\therefore \; |h\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (Random values)

Now, from the equation of RCP

$$E_R = (1 \cdot |C_{ox}|, i \cdot |C_{ox}|, 0) \quad \text{[Simplified.]}$$

We can rewrite it as,

$$|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{[Coefficients]}$$

Now, $\langle R|R\rangle = \bar{I}$

The calculation: $\frac{1}{2} (1^* \ i^*) \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$= \frac{1}{2} [1 + (-i)(i)]$$

$$= \frac{1}{2} [2]$$

$$= [1] = \bar{I}$$

Similarly for LCP,

$$|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Now, for horizontal and vertical polarizations—

$$|h\rangle = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix}, \quad |v\rangle = \frac{1}{\sqrt{c^2+d^2}}\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\therefore a^*c + b^*d = 0$$

$$\Rightarrow (a^* \; b^*)\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\Rightarrow \boxed{\langle h|v\rangle = 0}$$

[This will be zero]

Self inner product leading to zero means they are following orthonormality.

Now,

for RCP, $|r\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{i}{\sqrt{2}}|v\rangle$

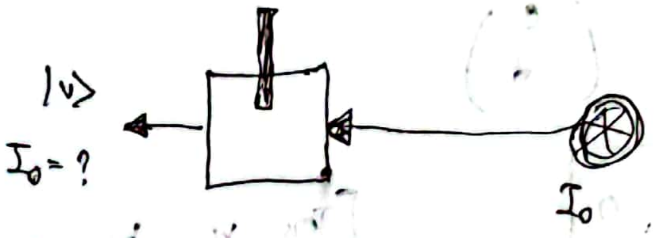
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

for LCP, just $-i$, Rest are same.

Again, for $|h\rangle = \dfrac{|r\rangle + |l\rangle}{\sqrt{2}}$

Now, if we pass a CP wave through a polarizer vertically,



The intensity then becomes half as

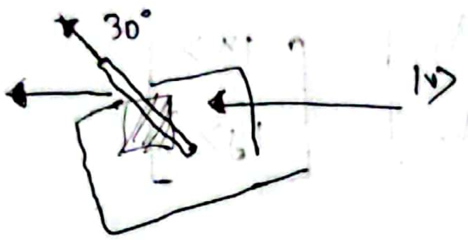
$$|r\rangle = \frac{1}{\sqrt{2}} \overset{c_1}{|h\rangle} + \frac{i}{\sqrt{2}} \overset{c_2}{|v\rangle} \quad , \quad \longrightarrow \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} = 50\%$$

As the horizontal state becomes non existent.

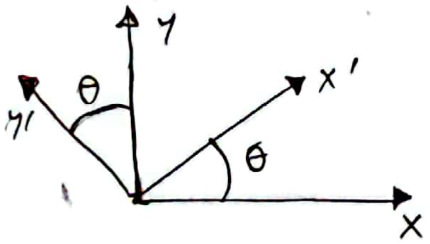
And it becomes vertical eigenstate.

These are vertically polarized.

Now, if we rotate a vertically polarized wave by 30°, and stop the polarizer,



For that we need the concept of Rotation Matrix.



$$R'' \begin{bmatrix} |x\rangle \\ |y\rangle \end{bmatrix} = \begin{bmatrix} |x'\rangle \\ |y'\rangle \end{bmatrix} \quad \text{—— (i)}$$ (Rotation Matrix)

Now to represent the initial state as the superposition of the final state,

$$|x\rangle = c_1' |x'\rangle + c_2 |y'\rangle$$

$$|y\rangle = c_1'' |x'\rangle + c_2'' |y''\rangle$$

(i) can be written as,

$$\begin{vmatrix} |x\rangle \\ |y\rangle \end{vmatrix} = R \begin{bmatrix} |x'\rangle \\ |y'\rangle \end{bmatrix}$$ [Inverse of Rotation Matrix]

It can finally be written as,

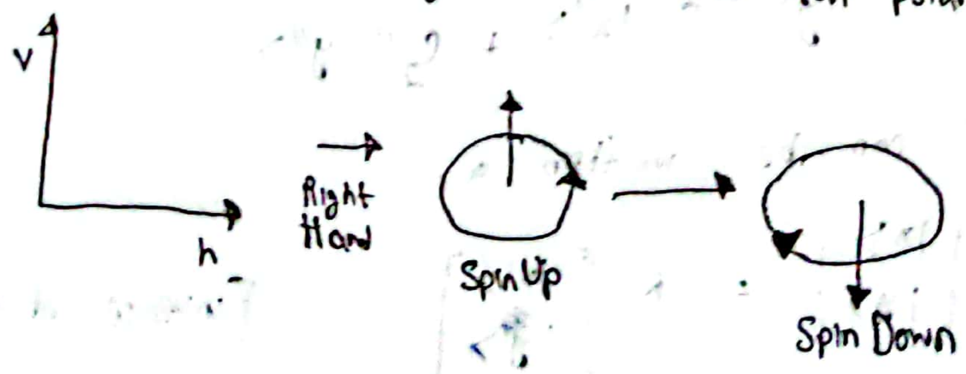
$$\begin{bmatrix} |x\rangle \\ |y\rangle \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} |x'\rangle \\ |y'\rangle \end{bmatrix}$$

Lec 2: Basic of Electron Spin & Their Matrices

- ### The Stern-Gerlach Experiment

There are two states that are talked about in this experiment. Up and down. As we have learnt previously from polarizations, the states vertical and horizontal are not the same as this.

But if we use right hand rule for polarizations,



In the experiment, the north pole of the magnet is towards the positive z axis and the south pole is towards the negative z axis.
And if we pass it through,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we pass the spin up through, then with 100% probability it will be in the positive z axis.
If we pass the spin down through, then with 100% probability it will be in the negative z axis.

Superpositions of states can be written as:

$$|+\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

The difference between these two states can be seen as the electrons have a 50:50 chance of hitting positions 'up' and 'down' on the screen.

Calculation: $P_{\uparrow} = |\langle\uparrow|+\rangle|^2$

$$= \left|\frac{\langle\uparrow|\uparrow\rangle}{\sqrt{2}} + \frac{\langle\uparrow|\downarrow\rangle}{\sqrt{2}}\right|^2$$

$$= \frac{1}{2} = 50\%$$

And, $P_{\downarrow} = |\langle\downarrow|+\rangle|^2$

$$= \left|\frac{\langle\downarrow|\uparrow\rangle}{\sqrt{2}} + \frac{\langle\downarrow|\downarrow\rangle}{\sqrt{2}}\right|^2$$

$$= \frac{1}{2} = 50\%$$

Now when we flip the magnets by $90°$, magnets are aligned along the horizontal axis, called x axis.
Each electron has a 50:50 chance of going to left and going to right. The same occurs when we create the electron in spin state $|\downarrow\rangle$.

When we create the electron in $|+\rangle$ and send it through x, we find it ends up on the left. Similarly, for $|-\rangle$, it ends up on the right.

So, $|+\rangle$ and $|-\rangle$ act the same in x as $|\uparrow\rangle$, $|\downarrow\rangle$ do in z ∴ Spin should be a quantity with a directional character which is a vector

$$S = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

Rotation Matrix for State Vector

Different than the other:

Real world: $R = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

State Vector: $R = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

The Spin Observable

Spin of an electron is described by a vector. Spin is a physical property with units.

Unit is: of angular momentum, Js, or kgm^2s^{-1}

Dirac's constant for electrons to express the magnitude of the spin:

$$\hbar = \frac{h}{2\pi} = 1.054572 \times 10^{-34}\ Js$$

$|+\rangle, |-\rangle$ is determined by the direction of an electron.

We may recognise $\hbar$ as Plank's constant. $\hbar$ is also called the reduced Planck constant. Value of the spin is in the z direction S_z of an electron in $|\uparrow\rangle$ is $+\frac{\hbar}{2}$. For $|\downarrow\rangle$ is $-\frac{\hbar}{2}$.

- Electromagnetic fields are quantized.

The Quantized nature,

$$\langle S_z \rangle = \frac{1}{2}\left(+\frac{\hbar}{2}\right) + \frac{1}{2}\left(-\frac{\hbar}{2}\right) = 0$$

As $\langle S_z \rangle = p\uparrow\left(+\frac{\hbar}{2}\right) + p\downarrow\left(-\frac{\hbar}{2}\right)$

The spin operator,

$$S_z = \frac{\hbar}{2}|\uparrow\rangle\langle\uparrow| - \frac{\hbar}{2}|\downarrow\rangle\langle\downarrow|$$

Derivation: Ch-4, Ch-3 A First Introduction to Quantum Physics

The matrix representation,

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Similar calculations for S_x, S_y.

Lec 3: Entangled State

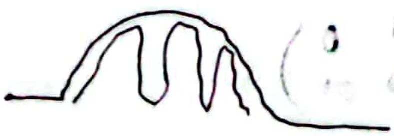
- Quantum Entanglement → entangled state
- Bell States
- Bell inequality

→ Wave Mechanics - older form
→ matrix/ Bra-ket - Newer form

Einstein's paper was written following the wave mechanics notations.

We have learnt single particle quantum mechanics for now. The concepts of Quantum Entanglement, Bell states, Bell inequalities are crucial for Quantum Computing.

For single particle Quantum Mechanics



Where the intensity for particles is the highest, we have tried to obtain the value for particle from $|\psi|^2$.

- Quantum Mechanics gets tougher for multi particle systems.

- For single particle Schrodinger's Equation,

$$H = \frac{p^2}{2m} + V \longrightarrow \hat{H}\Psi = E\Psi$$

For two particles,

$$H = \frac{p_1^2}{2m_1} + V_1 + \frac{p_2^2}{2m_2} + V_2$$

- But this not efficient for more particles.

- Important Books for this topic simplified:

→ Quantum mechanics for Pedestrians 2

→ AP French book – (Ch. 13)

→ Goswami Book – (Ch 9)

→ Robinett Book – (Ch 14)

* French Book for SCHRODINGER'S EQUATION FOR TWO NON INTERACTING PARTICLES → $\boxed{Pg\ 558}$

For non interacting particles,

$$E = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V_1(x_1) + V_2(x_2)$$

The difference between interacting and non-interacting particles can be observed in Hamiltonian equation.

If interacting,

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V(x_2 - x_1)$$

The potential V is dependend on the difference of these two co-ordinates, and thus called interacting particles.

For non interacting particles,

$$H = \frac{P_1^2}{2m_1} + V_1 + \frac{P_2^2}{2m_2} + V_2$$

This does not have the co-ordinates in the potential portion.

Einstein imagined a situation where two particles are interacting at first, then they strayed away so far from each other that there is no interaction among them. As there is no interaction, he stated if we still calculate, there would be problem.
Paper was written assuming non interacting particles.

Lackings of Einstein's Paper:

- Did not use the term 'Center of Mass Coordinates'.
- Despite not using that term, he used it and wrote the paper using those techniques.

In the equations of Hamiltonian, the x_1 and x_2 are variables. And these variables are independent. It can be expressed as three seperate functions,

$$\psi(x_1, x_2, t) = \psi_A(x_1) \cdot \psi_B(x_2) \cdot f(t)$$

If we place this in Schrodinger's equation,

$$\left[-\frac{h^2}{2m_1}\frac{d^2\psi_A}{dx_1^2} + V_1(x_1)\psi_A\right]\psi_B f$$

$$+\left[-\frac{h^2}{2m_2}\frac{d^2\psi_B}{dx^2} + V_2(x_2)\psi_B\right]\psi_A f = ih\,\psi_A\psi_B\frac{df}{dt}$$

Now, if we variable seperable technique, the initial assumption is correct, we get

$$(E_A + E_B)f = ih\frac{df}{dt}$$

where solution,

$$f(t) = e^{-i(E_A+E_B)it h}$$

The final Schrodinger's wave function ψ is,

$$\psi(x_1, x_2, t) = \psi_A(x_1) \cdot \psi_B(x_2)\, e^{-iEt/\hbar}$$

Now, we can get two different probabilities for dx_1 and dx_2 where,

$$dp = |\psi|^2 \, dx_1 \, dx_2 = |\psi_A(x_1)|^2 \, dx_1 \cdot |\psi_B(x_2)|^2 \, dx_2 \quad \text{—— (1)}$$

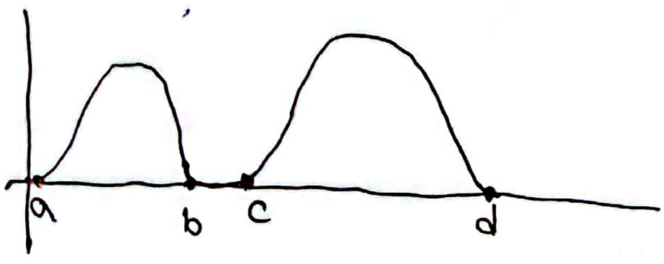
We know, for single particle ψ,

$\psi\psi^*$., where the physical meaning is :

If a particle is in a position, where it is between point a, b like :

then x, and $x + dx$ probability density can be figured out by using $\psi\psi^*$

To understand how to model for x_1, x_2 with $\psi\psi^*$, suppose,



suppose particle 1 is between a, b and partic[le]
2 is between c, d. And their wave functi[on]
are not overlapping.

Now the probabilites for achieving these two particles from a, b and c, d can be found in the equation we wrote for dP. Equation (1).

Discussion for beyond one dimension, multi dimension topic can be found in

Quantum Book : (Daniel Schroeder)

From the book,

The potential V can be:

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) + V_{12}(x_2 - x_1)$$

Part B: Introductory Quantum Computing

Lec 4:

1. **Quantum Computing:** Algorithms, Physics

let, $A|\psi\rangle = |\phi\rangle$

In Quantum Computing, the above equation,

$$|\psi\rangle \longrightarrow \boxed{\text{Quantum Gate}} \longrightarrow |\phi\rangle$$

let, $A|0\rangle$ be the input, we get, $A|0\rangle = |+\rangle$ as output.

where $|+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

Here, for input we gave a single state, but for output we get a superposition of a state.

Again, $A|1\rangle = |-\rangle \longrightarrow \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

Hadamard Matrix: $\hat{H} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Using Hadmard gate, we can convert a single bit input state to an output of superposition state.

Similarly,

Using Hadamard gates, we can convert a superposition state to a single bit state.

$A\,|+\rangle = |0\rangle$

Question: 1. Why the Hadamard Gate is not like conventional gates seen in Digital Logic Design (OR, AND, NOR, XOR etc)?

2. How do we design a gate B so we can get outputs like these for the given entangled states?

$|00\rangle \longrightarrow \boxed{B} \longrightarrow |00\rangle$

$|01\rangle \longrightarrow \boxed{B} \longrightarrow |01\rangle$

$|10\rangle \longrightarrow \boxed{B} \longrightarrow |11\rangle$

$|11\rangle \longrightarrow \boxed{B} \longrightarrow |10\rangle$

The gate 'B' is known as CNOT gates.

The equation for CNOT gate,

$$\hat{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

<u>Question 3:</u> Why does CNOT gate looks like this?

Quantum Computing Gates – wiki

<u>Lecture 5</u>

<u>Goal:</u> How to design any gates

let A be a gate with $|\psi_1\rangle, |\psi_2\rangle$ as inputs and $|\varphi_1\rangle, |\varphi_2\rangle$ be the corresponding gates.

$$|\psi_1\rangle \longrightarrow \boxed{A} \longrightarrow |\varphi_1\rangle$$

$$|\psi_2\rangle \longrightarrow \boxed{A} \longrightarrow |\varphi_2\rangle$$

Formula For Designing Any Gates

$$\boxed{\hat{A} = \sum_i |\text{output}\rangle_i \langle \text{input}|_i}$$

A is the gate.

Mathematical elaboration

$$\Rightarrow \hat{A} = |\varphi_1\rangle\langle\psi_1| + |\varphi_2\rangle\langle\psi_2|$$

[φ inputs, ψ outputs]

We know, $\hat{S}_x |+\rangle = \frac{1}{2}\hbar |+\rangle$

$\hat{S}_x |-\rangle = -\frac{1}{2}\hbar |-\rangle$

let the bases for S_z for $\varphi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\therefore \hat{A} = C_{00} |\varphi_1\rangle\langle\varphi_1| + C_{10} |\varphi_1\rangle\langle\varphi_2| + \dots$$

$$\hat{A} \text{ matrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}$$

The formula will even work for superpositions where,

$$\hat{S}_x |+\rangle = \frac{1}{2} \hbar |+\rangle$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Drawback of Formula: Output must be known.

Now, let $\hat{A} |\psi_1\rangle$, using the formula we can derive

$$= |\varphi_1\rangle \langle \psi_1| + |\varphi_2\rangle \langle \psi_2|$$

$$= |\varphi_1\rangle$$

[$\psi_2 | \psi_1$ must be orthogonal
$\langle \psi_2 | \psi_1 \rangle = 0$]

Now, S_x can be written, with formula,

$$\hat{S}_x = \frac{1}{2} \hbar |+\rangle \langle +| + \left(-\frac{1}{2} \hbar\right) |-\rangle \langle -|$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hadamard gate, using formula,

$$\hat{H}_A |0\rangle = |+\rangle \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\hat{H}_A |1\rangle = |-\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\therefore \hat{H}_A = \left[\, |+\rangle\langle 0| + |-\rangle \langle 1| \,\right]$$

Now, $\begin{bmatrix}a\\b\end{bmatrix} \begin{bmatrix}c & d\end{bmatrix} = A \otimes B = \begin{bmatrix}a\,[c \;\; d]\\ b\,[c \;\; d]\end{bmatrix} = \begin{pmatrix}ac & ad\\ bc & bd\end{pmatrix}$

$$\therefore \hat{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

* Now for $\hat{C}$, where,

$$\hat{C}|00\rangle = |00\rangle$$

$$\hat{C}|01\rangle = |01\rangle$$

$$\hat{C}|10\rangle = |11\rangle$$

$$\hat{C}|11\rangle = |10\rangle$$

Now, $|00\rangle = |0\rangle \otimes |0\rangle$

$|01\rangle = |0\rangle \otimes |1\rangle$

where, $|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\therefore \hat{C} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

Its matrix with indices [Formula]

$$\hat{C} = \begin{pmatrix} 1^{00} & 0^{01} & 0^{02} & 0^{03} \\ 0^{10} & 1^{11} & 0^{12} & 0^{13} \\ 0^{20} & 0^{21} & 0^{22} & 1^{23} \\ 0^{30} & 0^{31} & 1^{32} & 0^{33} \end{pmatrix}$$

Where the eqn.,

$$\hat{C} = \overset{00}{|00\rangle} + \langle 00| + \overset{11}{|01\rangle\langle 01|} + \overset{32}{|11\rangle\langle 10|} + \overset{23}{|10\rangle\langle 11|}$$

For example,

for $\hat{c}\,|10\rangle = |11\rangle$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

[Matches in multiplication]

- Now, for $\hat{A} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}$

Where Bases, $\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

And, $\hat{A} = C_{00}\,|\varphi_0\rangle\langle\varphi_0| + C_{01}\,|\varphi_0\rangle\langle\varphi_1| + C_{10}\,|\varphi_1\rangle\langle\varphi_0| + C_{11}\,|\varphi_1\rangle\langle\varphi_1|$

<u>Exception:</u>

Cases that does not follow this rule,

$\hat{S}_x$, $\hat{S}_y$ the spin operators.

If we want to calculate $\hat{S}_x$ like,

$$\hat{S}_x = C_{00} |+\rangle \langle +| + C_{01} |+\rangle \langle -| + C_{10} |-\rangle \langle +|$$
$$+ C_{11} |-\rangle \langle -| \quad \text{[In generic Formula]}$$

It won't even work, and not be equivalent.

Because for example, the product of say, the first term,

$$C_{00} |+\rangle \langle +| \neq \begin{bmatrix} C_{00} & 0 \\ 0 & 0 \end{bmatrix}$$

<u>Qubit:</u> Quantum Bit. Normal bits can either be 0 or 1. Due to superposition, both can be exist at the same time before measuring the output.

For Quantum Computing it is a drawback that we need to develop seperate algorithms for seperate tasks.

<u>Register in Quantum Computing:</u> It has an input state, if the input state is entangled with another state, if 2^n, then register can hold 2^{n+1} amount of data.

In quantum computing, versus with classical computing where we can work with $|0\rangle, |1\rangle$

in QC, we can work with $|a\rangle \otimes |b\rangle$

$= |ab\rangle$

where, $|00\rangle$ and $|11\rangle$ etc.

For a register $|a\rangle \otimes |b\rangle$ of two qubits, a possible basis could be $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. If we understand this as a binary notation, then the states are given by decimal notation : $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. Classical bits can be only one of these, qubits can be superposed.

Ex. We store a number 0-7 in a register. For this we need 3 bits (Generally, we need n bits to store one of the 2^n numbers between 0 and $2^{n}-1$)

Classical register can store <u>one of</u> these only

$0 = 000$; $1 = 001$, $2 = 010$, $3 = 011$, $4 = 100$
$5 = 101$, $6 = 110$, $7 = 111$.

Whereas qubits, 3 qubits can occupy following states

$0: |000\rangle$, $1: |001\rangle$, $2: |010\rangle$, $3: |011\rangle$,
$4: |100\rangle$, $5: |101\rangle$, $6: |110\rangle$, $7: |111\rangle$

This is before measurement. Taking the superposed states, Only for input this.

We can represent this as,

$$|q\rangle = \sum_{x,y,z \in \{0,1\}} c_{xyz} |xyz\rangle .$$

Here, the state vector allows us to store $2^3 = 8$ numbers at once. and 2^N numbers for N qubits (Here 3 qubits).
Measurement gives only one of the basis states. A measurement yields one from 0-7 not all at once.
The inputs are using superposition virtually.

The Deutsch Algorithm → Quantum Mech for Pedestrians 2

It calculates a boolean function $f: \{0,1\} \to \{0,1\}$

It was shown lastly if we get $|0\rangle$, $f(0)=f(1)$ else for $|1\rangle$, these are different. For the factor of 2 this works. But if we scale up, time depends polynomially. Classical computer: exponentially. Reduced time by half.

let, $$|z\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$$

The multiplication $$|z\rangle = \frac{|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle}{2\sqrt{2}}$$

In decimal, $$|z\rangle = \frac{|0\rangle+|1\rangle+|2\rangle+|3\rangle+|4\rangle+|5\rangle+|6\rangle+|7\rangle}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \sum_{k=0}^{7} |k\rangle$$

We can make it compact like this.

For n qubits,

$$|z\rangle = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} |k\rangle$$

* **Quantum Gates, Quantum Computers → Book**

Discussed: Hadamard, phase, CNOT gates..

For 2 qubit gates, CNOT gates can be used to implement these. We can equate a gate using a unitary operator (Ex: Fourier Transformation)

** Quantum Gates are reversible and the input can be found from the output. As it involves only unitary transformations. This is before measurement.

Final state is achieved by measuring through projective measurement.

Charm of Quantum Gates: Only three of these are needed to perform all computational operations. 1 qubit (two) gates and 2-qubit gate (one)

1 qubit Gates:

- Hadamard Gates : Described before
- Phase Shift Gates:

$$\varphi_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$$\Rightarrow \hat{\varphi}_\varphi |q\rangle = e^{iq\varphi} |q\rangle$$

or, $|q\rangle = |0\rangle$ OR, $|q\rangle = |1\rangle$

Again, $\hat{\varphi}_\varphi |0\rangle = |0\rangle$

$\hat{\varphi}_\varphi |1\rangle = e^{i\varphi} |1\rangle$

Using Formula, $\hat{\varphi}_\varphi = |0\rangle\langle 0| + e^{i\varphi} |1\rangle\langle 1|$

Quantum State Representation: Dancing with Qubits Book

Chapter 7:

- Even if uncontrolled phase enters, our probability calculations remains same.
- Can lead to problematic outputs for such uncontrolled phases.

* Bloch sphere representation in Quantum Computing → Wiki

Here, φ, θ here are azimuthal angle, and the other is the angle of z axis with respect to the radius.

For other spin: apart from S_x, S_y, S_z, in the book, they worked with S_θ. Where, General Formulation,

$$|\psi(\theta,\varphi)\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

θ can be $0°$–$360°$. Values for x, y different directions in book. (A first Introduction to Quantum Physics).

- Exercise → Dancing with Qubits

Q6.5 → 5

Lecture 6

Basic discussion of quantum chip design

$\longrightarrow$ Playing Qubits book, Chap 11: getting physical

Revision of previous lecture: qubits, registers.

* <u>Unitary Operator:</u> $U|\psi\rangle = |\psi'\rangle$, Example is fourier transform is one kind of unitary transform.

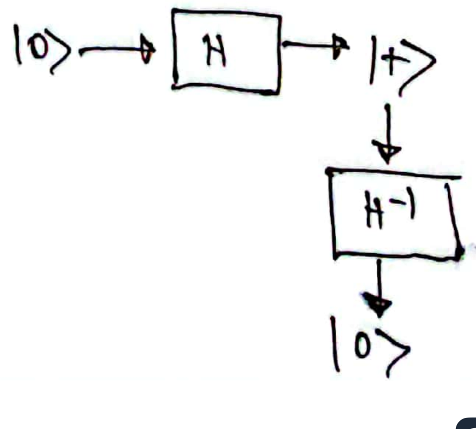
If we inverse in spherical coordinate system,

$$(\hat{U}^{-1}\hat{U})|\psi\rangle = U^{-1}|\psi'\rangle$$

$$\Rightarrow \hat{I}|\psi\rangle = U^{-1}|\psi'\rangle$$

$\rightleftharpoons |\psi\rangle$, this returns the initial. reversible via gate, <u>before measurement</u>.

For Hadamard gates,



As $H^{-1} = H$ for Hadamards, we get the initial $|0\rangle$ back by passing it through H gate initially, getting a superposition state and passing it to H^{-1} gate again. But if we measure, it won't be same

<u>For 1 qubit gate design</u>: Hadamard, phase gate.

More on Hadamard:

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|\psi\rangle \longrightarrow \boxed{H} \longrightarrow |\psi'\rangle$$

$$|0\rangle \longrightarrow \boxed{H} \longrightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \longrightarrow \boxed{H} \longrightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Chapter 9-12: Dancing with Qubits Book for physical implementations.

CNOT Gate Elaboration (Revision)

We have seen, $\hat{c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Where,

$$|00\rangle \rightarrow \boxed{c} \rightarrow |00\rangle$$
$$|01\rangle \rightarrow \boxed{c} \rightarrow |01\rangle$$
$$|10\rangle \rightarrow \boxed{c} \rightarrow |11\rangle$$
$$|11\rangle \rightarrow \boxed{c} \rightarrow |10\rangle$$

and the common formula, (Revision)

$$\boxed{\hat{A} = \sum_i |output\rangle_i \langle input|_i}$$

For $\hat{c} = \overset{00}{|00\rangle\langle 00|} + \overset{11}{|01\rangle\langle 01|} + \overset{32}{|11\rangle}\overset{23}{\langle 10|}$

$+ |10\rangle \langle 11|$

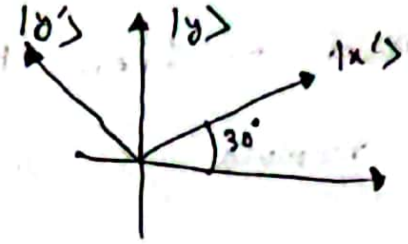
These were indexed previously.

- Revision of S_x, the exception case.

Why can we design all the gates this way?

We know $\hat{R}|x\rangle = |x'\rangle$

$\hat{R}|y\rangle = |y'\rangle$



Now,

$$(\hat{R}|x\rangle)\langle x| = |x'\rangle\langle x| \text{ ——— (i)}$$

$$(\hat{R}|y\rangle)\langle y| = |y'\rangle\langle y| \text{ ——— (ii)}$$

Now, (i) + (ii) $\rightarrow$

$$\hat{R}\left(|x\rangle\langle x| + |y\rangle\langle y|\right) \rightarrow \text{completeness Theorem, this is } \bar{\bar{I}}\text{, identity matrix.}$$

$$= |x'\rangle\langle x| + |y'\rangle\langle y|$$

$$\therefore \hat{R} = |x'\rangle\langle x| + |y'\rangle\langle y|$$

[Formula proved via Rotation Matrix.]

The advantage:

let $|xyz\rangle$, and we pass an entagled state $|xy\rangle$, like—

$$|xy\rangle \longrightarrow \boxed{\text{Gate}} \longrightarrow |xyz\rangle$$

Does it produce seperable state on entagled state?

If the output is entagled we can't seperate. Product state is seperable. If it follows $\boxed{ad=bc}$ then it is seperable, if it doesn't it is entangled. By using this we can determine before hand whether output would be of which kind.

And, $\hat{G}|\psi\rangle = |\phi_{final}\rangle$

we need to know this form. which is why the formula is so useful.

Hadamard and $\hat{\phi}$

for the inputs, we get $|0\rangle \longrightarrow |0\rangle$

$$|1\rangle \longrightarrow [e^{i\phi}|1\rangle]$$

And, $\hat{\phi} = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$

$$\begin{pmatrix}0\\1\end{pmatrix} \quad (1\ 0)$$

$$= \begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix}$$

Why the concept of Bloch Sphere.

In classical computing,

In quantum, let it be: $a|1\rangle + b|1\rangle$

a, b can be anything. It can take any angle of ϕ, $\theta \to 0° - 180°$. This has connection with spin. And it is represented by bloch sphere.

Peter Cock Book:

Bloch Sphere: The spin can be obtained in any direction. Until it is not measured, the spin does not have an exact value. It is a superposition of a state.

For bloch sphere, in quantum computing, if we want to generate state of any point, the sphere radius must be 1. And the general equation,

$$|\Psi(\theta, \phi)\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

New Topic:

2 Qubit Gate

- Concepts of CNOT gate.

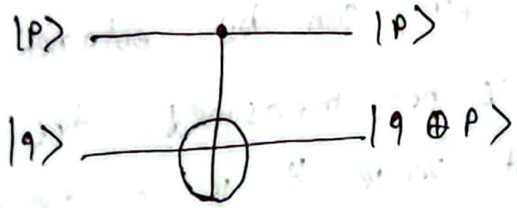
We have seen

$$|1\underline{0}\rangle \longrightarrow \boxed{c} \longrightarrow |1\underline{1}\rangle$$

$$|1\underline{1}\rangle \longrightarrow \boxed{c} \longrightarrow |1\underline{0}\rangle$$

Changing of these bits is done by an XOR operation.

The representation of the XOR:



If the q is not a single bit, but consists many qubits, then

$$\begin{bmatrix} |p\rangle \\ |q\rangle \end{bmatrix} \longrightarrow \begin{bmatrix} |p\rangle \\ |q \oplus f(p)\rangle \end{bmatrix}$$

Tensor Product:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} 1B & 2B & 3B \\ 4B & 5B & 6B \\ 7B & 8B & 9B \end{pmatrix} = \begin{pmatrix} a & b & 2a & 2b & 3a & 3b \\ c & d & 2c & 2d & 3c & 3d \\ 4a & 4b & 5a & 5b & 6a & 6b \\ 4c & 4d & 5c & 5d & 6c & 6d \\ 7a & 7b & 8a & 8b & 9a & 9b \\ 7c & 7d & 8c & 8d & 9c & 9d \end{pmatrix}$$

More concepts: Direct products, direct products on vector spaces, tensor sums (APPENDIX)

2 bit Qubit Gates

If the target bit is in state $|0\rangle$,

$$|p\rangle|0\rangle \rightarrow |p\rangle|p\rangle : p \in \{0,1\}$$

Copies do not exist but CNOT gate works as a copier only for $|0\rangle|0\rangle$ and $|1\rangle|0\rangle$. arbitrary states are not copied but entangled.

Kickback operation: with $m=1$, n is not specified, otherwise function f takes two values 0 and 1. For $|q\rangle$ we choose the superposition $|q\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. Then, $|0 \oplus f(p)\rangle = |f(p)\rangle$

briefly,

$$|p\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \rightarrow |p\rangle (-1)^{f(p)} \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Basic Ideas of Quantum Computer

- It can 'entangle' states, then the resultant can yield register together simultaneously contain 2^{N+M} values.

The Deotsch Algorithm review

- Previous lecture elaborated.

$$\rightarrow \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$\rightarrow \frac{1}{\sqrt{2}} \underbrace{\boxed{|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)}}_{\text{KickBack.}} + \underbrace{\boxed{|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)}}_{\text{KickBack}}$$

New p' state passed through Hadamand.

Then, $$\frac{\left[(-1)^{f(0)} + (-1)^{f(1)}\right]|0\rangle + \left[(-1)^{f(0)} - (-1)^{f(1)}\right]|1\rangle}{2}$$

Process: $(-1)^{f(0)}|0\rangle \longrightarrow \boxed{H} \longrightarrow (-1)^{f(0)}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

for $f(0) = f(1)$, we measure $|0\rangle$: otherwise $|1\rangle$.

It answers the question with one measurement.

Classical approach needs two.

Lecture 7:

Quantum Computing Explained → Math Based Book

Ch 1-8 covered except Ch 5. Not needed.

Ch 4 needed → for tensor concepts.

Ch 7: Entangled

- When is a state Entangled?

Not all states are entangled. When two systems are entangled the state of each composite system can only be described with reference to the other state.

Let $|\psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, The state is seperable

iff $ad = bc$

Ex. 7.2

Writing each state of Bell state a column vector,

$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$; $|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

Here, $|\beta_{00}| \Rightarrow a = d = 1/\sqrt{2}$, $b = c = 0$, $ad \neq bc$. $\therefore$ This state is entangled. Similar calculation for others.

2.3 Example

$H \otimes H$, H is hadamard matrix.

Is $H \otimes H |00\rangle$ entangled? Initial state $|00\rangle$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

for $|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, calculating $H \otimes H |00\rangle$:

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$ad = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$, $bc = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

$ad = bc$, Product state.

- Tensor Products Chapter : 4
- Ch 1-9 Examples
- Introductory book: Dancing with Qubits.
- Famous Algorithm, Shor. Algorithm