RESEARCH PAPER- Inverse-cubic law of index fluctuation distribution in Indian markets

What is the paper about (Abstract summary)

This paper researchers majorly talk about the distribution of fluctuations in market indicators, particularly focusing on the Indian financial market, one of the largest emerging markets globally. They analyzed data from the NSE and BSE, both at tick-by-tick and daily closing levels. The findings revealed that the cumulative distributions of index returns in the Indian market exhibit long tails consistent with a power law, characterized by an exponent $\alpha \approx 3$. This "inverse-cubic law" observed in both 1-minute and 1-day time scales is similar to patterns found in developed markets like the New York Stock Exchange (NYSE). These results provide strong evidence supporting the universality of market fluctuations' behavior across different markets.

All the keywords in the paper and the definition

- Index returns: The percentage change in the value of an index over a certain period of time.
- High-frequency data: Financial data that is recorded and analyzed at a very high frequency, such as tick-by-tick data.
- Market fluctuations: The changes in the value of financial assets over time.
- Cumulative distribution: A function that shows the probability that a random variable is less than or equal to a certain value.
- Power law: A functional relationship between two quantities, where one quantity varies as a power of the other.
- Inverse-cubic law: A power law with an exponent of approximately 3, which describes the distribution of index returns in the Indian financial market.
- Emerging market: A market that is in the process of becoming more developed and integrated into the global economy.

Mathematical formulae and logic:

The paper discusses the statistical properties of financial markets and the fluctuations in market indices. The main mathematical formula used in the paper is the logarithmic return, defined as:

$$R(t, \Delta t) \equiv \ln I(t + \Delta t) - \ln I(t)$$

where I(t) is the market index at time t and delta t is the time scale over which the fluctuation is observed. This formula calculates the change in the market index over a specific time scale.

$$P_c(r > x) \sim x^{-\alpha}$$

This formula represents the power law distribution of returns, where Pc(r > x) represents the probability of returns greater than x, and α is the exponent of the power law. The exponent α is found to be approximately 3, which is referred to as the "inverse-cubic law"

This formula defines the new rescaled volatility, σ(t), which is used to normalize the returns R(t) in order to compare the distributions. The formula removes the contribution of R(t) itself from the volatility to avoid underestimation of the tail of the normalized return distribution.

$$\sigma_{\Delta t}(t) = \sqrt{\frac{1}{N-1}\sum_{t' \neq t}\{R(t', \Delta t)\}^2 - \langle R(t', \Delta t)\rangle^2},$$

as described in Ref. [2]. The resulting *normalized* return is given by,

$$r(t, \Delta t) \equiv \frac{R - \langle R \rangle}{\sigma_{\Delta t}(t)}.$$

Interpretation of the data tables in the papers

Table 1
Comparison of the power-law exponent α of the cumulative distribution function for various index returns

Index	Δt	Power-law fit		Hill estimator	
		Positive	Negative	Positive	Negative
Nifty ('03–'04)	1 min	2.98 ± 0.09	3.37 ± 0.10	3.22 ± 0.03	3.47 ± 0.03
	5 min	4.42 ± 0.37	3.44 ± 0.21	4.51 ± 0.03	4.84 ± 0.03
	15 min	5.58 ± 0.88	3.96 ± 0.27	6.25 ± 0.03	4.13 ± 0.04
	30 min	5.13 ± 0.41	3.92 ± 0.45	5.65 ± 0.03	4.30 ± 0.03
	60 min	5.99 ± 1.52	4.42 ± 0.65	7.85 ± 0.03	5.11 ± 0.04
Nifty ('90–'06)	1 day	3.10 ± 0.34	3.18 ± 0.28	3.33 ± 0.14	3.37 ± 0.14
Sensex ('91–'06)	1 day	3.33 ± 0.77	3.45 ± 0.25	2.93 ± 0.15	3.84 ± 0.12

Power-law regression fits are done in the region $r \geq 2$. The Hill estimator is calculated using the bootstrap algorithm.

Table 1 presents the results of the analysis of various index returns for the NSE Nifty and BSE Sensex, focusing on the power-law exponent (α) obtained from both power-law regression fits and the Hill estimator. Here are the key interpretations:
Nifty ('03–'04) - 1 Min Returns:

- The power-law exponent α for the positive and negative tails of the 1-minute returns is approximately 2.98 and 3.37, respectively.
- As the time scale (1t) increases, the α values tend to increase, indicating a heavier tail in the distribution.

- The Hill estimator also supports these findings, with values around 3.22 for the positive tail and 3.47 for the negative tail.

2. Nifty ('90–'06) and Sensex ('91–'06) - Daily Returns:
   - The daily returns for both Nifty and Sensex show a power-law behavior with α values close to 3, indicating heavy tails.
   - The Hill estimator results support this observation, with values around 3.33 for Nifty and 3.45 for Sensex in the positive tail.

3. Effect of Time Scale (1t):
   - As the time scale increases from 1 minute to 1 day, α generally increases, suggesting that longer time scales lead to heavier tails.
   - This is consistent with the idea that longer time scales incorporate a broader range of market activities, resulting in more extreme events.

4. Intra-Day Variation:
   - The analysis of intra-day variation in Nifty's 1-minute returns indicates differences in the tail behavior between opening/closing hours and the intermediate period.
   - The power-law exponents for the opening/closing hours are close to 3, while for the intermediate period, the exponent is closer to 4, suggesting a potential U-shaped pattern in intra-day volatility.

5. Consistency Across Indices:
   - The analysis suggests that the nature of fluctuation distribution, as characterized by the power-law exponent, is consistent across different market indices (Nifty and Sensex) and time scales.

6. Comparison with Gaussian Behavior:
   - At longer time scales, the distribution gradually converges to a Gaussian behavior, which is in line with observations in financial markets where extreme events become less frequent over longer periods.

In summary, the findings support the presence of heavy tails in the distribution of financial market returns, consistent with observations in various markets worldwide. The analysis provides insights into the dynamic nature of market behavior across different time scales and highlights the consistency of these patterns across different indices in the Indian financial markets.

Section vise understanding

1) Introduction:The most extensively studied market-invariant features include the distributions of fluctuations in overall market indicators, particularly market indices. It shows the use of the market indices to get a macroscopic understanding of the market behavior. Early theories assumed the changes due to many independent features of Gaussian distributions based on

independent external shocks but the gaussian distributions were only valid for large time scales. Levy distributions were much more suitable but the tail decayed exponentially. It was found that the tails followed the “inverse power law”. In a few indices negative tail follows inverse cubic law and some other indices follow Gaussian and exponential distributions.

2) Data: For concluding that inverse power law has universality in all markets, the Indian market's dynamics over varying time frames and index compositions has been observed. The dataset focuses on Nifty index data (Jan 2003–Mar 2004) and includes extended analyses of Nifty (Jul 1990–May 2006), Sensex (Jan 1991–May 2006), and BSE 500 (Feb 1999–May 2006).

3) Distribution of index returns: To compare various distributions we need to normalize the returns by dividing them with the volatility. But due to normalization, very large single return will be bounded. To avoid this we remove the contribution of returns in the volatility. The TE and TP statistics will converge to zero when the distribution follows power law and exponential respectively.

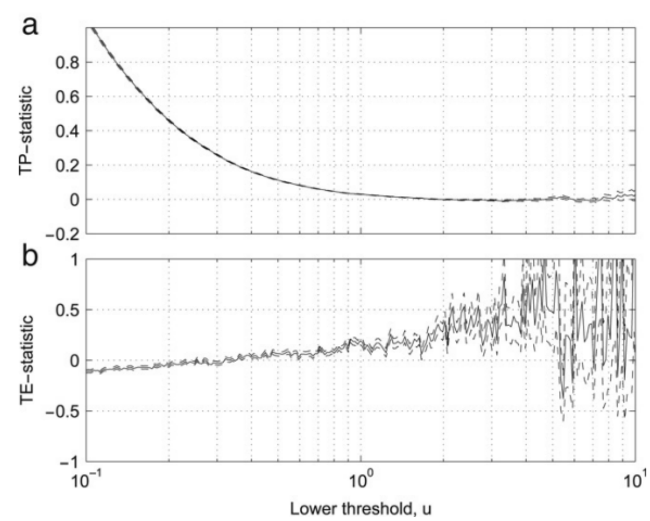

Fig. 2. (a) TP-statistic and (b) TE-statistic as functions of the lower cut-off u for positive returns of the NSE with time interval $\Delta t = 1$ min. The broken lines indicate plus or minus one standard deviation of the statistics.

TP-statistic suggests power-law for large u(cut off), while TE-statistic rejects exponential model for $u \geq 1$ and very low u, hinting at a possible exponential approximation in the intermediate range $2\times10^{-1} < u < 6\times10^{-1}$.

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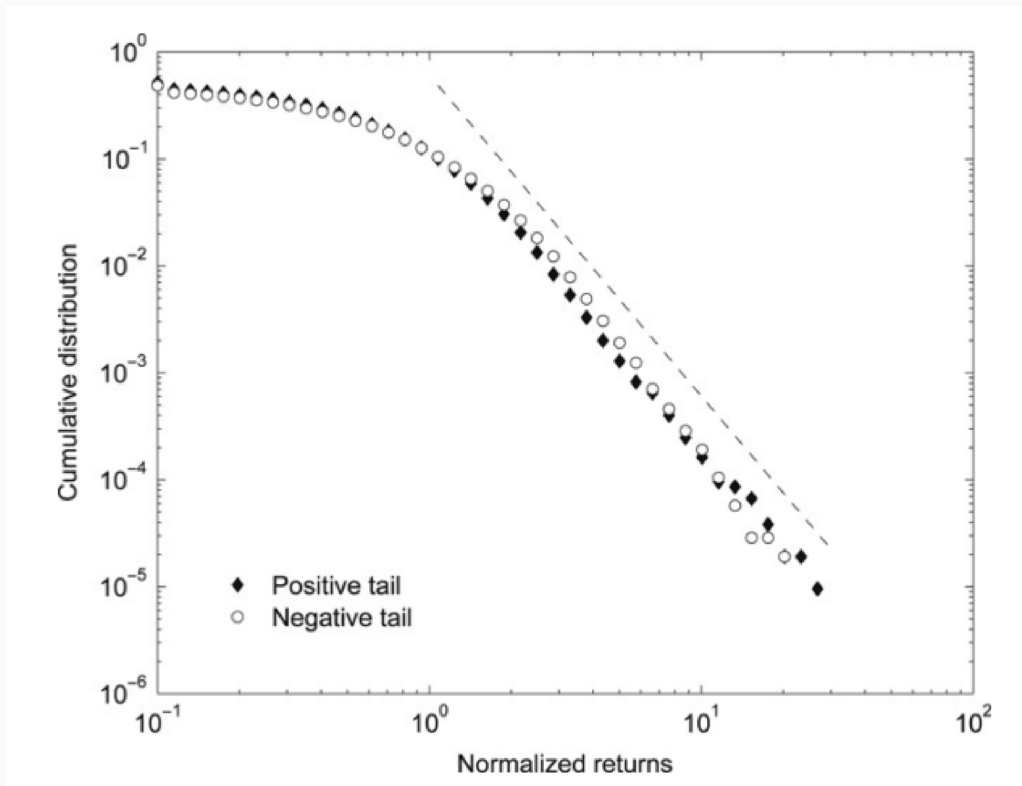

shows the cumulative distribution of the normalized returns for delta(t) = 1 min. For both positive and negative tails, there is an asymptotic power-law behavior.
For intraday, the tails follow the power law whereas the intermediate period follow an exponent of 4.

Comparison of the power-law exponent α of the cumulative distribution function for various index returns

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When we observed the variation of alpha for different time scales, we can interpret that even if the alphas increases initially from 1 min to 60 min, but for 1 day it follows the inverse cubic law i.e., the value of alpha is close to 3.

4) Conclusion: We cannot get accurate results from smaller datasets(individual stocks), they will be showing large deviations from the inverse cubic law.The study criticizes graphical fitting methods for being subjective and cut-off dependent, highlighting the need for more reliable statistical techniques. The research suggests that, despite differences in market dynamics, both the NSE and BSE in India exhibit remarkably consistent fluctuation behavior. This hints at

universal mechanisms shaping market fluctuations, independent of specific market details or economic conditions, offering insights into the physics of complex systems.