Assignment - 1

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Q.1 Bi-quadratic Interpolation

The expression for bi-quadratic interpolation

$$v(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij}\, x^i y^j$$

Expand the expression to get:

$$v(x,y) = a_{00} + a_{01} y^1 + a_{02} y^2 + a_{10} x^1 + a_{11} x^1 y^1 + a_{12} x^1 y^2 + a_{20} x^2 + a_{21} x^2 y^1 + a_{22} x^2 y^2$$

Matrix form for $v(x,y)$:

$$v(x,y) = \begin{bmatrix} 1 & y & y^2 & x & xy & xy^2 & x^2 & x^2y & x^2y^2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix}$$

There are 9 coefficients as we can see from the expression i.e- $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12},$

a_{20}, a_{21}, a_{22}.

To get value of 9 coefficients we need 9 equations (points) $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9$ that are as follows :-

$$\underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 1 & y_1 & y_1^2 & x_1 & x_1 y_1 & x_1 y_1^2 & x_1^2 & x_1^2 y_1 & x_1^2 y_1^2 \\ 1 & y_2 & y_2^2 & x_2 & x_2 y_2 & x_2 y_2^2 & x_2^2 & x_2^2 y_2 & x_2^2 y_2^2 \\ 1 & y_3 & y_3^2 & x_3 & x_3 y_3 & x_3 y_3^2 & x_3^2 & x_3^2 y_3 & x_3^2 y_3^2 \\ 1 & y_4 & y_4^2 & x_4 & x_4 y_4 & x_4 y_4^2 & x_4^2 & x_4^2 y_4 & x_4^2 y_4^2 \\ 1 & y_5 & y_5^2 & x_5 & x_5 y_5 & x_5 y_5^2 & x_5^2 & x_5^2 y_5 & x_5^2 y_5^2 \\ 1 & y_6 & y_6^2 & x_6 & x_6 y_6 & x_6 y_6^2 & x_6^2 & x_6^2 y_6 & x_6^2 y_6^2 \\ 1 & y_7 & y_7^2 & x_7 & x_7 y_7 & x_7 y_7^2 & x_7^2 & x_7^2 y_7 & x_7^2 y_7^2 \\ 1 & y_8 & y_8^2 & x_8 & x_8 y_8 & x_8 y_8^2 & x_8^2 & x_8^2 y_8 & x_8^2 y_8^2 \\ 1 & y_9 & y_9^2 & x_9 & x_9 y_9 & x_9 y_9^2 & x_9^2 & x_9^2 y_9 & x_9^2 y_9^2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{20} \\ a_{21} \\ a_{22} \end{bmatrix}}_{A}$$

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o) The 9 points are of form (x_i, y_i) where $i = 1$ to 9.

We have V, X and we can get the coefficient matrix.

$$\boxed{V = XA \quad \text{—— (1)}}$$

$$\Rightarrow A = X^{-1}V \quad \text{—— (2)}$$

Using eq. (2) we can get the coefficient matrix A.

Note :- o) In case of bi-quadratic interpolation, the method we follow is as follows :-

o) If we want pixel value at point (x, y) in input image, we find 9 points $(x_1, y_1), (x_2, y_2) \ldots (x_9, y_9)$ which are the 9 nearest neighbours and see the pixel value $V_1, V_2 \ldots V_9$ at those. Then we form V & X as shown above & calculate coefficient matrix A using $A = X^{-1}V$.

5) Now using this A, we can get the pixel value at point (x, y) in input grid using :-

$$V(x,y) = [1 \; y \; y^2 \; x \; xy \; xy^2 \; x^2 \; x^2y \; x^2y^2] \cdot A$$

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Q.2 $\underline{2 \times 2 \text{ image}}$:

$$\begin{array}{c} \text{origin} \\ \downarrow \end{array} \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

Interpolation factor $= c = 1.5$ on both X & Y

→ Dimension of input image $= 2 \times 2$

We know for interpolation factor of c on X & Y,

Dimension of output image $= (2 \times c) \times (2 \times c)$

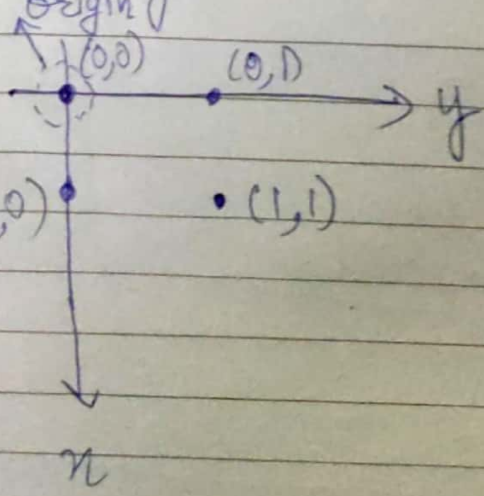
Here, input image $= (2 \times 2)$ & $c = 1.5$

Output image dimensions $= (2 \times 1.5) \times (2 \times 1.5)$

$\Rightarrow$ $\boxed{\text{" \quad " \quad " } = (3 \times 3)}$

→ Representing on a grid:

$\underline{\text{Input image}}$:-



pixel values:

$V(0,0) = 5$

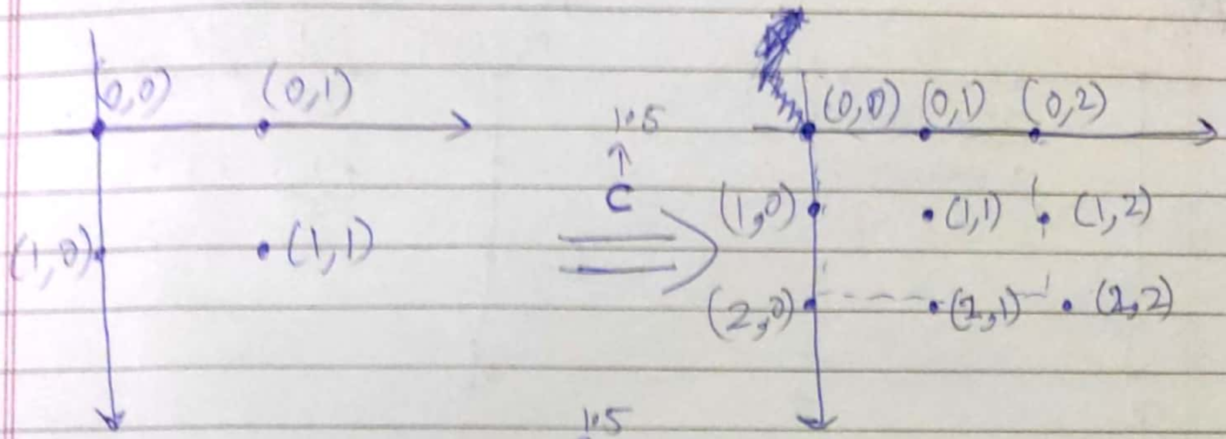
$V(0,1) = 10$

$V(1,0) = 10$

$V(1,1) = 20$

Scanned with CamScanner

→ Mapping for 3 coordinates



$$\underset{(2\times 2)}{\text{Input image}} \xRightarrow{\overset{1.5}{\uparrow}\, c} \underset{(3\times 3)}{\text{Output Image}}$$

Let's take 3 coordintes A^*, B^*, C^* in output grid which map to A', B', C' in input grid.

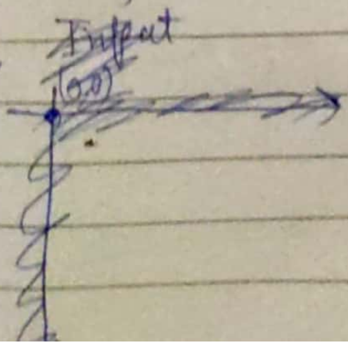
$A^*: (0, 0) \xrightarrow{c=1.5} A': (0/1.5, 0/1.5) = (0, 0)$

$B^*: (0, 1) \xrightarrow{c=1.5} B' = \left(0/1.5, 1/1.5\right) = (0, 0.667)$

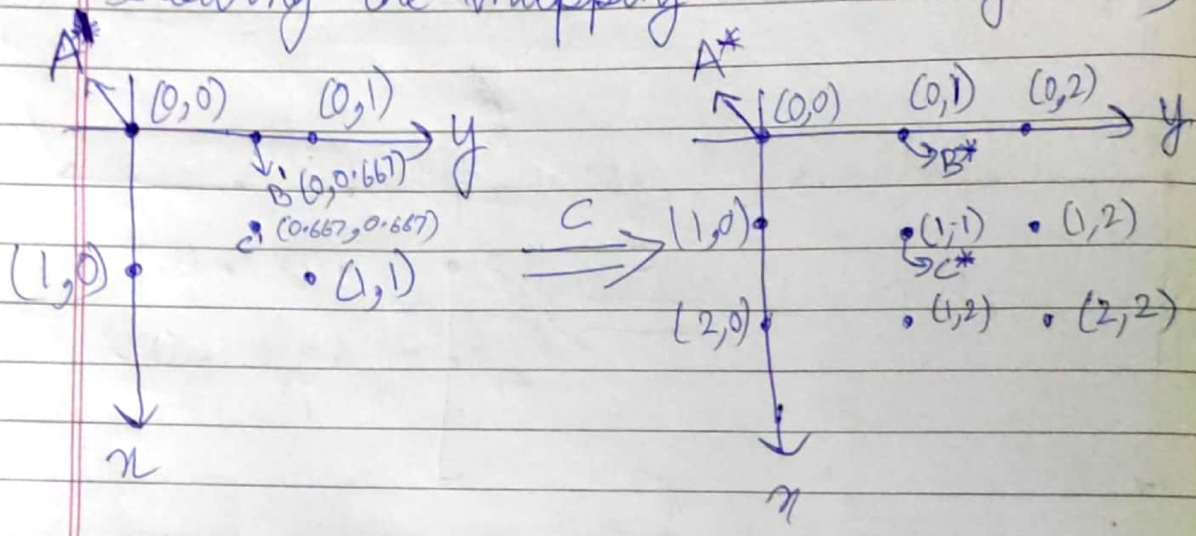
$C^*: (1, 1) \xrightarrow{c=1.5} C' = \left(1/1.5, 1/1.5\right) = (0.667, 0.667)$

(Note: for (x, y) in output grid, it will map to $(x/c, y/c)$)

Showing:



Showing the mapping on the grids,



→ For output coordinates (1,1) we want to find interpolated pixel value.

I will have to do a zero padding in the right & bottom direction.

After 0-padding, input image matrix,

$$\begin{bmatrix} 5 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ → 0 = padding

Proceeding, let $(x, y) = (1,1)$ in output image.

It will ~~map to~~ map (x', y') in input grid where :

$$(x', y') = (x/c , y/c)$$

$$\Rightarrow (x', y') = (1/1.5 , 1/1.5) = (0.667, 0.667)$$

Now find four nearest neighbours for (x', y') :-

1) $(x_1, y_1) = \left(\text{round}\left(\frac{x}{c}\right), \text{round}\left(\frac{y}{c}\right)\right)$

2) $(x_2, y_2) = \left(\text{round}\left(\frac{x+c}{c}\right), \text{round}\left(\frac{y}{c}\right)\right)$

3) $(x_3, y_3) = \left(\text{round}\left(\frac{x}{c}\right), \text{round}\left(\frac{y+c}{c}\right)\right)$

4) $(x_4, y_4) = \left(\text{round}\left(\frac{x+c}{c}\right), \text{round}\left(\frac{y+c}{c}\right)\right)$

$\Rightarrow$ 1) $(x_1, y_1) = \text{round}\left(\frac{1}{1.5}\right), \text{round}\left(\frac{1}{1.5}\right) = (1, 1)$

2) $(x_2, y_2) = \text{round}\left(\frac{2.5}{1.5}\right), \text{round}\left(\frac{1}{1.5}\right) = (2, 1)$

3) $(x_3, y_3) = \text{round}\left(\frac{1}{1.5}\right), \text{round}\left(\frac{2.5}{1.5}\right) = (1, 2)$

4) $(x_4, y_4) = \text{round}\left(\frac{2.5}{1.5}\right), \text{round}\left(\frac{2.5}{1.5}\right) = (2, 2)$

$\Rightarrow$ $V_1(x_1, y_1) = V(1,1) = 20$

$V_2(x_2, y_2) = V(2,1) = 0$

$V_3(x_3, y_3) = V(1,2) = 0$

$V_4(x_4, y_4) = V(2,2) = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} x_1 & y_1 & x_1y_1 & 1 \\ x_2 & y_2 & x_2y_2 & 1 \\ x_3 & y_3 & x_3y_3 & 1 \\ x_4 & y_4 & x_4y_4 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{A}$$

(bcoz for bilinear interpolation, $V(x,y) = ax + by + cxy + d$ i.e. $V = [x\ y\ xy\ 1]\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$)

Now, $V = XA$

$\Rightarrow A = X^{-1}V$

Thus here,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 4 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 & -1 \\ -2 & 1 & 2 & -1 \\ 1 & -1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -40 \\ -40 \\ 20 \\ 80 \end{bmatrix}$$

$\therefore$ We now have coefficients, (a, b, c, d)

$\therefore V(x,y) = ax + by + cxy + d = -40x - 40y + 20xy + 80$

For, $(x', y') = (0.667, 0.667)$

, $V(x', y') = (-40)(0.667) - 40(0.667) + 20(0.667)(0.667) + 80$

$\Rightarrow V(x', y') = 35.537$

Thus ~~as~~ $(1,1)$ in output maps to (x', y') in input grid, interpolated pixel value for $(1,1)$ is 35.537

Note: Explaining round():

round(0.5) → 0   round(1.6) = 2
round(1.5) → 1   round(1.3) = 1

basically for an integer x,

$$\text{round}(\underbrace{x.y}_{\text{number } x \text{ point } y}) = \begin{cases} x & \text{if } y \leq 5 \\ x+1 & \text{if } y > 5 \end{cases}$$

& similarly!

We are rounding towards bottom (rounding down) if number is less than or equal to $\underbrace{x.5}_{x \text{ point } 5}$ & upward to $(x+1)$ if number is greater than $x.5$.