A (Short) Reprise of High-Dimensional Expansion

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Preface

TODO

§1 Introduction: Definitions and Preliminaries

Definition 1.1. A simplicial complex is a hypergraph $G = (V, E)$ that is closed under *downward containment* i.e

$$S \in E, S' \subset S \implies S' \in E \tag{1}$$

Every $S \in E$ is called a face

We usually partition the faces of our simplicial complex according to their sizes.

Definition 1.2. Given a simplicial complex X

$$X = X(0) \cup X(1) \cup X(2) \cdots \cup X(d) \tag{2}$$

where $X(i)$ denotes the faces of size $i + 1$ - and is said to have a dimension i.

The notion of link is a rather important concept.

Definition 1.3. Given X a d- dimensional complex and $s \in X(i)$, the link of s is defined as follows:

$$X_s \triangleq \{t \in X \mid s \cup t \in X, s \cap t = \varnothing\} \tag{3}$$

It can be thought of as a generalization of the concept of the neighborhood of a graph, but for simplicial complexes.

Definition 1.4. Given X, a simplicial complex and $k < d$, for some non-negative k. The k-skeleton of X is the subspace of X that is the union of faces of dimension $\leqslant k$.

Definition 1.5. We say that a complex is a pure d-dimensional is every maximal face is of dimension exactly d.

Using these objects, we are ready to define the notion of expansion over simplicial complexes.

Definition 1.6. We say that X, a pure d-dimensional simplicial complex, is a λ-one-sided HDX (resp. λ-two-sided HDX) if

- 1-skeleton of X is a λ-one-sided expander (resp. λ-two-sided expander)
- $\forall i \leqslant d-2$, $s \in X(i)$, the 1-skeleton of X_s is a λ-one-sided expander (λ-two-sided expander)

Definition 1.7. A weighted pure simplicial complex is a pair (X, Π) where X is a pure d-dimensional simplicial complex endowed with a distribution $\Pi = (\pi_0, \pi_1, \ldots, \pi_d)$ over each level such that:

- π_i is a distribution over $X(i)$ for all $i = 0, 1, \ldots, d$
- π_d is arbitrary
- For all $0 \leq i \leq d$ and $\tau \in X(i)$, $\pi_i(\tau)$ is the probability that τ is selected via the following process:
  1. Draw a d-face σ from π_d
  2. Pick an i-face $\tau \subset \sigma$ where $|\tau| = i$

  Equivalently,

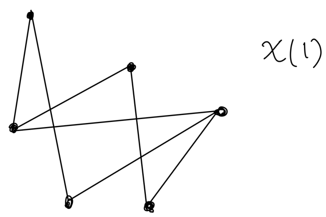
$$\pi_i(\tau) = \frac{1}{\binom{d}{i}} \sum_{\substack{\sigma \in X(d) \\ \tau \subset \sigma}} \pi_d(\sigma) \tag{4}$$

For any $0 \leq i < d$, π_i can be thought of as distribution induced by π_d. The weighted links of a weighted complex are themselves weighted complexes with distribution inherited from the global distribution.

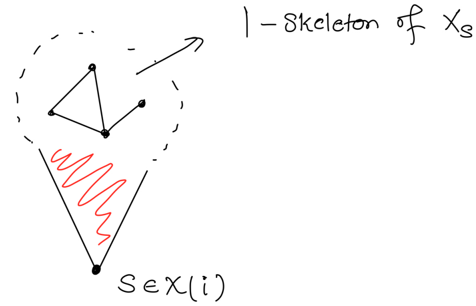
Definition 1.8. Let (X, Π) be a d-dimensional weighted simplicial complex. For all $0 \leq i \leq d$ and $\tau \in X(i)$, the weighted link (X_τ, π_τ) is defined as follows:

- $X_\tau = \{\sigma \mid \sigma \cup \tau \in X, \sigma \cap \tau = \varnothing\}$
- $\pi_{\tau,d-i}(\sigma) = \frac{\pi_d(\sigma \cup \tau)}{\sum_{\psi \in X_\tau(d-i)} \pi_d(\psi \cup \tau)}$

There is a notion of expansion of expansion over this general object, i.e. a weighted simplicial complex.



(a) 1-skeleton of a complex



(b) 1-skeleton of link corresponding to s

Definition 1.9 (Weighted Spectral Expander). A weighted simplicial complex (X, Π) is a (one-sided/two-sided) λ-local spectral expander if the underlying graph of every weighted i-link, for $0 \leq i \leq d-2$ is a (one-sided/two-sided) λ-spectral expander.

A central theme of these notes is to understand how we might construct two-sided HDXs with bounded degree.

Definition 1.10. We say that X is $(r_0, r_1 \ldots r_{d-1})$-regular if for $0 \leqslant i \leqslant d-1$ every $s \in X(i)$ is contained in r_i $(i+1)$-faces.

As it turns out, many important examples of HDXs are, in fact, not regular. However, assuming helps us avoid messy calculations, at least for now. Before we proceed, we discuss a few example

Example 1.11 (d-dimensional complete complex). The n-simplex, denoted as Δ^n is the n-dimensional simplicial complex that contains all subsets of $n+1$ elements. The d-dimensional complete complex on n vertices is the d-skeleton of Δ^{n-1}. Namely, these are all the subsets $[n]$ with size $d+1$. The 1-skeleton of every link is a complete graph and therefore a spectral expander, albeit with degree growing with n.

§2 Trickle-Down Theorem

One would think that to prove a certain simplicial complex is a HDX would require for us to show that every 1-skeleton of a link is an expander, which would certainly require a lot of work. In the seminal work of I. Oppenheim, [Opp18], they managed to show that if the $(d-2)$-faces of our complex is an expander then every link - notably of lower dimensions - is an expander too.

Proposition 2.1 (Trickle-Down, 2-dim). *If X is a 2-dimensional simplicial complex such that*

- *The graph $(X(0), X(1))$ is connected*
- *For all $v \in X(0)$, X_v - the link corresponding to v - is a one-sided λ-expander,*

then, $(X(0), X(1))$ *is a* $\frac{\lambda}{(1-\lambda)}$ *expander.*

Applied iteratively, we get,

Theorem 2.2 (Trickle-Down, d-dim). *Let X be a d-dimensional simplicial complex such that*

- 1*-skeleton of every link is connected (including the entire simplicial complex); and*
- *For all* $v \in X(d-2)$, X_v *is a one-sided* λ*-spectral expander*

then, X is a μ*-spectral expander where*

$$\mu = \frac{\lambda}{1-(d-1)\lambda} \tag{5}$$

Proof of Proposition 2.1. Let A denote the adjacency matrix of the 1-skeleton $(X(0), X(1))$. Suppose $f : X(0) \to \mathbb{R}$ is an eigenfunction with eigenvalue γ and assume $f \perp \mathbf{1}$. WLOG, assume that $\mathsf{E} f^2 = 1$. Observe,

$$\gamma = \langle f, Af \rangle = \mathsf{E}_{(u,v)\in X(1)} \left[f(u)f(v) \right] = \mathsf{E}_{w\in X(0)} \mathsf{E}_{u,v\in X_w(1)} \left[f(u)f(v) \right] \tag{6}$$

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Remark 2.1. *Note that there is also a trickle-down theorem for the smallest eigenvalue. For any* $v \in X(d-2)$, *if* A_v *(the adjacency matrix of* X_v*) has its smallest eigenvalue* $\geq \nu$ *then the 1-skeleton of X has its eigenvalue* $\geq \frac{\nu}{1-\nu}$. *Note that if* $\nu \leq 0$, *we have that* $\nu \leq \frac{\nu}{1-\nu}$. *Therefore, the smallest eigenvalue contracts (in magnitude).*

Remark 2.2. *One would naturally wonder how these definitions even differ? Well, the reason it's called a local-to-global theorem is every link is a complex itself. The theorem allows us to make a global statement, i.e. 1-skeleton of every link is a spectral expander, with a local observation, i.e. every* $(d-2)$*-link is a spectral expander.*

References

[1] Izhar Oppenheim. "Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps". In: *Discrete Comput Geom* 59.2 (Mar. 2018), pp. 293–330. ISSN: 1432-0444. DOI: 10.1007/s00454-017-9948-x. (Visited on 05/02/2025).