Practice Set
Double And Triple Integration.

1. Evaluate the double integrals:

   (a) $\iint_R x^2 \, dA$, where R is the region bounded by $y = x^2, y = x + 2$

   (b) $\iint_R (x^2 + y^2) \, dA$, where $R : 0 \le y \le \sqrt{1 - x^2}, 0 \le x \le 1$.

   (c) $\int_0^1 \int_2^{4-2x} dy \, dx$

   (d) $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y} dy \, dx$

2. Find the volume of the region under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy plane.

3. Use the given transformations to transform the integrals and evaluate them:

   (a) $u = x + 2y, v = x - y$ and $I = \int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} \, dA$

   (b) $u = x, v = xy, w = 3z$ and $I = \iiint_D (x^2 y + 3xyz) \, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x \le 2, \ 0 \le xy \le 2, \ 0 \le z \le 1\}$.

4. Evaluate the following volume integrals:

   (a) $\iiint_D (z^2 x^2 + z^2 y^2) \, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \le 1, -1 \le z \le 1\}$

   (b) $\iiint_D xyz \, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 \le 1, \ 0 \le z \le x^2 + y^2\}$

Practice Set
Double And Triple Integrals

1. Evaluate the double integrals

   (a) $\iint_R x^2 dA$ where R is the region bounded by $y = x^2, y = x + 2$

   The region is bounded by $y = x^2, y = x + 2$. Simple to notice that $y = x + 2$ is the upper curve and $y = x^2$ is the lower curve between the points of intersection $x = -1$ and $x = 2$. So it is enough to evaluate the iterated integral

   $$\int_{x=-1}^2 \left(\int_{x^2}^{x+2} x^2 dy \right) dx = \int_{-1}^2 x^2(x + 2 - x^2) dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^2 = \frac{163}{60}$$

   (b) $\iint_R (x^2 + y^2) dA$ where $R : 0 \le y \le \sqrt{1 - x^2}, 0 \le x \le 1$

   The region is the part of the unit disc $x^2 + y^2 \le 1$ that lies in the first quadrant. So the upper curve and lower curve can be easily identified as $y = \sqrt{1 - x^2}$ and $y = 0$ respectively. So it is enough to evaluate

   $$\int_{x=0}^1 \left(\int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy \right) dx = \int_{x=0}^1 \left(x^2 y - \frac{y^3}{3} \right) \Big|_0^{\sqrt{1-x^2}} dx = \pi/8.$$

   Second Method, It is very easy to do using polar co-ordinate: $x = r\cos\theta, y = r\sin\theta, 0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{2}$. Then

   $$I = \int_0^{\frac{\pi}{2}} \int_{r=0}^1 r^2 r \, dr \, d\theta = \frac{\pi}{8}.$$

   (c) $\int_0^1 \int_2^{4-2x} dy \, dx$

   The domain is bounded by $x = 0, y = 2$ and $y = 4 - 2x$. So it is easy to note that in the domain y ranges from 2 to 4 while x ranges from 0 to $4 - 2x$. So

   $$\int_0^1 \int_2^{4-2x} dy \, dx = \int_{y=2}^4 \left(\int_{x=0}^{(4-y)/2} dx \right) dy = \int_{y=2}^4 \frac{4-y}{2} dy = \frac{4y - y^2}{4} \Big|_2^4 = 1.$$

   Remark: Without changing the order of integration, we can easily solve this problem.

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(d) $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$

The domain is bounded by $x = 0$, $x = 2$, $y = 0$ and $y = 4 - x^2$. Hence

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx = \int_0^4 \left(\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\right) dy = \int_0^4 \frac{e^{2y}}{4-y}\left(\int_0^{\sqrt{4-y}} x\,dx\right) dy$$

$$= \int_0^4 \frac{e^{2y}}{2}\,dy = \frac{e^{2y}}{4}\,|_0^4 = \frac{1}{4} - \frac{e^8}{4}.$$

2. We use the formula $V = \iint_R f(x,y)\,dA$ where $f(x,y) \geq 0$ is a continuous real valued function defined over the domain R of the plane.

Here $f(x,y) = x^2 + y^2$ and the domain is bounded by $x = 0$, $y = 2 - x$ and $y = x$. So in this domain drawing a line parallel to y axis, it is easy to see that the domain may be described as $x \leq y \leq 2 - x$, $0 \leq x \leq 1$. In other words, the upper curve is $y = 2 - x$ and lower curve is $y = x$ between $x = 0$ and $x = 1$. Hence

$$V = \iint_R (x^2 + y^2)\,dA = \int_0^1 \left(\int_x^{2-x} (x^2 + y^2)\,dy\right) dx = \frac{4}{3}$$

3. Use the given transformations to transform the integrals and evaluate them:

(a) $u = x + 2y, v = x - y$. The Jacobian of the transformation is

$$J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3.$$

Therefore, $J(u,v) = -\frac{1}{3}$ and $|J| = 1/3$. The given domain is the triangle bounded by $y = x$, $y = 0$ and $x + 2y = 2$. The image of this triangle under the transformation is again a triangle bounded by $v = 0$, $v = u$ and $u = 2$. Hence

$$\int_0^{2/3} \int_y^{2-2y} (x+2y)e^{(y-x)}\,dA = \int_{u=0}^2 \left(\int_0^u ue^{-v}\frac{1}{3}\,dv\right) du = \frac{1}{3}(3e^{-2} + 1)$$

(b) The inverse of the given transformation is $x = u$, $y = \frac{v}{u}$ and $z = \frac{w}{3}$. So the

required Jacobian is

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$$

and the given integral is

$$\iiint_D (x^2 y + 3xyz)dV = \frac{1}{3} \int_{w=0}^{3} \int_{v=0}^{2} \int_{u=1}^{2} \left(v + \frac{vw}{u} \right) du \ dv \ dw = 2 + 3\log 2$$

4. (a) Using the cylindrical coordinates $x = r\cos\theta, y = r\sin\theta$ and $z = z$, the given domain may be represented as

$$D = \{(r, \theta, z) : \ 0 \le r \le 1, \ 0 \le \theta \le 2\pi, \ -1 \le z \le 1\}.$$

Hence the integral becomes

$$\iiint_D z^2(x^2 + y^2)dV = \int_{z=-1}^{1} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (z^2 r^2)r dr d\theta dz$$
$$\int_{z=-1}^{1} \int_{0}^{2\pi} z^2 \frac{1}{4} d\theta dz = \frac{\pi}{3}$$

(b) Again using the cylindrical coordinates

$$D = \{(r, \theta, z) : \ 0 \le z \le r^2, \ 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}.$$

Hence

$$\iiint_D xyz \ dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{r^2} zr^3 \cos\theta \sin\theta dz dr d\theta = 0$$

Good Luck!!!