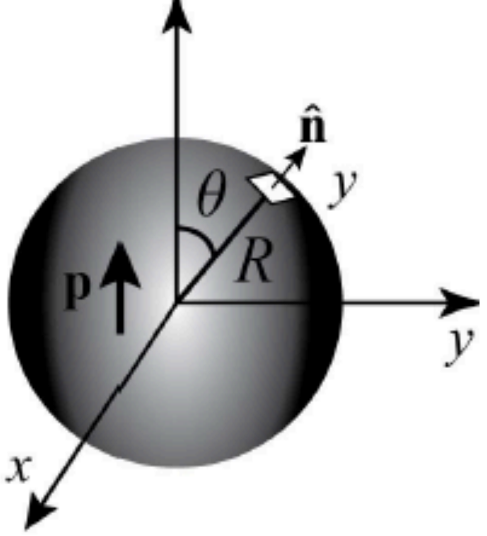
Lecture - 21

The Field of a Polarized Object:

Prob 4.10 (Griffiths, 3rd Ed.): Find the bound charges of a sphere of radius R, if its polarization is $\mathbf{P}(\mathbf{r}') = k\mathbf{r}$.



<u>Volume charge</u>

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) \quad = -\frac{1}{r^2} 3kr^2 \quad = -3k$$

<u>Surface charge</u>

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad = kR$$

Q: What is the electric field outside the sphere?

Volume and surface charge distributions are both symmetric with respect to the center of the sphere. So, the total charge can be thought of as being concentrated at the center

$$\text{Total charge } Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$$

$$= kR \times 4\pi R^2 \quad +(-3k) \times \frac{4\pi}{3} R^3$$

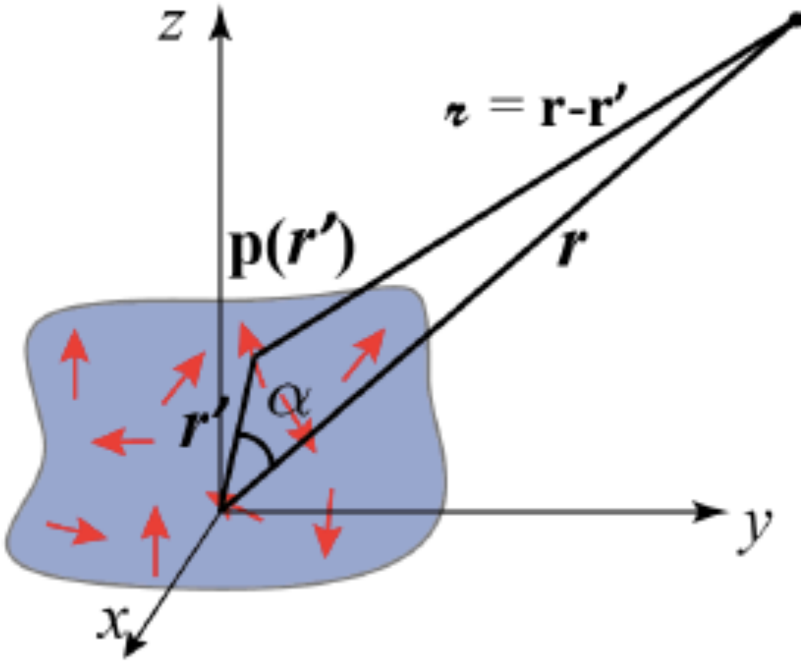
$$= 0$$

So the electric field outside the sphere is zero.

Questions 1:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\rho_b}{\mathscr{r}}\, d\tau'$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad \rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$$



Q: Is the decomposition unique?

Ans: Yes, it is. Because there is only one way in which an object can be divided into its surface and volume.

Questions 2:

Q: What happens to the Gauss's law when we have polarized objects?

Ans: We are going to answer this today.

Gauss's law in the presence of Dielectrics

Bound charges in a dielectric $\sigma_b = \mathbf{P}(\mathbf{r}) \cdot \hat{\mathbf{n}}$ $\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{r})$

In addition, there can be some free charge ρ_f in the dielectric as well

Total charge inside a dielectric is $\rho = \rho_b + \rho_f$

Gauss's law for the electric field is therefore

$$\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{r}) + \rho_f$$

Note: **E** is the total field, not just what is generated by the polarization

$$\boldsymbol{\nabla} \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\boldsymbol{\nabla} \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

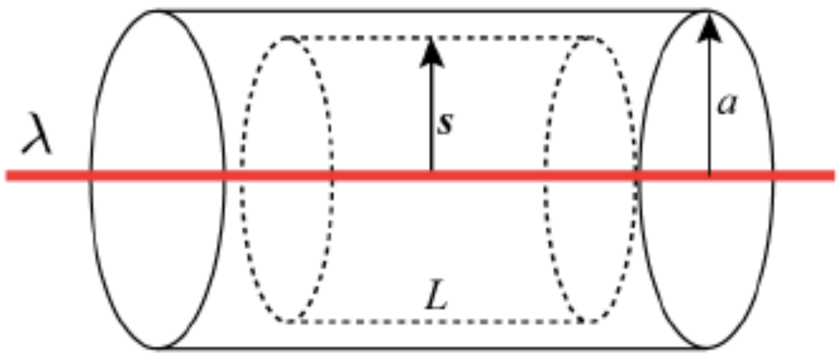
Define: $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$ **Electric displacement**

$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$ Differential form of Gauss's law in presence of a dielectric

$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$ Integral form of Gauss's law in presence of a dielectric

Gauss's law in the presence of Dielectrics

Ex. 4.4 (Griffiths, 3rd Ed.): Find the electric displacement.



$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{fenc}$$

$$\oint D\hat{\mathbf{s}} \cdot da\,\hat{\mathbf{s}} = Q_{fenc}$$

$$D(2\pi sL) = \lambda L$$

$$D = \frac{\lambda}{2\pi s} \quad \Rightarrow \quad \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

What is the electric field outside the dielectric ($s > a$) ??

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \qquad (\text{since } \mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \text{ and } \mathbf{P} = 0)$$

Electric Field Vs Electric Displacement

Similarity:

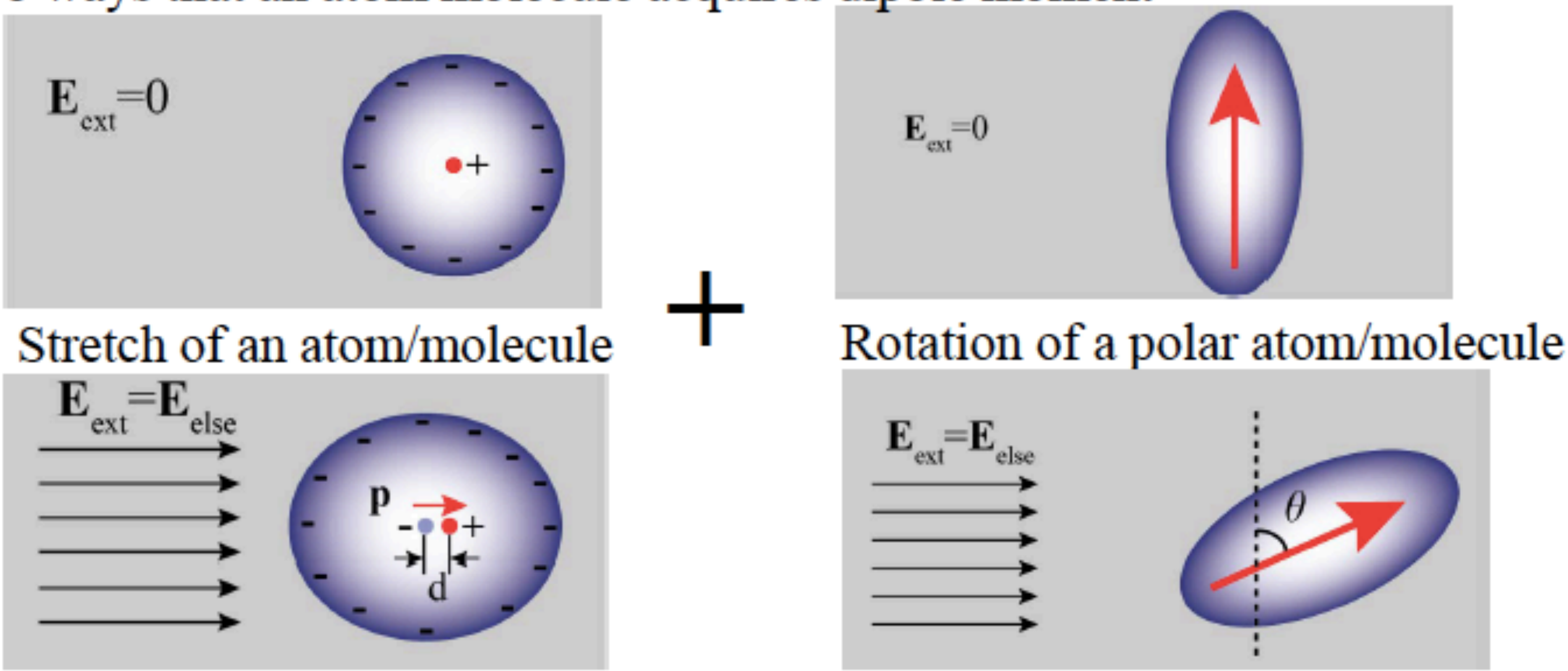
- Divergence and Gauss's Law: $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \iff \nabla \cdot \mathbf{D} = \rho_f$

Difference:

- Curl is not zero: $\nabla \times \mathbf{E} = 0$ But $\nabla \times \mathbf{D} = \epsilon_0(\nabla \times \mathbf{E}) + \nabla \times \mathbf{P} = \nabla \times \mathbf{P} \neq 0$
- $\mathbf{E} = -\nabla V$ But $\mathbf{D} \neq -\nabla f$

Linear Dielectrics

- Two ways that an atom/molecule acquires dipole moment



- Dipole moment **p** is a microscopic quantity. However, polarization **P** (dipole moment per unit volume) is a macroscopic quantity.

$$\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}}$$

- χ_e is called the electrical susceptibility and depends on the details (microscopic and macroscopic of the medium)

Linear Dielectrics

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

- Medium that obeys this equation is called the linear dielectric
- When polarization is proportional to the square or higher-order terms in **E**, the medium is called the nonlinear dielectric.
- At strong enough **E** every medium becomes nonlinear.
- In general χ_e is a tensor and is called the susceptibility tensor.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \quad = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$$\epsilon \equiv \epsilon_0 (1 + \chi_e)$$

ϵ is called the permittivity of the material.

$$\boxed{\mathbf{D} = \epsilon \mathbf{E}}$$

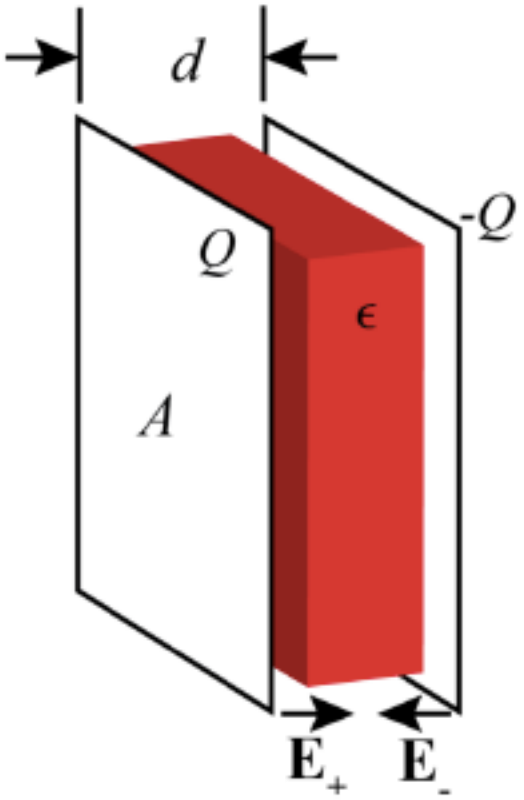
The electric displacement **D** is also proportional to **E**

relative permittivity or the dielectric constant

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = (1 + \chi_e)$$

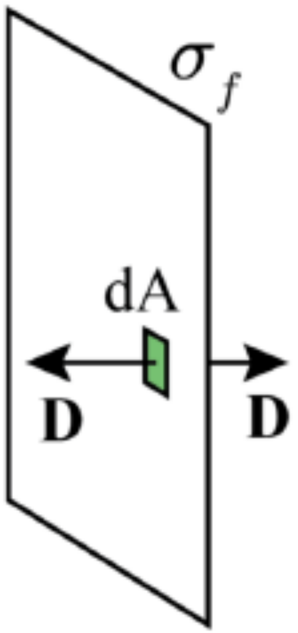
Capacitor with a dielectric filling

Q: For a parallel plate capacitor, what is the capacitance if the space between the plates is filled with a material of dielectric constant ϵ_r. (The vacuum capacitance is $C_{\text{vac}} = \frac{A\epsilon_0}{d}$)



Boundary condition on electric displacement is

$$D^{\perp}{}_{\text{above}} - D^{\perp}{}_{\text{below}} = \sigma_f \qquad D - (-D) = \sigma_f$$



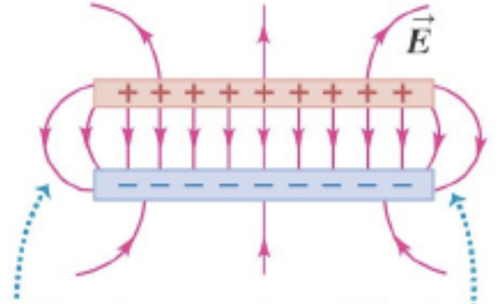
$$D = \frac{\sigma_f}{2} \qquad \text{Thus, } E = \frac{\sigma_f}{2\epsilon}$$

The electric field between the parallel plates

$$\mathbf{E} = \mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma_f}{2\epsilon} - \frac{(-\sigma_f)}{2\epsilon} = \frac{\sigma_f}{\epsilon} = \frac{Q}{A\epsilon}$$

The potential difference $V = -\int \mathbf{E}.d\mathbf{l} = E\,d = \frac{Q}{A\epsilon}d$

Capacitance $\quad C = \frac{Q}{V} = \frac{A\epsilon}{d} = \frac{A\epsilon_0}{d}\frac{\epsilon}{\epsilon_0} = C_{\text{vac}}\epsilon_r$



When the separation of the plates is small compared to their size, the fringing of the field is slight.

A metal shpere of radius a carries a charge Q. It is surrounded, out to radius b , by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).

Sol: Use the generalized Gauss's law

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \quad \text{for all points} \quad r > a$$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} & \text{for} \quad a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} & \text{for} \quad r > b \end{cases}$$

The metal sphere is equalpotential

$$V = -\int_\infty^a \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_b^a \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} + \frac{1}{\varepsilon_r a} - \frac{1}{\varepsilon_r b}\right)$$

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \frac{\varepsilon_0\chi_e Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi r^2}\left(\frac{\chi_e}{1+\chi_e}\right)\hat{\mathbf{r}}$$

volume bound charge $\rho_b = -\nabla \cdot \mathbf{P} = 0$

surface bound charge

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} & \text{at the outer surface} \\ \dfrac{-\varepsilon_0 \chi_e Q}{4\pi\varepsilon a^2} & \text{at the inner surface} \end{cases}$$

-The dielectric layer increases the maximum potential difference between the plates of a capacitor and allows to store more Q.