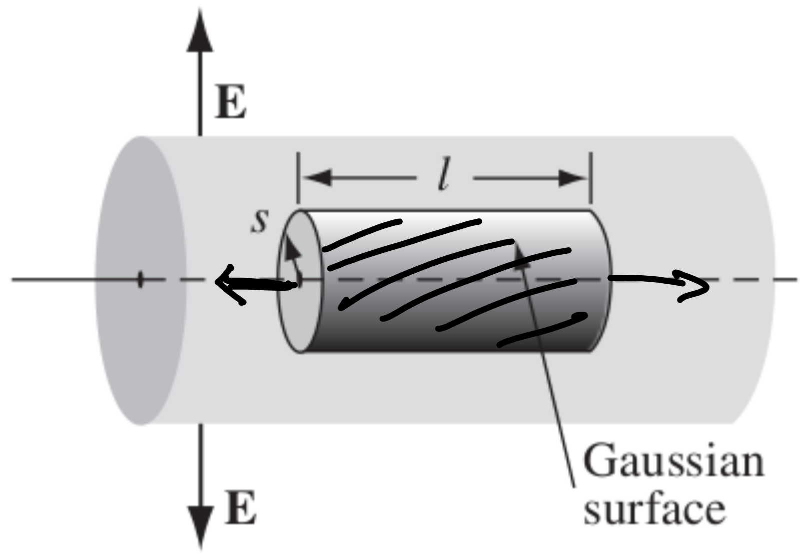
Examples of application of Gauss' law (cont.)

① Cylindrical symmetry:



→ Long cylinder carrying charge density, $\rho = ks$.

$\vec{E}_{inside} = ?$

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

The enclosed charge:

$$Q_{enc.} = \int_V \rho \, d\tau = \int_0^l \int_0^{2\pi} \int_0^s (ks')(s' \, ds' \, d\phi \, dz)$$

$$= 2\pi k l \int_0^s s'^2 \, ds'$$

$$= \frac{2}{3} \pi k l s^3$$

Looking at the symmetry:

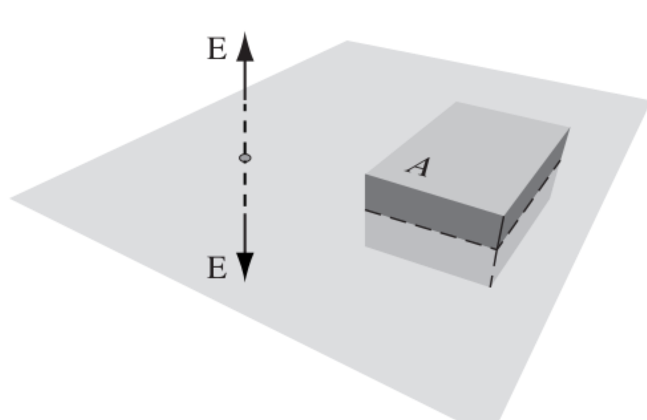
$$\int \underbrace{\vec{E}\cdot d\vec{a}} = |\vec{E}| \int da = |\vec{E}|\, 2\pi s l$$

(*) The two ends contribute nothing since $\vec{E} \perp d\vec{a}$.

Hence, $$|\vec{E}|\, 2\pi s l = \frac{1}{\epsilon_0}\,\frac{2}{3}\,\pi k l s^3$$

$$\Rightarrow \quad \vec{E} = \frac{1}{3\epsilon_0}\, k s^2\, \hat{s}$$

(*) Infinite plane carrying uniform charge density σ.



$\rightarrow$ Gaussian pillbox

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

$$Q_{enc.} = \sigma A$$

$\rightarrow$ Area of enclosed surface (equal to the area of the lid)

$\Rightarrow \vec{E}$ points outward.

from top and bottom surfaces,

$$\int \vec{E} \cdot d\vec{a} = 2A |\vec{E}|$$

$$\Rightarrow 2A |\vec{E}| = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

$\hookrightarrow$ unit vector pointing outward from surface.

Curl of $\vec{E}$

Take a point charge at origin

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Calculate line integral from point 'a' to 'b'

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical polar coordinate:

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\Rightarrow \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\,dr$$

Then $$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

→ for a closed path, $r_a = r_b$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$$

Apply Stoke's theorem: $\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l}$

$\Rightarrow \quad \vec{\nabla} \times \vec{E} = 0 \quad \equiv$ True for only electrostatics.

Electric Potential

The line integral is independent of path:

Define a fn: $V(\vec{r}) = - \int_{O}^{r} \vec{E} \cdot d\vec{l}$

Electric potential

O → reference point

⊛ Potential difference:

$$V(b) - V(a) = - \int_{O}^{b} \vec{E} \cdot d\vec{l} + \int_{O}^{a} \vec{E} \cdot d\vec{l}$$

$$= - \int_{O}^{b} \vec{E} \cdot d\vec{l} - \int_{a}^{O} \vec{E} \cdot d\vec{l}$$

$$= -\int_a^b \vec{E}\cdot d\vec{l}$$

Use fundamental theorem of gradient

$$V(b) - V(a) = \int_a^b (\vec{\nabla} V)\cdot d\vec{l}$$

$$\Rightarrow \int_a^b (\vec{\nabla} V)\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$$

$\equiv$ True for any two arbitrary points 'a' and 'b'

$$\Rightarrow \underline{\vec{E} = -\vec{\nabla} V}$$

⊗ Changing the reference point can change the potential.

$$V'(\vec{r}) = -\int_{O'}^{r} \vec{E}\cdot d\vec{l} = -\int_{O'}^{O} \vec{E}\cdot d\vec{l} - \int_{O}^{r} \vec{E}\cdot d\vec{l} = \underline{\underline{K}} + V(\vec{r})$$

$k \equiv$ const. independent of r.

- Potential diff. being a physical quantity should be independent of reference point.

$$V'(b) - V'(a) = V(b) - V(a)$$

- Obeys superposition rule. For a collection of charges,

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$$

$$\Rightarrow V = V_1 + V_2 + \cdots$$

- Unit : $\dfrac{N \cdot m}{C}$ or $\dfrac{J}{C}$ or volt