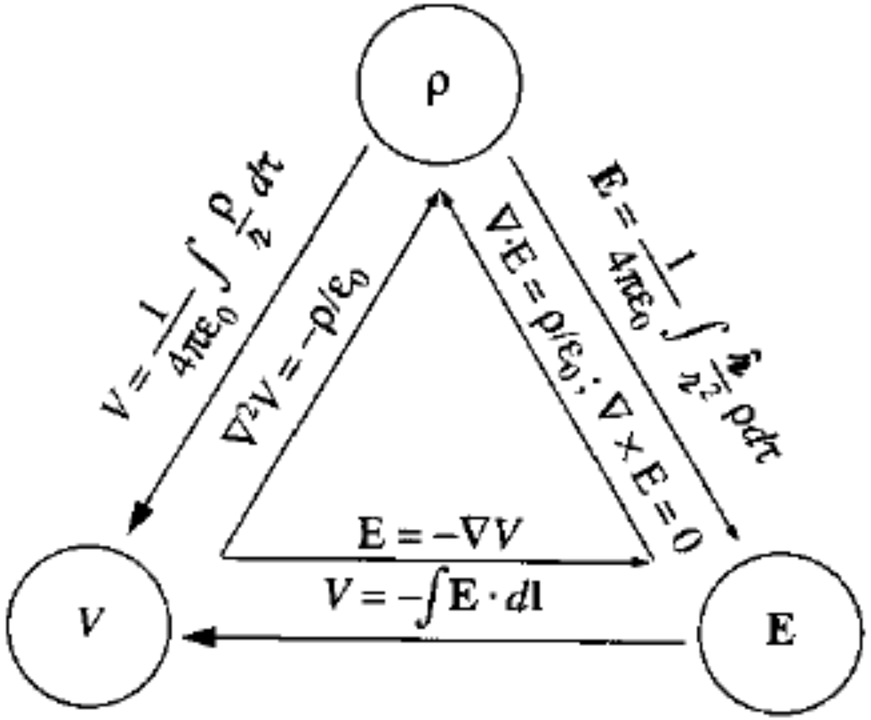
Lecture - 16

We have derived six formulas interrelating three fundamental quantities: ρ , $\mathbf{E}$ and V.

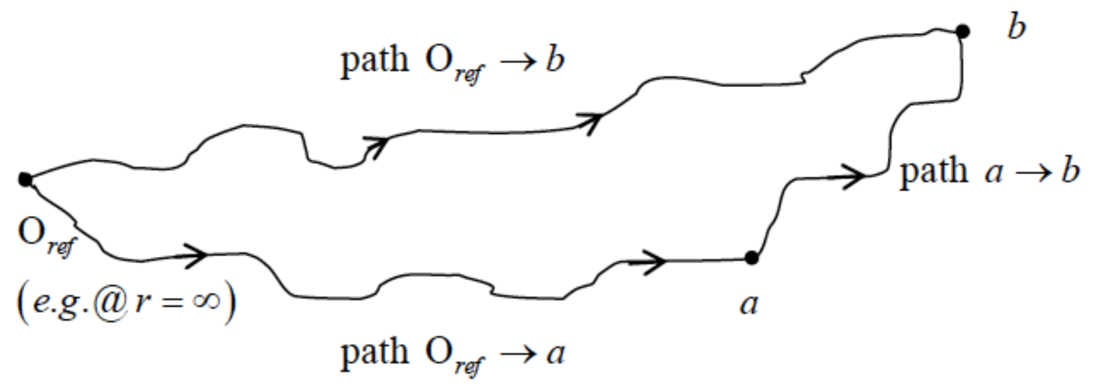


These equations are obtained from two observations:

- Coulomb's law: the fundamental law of electrostatics
- The principle of superposition: a general rule applying to all electromagnetic forces.

$$\boxed{V(\vec{r}) \equiv -\int_{O_{ref}}^{r} \vec{E}(\vec{r}) \cdot d\vec{\ell}}$$

$$\boxed{V(\vec{r}) \equiv \frac{1}{4\pi\varepsilon_o} \int_{v'} \frac{\rho(\vec{r}\,')}{\boldsymbol{r}} d\tau'}$$



$$\boxed{\Delta V_{ab} \equiv V(\vec{r} = b) - V(\vec{r} = a) = -\int_{a}^{b} \vec{E}(\vec{r}) \cdot d\vec{\ell}}$$

A test charge Q_T is moved from point <u>a</u> to point <u>b</u> in this electric field $\vec{E}(\vec{r})$. How much mechanical work W is done <u>on</u> the test charge Q_T in moving it (<u>slowly</u>) from point a to point b?

$$\vec{F}_C(\vec{r}) = Q_T\vec{E}(\vec{r})$$

The mechanical work done <u>on</u> the test charge Q_T along the path $a \rightarrow b$ is:

$$\boxed{W = \int_a^b \vec{F}_{mech}(\vec{r}) \bullet \vec{dl} = -Q_T\int_a^b \vec{E}(\vec{r}) \bullet \vec{dl}} \qquad V(b) - V(a) = -\int_a^b \vec{E}(\vec{r}) \bullet \vec{dl}$$

$$\boxed{\therefore\ W = Q_T\left[V(b) - V(a)\right] = Q_T\Delta V_{ab}}$$

Now, if point a is the reference point $\vec{r}_a = \infty$ where $V(\vec{r}_a) = V(\infty) = 0$ and point $\vec{r}_b = \vec{r}$

$$W = Q_T\left[V(\vec{r}) - \cancel{V(\infty)}\right] = Q_T V(\vec{r})$$

$P.E. = W = Q_T V(\vec{r})$ $\qquad$ $P.E.$ is linearly proportional to the <u>potential $V(r)$</u>

POTENTIAL ENERGY = amount of work W it takes to create the system (Joules).

$$W = P.E. = Q_T V(\vec{r}) = \text{Coulomb-Volts} = \text{Joules}$$

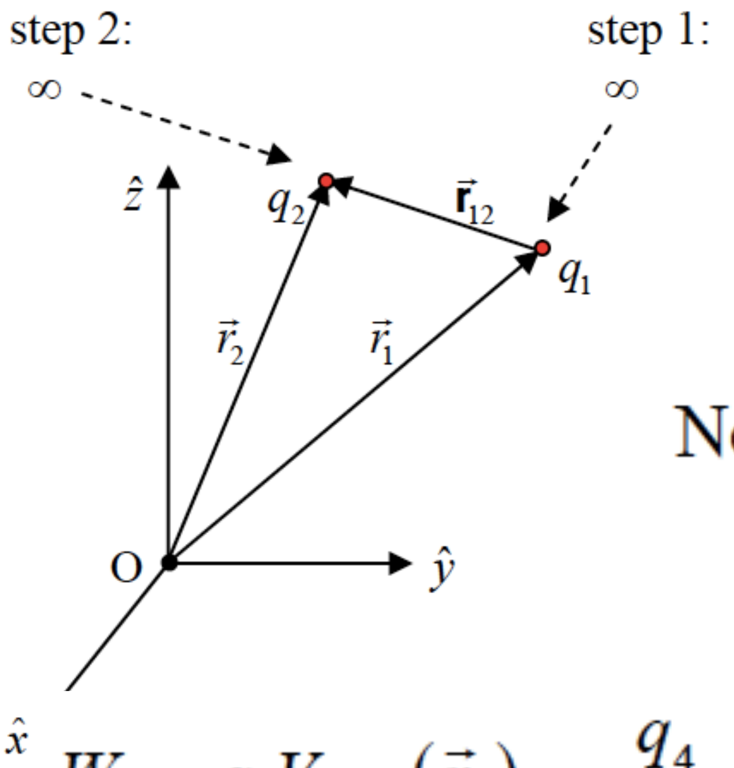
i.e. 1 Coulomb x 1 Volt = 1 Joule

Fundamental unit of electric charge: $1e = 1.602\times10^{-19}$ Coulombs $= 1.602\times10^{-19}\ C$

$$\therefore\ 1\ \underbrace{\text{electron volt}}_{=eV} = 1.602\times10^{-19}\ \text{Joules} \leftarrow \text{energy conversion factor for eV} \Leftrightarrow \text{Joules}$$

ELECTROSTATIC ENERGY OF ASSEMBLY OF A POINT CHARGE DISTRIBUTION

How much work does it take to assemble a collection of point charges – bringing them in from infinity, one by one? Bringing in the first charge q_1 takes NO work ($W_1 = 0$), since there is no electric field present, initially.



Now bring in the 2nd charge q_2 from infinity.

$$\boxed{W_2 = \frac{q_2}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{12}}\right) = q_2 V_1(\vec{r}_2)} \qquad \boxed{V_1(\vec{r}_2) = \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{12}}\right)}$$

Now bring in the 3rd charge from infinity.

$$W_3 = q_3 V_{1,2}(\vec{r}_3) = \frac{q_3}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{13}} + \frac{q_2}{\mathbf{r}_{23}}\right)$$

$$W_4 = q_4 V_{1,2,3}(\vec{r}_4) = \frac{q_4}{4\pi\varepsilon_o}\left(\frac{q_1}{\mathbf{r}_{14}} + \frac{q_2}{\mathbf{r}_{24}} + \frac{q_3}{\mathbf{r}_{34}}\right)$$

The *total* work necessary to assemble the first 4 charges is thus:

$$W_{TOT} = W_1 + W_2 + W_3 + W_4 = 0 + q_2 V_1(\vec{r}_2) + q_3 V_{1,2}(\vec{r}_3) + q_4 V_{1,2,3}(\vec{r}_4)$$

$$= \frac{1}{4\pi\varepsilon_o}\left(\frac{q_1 q_2}{\mathbf{r}_{12}} + \frac{q_1 q_3}{\mathbf{r}_{13}} + \frac{q_1 q_4}{\mathbf{r}_{14}} + \frac{q_2 q_3}{\mathbf{r}_{23}} + \frac{q_2 q_4}{\mathbf{r}_{24}} + \frac{q_3 q_4}{\mathbf{r}_{34}}\right)$$

$$W_{TOT} = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i\\ j>i}}^{N}\left(\frac{q_i q_j}{\mathbf{r}_{ij}}\right) = \frac{1}{8\pi\varepsilon_o}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\frac{q_i q_j}{\mathbf{r}_{ij}}\right)$$

double-counts pairs - but factor of 8 (vs. 4) takes care of this!

$$W_{TOT} = \frac{1}{2}\sum_{i=1}^{N} q_i \underbrace{\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{4\pi\varepsilon_o}\left(\frac{q_j}{\mathbf{r}_{ij}}\right)\right)}_{=V(\vec{r}_i)} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

For a discrete / discretized charge distribution:

$$\boxed{W_{TOT} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i) = \frac{1}{2}\sum_{i=1}^{N} q_i \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\pi\varepsilon_o}\left(\frac{q_i}{\mathbf{r}_{ij}}\right)\right)}$$

$$\sum_{i=1}^{N} q_i V(\vec{r}_i) \Rightarrow \int_V dq V(\vec{r}) = \int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

$$\boxed{W_{TOT} = \frac{1}{2}\oint_V \rho(\vec{r}) V(\vec{r}) d\tau}$$

$$\boxed{W_{TOT} = \frac{1}{2}\int_C \lambda(\vec{r}) V(\vec{r}) dl}$$

$$\boxed{W_{TOT} = \frac{1}{2}\int_S \sigma(\vec{r}) V(\vec{r}) dA}$$

$$\rho(r) = \varepsilon_o \vec{\nabla} \bullet \vec{E}$$

$$W_{TOT} = \frac{1}{2}\int_V \rho(\vec{r}) V(\vec{r}) d\tau = \frac{\varepsilon_o}{2}\int_V \left(\vec{\nabla} \bullet \vec{E}\right) V(\vec{r}) d\tau$$

$$W_{TOT} = \frac{\varepsilon_o}{2} \int\limits_{\substack{V \\ all \\ space}} E^2(\vec{r}) d\tau$$

Simplest kinds of electromagnetic properties:

- A.) conductor (of electricity)
- B.) $\updownarrow$ partial conductor/insulator
- C.) non-conductor $\Rightarrow$ insulator

Why materials conduct vs. do not conduct electricity depends on microscopic (i.e. quantum) structure of materials & temperature (i.e. thermal/internal energy).

One important property of a conductor is that: $\vec{E}_{ext}(\vec{r}) = E_0\hat{x}$

1) $\boxed{\vec{E}_{NET}(\vec{r}) \equiv 0 \text{ inside a conductor}}$

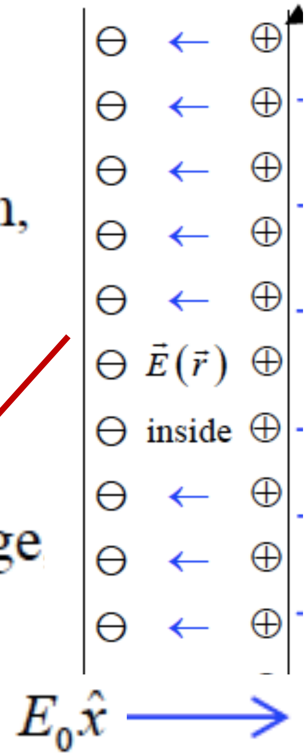
the free charges inside the conductor re-distribute themselves to <u>create</u>/<u>produce</u> $E_{\text{inside}}(\vec{r}) = 0$

The redistributed free charges pile up on the surface(s) of the conductor in such a way as to produce $\vec{E}_{\text{inside}}(\vec{r}) = 0$. These <u>*induced*</u> charges produce an internal electric field of their own, which <u>*exactly*</u> cancels the external field, $\vec{E}_{ext}(\vec{r})$!

$$\vec{E}_{\substack{\text{net}\\ \text{inside}}}(\vec{r}) = \vec{E}_{ext}(\vec{r}) + \vec{E}_{\substack{\text{induced}\\ \text{inside}}}(\vec{r}) = 0$$

$$\Rightarrow \vec{E}_{\substack{\text{induced}\\ \text{inside}}}(\vec{r}) = -\vec{E}_{ext}(\vec{r}) = -E_0\hat{x}$$

Induced surface charge



2) The volume free charge density, $\boxed{\rho_{inside}^{free}(\vec{r})=0}$ inside a conductor.

$$\vec{\nabla}\cdot\vec{E}_{inside}(\vec{r})=\rho_{inside}^{free}(\vec{r})/\varepsilon_o \quad \Rightarrow \rho_{inside}^{free}(\vec{r})=0,$$

3) Any *induced* charges on a conductor can ONLY reside on *surface(s)* of the conductor – as surface charge distributions, σ_{free}

4) The entire volume & surface of a conductor is an equipotential.

5) Just outside the surface of a conductor, $\vec{E}_{outside}(\vec{r}\,@\,\text{surface})$ is *perpendicular/normal* to the surface, i.e. $\vec{E}_{outside}(\vec{r}\,@\,surface)\parallel \hat{n}_{surface}$