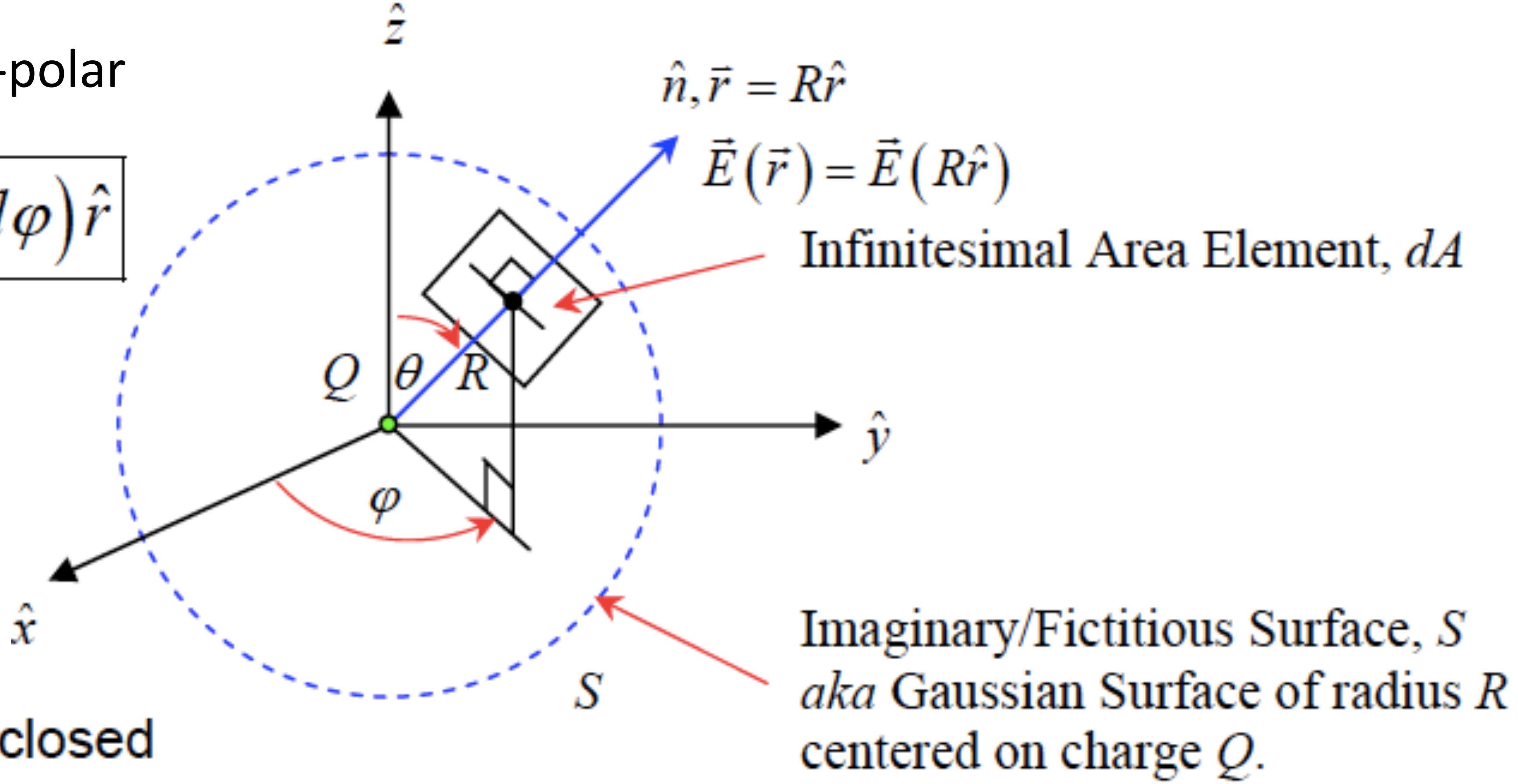
Lecture - 12

Gauss's law

Elemental area in Spherical-polar

$$\boxed{d\vec{A} = dA\hat{r} = \left(r^2 \sin\theta d\theta d\varphi\right)\hat{r}}$$



The total flux through this closed Gaussian surface is

$$\Phi_E = \oint_S \vec{E}(\vec{r}) \bullet d\vec{A} = \frac{Q}{4\pi\varepsilon_o} \int_S \left(\frac{1}{\cancel{r^2}}\hat{r}\right) \bullet \underbrace{\left(\cancel{r^2} \sin\theta d\theta d\varphi \hat{r}\right)}_{=d\vec{A}}$$

Thus: $$\Phi_E = \frac{Q}{4\pi\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\int_{\varphi=o}^{\varphi=2\pi}\sin\theta\, d\theta\, d\varphi\underbrace{\left(\hat{r}\bullet\hat{r}\right)}_{=1} = \frac{\cancel{2\pi}Q}{\underbrace{\cancel{4\pi}}_{2}\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\sin\theta\, d\theta$$

$$= \frac{\cancel{2}Q}{\cancel{2}\varepsilon_o} = \frac{Q}{\varepsilon_o}$$

<u>Gauss' Law (in Integral Form)</u>: $$\boxed{\Phi_E = \oint_s \vec{E}(\vec{r})\bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_o}}$$

Electric flux through closed surface S = (electric charge enclosed by surface S)/ ε_o

The circle on the integral sign indicates that the Gaussian surface must be enclosed.

Can we prove the above statement for arbitrary closed shape?

volume charge density $\rho(\vec{r}')$, then: $Q_{encl} = \int_v \rho(\vec{r}')d\tau'$

$$\Phi_E = \oint_S \vec{E}(\vec{r})\cdot d\vec{A} = \int_v \overbrace{\left(\vec{\nabla}\cdot\vec{E}(\vec{r})\right)} d\tau' = \frac{Q_{encl}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int_v \overbrace{\rho(\vec{r})}\, d\tau'$$

This relation holds for *any* volume v $\Rightarrow$ the *integrands* of $\int_v (\ \ )d\tau'$ *must* be equal

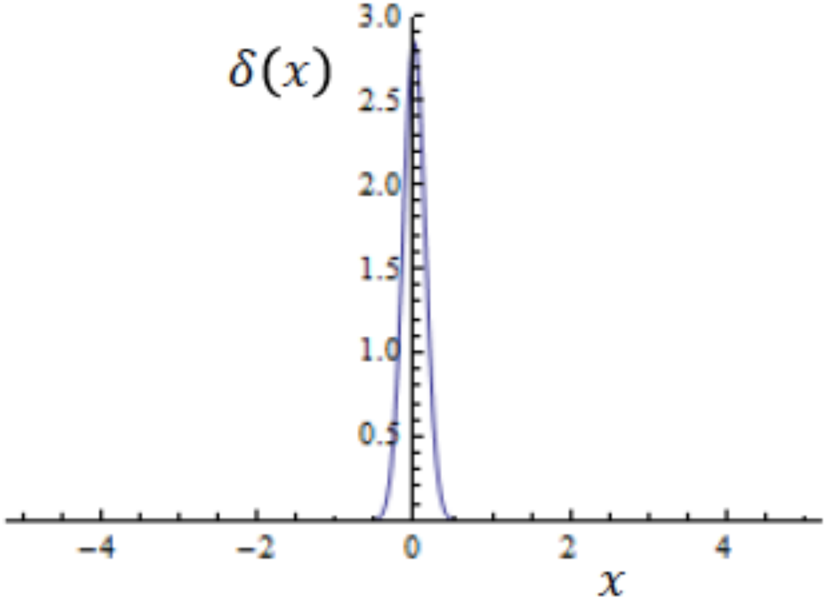
$\therefore$ Gauss' Law (in Differential Form): $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o}$

Dirac Delta Function

- Dirac delta function is a special function, which is defined as:

$$\delta(x) = 0, \qquad if\ x \neq 0$$
$$= \infty, \qquad if\ x = 0$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$



- Realization of a Dirac Delta function

$$\delta(x) = \lim_{s\to 0} \frac{1}{\sqrt{2\pi s^2}} \exp\left[-\frac{(x-a)^2}{2s^2}\right]$$

- Example: What is the charge density of a point charge q kept at the origin?

$$\rho(x) = q\delta(x);$$

$$\int_{-\infty}^{\infty} \rho(x)dx = \int_{-\infty}^{\infty} q\delta(x)dx = q$$

Important Properties of Dirac Delta Function

(1) $\delta(kx) = \dfrac{1}{|k|}\delta(x),$

(2) Dirac delta function centered at $x = a$ is defined as follows

$$\delta(x-a) = 0, \qquad if\ x \neq 0$$
$$= \infty, \qquad if\ x = a$$
$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

(3) If $f(x)$ is a continuous function of x

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx = f(a)$$

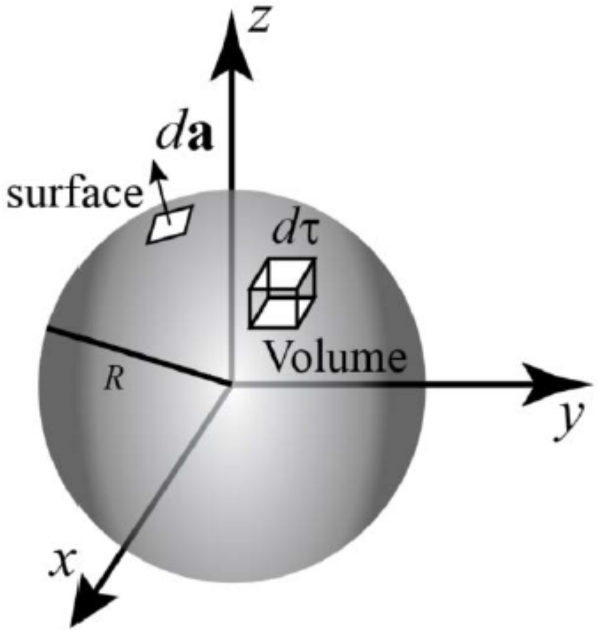
(4) 3D Dirac delta function is defined as:

$$\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{a}) = f(\boldsymbol{a})$$

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^{n}}\right) = \frac{2-n}{r^{(n+1)}} \qquad \mathbf{V} = \frac{\hat{\mathbf{r}}}{r^2} \qquad \nabla \cdot \mathbf{V} = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{0}{r^3} = 0$$

except at r=0 where it is 0/0, not defined

Let's calculate the divergence using the divergence theorem:



$$\int_{Vol} (\nabla \cdot \mathbf{V}) d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a}$$

Take the volume integral over a sphere of radius R and the surface integral over the surface of a sphere of radius R.

$$\oint_{Surf} \mathbf{V} \cdot d\mathbf{a} = \iint \left(\frac{\hat{\mathbf{r}}}{R^2}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \hat{\mathbf{r}}) = \iint \sin\theta \, d\theta \, d\phi = 4\pi$$

$$\boxed{\nabla \cdot \mathbf{V} = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi \, \delta(\mathbf{r})}$$