Tutorial Sheet 11

Taylor Series, Maxima, Minima in Multivariable.

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1. Suppose the gradient vector of the linear function $z = f(x, y)$ is $\nabla z = (5, -12)$. If $f(9, 15) = 17$, what is the value of $f(11, 11)$?

2. Find the quadratic Taylor's polynomial approximation of the function $f(x, y) = e^{-x^2-2y^2}$ near origin.

3. Find all the critical points of $f(x, y) = \sin x \sin y$ in the domain $-2 \leq x \leq 2,\ -2 \leq y \leq 2$.

4. Identify the local extreme points of the functions (a) $f(x, y) = 2x^2y - y^2 - 4x^2 + 3y$ (b) $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. Are the local maxima and minima also the global maxima and minima? Explain your answer.

5. Let $f(x, y) = (y - 4x^2)(y - x^2)$. Verify that $(0, 0)$ is a saddle point of f.

6. Let $f(x, y) = (x - y)^2$. Find all critical points of f and categorize them according as they are either saddle points or the location of local extreme values. Is the second derivative test useful in this case?

7. Find the maximum and minimum values of $f(x, y) = 3x + 4y$ subject to the constraint $x^2 + 4xy + 5y^2 = 10$.

Solutions to Tutorial Sheet 11

1. Suppose z $= ax + by + c$. Therefore, $5 = f_x(x, y) = a$ and $-12 = f_y(x, y) = b$. Since, $f(9, 15) = 17$, $c = 152$. $f(11, 11) = 75$

2.

$$
\begin{aligned}
f(x,y) &= e^{-x^2-2y^2}, & f(0,0) &= 1 \\
f_x(x,y) &= e^{-x^2-2y^2}(-2x), & f_x(0,0) &= 0 \\
f_y(x,y) &= e^{-x^2-2y^2}(-4y), & f_y(0,0) &= 0 \\
f_{xx}(x,y) &= -2e^{-x^2-2y^2} + 4x^2e^{-x^2-2y^2}, & f_{xx}(0,0) &= -2 \\
f_{yy}(x,y) &= -4e^{-x^2-2y^2} + 16y^2e^{-x^2-2y^2}, & f_{yy}(0,0) &= -4 \\
f_{xy}(x,y) &= 8xye^{-x^2-2y^2}, & f_{xy}(0,0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
f(x,y) &\approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{x^2}{2}f_{xx}(0,0) + xyf_{xy}(0,0) + \frac{y^2}{2}f_{yy}(0,0) \\
&= 1 - x^2 - 2y^2
\end{aligned}
$$

3. Given $f(x, y) = \sin x \sin y$, $-2 \le x \le 2$ and $-2 \le y \le 2$. $f_x = 0$ implies $\cos x \sin y = 0$ and $f_y = 0$ implies $\sin x \cos y = 0$. Thus $x = \pm(2n+1)\frac{\pi}{2}$ or $y = \pm n\pi$ and $x = \pm n\pi$ or $y = \pm(2n+1)\frac{\pi}{2}$ i.e. $(x, y) = (\pm(2n+1)\frac{\pi}{2}, \pm(2n+1)\frac{\pi}{2})$ and $(\pm n\pi, \pm n\pi)$. Thus critical points in the domain are given by $(0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$.

4. (a) Given $f(x, y) = 2x^2y - y^2 - 4x^2 + 3y$. Thus $f_x = 4x(y - 2) = 0$ implies $x = 0$ or $y = 2$ and $f_y = 0$ implies $2x^2 - 2y + 3 = 0$. Solving these equations, the critical points of f are given by $(0, \frac{3}{2})$ and $(\pm\frac{1}{\sqrt{2}}, 2)$. Also

$$D = f_{xx}f_{yy} - f_{xy}^2 = 16 - 8y - 16x^2.$$

Now it is easy to check that $(0, \frac{3}{2})$ is a point of local maxima and $(\pm\frac{1}{\sqrt{2}}, 2)$ are saddle points.

(b) Critical points are $(0, 0)$, $(0, 2)$, $(-1, 1)$ and $(1, 1)$. Easy to check that $(-1, 1)$ and $(1, 1)$ are saddle points, $(0, 0)$ is local maxima and $(0, 2)$ is local minima.

5. Given $f(x,y) = (y-4x^2)(y-x^2)$. Thus $f_x = 16x^3 - 10xy = 0$ implies $x = 0$ or $8x^2 = 5y$ and $f_y = -5x^2 + 2y = 0$ implies $2y = 5x^2$. Thus $(0,0)$ is the only critical point. As $D = 0$, the second derivative test fails. Note that along the parabola $y = 5x^2$, $f(x,y) > 0$ while along $y = 2x^2$, $f(x,y) < 0$. Thus $(0,0)$ is a saddle point.

6. Given $f(x,y) = (x-y)^2$. Thus $f_x = 0$ and $f_y = 0$ implies $x = y$. Note that as $D = 0$ at (x,x), the second derivative test fails. Also note that $f(x,y) \geq 0$ and $f(x,y) = 0$ at $x = y$. Thus (x,x) are points of local minimum.

7. Given $f(x,y) = 3x + 4y$ and $g(x,y) = x^2 + 4xy + 5y^2 - 10$. Thus

$$L(x,y,\lambda) = 3x + 4y + \lambda(x^2 + 4xy + 5y^2 - 10).$$

So $L_x = 0$ implies $2\lambda x + 4\lambda y + 3 = 0$ and $L_y = 0$ implies $4\lambda x + 10\lambda y + 4 = 0$. Solving these we get $x = \frac{-7}{2\lambda}$ and $y = \frac{1}{\lambda}$. Now $g(x,y) = 0$ gives $\lambda = \mp\frac{1}{2}\sqrt{\frac{13}{10}}$. Hence critical points are $(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}})$ and $(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}})$. Also $f(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}}) = -\sqrt{130}$ and $f(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}}) = \sqrt{130}$. Thus $(-7\sqrt{\frac{10}{13}}, 2\sqrt{\frac{10}{13}})$ and is point of minima and $(7\sqrt{\frac{10}{13}}, -2\sqrt{\frac{10}{13}})$ is point of maxima under the constraint $g(x,y)$.