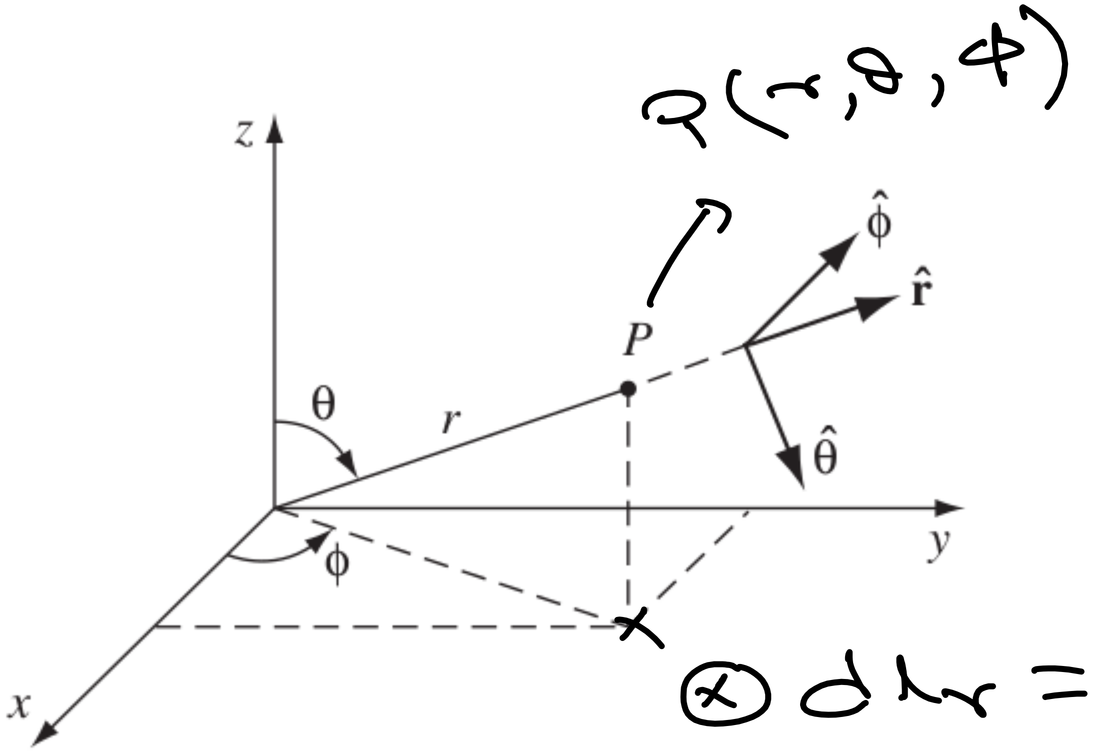
EPHY105L-7 7.10.21

Spherical polar coordinates



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

⊛ $\vec{A} = A_r\,\hat{r} + A_\theta\,\hat{\theta} + A_\phi\,\hat{\phi}$

⊛ $dl_r = dr, \quad dl_\theta = r\,d\theta, \quad dl_\phi = r\sin\theta\,d\phi$

→ Infinitesimal displacement:

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

→ Infinitesimal volume:

$$d\tau = dl_r\,dl_\theta\,dl_\phi$$

$$= r^2\sin\theta\,\underline{dr}\,\underline{d\theta}\,\underline{d\phi}$$

$$V = \int d\tau = \int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{\frac{4}{3}\pi R^3}$$

Limits

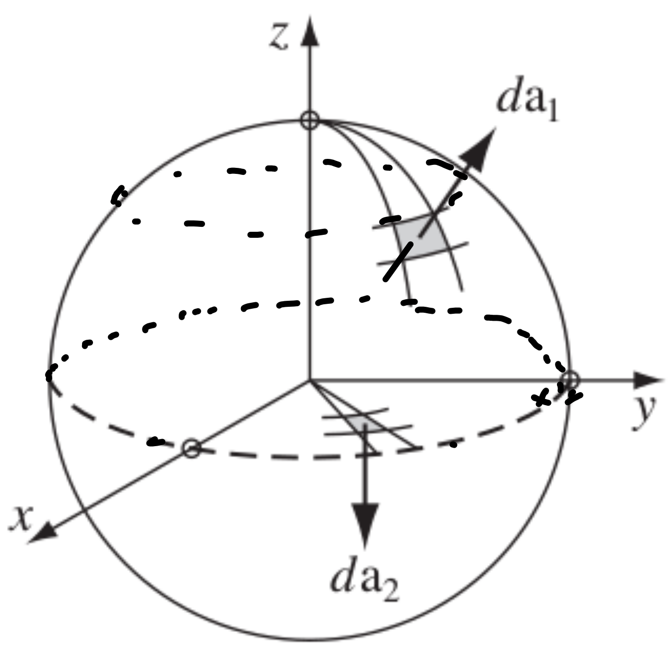
$r \equiv$ Ranging from 0 to ∞

$\theta \equiv$ Ranging from 0 to π

$\phi \equiv$ Ranging from 0 to 2π

→ Infinitesimal surface

→ Depends on geometry.



⊗ A surface element on the outer surface :

$$\underline{\underline{d\vec{a}_1}} = \underline{dl_\theta}\,\underline{dl_\phi}\,\hat{r}$$

$$= r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

⊗ Surface lies on the xy plane.

→ $\theta \equiv$ const.

$$d\vec{a}_2 = dl_r \, dl_\phi \, \hat{\theta} = r \, dr \, d\phi \, \hat{\theta}$$

(†) Now, we can write vector derivatives:

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right)$$

⋮

Gradient:

$$\underline{\underline{\nabla T}} = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}. \tag{1.71}$$

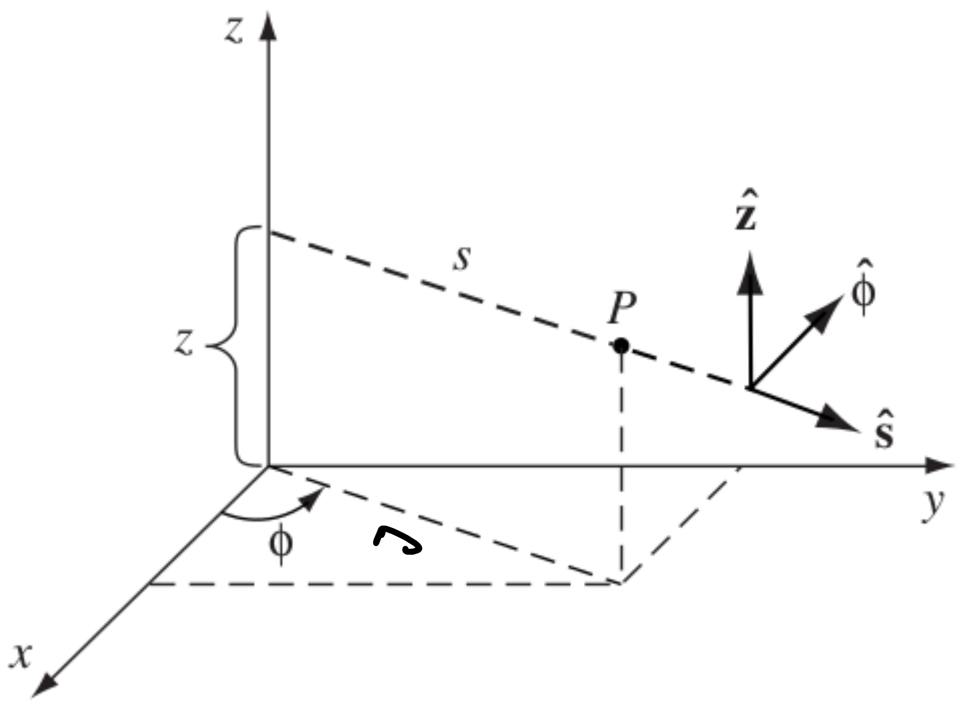
Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}. \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}. \tag{1.73}$$

Cylindrical Coordinates



$$P \equiv P(s, \phi, z)$$

$\phi \equiv$ Same as spherical polar

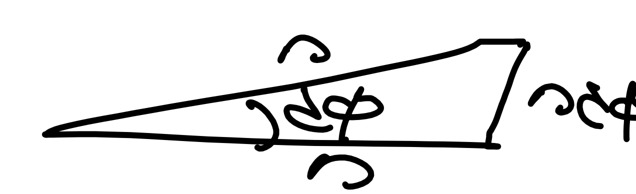
$z \equiv$ Same as Cartesian

$s \equiv$ Distance from z-axis.

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases}$$

$$\begin{aligned} \hat{s} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \\ \hat{z} &= \hat{z} \end{aligned}$$

(+) Infinitesimal displacements:



$$dl_s = ds$$

$$dl_\phi = s\,d\phi \quad , \quad dl_z = dz.$$

$\Rightarrow d\vec{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$

⊕ Volume element, $d\tau = r\,dr\,d\phi\,dz$

→ <u>Ranges</u>: $r \equiv$ Ranging from 0 to ∞

$\phi \equiv$ Ranging from 0 to 2π

$z \equiv$ Ranging from $-\infty$ to $+\infty$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$