Lecture - 29

—stationary charges experience no magnetic forces.

What sort of field exerts a force on charges at rest? *electric* fields

A changing magnetic field induces an electric field.

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \qquad \textbf{universal flux rule}:$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}.$$ **Faraday's law**, in integral form

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.}$$

Electrodynamics before Maxwell

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho & \text{(Gauss's law)} \\ \nabla \cdot \mathbf{B} = 0 & \text{(no name)} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's law)} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} & \text{(Ampere's law)} \end{cases}$$

A fatal inconsistency in Ampere's law

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{=0} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{\neq 0}$$

Ampere's law is incorrect for the nonsteady current.

The Electric and Magnetic Fields

Two distinct kinds of electric fields:

E (in static case): attributed to electric charges, using Coulomb's law.

E (in nonsteady case): associated with changing magnetic field, using Faraday's law.

Two distinct kinds of magnetic fields:

B (in static case): attributed to electric currents, using Ampere's law.

B (in nonsteady case): associated with changing electric field, using?

How Maxwell Fixed Ampere's Law

Applying the continuity equation and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial(\varepsilon_0 \nabla \cdot \mathbf{E})}{\partial t} = \nabla \cdot (-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

A new current $\mathbf{J}' = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \leftarrow$ kills off the extra divergence

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}') = \mu_0 \nabla \cdot (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = 0$$

When E is constant (electrostatic+magnetostatic), we will have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.

$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ plays a crucial role in the EM wave propagation.

Electric Analogy of Faraday's Law

Maxwell's term cures the defect in Ampere's law, and moreover, it has a certain aesthetic appeal.

Faraday's law

A changing magnetic field induces a electric field.

A changing electric field induces a magnetic field.

Maxwell called this extra term "the displacement current".

$$\mathbf{J}_{\mathrm{d}} \equiv \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

a misleading name, nothing to do with current

Maxwell's Equations

Maxwell's equations in the traditional way.

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}\rho & \text{(Gauss's law)} \\ \nabla \cdot \mathbf{B} = 0 & \text{(no name)} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{(Faraday's law)} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{(Ampere's law with Maxwell's correction)} \end{cases}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Another expression of the Maxwell's equations.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

The fields ($\mathbf{E}$ and $\mathbf{B}$) on the left

and the sources (ρ and $\mathbf{J}$) on the right.

Maxwell's equations tell you how sources produce fields; reciprocally, the Lorentz force law tells you how fields affect sources. ← **A nonlinear feedback**

Maxwell's Equations in Matter

When working with materials that are subject to electric and magnetic polarization, there is a more convenient way to write the Maxwell's equations.

Static case:

An electric polarization produces a bound charge: $\rho_b = -\nabla \cdot \mathbf{P}$

A magnetic polarization results in a bound current: $\mathbf{J}_b = \nabla \times \mathbf{M}$

Nonstatic case:

Any change in the electric polarization involves a flow of bound charge.

$$\mathbf{J}_p = \frac{dI}{da_\perp} = \frac{d\sigma_b}{dt}\frac{da_\perp}{da_\perp} = \frac{\partial \mathbf{P}}{\partial t} \quad \text{where } \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$$

polarization current

Now $\rho = \rho_{\mathrm{f}} + \rho_{\mathrm{b}} = \rho_{\mathrm{f}} - \nabla \cdot \mathbf{P}$

$$\mathbf{J} = \mathbf{J}_{\mathrm{f}} + \mathbf{J}_{\mathrm{b}} + \mathbf{J}_{\mathrm{p}} = \mathbf{J}_{\mathrm{f}} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Gauss's law: $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}(\rho_{\mathrm{f}} - \nabla \cdot \mathbf{P}) \;\Rightarrow\; \nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{\mathrm{f}}$

Ampere's law: $\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_{\mathrm{f}} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$$\Rightarrow \nabla \times (\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}) = \mathbf{J}_{\mathrm{f}} + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P})$$

Maxwell's Equations in Matter

In terms of free charges and currents, Maxwell's equations read

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{\mathrm{f}}$$

The constitutive relations:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{M} = \mu_0 \chi_{\mathrm{m}} \mathbf{H}$$

So

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \Rightarrow \quad \mathbf{B} = \mu_0 (1 + \chi_{\mathrm{m}}) \mathbf{H} = \mu \mathbf{H}$$

Differential form

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{f}}$$
$$\nabla \cdot \mathbf{B} = 0$$

Integral form

$$\left.\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{a} &= \rho_{\mathrm{f}} \\ \oint_S \mathbf{B} \cdot d\mathbf{a} &= 0\end{aligned}\right\} \text{ over any enclosed surface } S.$$

Differential form

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{\mathrm{f}}$$

Integral form

$$\left.\begin{aligned}\oint_P \mathbf{E} \cdot d\mathbf{l} &= -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{a} \\ \oint_P \mathbf{H} \cdot d\mathbf{l} &= \mathbf{J}_{\mathrm{f}} + \frac{\partial}{\partial t}\int_S \mathbf{D} \cdot d\mathbf{a}\end{aligned}\right\} \text{ for any surface } S \text{ bounded by the closed loop } P.$$