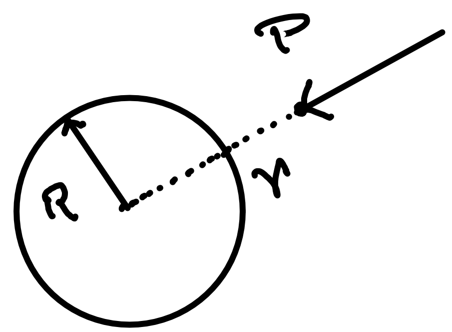
Electric potential

Ex:



Find potential inside and outside of a spherical shell carrying a uniform charge density

$q \equiv$ Total charge

→ Reference point $(O) \equiv \infty$

From Gauss' law, field outside,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Field inside the sphere, $\vec{E} = 0$

→ For point outside the sphere $(r \geq R)$

$$V(r) = -\int_O^r \vec{E} \cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0} \int_\infty^r \frac{q}{r'^2} dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

For points inside the the sphere $(r \leq R)$

$$V(r) = -\int_{0}^{r} \vec{E} \cdot d\vec{l}$$

$$= -\frac{1}{4\pi\epsilon_0} \int_{\infty}^{R} \frac{q}{r'^2} \, dr' - \int_{R}^{r} (0) \, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \equiv \text{Potential inside}$$

is non-zero although the electric field is zero.

⊛ <u>Poisson's & Laplace's eqn.</u>

$$\left.\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0 \end{aligned}\right\}$$ How do they look like in terms of V ?

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0} \equiv \text{Poisson's eq.}$$

→ If there is no charge distribution,

$$\nabla^2 V = 0 \equiv \text{Laplace's eq.}$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(-\vec{\nabla}V) = 0$$

<u>Potential of a charge distribution</u>

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}$$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

Setting the reference point at ∞, the potential for a point charge q at origin

$$V(r) = -\int_{-\infty}^{r} \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2}\, dr'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

$\mathscr{r} \equiv$ distance between q and 'r' then,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

→ From superposition principle,

$$\underbrace{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}}$$

(for a collection of charges)

[Superposition: if there is a collection of charges: $\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n$

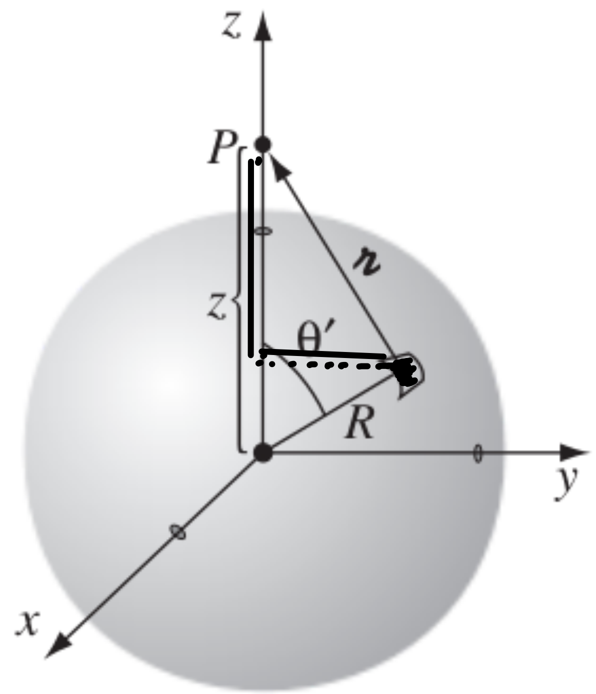
$$V = V_1 + V_2 + \cdots + V_n \text{]}$$

→ If we have a continuous charge distribution,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \, dq$$

- volume charge distribution: $dq = \rho \, d\tau'$
- surface " " $dq = \sigma \, da'$
- line " " $dq = \lambda \, dl'$

Ex: Potential of a uniformly charged spherical shell of radius R.



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{r}\, da'$$

$$r^2 = R^2 + z^2 - 2Rz\cos\theta'$$
(cosine law)

$$r^2 = (R\sin\theta')^2 + (z - R\cos\theta')^2$$
$$= R^2(\sin^2\theta' + \cos^2\theta') + z^2 - 2zR\cos\theta'$$
$$= R^2 + z^2 - 2Rz\cos\theta'$$

On the surface of a sphere, element of a surface area $= R^2 \sin\theta'\, d\theta' d\phi$

$$\Rightarrow 4\pi\epsilon_0 V(z) = \sigma \int_0^{2\pi}\int_0^{\pi} \frac{R^2 \sin\theta' \cdot d\theta'\, d\phi}{\left[R^2 + z^2 - 2Rz\cos\theta'\right]^{1/2}}$$

$$R^2 + z^2 - 2Rz\cos\theta' = x$$

$$\Rightarrow \quad 2Rz\sin\theta'\,d\theta' = dx$$

$$\Rightarrow \quad 4\pi\epsilon_0 V(z) = 2\pi R^2\sigma \left(\frac{1}{Rz}\left[R^2 + z^2 - 2Rz\cos\theta'\right]^{1/2}\right)\Bigg|_0^\pi$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

$$\Rightarrow \quad V(z) = \frac{R\sigma}{2\epsilon_0 z}\left((R+z) - (z-R)\right)$$

for points outside, $V(z) = \dfrac{R^2\sigma}{\epsilon_0 z}$

$(z > R)$

For points inside, $z < R$

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}\left[(R+z) - (R-z) \right]$$

$$= \frac{R\sigma}{\epsilon_0}$$

$q = 4\pi R^2 \sigma$ | $q = \int \sigma \, da'$

In terms of r and q

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad (r \geq R)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{R} \quad (r \leq R)$$