Solutions for Tutorial Sheet 6

1.(a) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} \quad \left(\because f(0) = 0\right)$$

$\forall\ h \in Q$, $$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$\forall\ h \notin Q$, $$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$$

$\therefore \quad f'(0) = 1$

1.(b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h}$$

$$= \lim_{k\to \infty} \frac{k}{e^{k^2}} \quad \left(\frac{\infty}{\infty}\right) \qquad \left(\text{let } \frac{1}{h} = k \quad \therefore \text{ as } h \to 0 \Rightarrow k \to \infty\right)$$

$$= \lim_{k\to\infty} \frac{1}{e^{k^2}.2k} = \frac{1}{\infty}$$

$$= 0$$

$\therefore \quad f'(0) = 0$

1) (c) $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{\cancel{h} \sin\frac{1}{h} - 0}{\cancel{h}}$

$= \lim_{h \to 0} \sin\frac{1}{h}$

$= \lim_{k \to \infty} \sin k$ (let $\frac{1}{h} = k$ $\therefore$ as $h \to 0$ $\Rightarrow k \to \infty$)

$\because$ this limit oscillates, this limit does not exist. $\Rightarrow$ f is not differentiable at 0.

1) (d) $L f'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0^-} \frac{e^x - e^0}{x}$

$= \lim_{x \to 0^-} \frac{e^x - 1}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0^-} \frac{e^x}{1} = 1$

$R f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0^+} \frac{e^{-x} - e^0}{x}$

$= \lim_{x \to 0^+} \frac{e^{-x} - 1}{x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0^+} \frac{-e^{-x}}{1} = -1$

$\because L f'(0) \neq R f'(0) \Rightarrow f$ is not differentiable at 0

2) (a) $f(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

When $x \neq 0$, $f'(x) = \frac{df}{dx} = \frac{d}{dx}\left(x^3 \sin \frac{1}{x}\right)$

$$= 3x^2 \sin \frac{1}{x} + x^3 \cos \frac{1}{x}\left(-\frac{1}{x^2}\right)$$

$$= 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$$

<u>Now:</u> $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h^3 \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0$$

$=$

[$\because$ We know: $\lim_{x \to e} f(x)\, g(x) = 0$ if $g(x)$ is bounded f^n and $\lim_{x \to e} f(x) = 0$]

$$\therefore f'(x) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Now, $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left\{3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}\right\}$

$$= \lim_{x \to 0} 3x^2 \sin \frac{1}{x} - \lim_{x \to 0} x \cos \frac{1}{x}$$

$$= 0 - 0 = 0 = f'(0)$$

$\Rightarrow f'$ is continuous at $x = 0$.

2) (b) $f(x) = \begin{cases} x^2\cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

When $x \neq 0$, $f'(x) = \frac{d}{dx}\left(x^2\cos\frac{1}{x}\right)$

$= 2x\cos\frac{1}{x} + \cancel{x^2}\left(-\sin\frac{1}{x}\right)\left(-\frac{1}{\cancel{x^2}}\right)$

$= 2x\cos\frac{1}{x} + \sin\frac{1}{x}$

Now, $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{h^2\cos\frac{1}{h} - 0}{h}$

$= \lim_{h\to 0} h\cos\frac{1}{h} = 0$

$\therefore f'(x) = \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Now, $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left\{2x\cos\frac{1}{x} + \sin\frac{1}{x}\right\}$

$= \lim_{x\to 0} 2x\cos\frac{1}{x} + \lim_{x\to 0}\sin\frac{1}{x}$

$=$ 0 + limit does not exist

$=$ limit does not exist

$\Rightarrow$ $f'(x)$ is not continuous at $x = 0$.

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3) (a) $\lim_{x\to 0} \frac{x - \tan x}{x^3}$ $\left(\frac{0}{0}\right)$ L- Hospital's Rule

$= \lim_{x\to 0} \frac{1 - \sec^2 x}{3x^2}$

$= \lim_{x\to 0} \frac{1 - (1 + \tan^2 x)}{3x^2}$

$= \lim_{x\to 0} - \frac{\tan^2 x}{3x^2}$

$= -\frac{1}{3} \times \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2$

$= -\frac{1}{3}$ $\because \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$

(b) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{e^x + e^x + xe^x + \frac{1}{(1+x)^2}}{2}$

$= \frac{1}{2} \times \left\{e^0 + e^0 + 0 \times e^0 + \frac{1}{1^2}\right\}$

$= \frac{1}{2} \times \{3\} = \frac{3}{2}$

3) (e) $\lim_{x \to \infty} \frac{e^x}{x^2} \left(\frac{\infty}{\infty}\right)$ L. Hospital's Rule

$= \lim_{x \to \infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x \to \infty} \frac{e^x}{2}$

$= \frac{e^\infty}{2} = \infty$

4) $f(x) = x^2 - 2x - 8$ on $[-1, 3]$

Now, $f(x)$ is contn on $[-1, 3]$

$f(x)$ is diffn on $(-1, 3)$

$f(-1) = (-1)^2 + 2 - 8 = -5$

$f(3) = 3^2 - 6 - 8 = -5$

$\therefore f(-1) = f(3)$

then from Rolle's theorem, $\exists\, c \in (-1, 3)$

s.t $f'(c) = 0$

$\Rightarrow 2c - 2 = 0$

$\Rightarrow c = 1 \in (-1, 3)$

5. on $[1,3]$, $f(x) = x + \frac{1}{x}$ is continuous.

on $(1,3)$, $f(x) = x + \frac{1}{x}$ is differentiable.

Now $f(1) = 1 + \frac{1}{1} = 2$

$f(3) = 3 + \frac{1}{3} = \frac{10}{3}$

So, $\frac{f(1) - f(3)}{1-3} = \frac{2 - \frac{10}{3}}{-2} = -\frac{\cancel{4}^{2}}{3} \times -\frac{1}{\cancel{2}}$

$= \frac{2}{3}$

Now $f'(x) = 1 - \frac{1}{x^2}$

$\therefore \quad f'(c) = \frac{f(1) - f(3)}{1-3}$

$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{2}{3}$

$\Rightarrow \quad \frac{1}{c^2} = 1 - \frac{2}{3} = \frac{1}{3}$

$\Rightarrow \quad c^2 = 3$

$\Rightarrow \quad c = \pm\sqrt{3}$

but only $c = \sqrt{3} = 1.73 \in (1,3)$

Hence Lagranges Mean Value theorem verified.

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6. Consider,

$$f(x) = \log(1+x) - \left\{x - \frac{x^2}{2}\right\}$$

$$\therefore f'(x) = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x} > 0 \quad \forall\ x>0$$

Hence $f(x)$ is an increasing function $\forall\ x>0$

Also,

$$f(0) = 0$$

Hence for $x>0$; $f(x) > 0$

Thus;

$$\log(1+x) - \left\{x - \frac{x^2}{2}\right\} > 0 \quad \forall\ x>0$$

$$\Rightarrow \log(1+x) > x - \frac{x^2}{2} \quad \forall\ x>0.$$

Similarly by considering the function.

$$F(x) = x - \frac{x^2}{2(1+x)} - \log(1+x).$$

it can be shown that

$$\log(1+x) < x - \frac{x^2}{2(1+x)} \quad \forall\ x>0.$$

7) (i) $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 0$

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0;\ x = 2.$

$\therefore$ critical points are 0 and 2.

Now, $f''(x) = 6x - 6$.

at $x = 0$; $f''(0) = 6 \times 0 - 6 = -6 < 0$

$\Rightarrow$ at $x = 0$; the function $f(x)$ has local maxima

at $x = 2$; $f''(2) = 6 \times 2 - 6 = 6 > 0$.

$\Rightarrow$ at $x = 2$; the function $f(x)$ has local minima

(ii) $f(x) = x^3 - 12x + 1$

$f'(x) = 0$

$\Rightarrow 3x^2 - 12 = 0$

$\Rightarrow 3(x^2 - 2^2) = 0$

$\Rightarrow 3(x-2)(x+2) = 0$

$\Rightarrow x = 2;\ x = -2$

$f''(x) = 6x$

$\therefore$ at $x = 2$

$f''(2) = 12 > 0$

at $x = -2$

$f''(-2) = -12 < 0$

$\therefore$ at $x = 2$; $f''(2) > 0 \Rightarrow x = 2$ is a local minima point

at $x = -2$; $f''(2) < 0 \Rightarrow x = -2$ is a local maxima point

(iii) $f(x) = 3x^3 - 9x^2 - 27x + 15$

$f'(x) = 9x^2 - 18x - 27 = 0$

$\Rightarrow \quad 9(x^2 - 2x - 3) = 0$

$\Rightarrow \quad 9(x^2 - 3x + x - 3) = 0$

$\Rightarrow \quad 9(x-3)(x+1) = 0.$

$\Rightarrow \quad x = 3; \quad x = -1$

$f''(x) = 18x - 18 = 18(x-1)$

at $x = 3$; $f''(3) = 18(3-1) = 36 > 0.$

at $x = -1$; $f''(-1) = -36 < 0$

$\Rightarrow$ at $x = 3$, $f(x)$ has local ~~maxima~~ minima

and at $x = -1$; $f(x)$ " " maxima