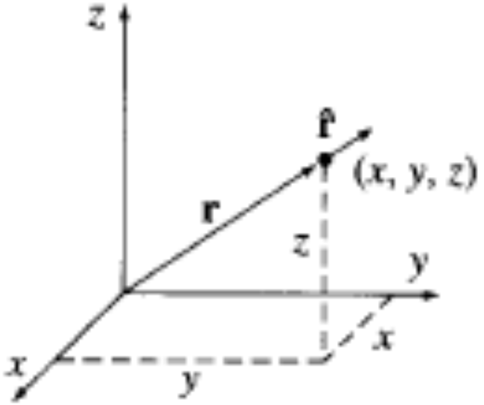
Lecture - 3

Vector Operator

Position vector: The vector to that point from the origin.

$$\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$



Its magnitude (the distance from the origin)

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} \equiv \sqrt{x^2 + y^2 + z^2}$$

Its direction unit vector (pointing radially outward)

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

The **infinitesimal displacement vector,** from (x, y, z) to ($x+dx$, $y+dy$, $z+dz$), is

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$

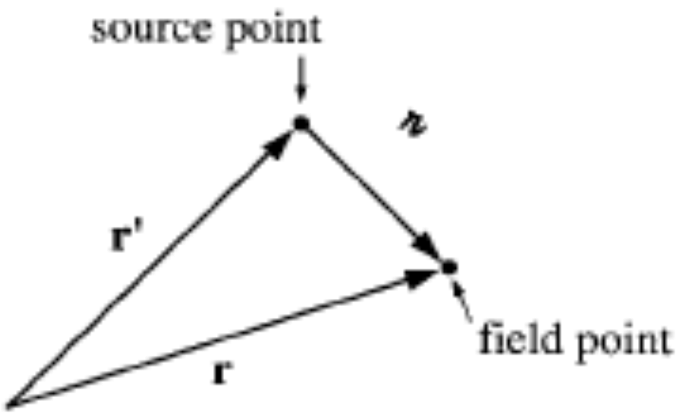
Vector Operator

In electrodynamics one frequently encounters problems involving two points:

A source point, **r**′, where an electric field is located

A field point, **r**, at which you are calculating the electric field

A short-hand notation for the separation vector from the source point to the field point is

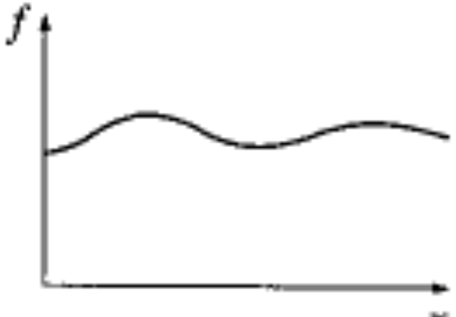


$$\mathscr{r} \equiv \mathbf{r} - \mathbf{r}', \text{ magnitude } \mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

unit vector in the direction form $\mathbf{r}'$ to $\mathbf{r}$ is $\hat{\mathscr{r}} = \dfrac{\mathscr{r}}{\mathscr{r}} = \dfrac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$

Ordinary derivative

Suppose we have a function of one variable, $f(x)$. What does the derivative, df/dx, do for us?



(a)

Ans: It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx.

$$df = \left(\frac{df}{dx}\right)dx$$

In words, if we change x by an amount dx, then, f changes by an amount df.

The derivative df/dx is the slope of the graph of f versus x.

Gradient

Suppose we have a function of three variables. What does the derivative mean in this case?

A mountain hill

$H(x, y, z)$

A theorem on partial derivatives states that

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz$$

$$= (\frac{\partial H}{\partial x}\hat{\mathbf{x}} + \frac{\partial H}{\partial y}\hat{\mathbf{y}} + \frac{\partial H}{\partial z}\hat{\mathbf{z}}) \cdot (dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}})$$

$$= (\nabla H) \cdot (d\mathbf{l})$$

The gradient of *H* is a vector quantity, with three components.

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Gradient

Geometrical interpretation: Like any vector, the gradient has magnitude and direction.

A dot product in abstract form is: $dH = \nabla H \cdot d\mathbf{l} = |\nabla H||d\mathbf{l}|\cos\theta$
where θ is the angle between ∇H and $d\mathbf{l}$.

The gradient ∇H points in the direction of maximum increase of the function H.

Analogous to the derivative of one variable, a vanishing derivative signals a maximum, a minimum, or an *inflection*.

Example   Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$

$$\text{Ans}: \nabla r = \frac{\partial r}{\partial x}\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\hat{\mathbf{z}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

The gradient has the formal appearance of a vector, ∇, "multiplying", a scalar H.

$$\nabla H = \underbrace{(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z})}_{\textbf{del}}H$$

∇ is a vector operator that acts upon H, not a vector that multiplies H.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\mathbf{r} \equiv (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

$$\text{(a) } \nabla \mathbf{r}^2 = ?$$

$$\begin{aligned}\nabla \mathbf{r}^2 &= \nabla[(x - x')^2 + (y - y')^2 + (z - z')^2] \\ &= 2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}} = 2\mathbf{r}\end{aligned}$$

$$\text{(b) } \nabla(1/\mathbf{r}) = \frac{-\nabla \mathbf{r}}{\mathbf{r}^2} = \frac{-\nabla\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$= -\tfrac{1}{2}[2(x - x')\hat{\mathbf{x}} + 2(y - y')\hat{\mathbf{y}} + 2(z - z')\hat{\mathbf{z}}]/\mathbf{r}^3 = -\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}$$

The Gradient Opertator

The gradient has the formal appearance of a vector, ∇, "multiplying", a scalar H.

$$\nabla H = (\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z})H$$

del

∇ is a vector operator that acts upon H, not a vector that multiplies H.

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

More about Gradient Operator

An ordinary vector **A** can be multiply in three ways:

1. Multiply a scalar a : $a\mathbf{A}$

2. Multiply another vector (dot product): $\mathbf{A}\cdot\mathbf{B}$

3. Multiply another vector (cross product): $\mathbf{A}\times\mathbf{B}$

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function H: ∇H (**Gradient**

2. On a vector function (dot product): $\nabla\cdot\mathbf{v}$ (**divergence**

3. On a vector function (cross product): $\nabla\times\mathbf{v}$ (**curl**