1. A particle of mass 4 units moves in a force field depending on time *t* given by $\vec{F} = 48t^2\hat{\imath} + (72t + 16)\hat{\jmath} - 24t\hat{k}$. Assuming that at $t = 0$ the particle is located at $\vec{r}_0 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and has velocity $\vec{v}_0 = 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k}$. Find the momentum and position at any time *t*. **4**

From Newton's second law, $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow 4\frac{d\vec{v}}{dt} = 48t^2\hat{\imath} + (72t + 16)\hat{\jmath} - 24t\hat{k}$

Till here 0.5 mark

$\Rightarrow \frac{d\vec{v}}{dt} = 12t^2\hat{\imath} + (18t + 4)\hat{\jmath} - 6t\hat{k} \Rightarrow \vec{v} = 4t^3\hat{\imath} + (9t^2 + 4t)\hat{\jmath} - 3t^2\hat{k} + c_1.$

Till here 1 mark

Since at $t = 0$, $\vec{v}_0 = 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k} \Rightarrow 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k} = c_1$

$\Rightarrow \vec{v}(t) = (4t^3 + 6)\hat{\imath} + (9t^2 + 4t + 15)\hat{\jmath} - (3t^2 + 8)\hat{k}.$

Till here 2 marks

Hence, momentum is $\vec{p}(t) = (16t^3 + 24)\hat{\imath} + (36t^2 + 16t + 60)\hat{\jmath} - (12t^2 + 32)\hat{k}$

Till here 2.5 marks

Since, $\vec{v} = \frac{d\vec{r}}{dt}$, hence $\frac{d\vec{r}}{dt} = (4t^3 + 6)\hat{\imath} + (9t^2 + 4t + 15)\hat{\jmath} - (3t^2 + 8)\hat{k} \Rightarrow \vec{r} = (t^4 + 6t)\hat{\imath} + (3t^3 + 2t^2 + 15t)\hat{\jmath} - (t^3 + 8t)\hat{k} + c_2.$

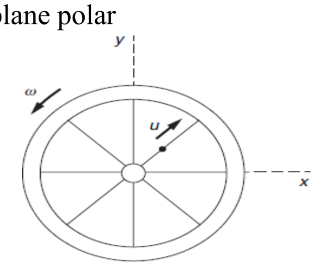
Till here 3 marks

At $t = 0, \vec{r}_0 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k} \Rightarrow c_2 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k}.$

So $\vec{r}(t) = (t^4 + 6t + 3)\hat{\imath} + (3t^3 + 2t^2 + 15t - 1)\hat{\jmath} - (t^3 + 8t - 4)\hat{k}.$

Till here 4 marks

Consider a bead moving along the spoke of a rotating wheel as shown in the figure. Assume both u and ω are constant. Calculate the velocity and acceleration of the bead in plane polar coordinates. **3**



Here, $\dot{r} = u; \dot{\theta} = \omega; \ddot{r} = 0; \ddot{\theta} = 0.$

Identifying the above conditions 1 Mark.

Thus, velocity in polar coordinate is $\vec{v} = u\hat{r} + r\omega\hat{\theta}.$

Till here 1.5 marks

However, in this case $r = ut$, hence, $\vec{v} = u\hat{r} + ut\omega\hat{\theta}.$

Till here 2 Marks

The acceleration can now be computed as,

$\vec{a} = -\omega^2 r\hat{r} + 2u\omega\hat{\theta} = -\omega^2 ut\hat{r} + 2u\omega\hat{\theta}$

Till here 3 Marks

Note: Correctly writing velocity and acceleration formula in plain polar coordinate you can award 0.5 each.

$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta},$

$a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}.$

Prob. 2 (a) Given that $\vec{F} = 3\hat{i} + z\hat{j} + y\hat{k}$ — (1)

To check whether $\vec{F}$ is conservative or not, calculate $\vec{\nabla} \times \vec{F}$

So,

Step 1 [0.5 Mark]

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & z & y \end{vmatrix} \longrightarrow 0.5 \text{ Mark}$$

Step 2 [1 mark]

$$= \hat{i}\left(\frac{\partial y}{\partial y} - \frac{\partial z}{\partial z}\right) - \hat{j}\left(\frac{\partial (y)}{\partial x} - \frac{\partial (3)}{\partial z}\right) + \hat{k}\left(\frac{\partial (z)}{\partial x} - \frac{\partial (3)}{\partial y}\right)$$

$$= 0$$

1.5 Mark

$\because$ $\vec{\nabla} \times \vec{F} = 0$, Force is conservative.

$\because$ $\vec{F} = -\vec{\nabla} V \longrightarrow 0.5$ Mark

Step 3 (0.5 mark)

From (1)

$$\frac{\partial V}{\partial x} = -3 \rightarrow \text{(I)}, \quad \frac{\partial V}{\partial y} = -z \rightarrow \text{(II)}, \quad \frac{\partial V}{\partial z} = -y \rightarrow \text{(III)}$$

Step 4 [1 mark]

On interating first term,

$$V(x,y,z) = -3x + f(y,z) \quad \text{— (2)}$$

From second term & eq^n (2)

$$\frac{\partial f}{\partial y} = -z$$

$$\Rightarrow f(y,z) = -zy + C$$

Hence $\boxed{V(x,y,z) = -3x - zy + C}$

1 Mark

Prob. 2(b)

Step 1 0.5 Mark

Stokes' theorem : $\boxed{\oint \vec{F}\cdot d\vec{\ell} = \int_S \text{Curl}\,\vec{F}\cdot d\vec{s}}$ 0.5 mark

Given that $\vec{F} = y\hat{i} - x\hat{j} + z\hat{k}$,

Line integral around the perimeter i.e. around circle, $x^2+y^2=a^2$
[X, Y Plane]

Step 2 0.5 Mark

$$\oint \vec{F}\cdot d\vec{r} = \oint (y\hat{i} - x\hat{j} + z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint (y\,dx - x\,dy) \quad \text{[Only in 2D plane]}$$

Step 3 1.5 Mark (1.5)

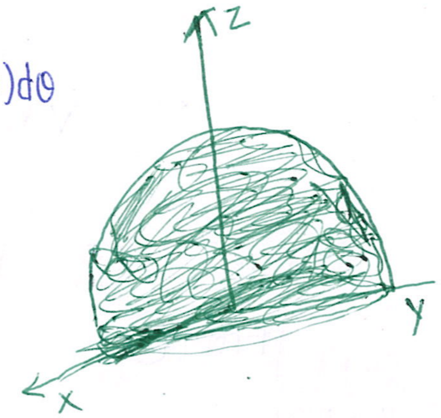
$\because$ $x = a\cos\theta$, $y = a\sin\theta$ } Note: One can either write θ or ϕ

$dx = -a\sin\theta\,d\theta$, $dy = a\cos\theta\,d\theta$

$$\therefore \oint (y\,dx - x\,dy) = -a^2\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= -a^2\int_0^{2\pi} d\theta$$

$$= -2\pi a^2 \quad \text{— (1)}$$



given that $d\vec{s} = a^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

Step 4 0.5 Mark

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix} = -2\hat{k}$$

Step 5 1 Mark

$$\int_S (\vec{\nabla}\times\vec{F})\cdot ds = -2\int \hat{k}\cdot(a^2\sin\theta\,d\theta\,d\phi)\,\hat{r}$$

$\because \hat{r}\cdot\hat{k} = |r||k|\cos\theta = \cos\theta$

$$= -2a^2\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= -2\pi a^2 \quad \text{— (2)}$$

From (1) & (2)

L.H.S = R.H.S

3\. A) $k = 225$ N/m (given)

a. mass not moving at $t = 1, 2, 3, 4$ seconds $\longrightarrow$ [1 mark] [0.25 marks for each t]

b. Energy $= \frac{1}{2}kx^2 \longrightarrow$ [0.5 mark]

$$= \frac{1}{2} \times 225 \times \left(\frac{7}{100}\right)^2$$

[Total 1 mark]

$$= 0.55 \text{ J}$$ $\longrightarrow$ [0.5 mark]

c. At $t = 1$ s, $V_x = 0$ and $x = -6$ cm

$$\therefore E_1 = \frac{1}{2}kx^2 = \frac{1}{2} \times 225 \times \left(\frac{6}{100}\right)^2 = 0.405 \text{ J}$$ [0.5 mark]

At $t = 4$ s, $V_x = 0$ and $x = 3$ cm

$$\therefore E_4 = \frac{1}{2}kx^2 = \frac{1}{2} \times 225 \times \left(\frac{3}{100}\right)^2 = 0.101 \text{ J}$$ [0.5 mark]

Energy lost $= E_1 - E_4 \approx 0.3$ J $\longrightarrow$ [0.5 mark]

This energy got converted to other forms of energy by nonconservative forces such as friction/air resistance $\longrightarrow$ [0.5 mark]

Total 2 marks

3. B) $T = 1.2\,s$ [$\overset{\text{Time}}{\text{period}}$ given]

Amplitude decay: $x = x_0 \, e^{-\frac{\gamma}{2}t}$ → [1 mark]

x ↓ amplitude at time t

x_0 ↓ amplitude at $t=0$

Ratio of amplitudes at $t=0$ and $t = 3T = 3.6\,s$ is

$$2 = \frac{x_0}{x(t=3.6)} = \frac{1}{e^{-3.6(\gamma/2)}}$$

$$\therefore \quad 2 = \frac{1}{e^{-1.8\gamma}}$$

$$\therefore \quad 1.8\gamma = \ln 2 = 0.69$$

$$\therefore \quad \gamma = 0.39\,s^{-1}$$ → [0.5 mark]

Quality factor $Q = \frac{\omega_0}{\gamma} = \frac{2\pi}{T}\cdot\frac{1}{\gamma} = \frac{2\pi}{1.2}\cdot\frac{1}{0.39} = 13.42$

≈ 13

→ [0.5 mark]

[Total 2 marks]