EPHY108L: Classical Mechanics

Tutorial 1 [QUESTIONS]

1. ►*A point P divides a line segment AB in the ratio $\lambda : \mu$ (see figure 7.5). If the position vectors of the points A and B are* **a** *and* **b**, *respectively, find the position vector of the point P.*

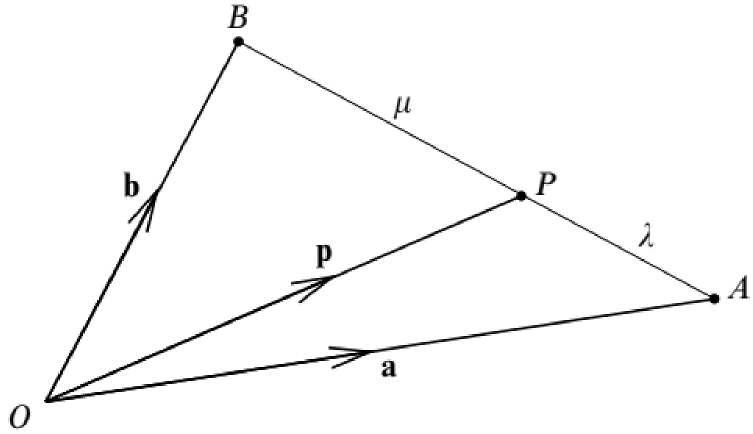


Figure 7.5 An illustration of the ratio theorem. The point P divides the line segment AB in the ratio $\lambda : \mu$.

2. ►*The vertices of triangle ABC have position vectors* **a**, **b** *and* **c** *relative to some origin O (see figure 7.6). Find the position vector of the centroid G of the triangle.*

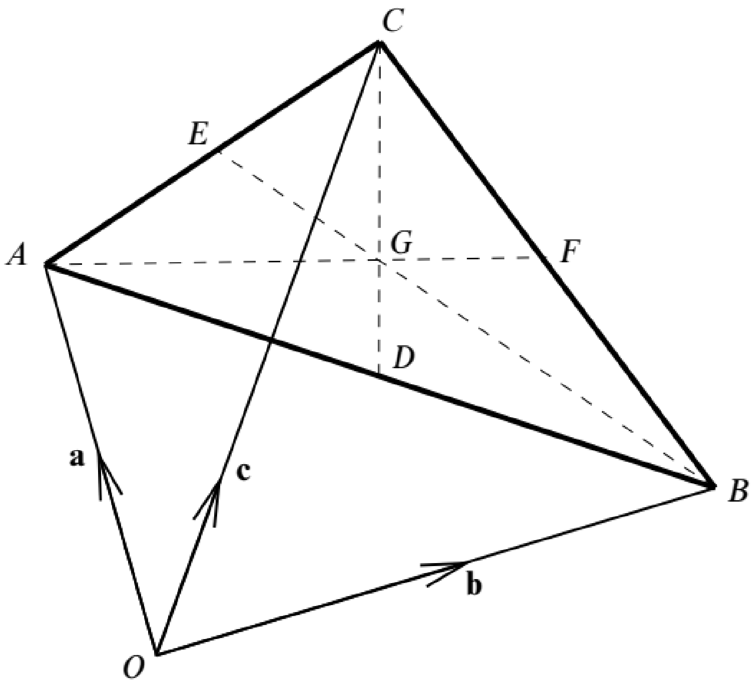


Figure 7.6 The centroid of a triangle. The triangle is defined by the points A, B and C that have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The broken lines CD, BE, AF connect the vertices of the triangle to the mid-points of the opposite sides; these lines intersect at the centroid G of the triangle.

3.

►*Two particles have velocities* $\mathbf{v}_1 = \mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ *and* $\mathbf{v}_2 = \mathbf{i} - 2\mathbf{k}$, *respectively. Find the velocity* $\mathbf{u}$ *of the second particle relative to the first.*

4.

►*Four non-coplanar points* A, B, C, D *are positioned such that the line* AD *is perpendicular to* BC *and* BD *is perpendicular to* AC. *Show that* CD *is perpendicular to* AB.

Hint: If vectors **a** and **b** are perpendicular then we have

$$\mathbf{a} \cdot \mathbf{b} = 0 \qquad (7.16)$$

5.

►*Find the angle between the vectors* $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

6.

►*Show that if* $\mathbf{a} = \mathbf{b} + \lambda\mathbf{c}$, *for some scalar* λ, *then* $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$.

7.

►*Find the area* A *of the parallelogram with sides* $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$.

8.

►*The position vector of a particle at time t in Cartesian coordinates is given by* $\mathbf{r}(t) = 2t^2\mathbf{i} + (3t-2)\mathbf{j} + (3t^2-1)\mathbf{k}$. *Find the speed of the particle at* $t = 1$ *and the component of its acceleration in the direction* $\mathbf{s} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

9.

►*The position vector of a particle in plane polar coordinates is* $\mathbf{r}(t) = \rho(t)\hat{\mathbf{e}}_\rho$. *Find expressions for the velocity and acceleration of the particle in these coordinates.*

10.

►*A particle of mass m with position vector* $\mathbf{r}$ *relative to some origin O experiences a force* $\mathbf{F}$, *which produces a torque (moment)* $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ *about O. The angular momentum of the particle about O is given by* $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$, *where* $\mathbf{v}$ *is the particle's velocity. Show that the rate of change of angular momentum is equal to the applied torque.*

11.

►*A small particle of mass m orbits a much larger mass M centred at the origin O. According to Newton's law of gravitation, the position vector* $\mathbf{r}$ *of the small mass obeys the differential equation*

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Show that the vector $\mathbf{r} \times d\mathbf{r}/dt$ *is a constant of the motion.*

12.

► *Find the element of area on the surface of a sphere of radius a, and hence calculate the total surface area of the sphere.*

13.

►*Find the gradient of the scalar field* $\phi = xy^2z^3$.

14.

►*For the function* $\phi = x^2y + yz$ *at the point* $(1, 2, -1)$*, find its rate of change with distance in the direction* $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. *At this same point, what is the greatest possible rate of change with distance and in which direction does it occur?*

15.

►*Find the divergence of the vector field* $\mathbf{a} = x^2y^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

16.

►*Find the Laplacian of the scalar field* $\phi = xy^2z^3$.

17.

►*Find the curl of the vector field* $\mathbf{a} = x^2y^2z^2\mathbf{i} + y^2z^2\mathbf{j} + x^2z^2\mathbf{k}$.

18.

►*Show that*

$$\nabla \times (\phi\mathbf{a}) = \nabla\phi \times \mathbf{a} + \phi\nabla \times \mathbf{a}.$$

19.

►*Express the vector field* $\mathbf{a} = yz\,\mathbf{i} - y\,\mathbf{j} + xz^2\,\mathbf{k}$ *in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.*

20.

►*Evaluate the line integral* $I = \int_C \mathbf{a} \cdot d\mathbf{r}$, *where* $\mathbf{a} = (x+y)\mathbf{i} + (y-x)\mathbf{j}$, *along each of the paths in the xy-plane shown in figure 11.1, namely*

(i) *the parabola* $y^2 = x$ *from* $(1,1)$ *to* $(4,2)$,
(ii) *the curve* $x = 2u^2 + u + 1$, $y = 1 + u^2$ *from* $(1,1)$ *to* $(4,2)$,
(iii) *the line* $y = 1$ *from* $(1,1)$ *to* $(4,1)$, *followed by the line* $x = 4$ *from* $(4,1)$ *to* $(4,2)$.

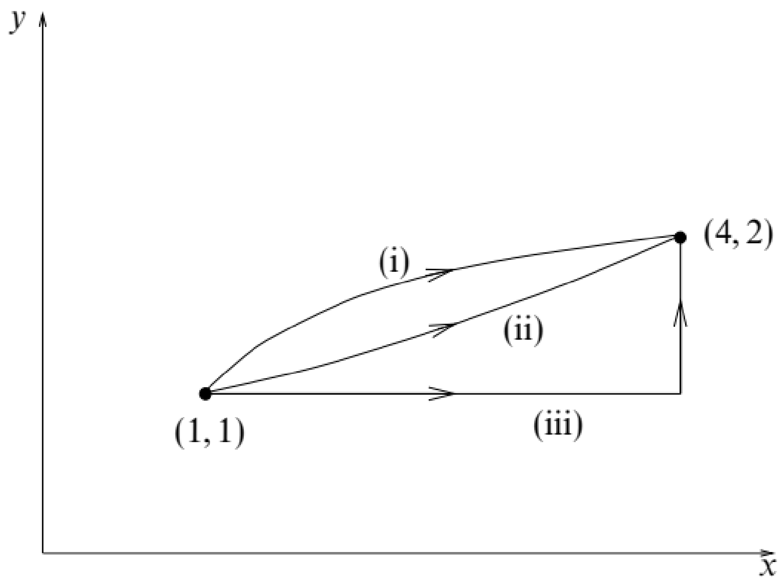


Figure 11.1 Different possible paths between the points $(1,1)$ and $(4,2)$.

21.

►*Evaluate the line integral* $I = \oint_C x\,dy$, *where* C *is the circle in the xy-plane defined by* $x^2 + y^2 = a^2$, $z = 0$.

22.

►*Evaluate the line integral* $I = \int_C (x-y)^2\,ds$, *where* C *is the semicircle of radius* a *running from* $A = (a, 0)$ *to* $B = (-a, 0)$ *and for which* $y \geq 0$.

23.

►*Evaluate the line integral* $I = \int_A^B \mathbf{a} \cdot d\mathbf{r}$, *where* $\mathbf{a} = (xy^2 + z)\mathbf{i} + (x^2y + 2)\mathbf{j} + x\mathbf{k}$, A *is the point* (c, c, h) *and* B *is the point* $(2c, c/2, h)$, *along the different paths*

(i) C_1, *given by* $x = cu$, $y = c/u$, $z = h$,
(ii) C_2, *given by* $2y = 3c - x$, $z = h$.

Show that the vector field $\mathbf{a}$ *is in fact conservative, and find* ϕ *such that* $\mathbf{a} = \nabla\phi$.

24.

►*Evaluate the surface integral* $I = \int_S \mathbf{a} \cdot d\mathbf{S}$, *where* $\mathbf{a} = x\mathbf{i}$ *and* S *is the surface of the hemisphere* $x^2 + y^2 + z^2 = a^2$ *with* $z \geq 0$.

25.

►*Find the vector area of the surface of the hemisphere* $x^2 + y^2 + z^2 = a^2$ *with* $z \geq 0$.

26.

►*Find the vector area of the surface of the hemisphere* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, *by evaluating the line integral* $\mathbf{S} = \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{r}$ *around its perimeter.*

27.

►*Given the vector field* $\mathbf{a} = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$, *verify Stokes' theorem for the hemispherical surface* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

28.

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k}.$$

Calculate $\nabla \times \mathbf{F}$ and deduce that $\mathbf{F}$ can be written $F = \nabla\phi$. Determine the form of ϕ.

29.

$\mathbf{F}$ is a vector field $xy^2\mathbf{i} + 2\mathbf{j} + x\mathbf{k}$, and L is a path parameterised by $x = ct$, $y = c/t$, $z = d$ for the range $1 \leq t \leq 2$. Evaluate (a) $\int_L \mathbf{F}\,dt$, (b) $\int_L \mathbf{F}\,dy$ and (c) $\int_L \mathbf{F} \cdot d\mathbf{r}$.

30.

A vector field $\mathbf{a}$ is given by $-zxr^{-3}\mathbf{i} - zyr^{-3}\mathbf{j} + (x^2+y^2)r^{-3}\mathbf{k}$, where $r^2 = x^2+y^2+z^2$. Establish that the field is conservative (a) by showing that $\nabla \times \mathbf{a} = \mathbf{0}$, and (b) by constructing its potential function ϕ.

31.

The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (3x^2yz + y^3z + xe^{-x})\mathbf{i} + (3xy^2z + x^3z + ye^x)\mathbf{j} + (x^3y + y^3x + xy^2z^2)\mathbf{k}.$$

Calculate (a) directly, and (b) by using Stokes' theorem the value of the line integral $\int_L \mathbf{F} \cdot d\mathbf{r}$, where L is the (three-dimensional) closed contour $OABCDEO$ defined by the successive vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 1)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 0)$.

32.

Find a unit vector, which lies in the xy plane, and which is perpendicular to $\mathbf{A}$ of previous problems. Similarly, find a unit vector which is perpendicular to $\mathbf{B}$, and lies in the xz plane.

33.

Calculate $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{A})$, for the vectors of the previous problem. Does the result obtained hold only for the given $\mathbf{A}$ and $\mathbf{B}$ vectors, or will it hold for any general vectors $\mathbf{A}$ and $\mathbf{B}$.

34.

Consider two distinct general vectors $\mathbf{A}$ and $\mathbf{B}$. Show that $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$ implies that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

35.

Position of a particle in the xy plane is given by

$$\mathbf{r}(t) = A\left(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}\right),$$

where α and A are constants. Calculate the velocity and acceleration of the particle, as functions of time t. Plot the vectors corresponding to $\mathbf{r}(0)$ and $\mathbf{v}(0)$.

36.

Acceleration of a particle in the xy plane is given by $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$, where $\mathbf{r}(t)$ denotes its position, and ω is a constant. If $\mathbf{r}(0) = a\hat{\mathbf{j}}$, and $\mathbf{v}(0) = a\omega\hat{\mathbf{i}}$ ($\mathbf{v}$ is the velocity), integrate the equation of motion to obtain the expression for $\mathbf{r}(t)$, in Cartesian coordinates. What is the the curve along which the particle is moving?

37.

Using known results from the Cartesian coordinate system, and the relation between plane-polar and Cartesian basis vectors, calculate: (a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$, (b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}}$, and (c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$

38.

A particle is moving along a circular path of radius a, with angular velocity given by $\omega(t) = \omega_0 + \alpha t$, where ω_0 and α are constants. Obtain the radial and tangential components of its velocity and acceleration.

39.

A particle is moving along the line $y = a$, with the velocity $\mathbf{v} = u\hat{\mathbf{i}}$, where u is a constant. Express its velocity in plane polar coordinates.

40.

A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r(t) = r_0 e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates.

For what values of β will the radial acceleration of the particle by zero?

41.

Consider a circle of radius a, with the origin of the plane polar coordinate system placed at a point on the circumference. The particle is moving along the circle with a constant speed u. Assume that $\theta(t = 0) = 0$

(a) What is the equation of the circle in this coordinate system?

(b) What is the value of $\dot{\theta}$ in terms of u and a?

(c) Write down the velocity of the particle in plane-polar coordinate system.

(d) What is the acceleration of the particle in plane-polar coordinate system?

42.

A particle moves along the curve $r = A\theta$, with $A = 1/\pi$ m/rad, and $\theta = \alpha t^2$, where α is a constant. Obtain the expressions for the velocity and the acceleration of this particle in plane polar coordinates.

(a) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

(b) At what angles do radial and tangential components of the acceleration have equal magnitude?

43.

Mass m rotates on a frictionless table, held to circular path by a string which passes through a hole in the table. The string is slowly pulled through the hole so that the radius of the circle changes from R_0 to R_1. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.

44.

A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed towards the origin with the magnitude B, and and inverse law repulsive force of magnitude A/x^2.

(a) Find the potential energy function $V(x)$

(b) Plot the potential energy as a function of x, and the total energy of the system, assuming that the maximum kinetic energy is $K_0 = \frac{1}{2}mv_0^2$.

(c) What is the point of equilibrium, i.e., the point where net force acting on the particle is zero.

45.

A particle of mass M is held fixed at the origin. The gravitational potential energy of another particle of m, in the field of the first mass, is given by

$$V(\mathbf{r}) = -\frac{GMm}{r},$$

where G is the gravitational constant, and r is the distance of mass m from the origin.

(a) What is the force acting on the particle of mass m?

(b) Calculate the curl of this force.

46.

Consider a 2D force field $\mathbf{F} = A(y^2\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}})$. Calculate the work done by this force in going around a closed path which is a square made up of sides of length a, lying in

the xy-plane, with two of its vertices located at the origin, and point (a, a). Find the answer by doing the line integral, as well as by using the Stokes' theorem. The path is traversed in a counter clockwise manner.

47.

Find the forces for the following potential energies

(a) $V(x, y, z) = Ax^2 + By^2 + Cz^2$

(b) $V(x, y, z) = A\ln(x^2 + y^2 + z^2)$

(c) $V(r, \theta) = A\cos\theta/r^2$ (r and θ are plane polar coordinates)

Above, A, B, and, C are constants.

48.

Determine whether each of the following forces is conservative. Find the potential energy function, if it exists. A, α, β are constants.

(a) $\mathbf{F} = A(3\hat{\mathbf{i}} + z\hat{\mathbf{j}} + y\hat{\mathbf{k}})$

(b) $\mathbf{F} = Axyz(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(c) $F_x = A\sin(\alpha y)\cos(\beta z)$, $F_y = -Ax\alpha\cos(\alpha y)\cos(\beta z)$, $F_z = Ax\sin(\alpha y)\sin(\beta z)$

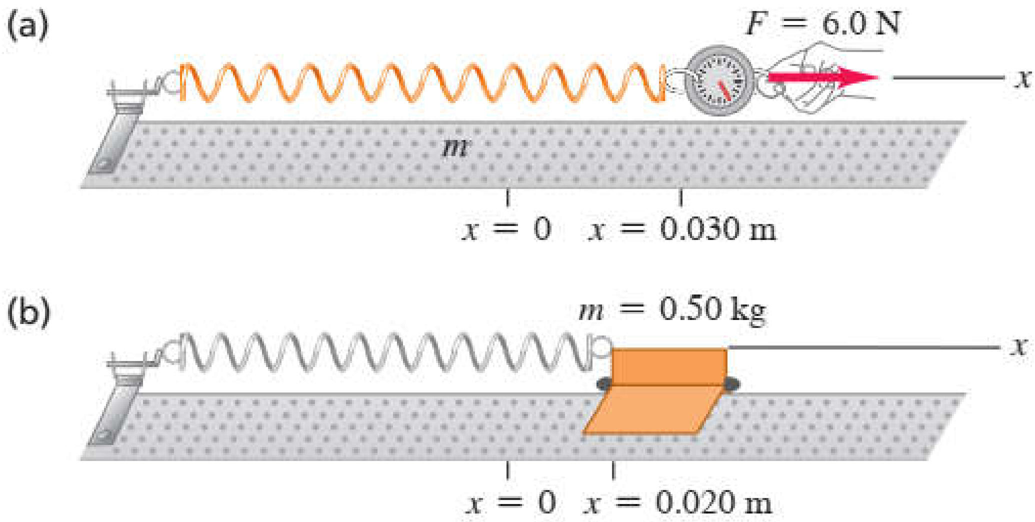
49.

An ultrasonic transducer used for medical diagnosis oscillates at 6.7 MHz $= 6.7 \times 10^6$ Hz. How long does each oscillation take, and what is the angular frequency?

50.

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (**Fig. 14.8a**) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50 kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (Fig. 14.8b). (a) Find the force constant k of the spring. (b) Find the angular frequency ω, frequency f, and period T of the resulting oscillation.

Figure 14.8 (a) The force exerted *on* the spring (shown by the vector F) has x-component $F_x = +6.0$ N. The force exerted *by* the spring has x-component $F_x = -6.0$ N. (b) A glider is attached to the same spring and allowed to oscillate.



51.

The shock absorbers in an old car with mass 1000 kg are completely worn out. When a 980 N person climbs slowly into the car at its center of gravity, the car sinks 2.8 cm. The car (with the person aboard) hits a bump, and the car starts oscillating up and down in SHM. Model the car and person as a single object on a single spring, and find the period and frequency of the oscillation.

52.

Find the period and frequency of a simple pendulum 1.000 m long at a location where $g = 9.800\ \mathrm{m/s^2}$.

53.

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

(b) What is the amplitude of the oscillation?

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

(d) What is the magnitude a_m of the maximum acceleration of the block?

(e) What is the phase constant ϕ for the motion?

(f) What is the displacement function $x(t)$ for the spring–block system?

54.

Many tall buildings have *mass dampers*, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say, eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

Suppose the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring–block system?

(b) What is the block's speed as it passes through the equilibrium point?