Lecture - 6

(c) **Volume integrals**: a line integral is an expression of the form

$$\int_{v} T d\tau,$$

where T is a scalar function, and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates, $d\tau = dxdydz$

For example, if T is a density of a substance, then the volume integral would give the total mass.

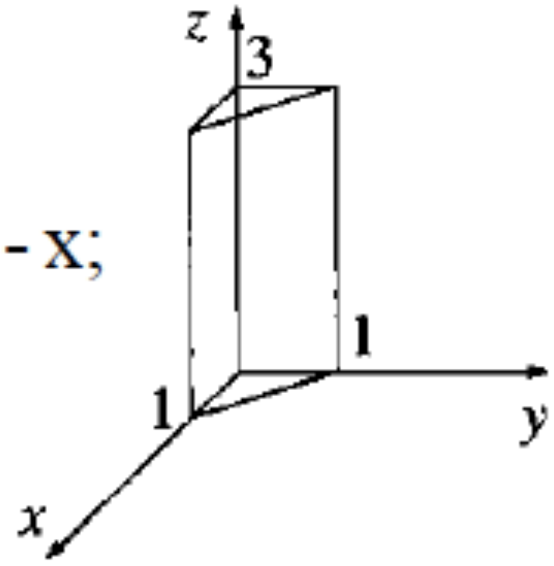
The volume integrals of vector functions:

$$\int \mathbf{v} d\tau = \int (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) d\tau$$

$$= \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

Calculate the volume integral of the function

$T = xyz^2$ over the prism in



Sol : Let's do z first (0 to 3); then y from 0 to 1 - x;
finally x from 0 to 1.

$$\iiint xyz^2\,dxdydz = \int_0^3 z^2 dz \left\{ \int_0^1 x\left(\int_0^{1-x} y\,dy\right)dx \right\}$$

$$= 9\left\{ \int_0^1 x\left(\frac{1}{2}(1-x)^2\right)dx \right\}$$

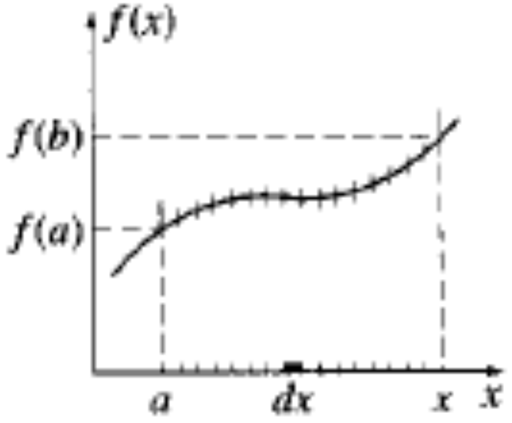
$$= 9\left(\frac{1}{2}\right)\left(\frac{1}{12}\right) = \frac{3}{8}$$

Fundamental theorem of calculus:

$$\int_a^b \frac{df}{dx}\,dx = \int_a^b df = f(b) - f(a)$$

Geometrical Interpretation: two ways to determine the total change in the function:

1. go step-by-step adding up all the tiny increments as you go
2. subtract the values at the ends.



The integral of a derivative over an interval is given by the value of the function at the end points (boundary).

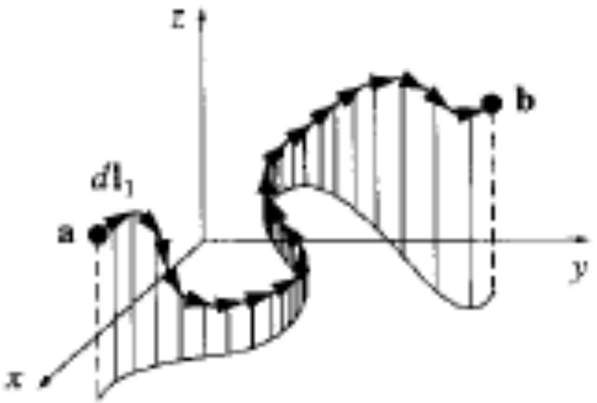
Fundamental theorem of Gradient

A scalar function of three variables $T(x, y, \mathbf{z})$ changes by a small amount.

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$

The total change in T in going from $\mathbf{a}$ to $\mathbf{b}$ along the path selected is:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$



Fundamental theorem for gradient.

Geometrical Interpretation: Measure the high of a skyscraper.

1. Measure the high of each floor and add them all up.
2. Place an altimeter at the top and the bottom, subtract the readings at the ends.

Fundamental theorem of Gradient

$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$ the right side of this equation makes no reference to the path---only to the end points.
Thus gradients have special property that their line integrals are path independent.

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b})$-$T(\mathbf{a})$=0.

A conservative force may be associated with a scalar potential energy function, whereas a non-conservative force cannot.