1. A particle of mass *2* units moves in a force field depending on time *t* given by $\vec{F} = 24t^2\hat{\imath} + (36t - 16)\hat{\jmath} - 12t\hat{k}$. Assuming that at $t = 0$ the particle is located at $\vec{r}_0 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and has velocity $\vec{v}_0 = 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k}$. Find the velocity and position at any time *t*. **2**

From Newton's second law, $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow 2\frac{d\vec{v}}{dt} = 24t^2\hat{\imath} + (36t - 16)\hat{\jmath} - 12t\hat{k} \Rightarrow \frac{d\vec{v}}{dt} = 12t^2\hat{\imath} + (18t - 8)\hat{\jmath} - 6t\hat{k} \Rightarrow \vec{v} = 4t^3\hat{\imath} + (9t^2 - 8t)\hat{\jmath} - 3t^2\hat{k} + c_1$.
Since at $t = 0$, $\vec{v}_0 = 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k} \Rightarrow 6\hat{\imath} + 15\hat{\jmath} - 8\hat{k} = c_1$
$\Rightarrow \vec{v}(t) = (4t^3 + 6)\hat{\imath} + (9t^2 - 8t + 15)\hat{\jmath} - (3t^2 + 8)\hat{k}$.

Since, $\vec{v} = \frac{d\vec{r}}{dt}$, hence $\frac{d\vec{r}}{dt} = (4t^3 + 6)\hat{\imath} + (9t^2 - 8t + 15)\hat{\jmath} - (3t^2 + 8)\hat{k} \Rightarrow \vec{r} = (t^4 + 6t)\hat{\imath} + (3t^3 - 4t^2 + 15t)\hat{\jmath} - (t^3 + 8t)\hat{k} + c_2$. At $t = 0, \vec{r}_0 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k} \Rightarrow c_2 = 3\hat{\imath} - \hat{\jmath} + 4\hat{k}$.
So $\vec{r}(t) = (t^4 + 6t + 3)\hat{\imath} + (3t^3 - 4t^2 + 15t - 1)\hat{\jmath} - (t^3 + 8t - 4)\hat{k}$.

2. A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r = r_0 e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates. For what values of β will the radial acceleration of the particle by zero? **4**

According to the question, $\dot{\theta} = \omega \Rightarrow \ddot{\theta} = 0$ and $r = r_0 e^{\beta t} \Rightarrow \dot{r} = r_0 \beta e^{\beta t}$ and $\ddot{r} = r_0 \beta^2 e^{\beta t}$.

Hence, $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = r_0 e^{\beta t}(\beta\hat{r} + \omega\hat{\theta})$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = r_0 e^{\beta t}(\beta^2 - \omega^2)\hat{r} + 2r_0\omega\beta e^{\beta t}\hat{\theta}$.

From the expression it is clear that the radial component of acceleration will vanish for $\beta = \pm\omega$.

2(a) Function f & g are given as $f = xyz$ & $g = x+y+z$. assuming a right-handed coordinate system, find the value of $\nabla\cdot\nabla(fg)$ at P [2, 0, 1]

2(a) $f = xyz$, $g = x+y+z$ [3 Marks]

$$fg = (xyz)(x+y+z)$$
$$= x^2yz + xy^2z + xyz^2$$

$$\nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2yz + xy^2z + xyz^2)$$

$$= \frac{\partial}{\partial x}(x^2yz + xy^2z + xyz^2)\hat{i}$$
$$+ \frac{\partial}{\partial y}(x^2yz + xy^2z + xyz^2)\hat{j}$$
$$+ \frac{\partial}{\partial z}(x^2yz + xy^2z + xyz^2)\hat{k}$$

$$= (2xyz + y^2z + yz^2)\hat{i} + (x^2z + 2xyz + xz^2)\hat{j}$$
$$+ (x^2y + xy^2 + 2xyz)\hat{k} \longrightarrow \vec{A}$$
(say)

$$\nabla\cdot\nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}$$

$$= 2yz + 0 + 0 + 0 + 2xz + 0 + 0 + 0 + 2xy$$

$$= 2(yz + xz + xy)$$

$$\nabla\cdot\nabla(fg) \text{ at point } (2,0,1) = 2[0\times1 + 2\times1 + 2\times0]$$

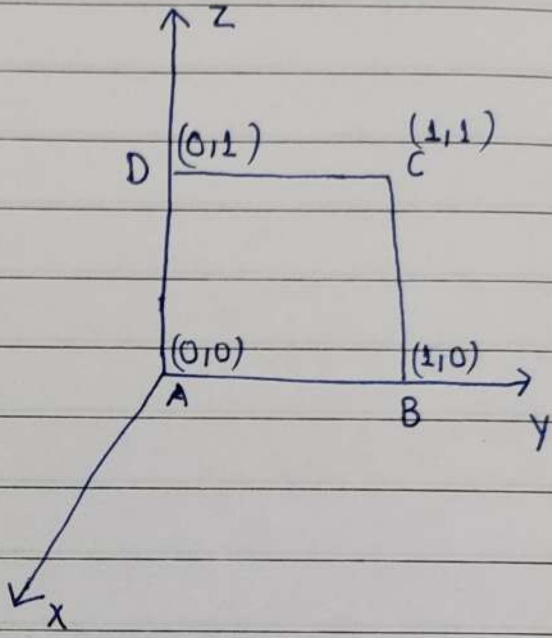
$$= 2\times2$$
$$= 4$$

2(B) Verify stokes' theorem for a given vector field.
$\vec{F} = (2xz + 3y^2)\hat{j} + (4yz^2)\hat{k}$ taken around a square (in yz-plane) having the vertices (0,0), (1,0), (1,1) & (0,1)

[4 Marks]

Stokes' theorem is given as

$$\boxed{\oint \vec{F}\cdot d\vec{l} = \int_S \text{curl}\,\vec{F}\cdot d\vec{S}}$$



$$\oint \vec{F}\cdot d\vec{l} = \int_{AB} \vec{F}\cdot d\vec{l} + \int_{BC} \vec{F}\cdot d\vec{l} + \int_{CD} \vec{F}\cdot d\vec{l} + \int_{DA} \vec{F}\cdot d\vec{l}$$

$$\int_{AB} \vec{F}\cdot d\vec{l} = \int_0^1 3y^2\,dy$$ Here $x=0, z=0$

$$= 3\left[\frac{y^3}{3}\right]_0^1 = 1$$

$$\int_{BC} \vec{F}\cdot d\vec{l} = \int_0^1 \vec{F}\cdot dz\,\hat{k}\Big|_{x=0,\,y=1} = \int_0^1 4z^2\,dz$$

$$= 4\left[\frac{z^3}{3}\right]_0^1 = \frac{4}{3}$$

$$\int_{CD} \vec{F}\cdot d\vec{l} = \int_0^1 \vec{F}\cdot dy\,(-\hat{j}) = -\int_0^1 3y^2\,dy\Big|_{x=0,\,z=1}$$

$$= -3\left[\frac{y^3}{3}\right]_0^1 = -1$$

$$\int_{DA} \vec{F}\cdot d\vec{l} = \int_0^1 \vec{F}\cdot dz(-\hat{k})\Big|_{x=0,\,y=0}$$

$$= 0$$

$$\oint \vec{F}\cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3} = L.H.S$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix}$$

$$= (4z^2 - 2x)\hat{i} + 2z\hat{k}$$

$$\int_S \text{Curl } \vec{F} \cdot d\vec{S} = \int_0^1\int_0^1 [(4z^2 - 2x)\hat{i} + 2z\hat{k}] \cdot dy\,dz\,\hat{i} \quad [x=0]$$

$$= \int_0^1\int_0^1 4z^2\,dy\,dz$$

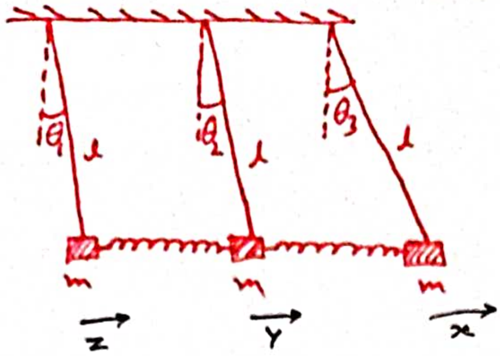
$$= \int_0^1 dy \int_0^1 4z^2\,dz$$

$$= [y]_0^1 \left[4\frac{z^3}{3}\right]_0^1$$

$$= \frac{4}{3} = R.H.S$$

Hence, $L.H.S = R.H.S$

3. A)



$$m\ddot{x} = -k(x-y) - \frac{mgx}{l}$$

$$m\ddot{y} = -k(y-x) - k(y-z) - \frac{mgy}{l}$$

$$m\ddot{z} = -k(z-y) - \frac{mgz}{l}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{k}{m}-\frac{g}{l} & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m}-\frac{g}{l} & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{k}{m}-\frac{g}{l} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{k}{m}-\frac{g}{l} & \frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m}-\frac{g}{l} & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{k}{m}-\frac{g}{l} \end{bmatrix}}_{M} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Eigenvectors of M give the normal modes
Eigenvalues of M lead to normal mode frequencies

Normal modes: $x+y+z$ $\quad$ $\theta_1+\theta_2+\theta_3$
$x-z$ $\quad$ or equivalently $\quad$ $\theta_1-\theta_3$
$x-2y+z$ $\quad$ $\theta_1-2\theta_2+\theta_3$

Corresponding normal mode frequencies : $\omega_0 \; , \; \sqrt{\omega_0^2 + \frac{k}{m}} \; , \; \sqrt{\omega_0^2 + \frac{3k}{m}}$

where $\omega_0 = \sqrt{\frac{g}{l}}$

$$M = -\frac{k}{m}\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{\substack{\text{eigenvectors} \\ \text{give normal} \\ \text{modes}}} - \frac{g}{l}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad \underbrace{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{matrix}}_{\text{Eigenvalues}} \begin{matrix} \to \omega_1 = \omega_0 \\ \to \omega_2 = \sqrt{\omega_0^2 + k/m} \\ \to \omega_3 = \sqrt{\omega_0^2 + 3k/m} \end{matrix}$$

Normalized eigenvectors are: $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

3. B) i) $m = 1\,kg$

$f = \frac{1}{2\pi}\,Hz \longrightarrow \omega = 2\pi f = 1\,rad/s$

$Q = 100$

$Q = \frac{\omega}{\gamma} \Rightarrow \gamma = \frac{\omega}{Q} = 0.01\,s^{-1}$

$k = m\omega^2 = 1\,N/m$

ii) Intensity $I = I_0 e^{-\gamma t}$, $f = 440\,Hz$

At, $t = 10\,s$, $I = \frac{I_0}{e}$ $\qquad \omega_0 = 2\pi f = 2\pi \times 440\,rad/s$

$$\frac{I_0}{e} = I_0 e^{-\gamma 10}$$

$$\Rightarrow \frac{1}{e} = \frac{1}{e^{10\gamma}}$$

$$\therefore 10\gamma = 1 \Rightarrow \gamma = 0.1$$

$$Q = \frac{\omega_0}{\gamma} = \frac{2\pi \times 440}{0.1} = 2\pi \times 4400$$