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Q1. (a) Consider a vector $\vec{A} = xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}$. Show that, for this vector $\vec{\nabla}.(\vec{\nabla}\times\vec{A}) = 0$. You need to show all the steps clearly.
(b) Consider a circular disc of radius R with surface charge density $\sigma = kr$, where k is a constant and r is the radial distance. Calculate the total charge inside the disc.
(c) We have a $+q$ charge placed at the origin. Now we bring another $+2q$ charge at the point (1,2,0). Calculate the electrostatic work done in the process.

3+2+2

Sol:
(a)

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cancel{xy} & \cancel{2yz} & \cancel{3x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} \longrightarrow 0.5 \text{ Mark}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}(3xz) - \frac{\partial}{\partial z}(2yz)\right] - \hat{y}\left[\frac{\partial}{\partial x}(3xz) - \frac{\partial}{\partial z}(xy)\right] + \hat{z}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right]$$

$\longrightarrow$ 1 Mark

$$= -2y\,\hat{x} - 3z\,\hat{y} - x\,\hat{z} \longrightarrow 2 \text{ Marks}$$

$$\vec{\nabla}.(\vec{\nabla}\times\vec{A}) = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(-3z) + \frac{\partial}{\partial z}(-x)$$

$$= 0 \longrightarrow 3 \text{ Marks}$$

(b)

$$\sigma = kr$$

$$Q = \int \sigma\, da = \int \sigma\, r\,dr\,d\phi \quad (\text{or}) \quad \int \sigma\, 2\pi r\, dr \longrightarrow 1 \text{ Mark}$$

$$= k\int_0^R r\,r\,dr \int_0^{2\pi} d\phi \quad (\text{or}) \; 2\pi k \int_0^R r^2\,dr$$

$$= 2\pi k \left.\frac{r^3}{3}\right|_0^R$$

$$\boxed{Q = 2\pi k \frac{R^3}{3}} \longrightarrow 2 \text{ Marks.}$$

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(C) $(0,0,0)$ $(1,2,0)$

$$r = \sqrt{(1-0)^2 + (2-0)^2 + 0^2} = \sqrt{5}$$ → 0.5 Mark.

Potential at $(1,2,0)$ due to $+q$ at the origin

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{5}}$$ → 1 Mark

Work done to bring $+2q$ charge to the point $(1,2,0)$

$$W = V \cdot 2q$$

$$\boxed{W = \frac{1}{4\pi\epsilon_0} \frac{2q^2}{\sqrt{5}}}$$ → 2 Marks

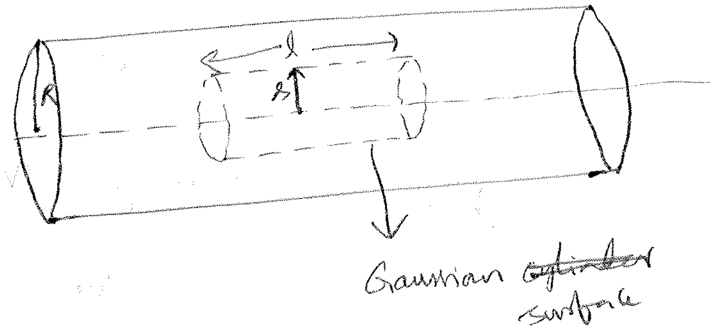
Q2. A long cylinder of radius R, carries a charge density $\rho = \frac{k}{s}$, where k is a constant and s is the distance from the axis. Using Gauss's law,
(a) find the electric field inside the cylinder, and
(b) find the electric field outside the cylinder.

4+3

Sol:

$\rho = \frac{k}{s}$

$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$ $\longrightarrow$ 0.5 Mark



(a) Inside: $Q_{enc} = \int \rho \, d\tau$ $\longrightarrow$ 1 mark

$= k \int \frac{1}{s} s \, ds \, d\phi \, dz$

$= k \int_0^s ds \int_0^{2\pi} d\phi \int_0^l dz$

$\Rightarrow Q_{enc} = 2\pi k s l$ $\longrightarrow$ 1.5 mark

$\oint \vec{E} \cdot d\vec{a} \rightarrow$ The edges of the Gaussian cylinder do not contribute to this integral.

$\Rightarrow \oint \vec{E} \cdot d\vec{a} = \int_S E \, da$ $\rightarrow$ $\vec{E}$ and $d\vec{a}$ are $\parallel$ on the Gaussian surface

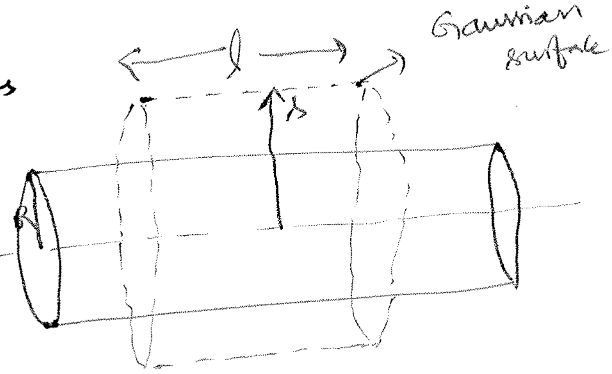
&

$= E \int_S da$ $\quad$ E is constant on the Gaussian surface

$= E \, 2\pi s l$ $\longrightarrow$ 3 Marks

$\Rightarrow E \, 2\pi s l = \frac{1}{\epsilon_0} 2\pi k s l$

$\Rightarrow \boxed{\vec{E} = \frac{k}{\epsilon_0} \hat{s}}$ $\longrightarrow$ 4 Marks



(b) Outside: $Q_{enc} = \int \rho \, d\tau = k \int_0^R \frac{1}{s} s \, ds \int_0^{2\pi} d\phi \int_0^l dz$

$= 2\pi k R l$ $\longrightarrow$ 1 mark

and $\oint \vec{E} \cdot d\vec{a} = E \, 2\pi s l$

$\Rightarrow E \, 2\pi s l = \frac{1}{\epsilon_0} 2\pi k R l$ $\Rightarrow \boxed{\vec{E} = \frac{k}{\epsilon_0} \frac{R}{s} \hat{s}}$

$\longrightarrow$ 3 Marks.

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Q3. (a) A magnetic dipole placed at the origin has magnetic dipole moment $\vec{m} = m_0\hat{z}$. Calculate the vector potential due to this dipole at the point (1,1,0).
(b) Calculate the magnetic field that gives rise to the vector potential $\vec{A} = y^2\hat{x} - x^2\hat{y}$. Find out the corresponding current density $\vec{J}$.
(c) Consider an infinitely long solenoid with circular cross-section of radius R having n turns per unit length and carrying a current I. A cylindrical rod of radius $a < R$ and made of a material of magnetic susceptibility χ_m is placed coaxial within the solenoid. Calculate the auxiliary field ($\vec{H}$) inside the rod using Ampere's law and from the result calculate the values of bound surface and volume current densities.

2+2+3

Sol:

(a)

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \quad \text{or} \quad \vec{A} = \frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}\,\hat{z}\times\hat{r}$$

$$\boxed{r^2 = 2} \longrightarrow 0.5 \text{ Mark}$$

$$= \frac{\mu_0}{4\pi}\frac{m_0}{r^2}\hat{\phi} \quad \because \theta = \frac{\pi}{2}$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\frac{m_0\,\hat{z}\times\hat{r}}{r^2} \longrightarrow 1 \text{ Mark}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{m_0}{2}\hat{\phi}}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{m_0}{2}\hat{\phi}} \longrightarrow 2 \text{ Marks}$$

(b)

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\Rightarrow \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -x^2 & 0 \end{vmatrix} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(-2x-2y)$$

$$= (-2x-2y)\hat{z} \longrightarrow 1 \text{ Mark}$$

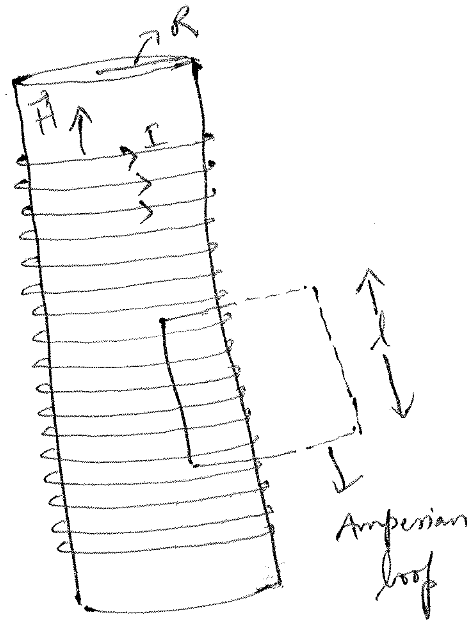
$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (-2x-2y) \end{vmatrix} = \hat{x}(-2) - \hat{y}(-2) + \hat{z}(0)$$

$$= -2\hat{x} + 2\hat{y}$$

$$\Rightarrow \boxed{\vec{J} = \frac{2}{\mu_0}(-\hat{x}+\hat{y})} \longrightarrow 2 \text{ Marks}$$

(c) $\oint \vec{H}\cdot d\vec{l} = I_{fenc}$

→ 0.5 Mark



$\Rightarrow Hl = nlI$

$\Rightarrow \boxed{\vec{H} = nI\hat{z}}$

→ 1 Mark

$\vec{M} = \chi_m \vec{H}$

$\boxed{\vec{M} = \chi_m n I \hat{z}}$

→ 2 Marks

$\vec{J}_b = \nabla \times \vec{M} = 0$ [$\because \vec{M}$ is constant]

→ 2.5 Marks

$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m n I \hat{z}\times\hat{s}$

$\hat{s}$ → unit vector along radial direction

or

$\hat{r}$

$= \chi_m n I \hat{\phi}$

→ 3 Marks.

Q4: A thick and long cylindrical wire of radius R is carrying a current with volume current density $\vec{J} = J_0(1 - \frac{s}{R})\hat{z}$, where s is the distance from the axis of the cylinder and J_0 is a constant.

(a) Find the total current inside the whole cylindrical wire.

(b) Calculate the magnetic field outside the cylinder at a distance s from the axis of the cylinder ($s > R$).

(c) Using the differential form of Ampere's law, find out the value of $\vec{\nabla} \times \vec{B}$ at a point on the axis of the wire.

3+2+2

Sol:

(a)

$$I = \int \vec{J} \cdot d\vec{a} \rightarrow 0.5 \text{ Mark}$$

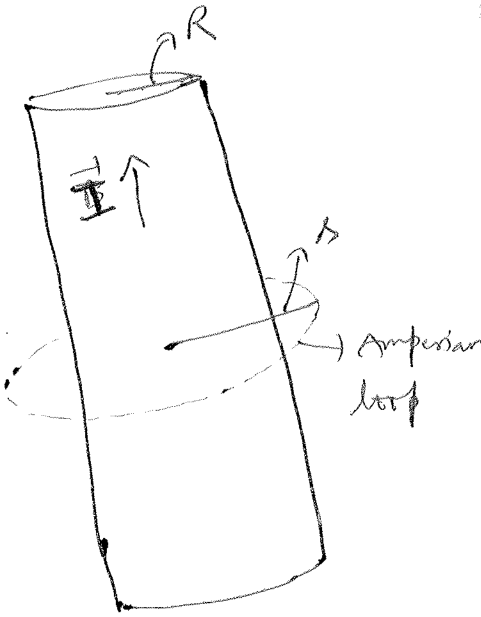
$$= J_0 \int \left(1 - \frac{s}{R}\right) s\, ds\, d\phi \quad \text{or} \quad J_0 \int \left(1 - \frac{s}{R}\right) 2\pi s\, ds \rightarrow 1 \text{ Mark}$$

$$= J_0 \int_0^R \left(s - \frac{s^2}{R}\right) ds \int_0^{2\pi} d\phi$$

$$= 2\pi J_0 \left(\frac{s^2}{2} - \frac{s^3}{3R}\right)\Big|_0^R$$

$$\boxed{I = 2\pi J_0 \left(\frac{R^2}{2} - \frac{R^2}{3}\right)} \rightarrow 3 \text{ Marks}$$

$$\Rightarrow \boxed{I = 2\pi J_0 \frac{R^2}{6} = \frac{\pi}{3} J_0 R^2}$$



(b)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \rightarrow 0.5 \text{ mark}$$

$$\Rightarrow B\, 2\pi s = \mu_0 \frac{\pi}{3} J_0 R^2 \rightarrow 1 \text{ Mark}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{6} J_0 \frac{R^2}{s} \hat{\phi}} \rightarrow 2 \text{ Marks}$$

(c)

$$\vec{J} = J_0 \left(1 - \frac{s}{R}\right) \hat{z}$$

on the axis of the wire $s = 0$ $\rightarrow$ 0.5 Marks

$$\Rightarrow \vec{J} = J_0 \hat{z}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \boxed{\mu_0 J_0 \hat{z}} \rightarrow 2 \text{ Marks.}$$

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Q5: Consider a thick spherical shell made of a linear dielectric material with inner radius R_1 and outer radius R_2. In the region $R_1 < r < R_2$, the displacement vector is given by $\vec{D} = \frac{2}{r^2}\hat{r}$.

(a) Find out the free charge density inside the shell.

(b) If the dielectric constant of the materials is 2, find the permittivity of the material and electric field $\vec{E}$ inside the shell?

(c) Consider a point charge Q placed at the center of a solid dielectric sphere of radius R and dielectric constant K. Obtain the bound surface charge density on the surface of the sphere.

2+2+3

Sol:

(a) $\vec{D} = \frac{2}{r^2}\hat{r}$

$\vec{\nabla}\cdot\vec{D} = \rho_f \rightarrow$ 1 Mark

$\Rightarrow \vec{\nabla}\cdot\left(2\frac{\hat{r}}{r^2}\right) = \rho_f$

$\Rightarrow \boxed{0 = \rho_f} \rightarrow$ 2 Marks

(b) $\epsilon = \epsilon_0\epsilon_r$ and dielectric constant $\epsilon_r = 2$

$\Rightarrow \epsilon = 2\epsilon_0 \rightarrow$ 1 Mark

$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{1}{2\epsilon_0}\frac{2}{r^2}\hat{r} = \frac{1}{\epsilon_0 r^2}\hat{r}$

$(R_1 < r < R_2) \rightarrow$ 2 Marks

$r < R_1$

$\vec{E} = 0 = \vec{D}$ (No free charge enclosed)

(* Both should be considered as correct answer.)

(c) $\oint \vec{D}\cdot d\vec{a} = Q_{fenc}$ $\rightarrow$ 0.5 Mark

$\Rightarrow D\,4\pi r^2 = Q$

$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$

and $\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r}$

$\rightarrow$ 1 Mark

or

$\vec{E} = \frac{1}{\epsilon_r}\vec{E}_{vac} = \frac{1}{K}\vec{E}_{vac}$ $\rightarrow$ 0.5 Mark

$\Rightarrow \vec{E} = \frac{1}{\epsilon_r}\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$

$\Rightarrow \vec{E} = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r}$ | Here $\epsilon = \epsilon_0\epsilon_r$, $\epsilon_r = K$

$\rightarrow$ 1 Mark

$\vec{P} = \epsilon_0\chi_e\vec{E} = \epsilon_0(K-1)\frac{Q}{4\pi\epsilon r^2}\hat{r} = \epsilon_0(K-1)\frac{Q}{4\pi\epsilon_0 K}\frac{\hat{r}}{r^2}$

$= \frac{1}{4\pi}\frac{(K-1)}{K}\frac{Q}{r^2}\hat{r} \rightarrow$ 2 marks

$\sigma_b = \vec{P}\cdot\hat{n} = \frac{1}{4\pi}\left(\frac{K-1}{K}\right)\frac{Q}{r^2}\hat{r}\cdot\hat{r}\Big|_{r=R} = \boxed{\frac{1}{4\pi}\left(\frac{K-1}{K}\right)\frac{Q}{R^2}}$

$\rightarrow$ 3 Marks