1. Write short answers:

   a. If the unit vectors in a spherical coordinate system are $\hat{r}, \hat{\theta}, \hat{\phi}$, then determine $\hat{\phi} \times \hat{r}$. **1**

   Ans: $\hat{\theta}$

   If the student writes $-\hat{\theta}$, then award 0.5

   b. If velocity of a particle is described as $\vec{v}(t) = e^t(t^2 - 2t)\hat{r}$, then determine the acceleration at $t = 1$. **2**

   Ans: $a(t) = \frac{dv}{dt} = [e^t(2t-2) + e^t(t^2 - 2t)]\hat{r} = e^t(t^2 + 2t - 2t - 2)\hat{r} = e^t(t^2 - 2)\hat{r}$.

   Till here 1. If vector sign is not there, then 0.5

   At $t = 1$, $a(t) = -e\,\hat{r} = -2.718\,\hat{r}$

   Till here 2 (any one answer can be treated as correct), if negative sign is missing then 1.5

   Note: In the question it is assumed that $\hat{r}$ is constant, but is the student considers $\hat{r}$ as function of time then,

   $a(t) = e^t(t^2 - 2)\hat{r} + e^t(t^2 - 2t)\frac{d\hat{r}}{dt} = e^t(t^2 - 2)\hat{r} + e^t(t^2 - 2t)\dot{\theta}\hat{\theta}$.

   If the students shows up to this, please award 2 marks.

   c. A particle of mass m is following a circular trajectory such that the angular velocity is $\omega\hat{j}$. The radial vector is $r\hat{\imath}$ then determine the centrifugal force. **2**

   Ans: Centrifugal force $= -m\vec{\Omega} \times \vec{\Omega} \times \vec{r}$ **Till here 0.5 (I mean writing the formula correctly, notation can be different)**

   $= -m\omega\hat{j} \times \omega\hat{j} \times r\hat{\imath}$, **Till here 1.0**

   $= m\omega\hat{j} \times \omega r\hat{k}$, **Till here 1.5**

   $= m\omega^2 r\hat{\imath}$. **Till here 2**

   If the answer is written with a -ve sign, then 1.5

   If magnitude is correct but vector is wrong, then 1. and vice versa

   If only answer written correctly then also 1

   d. If the angular velocity of a particle of mass m is defined as $\omega\hat{k}$ and linear velocity in the non-inertial frame is $\vec{v}_{rot} = v\hat{j}$, then determine the Coriolis force. **2**

   Ans: Coriolis force $= -2m\,\vec{\Omega} \times \vec{v}_{rot}$ **Till here 0.5 (I mean writing the formula correctly, notation can be different)**

   $= -2m\,\omega\hat{k} \times v\hat{j}$, **Till here 1**

   $= 2m\omega v\hat{\imath}$. **Till here 2**

   If the answer is written with a -ve sign, then 1.5

   If magnitude is correct but vector is wrong, then 1. and vice versa

   If only answer written correctly then also 1

e. A particle is rotating in the xy-plane, along a circular path in counter-clockwise direction, with angular speed ω, about z-axis. What is the position vector $\vec{r}(t)$? Given $\vec{r}(0) = a\hat{\imath}$. **2**

Ans: $r(t) = A(\cos\omega t\ \hat{\imath} + \sin\omega t\,\hat{\jmath})$, where A is any constant. **Till here 1**

Since, $r(0) = a\hat{\imath}$, hence, $A = a$ **Till here 1.5**

and $r(t) = a(\cos\omega t\,\hat{\imath} + \sin\omega t\,\hat{\jmath})$ **Till here 2**

f. Some general vector $\vec{A}$ precessing with constant angular velocity $\vec{\omega}$ about the axis in direction $\hat{n}$, then what will be $\frac{d\vec{A}}{dt}$? **2**

Ans: $\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$ **Award 2 marks**

If the answer is $\frac{d\vec{A}}{dt} = \vec{A} \times \vec{\omega}$ **then award 1 mark**

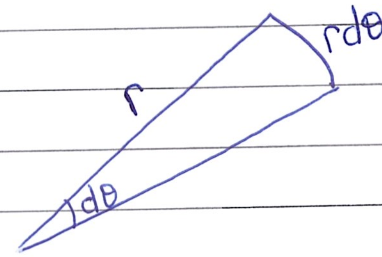
If the student considers $\vec{A} = a\cos\omega t\,\hat{\imath} + a\sin\omega t\,\hat{\jmath} \Rightarrow \frac{d\vec{A}}{dt} = -a\omega\sin\omega t\,\hat{\imath} + a\omega\cos\omega t\,\hat{\jmath}$, **then award 1mark**

g. The relation between inertial and a rotating frame is noted as $\left(\frac{d}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} + \vec{\Omega}\times$, where $\vec{\Omega}$ is the rotation velocity. Determine the inertial velocity ($\vec{v}_{in}$) in this case.**2**

Ans: $\vec{v}_{in} = \left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega}\times\vec{r} = \vec{v}_{rot} + \vec{\Omega}\times\vec{r}$. **Award 2 marks**

If the student writes $\vec{v}_{in} = \vec{\Omega}\times\vec{r}$, **Award 0.5 marks**

Q 2 (a) 4 Marks



As planet moves around the Sun, Let us consider the change in angular distance is $d\theta$

1 Mark

The area swept by the planet, $dA = \frac{1}{2} r \times r d\theta$

$$= \frac{1}{2} r^2 d\theta$$

1 Mark

→ Dividing both side by dt, we get

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} r^2 \dot{\theta} \qquad \because \dot{\theta} = \frac{d\theta}{dt}$$

1 Mark

$$\because \quad L = \mu r^2 \dot{\theta} = \text{constant}$$

$$\Rightarrow \quad \dot{\theta} = \frac{L}{\mu r^2} \quad (\text{constant})$$

1 Mark

So $\boxed{\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant}}$

Q 2(b) 5 Marks

Given that $\vec{F}(r) = -\frac{A}{r^3}\hat{r}$

1.5 Marks

Potential Energy (PE) can be calculated as

0.5 $$V(r) = -\int_{\infty}^{r} F(r)\,dr$$

1 $$= A\int_{\infty}^{r} \frac{dr}{r^3} = \left[-\frac{A}{2r^2}\right]_{\infty}^{r}$$

$$= -\frac{A}{2r^2}$$

Effective PE is given as

1 Mark $$V_{eff} = \frac{L^2}{2\mu r^2} + V(r)$$

0.5 Mark $$= \frac{L^2}{2\mu r^2} - \frac{A}{2r^2}$$

2 Marks

For circular orbit, the total Energy must be equal to minimum of Veff, which can be found as

$$\frac{dV_{eff}}{dr} = 0$$

$$\Rightarrow \frac{-L^2}{\mu r^3} + \frac{A}{r^3} = 0$$

$$\Rightarrow \boxed{L = \sqrt{\mu A}}$$

Q2 (c) **4 marks**

$f = xyz, \quad g = x+y+z$

$fg = (xyz)(x+y+z)$

$= x^2yz + xy^2z + xyz^2$ **0.5 mark**

0.5 mark $\nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2yz + xy^2z + xyz^2)$

$= \frac{\partial}{\partial x}(x^2yz + xy^2z + xyz^2)\hat{i}$

$+ \frac{\partial}{\partial y}(x^2yz + xy^2z + xyz^2)\hat{j}$

1.5 mark

$+ \frac{\partial}{\partial z}(x^2yz + xy^2z + xyz^2)\hat{k}$

$= (2xyz + y^2z + yz^2)\hat{i} + (x^2z + 2xyz + xz^2)\hat{j}$

$+ (x^2y + xy^2 + 2xyz)\hat{k} \rightarrow \vec{A}$ (say)

$\nabla \cdot \nabla(fg) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{A}$

1mark $= 2yz + 0 + 0 + 0 + 2xz + 0 + 0 + 0 + 2xy$

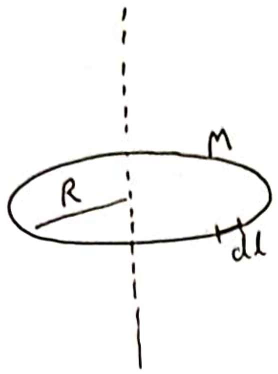
$= 2(yz + xz + xy)$

$\nabla \cdot \nabla(fg)$ at point $(2,0,1)$ $= 2[0\times1 + 2\times1 + 2\times0]$

0.5 mark $= 2\times2$

$= 4$

3.a) (i)



Moment of Inertia of a segment of length dl is

$$dI = R^2\, dm \longrightarrow 0.5$$

where dm is the mass of that ~~tg~~ line segment

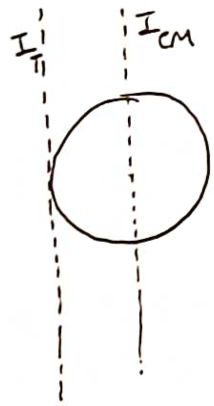
$$dm = \frac{M}{2\pi R}\, dl \longrightarrow 0.5$$

Moment of Inertia of the Ring

$$I = \int dI = \int R^2 \cdot \frac{M}{2\pi R}\, dl = \frac{MR^2}{(2\pi R)} \underbrace{\int dl}_{2\pi R} \longrightarrow 0.5$$

$$\therefore \boxed{I = MR^2} \longrightarrow 0.5$$

(ii)



Moment of Inertia through the center of mass using perpendicular axis theorem

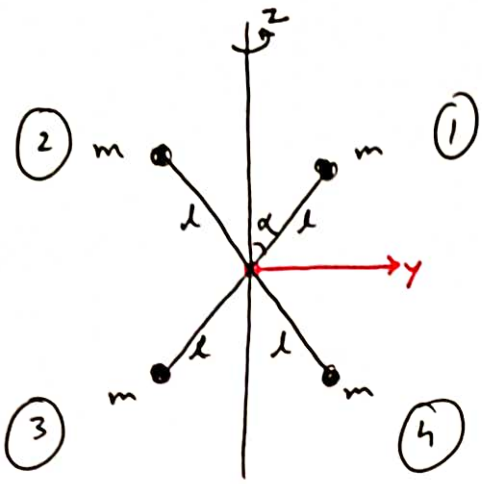
$$I_{CM} = \frac{1}{2} I = \frac{MR^2}{2} \quad — 1$$

Using parallel axis theorem, the moment of Inertia for rotation about the tangent line

$$I_T = MR^2 + I_{CM} = MR^2 + \frac{MR^2}{2} = \frac{3}{2} MR^2 \quad — 1$$

(iii) Moment of Inertia for rotation about the diameter is already calculated above as $I_{CM} = \frac{MR^2}{2}$ — 2

3. b)



Moment of Inertia tensor

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \rightarrow 0.5$$

Choose x-axis perpendicular to the plane of the masses

choose y-axis as shown. xyz-form a right handed system

The masses are now in the yz-plane

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) = 4ml^2 \longrightarrow 0.5$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2) = 4ml^2 \cos^2\alpha \longrightarrow 0.5$$

$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2) = 4ml^2 \sin^2\alpha \longrightarrow 0.5$$

$$\left.\begin{aligned} I_{xy} &= -\sum_i m_i x_i y_i = 0 \\ I_{yx} &= I_{xy} = 0 \end{aligned}\right\} 0.5$$

$$\left.\begin{aligned} I_{xz} &= -\sum_i m_i x_i z_i = 0 \\ I_{zx} &= I_{xz} = 0 \end{aligned}\right\} 0.5$$

$$0.5 \left\{\begin{aligned} I_{yz} &= -\sum_i m_i y_i z_i = -m\left(l^2 \sin\alpha\cos\alpha - l^2 \sin\alpha\cos\alpha + l^2 \sin\alpha\cos\alpha - l^2 \sin\alpha\cos\alpha\right) = 0 \\ I_{zy} &= I_{yz} = 0 \end{aligned}\right.$$

Moment of Inertia tensor

$$I = \begin{bmatrix} 4ml^2 & 0 & 0 \\ 0 & 4ml^2\cos^2\alpha & 0 \\ 0 & 0 & 4ml^2\sin^2\alpha \end{bmatrix} \rightarrow 0.5$$

Note: Student may choose axis differently. Please evaluate accordingly. Calculation of Matrix elements shall follow same marking Scheme.

3. c) (i) Equilibrium point corresponds to length of spring plus the extension of spring, since the mass is on an incline

Let the spring extension be x.

$$Kx = mg\sin\alpha \rightarrow 1$$

$$\Rightarrow x = \frac{mg\sin\alpha}{K} = \frac{14 \times \sin 40^\circ}{120} = 0.075\,m \rightarrow 0.5$$

The Block's equilibrium point is therefore $\underbrace{0.45 + 0.075}_{1} = 0.525\,m$

(ii) Period of oscillations

$$T = \frac{2\pi}{\omega} \quad \text{and} \quad \omega = \sqrt{\frac{K}{m}}$$

$$\therefore T = \underbrace{2\pi\sqrt{\frac{m}{K}}}_{1\ \cancel{0.5}} = \underbrace{2\pi\sqrt{\frac{14/9.8}{120}}}_{0.5} = 0.686\,s$$

Note: (i) → 2.5 marks
(ii) → 1.5 marks
} Total 4 marks question