EPHY105L - 6 18.10.21

<u>Fundamental Theorem</u> states that the integral of a derivative over a region is equal to the value of the function at the boundary.

⊛ Fundamental Theorem of gradient

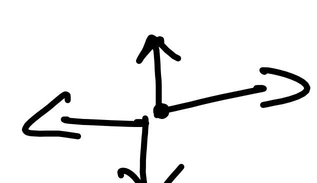
$$\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a)$$

$$\Leftrightarrow \oint (\vec{\nabla} T) \cdot d\vec{l} = 0$$

⊛ Fundamental Theorem of Divergence

Statement: $\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$

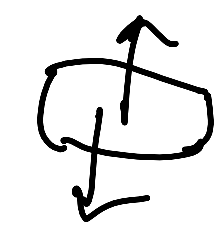
$\equiv$ Gauss' Theorem / Divergence theorem.

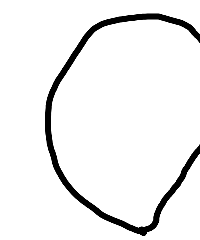


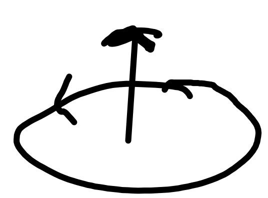
(x) Fundamental theorem for curls

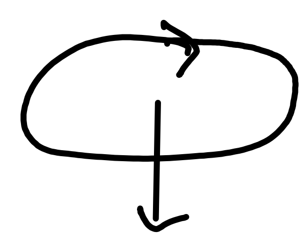
Statement : $$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

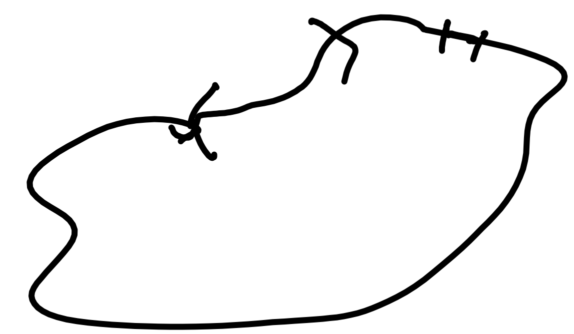
$\equiv$ Stoke's theorem.











$\rightarrow$ The consistency (of sign) of Stoke's theorem is given by the right-hand rule.

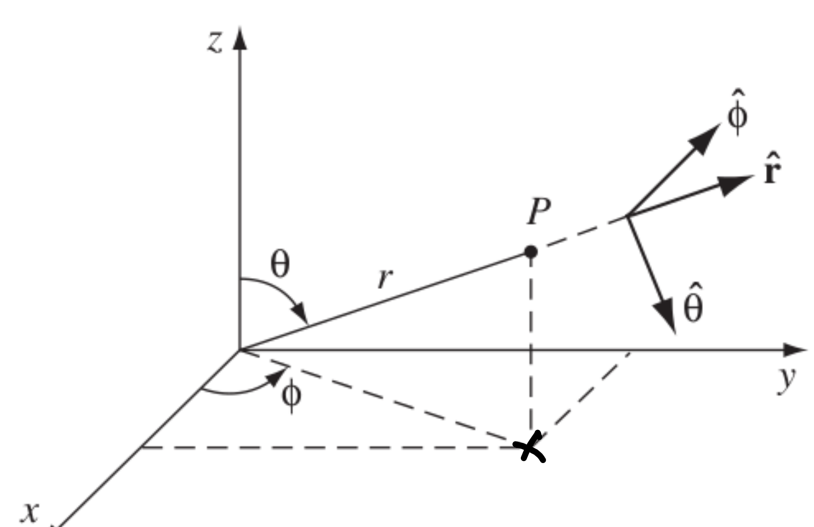
(x) Stoke's theorem states that $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ is equal to the line integral of $\vec{v}$ around the boundary.

<u>Corollary</u>: ⊛ $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ depends only on the boundary line, not on the particular surface chosen.

⊛ $\oint_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface since the boundary shrinks to a single point and line integral vanishes.

Curvilinear Coordinates

⊛ Spherical-Polar coordinates



Three coordinates ≡ (r, θ, ϕ)

$r \to$ Distance from origin

$\theta \rightarrow$ Polar angle $\equiv$ angle measured from z - axis.

$\phi \rightarrow$ Azimuthal angle $\equiv$ angle around from x-axis.

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

⊗ Unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ constitute an orthogonal basis just like $\hat{x}, \hat{y}, \hat{z}$

Any vector can be written as

$$\vec{A} = \underbrace{A_r}_{\text{radial}} \hat{r} + \underbrace{A_\theta}_{\text{Polar}} \hat{\theta} + \underbrace{A_\phi}_{\text{azimuthal}} \hat{\phi}$$

The unit vectors:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

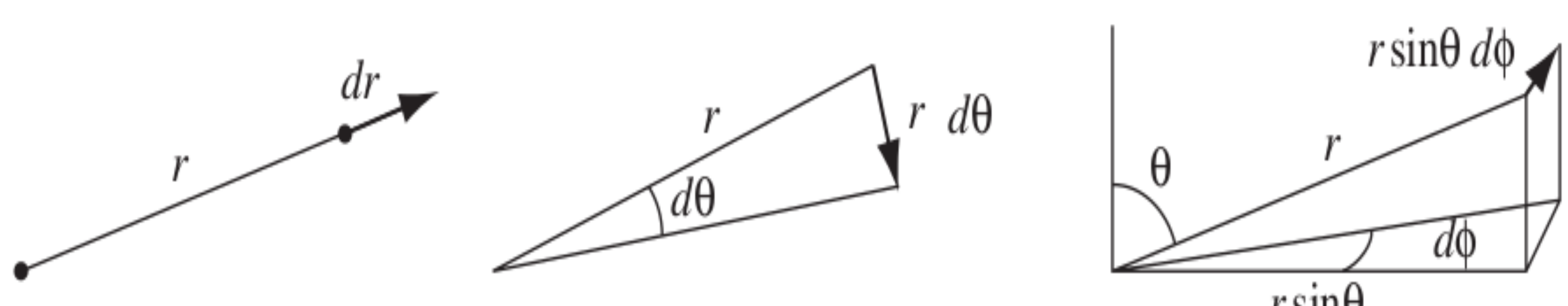
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

to get:

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}, \quad \hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}, \quad \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial\phi}}{\left|\frac{\partial\vec{r}}{\partial\phi}\right|}$$

⊗ Infinitesimal displacement along $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$



$$dl_r = dr$$

$$dl_\theta = r\, d\theta$$

$$dl_\phi = r \sin\theta\, d\phi$$

→ Infinitesimal displacement

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

→ Infinitesimal volume

$$d\tau = dl_r\, dl_\theta\, dl_\phi$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$