Solutions of Tutorial Sheet 9

Improper Integral

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1. (a) $\displaystyle\int_0^\infty e^{-x}\cos x dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x dx$

   Now take $I = \displaystyle\int_0^b e^{-x}\cos x dx = \frac{1}{2}(1-\cos b e^{-b} + \sin b e^{-b})$. So $\displaystyle\lim_{b\to\infty} I = \frac{1}{2}$.

   $\Rightarrow \displaystyle\int_0^\infty e^{-x}\cos x dx$ is convergent.

   (b) $\displaystyle\int_1^\infty \frac{dx}{x^2(1+e^x)} \le \int_1^\infty \frac{dx}{x^2}$ which is convergent .Hence by comparison test given improper integral is convergent.

   (c) $\displaystyle\int_1^\infty \frac{(x+1)dx}{x^{\frac{3}{2}}} = \int_1^\infty \frac{1}{\sqrt{x}}\,dx + \int_1^\infty x^{-3/2}dx$. The first integral on the right side diverges. Hence given integral diverges.

2. (a) Take $\ln x = t$ then $x = e^t$ and the integral becomes $\displaystyle\int_0^{\ln 2}\frac{e^{t/2}}{t}$. It is easy to see that integrand is $\ge \dfrac{1}{t}$ and the integral $\displaystyle\int_0^{\ln 2}\frac{1}{t}$ diverges.

   (b) $f(x) = \dfrac{\sin(\frac{1}{x})}{\sqrt{x}}$ and Take $g(x) = \dfrac{1}{\sqrt{x}}$. Then using comparison test, since $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent, we have $\displaystyle\int_0^1 \frac{\sin(\frac{1}{x})}{\sqrt{x}}dx$ is convergent.

   (c) Take $f(x) = \dfrac{\tan(x)}{x^{3/2}}$ and $g(x) = \tan x$. Then $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{f(x)}{g(x)} \in (0,\infty)$. Also as $\displaystyle\int_1^{\frac{\pi}{2}}\tan(x)dx$ is divergent so $\displaystyle\int_1^{\frac{\pi}{2}}\frac{\tan(x)}{x^{3/2}}dx$ is divergent.

3. (a) $\displaystyle\int_0^\infty x^{\frac{-1}{2}}e^{x^2}dx = \int_0^1\frac{e^{x^2}dx}{\sqrt{x}} + \int_1^\infty \frac{e^{x^2}dx}{\sqrt{x}}dx$. Now $\displaystyle\int_0^1\frac{e^{x^2}dx}{\sqrt{x}}$ is convergent, Since if we take $g(x) = \dfrac{1}{\sqrt{x}}$ then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)} = 1$ and $\displaystyle\int_0^1\frac{dx}{\sqrt{x}}$ is convergent But $\displaystyle\int_1^\infty\frac{e^{x^2}dx}{\sqrt{x}}$ is divergent. Since if we take $g(x) = \dfrac{1}{\sqrt{x}}$ then $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$ and $\displaystyle\int_1^\infty\frac{dx}{\sqrt{x}}$ is divergent. Hence the given integral is divergent.

(b) Note that

$$\int_{-\infty}^{\infty} \frac{x\ dx}{(x^2+1)} = \int_{-\infty}^{0} \frac{x\ dx}{(x^2+1)} + + \int_{0}^{\infty} \frac{x\ dx}{(x^2+1)}.$$

Now take $g(x) = \dfrac{1}{x}$ for both the integrals. Then use limit comparison test.

(c) Let $g(x) = \dfrac{1}{1+x^2}$ Then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$. Since $\int_0^{\infty} g(x)dx$ converges. So, by limit test $\int_0^{\infty} g(x)dx$ converges.

4. Clearly, for $p \leq 0$, $\int_0^1 \frac{\sin x}{x^p}dx$ exists as Riemann integrals. So, let $p > 0$. Then for $x > 0$, we have

$$\left|\frac{\sin x}{x^p}\right| = \left|\frac{\sin x}{x}\right| \frac{1}{x^{p-1}} \leq \frac{1}{x^{p-1}}.$$

From comparison test, it follows that $\int_0^1 \frac{\sin x}{x^p}dx$ converges for $p < 2$.

Now we will show that $\int_0^1 \frac{\sin x}{x^p}dx$ diverges for $p \geq 2$. Since $\frac{\sin x}{x}$ is decreasing in $(0,1]$, then for all $x \in (0,1]$, we have

$$\frac{\sin x}{x^p} = \frac{\sin x}{x}\frac{1}{x^{p-1}} \geq \frac{\sin 1}{x^{p-1}}.$$

Also, $\int_0^1 \frac{1}{x^p}dx$ diverges for $p-1 \geq 1$. Therefore, by Comparision test, it follows that $\int_0^1 \frac{\sin x}{x^p}$ diverges for $p \geq 2$.

5. Leibniz formula is to be used.

(a) Let $f(t) = \int_0^1 \frac{x^t - 1}{\ln x}dx$, then $\frac{df}{dt} = \int_0^1 x^t dx = \frac{1}{t+1}$.

$$\implies f(t) = \ln(t+1) + c.$$

Now $f(0) = 0 \Rightarrow c = 0$.

$$\Rightarrow \int_0^1 \frac{x^t - 1}{\ln x}dx = \ln(t+1).$$

(b) Let $f(t)=\int_0^\infty e^{-tx}\frac{\sin x}{x}dx$. Then $\frac{df}{dt}=-\int_0^\infty e^{-tx}\sin x dx=\frac{-1}{1+t^2}$.

$\Bigg($since

$$\begin{aligned}
I &= -\int_0^\infty e^{-tx}\sin x dx\\
&= -\left[-\sin x e^{-tx}\frac{1}{t}\Big|_0^\infty+\int_0^\infty e^{-tx}\frac{1}{t}\cos x dx\right]\\
&= -\frac{1}{t}\int_0^\infty e^{-tx}\cos x dx\\
&= -\frac{1}{t}\left[\cos x e^{-tx}\frac{1}{-t}\Big|_0^\infty+\int_0^\infty \frac{1}{t}e^{-tx}(-\sin x)dx\right]\\
&= -\frac{1}{t^2}-\frac{I}{t^2}.
\end{aligned}$$

$\therefore I=-\frac{1}{1+t^2}=f'(t).$ $\Bigg)$

$$\implies f(t)=-\tan^{-1}t+c.$$

By the second fundamental theorem, $f(a)-f(0)=\int_0^a f'(t)dt=\int_0^a\frac{-1}{1+t^2}dt$, taking $a\to\infty$, $\lim_{a\to\infty}f(a)-f(0)=-\pi/2$. Also,

$$0\le|f(a)|=\left|\int_0^\infty e^{-ax}\frac{sinx}{x}dx\right|\le C_1\int_0^\infty e^{-ax}dx \quad \text{as} \quad a\to\infty.$$

Therefore, $\lim\limits_{a\to\infty}f(a)=0$. Using this we get $c=\frac{\pi}{2}$ and hence $f(t)=\frac{\pi}{2}-\tan^{-1}t$.

6. (a) $I=\int_0^\infty e^{-x^2}dx$. Put $x^2=t\Rightarrow 2xdx=dt$.

$$\implies I=\int_0^\infty\frac{1}{2}e^{-t}t^{-\frac{1}{2}}dt$$

$$\therefore \quad I=\frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

(b) $I=\int_0^{\frac{\pi}{2}}\sqrt{\tan x}dx=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}x\cos^{-\frac{1}{2}}xdx=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$

(c) Let $I = \int_0^\infty x^{2/3} e^{-\sqrt{x}} dx$. Substitute $\sqrt{x} = t$.

$$\implies I = 2\int_0^\infty t^{\frac{7}{3}} e^{-t} dt.$$

Comparing it with the gamma function, $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$, we have $p = \frac{10}{3}$.

$$I = \Gamma\left(\frac{10}{3}\right).$$

7. Use $\Gamma(n+1) = n\Gamma(n)$, recursively.

(a) $\frac{3}{4}\sqrt{\pi}$

(b) $\frac{105}{16}\sqrt{\pi}$

(c) Use Euler's reflection formula:

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}.$$

Choose $p = -\frac{1}{2}$, so the answer is $-2\sqrt{\pi}$.