(1) Since A is skew-symmetric

$\therefore$ $A^T = -A$

Now $(A^2)^T = (AA)^T = A^T A^T = -A \cdot -A = A^2$

$\Rightarrow$ A^2 is symmetric.

(2) $W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 1 \}$

$\because$ W does not contain the null vector $(0,0,0)$

$\Rightarrow$ W is not a subspace of $\mathbb{R}^3$

(3) let $W = L\{(1,2), (3,4)\}$

Now dimension of $\mathbb{R}^2$ is 2.

and subset W contain 2 vectors.

Also $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 \neq 0 \Rightarrow (1,2)$ & $(3,4)$ L.I

$\Rightarrow$ $\{(1,2), (3,4)\}$ is a bases of $\mathbb{R}^2$

$\therefore$ $\mathbb{R}^2 = L\{(1,2), (3,4)\}$

(4) $\because$ A set of vectors containing the null vector θ in a vector space is linearly dependent

$\therefore$ Given set is L.D.

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(5) We know rank(A) + nullity(A) = no of columns

Now $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ rank (A) = 2

$\therefore$ rank(A) + null(A) = 4

$\Rightarrow$ null(A) = 4 - 2 = 2.

$\Rightarrow$ dimension of nullspace = 2.

(6) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(x_1, x_2) = (x_1 + x_2, x_2^2)$

let $\overset{\alpha}{(x_1, x_2)}$ & $\overset{\beta}{(y_1, y_2)} \in \mathbb{R}^2$ ~~$a, d \in \mathbb{R}$~~

Now $\cancel{a}\alpha + \cancel{a}\beta = \cancel{a}(x_1, x_2) + \cancel{d}(y_1, y_2)$

$= (\cancel{a}x_1 + \cancel{a}y_1, \cancel{a}x_2 + \cancel{a}y_2)$

$\Rightarrow$

Now $T(\alpha + \beta) = T(x_1 + y_1, x_2 + y_2)$

$= (x_1 + y_1 + x_2 + y_2, (x_2 + y_2)^2)$

$= (x_1 + x_2, x_2^2) + (y_1 + y_2, y_2^2)$

$+ (0, 2x_2y_2)$

$\neq T(\alpha) + T(\beta)$

$\Rightarrow$ T is not linear mapping.

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7) $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$

$$\text{Ker } T = \{(x_1, x_2) : T(x_1, x_2) = (0,0,0)\}$$

$$= \{(x_1, x_2) : (x_1, x_1 + x_2, x_2) = (0,0,0)\}$$

$$= \left\{(x_1, x_2) : \begin{array}{l} x_1 = 0 \\ x_1 + x_2 = 0 \Rightarrow x_2 = 0 \\ x_2 = 0 \end{array}\right\}$$

$$= \{(0,0)\}$$

We know the vector-space consisting only zero element, then dimension of that vector space is 0.

$$\therefore \dim(\text{Ker } T) = 0$$

(8) The characterstic equation of the given mat

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{-\lambda + \lambda^2\} - 1(-\lambda) = 0$$

$$\Rightarrow -\lambda(1-\lambda)^2 + \lambda = 0$$

$$\Rightarrow \lambda\{\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}\} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 2\lambda) = 0$$

$$\Rightarrow \lambda^2(\lambda - 2) = 0$$

$$\therefore \lambda = 0, \lambda = 2$$

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(9)

$$\begin{vmatrix} 2-\lambda & 2 & 0 & 0 \\ 2 & 1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 4-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)(3-\lambda)(4-\lambda) - 2\times 2(3-\lambda)(4-\lambda)$$

$$= (3-\lambda)(4-\lambda)\{2-\lambda-2\lambda+\lambda^2-4\}.$$

$$= (3-\lambda)(4-\lambda)(\lambda^2-3\lambda-2)$$

$\therefore$ all the eigen values are destinct

So $\exists$ unique eigen vector corresponding to every eigen value. Since, the eigen vector corresponding to destinct eigen values are linearly independent, So the no of L.I eigen vectors of the given matrix are 4.

(10) dimension of W = number of vareable − No of restriction

$= 5 - 1 = 4.$

<u>Alternative method:</u> $x_3 = x_2 - 3x_1$,

$\therefore W = \{(x_1, x_2, x_2-3x_1, x_4, x_5)\} = L\{(1,0,-3,0,0), (0,1,1,0,0), (0,0,0,1,0), (0,0,0,0,1)\}$

$\dim(W) = 4.$

(11) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 7 & 8 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 14 & 16 \end{vmatrix}$

$$\xrightarrow[\substack{R_3' \to R_3 - 3R_1 \\ R_4' \to R_4 - 4R_1}]{R_2' \to R_2 - 2R_1} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -4 \end{vmatrix}$$

$$\xrightarrow{R_4' \to R_4 - 2R_2} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$\therefore$ Rank of $A = 2$.

(12) $AX = b$

(Augmented matrix) $= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right)$$

Now, If $\lambda - 3 \neq 0 \Rightarrow$ Rank of the given matrix is 3

$\Rightarrow$ ~~the~~ system has unique value $\forall$ value of μ.

But If $\lambda - 3 = 0$, then Rank of $(A) = 2$.

Now Rank of (Augmented matrix) $= 2$ iff $\mu = 10$

$\therefore$ for $\lambda = 3$ & $\mu = 10$ system is consistent

But $\because$ Rank $(A) = 2 <$ no of variable $= 3$

$\Rightarrow$ It has infinite no of solution

$\therefore$ for $\lambda = 3$ & $\mu = 10$ system has infinite no of solution.

(13) $$\begin{vmatrix} a+d & a+d+k & a+d+e \\ c & c+b & e \\ d & d+k & d+c \end{vmatrix}$$

$$\xrightarrow{R_1' \to R_1 - R_3} \begin{vmatrix} a & a & a \\ c & c+b & c \\ d & d+k & d+e \end{vmatrix}$$

$$\xrightarrow{a} a \begin{vmatrix} 1 & 1 & 1 \\ c & c+b & c \\ d & d+k & d+e \end{vmatrix}$$

$$\underset{C_3' \to C_3 - C_1}{\xrightarrow{C_2' \to C_2 - C_1}} a \begin{vmatrix} 1 & 0 & 0 \\ c & b & 0 \\ d & k & e \end{vmatrix} = abe$$

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(14) $T(x_1, x_2, x_3) = (0,0,0)$

$$\Rightarrow \quad x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

$$x_1 - 2x_2 + 2x_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \cancel{x_3 = 0, \; 0 - 3x_2 + 2x_3 = 0 \Rightarrow x_2 = 0.}$$

$$\cancel{x_1 + x_2 - x_3} \qquad x_1 + x_2 - x_3 = 0$$

$$-3x_2 + 3x_3 = 0$$

$$\Rightarrow \quad x_2 = x_3$$

$$\therefore \quad x_1 + x_2 - x_2 = 0 \quad \Rightarrow \quad x_1 = 0$$

$$\therefore \quad \operatorname{Ker} T = \{(x_1, x_2, x_3)\} = \{(0, x_2, x_2)\}$$

$$= \{x_2 (0, 1, 1)\}$$

$$= L\{(0, 1, 1)\}$$

$$\Rightarrow \quad \dim \operatorname{Ker} T = 1.$$

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(15) $v_1 = (1,1,0)$

$$v_2 = (1,0,0) - \frac{\langle (1,0,0).(1,1,0)\rangle}{\langle (1,1,0).(1,1,0)\rangle}(1,1,0)$$

$$= (1,0,0) - \frac{1}{2}(1,1,0)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$$

$$= \frac{1}{2}(1,-1,0)$$

$$v_3 = (1,1,1) - \frac{\langle (1,1,1).(1,1,0)\rangle}{\langle (1,1,0).(1,1,0)\rangle}(1,1,0)$$

$$- \frac{\langle (1,1,1), \frac{1}{2}(1,-1,0)\rangle}{\langle \frac{1}{2}(1,-1,0).\frac{1}{2}(1,-1,0)\rangle}\frac{1}{2}(1,-1,0)$$

$$= (1,1,1) - \frac{\cancel{2}}{\cancel{2}}(1,1,0) - \cancel{\frac{0}{\frac{1}{4}\times 2}}^{=0}\frac{1}{2}(1,-1,0)$$

$$= (0,0,1)$$

$\therefore \{v_1, v_2, v_3\}$ is orthogonal.

$\Rightarrow \left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}$ is orthonormal.

Now $\|v_1\| = \sqrt{2}$

$\|v_2\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$.

$\|v_3\| = \sqrt{1} = 1$

$\therefore$ Orthonormal basis $= \left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{2}}(1,-1,0), (0,0,1)\right\}$

(16) We know that

~~dimension of (A) =~~

Rank of (A) + nullity of (A) = no of column

$\Rightarrow$ Rank of $A = 3 - 1 = 2$.

$\therefore$ Rank of $A = 2$

$\Rightarrow |A| = 0$.

$$\Rightarrow \begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$\Rightarrow k(-4+2) - 1(4+2) + 2(1+1) = 0$

$\Rightarrow -2k - 6 + 4 = 0$

$\Rightarrow -2k = 2 \quad \Rightarrow \quad k = -1$.

(17) $T(1,0,0,0) = (0,1,0,0)$

$= 0 \times (1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$

$T(0,1,0,0) = 0(1,0,0,0) + 0(0,1,0,0) + 1(0,0,1,0) + 0(0,0,0,1)$

$T(0,0,1,0) = 0(1,0,0,0) + 0(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$

$T(0,0,0,1) = 0(1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 0 \cdot (0,0,0,1)$

$\therefore$ for a given T the matrix is.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of this matrix $= 2$.

$\therefore$ Rank of $(T) = 2$.

(13) $x_1 + x_2 - x_3 = 0$

$x_2 + x_3 + x_4 = 0$.

$2x_1 + x_2 - 3x_3 - x_4 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_3' \to R_3 - 2R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\xrightarrow{R_3' \to R_3 + R_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = x_3 - x_2 = x_3 + x_3 + x_4$

$x_2 + x_3 + x_4 = 0 \Rightarrow x_2 = -x_3 - x_4 \quad = 2x_3 + x_4$.

$\therefore V = \{(x_1, x_2, x_3, x_4)\}$

$= \{(2x_3 + x_4, -x_3 - x_4, x_3, x_4)\}$

$= \{x_3(2, -1, 1, 0) + x_4(1, -1, 0, 1)\}$

$= L\{\underbrace{(2, -1, 1, 0), (1, -1, 0, 1)}_{L.I}\}$

$\therefore \{(2,-1,1,0), (1,-1,0,1)\}$ is a basis of V.

$\therefore \dim(V) = 2$

$\because$ they are L.I.

(19) $T(0,1,0) = (1,1,1)$

$= 1(0,1,0) + 1(0,0,1) + 1(1,0,0)$

$T(0,0,1) = (-1, 1, -1)$

$= 1(0,1,0) - 1(0,0,1) - 1(1,0,0)$

$T(1,0,0) = (1, 1, 0)$

$= 1(0,1,0) + 0(0,0,1) + 1(1,0,0)$

$\therefore$ matrix of $T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

(20) A is a 3×3 matrix whose all eigen valu are destinct $\Rightarrow$ A is diagonalizable.

$\Rightarrow \exists$ a non singular matrix P s.t

$P^{-1}AP =$ diagonal matrix.

$= \text{diag}(-1, 0, 1)$.

let $P = \begin{pmatrix} +1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$ $\therefore P^{-1} = \dfrac{(\text{adj } P)}{|P|}$

$= \frac{1}{6}\begin{pmatrix} 3 & +1 & 2 \\ -3 & 1 & 2 \\ 0 & -2 & 2 \end{pmatrix}^t$

$= \frac{1}{6}\begin{pmatrix} 3 & -3 & 0 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{pmatrix}$

Now, $\quad P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad A = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -3 & 0 \\ +1 & 1 & -2 \\ 2 & +2 & 2 \end{pmatrix}$$

$$\Rightarrow \quad 6A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -3 & 0 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\Rightarrow \quad 6A = \begin{pmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$