

Tutorial Sheet 10

Limit, Continuity and Differentiability of a Function of Several Variables

1. Examine the limits of the below mentioned functions as $(x, y) \rightarrow (0, 0)$.

$$(a) f_1(x, y) = \begin{cases} \frac{x^2+y^2}{x^2-y^2}, & x \neq y \\ 0, & x = y \end{cases}$$

$$(b) f_2(x, y) = xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right)$$

2. Find the repeated limits of the following functions.

$$(a) f(x, y) = \frac{3x - 4y}{5x + 8y}$$

$$(b) g(x, y) = \left(\frac{2+x}{3+y} \right) \left(\frac{y-x}{y+x} \right)$$

3. Examine the continuity of the following functions at $(0, 0)$.

$$(a) f_1(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^6}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise.} \end{cases}$$

$$(b) f_2(x, y) = \begin{cases} \frac{\sin^2(x-y)}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise.} \end{cases}$$

4. Find the first order partial derivatives of the following functions at the given points

$$(a) f(x, y) = (x^2 + xy)^3, \text{ at } (1, 0) \quad (b) g(x, y) = \left(\frac{x^2 + y^2}{xy} \right), \text{ at } (\sqrt{2}, \sqrt{2}).$$

5. Discuss the differentiability of the following functions at $(0, 0)$.

$$(a) f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ 0, & xy = 0 \end{cases} \quad (b) g(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$$

6. Let $f(x, y) = \left| |x| - |y| \right| - |x| - |y|$. Is f continuous at $(0, 0)$? Which directional derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Give reasons.

7. If $z = x^5 e^{9y}$, then find the value of dz .
8. Find the total differential of $z = x^3 y + xy$ at the point $(1, 2)$.
9. If $z = x^3 y + xy + 4$ and $x = \cos t$, $y = \sin 2t$, then compute $\frac{dz}{dt}$ and evaluate it at $t = \frac{\pi}{4}$.
10. Find the direction where the directional derivative is greatest for the function $f(x, y) = 3x^2 y^2 - x^4 - y^4$ at the point $(1, 2)$.
11. Let $\sin(xyz) = x + 3z + y$. Then find $\frac{\partial z}{\partial x}$.
12. Let $f(x, y) = \sqrt{x} \sin y$. Find the approximate value of $f(4.1, 0.2)$.
13. Given that $f(2, -3) = 6$, $f_x(2, -3) = 1.3$ and $f_y(2, -3) = -0.6$. Approximate the value of $f(2.1, -3.03)$.
14. An ant travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$, it is known that :

$$\frac{\partial z}{\partial x} = -7, \quad \frac{\partial z}{\partial y} = 2, \quad \frac{dx}{dt} = -1 \quad \text{and} \quad \frac{dy}{dt} = 3.$$

Find $\frac{dz}{dt}$ at time t_0 .

15. Let $\sin(xy) + y^2 + x = 5$. Find $\frac{dy}{dx}$.