Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 1

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1. Let, if possible , there exist a rational number $\frac{p}{q}$, where $q \neq 0$ and p, q are integers prime to each other (i.e. having no common factor) whose square is equal o 2,

$$\left(\frac{p}{q}\right)^2 = 2 \implies p^2 = 2q^2 \tag{1}$$

Now q is an integer and so is $2q^2$.
Thus, p^2 is an integer divisible by 2. As such p must be divisible by 2, for otherwise p^2 would not be divisible by 2.
Let, $p = 2m$, where m is an integer. Then fom (1),

$$2m^2 = q^2. \tag{2}$$

Thus, it follows that q is also divisible by 2.
Hence, p and q are both divisible by 2 which contradicts the hypothesis that p and q have no common facor.
Thus, there is no rational number whose square is 2.

2. Let, if possible, $\sqrt{8}$ be a rational number $\frac{p}{q}$, where $q \neq 0$ and p, q are integers prime to each other, So that

$$\sqrt{8} = \frac{p}{q}.$$

But $2 < \sqrt{8} < 3$. Therefore,

$$2 < \frac{p}{q} < 3 \implies 2q < p < 3q \implies 0 < p - 2q < q$$

Thus, $p - 2q$ is a positive integer less than q, so that $\sqrt{8}(p - 2q)$ is not an integer.
But,

$$\sqrt{8}(p-2q) = \frac{p}{q}(p-2q) = \frac{p^2}{q} - 2p$$

$$\implies \frac{p^2}{q2}q - 2p = 8q - 2p, \text{ which is an integer}$$

$$\implies \sqrt{8}(p-2q), \text{is an integer}$$

Thus, we arrive at a contradiction.
Hence, $\sqrt{8}$ is not a rational number.

3.
$$\mid \frac{x+3}{2x-6} \mid \leq 1 \implies -1 \leq \frac{x+3}{2x-6} \leq 1$$
Case 1: if $2x-6>0$ i.e.$x>3$, then
$$-1 \leq \frac{x+3}{2x-6} \leq 1 \iff -2x+6 \leq x+3 \leq 2x-6 \tag{3}$$
From (3), we get, $x \geq 1$ and $x \geq 9$ and $x>3$ simultaneously.
Therefore for case 1, solution set $S_1 = \{x \in \mathbf{R} : x \geq 9\}$
Case 2: if $2x-6<0$ i.e.$x<3$, then
$$-1 \leq \frac{x+3}{2x-6} \leq 1 \iff 2x-6 \leq x+3 \leq -2x+6 \tag{4}$$
From (3), we get, $x \leq 1$ and $x \leq 9$ and $x<3$ simultaneously.
Therefore for case 2, solution set $S_2 = \{x \in \mathbf{R} : x \leq 1\}$
So, the solution set is
$$\{x \in \mathbf{R} : x \geq 9\} \cup \{x \in \mathbf{R} : x \leq 1\}$$

4. (a)
$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$
Therefore, $sup\ S = 1$ and $inf\ S = 0$.
Here, $sup\ S = 1 \in S$ and $inf\ S = 0 \notin S$.
(b)
$$S = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...\right\}$$
Therefore, $sup\ S = 0$ and $inf\ S = -1$.
Here, $sup\ S = 0 \notin S$ and $inf\ S = -1 \in S$.
(c)
$$S = \left\{\frac{(-1)^n}{n} : n \in \mathbb{N}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, ...\right\}$$
$$\implies S = \left\{-1, -\frac{1}{3}, -\frac{1}{5}, ...\right\} \cup \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...\right\}$$
Therefore, $sup\ S = \frac{1}{2}$ and $inf\ S = -1$.
Here, $sup\ S = \frac{1}{2} \in S$ and $inf\ S = -1 \in S$.

(d)

$$S = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\} = \left\{0, 1 + \frac{1}{2}, 1 - \frac{1}{3}, 1 + \frac{1}{4}, 1 - \frac{1}{5}, ...\right\}$$

$$\implies S = \left\{0, 1 - \frac{1}{3}, 1 - \frac{1}{5}...\right\} \cup \left\{1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, ...\right\}$$

Therefore, $sup\ S = \frac{3}{2}$ and $inf\ S = 0$.
Here, $sup\ S = \frac{3}{2} \in S$ and $inf\ S = 0 \in S$.

(e)

$$S = \left\{x \in \mathbb{R} : x^2 < 1\right\} = \left\{x \in \mathbb{R} : -1 < x < 1\right\}$$

Therefore, $sup\ S = 1$ and $inf\ S = -1$.
Here, $sup\ S = 1 \notin S$ and $inf\ S = -1 \notin S$.

(f)

$$S = \left\{x \in \mathbb{R} : x^2 - 6x + 3 < 0\right\} = \left\{x \in \mathbb{R} : (x-3)^2 - 6 < 0\right\}$$

$$\implies S = \left\{x \in \mathbb{R} : (x-3)^2 < 6\right\} = \left\{x \in \mathbb{R} : -\sqrt{6} < (x-3) < \sqrt{6}\right\}$$

$$\implies S = \left\{x \in \mathbb{R} : -\sqrt{6} + 3 < x < \sqrt{6} + 3\right\}$$

Therefore, $sup\ S = \sqrt{6} + 3$ and $inf\ S = -\sqrt{6} + 3$.
Here, $sup\ S = \sqrt{6} + 3 \notin S$ and $inf\ S = -\sqrt{6} + 3 \notin S$.

(g)

$$S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\} = \left\{0 < \frac{1}{m} + \frac{1}{n} \leq 1 + 1 : m, n \in \mathbb{N}\right\}$$

$$\implies S = \left\{0 < \frac{1}{m} + \frac{1}{n} \leq 2 : m, n \in \mathbb{N}\right\}$$

Therefore, $sup\ S = 2$ and $inf\ S = 0$.
Here, $sup\ S = 2 \in S$ and $inf\ S = 0 \notin S$.

(h)

$$S = \left\{(-1)^m + \frac{1}{n} : m, n \in \mathbb{N}\right\} = \left\{-1 + \frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$$

$$\implies S = \left\{-1 < (-1)^m + \frac{1}{n} \leq 2 : m, n \in \mathbb{N}\right\}$$

Therefore, $sup\ S = 2$ and $inf\ S = -1$.
Here, $sup\ S = 2 \in S$ and $inf\ S = -1 \notin S$.

5. $x, y \in S \implies m \leq x \leq M, \qquad m \leq y \leq M.$
Therefore,
$$m - M \leq x - y \leq M - m \implies |x - y| \leq M - m.$$
This shows that the set T is bounded above, $M - m$ being an upper bound.

6. (a) Example of Bounded Set:
Let $S = \Big\{2, 5, 8, 9\Big\}$, then $2 \leq S \leq 9$,
$\implies S$ is bounded.
(b) Example of Not bounded Set:
The sets $\mathbb{I}, \mathbb{Q}, \mathbb{R}$ are not bounded.
(c) Example of Bounded below but not bounded above set:
The sets $\mathbb{N}$ is bounded below but not bounded above.
(d) Example of Bounded above but not bounded below set:
The interval $[-\infty, 1]$ is bounded above but not bounded below.