Name of Student: ..........................  Enrollment No. ..................
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# Mid Term Examination, Even Semester 2021-22

## BENNETT UNIVERSITY, GREATER NOIDA

POSSESSION OF MOBILES IN EXAM IS UFM PRACTICE

COURSE CODE : EMAT102L  MAX. DURATION: 1 Hour
COURSE NAME: Linear Algebra and Ordinary Differential Equations
COURSE CREDIT: 3-1-0  MAX. MARKS: 30

**Instructions:**

- All questions are mandatory and write your answer within the given space.
- It contains 10 pages.
- Calculators are not allowed.

1. Attempt any FIVE parts. Justify your answer. [$2 \times 5 = 10$]

(a) $W = \text{Span}(\{(1,2,-3),(-1,-2,3)\})$. Then $W$ describes line/subspace/1 dimensional subsp

(b) Check whether $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_1^2 + x_1^2 \leq 3\}$ is subspace of $\mathbb{R}^4$ or not.

$\Rightarrow x_1^2 \leq 1$

$x_1^2+x_2^2+x_3^2 \leq 3$

**Solution:**

Counter example: $(1,1,1,0) \in W$

but $(2,2,2,0) \notin W$.

NOT a subspace of $\mathbb{R}^4$.

(c) Investigate for what values of $\lambda$ and $\mu$ the following linear equations have an infinit number of solutions. $x + y + z = 6$, $x + 2y + 3\cancel{z} = 10$, $x + 2y + \lambda z = \mu$.

**Solution:**

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right)$$

$\Rightarrow \lambda - 3 = 0$ and $\mu - 10 = 0$

$\Rightarrow \boxed{\lambda = 3}$ and $\boxed{\mu = 10}$

* If someone considers the $2^{nd}$ $eq^n$ as $x+5y=10$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 5 & 0 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - \frac{R_2}{4}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & \lambda-\frac{3}{4} & \mu-7 \end{array}\right) \Rightarrow \boxed{\lambda = \frac{3}{4}} \boxed{\mu = 7}$$

(d) Let $\mathbb{R}^3$ be a vector space with respect to the usual addition and scalar multiplication operations. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 2x_2 \text{ or } x_3 = 3x_2\}$. Then check whether $S$ forms a subspace of $\mathbb{R}^3$ or not.

**Solution:** NOT a subspace of $\mathbb{R}^3$.

Counter example: $(2,1,0), (0,1,3) \in S$

$(2,1,0) + (0,1,3) = (2,2,3) \notin S$.

(e) Let $A$ be $3 \times 3$ matrix with real entries such that $\det(A)$ is 6 and trace of $A$ is 0. If $\det(A + 2I) = 0$, then find all the eigenvalues of $A$.

**Solution:**

$$\begin{cases} \det(A) = \lambda_1 \lambda_2 \lambda_3 = 6 \\ \text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 0 \end{cases}$$

$\det(A+2I) = 0$

$\Rightarrow \boxed{\lambda_1 = -2}$

$\Rightarrow \lambda_2 + \lambda_3 = 2$, $\lambda_2 \lambda_3 = -3$

$\Rightarrow \lambda_2 = -\frac{3}{\lambda_3}$

$\Rightarrow \frac{-3}{\lambda_3} + \lambda_3 = 2$

$\Rightarrow \lambda_3^2 - 2\lambda_3 - 3 = 0$

$\Rightarrow \lambda_3 = -1$ or $3$

$\therefore$ all the eigenvalues of $A$ are $\boxed{-1, -2, 3}$.

(f) Find the basis of the vector space of all $2 \times 2$ symmetric matrices over $\mathbb{R}$.

**Solution:**

$$\left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

2. Is it possible to find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ whose null space is $\{(x, y, z) \in \mathbb{R}^3 ; x + y = 0\}$? If yes, then find a linear transformation. [2 marks]

$T(x,y,z) = x+y$

Yes

Rank $(T) = 1$

Nullity $(T) = 2$.

3. Consider the vector space $V = \mathcal{P}_5[x]$, the set of all polynomials in variable $x$ and of degrees less than or equals to 5. Check if the set

$$S = \{-2 + 3x - 7x^2, -7 + 3x, 1 + x + x^3 + x^5, 3 - x^4\}$$

is linearly independent or dependent. Justify your answer. [3 marks]

**Solution:**

$\alpha_1(-2+3x-7x^2) + \alpha_2(-7+3x) + \alpha_3(1+x+x^3+x^5) + \alpha_4(3-x^4) = 0$.

Comparing coefficients of $x^k$ on both the sides, we have

$-2\alpha_1 - 7\alpha_2 + \alpha_3 + 3\alpha_4 = 0 \Rightarrow -7\alpha_2 = 0$

$3\alpha_1 + 3\alpha_2 + \alpha_3 = 0 \Rightarrow \boxed{\alpha_2 = 0}$

$-7\alpha_1 = 0 \Rightarrow \boxed{\alpha_1 = 0}$

$\boxed{\alpha_3 = 0}$ $\Rightarrow \boxed{\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0}$

$-\alpha_4 = 0 \Rightarrow \boxed{\alpha_4 = 0}$

linearly independent

[3 marks]

4. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**Solution:**

$|A - \lambda I| = 0$.

$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \boxed{\lambda = \pm i}$

to find corresponding eigenvectors:

$Ax = ix$.

$\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow x_2 = ix_1 \quad \Rightarrow$ eigenvector of $A$ corresponding to eigenvalue $\lambda = i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Similarly, $Ax = -ix$.

Solving we get eigenvector corresponding to $\lambda = -i$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation given by $T(x, y, z) = (2x + y + z, y - z, 3y)$. Find the matrix of $T$ corresponding to the ordered bases $\{(0,0,1), (1,0,0), (0,1,0)\}$ and $\{(1,1,0), (1,0,1), (1,0,0)\}$, respectively. [4 marks]

**OR**

Find all the possible values of $a$ and $b \in \mathbb{R}$ such that the following matrix $A$ can be reduces to the row reduced echelon form $A = \begin{pmatrix} 1 & a & 0 & 1 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Also write down the row reduced echelon form and the rank of $A$.

**Solution:**

$T(0,0,1) = (1,-1,0)$
$\quad = -(1,1,0) + 2(1,0,1) + 0\cdot(1,0,0)$

$T(1,0,0) = (2,0,0) = 0\cdot(1,1,0) + 0\cdot(1,0,1) + 2\cdot(1,0,0)$

$T(0,1,0) = (1,1,3) = 1\cdot(1,1,0) - 2(1,0,1) + 3\cdot(1,0,0)$

$\therefore$ the required matrix is $\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 2 & 3 \end{bmatrix}$.

OR

$a \in \mathbb{R}$, $b \overset{\mathbb{R}-\{0\}}{=} 1 \Rightarrow \text{Rank}(A) = 3$. $\begin{pmatrix} 1 & a & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$a \in \mathbb{R}$, $b = 0 \Rightarrow \text{Rank}(A) = 2$, $\begin{pmatrix} 1 & a & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathcal{M}_2(\mathbb{R})$, where $\mathcal{M}_2(\mathbb{R})$ is the vector space of all $2 \times 2$ matrices. Suppose [4 marks]

$$T(1,0,-1) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \quad T(0,2,0) = \begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}, \quad T(0,0,1) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

Then find $T(x,y,z)$.

**Solution:**

$$(x,y,z) = \alpha(1,0,-1) + \beta(0,2,0) + \gamma(0,0,1)$$

$$\boxed{\alpha = x} \qquad 2\beta = y \Rightarrow \boxed{\beta = y/2}$$

$$-\alpha + \gamma = z \Rightarrow \boxed{\gamma = x+z}$$

$$\therefore T(x,y,z) = x\,T(1,0,-1) + \frac{y}{2}\,T(0,2,0) + (x+z)\,T(0,0,1)$$

$$= x\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} + \frac{y}{2}\begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix} + (x+z)\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\boxed{T(x,y,z) = \begin{pmatrix} x + \frac{3}{2}y & 3x+2y \\ 2x + \frac{y}{2} + z & 4x - \frac{y}{2} + 2z \end{pmatrix}}$$

7. Solve the following system of linear equations using Gauss Jordan method, [4 marks]

$$x_1 + x_3 = 3, \quad 2x_1 + x_2 = 0, \quad x_2 + 2x_3 = 4.$$

Besides, mention the ranks of the coefficient matrix and the augmented matrix.

OR

A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 - x_2 + x_3, x_1 - 2x_2 + 2x_3).$$

Find Ker($T$), Range($T$) and their dimensions.

**Solution:**

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -6 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 4 & 10 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3/4} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 5/2 \end{array}\right) \xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5/2 \end{array}\right).$$

$$\therefore \boxed{x_1 = 1/2, \quad x_2 = -1, \quad x_3 = 5/2}$$

OR

$T(x_1, x_2, x_3) = (0,0,0)$

$$\Rightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow \begin{cases} x_1 + x_2 - x_3 = 0 \\ -3x_2 + 3x_3 = 0 \end{cases}$$

$$\Rightarrow \boxed{x_2 = x_3} \text{ and } \boxed{x_1 = 0}$$

$\therefore$ Null space$(T)$ = Ker$(T)$ = Span$\{(0,1,1)\}$.

To find range space of T :

$T(1,0,0) = (1,2,1), \quad T(0,1,0) = (1,-1,-2), \quad T(0,0,1) = (-1,1,2)$

$\therefore \text{Range}(T) = \text{Span}\{(1,2,1), (1,-1,-2), (-1,1,2)\}$

$= \underline{\text{Span}\{(1,2,1), (1,-1,-2)\}}$

$(\because \{(1,-1,-2), (-1,1,2)\}$ is LD$)$.

$\therefore \boxed{\text{Rank}(T) = 2}$ and $\boxed{\text{Nullity}(T) = 1}$