Lecture - 25

- The divergence of a magnetic field is zero.

$$\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r}) = 0$$

- The curl of a magnetic field: The Ampere's Law

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \quad \longleftrightarrow \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathbf{enc}}$$

- Magnetic Vector Potential

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

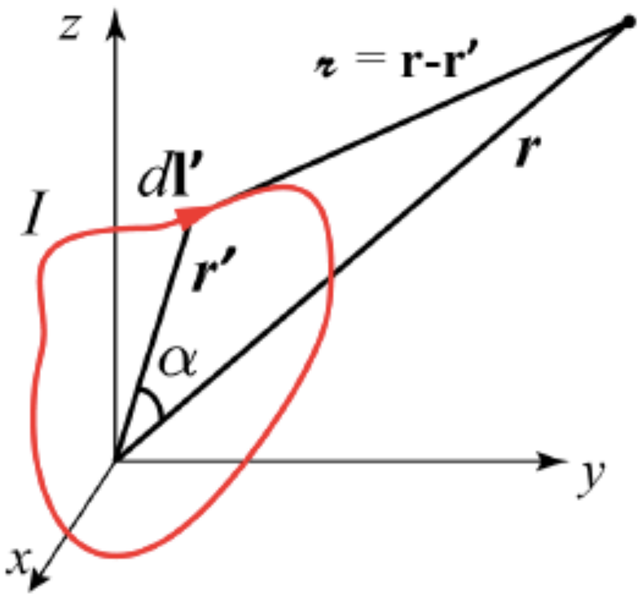
Recall: $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau'$

So, $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\, da'$

For line current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'}{\mathscr{r}}$



Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

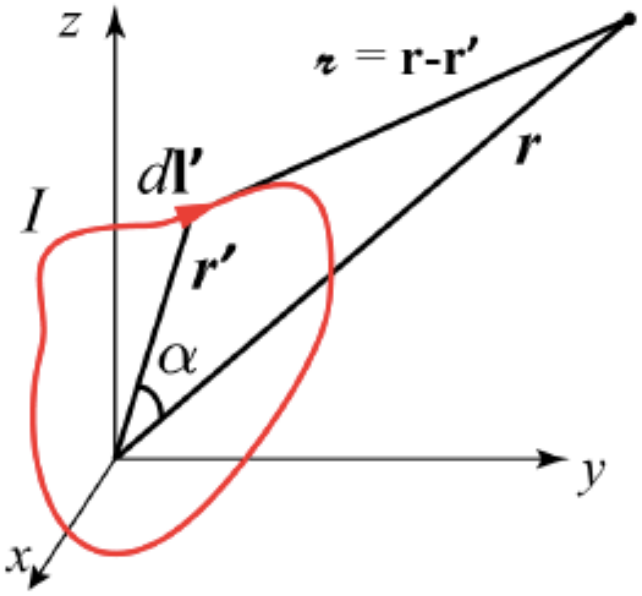
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint\frac{d\mathbf{l}'}{\mathscr{r}}$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n\, P_n(\cos\alpha)d\mathbf{l}'$$

$$= \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}'}_{\substack{\text{Monopole potential} \\ (\,1/r\text{ dependence})}} + \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}'}_{\substack{\text{Dipole potential} \\ (\,1/r^2\text{ dependence})}} + \underbrace{\frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint(r')^2\left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right)d\mathbf{l}'}_{\substack{\text{Quadrupole potential} \\ (\,1/r^3\text{ dependence})}} + \cdots$$

Monopole potential

$$\mathbf{A}_{\text{mono}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$



Dipole potential

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \;= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

$$= -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \nabla'(\hat{\mathbf{r}}\cdot\mathbf{r}')\times d\mathbf{a}'$$

[Corollary of Stokes Theorem: $\oint_{path} T\,d\mathbf{l} = -\int_{Surf} \nabla T \times d\mathbf{a}$]

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \hat{\mathbf{r}}\times d\mathbf{a}' \;= \frac{\mu_0}{4\pi}\frac{1}{r^2}\left(I\int d\mathbf{a}'\right)\times\hat{\mathbf{r}}$$

$$= \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{m} \equiv I\int d\mathbf{a}'}$$

Magnetic dipole moment

Magnetic field due to a magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Take $\mathbf{m} = m\,\hat{\mathbf{z}}$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A}_{\text{dip}}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Recall

$\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

Magnetic Vector Potential:

Prob. 5.23 (Griffiths, 3rd Ed.): What is the current density **J** that would produce the magnetic potential $\mathbf{A} = k\hat{\boldsymbol{\phi}}$

$$\mathbf{A} = k\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}}$$

$$\mathbf{J} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \frac{k}{\mu_0 s^2}\hat{\boldsymbol{\phi}}$$

What is Polarization? - dipole moment per unit volume

- The dipole moment is caused either by stretch of an atom/molecule or by rotation of polar molecules
- Polarization in the direction parallel to the applied electric field

What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion:
  (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).