**Solutions of Tutorial Sheet 10**

**Limit, Continuity and Differentiability of a Function of Several Variables**

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1. (a) Consider the path $y = mx$. Then $f_1(x, mx) = \frac{1+m^2}{1-m^2}$. So limit depends on $m$. Hence limit does not exist at $(0, 0)$.

   (b) Observe that

$$|f_2(x, y) - 0| = \left| xy \left( \frac{x^2 - y^2}{x^2 + y^2} \right) \right| \leq |xy| = |x| \cdot |y|$$
$$\leq \sqrt{x^2 + y^2} \cdot \sqrt{x^2 + y^2} < \delta^2 = \epsilon.$$

   Therefore, by choosing $\delta = \sqrt{\epsilon}$, then we have

$$\sqrt{x^2 + y^2} < \delta \implies |f_2(x, y) - 0| < \epsilon.$$

   So limit of the function exists at $(0, 0)$ and the value of the limit is zero.

2. (a) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x, y) = \dfrac{3}{5}$ and $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y) = -\dfrac{1}{2}$.

   (b) $\lim\limits_{x\to 0} \lim\limits_{y\to 0} g(x, y) = -\dfrac{2}{3}$ and $\lim\limits_{y\to 0} \lim\limits_{x\to 0} g(x, y) = \dfrac{2}{3}$.

3. (a) Take $x = my^3$ and then $f(x, y) = \frac{m}{1+m^2}$ which show that the function is not continuous at $(0, 0)$.

   (b) Let $\epsilon > 0$ be given. Now

$$\left| \frac{\sin^2(x - y)}{|x| + |y|} \right| \leq \frac{|x - y|^2}{|x| + |y|} \leq \frac{(|x| + |y|)^2}{|x| + |y|} = (|x| + |y|) \leq 2(x^2 + y^2)^{\frac{1}{2}} < \epsilon.$$

   If we take $\delta = \frac{\epsilon}{2}$. Then for every $\epsilon > 0$, there exist $\delta > 0$, such that

$$\sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0, 0)| < \epsilon.$$

4. Here $f(x, y) = (x^2 + xy)^3$.

   Hence $\left.\dfrac{\partial f}{\partial x}\right|_{(1,0)} = \lim\limits_{h\to 0} \dfrac{f(1 + h, 0) - f(1, 0)}{h} = \lim\limits_{h\to 0} \dfrac{(1 + h)^6 - 1}{h} = 6.$

   And $\left.\dfrac{\partial f}{\partial y}\right|_{(1,0)} = \lim\limits_{k\to 0} \dfrac{f(1, k) - f(1, 0)}{k} = \lim\limits_{h\to 0} \dfrac{(1 + k)^3 - 1}{k} = 3.$

5. (a) Both the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are 0 at point $(0,0)$. For differentiability $\triangle f = f(h,k) - f(0,0) = f(h,k)$, $df = hf_x(0,0) + kf_y(0,0) = 0$.
Consider $\displaystyle\lim_{\rho\to 0} \frac{\triangle f - df}{\rho} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h\sin\frac{1}{h} + k\sin\frac{1}{k}}{\sqrt{h^2+k^2}}$ fails to exist along $k = h$. Hence not differentiable.

(b) Both the partial derivatives are 0 at point $(0,0)$. Consider $\displaystyle\lim_{\rho\to 0}\frac{\triangle g - dg}{\rho} = \lim_{(x,y)\to(0,0)} \frac{g(x,y)}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2} = \frac{1}{2}$ by taking limit along the line $y = x$. Hence $g$ is not differentiable at $(0,0)$.

6. Given $\epsilon > 0$ we have to find a $\delta > 0$ such that for

$$0 < \sqrt{x^2+y^2} < \delta \implies |\ f(x,y) - 0\ | < \epsilon.$$

Consider

$$|\ f(x,y) - 0\ | \le ||\ x\ | - |\ y\ || + |\ x\ | + |\ y\ || \le 2(|\ x\ | + |\ y\ |) \le 4\sqrt{x^2+y^2}.$$

So take $\delta = \frac{\epsilon}{8}$ we have $|\ f(x,y) - 0\ | < \epsilon$. Hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$ and the function is continuous at (0,0).
Also both the the partial derivatives are 0 at point(0,0).
For differentiability, consider $\displaystyle\lim_{\rho\to 0}\frac{\triangle f - df}{\rho} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = -1$ along y=x.
Hence f is not differentiable at (0,0).
So, we can't apply the formula $D_u f = f_x(0,0)u_1 + f_y(0,0)u_2$. Direction derivative of the function exists only in the direction of $(1,0)$ and $(0,1)$ and this can be checked from the definition as

$$D_{\widehat{p}} f(0,0) = \lim_{t\to 0} \frac{f(t\rho_1, t\rho_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{|t|}{t}\left(||\rho_1| - |\rho_2|| - |\rho_1| - |\rho_2|\right)$$

exists only if $\rho = (1,0)$ or $(0,1)$.

7. $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$. Here $z_x = 5x^4e^{9y}$ and $z_y = 9x^5e^{9y}$. $\therefore dz = 5x^4e^{9y}dx + 9x^5e^{9y}dy$.

8. Here $z_x(1,2) = (3x^2y + y)\Big|_{(1,2)} = 8$ and $z_y(1,2) = (x^3 + x)\Big|_{(1,2)} = 2$. So $dz = 8dx + 2dy$.

9. By chain rule
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$
Here $z_x = 3x^2y + y$ and $z_y = x^3 + x$. Similarly, $x_t = \dfrac{\partial x}{\partial t} = -\sin t$ and $y_t = \dfrac{\partial y}{\partial t} = 2\cos 2t$. Now at $t = \frac{\pi}{4}$, we have $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $y = 1$. So the final answer is $\frac{-5}{2\sqrt{2}}$.

10. $f_x(1,2) = 20$, $f_y(1,2) = -20$. Directional derivative is greatest when pointing in the direction of the gradient $(20,-20)$. Hence, the direction is $\frac{1}{\sqrt{2}}\,\widehat{\imath} - \frac{1}{\sqrt{2}}\,\widehat{\jmath}$

11. Differentiating partially w.r.t. $x$ (and treating $z$ as a function of $x$; and $y$ as a constant), we get
$$\cos(xyz)\left(yz + xy\frac{\partial z}{\partial x}\right) = 1 + 3\frac{\partial z}{\partial x}.$$
Simplifying, we get
$$\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}.$$

12. We can approximate $f(4.1, 0.2)$ using $f(4,0) = 0$. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer. We let $\Delta z = f(4.1, 0.2) - f(4,0)$. The total differential $dz$ is approximately equal to $\Delta z$, so
$$f(4.1, 0.2) - f(4,0) \approx dz \implies f(4.1, 0.2) \approx dz + f(4,0).$$
To find $dz$, we need $f_x$ and $f_y$. $f_x(x,y) = \frac{\sin y}{2\sqrt{x}} \implies f_x(4,0) = 0$, and
$f_y(x,y) = \sqrt{x}\cos y \implies f_y(4,0) = 2$.
Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.2 with 0 gives $dy = 0.2$. Thus
$$dz(4,0) = f_x(4,0)(0.1) + f_y(4,0)(0.2) = 0(0.1) + 2(0.2) = 0.4.$$
$$\therefore, f(4.1, 0.2) \approx 0.4 + 0 = .4.$$

13. The total differential approximates how much $f$ changes from the point $(2,-3)$ to the point $(2.1, -3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have
$$dz = f_x(2,-3)dx + f_y(2,-3)dy = 1.3(0.1) + (-0.6)(-0.03) = 0.148.$$

The change in $z$ is approximately 0.148, so we approximate $f(2.1, -3.03) \approx 6.148$.

14. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt} = (-7)(-1) + 2(3) = 13.$

15. Let $f(x, y) = \sin(xy) + y^2 + x - 5$. Then $f_x(x, y) = y\cos(xy) + 1$ and $f_y(x, y) = x\cos(xy) + 2y$. Then

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y\cos(xy) + 1}{x\cos(xy) + 2y}.$$