End-Semester Examination 2022-23

Even Semester

EMAT 102L

Linear Algebra and ODE

Evaluation Scheme

Pg 1

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1) ch. eqn. $\lambda^2 - 2\sqrt{3}\lambda + 4 = 0$. [1 mark

$$\lambda = \frac{2\sqrt{3} \pm \sqrt{12-16}}{2}$$

$\Rightarrow \boxed{\lambda = \sqrt{3}+i}$ or $\boxed{\lambda = \sqrt{3}-i}$ [1 mark

2) $\lambda_1 + \lambda_2 = 10$, $\lambda_1\lambda_2 = 0$.

$\Rightarrow \boxed{\lambda_1 = 10}$, $\boxed{\lambda_2 = 0}$ (1 mark) ch. eqn. $\lambda^2 - 10\lambda = 0$.

$\Rightarrow A^2 - 10A = 0$.

eigenvalues of $A^2 - 10A + 2I$ are 2, 2.

$\therefore$ Trace of $A^2 - 10A + 2I$ is $\boxed{4}$. (1 mark)

3) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{x_1 + x_2 + x_3 = 0}$ (1 mark)

$\therefore$ eigenspace corresponding to $\lambda = 0$ is

$\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$. (1 mark)

4) $\dfrac{\langle v,u\rangle}{\langle u,u\rangle}\cdot u = \dfrac{\langle (1,2,3),(1,-1,1)\rangle}{\langle (1,-1,1),(1,-1,1)\rangle}(1,-1,1)$

formula (1)

$= \boxed{\frac{2}{3}(1,-1,1)}$ 2 marks

5) $\langle \sin x, \cos x\rangle = \int_0^{\pi/2} \sin x \cdot \cos x \, dx$

$= \frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx$

$= -\frac{1}{4} \cos 2x \Big]_0^{\pi/2}$

$= -\frac{1}{4}(-1-1) = \boxed{\frac{1}{2}}$ 2 marks

6) $2xy\,dx + x^2 dy = 0.$

$\frac{\partial M}{\partial y} = 2x, = \frac{\partial N}{\partial x} = 2x$ 1 mark

exact DE.

G. solⁿ. $\boxed{x^2 y = C}$ 1 mark

Alternatively:

$\frac{dy}{dx} = -\frac{2xy}{x^2} = \frac{-2y}{x}$

$\Rightarrow \frac{dy}{y} = -2\frac{dx}{x}$ 1 mark

$\Rightarrow \ln y = -2\ln x + \ln C$

$\Rightarrow \boxed{x^2 y = C}$ 1 mark

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7) $\frac{dy}{dx}+y=e^{-x}$. Linear 1^{st} order DE.

$I.F. = e^{\int dx} = e^{x}$ (1 mark

G. sol^n. $y \cdot e^{x} = \int e^{x} \cdot e^{-x}\,dx$

$\Rightarrow \boxed{y \cdot e^{x} = x + C}$ (1 mark

8) $M = \cos x \cdot \cos y$ $N = \alpha \sin x \cdot \sin y$

$\frac{\partial M}{\partial y} = -\sin y \cdot \cos x$. $\frac{\partial N}{\partial x} = \alpha \cos x \cdot \sin y$.

equating, $\boxed{\alpha = -1}$ (1 mark

General sol^n: $\boxed{\sin x \cdot \cos y = C}$ (1 mark

9) $(2x+1)\,dx + (3y^2+2)\,dy = 0$.

$\Rightarrow \boxed{x^2 + x + y^3 + 2y = C}$ 1 mark

$y(0) = 1 \Rightarrow 1+2 = C \Rightarrow \boxed{C = 3}$.

$\therefore$ sol^n: $\boxed{x^2 + x + y^3 + 2y = 3}$ (1 mark

10) aux. eq^n. $m^3 - m^2 + m - 1 = 0$.

$\Rightarrow (m^2+1)(m-1) = 0$

$\Rightarrow \boxed{m = i}$ or $\boxed{m = -i}$ or $\boxed{m = 1}$ 1 mark

G. sol^n. $\boxed{y = c_1 e^{x} + c_2 \cos x + c_3 \sin x}$ (1 mark

11) $\mathcal{L}(t^2+2t+3) = \mathcal{L}(t^2)+2\mathcal{L}(t)+3\,\mathcal{L}(1)$

$= \boxed{\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}}$ 2 mark

12) $(A-4I)x = 0 . \quad (x \neq 0)$

$\Rightarrow \begin{pmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

$\Rightarrow -3x_1+2x_2+3x_3 = 0$

$5x_3 = 0$

$2x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

$\Rightarrow \boxed{3x_1 = 2x_2}$ 1 mark

$\therefore$ An eigenvector corresponding to $\lambda = 4$ is

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}.$ 1 mark

13) aux. eqⁿ.

$$2m^3+m^2+2m+1=0.$$

$$\Rightarrow (m^2+1)(2m+1)=0.$$

$$\Rightarrow \boxed{m=-\frac{1}{2}} \text{ or } \boxed{m=i} \text{ or } \boxed{m=-i}$$ 1 mark

General solⁿ:

$$\boxed{y=c_1e^{-\frac{1}{2}\cdot x}+c_2\cos x+c_3\sin x.}$$ 1 mark

$y(0)=1$, $y'(0)=1$.

$$\Rightarrow \boxed{1=c_1+c_2} \checkmark$$

$$y'=-\frac{1}{2}c_1e^{-\frac{1}{2}x}-c_2\sin x+c_3\cos x.$$

$$\Rightarrow \boxed{1=-\frac{1}{2}c_1+c_3} \checkmark$$

$$y''=+\frac{1}{4}c_1e^{-\frac{1}{2}x}-c_2\cos x-c_3\sin x.$$

1 marks

$y''(0)=1$

$$\Rightarrow \boxed{1=\frac{1}{4}c_1-c_2} \checkmark$$

$$\boxed{c_1=\frac{8}{5}} \quad \boxed{c_2=\frac{-3}{5}} \quad \boxed{c_3=9/5}$$ 1 mark

$\therefore$ The solⁿ is $$\boxed{y=\frac{8}{5}e^{-\frac{1}{2}x}-\frac{3}{5}\cos x+\frac{9}{5}\sin x}$$ 1 mark

14) $\left(4x^{\alpha+1}y^{\beta+2} + 6x^{\alpha}\cdot y^{\beta+2}\right)dx + \left(5x^{\alpha+2}y^{\beta+1} + 8x^{\alpha+1}y^{\beta}\right)dy = 0.$

$$\frac{\partial M}{\partial y} = 4(\beta+2)x^{\alpha+1}y^{\beta+1} + 6(\beta+1)x^{\alpha}y^{\beta}$$ 1 mark

$$\frac{\partial N}{\partial x} = 5(\alpha+2)x^{\alpha+1}y^{\beta+1} + 8(\alpha+1)x^{\alpha}y^{\beta}$$ 1 mark

Comparing coefficients and considering the fact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for an exact DE, we have

$$4(\beta+2) = 5(\alpha+2) \text{ and } 6(\beta+1) = 8(\alpha+1)$$

$\Rightarrow \boxed{5\alpha - 4\beta + 2 = 0}$ and $\boxed{8\alpha - 6\beta + 2 = 0}$ 1 mark

$\Rightarrow \boxed{\alpha = 2}$ and $\boxed{\beta = 3}$ 1 mark

$\therefore$ The exact DE is

$$\boxed{\left(4x^3y^5 + 6x^2y^4\right)dx + \left(5x^4y^4 + 8x^3y^3\right)dy = 0}$$

General sol^n:

$$\boxed{x^4y^5 + 2x^3y^4 = c}$$ 1 mark

15) $v_1 = (1,1,1)$, $v_2 = (1,2,1)$, $v_3 = (2,-1,-1)$

$u_1 = v_1 \Rightarrow \boxed{u_1 = (1,1,1)}$ $\quad e_1 = \dfrac{u_1}{\|u_1\|}$

$\Rightarrow \boxed{e_1 = \frac{1}{\sqrt{3}}(1,1,1)}$ 1 mark

$u_2 = v_2 - \text{proj}_{u_1}(v_2) = v_2 - \dfrac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle}\cdot u_1$

$= (1,2,1) - \dfrac{\langle (1,2,1),(1,1,1)\rangle}{\langle (1,1,1),(1,1,1)\rangle}(1,1,1)$

$= (1,2,1) - \frac{4}{3}(1,1,1)$

$\boxed{u_2 = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)}$ 1 mark $\quad e_2 = \dfrac{u_2}{\|u_2\|}$

$\|u_2\| = \frac{1}{3}(\sqrt{1+4+1}) = \frac{\sqrt{2}}{\sqrt{3}}$ $\qquad e_2 = \frac{1}{\sqrt{2}}\left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

$\boxed{e_2 = \frac{1}{\sqrt{6}}(-1,2,-1)}$ 1 mark

$u_3 = v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3)$

$= v_3 - \dfrac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}\cdot u_1 - \dfrac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}\cdot u_2$.

Since $\langle v_3, u_1\rangle = 0$,

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 .$$

$$= (2,-1,-1) - \frac{\langle (2,-1,-1), \frac{1}{3}(-1,2,-1)\rangle}{\langle \frac{1}{3}(-1,2,-1), \frac{1}{3}(-1,2,-1)\rangle} \left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$= (2,-1,-1) + \frac{1}{2/3}\left(\frac{1}{3}(-1,2,-1)\right)$$

$$= (2,-1,-1) + \frac{1}{2}(-1,2,-1) .$$

$$= \left(\frac{3}{2}, 0, -\frac{3}{2}\right) \Rightarrow \boxed{u_3 = \frac{3}{2}(1,0,-1)}$$ 1 mark

$$e_3 = \frac{u_3}{\|u_3\|} \qquad \|u_3\| = \frac{3}{\sqrt{2}} \qquad \Rightarrow \boxed{e_3 = \frac{1}{\sqrt{2}}(1,0,-1)}$$ 1 mark

∴ The required orthonormal basis is

$$\left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(-1,2,-1), \frac{1}{\sqrt{2}}(1,0,-1) \right\} .$$

16) $|A-\lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (\lambda+2)(\lambda+2)(4-\lambda) = 0.$$

$$\Rightarrow \boxed{\lambda=-2} \text{ or } \boxed{\lambda=-2} \text{ or } \boxed{\lambda=4}$$ 1 mark

Eigenvector corresponding to $\lambda=-2$: $(A+2I)x = 0$. $(x \neq 0)$.

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{x_1 - x_2 + x_3 = 0}$$ ½ mark

$$E_{\lambda=-2}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ 1 mark

Eigenvector corresponding to $\lambda=4$: $(A-6I)x = 0$. $(x \neq 0)$

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left. \begin{array}{l} -x_1 - x_2 + x_3 = 0 \\ x_1 - 3x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \end{array} \right\} \Rightarrow \boxed{x_1 = x_2 = \frac{x_3}{2}}$$ $\frac{1}{2}$ mark

$$E_{\lambda=4}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$ 1 mark

$$\therefore \boxed{P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}}.$$

1 mark

$$P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

$\therefore$ A is diagonalizable.