Solution to Tutorial Set-6

1. Which of the following functions cannot represent a magnetic field?
   a) $\vec{F}_1 = x^2\hat{\imath} + 3xz^2\hat{\jmath} - 2xz\hat{k}$
   b) $\vec{F}_1 = xy\hat{\imath} + yz\hat{\jmath} + 2xz\hat{k}$
   c) $\vec{F}_3 = \frac{\alpha}{(x^2+y^2)}(-y\hat{\imath} + x\hat{\jmath})$

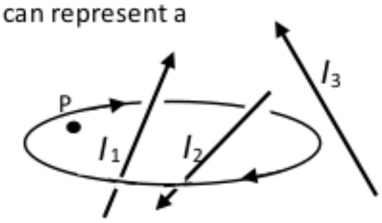
   Solution

   The necessary condition for a function to represent a magnetic field is that the divergence of the function goes to zero.

   a) $\vec{\nabla}\cdot\vec{F_1} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}2xz = 2x - 2x = 0 \Rightarrow \vec{F}_1$ can be a magnetic field.

   b) $\vec{\nabla}\cdot\vec{F_2} = \frac{\partial}{\partial x}xy + \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}2xz = y + z - 2x \neq 0 \Rightarrow \vec{F_2}$ cannot represent a magnetic field.

   c) $\vec{\nabla}\cdot\vec{F_3} = \frac{\partial}{\partial x}\left(\frac{-\alpha y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{\alpha x}{x^2+y^2}\right) = \frac{\alpha y\times 2x}{(x^2+y^2)^2} - \frac{\alpha x\times 2y}{(x^2+y^2)^2} = 0 \Rightarrow \vec{F_3}$ can represent a magnetic field.



2. Three wires are carrying currents I_1, I_2 and I_3 as shown in the figure.
   a) Write down the value of $\oint \vec{B}.d\vec{l}$ over the curved path shown.
   b) Draw paths over which we will get (i) zero value and (ii) maximum positive value of $\oint \vec{B}.d\vec{l}$.
   c) What will be the values of $\nabla.\vec{B}$ and $\nabla\times\vec{B}$ at the point P?
   d) State whether the following statement is true or false: "The magnetic field along the curved path shown in the figure depends only on currents I_1 and I_2."

   Solution

   a) Applying the idea of Amperean loop $\oint \vec{B}.d\vec{l} = \mu_0(I_2 - I_1)$. *Please note the direction of integration along the loop vis a vis the directions of the currents.*
   b) The integral value will be zero if it does not enclose any of the current component and it will be maximum when it will enclose I_1 and I_3 excluding I_2 (assuming $I_1 + I_3 > I_2$) .
   c) $\nabla.\vec{B} = 0$ and $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} = 0$ since $\vec{J} = 0$ at the point P.
   d) False, the magnetic field at point depends on all the currents present.

3. A long cylindrical wire of radius R carries a current I with a volume current density of $\vec{J} = \alpha r^2\hat{z}$ where r is the distance from the axis of the cylinder and $\hat{z}$ is the unit vector along the axis of the cylinder.
   a) Obtain the magnetic field in all regions.
   b) Obtain $\nabla.\vec{B}$ and $\nabla\times\vec{B}$ in all regions.

   Solution

   a) The magnetic field outside the wire can be evaluated by assuming an Amperean loop of radius r, where r>R, i.e $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B(2\pi r) = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$.

Since the current density is $\vec{J} = \alpha r^2 \hat{z}$, hence inside the wire if we consider an Amperean loop of radius r then the total current enclosed by the loop is $I_{encl} = \int_0^r \vec{J} \cdot d\vec{a} = \int_0^r \alpha r^2 \times 2\pi r dr = \frac{2\pi\alpha r^4}{4} = \frac{\pi\alpha r^4}{2}$. So, the magnetic field any point inside the wire will be $B(2\pi r) = \mu_0 \frac{\pi\alpha r^4}{2} \Rightarrow \vec{B} = \frac{\mu_0 \alpha r^3}{4}\hat{\phi}$.

b) $\vec{\nabla} \cdot \vec{B} = \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} = 0$ everywhere, since magnetic field does not have a ϕ dependency. This is consistent with the condition $\nabla . \vec{\boldsymbol{B}} = 0$ satisfied by the magnetic field. Now $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} = \mu_0 \alpha r^2 \hat{z}$ for r<R and zero for r>R.

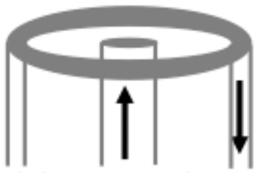
4. Consider a straight cylindrical region of thickness $(b-a)$ and having a circular cross section between inner radius a and outer radius b. A current I flows uniformly through the cross section of the cylinder.
   a) Calculate the magnetic field in all regions.
   b) Obtain $\nabla . \vec{\boldsymbol{B}}$ and $\nabla \times \vec{\boldsymbol{B}}$ in all regions.

Solution

a) Because of symmetry $\vec{B}$ will be azimuthal and depend only on r. Thus using $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$, we find that for r<a $\vec{B} = 0$.
In the region, a<r<b, the current density is $\frac{I}{\pi(b^2-a^2)}$. Thus, amount of current enclosed by an Amperean loop of radius r is $\frac{I\pi(r^2-a^2)}{\pi(b^2-a^2)}$. Hence the magnetic field $B = \frac{1}{2\pi r} \times \left(\frac{\mu_0 I\pi(r^2-a^2)}{\pi(b^2-a^2)}\right) = \mu_0 I \frac{r^2-a^2}{2\pi r(b^2-a^2)}$.
In the region r>b, the current enclosed is I hence $B = \frac{\mu_0 I}{2\pi r}$.

b) $\vec{\nabla} \cdot \vec{B}$ will be zero for all regions. In the region r<a, $\vec{\nabla} \times \vec{B} = 0$.
In the region a<r<b, $\vec{\nabla} \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{\mu_0 I}{\pi(b^2-a^2)}\hat{z} = \mu_0 \vec{J}$.
In the region r>b $\vec{\nabla} \times \vec{B} = 0$.

5. Consider a coaxial configuration as shown in the figure. The inner solid cylinder carries a current in the upward direction while the outer annular cylinder (tube) carries the same current in the downward direction. Calculate the magnetic field in all regions. The radius of the inner cylinder is a and the inner and outer radii of the outer annular cylinder are b and c respectively. Calculate $\nabla . \vec{\boldsymbol{B}}$ and $\nabla \times \vec{\boldsymbol{B}}$ in all regions.



Solution

Like the previous problems, we need to consider the Amperean loop appropriately to solve the problem. One can divide the total problem in four parts,

i. r<a

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \frac{\mu_0 I}{\pi a^2}\pi r^2 \Rightarrow B = \frac{\mu_0 I r}{2\pi a^2}$$

ii. a<r<b

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

iii. b<r<c

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \Rightarrow B(2\pi r) = \mu_0 I - \frac{\mu_0 I\pi(r^2-b^2)}{\pi(c^2-b^2)} \Rightarrow B = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I(r^2-b^2)}{2\pi r(c^2-b^2)}$$

iv. r>c

Total current in this case is zero as equal amounts of current are flowing in opposite directions. Thus B=0.

$\vec{\nabla} \cdot \vec{B} = 0$ in all regions.

However, $\vec{\nabla} \times \vec{B}$ depends on the current density at the point of evaluation.

In region (i), $\vec{\nabla} \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \times \frac{\mu_0 I r}{2\pi a^2}\right)\right)\hat{z} = \frac{\mu_0 I}{\pi a^2}\hat{z} = \mu_0 \vec{J}$.

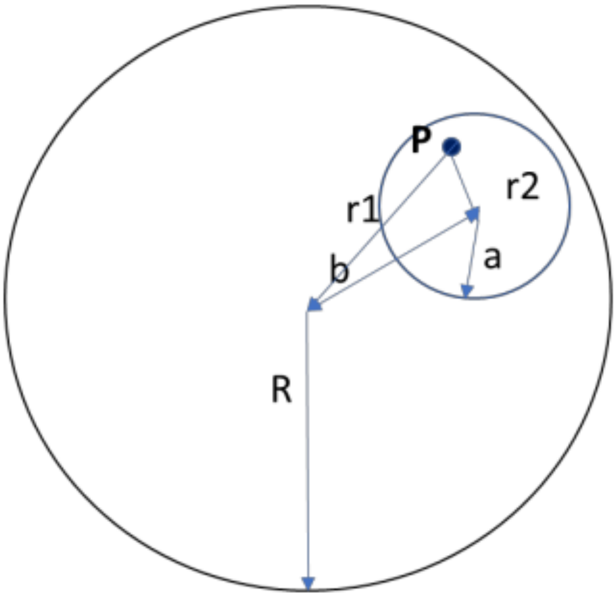
In region (ii), $\vec{\nabla} \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \times \frac{\mu_0 I}{2\pi r}\right)\right)\hat{z} = 0$.

In region (iii), $\vec{\nabla} \times \vec{B} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rB)\right)\hat{z} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \times \left(\frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I (r^2 - b^2)}{2\pi r (c^2 - b^2)}\right)\right)\right)\hat{z} = \frac{\mu_0 I}{\pi (c^2 - b^2)}\hat{z} = \mu_0 \vec{J}$.

In region (iv), $\vec{\nabla} \times \vec{B} = 0$ as B=0 itself.

6. A long cylindrical conductor of radius R has a cylindrical hole of radius a drilled out such that the axis of the hole is parallel to the axis of the cylinder. If b is the distance between the two axes and current I is passing through the remaining solid cylinder, show that the magnitude of the magnetic field is constant throughout the hole and is given by $\frac{\mu_0 I b}{2\pi (R^2 - a^2)}$.

Solution



To solve this problem, we need to apply principle of superposition, i.e we need to calculate the magnetic field inside the conductor without the hole and separately magnetic field inside the hole. In the drilled region, we need to assume that another current of same current density but in opposite direction is passing. Now the current density is $J = \frac{I}{\pi R^2 - \pi a^2}$.

The current density through the conducting cylinder is $J_1 = J$ and the current density through the drilled region is $J_2 = -J$.

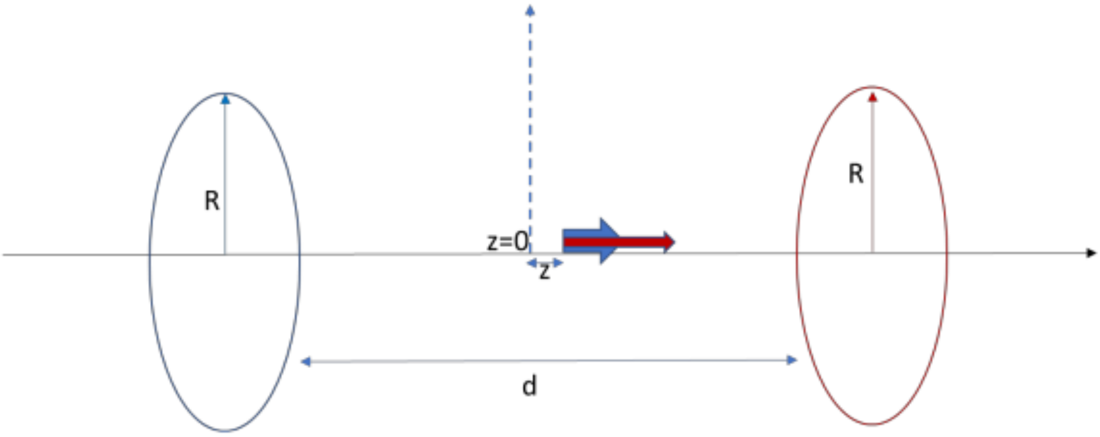
If magnetic field at P due to the conductor is denoted as B_1 and magnetic field at P due to the drilled region is B_2 then $B_1(2\pi r_1) = \mu_0 I = \mu_0 J \pi r_1^2 \Rightarrow B_1 = \frac{\mu_0 I r_1}{2\pi (R^2 - a^2)} = \frac{\mu_0}{2} J_1 r_1$.

Similarly, $B_2(2\pi r_2) = \mu_0 I = -\mu_0 J \pi r_2^2 \Rightarrow B_2 = -\frac{\mu_0 I r_2}{2\pi (R^2 - a^2)} = \frac{\mu_0}{2} J_2 r_2$. Since the current density and the radial direction is mutually perpendicular to each other therefore, $\vec{J_1} \times \vec{r_1} = J_1 r_1$ and $\vec{J_2} \times \vec{r_2} = J_2 r_2$. Applying the superposition principle we can write, $\vec{B} = \vec{B_1} + \vec{B_2} =$

$\frac{\mu_0}{2}\left(\vec{J_1}\times\vec{r_1}+\vec{J_2}\times\vec{r_2}\right)=\frac{\mu_0}{2}\left(\vec{J}\times\vec{r_1}-\vec{J}\times\vec{r_2}\right)=\frac{\mu_0}{2}\vec{J}\times(\vec{r_1}-\vec{r_2})=\frac{\mu_0}{2}\vec{J}\times\vec{b}$, since from the figure we can conclude that $\vec{r_1}-\vec{r_2}=\vec{b}$. Hence $B=\frac{\mu_0 Ib}{2\pi(R^2-a^2)}$.

7. One can produce a reasonably uniform magnetic field by using two parallel current carrying coils of radii R placed at a distance d apart. For this configuration calculate B along the axis as a function of z, the distance from a point midway between the coils and find out under what conditions both the first and second derivative of B with respect to z will vanish at a point midway between the coils. Find the corresponding magnetic field at that point. Such an arrangement is referred to as Helmholtz coil.

Solution



For a single coil of radius R carrying a current I, the magnetic field along the perpendicular axis through the center of the coil can be calculated by using Biot-Savarts law as $B=\frac{\mu_0 IR^2}{2(z^2+R^2)^{\frac{3}{2}}}$. So, from the figure, for two coils near the midpoint (z distance away from the midpoint) can be written as $\vec{B}=\vec{B_1}+\vec{B_2}=\frac{\mu_0 IR^2}{2\left(\left[\frac{d}{2}-z\right]^2+R^2\right)^{\frac{3}{2}}}\hat{z}+\frac{\mu_0 IR^2}{2\left(\left[\frac{d}{2}+z\right]^2+R^2\right)^{\frac{3}{2}}}\hat{z}$.

The first derivative, $\frac{dB}{dz}=-\frac{3}{2}\left[\frac{2\left(\frac{d}{2}-z\right)(-1)}{2\left(\left[\frac{d}{2}-z\right]^2+R^2\right)^{\frac{5}{2}}}+\frac{2\left(\frac{d}{2}+z\right)}{2\left(\left[\frac{d}{2}+z\right]^2+R^2\right)^{\frac{5}{2}}}\right]$. The derivative goes to zero if $z=0$, i.e at half the distance (d/2).

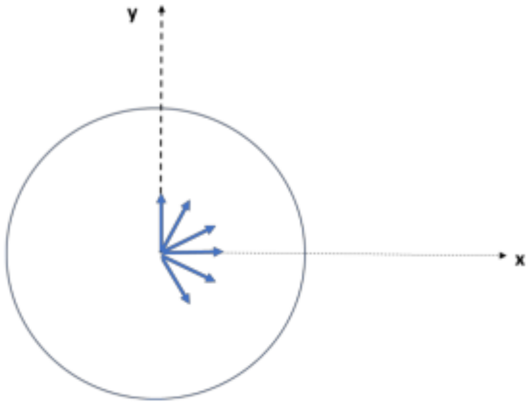
The second derivative, $\frac{d^2B}{dz^2}=-\frac{3}{2\left(\left[\frac{d}{2}-z\right]^2+R^2\right)^{\frac{5}{2}}}+\frac{3\left(\frac{d}{2}-z\right)(-5)\left(\frac{d}{2}-z\right)(-1)}{2\left(\left[\frac{d}{2}-z\right]^2+R^2\right)^{\frac{7}{2}}}-\frac{3}{2\left(\left[\frac{d}{2}+z\right]^2+R^2\right)^{\frac{5}{2}}}+\frac{3\left(\frac{d}{2}+z\right)(5)\left(\frac{d}{2}+z\right)}{2\left(\left[\frac{d}{2}+z\right]^2+R^2\right)^{\frac{7}{2}}}$. At the midpoint of the two coils, i.e at z=0, $\frac{d^2B}{dz^2}=-\frac{6}{2\left(\frac{d^2}{4}+R^2\right)^{\frac{5}{2}}}+\frac{30\left(\frac{d^2}{4}\right)}{2\left(\frac{d^2}{4}+R^2\right)^{\frac{7}{2}}}=\frac{30\left(\frac{d^2}{4}\right)-6\left(\frac{d^2}{4}+R^2\right)}{2\left(\frac{d^2}{4}+R^2\right)^{\frac{7}{2}}}=\frac{24\left(\frac{d^2}{4}\right)-6R^2}{2\left(\frac{d^2}{4}+R^2\right)^{\frac{7}{2}}}=\frac{6\left(d^2-R^2\right)}{2\left(\frac{d^2}{4}+R^2\right)^{\frac{7}{2}}}$. Hence the second derivative will be zero if d=R.

So, the magnetic field will be uniform at the middle of the two coils and the separation distance of the two coils are equal to the radius of each coil.

Hence the uniform magnetic field at z=0 and d=R is, $\vec{B}=\frac{\mu_0 IR^2}{\left(\left(\frac{R}{2}\right)^2+R^2\right)^{\frac{3}{2}}}\hat{z}=\frac{\mu_0 IR^2}{\left(\frac{5R^2}{4}\right)^{\frac{3}{2}}}=\left(\frac{4}{5}\right)^{\frac{3}{2}}\frac{\mu_0 I}{R}\hat{z}$.

8. A fine wire is used to wind six turns around an insulating sphere of radius a such that each turn makes an angle of 30° with the adjacent turn and that all the turns intersect each other at diametrically opposite points on the surface of the sphere. If a current I is passed through these turns, find the magnitude of $\vec{\boldsymbol{B}}$ at the center of the sphere. [Ans: $B \approx 1.93\mu_0 I/a$].

<u>**Solution**</u>



The magnetic field at the center of a current carrying loop can be evaluated using the Biot-Savart law and it reads: $B_0 = \frac{\mu_0 I}{2a}$. Since the wire is wounded six turns around an insulating sphere therefore we can consider the effective magnetic field is superposition of all the magnetic fields produced by each winding separated by 30°. So, the x and y component of the magnetic field can be written as,
$B_y = B_0 \cos 0^\circ + B_0 \cos 30^\circ + B_0 \cos 60^\circ + B_0 \cos 90^\circ + B_0 \cos 120^\circ + B_0 \cos 150^\circ = B_0 + B_0 \times \frac{\sqrt{3}}{2} + B_0 \times \frac{1}{2} + 0 - B_0 \times \frac{1}{2} - B_0 \times \frac{\sqrt{3}}{2} = B_0$,
and, $B_x = B_0 \sin 0^\circ + B_0 \sin 30^\circ + B_0 \sin 60^\circ + B_0 \sin 90^\circ + B_0 \sin 120^\circ + B_0 \sin 150^\circ = 0 + B_0 \times \frac{1}{2} + B_0 \times \frac{\sqrt{3}}{2} + B_0 + B_0 \times \frac{\sqrt{3}}{2} + B_0 \times \frac{1}{2} = B_0\left(1 + 1 + \sqrt{3}\right) = B_0(2 + \sqrt{3})$. So, the magnitude of the effective electric field is, $B = \sqrt{1 + \left(2 + \sqrt{3}\right)^2} B_0 \sim 1.93 \frac{\mu_0 I}{a}$.