Solution of Tutorial Sheet 7

1)(a) $\sum_{n=1}^{\infty} \frac{1}{n^n} x^n$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$
$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}}$$
$$= \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \frac{1}{n^{n\cdot\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\Rightarrow \frac{1}{R} = 0$$

$$\Rightarrow R = \infty$$

$\therefore$ Radius of convergence $(R) = \infty$.

Interval of convergence : $\because R = \infty$

$\Rightarrow$ The power series converges $\forall\, x \in \mathbb{R}$

1) (b) similar to 1(a) part.

1) (c) $\sum_{n=1}^{\infty} 4^n x^n$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (4^n)^{\frac{1}{n}} = 4$$

$\Rightarrow \frac{1}{R} = 4 \Rightarrow R = \frac{1}{4}$ $\therefore$ Radius of convergence $(R) = \frac{1}{4}$.

Interval of convergence : Series converges absolutely for $|x| < R = \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$

1) (d) $\sum \frac{1}{4^n} x^n$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{1}{4^n}\right)^{\frac{1}{n}} = \frac{1}{4}$

$\Rightarrow R = 4$

$\therefore$ Radius of convergence $(R) = 4$

Interval of convergence: The power series converges absolutely for $|x| < R = 4$ and diverges for $|x| > 4$.

For $x = 4$ and $x = -4$; we need to check seperately:

Now $x = 4$: $\sum \frac{1}{4^n} \cdot 4^n = \sum 1 \rightarrow$ divergent series

$x = -4$: $\sum \frac{1}{4^n} (-4)^n = \sum (-1)^n \rightarrow$ divergent series

$\therefore$ The power series converges absolutely for $|x| < 4$ and diverges for $|x| \geq 4$

1) (e) $\sum \frac{1}{3^n + 1} x^n$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^n + 1}{3^{n+1} + 1}$

$= \lim_{n\to\infty} \frac{1 + \frac{1}{3^n}}{3 + \frac{1}{3^n}} = \frac{1+0}{3+0} = \frac{1}{3}$.

$\therefore R = 3$

The power series converges absolutely for $|x| < 3$ and diverges for $|x| > 3$

1) (f) $\sum \frac{1}{n!}(x-3)^n$

$$\therefore \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$\Rightarrow R = \infty$

$\Rightarrow$ The power series converges $\forall x \in \mathbb{R}$.

1) (g) $\sum \frac{1}{n^p} x^n$

$$\therefore \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^p}{(n+1)^p}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^p = \left(\frac{1}{1+0}\right)^p = 1$$

$\therefore R = 1$

So, The power series converges absolutely $\forall |x| < R = 1$ & diverges $\forall |x| > 1$

~~Now: at $x = -1$: $\sum \frac{1}{n^p}(-1)^n = \sum \frac{1}{n^p} \rightarrow$ convergent if $p > 1$, divergent if $p \leq 1$~~

Now: at $x = 1$: $\sum \frac{1}{n^p} 1^n = \sum \frac{1}{n^p} \rightarrow \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases}$

at $x = -1$: $\sum \frac{1}{n^p}(-1^n) = \sum \frac{(-1)^n}{n^p} \rightarrow \begin{cases} \text{convergent if } p > 0 \\ \text{divergent if } p \leq 0 \end{cases}$

1) (h) $\sum \frac{n!}{n^n}(x+3)^n$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1!}{(n+1)^{n+1}} \times \frac{n^n}{n!}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n.$$

$$= \frac{1}{e}$$

$\Rightarrow \quad R = e$

$\therefore$ The power series converges absolutely $\forall\ |x+3| < R = e$ & diverges $\forall\ |x+3| > e$

1) (i) $\sum \frac{(-1)^n n}{4^n}(x+3)^n$

$$\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^n n}{4^n}\right|}$$

$$= \lim_{n\to\infty}\left(\frac{n}{4^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{(n)^{1/n}}{4} = \frac{1}{4}$$

$\therefore\ R = 4$

$\therefore$ The power series converges absolutely $\forall\ |x+3| < R = 4$ & diverges $\forall\ |x+3| > 4$

1)(j) $\sum \frac{2^n}{n} (4x-8)^n$

$= \sum \frac{2^n}{n} \cdot 4^n (x-2)^n$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| \frac{8^{n+1}}{n+1} \cdot \frac{n}{8^n} \right|$

$= \lim_{n\to\infty} \left| \frac{8}{1+\frac{1}{n}} \right| = \frac{8}{1+0} = 8$

$\therefore R = \frac{1}{8}$.

$\therefore$ The Power series converges absolutely $\forall\ |x-2| < R = \frac{1}{8}$ & diverges $\forall\ |x-2| > \frac{1}{8}$

1)(K) $\sum n!(2x+1)^n = \sum n!\, 2^n \left(x+\frac{1}{2}\right)^n$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{2^{n+1}}{2^n}$

$= \lim_{n\to\infty} 2(n+1) = \infty$

$\therefore R = 0$

$\Rightarrow$ The Power series only converges at $x = -\frac{1}{2}$ (i.e for $x \neq -\frac{1}{2}$ the power series diverges)

1)(l) $\sum_{n=1}^{\infty} \frac{(-4)^n}{(n+2)!} (x+3)^n$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-4)^{n+1}}{n+3!} \frac{n+2!}{-4^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-4}{n+3} \right|$$

$$= \lim_{n\to\infty} \frac{4}{n+3} = 0$$

$$\Rightarrow \frac{1}{R} = 0$$

$$\Rightarrow R = \infty$$

$\therefore$ The Power series converges $\forall\ x \in \mathbb{R}$.

1) (m) $\sum \frac{(x-2)^n}{10^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{10^n}} = \frac{1}{10}$$

$$\therefore \frac{1}{R} = \frac{1}{10} \Rightarrow R = 10$$

$\therefore$ The power series converges absolutely $\forall\ |x-2| < R = 10$ & diverges $\forall\ |x-2| \geq 10$

1) (n) $\sum (-1)^n (4x+1)^n = \sum (-1)^n 4^n \left(x+\frac{1}{4}\right)^n$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{|(-1)^n 4^n|} = \left|(-1)^{n \times \frac{1}{n}} 4^{n \times \frac{1}{n}}\right|$$

$$= 4$$

$$\therefore R = \frac{1}{4}$$

$\therefore$ The power series converges absolutely $\forall\ |x+\frac{1}{4}| < R = \frac{1}{4}$ & diverges $\forall\ |x+\frac{1}{4}| \geq \frac{1}{4}$

2) (a) $f(x) = \sin x$ & $c = 0$

Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!} (x-c)^n$

$$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \quad (\because c=0)$$

$f^{(0)}(x) = \sin x \Rightarrow f^{(0)}(0) = 0$

$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1$

$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x \Rightarrow f^{(6)}(0) = 0$

$\vdots \qquad \vdots$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\Rightarrow \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

2) (b) $f(x) = \cos x$ & $c = 0$

$$\therefore \; f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$f^{(0)}(x) = \cos x \implies f^{(0)}(0) = 1$

$f^{(1)}(x) = -\sin x \implies f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos x \implies f^{(2)}(0) = -1$

$f^{(3)}(x) = \sin x \implies f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin x \implies f^{(5)}(0) = 0$

$f^{(6)}(x) = -\cos x \implies f^{(6)}(0) = -1$

$\vdots \qquad\qquad\qquad\qquad \vdots$

$$\therefore \; \cos x = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$$= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

$$\Rightarrow \; \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

2) (c) $f(x) = e^{-x}$ and $c = 0$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\therefore \quad f^{(0)}(x) = e^{-x} \Rightarrow f^{(0)}(0) = 1$$

$$f^{(1)}(x) = -e^{-x} \Rightarrow f^{(1)}(0) = -1$$

$$f^{(2)}(x) = e^{-x} \Rightarrow f^{(2)}(0) = 1$$

$$f^{(3)}(x) = -e^{-x} \Rightarrow f^{(3)}(0) = -1$$

$$f^{(4)}(x) = e^{-x} \Rightarrow f^{(4)}(0) = 1$$

$$\vdots \qquad\qquad \vdots$$

$$f^{n}(x) = (-1)^n e^{-x} \qquad f^{n}(0) = (-1)^n$$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\Rightarrow \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

2) (d) $f(x) = \ln x$; $c = 2$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!} (x-c)^{n}$$

$$= \sum_{n=0}^{\infty} \frac{f^{n}(2)}{n!} (x-2)^{n}$$

$$f^{(0)}(x) = \ln(x) \quad \Rightarrow \quad f^{0}(2) = \ln 2$$

$$f^{(1)}(x) = \frac{1}{x} \quad \Rightarrow \quad f^{(1)}(2) = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{x^{2}} \quad \Rightarrow \quad f^{(2)}(2) = -\frac{1}{2^{2}}$$

$$f^{(3)}(x) = \frac{2}{x^{3}} \quad \Rightarrow \quad f^{(3)}(2) = \frac{2}{2^{3}}$$

$$f^{(4)}(x) = -\frac{2.3}{x^{4}} \quad \Rightarrow \quad f^{(4)}(2) = -\frac{2.3}{2^{4}}$$

$$f^{(5)}(x) = +\frac{2.3.4}{x^{5}} \quad \Rightarrow \quad f^{5}(2) = \frac{2.3.4}{2^{5}}$$

$\vdots$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^{n}} \quad \Rightarrow \quad f^{n}(2) = \frac{(-1)^{n+1}(n-1)!}{2^{n}}$$

$n = 1, 2, 3, \ldots$ $\qquad n = 1, 2, 3 \ldots$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(2)}{n!} (x-2)^{n}$$

$$= f(2) + \sum_{n=1}^{\infty} \frac{f^{n}(2)}{n!} (x-2)^{n}$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\, 2^{n}} (x-2)^{n}$$

$$\Rightarrow \ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^{n}} (x-2)^{n}$$

2) (e) $f(x) = \frac{1}{x^2}$ and $c = -1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^n(-1)}{n!}(x+1)^n$$

$f^{(0)}(x) = \frac{1}{x^2} \Rightarrow f^{(0)}(-1) = 1$

$f^{(1)}(x) = -\frac{2}{x^3} \Rightarrow f^{(1)}(-1) = 2$

$f^{(2)}(x) = \frac{2 \cdot 3}{x^4} \Rightarrow f^{(2)}(-1) = 2 \cdot 3$

$f^{(3)}(x) = -\frac{2 \cdot 3 \cdot 4}{x^5} \Rightarrow f^{(3)}(-1) = 2 \cdot 3 \cdot 4$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \Rightarrow f^n(-1) = \frac{(-1)^n (n+1)!}{(-1)^{n+2}}$

$$= \frac{\cancel{(-1)^n}(n+1)!}{\cancel{(-1)^n} \cdot (-1)^2}$$

$$= (n+1)!$$

$n = 0, 1, 2, \ldots$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n(-1)}{n!}(x+1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n$$

$$\Rightarrow \frac{1}{x^2} = \sum_{n=0}^{\infty} (n+1)(x+1)^n$$

2) (f) $f(x) = e^{-x}$ and $c = -4$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{n}(-4)}{n!} (x+4)^n$$

$f^{(0)}(x) = e^{-x} \Rightarrow f^{(0)}(-4) = e^{4}$

$f^{(1)}(x) = -e^{-x} \Rightarrow f^{(1)}(-4) = -e^{4}$

$f^{(2)}(x) = e^{-x} \Rightarrow f^{(2)}(-4) = e^{4}$

$f^{(3)}(x) = -e^{-x} \Rightarrow f^{(3)}(-4) = -e^{4}$

$\vdots \qquad \vdots$

$f^{(n)}(x) = (-1)^n e^{-x} \qquad f^{n}(-4) = (-1)^n e^{4}$

$$\therefore \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(-4)}{n!} (x+4)^n$$

$$\Rightarrow \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{4}}{n!} (x+4)^n$$