EPHY105L - 17 13.12.21

Gauss' law in presence of Dielectrics

Polarisation $\longrightarrow$ accumulation of bound charges

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \qquad \sigma_b = \vec{P} \cdot \hat{n}$$

$\rightarrow$ These bound charges give rise to electric field.

Total field = field due to bound charges + field due to free charges

Within a dielectric,

$$\rho = \rho_b + \rho_f$$

$\rho_f \rightarrow$ free charge density

Using Gauss' law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \epsilon_0 (\vec{\nabla} \cdot \vec{E}) = \rho = \rho_b + \rho_f$$

$$= -\vec{\nabla} \cdot \vec{P} + \rho_f$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho_f$$

$\rightarrow \vec{D} = \epsilon_0 \vec{E} + \vec{P} \equiv$ Electric displacement

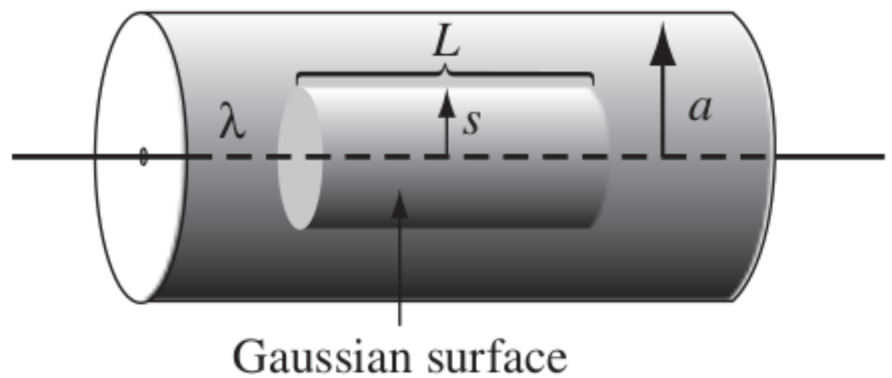
Hence, $\oint \vec{D} \cdot d\vec{a} = \int \rho_f \, d\tau$

Gauss' law for dielectrics.

$$= (Q_f)_{enc.}$$

(total free charge enclosed)

Ex:



A long straight wire carrying uniform line charge density λ, surrounded by a dielectric medium upto a radius 'a'.

→ Gaussian surface has radius ≡ s
length ≡ L

$$\oint \vec{D} \cdot d\vec{a} = (Q_f)_{enc.}$$

$$\Rightarrow \quad D(2\pi s L) = \lambda L$$

$$\Rightarrow \quad D = \frac{\lambda}{2\pi s}$$

$$\vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

Outside, $\vec{P} = 0$

from, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$= \epsilon_0 \vec{E}$

$$\Rightarrow \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \quad \text{for } s > a$$

For, $s < a$, we need the knowledge of $\vec{P}$ in order to calculate $\vec{E}$.

· <u>Linear Dielectrics</u>

A class of dielectrics for which the induced polarisation is proportional to the electric field.

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$\chi_e \equiv$ Proportionality const. (Electric susceptibility)

'ϵ_0' is brought outside to make χ_e dimensionless.

(*) Any material obeying $\vec{P} = \epsilon_0 \chi_e \vec{E}$ is called a linear dielectric.

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$= \epsilon_0 (1 + \chi_e) \vec{E}$$

$$= \epsilon \vec{E}$$

$\rightarrow \epsilon = \epsilon_0 (1 + \chi_e) \equiv$ Permittivity of a material

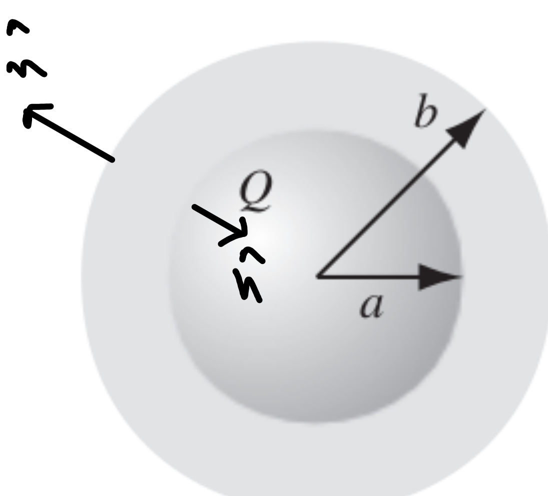
(*) In vacuum, there is no material,

$\chi_e = 0 \Rightarrow \epsilon = \epsilon_0 \equiv$ Permittivity of free space.

(*) $\epsilon_r = 1 + \chi_e \equiv$ Relative permittivity or Dielectric const.

$$= \frac{\epsilon}{\epsilon_0}$$

Ex:



A metal sphere of radius 'a' carrying a total charge 'Q'. It is surrounded by a dielectric material of permittivity 'ϵ'. Calculate the potential at the center.

→ In the region, $r > a$

$$\oint \vec{D} \cdot d\vec{a} = q$$

$$\Rightarrow \quad \vec{D} = \frac{q}{4\pi r^2} \hat{r}$$

Inside the sphere, $r < a$

$$\vec{D} = 0 = \vec{E} = \vec{P}$$

In the region, $a < r < b$

$$\vec{E} = \frac{q}{4\pi \epsilon r^2} \hat{r}$$

In the region, $r > b$

$$\vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \hat{r}$$

Potential at center,

$$V = - \int_{\infty}^{0} \vec{E} \cdot d\vec{l}$$

$$= - \int_{\infty}^{b} \frac{q}{4\pi\epsilon_0 r^2}\, dr - \int_{b}^{a} \frac{q}{4\pi\epsilon r^2}\, dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right)$$

Polarization,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$= \frac{\epsilon_0 \chi_e q}{4\pi\epsilon r^2}\, \hat{r} \quad \text{(in the dielectric)}$$

Bound charges:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \frac{\epsilon_0 \chi_e q}{4\pi \epsilon b^2} \quad \text{(at outer surface)}$$

$$= - \frac{\epsilon_0 \chi_e q}{4\pi \epsilon a^2} \quad \text{(at inner surface)}$$

(*) $\hat{n}$ points outward with respect to the dielectrics $\Rightarrow$

$$\left.\begin{aligned} \hat{n} &\equiv \hat{r} \quad \text{at } r = b \\ &= -\hat{r} \quad \text{at } r = a \end{aligned}\right\}$$