Solutions to Tutorial 7

1. The cylinder is uniformly magnetized along the axis, therefore magnetization can be written as $\vec{\boldsymbol{M}} = M_o\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{k}}$ is the unit vector in the direction of magnetization and parallel to the axis of the cylinder. Thus

- Bound volume current density is given by

$$\vec{\mathbf{J}}_{\mathbf{b}} = \vec{\nabla} \times \vec{\boldsymbol{M}} = 0 \text{ (as the magnetization vector is uniform)}$$

- Bound surface current density on the cylindrical surface is given by

$$\vec{\boldsymbol{K}}_{\mathrm{b}} = \vec{\boldsymbol{M}} \times \hat{\boldsymbol{n}} = M_o\hat{\boldsymbol{k}} \times \hat{\boldsymbol{r}} = M_o\hat{\boldsymbol{\phi}}$$

Thus such a magnetized cylinder can be considered as equivalent to a solenoid with an azimuthal current just like in a solenoid. The equivalent solenoid will have $NI = M$. Therefore the magnetic field outside the cylinder is zero and inside the cylinder it is given by

$$\vec{\boldsymbol{B}} = \mu_o M_o\hat{\boldsymbol{k}} = \mu_o\vec{\boldsymbol{M}}$$

2. In view of symmetry the magnetic field will be azimuthal and will be dependent only on the distance r from the axis of the wire.

By applying the modified Ampere's law, magnetic field inside the wire ($r < R$) can be obtained as follows:

$$\oint \vec{\boldsymbol{H}}.\vec{d\boldsymbol{l}} = I_{free} \Longrightarrow H2\pi r = \frac{I}{\pi R^2}\pi r^2 \Longrightarrow \vec{\boldsymbol{H}} = \frac{Ir}{2\pi R^2}\hat{\boldsymbol{\phi}}$$

The corresponding magnetic field is given by

$$\vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{H}} = \frac{\mu I r}{2\pi R^2}\hat{\boldsymbol{\phi}}$$

Magnetic field outside the wire ($r > R$)

$$\oint \vec{\boldsymbol{H}}.\vec{d\boldsymbol{l}} = I_{free} \Longrightarrow H2\pi r = I \Longrightarrow \vec{\boldsymbol{H}} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$$

and the magnetic field is given by

$$\vec{\boldsymbol{B}} = \mu_o\vec{\boldsymbol{H}} = \frac{\mu_o I}{2\pi r}\hat{\boldsymbol{\phi}}$$

Bound volume current density is given by

$$\vec{\boldsymbol{J}_{\boldsymbol{b}}} = \vec{\nabla} \times \vec{\boldsymbol{M}} = \chi_m\vec{\nabla} \times \vec{\boldsymbol{H}} = \chi_m\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{Ir}{2\pi R^2}\right)\hat{\boldsymbol{k}} = \frac{\chi_m I}{\pi R^2}\hat{\boldsymbol{k}}$$

Bound surface current density at the surface of the wire is given by

$$\vec{\boldsymbol{K}}_{\mathrm{b}} = \vec{\boldsymbol{M}} \times \hat{\boldsymbol{n}} = \chi_m\frac{IR}{2\pi R^2}\left(\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}}\right) = -\frac{\chi_m I}{2\pi R}\hat{\boldsymbol{k}} \qquad \text{at } (r = R)$$

The total bound current is given by

$$I_b = J_b \times \pi R^2 + K_b \times 2\pi R = \frac{\chi_m I}{\pi R^2} \times \pi R^2 - \frac{\chi_m I}{2\pi R} \times 2\pi R = 0$$

3. The configuration of the cable and the currents passing through the inner and outer tubes are shown in the figure below. Using modified Ampere's law for a point lying between two tubes at a distance r from the axis:

$$\oint \vec{\boldsymbol{H}}.\vec{d\boldsymbol{l}} = I_{free} \Longrightarrow \vec{\boldsymbol{H}} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$$

The magnetic field between the tubes can be estimated as

$$\vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{H}} = \mu_o(1+\chi_m)\frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$$

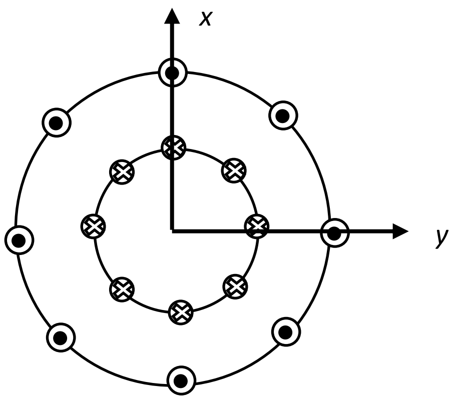
Bound volume current density is given by

$$\vec{\boldsymbol{J}_b} = \vec{\nabla}\times\vec{\boldsymbol{M}} = \chi_m\vec{\nabla}\times\vec{\boldsymbol{H}} = 0 \qquad \text{(Use cylindrical polar representation for } \vec{\nabla}\text{)}$$

Bound surface current density is given by

$$\vec{\boldsymbol{K}}_{\mathrm{b}}(\mathrm{r}=\mathrm{a}) = \vec{\boldsymbol{M}}\times\hat{\boldsymbol{n}} = \vec{\boldsymbol{M}}\times(-\hat{\boldsymbol{r}}) = -\chi_m\frac{I}{2\pi a}\left(\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{r}}\right) = \frac{\chi_m I}{2\pi a}\hat{\boldsymbol{k}}$$

$$\vec{\boldsymbol{K}}_{\mathrm{b}}(\mathrm{r}=\mathrm{b}) = \vec{\boldsymbol{M}}\times(\hat{\boldsymbol{r}}) = \chi_m\frac{I}{2\pi b}\left(\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{r}}\right) = -\frac{\chi_m I}{2\pi b}\hat{\boldsymbol{k}}$$



4. In this case the magnetic field will be along the axial direction and can only depend on the radial coordinate. The magnetic field inside the rod can be evaluated by assuming a rectangular Amperean loop i.e.

$$\oint\vec{\boldsymbol{H}}.\vec{d\boldsymbol{l}} = I_{free} \Longrightarrow \vec{\boldsymbol{H}} = NI\hat{\boldsymbol{k}};\ \vec{\boldsymbol{B}} = \mu\vec{\boldsymbol{H}} = \mu_o(1+\chi_m)NI\hat{\boldsymbol{k}}$$

The magnetic field between the rod and solenoid ($a < r < R$) is given by

$$\oint\vec{\boldsymbol{H}}.\vec{d\boldsymbol{l}} = I_{free} \Longrightarrow \vec{\boldsymbol{H}} = NI\hat{\boldsymbol{k}};\ \vec{\boldsymbol{B}} = \mu_o\vec{\boldsymbol{H}} = \mu_o NI\hat{\boldsymbol{k}}$$

Note that the $\boldsymbol{H}$ field is the same throughout while the $\boldsymbol{B}$ field is different between the two regions. This is similar to the case of a parallel plate capacitor with dielectric between the plates wherein the $\boldsymbol{D}$ vector was the same everywhere while the $\boldsymbol{E}$ vector was different.

Bound volume current density is given by

$$\vec{\boldsymbol{J}_b} = \vec{\nabla}\times\vec{\boldsymbol{M}} = \chi_m\vec{\nabla}\times\vec{\boldsymbol{H}} = 0 \qquad \text{(Use cylindrical polar representation for } \vec{\nabla}\text{)}$$

Bound surface current density is given by

$$\vec{\boldsymbol{K}}_{\mathrm{b}}(\mathrm{r}=\mathrm{a}) = \vec{\boldsymbol{M}}\times\hat{\boldsymbol{n}} = \vec{\boldsymbol{M}}\times\hat{\boldsymbol{r}} = \chi_m NI\left(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{r}}\right) = \chi_m NI\hat{\boldsymbol{\phi}}$$

5. Magnetic field due to a wire at distance r is given by $\vec{\boldsymbol{B}} = \frac{\mu_o I}{2\pi r}\hat{\boldsymbol{\phi}}$

Magnetic flux at a distance 'r' through the loop of length 'a' and width $'dr'$ is given by

$$d\phi_B = \frac{\mu_o I}{2\pi r}(adr) \Longrightarrow \phi_B = \frac{\mu_o Ia}{2\pi}\int_r^{r+a}\frac{1}{r}dr = \frac{\mu_o Ia}{2\pi}\ln\left(\frac{r+a}{r}\right)$$

Induced emf, $\epsilon = -\frac{d\phi_B}{dt} = \frac{\mu_o Ia}{2\pi}\left(\frac{1}{r+a}-\frac{1}{r}\right)\frac{dr}{dt} = \frac{\mu_o Ia^2}{2\pi r(r+a)}v \qquad \left(\frac{dr}{dt}=v\right)$

$$= \frac{\mu_o Ia^2}{2\pi b(b+a)}v \qquad (at\ r=b)$$

6. Given $N = 100;\ r = 20$ cm $= 0.2m;\ t = 0.2$ s; $B = 0.5$ T
Magnetic flux is given by $\phi_B = NBA = NB\pi r^2$

(i) $$\phi_{0.5} = 100 \times 0.5 \times 3.14 \times (0.2)^2 = 6.28;\ \phi_0 = 0$$
Therefore, Induced emf, $\epsilon = -\frac{\Delta\phi}{\Delta t} = \frac{\phi_{0.5}-\phi_0}{\Delta t} = \frac{6.28}{0.2} = 31.4V$
(ii) If the field is reverse in the direction, the change in magnetic flux is given by
$$\Delta\phi = \phi_{0.5} - (-\phi_{0.5}) = 2 \times \phi_{0.5}$$
Therefore, induced emf, $\epsilon = \frac{2\times 6.28}{0.2} = 62.8V$
(iii) Once the coil is rotated by 90^o, the axis of the coil becomes perpendicular to the magnetic field and the area vector becomes perpendicular to the direction of magnetic field. This will result in zero flux. Therefore, the change in magnetic flux before and after rotation is given by $\Delta\phi = 6.28$. Hence, induced emf, $\epsilon = 31.4V$

7. Given $R = 2$ mm $= 0.2 \times 10^{-3}$m; $l = 20$ cm $= 0.2$ m; $N = 100$;
$$I(t) = I_o sin(\omega t) = I_o sin(2\pi f t) = 5 sin(2\pi f t)$$
The induced EMF is defined as the -ve time rate of change of magnetic flux and it is also defined as line integral of electric field intensity, therefore
$$\oint \vec{\boldsymbol{E}}.d\vec{\boldsymbol{l}} = -d\phi/dt$$
Considering a loop of radius r $(r < R)$ inside the hollow solenoid, the magnetic flux through the loop is given by
$$\phi = NBA = \mu_o n I(t)\pi r^2 \Longrightarrow \frac{d\phi}{dt} = \mu_o \pi r^2 \frac{N}{L}\frac{dI(t)}{dt}$$
The induced electric field inside the solenoid is given by
$$|E|2\pi r = \mu_o \pi r^2 \frac{N}{L}\frac{dI(t)}{dt} = \mu_o \pi r^2 \frac{N}{L}\frac{dI_o \sin(2\pi f t)}{dt}$$
$$E = \frac{\mu_o N r}{2L}(2\pi f) I_o \cos(2\pi f t)$$
On substituting respective values, $E = 98.6 r cos(2\pi f t)$. The induced electric field will be having azimuthal direction.

8. Given $r = 1$ cm $= 0.01$ m; $N = 1000; \Delta t = 10^{-3}$ s; $I_1 = 0; I_2 = 1$ A
Electric field inside the solenoid is given by $E_{in} = \frac{\mu_o N r}{2L}\frac{\Delta I}{\Delta t} = 4\pi \times 10^{-4}$ V/m
Electric field outside the solenoid is given by $\oint E_{out}.dl = -\frac{d\phi}{dt} = \mu_o N\pi R^2 \frac{\Delta I}{\Delta t}$
$$|E_{out}|2\pi r = \mu_o N \pi R^2 \frac{\Delta I}{\Delta t} \Longrightarrow E_{out} = \frac{\mu_o N\pi R^2}{2r}\frac{\Delta I}{\Delta t}$$
On substituting respective values, $E_{out} = \pi \times 10^{-3}$ V/m