Lecture - 4

More about Gradient Operator

An ordinary vector **A** can be multiply in three ways:

1. Multiply a scalar a : a**A**
2. Multiply another vector (dot product): **A**·**B**
3. Multiply another vector (cross product): **A**×**B**

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function H: ∇H (**Gradient**
2. On a vector function (dot product): $\nabla \cdot$ **v** (**divergence**
3. On a vector function (cross product): $\nabla \times$ **v** (**curl**

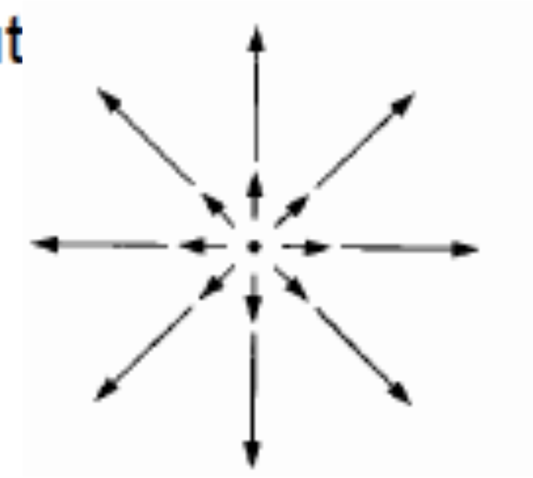
Divergence

Divergence of a vector **v** is:

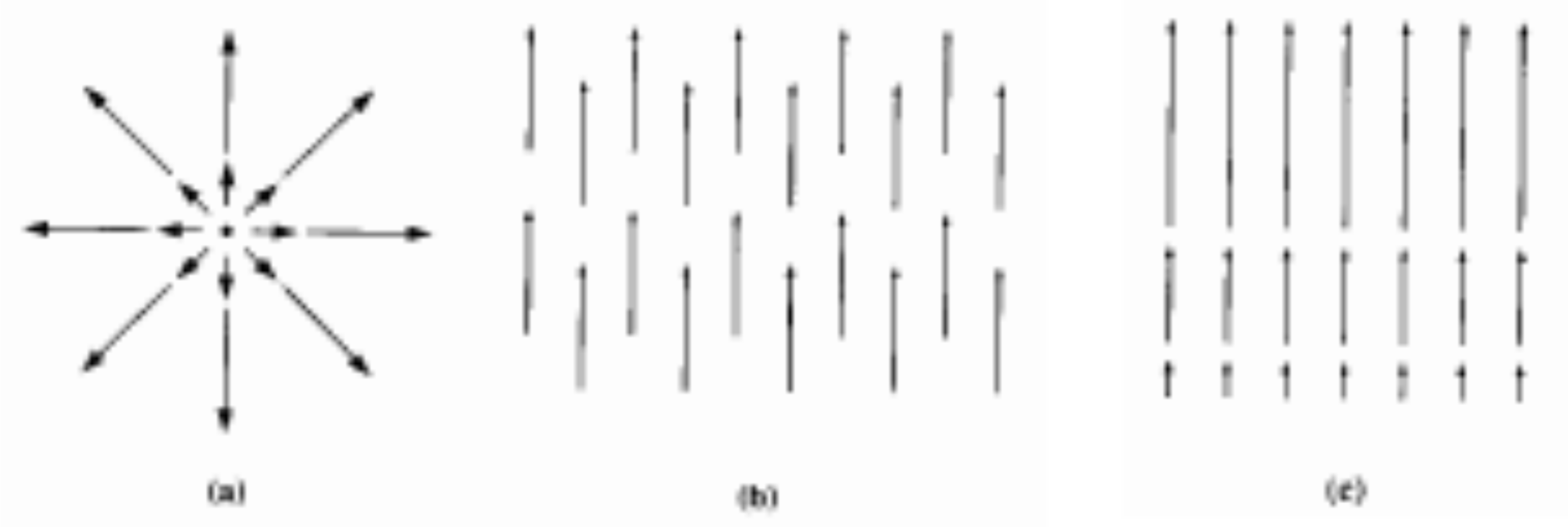
$$\nabla \cdot \mathbf{v} = (\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$\nabla \cdot \mathbf{v}$ is a measure of how much the vector **v** spread out from the point in question.



Divergence of a vector



$$\mathbf{v}_a = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \qquad \mathbf{v}_b = \hat{\mathbf{z}} \qquad \mathbf{v}_c = z\hat{\mathbf{z}}.$$

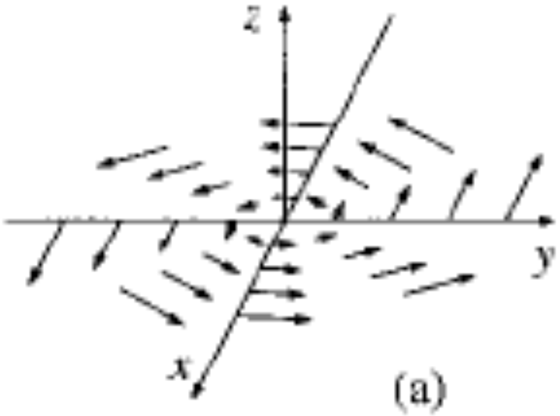
$$\nabla \cdot \mathbf{v}_a = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \qquad \nabla \cdot \mathbf{v}_b = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 1}{\partial z} = 0, \quad \nabla \cdot \mathbf{v}_c = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 1.$$

Curl of a vector

Curl of a vector $\mathbf{v}$ is:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ curl around the point in question.



(a)

$$\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v}_a = \hat{\mathbf{x}}\left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial(-y)}{\partial z} - \frac{\partial 0}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = 2\hat{\mathbf{z}}$$

Rules for sum, product etc

The sum rule:

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx} \qquad \nabla(f+g)=\nabla f+\nabla g$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B} \qquad \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}$$

The rule for multiplying by a constant:

$$\frac{d}{dx}(kf)=k\frac{df}{dx} \qquad \nabla(kf)=k\nabla f$$

$$\nabla\cdot(k\mathbf{A})=k\nabla\cdot\mathbf{A} \qquad \nabla\times(k\mathbf{A})=k\nabla\times\mathbf{A}$$

Rules for sum, product etc

The product rule: $\begin{cases} \text{scalar}: fg \\ \text{vector}: f\mathbf{A} \end{cases}$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx} \qquad \nabla(fg) = g\nabla f + f\nabla g$$

$$\nabla\cdot(f\mathbf{A}) = \nabla f\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}) \qquad \nabla\times(f\mathbf{A}) = \nabla f\times\mathbf{A} + f(\nabla\times\mathbf{A})$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} \qquad \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla \cdot \left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla \cdot \mathbf{A}) - \mathbf{A} \cdot \nabla g}{g^2}$$

$$\nabla \times \left(\frac{\mathbf{A}}{g}\right) = \frac{g(\nabla \times \mathbf{A}) - (\nabla g \times \mathbf{A})}{g^2} = \frac{g(\nabla \times \mathbf{A}) + \mathbf{A} \times \nabla g}{g^2}$$

$$\begin{cases} \text{scalar} : \mathbf{A} \cdot \mathbf{B} \\ \text{vector} : \mathbf{A} \times \mathbf{B} \end{cases}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Few others related to derivative

By applying ∇ twice, we can construct five species of second derivatives.

$$\text{Three first derivatives } \nabla T, \ \ \nabla \cdot \mathbf{v}, \ \ \nabla \times \mathbf{v}$$

(1) Divergence of gradient : $\nabla \cdot (\nabla T)$ ⟵ very important

(2) Curl of gradient : $\nabla \times (\nabla T)$ ⟵ always zero

(3) Gradient of divergence : $\nabla (\nabla \cdot \mathbf{v})$

(4) Divergence of curl : $\nabla \cdot (\nabla \times \mathbf{v})$ ⟵ always zero

(5) Curl of curl : $\nabla \times (\nabla \times \mathbf{v})$ ⟵ reduce to others