# Solutions - Tutorial Sheet 5

1. (a) $\lim\limits_{x \to c} \dfrac{\sin(x-c)}{x-c}$

$= \lim\limits_{z \to 0} \dfrac{\sin z}{z}$ $\left[\begin{array}{l} \text{let } x-c = z \\ \therefore \text{ as } x \to c \\ \Rightarrow z \to 0 \end{array}\right]$

$= 1$

But $\lim\limits_{x \to c} \dfrac{\sin(x-c)}{(x-c)} = 1 \neq f(c) = 0$

$\Rightarrow$ $f(x)$ is not continuous at $x = c$

$\Rightarrow$ $f(x)$ is discontinuous at $x = c$

Name of discontinuity = Removable discontinuity

(b) $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

$= \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x^2+2x+4)}{(x+2)\cancel{(x-2)}}$

$= \dfrac{4 + 2\times 2 + 4}{2+2} = \dfrac{\cancel{12}}{\cancel{4}} = 3$

$\because \lim\limits_{x \to 2} \dfrac{x^3-8}{x^2-4} = 3 = f(2)$

$\Rightarrow$ $f(x)$ is continuous at $x = 2$.

1.(c) $\lim_{x \to 0} \dfrac{1-\cos 4x}{x^2}$

$= \lim_{x \to 0} \dfrac{2\sin^2 2x}{x^2}$

$= \lim_{x \to 0} \dfrac{\sin^2 2x}{(2x)^2} \cdot 2 \cdot 4$

$= \left(\lim_{x \to 0} \dfrac{\sin 2x}{2x}\right)^2 \cdot 2 \cdot 4$

$= 1 \cdot 2 \cdot 4 = 8$

But $\lim_{x \to 0} \dfrac{1-\cos 4x}{x^2} = 8 \neq f(0) = 4$

$\Rightarrow$ $f(x)$ is discontinuous at $x = 0$

Name of discontinuity = Removable discontinuity

1.(d) $\lim_{x \to 0^+} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$

$= \lim_{x \to 0^+} \dfrac{1 - e^{-\frac{1}{x}}}{1 + e^{-1/x}}$

R.H.L $= \dfrac{1}{1} = 1$ $\left(\because x \to 0^+, +\frac{1}{x} \to +\infty \Rightarrow e^{-\frac{1}{x}} \to 0\right)$

$\lim_{x \to 0^-} \dfrac{e^{\frac{1}{x}} - 1}{e^{1/x} + 1}$ $\left(\because x \to 0^-, \frac{1}{x} \to -\infty \Rightarrow e^{\frac{1}{x}} \to 0\right)$

L.H.L $= \dfrac{0-1}{0+1} = -1$

$\therefore$ R.H.L $\neq$ L.H.L

$\Rightarrow$ $f(x)$ is descontinuous at $x = 0$

Name of descontinuity = Jump descontinuity.

2. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x^2 = 0 = $ L.H.L

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 5x - 4 = -4 = $ R.H.L

$\because$ L.H.L $\neq$ R.H.L

$\Rightarrow$ $f(x)$ is descontinuous at $x = 0$

Name of descontinuity = Jump descontinuity

Now, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5x - 4 = 1 = $ L.H.L

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x^2 - 3x = 1 = $ R.H.L

$\therefore$ L.H.L = R.H.L $= 1 = f(1) = 5 - 4 = 1$

$\Rightarrow$ $f(x)$ is continuous at $x = 1$.

Again, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 4x^2 - 3x = 16 - 6 = 10 = $ L.H.L

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x + 4 = 10 = $ R.H.L

$\therefore$ L.H.L = R.H.L $= f(2) = 3 \times 2 + 4 = 10 \Rightarrow f(x)$ continuous at $x = 2$

3. $$\lim_{x \to 0^+} \frac{x - |x|}{x}$$

$$= \lim_{x \to 0^+} \frac{\cancel{x} - \cancel{x}}{x} = 0 \quad (\because x \to 0^+,\ |x| = x)$$

$$\lim_{x \to 0^-} \frac{x - |x|}{x}$$

$$= \lim_{x \to 0^-} \frac{x + x}{x} = 2 \quad (\because x \to 0^-,\ |x| = -x)$$

$\therefore$ L.H.L $\neq$ R.H.L

$\Rightarrow$ $f(x)$ discontinuous at $x = 0$

Name of the discontinuity = Jump disconti

4. Let $f(x) = x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} - 119 \quad \forall x \in \mathbb{R}$.

Then $f: \mathbb{R} \to \mathbb{R}$ is continuous.

Now, $f(-2) < 0$ & $f(0) > 0$.

$\therefore \exists\ c \in (-2, 0)$ such that

$$f(c) = 0.$$

i.e $$c^{179} + \frac{163}{1 + c^2 + \sin^2 c} - 119 = 0.$$

$$\Rightarrow c^{179} + \frac{163}{1 + c^2 + \sin^2 c} = 119$$

5) (a) $f(x) = x^5 - 3x^2 + 1$. $x \in [0, 1]$

$f(0) = 1$, $f(1) = -1$ and $f$ is continuous.

Now apply IVT.

(b) $f(x) = \sin^2 x - 2\cos x + 1$, $x \in [0, \frac{\pi}{2}]$.

$f(0) = -1$, $f(\frac{\pi}{2}) = 2$ and $f$ is continuous

Now apply IVT.

6) $f(x) = \begin{cases} x & \text{if } x \text{ is rational.} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

This function continuous only at $x = 0$.

Otherwise it is discontinuous function.

7) $f(x) =$ constant : Constant function.

& polynomial function are continuous everywhere.