Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 2

1.
- (a) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = 3$
- (b) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{(3n+1)(n-2)}{n(n+3)} = 3$
- (c) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (-1)^n \left(\frac{2}{n+2}\right) = 0$
- (d) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$
- (e) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \sqrt{4n^2+n} - 2n = \lim\limits_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4}+2}$
- (f) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \sqrt{n^2+n} - \sqrt{n^2+1} = \lim\limits_{n\to\infty} \frac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}} = \frac{1}{2}$

2.
- (a) Since, $0 \leq \frac{1}{n}\sin^2 n \leq \frac{1}{n} \quad \forall n \in \mathbf{N}$ and since $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$
  Therefore by Sandwich theorem $\lim \frac{1}{n}\sin^2 n = 0.$

- (b) Since, $\frac{n}{(n+n)^2} \leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \leq \frac{n}{(n+1)^2} \quad \forall n \in \mathbf{N},$
  since $\lim\limits_{n\to\infty} \frac{n}{(n+n)^2} = 0.$ and $\lim\limits_{n\to\infty} \frac{n}{(n+1)^2} = 0.$

  Therefore by Sandwich theorem $\lim \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0.$

- (c) Since $\frac{n}{n^3+n} + \frac{2n}{n^3+n} + \cdots + \frac{n^2}{n^3+n} \leq \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n} \leq \frac{n}{n^3+1} + \frac{2n}{n^3+1} + \cdots + \frac{n^2}{n^3+1}$

  $\implies (1+2+3+...n)\frac{n}{n^3+n} \leq \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n} \leq (1+2+3+...n)\frac{n}{n^3+1}$

  $\implies \frac{n(n+1)}{2}\frac{n}{n^3+n} \leq \frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n} \leq \frac{n(n+1)}{2}\frac{n}{n^3+1}$

  Now, $\lim\limits_{n\to\infty} \frac{n(n+1)}{2}\frac{n}{n^3+n} = \frac{1}{2}$ and $\lim\limits_{n\to\infty} \frac{n(n+1)}{2}\frac{n}{n^3+1} = \frac{1}{2}$

  Therefore by Sandwich theorem $\lim \left[\frac{n}{n^3+1} + \frac{2n}{n^3+2} + \cdots + \frac{n^2}{n^3+n}\right] = \frac{1}{2}$

- (d) $\sqrt[n]{(a^n+b^n)} = b\Big\{(\frac{a}{b})^{\frac{1}{n}} + 1\Big\} > b \quad \forall n \in \mathbf{N}$
  Again, $0 < a < b \implies a^n < b^n \implies a^n + b^n < 2b^n \implies \sqrt[n]{(a^n+b^n)} < 2^{\frac{1}{n}}b$

  There fore, $b < \sqrt[n]{(a^n+b^n)} < 2^{\frac{1}{n}}b$

  Now, $\lim\limits_{n\to\infty} 2^{\frac{1}{n}}b = b$

Therefore by Sandwich theorem $\lim \sqrt[n]{(a^n + b^n)} = b$, where $0 < a < b$

3. $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$
   Now,
$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \geq \frac{2}{2n+2} - \frac{1}{n+1} = 0$$
$$\implies x_{n+1} \geq x_n \quad n \in \mathbf{N}$$
   $\implies$ {x$_n$} is monotonically increasing sequence.
   Also, $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \leq \frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n} = 1 \quad \forall n \in \mathbf{N}$

   $\implies$ the sequence $\{x_n\}$ is bounded above.

   Thus, $\{x_n\}$ is monotonically increasing and bounded above sequence.

   Therefore, using monotone convergence, $\{x_n\}$ is convergent.

4. Let $\epsilon > 0$ be given,
   Now,
$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$$
   If, we choose a natural number N such that $N > 2/\epsilon$.
   Then for $m, n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$, $\frac{1}{m} < \frac{\epsilon}{2}$.
   Hence,
$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   Therefore, we conclude that $\{\frac{1}{n}\}$ is a Cauchy sequence.

5. Let $\{x_n\} = \{ny^{n-1}\}$, where $y \in (0, 1)$
   Now,
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{y^n}{y^{n-1}} = \lim_{n\to\infty} \Big(1 + \frac{1}{n}\Big) y = y$$
   . Since, $0 < y < 1$, so $\{x_n\} = \{ny^{n-1}\} \to 0$.

6. $\{x_n\} = \{\frac{4^{3n}}{3^{4n}}\}$
   Now,
$$\lim_{n\to\infty} \sqrt[n]{x_n} = \frac{64}{81}$$
   . Since, $\frac{64}{81} < 1$, so $\{x_n\} = \{\frac{4^{3n}}{3^{4n}}\} \to 0$.

7. Let $\{x_n\} = \{n + 1\}$.
   Now,
$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{n+1} = \lim_{n\to\infty} \frac{1 + \frac{2}{n}}{1 + \frac{1}{n}} = 1$$
   . Since, $L = 1$ is a finite number, so $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{n+1} = 1$.

8. (a) False
   (b) False
   (c) False
   (d) True
   (e) False
   (f) True