EPHY105L

Solution to Tutorial Set - 2

(1) $(\hat{x}, \hat{y}, \hat{z}) \longrightarrow (\hat{r}, \hat{\theta}, \hat{\phi})$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

→ unit vector $\hat{r} = \dfrac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \left[\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right]^{1/2}$$

$$= \left[\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right]^{1/2}$$

$= 1$

Hence, $\hat{r} = \sin\theta \cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$ — (1)

→ unit vector $\hat{\theta} = \dfrac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|}$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\phi \, \hat{x} + r\cos\theta\sin\phi \, \hat{y} - r\sin\theta \, \hat{z}$$

$$\left| \frac{\partial \vec{r}}{\partial \theta} \right| = \left[r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta \right]^{1/2}$$

$$= \left[r^2\cos^2\theta\left(\cos^2\phi + \sin^2\phi\right) + r^2\sin^2\theta \right]^{1/2}$$

$$= r$$

Hence, $\hat{\theta} = \cos\theta\cos\phi \, \hat{x} + \cos\theta\sin\phi \, \hat{y} - \sin\phi \, \hat{z}$ — (2)

→ unit vector $\hat{\phi} = \dfrac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|}$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left[r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi\right]^{1/2}$$

$$= \left[r^2\sin^2\theta\left(\sin^2\phi + \cos^2\phi\right)\right]^{1/2}$$

$$= r\sin\theta$$

Hence, $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$ — 3

⊗ $(\hat{r}, \hat{\theta}, \hat{\phi}) \longrightarrow (\hat{x}, \hat{y}, \hat{z})$

Step 1: 1 $\times \sin\theta$ + 2 $\times \cos\theta$

$\rightarrow \hat{r}\sin\theta + \hat{\theta}\cos\theta = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ — 4

Step 2: 3 $\times \cos\phi$ + 4 $\times \sin\phi$

$\rightarrow \hat{\phi}\cos\phi + \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi$

$$= \cos^2\phi\,\hat{y} + \sin^2\phi\,\hat{y} = \hat{y}$$

$$\Rightarrow \hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

Step 3: Replace $\hat{y}$ in (3)

$$\rightarrow \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\left(\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi\right)$$

$$\Rightarrow \sin\phi\,\hat{x} = \hat{r}\sin\theta\sin\phi\cos\phi + \hat{\theta}\cos\theta\sin\phi\cos\phi + \hat{\phi}\left(\cos^2\phi - 1\right)$$

$$= \hat{r}\sin\theta\sin\phi\cos\phi + \hat{\theta}\cos\theta\sin\phi\cos\phi - \hat{\phi}\sin^2\phi$$

$$\Rightarrow \hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

Step 4: $\dfrac{(1)}{\sin\theta} - \dfrac{(2)}{\cos\theta}$

$$\rightarrow \frac{\hat{r}}{\sin\theta} - \frac{\hat{\theta}}{\cos\theta} = \frac{\cos\theta}{\sin\theta}\hat{k} + \frac{\sin\theta}{\cos\theta}\hat{k}$$

$$= \frac{1}{\sin\theta\cos\theta}\hat{k}$$

$$\Rightarrow \quad \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

An alternate solution

If the expressions for $\hat{x}$, $\hat{y}$ and $\hat{z}$ are given, i.e,

$$\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

We can write,

$$\hat{r} = \alpha_1\hat{x} + \alpha_2\hat{y} + \alpha_3\hat{z}$$

$$\hat{\theta} = \beta_1\hat{x} + \beta_2\hat{y} + \beta_3\hat{z}$$

$$\hat{\phi} = \gamma_1\hat{x} + \gamma_2\hat{y} + \gamma_3\hat{z}$$

$$\Rightarrow \alpha_1 = \hat{r}\cdot\hat{x} = \sin\theta\cos\phi$$

$$\alpha_2 = \hat{r}\cdot\hat{y} = \sin\theta\sin\phi$$

$$\alpha_3 = \hat{r}\cdot\hat{z} = \cos\theta$$

$$\beta_1 = \hat{\theta}\cdot\hat{x} = \cos\theta\cos\phi$$

$$\beta_2 = \hat{\theta}\cdot\hat{y} = \cos\theta\sin\phi$$

$$\beta_3 = \hat{\theta}\cdot\hat{z} = -\sin\theta$$

$$\gamma_1 = \hat{\phi}\cdot\hat{x} = -\sin\phi$$

$$\gamma_2 = \hat{\phi}\cdot\hat{y} = \cos\phi$$

$$\gamma_3 = \hat{\phi}\cdot\hat{z} = 0$$

Similarly, given that

$$\hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$
$$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$

we can write,

$$\hat{x} = \alpha_1\hat{r} + \alpha_2\hat{\theta} + \alpha_3\hat{\phi}$$
$$\hat{y} = \beta_1\hat{r} + \beta_2\hat{\theta} + \beta_3\hat{\phi}$$
$$\hat{z} = \gamma_1\hat{r} + \gamma_2\hat{\theta} + \gamma_3\hat{\phi}$$

$\Rightarrow \alpha_1 = \hat{x}\cdot\hat{r} = \sin\theta\cos\phi$

$\alpha_2 = \hat{x}\cdot\hat{\theta} = \cos\theta\cos\phi$

$\alpha_3 = \hat{x}\cdot\hat{\phi} = -\sin\phi$

$\beta_1 = \hat{y}\cdot\hat{r} = \sin\theta\sin\phi$

$\beta_2 = \hat{y}\cdot\hat{\theta} = \cos\theta\sin\phi$

$\beta_3 = \hat{y}\cdot\hat{\phi} = \cos\phi$

$\gamma_1 = \hat{z}\cdot\hat{r} = \cos\theta$

$\gamma_2 = \hat{z}\cdot\hat{\theta} = -\sin\theta$

$\gamma_3 = \hat{z}\cdot\hat{\phi} = 0$

② $(\hat{x}, \hat{y}, \hat{z}) \longrightarrow (\hat{\rho}, \hat{\phi}, \hat{z})$

$$\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}\right\} \vec{r} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\,\hat{z}$$

$\rightarrow$ unit vector $\hat{\rho} = \dfrac{\frac{\partial \vec{r}}{\partial \rho}}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|}$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\left|\frac{\partial \vec{r}}{\partial \rho}\right| = \left[\cos^2\phi + \sin^2\phi\right]^{1/2} = 1$$

Hence, $\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ — ①

$\rightarrow$ unit vector $\hat{\phi} = \dfrac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$

$$\frac{\partial \vec{r}}{\partial \phi} = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left[s^2 \sin^2\phi + s^2 \cos^2\phi \right]^{1/2}$$
$$= s$$

Hence, $\hat{\phi} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y}$ — (2)

$\Rightarrow$ unit vector $\hat{z} = \dfrac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \hat{z}$ — (3)

(x) $(\hat{s}, \hat{\phi}, \hat{z}) \longrightarrow (\hat{x}, \hat{y}, \hat{z})$

Step 1: (1) $\times \sin\phi$ + (2) $\times \cos\phi$

$\rightarrow \hat{s}\sin\phi + \hat{\phi}\cos\phi = (\sin^2\phi + \cos^2\phi)\,\hat{y}$

$\Rightarrow \hat{y} = \hat{s}\sin\phi + \hat{\phi}\cos\phi$

Step 2: (1) $\times \cos\phi$ − (2) $\times \sin\phi$

$\rightarrow \hat{s}\cos\phi - \hat{\phi}\sin\phi = (\cos^2\phi + \sin^2\phi)\,\hat{x}$

$\Rightarrow \hat{x} = \hat{s}\cos\phi - \hat{\phi}\sin\phi$

** following the same steps in the previous problem you also can find one set of unit vectors given the other set.

(3).

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} r &= \left[x^2 + y^2 + z^2\right]^{1/2} \\ \phi &= \tan^{-1}(y/x) \\ \theta &= \cos^{-1}(z/r) \end{aligned}$$

(a) $x = 10,\ y = 0,\ z = 0$

$$\Rightarrow \quad r = [100 + 0 + 0]^{1/2} = 10$$

$$\theta = \cos^{-1}(0/10) = \frac{\pi}{2}$$

$$\phi = \tan^{-1}(0/10) = 0$$

$\rightarrow$ unit vector $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$$= \hat{x}$$

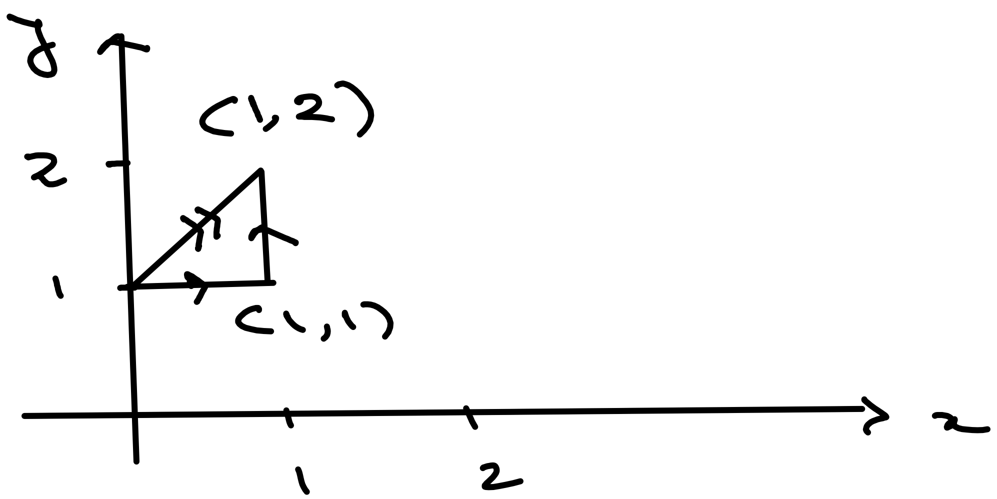
*** Follow the same steps for the rest.

(4) (a)

$$\left.\begin{aligned} r &= 5 \\ \theta &= \frac{\pi}{2} \\ \phi &= \frac{\pi}{4} \end{aligned}\right\} \quad \begin{aligned} x &= r \sin\theta \cos\phi = 5/\sqrt{2} \\ y &= r \sin\theta \sin\phi = 5/\sqrt{2} \\ z &= 0 \end{aligned}$$

*** Follow the same steps for the rest.

(5)



$$\vec{F} = (x^2 - y)\hat{x} + (y^2 + x)\hat{y}$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$$

$$\vec{F}\cdot d\vec{l} = (x^2 - y)dx + (y^2 + x)dy$$

(a) Along straight line from $(0,1)$ to $(1,2)$

Straight line described by

$$y = \frac{2-1}{1-0}x + 1 = x + 1$$

$$\Rightarrow dy = dx$$

Hence, $\vec{F}\cdot d\vec{l} = (x^2 - (x+1))\,dx + ((x+1)^2 + x)\,dx$

$= (x^2 - x - 1 + x^2 + 2x + 1 + x)\, dx$

$= (2x^2 + 2x)\, dx$

Line integral, $\int \vec{F} \cdot d\vec{l} = 2\int_0^1 (x^2 + x)\, dx$

$= 2\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$

$= 2\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{3}$

(b) Along straight line from $(0,1)$ to $(1,1)$

$\rightarrow y = 1 \equiv$ constant $\Rightarrow dy = 0$

Hence, $\int \vec{F} \cdot d\vec{l} = \int_0^1 (x^2 - 1)\, dx$

$= \left[\frac{x^3}{3} - x\right]_0^1 = \left(\frac{1}{3} - 1\right) = -\frac{2}{3}$

Along straight line from $(1,1)$ to $(1,2)$

$\rightarrow x = 1 \equiv$ constant $\Rightarrow dx = 0$

Hence, $\int \vec{F} \cdot d\vec{l} = \int_1^2 (y^2 + 1)\, dy$

$$= \left[\frac{y^3}{3} + y \right]_1^2$$

$$= \left(\frac{8}{3} + 2 - \frac{1}{3} - 1 \right)$$

$$= \frac{10}{3}$$

Hence, $\int \vec{F} \cdot d\vec{l} = -\frac{2}{3} + \frac{10}{3} = \frac{8}{3}$

(6) $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$

$d\vec{a} = dx\, dy\, \hat{z}$

$\Rightarrow \vec{F} \cdot d\vec{a} = z\, dx\, dy$

$\Rightarrow \int \vec{F} \cdot d\vec{a} = \int z\, dx\, dy = 10 \int dx\, dy$

The area is the area of a circular disc of radius $r = 5$ unit

In spherical polar coordinate, the surface integral reduces to

$$\int_0^5 \int_0^{2\pi} r \, dr \, d\phi = \pi (5)^2$$

$$\Rightarrow \int \vec{F} \cdot d\vec{a} = 10 \pi (5)^2$$
$$= 250\pi$$

(7) Volume integral $= \int T \, d\tau$

$$= \int 8xyz \, dx \, dy \, dz$$

Given the configuration, x, y, z all varies from 0 to 1. Hence

$$\int T \, d\tau = 8 \int_0^1 \int_0^1 \int_0^1 xyz \, dx \, dy \, dz$$

$$= 8 \int_0^1 \int_0^1 \left[\frac{x^2}{2} yz \right]_0^1 dy \, dz$$

$$= 8 \int_0^1 \int_0^1 \frac{1}{2} yz \, dy \, dz$$

$$= 4 \int_0^1 \left[\frac{y^2}{2} z \right]_0^1 dz$$

$$= 2 \int_0^1 z \, dz$$

$$= 2 \left[\frac{z^2}{2} \right]_0^1$$

$$= 1$$

(8) An infinitesimal surface element on the surface of a sphere $= R^2 \sin\theta \, d\theta \, d\phi$

($R \equiv$ radius of the sphere)

The direction of the area vector $\equiv \hat{r}$

Hence, $d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

Area of the sphere from $\theta = 0$ to $\theta = \theta_0$

$$A = \int_0^{2\pi} d\phi \int_0^{\theta_0} d\theta \, R^2 \sin\theta$$

$$= -2\pi R^2 \cos\theta \Big|_0^{\theta_0}$$

$$= 2\pi R^2 (1 - \cos\theta_0)$$

for $\theta_0 = 90^\circ$, $\quad A = 2\pi R^2 (1 - 0)$

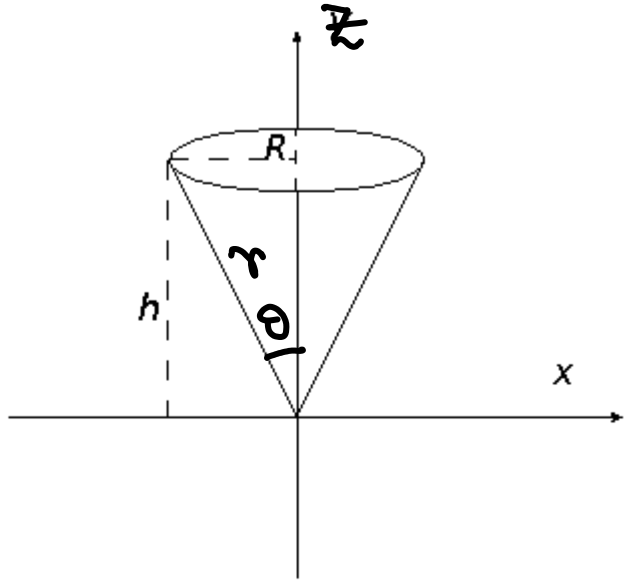
(surface area of one hemisphere) $\quad = 2\pi R^2$

for $\theta_0 = 180^\circ$, $\quad A = 2\pi R^2 (1 - (-1))$

$\quad = 4\pi R^2$

(surface area of a sphere)

(9)



$$\sin\theta = \frac{R}{\sqrt{R^2+h^2}}$$

$$\left[\begin{array}{l} r^2 = R^2 + h^2 \\ \Rightarrow r = \sqrt{R^2+h^2} \end{array}\right]$$

An infinitesimal area on the surface of the cone:

$$da = r\sin\theta \, dr \, d\phi$$

$\Rightarrow$ The area of the curved surface:

$$\int_0^{2\pi} d\phi \int_0^{\sqrt{R^2+h^2}} r\sin\theta \, dr = 2\pi \frac{R}{\sqrt{R^2+h^2}} \left.\frac{r^2}{2}\right|_0^{\sqrt{R^2+h^2}}$$

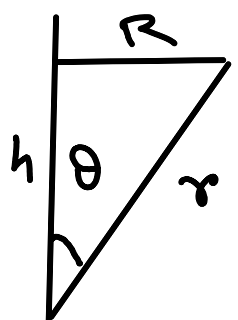
$$= \pi R\sqrt{R^2+h^2}$$

(*) for a closed cone, total surface area $= \pi R^2 + \pi R\sqrt{h^2+R^2}$

$$\text{Volume} = \int_0^{2\pi} d\phi \int_0^{\tan^{-1}\left(\frac{R}{h}\right)} d\theta \int_0^{h\sec\theta} r^2 \sin\theta \, dr$$

Note that,

$\cos\theta = \frac{h}{r}$



$\tan\theta = \frac{R}{h}$

$$= 2\pi \int_0^{\tan^{-1}\left(\frac{R}{h}\right)} \left[\frac{r^3}{3}\right]_0^{h\sec\theta} \sin\theta \, d\theta$$

$$= \frac{2\pi}{3} \int_0^{\tan^{-1}\left(\frac{R}{h}\right)} h^3 \sec^3\theta \sin\theta \, d\theta$$

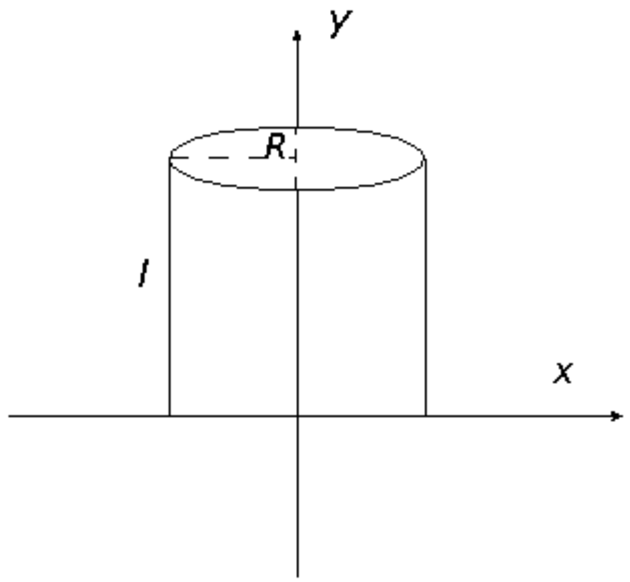
$$= \frac{2\pi h^3}{3} \int_0^{\tan^{-1}\left(\frac{R}{h}\right)} \sec^2\theta \tan\theta \, d\theta$$

Let
$x = \tan\theta$
$\Rightarrow dx = \sec^2\theta \, d\theta$

$$= \frac{2\pi h^3}{3} \int_0^{R/h} x \, dx = \frac{2\pi h^3}{3} \frac{R^2}{2h^2}$$

$$= \frac{1}{3} \pi R^2 h$$

(10)



In cylindrical coordinate, infinitesimal surface element:

$$ds = R\,d\phi\,dz$$

Total curved surface area $= \int_0^l \int_0^{2\pi} R\,d\phi\,dz$

$= 2\pi R l$

* For a closed cylinder, total surface area

$= 2\pi R l + 2\pi R^2$