Mid-term

1)

(a) $\displaystyle\int_{1,0,1}^{2,0,2} \vec{\nabla} f \cdot \vec{dl} = \quad \Big| \quad f = x^2 + y^2 + z^2$

Method - 1

$$\vec{\nabla} f \cdot \vec{dl} = \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right) \cdot \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= df \longrightarrow (1 \text{ mark})$$

$$\int_{1,0,1}^{2,0,2} df = f(2,0,2) - f(1,0,1)$$

$$= \underline{8 - 2 = 6} \longrightarrow (3 \text{ marks})$$

$$\boxed{\begin{aligned} f(2,0,2) &= 4 + 0 + 4 = 8 \\ f(1,0,1) &= 1 + 0 + 1 = 2 \end{aligned}}$$

Method - 2

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$= 2x\,\hat{x} + 2y\,\hat{y} + 2z\,\hat{z} \quad \text{— } (0.5 \text{ mark})$$

$$\vec{\nabla} f \cdot \vec{dl} = \left(2x\,dx + 2y\,dy + 2z\,dz\right) \quad \text{— } (1 \text{ mark})$$

$$\int_{1,0,1}^{2,0,2} \vec{\nabla} f \cdot \vec{dl} = \int_1^2 2x\,dx + \int_0^0 2y\,dy + \int_1^2 2z\,dz$$

$$= x^2 \Big|_1^2 + z^2 \Big|_1^2$$

$$= \underline{3 + 3 = 6}$$ —— (3 marks)

Note: In this problem $x = z$
$dx = dz$, $dy = 0$

Hence,

$$\int (2x\,dx + 2y\,dy + 2z\,dz)$$

$$= \int_1^2 4x\,dx \quad \text{or} \quad \int_1^2 4z\,dz$$

$$= \underline{\underline{6}}$$

1. (b)

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

For this

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$ —— (1 mark)

$$(x, y, z) = (1, 1, 0)$$

~~For this~~ $\Rightarrow r = \sqrt{2}$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{2}}\right) = \pi/2$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$ —— (2 marks)

(Q2)
(a) $\vec{E} = (axy+z)\,\hat{x} + 3x^2\,\hat{y} + x\,\hat{z}$

$\vec{\nabla}\times\vec{E}$ should be equal to zero for an electrostatic field. $\longrightarrow$ 0.5 Mark

$$\Rightarrow \vec{\nabla}\times\vec{E} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+z & 3x^2 & x \end{vmatrix} = \hat{x}\left(\frac{\partial x}{\partial y} - \frac{\partial}{\partial z}(3x^2)\right) - \hat{y}\left(\frac{\partial x}{\partial x} - \frac{\partial(axy+z)}{\partial z}\right) + \hat{z}\left(\frac{\partial(3x^2)}{\partial x} - \frac{\partial}{\partial y}(axy+z)\right)$$

$\longrightarrow$ 1 Mark

$$= \hat{x}(0-0) - \hat{y}(1-1) + \hat{z}(6x - ax) = 0$$

$$\Rightarrow 6x - ax = 0$$

$$\Rightarrow \boxed{a = 6}$$ $\longrightarrow$ 2 Marks

(b) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0(\vec{\nabla}\cdot\vec{E})$ $\longrightarrow$ 0.5 Mark

$$\Rightarrow \rho = \epsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$ $\longrightarrow$ 1 mark

$$= \epsilon_0(ay + 0 + 0)$$

$$\Rightarrow \boxed{\rho = \epsilon_0 a y}$$ $\longrightarrow$ ~~1~~.5 mark

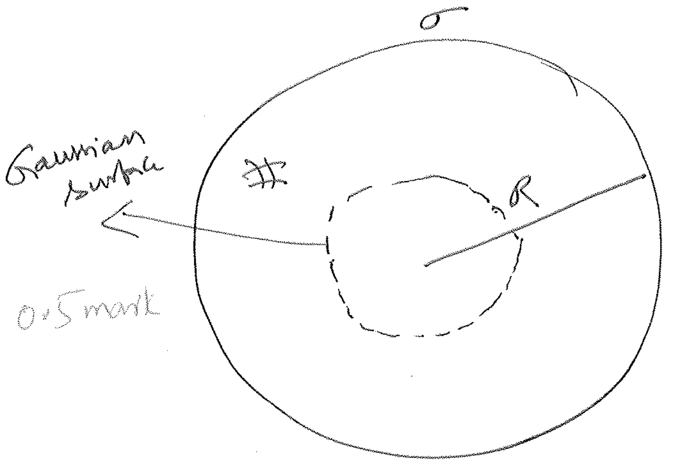
At $(2,2,2)$ $\boxed{\rho = 2\epsilon_0 a \text{ or } 12\epsilon_0}$ $\longrightarrow$ 2 mark

(c) C/m^3 or Coulomb/meter^3 $\longrightarrow$ 1 mark

(Q3)

(a) $\sigma =$ constant

Inside the shell:



Gauss's Law

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \rightarrow 0.5 \text{ mark}$$

Here $Q_{enc} = 0$

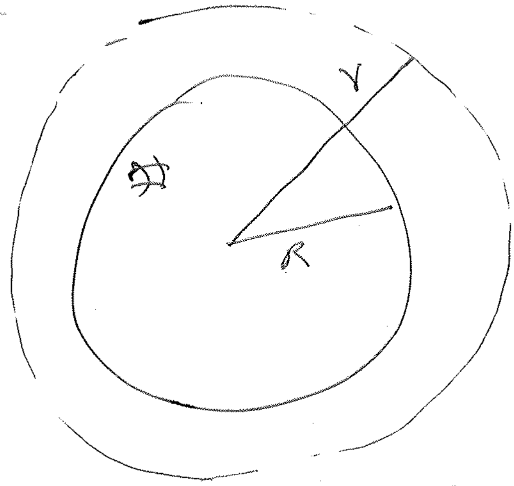
$\Rightarrow \oint \vec{E}\cdot d\vec{a} = 0,$

and $\vec{E}\cdot d\vec{a} = E\,da$ and E is constant on the Gaussian surface

$\Rightarrow EA = 0$

$$\Rightarrow \boxed{\vec{E} = 0} \rightarrow 1.5 \text{ mark}$$

Outside the shell:



$Q_{enc} = \sigma 4\pi R^2 \rightarrow 2 \text{ marks}$

and $\oint \vec{E}\cdot d\vec{a} = \oint E\,da$

$= E\oint da$ [$\because$ E is constant on the Gaussian surface]

$= E 4\pi r^2$

$\Rightarrow E 4\pi r^2 = \frac{\sigma 4\pi R^2}{\epsilon_0}$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{r^2} \rightarrow 2.5 \text{ marks}$$

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{r^2}\hat{r}} \rightarrow 3 \text{ marks}$$

or

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}$$ where $Q = \sigma 4\pi R^2$

(b)

$$V = -\int_{\infty}^{\vec{r}} \vec{E}\cdot\vec{dl}$$

Inside the shell: $\vec{E}_{in} = 0$ $(r < R)$

$\Rightarrow V = \text{constant}$ → 0.5 marks

$$V = -\int_{\infty}^{R} \vec{E}_{out}\cdot\vec{dl}$$

$\vec{E}_{out} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$ and $\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\Rightarrow V = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = -\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_{\infty}^{R}$$

$$\Rightarrow \boxed{V = \frac{Q}{4\pi\epsilon_0}\frac{1}{R}}$$ → 1 mark

Outside the shell: $V = -\int_{\infty}^{r} \vec{E}_{out}\cdot\vec{dl}$

$$\Rightarrow V = -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{1}{r^2}dr$$

$$\Rightarrow V = \frac{-Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_{\infty}^{r}$$

$$\Rightarrow \boxed{V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}}$$ → 2 marks

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Alternative: potential can be calculated from the charge distribution $V_{inside} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$ and $V_{outside} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

$\vec{E} = -\vec{\nabla}V \Rightarrow \vec{E}_{inside} = 0$ and $\vec{E}_{outside} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$

Q4) (a) The induced charge on the outer surface is uniformly distributed

$$\Rightarrow \sigma = \frac{-Q}{4\pi R^2} \longrightarrow \text{0.5 mark}$$

$$\Rightarrow \sigma = \frac{-1\,nC}{4\pi(0.1)^2\,m^2} = \frac{1}{4\pi \times 0.01}\ nC/m^2$$

$$\Rightarrow \boxed{\sigma = \frac{-100}{4\pi}\ nC/m^2} \longrightarrow \text{1 mark}$$

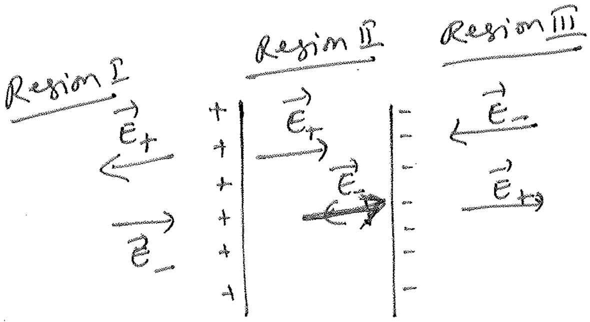
Sign is important $\sigma = +\frac{Q}{4\pi R^2} \longrightarrow$ 0 mark

(b)

$$C = \frac{Q}{V} = \frac{0.03}{6}\ C/V = 0.005\ C/V \longrightarrow \text{1 mark}$$

$$V = \frac{Q}{C} = \frac{2}{0.005}\ V = \boxed{400\ V} \longrightarrow \text{2 mark}$$

(c)



(i) To the left:

$$\vec{E}_+ = -\frac{\sigma}{2\epsilon_0}\hat{n}$$

$$\vec{E}_- = \frac{\sigma}{2\epsilon_0}\hat{n}$$

$$\Rightarrow \vec{E} = \vec{E}_+ + \vec{E}_- = 0 \longrightarrow \text{0.5 marks}$$

(ii) Between the planes: $\vec{E}_+ = \frac{\sigma}{2\epsilon_0}\hat{n}$ and $\vec{E}_- = \frac{\sigma}{2\epsilon_0}\hat{n}$

$$\Rightarrow \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\sigma}{\epsilon_0}\hat{n} \longrightarrow \text{1.5 marks}$$

(iii) To the right: $\vec{E}_+ = \frac{\sigma}{2\epsilon_0}\hat{n}$ and $\vec{E}_- = -\frac{\sigma}{2\epsilon_0}\hat{n}$

$$\Rightarrow \vec{E} = \vec{E}_+ + \vec{E}_- = 0 \longrightarrow \text{2 marks}$$