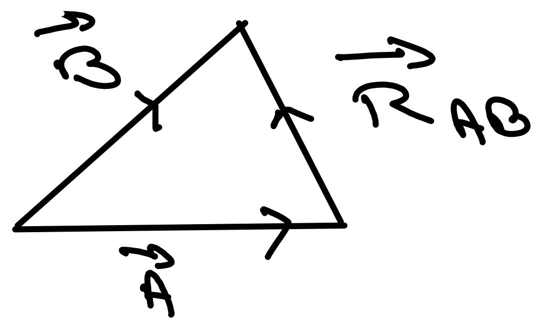
EPHY105L

Solution to Tutorial Set-1

(1) 

$$\vec{A} = 6\hat{x} - \hat{y} + 2\hat{z}$$
$$\vec{B} = -2\hat{x} + 3\hat{y} - 4\hat{z}$$
$$\vec{C} = -3\hat{x} + \hat{y} + 5\hat{z}$$

$$\vec{A} + \vec{R}_{AB} = \vec{B}$$

$$\Rightarrow \vec{R}_{AB} = \vec{B} - \vec{A}$$

Similarly, $\vec{R}_{AC} = \vec{C} - \vec{A}$

(a) $$\vec{R}_{AB} = -8\hat{x} + 4\hat{y} - 6\hat{z}$$

$$\vec{R}_{AC} = -9\hat{x} + 2\hat{y} + 3\hat{z}$$

(b) θ_{BAC} : $$\vec{R}_{AB} \cdot \vec{R}_{AC} = |\vec{R}_{AB}||\vec{R}_{AC}| \cos\theta_{BAC}$$

$$\Rightarrow \cos\theta_{BAC} = \frac{\vec{R}_{AB} \cdot \vec{R}_{AC}}{|\vec{R}_{AB}|\,|\vec{R}_{AC}|}$$

$$= \frac{72 + 8 - 18}{\sqrt{64+16+36}\ \sqrt{81+4+9}}$$

$$= \frac{62}{\sqrt{116} \cdot \sqrt{94}}$$

$$\Rightarrow \theta_{BAC} = \cos^{-1}\left(\frac{62}{\sqrt{116} \cdot \sqrt{94}}\right)$$

$$= 53.6^{\circ}$$

2

$$\vec{a} = \hat{x} + 3\hat{y}$$

$$\vec{b} = \hat{x} - 3\hat{y}$$

$z = 0$ for both.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & +3 & 0 \\ 1 & -3 & 0 \end{vmatrix} = 6\hat{z}$$

The area of parallelogram $= |\vec{a} \times \vec{b}|$

$= 6$ unit.

(3) $\vec{u} = \hat{x} + 3\hat{y}$

$\vec{v} = \hat{x} - 3\hat{y}$

$\vec{w} = -\hat{x} - \hat{y} - \hat{z}$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & 3 & 0 \\ 1 & -3 & 0 \\ -1 & -1 & -1 \end{vmatrix}$$

$= 6$

The volume of a parallelopiped $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

$= 6$ unit.

(4) From P, Q and R we can construct two vectors: $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$$\overrightarrow{PQ} = 2\hat{y} + 3\hat{z}$$

$$\overrightarrow{PR} = \hat{x} + 2\hat{y} + 2\hat{z}$$

A vector normal to the plane:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 2 & 3 \\ 1 & 2 & 2 \end{vmatrix} = -2\hat{x} + 3\hat{y} - 2\hat{z}$$

(5) a) $f(x, y, z) = x^2 + y^2 + z^2 - 9$

$\vec{\nabla} f = \hat{x}(2x) + \hat{y}(2y) + \hat{z}(2z)$

at $(2,-1,2)$, $\vec{\nabla} f = 4\hat{x} - 2\hat{y} + 4\hat{z}$

b) $f(x,y,z) = x^2 + y^2 - z - 3$

$\vec{\nabla} f = \hat{x}(2x) + \hat{y}(2y) - \hat{z}$

at $(2,-1,2)$, $\vec{\nabla} f = 4\hat{x} - 2\hat{y} - \hat{z}$

6 $f(x,y,z) = x^2 y z^3$

$\vec{\nabla} f = \hat{x}(2xyz^3) + \hat{y}(x^2 z^3) + \hat{z}(3x^2 y z^2)$

at $(2,1,-1)$, $\vec{\nabla} f = -4\hat{x} - 4\hat{y} + 12\hat{z}$

Maximum directional derivative $= |\vec{\nabla} f|$

$= \sqrt{16+16+144}$

$\simeq 13.3$

⑦ a) $\vec{F}_1 = x\hat{x} - y\hat{y}$

$$\vec{\nabla}\cdot\vec{F}_1 = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) = 0$$

b) $\vec{F}_2 = \hat{z}\, z$

$$\vec{\nabla}\cdot\vec{F}_2 = \frac{\partial}{\partial z}(z) = 1$$

c) $\vec{F}_3 = \alpha\,(\hat{x}\, x + \hat{y}\, y + \hat{z}\, z)$

$$\vec{\nabla}\cdot\vec{F}_3 = \alpha\,(1 + 1 + 1) = 3\alpha$$

d) $\vec{F}_4 = \beta \dfrac{(\hat{x}\, x + \hat{y}\, y + \hat{z}\, z)}{(x^2 + y^2 + z^2)^{3/2}}$

To find $\vec{\nabla}\cdot\vec{F}_4$,

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} + \left(-\frac{3}{2}\right)\frac{x \cdot 2x}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{1}{r^3} - \frac{3x^2}{r^5}$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)$

$$= \frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{1}{r^3} - \frac{3z^2}{r^5}$$

Hence,

$$\vec{\nabla} \cdot \left(\beta \frac{\vec{r}}{r^3}\right) = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}$$

$$= \frac{3}{r^3} - \frac{3}{r^5} \cdot r^2$$

$$= 0$$

8 a) $\vec{F}_1 = \hat{x}\,\alpha y$

$$\vec{\nabla} \times \vec{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y & 0 & 0 \end{vmatrix} = \hat{z}\left(0 - \frac{\partial}{\partial y}(\alpha y)\right)$$

$$= -\hat{z}\,\alpha$$

b) $\vec{F}_2 = \hat{x}\,\alpha x + \hat{y}\,\beta y^2$

$$\vec{\nabla} \times \vec{F}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x & \beta y^2 & 0 \end{vmatrix} = 0$$

c) $\vec{F}_3 = \hat{x}\, x^2 + \hat{y}\, 3xz^2 - \hat{z}\, 2xz$

$$\vec{\nabla} \times \vec{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{x}\left(- \quad 6xz\right) + \hat{y}\left(+\ 2z\right) + \hat{z}\left(3z^2\right)$$

$$= -\hat{x}\ 6xz + \hat{y}\ 2z + \hat{z}\ 3z^2$$

(9) $f(x, y, z) = \alpha x y^2$

a) $\vec{\nabla} f = \hat{x}\left(\alpha y^2\right) + \hat{y}\left(2\alpha x y\right)$

b) $$\vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y^2 & 2\alpha x y & 0 \end{vmatrix} = \hat{z}\left(2\alpha y - 2\alpha y\right)$$

$$= 0$$

(10) $\vec{G} = \hat{x}\, x^2 + \hat{y}\, 3xz^2 - \hat{z}\, 2xz$

a) $\vec{\nabla} \times \vec{G} = -\hat{x}\, 6xz + \hat{y}\, 2z + \hat{z}\, 3z^2$

(See solution to 8(e))

b) $\vec{A} = \vec{\nabla} \times \vec{G}$

$$\vec{\nabla} \cdot \vec{A} = -\frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2)$$

$$= -6z + 6z$$

$$= 0$$

(11) $f(x, y, z) = x^3 + y^2 + z$

Divergence of curl of a vector is always zero. $\vec{\nabla} f$ is a vector. Hence,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\nabla} f) = 0$$

***You can check the result explicitly