Lecture - 7

Fundamental theorem of Gradient

$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$ the right side of this equation makes no reference to the path---only to the end points.
Thus gradients have special property that their line integrals are path independent.

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b})$-$T(\mathbf{a})$=0.

A conservative force may be associated with a scalar potential energy function, whereas a non-conservative force cannot.

Potential energy defined in terms of work done by the associated conservative force.

$$U_B - U_A = -\int_A^B \mathbf{F}_c \cdot d\mathbf{s}$$

*Conservative forces tend to ***minimize*** the potential energy within any system: It allowed to, an apple falls to the ground and a spring returns to its natural length.

Non-conservative force does not imply it is dissipative, for example, magnetic force, and also does not mean it will decrease the potential energy, such as hand force.

The distinction between conservative and non-conservative forces is best stated as follows:

A conservative force may be associated with a scalar potential energy function, whereas a non-conservative force cannot.

$$U_B - U_A = -\int_A^B \mathbf{F}_c \cdot d\mathbf{s}$$

$$\mathbf{F}_c = -\nabla U$$

How can we find a conservative force if the associated potential energy function is given?

A conservative force can be derived from a scalar potential energy function.

$$\mathbf{F}_c = -\nabla U$$

The negative sign indicates that the force points in the direction of ***decreasing*** potential energy.

$$\textbf{Gravity} \quad U_g = mgy; \qquad F_y = -\frac{dU_g}{dy} = -mg$$

$$\textbf{Spring} \quad U_{sp} = \frac{1}{2}kx^2; \qquad F_x = -\frac{dU_{sp}}{dx} = -kx$$

The fundamental theorem for divergences states that:

$$\int_{v} (\nabla \cdot \mathbf{v}) d\tau = \oint_{S} \mathbf{v} \cdot d\mathbf{a}$$

The integration of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume)

This theorem has at least three special names: **Gauss's theorem, Green's theorem,** or the **divergence theorem.**

Geometrical Interpretation: Measure the total amount of fluid passing out through the surface, per unit time.

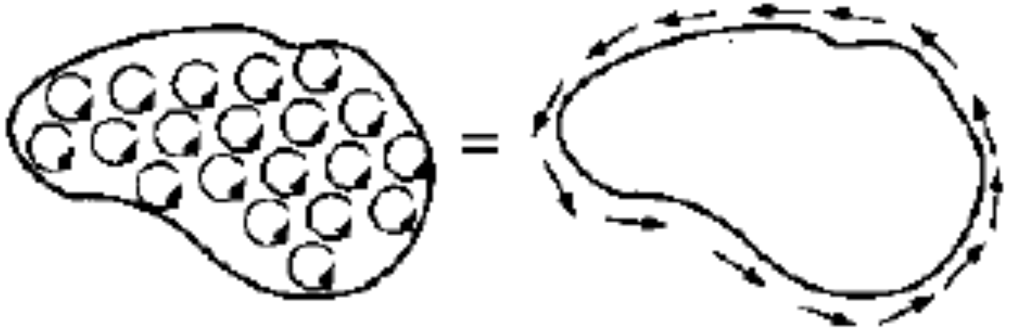
1. Count up all the faucets, recording how much each put out.
2. Go around the boundary, measuring the flow at each point, and add it all up.

The fundamental theorem for curls---**Stokes' theorem**---states that:

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

The integration of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (in this case the perimeter of the patch).

Geometrical Interpretation:
Measure the "twist" of the vectors $\mathbf{v}$; a region of high curl is a whirlpool.



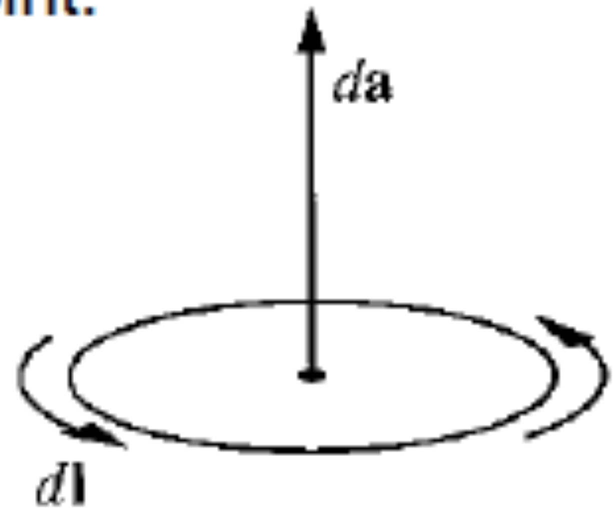
Stokes' theorem
(Transformation between surface integrals and line integrals)

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}$$

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary lines, not on the particular surface used.

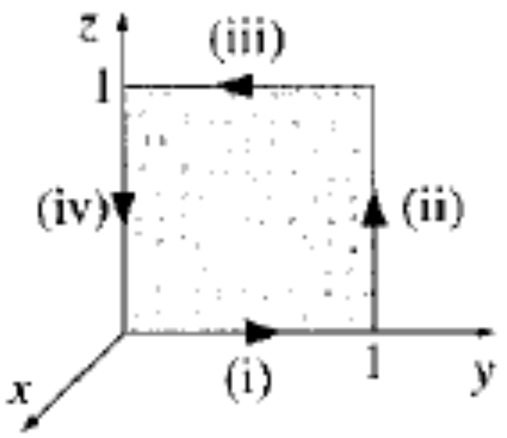
Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line shrinks down to a point.

These corollaries are analogous to those for the gradient theorem.



Example Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$

Check Stokes' theorem for the square surface shown below.



Sol : $\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}; \quad d\mathbf{a} = dydz\hat{\mathbf{x}}$

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2 dydz = \frac{4}{3}$$

The line integral of the four segments

(i) $x = 0, \quad z = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2\,dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 4z^2\,dz, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2\,dz = \frac{4}{3},$

(iii) $x = 0, \quad z = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2\,dy = -1,$

(iv) $x = 0, \quad y = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 0, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0\,dz = 0.$

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$