EPHY108L: Classical Mechanics

Tutorial 1 [SOLUTIONS]

1.

As is conventional for vector geometry problems, we denote the vector from the point A to the point B by $\mathbf{AB}$. If the position vectors of the points A and B, relative to some origin O, are $\mathbf{a}$ and $\mathbf{b}$, it should be clear that $\mathbf{AB} = \mathbf{b} - \mathbf{a}$.

Now, from figure 7.5 we see that one possible way of reaching the point P from O is first to go from O to A and to go along the line AB for a distance equal to the the fraction $\lambda/(\lambda + \mu)$ of its total length. We may express this in terms of vectors as

$$\begin{aligned}
\mathbf{OP} = \mathbf{p} &= \mathbf{a} + \frac{\lambda}{\lambda + \mu}\mathbf{AB} \\
&= \mathbf{a} + \frac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a}) \\
&= \left(1 - \frac{\lambda}{\lambda + \mu}\right)\mathbf{a} + \frac{\lambda}{\lambda + \mu}\mathbf{b} \\
&= \frac{\mu}{\lambda + \mu}\mathbf{a} + \frac{\lambda}{\lambda + \mu}\mathbf{b}, \qquad (7.6)
\end{aligned}$$

which expresses the position vector of the point P in terms of those of A and B. We would, of course, obtain the same result by considering the path from O to B and then to P. ◄

2.

From figure 7.6, the points D and E bisect the lines AB and AC respectively. Thus from the ratio theorem (7.6), with $\lambda = \mu = 1/2$, the position vectors of D and E relative to the origin are

$$\mathbf{d} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b},$$
$$\mathbf{e} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{c}.$$

Using the ratio theorem again, we may write the position vector of a general point on the line CD that divides the line in the ratio $\lambda : (1-\lambda)$ as

$$\begin{aligned}\mathbf{r} &= (1-\lambda)\mathbf{c} + \lambda\mathbf{d},\\ &= (1-\lambda)\mathbf{c} + \tfrac{1}{2}\lambda(\mathbf{a}+\mathbf{b}),\end{aligned} \tag{7.7}$$

where we have expressed $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Similarly, the position vector of a general point on the line BE can be expressed as

$$\begin{aligned}\mathbf{r} &= (1-\mu)\mathbf{b} + \mu\mathbf{e},\\ &= (1-\mu)\mathbf{b} + \tfrac{1}{2}\mu(\mathbf{a}+\mathbf{c}).\end{aligned} \tag{7.8}$$

Thus, at the intersection of the lines CD and BE we require, from (7.7), (7.8),

$$(1-\lambda)\mathbf{c} + \tfrac{1}{2}\lambda(\mathbf{a}+\mathbf{b}) = (1-\mu)\mathbf{b} + \tfrac{1}{2}\mu(\mathbf{a}+\mathbf{c}).$$

By equating the coefficents of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we find

$$\lambda = \mu, \qquad \tfrac{1}{2}\lambda = 1-\mu, \qquad 1-\lambda = \tfrac{1}{2}\mu.$$

These equations are consistent and have the solution $\lambda = \mu = 2/3$. Substituting these values into either (7.7) or (7.8) we find that the position vector of the centroid G is given by

$$\mathbf{g} = \tfrac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}). \blacktriangleleft$$

3.

The required relative velocity is given by

$$\begin{aligned}\mathbf{u} = \mathbf{v}_2 - \mathbf{v}_1 &= (1-1)\mathbf{i} + (0-3)\mathbf{j} + (-2-6)\mathbf{k}\\ &= -3\mathbf{j} - 8\mathbf{k}. \blacktriangleleft\end{aligned}$$

4.

Denote the four position vectors by **a**, **b**, **c**, **d**. As none of the three pairs of lines actually intersect, it is difficult to indicate their orthogonality in the diagram we would normally draw. However, the orthogonality can be expressed in vector form and we start by noting that, since $AD \perp BC$, it follows from (7.16) that

$$(\mathbf{d}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b}) = 0.$$

Similarly, since $BD \perp AC$,

$$(\mathbf{d}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a}) = 0.$$

Combining these two equations we find

$$(\mathbf{d}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b}) = (\mathbf{d}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a}),$$

which, on mutliplying out the parentheses, gives

$$\mathbf{d}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}-\mathbf{d}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b} = \mathbf{d}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-\mathbf{d}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{a}.$$

Cancelling terms that appear on both sides and rearranging yields

$$\mathbf{d}\cdot\mathbf{b}-\mathbf{d}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{a} = 0,$$

which simplifies to give

$$(\mathbf{d}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a}) = 0.$$

From (7.16), we see that this implies that CD is perpendicular to AB. ◄

5.

the cosine of the angle θ between **a** and **b** is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

$$\mathbf{a}\cdot\mathbf{b} = 1\times 2 + 2\times 3 + 3\times 4 = 20,$$

$$|\mathbf{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{2^2+3^2+4^2} = \sqrt{29}.$$

Thus,

$$\cos\theta = \frac{20}{\sqrt{14}\sqrt{29}} \approx 0.9926 \qquad \Rightarrow \qquad \theta = 0.12 \text{ rad.} ◄$$

6.

$$\mathbf{a}\times\mathbf{c} = (\mathbf{b}+\lambda\mathbf{c})\times\mathbf{c} = \mathbf{b}\times\mathbf{c}+\lambda\mathbf{c}\times\mathbf{c}.$$

$\mathbf{c}\times\mathbf{c} = \mathbf{0}$ and so

$$\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}.$$

7.

The vector product $\mathbf{a} \times \mathbf{b}$ is given in component form by

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (2 \times 6 - 3 \times 5)\mathbf{i} + (3 \times 4 - 1 \times 6)\mathbf{j} + (1 \times 5 - 2 \times 4)\mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.\end{aligned}$$

Thus the area of the parallelogram is

$$A = |\mathbf{a} \times \mathbf{b}| = \sqrt{(-3)^2 + 6^2 + (-3)^2} = \sqrt{54}. \blacktriangleleft$$

8.

The velocity and acceleration of the particle are given by

$$\begin{aligned}\mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} = 4t\mathbf{i} + 3\mathbf{j} + 6t\mathbf{k}, \\ \mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} = 4\mathbf{i} + 6\mathbf{k}.\end{aligned}$$

The speed of the particle at $t = 1$ is simply

$$|\mathbf{v}(1)| = \sqrt{4^2 + 3^2 + 6^2} = \sqrt{61}.$$

The acceleration of the particle is constant (i.e. independent of t), and its component in the direction $\mathbf{s}$ is given by

$$\mathbf{a} \cdot \hat{\mathbf{s}} = \frac{(4\mathbf{i} + 6\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{5\sqrt{6}}{3}. \blacktriangleleft$$

9.

the velocity of the particle is given by

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\,\dot{\hat{\mathbf{e}}}_\rho = \dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\phi}\,\hat{\mathbf{e}}_\phi,$$

and acceleration is given by

$$\begin{aligned}\mathbf{a}(t) &= \frac{d}{dt}(\dot{\rho}\,\hat{\mathbf{e}}_\rho + \rho\dot{\phi}\,\hat{\mathbf{e}}_\phi) \\ &= \ddot{\rho}\,\hat{\mathbf{e}}_\rho + \dot{\rho}\,\dot{\hat{\mathbf{e}}}_\rho + \rho\dot{\phi}\,\dot{\hat{\mathbf{e}}}_\phi + \rho\ddot{\phi}\,\hat{\mathbf{e}}_\phi + \dot{\rho}\dot{\phi}\,\hat{\mathbf{e}}_\phi \\ &= \ddot{\rho}\,\hat{\mathbf{e}}_\rho + \dot{\rho}(\dot{\phi}\,\hat{\mathbf{e}}_\phi) + \rho\dot{\phi}(-\dot{\phi}\,\hat{\mathbf{e}}_\rho) + \rho\ddot{\phi}\,\hat{\mathbf{e}}_\phi + \dot{\rho}\dot{\phi}\,\hat{\mathbf{e}}_\phi \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\mathbf{e}}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\hat{\mathbf{e}}_\phi. \blacktriangleleft\end{aligned}$$

10.

The rate of change of angular momentum is given by

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}).$$

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d\mathbf{r}}{dt} \times m\mathbf{v} \;+\; \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) \\
&= \mathbf{v} \times m\mathbf{v} \;+\; \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) \\
&= \mathbf{0} \;+\; \mathbf{r} \times \mathbf{F} = \mathbf{T},
\end{aligned}
$$

where in the last line we use Newton's second law, namely $\mathbf{F} = d(m\mathbf{v})/dt$. ◄

11.

Forming the vector product of the differential equation with $\mathbf{r}$, we obtain

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^2}\mathbf{r} \times \hat{\mathbf{r}}.$$

Since $\mathbf{r}$ and $\hat{\mathbf{r}}$ are collinear, $\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0}$ and therefore we have

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{0}.$$

However,

$$\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0},$$

Integrating we get the required result.

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{c},$$

where $\mathbf{c}$ is a constant vector.

12.

We can represent a point $\mathbf{r}$ on the surface of the sphere in terms of the two parameters θ and ϕ:

$$\mathbf{r}(\theta, \phi) = a \sin\theta \cos\phi\, \mathbf{i} + a \sin\theta \sin\phi\, \mathbf{j} + a \cos\theta\, \mathbf{k},$$

where θ and ϕ are the polar and azimuthal angles respectively. At any point P, vectors tangent to the coordinate curves $\theta = \text{constant}$ and $\phi = \text{constant}$ are

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= a \cos\theta \cos\phi\, \mathbf{i} + a \cos\theta \sin\phi\, \mathbf{j} - a \sin\theta\, \mathbf{k}, \\
\frac{\partial \mathbf{r}}{\partial \phi} &= -a \sin\theta \sin\phi\, \mathbf{i} + a \sin\theta \cos\phi\, \mathbf{j}.
\end{aligned}
$$

A normal $\mathbf{n}$ to the surface at this point is then given by

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= a^2\sin\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}),$$

which has a magnitude of $a^2\sin\theta$. Therefore, the element of area at P is,

$$dS = a^2\sin\theta\,d\theta\,d\phi,$$

and the total surface area of the sphere is given by

$$A = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ a^2\sin\theta = 4\pi a^2.$$

This familiar result can, of course, be proved by much simpler methods!

13.

$$\nabla\phi = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}. \blacktriangleleft$$

14.

$$\begin{aligned}\nabla\phi &= 2xy\mathbf{i} + (x^2+z)\mathbf{j} + y\mathbf{k}, \\ &= 4\mathbf{i} + 2\mathbf{k} \qquad \text{at the point } (1,2,-1).\end{aligned}$$

The unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, so the rate of change of ϕ with distance s in this direction is,

$$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{\mathbf{a}} = \frac{1}{\sqrt{14}}(4+6) = \frac{10}{\sqrt{14}}.$$

From the above discussion, at the point $(1,2,-1)$ $d\phi/ds$ will be greatest in the direction of $\nabla\phi = 4\mathbf{i} + 2\mathbf{k}$ and has the value $|\nabla\phi| = \sqrt{20}$ in this direction. ◄

15.

the divergence of $\mathbf{a}$ is given by

$$\nabla \cdot \mathbf{a} = 2xy^2 + 2yz^2 + 2x^2z = 2(xy^2 + yz^2 + x^2z). \blacktriangleleft$$

16.

the Laplacian of ϕ is given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2xz^3 + 6xy^2z. \blacktriangleleft$$

17.

The curl of $\mathbf{a}$ is given by

$$\nabla\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^2z^2 & y^2z^2 & x^2z^2 \end{vmatrix} = -2\left[y^2z\mathbf{i} + (xz^2 - x^2y^2z)\mathbf{j} + x^2yz^2\mathbf{k}\right]. \blacktriangleleft$$

18.

The x-component of the LHS is

$$\begin{aligned}\frac{\partial}{\partial y}(\phi a_z) - \frac{\partial}{\partial z}(\phi a_y) &= \phi\frac{\partial a_z}{\partial y} + \frac{\partial\phi}{\partial y}a_z - \phi\frac{\partial a_y}{\partial z} - \frac{\partial\phi}{\partial z}a_y, \\ &= \phi\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \left(\frac{\partial\phi}{\partial y}a_z - \frac{\partial\phi}{\partial z}a_y\right), \\ &= \phi(\nabla\times\mathbf{a})_x + (\nabla\phi\times\mathbf{a})_x,\end{aligned}$$

where, for example, $(\nabla\phi\times\mathbf{a})_x$ denotes the x-component of the vector $\nabla\phi\times\mathbf{a}$. Incorporating the y- and z- components, which can be similarly found, we obtain the stated result. ◀

19.

$$\begin{aligned}\mathbf{i} &= \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\ \mathbf{j} &= \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \\ \mathbf{k} &= \hat{\mathbf{e}}_z.\end{aligned}$$

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z,$$

$$\begin{aligned}\mathbf{a} &= z\rho\sin\phi\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi) - \rho\sin\phi\,(\sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi) + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z \\ &= (z\rho\sin\phi\cos\phi - \rho\sin^2\phi)\,\hat{\mathbf{e}}_\rho - (z\rho\sin^2\phi + \rho\sin\phi\cos\phi)\,\hat{\mathbf{e}}_\phi + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z.\end{aligned}$$

Substituting into the expression for $\nabla\cdot\mathbf{a}$ given in table 10.2,

$$\begin{aligned}\nabla\cdot\mathbf{a} &= 2z\sin\phi\cos\phi - 2\sin^2\phi - 2z\sin\phi\cos\phi - \cos^2\phi + \sin^2\phi + 2z\rho\cos\phi \\ &= 2z\rho\cos\phi - 1.\end{aligned}$$

Alternatively, and much more quickly in this case, we can calculate the divergence directly in Cartesian coordinates. We obtain

$$\nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2zx - 1,$$

which on substituting $x = \rho\cos\phi$ yields the same result as the calculation in cylindrical polars. ◀

Finally, we note that similar results can be obtained for (two-dimensional) polar coordinates in a plane by omitting the z-dependence. For example, $(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2$, while the element of volume is replaced by the element of area $dA = \rho\,d\rho\,d\phi$.

$$\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}$$

Table 10.2 Vector operators in cylindrical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

20.

Since each of the paths lies entirely in the xy-plane, we have $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$. We can therefore write the line integral as

$$I = \int_C \mathbf{a} \cdot d\mathbf{r} = \int_C [(x+y)\,dx + (y-x)\,dy]. \tag{11.3}$$

We must now evaluate this line integral along each of the prescribed paths.

Case (i). Along the parabola $y^2 = x$ we have $2y\,dy = dx$. Substituting for x in (11.3) and using just the limits on y, we obtain

$$I = \int_{(1,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy] = \int_1^2 [(y^2+y)2y + (y-y^2)]\,dy = 11\tfrac{1}{3}.$$

Note that we could just as easily have substituted for y and obtained an integral in x, which would have given the same result.

Case (ii). The second path is given in terms of a parameter u. We could eliminate u between the two equations to obtain a relationship between x and y directly and proceed as above, but it is usually quicker to write the line integral in terms of the parameter u. Along the curve $x = 2u^2 + u + 1$, $y = 1 + u^2$ we have $dx = (4u+1)\,du$ and $dy = 2u\,du$.

Substituting for x and y in (11.3) and writing the correct limits on u, we obtain

$$\begin{aligned} I &= \int_{(1,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy] \\ &= \int_0^1 [(3u^2+u+2)(4u+1) - (u^2+u)2u]\,du = 10\tfrac{2}{3}. \end{aligned}$$

Case (iii). For the third path the line integral must be evaluated along the two line segments separately and the results added together. First, along the line $y = 1$ we have $dy = 0$. Substituting this into (11.3) and using just the limits on x for this segment, we obtain

$$\int_{(1,1)}^{(4,1)} [(x+y)\,dx + (y-x)\,dy] = \int_1^4 (x+1)\,dx = 10\tfrac{1}{2}.$$

Next, along the line $x = 4$ we have $dx = 0$. Substituting this into (11.3) and using just the limits on y for this segment, we obtain

$$\int_{(4,1)}^{(4,2)} [(x+y)\,dx + (y-x)\,dy] = \int_1^2 (y-4)\,dy = -2\tfrac{1}{2}.$$

The value of the line integral along the whole path is just the sum of the values of the line integrals along each segment, and is given by $I = 10\tfrac{1}{2} - 2\tfrac{1}{2} = 8$. ◀

21.

Adopting the usual convention mentioned above, the circle C is to be traversed in the anticlockwise direction. Taking the circle as a whole means x is not a single-valued function of y. We must therefore divide the path into two parts with $x = +\sqrt{a^2 - y^2}$ for the semicircle lying to the right of $x = 0$, and $x = -\sqrt{a^2 - y^2}$ for the semicircle lying to the left of $x = 0$. The required line integral is then the sum of the integrals along the two semicircles. Substituting for x, it is given by

$$\begin{aligned} I = \oint_C x\, dy &= \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy + \int_{a}^{-a} \left(-\sqrt{a^2 - y^2}\right)\, dy \\ &= 4 \int_0^a \sqrt{a^2 - y^2}\, dy = \pi a^2. \end{aligned}$$

Alternatively, we can represent the entire circle parametrically, in terms of the azimuthal angle ϕ, so that $x = a\cos\phi$ and $y = a\sin\phi$ with ϕ running from 0 to 2π. The integral can therefore be evaluated over the whole circle at once. Noting that $dy = a\cos\phi\, d\phi$, we can rewrite the line integral completely in terms of the parameter ϕ and obtain

$$I = \oint_C x\, dy = a^2 \int_0^{2\pi} \cos^2\phi\, d\phi = \pi a^2. \blacktriangleleft$$

22.

The semicircular path from A to B can be described in terms of the azimuthal angle ϕ (measured from the x-axis) by

$$\mathbf{r}(\phi) = a\cos\phi\,\mathbf{i} + a\sin\phi\,\mathbf{j},$$

where ϕ runs from 0 to π. Therefore the element of arc length is given, from section 10.3, by

$$ds = \sqrt{\frac{d\mathbf{r}}{d\phi} \cdot \frac{d\mathbf{r}}{d\phi}}\, d\phi = a(\cos^2\phi + \sin^2\phi)\, d\phi = a\, d\phi.$$

Since $(x - y)^2 = a^2(1 - \sin 2\phi)$, the line integral becomes

$$I = \int_C (x - y)^2\, ds = \int_0^{\pi} a^3(1 - \sin 2\phi)\, d\phi = \pi a^3. \blacktriangleleft$$

23.

Expanding out the integrand, we have

$$I = \int_{(c,c,h)}^{(2c,c/2,h)} \left[(xy^2 + z)\,dx + (x^2y + 2)\,dy + x\,dz\right], \tag{11.7}$$

which we must evaluate along each of the paths C_1 and C_2.

(i) Along C_1 we have $dx = c\,du$, $dy = -(c/u^2)\,du$, $dz = 0$, and on substituting in (11.7) and finding the limits on u, we obtain

$$I = \int_1^2 c\left(h - \frac{2}{u^2}\right)\,du = c(h-1).$$

(ii) Along C_2 we have $2\,dy = -dx$, $dz = 0$ and, on substituting in (11.7) and using the limits on x, we obtain

$$I = \int_c^{2c} \left(\tfrac{1}{2}x^3 - \tfrac{9}{4}cx^2 + \tfrac{9}{4}c^2x + h - 1\right)\,dx = c(h-1).$$

Hence the line integral has the same value along paths C_1 and C_2. Taking the curl of **a**, we have

$$\nabla \times \mathbf{a} = (0-0)\mathbf{i} + (1-1)\mathbf{j} + (2xy - 2xy)\mathbf{k} = \mathbf{0},$$

so **a** is a conservative vector field, and the line integral between two points must be independent of the path taken. Since **a** is conservative, we can write $\mathbf{a} = \nabla\phi$. Therefore, ϕ must satisfy

$$\frac{\partial \phi}{\partial x} = xy^2 + z,$$

which implies that $\phi = \frac{1}{2}x^2y^2 + zx + f(y,z)$ for some function f. Secondly, we require

$$\frac{\partial \phi}{\partial y} = x^2y + \frac{\partial f}{\partial y} = x^2y + 2,$$

which implies $f = 2y + g(z)$. Finally, since

$$\frac{\partial \phi}{\partial z} = x + \frac{\partial g}{\partial z} = x,$$

we have $g = \text{constant} = k$. It can be seen that we have explicitly constructed the function $\phi = \frac{1}{2}x^2y^2 + zx + 2y + k$. ◄

24.

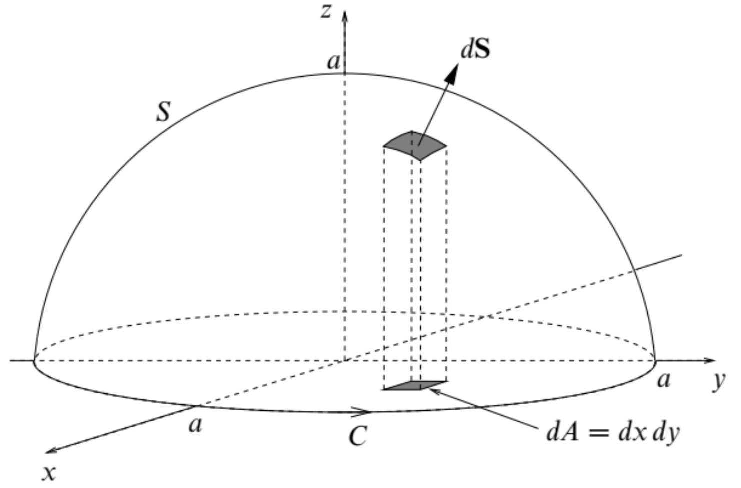


Figure 11.7 The surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

The surface of the hemisphere is shown in figure 11.7. In this case dS may be easily expressed in spherical polar coordinates as $dS = a^2 \sin\theta \, d\theta \, d\phi$, and the unit normal to the surface at any point is simply $\hat{\mathbf{r}}$. On the surface of the hemisphere we have $x = a \sin\theta \cos\phi$ and so

$$\mathbf{a} \cdot d\mathbf{S} = x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\; dS = (a \sin\theta \cos\phi)(\sin\theta \cos\phi)(a^2 \sin\theta \, d\theta \, d\phi).$$

Therefore, inserting the correct limits on θ and ϕ, we have

$$I = \int_S \mathbf{a} \cdot d\mathbf{S} = a^3 \int_0^{\pi/2} d\theta \, \sin^3\theta \int_0^{2\pi} d\phi \, \cos^2\phi = \frac{2\pi a^3}{3}.$$

We could, however, follow the general prescription above and project the hemisphere S onto the region R in the xy-plane that is a circle of radius a centred at the origin. Writing the equation of the surface of the hemisphere as $f(x, y) = x^2 + y^2 + z^2 - a^2 = 0$ and using (11.10), we have

$$I = \int_S \mathbf{a} \cdot d\mathbf{S} = \int_S x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\; dS = \int_R x\,(\mathbf{i} \cdot \hat{\mathbf{r}})\, \frac{|\nabla f|\, dA}{\partial f/\partial z}.$$

Now $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{r}$, so on the surface S we have $|\nabla f| = 2|\mathbf{r}| = 2a$. On S we also have $\partial f/\partial z = 2z = 2\sqrt{a^2 - x^2 - y^2}$ and $\mathbf{i} \cdot \hat{\mathbf{r}} = x/a$. Therefore, the integral becomes

$$I = \iint_R \frac{x^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

Although this integral may be evaluated directly, it is quicker to transform to plane polar coordinates:

$$\begin{aligned} I &= \iint_{R'} \frac{\rho^2 \cos^2\phi}{\sqrt{a^2 - \rho^2}}\; \rho \, d\rho \, d\phi \\ &= \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^a \frac{\rho^3 \, d\rho}{\sqrt{a^2 - \rho^2}}. \end{aligned}$$

Making the substitution $\rho = a \sin u$, we finally obtain

$$I = \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi/2} a^3 \sin^3 u \, du = \frac{2\pi a^3}{3}. \blacktriangleleft$$

25.

$$\mathbf{S} = \iint_S a^2 \sin\theta \, \hat{\mathbf{r}} \, d\theta \, d\phi.$$

Now, since $\hat{\mathbf{r}}$ varies over the surface S, it also must be integrated. This is most easily achieved by writing $\hat{\mathbf{r}}$ in terms of the constant Cartesian basis vectors. On S we have

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},$$

so the expression for the vector area becomes

$$\begin{aligned}\mathbf{S} = {} & \mathbf{i}\left(a^2\int_0^{2\pi}\cos\phi\,d\phi\int_0^{\pi/2}\sin^2\theta\,d\theta\right) + \mathbf{j}\left(a^2\int_0^{2\pi}\sin\phi\,d\phi\int_0^{\pi/2}\sin^2\theta\,d\theta\right)\\ & + \mathbf{k}\left(a^2\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right)\\ = {} & \mathbf{0} + \mathbf{0} + \pi a^2\mathbf{k} = \pi a^2\mathbf{k}.\end{aligned}$$

Note that the magnitude of $\mathbf{S}$ is the projected area, of the hemisphere onto the xy-plane, and not the surface area of the hemisphere. ◄

26.

The perimeter C of the hemisphere is the circle $x^2 + y^2 = a^2$, on which we have

$$\mathbf{r} = a\cos\phi\,\mathbf{i} + a\sin\phi\,\mathbf{j}, \qquad d\mathbf{r} = -a\sin\phi\,d\phi\,\mathbf{i} + a\cos\phi\,d\phi\,\mathbf{j}.$$

Therefore the cross product $\mathbf{r}\times d\mathbf{r}$ is given by

$$\mathbf{r}\times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi & a\sin\phi & 0 \\ -a\sin\phi\,d\phi & a\cos\phi\,d\phi & 0 \end{vmatrix} = a^2(\cos^2\phi + \sin^2\phi)\,d\phi\,\mathbf{k} = a^2\,d\phi\,\mathbf{k},$$

and the vector area becomes

$$\mathbf{S} = \tfrac{1}{2}a^2\mathbf{k}\int_0^{2\pi} d\phi = \pi a^2\,\mathbf{k}. \; ◄$$

27.

Let us first evaluate the surface integral

$$\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S}$$

over the hemisphere. It is easily shown that $\nabla \times \mathbf{a} = -2\,\mathbf{k}$, and the surface element is $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$ in spherical polar coordinates. Therefore

$$\begin{aligned}
\int_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \left(-2a^2 \sin\theta\right) \hat{\mathbf{r}} \cdot \mathbf{k} \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \left(\frac{z}{a}\right) d\theta \\
&= -2a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = -2\pi a^2.
\end{aligned}$$

We now evaluate the line integral around the perimeter curve C of the surface, which

is the circle $x^2 + y^2 = a^2$ in the xy-plane. This is given by

$$\begin{aligned}
\oint_C \mathbf{a} \cdot d\mathbf{r} &= \oint_C (y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) \\
&= \oint_C (y\,dx - x\,dy).
\end{aligned}$$

Using plane polar coordinates, on C we have $x = a\cos\phi$, $y = a\sin\phi$ so that $dx = -a\sin\phi\,d\phi$, $dy = a\cos\phi\,d\phi$, and the line integral becomes

$$\oint_C (y\,dx - x\,dy) = -a^2 \int_0^{2\pi} (\sin^2\phi + \cos^2\phi)\,d\phi = -a^2 \int_0^{2\pi} d\phi = -2\pi a^2.$$

Since the surface and line integrals have the same value, we have verified Stokes' theorem in this case. ◄

28.

With $\mathbf{F}$ as given, we calculate the curl of $\mathbf{F}$ to see whether or not it is the zero vector:

$$\nabla \times \mathbf{F} = (4yz - 4yz,\ 2x - 2x,\ 0 - 0) = \mathbf{0}.$$

The fact that it is implies that $\mathbf{F}$ can be written as $\nabla\phi$ for some scalar ϕ.

The form of $\phi(x, y, z)$ is found by integrating, in turn, the components of $\mathbf{F}$ until consistency is achieved, i.e. until a ϕ is found that has partial derivatives equal to the corresponding components of $\mathbf{F}$:

$$\begin{aligned}
2xz = F_x = \frac{\partial \phi}{\partial x} \quad &\Rightarrow \quad \phi(x, y, z) = x^2 z + g(y, z),\\
2yz^2 = F_y = \frac{\partial}{\partial y}[x^2 z + g(y, z)] \quad &\Rightarrow \quad g(y, z) = y^2 z^2 + h(z),\\
x^2 + 2y^2 z - 1 = F_z \quad &= \quad \frac{\partial}{\partial z}[x^2 z + y^2 z^2 + h(z)]\\
&\Rightarrow \quad h(z) = -z + k.
\end{aligned}$$

Hence, to within an unimportant constant, the form of ϕ is

$$\phi(x, y, z) = x^2 z + y^2 z^2 - z.$$

29.

Although all three integrals are along the same path L, they are not necessarily of the same type. The vector or scalar nature of the integral is determined by that of the integrand when it is expressed in a form containing the infinitesimal dt.

(a) This is a vector integral and contains three separate integrations. We express each of the integrands in terms of t, according to the parameterisation of the integration path L, before integrating:

$$\begin{aligned}\int_L \mathbf{F}\,dt &= \int_1^2 \left(\frac{c^3}{t}\mathbf{i} + 2\mathbf{j} + ct\,\mathbf{k}\right)\,dt \\ &= \left[c^3 \ln t\,\mathbf{i} + 2t\,\mathbf{j} + \frac{1}{2}ct^2\,\mathbf{k}\right]_1^2 \\ &= c^3 \ln 2\,\mathbf{i} + 2\,\mathbf{j} + \frac{3}{2}c\,\mathbf{k}.\end{aligned}$$

(b) This is a similar vector integral but here we must also replace the infinitesimal dy by the infinitesimal $-c\,dt/t^2$ before integrating:

$$\begin{aligned}\int_L \mathbf{F}\,dy &= \int_1^2 \left(\frac{c^3}{t}\mathbf{i} + 2\mathbf{j} + ct\,\mathbf{k}\right)\left(\frac{-c}{t^2}\right)\,dt \\ &= \left[\frac{c^4}{2t^2}\mathbf{i} + \frac{2c}{t}\mathbf{j} - c^2 \ln t\,\mathbf{k}\right]_1^2 \\ &= -\frac{3c^4}{8}\mathbf{i} - c\,\mathbf{j} - c^2 \ln 2\,\mathbf{k}.\end{aligned}$$

(c) This is a scalar integral and before integrating we must take the scalar product of $\mathbf{F}$ with $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ to give a single integrand:

$$\begin{aligned}\int_L \mathbf{F}\cdot d\mathbf{r} &= \int_1^2 \left(\frac{c^3}{t}\mathbf{i} + 2\mathbf{j} + ct\,\mathbf{k}\right)\cdot\left(c\,\mathbf{i} - \frac{c}{t^2}\mathbf{j} + 0\,\mathbf{k}\right)dt \\ &= \int_1^2 \left(\frac{c^4}{t} - \frac{2c}{t^2}\right)\,dt \\ &= \left[c^4 \ln t + \frac{2c}{t}\right]_1^2 \\ &= c^4 \ln 2 - c.\end{aligned}$$

30.

We are told that

$$\mathbf{a} = -\frac{zx}{r^3}\mathbf{i} - \frac{zy}{r^3}\mathbf{j} + \frac{x^2 + y^2}{r^3}\mathbf{k},$$

with $r^2 = x^2 + y^2 + z^2$. We will need to differentiate r^{-3} with respect to x, y and z, using the chain rule, and so note that $\partial r/\partial x = x/r$, etc.

(a) Consider $\nabla \times \mathbf{a}$, term-by-term:

$$
\begin{aligned}
[\nabla \times \mathbf{a}]_x &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-zy}{r^3}\right) \\
&= \frac{-3(x^2+y^2)y}{r^4\,r} + \frac{2y}{r^3} + \frac{y}{r^3} - \frac{3(zy)z}{r^4\,r} \\
&= \frac{3y}{r^5}(-x^2-y^2+x^2+y^2+z^2-z^2) = 0; \\
[\nabla \times \mathbf{a}]_y &= \frac{\partial}{\partial z}\left(\frac{-zx}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{r^3}\right) \\
&= \frac{3(zx)z}{r^4\,r} - \frac{x}{r^3} - \frac{2x}{r^3} + \frac{3(x^2+y^2)x}{r^4\,r} \\
&= \frac{3x}{r^5}(z^2-x^2-y^2-z^2+x^2+y^2) = 0; \\
[\nabla \times \mathbf{a}]_z &= \frac{\partial}{\partial x}\left(\frac{-zy}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{-zx}{r^3}\right) \\
&= \frac{3(zy)x}{r^4\,r} - \frac{3(zx)y}{r^4\,r} = 0.
\end{aligned}
$$

Thus all three components of $\nabla \times \mathbf{a}$ are zero, showing that $\mathbf{a}$ is a conservative field.

(b) To construct its potential function we proceed as follows:

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{-zx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \phi = \frac{z}{(x^2+y^2+z^2)^{1/2}} + f(y,z), \\
\frac{\partial \phi}{\partial y} &= \frac{-zy}{(x^2+y^2+z^2)^{3/2}} = \frac{-zy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial f}{\partial y} \Rightarrow f(y,z) = g(z), \\
\frac{\partial \phi}{\partial z} &= \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{1}{(x^2+y^2+z^2)^{1/2}} + \frac{-z\,z}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial z} \\
&\Rightarrow g(z) = c.
\end{aligned}
$$

Thus,

$$\phi(x,y,z) = c + \frac{z}{(x^2+y^2+z^2)^{1/2}} = c + \frac{z}{r}.$$

The very fact that we can construct a potential function $\phi = \phi(x,y,z)$ whose derivatives are the components of the vector field shows that the field is conservative.

31.

(a) This calculation is a piece-wise evaluation of the line integral, made up of a series of scalar products of the length of a straight piece of the contour and the component of **F** parallel to it (integrated if that component varies along the particular straight section).

On OA, $y = z = 0$ and $F_x = xe^{-x}$;

$$I_1 = \int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = 1 - 2e^{-1}.$$

On AB, $x = 1$ and $y = 0$ and $F_z = 0$; the integral I_2 is zero.

On BC, $x = 1$ and $z = 1$ and $F_y = 3y^2 + 1 + ey$;

$$I_3 = \int_0^1 (3y^2 + 1 + ey)\,dy = 1 + 1 + \tfrac{1}{2}e.$$

On CD, $x = 1$ and $y = 1$ and $F_z = 1 + 1 + z^2$;

$$I_4 = \int_1^0 (1 + 1 + z^2)\,dz = -1 - 1 - \tfrac{1}{3}.$$

On DE, $y = 1$ and $z = 0$ and $F_x = xe^{-x}$;

$$I_5 = \int_1^0 xe^{-x}\,dx = -1 + 2e^{-1}.$$

On EO, $x = z = 0$ and $F_y = ye^0$;

$$I_6 = \int_1^0 ye^0\,dy = -\tfrac{1}{2}.$$

Adding up these six contributions shows that the complete line integral has the value $\dfrac{e}{2} - \dfrac{5}{6}$.

(b) As a simple sketch shows, the given contour is three-dimensional. However, it is equivalent to two plane square contours, one $OADEO$ (denoted by S_1) lying in the plane $z = 0$ and the other $ABCDA$ (S_2) lying in the plane $x = 1$; the latter is traversed in the negative sense. The common segment AD does not form part of the original contour but, as it is traversed in opposite senses in the two constituent contours, it (correctly) contributes nothing to the line integral.

To use Stokes' theorem we first need to calculate

$$\begin{aligned}
(\nabla \times \mathbf{F})_x &= x^3 + 3y^2x + 2yxz^2 - 3xy^2 - x^3 = 2yxz^2,\\
(\nabla \times \mathbf{F})_y &= 3x^2y + y^3 - 3x^2y - y^3 - y^2z^2 = -y^2z^2,\\
(\nabla \times \mathbf{F})_z &= 3y^2z + 3x^2z + ye^x - 3x^2z - 3y^2z = ye^x.
\end{aligned}$$

Now, S_1 has its normal in the positive z-direction and so only the z-component of $\nabla \times \mathbf{F}$ is needed in the first surface integral of Stokes' theorem. Likewise only the x-component of $\nabla \times \mathbf{F}$ is needed in the second integral, but its value must be subtracted because of the sense in which its contour is traversed:

$$\begin{aligned}
\int_{OABCDEO} (\nabla \times \mathbf{F}) \cdot d\mathbf{r} &= \int_{S_1} (\nabla \times \mathbf{F})_z\,dx\,dy - \int_{S_2} (\nabla \times \mathbf{F})_x\,dy\,dz\\
&= \int_0^1 \int_0^1 ye^x\,dx\,dy - \int_0^1 \int_0^1 2y \times 1 \times z^2\,dy\,dz\\
&= \frac{1}{2}(e-1) - 2\,\frac{1}{2}\,\frac{1}{3} = \frac{e}{2} - \frac{5}{6}.
\end{aligned}$$

As they must, the two methods give the same value.

32.

Soln: Let the (non-unit) vector $\perp$ to $\mathbf{A}$ be $\mathbf{C} = \pm(c\hat{\mathbf{i}} + d\hat{\mathbf{j}})$, where c and d are constants to be determined Thus

$$\begin{aligned} \mathbf{A} \cdot \mathbf{C} &= \pm(2c - d) = 0 \\ \implies d &= 2c \\ \implies \mathbf{C} &= \pm c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \\ \implies \mathbf{C} &= \pm \frac{c(\hat{\mathbf{i}} + 2\hat{\mathbf{j}})}{\sqrt{c^2 + 4c^2}} = \pm \frac{1}{\sqrt{5}} \left(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right) \end{aligned}$$

Following a similar procedure, we obtain unit vector $\hat{\mathbf{D}}$, which lies in the xz plane, and is perpendicular to $\mathbf{B}$

$$\hat{\mathbf{D}} = \pm \frac{1}{\sqrt{5}} \left(2\hat{\mathbf{i}} + \hat{\mathbf{k}}\right)$$

33.

Soln: In the previous problem we computed

$$\begin{aligned} \mathbf{B} \times \mathbf{A} &= \left(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 3\hat{\mathbf{k}}\right) \\ \therefore \mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) &= 2 + 7 - 9 = 0 \end{aligned}$$

The result is general because by definition $\mathbf{B} \times \mathbf{A}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, therefore

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{B} \cdot (\mathbf{B} \times \mathbf{A}) = \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$$

34.

Soln: We have

$$\begin{aligned} |\mathbf{A} + \mathbf{B}| &= |\mathbf{A} - \mathbf{B}| \\ \implies |\mathbf{A} + \mathbf{B}|^2 &= |\mathbf{A} - \mathbf{B}|^2 \\ \implies (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) &= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \\ \implies A^2 + B^2 + 2\mathbf{A} \cdot \mathbf{B} &= A^2 + B^2 - 2\mathbf{A} \cdot \mathbf{B} \\ \implies 4\mathbf{A} \cdot \mathbf{B} &= 0 \\ \implies \mathbf{A} &\perp \mathbf{B} \end{aligned}$$

35.

Soln: By defintion

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = A\alpha\left(e^{\alpha t}\hat{\mathbf{i}} - e^{-\alpha t}\hat{\mathbf{j}}\right)$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = A\alpha^2\left(e^{\alpha t}\hat{\mathbf{i}} + e^{-\alpha t}\hat{\mathbf{j}}\right) = \alpha^2\mathbf{r}$$

Note that

$$\mathbf{r}(0)\cdot\mathbf{v}(0) = A^2\alpha\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)\cdot\left(\hat{\mathbf{i}} - \hat{\mathbf{j}}\right) = 0,$$

so $\mathbf{r}(0)$ and $\mathbf{v}(0)$ can be plotted as two mutually perpendicular vectors.

36.

Soln: The acceleration equation $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$ is equivalent to the following two differential equations in Cartesian coordinates

$$\frac{d^2x}{dt^2} = -\omega^2 x$$
$$\frac{d^2y}{dt^2} = -\omega^2 y.$$

We will integrate the x equation, and the same procedure will apply to the y equation. Multiply on both sides by $2\frac{dx}{dt}$

$$\begin{aligned}
2\frac{dx}{dt}\frac{d^2x}{dt^2} &= -\omega^2 2x\frac{dx}{dt} \\
\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 &= -\omega^2\frac{dx^2}{dt} \\
\implies \left(\frac{dx}{dt}\right)^2 &= -\omega^2x^2 + c^2 \quad (c \text{ is a constant}) \\
\implies \frac{dx}{dt} &= \pm\sqrt{c^2 - \omega^2x^2} \\
\pm\frac{dx}{\omega\sqrt{\alpha^2 - x^2}} &= dt \\
\int\frac{dx}{\sqrt{\alpha^2 - x^2}} &= \pm\,\omega\int dt + C \quad (\alpha \text{ and } C \text{ are constants}) \\
\sin^{-1}\left(\frac{x}{\alpha}\right) &= C \pm \omega t \\
x &= \alpha\sin\left(C \pm \omega t\right) \\
x &= A\sin\omega t + B\cos\omega t.
\end{aligned}$$

Similarly,

$$y = C\sin\omega t + D\cos\omega t,$$

above A, B, C, and D are constants of integration to be determined by initial conditions, which are

$$x(0) = 0$$
$$v_x(0) = a\omega$$
$$y(0) = a$$
$$v_y(0) = 0.$$

Using the fact that

$$v_x(t) = \frac{dx}{dt} = A\omega\cos\omega t - B\omega\sin\omega t$$
$$v_y(t) = \frac{dy}{dt} = C\omega\cos\omega t - D\omega\sin\omega t,$$

application of x initial conditions gives

$$\begin{aligned} x(0) &= B = 0 \\ v_x(0) &= A\omega = a\omega \\ \Longrightarrow\ A &= a \end{aligned}$$

and y initial conditions yield

$$\begin{aligned} y(0) &= D = a \\ v_y(0) &= C\omega = 0 \\ \Longrightarrow\ C &= 0. \end{aligned}$$

Thus the final solution is

$$\begin{aligned} x(t) &= a\sin\omega t \\ y(t) &= a\cos\omega t \\ \mathbf{r}(t) &= a\left(\sin\omega t\hat{\mathbf{i}} + \cos\omega t\hat{\mathbf{j}}\right). \end{aligned}$$

And the curve is a circle of radius a, centered at the origin, because

$$x^2 + y^2 = a^2(\cos^2\omega t + \sin^2\omega t) = a^2.$$

37.

Soln: Here we use $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$, so that

(a) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) \times \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) = \cos^2\theta\hat{\mathbf{k}} + \sin^2\theta\hat{\mathbf{k}} = \hat{\mathbf{k}}$

(b) $\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}\right) \times \hat{\mathbf{k}} = \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{i}} = \hat{\mathbf{r}}$

(c) $\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\mathbf{k}} \times \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) = \cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{i}} = \hat{\boldsymbol{\theta}}$

38.

Soln: Here we have

$$\begin{aligned} r &= a \\ \dot{r} &= 0 \\ \ddot{r} &= 0, \end{aligned}$$

and

$$\begin{aligned} \dot{\theta} &= \omega(t) = \omega_0 + \alpha t \\ \ddot{\theta} &= \alpha, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = a\left(\omega_0 + \alpha t\right)\hat{\boldsymbol{\theta}} \\ \mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = -a\left(\omega_0 + \alpha t\right)^2\hat{\mathbf{r}} - a\alpha\hat{\boldsymbol{\theta}} \end{aligned}$$

39.

Soln: Because $\hat{\mathbf{i}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$, we get

$$\mathbf{v} = u\cos\theta\hat{\mathbf{r}} - u\sin\theta\hat{\boldsymbol{\theta}}$$

40.

Soln: Here we have

$$\begin{aligned} \dot{\theta} &= \omega \\ \ddot{\theta} &= 0 \\ \dot{r} &= r_0\beta e^{\beta t} \\ \ddot{r} &= r_0\beta^2 e^{\beta t}, \end{aligned}$$

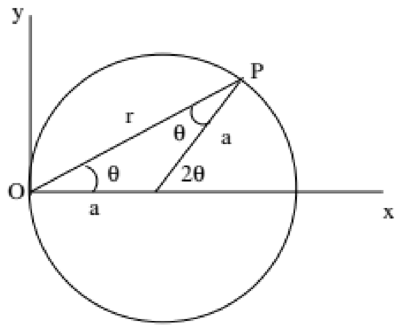
so that

$$\begin{aligned} \mathbf{v} &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta\hat{\mathbf{r}} + \omega\hat{\boldsymbol{\theta}}\right) \\ \mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} = r_0 e^{\beta t}\left(\beta^2 - \omega^2\right)\hat{\mathbf{r}} + 2r_0\omega\beta e^{\beta t}\hat{\boldsymbol{\theta}}. \end{aligned}$$

It is obvious from the expression of acceleration that its radial component will vanish when $\beta = \pm\omega$.

41.

Soln:



It is obvious from the figure above that equation of the circle is

$$r = 2a\cos\theta$$

Soln: From the figure it is obvious that

$$\begin{aligned} 2\dot{\theta} &= \frac{u}{a} \\ \implies \dot{\theta} &= \frac{u}{2a} \\ \theta &= \frac{ut}{2a} \end{aligned}$$

Soln: From equation of the circle we obtain

$$\dot{r} = \frac{d}{dt}(2a\cos\theta) = -2a\sin\theta\dot{\theta} = -u\sin\theta,$$

so that

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = -u\sin\theta\hat{\mathbf{r}} + 2a\cos\theta\left(\frac{u}{2a}\right)\hat{\boldsymbol{\theta}} = -u\sin\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} + u\cos\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}$$

Soln: We have

$$\ddot{\theta} = \frac{d}{dt}\left(\frac{u}{2a}\right) = 0$$
$$\ddot{r} = \frac{d}{dt}(-u\sin\theta) = -u\cos\theta\dot{\theta} = -\frac{u^2}{2a}\cos\theta,$$

so that

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\ &= \left(-\frac{u^2}{2a}\cos\theta - 2a\cos\theta\frac{u^2}{4a^2}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\theta\hat{\boldsymbol{\theta}} \\ &= -\frac{u^2}{a}\cos\left(\frac{ut}{2a}\right)\hat{\mathbf{r}} - \frac{u^2}{a}\sin\left(\frac{ut}{2a}\right)\hat{\boldsymbol{\theta}}\end{aligned}$$

42.

Soln: We have

$$\begin{aligned} & r = A\theta = A\alpha t^2 \\ \Longrightarrow\ & \dot{r} = A\dot{\theta} = 2A\alpha t \\ \Longrightarrow\ & \ddot{r} = 2A\alpha,\end{aligned}$$

and

$$\dot{\theta} = 2\alpha t$$
$$\ddot{\theta} = 2\alpha.$$

Therefore, the expression for velocity is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} = 2A\alpha t\hat{\mathbf{r}} + 2A\alpha^2 t^3\hat{\boldsymbol{\theta}}.$$

Expression for acceleration is given below.

$$\begin{aligned}\mathbf{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} \\ &= 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + A\alpha^2\left(8t^2 - 2t^2\right)\hat{\boldsymbol{\theta}} \\ & 2A\alpha\left(1 - 2\alpha^2 t^4\right)\hat{\mathbf{r}} + 6A\alpha^2 t^2\hat{\boldsymbol{\theta}}\end{aligned}$$

It is obvious from above that the radial component of the acceleration vanishes for $t^2 = 1/\alpha\sqrt{2}$, for which $\theta = \alpha(1/\alpha\sqrt{2}) = 1/\sqrt{2}$.

Soln: The two components will be equal in magnitude when

$$2A\alpha\left(1-2\alpha^2t^4\right) = \pm 6A\alpha^2t^2$$
$$2\alpha^2t^4 \pm 3\alpha t^2 - 1 = 0.$$

This equation has two possible solutions for t^2

$$t^2 = \frac{\pm 3+\sqrt{17}}{4\alpha}$$
$$\Longrightarrow \theta = \alpha t^2 = \frac{\pm 3+\sqrt{17}}{4} \text{ radians}$$

43.

Soln: We know that the radial component of acceleration is given by

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

Here the mass is being pulled very slowly so $\ddot{r} \approx 0$, therefore, the only acceleration experienced by the mass is the centripetal acceleration

$$a_r \approx -r\dot{\theta}^2 = -r\omega^2.$$

If the particle is pulled slowly, the force applied is the tension in the string which should be the centripetal force

$$F = T = -m\omega^2 r,$$

so that the work done will be

$$W = \int_{R_0}^{R_1} F dr = -m \int_{R_0}^{R_1} \omega^2 r dr.$$

r dependence of ω can be calculated using the fact that the angular momentum of the particle about the center of the circle will be conserved because F being radial, does not impart any torque to the particle, with respect to the center. Assuming that the initial angular velocity of the mass was ω_0 corresponding to the radius R_0, conservation of angular momentum ($I\omega = mr^2\omega$), implies

$$mR_0^2\omega_0 = mr^2\omega$$
$$\implies \omega(r) = \frac{R_0^2\omega_0}{r^2},$$

using this we have

$$W = -\int_{R_0}^{R_1} \omega^2 r dr = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{rdr}{r^4} = -m\omega_0^2 R_0^4 \int_{R_0}^{R_1} \frac{dr}{r^3} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right)$$

Increase in kinetic energy of the particle will be

$$\Delta K = \frac{1}{2} I_1\omega_1^2 - \frac{1}{2} I_0\omega_0^2 = \frac{1}{2}\left\{mR_1^2\left(\frac{R_0^4\omega_0^2}{R_1^4}\right) - mR_0^2\omega_0^2\right\} = \frac{1}{2} m\omega_0^2 R_0^4 \left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right),$$

which is same as the expression for W above.

44.

Soln: The force is given by

$$\mathbf{F} = \left(-B + \frac{A}{x^2}\right)\hat{\mathbf{i}}, \text{ for } x \geq 0.$$

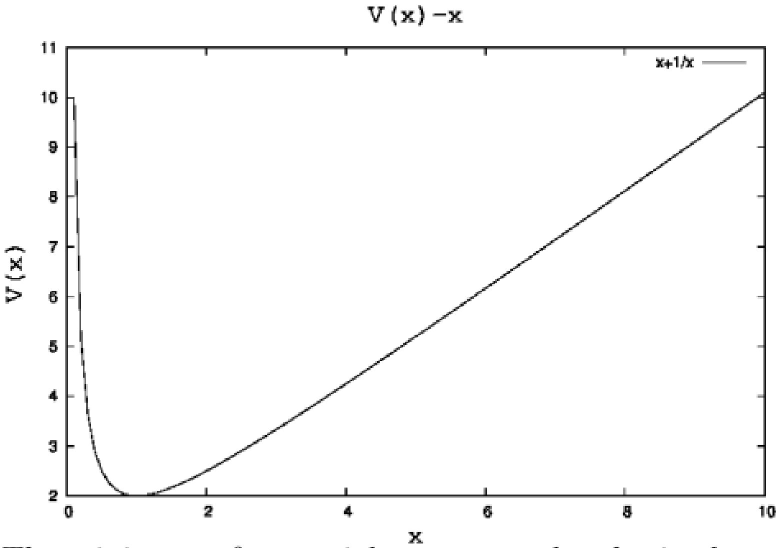
If $U(x)$ is the potential energy, then for $x \geq 0$

$$-\frac{dV}{dx} = -B + \frac{A}{x^2}$$
$$\implies V(x) = Bx + \frac{A}{x} + C,$$

where C is a constant, which we can set to zero so

$$V(x) = Bx + \frac{A}{x}$$

Soln: The plot of the potential energy for $A = B = 1$ is



The minimum of potential energy can be obtained

$$\frac{dV}{dx} = B - \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

for which $V = B\sqrt{\frac{A}{B}} + A\sqrt{\frac{B}{A}} = 2\sqrt{AB}$. Because total energy is conserved, therefore, $E = K + V =$ constant. Thus, when we have maximum kinetic energy,

we will have minimum potential energy, implying that $E = K_{max} + V_{min} = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$. Therefore, total energy of the particle can be represented by a horizontal line in the plot, corresponding to $V(x) = \frac{1}{2}mv_0^2 + 2\sqrt{AB}$.

Soln: Point of equilibrium is obtained by

$$F = -B + \frac{A}{x^2} = 0$$
$$\implies x = \sqrt{\frac{A}{B}},$$

which is the same point where the potential energy is minimum.

45.

Soln: Here

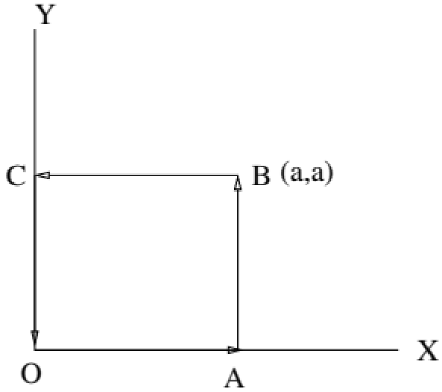
$$V(\mathbf{r}) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$
$$\implies \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$$
$$= GMm\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\hat{\mathbf{k}}\right)$$
$$= -GMm\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{GMm\mathbf{r}}{r^3} = -\frac{GMm\hat{\mathbf{r}}}{r^2}$$

Soln: Because for this force a potential energy function exists, its curl must vanish. We calculate it as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$
$$= -3\left(\frac{yz - yz}{(x^2+y^2+z^2)^{5/2}}\right)\hat{\mathbf{i}} + \cdots = 0$$

46.

Soln: Let us first calculate the line integral along the given path



The work done will be

$$W = \oint \mathbf{F}\cdot d\mathbf{r} = \int_{OA}\mathbf{F}\cdot d\mathbf{r} + \int_{AB}\mathbf{F}\cdot d\mathbf{r} + \int_{BC}\mathbf{F}\cdot d\mathbf{r} + \int_{CO}\mathbf{F}\cdot d\mathbf{r}$$
$$= A(0)\int_0^a dx + 2A(a^2)\int_0^a dy + Aa^2\int_a^0 dx + 2(0)\int_a^0 dy = 2Aa^3$$

To verify Stokes theorem we need to compute

$$\int (\nabla\times\mathbf{F})\cdot d\mathbf{S}.$$

Here

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ay^2 & 2Ax^2 & 0 \end{vmatrix} = (4Ax - 2Ay)\hat{\mathbf{k}}$$

and

$$d\mathbf{S} = dxdy\hat{\mathbf{k}},$$

so that

$$\int (\nabla\times\mathbf{F})\cdot d\mathbf{S} = 4A\int_0^a xdx\int_0^a dy - 2A\int_0^a dx\int_0^a ydy$$
$$= 4A(\frac{a^2}{2})a - 2Aa(\frac{a^2}{2}) = 2Aa^3.$$

Thus we get the same value of work done by computing the line integral, and by using the Stokes theorem.

47.

Soln:

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$
$$= -2Ax\hat{\mathbf{i}} - 2By\hat{\mathbf{j}} - 2Cz\hat{\mathbf{k}}$$

Soln:

$$\mathbf{F} = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} - \frac{\partial V}{\partial z}\hat{\mathbf{k}} = \frac{2A}{(x^2+y^2+z^2)}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)$$

Soln:

$$V = \frac{A\cos\theta}{r^2} = \frac{Ax}{r^3} = \frac{Ax}{(x^2+y^2)^{3/2}}$$

$$\implies \mathbf{F} = -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} = -A\left(\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}\right)\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}}$$

$$= \frac{A(2x^2-y^2)}{(x^2+y^2)^{5/2}}\hat{\mathbf{i}} + \frac{A3xy}{(x^2+y^2)^{5/2}}\hat{\mathbf{j}}$$

48.

Soln: First we compute the curl of **F**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3A & Az & Ay \end{vmatrix} = 0.$$

Therefore, the force is conservative and it is possible to obtain a potential energy function for it using

$$-\frac{\partial V}{\partial x} = 3A$$
$$-\frac{\partial V}{\partial y} = Az$$
$$-\frac{\partial V}{\partial z} = Ay.$$

On integrating the first equation above we have

$$V(x,y,z) = -3Ax + f(y,z),$$

which on substitution in the second equation yields

$$-\frac{\partial f}{\partial y} = Az$$
$$\implies f(y,z) = -Ayz + C$$
$$\implies V(x,y,z) = -3Ax - Ayz + C,$$

where C is a constant. Note that this expression for V satisfies the third equation above, implying that the solution is complete.

Soln:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Axyz & Axyz & Axyz \end{vmatrix} = A(xz-xy)\hat{\mathbf{i}} + A(xy-yz)\hat{\mathbf{j}} + A(yz-xz)\hat{\mathbf{i}} \neq 0,$$

Soln:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A\sin(\alpha y)\cos(\beta z) & -Ax\alpha\cos(\alpha y)\cos(\beta z) & Ax\sin(\alpha y)\sin(\beta z) \end{vmatrix}$$

$$= A(x\alpha\cos(\alpha y)\sin(\beta z) - x\alpha\beta\cos(\alpha y)\sin(\beta z))\hat{\mathbf{i}}$$
$$+ A(-x\beta\sin(\alpha y)\sin(\beta z) - x\alpha\cos(\alpha y)\sin(\beta z))\hat{\mathbf{j}}$$
$$+ A(0 - \alpha\cos(\alpha y)\cos(\beta z)\hat{\mathbf{k}}$$
$$\neq 0,$$

hence, for this force also, no potential energy function exists.

49.

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \text{Hz}} = 1.5 \times 10^{-7}\ \text{s} = 0.15\ \mu\text{s}$$

$$\omega = 2\pi f = 2\pi(6.7 \times 10^6\ \text{Hz})$$
$$= (2\pi\ \text{rad/cycle})(6.7 \times 10^6\ \text{cycle/s}) = 4.2 \times 10^7\ \text{rad/s}$$

50.

EXECUTE (a) When $x = 0.030$ m, the force the spring exerts on the spring balance is $F_x = -6.0$ N. From Eq. (14.3),

$$k = -\frac{F_x}{x} = -\frac{-6.0\ \text{N}}{0.030\ \text{m}} = 200\ \text{N/m} = 200\ \text{kg/s}^2$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{kg/s}^2}{0.50\ \text{kg}}} = 20\ \text{rad/s}$$

$$f = \frac{\omega}{2\pi} = \frac{20\ \text{rad/s}}{2\pi\ \text{rad/cycle}} = 3.2\ \text{cycle/s} = 3.2\ \text{Hz}$$

$$T = \frac{1}{f} = \frac{1}{3.2\ \text{cycle/s}} = 0.31\ \text{s}$$

51.

EXECUTE When the force increases by 980 N, the spring compresses an additional 0.028 m, and the x-coordinate of the car changes by -0.028 m. Hence the effective force constant (including the effect of the entire suspension) is

$$k = -\frac{F_x}{x} = -\frac{980\ \text{N}}{-0.028\ \text{m}} = 3.5 \times 10^4\ \text{kg/s}^2$$

The person's mass is $w/g = (980\ \text{N})/(9.8\ \text{m/s}^2) = 100\ \text{kg}$. The *total* oscillating mass is $m = 1000\ \text{kg} + 100\ \text{kg} = 1100\ \text{kg}$. The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{1100\ \text{kg}}{3.5 \times 10^4\ \text{kg/s}^2}} = 1.11\ \text{s}$$

The frequency is $f = 1/T = 1/(1.11\ \text{s}) = 0.90\ \text{Hz}$.

52.

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000\ \text{m}}{9.800\ \text{m/s}^2}} = 2.007\ \text{s}$$

and

$$f = \frac{1}{T} = \frac{1}{2.007\ \text{s}} = 0.4982\ \text{Hz}$$

53.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ \text{N/m}}{0.68\ \text{kg}}} = 9.78\ \text{rad/s}$$

≈ 9.8 rad/s.

$$f = \frac{\omega}{2\pi} = \frac{9.78 \text{ rad/s}}{2\pi \text{ rad}} = 1.56 \text{ Hz} \approx 1.6 \text{ Hz}.$$

$$T = \frac{1}{f} = \frac{1}{1.56 \text{ Hz}} = 0.64 \text{ s} = 640 \text{ ms}.$$

With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm}. \qquad \text{(Answer)}$$

The maximum speed v_m is the velocity amplitude ωx_m

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m})$$
$$= 1.1 \text{ m/s}.$$

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

$$x(t) = x_m \cos(\omega t + \phi) \qquad \text{(displacement)},$$

$$\phi = 0 \text{ rad}.$$

$$x(t) = x_m \cos(\omega t + \phi)$$
$$= (0.11 \text{ m}) \cos[(9.8 \text{ rad/s})t + 0]$$
$$= 0.11 \cos(9.8t),$$

where x is in meters and t is in seconds.

54.

The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position

$x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$k = m\omega^2 = m(2\pi f)^2$$
$$= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2$$
$$= 1.073 \times 10^9 \text{ N/m}.$$

We can now evaluate E as

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$
$$= 0 + \tfrac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2$$
$$= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J}. \qquad \text{(Answer)}$$

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$2.147 \times 10^7 \text{ J} = \tfrac{1}{2}(2.72 \times 10^5 \text{ kg})v^2 + 0,$$

or $$v = 12.6 \text{ m/s}. \qquad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.