

## Del Operator

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

↪ Operator, only has meaning when it operates on something.

## Operations

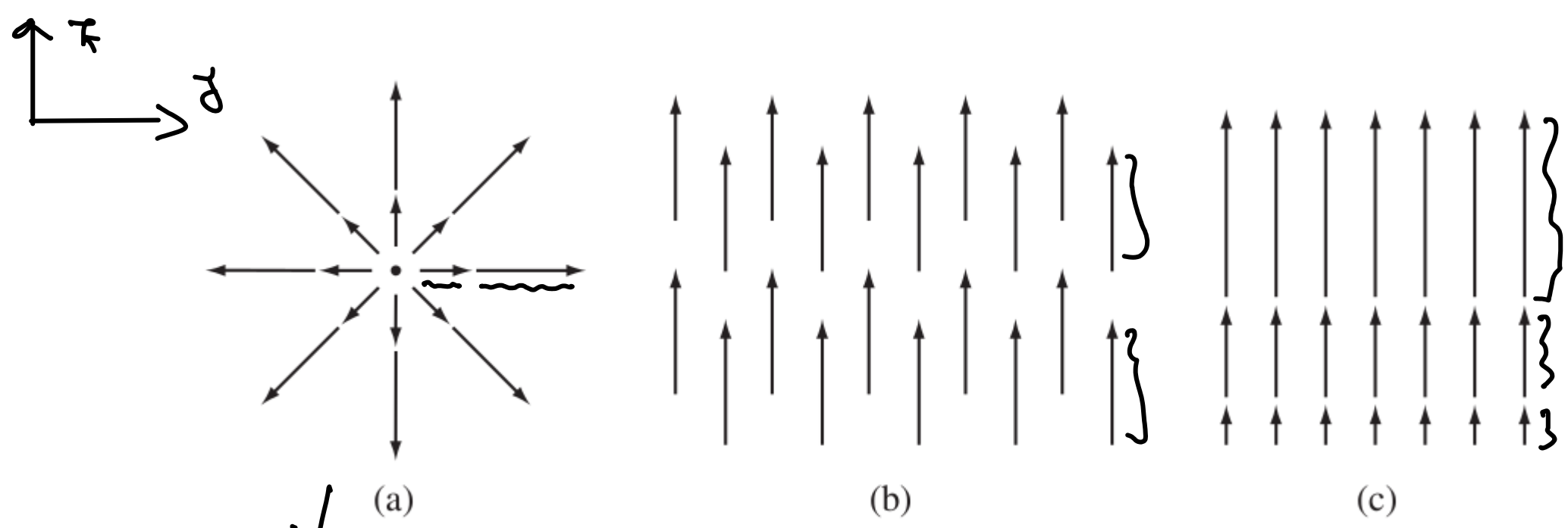
$$\vec{\nabla} f \equiv \text{Gradient}$$

$$\vec{\nabla} \cdot \vec{A} \equiv \text{Divergence}$$

$$\vec{\nabla} \times \vec{A} \equiv \text{Curl}$$

## Divergence

$\vec{\nabla} \cdot \vec{A} \equiv$  measurement of how the vector spreads out



$$\vec{A} = \alpha \hat{x} + \alpha \hat{y}$$

$$\vec{\nabla} \cdot \vec{A} \neq 0$$

$$\vec{A} = c \hat{z}$$

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\vec{A} = \alpha \hat{z}$$

$$\vec{\nabla} \cdot \vec{A} \neq 0$$

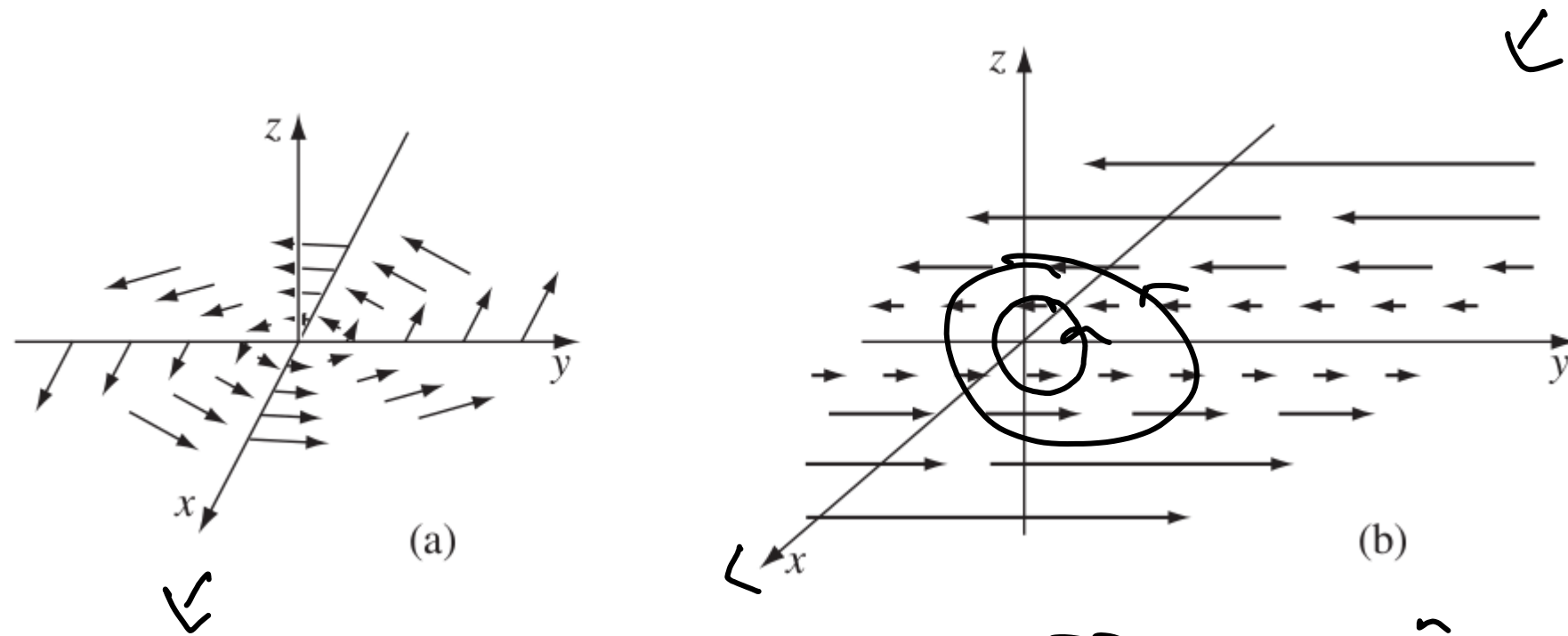
$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

is a scalar quantity.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \hat{y} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

→ Measurement of how much the vector  
circles around a point.



$$\vec{A} = -y\vec{i} + x\vec{j}$$

$$\vec{\nabla} \times \vec{A} \neq 0$$

$$\vec{A} = z\vec{j}$$

$$\vec{\nabla} \times \vec{A} \neq 0$$

## Product Rules

$$\vec{\nabla} \cdot (\vec{f} + \vec{g}) = \vec{\nabla} \cdot \vec{f} + \vec{\nabla} \cdot \vec{g}$$

$$\vec{\nabla} \cdot (\vec{f} \times \vec{g}) = \vec{\nabla} \cdot \vec{f} \times \vec{g} + \vec{\nabla} \cdot \vec{g} \times \vec{f}$$

$$\vec{\nabla} \times (\vec{f} + \vec{g}) = \vec{\nabla} \times \vec{f} + \vec{\nabla} \times \vec{g}$$

$$\rightarrow \vec{\nabla} (kf) = k \vec{\nabla} f + f \vec{\nabla} k$$

$$\rightarrow \vec{\nabla} \cdot (k\vec{A}) = k (\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} k)$$

$$\vec{\nabla} \times (k\vec{A}) = k (\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} k)$$

Accordingly, there are *six* product rules, two for gradients:

$$(i) \quad \nabla(fg) = f\nabla g + g\nabla f,$$

$$(ii) \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A},$$

two for divergences:

$$(iii) \quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f),$$

$$(iv) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}), \quad \leftarrow$$

and two for curls:

$$(v) \quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f),$$

$$(vi) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$$

## Second Derivative

①  $\nabla \cdot (\nabla \tau) \equiv$  Divergence of a gradient

②  $\nabla \times (\nabla \tau) \equiv$  Curl of a gradient

③  $\nabla (\nabla \cdot \vec{r}) \equiv$  Gradient of a divergence

④  $\nabla \cdot (\nabla \times \vec{r}) \equiv$  Divergence of a curl

⑤  $\nabla \times (\nabla \times \vec{r}) \equiv$  Curl of a curl

$$\begin{aligned} \textcircled{1} \quad \nabla \cdot (\nabla \tau) &= \nabla^2 \tau \\ &= \frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2} \end{aligned}$$

↳ Laplacian of a scalar.

$$\textcircled{2} \quad \nabla \times (\nabla \tau) = 0$$

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③

$$\frac{1}{T} \left( \frac{\partial \ln Z}{\partial \beta} \right)$$

$$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}$$

(important)

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ \parallel & \parallel & \parallel \\ \parallel & \parallel & \parallel \\ \parallel & \parallel & \parallel \\ \parallel & \parallel & \parallel \end{array}$$

Handwritten mathematical derivation showing the simplification of a complex fraction:

$$\frac{2c^2r^2 + 2c^2r^2 + 2c^2r^2}{(2c^2r^2 + 2c^2r^2 + 2c^2r^2) \cdot (2c^2r^2 + 2c^2r^2 + 2c^2r^2)}$$

The numerator is simplified to  $6c^2r^2$ .

The denominator is simplified to  $6c^2r^2 \cdot 6c^2r^2$ .

The final result is  $\frac{1}{6c^2r^2}$ .

$$\begin{array}{r} \text{H}_2 \\ \text{H}_2 \\ \text{H}_2 \end{array} \quad \begin{array}{r} \text{H}_2 \\ \text{H}_2 \\ \text{H}_2 \end{array} \quad \begin{array}{r} \text{H}_2 \\ \text{H}_2 \\ \text{H}_2 \end{array}$$

$\frac{z_c}{1c}$	$\frac{z_c}{1c}$	$\frac{z_c}{1c}$
$z_c/c$	$z_c/c$	$z_c/c$
$1c$	$1c$	$1c$

④

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\frac{\partial}{\partial x_1} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} - \frac{\partial}{\partial x_2} \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} + \frac{\partial}{\partial x_3} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$= \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) + \left( \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right) + \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} \right)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = \frac{\partial}{\partial x_1} \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} \right) + \frac{\partial}{\partial x_3} \left( \frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2} \right)$$

$$= \frac{\partial^2 A_2}{\partial x_1^2} - \frac{\partial^2 A_1}{\partial x_1 \partial x_2} + \frac{\partial^2 A_3}{\partial x_1 \partial x_2} - \frac{\partial^2 A_1}{\partial x_2^2} + \frac{\partial^2 A_3}{\partial x_2 \partial x_1} - \frac{\partial^2 A_1}{\partial x_3^2} + \frac{\partial^2 A_2}{\partial x_3 \partial x_1} - \frac{\partial^2 A_3}{\partial x_3 \partial x_2} + \frac{\partial^2 A_2}{\partial x_3 \partial x_2} - \frac{\partial^2 A_3}{\partial x_2 \partial x_3}$$

$$= 0$$

$$\begin{aligned}
 \textcircled{5} \quad \nabla_\mu \times (\nabla_\mu \times \mathbf{D}) &= \nabla_\mu (\nabla_\mu \cdot \mathbf{D}) - \nabla^2 \mathbf{D} \\
 &\equiv \textcircled{3} \quad \nabla_\mu (\nabla^2 \mathbf{A}_\mu) + \nabla^2 \mathbf{A}_0 \\
 &\quad + \nabla_\mu^2 (\nabla^2 \mathbf{A}_\mu)
 \end{aligned}$$

Link to the recording:

<https://bennettu.sharepoint.com/sites/EPHY105L-Odd2021/Shared%20Documents/Forms/AllItems.aspx?id=%2Fsites%2FEPHY105L%2DOdd2021%2FShared%20Documents%2FGeneral%2FRecordings%2FEPHY105L%20Theory%20Class%2D20211007%5F133314%2DMeeting%20Recording%2Emp4&parent=%2Fsites%2FEPHY105L%2DOdd2021%2FShared%20Documents%2FGeneral%2FRecordings%2FEPHY105L%20Theory%20Class%2D20211007%5F133314%2DMeeting%20Recording%2Emp4>