Ordinary Differential Equations(EMAT102L) (Lecture-1)

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Books

Texts/References

1. S. L. Ross, Differential Equations, John Wiley & Sons, Inc., 2004.
2. E.A.Coddington, An introduction to Ordinary Differential Equations, Prentice Hall India, 1995.
3. E.Kreyszig, Advanced Engineering Mathematics, John Wiley & Son Inc.,2011.

Outline of the Lecture

We will learn

- Differential Equation
- Examples
- Classification of Differential Equations
  - Type: ODE/PDE
  - Order
  - Linear/Nonlinear
- Solution of a DE
  - General, Particular and Singular Solution

Differential Equations

An equation containing derivatives is called a **differential equation**.

Differential equations arise naturally in the study of various physical phenomena and problems in Science and Engineering.

Newton's Equation of Motion

For example the motion of a particle of mass m under the influence of a force can be represented by the Newton's equation of motion

$$m\frac{d^2x}{dt^2} = F \tag{1}$$

where

- $x(t)$: displacement of the particle
- $\frac{d^2x}{dt^2}$: acceleration of the particle.
- m : mass of the object.
- F : net force exerted on the object.

(Here the time t is the independent variable and x denotes the dependent variable (the solution) of the equation.)

The equation (1) is a differential equation, which is the mathematical model of a falling object. This is an example to show how differential equations arise in nature.

How to solve this equation?

To solve this, we need to find a function $x(t)$ which satisfies this equation.

We will learn this soon in this course!

As an another example, consider the population growth of a specie. If

- $y(t)$ represents the population of a specie at time t.
- b and d denote the birth and death rate respectively of the specie.

then by assuming that the rate of change of the population y at time t is proportional to the population at t, i.e,

$$\frac{dy}{dt} \propto y(t),$$

we get the mathematical model

Population Growth Model

$$\frac{dy}{dt} = (b-d)y(t) \tag{2}$$

The equations (1) and (2) are simple examples of ordinary differential equations.

Differential Equations:Formal definition

Definition

An equation involving derivatives of one or more dependent variables w.r.t. one or more independent variables is called a **differential equation**.

Examples

1. $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0.$
2. $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$
3. $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$
4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

Classification by type

A differential equation can be classified based on its type as Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE).

Ordinary Differential Equation

A differential equation involving derivatives of one or more dependent variables w.r.t. a single independent variable is called an **ordinary differential equation(ODE)**.

Examples

(i) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$. (ii) $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$.

Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables w.r.t. more than one independent variables is called **partial differential equation(PDE)**.

Examples

(i) $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

In this course we will deal with ODE's only.

Definition

The **order** of a differential equation(ODE or PDE) is the order of the highest derivative appearing in the equation.

Degree of the differential equation

The **degree** of a differential equation is the power of the highest order derivative involved in the differential equation(after the differential equation has been made free from radicals and fractions as far as derivatives are concerned).

Examples

1. $x\left(\dfrac{dy}{dx}\right)^2 + y^2 = 1$. (Order=1, degree=2).
2. $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$. (Order=2, degree=1)
3. $1 + \left(\dfrac{d^2y}{dx^2}\right)^{1/2} = \dfrac{dy}{dx}$. After rationalizing, we get $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx} - 1\right)^2$.
   This equation has order 2 and degree 1. Note that the degree is not $1/2$.
4. $y\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^3 + y^2 = e^x$. (Order=2, degree=1).
5. $\dfrac{d^ny}{dx^n} + x\dfrac{d^{n-1}y}{dx^{n-1}} + x^2\dfrac{d^{n-2}y}{dx^{n-2}} + \cdots x^n y = \sin x$. (Order=n, degree=1).
6. $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$. (Order=4, degree=1)
7. $(y')^2 + y = 0$. (Order=1, degree=2)

General Form of an ODE

The general form of an ordinary differential equation is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots \frac{d^ny}{dx^n}\right) = 0$$

or

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0$$

where f is any function, x denotes the independent variable, y denotes the dependent variable.

Definition

If every term in a differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$ is linear in $y, y', y'', \cdots, y^{(n)}$. i.e, If a differential equation is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x) \tag{3}$$

where $a_0(x), a_1(x) \cdots a_n(x)$ and $g(x)$ are continuous functions of x and $a_0(x) \neq 0$ for any x. Then equation (3) is called the n^{th} order **linear differential equation**. If it is not linear, then the differential equation is called **nonlinear ordinary differential equation**.

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Note that in a linear differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$,

- *the dependent variable y and its various derivatives occur in the first degree only. For instance, expressions like* y^2, xy^2, $\left(\frac{dy}{dx}\right)^2$ *should not appear in the differential equation.*
- *There should not be products of dependent variable and any of its derivatives. For instance, expressions like* $y\frac{dy}{dx}$ *should not appear in the differential equation.*
- *No transcendental functions of the dependent variable and its derivatives should occur in the differential equation. For instance, expressions like* $\sin y$, e^y, $\log y$ *should not appear in the equation.*

Examples

- $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0.$ (Linear ODE)
- $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y^2 = 0.$ (Nonlinear ODE)
- $\dfrac{d^4y}{dx^4} + x^2\dfrac{d^3y}{dx^3} + x^3\dfrac{dy}{dx} = xe^x.$ (linear ODE)
- $\dfrac{d^2y}{dx^2} + 5y\left(\dfrac{dy}{dx}\right)^3 + 6y = 0.$ (Nonlinear ODE)
- $y\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right) + 6y = 0.$ (Nonlinear ODE)

Definition

A function $y = f(x)$ is called a solution of a differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$ on any interval I if

- $y = f(x)$ is differentiable (as many times as the order of the equation) on I.
- y satisfies the differential equation for all $x \in I$.

The curve (the graph) of $y = f(x)$ is called a **solution curve**.

Example

Show that $y = ce^{-2x}$ is a solution of $y' + 2y = 0$ on $\mathbb{R}$ for a constant $c \in \mathbb{R}$.

Solution: By direct differentiation, we have

$$y' = -2ce^{-2x} = -2y$$

.

$$\Rightarrow y' + 2y = 0.$$

Example

Show that for any constant $a \in \mathbb{R}$, $y = \dfrac{a}{1-x}$ is a solution of $(1-x)y' - y = 0$ on $(-\infty, 1)$ or on $(1, \infty)$.

Solution:

$$\frac{dy}{dx} = \frac{a}{(1-x)^2}$$

$$\Rightarrow (1-x)y' - y = 0$$

$\Rightarrow y = \dfrac{a}{1-x}$ is a solution of $(1-x)y' - y = 0$ on $(-\infty, 1)$ or on $(1, \infty)$.

Note that y is not a solution on any interval containing 1.

Example

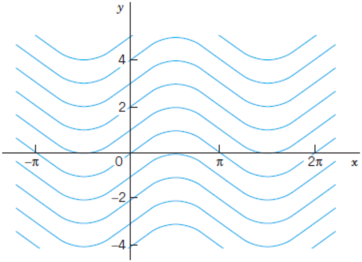
Consider the ODE

$$\frac{dy}{dx} = \cos x$$

The above ODE can be solved directly by integration on both sides. We obtain

$$y = \int \cos x + c = \sin x + c$$

where c is an arbitrary constant. This is a **family of solutions**. Each value of c gives one of these curves. The following figure shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.



General Solution and Particular Solution

To start with, let us try to understand a structure of a first order differential equation of the form

$$f(x, y, y') = 0 \tag{4}$$

One parameter family of solutions is given by $g(x, y, c) = 0$.

This one parameter family of solutions is called general solution of given ODE.

General Solution

Solution containing an arbitrary constant is called a general Solution of ODE.

For example, $y = x^2 + c$ is a general solution of the ODE $\dfrac{dy}{dx} = 2x$.

Particular Solution

Solution corresponding to a particular value of constant is called a particular solution of ODE.

For example, $y = x^2$ is a particular solution of the ODE $\dfrac{dy}{dx} = 2x$. This solution is obtained from the general solution $y = x^2 + c$ by assigning the value 0 to the constant c.

Singular Solution

If a solution to an ODE cannot be obtained from a general solution, then it is called a **singular solution**.

Example

$$\frac{dy}{dx} = (y-3)^2 \tag{5}$$

Then

$$\int \frac{dy}{(y-3)^2} = \int dx$$

$$\Rightarrow y - 3 = -\frac{1}{x+c}$$

.

$$y = 3 - \frac{1}{x+c} \tag{6}$$

where c is an arbitrary constant. We note that $y = 3$ is also a solution of (5). But no value of c in (6) gives $y = 3$.
Thus the solution $y = 3$ is a singular solution.


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-2)

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Outline of the Lecture

We will learn

- Initial Value Problems
- Formation of differential equations
- Separable Equations

Initial Value Problems

A first order ODE can be expressed as $F(x,y,y') = 0$ or $\dfrac{dy}{dx} = f(x,y)$.

Initial value problem (IVP)

A differential equation along with an initial condition is called an initial value problem (IVP), i.e.,

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Geometrically, the IVP is to find an integral curve of the DE that passes through the point (x_0, y_0).

Example

Given an amount of a radioactive substance, say 0.5 gm, find the amount present at any later time.

Solution:
Physical Information. Experiments show that at each instant a radioactive substance decomposes and is thus decaying in time proportional to the amount of substance present.
We solve the given problem in three steps.
Step 1: Setting up a mathematical model of the physical process. Let us denote by $y(t)$ the amount of substance present at any time t. We know by the physical law, $\frac{dy}{dt}$ is proportional to $y(t)$. This gives the first-order ODE

$$\frac{dy}{dt} = -ky$$

where the constant k is positive and negative sign taken due to decay.
Initially at time $t = 0$, amount is 0.5gm, i.e., $y(0) = 0.5$gm.

Step 2: Mathematical solution. Now we have the mathematical model of the physical process is the initial value problem

$$\frac{dy}{dt} = -ky(t), y(0) = 0.5$$

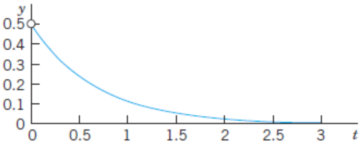
Solution of the above initial value problem is

$$y(t) = ce^{-kt}$$

Determine the value of c using the initial condition $y(0) = 0.5$, we get $c = 0.5$. Thus we have

$$y(t) = 0.5e^{-kt}.$$

Step 3: Interpretation of the solution The function $y(t) = 0.5e^{-kt}$ gives the amount of radioactive substance at time t. It starts from the correct initial amount and decreases with time because k is positive. The limit of y as $t \to \infty$ is zero.



Solution of IVP

What can we say about the solutions of the following IVPs?

- $\dfrac{dy}{dx} = \dfrac{2y}{x},\ y(2) = 4,\ (y = x^2$, Unique Solution)
- $\dfrac{dy}{dx} = \dfrac{2y}{x},\ y(0) = 4$ (No Solution)
- $\dfrac{dy}{dx} = \dfrac{2y}{x},\ y(0) = 0$ (Infinitely Many Solutions)

Thus we observe that an initial value problem can have unique, infinitely many solutions or no solution.

Problem

Which of the following IVP's have unique solution, Infinitely many solution or unique solution?

- $\dfrac{dy}{dx} = 2x, y(0) = 0$
- $x\dfrac{dy}{dx} = y - 1, y(0) = 1$
- $\left|\dfrac{dy}{dx}\right| + |y| = 0, y(0) = 1$

Suppose we are given a family of curves containing n arbitrary constants. Then we can obtain an nth order differential equation whose solution is the given family as follows.

Working Rule to form the differential equation from the given equation containing 'n' arbitrary constants:

- **Step I.** Write the equation of the given family of curves.
- **Step II.** Differentiate the equation of step I, n times so as to get n additional equations containing the n arbitrary constants and derivatives.
- **Step III.** Eliminate n arbitrary constants from the equations obtained in step I and step II. Thus we obtain the required differential equation involving a derivative of nth order.

Example

Find the differential equation for a family of circles with center at $(1,0)$ and arbitrary radius 'a'.

Solution

Equation of family of of circles with center at $(1,0)$ and arbitrary radius 'a' is

$$(x-1)^2 + y^2 = a^2,$$

where a is a constant. Differentiating the above equation w.r.t. 'x', we get

$$\begin{aligned} 2(x-1) + 2y.\frac{dy}{dx} &= 0 \\ y.\frac{dy}{dx} + (x-1) &= 0 \end{aligned}$$

which is the required differential equation.

Example

Find the differential equation corresponding to the family of curves $(x-c)^2 + y^2 = 1$, where c is a constant.

Solution

Differentiating the given equation w.r.t. 'x', we get

$$(x-c) + yy' = 0 \Rightarrow x - c = -yy'$$

Substituting the value of $x - c$ in the given family of curves, we get

$$(yy')^2 + y^2 = 1.$$

Example

Find the differential equation of all circles of radius '2'.

Solution: The equation of all circles of radius '2' is given by

$$(x-h)^2+(y-k)^2=4, \tag{1}$$

where h and k are taken to be arbitrary constants.
Differentiating the above equation w.r.t.'x', we get

$$(x-h)+(y-k)y'=0 \tag{2}$$

Differentiating again w.r.t.'x', we get

$$1+(y')^2+(y-k)y''=0$$

or

$$y-k=-\frac{\{1+(y')^2\}}{y''} \tag{3}$$

Putting this value of $(y-k)$ in (2), we get

$$x-h=-(y-k)y'=\frac{(1+y'^2)}{y''}.y'. \tag{4}$$

Substituting the values of $x-h$ and $y-k$ from (3) and (4) in (1), we get

$$\frac{\{1+(y')^2\}^2(y')^2}{(y'')^2}+\frac{\{1+(y')^2\}^2}{(y'')^2}=4 \ or \ \{(1+(y')^2\}^3=4(y'')^2.$$

We will now discuss different methods of finding the solutions of first order ODEs. These methods are described as below:

- Separation of Variables(Separable Equations)
- Reducible to Separable Equation
- Homogeneous Equation(Reducible to Separable)
- Equations reducible to Homogeneous
- Exact Differential Equation
- Reducible to Exact Differential Equation(Integrating Factors)
- Linear Differential Equation
- Reducible to Linear Differential Equation(Bernoulli's Equation)

Definition

Separable Equation: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration: Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides, we get

$$\int \frac{dy}{h(y)} = \int g(x)dx + c$$

$\Rightarrow H(y) = G(x) + c$, where c is a constant of integration.

Example

Solve $y' = y(y-1)$.

Example

Solve $y' = y(y-1)$.

Solution.

$$\frac{dy}{y(y-1)} = dx$$

$$\Rightarrow \int \frac{dy}{y(y-1)} = \int dx$$

$$\Rightarrow \int \left(\frac{-1}{y} + \frac{1}{y-1} \right) dy = x + c$$

$$\log\left(\frac{y-1}{y}\right) = x + c$$

$$\frac{y-1}{y} = e^{x+c}$$

$$y = \frac{1}{1-e^{x+c}}$$

where c is a constant of integration.

Example

Solve the differential equation $y\dfrac{dy}{dx} = (\cos^2 3x)(y+1), y(0) = 0.$

Example

Solve the differential equation $y\dfrac{dy}{dx} = (\cos^2 3x)(y+1), y(0) = 0.$

Separating the variables and integrating, we get

$$\int \frac{y}{y+1}dy = \int \cos^2 3x dx + c.$$
$$\Rightarrow \int dy - \int \frac{dy}{y+1} = \int \left(\frac{\cos 6x + 1}{2}\right) dx + c.$$
$$\Rightarrow y - \log(y+1) = \frac{1}{2}\left(\frac{\sin 6x}{6} + x\right) + c$$

is the general solution.
The initial condition $y(0) = 0$ gives $c = 0$.
So, the required solution is $y - \log(y+1) = \dfrac{1}{2}\left(\dfrac{\sin 6x}{6} + x\right)$.

Example

Solve

$$e^x \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

Example

Solve

$$e^x \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

This equation can be rewritten as $\dfrac{dy}{dx} = e^{-x}e^{-y} + e^{-3x-y}$, which is the same as

$$\frac{dy}{dx} = e^{-y}(e^{-x} + e^{-3x}).$$

This equation is now in separable variables form.

$$\frac{dy}{e^{-y}} = (e^{-x} + e^{-3x})dx$$

Integrating, we get the required solution as

$$e^y = -e^{-x} - \frac{e^{-3x}}{3} + c,$$

where c is a constant of integration.

Some Problems for Practice

1. $$\frac{dy}{dx} + \sqrt{\frac{(1+y^2)}{(1+x^2)}} = 0$$

   Answer: $\sinh^{-1} x + \sinh^{-1} y = c$, where c is a constant.

2. $$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

   Answer: $e^y = \dfrac{x^3}{3} + e^x + c$

3. $$y - x\frac{dy}{dx} = a(y^2 + \frac{dy}{dx})$$

   Answer: $y = c(a+x)(1-ay)$

4. Find the equation of the curve passing through $(1,1)$, whose differential equation is

   $$(y - yx)dx + (x + xy)dy = 0$$

   Answer: $\log|xy| + (y - x) = 0$

Example

Consider the IVP

$$\frac{dy}{dx} = 3y^{2/3}; y(0) = 0$$

If $y \neq 0$, then

$$\frac{dy}{y^{2/3}} = 3dx \Rightarrow 3y^{1/3} = 3(x+c) \Rightarrow y = (x+c)^3$$

Use initial condition $y(0) = 0$, we get $c = 0$, i.e, $y = x^3$.
Observe that the constant solution $y = 0$ is lost while solving the IVP

$$\frac{dy}{dx} = 3y^{2/3}; y(0) = 0$$

by separable variables method.
Recall that such solutions are called Singular solutions of the given ODE.


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-3)

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Outline of the Lecture

We will learn

- Equations reducible to Separable Equations
- Homogeneous Equations
- Equations Reducible to Homogeneous Equation

Recall that

Definition

Separable Equation: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration: Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides, we get

$$\int \frac{dy}{h(y)} = \int g(x)dx + c$$

$\Rightarrow H(y) = G(x) + c$, where c is a constant of integration.

Consider the differential equation

$$\frac{dy}{dx} = f(x,y)$$

If $f(x,y)$ **is of the form** $g(ax+by+c)$, then by putting $r = ax+by+c$ we get

$$\frac{dr}{dx} = a + b\frac{dy}{dx} = a + b.g(r)$$

$$\Rightarrow \qquad \frac{dr}{a+bg(r)} = dx$$

On integrating both sides and replacing r in terms of x and y, we get the solution.

Example

Solve $\dfrac{dy}{dx} = (4x + y + 1)^2$

Solution: Put $r = 4x + y + 1$, then

$$\frac{dr}{dx} = 4 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dr}{dx} - 4$$

So, from the given equation, we get

$$\frac{dr}{dx} - 4 = r^2$$

$$\int \frac{dr}{4 + r^2} = \int dx + c \Rightarrow \frac{1}{2} \tan^{-1}\left(\frac{r}{2}\right) = x + c$$

$$\Rightarrow 4x + y + 1 = 2\tan(2x + c_1)$$

where c_1 is an arbitrary constant.

Example

Solve $\dfrac{dy}{dx} = x\tan(y-x)+1.$

Put $y-x=r \Rightarrow \dfrac{dy}{dx}-1=\dfrac{dr}{dx}.$
Then from the given equation, we get

$$1+\frac{dr}{dx} = x\tan r + 1 \Rightarrow \frac{dr}{dx} = x\tan r$$

$$\Rightarrow \frac{dr}{\tan r} = xdx$$

$$\Rightarrow \int \cot r dr = \int xdx + c$$

$$\Rightarrow \log|\sin r| = \frac{x^2}{2} + c$$

$$\Rightarrow \log|\sin(y-x)| = \frac{x^2}{2} + c$$

A class of differential equations can be reduced to separable equations by using change of variables.

Definition

A function $f(x,y)$ is said to be **homogeneous** of degree n if $f(kx,ky) = k^n f(x,y)$ for all (x,y) in the domain and for all $k \in \mathbb{R}$.

Examples

1. $f(x,y) = x^2 + y^2$ is homogeneous of degree 2 as $f(kx,ky) = (kx)^2 + (ky)^2 = k^2(x^2+y^2) = k^2 f(x,y)$
2. $f(x,y) = tan^{-1}(\frac{y}{x})$ is homogeneous of degree 0.
3. $f(x,y) = \dfrac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4. $f(x,y) = x^2 + xy + 1$ is NOT homogeneous.

Definition

A first order DE of the form

$$M(x,y)dx + N(x,y)dy = 0 \ or \ \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

is said to be **homogeneous** if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.
Such equations can be reduced to **separable equations** by transformation

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting it in the above equation, we obtain,

$$v + x\frac{dv}{dx} = -\frac{M(x,vx)}{N(x,vx)}$$

We can solve this by separable method. Put $v = \dfrac{y}{x}$ to obtain the required solution.

Example

Find the general solution of

$$2xyy' - y^2 + x^2 = 0$$

Solution:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{-v^2 - 1}{2v}$$

$$\frac{2v}{v^2 + 1}dv = -\frac{1}{x}dx$$

On integrating,

$$\log|v^2+1| = -\log|x| + \log c$$

$$v^2+1 = \frac{c}{x}$$

Put $v = \dfrac{y}{x}$, we get

$$y^2+x^2 = cx$$

This can be rewritten as

$$\left(x-\frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

This represent a family of circles with centre $\left(\dfrac{c}{2}, 0\right)$ and radius $\dfrac{c}{2}$.

Example

Solve $x^2ydx - (x^3 + y^3)dy = 0$.

Solution: The given differential equation can be rewritten as $\dfrac{dy}{dx} = \dfrac{x^2y}{x^3 + y^3}$.

Let $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
Putting this in the given equation, we get

$$v + x\frac{dv}{dx} = \frac{v}{1 + v^3}.$$

Or in other words,

$$\left(\frac{1 + v^3}{v^4}\right) dv = -\frac{dx}{x}$$

which is now in separable variables form.

DE reducible to homogeneous DE

For solving differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where a_1,a_2, b_1,b_2, c_1,c_2 are constants.

- If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, then use the substitution $x = X + h$ and $y = Y + k$, where h and k are choosen such that

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

  This condition changes the given differential equation into homogeneous equation in X and Y.

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

  Now consider $Y = VX$ and solve as before.

- If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$, then use the substitution $z = a_1x + b_1y$. This transformation reduces the given DE to a separable equation in the variables x and z.

Solve

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\dfrac{1}{2} = \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} = 2$.
Put $x = X+h$, $y = Y+k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}$$

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$$

Choose h and k such that

$$h+2k-3=0, 2h+k-3=0$$

$$\Rightarrow h=1, k=1$$

$$\Rightarrow x = X+1, y = Y+1$$

So, the given equation becomes

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

which is a Homogeneous differential equation.

Example(cont.)

Put $Y = VX$, we get

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{1+2V}{2+V} \Rightarrow X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

Separating the variables, we obtain

$$\frac{dX}{X} = \frac{2+V}{1-V^2}dV$$

$$\Rightarrow \log X = \log\left(\frac{1+V}{1-V}\right) - \frac{1}{2}\log(1-V^2) + \log c$$

$$\log\left(\frac{X}{c}\right) = \log\left(\frac{X+Y}{X-Y}\right) - \log\left(\frac{\sqrt{X^2-Y^2}}{X}\right) = \log\left(\frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}\right)$$

$$\frac{X}{c} = \frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}$$

$$\Rightarrow \frac{X}{c} = \frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}$$

$$\Rightarrow (X-Y)^{3/2} = c\sqrt{X+Y}$$

$$\Rightarrow (x-1-y+1)^{3/2} = c(x-1+y-1)^{1/2}$$

Example

Solve

$$\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$.
Use the substitution $x + y = z$. Then we have,

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

Substituting the value of $x + y$ and $\dfrac{dy}{dx}$ in the given equation, we get

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$$\Rightarrow \frac{dz}{dx} = \frac{2(z-1)}{z-6}$$

$$\Rightarrow 2dx = \frac{z-6}{z-1}dz = \left(1 - \frac{5}{z-1}\right)dz$$

Example(cont.)

$$\Rightarrow 2x = z - 5\log(z-1) + c$$
$$\Rightarrow 2x = x + y - 5\log(x+y-1) + c$$
$$\Rightarrow 5\log(x+y-1) = y - x + c$$

Example

Solve $\dfrac{dy}{dx} = \dfrac{x+y-4}{x-y-6}$.

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $1 = \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} = -1$. Put $x = X+h$, $y = Y+k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X+Y+(h+k-4)}{X-Y+(h-k-6)} \tag{1}$$

If h and k are such that $h+k-4=0$ and $h-k-6=0$, then (1) becomes

$$\frac{dY}{dX} = \frac{X+Y}{X-Y}$$

which is a homogeneous DE. We can easily solve the system

$$h+k=4$$

$$h-k=6$$

of linear equations to determine the constants h and k.

Example

Solve $\dfrac{dy}{dx} = \dfrac{x+y-4}{3x+3y-5}$.

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$. Use the substitution $z = x+y$. Then we have

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}.$$

Putting these in the given DE, we get

$$\frac{dz}{dx} - 1 = \frac{z-4}{3z-5},$$

or in other words,

$$\frac{3z-5}{4z-9}dz = dx.$$

This equation is now in variable separable form.

Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-4)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Exact Differential Equation
- Solution of Exact Differential Equation

Definition

Differential of a function of 2 variables: If $F(x,y)$ is a function of two variables with continuous first order partial derivatives in a region R of the xy-plane, then its differential dF is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

In the special case when $F(x,y) = c$, where c is a constant, we have

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

So given a one-parameter family of functions $F(x,y) = c$, we can generate a first order ODE by computing the differential on both sides of the equation $F(x,y) = c$.

Definition

A differential expression

$$M(x,y)dx + N(x,y)dy \tag{1}$$

is called an **exact differential** in a region R of the xy-plane if it there exists a function F of two variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in R$. That means, expression (1) is an **exact differential** in R if there exists a function F such that $\dfrac{\partial F}{\partial x} = M(x,y)$ and $\dfrac{\partial F}{\partial y} = N(x,y)$ for all $(x,y) \in R$.

Exact Differential Equation

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact differential equation**.

Examples

1. $x^2y^3dx + x^3y^2dy = 0$ is an exact differential equation since $x^2y^3dx + x^3y^2dy = d\left(\frac{x^3y^3}{3}\right)$.
2. $ydx + xdy = 0$ is an exact differential equation since $ydx + xdy = d(xy)$.
3. $\frac{ydx - xdy}{y^2} = 0$ is an exact differential equation since $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

Theorem

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{2}$$

Let $M(x,y)$ *and* $N(x,y)$ *be continuous and have continuous first order partial derivatives for all points* (x,y) *in a rectangular domain R. Then the necessary and sufficient condition for (2) to be an exact differential equation is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 1.

Consider the equation $y^2dx + 2xydy = 0$.
Here $M = y^2$ and $N = 2xy$. So,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ the given equation is an exact equation.

Example 2.

Consider the equation $ydx + 2xdy = 0$.
Here $M = y$ and $N = 2x$. So,

$$\frac{\partial M}{\partial y} = 1 \quad and \quad \frac{\partial N}{\partial x} = 2.$$

$\Rightarrow$ the given equation is **not** an exact equation.

Example 3.

Consider the equation $(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0$
Here $M = 2x\sin y + y^3e^x$ and $N = (x^2\cos y + 3y^2e^x)$. So,

$$\frac{\partial M}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ the given equation is an exact equation.

Let us assume that the differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{3}$$

is exact in rectangular domain R. Then a one parameter family of solutions of this differential equation is given by

$$F(x, y) = c$$

where F is a function such that $\frac{\partial F}{\partial x}(x, y) = M(x, y)$ and $\frac{\partial F}{\partial y}(x, y) = N(x, y)$ for all $(x, y) \in R$ and c is an arbitrary constant.

For a given exact DE, $M(x,y)dx + N(x,y)dy = 0$, the function $F(x,y)$ can be found either by inspection or by the following procedure:

- **Step 1.** Integrate $\dfrac{\partial F}{\partial x} = M(x,y)$ with respect to x to obtain

$$F(x,y) = \int M(x,y)dx + \phi(y),$$

  where $\phi(y)$ is a constant of integration.

- **Step 2.** To determine the function $\phi(y)$, differentiate the above equation with respect to y, to obtain

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$

- **Step 3.** Use the condition

$$\frac{\partial F}{\partial y}(x,y) = N(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$

  Determine $\phi(y)$ and hence the function $F(x,y)$.

Example

Solve the equation

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution:
To check whether the equation is exact or not:
Comparing with $Mdx + Ndy = 0$, we get

$$M = (3x^2 + 4xy) \ and \ N = (2x^2 + 2y)$$

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$$

So, the given DE is exact.
Solution of exact differential equation: We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = (3x^2 + 4xy)$$

$$\frac{\partial F}{\partial y} = N(x, y) = (2x^2 + 2y)$$

Step 1. Integrate $\frac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (3x^2 + 4xy)dx + \phi(y)$$

$$\Rightarrow F(x,y) = x^3 + 2x^2y + \phi(y).$$

Step 2. Find the unknown function $\phi(y)$ using the condition $\frac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy} = 2x^2 + 2y$$

$$\frac{d\phi(y)}{dy} = 2y \Rightarrow \phi(y) = y^2 + c_0$$

where c_0 is an arbitrary constant.

$$So, F(x,y) = x^3 + 2x^2y + y^2 + c_0.$$

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

Combining the constant c_1 and c_0, we get

$$x^3 + 2x^2y + y^2 = c$$

where $c = c_1 - c_0$ is an arbitrary constant.

Thank You

Ordinary Differential Equations(EMAT102L) (Lecture-5)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Exact Differential Equation(cont.)
- How to convert a non-exact DE to an exact DE?
- Integrating Factors
- Examples

Example-1

Solve the DE by method of inspection

$$y + x\frac{dy}{dx} = 0$$

Solution:

$$d(xy) = ydx + xdy = 0.$$

$\Rightarrow xy = c$ is the solution of the given DE.

Example-2

Solve the DE by method of inspection

$$(2x + y^2)dx + 2xydy = 0$$

Solution:

$$(2x + y^2)dx + 2xydy = 0$$

$$\Rightarrow 2xdx + (y^2dx + 2xydy) = 0$$

$$\Rightarrow d(x^2) + d(xy^2) = 0$$

$$\Rightarrow x^2 + xy^2 = c.$$

Example

Solve the equation

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution:

To check whether the equation is exact or not:

Comparing with $Mdx + Ndy = 0$, we get

$$M = (3x^2 + 4xy) \ and \ N = (2x^2 + 2y)$$

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$$

So, the given DE is exact.

Solution of exact differential equation: We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = (3x^2 + 4xy)$$

$$\frac{\partial F}{\partial y} = N(x, y) = (2x^2 + 2y)$$

Example(cont.)

Step 1. Integrate $\dfrac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (3x^2 + 4xy)dx + \phi(y)$$

$$\Rightarrow F(x,y) = x^3 + 2x^2y + \phi(y).$$

Step 2. Find the unknown function $\phi(y)$ using the condition $\dfrac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy} = 2x^2 + 2y$$

$$\frac{d\phi(y)}{dy} = 2y \Rightarrow \phi(y) = y^2 + c_0$$

where c_0 is an arbitrary constant.

$$So, F(x,y) = x^3 + 2x^2y + y^2 + c_0.$$

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

Combining the constant c_1 and c_0, we get

$$x^3 + 2x^2y + y^2 = c$$

where $c = c_1 - c_0$ is an arbitrary constant.

The same differential equation can be solved by the method of grouping also.

Solve the differential equation by the method of grouping

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution: Writing the given equation in the form

$$3x^2dx + (4xydx + 2x^2dy) + 2ydy = 0$$

We can write this as

$$d(x^3) + d(2x^2y) + d(y^2) = d(c)$$

where c is an arbitrary constant.

$$\Rightarrow d(x^3 + 2x^2y + y^2) = d(c)$$

$$x^3 + 2x^2y + y^2 = c$$

is the required solution.

Example

Solve the differential equation

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0.$$

Solution: Comparing with $Mdx + Ndy = 0$, we get

$$M = (y\cos x + 2xe^y)\ and\ N = (\sin x + x^2e^y - 1)$$

Check whether the given equation is exact or not:

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y = \frac{\partial N}{\partial x}$$

So, the given DE is exact.
Solution of exact differential equation:
We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = y\cos x + 2xe^y$$

$$\frac{\partial F}{\partial y} = N(x, y) = y\sin x + x^2e^y - 1$$

Step 1. Integrate $\dfrac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (y\cos x + 2xe^y)dx + \phi(y)$$

$$\Rightarrow F(x,y) = y\sin x + x^2e^y + \phi(y).$$

Step 2. Find $\phi(y)$ using the condition $\dfrac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = \sin x + x^2e^y + \phi'(y) = \sin x + x^2e^y - 1$$

$$\phi'(y) = -1 \Rightarrow \phi(y) = -y + c_0$$

So,

$$F(x,y) = y\sin x + x^2e^y - y + c_0.$$

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$
\begin{aligned}
F(x,y) &= c_1 \\
y\sin x + x^2 e^y - y + c_0 &= c_1 \\
y\sin x + x^2 e^y - y &= c_1 - c_0 = c \\
y\sin x + x^2 e^y - y &= c
\end{aligned}
$$

Solution of an exact differential equation

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact, then the solution of this exact differential equation is given by

$$\int\limits_{\text{treating y constant}} M(x,y)dx + \int (\text{ terms of N not containing } x)dy = c$$

where c is an arbitrary constant.

Problems for Practice

Problem 1.

Solve the differential equation

$$x(1+2y)+(x^2-y)\frac{dy}{dx}=0.$$

Problem 2.

Find the values of l and m such that the equation

$$ly^2+mxy\frac{dy}{dx}=0$$

is exact. Also find its general solution.

Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE $ydx - xdy = 0$ is clearly not exact.

But observe that if we multiply both sides of this DE by $\dfrac{1}{y^2}$, the resulting ODE becomes

$$\frac{dx}{y} - \frac{x}{y^2}dy = 0$$

which is exact.

Definition

It is sometimes possible that even though the original first order DE

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact, but we can multiply both sides of this DE by some function (say, $\mu(x,y)$) so that the resulting DE

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

becomes exact. Such a function/factor $\mu(x,y)$ is known as an **integrating factor** for the original DE $M(x,y)dx + N(x,y)dy = 0$.

Example

Consider the differential equation

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

Here

$$M = (3y + 4xy^2) \; and \; N = (2x + 3x^2y)$$
$$\frac{\partial M}{\partial y} = 3 + 8xy \neq 2 + 6xy = \frac{\partial N}{\partial x}$$

So, the given DE is not exact. But if we multiply the given equation by $\mu(x, y) = x^2y$, then the given equation becomes

$$(3x^2y^2 + 4x^3y^3)dx + (2x^3y + 3x^4y^2)dy = 0$$

Now, this equation is exact, Since

$$\frac{\partial(\mu M)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial(\mu N)}{\partial x}$$

Hence $\mu(x, y) = x^2y$ is an **integrating factor** for the given DE.

Integrating Factors

Suppose the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

is not exact and that $\mu(x,y)$ is an **integrating factor** of it.
Then the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \tag{2}$$

is exact.
Now, using the criterion for exactness, equation (2) is exact iff

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Thus

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

That is, $\mu(x,y)$ satisfies the differential equation.

$$(\mu_y M - \mu_x N) + (M_y - N_x)\mu = 0 \tag{3}$$

Hence $\mu(x,y)$ is an integrating factor of given differential equation (1) iff it is a solution of the DE (3).
This is a PDE. So, we are in no position to attempt such an equation.

Let us instead attempt to determine integrating factors of certain special types.
Case I: Suppose μ is a function of x alone. That is, $\mu = \mu(x)$, $\mu_y = 0$. Then, the DE above reduces to

$$\mu_x N = (M_y - N_x)\mu$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

If further, $\dfrac{M_y - N_x}{N}$ is a function of x, $i.e$, $\dfrac{M_y - N_x}{N} = f(x)(say)$, then the above DE is separable. We try to solve it to find $\mu(x)$.

$$\mu(x) = e^{\int f(x)dx}$$

Case II: Suppose μ is a function of y alone in the DE

$$(\mu_y M - \mu_x N) + (M_y - N_x)\mu = 0$$

That is, $\mu = \mu(y)$, $\mu_x = 0$. Then the differential equation reduces to

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

If further, $\dfrac{N_x - M_y}{M}$ is a function of y, $i.e$, $\dfrac{N_x - M_y}{M} = f(y)(say)$, then the above DE is separable. We try to solve it to find $\mu(y)$.

$$\mu(y) = e^{\int f(y)dy}$$

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

Rule 1

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$(function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for the given differential equation.

Example

Solve the differential equation $(2x^2 + y)dx + (x^2y - x)dy = 0$.

Solution: Here $M = (2x^2 + y)$ and $N = (x^2y - x)$.
$\Rightarrow M_y = 1$ and $N_x = 2xy - 1$
So, the given equation is not exact.
We observe that

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{(x^2y - x)} = \frac{2 - 2xy}{x(xy - 1)} = \frac{-2}{x} = f(x)(say)$$

which depends upon x only, so integrating factor is

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2}$$

Multiplying the given ODE by I.F., we get

$$\left(2 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

which is an exact DE.

Example(cont.)

Solution of ODE:

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2}, \ \ \frac{\partial F}{\partial y} = y - \frac{1}{x}$$

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2} \Rightarrow F(x, y) = 2x - \frac{y}{x} + \phi(y)$$

To find unknown function $\phi(y)$, use the condition $\dfrac{\partial F}{\partial y} = N(x, y)$,

$$\frac{\partial F}{\partial y} = y - \frac{1}{x} \Rightarrow -\frac{1}{x} + \phi'(y) = y - \frac{1}{x} \Rightarrow \phi(y) = \frac{y^2}{2} + c_0$$

Solution of exact ODE is

$$2x - \frac{y}{x} + \frac{y^2}{2} = c$$

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

Rule 2

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$(function of y-alone), then $e^{\int f(y)dy}$ is an integrating factor for (1).

Example

Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get that

$$M = (y^4 + 2y) \text{ and } N = (xy^3 + 2y^4 - 4x)$$
$$\therefore \frac{\partial M}{\partial y} = 4y^3 + 2 \text{ and } \frac{\partial N}{\partial x} = y^3 - 4$$

Thus $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.
$\therefore$ the given equation is not exact.
Here

$$\frac{N_x - M_y}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = \frac{-3}{y} = f(y)(say)$$

$\therefore$ the integrating factor is $e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{\log y^{-3}} = y^{-3} = \dfrac{1}{y^3}$.

Example(cont.)

Multiplying the given ODE by I.F., we get

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0 \tag{4}$$

Now for this equation

$$\frac{\partial M}{\partial y}=1-\frac{4}{y^3}=\frac{\partial N}{\partial x}$$

The equation (4) is exact. Hence the required solution is

$$\left(y+\frac{2}{y^2}\right)x+y^2=c$$

where c is an arbitrary constant.

Rules to remember (for finding integrating factors)

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

Rule 1

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$(function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for (1).

Rule 2

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$(function of y-alone), then $e^{\int f(y)dy}$ is an integrating factor for (1).

Problem for Practice

Problem

Solve the differential equation

$$y(2xy + e^x)dx - e^x dy = 0$$


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-6)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Linear Equation
- Bernoulli's Equation(Reducible to Linear Equation)

Recall that a **first order linear ODE** is one in which the dependent variable and its first order derivative occur in the first degree only. That is, a first order linear ODE has the form

$$a_0(x)\frac{dy}{dx} + a_1(x)y = g(x) \tag{1}$$

where $a_0(x) \neq 0$ and $a_0(x), a_1(x), g(x)$ are continuous in an interval I.

Definition

A first order linear ODE (of the above form (1)) is called **homogeneous** if $g(x) = 0$ and **non-homogeneous** otherwise.

Definition

By dividing both sides of equation (1) by the leading coefficient $a_0(x)$, we obtain a more useful form of the above first order linear ODE, called the **standard form**, given by

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

where $P(x) = \dfrac{a_1(x)}{a_0(x)}$, $Q(x) = \dfrac{a_2(x)}{a_0(x)}$.
Equation (2) is called the **standard form** of a first order linear ODE.

Note that a linear ODE can be converted into an exact ODE by using integrating factor

$$\mu = e^{\int P(x)dx}$$

Theorem

The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu(x) = e^{\int P(x)dx}$$

A one-parameter family of solutions of this equation is

$$y \times I.F. = \int Q(x) \times I.F.dx + c$$

or

$$\left[ye^{\int P(x)dx}\right] = \int Q(x)e^{\int P(x)dx}dx + c$$

or

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + c\right)$$

First order linear ODEs

A first order linear differential equation in the dependent variable x and independent variable y is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Then it has an integrating factor of the form

$$\mu(y) = e^{\int P(y)dy}$$

A one-parameter family of solutions of this equation is

$$x \times I.F. = \int Q(y) \times I.F. dy + c$$

or

$$\left[xe^{\int P(y)dy}\right] = \int Q(y)e^{\int P(y)dy}dy + c$$

or

$$x = e^{-\int P(y)dy}\left(\int Q(y)e^{\int P(y)dy}dy + c\right)$$

Example

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$.

The standard form of this ODE is

$$\frac{dy}{dx} + \left(\frac{-4}{x}\right) y = x^5 e^x.$$

On comparing with $\dfrac{dy}{dx} + P(x)y = Q(x)$, we get

$$P(x) = \frac{-4}{x} \text{ and } Q(x) = x^5 e^x.$$

Integrating Factor (I.F) $= e^{\int P(x)dx} = e^{\int \frac{-4}{x} dx} = \dfrac{1}{x^4}$

Solution of the given ODE is given by

$$y \times I.F. = \int Q(x) \times I.F. dx + c$$

$$\Rightarrow y \times \frac{1}{x^4} = \int x^5 e^x . \frac{1}{x^4} dx + c \Rightarrow \frac{y}{x^4} = \int x e^x dx + c$$

$$\Rightarrow \frac{y}{x^4} = x e^x - e^x + c \Rightarrow y = x^5 e^x - x^4 e^x + c x^4$$

Example

Consider the differential equation

$$y^2dx + (3xy - 1)dy = 0$$

Solution: Solving for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{y^2}{1 - 3xy}$$

which is not linear in y.
Writing the above equation in the form

$$\frac{dx}{dy} = \frac{1 - 3xy}{y^2}$$

or

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}$$

Now the above equation is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Which is linear in x.

Thus $I.F = e^{\int P(y)dy} = e^{\int \frac{3}{y}dy} = y^3$.
So, the solution of the above ODE is

$$x \times I.F. = \int Q(y) \times I.F.dy + c$$

$$\Rightarrow x \times y^3 = \int \frac{1}{y^2}.y^3 dy + c$$

$$\Rightarrow x.y^3 = \frac{y^2}{2} + c$$

$$\Rightarrow x = \frac{1}{2y} + \frac{c}{y^3}$$

Problems for Practice

Example 1.

Solve $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right) y = e^{-2x}$.

Example 2.

Solve the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 5x^2.$$

Example 3.

Solve the differential equation

$$y' \cos x - y \sin x = \sec^2 x.$$

Example 4.

Solve the differential equation

$$(x + 2y^3)dy - ydx = 0.$$

Equations reducible to linear DE: Bernoulli's DE

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Qy^n \tag{3}$$

where n is any real number, is called **Bernoulli's differential equation** named after the **Swiss mathematician James Bernoulli(1654-1705)**.

Note that when $n = 0$ or 1, Bernoulli's DE is a linear DE.
<u>Method of Solution</u>: Multiply by y^{-n} throughout the DE (4) to get

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \tag{4}$$

Use the substitution $z = y^{1-n}$. Then $\dfrac{dz}{dx} = (1-n)\dfrac{1}{y^n}\dfrac{dy}{dx}$.
Substituting in equation (4), we get $\dfrac{1}{1-n}\dfrac{dz}{dx} + P(x)z = Q(x)$, which is a linear DE.

Example

Solve the Bernoulli's DE $\dfrac{dy}{dx} + y = xy^3$.

Solution: Multiplying the above equation throughout by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

Putting $z = \dfrac{1}{y^2}$, we get $\dfrac{dz}{dx} - 2z = -2x$, which is a linear DE.
Integrating Factor (I.F.) = $e^{-\int 2dx} = e^{-2x}$.
Therefore the solution is

$$z.e^{-2x} = \left[-2\int xe^{-2x}dx + c\right]$$

$$z = e^{2x}\left[-2\int xe^{-2x}dx + c\right] = x + \frac{1}{2} + ce^{2x}.$$

Putting back $z = \dfrac{1}{y^2}$ in this, we get the final solution

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$$

Problems for practice

Example

Solve the Bernoulli equation $y' + xy - 2xy^2 = 0$

Example

Solve the Bernoulli equation $x^3y' = x^2y - y^4 \cos x, y(0) = 1$


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-10 and 11)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Second Order Linear Differential Equation
- Solution of Second Order DE
- Linearly Dependent/Independent Functions
- Wronskian
- Abel's Formula

Second Order Linear ODE

The general form of a second order differential equation is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), x \in I$$

Here I is an interval contained in R and the functions $a_0(x)$, $a_1(x)$, $a_2(x)$ and F are real valued continuous functions defined on I and $a_0(x) \neq 0$.

The above equation is called **homogeneous** if $F(x) = 0$ for all x otherwise it is called **nonhomogeneous**.

Examples

$y'' - y = 0$ **(Linear, Homogeneous)**

$y'' + y' + y = \sin x$ **(Linear, Nonhomogeneous)**

$y'' + 3xy' + x^3y = e^x$ **(Linear, Nonhomogeneous)**

Solution of Second Order ODE

Consider the second order ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), x \in I \tag{1}$$

A function y defined on an interval I is called a solution of the second order ODE if

- y is twice differentiable.
- y satisfies equation (1).

Examples

1. e^x, e^{-x} are solutions of $y'' - y = 0$.
2. $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$.

Consider the initial value problem (IVP) for a second order linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x),\; y(x_0) = c_0,\; y'(x_0) = c_1$$

Existence and Uniqueness Theorem for Second Order IVP

If $a_0(x)$, $a_1(x)$, $a_2(x)$ and $F(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x)$ in the interval I.

Note: This is the sufficient condition only.

Consider the initial value problem (IVP) for a second order Homogeneous linear ODE

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0,\ \ y(x_0)=0,\ \ y'(x_0)=0$$

Existence and Uniqueness Theorem for Second Order IVP

If $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous functions on an interval I where $a_0(x)\neq 0$ and $x_0\in I$, then the above initial value problem has a **unique solution** $y(x)=0$ for all x in the interval I.

Note: This is the sufficient condition only.

Superposition Principle

Consider the second order Homogeneous linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{2}$$

If y_1, y_2 are two solutions of the linear second order homogeneous differential equation (2), then

$$c_1y_1 + c_2y_2, c_1, c_2 \in \mathbb{R}$$

is also a solution of the above equation. That is, any linear combination of solutions of the homogeneous linear differential equation (2) is also a solution of (2).

Linearly Dependent Functions

The functions $f(x)$ and $g(x)$ are said to be **linearly dependent** on an interval I if there exist constants a, b**, not all zero**, such that

$$af(x) + bg(x) = 0$$

for every $x \in I$.

Linearly Independent Functions

The functions $f(x)$ and $g(x)$ are said to be **linearly independent** on an interval I if there exist constants a, b such that

$$af(x) + bg(x) = 0 \ \forall \ x \in I \Rightarrow a = b = 0$$

for every $x \in I$.

Examples

1. The functions x and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$. For there exist constants c_1 and c_2, not both zero, such that

$$c_1 x + c_2(2x) = 0$$

for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2$, $c_2 = -1$.

2. The functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$

Examples

1. The functions x and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$.
   For there exist constants c_1 and c_2, not both zero, such that
   $$c_1 x + c_2(2x) = 0$$
   for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2$, $c_2 = -1$.
2. The functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ are linearly dependent on the interval $(-\infty, \infty)$ because $f_1(x)$ is a constant multiple of $f_2(x)$.
3. The functions x and x^2 are linearly independent on $0 \leq x \leq 1$.
   Since $c_1 x + c_2 x^2 = 0$ for all x on $0 \leq x \leq 1$ implies that both $c_1 = 0$ and $c_2 = 0$.
4. The functions $f_1(x) = x$ and $f_2(x) = |x|$ are linearly independent on $(-\infty, \infty)$.
   Neither of the functions is a constant multiple of the other on $(-\infty, \infty)$ but linearly dependent on $(0, \infty)$ and $(-\infty, 0)$.

Definition

The **Wronskian** of two differentiable functions $f(x)$ and $g(x)$ is defined by

$$W(f,g) = W(f,g)(x) = det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} = f(x)g'(x) - f'(x)g(x)$$

Theorem

Let y_1, y_2 be two solutions of the homogeneous linear Second order DE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

on an interval I. Then the set of solutions $\{y_1, y_2\}$ is ***linearly independent*** *on I if and only if*

$$W(y_1, y_2) \neq 0$$

for every x in the interval I.

Theorem

The Wronskian $W(y_1, y_2)$ of two solutions y_1, y_2 of (1) is either identically zero or never zero on the interval.

Example

Show that the solutions $\sin x$ and $\cos x$ of $y'' + y = 0$ are linearly independent.

Solution: Here $W(\sin x, \cos x)$=

$$\det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all real x. Thus $W(\sin x, \cos x) \neq 0$ for all real x.
So, we conclude that $\sin x$ and $\cos x$ are linearly independent solutions of the given differential equation on every real interval.

Result 1.

If y_1 and y_2 have a common zero at point x_0 in the interval $[a, b]$, then y_1 and y_2 are linearly dependent.

Solution: Since y_1 and y_2 have common zero at $x_0 \in [a, b]$,

$$\Rightarrow y_1(x_0) = y_2(x_0) = 0$$

So,

$$W(y_1, y_2)(x_0) = \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0$$

$$\Rightarrow W(y_1, y_2)(x_0) = 0 \text{ for some point } x_0 \in [a, b].$$

$\Rightarrow$ y_1 and y_2 are linearly dependent.
Note: Here y_1 and y_2 are the solutions of the same differential equation.

Result 2.

If y_1 and y_2 have a relative maxima or minima at some common point $x_0 \in [a, b]$, then y_1 and y_2 are linearly dependent.

Solution: Since y_1 and y_2 have a relative maxima or minima at some common point $x_0 \in [a, b]$,

$$\Rightarrow y_1'(x_0) = y_2'(x_0) = 0$$

So,

$$W(y_1, y_2)(x_0) = \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0$$

$$\Rightarrow W(y_1, y_2)(x_0) = 0 \text{ for some point } x_0 \in [a, b].$$

$\Rightarrow y_1$ and y_2 are linearly dependent.

Note: Here y_1 and y_2 are the solutions of the same differential equation.

Definition

If $\{y_1, y_2\}$ are two linearly independent solutions of the homogeneous linear second order DE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{3}$$

where $a_0(x) \neq 0$, $a_i(x), i = 1, 2$ are continuous functions on an interval I, then the set $\{y_1, y_2\}$ is said to be the **fundamental set of solutions** on the interval I.

Theorem

There exists a fundamental set of solutions(Linearly independent solutions) for the homogeneous linear second order DE (3) on an interval I.

Theorem

Let $\{y_1, y_2\}$ be a fundamental set of solutions for the homogeneous linear second order DE (3) on an interval I. Then the general solution of the equation (3) on the interval I is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1, c_2 are arbitrary constants.

Note:

- For a homogeneous linear second order ODE, if we know two linearly independent solutions, then every solution can be obtained with the linear combination of these two linearly independent solutions.
- That is, if y_1, y_2 are two linearly independent solutions of the homogeneous linear second order DE, then the general solution $y(x)$ can be written as the linear combination of these solutions. i.e,

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

  where c_1 and c_2 are arbitrary constants.

Example

1. The functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.

Example

1. The functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.
   - Here Wronskian $W(e^{3x}, e^{-3x}) = -6 \neq 0$ for every $x \in (-\infty, \infty)$.
   - So the solutions y_1, y_2 are linearly independent on $(-\infty, \infty)$.
   - Hence we can conclude that $\{y_1, y_2\}$ is a fundamental set of solutions.
   - Therefore $y(x) = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on $(-\infty, \infty)$.

Abel's Theorem

If y_1 and y_2 are solutions of the DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

where $a_0(x) \neq 0$, $a_i(x), i = 1, 2$ are continuous functions on an open interval I, then the Wronskian $W(y_1, y_2)(x)$ is given by

$$W(y_1, y_2)(x) = c \exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right],$$

where c is a certain constant that depends on y_1 and y_2, but not on x.
Further, $W(y_1, y_2)(x)$ is either zero for all $x \in I$ (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Problem

Let y_1 and y_2 be two linearly independent solutions of

$$y'' + (\sin x)y = 0 \text{ in } [0, 1]$$

Let $g(x) = W(y_1, y_2)$, then show that $g'(x) = 0$.

Solution: Here $a_0(x) = 1, a_1(x) = 0, a_2(x) = \sin x$.
Therefore, by Abel's formula,

$$g(x) = W(y_1, y_2) = c \exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right] = ce^0 = c.$$

$$\Rightarrow g'(x) = 0.$$


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-12)

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We will learn

- Method of Reduction of Order

Method of Reduction of Order

According to this method, if one non-zero solution of a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

is known, then by making the appropriate transformation we may reduce the given equation to another homogeneous linear equation that is one order lower than the original.

Consider the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad (1)$$

Derivation

- Suppose $f(x)$ is a known non-trivial solution of the given equation, then we can determine a solution $g(x)$ using this method such that $\{f, g\}$ forms a fundamental set of solutions of (1). i.e, $f(x)$ and $g(x)$ are linearly independent.
- For this, use the transformation $g(x) = f(x)v$, where f is the known solution of (1) and v is a function of x to be determined.
- Then differentiating, we obtain

$$g' = f(x)v' + f'(x)v$$

$$g'' = f(x)v'' + 2f'(x)v' + f''(x)v.$$

Since g is a solution of (1), therefore substituting the values of g, g', g'' in (1), we obtain

$$\Rightarrow a_0(x)(v''f(x) + 2f'(x)v' + f''(x)v) + a_1(x)(f(x)v' + f'(x)v) + a_2(x)f(x)v = 0$$

$$a_0(x)f(x)v'' + \{2a_0(x)f'(x) + a_1(x)f(x)\}v' + \{a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)\}v = 0$$

Since f is a solution of (1), the coefficient of v is zero. So, the last equation becomes

$$a_0(x)f(x)v'' + \{2a_0(x)f'(x) + a_1(x)f(x)\}v' = 0.$$

Put $w = \dfrac{dv}{dx}$, then the above equation becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

$$\Rightarrow \frac{w'}{w} = -\frac{2a_0(x)f'(x) + a_1(x)f(x)}{a_0(x)f(x)} = -\frac{2f'(x)}{f(x)} - \frac{a_1(x)}{a_0(x)}$$

- Integrating, we obtain

$$\log|w| = -\log[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}dx + \log|c|$$

$$\Rightarrow w = \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}$$

Since $w = \dfrac{dv}{dx}$, so the above equation becomes

$$v = \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}dx.$$

Method of Reduction of Order(cont.)

- Therefore the new solution is

$$g(x) = f(x)v(x) = f(x)\frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}$$

Also, the original known solution $f(x)$ and the new solution $g(x) = v(x)f(x)$ are linearly independent. Since

$$\begin{aligned}
W(f,g)(x) &= W(f,fv)(x) = f(x)(f(x)v)' - f'(x)(f(x)v) \\
&= f(x)(v'(x)f(x) + v(x)f'(x)) - v(x)f(x)f'(x) \\
&= [f(x)]^2 v' \\
&= [f(x)]^2 \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2} \\
&= e^{-\int \frac{a_1}{a_0}dx} \neq 0.
\end{aligned}$$

- The general solution of the ODE (1) is

$$y(x) = c_1 f(x) + c_2 g(x), c_1, c_2 \text{ are arbitrary constants}$$

Hypothesis

Let f be a known non-trivial solution of the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

then we can determine a solution $g(x)$ by the **Method of reduction of order** such that $\{f, g\}$ forms a fundamental set of solutions of (1). i.e, $f(x)$ and $g(x)$ are linearly independent.

- **Conclusion 1.** The transformation $g(x) = f(x)v$ reduces the equation (1) to the first order homogeneous linear order differential equation

  $$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0 \tag{2}$$

  in the dependent variable w, where $w = v'$.

- **Conclusion 2.** The particular solution
$$w = \frac{e^{-\int \frac{a_1}{a_0} dx}}{[f(x)]^2}$$
of equation (2) gives rise to the function v, where
$$v = \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{[f(x)]^2} dx.$$
The function g defined by $g(x) = f(x)v(x)$ is then a solution of the second order equation (1).
- **Conclusion:** The original known solution $f(x)$ and the new solution $g(x) = v(x)f(x)$ are linearly independent. Hence, the general solution of (1) can be expressed as
$$y(x) = c_1 f(x) + c_2 g(x).$$

Example

Given that $y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

find a linearly independent solution by reducing the order.

Solution: Given differential equation is

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Observe that $y = x$ satisfies the given differential equation.

- Let us use the transformation $g(x) = f(x)v = xv$.
- Here

$$v(x) = \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}dx = \int \frac{e^{-\int \frac{-2x}{x^2+1}dx}}{x^2}dx$$
$$= \int \frac{x^2+1}{x^2}dx = \int \left(1+\frac{1}{x^2}\right)dx = x - \frac{1}{x}.$$

Example(cont.)

- $g(x) = f(x)v(x) = x\left(x - \frac{1}{x}\right) = x^2 - 1.$
  Also f and g are linearly independent, since $W(x, x^2 - 1) \neq 0$.
- Thus the general solution of the ODE is

$$y(x) = c_1 f(x) + c_2 g(x) = c_1 x + c_2(x^2 - 1), c_1, c_2 \text{ are arbitrary constants}$$

Example

If e^x is one of the solutions of homogeneous equation

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$$

find a linearly independent solution by reducing the order.


Thank You

Ordinary Differential Equations(EMAT102L)
(Lecture-13 and 14)

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We will learn

- Higher Order Differential Equations
- Results Related to Higher Order Differential Equations
- Homogeneous Linear Differential Equation with constant coefficients

The general form of an n-th order linear differential equation is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \tag{1}$$

where the coefficients $a_i(x); i = 0, 1, \cdots n$ and $F(x)$ are continuous and $a_0(x) \neq 0$ for every $x \in I$.

The above equation is said to be **homogeneous** if $F(x) = 0$ and **nonhomogeneous** if $F(x) \neq 0$.

Initial Value Problem for n th order Linear Differential Equation

Consider the initial value problem (IVP) for an nth order linear nonhomogeneous ODE

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x),$$

with the initial conditions $y(x_0) = c_0,\ y'(x_0) = c_1, \cdots y^{n-1}(x_0) = c_n$.

Existence and Uniqueness Theorem for an nth order linear nonhomogeneous IVP

If $a_0(x)$, $a_1(x)$, $a_2(x)$, $\cdots$ $a_n(x)$ and $F(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x)$ in the interval I.

Note: This is the sufficient condition only.

Consider the initial value problem (IVP) for an nth order Homogeneous linear ODE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0,$$

with the initial conditions $y(x_0) = 0,\ y'(x_0) = 0, \cdots y^{n-1}(x_0) = 0$

Existence and Uniqueness Theorem for nth Order Homogeneous Linear IVP

If $a_0(x), a_1(x), a_2(x), \cdots\ a_n(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x) = 0$ for all x in the interval I.

Note: This is the sufficient condition only.

In the following theorem, we observe that the sum of two or more solutions of a homogeneous linear DE is also a solution.

Theorem

Superposition principle-Homogeneous equations: *Let $y_1, y_2, \cdots y_n$ be solutions of the n-th order homogeneous DE*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

where the $c_i; i = 1, 2, \cdots n$ are arbitrary constants, is also a solution to (2) on the same interval.

Linear dependent/independent functions

Definition

A set of functions $f_1(x), f_2(x) \cdots f_n(x)$ are said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, \cdots c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every $x \in I$. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

In other words, a set of functions $f_1(x), f_2(x) \cdots f_n(x)$ is **linearly independent** on an interval if the only constants for which

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \;\; \forall \;\; x \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Example

- The functions $f_1(x) = \cos^2 x, f_2(x) = \sin^2 x, f_3(x) = \sec^2 x, f_4(x) = tan^2x$ are linearly dependent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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- The functions $f_1(x) = \cos^2 x, f_2(x) = \sin^2 x, f_3(x) = \sec^2 x, f_4(x) = tan^2 x$ are linearly dependent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
  Since
  $$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$
  for $c_1 = c_2 = c_4 = 1, c_3 = -1$.
  We used here $\sin^2 x + \cos^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$.

Definition

Suppose each of the functions $f_1(x), f_2(x) \cdots f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \cdots f_n) = det \begin{pmatrix} f_1 & f_2 & f_3 & . & . & . & f_n \\ f_1' & f_2' & f_3' & . & . & . & f_n' \\ f_1'' & f_2'' & f_3'' & . & . & . & f_n'' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & . & . & . & f_n^{(n-1)} \end{pmatrix}$$

where the prime denote derivatives, is called the **Wronskian** of the functions $f_1, \cdots f_n$.

Theorem

Let $y_1, y_2, \cdots y_n$ be n solutions of the homogeneous linear n-th order DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{3}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$. Then the set of solutions $\{y_1, y_2 \cdots y_n\}$ is ***linearly independent*** *on I if and only if*

$$W(y_1, y_2, \cdots y_n) \neq 0$$

for every x in the interval I.

Theorem

The Wronskian $W(y_1, y_2, \cdots y_n)$ of n solutions $y_1, y_2 \cdots y_n$ of (3) is either identically zero or never zero on the interval.

Recall the homogeneous linear n-th order DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{4}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$.

Fundamental Set of Solutions

Any set $\{y_1, y_2, \cdots y_n\}$ of n linearly independent solutions of the homogeneous linear n-th order DE (4) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem

There exists a fundamental set of solutions for the homogeneous linear n-th order DE (4) on an interval I.

Theorem

Let $\{y_1, y_2, \cdots y_n\}$ be a fundamental set of solutions for the homogeneous linear n-th order DE (4) on an interval I. Then the general solution of the equation (4) on the interval I is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_i, i = 1, 2, \cdots n$ are arbitrary constants.

Example

The functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$, and $y_3(x) = e^{3x}$ satisfy the DE

$$y''' - 6y'' + 11y' - 6y = 0.$$

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$$y''' - 6y'' + 11y' - 6y = 0.$$

Since $W(e^x, e^{2x}, e^{3x}) = 2e^{6x} \neq 0$ for every real x.
Therefore the functions y_1, y_2, y_3 form a fundamental set of solutions on $(-\infty, \infty)$.
Thus the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Theorem

Consider the nonhomogeneous 2^{nd} order linear ODE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) : \quad a < x < b. \qquad (NH)$$

where $a_i(x);\ i = 0, 1, 2, \cdots n$ are continuous function on (a, b) and $a_0(x) \neq 0$ for any $x \in (a, b)$. Let

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (H)$$

be the corresponding homogeneous equation.
If $y_c(x)$ is a solution of (H) and $y_p(x)$ is a solution of (NH) then

$$y(x) = y_c(x) + y_p(x)$$

is a solution of (NH).

Solution of a Linear Nonhomogeneous Equation(cont.)

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

and since $y_p(x)$ is a solution of (NH) we get

$$a_0(x)\frac{d^n y_p}{dx^n} + a_1(x)\frac{d^{n-1} y_p}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_p}{dx} + a_n(x)y_p = F(x) \tag{6}$$

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

and since $y_p(x)$ is a solution of (NH) we get

$$a_0(x)\frac{d^n y_p}{dx^n} + a_1(x)\frac{d^{n-1} y_p}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_p}{dx} + a_n(x)y_p = F(x) \tag{6}$$

Adding equations (5) and (6), we get

$$a_0(x)\frac{d^n (y_c+y_p)}{dx^n} + a_1(x)\frac{d^{n-1} (y_c+y_p)}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d(y_c+y_p)}{dx} + a_n(x)(y_c+y_p) = F(x)$$

which implies that the function $y_c + y_p$ is also a solution of (NH).

General solution of (NH):

From the last Theorem, we can conclude the following.
If $y_1, y_2, \cdots y_n$ are n linearly independent solutions of (H) and $y_p(x)$ is a solution of (NH). Then the general solution of (NH) can be expressed as

$$y(x) = \sum_{i=1}^{n} c_i \, y_i(x) + y_p(x) = y_c(x) + y_p(x)$$

where the first term $y_c(x)$ is called the **complementary function** and $y_p(x)$ is called the **particular solution** or **particular integral** of (NH).

Example: If $y = x$ is the solution of the nonhomogeneous equation

$$\frac{d^2y}{dx^2} + y = x.$$

and $y = \sin x$ is a solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Then by the previous Theorem, the sum

$$\sin x + x$$

is the solution of the given nonhomogeneous equation.

Solution of *n*th order homogeneous linear equation with constant coefficients

Consider the *n*th order homogeneous linear equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0 \tag{7}$$

where $a_0 \neq 0$, a_1,$a_2 \cdots\ a_n$ are real constants.

Solution of nth order homogeneous linear equation with constant coefficients

Consider the nth order homogeneous linear equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{7}$$

where $a_0 \neq 0$, a_1,a_2 $\cdots$ a_n are real constants.

- Now, we will try to find the general solution of the above equation .
- We need to find a function which can be the solution of the above equation?

Solution of nth order homogeneous linear equation with constant coefficients

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where $a_0 \neq 0$, $a_1, a_2 \cdots a_n$ are real constants.

- Now, we will try to find the general solution of the above equation .
- We need to find a function which can be the solution of the above equation?
- For this, we need a function such that its derivatives are constant multiples of itself.
- Do we know any function f having this property

$$\frac{d^k f(x)}{dx^k} = cf(x) \; \forall \; x$$

Homogeneous Linear equation with constant coefficients

- **Answer:** Yes, exponential function e^{mx}, where m is a constant such that

$$\frac{d^k}{dx^k}(e^{mx}) = m^k e^{mx}$$

- Thus we seek solution of (7) of the form $y = e^{mx}$, where the constant m will be choosen such that e^{mx} does satisfy the equation.

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- Thus we seek solution of (7) of the form $y = e^{mx}$, where the constant m will be choosen such that e^{mx} does satisfy the equation.
- Assume that $y = e^{mx}$ is a solution of equation (7) for certain m, we have

$$y' = me^{mx}, \frac{d^2y}{dx^2} = m^2 e^{mx}, \cdots \frac{d^ny}{dx^n} = m^n e^{mx}.$$

- Substituting in (7), we get

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$
$$(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) e^{mx} = 0$$

- Since $e^{mx} \neq 0$, we obtain the polynomial equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{8}$$

- This equation is called the **auxiliary equation** or **characteristic equation** of the given differential equation (7).
- We note that if $y = e^{mx}$ is a solution of (7), then the constant m should satisfy the equation (8).
- Hence to solve (7), we write the auxiliary equation (8) and solve it for m.
- Since equation (8) is a polynomial of degree n. Therefore it has n roots(real or complex).
- Thus three cases arise according as the roots of the auxiliary equation (8).
  (i) The roots are real and distinct.
  (ii) The roots are real and repeated.
  (iii) The roots are complex.

Case I: The roots are real and distinct

Suppose the roots of the auxiliary equation (8) are n distinct real numbers say

$$m_1, m_2, \cdots m_n.$$

Then

$$e^{m_1 x}, e^{m_2 x}, \cdots e^{m_n x}$$

are distinct n solutions of (7).
Also, these n solutions are linearly independent.($\because W(e^{m_1 x}, e^{m_2 x}, \cdots e^{m_n x}) \neq 0$)
Thus the general solution of (7) can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

where c_1, c_2, $\cdots$ c_n are arbitrary constants.

Example

Consider the differential equation $4y'' - 20y' + 24y = 0$.

The auxiliary equation is

$$4m^2 - 20m + 24 = 0.$$

$$\Rightarrow m_1 = 2, m_2 = 3.$$

That is, the roots are real and distinct.
$\therefore$ The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

where c_1 and c_2 are arbitrary constants.
Verify that e^{2x} and e^{3x} are linearly independent.
$\because$ their Wronskian is $\neq 0$

Case II- If the roots are real and repeated.

We will study this case by considering a simple example.
Consider the differential equation

$$y'' - 6y' + 9y = 0.$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$\Rightarrow (m-3)^2 = 0.$$

The roots of this equation are

$$m_1 = 3, m_2 = 3$$

which are real but not distinct.
Corresponding to the root m_1, we have the solution e^{3x}, and corresponding to m_2, we have the same solution e^{3x}.

Case II- If the roots are real and repeated(cont.).

- We can write the combination $c_1e^{3x} + c_2e^{3x} = (c_1 + c_2)e^{3x} = C_3e^{3x}$, which is involving only one arbitrary constant.

Case II- If the roots are real and repeated(cont.).

- We can write the combination $c_1e^{3x} + c_2e^{3x} = (c_1 + c_2)e^{3x} = C_3e^{3x}$, which is involving only one arbitrary constant.
- So
$$y = C_3e^{3x}$$
is not the general solution of the given differential equation.
- We need to find another linearly independent solution.
- But how shall we proceed to do so?

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- So
$$y = C_3e^{3x}$$
is not the general solution of the given differential equation.
- We need to find another linearly independent solution.
- But how shall we proceed to do so?
- Using the method of reduction of order, we find that the another linearly independent solution is
$$xe^{3x}.$$
- Thus the general solution of the given equation is
$$y = c_1e^{3x} + c_2xe^{3x}$$
$$y = (c_1 + c_2x)e^{3x}$$

Theorem

Consider the n th- order homogeneous linear differential equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_ny=0$$

with constant coefficients.

1. If the auxiliary equation $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0$ has the real root m occurring k times, then the part of the general solution of the given equation corresponding to this k fold repeated root is

$$y=(c_1+c_2x+c_2x^2+\cdots+c_kx^{k-1})e^{mx}.$$

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$$y=(c_1+c_2x+c_2x^2+\cdots+c_kx^{k-1})e^{mx}.$$

2. If, further, the remaining roots of the auxiliary equation are the distinct real numbers $m_{k+1},\cdots m_n$, the the general solution of the given equation is

$$y=(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})e^{mx}+c_{k+1}e^{m_{k+1}x}+\cdots+c_ne^{m_nx}.$$

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$$y=(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})e^{mx}+c_{k+1}e^{m_{k+1}x}+\cdots+c_ne^{m_nx}.$$

Example

Find the general solution of

$$y''' - y'' - y' + y = 0$$

The auxiliary equation is

$$m^3 - m^2 - m + 1 = 0$$

The roots of the auxiliary equation are $1, 1, -1$.
The general solution is

$$y = (c_1 + c_2x)e^x + c_3e^{-x}.$$

Example

If the roots of the auxiliary equation are $2, 2, 2, -1$. Then, the general solution of corresponding DE is

$$y = (c_1 + c_2 x + c_3 x^2) e^{2x} + c_4 e^{-x}.$$

Case III: If the roots are Conjugate Complex

- Suppose that the auxiliary equation has the complex root $a + ib$ (a, b real, $i^2 = -1$, $b \neq 0$) which is nonrepeated.
- Then since the coefficients are real, the conjugate complex number $a - ib$ is also a nonrepeated root.
- Therefore the corresponding part of the general solution is

$$k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x},$$

  where c_1 and c_2 are arbitrary constants.
- It is desirable to replace these by two real independent solutions.

Case III: If the roots are Complex(cont.)

For this, consider

$$
\begin{aligned}
k_1e^{(a+ib)x} + k_2e^{(a-ib)x} &= k_1e^{ax}e^{ibx} + k_2e^{ax}e^{-ibx} \\
&= e^{ax}[k_1e^{ibx} + k_2e^{-ibx}] \\
&= e^{ax}[k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)] \\
&\quad (\text{ Using Euler's Formula } e^{i\theta} = \cos\theta + i\sin\theta.) \\
&= e^{ax}[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx] \\
&= e^{ax}[c_1\cos bx + c_2\sin bx]
\end{aligned}
$$

where $c_1 = k_1 + k_2$, $c_2 = i(k_1 - k_2)$ are two new arbitrary constants.
Thus the general solution corresponding to the nonrepeated conjugate complex roots $a \pm ib$ is

$$e^{ax}[c_1\cos bx + c_2\sin bx].$$

Theorem

Consider the n th- order homogeneous linear differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

with constant coefficients.

1. If the auxiliary equation $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0$. has the conjugate complex roots $a + ib$ and $a - ib$, neither repeated, then the corresponding part of the general solution of the given differential equation is

$$y = e^{ax}[c_1 \cos bx + c_2 \sin bx].$$

2. If, however, $a + ib$ and $a - ib$ are each k-fold roots of the auxiliary equation, then the general solution of the given equation is

$$y = e^{ax}[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) \cos bx + (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1}) \sin bx].$$

Example

Find the general solution of

$$y'' + y = 0$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 0$, $b = 1$.
The general solution is

$$y = e^{0x}(c_1 \cos x + c_2 \sin x).$$

$$y = (c_1 \cos x + c_2 \sin x)$$

Example

Find the general solution of

$$y'' - 6y' + 25y = 0$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

$$\Rightarrow m = 3 \pm 4i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 3$, $b = 4$.
The general solution is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Example

If the roots of the auxiliary equation are $1+2i, 1-2i, 1+2i, 1-2i$, then the general solution of corresponding DE is

$$y = e^{x}[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x].$$

Example

Solve the initial value problem

$$y'' - 6y' + 25y = 0, y(0) = -3, y'(0) = -1$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

$$\Rightarrow m = 3 \pm 4i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 3$, $b = 4$.
The general solution is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Since

$$y(0) = -3 \Rightarrow c_1 = -3.$$

$$y'(0) = -1 \Rightarrow c_2 = 2.$$

Thus the solution is

$$y = e^{3x}(2\cos 4x - 3\sin 4x).$$


Thank You

Method of variation of parameters

- Method of Undetermined Coefficients is a restricted method which cannot apply to find particular integral for such type of equations like

$$\frac{d^2y}{dx^2} + y = \tan x.$$

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- Variation of Parameters is used to find Particular integral.
- It applies in all cases (including variable coefficients) in which the complementary function of the given differential equation is known.

Method of variation of parameters

- Method of Undetermined Coefficients is a restricted method which cannot apply to find particular integral for such type of equations like

$$\frac{d^2y}{dx^2} + y = \tan x.$$

- Variation of Parameters is used to find Particular integral.
- It applies in all cases (including variable coefficients) in which the complementary function of the given differential equation is known.

Method of variation of parameters

Consider the second order non-homogeneous linear equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x), \tag{1}$$

where $a_0(x) \neq 0$ and $a_0(x), a_1(x), a_2(x), F(x)$ are continuous in $[a, b]$.

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- Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

  $$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

  Then the complementary function of the given equation is

  $$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

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Then the complementary function of the given equation is
$$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

- In the **method of variation of parameters**, we replace the arbitrary constants c_1 and c_2 in the complementary function by respective function $A(x)$ and $B(x)$ which will be determined so that the resulting function
$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \tag{2}$$
is the particular integral of equation (1).

- Now, $y_p(x)$ contains two functions $A(x)$ and $B(x)$ which are to be determined.

Method of variation of parameters(cont.)

- Now, $y_p(x)$ contains two functions $A(x)$ and $B(x)$ which are to be determined.
- Differentiating the above equation, we get

$$y_p'(x) = A'y_1 + Ay_1' + B'y_2 + By_2' = (A'y_1 + B'y_2) + (Ay_1' + By_2') \tag{3}$$

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- If we differentiate this equation again, then these equations would contain the second derivative A'' and B'' of the unknown functions. In order to avoid the second order derivatives, we impose the condition

$$A'y_1 + B'y_2 = 0.$$

With this condition, (3) reduces to

$$y_p'(x) = Ay_1' + By_2',$$

we obtain

$$y_p''(x) = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

- Since $y_p(x)$ satisfies the given equation. Therefore, substituting the expressions for $y_p(x),\ y_p'(x)$ and $y_p''(x)$ in equation (1), we obtain

$$
\begin{aligned}
& a_0(x)[Ay_1'' + A'y_1' + By_2'' + B'y_2'] + a_1(x)[Ay_1' + By_2'] + a_2(x)[Ay_1 + By_2] = F(x) \\
& \Rightarrow a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1] \\
& \qquad\qquad + B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = F(x)
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$$\Rightarrow a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1]$$
$$+B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = F(x)$$

- Since, $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding homogeneous equation, we obtain

$$a_0(x)[A'y_1' + B'y_2'] = F(x),\ or\ \ A'y_1' + B'y_2' = \frac{F(x)}{a_0(x)}.$$

- Thus we have two imposed conditions.

$$\begin{aligned} A'y_1 + B'y_2 &= 0 \\ A'y_1' + B'y_2' &= \frac{F(x)}{a_0(x)} \end{aligned}$$

These conditions require that the function $A(x)$ and $B(x)$ be chosen such that the above system of equations is satisfied.

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- Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation. Therefore the determinant of coefficients of this system is

$$\text{Wronskian } W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_2y_1' \neq 0$$

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  Hence the system has a unique solution. Solving the system of equations, we get

$$A' = -\frac{F(x)y_2}{a_0(x)(y_1y_2' - y_2y_1')}, \quad B' = \frac{F(x)y_1}{a_0(x)(y_1y_2' - y_2y_1')}$$

Method of variation of parameters(cont.)

- Therefore

$$A' = -\frac{F(x)y_2}{a_0(x)W(y_1, y_2)}, \quad B' = \frac{F(x)y_1}{a_0(x)W(y_1, y_2)}$$

Integrating, we obtain

$$A(x) = -\int \frac{F(x)y_2}{a_0(x)W(y_1, y_2)}dx, \quad B(x) = \int \frac{F(x)y_1}{a_0(x)W(y_1, y_2)}dx \tag{4}$$

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- Thus we have particular integral y_p of the given equation is defined by

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x).$$

where $A(x)$ and $B(x)$ are given by (4).

Consider the second order non-homogeneous linear equation with variable coefficients

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- **Step 1.** Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

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- **Step 2.** Find Wronskian of functions y_1 and y_2.

  $$W = W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'.$$

- **Step 2.** Let $y_p = A(x)y_1 + B(x)y_2$, where

$$A(x) = -\int \frac{F(x)y_2}{a_0(x)W}dx, B(x) = \int \frac{F(x)y_1}{a_0(x)W}dx,$$

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Working Rule(cont.)

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- **Step 4.** Thus, we have the general solution

$$y(x) = y_c + y_p = c_1y_1 + c_2y_2 + A(x)y_1 + B(x)y_2.$$

Example

Consider the differential equation

$$y'' + 4y' + 4y = e^{-2x} sin\ x.$$

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Solution:

- Here $a_0(x) = 1$, $a_1(x) = 4$, $a_2(x) = 4$, $F(x) = e^{-2x} \sin x$.
  The auxiliary equation of the corresponding homogeneous equation is

  $$m^2 + 4m + 4 = 0.$$

  The characteristic roots are $m = -2, -2$.

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Example(cont.)

- The Wronskian of $y_1,\ y_2$ is given by

$$W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}(1-2x) \end{pmatrix} = e^{-4x}$$

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$$\begin{aligned} A(x) &= - \int \frac{F(x)y_2(x)}{a_0(x)W}dx = -\int x\sin x dx = x\cos x - \sin x \\ B(x) &= \int \frac{F(x)y_1(x)}{a_0(x)W}dx = \int \sin x dx = -\cos x \end{aligned}$$

- Thus the particular integral is

$$\begin{aligned} y_p(x) &= A(x)e^{-2x} + B(x)xe^{-2x} \\ &= (x\cos x - \sin x)e^{-2x} + (-\cos x)xe^{-2x} \\ &= -e^{-2x}\sin x. \end{aligned}$$

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- The general solution is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= c_1e^{-2x} + c_2xe^{-2x} - e^{-2x}\sin x. \end{aligned}$$

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- The Wronskian of y_1, y_2 is given by

  $$W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1$$

Example(cont.)

- By the method of the variation of parameters, we write the particular integral as

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- **To find $A(x)$ and $B(x)$:**

$$\begin{aligned}
A(x) &= -\int \frac{F(x)y_2(x)}{a_0(x)W}dx = -\int \tan x \sin x dx \\
&= -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx \\
&= \sin x - \log(\sec x + \tan x). \\
B(x) &= \int \frac{F(x)y_1(x)}{a_0(x)W}dx = \int \tan x \cos x dx = -\cos x.
\end{aligned}$$

Example(cont.)

- Thus the particular integral is

$$
\begin{aligned}
y_p &= A(x)\cos x + B(x)\sin x = \sin x\cos x - \cos x\log(\sec x + \tan x) - \sin x\cos x \\
&= -\cos x\log(\sec x + \tan x).
\end{aligned}
$$

- The general solution is

$$
y(x) = y_c + y_p = c_1\cos x + c_2\sin x + A(x)\cos x + B(x)\sin x
$$
$$
\Rightarrow y(x) = c_1\cos x + c_2\sin x - \cos x\log(\sec x + \tan x)
$$