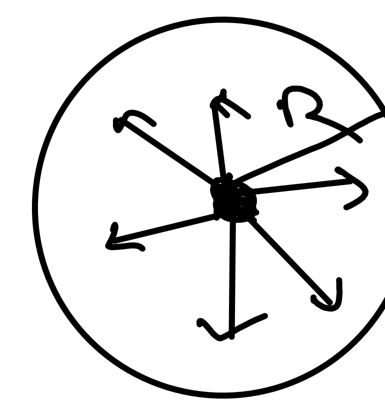
Volume integral

Volume integral : $\int_V T \, d\tau$

Fundamental th: $\int_V (\vec{\nabla} \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$

$\int \vec{v} \cdot d\vec{a}$ = flux.



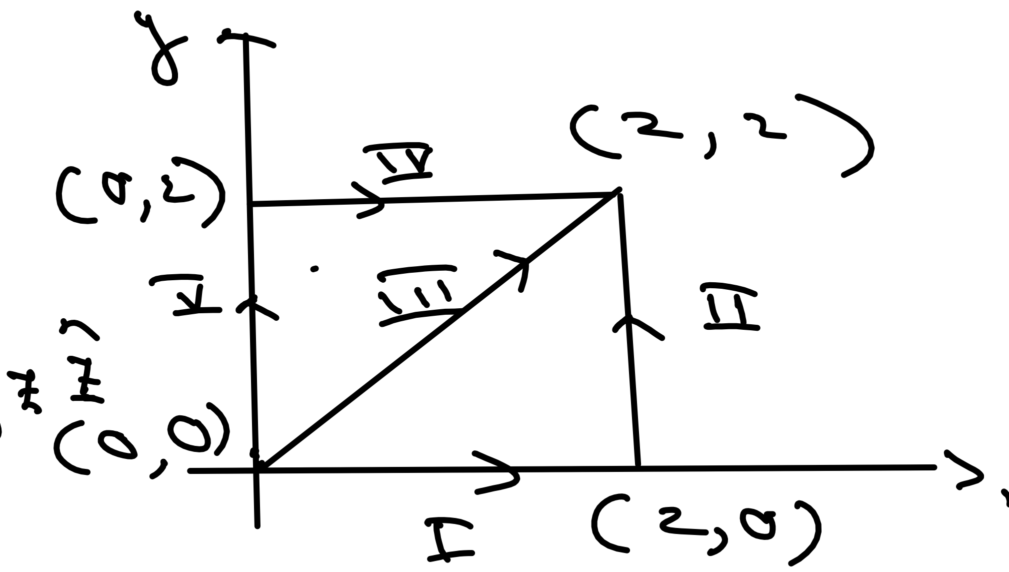
$$\int d\tau = \int_0^{L} dx \int_0^{L} dy \int_0^{L} dz$$

$$= r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\int_V d\tau = \int_0^{R} \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi R^3$$

Line integral

$$\int (\vec{F} \cdot d\vec{l})$$



$$\vec{F} = x^2 \hat{x} + y^2 x \hat{y} + xyz \hat{z}$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

Along I $\quad d\vec{l} = dx\,\hat{x}$

$$\vec{F} \cdot d\vec{l} = x^2\, dx$$

$$\int_I x^2\, dx = \int_0^2 x^2\, dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$

Along II $\quad d\vec{l} = dy\,\hat{y}$

$$\vec{F} \cdot d\vec{l} = y^2 x\, dy$$

$$\int_{II} y^2 x\, dy = \int_0^2 2y^2\, dy = 2\,\frac{y^3}{3}\Big|_0^2 = \frac{16}{3}$$

I + II $\int = \frac{24}{3} = 8$

$y = 2x$

$dy = 2dx$

III $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$\vec{f}\cdot d\vec{l} = x^2\,dx + y^2x\,dy$

$\begin{cases}(0,0) \\ \downarrow \\ (2,2)\end{cases}$

$\int (x^2\,dx + y^2x\,dy)$

$= \int_0^2 (x^2\,dx + x^3\,dx)$

$= \frac{x^3}{3}\Big|_0^2 + \frac{x^4}{4}\Big|_0^2$

$= \frac{8}{3} + \frac{16}{4}$

$= \frac{8}{3} + 4$

$y = mx + \cancel{c}^{\,0}$

$= \frac{y_2 - y_1}{x_2 - x_1}x$

$= x$

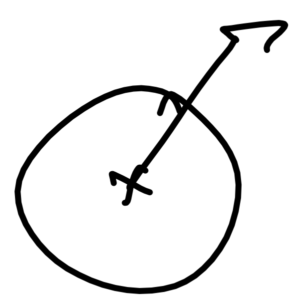
$y = x$

$dy = dx$

Surface integral

On the surface of a sphere,

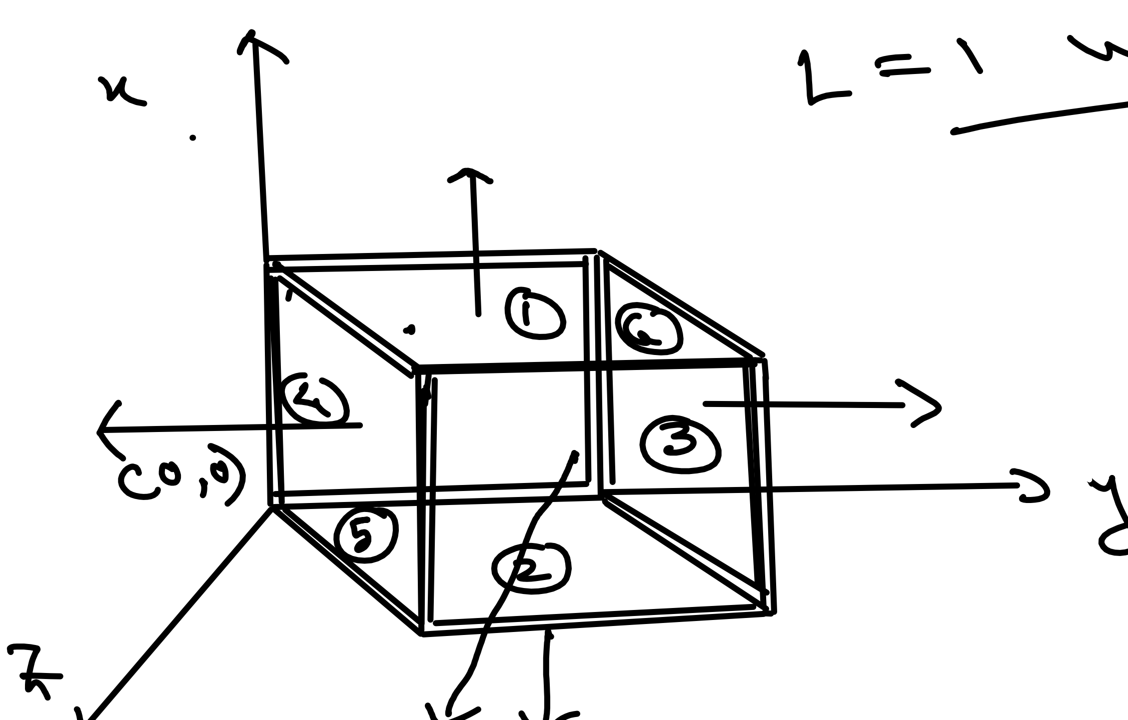
$$d\vec{a} = dl_\theta \, dl_\phi \, \hat{r}$$



$$= r^2 \sin\theta \, d\theta \, d\phi$$

$$\vec{F} = \frac{A}{r^2} \hat{r}$$

$$\int \vec{F} \cdot d\vec{a} = \int \frac{A}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$



L = 1 unit

$$\vec{A} = x^2 \hat{x} + y^2 x \hat{y} + xyz \hat{z}$$

$$d\vec{a}_1 = dy \, dz \, \hat{x} \quad (x = 1)$$

$$d\vec{a}_2 = - dy \, dz \, \hat{x} \quad (x = 0)$$

$$d\vec{a}_3 = dx \, dy \, \hat{z} \quad (z = 1)$$

$$\oint \vec{A}\cdot d\vec{a}$$

$$= \int \vec{A}\cdot d\vec{a}_1 + \int \vec{A}\cdot d\vec{a}_2$$
$$+ \int A\cdot d\vec{a}_3 + \int \vec{A}\cdot d\vec{a}_4$$
$$+ \int \vec{A}\cdot d\vec{a}_5 + \int \vec{A}\cdot d\vec{a}_6$$

$$d\vec{a}_4 = -dx\,dy\,\hat{z} \quad (z=0)$$
$$d\vec{a}_5 = \underline{-dx\,dz\,\hat{y}} \quad (y=0)$$
$$d\vec{a}_6 = dx\,dz\,\hat{y} \quad (y=1)$$

$$\int \vec{A}\cdot d\vec{a}_1 = \int x^2\,dy\,dz = 1\int_0^L dy\int_0^L dz$$

$$= L^2$$

$$\int \vec{A}\cdot d\vec{a}_5 = -\int y^2 x\,dx\,dz = 0$$

$$\left(\oint_S \vec{A}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{A})\,d\tau\right)$$

$$\int d\tau = \int_0^L dx\int_0^L dy\int_0^L dz$$

$$\vec{\nabla}\cdot\vec{A} = 2x + 2yx + xy$$

$$\vec{A} = x^2\,\hat{x} + y^2 x\,\hat{y} + xyz\,\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \left(\underline{\underline{\hat{x}}}\frac{\partial}{\partial x} + \underline{\underline{\hat{y}}}\frac{\partial}{\partial y} + \underline{\underline{\hat{z}}}\frac{\partial}{\partial z}\right)\cdot\left(x^2\,\underline{\underline{\hat{x}}} + y^2 x\,\hat{y} + xyz\,\underline{\underline{\hat{z}}}\right) =$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2 x) + \frac{\partial}{\partial z}(xyz)$$

$$= 2x + 2yx + xy$$