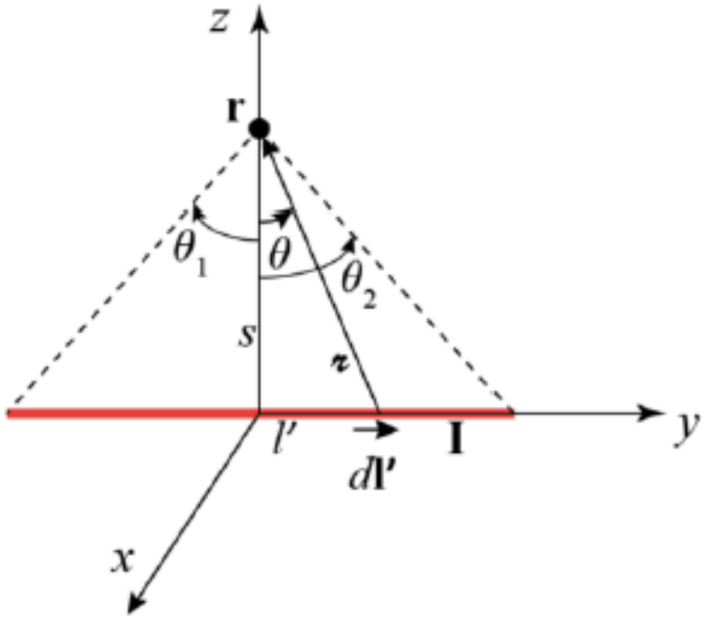
Lecture - 24

The Biot-Savart Law

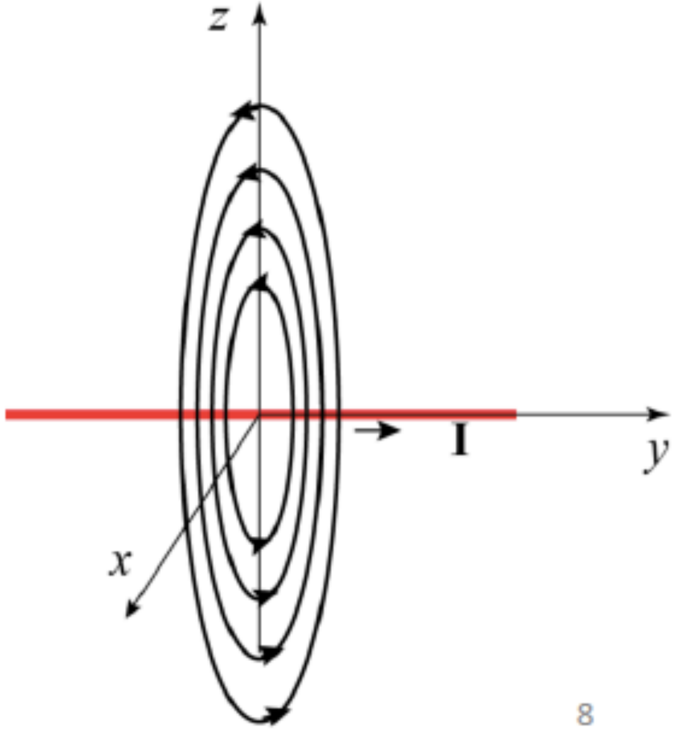
Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to a long straight wire carrying a steady current I .



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

Field due to an infinite wire ?



$$\theta_1 = \frac{\pi}{2} \qquad\qquad \theta_2 = \frac{\pi}{2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

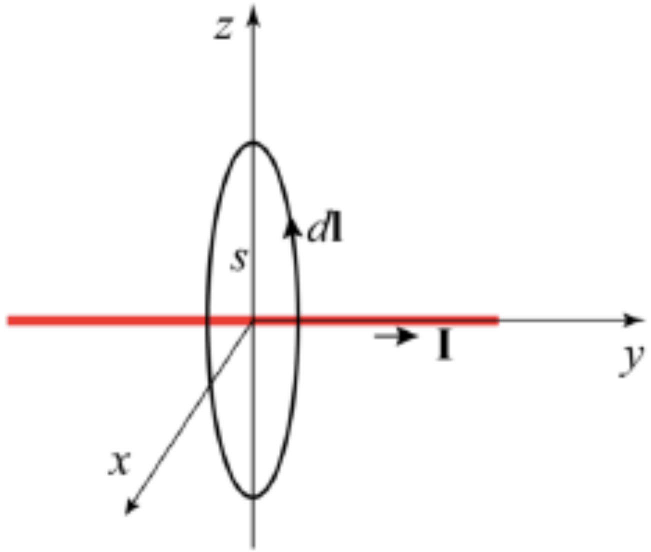
$$= \frac{\mu_0 I}{4\pi s} (1 + 1)\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\mathbf{x}}$$

The Divergence and Curl of B

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

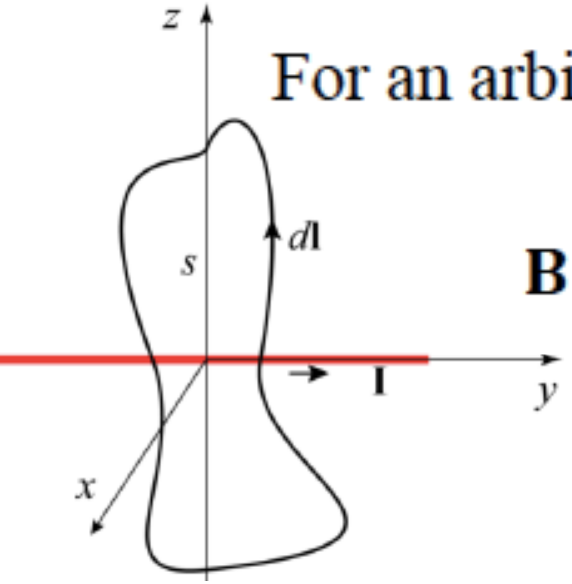
What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$



$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

$$\boxed{B(s) = \frac{\mu_0 I}{2\pi s}}$$

The line integral is independent of *s*



For an arbitrary path enclosing the current carrying wire

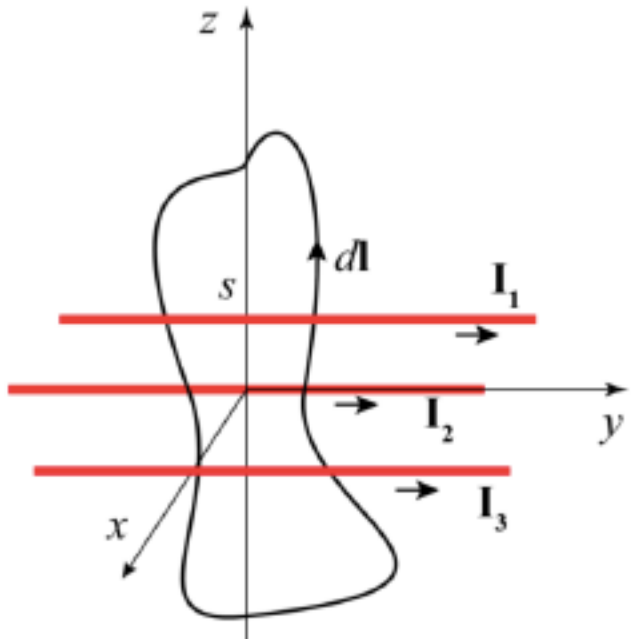
$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}$$

cylindrical coordinate

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \cdot (ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

If the path encloses more than one current carrying wire



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 = \mu_0 I_{\text{enc}}$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

It is valid in general

The Ampere's Law

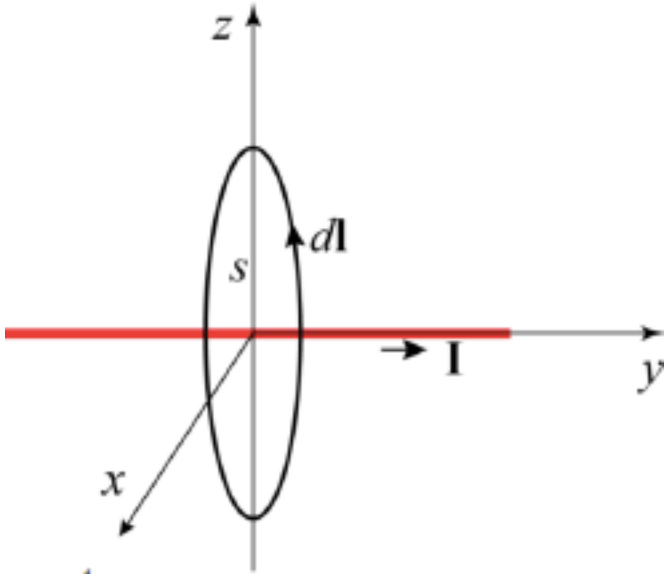
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$ Ampere's law in differential form

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$ Ampere's law in integral form

- Ampere's law is analogous to Gauss's law
- Ampere's law makes the calculation of magnetic field very easy if there is symmetry.
- If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.

The Ampere's Law

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to an infinitely long straight wire carrying a steady current I .



Using Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$

Make an Amperian loop of radius s enclosing the current

Since it is an infinite wire, the magnetic field must be circularly symmetric. Therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} = B\oint dl = B2\pi s = \mu_0 I_{\mathbf{enc}} = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

Magnetostatics and Electrostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\rho(\mathbf{r}')d\tau$$

Coulomb's Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

Biot-Savart Law

$$\mathbf{F}_{\mathbf{elec}} = \mathrm{Q}\mathbf{E}$$

Electric Force

$$\mathbf{F}_{\mathbf{mag}} = \mathrm{Q}(\mathbf{v} \times \mathbf{B})$$

Magnetic Force

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss's Law

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

No Name

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

No Name

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{J}$$

Amperes's Law

Vector Potential

If the divergence of a vector field $\mathbf{F}$ is zero everywhere, $(\boldsymbol{\nabla} \cdot \mathbf{F} = 0)$, then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\boldsymbol{\nabla} \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$$

$\mathbf{F}$ is the curl of a vector function: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$

Magnetic Vector Potential

The vector potential is not unique. A gradient $\boldsymbol{\nabla} V$ of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

What happens to the Ampere's Law ?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$
$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

- This is not in a very nice form.
- Ampere's law in terms of **B** seems better

- However, if we can ensure that $\nabla \cdot \mathbf{A} = 0$, we can have it in a nice form.
- This can be done since we know that a $\nabla\lambda$ can be added to **A** without changing **B**

Suppose we start with $\mathbf{A_0}$, such that, $\mathbf{B} = \nabla \times \mathbf{A_0}$ but, $\nabla \cdot \mathbf{A_0} \neq \mathbf{0}$.

Then, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A_0}) - \nabla^2 \mathbf{A_0} = \mu_0 \mathbf{J}$

Re-define by adding $\nabla\lambda$: $\mathbf{A_0} + \nabla\lambda \equiv \mathbf{A}$ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2 \lambda = 0$

Then $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \Rightarrow -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

Thus, one can always redefine the vector potential such that $\nabla \cdot \mathbf{A} = 0$

Recall: $\nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$

So, $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} da'$

For line current: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'}{\mathscr{r}}$

- The divergence of a magnetic field is zero.

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

- The curl of a magnetic field: The Ampere's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \longleftrightarrow \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathbf{enc}}$$

- Magnetic Vector Potential

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Recall: $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$

So, $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da'$

For line current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} I \displaystyle\int \dfrac{d\mathbf{l}'}{\mathscr{r}}$