Lecture – 19-20

Most everyday objects belong to one of two large classes: **conductors** and **insulators** (or **dielectrics**)

Conductors : Substances contains an “unlimited” supply of charges that are free to move about through the material.

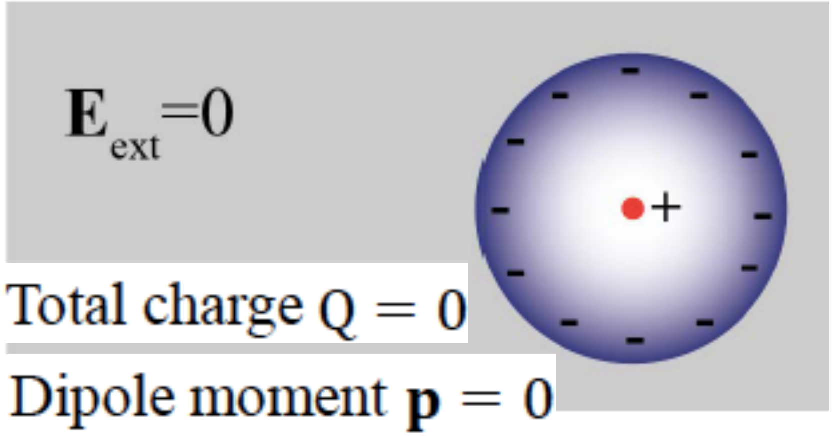
Dielectrics : all charges are attached to specific atoms or molecules. All they can do is move a bit within the atom or molecule.

Dielectrics : Microscopic displacements are not as dramatics as the wholesale rearrangement of charge in conductor, but their **cumulative effects** account for the characteristic behavior of dielectric materials.

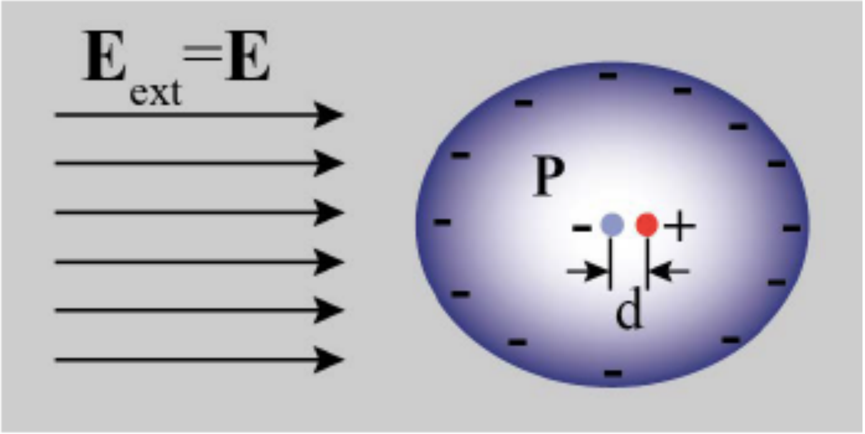
Electrostatics in Matter (Electric Fields in Matter)

Polarization (Induced Dipoles)

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$?



Although the atom as a while is electrically neutral, there is a *positively* charged core (the nucleus) and a *negatively* charged electron cloud surrounding it.



The electric fields *pull* the electron cloud and the nuclear *apart*, their mutual attraction drawing them together - reach balance, leaving the atom polarized.

The atom or molecule now has a tiny dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$ and is proportional to the field.

$$\boxed{\mathbf{p} = \alpha \mathbf{E},}$$

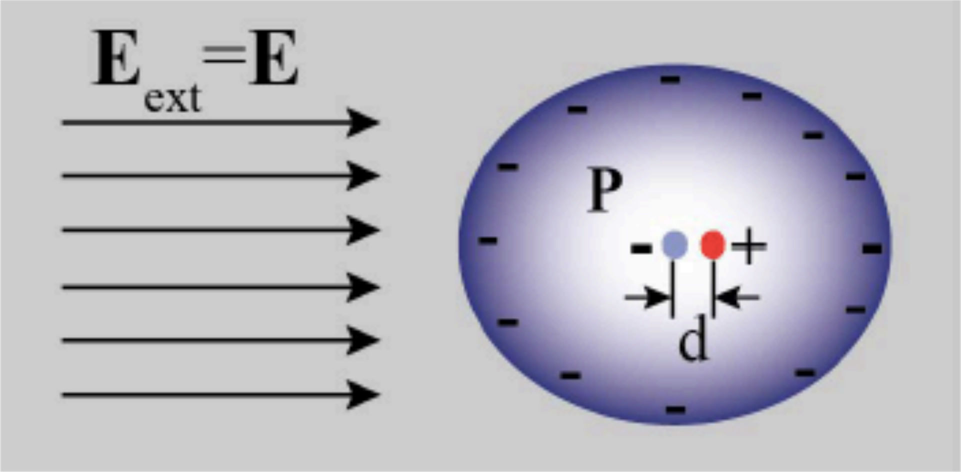
α = atomic polarizability

$$\boxed{\mathbf{p} = \boldsymbol{\alpha}\,\mathbf{E}}$$

- $\boldsymbol{\alpha}$ is called the atomic polarizability
- $\boldsymbol{\alpha}$ depends on the detailed structure of the atom
- $\boldsymbol{\alpha}$ is determined experimentally

Polarization (Induced Dipoles)

Ex. 4.1 (Griffiths, 3rd Ed.): An atom can be considered as a point nucleus of charge $+q$ surrounded by a uniformly charged spherical electron cloud of charge $-q$ and radius a ? Find the atomic polarizability.



Here, we are assuming that the electron cloud remains spherical in shape even in the presence of the external electric field.

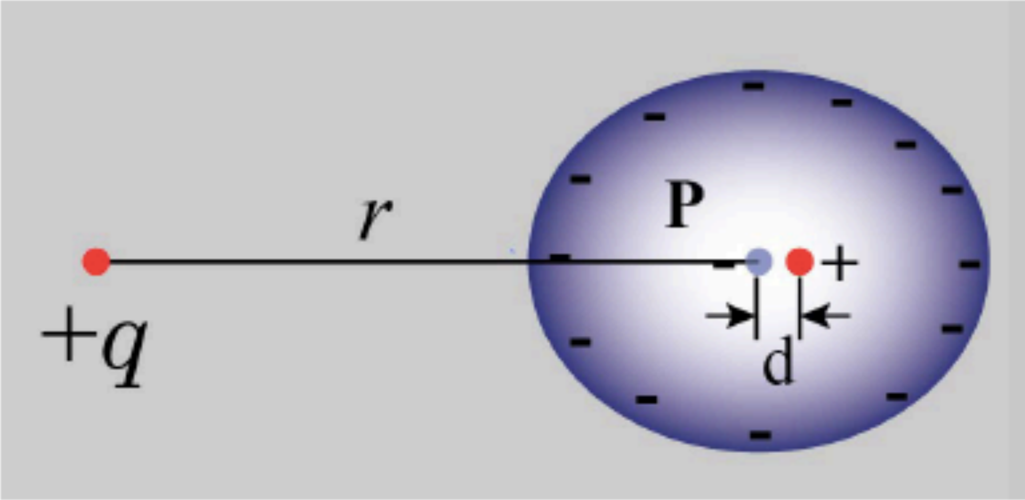
In equilibrium, the field at the nucleus due to the electron cloud is

$$E_{\text{elec}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} = E \quad \Rightarrow \quad d = \frac{4\pi\epsilon_0\, a^3 E}{q}$$

So the atomic polarizability $\alpha = \frac{p}{E} = \frac{qd}{E} = \frac{4\pi\epsilon_0\, a^3 E}{E} = 4\pi\epsilon_0\, a^3$

Polarization (Induced Dipoles)

Ex. 4.4 (Griffiths, 3rd Ed.): A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.



The field due to charge q is

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The induced dipole moment is

$$\mathbf{p} = \alpha\,\mathbf{E} = \frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The field due to this dipole at the location $(\theta = \pi)$ of the charge is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0 r^3}\frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}(-2\,\hat{\mathbf{r}})$$

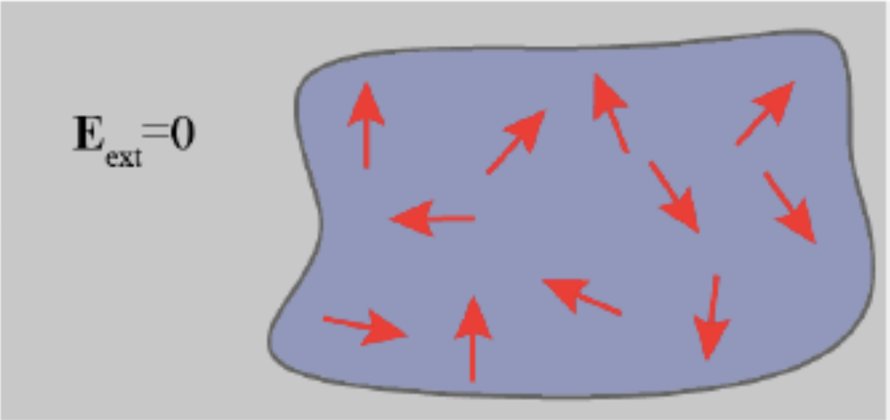
Therefore, the force is $\quad \mathbf{F} = q\mathbf{E}_{\text{dip}}(\mathbf{r}) = -2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^5}\,\hat{\mathbf{r}}$

Polarization (Permanent Dipoles)

A neutral atom has no dipole moment to begin with but some molecules (polar molecules) have permanent dipole moment, even without the external electric field.

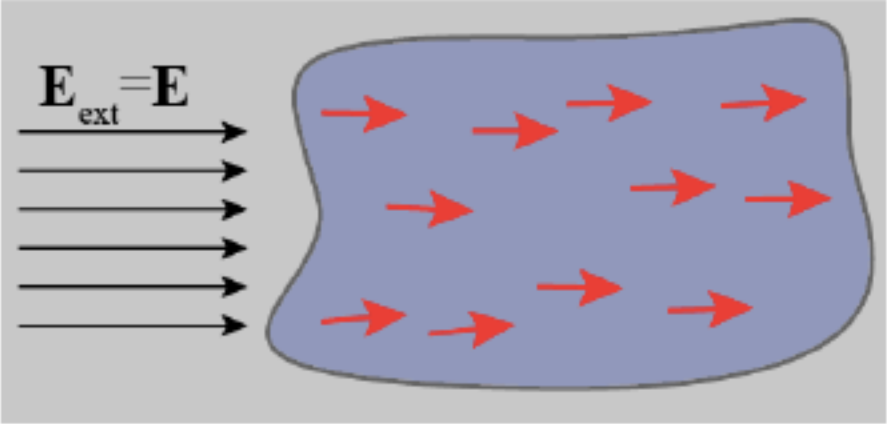
<u>Polar molecules</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} = 0$;



<u>Polar molecules in an electric field</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} \neq 0$;

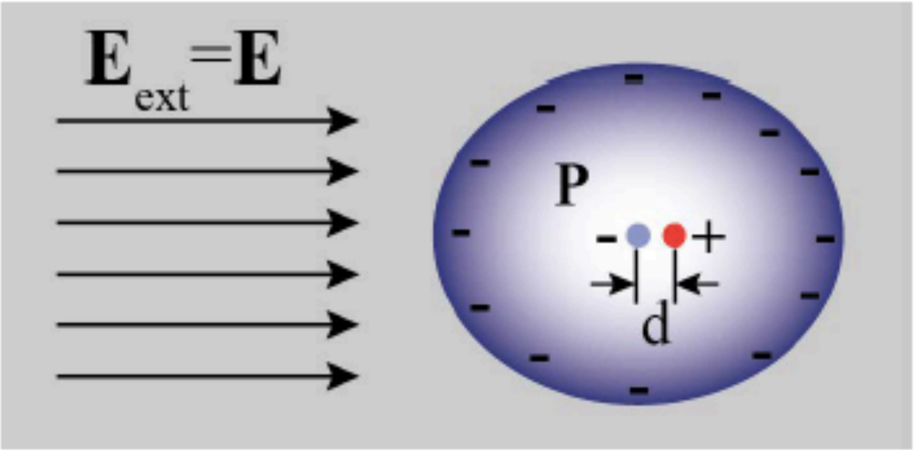


Molecules already had dipole moments
The dipole moments have now become aligned

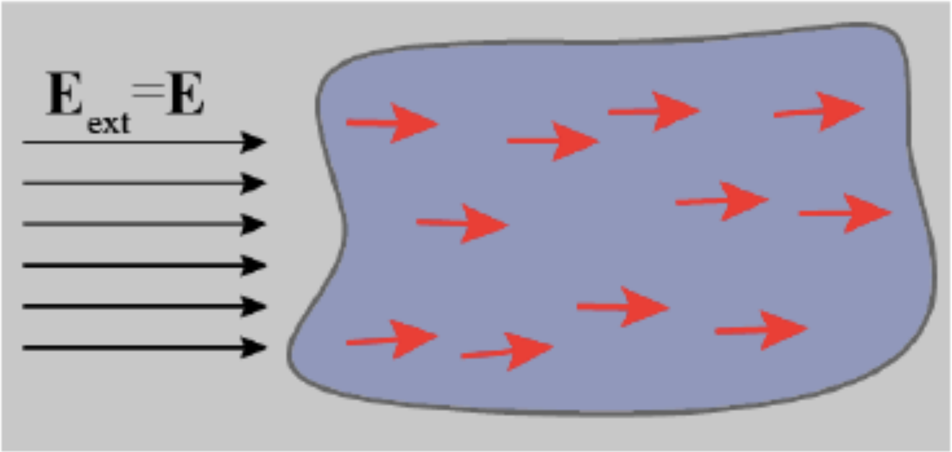
Polarization in a medium

Two mechanism for polarization in a medium

Induced Dipoles



Alignment of permanent dipoles



The two mechanisms need not be independent.

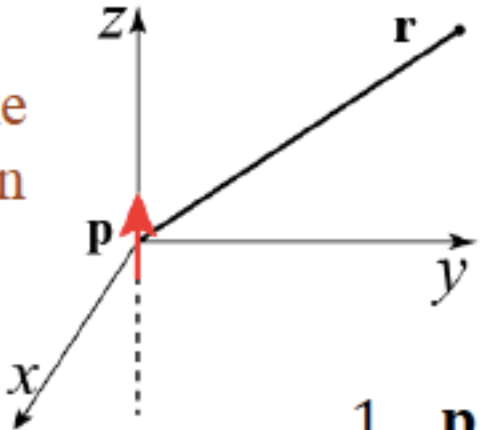
A permanent dipole may also get some dipole moment induced.

Total dipole moment of per unit volume is called the polarization **P**.

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$

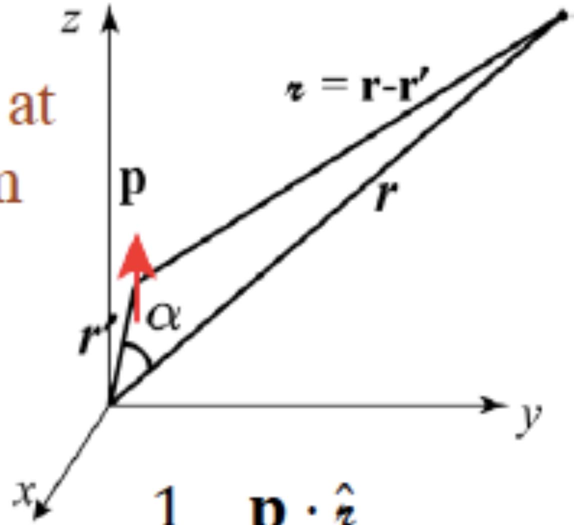
The Field of a Polarized Object:

Point dipole at the origin



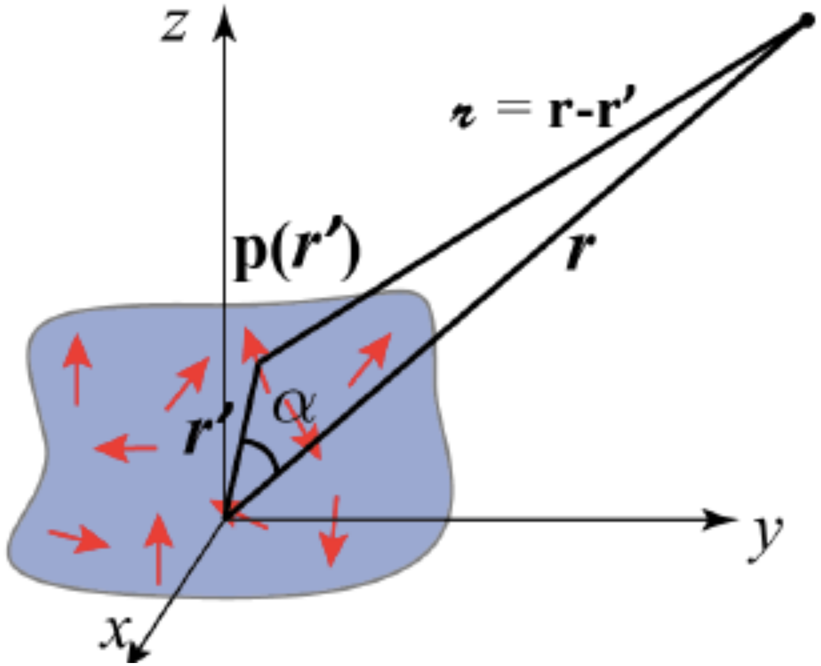
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Point dipole at $\mathbf{r}'$ away from the center



$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

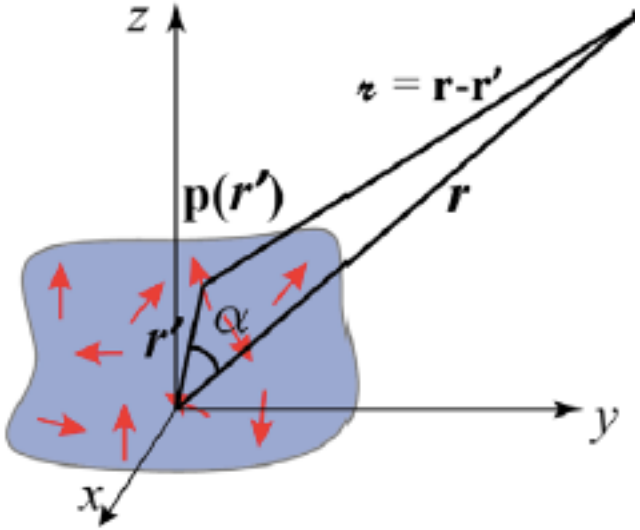
Continuous distribution of dipoles



$$V_{\text{dip}}(\mathbf{r}) = \int_{\mathcal{V}} \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

$$\mathbf{p} = \mathbf{P}(\mathbf{r}')\,\mathbf{d}\boldsymbol{\tau}'$$

The Field of a Polarized Object:



$$V_{dip}(\mathbf{r}) = \int_{vol} \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' = \int_{vol} \frac{1}{4\pi\epsilon_0} \mathbf{P}(\mathbf{r}') \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

$$= \int_{vol} \frac{1}{4\pi\epsilon_0} \mathbf{P}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{\mathscr{r}}\right) d\tau' \qquad \left[\text{ Using } \nabla' \left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \int_{vol} \nabla' \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{\mathscr{r}}\right) d\tau' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{1}{\mathscr{r}} \nabla' \cdot \mathbf{P}(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$= \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\mathbf{P}(\mathbf{r}')}{\mathscr{r}} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{1}{\mathscr{r}} \nabla' \cdot \mathbf{P}(\mathbf{r}')\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'}}{\mathscr{r}} da' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\sigma_b}{\eta}\, da' + \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\rho_b}{\eta} d\tau'}$$

Surface charge density

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'}$$

Volume charge density

$$\rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$$

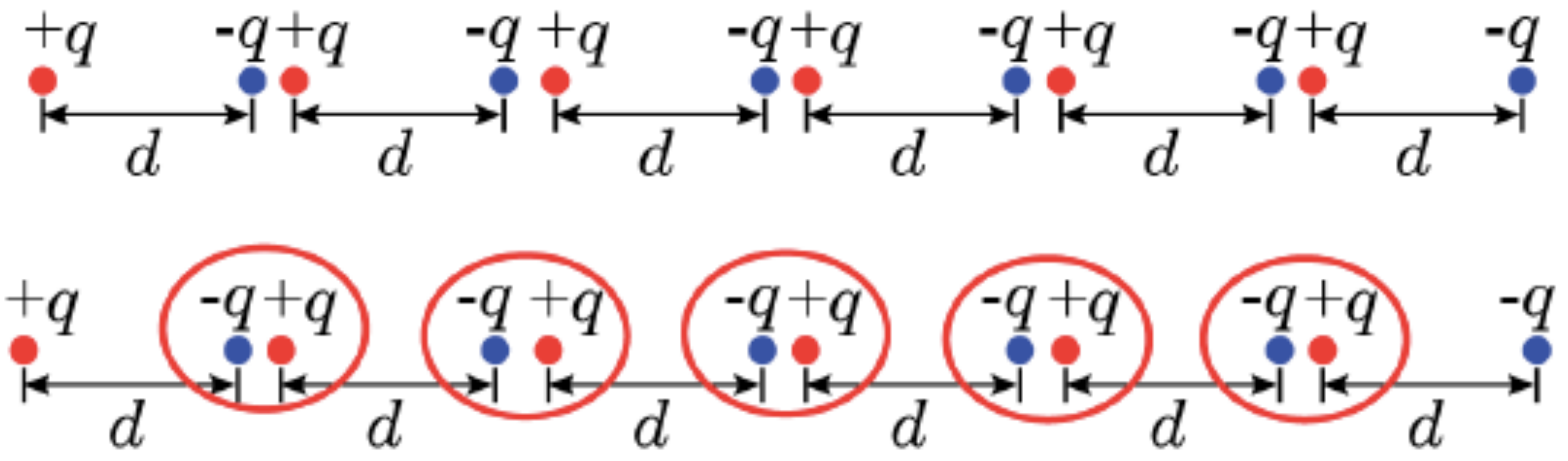
Potential due to a polarized object can be thought of as the sum of the potentials due to a surface charge $\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'}$ and a volume charge $\rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$.

Are bound charges real?

These bound charges are not just mathematical constructs. They do appear on the surface and in the volume of the dielectric.

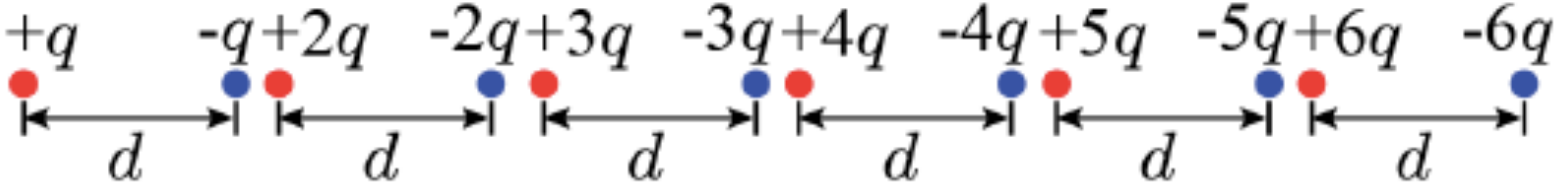
- **Uniform Polarization in one-dimension**

$$\boxed{\sigma_b \neq 0 \qquad \rho_b = 0}$$



- **Non-Uniform Polarization in one-dimension**

$$\boxed{\sigma_b \neq 0 \qquad \rho_b \neq 0}$$



Prob 4.14 (Griffiths, 3rd Ed.): Prove that the total bound charge is zero.

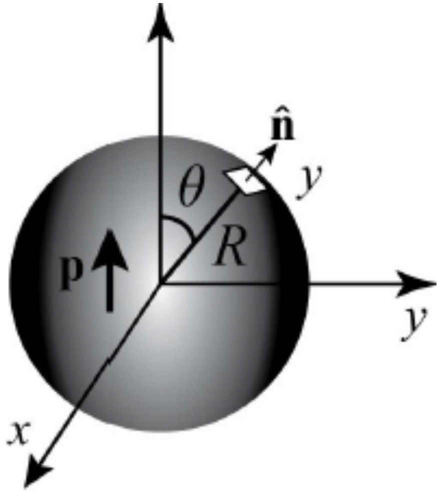
Total charge $$Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$$

$$= \oint_{surf} \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}}' da' - \int_{vol} \boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\tau'$$

$$= \oint_{surf} \mathbf{P}(\mathbf{r}') \cdot d\mathbf{a}' - \int_{vol} \boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\tau'$$

$$= \int_{vol} \boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\tau' - \int_{vol} \boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\tau' = 0$$

Ex. 4.2 (Griffiths, 3rd Ed.): Find the electric field of a sphere of radius R, if it is uniformly polarized $\mathbf{P}(\mathbf{r}') = P\,\hat{\mathbf{z}}$.



$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}') = -\nabla' \cdot (P\,\hat{\mathbf{z}}) = 0$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} = P\cos\theta$$

The potential due to a uniformly polarized sphere is equal to the potential due to a spherical surface charge density $\sigma_b = P\cos\theta$

For $r \geq R$
$$V(r,\theta) = \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta = \frac{R^3}{3\epsilon_0}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{3\epsilon_0}\frac{\left(\frac{4\pi}{3}\right)R^3}{\left(\frac{4\pi}{3}\right)}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

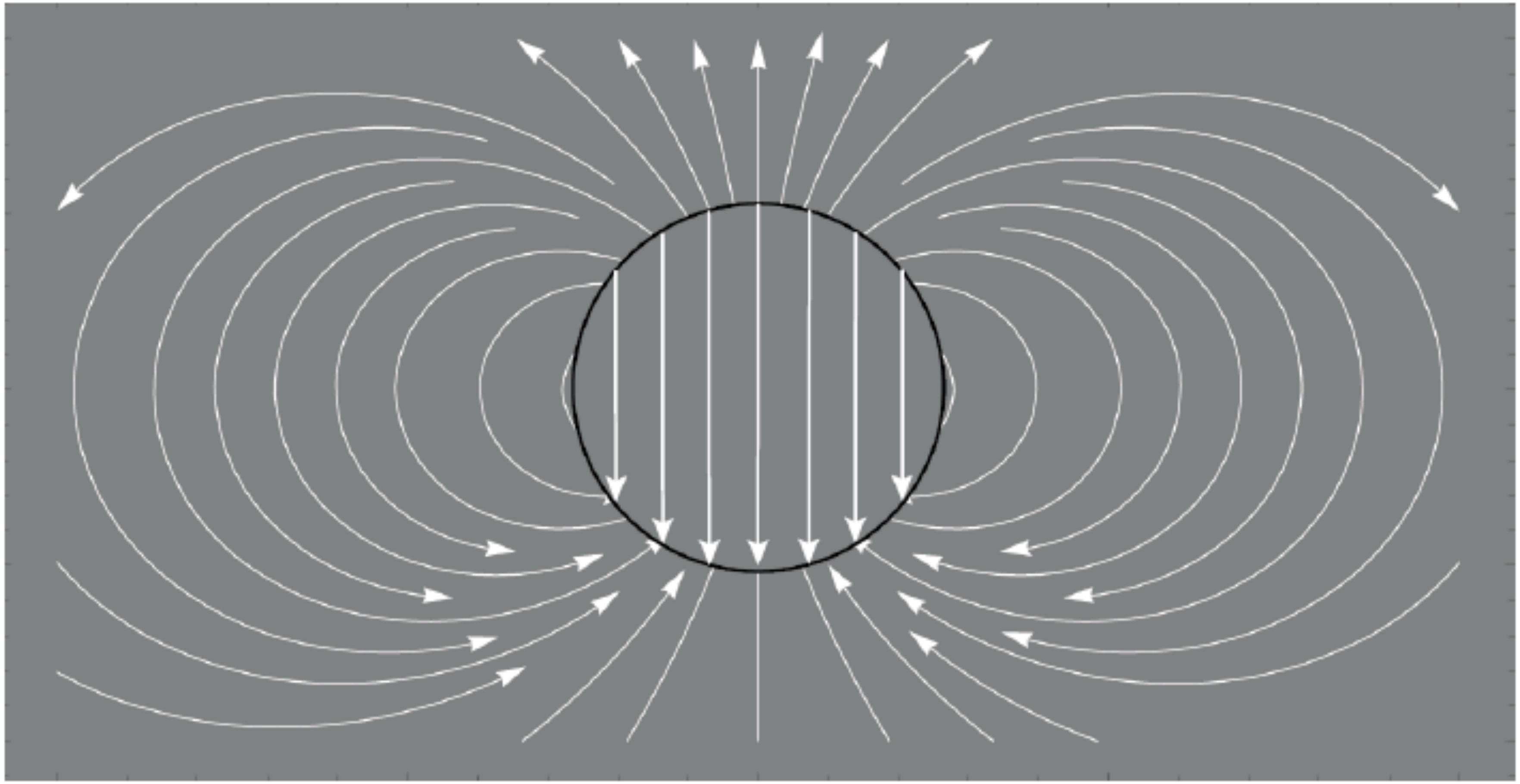
$$\boxed{\mathbf{p} = \left(\frac{4\pi}{3}\right)R^3\mathbf{P}}$$ total dipole moment of the sphere

$$\Rightarrow \quad \mathbf{E} = -\nabla V = \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

For $r \leq R$
$$V(r,\theta) = \frac{P}{3\epsilon_0}r\cos\theta \Rightarrow \mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$$

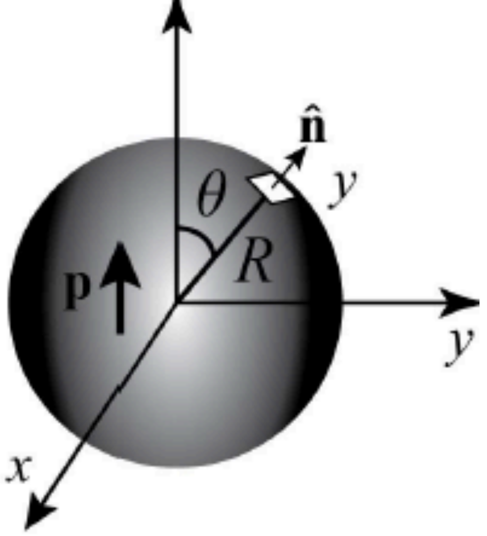
For $r \geq R$ $$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

For $r \leq R$ $$\mathbf{E} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$$



The Field of a Polarized Object:

Prob 4.10 (Griffiths, 3rd Ed.): Find the bound charges of a sphere of radius R, if its polarization is $\mathbf{P}(\mathbf{r}') = k\mathbf{r}$.



<u>Volume charge</u>

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) \quad = -\frac{1}{r^2}3kr^2 \quad = -3k$$

<u>Surface charge</u>

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad = kR$$

Q: What is the electric field outside the sphere?

Volume and surface charge distributions are both symmetric with respect to the center of the sphere. So, the total charge can be thought of as being concentrated at the center

$$\text{Total charge } Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$$

$$= kR \times 4\pi R^2 \quad +(-3k) \times \frac{4\pi}{3}R^3$$

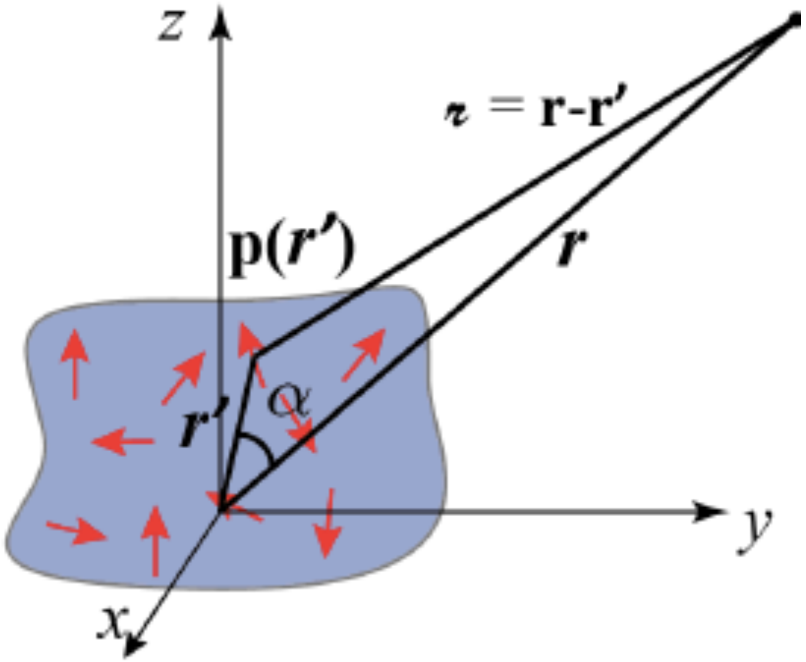
$$= 0$$

So the electric field outside the sphere is zero.

Questions 1:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\rho_b}{\mathscr{r}}\, d\tau'$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad \rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$



Q: Is the decomposition unique?

Ans: Yes, it is. Because there is only one way in which an object can be divided into its surface and volume.

Questions 2:

Q: What happens to the Gauss's law when we have polarized objects?

Ans: We are going to answer this today.