Lecture - 5

Few others related to derivative

By applying ∇ twice, we can construct five species of second derivatives.

$$\text{Three first derivatives } \nabla T, \quad \nabla \cdot \mathbf{v}, \quad \nabla \times \mathbf{v}$$

(1) Divergence of gradient : $\nabla \cdot (\nabla T)$ ⟵ very important

(2) Curl of gradient : $\nabla \times (\nabla T)$ ⟵ always zero

(3) Gradient of divergence : $\nabla (\nabla \cdot \mathbf{v})$

(4) Divergence of curl : $\nabla \cdot (\nabla \times \mathbf{v})$ ⟵ always zero

(5) Curl of curl : $\nabla \times (\nabla \times \mathbf{v})$ ⟵ reduce to others

$$(1)\ \nabla\cdot(\nabla T)=(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z})\cdot(\hat{\mathbf{x}}\frac{\partial T}{\partial x}+\hat{\mathbf{y}}\frac{\partial T}{\partial y}+\hat{\mathbf{z}}\frac{\partial T}{\partial z})$$

$$=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\nabla^2 T \longleftarrow \text{ the Laplacian of } T$$

$$(4)\ \nabla\cdot(\nabla\times\mathbf{v})=\hat{\mathbf{x}}\frac{\partial}{\partial x}(\hat{\mathbf{x}}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}))+\hat{\mathbf{y}}\frac{\partial}{\partial y}(\hat{\mathbf{y}}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}))+\hat{\mathbf{z}}\frac{\partial}{\partial z}(\hat{\mathbf{z}}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}))$$

$$=\frac{\partial}{\partial x}(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y})$$

$$=0 \longleftarrow \text{ always zero}$$

Integral Calculus

In electrodynamics, the **line** (or **path**) integrals, **surface** integrals (or **flux**), and **volume** integrals are the most important integrals.

(a) **Line integrals**: a line integral is an expression of the form

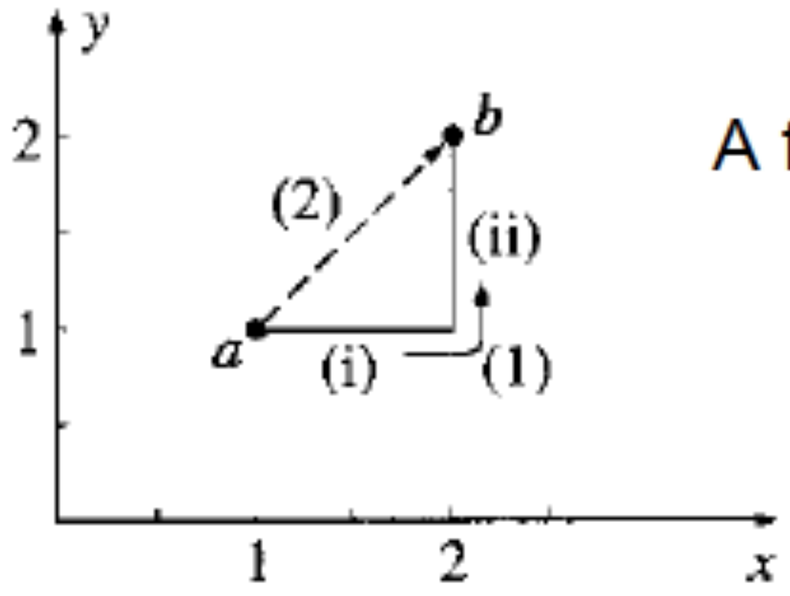
$$\int_{\mathbf{a}\,\mathcal{P}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l},$$

Put a circle on the integral, in the path in question forms a closed loop.

$$\oint \mathbf{v} \cdot d\mathbf{l}$$

The value of a line integral depends critically on the particular path taken from **a** to **b**, but there is an important special class of vector functions for which ***the line integral is independent of the path, and is determined entirely by the end points***, e.g.

$$W = \int_{\mathbf{a}\,\mathcal{P}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$$



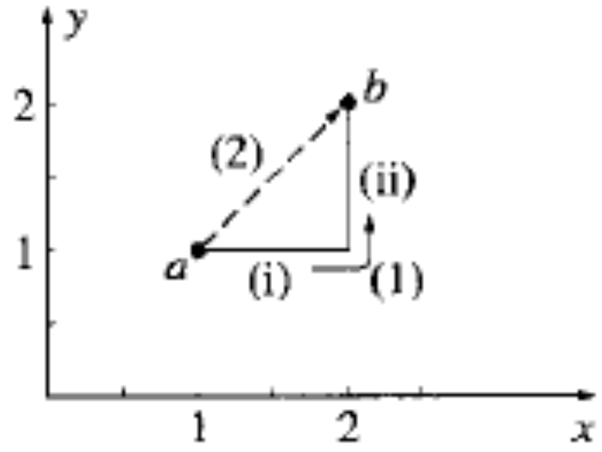
A force that has this property is called conservative.

Exercise

Calculate the line integral of the function

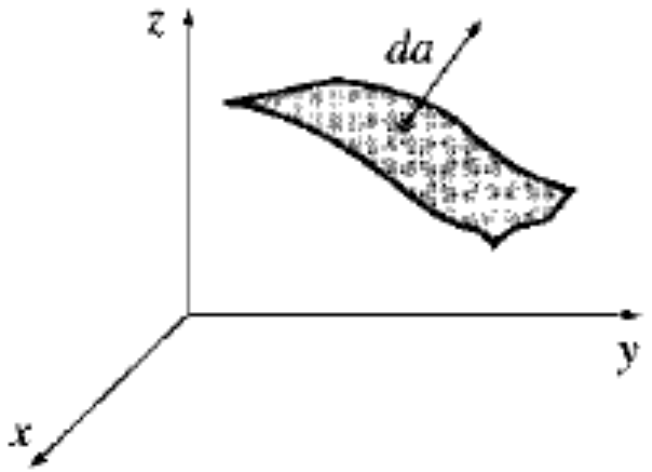
$\mathbf{v} = y^2\hat{\mathbf{x}} + 2x(y+1)\hat{\mathbf{y}}$, from the point **a**=(1,1,0) to the point **b**=(2,2,0), along the paths (1) and (2) What is the loop integral that goes from **a** to **b** along (1) and returns to **a** along (2)?

The strategy here is to get everything in terms of one variable.



(b) **Surface integrals**: a line integral is an expression of the form

$$\int_S \mathbf{v} \cdot d\mathbf{a},$$



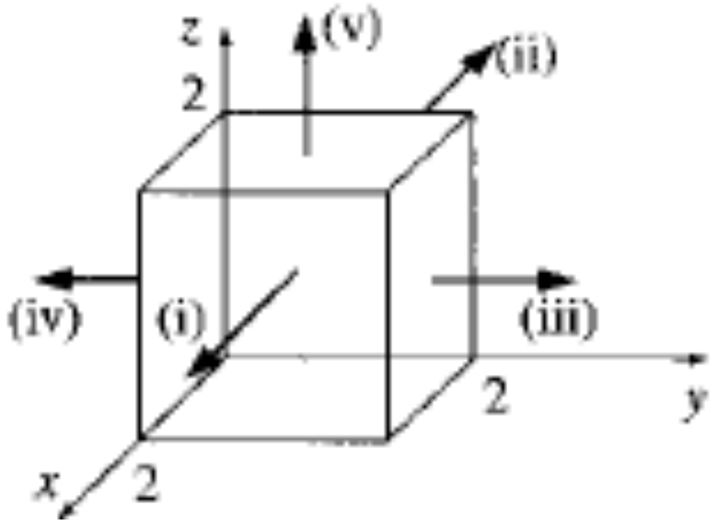
where $\mathbf{v}$ is a vector function, and $d\mathbf{a}$ is the infinitesimal patch of area, with direction perpendicular to the surface.

The value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, *and is determined entirely by the boundary.*

Exercise

Calculate the surface integral of the function

$\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (2+x)\,\hat{\mathbf{y}} + y(z^2 - 3)\,\hat{\mathbf{z}}$ over five sides of the cubical box. Let "upward and outward" be the positive direction, as indicated by the arrow.



$$\text{Sol : Taking the sides one at a time :}$$

$$(1)\ x = 2,\ \ d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}, \quad \mathbf{v}\cdot d\mathbf{a} = 2xz\,dy\,dz = 4z\,dy\,dz$$

$$\int \mathbf{v}\cdot d\mathbf{a} = 4\int_0^2 dy \int_0^2 z\,dz = 16$$