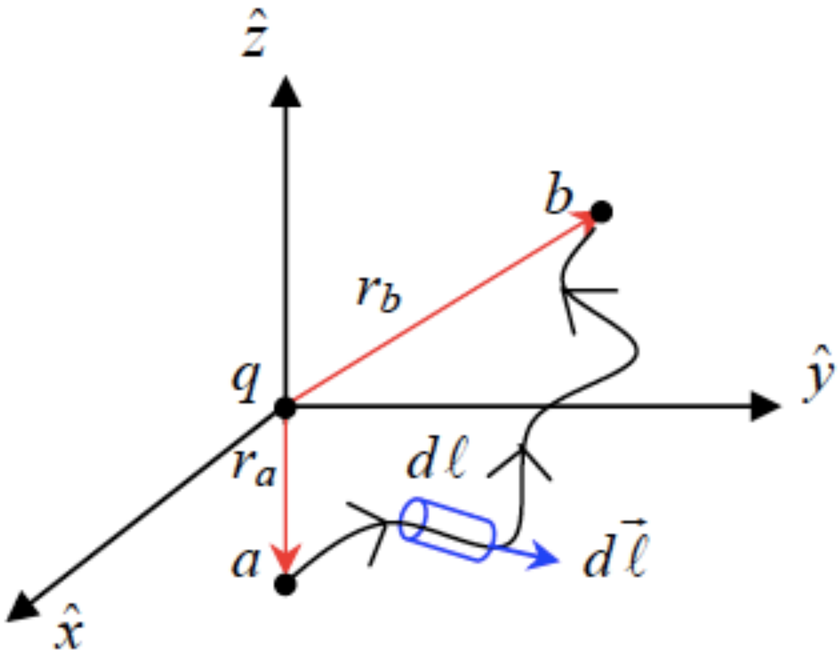
Lecture - 15



In spherical coordinates: $d\vec{\ell} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi}$

$$\vec{E}(\vec{r})\bullet d\vec{\ell} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)\hat{r}\bullet\left\{dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\varphi\hat{\varphi}\right\}$$

$$\vec{E}(\vec{r})\bullet d\vec{\ell} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r^2}\right)dr$$

$$\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell} = \frac{1}{4\pi\varepsilon_o}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\varepsilon_o}\left(\frac{q}{r}\right)\Bigg|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_o}\left(\frac{q}{r_a} - \frac{q}{r_b}\right) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

r_a = distance from origin O to point <u>*a*</u>. r_b = distance from origin O to point <u>*b*</u>.

The line integral $\int \vec{E}(\vec{r})\bullet d\vec{\ell}$ around a <u>*closed*</u> contour C is zero!

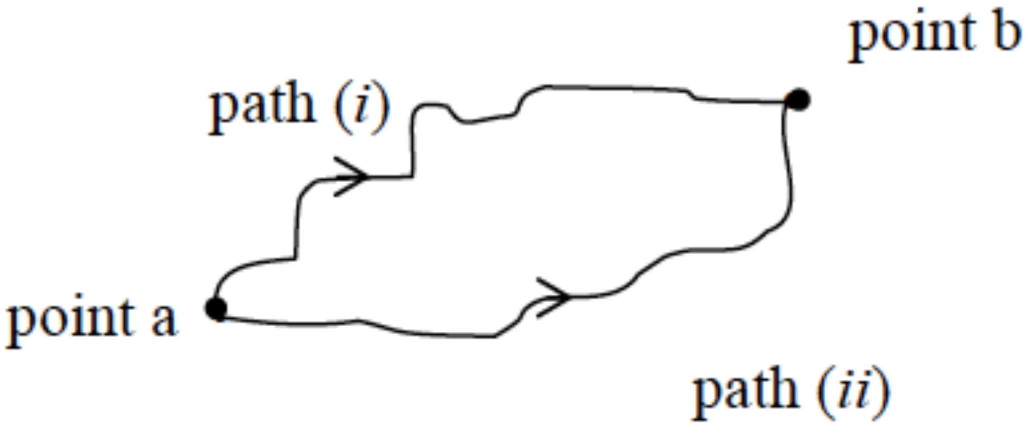
$$\boxed{\int_S \left(\vec{\nabla}\times\vec{E}(\vec{r})\right)\bullet d\vec{A} = \oint_C \vec{E}(\vec{r})\bullet d\vec{\ell} = 0}$$

arbitrary closed surface S

arbitrary closed contour C (on S)

$$\boxed{\oint_C \vec{E}(\vec{r})\bullet d\vec{\ell} = 0}$$

$$\boxed{\vec{\nabla}\times\vec{E}(\vec{r}) = 0}$$



$$\boxed{\underbrace{\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}}_{\text{path }(i)} = \underbrace{\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}}_{\text{path }(ii)} = \underbrace{\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}}_{\text{any path}}}$$

because $\int_a^b \vec{E}(\vec{r})\bullet d\vec{\ell}$ is *independent* of the path taken from point $a \to b$.

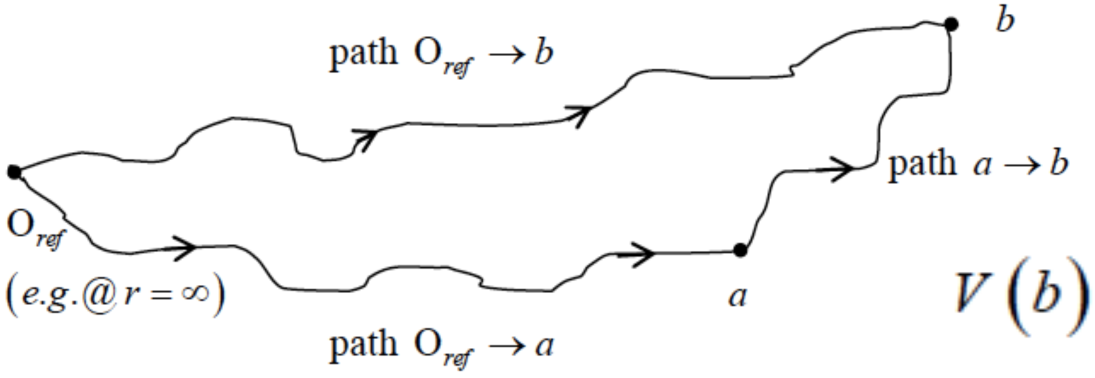
We now define a *scalar point function*, $V(\vec{r})$ known as the electric potential,

$$\boxed{V(\vec{r}) \equiv -\int_{O_{ref}}^{r} \vec{E}(\vec{r})\bullet d\vec{\ell}}$$

Electric Potential (integral version)	$V(\vec{r}) \equiv -\int_{O_{ref}}^{r} \vec{E}(\vec{r}) \bullet d\vec{\ell}$

By convention, the point $r = O_{ref}$ is taken to be a *standard reference point* of electric potential, $V(\vec{r})$ where $V(\vec{r} = O_{ref}) = 0$ (usually $r = \infty$).

$V(\vec{r})$ depends only on point $\vec{r}$.



$$V(b) - V(a) = \left(-\int_{O_{ref}}^{b} \vec{E}(\vec{r}) \bullet d\vec{\ell}\right) - \left(-\int_{O_{ref}}^{a} \vec{E}(\vec{r}) \bullet d\vec{\ell}\right)$$

$$= -\int_{O_{ref}}^{b} \vec{E}(\vec{r}) \bullet d\vec{\ell} + \int_{O_{ref}}^{a} \vec{E}(\vec{r}) \bullet d\vec{\ell}$$

$$= -\int_{O_{ref}}^{b} \vec{E}(\vec{r}) \bullet d\vec{\ell} - \int_{a}^{O_{ref}} \vec{E}(\vec{r}) \bullet d\vec{\ell}$$

$$= -\int_{a}^{O_{ref}} \vec{E}(\vec{r}) \bullet d\vec{\ell} - \int_{O_{ref}}^{b} \vec{E}(\vec{r}) \bullet d\vec{\ell}$$

$$\boxed{\Delta V_{ab} \equiv V(\vec{r} = b) - V(\vec{r} = a) = -\int_{a}^{b} \vec{E}(\vec{r}) \bullet d\vec{\ell}}$$

The <u>fundamental theorem for gradients</u> states that:

$$\text{Potential difference: } \Delta V_{ab} \equiv V(r=b) - V(r=a) = \int_a^b \vec{\nabla} V(\vec{r}) \bullet d\ell = -\int_a^b \vec{E}(\vec{r}) \bullet d\vec{\ell}$$

This is true for <u>any</u> end-points <u>a</u> & <u>b</u> (and any contour from a → b). Thus the two *<u>integrands</u>* ***<u>must</u>*** be equ

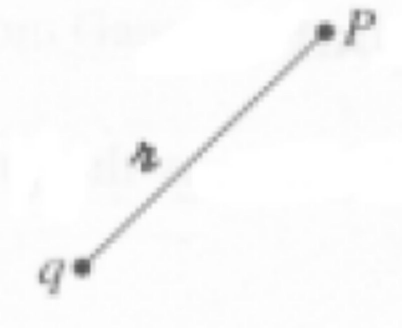
$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$	Differential Version

It is often easier to analyze a physical situation in terms of ***potential***, which is a ***scalar***, rather than the ***electric field strength***, which is a ***vector***.

$\Rightarrow$ Knowing $V(\vec{r})$ enables you to specify/calculate $\vec{E}(\vec{r})$!!

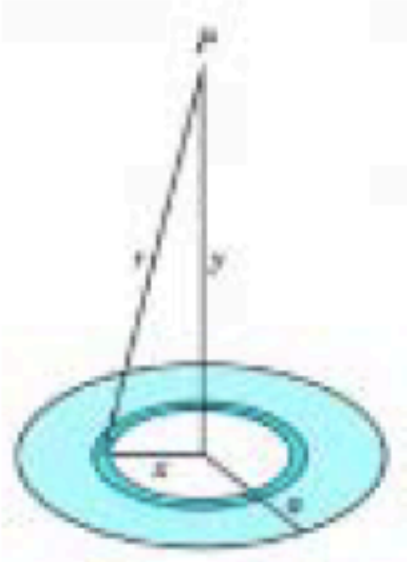
Setting the reference point at infinity, the potential of a point charge q **at the origin** is

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$



$$\boxed{V(\vec{r}) \equiv \frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho(\vec{r}')}{\boldsymbol{r}}d\tau'}$$

A nonconducting disk of radius a has a uniform surface charge density σ C/m². What is the potential at a point on the axis of the disk at a distance from its center.



Solution:

$$dV = \frac{dq}{4\pi\varepsilon_0 r}, \quad dq = \sigma(2\pi x dx)$$

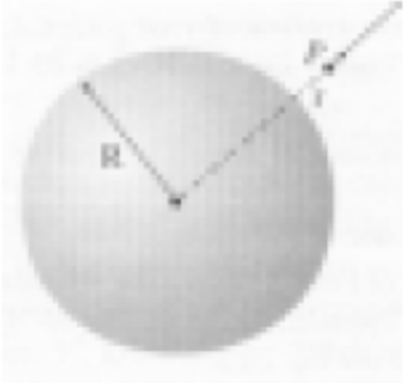
$$dV = \frac{\sigma\pi}{4\pi\varepsilon_0\sqrt{x^2+y^2}}dx^2$$

$$V = \int_0^a \frac{\sigma\pi}{4\pi\varepsilon_0\sqrt{x^2+y^2}}dx^2$$

$$= \frac{\sigma}{2\varepsilon_0}\left[(x^2+y^2)^{0.5} - y\right]_0^a = \frac{\sigma}{2\varepsilon_0}\left[(a^2+y^2)^{0.5} - y\right]$$

Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge. Set the reference point at infinity.

$$\begin{cases} \text{Inside } (r < R) & E = 0 \\ \text{outside } (r > R) & E = \dfrac{q}{4\pi\varepsilon_0 r^2} \end{cases}$$



$$V(r) = -\int_{\infty}^{r} \mathbf{E} \cdot d\ell = \frac{q}{4\pi\varepsilon_0 r} \quad (r > R)$$

$$\text{and } V(r) = \frac{q}{4\pi\varepsilon_0 R} \quad (r < R)$$

The electric field can be written as the gradient of a scalar potential.

$$\mathbf{E} = -\nabla V$$

What do the fundamental equations for E looks like, in terms of V?

$$\text{Gauss's law } \nabla \cdot \mathbf{E} = -(\nabla \cdot \nabla V) = \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0}}$$

$$\text{Curl law} \quad \nabla \times \mathbf{E} = -(\nabla \times \nabla V) = 0$$

$$\nabla \times \mathbf{E} = 0 \text{ permits } \mathbf{E} = -\nabla V;$$

$$\text{in turn, } \mathbf{E} = -\nabla V \text{ guarantees } \nabla \times \mathbf{E} = 0$$

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{\rho_{encl}(\vec{r})}{\varepsilon_o} \Leftarrow \text{Poisson's Equation}}$$

In regions of space where the volume charge density, $\rho(\vec{r}) = 0$, then Poisson's equation $\Rightarrow$ Laplace's Equation $\nabla^2 V(\vec{r}) = 0 \Leftarrow$ linear homogenous 2^{nd} order differential equation.

Cartesian Coordinates: $$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Cylindrical Coordinates: $$\boxed{\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}}$$

Spherical Coordinates: $$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}}$$

$$\left\{\begin{array}{l}\text{charge distribution} \\ q, \sum_{i=1}^{N} q_i, \lambda, \sigma, \rho\end{array}\right\}$$

$$V(\vec{r}) = \left\{\begin{array}{l} \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathbf{r}}, \ \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathbf{r}_i}, \text{ or} \\ \frac{1}{4\pi\varepsilon_o}\int_C \frac{\lambda(r')\,d\ell'}{\mathbf{r}}, \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(r')\,dA'}{\mathbf{r}} \\ \frac{1}{4\pi\varepsilon_o}\int_V \frac{\rho(r')\,d\mathbf{r}'}{\mathbf{r}} \end{array}\right\} \Rightarrow \ \vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

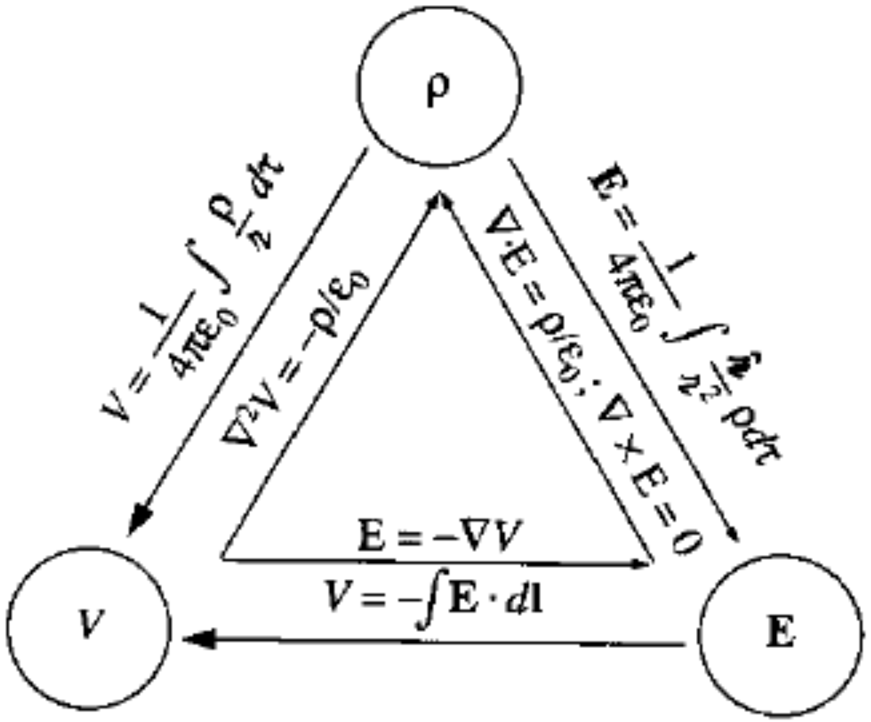
$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o} \text{ to find } \rho(\vec{r}).$$

$$\vec{E}(\vec{r}) = \left\{\begin{array}{l} \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathbf{r}^2}, \ \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{N}\frac{q_i}{\mathbf{r}_i^2}\hat{\mathbf{r}} \text{ or} \\ \frac{1}{4\pi\varepsilon_o}\int_C \frac{\lambda(r')\hat{\mathbf{r}}\,d\ell'}{\mathbf{r}^2}, \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(r')\hat{\mathbf{r}}\,dA'}{\mathbf{r}^2} \\ \frac{1}{4\pi\varepsilon_o}\int_V \frac{\rho(r')\hat{\mathbf{r}}\,d\tau'}{\mathbf{r}^2} \end{array}\right\}$$

$$V(\vec{r}) = -\int_c \vec{E}(\vec{r})\bullet d\vec{\ell}' \text{ to find } V(\vec{r})$$

$$\vec{\nabla}\bullet\vec{E}(\vec{r}) = \rho(\vec{r})/\varepsilon_o \text{ to find } \rho(\vec{r}).$$

We have derived six formulas interrelating three fundamental quantities: ρ , $\mathbf{E}$ and V.



These equations are obtained from two observations:

•Coulomb’s law: the fundamental law of electrostatics

•The principle of superposition: a general rule applying to all electromagnetic forces.