Lecture - 26

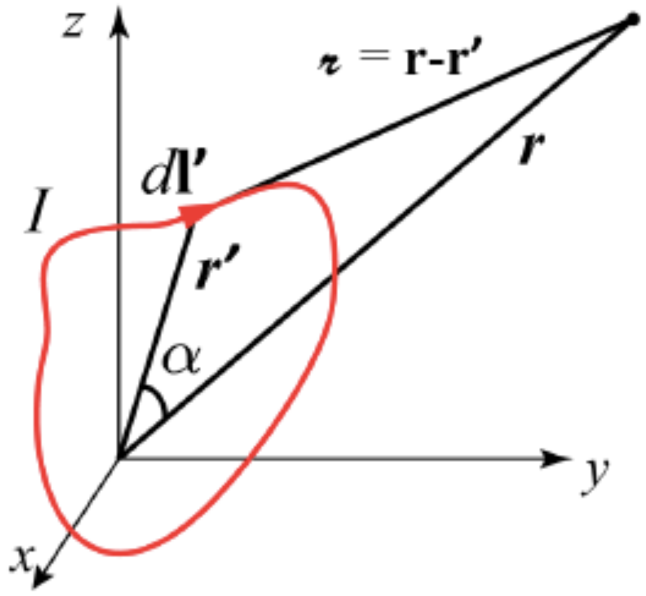
Monopole potential

$$\mathbf{A}_{\text{mono}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$

Dipole potential

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r' \cos\alpha \, d\mathbf{l}' \;= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') d\mathbf{l}'$$

$$= -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \nabla'(\hat{\mathbf{r}} \cdot \mathbf{r}') \times d\mathbf{a}'$$



$$\left[\text{Corollary of Stokes Theorem:} \oint_{path} T d\mathbf{l} = -\int_{Surf} \nabla T \times d\mathbf{a}\right]$$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \hat{\mathbf{r}} \times d\mathbf{a}' \;= \frac{\mu_0}{4\pi}\frac{1}{r^2}\left(I\int d\mathbf{a}'\right) \times \hat{\mathbf{r}}$$

$$= \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{m} \equiv I\int d\mathbf{a}'}$$

Magnetic dipole moment

Magnetic field due to a magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Take $\mathbf{m} = m\,\hat{\mathbf{z}}$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A}_{\text{dip}}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Recall

$$\mathbf{p} = p\hat{\mathbf{z}}$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

Magnetic Vector Potential:

Prob. 5.23 (Griffiths, 3rd Ed.): What is the current density **J** that would produce the magnetic potential $\mathbf{A} = k\hat{\boldsymbol{\phi}}$

$$\mathbf{A} = k\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}}$$

$$\mathbf{J} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \frac{k}{\mu_0 s^2}\hat{\boldsymbol{\phi}}$$

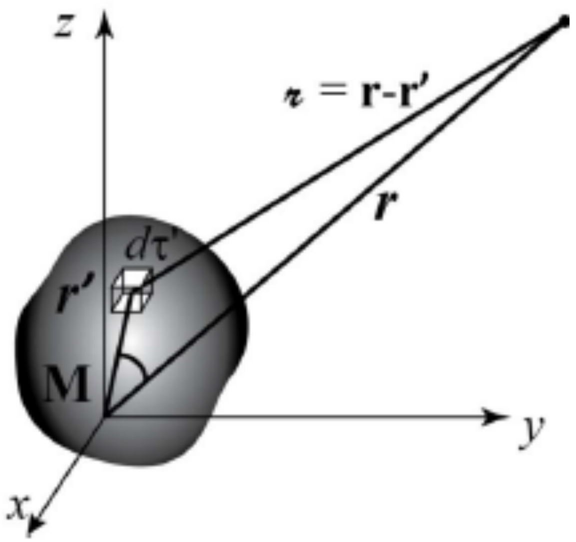
What is Polarization? - dipole moment per unit volume

- The dipole moment is caused either by stretch of an atom/molecule or by rotation of polar molecules
- Polarization in the direction parallel to the applied electric field

What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion:
  (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).

The Field of a Magnetized Object:



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{vol}\left[\mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

[Using $\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]d\tau' - \frac{\mu_0}{4\pi}\int_{vol}\nabla'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'$$

[Using $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

$\int_{vol}(\nabla\times\mathbf{V})d\tau = -\oint_{surf}\mathbf{V}\times d\mathbf{a}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}d\mathbf{a}'$$

$\mathbf{J}_b(\mathbf{r}') = \nabla'\times\mathbf{M}(\mathbf{r}')$ Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}$ Surface current

Ampere's law in magnetized material:

$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$

Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$

Surface current

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$$

Define: $\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M})$

$\nabla \times \mathbf{H} = \mathbf{J}_f$ Ampere's law in magnetized material (differential form)

$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}}$ Ampere's law in magnetized material (integral form)

Magnetic Susceptibility and Permeability

$$\mathbf{M} = \chi_m \, \mathbf{H}$$

χ_m is called the magnetic susceptibility

χ_m is a dimensionless quantity

χ_m is positive for paramagnetic and negative for diamagnets

The magnetic field thus becomes

$$\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi_m\mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H}$$

So, $\mathbf{B} \equiv \mu\mathbf{H}$

$\mu \equiv \mu_0(1 + \chi_m)$

is called the permeability of the material