Algebraic Geometry : Homework 5

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Exercise 2: When $n = 1$, the Veronese embedding is given by

$$\nu_d([x : y]) = [x^d : x^{d-1}y : \cdots : y^{d-1}x : y^d].$$

Show that μ_d is an isomorphism onto its image, and that the image is the projective variety $V = V_p(I)$ where $I \subset k[z_1, ..., z_{d+1}]$ is the homogeneous ideal $I := \langle z_i z_j - z_k z_l \mid i + j = k + l \rangle$.

Solution : First, let's show that ν_d is an isomorphism onto its image. It is clear from the definition of ν_d that is regular. Since x and y cannot be both zero for $[x : y] \in \mathbb{P}^1(k)$, then x^d and y^d cannot be both zero. Hence, we can define $\mu : \text{Im}(\nu_d) \to \mathbb{P}^1(k)$ by $\mu([x_1 : x_2 : \cdots : x_N]) = [x_1 : x_2]$. If $x_1 = 0$, then we must have $x_N \neq 0$ (from the previous remark) and so we define $\mu([x_1 : x_2 : \cdots : x_N])$ as $[x_{N-1} : x_N]$ instead. Notice that if both x_1 and x_2 are nonzero, then the two definitions are coincide since $[x^d : x^{d-1}y] = [\frac{y^{d-1}}{x^{d-1}}x^d : \frac{y^{d-1}}{x^{d-1}}x^{d-1}y] = [y^{d-1}x : y^d]$. Hence, from this observation, it follows easily that μ is a morphism from the image of ν_d to $\mathbb{P}^1(k)$. Finally, given $[x : y] \in \mathbb{P}^1(k)$, if $x \neq 0$, we have

$$(\mu \circ \nu_d)([x : y]) = \mu([x^d : x^{d-1}y : \cdots : y^d]) = [x^d : x^{d-1}y] = [x : y].$$

The case $y \neq 0$ is similar. Proving that $\nu_d \circ \mu = 1$ follows from the fact that $\mu \circ \nu_d = 1$ and that the domain of μ is the image of ν_d $((\nu_d \circ \mu)(\nu_d([x : y])) = \nu_d([x : y])$ for every $\nu_d([x : y])$ in the domain of μ). Therefore, ν_d is an isomorphism onto its image.

Next, let's prove that the image of ν_d is $V_p(I)$ where $I := \langle z_i z_j - z_k z_l \mid i + j = k + l \rangle$. If we let V be the image of ν_d, then clearly $V \subset V_p(I)$ since given a point $[x^d : x^{d-1}y : \cdots : y^d] \in V$, we have

$$\begin{aligned} x^{d-i+1}y^{i-1}x^{d-j+1}y^{j-1} &= x^{2d-(i+j)+2}y^{i+j-2} \\ &= x^{2d-(k+l)+2}y^{k+l-2} \\ &= x^{d-k+1}y^{k-1}x^{d-l+1}y^{l-1}. \end{aligned}$$

Conversely, let $[z_1 : \cdots : z_{d+1}] \in V_p(I)$ and consider the case where $z_1 \neq 0$, then we can write the point as $[1 : \cdots : z_{d+1}]$. By the defining property of I, we have that $z_3 = z_3 z_1 = z_2 z_2 = z_2^2$. From this result, we have that $z_4 = z_4 z_1 = z_3 z_2 = z_2^2 z_2 = z_2^3$. If we continue this process, then we can show by induction that

$$[z_1 : \cdots : z_{d+1}] = [1 : z_2 : z_2^2 : \cdots : z_2^d] = \nu_d([1 : z_2]) \in V.$$

Next, if $z_1 = 0$, then $z_2^2 = z_3 z_1 = 0$ implies that $z_2 = 0$. Similarly, $z_3^2 = z_4 z_2 = 0$ implies that $z_3 = 0$. If we repeat this process, then by induction, $z_i = 0$ for all $i \leq d$ (the case $i = d+1$ doesn't exist because we cannot write $2(d+1)$ as a sum of two integers $a, b \leq d+1$ where one of them is strictly less than $d+1$). Hence, we have

$$[z_1 : \cdots : z_{d+1}] = [0 : \cdots : 0 : z_{d+1}] = [0 : \cdots : 0 : 1] = \nu_d([0 : 1]) \in V.$$

Therefore, since we covered all possible cases, we have that $V = V_p(I)$.