MATH 458 Notes : Honours Differential Geometry

Samy Lahlou

These notes are based on lectures given by Professor Jean Pierre Mutanguha at McGill University during the Winter 2026 semester. These lectures will introduce the local & global theory of curves and surfaces in two and three dimensions. As a disclaimer, it is more than possible that I made some mistakes. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

Contents

1 Preliminaries

1.1 Linear Algebra

In this course, we will focus on the vector space

$$\mathbb{R}^n = \{(x_1, ..., x_n) \ : \ x_i \in \mathbb{R}\},$$

especially $\mathbb{R}^2$ and $\mathbb{R}^3$ even if everything that will be covered can be generalized for $\mathbb{R}^n$ where $n \geq 2$ is arbitrary. We will also care about the standard dot product on $\mathbb{R}^n$, defined by

$$\mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i.$$

This is an example of an *inner product*, i.e., a symmetric function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, bilinear, and positive definite. Notice that for any invertible linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$, we can define a new inner product $b_T : (v, w) \mapsto Tv \cdot Tw$. It turns out that every inner product on $\mathbb{R}^n$ is of this form for some linear transformation T. Using the inner product, we can define the usual notions of Euclidean norm and Euclidean distance in the vector space $\mathbb{R}^n$:

$$\|v\| = \sqrt{v \cdot v}$$

and

$$d(v, w) = \|v - w\|.$$

The goal now is to define what it means for a transformation to be a rigid motion. Intuitively, a rigid motion is like a translation or a rotation, it preserves distances. This motivates the following definition.

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if the following conditions are satisfied:

- $T(v) \cdot T(w) = v \cdot w$ for all $v, w \in \mathbb{R}^n$,
- $\|T(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$,
- $d(T(v), T(w)) = d(v, w)$ for all $v, w \in \mathbb{R}^n$.

It turns out that the three conditions in this definition are equivalent. This should not be surprising since the notions of norm and distance are both defined in terms of the dot product. Let's prove this.

Proposition 1.1.1. The following are equivalent:

(1) $T(v) \cdot T(w) = v \cdot w$ for all $v, w \in \mathbb{R}^n$,

(2) $\|T(v)\| = \|v\|$ for all $v \in \mathbb{R}^n$,

(3) $d(T(v), T(w)) = d(v, w)$ for all $v, w \in \mathbb{R}^n$.

Proof. Condition (1) implies condition (2) because if T preserves the dot product, then $T(v) \cdot T(v) = v \cdot v$ for all $v \in V$. Taking the square root on both sides gives us precisely condition (2). Condition (2) implies condition (3) because replacing v with $v - w$ in condition (2) gives us $\|T(v - w)\| = \|v - w\|$ which is equivalent to $d(T(v), T(w)) = d(v, w)$. Similarly, condition (3) implies condition (2) by taking $w = 0$.

Finally, we need to show that conditions (2) and (3) imply condition (1). To do so, let $v, w \in V$, then $d(T(v), T(w)) = d(v, w)$. Squaring both sides and using the definition of the distance gives us

$$(v - w) \cdot (v - w) = (T(v) - T(w)) \cdot (T(v) - T(w)).$$

By symmetry and bilinearity of the dot product, we can expand both sides to get

$$v \cdot v - 2(v \cdot w) + w \cdot w = T(v) \cdot T(v) - 2(T(v) \cdot T(w)) + T(w) \cdot T(w).$$

By condition (2), $v \cdot v = T(v) \cdot T(v)$ and $w \cdot w = T(w) \cdot T(w)$, hence, after cancelling the terms:

$$v \cdot w = T(v) \cdot T(w)$$

which is exactly condition (1). ∎

For the moment, an orthogonal transformation is not exactly what we have in mind when we talk about rigid motions. For example, the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-x, y)$ is an orthogonal transformation, but it flips the plane so it is not a rigid motion. The following proposition makes the phenomenon more precise.

Proposition 1.1.2. If T is an orthogonal transformation, then $\det(T) = \pm 1$.

Proof. Let $e_1, ..., e_n$ be the standard basis in $\mathbb{R}^n$. Let $A = (a_{ij})_{ij}$ be the matrix representation of the orthogonal transformation T, then $Te_i = \sum_k a_{ik} e_k$. Since T is orthogonal, then it preserves the inner product. It follows that $Te_i \cdot Te_j = \delta_{ij}$. But on the other hand, we have that $Te_i \cdot Te_j = \sum_k a_{ik} a_{jk}$. Next, consider the matrix AA^T, its coefficient with index ij is equal to $\sum_k a_{ik} b_{kj}$ where $A^T = (b_{ij})_{ij}$. But since $b_{ij} = a_{ji}$ holds for all i and j, then the coefficient with index ij in AA^T is

$$\sum_k a_{ik} a_{jk} = Te_i \cdot Te_j = \delta_{ij}.$$

In other words, $AA^T = I_n$. Taking the determinant on both sides gives us $\det(AA^T) = 1$. Since the determinant preserves the product of matrices, and $\det(A^T) = \det(A)$, then the last equation is equivalent to $\det(A)^2 = 1$. Therefore, $\det(T) = \det(A) = \pm 1$. ∎

We can think of a transformation with negative determinant in R^2 as a transformation that flips the plane. More generally, such a transformation can be thought as changing the orientation of the space. This motivates the next definition.

Definition. A linear transformation T is *orientation-preserving* if $\det(T) > 0$.

An orthogonal orientation-preserving transformation can be thought as rotations about the origin. Hence, if we combine the last two definitions, it would be tempting to say that we would get the definition of a rigid motion. However, our definitions only apply to linear transformations, and not translations for example. This is precisely what is the left to add in our definition of a rigid motion.

Definition. A function $M : \mathbb{R}^n \to \mathbb{R}^n$ is a *rigid motion* if there is an element $a \in \mathbb{R}^n$ and an orthogonal orientation-preserving transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $M(v) = T(v) + a$ for all $v \in \mathbb{R}^n$. Equivalently, a rigid motion is an affine linear map where linear part is orthogonal and orientation-preserving.

The notion of rigid motion is really important in this course because it lets us focus on the important properties of curves and surfaces. For example, the curvature of a curve at a point is invariant under rigid motions. This shows that the curvature is a property that doesn't depend on where the curve is exactly in the space. More generally, the properties we are going to study are these properties that are invariant under rigid motions. To avoid saying that a property is invariant up to rigid motions many times, we will replace sentences like

"The length of a regular curve $\mathcal{C} \subset \mathbb{R}^n$ is defined in terms of the length of its parametrization. The length of a curve is invariant up to rigid motions."

with

"The length of a regular curve $\mathcal{C} \subset \mathbb{E}^n$ is defined in terms of the length of its parametrization."

In short, $\mathbb{R}^n$ is replaced with $\mathbb{E}^n$ whenever the stated property is invariant under rigid motions.

1.2 Topology

In the previous section, we defined the notion of rigid motion, in this section, we define the basic notions of topology required for this course. The most basic notion of topology is the notion of an open set. An open set can be seen as a generalization of an open interval in $\mathbb{R}$. To make this precise, let's first define what disks are in $\mathbb{R}^n$.

Definition. Given $\epsilon > 0$ and $p \in \mathbb{R}^n$, we define the *open disk* (or simply *disk*) as

$$\mathbb{D}_\epsilon(p) = \{q \in \mathbb{R}^n \ : \ d_{\mathbb{E}}(p,q) < r\}.$$

We also define the closed *closed disk* as

$$\overline{\mathbb{D}_\epsilon(p)} = \{q \in \mathbb{R}^n \ : \ d_{\mathbb{E}}(p,q) \leq r\}.$$

Even if open and closed disks seem very similar by looking at their definitions, they are in fact very different. To see this difference, suppose that a certain property holds precisely for the points in the open disk $\mathbb{D}_\epsilon(0)$, then we can say that whenever a point p satisfies this property, then there is a small open disk around p such that every point in this smaller disk satisfies this property. However, this is not the case for the closed disk because if we take a point on the boundary, any disk around that point will have elements inside and outside the closed disk. From this, we can now define the what it means for a general set to be open.

Definition. A set $U \subset \mathbb{R}^n$ is *open* if for all $p \in U$, there is a $\epsilon > 0$ such that $\mathbb{D}_\epsilon(p) \subset U$. A set $A \subset \mathbb{R}^n$ is *closed* if its complement is open.

The open sets really capture the notion of interior. Every point is at the interior of the set, no point is alone, or on the boundary. The open sets satisfy the following properties.

Proposition 1.2.1. (a) Let $\{U_i\}_i$ be an arbitrary collection of open subsets of $\mathbb{R}^n$, then $\bigcup_i U_i$ is an open set.

(b) Let $\{U_1, ..., U_n\}$ be a finite collection of open subsets of $\mathbb{R}^n$, then $\bigcap_{i=1}^n U_i$ is an open set.

The notion of disk is also useful to talk about boundedness. In $\mathbb{R}$, a set is bounded if every point satsfies $|x| < M$ for some uniform upper bound M. Equivalently, a set is bounded if it is contained in the interval $(-M, M) = \mathbb{D}_M(0)$. This motivates the following more general definition.

Definition. A set is *bounded* if it is contained in a disk (open or closed).

We see that topology is giving us tools to characterize sets. We defined the notion of open, closed, and bounded sets. There is one last type of sets which will be important for this course that we need to define.

Definition. A set $C \subseteq \mathbb{R}^n$ is *compact* if whenever there is a collection of open sets $\{U_i\}_i$ such that $C \subseteq \bigcup_i U_i$, then there is a subcollection $\{U_1, ..., U_n\}$ such that $C \subseteq \bigcup_{i=1}^n U_i$.

The notion of compactness may seem very obscure and useless, but it turns out that compact sets are, in a sense, a generalization of finite sets. Morever, compact sets are very nice to work with for their numerous properties. Thankfully, the following theorem will give us a very useful equivalent definition for compact sets in $\mathbb{R}^n$.

Theorem 1.2.2 (Heine-Borel)**.** A subset $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

As always in mathematics, after adding more structure to a set, we need to define what it means for a function interact well with that new structure. In particular, we need to define what it means for a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ to preserve open sets. This motivates the following definition.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* if $f^{-1}(U) \subseteq \mathbb{R}^n$ is open for all open subsets $U \subseteq \mathbb{R}^m$. If f is continuous, bijective, and its inverse is also continuous, then f is a *homeomorphism*.

Here are important properties of continuous functions:

Proposition 1.2.3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function.

(a) $f^{-1}(A) \subseteq \mathbb{R}^n$ is closed for all closed sets $A \subseteq \mathbb{R}^m$.

(b) If $C \subseteq \mathbb{R}^n$, then $f(C) \subseteq \mathbb{R}^m$ is compact.

We will not need more topology than what is contained in this section.

1.3 Vector Calculus

In this section, we will state the most important theorem from Vector Calculus: the Inverse Function Theorem. But first, let's define the notion of differentiability for multivariable functions.

Definition. Given a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and a point $a \in U$ where U is open, we say that f is differentiable at p if there is a linear transformation $df_p : \mathbb{R}^n \to \mathbb{R}^m$, called the *derivative of f at p*, that satisfies

$$\lim_{x \to p} \frac{\|f(x) - f(p) - df_p(x-p)\|}{\|x - p\|} = 0.$$

The matrix associated to df_p is called the *Jacobian of f at p* and is denoted by $J_f(p)$.

The intuitive way of thinking about the derivative is to view it as the linear approximation of the function at a given point (up to translation). Hence, it should be no surprise that it satisfies most of the properties of the usual derivative:

Proposition 1.3.1. Let $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be two functions differentiable at $p \in U$.

(a) The function f is continuous at p.

(b) The function λf is differentiable at p, and $d(\lambda f)_p = \lambda df_p$ for all $\lambda \in \mathbb{R}$.

(c) The function $f + g$ is differentiable at p, and $d(f+g)_p = df_p + dg_p$.

(d) The function fg is differentiable at p, and $d(fg)_p = g(p)df_p + f(p)dg_p$.

(e) If $g(p) \neq 0$, then the function f/g is differentiable at p, and

$$d\left(\frac{f}{g}\right)_p = \frac{g(p)df_p - f(p)dg_p}{g(p)^2}.$$

The formulas above should be compared to their one-variable analogue to see that they are really the same. Another important property that really deserves its own theorem is the Chain Rule.

Theorem 1.3.2 (Chain Rule)**.** Let $f : U \subset \mathbb{R}^n \to \tilde{U} \subset \mathbb{R}^m$ and $g : \tilde{U} \to \mathbb{R}^k$ be differentiable at $p \in U$ and $f(p) \in \tilde{U}$ respectively, then $g \circ f$ is differentiable at p with

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

In terms of the Jacobian:

$$J_{g \circ f}(p) = J_g(f(p)) \cdot J_f(p).$$

The notion of derivatives lets us classify points in the domain and in the codomain in the following way:

Definition. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. The point $p \in U$ is a *critival point* of f if df_p is not onto R^m (i.e., $\mathrm{rank}(J_f(p)) < m$). The point $r \in \mathbb{R}^m$ is a *regular value* if it is not the image of a critical point.

In a first multivariable calculus class, we generalize the notion of derivatives by introducing the notion of partial derivatives of a function. The following theorem shows that partial derivatives and Jacobians are related:

Proposition 1.3.3. Let $f = (f_1, ..., f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $p \in U$, then all the partial derivatives of f exist and

$$J_f(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial x_n}(p) \end{pmatrix}$$

where the entry in the ith row and jth column is equal to the ith partial derivative of f with respect to x_j. Conversely, if all the partial derivatives of f exist and are continuous at p, then f is differentiable at p.

As it can be seen from the second part of the previous proposition, the fact that all the partial derivatives of a function exist and continuous at a point (or on an open set) is useful, and hence, it derserves its own name.

Definition. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be C^1 if all its partial derivatives exist and are continuous on U. More generally, f is said to be C^2 if all its partial derivatives of the kth order exist and are continuous on U. By convention, we say that f is C^0 if it is continuous. Finally, f is said to be C^∞, or *smooth*, if it is C^k for all k.

Using this notation, it is clear that any C^{k+1} function is also C^k. This also holds for the case $k = 0$, i.e., every C^1 function is continuous. This follows from the fact that every C^1 function is differentiable, and every differentiable function is continuous. We are now finally able to state the Inverse Function Theorem.

Theorem 1.3.4 (Inverse Function Theorem)**.** Suppose $f : U' \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a C^k function whose derivative at $p \in U'$ is invertible, then there are open sets $U \subseteq U'$ and $V \subseteq \mathbb{R}^n$ such that $p \in U$, $f(U) = V$, and the restriction $f|_U : U \to V$ is a C^k bijection whose inverse $g : V \to U$ is C^k with

$$J_g(f(p)) = J_f(p)^{-1}.$$

The one-variable analogue of this theorem is simply the fact that if a function f has a non-zero derivative at a point p in its domain, then the function has an inverse that satisfies $(f^{-1})'(f(p)) = 1/f'(p)$. Writing $y = f(x)$ and using the Leibniz notation gives us the more natural equation

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.$$

The Inverse Function Theorem is probably the theorem that we will use the most in what follows.

2 Curves

2.1 Paths

The goal of this chapter is to define and understand the different properties of curves that are invariant under rigid motions. But first, we need to talk about paths which are simpler objects. In a sense, curves are formed by paths which means that it is very useful to first understand the concept of paths well, and then move on to curves.

Definition. A *path* in $\mathbb{R}^n$ is a continuous function $\gamma : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval (closed or open, bounded or unbounded). The path is said to be *compact* if I is compact.

Since paths will be used everywhere in this chapter, there are many many new quantities and terminology that need to be defined. First, most paths that we will consider will be differentiable at least once.

Definition. A path $\gamma : I \to \mathbb{R}^n$ is said to be C^k if the function γ is a C^k function.

Next, it is useful to visualize a path as a function of time. Hence, we will also visualize the first and second derivatives of γ in this way by renaming the derivatives.

Definition. If the path $\gamma : I \to \mathbb{R}^n$ is C^2, then we call $\dot{\gamma} : I \to \mathbb{R}^n$ its *velocity* and $\|\dot{\gamma}\| : I \to \mathbb{R}^n$ its *speed*. Moreover, if the path is C^2, then we call $\ddot{\gamma} : I \to \mathbb{R}^n$ its *acceleration*.

This lets us define a very important type of path.

Definition. Let $\gamma : I \to \mathbb{R}^n$ be a C^1 path, then it is said to be *regular* if its velocity is never zero, i.e., if $\dot{\gamma}(t) \neq 0$ for all $t \in I$.

Now that we have defined the main terminology and properties of path, we can start proving that they are all preserved under rigid motions.

Proposition 2.1.1. If $\gamma : I \to \mathbb{R}^n$ is a (regular) C^k path and $M : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion, then $\tilde{\gamma} = M \circ \gamma : I \to \mathbb{R}^n$ is a (regular) C^k path.

Proof. Write $\gamma = (\gamma_1, ..., \gamma_n)$ where $\gamma_i : I \to \mathbb{R}$. Assuming that γ is C^k implies that γ_i is k times differentiable and $\gamma_i^{(k)}$ is continuous. If $M = \vec{a} + T$ with $T = (m_{ij})_{ij}$, then

$$\tilde{\gamma} = \left(a_1 + \sum_{j=1}^{n} m_{1j}\gamma_j,\ a_2 + \sum_{j=1}^{n} m_{2j}\gamma_j,\ ...,\ a_m + \sum_{j=1}^{n} m_{nj}\gamma_j \right).$$

Since each component of $\tilde{\gamma}$ is a linear combination of k times differentiable functions with continuous kth derivatives, then each component of $\tilde{\gamma}$ is k times differentiable functions with continuous kth derivatives. Thus, $\tilde{\gamma}$ is C^k.

Next, suppose γ is regular and let $t \in I$. By differentiating $\tilde{\gamma}$ component-wise, we get that $\dot{\tilde{\gamma}}(t) = T(\dot{\gamma}(t))$. By our assumption on γ, the vector $\dot{\gamma}(t)$ is non-zero, and hence, $T(\dot{\gamma}(t))$ is also non-zero since T is invertible. Thus, $\tilde{\gamma}$ is regular. ■

By looking at the proof, we see that we didn't need M to be a rigid motion but just a function of the form $T + \vec{a}$ where T is an invertible linear transformation. Hence, we could generalize the previous proposition but we will not since it will not be useful.

Proposition 2.1.2. If $\gamma : I \to \mathbb{R}^n$ is a C^k path, $M : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion and $\tilde{\gamma} = M \circ \gamma : I \to \mathbb{R}^n$, then $\left\|\dot{\tilde{\gamma}}\right\| = \|\dot{\gamma}\|$, i.e., speed is preserved by rigid motions.

Proof. First, write $M = T + \vec{a}$ where T is an orthogonal transformation. As we saw in the proof of the previous proposition, we have $\dot{\tilde{\gamma}}(t) = T(\dot{\gamma}(t))$ for all $t \in I$. Hence, $\left\|\dot{\tilde{\gamma}}(t)\right\| = \|T(\dot{\gamma}(t))\|$. But since T is orthogonal, then $\left\|\dot{\tilde{\gamma}}(t)\right\| = \|\dot{\gamma}(t)\|$. Therefore, γ and $\tilde{\gamma}$ have the same speed. ■

We will finish this subsection by defining the length of a path.

Definition. Let $\gamma : I \to \mathbb{R}^n$ be a C^1 path where I is a bounded interval. The *length* of γ is defined as

$$l(\gamma) = \int_I \|\dot{\gamma}(t)\| \, dt.$$

From what we proved earlier, we can directly prove with a useful proposition.

Proposition 2.1.3. The length of a path is preserved under rigid motions.

Proof. Simply notice that the length of a curve only depends on its speed, and that we already proved that speed is preserved under rigid motions. ■

It turns out that as a direct corollary of the fact that length are preserved under rigid motions, we can prove a useful generalization of the triangle inequality in $\mathbb{R}^2$.

Corollary. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a C^1 path with $\gamma(a) = p$ and $\gamma(b) = q$, then

$$\ell(\gamma) \geq d(p, q)$$

with equality holding if and only if γ is a line that never changes direction.

Proof. We can assume that p is at the origin and that q is an the x-axis because the length is unchanged under rigid motions, as well as the fact that γ is C^1. Once this assumption made, we can write $\gamma = (x, y)$ where x and y are the components of γ. By our assumptions, $x(a) = p$ and $x(b) = q$. Hence:

$$l(\gamma) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \geq \int_a^b \left|\frac{dx}{dt}\right| dt \geq \left|\int_a^b \frac{dx}{dt}\right| dt = |x(b) - x(a)| = d(p, q).$$

This proves the first part of the theorem. Now, notice that if both quantities were equal, then $(dy/dt)^2 \equiv 0$ and $dx/dt \geq 0$. Equivalently, this means that y is constant and that x is increasing. Since $\gamma(a)$ is on the x-axis and y is constant, then the image of γ is a subset of the x-axis, and hence, a line. Since x is increasing, then this line never changes direction. ■

2.2 Reparametrizations of paths

We usually think of a path as its image. However, many distinct paths have the same image, and hence, many common properties. This is the subject of this section.

Definition. Given two paths $\gamma : I \to \mathbb{R}^n$ and $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$, then $\tilde{\gamma}$ is a *reparametrization* of γ if there is a homeomorphism $t : \tilde{I} \to I$ such that $\tilde{\gamma} = \gamma \circ t$.

Intuitively, reparametrizing a path is the same as changing the speed of the path, or even the orientation of the original path. For example, if $\gamma : [a, b] \to \mathbb{R}^n$ is a path that goes from $\gamma(a)$ to $\gamma(b)$, then $\tilde{\gamma} = \gamma \circ t$ with $t : [a, b] \to [a, b]$ defined by $t(s) = b + a - s$ is the exact same path as γ, except that it now goes from $\gamma(b)$ to $\gamma(a)$. This motives the following definition.

Definition. A change of variable t is *orientation-preserving* if it is increasing, and *orientation-reversing* if it is decreasing.

As we did in the previous section, we need to be sure that some essential properties of a path are not lost after a reparametrization.

Proposition 2.2.1. Suppose γ is (regular) C^k path and $t : \tilde{I} \to I$ is a (regular) C^k bijection, then $\tilde{\gamma} = \gamma \circ t$ is a (regular) C^k reparametrization, i.e., reparametrization preserves the differentiability and regularity.

Proof. By the chain rule, $\tilde{\gamma}$ is a composition of two C^k functions so it is a C^k function. If γ and t are regular, then $\dot{\gamma}(t) \neq 0$ for all $t \in I$ and $t'(s) \neq 0$ for all $s \in \tilde{I}$. By the chain rule, $\dot{\tilde{\gamma}}(s) = t'(s) \cdot \dot{\gamma}(t(s)) \neq 0$ for all $s \in \tilde{I}$ since both factors are non-zero. Thus, $\tilde{\gamma}$ is regular. ∎

Since a reparametrization of a path have the same image as the path, it can be guessed that they have the same length. It turns out that we can easily prove this remark under the assumption that the change of variable is C^1.

Proposition 2.2.2. Suppose $\gamma : I \to \mathbb{R}^n$ is a C^1 path, $t : \tilde{I} \to I$ is a C^1 homeomorphism and $\tilde{\gamma} = \gamma \circ t$, then $\ell(\tilde{\gamma}) = \ell(\gamma)$, i.e., reparametrization preserves the length.

Proof. First, notice that t is a homeomorphism between two intervals, hence, t is either increasing or decreasing. Let c be a constant equal to 1 is t is increasing, and -1 is t is decreasing, then $|t'| = c \cdot t'$. Moreover, if t is used as a change of variable in an integral, then multiplying the integral by c will make it stay in the right orientation. It follows that

$$\ell(\tilde{\gamma}) = \int_{\tilde{I}} \left\|\dot{\tilde{\gamma}}(s)\right\| ds = c \int_{\tilde{I}} t'(s) \cdot \|\dot{\gamma}(t(s))\| \, ds = \int_I \|\dot{\gamma}(t)\| \, dt = \ell(\gamma).$$

∎

As we see from the previous propositions, reparametrizing a path don't change some key properties. There are many situations where reparametrizing a path can be useful, it lets us choose an appropriate parametrization depending on the context.

Definition. **TODO**

TODO

(a) define arclength reparametrization

(b) prove that it always exists

(c) prove that for a constant speed path, the acceleration is perpendicular to the velocity.

TODO

2.3 Curves

Definition. A subset $\mathcal{C} \subset \mathbb{E}^n$ is a (C^k) curve if is connected and for all points $p \in G$, there exists a compact neighborhood N_p of p and a one-to-one compact (regular C^k) parametrized curve $\gamma : I \to \mathbb{E}^n$ such that $\gamma(I) = \mathcal{C} \cap N_p$.

From this definition, we say that the unit circle is a curve which is not the image of a single parametrized curve. The logarithmic spiral is a curve, even if we add the origin.

Some examples of curves. [desmos **TODO**]

The change of parameter t is orientation preserving if it is increasing, otherwise, it is orientation-reversing.

Suppose γ is a regular C^k path and $t : \tilde{I} \to I$ is a C^k bijection with C^k inverse (which is equivalent to be a C^k bijection with the derivative nonzero everywhere by the Inverse Function Theorem), then $\tilde{\gamma}$ is a C^k reparametrization of γ.

We can reparametrize the logarithmic spiral so that the domain is finite (using the tangent reparametrization). We use this reparametrization to show that the logarithmic spiral with the additional zero point is a curve by making the finite domain of the reparametrization compact (by adding a point).

Theorem 2.3.1. Given a connected subset $\mathcal{C} \subset \mathbb{R}^n$, then $\mathcal{C}$ is a regular C^k curve if and only if $\mathcal{C}$ is the image of a regular C^k path $\gamma : I \to \mathbb{R}^n$ satisfying one the following definition: γ is injective with a continuous inverse that is C^k, or $I = \mathbb{R}$, γ is periodic and the restriction of γ to any compact interval shorter than the period is one-to-one.

Any regular C^k path is called a global parametrization of the image curve.

Definition. Arclength parametrization. $s(t) = \int_{t_0}^{t} \|\gamma dot(t)\| \, dt$. By FTC, $s'(t) > 0$ on I, so s is strictly increasing. Set $\tilde{I} = s(I)$, then $s : I \to \tilde{I}$ is an orientation-preserving C^1 bijection with $s' > 0$. By the Inverse Function Theorem, $t = s^{-1} : \tilde{I} \to I$ is an orientation preserving C^1 bijection and $t'(s) = 1/\|gammadot(t(s))\|$.

[**TODO**change notation for speed and velocity by replacing ν with gamma dot. **TODO**]

What is the length of a curve ?
exercise: the last two definitions of length agree.

Let γ be a path and $\tilde{\gamma}$ a reparametrization with constant speed c.

Lemma 2.3.2. $\ddot{\tilde{\gamma}} \perp \dot{\tilde{\gamma}}$.

Proof. First, notice that

$$\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} = \left\|\dot{\tilde{\gamma}}\right\|^2 = c.$$

Hence, if we take the derivative, we get

$$\frac{d}{dt}(\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}}) = 0.$$

But notice that the product rule gives us

$$\frac{d}{dt}(\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}}) = \ddot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} + \dot{\tilde{\gamma}} \cdot \ddot{\tilde{\gamma}} = 2(\ddot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}})$$

Thus, $\ddot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} = 0$ which directly implies that $\ddot{\tilde{\gamma}} \perp \dot{\tilde{\gamma}}$. ■

2.4 Curvature

Given a regular C^2 path $\gamma : I \to \mathbb{R}^n$, we can find an orientation-preserving change of parameters $t : \tilde{I} \to I$ and define $\tilde{\gamma} = \gamma \circ t : \tilde{I} \to \mathbb{R}^n$ such that $\tilde{\gamma}$ has unit speed. Let $s = t^{-1} : I \to \tilde{I}$.

Definition. Define the *curvature* of γ at time t to be $\kappa_\gamma(t) = \left\|\ddot{\tilde{\gamma}}(s(t))\right\|$.

In the homework, we prove that the curvature is well-defined.

Given a regular C^2 curve $\mathcal{C} \subset \mathbb{R}^n$, and a point $p \in \mathcal{C}$, the Classification Theorem implies that there is a global regular C^2 parameter $\gamma : I \to \mathbb{R}^n$ of $\mathcal{C}$. So there is some time $t \in I$ such that $\gamma(t) = p$. Define the curvature of $\mathcal{C}$ at p to be the curvature of γ at time t.

This is well-defined by the fact that every two global parametrization of a regular curve are related by a change of parameter, and hence, it follows from the well-definedness of the curvature for paths.

Exercise: Curvature is preserved by rigid-motions of $\mathbb{R}^n$, i.e., $\kappa_\gamma = \kappa_{M \circ \gamma}$ where M is a rigid motion of $\mathbb{R}^n$.

Proposition 2.4.1.

$$\kappa_\gamma = \frac{1}{\|\dot{\gamma}\|^2}\left\|\ddot{\gamma} - \left(\frac{\ddot{\gamma} \cdot \dot{\gamma}}{\dot{\gamma} \cdot \dot{\gamma}}\right)\dot{\gamma}\right\| = \frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2}$$

Proof. **TODO** ■

Definition. Let $\gamma : I \to \mathbb{R}^n$ be a regular path. Define the unit tangent vector as

$$T(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}.$$

Definition. Let $\gamma : I \to \mathbb{R}^n$ be a C^2 regular path with nonzero curvature. Define the unit normal vector as

$$N(t) = \frac{\ddot{\gamma}(t)^{\perp}}{\|\ddot{\gamma}(t)^{\perp}\|}.$$

Definition. The osculating plane at time t: contains $\gamma(t)$ and spanned by $\{\dot{\gamma}(t), \ddot{\gamma}(t)\}$ assuming the curvature is nonzero.

The osculating circle at time t: circle in the osculating plane of radius $1/\kappa_\gamma(t)$, and center $\gamma(t) + N(t)/\kappa_\gamma(t)$. The path formed by the center of the osculating circles is called the evolute of γ.

Exercise: The curvature of a circle of radius r is $1/r$.

Jeudi 22 Janvier:

Definition. Let γ be a regular curve with tangent vector function $T(t)$, there is a unique function θ such that $T = \rho \circ \theta$ where $\rho(\theta) = (\cos\theta, \sin\theta)$. This function $\theta(t)$ is called the angle function.

2.5 Space paths

Let's move to dimension 3. Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2 path with positive curvature $\kappa > 0$. Since it has positive curvature, the tangent and normal vectors are always defined. Since we are working in $\mathbb{R}^3$, we can define a third vector to form a basis.

Definition. We define the *binormal* vector $B(t)$ as the tangent vector crossed with the normal vector, in other words, $B(t) = T(t) \times N(t)$. These three vectors form a basis which we call the *Frenet Frame*.

Now, we assume that the curve is C^3 and parametrized by arclength. Hence, $T = \dot{\gamma}$ and $\frac{dT}{ds} = \|\ddot{\gamma}\| \cdot N = \kappa \cdot \kappa$. Since $\|B\| \equiv 1$, then $\frac{dB}{ds} \bullet B \equiv 0$. Since $B = T \times N$, then

$$\frac{dB}{ds} = \frac{dT}{ds} \times N + T \times \frac{dN}{ds}.$$

But $\frac{dT}{ds} = \kappa \cdot \kappa$ so $\frac{dT}{ds} \times N = \kappa \cdot N \times N = 0$. Hence,

$$\frac{dB}{ds} = T \times \frac{dN}{ds}.$$

It follows that $\frac{dB}{ds} \cdot T \equiv 0$. Since dB/ds is perpendicular to B and T, then it must be parallel to N. We define the torsion of γ, $\tau(s)$ as the constant such that

$$\frac{dB}{ds}(s) = -\tau(s)N(s),$$

or in other words,

$$\tau(s) = -\frac{dB}{ds}(s) \cdot N(s).$$

Since $\|N\| = 1$, then $\frac{dN}{ds} \cdot N \equiv 0$. Since $T \cdot N \equiv 0$, then $T \cdot \frac{dN}{ds} = -\kappa$, and similrly, $B \cdot \frac{dN}{ds} = \tau$. Hence,

$$\frac{dN}{ds} = -\kappa T + 0 \cdot N + \tau B.$$

We can do this for all the otehr vectors. This gives us the *Frenet Equations*:

$$\begin{aligned} T' &= 0 \cdot T + \kappa N + 0 \cdot B, \\ N' &= -\kappa T + 0 \cdot N + \tau B, \\ B' &= 0 \cdot T - \tau N + 0 \cdot B. \end{aligned}$$

Theorem 2.5.1 (Fundamental Theorem of Space Paths). Let $I \subseteq \mathbb{R}$ be an interval with basepoint $s_0 \in I$. Suppose $\tau : I \to \mathbb{R}$ is a C^{k-3} function, and $\kappa : I \to \mathbb{R}_{>0}$ is a C^{k-2} function. Then for any initial position p_0, initial velocity v_0, and initial normal n_0 in $\mathbb{R}^3$ such that $\|v_0\| = \|n_0\| = 1$ and $v_0 \cdot n_0 = 1$, there is a unique regular C^k path $\gamma : I \to \mathbb{R}^3$ parametrized by arclength and satisfying the following conditions:

$$\kappa_\gamma = \kappa, \qquad \tau_\gamma = \tau, \qquad \gamma(s_0) = p_0, \qquad \dot\gamma(s_0) = v_0, \qquad \ddot\gamma(s_0)/\|\ddot\gamma(s_0)\| = n_0.$$

Proof. If we look at the Frenet Equations and the given values in the statement of theorem, we can recognize that we have a 1st order linear IVP. By the Picard-Lindelöf Theorem, there is a unique solution $T, N, B : I \to \mathbb{R}^3$. Let's find γ such that T, N, B are the corresponding T, N, B of γ. To do this, let's show that they form an orthonormal basis. By the Frenet Equations,

$$\begin{aligned} \frac{d}{ds}(T \cdot N) &= \kappa(N \cdot N) - \kappa(T \cdot T) + \tau(T \cdot B), \\ \frac{d}{ds}(T \cdot N) &= \kappa(N \cdot B) - \tau(T \cdot N), \\ \frac{d}{ds}(N \cdot B) &= -\kappa(T \cdot B) + \kappa(B \cdot B) - \tau(N \cdot N), \\ \frac{d}{ds}(T \cdot T) &= 2\kappa(T \cdot T), \\ \frac{d}{ds}(N \cdot N) &= -2\kappa(T \cdot N) + 2\tau(N \cdot B), \\ \frac{d}{ds}(B \cdot B) &= -2\tau(N \cdot B). \end{aligned}$$

This is a system of six first order ODEs with initial values $0, 0, 0, 1, 1, 1$ which has a unique solution again. Since the constant functions $0, 0, 0, 1, 1, 1$ also solve this IVP, then by uniqueness,

$$T \cdot N = T \cdot B = N \cdot B \equiv 0, \qquad T \cdot T = N \cdot N = B \cdot B \equiv 1.$$

Therefore, T, N, B form an orthogonal basis. **TODO** ■

TODO

This section ends with Hopf's Umlaufatz, and its pictorial proof (homotopy theory). **TODO**

3 Surfaces

Definition. A subset of $\mathbb{R}^n$ is a regular C^k surface ($n \geq 2$, $1 \leq k \leq \infty$) if for all points $p \in S$, there are open subsets $U \subseteq \mathbb{R}^2$, $p \in V \subseteq \mathbb{R}^n$, and a C^k homemorphism $\varphi : U \to V \cap S \subseteq \mathbb{R}^n$ such that $D_q\varphi : \mathbb{R}^2 \to \mathbb{R}^n$ is one-to-one (has rank 2) for all $q \in U$.

Proposition 3.0.1 (Graphs are surfaces)**.** If $U \subseteq \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a C^k function, then the graph of f in $\mathbb{R}^3$ is a regular C^k surface.

Proof. Use $V = \mathbb{R}^3$, $S = \text{graph}(f) \subseteq \mathbb{R}^3$ and $\varphi : U \to \mathbb{R}^3 : (x, y) \mapsto (x, y, f(x, y))$. Note that φ is a C^k bijection from U to S with a continuous inverse $\pi : (x, y, z) \mapsto (x, y)$. The jacobian of ϕ is

$$J_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

which has rank 2 for any input it is given (the two columns are always linearly independant, for any the value for the partial derivatives). ∎

For example, the set defined by $z = x^2 + y^2$ is a regular C^∞ surface.

Proposition 3.0.2 (Level sets are surfaces)**.** Suppose $\tilde{V} \subseteq \mathbb{R}^3$ is open, $F : \tilde{V} \to \mathbb{R}$ is a C^k function, and $a \in \mathbb{R}$ is a regular value, then $F^{-1}(a)$ is a regular C^k surface.

Proof. Given $p \in F^{-1}(a)$, we have $\nabla F(p) \neq 0$. Without loss of generality, we can assume that $\frac{\partial F}{\partial z}(p) \neq 0$. Define $G : \tilde{V} \to \mathbb{R}^3$ by $G(x, y, z) = (x, y, F(x, y, z))$ and compute its Jacobian:

$$J_G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x}(p) & \frac{\partial F}{\partial y}(p) & \frac{\partial F}{\partial z}(p) \end{pmatrix}$$

Since $\det J_g(p) = \frac{\partial F}{\partial z}(p) \neq 0$, then we can apply the Inverse Function Theorem. From this, we get that there is a C^k homeomorphism $H : \hat{U} \to V \subseteq \tilde{V}$ which is inverse to G. Let U be the projection on $\mathbb{R}^2$ of the set $\hat{S} = \{z = a\}$ and define the homemorphism $\psi : U \to \hat{S} \cap \hat{U} : (x, y) \mapsto (x, y, a)$. To conclude, define the C^k homeomorphism $\varphi = H \circ \psi : U \to F^{-1}(a) \cap V$. Its Jacobian is $J_\varphi(q) = J_H(\psi(q))J_\psi(q)$

$$J_\varphi(q) = J_H(\psi(q))J_\psi(q) = J_G(H(\psi(q)))^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since the first matrix is invertoble and the second matrix as rank 2, then their multiplication has rank 2. ∎

Notice that the theorem for level sets is a generalization of the theorem for graphs. For example, $x^2 + y^2 + z^2 = 1$ is a regular C^∞ surface. Some other examples: hyperboloid with 2-sheets given by the equation $x^2 + y^2 - z^2 = -1$, 1-sheet $x^2 + y^2 - z^2 = 1$. These examples show that surfaces can be disconnected.

For the moment, everything done on surfaces in $\mathbb{R}^3$ applies to curves in $\mathbb{R}^2$ (curves with no boundary). In general ($1 \leq m \leq n$, $1 \leq k \leq \infty$), we define the general notion of manifold as follows:

Definition (Manifold)**.** A subset M of $\mathbb{R}^n$ is a regular C^k m-manifold if for all points $p \in M$, there are open subsets $U \subseteq \mathbb{R}^m$, $p \in V \subseteq \mathbb{R}^n$, and a C^k homemorphism $\varphi : U \to V \cap S \subseteq \mathbb{R}^n$ such that $D_q\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is has rank m for all $q \in U$.

A curve is a 1-manifold, a surface is a 2-manifold.

Theorem 3.0.3 (Surfaces = local graphs)**.** The subset $S \subseteq \mathbb{R}^3$ is a regular C^k surface if and only if for all points $p \in S$, there are open subsets $U \subseteq \mathbb{R}^2$, $V \subseteq \mathbb{R}^3$ with $p \in V$ such that $V \cap S$ is the graph of a C^k function $U \to \mathbb{R}$, i.e., it can be described by an equation of the form $z = f(x, y)$, $y = g(x, z)$, or $x = h(y, z)$.

Proof. The direct implication will be proved in the notes, the reverse implication follows from the fact that every graph is a surface. ■

For example, the unit sphere is not the graph of a function, but it does locally. Exercise: Show that $z^2 = x^2 + y^2$ and $z = \sqrt{x^2 + y^2}$ are NOT surfaces.

Proposition 3.0.4. Suppose $S \subseteq \mathbb{R}^3$ is a regular C^k surface, and $U_1, U_2 \subseteq \mathbb{R}^2$ are open subsets with parametrization $\varphi_i : U_i \to S$ which are C^k and satisfy the regularity conditions, then $g = \varphi_2^{-1} \circ \varphi_1$ and $h = \varphi_1^{-1} \circ \varphi_2$ are C^k homeomorphisms. Hence, we can say $\varphi_2 = \varphi_1 \circ h$ is a reparametrization of φ_1 using the change of parameters $h : U_2 \to U_1$.

Definition. We call $\varphi : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ a regular C^k *surface patch* if φ is C^k and $d\varphi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for all $q \in U$. This is analogous to path.

Remark that there is NO "arc length reparametrization" of surface patches ! We will look at Gauss' Theorema Egregium and Isothermal coordinates.

Definition. Fix $q \in U$. The *tangent space* $T_q\varphi$ to φ at q is the image of the derivative of φ at q.

Proposition 3.0.5. If $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^3$ is a regular C^k reparametrization of $\varphi : U \to \mathbb{R}^3$ using the change of parameters $h : \tilde{U} \to U$, then $T_{\tilde{q}}\tilde{\varphi} = T_q\varphi$ where $q = h(\tilde{q})$.

Proof. By definition, $T_{\tilde{q}}\tilde{\varphi} = d\tilde{\varphi}_{\tilde{q}}(\mathbb{R}^2)$. By the chain rule:

$$d\tilde{\varphi}_{\tilde{q}}(\mathbb{R}^2) = (d\varphi_{h(\tilde{q})} \circ dh_{\tilde{q}})(\mathbb{R}^2).$$

But since $h : \tilde{U} \subseteq \mathbb{R}^2 \to U \subseteq \mathbb{R}^2$ is invertible, then $dh_{\tilde{q}}(\mathbb{R}^2) = \mathbb{R}^2$. Therefore,

$$T_{\tilde{q}}\tilde{\varphi} = d\varphi_{h(\tilde{q})}(dh_{\tilde{q}}(\mathbb{R}^2)) = T_q\varphi.$$

■

Hence, we can now define the tangent space T_pS of a surface. As an exercise, we can show that the tangent space at a point p in a curve in $\mathcal{C} \subseteq S$ is a one dimensional subspace of the tangent space at p of the surface S.

Example: Let $\gamma : (0, 1) \to \mathbb{R}^3$ be a regular C^2 path with $\kappa > 0$ ($\iff$ $\{\dot{\gamma}, \ddot{\gamma}\}$ is linearly independent). Define $\varphi : \mathbb{R} \times (0, 1) \to \mathbb{R}^3$ by $\varphi(s, t) = \gamma(t) + s\dot{\gamma}$, then

$$J_\varphi(s, t) = \begin{pmatrix} | & | \\ \dot{\gamma}(t) & \dot{\gamma}(t) + s\ddot{\gamma}(t) \\ | & | \end{pmatrix}.$$

The columns are linearly independent if and only if $s \neq 0$. Therefore, φ is a surface patch on the open subset $\{(s,t) \ : \ s \neq 0, \ 0 < t < 1\}$.

Example: Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at 0. Let $p = \frac{1}{\sqrt{3}}(1,1,1) \in \mathbb{S}^2$. We can parametrize the top hemisphere of this surface by the graph of the function $z = \sqrt{1-x^2-y^2}$, hence, $\varphi_N : U \to \mathbb{S}^2$ is given by $\varphi(x,y) = (x,y,\sqrt{1-x^2-y^2})$ where U is the unit disk centered at the origin. Thus, we can write $p = \varphi(q)$ where $q = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Using the definition of the tangent space, we get

$$J_q\varphi_N = \mathrm{Im}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right).$$

Definition (Unit Normal)**.** We say that $n \in \mathbb{R}^3$ is a *unit normal* to S at $p \in S$ if $\|n\| = 1$ and $n \perp T_pS$. There are exactly two unit normal to S at p.

Definition (Orientation of a Surface)**.** An orientation on S is a continuous function $N : S \to \mathbb{S}^2$ such that $N(p)$ is a unit normal to S at p for all $p \in S$. We say that S is orientable if there exists an orientation on S.

Remark: If S is connected and orientable, then there are exactly two orientations.

Proposition 3.0.6. Level sets are orientable.

Proof. **TODO**Use the gradient. ∎

TODO

Non-example: There are non orientable surfaces. More precisely, define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by $\gamma(t) = (2\cos t, 2\sin t, 0)$, and define $\varphi : (-1,1) \times \mathbb{R} \to \mathbb{R}^3$ by $\varphi(s,t) = \gamma(t) + s\cos(t/2)N(t) + s\sin(t/2)B(t)$. Here, $N(t) = (-\cos t, -\sin t, 0)$ and $B(t) = (0,0,1)$. The Mobius band is defined as the image of φ. We still need to verify the regularity conditions by computing the Jacobian etc [**TODO**]. Let's now verify that it is not orientable by focusing on the central circle at $s = 0$:

$$J_\varphi(0,t) = \begin{pmatrix} -\cos(t)\cos(t/2) & -2\sin(t) \\ -\sin(t)\cos(t/2) & -2\cos(t) \\ \sin(t/2) & 0 \end{pmatrix}.$$

Since the first column gives us the tangent vector to the strip at t, and the second column gives us the tangent to the circle at t, then these two vectors span the tangent plane at t. Hence, if we take their cross product, we get a normal vector:

$$N(t) = \frac{1}{2} d\varphi_{(0,t)}(e_1) \times d\varphi_{(0,t)}(e_2) = \begin{pmatrix} -\cos(t)\sin(t/2) \\ -\sin(t)\sin(t/2) \\ -\cos(t/2) \end{pmatrix}.$$

But now, we see that at $t = 0$ and at $t = 2\pi$, we get two opposite vectors, hence the surface is not orientable using this potential orientation we just defined. But since this function of normal vector is continuous everywhere except at $t = 0$, then using the fact that there are only two possible orientability for a surface, we get that we actually proved that the surface is not orientable at all, not just for the function we just defined [**TODO**].

Definition (Differentiability of a function defined on a surface)**.** Let $S \subseteq \mathbb{R}^3$ be a regular C^k surface. The function $f : S \to \mathbb{R}^n$ is C^k if for all $p \in S$, there is a local parametrization $\varphi : U \to S$ near p such that $f \circ \varphi : U \to \mathbb{R}^n$ is $\mathbb{C}^k$. The expression $f \circ \varphi$ is sometimes called f in local coordinates.

Lemma 3.0.7. Let $S, \tilde{S} \subseteq \mathbb{R}^3$ be regular C^k surfaces. A function $f : \tilde{S} \to S$ is C^k if and only if for all $\tilde{p} \in \tilde{S}$, there are local parametrizations $\tilde{\varphi} : \tilde{U} \to \tilde{S}$ near $\tilde{p}$ and $\varphi : U \to S$ near $f(\tilde{p})$ such that $\varphi^{-1} \circ f \circ \tilde{\varphi} : \tilde{U} \to U$ is C^k.

Exercise: $f : \tilde{S} \to S$ a C^1 function, then there is a well-defined derivative $df_{\tilde{p}} : T_{\tilde{p}}\tilde{S} \to T_{f(\tilde{p})}S$. [**TODO**]

Exercise: Take the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Define the stereographic projection $\pi_N : \mathbb{R}^2 \to \mathbb{S}^2$ based at the north pole $N = (0, 0, 1)$ by etc. Fix a constant $c \in \mathbb{R}^2$ and let $M_c : \mathbb{R}^2 \to \mathbb{R}^2$ be the rigid motion that translates by c: $M_c(p) = p + c$. From this, define $f_c : \mathbb{S}^2 \to \mathbb{S}^2$ by

$$f_c(p) = \begin{cases} N & p = N, \\ \pi_N \circ M_c \circ \pi_n^{-1}(p) & p \neq N. \end{cases}$$

The exercise is to show that f_c is smooth (especially at N) [**TODO**].

Definition. Let $S \subseteq \mathbb{R}^3$ be a regular surface. The *First Fundamental Form* assigns to each $p \in S$ the quadratic form $\|\cdot\|^2 : T_pS \to \mathbb{R}$.

Definition. Suppose $f : \tilde{S} \to S$ is C^1, we say that it is a *local isometry* if for all vectors $\tilde{v} \in T_{\tilde{p}}\tilde{S}$, $\|\tilde{v}\|^2 = \|df_{\tilde{p}}(\tilde{v})\|^2$.

Definition. The function $f : \tilde{S} \to S$ is an *isometry* if it is a bijective local isometry.

Exercise: If $f : \tilde{S} \to S$ is a restriction of a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$, then f is an isometry [**TODO**].

In local coordinates, there is some $\varphi : U \to S$ which is a local parametrization near a point $p \in S$. For each $q \in U$, we get a a quadratic form $I_q : \mathbb{R}^2 \to \mathbb{R}$ that sends $u \in \mathbb{R}^2$ to $\|d\varphi_q(u)\|^2$. We can define an inner product on $\mathbb{R}^2$ for each $q \in U$:

$$\langle u_1, u_2 \rangle_q = \frac{1}{4}(I_q(u_1 + u_2) - I_q(u_1 - u_2)).$$

Exercise: Check that this is indeed an inner product [**TODO**].