Solutions to Linear Algebra Done Right (4th Ed) - Sheldon Axler

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Preface

The goal of this document is to share my personal solutions to the exercises in the Fourth Edition of Linear Algebra Done Right by Sheldon Axler during my reading. As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Vector Spaces

1A $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1
Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib \quad \text{and} \quad \beta = c + id$$

where $a, b, c, d \in \mathbf{R}$, then

$$\begin{aligned}\alpha + \beta &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \\ &= (c + a) + i(d + b) \\ &= (c + id) + (a + ib) \\ &= \beta + \alpha\end{aligned}$$

which proves that addition is commutative in $\mathbf{C}$ using the fact that it is commutative in $\mathbf{R}$.

Exercise 2
Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}(\alpha + \beta) + \lambda &= [(a + ib) + (c + id)] + (e + if) \\ &= [(a + c) + i(b + d)] + (e + if) \\ &= ([a + c] + e) + i([b + d] + f) \\ &= (a + [c + e]) + i(b + [d + f]) \\ &= (a + ib) + [(c + e) + i(d + f)] \\ &= (a + ib) + [(c + id) + (e + if)] \\ &= \alpha + (\beta + \lambda)\end{aligned}$$

which proves that addition is associative in $\mathbf{C}$ using the fact that it is associative in $\mathbf{R}$.

Exercise 3
Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
(\alpha\beta)\lambda &= [(a+ib)(c+id)](e+if) \\
&= [(ac-bd)+i(ad+bc)](e+if) \\
&= ([ac-bd]e-[ad+bc]f)+i([ac-bd]f+[ad+bc]e) \\
&= (ace-bde-adf-bcf)+i(acf-bdf+ade+bce) \\
&= (a[ce-fd]-b[cf+de))+i(a[cf+de]+b[ce-fd]) \\
&= (a+ib)[(ce-fd)+i(cf+de)] \\
&= (a+ib)[(c+id)(e+if)] \\
&= \alpha(\beta\lambda)
\end{aligned}$$

which proves that multiplication is associative in $\mathbf{C}$ using the fact that multiplication is associative and addition is commutative in $\mathbf{R}$.

Exercise 4
Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
\lambda(\alpha+\beta) &= (e+if)[(a+ib)+(c+id)] \\
&= (e+if)[(a+c)+i(b+d)] \\
&= [e(a+c)-f(b+d)]+i[e[b+d]+f[a+c]] \\
&= (ea+ec-fb-fd)+i(eb+ed+fa+fc) \\
&= [(ea-fb)+i(eb+fa)]+[(ec-fd)+i(ed+fc)] \\
&= [(e+if)(a+ib)]+[(e+if)(c+id)] \\
&= \lambda\alpha+\lambda\beta
\end{aligned}$$

which proves the distributivity in $\mathbf{C}$ using the distributivity in $\mathbf{R}$.

Exercise 5
Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution
Let $\alpha = a + ib$ and consider $\beta = (-a) + i(-b)$, then we get

$$\begin{aligned}\alpha + \beta &= (a + ib) + ([-a] + i[-b]) \\ &= (a + [-a]) + i(b + [-b]) \\ &= 0 + i0 \\ &= 0\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha+\beta_1 = 0$ and $\alpha + \beta_2 = 0$, this implies that $\alpha + \beta_1 = \alpha + \beta_2$. If we add β_1 on both sides, we get

$$\begin{aligned}\beta_1 + (\alpha + \beta_1) = \beta_1 + (\alpha + \beta_2) &\implies (\beta_1 + \alpha) + \beta_1 = (\beta_1 + \alpha) + \beta_2 \\ &\implies (\alpha + \beta_1) + \beta_1 = (\alpha + \beta_1) + \beta_2 \\ &\implies 0 + \beta_1 = 0 + \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 6
Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution
Let $\alpha = a + ib \neq 0$, then notice that we must have $a^2 + b^2 \neq 0$. Hence, consider

$$\beta = \left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)$$

Thus, we get

$$\begin{aligned}\alpha\beta &= (a + ib)\left[\left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)\right] \\ &= \left(a\left(\frac{a}{a^2 + b^2}\right) - b\left(-\frac{b}{a^2 + b^2}\right)\right) + i\left(a\left(-\frac{b}{a^2 + b^2}\right) + b\left(\frac{a}{a^2 + b^2}\right)\right) \\ &= \frac{a^2 + b^2}{a^2 + b^2} + i\frac{-ab + ba}{a^2 + b^2} \\ &= 1 + i0 \\ &= 1\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$, this implies that $\alpha\beta_1 = \alpha\beta_2$. If we multiply by β_1 on both sides, we get

$$\begin{aligned}\beta_1(\alpha\beta_1) = \beta_1(\alpha\beta_2) &\implies (\beta_1\alpha)\beta_1 = (\beta_1\alpha)\beta_2 \\ &\implies (\alpha\beta_1)\beta_1 = (\alpha\beta_1)\beta_2 \\ &\implies 1 \cdot \beta_1 = 1 \cdot \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 7
Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution
This is pretty straightforward:

$$\begin{aligned}
\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{(-1+\sqrt{3}i)^3}{2^3} \\
&= \frac{(-1)^3+3(-1)^2(\sqrt{3}i)+3(-1)^1(\sqrt{3}i)^2+(\sqrt{3}i)^3}{8} \\
&= \frac{-1+3\sqrt{3}i+3\cdot 3-3(\sqrt{3}i)}{8} \\
&= \frac{8}{8} \\
&= 1
\end{aligned}$$

Exercise 8
Find two distinct square roots of i.

Solution
Consider $\alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\beta = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Hence,

$$\begin{aligned}
\alpha^2 &= \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 \\
&= \left(\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\frac{\sqrt{2}}{2}\cdot i\frac{\sqrt{2}}{2} + \left(i\frac{\sqrt{2}}{2}\right)^2 \\
&= \frac{2}{4} + i - \frac{2}{4} \\
&= i
\end{aligned}$$

and

$$\begin{aligned}
\beta^2 &= \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^2 \\
&= \left(-\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\left(-\frac{\sqrt{2}}{2}\right)\cdot\left(-i\frac{\sqrt{2}}{2}\right) + \left(-i\frac{\sqrt{2}}{2}\right)^2 \\
&= \frac{2}{4} + i - \frac{2}{4} \\
&= i
\end{aligned}$$

Therefore, α and β are two distinct square roots of i.

Exercise 9
Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution
First, suppose that such an element x exists, then there exist $a, b, c, d \in \mathbf{R}$ such that $x = (a, b, c, d)$ and

$$(4 + 2a, -3 + 2b, 1 + 2c, 7 + 2d) = (5, 9, -6, 8)$$

But notice that this is equivalent to the following system of equations:

$$\begin{cases} 4 + 2a = 5 \\ -3 + 2b = 9 \\ 1 + 2c = -6 \\ 7 + 2d = 8 \end{cases}$$

which implies that

$$\begin{cases} a = \frac{1}{2} \\ b = 6 \\ c = \frac{7}{2} \\ d = \frac{1}{2} \end{cases}$$

Therefore, we get that $x = (\frac{1}{2}, 6, \frac{7}{2}, \frac{1}{2}) \in \mathbf{R}^4$ is indeed a solution to our original equation.

Exercise 10
Explain why there is does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution
By contradiction, suppose there exists a complex number $\lambda = a + ib$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

Then, we would get the following system of equation:

$$\begin{cases} \lambda(2 - 3i) = 12 - 5i \\ \lambda(5 + 4i) = 7 + 22i \\ \lambda(-6 + 7i) = -32 - 9i \end{cases}$$

which is equivalent to

$$\begin{cases} \lambda = 3 + 2i \\ \lambda = 3 + 2i \\ \lambda = \frac{129}{85} + i\frac{278}{85} \end{cases}$$

We clearly have a contradiction since $3 + 2i \neq \frac{129}{85} + i\frac{278}{85}$. Therefore, there doesn't exist such a complex number λ.

Exercise 11
Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution
First, write

$$x = (x_1, ..., x_n), \quad y = (y_1, ..., y_n) \quad \text{and} \quad z = (z_1, ..., z_n)$$

Since addition is commutative in $\mathbf{F}$, we get

$$\begin{aligned}
(x + y) + z &= [(x_1, ..., x_n) + (y_1, ..., y_n)] + (z_1, ..., z_n) \\
&= (x_1 + y_1, ..., x_n + y_n) + (z_1, ..., z_n) \\
&= ([x_1 + y_1] + z_1, ..., [x_n + y_n] + z_n) \\
&= (x_1 + [y_1 + z_1], ..., x_n + [y_n + z_n]) \\
&= (x_1, ..., x_n) + (y_1 + z_1, ..., y_n + z_n) \\
&= (x_1, ..., x_n) + [(y_1, ..., y_n) + (z_1, ..., z_n)] \\
&= x + (y + z)
\end{aligned}$$

which proves that addition is associative in $\mathbf{F}^n$.

Exercise 12
Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution
First, write $x = (x_1, ..., x_n)$. Using associativity of multiplication in $\mathbf{F}$, we get

$$\begin{aligned}
(ab)x &= (ab)(x_1, ..., x_n) \\
&= ((ab)x_1, ..., (ab)x_n) \\
&= (a(bx_1), ..., a(bx_n)) \\
&= a(bx_1, ..., bx_n) \\
&= a[b(x_1, ..., x_n)] \\
&= a(bx)
\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Exercise 13
Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution
Let $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Hence,

$$\begin{aligned}
1x &= 1(x_1, ..., x_n) \\
&= (1 \cdot x_1, ..., 1 \cdot x_n) \\
&= (x_1, ..., x_n) \\
&= x
\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$.

Exercise 14
Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$.

Solution
Let $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$ with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
\lambda(x+y) &= \lambda[(x_1, ..., x_n) + (y_1, ..., y_n)] \\
&= \lambda(x_1 + y_1, ..., x_n + y_n) \\
&= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) \\
&= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) \\
&= \lambda x + \lambda y
\end{aligned}$$

which proves the desired formula.

Exercise 15
Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution
Let $a, b \in \mathbf{F}$ and $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
(a+b)x &= (a+b)(x_1, ..., x_n) & &= ((a+b)x_1, ..., (a+b)x_n) \\
&= (ax_1 + bx_1, ..., ax_n + bx_n) \\
&= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) \\
&= a(x_1, ..., x_n) + b(x_1, ..., x_n) \\
&= ax + bx
\end{aligned}$$

which proves the desired formula.

1B Definition of Vector Space

Exercise 1
Prove that $-(-v) = v$ for every $v \in V$.

Solution
Let $v \in V$, by definition, we know that by definition, $-v$ is defined as the only vector in V satisfying

$$v + (-v) = 0$$

which is equivalent to

$$(-v) + v = 0$$

by commutativity of addition in V. However, notice that by definition, $-(-v)$ is the unique vector satisfying

$$(-v) + [-(-v)] = 0$$

But since v itself also satisfies this equation, we get $-(-v) = v$ by uniqueness.

Exercise 2
Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution
Suppose that $a \neq 0$, then by properties of $\mathbf{F}$, the inverse a^{-1} exists. Hence, if we multiply by a^{-1} on both sides, we get

$$\begin{aligned} av = 0 &\implies a^{-1}(av) = a^{-1}0 \\ &\implies (a^{-1}a)v = 0 \\ &\implies 1v = 0 \\ &\implies v = 0 \end{aligned}$$

Therefore, we either have $a = 0$ or $v = 0$.

Exercise 3
Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution
By properties of vector spaces, since $v \in V$, then $-v \in V$. Similarly, since w and $-v$ are in V, then $w + (-v) \in V$. Finally, since $w + (-v) \in V$, then $3^{-1}(w + (-v)) \in V$. Thus, define x_0 as the vector $3^{-1}(w + (-v))$ in V. Notice that

$$\begin{aligned} v + 3x_0 &= v + 3[3^{-1}(w + (-v))] \\ &= v + (3 \cdot 3^{-1})(w + (-v)) \\ &= v + 1(w + (-v)) \\ &= v + (w + (-v)) \\ &= v + ((-v) + w) \\ &= (v + (-v)) + w \\ &= 0 + w \\ &= w \end{aligned}$$

which shows that the equation has at least one solution. To prove uniqueness, let $x_1 \in V$ be an arbitrary solution to the equation, then we get

$$\begin{aligned} v + 3x_1 = w &\implies (-v) + (v + 3x_1) = (-v) + w \\ &\implies ((-v) + v) + 3x_1 = w + (-v) \\ &\implies 0 + 3x_1 = w + (-v) \\ &\implies 3x_1 = w + (-v) \\ &\implies 3^{-1}(3x_1) = 3^{-1}(w + (-v)) \\ &\implies (3^{-1}3)x_1 = x_0 \\ &\implies 1x_1 = x_0 \\ &\implies x_1 = x_0 \end{aligned}$$

which proves that x_0 is the unique solution to the equation.

Exercise 4
The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space. Which one?

Solution
The empty set doesn't satisfy the axiom that states that there must be an additive identity since the empty set is empty by definition.

Exercise 5
Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here, the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution
We already know that the axioms of a vector space imply that $0v = 0$ for all $v \in V$. Hence, it suffices to prove that if we assume the axioms of a vector space without the additive inverse condition, then we can prove the additive inverse condition if we also assume the property that $0v = 0$ for all $v \in V$. Let $v \in V$, the by the distributive condition, we get

$$\begin{aligned} 0v = 0 &\implies (1 + (-1))v = 0 \\ &\implies 1v + (-1)v = 0 \\ &\implies v + (-1)v = 0 \end{aligned}$$

which proves that v has an additive inverse for all $v \in V$.

Exercise 6
Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the

notation. Specifically, the sum and product of two reals numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty = \infty \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty \\ \infty + (-\infty) = (-\infty) + \infty = 0 \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution
With these operations of addition and scalar multiplication, $\mathbf{R} \cup \{\infty, -\infty\}$ cannot be a vector space since

$$((-\infty) + \infty) + \infty = 0 + \infty = \infty$$

and

$$(-\infty) + (\infty + \infty) = (-\infty) + \infty = 0$$

which proves that addition isn't associative under this operation.

Exercise 7
Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution
For any f and g in V^S, define $f + g : S \to V$ by $s \mapsto f(s) + g(s)$ for all $s \in S$. Similarly, for all $\alpha \in \mathbf{F}$ and $f \in V^S$, define $\alpha f : S \to V$ by $s \mapsto \lambda f(s)$ for all $s \in S$. With these definitions, let's prove that V^S is a vector space.

- (**commutativity**) Let $f, g \in V^S$, let's show that $f + g = g + f$. Let $s \in S$, then by commutativity in V, we obviously have

  $$(f + g)(s) = f(s) + g(s) = g(s) + f(s) = (g + f)(s)$$

  Since it holds for all s, then $f + g = g + f$.

- (**associativity**) Let $f, g, h \in V^S$ and $s \in S$, then by associativity in V, we have

  $$\begin{aligned} [(f + g) + h](s) &= (f + g)(s) + h(s) \\ &= [f(s) + g(s)] + h(s) \\ &= f(s) + [g(s) + h(s)] \\ &= f(s) + (g + h)(s) \\ &= [f + (g + h)](s) \end{aligned}$$

Since it holds for all $s \in S$, then $(f+g)+h = f+(g+h)$.
Let now $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, then by associativity in V, we get:

$$\begin{aligned}[(ab)f](s) &= (ab)f(s) \\ &= a(bf(s)) \\ &= a(bf)(s) \\ &= [a(bf)](s)\end{aligned}$$

Since it holds for all $s \in S$, then $(ab)f = a(bf)$.

- (**additive identity**) Let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$ and $s \in S$, we have

$$(f + 0_{V^S})(s) = f(s) + 0_{V^S}(s) = f(s) + 0 = f(s)$$

Since it holds for all $s \in S$, then $f + 0_{V^S} = f$ for all $f \in V^S$.

- (**additive inverse**) Again, let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$, we can define the function $g = (-1)f \in V^S$. Hence, for all $s \in S$, we get

$$\begin{aligned}(f+g)(s) &= f(s) + g(s) \\ &= f(s) + (-1)f(s) \\ &= f(s) + (-f(s)) \\ &= 0 \\ &= 0_{V^S}(s)\end{aligned}$$

Since it holds for all $s \in S$, then $f + g = 0_{V^S}$.

- (**multiplicative identity**) Let $f \in V^S$, then for all $s \in S$, we have

$$(1f)(s) = 1f(s) = f(s)$$

Since it holds for all $s \in S$, then $1f = f$.

- (**distributive property**) Let $f, g \in V^S$, $a \in \mathbf{F}$ and $s \in S$, then

$$\begin{aligned}[a(f+g)](s) &= a(f+g)(s) \\ &= a(f(s) + g(s)) \\ &= af(s) + ag(s) \\ &= (af)(s) + (ag)(s) \\ &= (af + ag)(s)\end{aligned}$$

Since it holds for all $s \in S$, then $a(f+g) = af + ag$. Similarly, for all $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, we have

$$\begin{aligned}[(a+b)f](s) &= (a+b)f(s) \\ &= af(s) + bf(s) \\ &= (af)(s) + (bf)(s) \\ &= (af + bf)(s)\end{aligned}$$

Since it holds for all $s \in S$, then $(a+b)f = af + bf$.

Therefore, V^S is a vector space under these definitions.

Exercise 8
Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
$$(a + ib)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with these definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution

- (**commutativity**) Let $u_1, v_1, u_2, v_2 \in V$, then by commutativity in V, we have
$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) &= (u_1 + u_2) + i(v_1 + v_2)\\ &= (u_2 + u_1) + i(v_2 + v_1)\\ &= (u_2 + iv_2) + (u_1 + iv_1)\end{aligned}$$
which proves that addition is commutative.
- (**associativity**) Let $u_1, v_1, u_2, v_2, u_3, v_3 \in V$, then by associativity in V, we have
$$\begin{aligned}[(u_1 + iv_1) + (u_2 + iv_2)] + (u_3 + iv_3) &= [(u_1 + u_2) + i(v_1 + v_2)] + (u_3 + iv_3)\\ &= ([u_1 + u_2] + u_3) + i([v_1 + v_2] + v_3)\\ &= (u_1 + [u_2 + u_3]) + i(v_1 + [v_2 + v_3])\\ &= (u_1 + iv_1) + [(u_2 + u_3) + i(v_2 + v_3)]\\ &= (u_1 + iv_1) + [(u_2 + iv_2) + (u_3 + iv_3)]\end{aligned}$$
Let now $a, b, c, d \in \mathbf{R}$ and $u, v \in V$, then we get:
$$\begin{aligned}&[(a + bi)(c + di)](u + iv)\\ &\qquad = [(ac - bd) + i(ad + bc)](u + iv)\\ &\qquad = [(ac - bd)u - (ad + bc)v] + i[(ac - bd)v + (ad + bc)u]\\ &\qquad = [acu - bdu - adv - bcv] + i[acv - bdv + adu + bcu]\\ &\qquad = [a(cu - dv) - b(cv + du)] + i[a(cv + du) + b(cu - dv)]\\ &\qquad = (a + ib)[(cu - dv) + i(cv + du)]\\ &\qquad = (a + ib)[(c + id)(u + iv)]\end{aligned}$$
which proves the associativity condition.

- (**additive identity**) For all $u, v \in V$,

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv$$

  which proves that $0 + i0$ is an additive identity.

- (**additive inverse**) Let $u, v \in V$, then since $(-u), (-v) \in V$, we get

$$(u + iv) + ([-u] + i[-v]) = (u + [-u]) + i(v + [-v]) = 0 + i0$$

  which proves that every element has an additive inverse.

- (**multiplicative identity**) Let $u, v \in V$, then

$$(1 + i0)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv$$

  which proves that $1 = 1 + i0$ is a multiplicative identity.

- (**distributive property**) Let $a, b \in \mathbf{R}$ and $u_1, v_1, u_2, v_2 \in V$, then

$$\begin{aligned}
&(a + ib)[(u_1 + iv_1) + (u_2 + iv_2)] \\
&\qquad = (a + ib)([u_1 + u_2] + i[v_1 + v_2]) \\
&\qquad = (a[u_1 + u_2] - b[v_1 + v_2]) + i(a[v_1 + v_2] + b[u_1 + u_2]) \\
&\qquad = (au_1 + au_2 - bv_1 - bv_2) + i(av_1 + av_2 + v_1 + bv_2) \\
&\qquad = ([au_1 - bv_1] + [au_2 - bv_2]) + i([av_1 + bu_1] + [av_2 + bu_1]) \\
&\qquad = [(au_1 - bv_1) + i(av_1 + bu_1)] + [(au_2 - bv_2) + i(av_2 + bu_2)] \\
&\qquad = [(a + ib)(u_1 + iv_1)] + [(a + ib)(u_2 + iv_2)]
\end{aligned}$$

  Similarly, for all $a, b, c, d \in \mathbf{R}$, and $u, v \in \mathbf{R}$, we have

$$\begin{aligned}
&[(a + ib) + (c+id)](u + iv) \\
&\qquad = ([a + c] + i[b + d])(u + iv) \\
&\qquad = ([a + c]u - [b + d]v) + i([a + c]v + [b + d]u) \\
&\qquad = (au + cu - bv - dv) + i(av + cv + bu + du) \\
&\qquad = ([au - bv] + [cu - dv]) + i([av + bu] + [cv + du]) \\
&\qquad = [(au - bv) + i(av + bu)] + [(cu - dv) + i(cv + du)] \\
&\qquad = (a + ib)(u + iv) + (c + id)(u + iv)
\end{aligned}$$

  which proves the distributive property.

Therefore, $V_{\mathbf{C}}$ is a vector space under these definitions.

1C Subspaces

Exercise 1
For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution

(a) First, define
$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$$
Let's prove that it is indeed a subspace of $\mathbf{F}^3$. Since $0 + 2 \cdot 0 + 3 \cdot 0 = 0$, then $0 = (0, 0, 0) \in U$. Now, let $x, y \in U$ be two arbitrary elements where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then by definition:
$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 0 \\ y_1 + 2y_2 + 3y_3 & = 0 \end{cases}$$
Adding the two equations gives us
$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0 + 0 = 0$$
which proves that $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U$. Similarly, let $x = (x_1, x_2, x_3)$ be an arbitrary element in U and α an arbitrary scalar in $\mathbf{F}$, then by definition of U:
$$x_1 + 2x_2 + 3x_3 = 0$$
Multiplying by α on both sides gives us
$$(\alpha x_1) + 2(\alpha x_2) + 3(\alpha x_3) = \alpha \cdot 0 = 0$$
which proves that $\alpha x \in U$. Therefore, U is a subspace of $\mathbf{F}^3$.

(b) Since $0 = (0, 0, 0)$ doesn't satisfy $x_1 + 2x_2 + 3x_3 = 4$, then the set of such vectors cannot be a subspace since it doesn't contain the zero vector.

(c) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$ and notice that that both $x = (1, 1, 0)$ and $y = (0, 0, 1)$ are in U. However, $x + y$ is obviously not in U since $x + y = (1, 1, 1)$ and $1 \cdot 1 \cdot 1 = 1$. Therefore, U is not a subspace of $\mathbf{F}^3$.

(d) Define $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$ and let's show that it is a subspace of $\mathbf{F}^3$. First, since $0 = 5 \cdot 0$, then $0 = (0, 0, 0) \in U$. To prove that U is closed under addition, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two arbitrary elements of U, then by definition:
$$\begin{cases} x_1 = 5x_3 \\ y_1 = 5y_3 \end{cases}$$

By adding the two equations together, we get

$$x_1 + y_1 = 5(x_3 + y_3)$$

Thus, $x+y = (x_1+y_1, x_2+y_2, x_3+y_3) \in U$. Finally, to prove that U is closed under scalar multiplication, let $x = (x_1, x_2, x_3)$ be an element of U and $\alpha \in \mathbf{F}$, then

$$\begin{aligned} x_1 = 5x_3 &\implies \alpha x_1 = \alpha(5x_3) \\ &\implies \alpha x_1 = 5(\alpha x_3) \end{aligned}$$

Thus, $\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \in U$. Therefore, U is a subspace of $\mathbf{F}^3$.

Exercise 2

Verify all assertions about subspaces in Example 1.35:

(a) If $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval [0,1] is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The set of differentiable real-valued functions f on the interval (0,3) such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

Solution

(a) Define $U_b = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$ for all $b \in \mathbf{F}$ and suppose first that U is a subspace of $\mathbf{F}^4$, then it must contain the zero vector. Hence, since $(0, 0, 0, 0) \in U$, then by definition:

$$0 = 5 \cdot 0 + b$$

which is equivalent to $b = 0$.
For the converse, let's show that U_0 is a subspace of $\mathbf{F}^4$. Since $0 = 5 \cdot 0$, then $0 = (0, 0, 0, 0) \in U_0$. If $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are arbitrary elements of U_0, then $x_3 = 5x_4$ and $y_3 = 5y_4$. By adding these two equations and by distributivity, we get

$$x_3 + y_3 = 5(x_4 + y_4)$$

which implies that $x+y \in U_0$. Similarly, if $x = (x_1, x_2, x_3, x_4) \in U_0$ and $\alpha \in \mathbf{F}$, then we get

$$x_3 = 5x_4 \implies \alpha x_3 = 5(\alpha x_4)$$

which implies that $\alpha x \in U_0$. Thus, U_0 is a subspace of $\mathbf{F}^4$. Therefore, U_b is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) Let C denote the set of real-valued continuous functions on the interval [0,1] and $0_{\mathbf{R}^{[0,1]}}$ the zero function which acts as the additive identity in $\mathbf{R}^{[0,1]}$. Since the constant zero function is continuous, then $0_{\mathbf{R}^{[0,1]}} \in C$. Similarly, since the sum of two continuous functions is continuous and the multiplication of a continuous function with a scalar is still continuous, then C is closed under addition and scalar multiplication. Therefore, C is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The proof is similar to part (b). The constant zero function is differentiable on $\mathbf{R}$. Moreover, differentiable functions are closed under addition and scalar multiplication. Therefore, the set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) Define $U_b = \{f : (0,3) \to \mathbf{R} \text{ differentiable } : f'(2) = b\}$ for all $b \in \mathbf{R}$. Suppose that U_b is a subspace of $\mathbf{R}^{(0,3)}$, then we must have $0_{(0,3)} \in U_b$ where $0_{(0,3)}$ denotes the constant zero function on (0,3). By definition of U_b, it implies that $0'_{(0,3)}(2) = b$. However, we know that $0'_{(0,3)}(2) = 0$. Thus, $b = 0$.

Conversely, let's show that U_0 is a subspace of $\mathbf{R}^{(0,3)}$. First, the constant zero function $0_{\mathbf{R}^{(0,3)}}$ on (0,3) which acts as the additive identity in $\mathbf{R}^{(0,3)}$, is differentiable on (0,3) and its derivative at 2 is 0. Hence, $0_{\mathbf{R}^{(0,3)}} \in U_0$. Now, let $f, g \in U_0$, then $f + g$ is differentiable on (0,3) and

$$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0$$

so $f + g \in U_0$. Similarly, for any $f \in U_0$ and $\alpha \in \mathbf{F}$, the function αf is still differentiable on (0,3) and

$$(\alpha f)'(2) = \alpha f'(2) = \alpha \cdot 0 = 0$$

so $\alpha f \in U_0$. Thus, U_0 is a subspace of $\mathbf{R}^{(0,3)}$. Therefore, U_b is a subspace if and only if $b = 0$.

(e) Let S be the set of sequences of complex numbers with limit 0. Since the additive identity $(0, 0, ...)$ of C^∞ converges to 0, then it is in S. Let $(a_n)_n, (b_n)_n \in S$, then

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0$$

which implies

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0 + 0 = 0$$

Thus, $(a_n)_n + (b_n)_n \in S$. Similarly, for all $(a_n)_n \in S$ and $\alpha \in \mathbf{C}$, we have

$$\lim_{n\to\infty} \alpha a_n = \alpha \lim_{n\to\infty} a_n = \alpha \cdot 0 = 0$$

so $\alpha(a_n)_n \in S$. Therefore, S is a subspace of $\mathbf{C}^\infty$.

Exercise 3

Show that the set of differentiable real-valued functions f on the interval (-4, 4) such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Solution
Define the set $U = \{f : (-4,4) \to \mathbf{R} \text{ differentiable } : f'(-1) = 3f(2)\}$ and let's show that it is a subspace of $\mathbf{R}^{(-4,4)}$. First, denote by f_0 to constant zero function on $(-4,4)$ which is also the additive identity in $\mathbf{R}^{(-4,4)}$. We know that f_0 is differentiable on $(-4,4)$ with $f_0' = f_0$. Hence, $f_0'(-1) = 0 = 3f_0(2)$ which proves that $f_0 \in U$.
To show that it is closed under addition, let $f, g \in U$, then by definition, f and g are differentiable on (-4, 4) and

$$\begin{cases} f'(-1) = 3f(2) \\ g'(-1) = 3g(2) \end{cases}$$

If we add these two equations, we get

$$(f+g)'(-1) = 3(f+g)(2)$$

which proves that $f+g \in U$ since $f+g$ is differentiable on (-4, 4).
To prove that it is closed under scalar multiplication, let $f \in U$ and $\alpha \in \mathbf{R}$, then

$$\begin{aligned} f'(-1) = 3f(2) &\implies \alpha f'(-1) = \alpha \cdot 3f(2) \\ &\implies (\alpha f)'(-1) = 3(\alpha f)(2) \end{aligned}$$

which proves that $\alpha f \in U$ since αf is differentiable on (-4, 4). Therefore, U is a subspace of $\mathbf{R}^{(-4,4)}$.

Exercise 4
Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Solution
Let $b \in \mathbf{R}$ and define $I = \{f : [0,1] \to \mathbf{R} \text{ continuous } : \int_0^1 f = b\}$. Suppose that I is a subspace of $\mathbf{R}^{[0,1]}$, then the additive identity $0 : x \mapsto 0$ must be in I so $\int_0^1 0 = b$. But we know that $\int_0^1 0 = 0$ so it follows that $b = 0$.
Conversly, let's show that I is a subspace of $\mathbf{R}^{[0,1]}$ when $b = 0$. First, the additive identity 0 is obviously continuous with $\int_0^1 0 = 0$ so $0 \in I$. Now, let $f, g \in I$, then f and g are continuous and

$$\int_0^1 f = \int_0^1 g = 0$$

It follows that $f+g$ is a continuous function that satisfies

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0$$

Hence, $f+g \in I$. Similarly, if $f \in I$ and $\alpha \in \mathbf{R}$, then f is continuous and

$$\int_0^1 f = 0$$

which implies that αf is also continuous and

$$\int_0^1 (\alpha f) = \alpha \int_0^1 f = \alpha \cdot 0 = 0$$

Hence, $\alpha f \in I$. Therefore, I is a subspace of $\mathbf{R}^{[0,1]}$.

Exercise 5
Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution
No, it isn't because it is not closed under scalar multiplication since the scalars are complex numbers. For example, $(1,1) \in \mathbf{R}^2$ but $i(1,1) = (i,i) \notin \mathbf{R}^2$. Therefore, $\mathbf{R}^2$ is not a subspace of the complex vector space $\mathbf{C}^2$.

Exercise 6

(a) Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = c^3\}$ a subspace of $\mathbf{C}^3$?

Solution

(a) In $\mathbf{R}$, the function $x \mapsto x^3$ is bijective so if we define $I = \{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$, then we actually have $I = \{(a,b,c) \in \mathbf{R}^3 : a = c\}$. Hence, it is easier now to show that I is a subspace of $\mathbf{R}^3$. Obviously, $(0,0,0) \in I$ since $0 = 0$. Moreover, if (x_1,x_2,x_3) and (y_1,y_2,y_3) are in I, then $x_1 = x_3$ and $y_1 = y_3$ which implies that $x_1 + y_1 = x_3 + y_3$. Hence, $(x_1+y_1, x_2+y_2, x_3+y_3)$ in in I. Similarly, for $(x_1,x_2,x_3) \in I$ and $\alpha \in \mathbf{R}$, we must have $x_1 = x_3$ which implies that $\alpha x_1 = \alpha x_3$. Thus, $(\alpha x_1, \alpha x_2, \alpha x_3) \in I$. Therefore, I is a subspace of $\mathbf{R}^3$.

(b) If we let $I = \{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$, notice that $(\frac{-1+\sqrt{3}i}{2}, 0, 1)$ and $(\frac{-1-\sqrt{3}i}{2}, 0, 1)$ are both elements of I. However, their sum is not in I since

$$\left(\frac{-1+\sqrt{3}i}{2}, 0, 1\right) + \left(\frac{-1-\sqrt{3}i}{2}, 0, 1\right) = (-1, 0, 2) \notin I$$

Therefore, it is not a subspace of $\mathbf{C}^3$ since it is not closed under addition.

Exercise 7
Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

Solution
Consider the set $U = \{(k,k) : k \in \mathbf{Z}\}$ which is obviously closed under addition and taking inverses. Notice that U is not a subspace because it is not closed under scalar multiplication: $(1,1) \in U$ and $\pi \in \mathbf{R}$ but $\pi(1,1) = (\pi,\pi) \notin U$.

Exercise 8
Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar

multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution

Consider the set $U = \{(x, y) \in \mathbf{R}^2 : xy \geq 0\}$, let's first show that it is closed under scalar multiplication. Given $(x, y) \in U$ and $\alpha \in \mathbf{R}$, we know by definition of U tht $xy \geq 0$. Moreover, since α is a real number, then $\alpha^2 \geq 0$. Hence,

$$(\alpha x)(\alpha y) = \alpha^2 xy \geq 0$$

Thus, $(\alpha x, \alpha y) \in U$ so U is indeed closed under scalar multiplication. To show that U is not a subspace, consider the elements $(-1, 0)$ and $(0, 1)$ in U and notice that their addition cannot be in U since $(-1) \cdot 1 \ngeq 0$. Thus, U is not closed under addition which proves that it is not a subspace.

Exercise 9

A function $f : \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x + p) = f(x)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution

Let's prove that this set is not a subspace of $\mathbf{R}^{\mathbf{R}}$ by showing that it is not closed under addition. To do so, consider the functions $x \mapsto \cos(x)$ and $x \mapsto \cos(\pi x)$ defined on $\mathbf{R}$. Obviously, both are periodic since the first one has period 2π and the second one has period 2. Consider their sum $f : \cos(x) + \cos(\pi x)$ and suppose by contradiction that there exists a $p > 0$ such that

$$f(x) = f(x + p) \tag{1}$$

for all $x \in \mathbf{R}$. Notice that

$$\begin{aligned} f(x) = 2 &\implies \cos(x) + \cos(\pi x) = 2 \\ &\implies \cos(x) = 1 \quad \text{and} \quad \cos(\pi x) = 1 \\ &\implies x \in 2\pi\mathbf{Z} \quad \text{and} \quad x \in 2\mathbf{Z} \\ &\implies x = 0 \end{aligned}$$

Hence, f is equal to 2 if and only if $x = 0$. Thus, if we plug-in $x = 0$ in equation (1), we get

$$f(p) = f(0) = 2$$

which implies that $p = 0$, a contradiction since $p > 0$. Therefore, f is not periodic which proves that periodic functions are not closed under addition. With a similar argument, periodic functions are not closed under multiplication either.

Exercise 10

Suppose V_1 and V_2 are subspaces of V. Prove that $V_1 \cap V_2$ is a subspace of V.

Solution

Let's show that $V_1 \cap V_2$ satisfies the three subspace conditions:

- (**additive identity**) Since V_1 and V_2 are subspaces, then they both contain the additive identity 0 of V. It follows that $0 \in V_1 \cap V_2$ since it is contained in both sets.

- (**closed under addition**) Let u and v be two vectors in $V_1 \cap V_2$, then u and v must be contained in V_1. Since V_1 is a subspace, then it is closed under addition so $u+v$ must also be an element of V_1. Similarly, u and v are contained in V_2 so for the same reasons, $u+v$ must be an element of V_2. Thus, $u+v \in V_1 \cap V_2$ since $u+v \in V_1$ and $u+v \in V_2$.

- (**closed under scalar multiplication**) Let $a \in \mathbf{F}$ and $u \in V_1 \cap V_2$, then u must be contained in V_1. Since V_1 is a subspace, then it is closed under scalar multiplication so au must also be an element of V_1. Similarly, u is contained in V_2 so for the same reasons, au must be an element of V_2. Thus, $au \in V_1 \cap V_2$ since $au \in V_1$ and $au \in V_2$.

Therefore, $V_1 \cap V_2$ is a subspace of V.

Exercise 11
Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution
Let $\{V_i\}_{i\in I}$ be an arbitrary collection of subspaces of V, let's show that $\cap_{i\in I} V_i$ is also a subspace of V by proving the three subspace conditions:

- (**additive identity**) Since V_i is a subspace of V, then $0 \in V_i$ for all $i \in I$. It follows that $0 \in \cap_{i\in I} V_i$.

- (**closed under addition**) Let u and v be two vectors in $\cap_{i\in I} V_i$, then u and v must be contained in V_i for all $i \in I$. For any $i \in I$, V_i is a subspace so it is closed under addition, hence $u+v \in V_i$. It follows that $u+v \in \cap_{i\in I} V_i$.

- (**closed under scalar multiplication**) Let $a \in \mathbf{F}$ and $v \in \cap_{i\in I}$. For all $i \in I$, since $u \in V_i$ and V_i is a subspace, then $au \in V_i$. It follows that $au \in \cap_{i\in I} V_i$ since $au \in V_i$ for all $i \in I$.

Therefore, $\cap_{i\in I} V_i$ is a subspace of V.

Exercise 12
Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution
Let V_1 and V_2 be subspaces of V. If $V_1 \subset V_2$ or $V_2 \subset V_1$, then $V_1 \cup V_2$ must be a subspace of V as well. To show the converse, suppose now that $V_1 \cup V_2$ is a subspace of V and that $V_1 \not\subset V_2$. Then there exists a vector $u_1 \in V_1$ such that $u_1 \notin V_2$. Let's prove that $V_2 \subset V_1$ in that case. Let $v \in V_2$ be arbitrary, since u_1 and v are both vectors in $V_1 \cup V_2$, then $u_1 + v \in V_1 \cup V_2$ since it is a subspace. But this implies that $u_1 + v$ is either in V_1 or in V_2. If $u_1 + v \in V_2$, then we must have

$$u_1 = (u_1 + v) - v \in V_2$$

since $v \in V_2$ and V_2 is a subspace. A contradiction since $u_1 \notin V_2$. It follows that $u_1 + v \in V_1$. But again, since V_1 is a subspace and $u_1 \in V_1$, then

$$v = (u_1 + v) - u_1 \in V_1$$

which proves that $V_2 \subset V_1$. Therefore, if $V_1 \cup V_2$ is a subspace, then we either have $V_1 \subset V_2$ or $V_2 \subset V_1$.

Exercise 13
Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution
Let V_1, V_2 and V_3 be three subspaces of V. Obviously, if one contains the other two, then $V_1 \cup V_2 \cup V_3$ is also a subspace of V. To show the converse, suppose that $V_1 \cup V_2 \cup V_3$ is a subspace of V. **TODO**

Exercise 14
Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Solution
By definition, we have

$$\begin{aligned} U + W &= \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} + \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \\ &= \{(x + y, -x + y, 2x + 2y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \\ &= \{(x + y, -x + y, 2(x + y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \end{aligned}$$

From this expression, let's prove that

$$U + W = \{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$$

Obviously, $U + W \subset \{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$ because for any vector $(x + y, -x + y, 2(x + y) \in \mathbf{F}^3$, if we let $a = x + y$ and $b = -x + y$, we can rewrite this vector as $(a, b, 2a)$ which is in $\{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$. Similarly, given an arbitrary vector $(a, b, 2a) \in \mathbf{F}^3$, if we let

$$x = \frac{a - b}{2} \qquad \text{and} \qquad y = \frac{x + y}{2}$$

then we can rewrite the vector as $(x + y, -x + y, 2(x + y))$ which is obviously in $U + W$. It follows that the sets are equal. Without symbols, this just means that $U + W$ is precisely the set of vectors in V such that the third component is twice the first component.

Exercise 15
Suppose U is a subspace of V. What is $U + U$?

Solution
Let's show that $U = U + U$. An arbitrary element in $U + U$ is of the form $x + y$ where x and y are in U. Since U is a subspace, then it is closed under addition

which implies that $x + y \in U$. It follows that $U + U \subset U$.
For the reverse inclusion, take an arbitrary $u \in U$ and notice that we can write $u = u + 0$. Again, since U is a subspace of V, then $0 \in V$. Thus, in the expression $u+0$, both vectors are in U. It follows that $u = u+0 \in U+U$. Therefore, $U = U+U$.

Exercise 16
Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution
Let U and W be subspaces of V. Then by commutativity of addition in V, we get

$$\begin{aligned} U + W &= \{u + w : u \in U \text{ and } w \in W\} \\ &= \{w + u : w \in W \text{ and } u \in U\} \\ &= W + U \end{aligned}$$

Therefore, the operation of addition on subspaces of V is commutative.

Exercise 17
Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2 and V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Solution
Let V_1, V_2 and V_3 are subspaces of V and let's show that

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$$

First, take an arbitrary $x + y \in (V_1 + V_2) + V_3$ where $x \in V_1 + V_2$ and $y \in V_3$. Since $x \in V_1 + V_2$, then there exist vectors $a \in V_1$ and $b \in V_2$ such that $x = a + b$. It follows from the associativity of addition in V that

$$x + y = (a + b) + y = a + (b + y)$$

Since $b \in V_2$ and $y \in V_3$, then $b+y \in V_2+V_3$. Hence, $a+(b+y) \in V_1+(V_2+V_3)$ using the fact that $a \in V_1$. Thus, the arbitrary $x + y \in (V_1 + V_2) + V_3$ is in $V_1 + (V_2 + V_3)$ as well so

$$(V_1 + V_2) + V_3 \subset V_1 + (V_2 + V_3)$$

The reverse inclusion has the same proof. The desired equality follows.

Exercise 18
Does the operation of addition on subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution
First, let's show that indeed, the operation of addition on subspaces of V has an

additive identity. Define $I = \{0\}$, the subspace of V containing the zero vector only. Take an arbitrary subspace U of V and notice that

$$\begin{aligned} U + I &= \{u + i : u \in U \text{ and } i \in I\} \\ &= \{u + 0 : u \in U \text{ and } i \in I\} \\ &= \{u : u \in U\} \\ &= U \end{aligned}$$

By commutativity of addition of subspaces of V, we also have $I + U = U$. Therefore, I is an additive identity for the addition on subspaces of V.
Concerning additive inverses, let's determine which subspaces of V have an additive inverse by taking an arbitrary subspace U of V and supposing that there is a subspace W of V such that $V + W = I$. Since W is a subspace of V, then $0 \in W$. It follows that for all $u \in U$,

$$u = u + 0 \in U + W = I = \{0\}$$

In other words, $U = \{0\} = I$. Since I obviously has an additive inverse (itself), then the unique subspace having an additive inverse is I.

Exercise 19
Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Solution
Consider the following counterexample. Let $V_1 = U = V$ and $V_2 = \{0\}$. We know from Exercise 15 of this section that

$$V_1 + U = V + V = V$$

Moreover, from Exercise 19, we also have

$$V_2 + U = \{0\} + V = V$$

Thus,

$$V_1 + U = V_2 + U$$

but $V_1 \neq V_2$.

Exercise 20
Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution
Consider the subspace

$$W = \{(0, a, 0, b) \in \mathbf{F}^4 : a, b \in \mathbf{F}\}$$

and the sum $U+W$. First, let's show that the sum is direct by proving that $(0,0,0,0)$ has a unique representation in this sum. Suppose $(x,x,y,y) \in U$ and $(0,a,0,b) \in W$ satisfy

$$(0,0,0,0) = (x,x,y,y) + (0,a,0,b)$$

This is equivalent to the system of equation

$$\begin{cases} x = 0 \\ a + x = 0 \\ y = 0 \\ b + y = 0 \end{cases}$$

which clearly has the following unique solution

$$\begin{cases} x = 0 \\ a = 0 \\ y = 0 \\ b = 0 \end{cases}$$

Therefore, in $U+W$, the zero vector can only be written as the sum of two zero vectors. It follows that the sum is direct.
Let's now show that $U \oplus W = \mathbf{F}^4$ by taking an arbitrary vector (x_1, x_2, x_3, x_4). Consider the vectors

$$u = (x_1, x_1, x_3, x_3) \in U \quad \text{and} \quad w = (0, x_2 - x_1, 0, x_4 - x_3) \in W$$

and notice that

$$\begin{aligned} u + w &= (x_1, x_1, x_3, x_3) + (0, x_2 - x_1, 0, x_4 - x_3) \\ &= (x_1, x_1 + x_2 - x_1, x_3, x_3 + x_4 - x_3) \\ &= (x_1, x_2, x_3, x_4) \end{aligned}$$

which shows that $(x_1, x_2, x_3, x_4) \in U \oplus W$. Thus, $\mathbf{F}^4 \subset U \oplus W$. Since U and W are subspaces of $\mathbf{F}^4$, then $U \oplus W$ must also be a subspace of $\mathbf{F}^4$: $U \oplus W \subset \mathbf{F}^4$. Therefore, we get $U \oplus W = \mathbf{F}^4$.

Exercise 21
Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution
Consider the subspace

$$W = \{(0,0,a,b,c) \in \mathbf{F}^5 : a,b,c \in \mathbf{F}\}$$

and consider the sum $U+W$. Let's first prove that it is actually a direct sum by focusing on the vector zero. Let $(x, y, x+y, x-y, 2x) \in U$ and $(0,0,a,b,c) \in W$ be two vectors such that

$$(0,0,0,0) = (x, y, x+y, x-y, 2x) + (0,0,a,b,c)$$

This translates to the following system of equation:

$$\begin{cases} x = 0 \\ y = 0 \\ x + y + a = 0 \\ x - y + b = 0 \\ 2x + c = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} x = 0 \\ y = 0 \\ a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore, since the zero vector can only be written as the sum of two zero vectors, the sum is direct.
Let's now show that $U \oplus W = \mathbf{F}^5$. Obviously, since $U \oplus W$ is a subspace of $\mathbf{F}^5$, we have $U \oplus W \subset \mathbf{F}^5$. Moreover, for any $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$\begin{aligned} &(x_1, x_2, x_3, x_4, x_5) \\ &\quad = (x_1, x_2, [x_1 + x_2] + [x_3 - x_2 - x_1], [x_1 - x_2] + [x_4 + x_2 - x_1], 2x_1 + [x_5 - 2x_1]) \\ &\quad = (x_1, x_2, x_1 + x_2, x_1 - x_2, 2x_1) + (0, 0, x_3 - x_2 - x_1, x_4 + x_2 - x_1, x_5 - 2x_1) \\ &\quad = (x, y, x + y, x - y, 2x) + (0, 0, a, b, c) \\ &\quad \in U \oplus W \end{aligned}$$

where $x = x_1$, $y = x_2$, $a = x_3 - x_2 - x_1$, $b = x_4 + x_2 - x_1$ and $c = x_5 - 2x_1$. Thus, $\mathbf{F}^5 \subset U \oplus W$. Therefore, $U \oplus W = \mathbf{F}^5$.

Exercise 22
Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution
Consider the subspaces

$$\begin{aligned} W_1 &= \{(0, 0, a, 0, 0) \in \mathbf{F}^5 : a \in \mathbf{F}\} \\ W_2 &= \{(0, 0, 0, b, 0) \in \mathbf{F}^5 : b \in \mathbf{F}\} \\ W_3 &= \{(0, 0, 0, 0, c) \in \mathbf{F}^5 : c \in \mathbf{F}\} \end{aligned}$$

and their sum $U + W_1 + W_2 + W_3$. Let's first prove that it is actually a direct sum by focusing on the zero vector. Let

$$\begin{aligned} u &= (x, y, x + y, x - y, 2x) \in U \\ w_1 &= (0, 0, a, 0, 0) \in W_1 \\ w_2 &= (0, 0, 0, b, 0) \in W_2 \\ w_3 &= (0, 0, 0, 0, c) \in W_3 \end{aligned}$$

be arbitrary vectors in their respective sets such that

$$(0,0,0,0,0) = u + w_1 + w_2 + w_3$$

This can be rewritten into the following system of equation:

$$\begin{cases} x = 0 \\ y = 0 \\ x + y + a = 0 \\ x - y + b = 0 \\ 2x + c = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} x = 0 \\ y = 0 \\ a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Hence, $u = w_1 = w_2 = w_3 = (0,0,0,0,0)$. Therefore, since the zero vector can only be written as the sum of zero vectors, the sum is direct.
Let's now show that $U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbf{F}^5$. Obviously, since $U \oplus W_1 \oplus W_2 \oplus W_3$ is a subspace of $\mathbf{F}^5$, we have $U \oplus W_1 \oplus W_2 \oplus W_3 \subset \mathbf{F}^5$. Moreover, for any $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$\begin{aligned}
&(x_1, x_2, x_3, x_4, x_5) \\
&\quad = (x_1, x_2, [x_1 + x_2] + [x_3 - x_2 - x_1], [x_1 - x_2] + [x_4 + x_2 - x_1], 2x_1 + [x_5 - 2x_1]) \\
&\quad = (x_1, x_2, x_1 + x_2, x_1 - x_2, 2x_1) + (0, 0, x_3 - x_2 - x_1, x_4 + x_2 - x_1, x_5 - 2x_1) \\
&\quad = (x, y, x + y, x - y, 2x) + (0, 0, a, b, c) \\
&\quad = (x, y, x + y, x - y, 2x) + (0, 0, a, 0, 0) + (0, 0, 0, b, 0) + (0, 0, 0, 0, c) \\
&\quad \in U \oplus W
\end{aligned}$$

where $x = x_1$, $y = x_2$, $a = x_3 - x_2 - x_1$, $b = x_4 + x_2 - x_1$ and $c = x_5 - 2x_1$. Thus, $\mathbf{F}^5 \subset U \oplus W_1 \oplus W_2 \oplus W_3$. Therefore, $U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbf{F}^5$.

Exercise 23
Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Solution
Consider the following counterexample:

$$\begin{aligned}
V &= \mathbf{R}^2 \\
V_1 &= \{(0, x) : x \in \mathbf{R}\} \\
V_2 &= \{(x, x) : x \in \mathbf{R}\} \\
U &= \{(x, 0) : x \in \mathbf{R}\}
\end{aligned}$$

I will not prove that V_1, V_2 and U are subspaces of V because it is not goal of this exercice. Notice that

$$V_1 + U = \{(x, y) : x, y \in \mathbf{R}\} = \mathbf{R}^2 = V$$

Moreover, for any arbitrary $u = (0, y) \in V_1$ and $y = (x, 0) \in U$, if

$$(0, 0) = u + v = (x, y)$$

then it follows that $u = v = (0, 0)$ since $x = y = 0$. Hence, $V_1 \oplus U = V$. Similarly, let's show that $V_2 \oplus U = V$. To do so, let's first prove that $V_2 + U = V$. Since $V_2 + U \subset V$, it suffices to prove that $V \subset V_2 + U$. Let (a, b) be an arbitrary element in V, then we have

$$(x, y) = (x - y, 0) + (y, y) \in V_2 + U$$

Hence, $V_2 + U = V$. To prove that the sum is direct, let $(x, x) \in V_2$ and $(y, 0) \in U$ such that

$$(x + y, x) = (x, x) + (y, 0) = (0, 0)$$

Since it follows that $x = 0$ and $y = 0$, then it follows that the zero vector can only be written as a sum of two zero vectors in $V_2 + U$. Thus, $V_2 \oplus U = V$. However, notice that $V_1 \neq V_2$ since $(1, 1) \in V_2$ but $(1, 1) \notin V_1$.

Exercise 24

A function $f : \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Solution

First, let's show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$. Since $V_e \oplus V_o \subset \mathbf{R}^{\mathbf{R}}$, then it suffices to prove that $\mathbf{R}^{\mathbf{R}} \subset V_e \oplus V_o$. Given an arbitrary function $f \in \mathbf{R}^{\mathbf{R}}$, define

$$f_e(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in \mathbf{R}$. Notice that for all $x \in \mathbf{R}$, we have

$$\begin{aligned} f_e(-x) &= \frac{f((-x)) + f(-(-x))}{2} \\ &= \frac{f(-x) + f(x)}{2} \\ &= f_e(x) \end{aligned}$$

and

$$\begin{aligned} f_o(-x) &= \frac{f((-x)) - f(-(-x))}{2} \\ &= \frac{f(-x) - f(x)}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= f_o(x) \end{aligned}$$

which proves that $f_e \in V_e$ and $f_o \in V_o$. Moreover, for all $x \in \mathbf{R}$

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

so $f = f_e + f_o \in V_e + V_o$. Therefore, $\mathbf{R}^{\mathbf{R}} = V_e + V_o$ since we just proved that $\mathbf{R}^{\mathbf{R}} \subset V_e + V_o$. Let's now show that the sum is direct by proving that the zero function can be represented as $f_e + f_o$ where $f_e \in V_e$ and $f_o \in V_o$ only when $f_e = f_o \equiv 0$. To prove this, consider two arbitray functions $f_e \in V_e$ and $f_o \in V_o$ such that

$$f_e(x) + f_o(x) = 0$$

for all $x \in \mathbf{R}$. Then, given any $y \in \mathbf{R}$, we have

$$f_e(y) + f_o(y) = 0$$

and

$$f_e(-y) + f_o(-y) = 0 \implies f_e(y) - f_o(y) = 0$$

by plugging-in $x = y$ and $x = -y$ into our previous equation. Adding the two equations gives us

$$\begin{aligned} [f_e(y) + f_o(y)] + [f_e(y) - f_o(y)] = 0 &\implies 2f_e(y) = 0 \\ &\implies f_e(y) = 0 \end{aligned}$$

It follows that $f_o(y) = 0$ as well since $f_e(y) + f_o(y) = 0$. Thus, since it holds for all $y \in \mathbf{R}$, then $f_e = f_o \equiv 0$. Therefore, $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Chapter 2

Finite-Dimensional Vector Spaces

2A Span and Linear Independence

TODO

2B Basis

[Coming soon...]

2C Dimension

[Coming soon...]