Algebraic Geometry : Homework 3

Samy Lahlou

Exercise 1: Let H_1, ..., H_m be hyperplanes in $\mathbb{P}^n(k)$. If $m \leq n$, show that

$$H_1 \cap \cdots \cap H_m \neq \varnothing.$$

Solution : By definition of a hyperplane, we can write $H_i = V(F_i)$ where

$$F_i = a_1^{(i)} X_1 + \cdots + a_{n+1}^{(i)} X_{n+1}$$

for all $1 \leq i \leq m$. It follows that a point $p = (x_1, ..., x_{n+1}) \in k^{n+1}$ satisfies $F_i(p) = 0$ for all i if and only if the coordinates of that point satisfy the following system of equations:

$$\begin{cases} a_1^{(1)} x_1 + \cdots + a_{n+1}^{(1)} x_{n+1} & = 0 \\ a_1^{(2)} x_1 + \cdots + a_{n+1}^{(2)} x_{n+1} & = 0 \\ & \vdots \\ a_1^{(m)} x_1 + \cdots + a_{n+1}^{(m)} x_{n+1} & = 0 \end{cases}$$

If we define the linear transformation $T : k^{n+1} \to k^m$ that acts as follows:

$$T(x_1, ..., x_{n+1}) = \begin{pmatrix} a_1^{(1)} x_1 + \cdots + a_{n+1}^{(1)} x_{n+1} \\ a_1^{(2)} x_1 + \cdots + a_{n+1}^{(2)} x_{n+1} \\ \vdots \\ a_1^{(m)} x_1 + \cdots + a_{n+1}^{(m)} x_{n+1} \end{pmatrix},$$

then we can apply the rank-nullity theorem to get that $\dim \ker T + \dim \mathrm{im}\ T = n + 1$. Since the image of T is a subspace of k^m, then $\dim \ker T \geq n + 1 - m > 0$ (since $m \leq n$). It follows that the kernel of T contains nonzero vectors $(x_1, ..., x_{n+1})$. But by construction, $\ker T = \{p \in k^{n+1} \mid F_i(p) = 0 \text{ for all } i\}$, and hence, there is a nontrivial element $p = (x_1, ..., x_{n+1})$ that satisfies $F_i(p) = 0$ for all i. Since these equations are all linear, then it follows that the point $P = [x_1 : ... : x_{n+1}] \in \mathbb{P}^n(k)$ is in $H_1 \cap \cdots \cap H_m$. Therefore,

$$H_1 \cap \cdots \cap H_m \neq \varnothing.$$