Algebraic Geometry : Homework 5

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Exercise 1: Let $X = V(x^3 - y^2) \subset k^2$. Is the map

$$f : k \to X, \qquad f(t) = (t^2, t^3)$$

a finite map?

Solution : Clearly, f is a regular map that satisfies the condition that the image is dense in the codomain (since it is surjective). If we consider the pullback $f^* : k[X] \to k[k] = k[t]$, then it suffices to show that $k[t]$ is integral over $k[X]$ to prove that f is a finite map. First, since $\overline{f(k)}$ is dense in X, then we can view $k[X]$ as a subring of $k[t]$ by the injectivity of the pullback. Since $f^*(\overline{x}) = t^2$ and $f^*(\overline{y}) = t^3$, then $k[X] = f^*(k[X]) = k[t^2, t^3]$. Hence, we have to prove that $k[t]$ is integral over $k[t^2, t^3]$. Since the integral elements of a ring extension form a ring, then it suffices to prove that t is an integral element. To do this, simply notice that t satsfies the monic polynomial $u^2 - t^2 \in k[t^2, t^3][u]$. Therefore, f is a finite map.