MATH 570 Notes : Higher Algebra 1

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These notes are based on lectures given by Professor Eyal Goren at McGill University in Fall 2025. The subject of these lectures is **TODO**. As a disclaimer, it is more than possible that I made some mistakes. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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1 Categories and Functors

1.1 Definitions

Definition. A category $\underline{C}$ is

1. a collection of objects $\text{ob}(\underline{C})$,
2. for any $A, B \in \text{ob}(\underline{C})$ a set $\text{Mor}_{\underline{C}}(A, B)$ with an associative composition law,
3. For all $A \in \text{ob}(\underline{C})$, there is a morphism 1_A such that for all $f \in \text{Mor}_{\text{ob}(\underline{C})}(A, B)$, we have $f \circ 1_A = f$ and $1_B \circ f = f$.
4.

Definition. A morphism $f \in \text{Mor}(A, B)$ is an isomorphism if there exists a $g \in Mor(A, B)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

1.2 Initial and Final Objects

Definition. An object $A \in \text{ob}(\underline{C})$ is initial (final) if for any object $B \in \text{ob}(\underline{C})$, $\text{Mor}(A, B)$ has only one element ($\text{Mor}(B, A)$ has only one element). A is a zero object if it's both initial and final.

Proposition 1.2.1. *An initial object (if it exists) is unique up to a unique isomorphism (similar, final).*

Proof. Suppose $A, A' \in \text{Ob}(\underline{C})$ are initial, let $f \in \text{Mor}(A, A')$ and $g \in \text{Mor}(A', A)$ be the unique such morphisms, then $g \circ f \in \text{Mor}(A, A) = \{1_A\}$ and so $g \circ f = 1_A$. Similarly, $f \circ g = 1_{A'}$. It follows that A and A' are isomorphic. ∎

Example:

- Sets: $\text{Ob}(\underline{C})$ are sets, $\text{Mor}(A, B)$ are functions from A to B. The empty set is the unique initial object. The singletons are precisely the final objects. It follows that zero objects don't exist.
- ${}_R\text{Mor}$ (Mor_R): R is a ring (always with 1 and is associative), $\text{Ob}({}_R\text{Mor})$ are left R-modules M, functions are R-modules homomorphisms $f : M \to N$. The zero-module $\{0\}$ is the unique initial object and also the unique final object. Hence, it is a zero object in that category.
- Gps (AbGps): The objects are (abelian) groups, the morphisms are (abelian) group homomorphisms, as for the previous example, there is a unique zero-object: the group $\{1\}$.

1.3 Functors

Definition (Covariant and Contravariant Functors)**.** A covariant (contravariant) functor $F : \underline{C} \to \underline{D}$ is the following:

1. For any object $A \in \text{Ob}(\underline{C})$, $FA \in \text{Ob}(\underline{D})$.

2. For any morphism $f\mathrm{Mor}_{\underline{C}}(A,B)$, we have a morphism $Ff \in \mathrm{Mor}_{\underline{D}}(FA,FB)$ ($Ff \in \mathrm{Mor}_{\underline{D}}(FB,FA)$) such that $F1_A = 1_{FA}$ and $F(g \circ f) = Fg \circ Ff$ ($F(g\circ f) = Ff \circ Fg$).

Definition (Faithful)**.** We say that F is faithful if whenever $Ff = Fg$ for some $f, g \in \mathrm{Mor}(A,B)$, then $f = g$.

Definition (Full)**.** We say that F is full if given any $h \in \mathrm{Mor}(FA,FB)$, there exists a $f \in \mathrm{Mor}(A,B)$ such that $Ff = h$.

Definition (Essentially Surjective)**.** We say that F is full if any $C \in \mathrm{Ob}(\underline{D})$ is isomorphic to FA for some $A \in \mathrm{Ob}(\underline{C})$.

Example:

- Forgetful functors: for example, the functor $F : \mathrm{Gps} \to \mathrm{Sets}$ defined by $FA = A$, $Ff = f$ forgets the group structure of the objects. This functor is not full but it is faithful. With some logic, we can prove that it is also essentially surjective.
- Consider the functor $F : \mathrm{Rings} \to \mathrm{Gps}$ such that $FR = R^*$ and $Ff = f|_{R^*}$. Is it faithful, full, essentially surjective ?
- If k is a field, then ${}_k\mathrm{Mod}$ is the same as the category of k-vector spaces where the morphisms are the k-linear maps. From this categroy, we can consider the contravariant functor $F :_k \mathrm{VSp} \to_k \mathrm{VSp}$ that sends V to its dual and homomorphisms to their transpose.
- The category $\mathrm{Rep}(G)$, where G is a fixed finite group, is the category of finite linear complex representations of G.
- We can define the functor $F : \mathrm{FinGps} \to \mathrm{Rings}$ by $FG = k[G]$ where k is a field.

Next time: we'll see that the category of representations of G is equivalent to the category ${}_{\mathbb{C}[G]}\mathrm{Mod}$.

1.4 Morphisms of Functors

Let $F, G : \underline{C} \to \underline{D}$ be two functors of the same variance. A morphism of functors $\varphi : F \to G$ is a collection of marphisms in $\underline{D}$ such that

$$commutative diagram$$

(inverse the arrows if F and G are contravariant).

Example:

- Let $\underline{C} = \underline{D} = {}_k\mathrm{VSp}$ and $F : \underline{C} \to \underline{C}$ be the duality functor, then **TODO**

1.5 Equivalence of categories

Two categories $\underline{C}$ and $\underline{D}$ are called (anti) equivalent if there are co(ntra)variant functors $F : \underline{C} \to \underline{D}$ and $G : \underline{D} \to \underline{C}$ such that $F \circ G \cong 1_{\underline{D}}$ and $G \circ F \cong 1_{\underline{C}}$ (isomorphic means all the φ_A are isomorphisms).

Example:

- Let G be a finite group, then the categories $\mathrm{Rep}(G)$ and f.g.$_{\mathbb{C}[G]}$ are equivalent. **TODO**

- Let k be a field and let $\underline{C}$ be the category composed of the vector spaces k^0, k^1, k^2, ..., then $\mathrm{Mor}(k^a, k^b) = \mathrm{M}_{ab}(k)$. Now, if we let $\underline{D} = \text{f.d.}_k\text{VSp}$ be the category of finite dimensional vector spaces, then this category is uncountable whereas the previous one is countable. Let $F : \underline{C} \to \underline{D}$ be the functor defined by $Fk^a = k^a$ and FM is the linear map that maps x to Mx, then F is full, faithful and essentially surjective. To define $G : \underline{D} \to \underline{C}$, choose for any vector space V an isomorphism $i_v : V \to k^{\dim(V)}$, this induces an isomorphism between $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(k^{\dim(V)}, k^{\dim(W)})$ by the map $T \mapsto i_w T i_V^{-1}$. Thus, we can define G by $GV = k^{\dim(V)}$ and $GT = i_w T i_V^{-1}$. If we choose i_{k^n} to be the identity, then we get $GF = 1$ and $FG \cong 1$.

Theorem 1.5.1. *A functor $F : \underline{C} \to \underline{D}$ is an (anti) equivalence of categories if and only if F is full, faithfull and essentially surjective.*

Proof. **TODO** ∎

Theorem 1.5.2 (Morita's Theorem)**.** *Let R be a ring and $n \geq 1$ be an integer. The categories ${}_R Mod$ and ${}_{M_n(R)}Mod$ are equivalent.*

Proof. **TODO** ∎