Fourier Analysis :
The Catalyst of Modern Analysis

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Abstract

We present in this paper the history of the fundamental concepts, theorems and definitions that we use in modern Analysis that were motivated by the study of Fourier Analysis. This will be done chronologically with an emphasis on some mathematicians and papers that had a great impact on the subject.

1 Early Stages of Fourier Analysis

1.1 Introduction

Our story begins in 1747 when Jean Le Rond D'Alembert $(1717 - 1783)$, a French mathematician, derives the *wave equation* [8].

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

The wave equation is a differential equation which governs the vibration of a string with two fixed endpoints. In other words, if you have a function $u(x, t)$ that solves the wave equation (1) such that $u(x, 0)$ corresponds to the initial position of a string, then you can predict the behavior of the string at any time.

D'Alembert claimed that he had found the solution to the wave equation (1). He expresses it as a sum of two traveling waves going in opposite directions [16]:

$$u(x, t) = \frac{1}{2}[f(x + t) + f(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy, \tag{2}$$

where $f(x)$ and $g(x)$ are arbitrary functions. Since $f(x)$ and $g(x)$ are arbitrary, D'Alembert considered that he could represent any initial position of the string, and therefore, has found the *general* solution to the wave equation.

Not long after D'Alembert published his solution, the notorious Swiss mathematician Leonhard Euler $(1707 - 1783)$ objected to D'Alembert's claim to have found the general solution. To understand his objection, we first need to know what D'Alembert and Euler meant by *functions*.

1.2 What is a function ?

At that time, a function is understood to be a *formula* or an *analytic expression* where one can use addition, multiplication, composition with some special functions like $\cos(x)$, $\sin(x)$, e^x... For example

$$y = x^2 \cos\left(\frac{1}{x}\right) + 3e^{-x^3},$$

fits the criteria to be called a function. With this definition of functions, it was believed, and admitted, that every function could be represented by a graph but not every graph has an associated function.

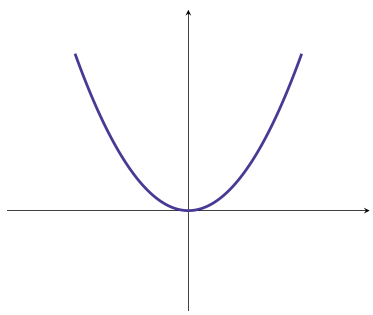

Figure 1:
Graph of the function x^2

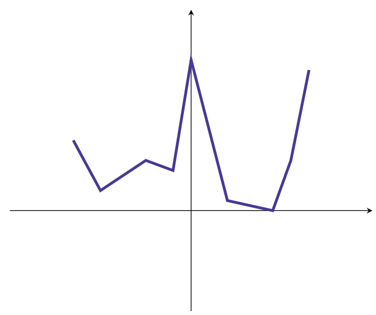

Figure 2:
Graph not associated to a function

Knowing this, we are now able to understand Euler's objection. Any graph models a possible initial position of a string (as long as the end points are fixed on the horizontal axis). But if we consider D'Alembert's solution (2), the initial position of the string (when $t = 0$) is given by $u(x, 0) = f(x)$. Therefore, D'Alembert's claim that you can model any initial position of the string using a function $f(x)$ cannot be true since, as we said earlier, not every graph represents a function. This is Euler's objection.

1.3 Bernoulli's solution to the Wave Equation

A few years later, in 1755, the Swiss mathematician Daniel Bernoulli (1700 – 1782) also claimed to have solved the wave equation (1), but this time using a different technique. His solution is of the form of an infinite sum of *standing waves* (see Figure 3) [16].

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(mt) + B_m \sin(mt)) \sin(mx), \tag{3}$$

where A_m and B_m are coefficients that are determined by the initial conditions of the problem (that is, the initial position of the string).

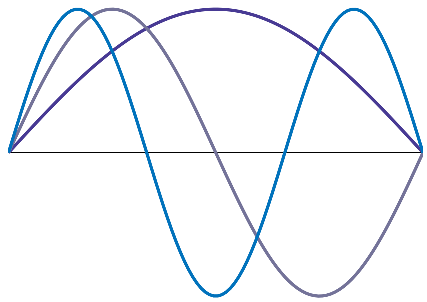

Figure 3: Overlapping standing waves

Similarly to D'Alembert, Euler objected. To understand why, let's consider the initial position of the string according to Bernoulli's solution by plugging $t = 0$ in equation (3):

$$u(x,0) = \sum_{m=1}^{\infty} A_m \sin(mx)$$

According to Bernoulli, such sum could represent any graph. To prove that Bernoulli was wrong, Euler gave the following argument: Suppose that the initial position of the wave is given by the graph of a function $h(x)$ (expressed using a formula) on the interval $[0, 2\pi]$. Take for example $h(x) = x(2\pi - x)$. Bernoulli's claim implies that there exist coefficients A_m such that

$$x(2\pi - x) = \sum_{m=1}^{\infty} A_m \sin(mx),$$

on $[0, 2\pi]$. But according to Euler, if two functions are equal on an interval, they must be equal on the whole real line. In other words, there is only one way to extend a graph associated with a function.

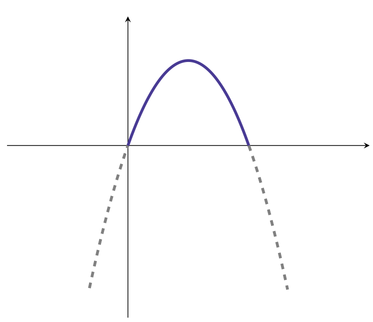

Figure 4:
Graph of the usual function $x(2\pi - x)$

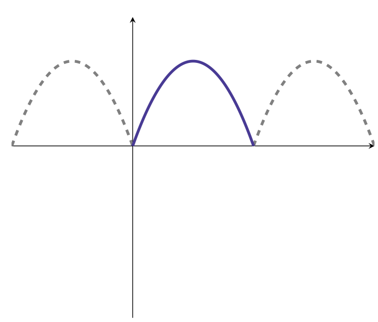

Figure 5:
Graph of the function $x(2\pi - x)$ as a 2π-periodic function

But since the sine function is 2π-periodic, so is $\sum_{m=1}^{\infty} A_m \sin(mx)$. Therefore, if you were to extend this infinite sum on the real line you will have a 2π periodic graph. On the other hand, if you were to extend $h(x) = x(2\pi - x)$ on the real line, you will simply have a parabola. Since, according to Euler, such extension is unique, we have a contradiction. Therefore, Bernoulli's infinite sum couldn't represent any function $h(x)$. We conclude that equation (3) couldn't be the general solution. Little did they know, Bernoulli was closer to the truth than Euler.

1.4 Euler's work on Trigonometric Series

Later, in 1777, Euler studied functions that can be represented by cosine series [9], functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx),$$

where the coefficients a_n depend on $f(x)$. Euler's goal was to find a way to compute a_n knowing $f(x)$. He was the first mathematician to found a formula for the coefficient a_n by performing a long computation involving repeated use of trigonometric identities and taking limits. Ultimately, he finds

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx. \tag{4}$$

Only after he derived equation (4), he noticed that this derivation could have been done in only three steps using the orthogonality properties of the *cosine* function:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

It is important to note that Euler always believed that equation (4) only applied to functions expressed by a cosine series. He didn't notice that he could use his formula to contradict his own objection to Bernoulli's claim.

1.5 The Birth of Fourier Analysis

The question of the general solution of the wave equation stayed unanswered for nearly 50 years. In 1822, the French mathematician Jean-Baptiste Joseph Fourier (1768 - 1830) published his famous book *Théorie Analytique de la Chaleur* [12]. In this paper, Fourier studies heat propagation by deriving the *heat equation*

$$\frac{\partial f}{\partial t} = k \frac{\partial^2 f}{\partial x^2} \tag{5}$$

Fourier provides a general solution to this differential equation. To solve his problem with given initial conditions, he must find the coefficients a_n and b_n of the following expression

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{6}$$

where $f(x)$ models the initial heat distribution. In a very unrigorous 31 pages long derivation, Fourier finally finds the following formulas for a_n and b_n, in terms of $f(x)$:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

But unfortunately, Fourier was not aware of Euler's nearly identical formula (4) of nearly 50 years ago. After stating these results, Fourier states what we'll call **Fourier's Theorem**:

"This theorem and the previous one are suitable for all possible functions, whether we can express their nature by known means of analysis, or whether they correspond to curves drawn arbitrarily." _ *page 241.*

In other words, Fourier states that any function or graph can be expressed as a trigonometric series as in equation (6). Given a function $f(x)$, we call the right hand side of equation (6) its *Fourier Series*. This very bold statement, and many other ones, were not proven in any way by Fourier. However, Fourier's book had a huge impact on the mathematical community of his time which led many other mathematicians to attempt to prove Fourier's Theorem.

2 Dirichlet's 1829 paper

2.1 First Proof Attempts

The first proof attempt of Fourier's Theorem was made by the French mathematician Siméon Denis Poisson (1781 - 1840) in 1820 [14]. Because of its lack of rigor, Poisson's proof was not accepted. Moreover, Abel's Limit Theorem (published in 1826) contradicts some parts of Poisson's proof.

Similarly, the French mathematician Augustin Louis Cauchy (1789 - 1857) tried to prove Fourier's Theorem in 1827 but used some term-by-term integration and only showed convergence of the series instead of convergence to the original function [7]. Additionally, the class of functions to which his proof applies wasn't specified. Again, in 1827, Cauchy proposed a second proof, which was shown to be erroneous because of a bad use of the Limit Comparison Test [5].

2.2 Dirichlet's Proof

The first valid and accepted proof was published by the German mathematician Peter Lejeune Dirichlet in 1829 [10]. In his paper, Dirichlet started by pointing out that Cauchy's second proof was wrong. Cauchy used the fact that given two sequences (a_n) and (b_n), if the limit of their quotient is 1, then $\sum a_n$ converges if and only if $\sum b_n$ converges. To show why it was false, Dirichlet gave the following counterexample:

$$a_n = \frac{(-1)^n}{\sqrt{n}}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right), \qquad b_n = \frac{(-1)^n}{\sqrt{n}}$$

This counterexample explains why we need to assume that both sequences must be positive in the modern Limit Comparaison Test that we learn in Calculus:

Theorem (Limit Comparaison Test)**.** *Given two positive sequences (a_n) and (b_n), if the limit of their quotient is 1, then $\sum a_n$ converges if and only if $\sum b_n$ converges.*

After this counterexample, Dirichlet starts the set up for his proof of Fourier's Theorem. He begins by defining a class of functions on which his proof will apply. These functions must satisfy the three following conditions:

(i) The function must be integrable.

(ii) The function must have finitely many maxima and minima.

(iii) If the function has a jump discontinuity at a point, then its value at this point must be the average of its left and right limits.

Dirichlet, with these clear conditions, gives a proof of Fourier's Theorem that surpassed all of the previous attempts by its rigor. In his proof, Dirichlet makes use of the following trigonometric identity:

$$\frac{1}{2} + \cos(x) + \cos(2x) + ... + \cos(nx) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{x}{2})},$$

in which the right hand side is now called the *Dirichlet Kernel*.

After his proof, Dirichlet discusses his three conditions and points out that his first condition (i) is non trivial by giving an example of a function that is not subject to integration. To do so, he defines the function $\varphi(x)$ which is equal to a constant c when x is rational, and to a distinct constant d when x is irrational.

For Dirichlet, this function is not subject to integration since the area under its curve don't make any sense. When $c = 1$ and $d = 0$, we call it *Dirichlet's function*. Notice that with this function, Dirichlet shows that the concept of function in the sense of D'Alembert, Euler and Bernoulli needs to be generalized. As he writes in a paper in 1837 [11]:

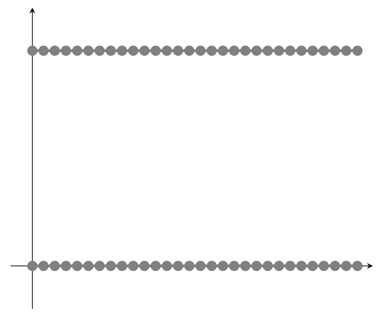

Figure 6: Graph of the Dirichlet Function

"It is not necessary that y be subject to the same rule as regards x throughout the interval, indeed one need not even be able to express the relation through mathematical operations" _ Dirichlet, 1837

Dirichlet set the tone for how Mathematical Analysis should be done (even though he is more remembered for his work on Number Theory). His taste for rigor also led him to prove rigorously Abel's Limit Theorem. His proof of Fourier's Theorem was widely accepted in the community and his influence on mathematics is nearly unmatched.

3 Riemann's Integral and functions

3.1 Introduction

Dirichlet's proof of Fourier's Theorem was widely accepted in the mathematical community. Even if its proof only focuses on a specific class of functions (the functions satisfying conditions (i), (ii) and (iii)), it turns out that any function that occurs in nature is in this class. Hence, physicists were able to use Fourier's Theorem without worrying about convergence. Thanks to Dirichlet, the problem of proving Fourier's Theorem was solved in the practical case.

In 1854, the German mathematician Georg Friedrich Bernhard Riemann $(1826 - 1866)$ writes his paper *On the possibility of representing a function by a trigonometric series* [15]. It was published after his death in 1867 by his friend and mathematician Richard Dedekind, which we'll talk about in the following chapter. In this paper, Riemann argues that Dirichlet's proof is sufficient for the practical case, but the problem of proving Fourier's Theorem in the general case is the still worth studying. To justify that, he gives the two following reasons. First, in addition to Phyiscs, the use of Fourier Series became more popular in pure Mathematics such as Number Theory. Hence, it would be a mistake to limit ourselves to physics applications only. Secondly, like Dirichlet pointed out in his 1829 paper, trying to prove Fourier's Theorem rigorously leads to profound questions about the foundations of Infinitesimal Calculus.

3.2 Riemann's Rearrangement Theorem

The first result Riemann presents concerns the convergence of series. As Riemann said in his paper, Dirichlet had already observed that *"[...] infinite series are divided into two classes, depending on whether or not they remain convergent when all their terms are made positive."* Notice that these two classes correspond exactly to absolutely convergent and conditionally convergent series. These classes were defined to explain the fact that changing the order of the terms of some convergent series changes the value to which its converges. This is what Riemann proved in the paper.

Theorem (Riemann's Rearrangement Theorem). *Let $\sum a_n$ be a conditionally convergent series of real numbers. For any real number C, there exists a permutation $p : \mathbb{N} \to \mathbb{N}$ such that $\sum a_{p(n)} = C$.*

Riemann claims that absolutely convergent series can be treated as finite sums in the sense that rearranging the terms will not affect the value of the series. However his claim was left unproved.

As Dirichlet proved that Cauchy's use of the Limit Comparison Test was wrong, it was now Riemann's turn to contradict Cauchy's proof attempt of Fourier's Theorem. His Rearrangement Theorem shows that it is necessary to prove that the Fourier Series of a function (6) converges to the function instead of simply proving its convergence since Fourier Series may be conditionally convergent.

3.3 The Riemann Integral

Riemann begins his 4^{th} chapter by mentioning that Integral Theory is still very uncertain. He then asks a clever question:

Also zuerst: Was hat man unter $\int_a^b f(x)\, dx$ zu verstehen?

which translates to "*But first, what do we mean by $\int_a^b f(x)dx$?*"

Until that point in time, mathematicians have been talking about integrals recurrently, however the integral had no robust definition. Before Riemann's 1867 paper, the closest definition we had was given by Cauchy in his 1823 book [6]. However, his integral wasn't widely accepted since it only applies to continuous functions or functions with finitely many discontinuities. This left the door open for other mathematicians to broaden the definition of the integral and better accommodate it to Dirichlet's definition of a function and handle infinitely many discontinuities.

The Riemann integral is born on page 34 and can indeed handle infinitely many discontinuities. To illustrate the power of his new integral, Riemann gives a function with infinitely many discontinuities that is still integrable: Riemann's Pathological Function.

He first defines the function $x \mapsto (x)$ (Figure 7) where (x) denotes the periodic function equal to the identity function on $[-\frac{1}{2}, \frac{1}{2})$ with period 1.

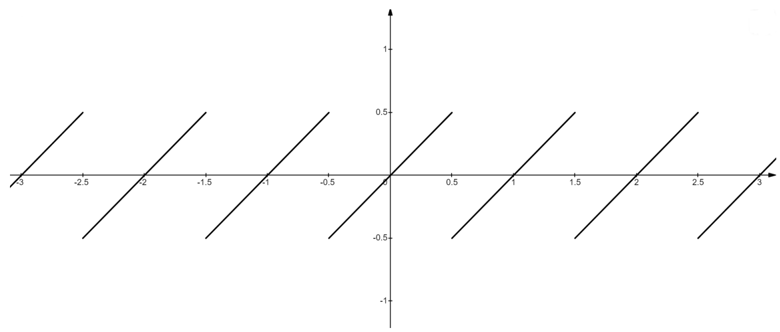

Figure 7: Graph of the function $x \mapsto (x)$

Then he defines the function $f(x)$ (Figure 8) as follows:

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + ... = \sum_{1}^{\infty} \frac{(nx)}{n^2}$$

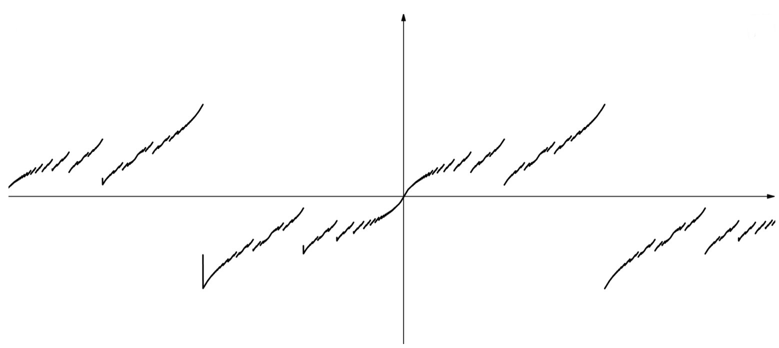

Figure 8: Graph of Riemann's Pathological function

It is easy to prove that this function has infinitely many discontinuities and that it has infinitely many maxima and minima, which is easy to see from Figure 8. However, he still manages to show that $f(x)$ is integrable by his definition. Therefore, $f(x)$ is integrable and has infinitely many maxima and minima. Does this ring a bell?

If you remember Dirichlet's conditions, here we have a function that satisfies condition (i) but does not satisfy condition (ii). This proves that condition (ii) cannot imply condition (i). However Riemann proved rigorously that the converse is true. Namely, if a function has finitely many maxima and minima, then it is integrable. This shows that condition (i) doesn't need to be cited in Dirichlet's conditions, conditions (i) and (iii) suffice.

Another primary result of Fourier Analysis was presented in this paper: The so-called Riemann-Lebesgue Lemma.

Theorem (Riemann-Lebesgue Lemma). *If $f(x)$ is integrable (by Riemann's Definition), then*

$$\int_{-\pi}^{\pi} f(x)sin(n(x-a)) \to 0$$

as $n \to \infty$ and where a in a real number.

Riemann is the first to state it clearly and prove it. This proposition will turn out to be a useful tool for Riemann in his paper because it states that Fourier coefficients of a function tend to zero as n goes to infinity. This is the reason way this proposition is now known as a Lemma and not a Theorem. The use of the Riemann-Lebesgue Lemma is unprecedent in today's Fourier Analysis.

3.4 Funky Functions

The end of Riemann's paper is filled with what we'll call "funky functions". They each test the limits of Dirichlet's conditions in a different way. We have already seen a prime example of such functions with Riemann's Pathological function (Figure 8).

As for the previous function, Riemann proves that the whole class of functions of the form

$$f(x) = \frac{d(x^\nu \cos\frac{1}{x})}{dx}, \quad (0 < \nu < \frac{1}{2})$$

have infinitely many maxima and minima and are integrable. However, he also proves that they all have a divergent Fourier Series.

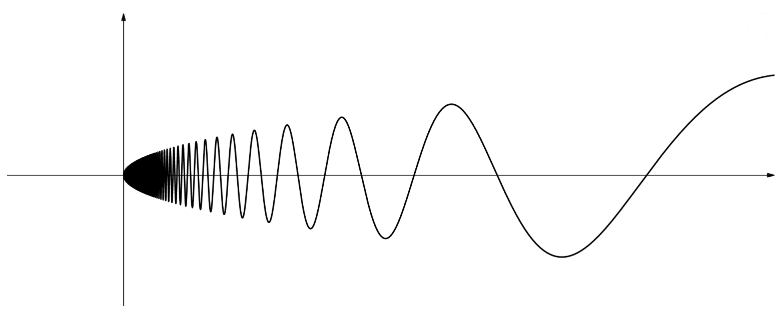

Figure 9: Graph of the function $x \mapsto x^\nu \cos\frac{1}{x}$ with $\nu = \frac{1}{3}$

The second function is a slight modification of his Pathological function (Figure 8). It is a non-integrable function which still has a Fourier series that converges and diverges on a dense subset of $\mathbb{R}$. He finds its Fourier Series by crafting a trigonometric series directly from the expression of the function instead of integrating the function to find the Fourier coefficients:

$$f(x) = \sum_{1}^{\infty} \frac{(nx)}{n}$$

Here, $x \mapsto (x)$ denotes the function we defined earlier in section 3.3 and is shown in Figure 7.

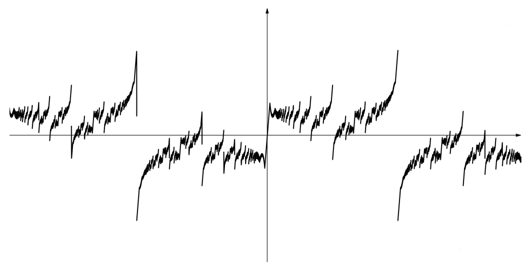

Figure 10: Graph of the function $x \mapsto \sum_1^\infty \frac{(nx)}{n}$

The third and final function is a Fourier Series which converges on a dense subset of $\mathbb{R}$ and has Fourier coefficients which do not converge to zero.

$$f(x) = \sum_1^\infty \sin((n!)x\pi)$$

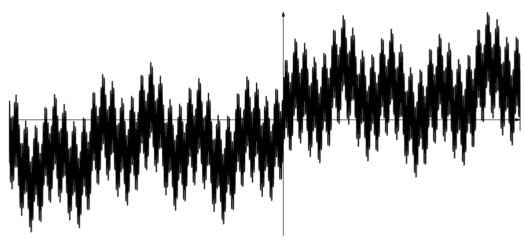

Figure 11: Graph of the function $x \mapsto \sum_1^\infty \sin((n!)x\pi)$

One may think that this function violates the Riemann-Lebesgue Lemma but since the coefficients do not result from the integrable formula for the Fourier coefficients, the theorem does not apply.

In the 19th century, Mathematical Analysis takes a radical turn. A new trend started emerging: imagining weird behaving functions to push the limits of theorems and definitions. Riemann's paper is a paramount example of this. Notice that Dirichlet's function is perfectly aligned with this trend. Other famous examples can be cited from this period: Thomae's function (1872) which is continuous only on the irrationals, and Weierstrass's function (1875) which is continuous but nowhere differentiable. This naturally lead to the study of sets of discontinuities of weird behaving function and ultimately to sets in general.

4 Cantor's study of sets

4.1 The Uniqueness Theorem

The mathematician who pushed the concept of sets to another level is Georg Cantor (1845 – 1918) in the late 19th century. As he was working at the University of Halle, Cantor heard about the uniqueness problem for trigonometric series in 1869 by his colleague, Eduard Heine, who was working on it. Their goal was to show that if a function has a representation as a trigonometric series, then such a representation is unique (i.e, it cannot be represented by two trigonometric series with different coefficients). It is equivalent to show that the only trigonometric series that represents the constant function 0 on the interval $[0, 2\pi]$ is the one where all the coefficients are zero (and this is what Cantor will end up proving).

After only one year, Cantor published in 1870 [1] a proof of the Uniqueness Theorem for trigonometric series where he assumes the convergence of the series for all values of x taken between 0 and 2π. One year later, in 1871 [2], Cantor improves his theorem and shows that it still holds even if you don't necessarily assume the convergence of the series for finitely many points.

Nevertheless, Cantor is convinced that this is too restrictive and that the theorem would still hold even if the convergence is not assumed on infinitely many points. However, those "infinitely many points" cannot be arbitrarily distributed. For example, if you don't assume the convergence on the infinitely many points that constitute the interval $[0, 2\pi]$, then the theorem obviously doesn't hold anymore. Thus, Cantor needs to describe precisely the *systems of points* (he would use the term *set* only a decade later) that may be infinite and on which the convergence of the series can be ignored.

This is exactly what Cantor did the following year, in 1872 [3]. Cantor knew that he had to be rigorous enough to be able to make such an improvement to his theorem. Hence, he starts from the very beginning by defining the real numbers from the rational numbers. To do so, he considers rational Cauchy sequences and defines what we would now call an equivalence relation defined as follows

$$(a_n) \sim (b_n) :\iff \lim(a_n - b_n) = 0$$

Hence, the collection of rational Cauchy sequences can be divided into subcollections of sequences equivalent to one another, what we now call equivalence classes. Then, a real number would be one equivalence class, and the real numbers as a whole would be the collection of all real numbers (i.e. the collection of all equivalence classes).

After that, Cantor defines the notions of *neighborhoods*, *limit points* and *isolated points* [3].

Definition. *"I call neighborhood of a point any interval in which this point is contained."*

Definition. *"By limit point of a point system P, I mean a point of the line such that in his neighborhood, there is infinitely many points of the system P."*

Definition. *"We call isolated point of P any point that, in P, is not at the same time a limit point of P."*

These definitions are very similar to the ones we use today in topology. To make everything clearer, notice that if our system P contains the elements

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ ...$$

we can visualize this system as follows:

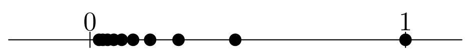

Figure 11: Illustration of the system containing the points $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ ...$

The point 0 is a limit point of P (even if it is not contained in P) and any element of P is an isolated point.

From these notions, he defines from a given system of points P the *derived system* P' of P.

Definition. *'The derived system of P, called P', is the system of the limit points of P."*

In our previous example, P' would be the system containing the point 0. Applying this operation n times to a system P gives what he calls the nth derived system $P^{(n)}$ from P.

After all these definitions, Cantor is finally able to state and prove his theorem. Here is how he stated it in his paper:

Theorem (Cantor)**.** *If an equation is of the form*

$$0 = C_0 + C_1 + C_2 + ... + C_n + ...$$

where $C_0 = \frac{1}{2}d_0$ *and*

$$C_n = c_n \sin(nx) + d_n \cos(nx)$$

holds for all values of x in $[0, 2\pi]$, except on a system P of the ν-th species where ν is a whole number as large as we want, I say that we will have

$$d_0 = 0, \quad c_n = d_n = 0$$

4.2 A General Theory of Sets

However, a disturbing question arises: Why would this theorem hold for some infinite systems and not others ?

This question led to a correspondence between Cantor and the mathematician Richard Dedekind (1831 - 1916) in 1873 in which Cantor famously asks if it possible to find a one-to-one correspondence between the natural numbers and the real numbers. Dedekind answered that he saw no objections to the existence of such bijection.

However, one year later, in 1874, Cantor proves, using what we now call the Nested Interval Property, that such bijection is indeed impossible. Cantor is beginning to explore the idea of different *kinds* of infinity. Indeed, he just proved that there exists two different infinities.

He would then ask Dedekind if it is possible to find a bijection between a line and a plane and again, 3 years later, even if the mathematical community accepted without a proof that such correspondence is impossible, Cantor showed that it is possible to find one-to-one correspondence between the line and any n-dimensional space.

Combining all of these results about what he now calls *sets*, and after creating more tools to explore his new ideas (such as Transfinite Numbers and their Transfinite Arithmetic), Cantor publishes all of these results in a 1883 paper called *Foundations of a General Theory of Sets* [4].

4.3 Derived Systems and Sets of Measure Zero

If you now about Measure Theory, the way Cantor's last Uniqueness Theorem is stated may ring a bell. Cantor was trying to determine which systems of points were considered negligible. Today, from a measure theoretic point of view, we know that such systems probably correspond to sets of measure zero. It turns out that the systems Cantor described are indeed sets of measure zero. Let's prove it.

Proof. Let P be an arbitrary of points. By definition, P' is the system of isolated points. As we saw in the previous example, the points in P may not be in P', in other words, it may not be true that $P \subset P'$. However, the points in P that are not in P' are precisely the isolated points of P. Hence:

$$P \setminus \{\text{isolated points of } P\} \subset P'$$

But we know that a set cannot have more than countably many isolated points (we leave it as an exercise to the reader). Thus, if we denote by λ the Lebesgue measure on the real line, we get the following inequality:

$$0 \leq \lambda(P) = \lambda(P \setminus \{\text{isolated points of } P\}) \leq \lambda(P')$$

By induction, we get

$$0 \leq \lambda(P) \leq \lambda(P^{(k)})$$

for all positive integers k. If there is a positive integer n such that $P^{(n)}$ is finite, then $\lambda(P^n) = 0$ and it follows that

$$\lambda(P) = 0$$

by plugging-in $k = n$. Therefore, the systems described by Cantor in the final statement of his Uniqueness Theorem are sets of measure zero.

5 What about now ?

As it was said at the end of our discussion on Riemann, the study of sets in Analysis became more and more common at the end of the century. Even if we decided to focus on Cantor, other mathematicians started to define some way of describing sets. One such mathematician is Giuseppe Peano $(1858 - 1932)$ who defined to notion of inner and outer content of a set [13]. His work was then continued by the French mathematicians Camille Jordan $(1838-1922)$ and especially Emile Borel $(1871 - 1956)$ who created the notion of *measure*.

From this new theory of measures, Lebesgue extended Borel's work by creating a whole new theory of functions and integration. Lebesgue's integral is still used today as the standard integral in Analysis because of its generality and also because of all the nice convergence theorems that Riemann's integral lacks of. It also turns out that today's most advanced results on the original goal of proving Fourier's Theorem and improving Dirichlet's conditions are stated in the language of Lebesgue's Analysis. Two such results are the Riesz-Fischer Theorem, proved in 1907, and Carleson's Theorem, proved in 1966.

Theorem (Riesz-Fischer Theorem)**.** *A function is in L^2 if and only if its Fourier Series converges in the sense of L^2.*

Theorem (Carleson's Theorem)**.** *Any function in L^2 has Fourier Series that converges almost everywhere.*

We find that the main takeaway of this paper is how a simple idea coming from a single person can impact a whole research field. Fourier Analysis is now a key component in Mathematics, Physics, Quantum Computation and even Music Softwares. However, it would be a mistake to think that this is a one in a million phenomenon. This happened and will happen a numerous amount of times in Mathematics. One can think of Euclid's famous fifth postulate, which preoccupied the minds of mathematicians for centuries and led to the study of *non-euclidean geometry*. Another example would be Fermat's Last Theorem which led to the development of *Algebraic Number Theory*. This is where our story ends.

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