Higher Algebra 1 : Assignment 5

Samy Lahlou

Exercise 27: Let R be a commutative ring. Prove that the open sets $D(h)$ form a basis for the topology on $\mathrm{Spec}(R)$ as h varies over R. Prove further that if $f : R \to S$ is a ring homomorphism, and $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the induced map, then $(f^*)^{-1}(D(h)) = D(f(h))$. Conclude this way another proof that f^* is continuous.

Solution : Let U be an open subset of $\mathrm{Spec}(R)$, then $U = \mathrm{Spec}(R) \setminus V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R. Since $\mathfrak{p}$ is clearly equal to the ideal generated by its elements, we have the equality $\mathfrak{p} = \sum_{h\in\mathfrak{p}}(h)$. It follows that

$$\begin{aligned} U &= \mathrm{Spec}(R) \setminus V(\mathfrak{p}) \\ &= \mathrm{Spec}(R) \setminus \bigcap_{a\in\mathfrak{p}} V((h)) \\ &= \bigcup_{h\in\mathfrak{p}} [\mathrm{Spec}(R) \setminus V((h))] \\ &= \bigcup_{h\in\mathfrak{p}} D(h). \end{aligned}$$

Therefore, by definition, the $D(h)$'s form a basis for the topology on $\mathrm{Spec}(R)$.

Let's now prove that $(f^*)^{-1}(D(h)) = D(f(h))$. If we let $\mathfrak{p}$ be a prime ideal of S, then

$$\begin{aligned} [\mathfrak{p}] \in (f^*)^{-1}(D(h)) &\iff f^*([\mathfrak{p}]) \in D(h) \\ &\iff [f^{-1}(\mathfrak{p})] \in D(h) \\ &\iff h \notin f^{-1}(\mathfrak{p}) \\ &\iff f(h) \notin \mathfrak{p} \\ &\iff [\mathfrak{p}] \in D(f(h)) \end{aligned}$$

which proves the equality between the two sets. The continuity of f^* follows from the fact that the inverse image preserves arbitrary unions and the inverse image of any element of the basis is open (as we just showed).

Exercise 28: Let $X = \{x, y\}$ a set with two points. There are (basically) 3 different topologies on X; call them $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, where $\mathcal{T}_1$ is the discrete topology, $\mathcal{T}_2$ has $\{x\}$ as an open set (but not $\{y\}$) and $\mathcal{T}_3$ is the trivial topology.

(a) Determine for which i, can the topological space $(X, \mathcal{T}_i)$ be the spectrum of a commutative ring R_i.

(b) Suppose that for $i \neq j$ both $(X, \mathcal{T}_i) = \mathrm{Spec}(R_i)$ and $(X, \mathcal{T}_j) = \mathrm{Spec}(R_j)$. Suppose also that the identity map $(X, \mathcal{T}_i) = \mathrm{Spec}(R_i) \to (X, \mathcal{T}_j) = \mathrm{Spec}(R_j)$ is continuous. Can you find examples of rings R_i, R_j such that this map is induced from a ring homomorphism $R_j \to R_i$, or is that not possible ?

Solution :

(a) Since $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{[(2)], [(3)]\}$, then $V((2)) = \{[2]\}$ and $V((3)) = \{[3]\}$ using the fact that none of the ideals contain the other. It follows that the two singeltons in $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z})$ are closed, and hence, the two singletons are open. Thus, the topology on $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z})$ is the discrete topology. Hence, it holds for $i = 1$.

For $i = 2$, it suffices to use the example we saw in class where $R_2 = \mathbb{Z}_{(p)}$ for a prime p. In that case, $\mathrm{Spec}(R_2) = \{[0], [(p)]\}$ where $\{[0]\}$ is open and $\{[(p)]\}$ is closed. Hence, it holds for $i = 2$.

Finally, if a ring R_3 has precisely two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $\mathrm{Spec}(R_3)$ has the trivial topology, then none of $\{[\mathfrak{p}]\}$ and $\{[\mathfrak{q}]\}$ are closed. It follows that $V(\mathfrak{p}) = V(\mathfrak{q}) = \mathrm{Spec}(R_3)$. Hence, $[\mathfrak{p}] \in V(\mathfrak{q})$ and $[\mathfrak{q}] \in V(\mathfrak{p})$ which implies that $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$, and hence, $\mathfrak{p} = \mathfrak{q}$, a contradiction. Thus, it doesn't hold for $i = 3$.

(b) Define R_1 to be a commutative ring such that $\mathrm{Spec}(R_1) = \{[\mathfrak{p}_1], [\mathfrak{q}_1]\} = \{x, y\}$ where the topology is the discrete topology, and let R_2 be a commutative ring such that $\mathrm{Spec}(R_2) = \{[\mathfrak{p}_2], [\mathfrak{q}_2]\} = \{x, y\}$ where $\{[\mathfrak{p}_2]\}$ is open but not $\{[\mathfrak{q}_2]\}$. In other words, $\mathfrak{q}_2 \supset \mathfrak{p}_2$ while no such inclusion relation exist between $\mathfrak{p}_1$ and $\mathfrak{q}_1$.

Let's show that any homomorphism $f : R_1 \to R_2$ cannot induce the identity function. By contradiction, suppose that there is such a homomorphism $f : R_1 \to R_2$ such that $f^* : \mathrm{Spec}(R_2) \to \mathrm{Spec}(R_1) = id$, then $f^*([\mathfrak{p}_2]) = [\mathfrak{p}_1]$ and $f^*([\mathfrak{q}_2]) = [\mathfrak{q}_1]$, which implies that $f^{-1}(\mathfrak{p}_2) = \mathfrak{p}_1$ and $f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$. However, $\mathfrak{p}_2 \subset \mathfrak{q}_2$ contains so for all $x \in \mathfrak{p}_1 = f^{-1}(\mathfrak{p}_2)$, $f(x) \in \mathfrak{p}_2 \subset \mathfrak{q}_2$. It follows that $x \in f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$, and hence, $\mathfrak{p}_1 \subset \mathfrak{q}_1$. But this is a contradiction with the fact that there are no inclusion relations between $\mathfrak{p}_1$ and $\mathfrak{q}_1$. Therefore, such a function f cannot exist.

Let's now show that the only other possible case cannot happen. Suppose there is a homomorphism $f : R_2 \to R_1$ such that $f^* : \mathrm{Spec}(R_1) \to \mathrm{Spec}(R_2) = id$ is continuous, then $f^*([\mathfrak{p}_1]) = [\mathfrak{p}_2]$ and $f^*([\mathfrak{q}_1]) = [\mathfrak{q}_2]$. Since there are no inclusion relations between $\mathfrak{p}_1$ and $\mathfrak{q}_1$, then there is an element $s \in \mathfrak{p}_1 \setminus \mathfrak{q}_1 \subset \mathfrak{p}_1 = f^{-1}(\mathfrak{p}_2)$. It follows that $s = f(x)$ where $x \in \mathfrak{p}_2 \subset \mathfrak{q}_2 = f^{-1}(\mathfrak{q}_1)$, and hence, $s = f(x) \in \mathfrak{q}_1$ which contradicts the definition of s.

In conclusion, the identity map from $(X, \mathcal{T}_i) = \mathrm{Spec}(R_i)$ to $(X, \mathcal{T}_j) = \mathrm{Spec}(R_j)$ for $i \neq j$ is never induced by a homomorphism from R_j to R_i.

Exercise 29:

(a) Let X be the real numbers with the usual topology and let $\underline{\mathbb{Z}}$ the sheaf of locally constant integer-valued functions on X. Calculate the ring of germs at every point of $(X, \underline{\mathbb{Z}})$ and note that X is not a locally ringed space.

(b) Do the same when $X = \{0\} \cup \{2^{-n} : n = 1, 2, 3, \dots\}$ (with the topology induced from $\mathbb{R}$).

Solution :

(a) Let x be a real number, then $\mathcal{O}_{X,x}$ is the set of pairs (U, f), where U is an open set containing x and f is an integer-valued locally-constant function on U, with the equivalence relation that associates two pairs (U, f) and (V, g) when $f = g$ on an open subset of $U \cap V$. Let's show that $\mathcal{O}_{X,x} = \mathbb{Z}$.

First, let I be an open interval containing x, then for all $n \in \mathbb{Z}$, the element (I, n) is clearly in $\mathcal{O}_{X,x}$. Moreover, for $m \neq n$, (I, n) is not isomorphic to (I, m) since the two constant functions disagree everywhere on I (and hence, on any open subset of the intersection).

Now, let (U, f) be an arbitrary element of the ring of germs at x, then there is an interval $J \subset U$ around x on which the function f must be constant. If we let n be the value of f on J, then it is clear that (U, f) is equivalent to (I, n) since $f \equiv n$ on $I \cap J$. Therefore, we have just proved that the ring of germs at x is precisely the ring of integers.

Since the ring of integers is not a local ring ((2) and (3) are distinct maximal ideals for example), then X is not a locally ringed space.

(b) Notice that the locally-constant functions on X are precisely the functions which are constant for all $x \leq 2^{-n_0}$ in X for some $n_0 \geq 1$, and completely arbitrary for $x > 2^{-n_0}$. As for part (a), let's show that for all $x \in X$, the ring of germs at x is the ring of integers.

First, for $x \in X \setminus \{0\}$, we have that $(\{x\}, n)$ is an element of the ring of germs for all integer n (notice that the singleton $\{x\}$ is open). Moreover, it is clear that the two pairs $(\{x\}, n)$ and $(\{x\}, m)$ are not equivalent when $n \neq m$. Next, let (U, f) be an arbitrary element, then if we let $n = f(x)$, then this pair is equivalent to the pair $(\{x\}, n)$. It follows that the ring of germs is precisely the ring of integers.

Next, consider the case $x = 0$. First, as before, we have that for all integers n, the incongruent pairs (X, n) are all in the ring of germs. Now, for any pair (U, f) in the ring, we know that f must be constant on a neighborhood V of 0. If we let $n = f(0)$, then the pair is equivalent to (X, n). Therefore, the ring of germs is equal to the ring of integers.

Therefore, X is not a locally ringed space.