MATH 570 Notes : Higher Algebra 1

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These notes are based on lectures given by Professor Eyal Goren at McGill University in Fall 2025. The subject of these lectures is **TODO**. As a disclaimer, it is more than possible that I made some mistakes. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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1 Categories and Functors

1.1 Definitions

Definition. A category $\underline{C}$ is

1. a collection of objects $\mathrm{ob}(\underline{C})$,
2. for any $A, B \in \mathrm{ob}(\underline{C})$ a set $\mathrm{Mor}_{\underline{C}}(A, B)$ with an associative composition law,
3. For all $A \in \mathrm{ob}(\underline{C})$, there is a morphism 1_A such that for all $f \in \mathrm{Mor}_{\mathrm{ob}(\underline{C})}(A, B)$, we have $f \circ 1_A = f$ and $1_B \circ f = f$.
4.

Definition. A morphism $f \in \mathrm{Mor}(A, B)$ is an isomorphism if there exists a $g \in Mor(A, B)$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

1.2 Initial and Final Objects

Definition. An object $A \in \mathrm{ob}(\underline{C})$ is initial (final) if for any object $B \in \mathrm{ob}(\underline{C})$, $\mathrm{Mor}(A, B)$ has only one element ($\mathrm{Mor}(B, A)$ has only one element). A is a zero object if it's both initial and final.

Proposition 1.2.1. *An initial object (if it exists) is unique up to a unique isomorphism (similar, final).*

Proof. Suppose $A, A' \in \mathrm{Ob}(\underline{C})$ are initial, let $f \in \mathrm{Mor}(A, A')$ and $g \in \mathrm{Mor}(A', A)$ be the unique such morphisms, then $g \circ f \in \mathrm{Mor}(A, A) = \{1_A\}$ and so $g \circ f = 1_A$. Similarly, $f \circ g = 1_{A'}$. It follows that A and A' are isomorphic. ∎

Example:

- Sets: $\mathrm{Ob}(\underline{C})$ are sets, $\mathrm{Mor}(A, B)$ are functions from A to B. The empty set is the unique initial object. The singletons are precisely the final objects. It follows that zero objects don't exist.
- ${}_R\mathrm{Mor}$ (Mor_R): R is a ring (always with 1 and is associative), $\mathrm{Ob}({}_R\mathrm{Mor})$ are left R-modules M, functions are R-modules homomorphisms $f : M \to N$. The zero-module $\{0\}$ is the unique initial object and also the unique final object. Hence, it is a zero object in that category.
- Gps (AbGps): The objects are (abelian) groups, the morphisms are (abelian) group homomorphisms, as for the previous example, there is a unique zero-object: the group $\{1\}$.

1.3 Functors

Definition (Covariant and Contravariant Functors)**.** A covariant (contravariant) functor $F : \underline{C} \to \underline{D}$ is the following:

1. For any object $A \in \mathrm{Ob}(\underline{C})$, $FA \in \mathrm{Ob}(\underline{D})$.

2. For any morphism $f\mathrm{Mor}_{\underline{C}}(A, B)$, we have a morphism $Ff \in \mathrm{Mor}_{\underline{D}}(FA, FB)$ ($Ff \in \mathrm{Mor}_{\underline{D}}(FB, FA)$) such that $F1_A = 1_{FA}$ and $F(g \circ f) = Fg \circ Ff$ ($F(g \circ f) = Ff \circ Fg$).

Definition (Faithful)**.** We say that F is faithful if whenever $Ff = Fg$ for some $f, g \in \mathrm{Mor}(A, B)$, then $f = g$.

Definition (Full)**.** We say that F is full if given any $h \in \mathrm{Mor}(FA, FB)$, there exists a $f \in \mathrm{Mor}(A, B)$ such that $Ff = h$.

Definition (Essentially Surjective)**.** We say that F is full if any $C \in \mathrm{Ob}(\underline{D})$ is isomorphic to FA for some $A \in \mathrm{Ob}(\underline{C})$.

Example:

- Forgetful functors: for example, the functor $F : \mathrm{Gps} \to \mathrm{Sets}$ defined by $FA = A$, $Ff = f$ forgets the group structure of the objects. This functor is not full but it is faithful. With some logic, we can prove that it is also essentially surjective.

- Consider the functor $F : \mathrm{Rings} \to \mathrm{Gps}$ such that $FR = R^*$ and $Ff = f|_{R^*}$. Is it faithful, full, essentially surjective ?

- If k is a field, then ${}_k\mathrm{Mod}$ is the same as the category of k-vector spaces where the morphisms are the k-linear maps. From this categroy, we can consider the contravariant functor $F :_k \mathrm{VSp} \to_k \mathrm{VSp}$ that sends V to its dual and homomorphisms to their transpose.

- The category $\mathrm{Rep}(G)$, where G is a fixed finite group, is the category of finite linear complex representations of G.

- We can define the functor $F : \mathrm{FinGps} \to \mathrm{Rings}$ by $FG = k[G]$ where k is a field.

Next time: we'll see that the category of representations of G is equivalent to the category ${}_{\mathbb{C}[G]}\mathrm{Mod}$.