Motivating L-functions

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Preface

This document is my report for my Summer 2025 NSERC under the supervision of Professor Henri Darmon at McGill University. My goal with this document is to motivate the concept of L-functions and highlight some of its applications to show the deep connections between L-functions and Number Theory. To motivate the theory, I will first deeply focus on the historical motivations, and then present some more modern results such as the connections with elliptic curves and modular forms, or more generally, the Langlands program.

The first chapter will focus on Leonhard Euler. I believe that the historical motivations for the theory of L-functions can be traced back to the work of Euler more than any other mathematician. I will first present his summation formula which he used extensively. After that, I will present his solution(s) to the Basel Problem but also the link he made with some particular series and the prime numbers. I will then finish the chapter with important results that involve the Bernoulli numbers and what will become the functional equations of some L-functions. I will add more informations about the other chapters once I will finish writing them.

Beside the structure of this report, there are three things I find very important to keep in mind while reading this document:

- My goal is to preserve the authenticity of the results that will be presented. Hence, I will use the original notation and the original terms as much as possible. For example, in the first chapter, I will avoid using the symbol $\sum a_n$ for sums and use $a_1 + a_2 + a_3 + \dots$ instead. Similarly, I will not talk about *real numbers* or *sets* as this notion didn't exist until the very end of the 19th century. Moreover, I will also state and prove the theorems that are presented as they were stated and proved. Thus, the proofs that I will present will always be the original proofs with very little modifications.

- The Bernoulli numbers will be mentionned for the first time in section 1.4 and will probably be used a lot in the rest of the document. It is important to keep in mind that in modern number theory books, the Bernoulli number B_1 is sometimes defined as $+1/2$ and sometimes defined as $-1/2$. In this document, I will assume that $B_1 = +1/2$ for very good reasons that I will expose in section 1.4.

- I made the decision of adding exercises at the end of each section because I strongly believe that it really helps understanding the content of the section. There are two types of exercises: the first type will be exercises outlining some results from papers that are presented in the section but that I didn't get time to present or prove, the second type will be exercises that outline a rigorous or alternative proof of a result that appears in the section. Some important results have non-rigorous or false proofs, hence, the second type of exercise will help using these results in the latter chapters without having doubts about their validity.

The prerequisites for this document would be some familiarity with Real and Complex Analysis, Abstract Algebra (especially Group Theory) and Elementary Number Theory (properties of prime numbers, congruences, ...). Besides these prerequisites, the document contains appendices that should make it self-contained.

It is more than possible that I made some mistakes. Feel free to correct me or ask me anything about the content of this document at the following email address : samy.lahloukamal@mail.mcgill.ca.

Contents

Chapter 1

Euler's Marvelous Series

1.1 Euler's Summation Formula

The Calculus developed by Sir Isaac Newton (1643 - 1727) and Gottfried Wilhelm Leibniz (1646 - 1716) at the end of the 17th century made the subject of infinite series very popular and useful in mathematics. Before this era, the concept of infinite sums was already encountered in different places in the world. For example, in a treatise written by Archimedes of Syracuse (287 BC - 212 BC) in the 3rd century BC called *Quadrature of the Parabola* [19], there is a visual proof that

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

using embedded squares. Next, the decimal representation of numbers, which was introduced in Europe during the 13th century, is simply an application of infinite sums in disguise. For example, the fact that $1/3$ has the decimal expansion $0.333333\ldots$ can be reinterpreted as saying that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \ldots$$

In the 14th century, the French mathematician Nicole Oresme (1320 - 1382) showed that the infinite sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

has an infinite value in the sense that it exceeds any finite quantity. This sum is now called the Harmonic Series. In the same way as Archimedes, Nicole Oresme used a geometric argument to find the following results:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2$$
$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \ldots = 2$$

More informations about the work of Nicole Oresme can be found in the article *Mathematical Concepts and Proofs from Nicole Oresme* [1].

Later, between the 14th and 15th century, members of the Kerala school of astronomy and mathematics, in India, found representations of the sine and the cosine of an angle as an infinite sum. They also found an infinite sum representation of the arctangent of a given quantity. In modern notation, these results can be written as follows:

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \tag{1.1.1}$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \tag{1.1.2}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \tag{1.1.3}$$

These three equations are sometimes called Madhava Series in reference to the Indian mathematician Madhava of Sangamagrama (1340 - 1425), a member of the Kerala school to which these results are attributed. Moreover, by plugging-in $x = 1$ in equation (1.1.3), the following equation is obtained:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots \tag{1.1.4}$$

This equation was later rediscovered independently by Leibniz which is the reason why people usually call equation (1.1.4) the Leibniz Series. More informations about this result can be found in the article *The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha* [23]. Compared to the previously discussed results, this one has a particular importance since it involves the constant π even though the infinite sum on the right hand side doesn't seem more complicated than the ones discussed above. This result is a first hint that some seemingly simple infinite sums can have unexpected behaviors.

Finally, in 1650, the Italian mathematician Pietro Mengoli (1626 - 1686) publishes his book *Novœ quadraturœ arithmeticœ, seu de additione fractionum* [21] in which he proves various results about infinite sums. For example, he proves that the Harmonic Series is infinite, and also finds the values of

$$\frac{1}{1 \cdot (1+r)} + \frac{1}{2 \cdot (2+r)} + \frac{1}{3 \cdot (3+r)} + \frac{1}{4 \cdot (4+r)} + \dots$$

where r is any integer between 1 and 10. However, he was unable to find the value to which the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

converges to (which is the case $r = 0$ of the previous sums). Thus, we see through these examples that before the works of Newton and Leibniz, infinite sums already made their appearance in various contexts in time.

Even though some specific examples of infinite sums were already studied in the previous centuries, they really became central in mathematics when the tools of calculus became available. For example, with his method of fluxions, Newton rediscovered the Madhava Series for the sine and cosine functions and was able to solve differential equations. The tools of calculus gave the perfect framework for what is begining to be called series or infinite series.

Later in that period, in 1689, the Swiss mathematician Jakob Bernoulli (1655 - 1705) wrote his *Tractatus de seriebus infinitis*, a treatise on infinite series in which he discusses the limiting values of various series such as geometric series, telescoping series, the harmonic series and other types of series. For example, he derives the following formula for geometric series:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}, \qquad -1 < r < 1. \tag{1.1.5}$$

He also studied some more specific examples such as

$$\begin{aligned} &1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \dots \\ &= \frac{2}{1(1+1)} + \frac{2}{2(2+1)} + \frac{2}{3(3+1)} + \frac{2}{4(4+1)} + \cdots = 2 \end{aligned} \tag{1.1.6}$$

using the fact that it is a telescoping series. As his predecessors, he gave a new proof of the divergence of the Harmonic Series. Finally, when considering the series of the reciprocals of the squares

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots,$$

he was able to show that it must have a finite limiting value, i.e., that the series converges, by using the inequality

$$\frac{1}{n^2} \leq \frac{2}{n(n+1)}$$

and combining it with equation (1.1.6) to obtain

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \leq 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots = 2 < \infty.$$

However, he was unable to find the precise value to which the series converges to and wrote in his *Tractatus* that "great will be our gratitude" if anyone finds and communicates this limiting value. This became known as the Basel Problem (since the mathematician wrote from Basel in Switzerland) and it remained unsolved for decades.

Notice that this series converges very slowly since the individual terms don't approach zero fast enough. This makes the problem even harder since a good strategy for finding the value of a series is to find the sum of the first 20 terms (for example) and guess the limit of the series from this approximation. However, with the series of the reciprocals of the squares, taking the sum of the first 200 terms gives an approximation which is only correct for one decimal.

The first mathematician to make some significant progress on the Basel Problem is a young Swiss mathematician who would soon become the most prolific mathematician of all time. This mathematician is obviously the great Leonhard Euler.

Enter Euler

Born in the town of Basel on the 15th of April in 1707, Leonhard Euler is the son of the pastor Paul Euler. From a young age, Leonhard received schooling in mathematics from different persons. First, from his father who had taken courses in mathematics from Jakob Bernoulli at the University of Basel. Then, when attending the University of Basel in 1720 at the age of 13, from Johann Bernoulli (1667 - 1748), Jakob Bernoulli's younger brother. Johann Bernoulli had a great influence on Euler for three reasons, he was one of the top mathematician of his time, they met every saturday afternoon to discuss about mathematics, and because he helped Euler get his father's consent to become a mathematician instead of a pastor. In 1727, Euler joined Daniel Bernoulli, Johann Bernoulli's son, to take a position in the department of mathematics at the Imperial Russian Academy of Sciences in Saint Petersburg. It is here that most of the papers presented in this chapter will be written.

Euler's first contribution to the Basel Problem can be found in his article *De summatione innumerabilium progressionum* [7], written in 1731 and published in 1738. In this article, using many integral tricks and algebraic manipulations, Euler was able to obtain the formula

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\left(\frac{1}{2^0\cdot 1^2}+\frac{1}{2^1\cdot 2^2}+\frac{1}{2^2\cdot 3^2}+\frac{1}{2^3\cdot 4^2}+\dots\right)+(\ln 2)^2 \tag{1.1.7}$$

which relates the series of the reciprocals of the squares to a series which converges much faster to which is added a constant term that can be computed very precisely. With only 20 terms of the right hand side series, Euler is able to obtain the following approximation:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=1.644934$$

He notices himself that such an approximation can only be obtained by adding more than a thousand terms of the series on the left hand side. Using computers, it turns out that such an approximation would require to sum more than 15 millions terms of the original series, which makes this result already very remarkable. Again, this shows how slowly the original series converges.

Three years later, in the article *Methodus universalis serierum convergentium summas quam proxime inveniendi* [11] written in June 1735 and published in 1741, Euler found another way of approximating the sum of the reciprocals of the squares. He presented a general geometric method for approximating series using integrals. At the end of the paper, he applied his method to the series of the reciprocals of the square.

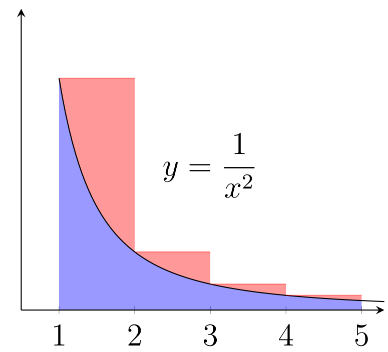

Figure 1.1: Visual interpretation of
$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

He considered Figure 1.1 in which the blue and red area represents the value of the sum of the reciprocals of the squares. We can see that this sum is greater than the blue area under the curve $y = \frac{1}{x^2}$ between $x = 1$ and $x = \infty$. But Euler's goal was to approximate the red area above the curve. Using some calculus and geometry, he was able to derive the following approximation of the total shaded area:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = 1.644920$$

which is only true for the first four decimals. It seems like this method is worst than the previous one since it only gives an approximation true for the first four decimals. However, Euler was on track to develop a new very powerful method.

The Summation Formula

As for the previous method, Euler's goal was to approximate the difference between a series and the integral of the general term of the series. In his paper *Methodus generalis summandi progressiones* [8], written in 1732 and published in 1738, Euler mentions without proof such a formula that links a series to its corresponding integral. He then wrote another article called *Inventio summae cuiusque seriei ex dato termino generali* [10], written in October 1735 and published in 1741, to go over the proof of his formula and apply it to approximate some series. Let's dive into this last paper to understand his formula.

The paper starts with an important preliminary result. Take a function y which can be expanded as follows:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots,$$

then for any α and by the Binomial Theorem:

$$\begin{aligned} y(x+\alpha) = \; & a_0 \\ & + (a_1 x \quad + a_1\alpha) \\ & + \left(a_2x^2 + 2a_2x\alpha \quad + a_2\alpha^2\right) \\ & + \left(a_3x^3 + 3a_3x^2\alpha + 3a_3x\alpha^2 \quad + a_3\alpha^3\right) \\ & + \left(a_4x^4 + 4a_4x^3\alpha + 6a_4x^2\alpha^2 + 4a_4x\alpha^3 + a_4\alpha^4\right) \\ & + \dots \end{aligned}$$

Now, if we sum the right hand sum column by column, we obtain

$$\begin{aligned} y(x+\alpha) = & (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots) \\ & + \frac{\alpha}{1}(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) \\ & + \frac{\alpha^2}{1\cdot 2}(2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \dots) \\ & + \frac{\alpha^3}{1\cdot 2\cdot 3}(3\cdot 2a_3 + 4\cdot 3\cdot 2a_4x + \dots) \\ & + \dots \end{aligned}$$

Finally, we can recognize that the expressions inside the prentheses are simply the successive derivatives of y. It follows that

$$y(x+\alpha) = y(x) + \frac{\alpha dy}{1\cdot dx} + \frac{\alpha^2 d^2y}{1\cdot 2\cdot dx^2} + \frac{\alpha^3 d^3y}{1\cdot 2\cdot 3\cdot dx^3} + \frac{\alpha^4 d^4y}{1\cdot 2\cdot 3\cdot 4\cdot dx^4} + \dots \quad (1.1.8)$$

Euler attributes this result to the mathematician Brook Taylor (1685 - 1731) who is now known for his work on Taylor Series. After obtaining this formula, Euler introduced the main problem of the article. Suppose we are given a function $f(x)$ and we define the new function

$$S(x) = f(1) + f(2) + f(3) + \cdots + f(x), \quad (1.1.9)$$

How can we find a simpler expression of the function $S(x)$? For example, if $f(x) = x$, then $S(x)$ is simply $x(x+1)/2$. First, he noticed that by definition of $S(x)$, we have

$$S(x-1) = S(x) - f(x)$$

and so

$$f(x) = S(x) - S(x-1). \quad (1.1.10)$$

Moreover, using equation (1.1.8) with $y = S$ and $\alpha = -1$, he obtained

$$S(x-1) = S(x) - \frac{dS}{1\cdot dx} + \frac{d^2S}{1\cdot 2\cdot dx^2} - \frac{d^3S}{1\cdot 2\cdot 3\cdot dx^3} + \frac{d^4S}{1\cdot 2\cdot 3\cdot 4\cdot dx^4} + \dots$$

Even if $S(x)$ was defined for positive integers only, Euler assumed that it could be treated as an infinitely differentiable function. This can be explained by the fact that our goal is to find a nice function that interpolates the partial sums of $f(x)$. Hence, we are not defining $S(x)$ as the partial sums of $f(x)$, instead, we are trying to deduce the formula of such a function $S(x)$ if it exists. After this last equation, he substituted it into equation (1.1.10) to get

$$f(x) = \frac{dS}{1 \cdot dx} - \frac{d^2S}{1 \cdot 2 \cdot dx^2} + \frac{d^3S}{1 \cdot 2 \cdot 3 \cdot dx^3} - \frac{d^4S}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} + \dots \tag{1.1.11}$$

However, Euler notices that the goal is to find a new expression of $S(x)$, not of $f(x)$. The last equation expresses $f(x)$ in terms of $S'(x)$ and its derivative. Hence, his next step is to invert this equation, i.e., to write $S'(x)$ in terms of $f(x)$ and its derivatives. To do so, he assumes that

$$\frac{dS}{dx} = \alpha_0 f(x) + \alpha_1 \frac{df}{dx} + \alpha_2 \frac{d^2f}{dx^2} + \alpha_3 \frac{d^3f}{dx^3} + \alpha_4 \frac{d^4f}{dx^4} + \dots \tag{1.1.12}$$

and so now the goal is to find the coefficients $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots$. To determine these coefficients, Euler differentiated both sides of equation (1.1.12) to obtain:

$$\begin{aligned}
\frac{dS}{dx} &= \alpha_0 f(x) + \alpha_1 \frac{df}{dx} + \alpha_2 \frac{d^2f}{dx^2} + \alpha_3 \frac{d^3f}{dx^3} + \alpha_4 \frac{d^4f}{dx^4} + \dots \\
\frac{d^2S}{dx^2} &= \alpha_0 \frac{df}{dx} + \alpha_1 \frac{d^2f}{dx^2} + \alpha_2 \frac{d^3f}{dx^3} + \alpha_3 \frac{d^4f}{dx^4} + \dots \\
\frac{d^3S}{dx^3} &= \alpha_0 \frac{d^2f}{dx^2} + \alpha_1 \frac{d^3f}{dx^3} + \alpha_2 \frac{d^4f}{dx^4} + \dots \\
\frac{d^4S}{dx^4} &= \alpha_0 \frac{d^3f}{dx^3} + \alpha_1 \frac{d^4f}{dx^4} + \dots \\
\frac{d^5S}{dx^5} &= \alpha_0 \frac{d^4f}{dx^4} + \dots
\end{aligned}$$

He then substituted these equations into equation (1.1.11) to get:

$$\begin{aligned}
f(x) = \frac{1}{1} &\left(\alpha_0 f(x) + \alpha_1 \frac{df}{dx} + \alpha_2 \frac{d^2f}{dx^2} + \alpha_3 \frac{d^3f}{dx^3} + \alpha_4 \frac{d^4f}{dx^4} + \dots \right) \\
- \frac{1}{2} &\left(\alpha_0 \frac{df}{dx} + \alpha_1 \frac{d^2f}{dx^2} + \alpha_2 \frac{d^3f}{dx^3} + \alpha_3 \frac{d^4f}{dx^4} + \dots \right) \\
+ \frac{1}{6} &\left(\alpha_0 \frac{d^2f}{dx^2} + \alpha_1 \frac{d^3f}{dx^3} + \alpha_2 \frac{d^4f}{dx^4} + \dots \right) \\
- \frac{1}{24} &\left(\alpha_0 \frac{d^3f}{dx^3} + \alpha_1 \frac{d^4f}{dx^4} + \dots \right) \\
+ \frac{1}{120} &\left(\alpha_0 \frac{d^4f}{dx^4} + \dots \right) \\
- \dots &
\end{aligned}$$

Next, by expanding the right hand side and grouping the terms by their respective derivatives of $f(x)$, Euler obtained the following equation:

$$\begin{aligned} f(x) &= \alpha_0 f(x) \\ &+ \left(\alpha_1 - \frac{\alpha_0}{2}\right)\frac{df}{dx} \\ &+ \left(\alpha_2 - \frac{\alpha_1}{2} + \frac{\alpha_0}{6}\right)\frac{d^2f}{dx^2} \\ &+ \left(\alpha_3 - \frac{\alpha_2}{2} + \frac{\alpha_1}{6} - \frac{\alpha_0}{24}\right)\frac{d^3f}{dx^3} \\ &+ \left(\alpha_4 - \frac{\alpha_3}{2} + \frac{\alpha_2}{6} - \frac{\alpha_1}{24} + \frac{\alpha_0}{120}\right)\frac{d^4f}{dx^4} \\ &+ \dots \end{aligned}$$

Finally, Euler deduced from this equation that α_0 must be 1 and that all the other coefficients in front of the derivatives of $f'(x)$ must be zero. It follows that

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= \frac{\alpha_0}{2} \\ \alpha_2 &= \frac{\alpha_1}{2} - \frac{\alpha_0}{6} \\ \alpha_3 &= \frac{\alpha_2}{2} - \frac{\alpha_1}{6} + \frac{\alpha_0}{24} \\ \alpha_4 &= \frac{\alpha_3}{2} - \frac{\alpha_2}{6} + \frac{\alpha_1}{24} - \frac{\alpha_0}{120} \\ & etc \dots \end{aligned}$$

which implies that each α_n can be determined by its predecessors. Euler used these formulas to compute the first α_n's:

$$\alpha_0 = 1 \qquad \alpha_1 = \frac{1}{2} \qquad \alpha_2 = \frac{1}{12} \qquad \alpha_3 = 0 \qquad \alpha_4 = -\frac{1}{720} \qquad \alpha_5 = 0 \qquad \alpha_6 = \frac{1}{30240}$$

from which he finally obtained

$$S'(x) = f(x) + \frac{df}{2 \cdot dx} + \frac{d^2f}{12 \cdot dx^2} - \frac{d^4f}{720 \cdot dx^4} + \frac{d^6f}{30240 \cdot dx^6} - \dots \tag{1.1.13}$$

Therefore, the last step is simply to integrate this equation to obtain a new expression for $S(x)$:

$$\boxed{S(x) = \int f(x)dx + \frac{f(x)}{2} + \frac{df}{12 \cdot dx} - \frac{d^3f}{720 \cdot dx^3} + \frac{d^5f}{30240 \cdot dx^5} - \dots + C} \tag{1.1.14}$$

where C is the constant of integration that makes $S(0) = 0$, and where the coefficient in front of the nth derivative of $f(x)$ is α_{n+1}. From the condition that $S(0) = 0$ we have that our integration constant is

$$C = -\left[\int f(x)dx + \frac{f(x)}{2} + \frac{df}{12 \cdot dx} - \frac{d^3 f}{720 \cdot dx^3} + \frac{d^5 f}{30240 \cdot dx^5} - \dots\right]_{x=0} \tag{1.1.15}$$

since adding this constant on the right hand side of equation (1.1.14) and evaluating the expression on the right at $x = 0$ would give $S(0) = 0$.

This is the now famous Euler Summation Formula. There is a lot to say about this derivation. First, we can clearly see how low were the standards in terms of rigor at that time. Such a proof would never be accepted today. However, since Euler, other proofs were provided for this formula so we can rest our mind and be sure of its validity. The usefulness of this formula may not be obvious for the moment since it seems like Euler have made the problem harder. Our first expression for $S(x)$ was a simple finite sum involving no derivatives. This new expression of $S(x)$ is an infinite sum involving derivatives of $f(x)$ of arbitrarily high orders. At least it is clear now that $S(x)$ is defined over way more numbers than just the positive integers, and that it is differentiable. But to really understand how powerful this formula is, let's see it in action.

The Formula in Action

For his first example, Euler took the function $f(x) = x$. Notice that with this function, the constant C is simply equal to $-1/12$ since the only non-zero term in the formula for C is the one involving the first derivative of $f(x)$. Thus, we get

$$S(x) = \int x dx + \frac{x}{2} + \frac{1}{12} - 0 + 0 - \dots + C = \frac{x(x+1)}{2}$$

which is indeed the correct formula. Similarly, with $f(x) = x^2$, he obtained

$$S(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} = \frac{x(x+1)(2x+1)}{6}$$

which is, again, the correct formula. Since taking the successive derivatives of the function $f(x) = x^m$ is an easy task, we can see how Euler's formula can be used to recover all the formulas for the sum of the first n powers of m. And this is exactly what Euler did, from his formula, he deduced the formula for the specific case $f(x) = x^m$ and from that, he deduced the equivalent of the two previous equations (which were the case $m = 1$ and $m = 2$) for the cases $m = 3, 4, 5, ..., 15, 16$. In Figure 1.2, the symbole $\int x^m$ denotes the sum $1^m + 2^m + 3^m + \dots + x^m$.

After these examples, Euler finished his paper by returning on the original problem of approximating the sum of the reciprocals of the squares. Euler let $f(x) = \frac{1}{x^2}$ and noticed that he could not apply the formula directly since computing the constant C would require dividing by 0. Hence, Euler split the sum as follows:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \left(1 + \frac{1}{4} + \dots + \frac{1}{100}\right) + \left(\frac{1}{121} + \frac{1}{144} + \frac{1}{169} + \dots\right)$$

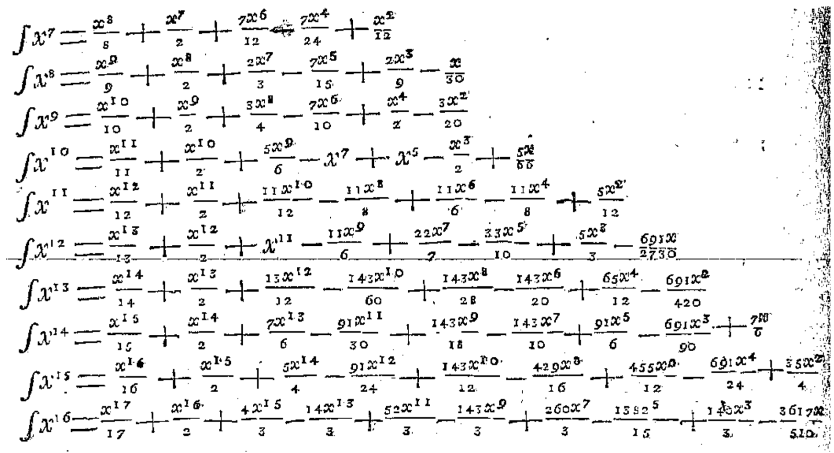

Figure 1.2: Extract from Euler's paper

He computed by hand the first ten terms of the series to get

$$1 + \frac{1}{4} + \cdots + \frac{1}{81} + \frac{1}{100} = 1.549767731166540 \tag{1.1.16}$$

and then used his summation formula to approximate the remaining terms:

$$S(x) = \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \cdots + \frac{1}{x^2}. \tag{1.1.17}$$

In the previous equation, the initial condition now becomes $S(10) = 0$ instead of $S(0) = 0$ because the first term has index $x = 11$. Therefore, the constant term is

$$C = -\int f(x)dx\Big|_{x=10} - \frac{f(x)}{2}\Big|_{x=10} - \frac{df}{12 \cdot dx}\Big|_{x=10} + \frac{d^3 f}{720 \cdot dx^3}\Big|_{x=10} - \frac{d^5 f}{30240 \cdot dx^5}\Big|_{x=10} + \dots \tag{1.1.18}$$

Euler then computed the first successive derivatives of $f(x)$ to obtain

$$\int f(x)dx = -\frac{1}{x} \qquad \frac{df}{dx} = -\frac{2}{x^3} \qquad \frac{d^3 f}{dx^3} = -\frac{2 \cdot 3 \cdot 4}{x^5} \qquad \frac{d^5 f}{dx^5} = -\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{x^7}$$

and so plugging this into equation (1.1.18) gives

$$C = \frac{1}{10} - \frac{1}{200} + \frac{1}{6000} - \frac{1}{3000000} + \frac{1}{420000000} - \frac{1}{30000000000} + \dots$$

which converges really fast, and hence, can be approximated really well. Therefore, using his summation formula, he obtained

$$\frac{1}{11^2} + \frac{1}{12^2} + \cdots + \frac{1}{x^2} = \left(-\frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \dots \right) + C \tag{1.1.19}$$

He then combined equations (1.1.16) and (1.1.19) to get

$$1+\frac{1}{4}+\cdots+\frac{1}{x^2}=1.549767731166540+C+\left(-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\dots\right)$$

Finally, letting x go to infinity gives

$$\boxed{1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=1.64493406684822643647} \tag{1.1.20}$$

which is an apprixomation with twenty correct decimals! It was mentioned earlier in this section that to obtain an approximation correct to six decimals, it would require taking the sum of more than the first 15 million terms of the original series. Similarly, to get an approximation with 20 correct decimals, it is required to sum more than the first 10^{20} terms of the original series. The last two examples he provided in his paper are similar approximations of the sum of the inverse of the cubes and the sum of the inverse of the biquadrates:

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\ldots=1.202056903159594 \tag{1.1.21}$$

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots=1.0823232337110824 \tag{1.1.22}$$

With this single summation formula and its countless applications, Euler clearly stands as one of the most ingenious mathematician of his time. However, you may guess that this is only the begining. Euler went further than that. It turns out that only two months after writting his paper *Inventio summae...* which we just presented, Euler solved the Basel Problem. Not only he recognized the exact value of the number he approximated in equation (1.1.20), but he also found a way to prove it. The next section will be focused on Euler's paper which contains his solution to the Basel Problem.

The Euler Summation Formula which was introduced in this section will also appear in the next sections. Euler used it a lot throughout his mathematical career because it turns out that there is still a lot to say about this formula, especially concerning the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots$ which will have a great importance later.

Exercises

Exercise 1.1.1. In this exercise, the sequence $H_n=\sum_{k=1}^{n}\frac{1}{k}$ denotes the sequence of partial sums of the Harmonic Series. In the article *De summatione innumerabilium progressionum* written in 1731, Euler interpolated the sequence H_n using the function

$$H(x)=\int_0^1 \frac{1-t^x}{1-t}dt.$$

(a) Prove that $H(n)=H_n$ for all $n\geq 1$.

(b) Find the value of $H(1/2)$.

(c) Prove that $H(x+1) = H(x) + \frac{1}{x+1}$ for all $x > 0$.

(d) Deduce a general formula for $H(n + \frac{1}{2})$.

Exercise 1.1.2. In the same paper as the one mentioned in the previous exercise, Euler proves that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \left(\frac{1}{2^0 \cdot 1^2} + \frac{1}{2^1 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^3 \cdot 4^2} + \dots \right) + (\ln 2)^2$$

as a corollary of a more general method. To make Euler's proof easier to understand, this exercise outlines Euler's argument applied to the specific case of the sum of the reciprocals of the squares. This version of Euler's proof comes from the article *Euler and the Zeta Function* written by Raymond Ayoub.

(a) Show that
$$-\frac{\ln(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \dots$$

(b) Deduce the following new expression of the sum of the reciprocals of the squares
$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} + \dots = -\int_0^1 \frac{\ln(1-x)}{x} dx.$$

(c) Split the integral on right hand side at $x = \frac{1}{2}$ and define
$$I_1 = -\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx \qquad I_2 = -\int_{\frac{1}{2}}^1 \frac{\ln(1-x)}{x} dx.$$

so that the sum of the reciprocals of the squares is equal to $I_1 + I_2$. Use part (a) to find
$$I_1 = \frac{1}{2^1 \cdot 1^2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{2^3 \cdot 3^2} + \frac{1}{2^4 \cdot 4^2} + \dots$$

(d) In the integral I_2, make the change of variable $u = 1 - x$ and expand the resulting denominator in a power series. From this, make an integration by parts term-by-term to obtain
$$I_2 = I_1 + (\ln 2)^2.$$

(e) Deduce the desired formula from part (c) and part (d).

Exercise 1.1.3. Use Taylor's Formula (equation (1.1.8)) to derive the following identities:

(a) $e^{a+b} = e^a e^b$

(b) $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

(c) $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Exercise 1.1.4. Use Euler's Summation Formula (equation (1.1.14)) to find the formula for the sum of the first n cubes. By induction, prove that your formula is correct for all positive integers.

Exercise 1.1.5. Use Euler's Summation Formula (equation (1.1.14)) to find the general formula for the sum of the n first powers of m.

Exercise 1.1.6. In the 1735 paper *Inventio summae ...*, we saw how Euler applied his summation formula to various sums and series. One of the example he studied in his paper is the case $f(x) = \frac{1}{x}$. Read how Euler approximated the sum of the reciprocals of the squares and follows these exact same steps to conclude that for large enough values of x, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} = \ln(x) + c$$

where c is a constant. Find a way of approximating this constant c that follows his method for approximating the sum of the reciprocals of the squares.

1.2 Solving the Basel Problem

We ended last section with Euler's astonishing approximation of the sum of the reciprocals of the squares. However, an approximation is still an approximation, the Basel Problem is still far from being solved. Or is it ? As it was said at the end of the previous section, from his approximation, Euler was finally able to see the light and understand the true nature of the sum of the reciprocals of the squares. But what could he notice from these 20 mysterious decimals ?

At that time, the values of both the Leibniz Series and the Alternating Harmonic Series were known:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4} \tag{1.2.1}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2 \tag{1.2.2}$$

Thus, by squaring both sides of both equations could imply that the sum of the reciprocals of the squares can be written in terms of π^2, $(\ln 2)^2$, or both. Moreover, as we saw in the previous section, Euler already managed to write the sum of the reciprocals of the squares in terms of $(\ln 2)^2$ in equation (1.1.7). These results narrow our possible guesses a lot. This is probably what Euler had in mind because after approximating the desired sum up to 20 decimals, he quickly recognized this value to be... $\pi^2/6$! But this is not even the most surprising part, there is still an important question that remains unanswered: why would the sum of the reciprocals of the squares converge to $\pi^2/6$?

Euler came up and wrote his (first) proof of this result in December 1735 in the paper *De summis serierum reciprocarum* [4] published in 1740. The goal of this section is to understand Euler's proof in details. Let's begin.

The Proof

First, for a fixed y between -1 and 1, he considered the equation $y = \sin(s)$ where s is a variable. Using the series expansion of the sine function, this equation can be rewritten as

$$y = s - \frac{s^3}{1 \cdot 2 \cdot 3} + \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \ldots \tag{1.2.3}$$

Now, Euler noticed that if y is supposed to be non-zero, then dividing by both sides by y and putting all the terms on one side of the equation leads to the key equation

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1 \cdot 2 \cdot 3 \cdot y} - \frac{s^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot y} + \ldots \tag{1.2.4}$$

Next, he viewed this equation as an infinite-degree polynomial equation in s, and hence treated the right hand side of equation (1.2.4) as a regular polynomial. Recall that given a polynomial $P(s)$ of finite degree n with roots $a_1, ..., a_n$ and such that $P(0) = 1$, then we can write $P(s)$ as

$$P(s) = \left(1 - \frac{s}{a_1}\right)\left(1 - \frac{s}{a_2}\right)\dots\left(1 - \frac{s}{a_n}\right). \tag{1.2.5}$$

This comes from the fact that the expression on the right hand side of equation (1.2.5) is a polynomial of degree n that has the same roots as $P(s)$ and that has the same value at 0. Since both polynomial coincide on $n+1$ points, then they must be strictly equal. Thus, if we denote by A, B, C, D, ... the roots of equation (1.2.4), then Euler extended the previous principle to the infinite case to obtain the following important equation:

$$1 - \frac{s}{y} + \frac{s^3}{1\cdot 2\cdot 3\cdot y} - \frac{s^5}{1\cdots 5\cdot y} + \dots = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\dots \tag{1.2.6}$$

Finally, he defined A to be the least positive root of the equation $y = \sin(s)$, and observed that the roots of the equation are precisely the sequence A, $\pi - A$, $-\pi - A$, $2\pi + A$, $-2\pi + A$, $3\pi - A$, $-3\pi - A$, ... which implies that the previous equation can be rewritten as

$$\begin{aligned} 1 - \frac{s}{y} &+ \frac{s^3}{1\cdot 2\cdot 3\cdot y} - \frac{s^5}{1\cdots 5\cdot y} + \dots \\ &= \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{\pi - A}\right)\left(1 - \frac{s}{-\pi - A}\right)\left(1 - \frac{s}{2\pi + A}\right)\left(1 - \frac{s}{-2\pi + A}\right)\dots \end{aligned} \tag{1.2.7}$$

From this equation, Euler observed that by expanding the infinite product on the right hand side, the coefficient in front of s on the left hand side is equal to the sum of the terms of the sequence $-\frac{1}{A}$, $-\frac{1}{\pi - A}$, ..., the coefficient in front of s^2 on the left hand side is equal to the sum of the factors of two terms in the same sequence, and more generally, that the coefficient in front of s^n on the left hand side is equal to the sum of the factors of n elements in the sequence $-\frac{1}{A}$, $-\frac{1}{\pi - A}$, ... It follows that

$$\frac{1}{y} = \frac{1}{A} + \frac{1}{\pi - A} + \frac{1}{-\pi - A} + \frac{1}{2\pi + A} + \frac{1}{-2\pi + A} + \dots \tag{1.2.8}$$

With this equation in hand, Euler considered the case where $y = 1$, from which he concluded that the least positive root of the equation $\sin(s) = 1$ is $A = \pi/2$. Thus, by plugging-in $y = 1$ and $A = \pi/2$ in equation (1.2.8), he obtained

$$1 = \frac{2}{\pi} + \frac{2}{\pi} - \frac{2}{3\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{5\pi} + \dots \tag{1.2.9}$$

which is equivalent to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \tag{1.2.10}$$

Euler knew that his method was not perfectly rigorous but this first result was a way for him to confirm that his method works since it recovers some known formulas. Next, Euler recalled one last crucial fact about series. Given a series $\alpha = a + b + c + d + \dots$, if we let $\beta = ab + ac + ad + \dots + bc + bd + \dots + cd + \dots$ be the series of factors from two terms of the sequence $a, b, c, d, \dots$, then

$$a^2 + b^2 + c^2 + d^2 + \dots = \alpha^2 - 2\beta. \tag{1.2.11}$$

If we apply this formula to the sequence $-\frac{1}{\pi/2}, -\frac{1}{\pi-(\pi/2)}, \dots$, then by a previous result and by a previous observation, Euler concluded that $\alpha = -1$ and β is simply equal to the coefficient in front of s^2 in the left hand side of equation (1.2.6) and so $\beta = 0$. Therefore, we obtain

$$\left(-\frac{2}{\pi}\right)^2 + \left(-\frac{2}{\pi}\right)^2 + \left(\frac{2}{3\pi}\right)^2 + \left(\frac{2}{3\pi}\right)^2 + \dots = (-1)(-1) - 2 \cdot 0 = 1$$

which is equivalent to

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}. \tag{1.2.12}$$

This equation is very close to the series we are interested in, only the even terms are missing. Euler's last trick was to notice that if we let

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots,$$

then

$$\begin{aligned} S - \frac{\pi^2}{8} &= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \\ &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \\ &= \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) \\ &= \frac{1}{4}S \end{aligned}$$

Thus, solving this equation for S gives us

$$S = \left(1 - \frac{1}{4}\right)^{-1} \frac{\pi^2}{8} = \frac{4}{3}\frac{\pi^2}{8}$$

and so

$$\boxed{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}} \tag{1.2.13}$$

There is a lot to say about this proof since it is as creative and ingenious as unrigorous. First, since the publication of this proof, other proofs were published and some are as rigorous as one can be. Moreover, most of the formulas used in the previous proof turns out to be true even if Euler's justifications may not be really convincing since formulas that are true in the finite case might not directly extend to the infinite case. But Euler didn't stop there at all, this was only the begining of Euler's investigation of this curious series.

More Results

In the same article, Euler considered a generalization of equation (1.2.11). If we consider again the series $a+b+c+d+\ldots$, then let P_n be equal to the series where each term is taken to the nth power, and then let α_n be equal to the series of factors of n terms of the original series, then Euler deduced the following relations:

$$\begin{aligned}
P_1 &= \alpha_1 \\
P_2 &= P_1\alpha_1 - 2\alpha_2 \\
P_3 &= P_2\alpha_1 - P_1\alpha_2 + 3\alpha_3 \\
P_4 &= P_3\alpha_1 - P_2\alpha_2 + P_1\alpha_3 - 4\alpha_4 \\
P_5 &= P_4\alpha_1 - P_3\alpha_2 + P_2\alpha_3 - P_1\alpha_4 + 5\alpha_5 \\
& etc\ldots
\end{aligned}$$

Notice that the first relation is trivial and the second relation is the same equation (1.2.11). As before, he recalled that α_n is simply equal to the coefficient in front of s^n on the left hand side of equation (1.2.6). From this, Euler applied the third relation to the sequence $-\frac{1}{\pi/2}, -\frac{1}{\pi-(\pi/2)}, \ldots.$ From the previous observations and results, we have $P_1 = \alpha_1 = -1$, $P_2 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 1/6$. Thus, the third relation applied to these values gives us

$$\left(-\frac{2}{\pi}\right)^3 + \left(-\frac{2}{\pi}\right)^3 + \left(\frac{2}{3\pi}\right)^3 + \left(\frac{2}{3\pi}\right)^3 + \cdots = P_3 = -\frac{1}{2}$$

which is equivalent to

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32} \tag{1.2.14}$$

Next, in the same way, Euler applied the relations for P_4, P_5, P_6, ... to obtain the following series:

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}. \tag{1.2.15}$$

$$\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536} \tag{1.2.16}$$

$$\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960}. \tag{1.2.17}$$

$$\frac{1}{1^7} - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \cdots = \frac{61\pi^7}{184320} \tag{1.2.18}$$

$$\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\cdots=\frac{17\pi^8}{161280}. \tag{1.2.19}$$

etc...

There are a few things to observe from this enumeration. First, it is easy to convince ourselves that we can repeat this process such that for all natural numbers n, we obtain the exact value of the series

$$\frac{1}{1^n}+\frac{(-1)^n}{3^n}+\frac{1}{5^n}+\frac{(-1)^n}{7^n}+\ldots \tag{1.2.20}$$

Moreover, it seems like the exact value will always be a rational multiple of π^n (the reader is encouraged to try to prove it). From these exact values, Euler focused on finding the exact values of series of the form

$$\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots$$

as he did for the special case $n=2$. It turns out that when n is an even number, we can easily deduce the value of the series of the reciprocals of the powers of n using the series of the reciprocals of the odd numbers to the power of n with the exact same technique as the case $n=2$. This comes from the fact that if we let $n=2k$ be an arbitrary positive even number,

$$S=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots,$$

and

$$T=\frac{1}{1^{2k}}+\frac{(-1)^{2k}}{3^{2k}}+\frac{1}{5^{2k}}+\frac{(-1)^{2k}}{7^{2k}}+\cdots=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\ldots,$$

then

$$S-T=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\cdots=\frac{1}{2^n}S$$

and so

$$S=\frac{2^n}{2^n-1}T$$

which means that if know T, we automatically know S. From this, Euler obtained

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{6} \tag{1.2.21}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{90} \tag{1.2.22}$$

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\ldots=\frac{\pi^6}{945} \tag{1.2.23}$$

$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\ldots=\frac{\pi^8}{9450} \tag{1.2.24}$$

$$\frac{1}{1^{10}}+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\ldots=\frac{\pi^{10}}{93555} \tag{1.2.25}$$

and admitted that as the powers become larger, the work needed to compute these exact values becomes longer. However, when n is odd, the fact that the series (1.2.20) is alternating makes it impossible to use the same technique we used for $n = 2$. For example, Euler didn't manage to find a way to deduce the exact value of the series

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots$$

or any other series of this form where the power is a positive odd number. None of the techniques and ideas he used to solve the Basel Problem work for the odd powers.

What about $y \neq 1$?

These last results clearly show that Euler did way more than simply solving the Basel Problem. He generalized the problem and partially solved the general version. However, we are talking about Euler so it should not be surprising that he didn't stop there in this single article. If we look back to the key equation (1.2.8), we can notice that for the moment, we only studied the case where $y = 1$. What if we plug-in other non-zero values of y between -1 and 1 ? Euler fixed $y = \sqrt{2}/2$, which implies that the least $A > 0$ such that $\sin(A) = y$ is $A = \pi/4$. Thus, equation (1.2.8) gives us

$$\frac{2}{\sqrt{2}} = \frac{4}{\pi} + \frac{4}{3\pi} - \frac{4}{5\pi} - \frac{4}{7\pi} + \frac{4}{9\pi} + \frac{4}{11\pi} - \frac{4}{13\pi} - \frac{4}{15\pi} + \dots$$

which is equivalent to

$$\frac{\pi}{2\sqrt{2}} = \frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \dots \tag{1.2.26}$$

Euler observed that this result was already published by Newton. As he did for $y = 1$, from the series he obtained, he derived a second time that the series of the reciprocals of the squares is equal to $\pi^2/6$. Next, Euler fixed $y = \sqrt{3}/2$, and so in this case, $A = \pi/3$. Thus, equation (1.2.8) gives us

$$\frac{2}{\sqrt{3}} = \frac{3}{\pi} + \frac{3}{2\pi} - \frac{3}{4\pi} - \frac{3}{5\pi} + \frac{3}{7\pi} + \frac{3}{8\pi} - \dots$$

which is equivalent to

$$\frac{2\pi}{3\sqrt{3}} = \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \dots \tag{1.2.27}$$

Again, Euler derived from this series that the series of the reciprocals of the squares is equal to $\pi^2/6$. Finally, Euler considered the case $y = 0$. We cannot apply this case to equation (1.2.4) since we divide by y but we can plug-in $y = 0$ into equation (1.2.3) and divide by zero on both sides to get

$$0 = 1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdots 5} - \dots \tag{1.2.28}$$

which is equivalent to the equation

$$\frac{\sin(s)}{s} = 0 \tag{1.2.29}$$

Euler noticed that in this case, we can again apply our factorization into an infinite product since the constant coefficient of the infinite polynomial on the right hand side of equation (1.2.28) has a constant coefficient of 1. The roots of equation (1.2.28) are precisely the roots of equation (1.2.29), and so the roots are the non-zero integer multiples of π. Thus, we obtain

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdots 5} - \cdots = \left(1 - \frac{s}{\pi}\right)\left(1 + \frac{s}{\pi}\right)\left(1 - \frac{s}{2\pi}\right)\left(1 + \frac{s}{2\pi}\right)\ldots$$

which Euler rewrote as

$$1 - \frac{s^2}{1 \cdot 2 \cdot 3} + \frac{s^4}{1 \cdots 5} - \cdots = \left(1 - \frac{s^2}{\pi^2}\right)\left(1 - \frac{s^2}{2^2\pi^2}\right)\left(1 - \frac{s^2}{3^2\pi^2}\right)\ldots \tag{1.2.30}$$

From this last equation, Euler noticed that by expanding the infinite product on the right hand side and comparing the coefficients in front of s^2 on both sides of the equation, he would obtain

$$-\frac{1}{6} = -\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \frac{1}{4^2\pi^2} - \ldots$$

which is again equivalent to

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

Therefore, in total, Euler derived four times his now famous identity in this single paper. This finishes our study of Euler's 1735 paper on infinite series.

After presenting and publishing his paper, Euler became very popular in the mathematical community and even became the leading mathematician of his period. The solution to the Basel Problem is still one of the most unexpected equation in mathematics for no one would expect the constant π, nor its square, to appear in this context. We can clearly see in his article that by solving the Basel Problem, instead of moving on to something else, Euler opened the door to a whole new family of series with countless surprising properties. This article is only the begining of Euler's very deep investigation of the series of the form

$$\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots$$

and this investigation will lead to some major results and open problems in mathematics that will have a great impact on the centuries following Euler's work. The goal of the next sections will be to follow Euler's exploration of this new world he discovered. We will see the links that Euler created between these series and other well known mathematical objects.

Exercises

Exercise 1.2.1. In the paper presented in this section, Euler plugged-in the values $y = 1, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$ into the key equation (1.2.4) to rederive the Leibniz Series as well as equations (1.2.26) and (1.2.27). What series would he obtain with $y = \frac{1}{2}$?

Exercise 1.2.2. To prove his famous identity (1.2.13), Euler started by considering the equation $y = \sin(s)$, fixed different values of y, and for each value of y transformed the equation as an equality between a series and an infinite product. What would you get if you follow this method but starting with the equation $y = \cos(s)$ and fixing $y = 0$?

Exercise 1.2.3. In 1741, Euler wrote the article *Démonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + ...* [12] in which he presents a completely different proof of his famous identity. This exercise outlines William Dunham's reformulation of Euler's proof from 1741 which can be found in Chapter 3 of his book *Euler : The Master of Us All.*

(a) Prove the identity
$$\frac{1}{2}(\sin^{-1} x)^2 = \int_0^x \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt$$
using a change of variable.

(b) Using Newton's Generalized Binomial Theorem, find the Taylor series of the function $\sin^{-1} t$.

(c) Prove the relation
$$\int_0^1 \frac{t^{n+2}}{\sqrt{1-t^2}} dt = \frac{n+1}{n+2} \int_0^1 \frac{t^n}{\sqrt{1-t^2}} dt$$
for all integers $n \geq 1$.

(d) Conclude that
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$
by evaluating $\int_0^1 \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt$ in two different ways.

(e) Deduce that the sum of the reciprocals of the squares is equal to $\pi^2/6$.

1.3 Series and Products of Prime Numbers

In this section, we explore the link that Euler established between two seemingly very unrelated fields of mathematics: the series he studied in his 1735 paper (which is presented in the previous section) and number theory. Such a correspondence is unexpected since on one hand, the study of series is purely analytic, and hence, deals with continuous objects, while on the other hand, the theory of numbers is purely discrete.

This link was established in the paper *Variae observationes circa series infinitas* [6] written in 1737 and published in 1744. This paper contains a very large number of theorems about series and infinite products. More specifically, the goal of this paper is to study series where the terms are not generated by a formula but with a more intricate rule, as we will see later. Even though there is a lot of very interesting results and theorems, we will only be interested in a few.

Different Kinds of Infinite

However, before diving into the paper, we first need to understand the distinction Euler made between "∞" and "$\ln(\infty)$". Using our knowledge of limits, it would be tempting to understand $\ln(\infty)$ as the limit of $\ln(n)$ as n goes to infinity, and hence obtain $\ln(\infty) = \infty$. However, Euler treated these two *values* differently. For Euler, these symbols contain an additional information about the way a series (or any sequence in general) approaches its limit. If a series is equal to $\ln(\infty)$, then it diverges to infinity very slowly, as slowly as the logarithm function diverges to infinity. Hence, this notation indicates the rate at which a function diverges. He calls the symbol ∞ the *absolute infinite*, and it is to be viewed as the rate of divergence of a sequence which diverges to infinity at the same rate as its input. For example, Euler showed in his paper *De Progressionibus Harmonicis Observationes* [9], written in 1734, that the Harmonic Series is equal to $\ln(\infty)$. To do so, he recalled the polynomial expansion of the logarithm function to obtain the equation

$$\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots$$

which implies that applying this formula for $n = 1, ..., k$ and isolating for the $\frac{1}{n}$ term gives the following equations:

$$\begin{array}{llllll}
\frac{1}{1} = \ln\left(\frac{2}{1}\right) & + \frac{1}{2\cdot 1^2} & - \frac{1}{3\cdot 1^2} & + \frac{1}{4\cdot 1^2} & - \frac{1}{5\cdot 1^2} & + \dots \\
\frac{1}{2} = \ln\left(\frac{3}{2}\right) & + \frac{1}{2\cdot 2^2} & - \frac{1}{3\cdot 2^2} & + \frac{1}{4\cdot 2^2} & - \frac{1}{5\cdot 2^2} & + \dots \\
\vdots \quad \vdots & \vdots & \vdots & \vdots & \vdots & \\
\frac{1}{k} = \ln\left(\frac{k+1}{k}\right) & + \frac{1}{2\cdot k^2} & - \frac{1}{3\cdot k^2} & + \frac{1}{4\cdot k^2} & - \frac{1}{5\cdot k^2} & + \dots
\end{array}$$

Next, by taking the sum on both sides column by column, and using the multiplicative property of the logarithm, Euler obtained

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} = \ln(k+1) &+ \frac{1}{2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{k^2}\right) \\
&- \frac{1}{3}\left(1+\frac{1}{2^3}+\cdots+\frac{1}{k^3}\right) \\
&+ \frac{1}{4}\left(1+\frac{1}{2^4}+\cdots+\frac{1}{k^4}\right) \\
&\quad etc\dots
\end{aligned}$$

Then, by letting k go to infinity, he obtained

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \ln(\infty)+C \tag{1.3.1}$$

where C is defined by

$$C = \frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) - \frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\dots\right) + \frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right) - \dots$$

Euler observed that the series defining C converges (we can convince ourselves that this is true using the convergence of the series of the reciprocals and the Alternating Series Test) and even approximated it to be $C \approx 0.577218$. Since C is a constant, Euler deduced from equation (1.3.1) the following one:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \ln(\infty) \tag{1.3.2}$$

since adding a constant does not change the rate at which the series diverges. Concerning the constant C defined above, Euler considered it to be of great interest since it appears in a lot of other results. Notice that it is the same constant that appeared in Exercise 1.1.6 which means that Euler also computed in 1735 with greater precision this time. Today, this constant is denoted by the greek letter γ, and it is called the Euler-Mascheroni constant. This constant is very mysterious and we know very little about it even though it comes up in a lot of different places. We don't know yet for sure if it is irrational.

Today, these distinct *infinites* are replaced by the notion of asymptotic behaviors. For example, with the standard modern notation, we would write equation (1.3.1) as

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + o(1). \tag{1.3.3}$$

For more informations and a more rigorous treatment of asymptotic behaviors of sequences and functions, I recommend reading Appendix B which is entirely dedicated to this subject of great importance for the next chapters.

Euler's Infinite Products

We are now ready to understand the theorems that will be interesting for us in the paper. The first six theorems of the paper are dedicated to finding the limits of various series where the terms follow intricate rules. With a modern notation, Euler studied subsets A of the natural numbers and the corresponding series

$$\sum_{\substack{n=1 \\ n \in A}}^{\infty} \frac{f(n)}{n}$$

where $f : A \to \{\pm 1\}$ is a function that decides the sign of each term. For example, the first theorem of the paper states that if A is the set of all numbers of the form $m^n - 1$, and f puts a positive sign to each element of the set, then the corresponding series is equal to

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots = 1.$$

Euler attributes this theorem to Christian Goldbach (1690 - 1764), a Prussian mathematician. In a similar way, the third theorem of the paper states that if A is the set of multiples of 4 that are one less or one more than a power of an odd number, and f puts a positive sign to elements that exceed a power by a unit and a minus sign to the other elements, then the corresponding series is equal to

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \dots$$

Finally, the sixth theorem states that if A is the set of numbers that are one less than squares that can also be written as another power, and f gives a positive sign to each element of A, then the corresponding series is equal to

$$\frac{7}{4} - \frac{\pi^2}{6} = \frac{1}{15} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \frac{1}{624} + \dots$$

As mentionned above, even though these results are very surprising, it is the next theorem that will be interesting for our story and that we will discuss in more details. His seventh theorem is the following:

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \ldots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot \ldots} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \tag{1.3.4}$$

where, on the left hand side, the numerator is the product of all prime numbers and the denominator is the product of the numbers that are one less than the numbers in the numerator, and the right hand side is the Harmonic Series. Right before stating his seventh theorem, Euler points out that infinite products are "not less admirable" than infinite sums. Hence, equation (1.3.4) relates the Harmonic Series to its analoguous, and not less interesting, infinite product. Let's take a look at the proof of equation (1.3.4) that Euler proposed.

The first step of his proof is to let x be equal to the Harmonic Series and to notice that

$$\frac{1}{2}x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots$$

is the series of the reciprocals of the even numbers. From that, he deduced that

$$\frac{1}{2}x = x - \frac{1}{2}x = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \ldots\right) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots$$

is the series of the reciprocals of the odd numbers. Next, he divides both sides of the previous equation by 3 to get

$$\frac{1}{2} \cdot \frac{1}{3}x = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \ldots$$

which implies that

$$\begin{aligned}
\frac{1}{2} \cdot \frac{2}{3}x &= \frac{1}{2}x - \frac{1}{2} \cdot \frac{1}{3}x \\
&= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\right) - \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \frac{1}{21} + \ldots\right) \\
&= 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \ldots
\end{aligned}$$

is the series of the reciprocals of the numbers that are not divisible by 2 or 3. In the same way, he concluded that

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \ldots$$

is the series of the reciprocals of the numbers that are not divisible by 2, 3 or 5. Thus, by extending this pattern to infinity, he concluded that

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot \ldots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \ldots}x = 1$$

and so

$$\frac{2\cdot 3\cdot 5\cdot 7\cdot 11\cdot\ldots}{1\cdot 2\cdot 4\cdot 6\cdot 10\cdot\ldots}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

The lack of rigor in this proof makes it hard to learn anything from these manipulations since the only trick used in the proof is, by today's standards, *illegal*. However, I still chose to present this proof because right after this theorem, Euler generalized his result to obtain a new theorem which is of great interest for our broader study of L-functions. His eighth theorem is the following:

$$\boxed{\frac{2^n}{2^n-1}\cdot\frac{3^n}{3^n-1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n-1}\cdots=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots} \tag{1.3.5}$$

where the infinite product on the left hand side is taken over all the prime numbers. It is this formula that creates a first link between the prime numbers and the series on the right hand side of the equation. Notice that if we plug-in $n=1$, we get Euler's previous theorem.

To prove equation (1.3.5), the technique is precisely the same as before. Euler let x be equal to the series on the left hand side of equation (1.3.5) and divided it by 2^n to obtain

$$\frac{1}{2^n}x=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\ldots$$

Thus, by looking at the difference between this new series with x, he obtained

$$\frac{2^n-1}{2^n}x=\left(\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\ldots\right)-\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots\right)=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\ldots$$

where the sum on the right hand side is taken over all the odd numbers. Again, by extending this process to infinity, he concluded in the same way as before that

$$\left(\frac{2^n-1}{2^n}\cdot\frac{3^n-1}{3^n}\cdot\frac{5^n-1}{5^n}\cdot\frac{7^n-1}{7^n}\cdot\ldots\right)x=1$$

from which he easily deduced equation (1.3.5).

Notice that even though this proof is nearly exactly the same as the previous one, it is way more rigorous by today's standard since in the case $n>1$, all the series involved in the proof are now convergent (by the p-series test). From this theorem, Euler deduced that with $n=2$ and using equation (1.2.13), he obtained

$$\frac{4\cdot 9\cdot 25\cdot 49\cdot\ldots}{3\cdot 8\cdot 24\cdot 48\cdot\ldots}=\frac{\pi^2}{6}. \tag{1.3.6}$$

which seems far from obvious at first sight since it relates an infinite product of squares of prime numbers with the square of the constant π. We can obtain similar formulas if we let n be any other positive even number using the fact that Euler found a way to find a closed formula for the series on the right hand side of equation (1.3.5). Moreover, by writing equation (1.3.5) as

$$\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots\right)^{-1}=\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\dots$$

and by expanding the right hand side, we obtain

$$\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots\right)^{-1}=1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\dots \tag{1.3.7}$$

where the sum on the right hand side is taken over all the square-free integers, and the sign of each term is determined by the number of prime numbers dividing the term. For example, if we plug-in $n=2$, we obtain

$$\frac{6}{\pi^2}=1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}+\frac{1}{10^2}-\dots \tag{1.3.8}$$

As we just showed with equations (1.3.6) and (1.3.8), we can deduce a lot of surprising formulas from equation (1.3.5). Euler spent the next ten theorems exploring the conse-quences of his product formula.

The Series of the Reciprocals of the Prime Numbers

Finally, after all of these results, Euler states his final theorem of the paper, the nineteenth theorem:

$$\boxed{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\dots=\ln(\ln(\infty))} \tag{1.3.9}$$

where the left hand side is the sum of the reciprocals of the prime numbers. This theorem is of great importance because, as Euler points it out himself, not only it proves that there are infinitely many prime numbers as Euclid proved it nearly two millenials before, but it also shows that the prime numbers are, in a sense, infinitely more numerous than the squares. The argument is that there are obviously infinitely many squares in the natural numbers, but the squares are so sparse that the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\dots$$

converges. However, there are infinitely many prime numbers, and equation (1.3.9) tells us that the sum of their reciprocals is infinite. Thus, the prime numbers are more dense since the series of their reciprocals diverges. Let's take a look at how Euler proved it.

First, he let

$$\begin{aligned}
\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots &= A \\
\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \ldots &= B \\
\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \ldots &= C \\
\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \ldots &= D \\
&etc\ldots
\end{aligned}$$

where the sums on the left hand side are taken over the prime numbers. Then, he observed that by dividing both sides of the second equation by 2, the third equation by 3, the fourth equation by 4, etc..., he would obtain

$$\begin{aligned}
A &= \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots \\
\frac{1}{2}B &= \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1}{2}\cdot\frac{1}{5^2} + \frac{1}{2}\cdot\frac{1}{7^2} + \frac{1}{2}\cdot\frac{1}{11^2} + \ldots \\
\frac{1}{3}C &= \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{3}\cdot\frac{1}{11^3} + \ldots \\
\frac{1}{4}D &= \frac{1}{4}\cdot\frac{1}{2^4} + \frac{1}{4}\cdot\frac{1}{3^4} + \frac{1}{4}\cdot\frac{1}{5^4} + \frac{1}{4}\cdot\frac{1}{7^4} + \frac{1}{4}\cdot\frac{1}{11^4} + \ldots \\
&etc\ldots
\end{aligned}$$

By taking the sum on both sides column by column, we get

$$\begin{aligned}
A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \ldots &= \left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4} + \ldots\right) \\
&+ \left(\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3^2} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{4}\cdot\frac{1}{3^4} + \ldots\right) \\
&+ \left(\frac{1}{5} + \frac{1}{2}\cdot\frac{1}{5^2} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{4}\cdot\frac{1}{5^4} + \ldots\right) \\
&+ \left(\frac{1}{7} + \frac{1}{2}\cdot\frac{1}{7^2} + \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{4}\cdot\frac{1}{7^4} + \ldots\right) \\
&+ \quad etc\ldots \\
&= \ln\left(\frac{1}{1-\frac{1}{2}}\right) + \ln\left(\frac{1}{1-\frac{1}{3}}\right) + \ln\left(\frac{1}{1-\frac{1}{5}}\right) + \ln\left(\frac{1}{1-\frac{1}{7}}\right) + \ldots \\
&= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{7}{6}\right) + \ldots \\
&= \ln\left(\frac{2\cdot 3\cdot 5\cdot 7\cdot\ldots}{1\cdot 2\cdot 4\cdot 6\cdot\ldots}\right)
\end{aligned}$$

But now, in the last expression of the equation, Euler recalled his seventh theorem (equation (1.3.4)) to deduce the following equation:

$$A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots=\ln\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right) \tag{1.3.10}$$

Since the Harmonic Series diverges, then the right hand side of the equation diverges, and certainly the left hand side diverges as well. Next, Euler claims that the expression

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\dots$$

is finite. To see why, we can notice that

$$B=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\leq\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots,$$

where the series on the right hand side is the sum of the reciprocals of the squares without the first term. By interpreting the series on the right hand side as the area of the rectangles having width 1 and height equal to the term of the series as in Figure 1.3 and Figure 1.4, we obtain the inequality

$$B\leq\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\leq\int_1^\infty\frac{1}{x^2}dx=1.$$

Similarly, with the same argument, we obtain

$$C\leq\frac{1}{2}\qquad\qquad D\leq\frac{1}{3}\qquad\qquad E\leq\frac{1}{4}\qquad\qquad etc...$$

and so it follows that

$$\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\cdots\leq\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3}+\cdots=1<\infty$$

using equation (1.1.6). Therefore, the expression $\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\dots$ is a constant so in equation (1.3.10), since the left hand side diverges, then A must be the only infinite term.

If we look at equation (1.3.10) again and use the fact that everything except A is a constant on the left hand side, we obtain the simpler equation

$$A=\ln\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right). \tag{1.3.11}$$

Finally, by using the definition of A and the fact that the Harmonic Series is equal to $\ln(\infty)$ (see equation (1.3.2)), Euler obtained

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots=\ln(\ln(\infty)). \tag{1.3.12}$$

For Euler, this was a striking result. Here is what he had to say about this discovery at the beginning of an article he wrote in 1775 called *De summa seriei ex numeris primis formatae [...]* [17]:

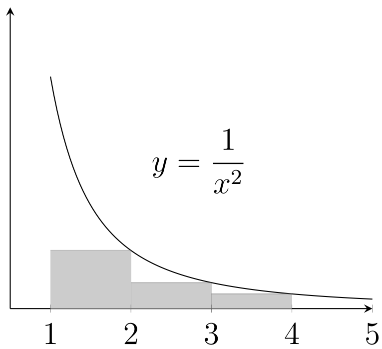

Figure 1.3: Visual interpretation of $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

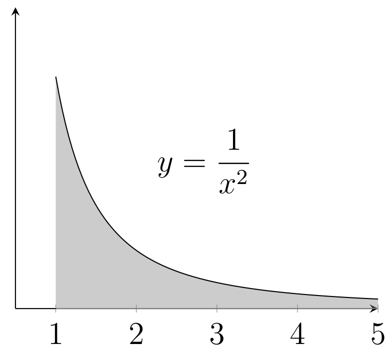

Figure 1.4: Visual interpretation of $\int_1^\infty \frac{1}{x^2} dx$

> Even as Euclid had demonstrated that the multitude of prime numbers is infinite, many years ago I also showed that the sum of the series of the reciprocals of the primes [...] is infinitely large; more precisely, I showed that it has the magnitude of the logarithm of the harmonic series [...] which seems not just a little remarkable, since commonly the harmonic series is counted as the smallest kind of infinite. [1]

The divergence of the reciprocals of the prime number can be more rigorously stated, with a modern notation, as follows:

$$\sum_{\substack{p \leq n \\ p \text{ prime}}} \frac{1}{p} = \ln(\ln(n)) + M + o(1) \tag{1.3.13}$$

where M is a constant called that Meissel-Martens constant. This constant is the analoguous of the Euler constant γ for the series of the reciprocals of the prime numbers.

Euler's paper finishes with the proof of the nineteenth theorem. Again, if we forget about rigor, the proof is full of creativity, and it really shows how easy it was for Euler to use all of these seemingly complicated formulas. We often say that Euler's identity:

$$e^{i\pi} + 1 = 0,$$

is beautiful because it links a lot of different concepts in one single equation. However, by looking at Euler's use of logarithms, series, integrals, infinite products, prime numbers, telescoping series, etc $\dots$, in the previous proof and in his whole paper, it is clear that Euler's identity is, by far, not the only of Euler's work to display such deep links between these different mathematical objects.

At the beginning of this section, I mentionned that the paper we studied would create a link between series of the reciprocals of the powers, and prime numbers. Now

[1] Translated from the Latin by Jordan Bell, Department of Mathematics, University of Toronto, Toronto, Ontario, Canada.

that we went over the theorems presented in the paper, we see that this link was established with equation (1.3.5). However, for the moment, this link may not seem very surprising. After all, this equation can be interpreted as another way of saying that any natural number can be uniquely written as a product of prime numbers, and that's it. The only original thing about this equation is that it involves series and infinite products. However, this precise equation will turn out to have a great importance in the future of Number Theory. Precisely one century after Euler's Paper, the mathematician Peter Lejeune Dirichlet would use this same equation to prove his famous theorem on arithmetic progressions, and hence, be one of the founder of Analytic Number Theory.

The second important result of the paper, is equation (1.3.9). Again, this result can be interpreted as another way of saying that there are infinitely many primes numbers. What is original about this equation is, again, that it is stated using series, and that it also carries an additional information about the density of the prime number as a subset of the natural numbers. Euler's quote from earlier in this section is from a paper he wrote in 1775 in which he extended his work on the series of the reciprocals of the prime numbers. In this paper, he proved that the series of the reciprocals of the prime numbers of the form $4n+1$ diverges and that the series of the reciprocals of the prime numbers of the form $4n-1$ diverges as well. He even conjectured that the series of the reciprocals of the prime numbers of the form $100n+1$ diverges. As a direct corollary, we have that there are infinitely many prime numbers of the form $4n+1$ and of the form $4n-1$. These theorems will be extended into a more general theorem proved by Dirichlet: there are infinitely many prime numbers of the form $an+b$ where a and b have no common divisors. Let's end this section with a quote by William Dunham that summerizes well our previous discussion on Euler's work.

> Those familiar with the prime number theorem may forget how wondrous a thing it is, linking primes to the natural logarithm function. Yet this is precisely the sort of connection - between discrete and continuous - that Euler first perceived [...]. If Euler does not quite deserve to be called the "parent" of analytic number theory, let us at least credit him with being its obvious grandparent. [2]

Exercises

Exercise 1.3.1. The goal of this exercise is to prove that the infinite product $\prod_p (1-p^{-s})^{-1}$ converges for all real numbers $s > 1$. To prove that it converges, we will mostly rely on properties of the logarithm since it will be used to convert products into a sums.

(a) Using the Taylor expansion of $\ln(1+x)$, prove that

$$\ln(1+x) = x + R(x)$$

[2] Quote from the end of Chapter 4, Euler : The Master of Us All [3].

for all $x \in (-1, 1)$ where

$$R(x) = -\frac{x^2}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2x^n}{n+2}.$$

(b) Show that $|R(x)| \leq x^2$ when $|x| < 1/2$.

(c) Use part (b) to prove that

$$|\ln(1+x)| \leq 2|x|$$

provided $|x| < 1/2$.

(d) From the inequality proved in part (c), show that the sequence $\prod_{n=1}^{N}(1 + a_n)$ converges if we suppose that the series $\sum_{n=1}^{\infty} |a_n|$ converges. Moreover, show that the product converges to 0 only if one of its term is 0.

(e) Conclude that the infinite product $\prod_p (1 - p^{-s})^{-1}$ converges.

Exercise 1.3.2. The goal of this exercise is to prove Euler's Product Fomula (1.3.5) rigorously for all real numbers $s > 1$. If we fix $s > 1$, then by the p-series test and Exercise 1.3.1, we know that both $\sum_{n=1}^{\infty} n^{-s}$ and $\prod_p (1 - p^{-s})^{-1}$ converge.

(a) Let $N \leq M$ be two positive integers. Argue that if $n \leq N$ and $n = p_1^{e_1} \cdot \ldots \cdot p_k^{e_k}$, then $p_i \leq N$ and $e_i \leq M$ for all $i \in [\![1, k]\!]$.

(b) Using part (a), deduce that

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \sum_{k=1}^{M} \frac{1}{p^{ks}}.$$

(c) Conclude with the following inequality:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p} \left(\frac{1}{1 - p^{-s}} \right).$$

(d) To prove the reverse inequality, take positive integers N and M and prove that

$$\prod_{p \leq N} \sum_{k=1}^{M} \frac{1}{p^{ks}} \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(e) From part (d), deduce that

$$\prod_{p} \left(\frac{1}{1 - p^{-s}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and conclude that the two values are equal.

1.4 The Bernoulli Numbers

As it was mentionned in the previous sections, Euler really became passionate about series of the form

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \dots \tag{1.4.1}$$

He wrote numerous articles about these series and found several other interesting results other than the ones presented in the two previous sections. In this section, we will study one of these results that Euler found which relates these series with an important sequence of numbers discovered by Jakob Bernoulli. To understand this result, recall that in his 1735 paper, in which he found the value of the series (1.4.1) with $n = 2$, he also found a way to deduce all the values of the series (1.4.1) where n is an even number. However, even if the method worked really well, it was time consuming since it was recursive: to find the value of the series for some even number n, it is required to first find the values of the series for all even numbers smaller than n.

A few years later, in his paper *De seriebus quibusdam considerationes* [13], written in 1739 and published in 1750, Euler found a better method: a general formula for computing the series (1.4.1) which only depends on n when n is even. However, to understand this general formula, we first need to learn about this sequence of the numbers that Jakob Bernoulli found a few decades before.

Bernoulli's Formula

In his famous book *Ars Conjectandi* [2], published posthumously in 1713, Jakob Bernoulli consider sums of the form

$$1^m + 2^m + 3^m + \cdots + n^m.$$

We all have probably seen before that for $m = 1$, $m = 2$ and $m = 3$, we have the following formulas

$$\begin{aligned} 1^1 + 2^1 + 3^1 + \cdots + n^1 &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \cdots + n^3 &= \left[\frac{n(n+1)}{2}\right]^2 \end{aligned}$$

which have been known for more than a millenia. With some clever algebraic manipulations, we can find similar formulas for all the higher values of m. However, only by looking at the three previous equations, an important question arises: what relates these three formulas ? They have some similarities, but not enough to be able to find their general form (if there is one). It is this question that Bernoulli answered in a chapter of his book.

First, let's rewrite these formulas as well as the formulas for some higher terms of m, and let's expand them as polynomials in n:

$$\begin{aligned}
\int n^0 &= 1 \cdot n \\
\int n^1 &= \frac{1}{2}n^2 + \frac{1}{2}n \\
\int n^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\int n^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0 \cdot n \\
\int n^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + 0 \cdot n^2 - \frac{1}{30}n \\
\int n^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + 0 \cdot n^3 - \frac{1}{12}n^2 + 0 \cdot n
\end{aligned}$$

where $\int n^m$ denotes the sum $1^m + \cdots + n^m$. Written in this way, it seems like there is a lot of patterns to spot. The first term of the sum of the powers of m is always $\frac{n^{m+1}}{m+1}$. Notice that this is really the discrete analogue of the fact that

$$\int_0^n x^m dx = \frac{1}{m+1}x^{m+1}$$

since we can view a sum as a discrete version of the integral. Therefore, the remaining terms in the above formulas can be seen as the correcting terms from this integral-sum correspondence. But this is not the only pattern we can find by looking at the above formulas, for example, the fourth column only contains 0's, the second column only contains $\frac{1}{2}$'s, and so on. But all of these patterns does not let us predict exactly the next formula, the formula for the sum of the n first powers of 6. With our observations, we could guess that this formula might look like

$$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + An^5 + 0 \cdot n^4 + Bn^3 + 0 \cdot n^2 + Cn$$

where the coefficients A, B and C could not be determined by our observations. Using some pretty complicated algebraic manipulations, we get that the true formula is the following:

$$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 + 0 \cdot n^4 + -\frac{1}{6}n^3 + 0 \cdot n^2 + \frac{1}{42}n$$

which means that we were close. The question now is the following: is there an ultimate pattern that would let us find these formulas easily ? It is exactly this pattern that Bernoulli found and presented in his book.

Without going into the details, the key is to look at the coefficient in front of n in each formula. We can see that starting from $m = 3$, the the coefficients in front of n

when m is odd seems to always be zero. Bernoulli didn't prove this but he still used it when he defined his sequence. From this observation, he defined the sequence

$$b_1 = \frac{1}{6} \qquad b_2 = -\frac{1}{30} \qquad b_3 = \frac{1}{42} \qquad b_4 = -\frac{1}{30} \qquad b_5 = \frac{5}{66}$$

where b_m is the coefficient in front of n in the polynomial expression of $\int n^{2m}$. These numbers are the first Bernoulli numbers. What Bernoulli noticed, but didn't prove, is that we only need these numbers to recover all the formulas we found and all the formulas for higher powers. Here is Bernoulli's general formula:

$$\begin{aligned}\int n^m = \frac{1}{m+1}n^{m+1} + \frac{1}{2}n^m + \frac{m}{2}b_1 n^{m-1} + \frac{m(m-1)(m-2)}{2\cdot 3\cdot 4}b_2 n^{m-3} \\ + \frac{m(m-1)(m-2)(m-3)(m-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}b_3 n^{m-5} + \dots\end{aligned}$$

This formula looks complicated but if we look at it more closely, we can notice that if we factorize by $\frac{1}{m+1}$, then the coefficients simply becomes the binomial coefficients. Thus, with a more modern notation, we can rewrite Bernoulli's formula as follows:

$$\int n^m = \frac{1}{m+1}\left[n^{m+1} + \frac{m+1}{2}n^m + \binom{m+1}{2}b_1 n^{m-1} + \binom{m+1}{4}b_2 n^{m-3} + \dots\right] \tag{1.4.2}$$

which now looks a bit simpler. Therefore, thanks to Bernoulli's formula, we get that the problem of finding the formula for the sum of the first mth power simply gets reduced to the problem of finding the first $m/2$ Bernoulli numbers. This seems easier but we quickly run into a problem: we defined and find the Bernoulli numbers by first finding the desired formulas, but we are using the Bernoulli numbers to find these desired formulas, is there another way of finding the Bernoulli numbers without having to find the desired formulas first ? Fortunately, the answer is yes, it suffices to plug-in $n = 1$ in equation (1.4.2), to obtain the new equation

$$1 = \frac{1}{m+1}\left[1 + \frac{m+1}{2} + \binom{m+1}{2}b_1 + \binom{m+1}{4}b_2 + \dots\right]$$

which, for a given m, relates the Bernoulli number with the highest index directly to those with an index smaller. For example, if we plug-in $2m$ instead of m, then we get

$$1 = \frac{1}{2m+1}\left[1 + \frac{2m+1}{2} + \binom{2m+1}{2}b_1 + \binom{2m+1}{4}b_2 + \dots + \binom{2m+1}{2m}b_m\right]$$

which we can rewrite as

$$b_m = 1 - \frac{1}{2m+1}\left[1 + \frac{2m+1}{2} + \binom{2m+1}{2}b_1 + \binom{2m+1}{4}b_2 + \dots\right] \tag{1.4.3}$$

This equation is precisely what we need since it lets us compute each Bernoulli number using the previous ones. Therefore, the problem of finding a general pattern in the formulas of the sum of the powers of a given positive integer is solved thanks to this mysterious sequence discovered by Jakob Bernoulli.

But as it was just said, this sequence is really mysterious. If we look back to Bernoulli's formula, it is unclear why the first terms of the formula do not seem to obey the same pattern as the ones that involve the Bernoulli numbers. Similarly, we saw that the Bernoulli numbers arise from looking at the last coefficient in the polynomial expression of $\int n^m$. But by looking at these last coefficients, we observe a similar irregularity, the first three coefficients are non-zero but after that, they alternate between being zero and non-zero. Moreover, the signs of these coefficients doesn't seem to follow clear pattern as well. By solving the problem of finding a general formula for $\int n^m$, it seems like Jakob Bernoulli opened the door to a deeper problem. Fortunately, the Great Euler would bring more light to this sequence. For more informations about the Bernoulli numbers, I strongly recommend the excellent Youtube video *Power sum MASTER CLASS* [20] which is an excellent introduction to the Bernoulli numbers and their applications.

Two Proofs From Euler

As we will see, Euler really made the study of the Bernoulli numbers central in the theory of series. Even though most of the results that we will consider in this section were found by Euler between 1732 and 1739, Euler only made the connection between the numbers he was studying and the Bernoulli numbers in 1740 when he wrote his book *Institutiones Calculi differentialis* [5], published in 1755. This is the reason why we will focus on the *Institutiones* to understand Euler's contribution to the study of the Bernoulli numbers.

In chapter 5 of part 2 of the *Institutiones*, Euler started by rederiving his summation formula which we discussed in section 1.1. As a reminder, given a function $f(x)$ and defining the function $S(x)$ as

$$S(x) = f(a) + f(a+1) + f(a+2) + \cdots + f(x) \tag{1.4.4}$$

where a is any number, then Euler derived a continuous formula for $S(x)$ which interpolates the first definition of $S(x)$ which is only valid for a discrete set of values of x:

$$S(x) = \alpha_0 \int f(x)dx + \alpha_1 f(x) + \alpha_2 \frac{df}{dx} + \alpha_3 \frac{d^2 f}{dx^2} + \alpha_4 \frac{d^3 f}{dx^3} + \cdots + C \tag{1.4.5}$$

where C is a constant that makes $S(a-1) = 0$, and the $\alpha_n's$ satisfy the following formulas:

$$\begin{aligned}
1 &= \alpha_0 \\
0 &= \frac{\alpha_1}{1} - \frac{\alpha_0}{2} \\
0 &= \frac{\alpha_2}{1} - \frac{\alpha_1}{2} + \frac{\alpha_0}{6} \\
0 &= \frac{\alpha_3}{1} - \frac{\alpha_2}{2} + \frac{\alpha_1}{6} - \frac{\alpha_0}{24} \\
0 &= \frac{\alpha_4}{1} - \frac{\alpha_3}{2} + \frac{\alpha_2}{6} - \frac{\alpha_1}{24} + \frac{\alpha_0}{120} \\
&\quad etc\ldots
\end{aligned}$$

from which we obtain

$$\alpha_0 = 1 \qquad \alpha_1 = \frac{1}{2} \qquad \alpha_2 = \frac{1}{12} \qquad \alpha_3 = 0 \qquad \alpha_4 = -\frac{1}{720} \qquad \alpha_5 = 0$$

$$\alpha_6 = \frac{1}{30240} \qquad \alpha_7 = 0 \qquad \alpha_8 = -\frac{1}{1209600} \qquad \alpha_9 = 0 \qquad \alpha_{10} = \frac{1}{47900160} \qquad \alpha_{11} = 0$$

Euler focused his attention on this sequence and the recursive formulas that relates its terms. By looking at the terms of the sequence, Euler noticed two things: that the odd terms are all 0 except α_1, and the even terms have alternating signs starting from α_2. Let's look at Euler's proof of these two facts.

First, looking at the ascending factorials in the numerator and the descending indices in each of the recursive formulas, Euler could have noticed the similarity with the formula of the terms resulting from the multiplication of two series (which is now mysteriously known as the Cauchy Product). He considered the following function:

$$V(u) = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + \ldots \tag{1.4.6}$$

and computed the following series expansion:

$$\frac{1 - e^{-u}}{u} = 1 - \frac{1}{2}u + \frac{1}{6}u^2 - \frac{1}{24}u^3 + \frac{1}{120}u^4 - ... \tag{1.4.7}$$

from the series exponential of the exponential function. Then, he simply multiplied both series to obtain:

$$\frac{1 - e^{-u}}{u} V(u) = \alpha_0 + \left(\frac{\alpha_1}{1} - \frac{\alpha_0}{2}\right) u + \left(\frac{\alpha_2}{1} - \frac{\alpha_1}{2} + \frac{\alpha_0}{6}\right) u^2 + \ldots$$

But since the coefficients in front of the powers of u are simply the recursive formulas he found above for the α_n's, he concluded that

$$\frac{1 - e^{-u}}{u} V(u) = 1$$

and so

$$\frac{u}{1-e^{-u}} = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + \dots \tag{1.4.8}$$

From this equation, he substracted both sides by $u/2$ to obtain

$$\alpha_0 + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + \dots = \frac{u}{1-e^{-u}} - \frac{u}{2} = \frac{u(e^{\frac{1}{2}u} + e^{-\frac{1}{2}u})}{2(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})} \tag{1.4.9}$$

and then expanded the numerator and denominator into series to get

$$\alpha_0 + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4 + \dots = \frac{1 + \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} + \dots}{1 + \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} + \dots} \tag{1.4.10}$$

Finally, Euler argued that if the right hand side were to be expanded as a series, then it would only contain even powers of u since both the numerator and denominator only contain even powers (Exercise 1.4.1). Therefore, the odd coefficients on the left hand side must all be equal to zero.

To prove his second observation on the behavior of the α_n's, Euler rewrote equation (1.4.10) using the fact that the odd coefficients are zero as follows :

$$\frac{1 + \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} + \frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \dots}{1 + \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} + \frac{u^6}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} + \dots} = 1 + \alpha_2 u^2 + \alpha_4 u^4 + \alpha_6 u^6 + \dots \tag{1.4.11}$$

He then multiplied both sides of the previous equation by the numerator of the fraction on the left hand side, and expanded the product of series to obtain an equality between two series. From this equality between two series, he obtained the following formulas for the α_n's:

$$\begin{aligned}
\alpha_2 &= \frac{1}{2\cdot 4} - \frac{1}{4\cdot 6} \\
\alpha_4 &= \frac{1}{2\cdot 4\cdot 6\cdot 8} - \frac{\alpha_2}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} \\
\alpha_6 &= \frac{1}{2\cdot 4\cdot 6\cdots 12} - \frac{\alpha_4}{4\cdot 6} - \frac{\alpha_2}{4\cdot 6\cdot 8\cdot 10} - \frac{1}{4\cdot 6\cdots 14}
\end{aligned}$$

In modern notation, these formulas can be rewritten as follows:

$$\alpha_{2n} = \frac{1}{2^{2n}(2n)!} - \sum_{k=1}^{n} \frac{\alpha_{2(n-k)}}{2^{2k}(2k+1)!} \tag{1.4.12}$$

From these formulas, Euler then stated that equation (1.4.11) still holds if we make all the series involved in the equation alternate signs (Exercise 1.4.3). Thus, he obtained

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} + \dots}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^6}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} + \dots} = 1 - \alpha_2 u^2 + \alpha_4 u^4 - \alpha_6 u^6 + \dots \tag{1.4.13}$$

From this equation, Euler divided both sides by u and rewrote the equation as follows:

$$\frac{1}{2}\cdot\frac{1-\frac{(\frac{u}{2})^2}{1\cdot 2}+\frac{(\frac{u}{2})^4}{1\cdot 2\cdot 3\cdot 4}-\frac{(\frac{u}{2})^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\dots}{(\frac{u}{2})-\frac{(\frac{u}{2})^3}{1\cdot 2\cdot 3}+\frac{(\frac{u}{2})^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{(\frac{u}{2})^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\dots}=\frac{1}{u}-\alpha_2 u+\alpha_4 u^3-\alpha_6 u^5+\dots \tag{1.4.14}$$

On the left hand side, he recognized the numerator to be the series expansion of the function $\cos(\frac{1}{2}u)$, and similarly he recognized the denominator to be the series expansion of the function $\sin(\frac{1}{2}u)$. Thus, the previous equation can be rewritten as

$$\frac{1}{2}\cot\left(\frac{1}{2}u\right)=\frac{1}{u}-\alpha_2 u+\alpha_4 u^3-\alpha_6 u^5+\dots \tag{1.4.15}$$

Next, towards his goal, Euler defined the new sequence

$$A_1=\alpha_2 \qquad A_2=-\alpha_4 \qquad A_3=\alpha_6 \qquad A_4=-\alpha_8 \qquad \textit{etc...}$$

for which it suffices to show that all the terms are positive to prove the alternating property of the even α_n's. With this new sequence, Euler defined a new function s

$$s=\frac{1}{2}\cot\left(\frac{1}{2}u\right)=\frac{1}{u}-A_1 u-A_2 u^3-A_3 u^5-A_4 u^7-\dots \tag{1.4.16}$$

and immediately derived that this function satisfies the following differential equation:

$$\frac{4ds}{du}+1+4s^2=0. \tag{1.4.17}$$

But since we have an expression of s in terms of the A_n's, then plugging this expression into the differential equation leads to

$$\begin{aligned}\frac{4ds}{du}&=-\frac{4}{u^2}-4A_1-4\cdot 3A_2u^2-4\cdot 5A_3u^4-4\cdot 7A_4u^6-\dots\\ 4s^2&=\frac{4}{u^2}-8A_1+(4A_1^2-8A_2)u^2+(8A_1A_2-8A_3)u^4+\dots\end{aligned}$$

and so

$$0=\frac{4ds}{du}+1+4s^2=(1-12A_1)+(4A_1^2-20A_2)u^2+(8A_1A_2-28A_3)u^4+\dots$$

From this equation, Euler concluded that all the coefficients must zero and so he obtained the following recursive equations

$$
\begin{aligned}
A_1 &= \frac{1}{12} \\
A_2 &= \frac{A_1A_1}{5} \\
A_3 &= \frac{2A_1A_2}{7} \\
A_4 &= \frac{2A_1A_3 + A_2A_2}{9} \\
A_5 &= \frac{2A_1A_4 + 2A_2A_3}{11} \\
A_6 &= \frac{2A_1A_5 + 2A_2A_4 + A_3A_3}{13} \\
A_7 &= \frac{2A_1A_6 + 2A_2A_5 + 2A_3A_4}{15} \\
A_8 &= \frac{2A_1A_7 + 2A_2A_6 + 2A_3A_5 + A_4A_4}{17} \\
&\quad etc\ldots
\end{aligned}
$$

from which he concluded that the A_n's must be positive, and so the sequence $\alpha_2, \alpha_4, \alpha_6, \ldots$ must have alternating signs. Therefore, Euler proved his two observations about the sequence of α_n's. But there is one last observation that would have an important impact on everything that will follow. Looking back at the sequence of A_n's, Euler noticed that the denominators are growing very fast and so he decided to rewrite each term of the sequence as follows:

$$A_1 = \frac{1}{6} \cdot \frac{1}{1 \cdot 2} \qquad A_2 = \frac{1}{30} \cdot \frac{1}{1 \cdots 4} \qquad A_3 = \frac{1}{42} \cdot \frac{1}{1 \cdots 6} \qquad A_4 = \frac{1}{30} \cdot \frac{1}{1 \cdots 8}$$

and from these new expressions, he finally recognized that the first factors are precisely the Bernoulli numbers (more precisely, their absolute value). But before looking at the consequences of this observation, we first need to recall the definition of the Bernoulli numbers and make some modifications.

Defining the Bernoulli Numbers

As we saw earlier in this section, the Bernoulli numbers were originally defined by Jakob Bernoulli as the coefficients in front of n in the polynomial expression of $\int n^{2m}$ for $m \geq 1$. But this definition assumed that the coefficient in front of n in the polynomial expression of $\int n^{2m+1}$ is zero for $m \geq 1$. Moreover, this definition is not very satisfying in the sense that it makes Bernoulli's formula irregular: the first term in the formula don't seem to follow the same pattern as the remaining terms. One way to correct this issue is to define the mth Bernoulli number B_m as the coefficient in front of n in the polynomial expression of $\int n^m$ for $m \geq 0$. Notice that this new definition is not so much different from the original one since except for the first two terms, all the additional Bernoulli numbers in the second definition are zero. From this definition, we get that the first Bernoulli numbers are:

$$B_0 = 1 \qquad B_1 = \frac{1}{2} \qquad B_2 = \frac{1}{6} \qquad B_3 = 0 \qquad B_4 = -\frac{1}{30} \qquad B_5 = 0$$

$$B_6 = \frac{1}{42} \qquad B_7 = 0 \qquad B_8 = -\frac{1}{30} \qquad B_9 = 0 \qquad B_{10} = \frac{5}{66} \qquad B_{11} = 0$$

These new Bernoulli numbers can be used to express the original Bernoulli numbers as follows: $b_m = B_{2m}$ for all $m \geq 1$. This new definition of the Bernoulli numbers lets us rewrite Bernoulli's formula as follows:

$$\int n^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \cdot n^{m+1-k} \tag{1.4.18}$$

which is way simpler than the first formulation given by Jakob Bernoulli since all the terms in the formula seem to follow a clear pattern. Another interesting consequence of this new definition is that Euler's observation that $A_n = |b_n|/(2n)$ for all $n \geq 1$ can be rewritten as

$$\alpha_n = \frac{B_n}{n!} \tag{1.4.19}$$

for all $n \geq 0$ which is much more simpler and nicer (Exercise 1.4.4). It turns out that this new definition of the Bernoulli numbers make all the formulas involving these numbers much more nicer. Therefore, it will be better to think of the Bernoulli numbers as the sequence of B_n's and not the original b_n's.

Now, let's reinterpret Euler's result about the α_n's in terms of the Bernoulli numbers using equation (1.4.19). First, Euler proved two important properties of the α_n's: $\alpha_{2n+1} = 0$ for all $n \geq 1$ and $(-1)^{n+1}\alpha_{2n} \geq 0$ for all $n \geq 1$. Since $B_n = n! \cdot \alpha_n$, then the B_n's have the same signs and zeros as the α_n's and so we get that $B_{2n+1} = 0$ for all $n \geq 1$ and $(-1)^{n+1}B_{2n} \geq 0$ for all $n \geq 1$ for free. This proves Bernoulli's observation which motivated the first definition of the Bernoulli numbers.

Next, recall that from Bernoulli's formula, we deduced the recursive formula (1.4.3) by plugging-in $n = 1$. Moreover, in his investigation of the α_n's, Euler found some other recursive formulas such as the one defining the α_n's, the one proving that the A_n's are positive, and equation (1.4.12). Using the correspondence between the α_n's and the B_n's, from the previously mentioned recursive formulas, we can deduce recursive formulas for the Bernoulli numbers. From equation (1.4.18) and by plugging-in $n = 1$, we obtain:

$$B_n = 1 - \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \tag{1.4.20}$$

for all $n \geq 0$. From the recursive definition of the α_n's, we obtain

$$B_n = \frac{1}{n+1} \sum_{k=0}^{n-1} (-1)^{n+k+1} \binom{n+1}{k} B_k \tag{1.4.21}$$

for all $n \geq 1$. From equation (1.4.12), we get

$$B_{2n} = \frac{1}{2^{2n}}\left[1 - \frac{1}{2n+1}\sum_{k=0}^{n-1} 2^{2k}\binom{2n+1}{2k} B_{2k}\right] \tag{1.4.22}$$

for all $n \geq 1$. Finally, from the recursive formula of the A_n's used to show that the A_n's are all positive, we get

$$B_{2n} = -\frac{1}{2n+1}\sum_{k=1}^{n-1}\binom{2n}{2k} B_{2k}B_{2(n-k)} \tag{1.4.23}$$

for all $n \geq 2$. These formulas can be really useful for computing the Bernoulli numbers, but also for deriving some of their key properties in the same way as Euler did.

One trick that Euler introduced in the two previous proofs that turned out to be very important was to relate the α_n's to the coefficients in the series expansion of some functions. From this, he was able to derive the recursive formulas and to deduce the desired properties of the α_n's. From equation (1.4.8), we get

$$\boxed{\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = B_0 + B_1 x + B_2\frac{x^2}{2} + B_3\frac{x^3}{6} + \dots} \tag{1.4.24}$$

and from equation (1.4.15) we obtain

$$\frac{1}{2}\cot\left(\frac{1}{2}x\right) = \sum_{n=0}^{\infty}(-1)^n \frac{B_{2n}}{(2n)!}x^{2n-1} = \frac{1}{x} - \frac{B_2}{2}x + \frac{B_4}{24}x^3 - \frac{B_6}{720}x^5 + \dots \tag{1.4.25}$$

As Euler noticed, we can recover all the Bernoulli numbers only from the function $x/(1-e^{-x})$. We call it the *generating function* of the Bernoulli numbers. This function is important because it is now used as the definition of the Bernoulli numbers. This comes from the fact that from equation (1.4.24), we can recover the recursive formula (1.4.21) from which we can recover all the properties of the Bernoulli numbers. Moreover, defining the Bernoulli numbers in this way makes the definition more compact.

In the past century, it seems like the definition of the Bernoulli numbers changed slightly. Instead of using $x/(1-e^{-x})$ as the generating function, some textbooks started to define the Bernoulli numbers as follows:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = B_0 + B_1 x + B_2\frac{x^2}{2} + B_3\frac{x^3}{6} + \dots$$

which is not very different from the previous generating function since

$$\frac{x}{e^x - 1} = \frac{(-x)}{1 - e^{-(-x)}}$$

which shows that this new generating function is just the first one evaluated at $-x$. From this, it follows that the new generating function must have the same even Bernoulli numbers since evaluating a function at $-x$ does not change the coefficients in front of the even powers of x. Moreover, all the odd Bernoulli numbers except B_1 are zero and so they are unchanged as well. Therefore, the only difference between this new generating function and the one found by Euler is that according to it, $B_1 = -\frac{1}{2}$ instead of $B_1 = \frac{1}{2}$. This means that in the past century, some people started using an alternative definition of the Bernoulli numbers in which B_1 is negative instead of positive. After searching an answer for weeks and reading numerous papers, I really can't seem to find a reason for this change. This new definition, which is starting to be used in major Number Theory textbooks, would have been justified if it made some of the formulas involving the Bernoulli numbers better looking. But this is false, these new Bernoulli numbers actually make most of the formulas in which they appear worst.

This seems to be a very minor sign change but it turns out to be very important. For more informations about the comparaison between the two generating functions of the Bernoulli numbers, I strongly recommend reading the *Bernoulli Manifesto* written by Peter Luschny. The *Manifesto*, written as an open letter addressed to the American mathematician Donald Knuth (1938 - ...), explains in details and with important arguments why we should use equation (1.4.24) as the definition of the Bernoulli numbers. I strongly recommend reading Donald Knuth's answer which can be read at the end of the *Manfiesto*. Therefore, to be clear about this problem of defining the Bernoulli numbers, it will be important to keep in mind that for the rest of this report, I will use equation (1.4.24) as my definition of the Bernoulli numbers for historical and practical reasons.

Understanding the Bernoulli Numbers

When Euler noticed the correspondence between the α_n's and the Bernoulli numbers, the first formula he rewrote in terms of the Bernoulli numbers was his summation formula. This makes sense since the summation formula plays a central role here as it is from this formula that Euler obtained the sequence of α_n's. Recall that until this point, Euler wrote the summation formula as follows:

$$S(x) = \alpha_0 \int f(x)dx + \alpha_1 f(x) + \alpha_2 \frac{df}{dx} + \alpha_3 \frac{d^2 f}{dx^2} + \alpha_4 \frac{d^3 f}{dx^3} + \cdots + C$$

where $S(x)$ represents the sum of the $f(i)$'s with i ranging from $i = a + 1$ to $i = x$, and where

$$C = -\left[\alpha_0 \int f(x)dx + \alpha_1 f(x) + \alpha_2 \frac{df}{dx} + \alpha_3 \frac{d^2 f}{dx^2} + \alpha_4 \frac{d^3 f}{dx^3} + \dots\right]_{x=a}.$$

Thus, if we replace the α_n's with the B_n's, distribute the constant C and use a modern notation, we obtain

$$\boxed{\sum_{i=a+1}^{x} f(i) = \sum_{k=0}^{\infty} \frac{B_k}{k!}[f^{(k-1)}(x) - f^{(k-1)}(a)]} \tag{1.4.26}$$

where $f^{(-1)}(x)$ denotes any antiderivative of $f(x)$. This way of writing the summation formula is the way we write it today. An interesting fact about the summation formula, written in this way, is that if we plug-in $a = 0$, $x = n$ and $f(x) = x^m$, we obtain Bernoulli's formula (1.4.18) which is equivalent to Bernoulli's original formula (1.4.2). Since Euler himself noticed this when he found the link between the α_n and the Bernoulli numbers, then it follows that Euler gave the first proof of Bernoulli's formula.

In accordance with Euler's original goal when creating his summation formula, let's rewrite it as follows:

$$\sum_{t=a+1}^{x} f(t) - \int_a^x f(t)dt = \sum_{k=1}^{\infty} \frac{B_k}{k!}[f^{(k-1)}(x) - f^{(k-1)}(a)]. \tag{1.4.27}$$

Written in this way, it is clear that the summation formula can be viewed as a bridge between the discrete and the continuous. It provides a formula for the error between the value of a sum and its corresponding integral, it lets us compute one from the another. Visually, the summation formula gives you a precise expression for the red area representing the error in Figure 1.5.

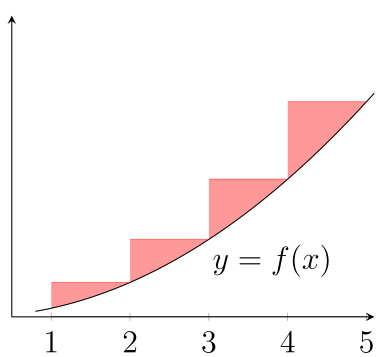

Figure 1.5: Visual interpretation of Equation (1.4.27) with $a = 1$ and $x = 5$

When discussing Bernoulli's Formula at the begining of this section, it was mentioned that the polynomial expression of $\int n^m$ was of the form

$$\int n^m = \int_0^n x^m dx + \text{error}.$$

Comparing this with the above discussion on the Euler Summation Formula clearly shows that the summation formula is a generalization of Bernoulli's Formula. Therefore, the Bernoulli numbers don't encapsulate the bridge between $\int n^m$ and $\int_0^n x^m dx$ but the more general bridge between $\sum f(t)$ and $\int f(t)dt$. It is clear now that the Bernoulli numbers are of high importance.

The General Formula

It seems like the section can end here, but remember that the goal of this section, as it was presented at the very begining of it, is to understand one of Euler's greatest discovery. As we have seen, Euler already made major discoveries by generalizing Bernoulli's Formula and proving important properties of the the now called Bernoulli numbers. However, there is one result that remains to be studied and which will we have a great importance in the future.

Recall that when Euler solved the Basel Problem in 1735, he also found an general method for finding the following formulas:

$$\begin{aligned}
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots &= \frac{\pi^2}{6}\\
\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots &= \frac{\pi^4}{90}\\
\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\ldots &= \frac{\pi^6}{945}\\
\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\ldots &= \frac{\pi^8}{9450}\\
\frac{1}{1^{10}}+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\ldots &= \frac{\pi^{10}}{93555}\\
\frac{1}{1^{12}}+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\ldots &= \frac{691\cdot\pi^{12}}{638512875}
\end{aligned}$$

but as he explained it himself in his 1735 article, when the exponents gets larger, his method would take too much time to apply.

In his book *Institutiones Calculi differentialis* written in 1740 and which we studied earlier in this section, after studying the link between the Bernoulli numbers and his own results, Euler would introduce a new way of computing the series above which would take a considerable less amount of time compared to his first method. His new method is simple: he recognized a pattern in the rational coefficients in front of the even powers of π, and hence found a general formula. Let's look at Euler's proof of this formula.

First, Euler recalled his infinite factoring trick he introduced in his 1735 paper, and used it to write the sine function as follows:

$$\sin(x) = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots$$

By replacing x with πu and taking the natural logarithm on both sides, he obtained

$$\ln(\sin(\pi u)) = \ln(u) + \ln(1-u) + \ln(1+u) + \ln\left(1-\frac{u}{2}\right) + \ln\left(1+\frac{u}{2}\right) + \ldots$$

from which he derived

$$\pi\cot(\pi u) = \frac{1}{u} - \frac{1}{1-u} + \frac{1}{1+u} - \frac{1}{2-u} + \frac{1}{2+u} - \dots$$

by taking the derivative. Combining the terms in pairs gives

$$\pi\cot(\pi u) = \frac{1}{u} - \frac{2u}{1-u^2} - \frac{2u}{4-u^2} - \frac{2u}{9-u^2} - \frac{2u}{16-u^2} - \dots$$

which can be rearranged into

$$\frac{1}{2u^2} - \frac{\pi}{2u}\cot(\pi u) = \frac{1}{1-u^2} + \frac{1}{4-u^2} + \frac{1}{9-u^2} + \frac{1}{16-u^2} + \dots$$

Then, Euler expanded each term in the right hand side of the last equation as follows:

$$\begin{aligned}
\frac{1}{1-u^2} &= 1 + u^2 + u^4 + u^6 + u^8 + \dots \\
\frac{1}{4-u^2} &= \frac{1}{2^2} + \frac{u^2}{2^4} + \frac{u^4}{2^6} + \frac{u^6}{2^8} + \frac{u^8}{2^{10}} + \dots \\
\frac{1}{9-u^2} &= \frac{1}{3^2} + \frac{u^2}{3^4} + \frac{u^4}{3^6} + \frac{u^6}{3^8} + \frac{u^8}{3^{10}} + \dots \\
\frac{1}{16-u^2} &= \frac{1}{4^2} + \frac{u^2}{4^4} + \frac{u^4}{4^6} + \frac{u^6}{4^8} + \frac{u^8}{4^{10}} + \dots
\end{aligned}$$

Finally, by letting

$$\begin{aligned}
\mathfrak{a} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \\
\mathfrak{b} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots \\
\mathfrak{c} &= 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots \\
\mathfrak{d} &= 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \dots
\end{aligned}$$

and by taking the sum columns by columns of the serie expansions he find for the fractions above, he obtained

$$\frac{1}{2u^2} - \frac{\pi}{2u}\cot(\pi u) = \mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \mathfrak{e}u^8 + \dots \tag{1.4.28}$$

What Euler managed to do is very impressive but the end goal seems preety far for moment. Or is it ? It turns out that the proof is nearly finished since Euler then recalled from equation (1.4.15) that

$$\frac{1}{2}\cot\left(\frac{1}{2}u\right) = \frac{1}{u} - \alpha_2 u + \alpha_4 u^3 - \alpha_6 u^5 + \alpha_8 u^7 - \alpha_{10} u^9 + \dots$$

Replacing $\frac{1}{2}u$ with πu yields the new equation

$$\frac{1}{2}\cot(\pi u) = \frac{1}{2\pi u} - 2\alpha_2\pi u + 2^3\alpha_4\pi^3u^3 - 2^5\alpha_6\pi^5u^5 + 2^7\alpha_8\pi^7u^7 - \dots$$

and multiplying both sides by $\frac{\pi}{u}$ gives

$$\frac{\pi}{2u}\cot(\pi u) = \frac{1}{2u^2} - 2\alpha_2\pi^2 + 2^3\alpha_4\pi^4u^2 - 2^5\alpha_6\pi^6u^4 + 2^7\alpha_8\pi^8u^6 - \dots$$

By rearranging this last equation, Euler obtained

$$\frac{1}{2u^2} - \frac{\pi}{2u}\cot(\pi u) = 2\alpha_2\pi^2 - 2^3\alpha_4\pi^4u^2 + 2^5\alpha_6\pi^6u^4 - 2^7\alpha_8\pi^8u^6 + \dots \tag{1.4.29}$$

Finally, by comparing the coefficients of both series in equations (1.4.28) and (1.4.29), Euler concluded with the following equations:

$$\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots &= +2\alpha_2\pi^2 \\
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots &= -2^3\alpha_4\pi^4 \\
1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots &= +2^5\alpha_6\pi^6 \\
1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \dots &= -2^7\alpha_8\pi^8
\end{aligned}$$

Using the new Bernoulli numbers notation and a more modern notation, we can rewrite this general formula as follows:

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = (-1)^{k+1}\frac{2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}.} \tag{1.4.30}$$

This formula is one of Euler's most celebrated result. It really encapsulates the idea that not only did Euler solve the Basel Problem, but also solved it for all exponents that are even powers and recognized the pattern to generate a general formula. This result is probably beyond what anyone could have expected. Notice that by plugging-in $k = 1$, we indeed get that the sum of the reciprocals of the squares is equal to $\frac{\pi^2}{6}$. It follows that up to this point, this is the third proof of Euler's identity where the first one is the one presented in section 1.2, the second one is the one presented in Exercise 1.2.3, and the third one is the special case $k = 1$ of equation (1.4.30).

Let's play a little bit with this general formula. First, using the fact that Euler proved that the B_{2k}'s have alternating signs, we can remove the factor of $(-1)^{k+1}$ by rewriting the equation as follows:

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{2^{2k-1}|B_{2k}|}{(2k)!}\pi^{2k}. \tag{1.4.31}$$

Moreover, if we use back the α_n's instead of the B_n's, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1}|\alpha_{2k}|\pi^{2k}. \tag{1.4.32}$$

Finally, to make the formula even more simple, we can rewrite it as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{2}|\alpha_{2k}|(2\pi)^{2k}. \tag{1.4.33}$$

Again, unexpectedly, it follows that the pattern behind the values of the sum of the reciprocals of the even powers is the same as the sequence of α_{2k}'s or as the B_{2k}'s. Thus, this is another hint pointing to the fact that the Bernoulli numbers deserve the greatest attention.

Before concluding this section, there is one last thing that should be mentioned about this formula: what about the odd exponents ? By looking at equation (1.4.33), it is tempting to generalize it to the following even more general formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^{k}} = \frac{1}{2}|\alpha_{k}|(2\pi)^{k}$$

However, this is unfortunately impossible since for all odd values of $k \geq 3$, we have that $\alpha_k = 0$ and so it would imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^{k}} = 0.$$

Therefore, Euler was again not able to solve the Basel Problem for the odd exponents. Fortunately, Euler didn't stop his work on these series. The next and final section of this chapter will explore some of Euler's latest contributions to this subject.

Exercises

Exercise 1.4.1. Euler proved that the odd α_n's are all zero, except for α_1, by showing that if the following ratio is to be expanded as a series:

$$\frac{1 + \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} + \ldots}{1 + \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} + \ldots} = c_0 + c_1 u + c_2 u^2 + c_3 u^3 + \ldots,$$

then all the odd terms on the right hand side are zero. This exercise outlines a proof that Euler could have given.

(a) Multiply by the denominator of the ratio on both sides of the equation and expand the product of the two series on the right hand side.

(b) Compare the coefficients one-by-one on both sides of the equation and conclude that all odd coefficients must be zero.

Exercise 1.4.2. To show that the function $V(u) - \frac{u}{2}$ only has even powers of u in its series expansion, Euler wrote it as a ratio of two series with only even powers and then stated that the ratio of two series with only even powers must also be written as a series with only even powers. This exercise outlines an easier and more general proof that is standard today.

(a) Show that if $f(u)$ is a function that satisfies $f(-u) = f(u)$, then its series expansion must only contain even powers of x.

(b) If we let $g(u) = V(u) - \frac{u}{2}$, then show that $g(-u) = g(u)$.

(c) Deduce that $V(u) - \frac{u}{2}$ only has even powers of u in its series expansion.

Exercise 1.4.3. From the following equation:

$$\frac{a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots}{b_0 + b_2x^2 + b_4x^4 + b_6x^6 + \dots} = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \dots,$$

deduce that

$$\frac{a_0 - a_2x^2 + a_4x^4 - a_6x^6 + \dots}{b_0 - b_2x^2 + b_4x^4 - b_6x^6 + \dots} = c_0 - c_2x^2 + c_4x^4 - c_6x^6 + \dots,$$

using techniques similar to Exercise 1.4.1. [Hint: Find a recursive formula for the c_n's in both cases and relate the two formulas.]

Exercise 1.4.4. Show that $\alpha_m = B_m/m!$ for all $m \geq 0$ using Euler's Summation Formula by taking the function $f(x) = x^m$ and looking at the coefficient in front of x in the resulting polynomial expression of $S(x)$.

Exercise 1.4.5. Given two positive sequences a_n and b_n, if the limit of their ratio is equal to 1, then we say that they are equivalent and we denote it by $a_n \sim b_n$. Using Euler's general formula (1.4.30) and its reformulations, prove the following equivalences:

(a) $|B_{2n}| \sim 2(2n)!/(2\pi)^{2n}$

(b) $|\alpha_{2n}| \sim 2/(2\pi)^{2n}$

Exercise 1.4.6. Prove that the sequence $(|B_{2n}|)_n$ diverges to infinity and that the sequence $(\alpha_n)_n$ converges to 0 as n goes to infinity.

1.5 The Odd Number Problem

It seems like there is not much left to be discovered for Euler. He not only solved the Basel Problem, but solved it for all even numbers (section 1.2), created a link between these series and the prime numbers (section 1.3), and even found a link between the values of these series and the coefficients of the formula he used in the first place to approximate these numbers (section 1.1 and section 1.4). But looking back at these results, there is one clear missing piece: what is the value of the series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\dots$$

when n is an odd number ? Since the very begining of Euler's work on these series, he tried solving this problem. In section 1.1, he approximated the series of the reciprocals of the squares very precisely but also did the same for the series of the reciprocals of the cubes:

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\dots=1.202056903159594$$

When solving the Basel Problem in section 1.2, he solved it for all even powers of n with one general and powerful trick which turned out to be ineffective for odd values of n. He was still able to obtain the following series:

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\dots=\frac{\pi^3}{32}.$$

When establishing his product formula (1.3.5) in section 1.3, Euler obtained the equation

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\dots=\frac{2^3}{2^3-1}\cdot\frac{3^3}{3^3-1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3-1}\cdots$$

which is not useful for finding the exact value of the series of the reciprocals of the cubes. Finally, when discovering that the Bernoulli numbers encode the values of the series associated with even values of n in section 1.4, he was unable to generalize this result since he proved himself that the Bernoulli numbers are zero for all odd values of $n \geq 3$. In each of the articles containing these results, Euler always mentions that unfortunately, none of his efforts were able to lead him to the value of the series where n is odd.

Euler attempted to find the value of the series of the reciprocals of the cubes way more times than just the ones presented above. For example, in his 44 pages long article *De seriebus quibusdam considerationes* [14], written in 1739 and published in 1750, Euler tries everything he can to find the value of this series. This article contains most of Euler's results presented in section 1.4, but at that time, Euler didn't notice that the numbers he was studying were the Bernoulli numbers. In the article, Euler manages to find many surprising equivalent expressions of the sum of the reciprocals of the cubes such as the following one:

$$\begin{aligned}1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots=\frac{\pi^2}{6}\ln(2)&-\frac{1}{2^2}\left(\frac{1}{2\cdot 2}\right)\\&-\frac{1}{3^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}\right)\\&-\frac{1}{4^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\right)\\&-\frac{1}{5^2}\left(\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 8}\right)\\&\quad etc...\end{aligned}$$

However, despite all of his efforts, the article ends with the following (very sad) paragraph which probably captures Euler's feeling after searching this value for 10 years.

> But because, no matter how we transform this series, we are not able to reduce it to a simple series, whose sum is known, we stop our attempts here, contented by these many expressions equivalent to the propounded series $1-\frac{1}{2^3}+\frac{1}{3^2}-\frac{1}{4^3}+\frac{1}{5^2}-etc.$ [3]

However, this was not Euler's final attempt, there is another attempt that really deserves our attention. In his article *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques* [15], written in 1749 and published in 1768, Euler finds a curious formula which will turn out to have a great importance in the study of L-functions. Let's take a look at this article.

On Divergent Series

First, let's recall that the alternating and non-alternating series of the reciprocals of the nth powers are related by this equation:

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots=\frac{2^{n-1}}{2^{n-1}-1}\left(1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\dots\right). \tag{1.5.1}$$

Therefore, finding the series on the left hand side is equivalent to finding the series on the right hand side. This explains why Euler will study series of the following form in his article:

$$\frac{1}{1^n}-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\dots$$

which he will call series of the *first species*. Curiously, he will then define series of the *second species* to be the series of the form

$$1^n-2^n+3^n-4^n+5^n-6^n+\dots$$

[3] This translation was made by Alexander Aycock.

This is surprising because the series of the second species are obviously divergent (assuming n to be positive). But Euler didn't simply assume that these series could be manipulated as if they were convergent, he knew that it would lead to some contradictions. In his article *De seriebus divergentibus* [18] written in 1746 and published in 1760, the first 12 sections are precisely dedicated to making sense of diverging series and the values that can be assigned to these series. Without going into the details of this paper, Euler's conclusion is that the sum of a series can be defined as the finite expression from which the series is generated. For example, since the series

$$1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

is generated by the finite expression

$$\frac{1}{1-x},$$

then by plugging-in $x = -1$, Euler concluded that

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

However, he also raised an important matter: by plugging-in $x = 2$, he obtained

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots = -1$$

which seems to contradict the laws of mathematics since one can never obtain a negative value by adding positive quantities. He mentioned that some mathematicians made sense of this using the following argument: the sequence

$$\frac{1}{4}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{1}{1}, \quad \frac{1}{0}, \quad \frac{1}{-1}, \quad \frac{1}{-2}, \quad \frac{1}{-3}, \quad \frac{1}{-4}, \quad etc$$

has increasing positive terms and also increasing negative terms, and so if this sequence is seen as being increasing, then we obtain the inequality $-\frac{1}{2} > \infty$ which explains the equation above. Thus, the negative numbers would be the numbers greater than ∞ and smaller than 0 while the positive numbers are the numbers greater than 0 and smaller than ∞. Visually, this means that we can visualize the real numbers not as an infinite straight line but as a circle (Figure 1.6). But Euler rejected this argument since, according to him, it would imply that -1 behaves differently if it is obatined as $(a+1) - a$ or as $\frac{1}{-1}$. Therefore, Euler would only consider alternating diverging series to avoid these difficulties. This also explains why Euler focuses on alternating series in his 1749 article.

Returning to the main article, Euler considered the equation

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \frac{1}{1+x},$$

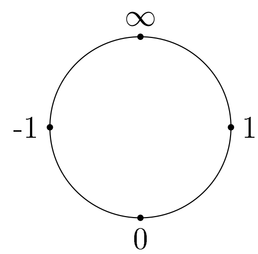

Figure 1.6: Circle representation of the real numbers

from which he obtained the following ones by successively multiplying both sides by x and taking the derivative:

$$\begin{aligned}
1 - x + x^2 - x^3 + \ldots &= \frac{1}{1+x}, \\
1 - 2x + 3x^2 - 4x^3 + \ldots &= \frac{1}{(1+x)^2}, \\
1 - 2^2x + 3^2x^2 - 4^2x^3 + \ldots &= \frac{1-x}{(1+x)^3}, \\
1 - 2^3x + 3^3x^2 - 4^3x^3 + \ldots &= \frac{1-4x+x^2}{(1+x)^4}, \\
1 - 2^4x + 3^4x^2 - 4^4x^3 + \ldots &= \frac{1-11x+11x^2-x^3}{(1+x)^5}, \\
1 - 2^5x + 3^5x^2 - 4^5x^3 + \ldots &= \frac{1-26x+66x^2-26x^3+x^4}{(1+x)^6}.
\end{aligned}$$

Hence, by his study of diverging series, he obtained the following values for the series of the second species by taking $x = 1$:

$$\begin{aligned}
1 - 2^0 + 3^0 - 4^0 + 5^0 - 6^0 + \ldots &= \frac{1}{2}, \\
1 - 2^1 + 3^1 - 4^1 + 5^1 - 6^1 + \ldots &= \frac{1}{4}, \\
1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots &= 0, \\
1 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \ldots &= -\frac{2}{16}, \\
1 - 2^4 + 3^4 - 4^4 + 5^4 - 6^4 + \ldots &= 0, \\
1 - 2^5 + 3^5 - 4^5 + 5^5 - 6^5 + \ldots &= +\frac{16}{64}, \\
1 - 2^6 + 3^6 - 4^6 + 5^6 - 6^6 + \ldots &= 0, \\
1 - 2^7 + 3^7 - 4^7 + 5^7 - 6^7 + \ldots &= -\frac{272}{256}.
\end{aligned}$$

By looking at these values, we can clearly recognize the pattern of the Bernoulli numbers in the signs and the zeros of the sequence generated by these values. But to prove this, Euler would have to use his summation formula (section 1.1) in a new way.

The Infinite Summation Formula

Recall that at first, Euler's summation formula was developed to interpolate the sequence of partial sums of a general function. In other words, it interpolates the function defined by

$$f(a)+f(a+b)+f(a+2b)+f(a+3b)+\cdots+f(x)$$

where a and b are two numbers (b is assumed to be positive). However, Euler would now focus on a new problem: extending the function

$$f(x)-f(x+a)+f(x+2)-f(x+3)+f(x+4)-\ldots$$

where the sum is infinite. This time, the variable x is not related to the number of terms in the summation but indicates the value of the first index. This seemingly distinct problem will turn out to be really similar to the original one.

First, Euler would let

$$S(x)=f(x)+f(x+a)+f(x+2)+f(x+3)+\ldots$$

and use Taylor's formula (Equation (1.1.8)) to obtain

$$S(x+a)=S(x)+\frac{adS}{1dx}+\frac{a^2ddS}{1\cdot 2dx^2}+\frac{a^3d^3S}{1\cdot 2\cdot 3dx^3}+\frac{a^4d^4S}{1\cdot 2\cdot 3\cdot 4dx^4}+\ldots$$

from which he obtained

$$-f(x)=\frac{adS}{1dx}+\frac{a^2ddS}{1\cdot 2dx^2}+\frac{a^3d^3S}{1\cdot 2\cdot 3dx^3}+\frac{a^4d^4S}{1\cdot 2\cdot 3\cdot 4dx^4}+\ldots \tag{1.5.2}$$

by using the fact that $S(x+a)-S(x)=-f(x)$ from the definition of $S(x)$. Next, in view of inverting equation (1.5.2), he wrote

$$\frac{dS}{dx}=c_0f(x)+c_1\frac{df}{dx}+c_2\frac{d^2f}{dx^2}+c_3\frac{d^3f}{dx^3}+\ldots \tag{1.5.3}$$

By plugging the previous equation into equation (1.5.2), he obtained

$$\begin{aligned}-f(x)=&\frac{a}{1}\left(c_0f(x)+c_1\frac{df}{dx}+c_2\frac{d^2f}{dx^2}+c_3\frac{d^3f}{dx^3}++\ldots\right)\\&+\frac{a^2}{2}\left(c_0\frac{df}{dx}+c_1\frac{d^2f}{dx^2}+c_2\frac{d^3f}{dx^3}+\ldots\right)\\&+\frac{a^3}{6}\left(c_0\frac{d^2f}{dx^2}+c_1\frac{d^3f}{dx^3}+\ldots\right)\\&+\frac{a^4}{24}\left(c_0\frac{d^3f}{dx^3}+\ldots\right)\end{aligned}$$

which he rewrote as

$$\begin{aligned}-f(x) = &\frac{ac_0}{1}f(x)\\&+\left(\frac{ac_1}{1}+\frac{a^2c_0}{2}\right)\frac{df}{dx}\\&+\left(\frac{ac_2}{1}+\frac{a^2c_1}{2}+\frac{a^3c_0}{6}\right)\frac{d^2f}{dx^2}\\&+\left(\frac{ac_3}{1}+\frac{a^2c_2}{2}+\frac{a^3c_1}{6}+\frac{a^4c_0}{24}\right)\frac{d^3f}{dx^3}\\&+\dots\end{aligned}$$

Next, by comparing the coefficients in front of the derivatives of f on both sides, Euler obtained the following equations:

$$\begin{aligned}c_0 &= -\frac{1}{a}\\c_1 &= -\frac{ac_0}{2}\\c_2 &= -\frac{ac_1}{2}-\frac{a^2c_0}{6}\\c_3 &= -\frac{ac_2}{2}-\frac{a^2c_1}{6}-\frac{a^3c_0}{24}\end{aligned}$$

which he used to get

$$c_0 = -\frac{1}{a}, \qquad c_1 = \frac{1}{2}, \qquad c_2 = -\frac{a}{12}, \qquad c_3 = 0, \qquad c_4 = \frac{a^4}{720}, \qquad c_5 = 0, \quad etc...$$

Then, Euler noticed that the sequence of c_n's could be written in terms of the sequence of α_n's found for its original summation formula (section 1.1): $c_n = (-a)^{n-1}\alpha_n$ for all $n \geq 0$ (Exercise 1.5.2). Thus, plugging everything in equation (1.5.3) and integrating both sides gives:

$$\begin{aligned}&f(x)+f(x+a)+f(x+2a)+f(x+3a)+\dots\\&= -\frac{\alpha_0}{a}\int f(x)dx+\alpha_1 f(x)-\frac{a\alpha_2 df}{dx}+\frac{a^2\alpha_3 d^2f}{dx^2}-\frac{a^3\alpha_4 d^3f}{dx^3}+\dots\end{aligned}$$

From this equation, Euler replaced a with $2a$ and multiplied both sides by 2 to obtain

$$\begin{aligned}&2f(x)+2f(x+2a)+2f(x+4a)+2f(x+6a)+\dots\\&= -\frac{\alpha_0}{a}\int f(x)dx+2\alpha_1 f(x)-\frac{2^2a\alpha_2 df}{dx}+\frac{2^3a^2\alpha_3 d^2f}{dx^2}-\frac{2^4a^3\alpha_4 d^3f}{dx^3}+\dots\end{aligned}$$

Finally, he subtracted from this equation the one above to obtain the desired formula:

$$f(x) - f(x+a) + f(x+2a) - f(x+3a) + \dots$$
$$= \alpha_1 f(x) - \frac{(2^2-1)a\alpha_2 df}{dx} + \frac{(2^3-1)a^2\alpha_3 d^2 f}{dx^2} - \frac{(2^4-1)a^3\alpha_4 d^3 f}{dx^3} + \dots$$

From this general formula, he considered the special case where $f(x) = x^m$ and $a = 1$:

$$x^m - (x+1)^m + (x+2)^m - (x+3)^m + \dots$$
$$= \alpha_1 x^m - m(2^2-1)\alpha_2 x^{m-1} + m(m-1)(2^3-1)\alpha_3 x^{m-2} - m(m-1)(m-2)(2^4-1)\alpha_4 x^{m-3} + \dots$$

Notice that the series at the bottom in the above equation has finitely many terms since all the terms will vanish after the first $m+1$ terms. To get the values of the series of the second species, Euler noticed that instead of plugging $x = 1$, it is more convenient and way easier to plug-in $x = 0$ since the series becomes

$$0^m - 1^m + 2^m - 3^m + 4^m - 5^m + \dots$$

and only the constant term in the bottom polynomial will be non-zero. From these observations and by multiplying both sides of the equation by -1 when $m > 0$, Euler obtained the following values

$$\begin{aligned}
&1 - 2^0 + 3^0 - 4^0 + 5^0 - 6^0 + \dots = \alpha_1,\\
&1 - 2^1 + 3^1 - 4^1 + 5^1 - 6^1 + \dots = +1 \cdot (2^2-1)\alpha_2,\\
&1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \dots = -1 \cdot 2 \cdot (2^3-1)\alpha_3,\\
&1 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \dots = +1 \cdot 2 \cdot 3 \cdot (2^4-1)\alpha_4,\\
&1 - 2^4 + 3^4 - 4^4 + 5^4 - 6^4 + \dots = -1 \cdot 2 \cdot 3 \cdot 4 \cdot (2^5-1)\alpha_5,\\
&1 - 2^5 + 3^5 - 4^5 + 5^5 - 6^5 + \dots = +1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (2^6-1)\alpha_6,\\
&1 - 2^6 + 3^6 - 4^6 + 5^6 - 6^6 + \dots = -1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot (2^7-1)\alpha_7,\\
&1 - 2^7 + 3^7 - 4^7 + 5^7 - 6^7 + \dots = +1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (2^8-1)\alpha_8
\end{aligned}$$

which are consistent with the values found above. Using the fact that $\alpha_n = 0$ for all odd integers $n \geq 3$, we can remove the minus signs on the right hand side of the previous equations to obtain

$$\begin{aligned}
&1 - 2^0 + 3^0 - 4^0 + 5^0 - 6^0 + \dots = 1 \cdot (2^1-1)\alpha_1,\\
&1 - 2^1 + 3^1 - 4^1 + 5^1 - 6^1 + \dots = 1 \cdot (2^2-1)\alpha_2,\\
&1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \dots = 1 \cdot 2 \cdot (2^3-1)\alpha_3,\\
&1 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \dots = 1 \cdot 2 \cdot 3 \cdot (2^4-1)\alpha_4,\\
&1 - 2^4 + 3^4 - 4^4 + 5^4 - 6^4 + \dots = 1 \cdot 2 \cdot 3 \cdot 4 \cdot (2^5-1)\alpha_5,\\
&1 - 2^5 + 3^5 - 4^5 + 5^5 - 6^5 + \dots = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (2^6-1)\alpha_6,\\
&1 - 2^6 + 3^6 - 4^6 + 5^6 - 6^6 + \dots = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot (2^7-1)\alpha_7,\\
&1 - 2^7 + 3^7 - 4^7 + 5^7 - 6^7 + \dots = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot (2^8-1)\alpha_8
\end{aligned}$$

which makes the pattern even clearer. Using a modern notation, we can rewrite these equations as follows:

$$\sum_{n=1}^{\infty}(-1)^{n+1}n^m = m!(2^{m+1}-1)\alpha_{m+1}. \tag{1.5.4}$$

Therefore, using his infinite summation formula, Euler was able to find the general formula for the series of the second species. Again, this general formula involves the α_n's which are closely related to the Bernoulli numbers.

A Curious Ratio Between Two Series

Euler recalled the general formula for the series of the first species he found before (section 1.4) which also involves the α_n's:

$$\begin{aligned}
1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots &= +2\alpha_2\pi^2\\
1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots &= -2^3\alpha_4\pi^4\\
1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\ldots &= +2^5\alpha_6\pi^6\\
1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\ldots &= -2^7\alpha_8\pi^8
\end{aligned}$$

Then, using the fact that $\alpha_n = 0$ for all odd integers $n \geq 3$ and the fact that the sum

$$\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots$$

is nonzero for all positive integers n, he obtained the following ratios by dividing the series of the second species by the corresponding series of the first species:

$$\begin{aligned}
\frac{1-2+3-4+5-6+\&c.}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\&c.} &= +\frac{1(2^2-1)}{(2-1)\pi^2}\\
\frac{1-2^2+3^2-4^2+5^2-6^2+\&c.}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\&c.} &= 0\\
\frac{1-2^3+3^3-4^3+5^3-6^3+\&c.}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\&c.} &= -\frac{1\cdot 2\cdot 3(2^4-1)}{(2^3-1)\pi^4}\\
\frac{1-2^4+3^4-4^4+5^4-6^4+\&c.}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\&c.} &= 0\\
\frac{1-2^5+3^5-4^5+5^5-6^5+\&c.}{1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\&c.} &= +\frac{1\cdot 2\cdot\cdot 5(2^6-1)}{(2^5-1)\pi^6}
\end{aligned}$$

in which the α_n's cancel out. From this, Euler rewrote these equations in the following single formula:

$$\boxed{\frac{1-2^{n-1}+3^{n-1}-4^{n-1}+...}{1-\dfrac{1}{2^n}+\dfrac{1}{3^n}-\dfrac{1}{4^n}+...}=\frac{-1\cdot 2\cdots(n-1)(2^n-1)}{(2^{n-1}-1)\pi^n}\cos\left(\frac{n\pi}{2}\right)} \tag{1.5.5}$$

for all integers $n \geq 2$. This formula seems strange but it is really just a mix between the two general formulas Euler found before. Moreover, the additional cosine factor is just here to make the signs alternate and be equal to 0 when n is odd. However, Euler's surprising observation is that equation (1.5.5) is true not just for the integers $n \geq 2$ but for all values of n (in the sense that it holds for all real numbers n). This is surprising because for $n = 1$, we know from earlier that

$$1-1+1-1+1-1+...=\frac{1}{2}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2$$

and so the left hand side of equation (1.5.5) becomes

$$\frac{1-1+1-1+1-1+...}{1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\dfrac{1}{6}+\ldots}=\frac{1}{2\ln 2}$$

which seems to contradict Euler's observation since the right hand of the equation involves no natural logarithm. When evaluating the right hand side of equation (1.5.5) at $n = 1$, we get that $1 \cdot 2 \cdots (n-1) = 1$ (since $0! = 1$), $2^n - 1 = 1$ and so we are only left with $-1/\pi$ multiplied by

$$\frac{\cos\left(\frac{n\pi}{2}\right)}{2^{n-1}-1}$$

in which both the numerator and the denominator vanish. Euler then considered the variable n to be continuous and stated that this ratio is equal to the ratio of the differentials of the numerator and denominator (which is now known as l'Hopital's Rule). Since the differential of the numerator is $-\frac{\pi dn}{2}\sin(\frac{n\pi}{2})$ and the differential of the denominator is $2^{n-1}dn\ln 2$, then at $n = 1$ we obtain

$$-\frac{1}{\pi}\frac{\cos\left(\frac{n\pi}{2}\right)}{2^{n-1}-1}=-\frac{1}{\pi}\frac{-\frac{\pi}{2}\sin(\frac{n\pi}{2})}{2^{n-1}\ln 2}=\frac{1}{2\ln 2}.$$

Therefore, Euler proved that his formula holds for the case $n = 1$. In a similar way, he also proved that it also holds for the case $n = 0$ (Exercise 1.5.4).

For Euler, proving that the formula holds for all integers $n \geq 2$, as well as for the non-trivial cases $n = 0$ and $n = 1$ is already a strong proof that his observation is true since, as he said in his article, it seems impossible for a false supposition to lead to such proofs. But even after these proofs, he was still willing to give more proofs that his observation is true. Here, it is important to notice that during Euler's time, a proof was simply an argument in favor of a conjecture. Thus, from Euler's point of view, he already gave two proofs of his observation with the cases $n = 0$ and $n = 1$.

To make his formula even more certain, Euler then proved that it also holds for the negative integers. To do so, he let n be a negative integer and defined the positive integer $m = -(n-1)$. Since the formula holds for the integer m, then we have

$$\frac{1 - 2^{-n} + 3^{-n} - 4^{-n} + ...}{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + ...} = \frac{-1 \cdot 2 \cdots (-n)(2^{1-n} - 1)}{(2^{-n} - 1)\pi^{1-n}} \cos\left(\frac{(1-n)\pi}{2}\right).$$

Next, Euler denoted by $[\lambda]$ the product $1 \cdot 2 \cdot 3 \cdots \lambda$ (which we now denote by $\lambda!$) and recalled the following formula which he proved before:

$$[\lambda][-\lambda] = \frac{\pi\lambda}{\sin(\pi\lambda)}. \tag{1.5.6}$$

This formula is strange since the product $1 \cdot 2 \cdot 3 \cdots (-\lambda)$ doesn't intuitively make sense when λ is a positive integer. However, a few years before this paper, Euler was able to interpolate the function $[\lambda]$ so that λ can be any number, not just the positive integers. Therefore, equation (1.5.6) is valid in the sense of Euler's interpolation. More informations about interpolations of the function $[\lambda]$ in Appendix A.

In equation (1.5.6), Euler took $\lambda = n$ to obtain

$$1 \cdot 2 \cdot 3 \cdots (-n) = [-n] = \frac{\pi n}{[n]\sin(\pi n)} = \frac{\pi}{1 \cdot 2 \cdots (n-1)\sin(\pi n)}$$

which he substituted in the equation above to get

$$\frac{1 - 2^{-n} + 3^{-n} - 4^{-n} + ...}{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + ...} = -\frac{(2^{1-n} - 1)\pi^n}{1 \cdot 2 \cdots (n-1)(2^{-n} - 1)\sin(\pi n)} \sin\left(\frac{n\pi}{2}\right)$$

using the fact that $\cos(\frac{(1-n)\pi}{2}) = \sin(\frac{n\pi}{2})$. Then, he simplified the expression on the right hand side by multiplying the numerator and the denominator by 2^n and by using the fact that $2\sin(\frac{n\pi}{2})\cos(\frac{n\pi}{2}) = \sin(n\pi)$ to obtain

$$\frac{1 - 2^{-n} + 3^{-n} - 4^{-n} + ...}{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + ...} = -\frac{(2^{n-1} - 1)\pi^n}{1 \cdot 2 \cdots (n-1)(2^n - 1)\cos\left(\frac{n\pi}{2}\right)}.$$

Finally, taking the inverse on both sides gives the equation

$$\frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + ...}{1 - 2^{-n} + 3^{-n} - 4^{-n} + ...} = \frac{-1 \cdot 2 \cdots (n-1)(2^n - 1)}{(2^{n-1} - 1)\pi^n} \cos\left(\frac{n\pi}{2}\right)$$

which is precisely the desired result. Therefore, the formula is true for all (positive and negative) integers.

But there is still one last case that Euler considered: the case $n = \frac{1}{2}$. If we plug-in this value of n in the left hand side of equation (1.5.5), we get

$$\frac{1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + ...}{1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + ...} = 1.$$

For the right hand side of the equation, we obtain

$$\frac{-1 \cdot 2 \cdots (n-1)(2^n - 1)}{(2^{n-1} - 1)\pi^n} \cos\left(\frac{n\pi}{2}\right) = -\frac{[-\frac{1}{2}](\sqrt{2} - 1)}{(\frac{1}{\sqrt{2}} - 1)\sqrt{\pi}} \cos\left(\frac{\pi}{4}\right) = \frac{[-\frac{1}{2}]}{\sqrt{\pi}} = 1$$

where the last equality comes from the fact that Euler proved before that $[-\frac{1}{2}] = \sqrt{\pi}$ (Appendix A). Therefore, both sides of equation (1.5.5) are still equal if we replace n with $\frac{1}{2}$. Hence, for Euler, the formula is now true with *the highest degree of certainty* (using his words) since it holds not only for the integers but also for some fractions.

The Odd Powers

But Euler didn't end his paper on these proofs, he used his brand new formula to attack the original probem of finding the sum of the reciprocals of the integers raised to an odd power. As we have seen earlier (equation (1.5.1)), this problem is equivalent to finding the value of the series

$$1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + ...$$

where n is an odd number. Thus, he rewrote his ratio formula as follows:

$$1 - \frac{1}{2^{2\lambda+1}} + \frac{1}{3^{2\lambda+1}} - \frac{1}{4^{2\lambda+1}} + \dots$$
$$= \frac{(2^{2\lambda} - 1)\pi^{2\lambda+1}}{-1 \cdot 2 \cdots (2\lambda)(2^{2\lambda+1} - 1)} \cdot \frac{1 - 2^{2\lambda} + 3^{2\lambda} - 4^{2\lambda} + ...}{\cos(\frac{(2\lambda+1)\pi}{2})}.$$

However, he quickly made the observation that both the numerator and the denominator of the second factor on the bottom of the previous equation are zero. Therefore, after substituting this ratio with the ratio of the respective differentials (l'Hopital's Rule), he obtained

$$1 - \frac{1}{2^{2\lambda+1}} + \frac{1}{3^{2\lambda+1}} - \frac{1}{4^{2\lambda+1}} + \dots$$
$$= \frac{2(2^{2\lambda} - 1)\pi^{2\lambda}}{1 \cdot 2 \cdots (2\lambda)(2^{2\lambda+1} - 1)} \cdot \frac{1 \ln 1 - 2^{2\lambda} \ln 2 + 3^{2\lambda} \ln 3 - 4^{2\lambda} \ln 4 + ...}{\cos(\lambda\pi)}.$$

using the identity $\cos(\frac{(2\lambda+1)\pi}{2}) = -\sin(\lambda\pi)$. Therefore, the problem is reduced to the one of finding the value of the series

$$1 \ln 1 - 2^{2\lambda} \ln 2 + 3^{2\lambda} \ln 3 - 4^{2\lambda} \ln 4 + ...$$

which seems harder. Indeed, Euler said in his article that he can think of no method that would help him find the value of this series.

Continuing on this road, he considered the sum

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$$

which is related to the alternating sum studied above by the equation

$$1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots=\frac{2^n-1}{2(2^{n-1}-1)}\left(1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\dots\right).$$

Hence, finding the sum on the left hand side is equivalent to finding the sum on the right hand side. From the previous equation, he was able to rewrite the result above as follows:

$$\begin{aligned}&1+\frac{1}{3^{2\lambda+1}}+\frac{1}{5^{2\lambda+1}}+\frac{1}{7^{2\lambda+1}}+\dots\\ &=-\frac{\pi^{2\lambda}}{1\cdot 2\cdots(2\lambda)\cos(\lambda\pi)}\cdot(2^{2\lambda}\ln 2-3^{2\lambda}\ln 3+4^{2\lambda}\ln 4-...)\end{aligned}$$

which makes it simpler but still inaccessible.

After that, Euler stated without proof a similar formula as the one he *proved* above:

$$\boxed{\frac{1-3^{n-1}+5^{n-1}-7^{n-1}+\dots}{1-3^{-n}+5^{-n}-7^{-n}+\dots}=\frac{1\cdot 2\cdot 3\cdots(n-1)2^n}{\pi^n}\sin\left(\frac{n\pi}{2}\right).} \tag{1.5.7}$$

From this formula, in the same way as above, he was able to find

$$1-\frac{1}{3^{2\lambda}}+\frac{1}{5^{2\lambda}}-\frac{1}{7^{2\lambda}}+\dots=\frac{-\pi^{2\lambda-1}(3^{2\lambda-1}\ln 3-5^{2\lambda-1}\ln 5+7^{2\lambda-1}\ln 7-\dots)}{1\cdot 2\cdot 3\cdots(2\lambda-1)2^{2\lambda-1}\cos(\lambda\pi)}$$

but unfortunately, he was unable to link this result to the previous ones in a way that would help him determine the sum of the reciprocals of the cubes or any other odd power. Again, Euler was unable to determine these mysterious values. These last formulas conclude Euler's article.

Compared to the other papers that were presented in this chapter, it seems that the one presented in this section is disapointing in the sense that it shows Euler failing his goal of finding the sum of the reciprocals of the cubes and other odd powers. However, even though the paper ends on a failure, it still shows how creative and persistent Euler was. Moreover, it turns out that the formulas found in this article will have a great importance in the future. These formulas will be brought back to the main stage more than a century later by another master of mathematics that will also have a complete chapter dedicated to him. But for the moment, lets keep our focus on Euler.

Euler's Legacy

The previously presented paper was not Euler's last attempt in finding the sum of the reciprocals of the nth powers with n an odd integer. In 1772, in his article *Exercitationes analyticae* [16] published in 1773, Euler proved that

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\cdots=\frac{\pi^2}{4}\ln 2+\int_0^{\frac{\pi}{2}}\varphi\ln(\sin(\varphi))d\varphi$$

but this was not enough since he was unable to find a closed form for the integral on the right hand side of the equation. It turns out that Euler never found the sum of the reciprocals of the cubes. He died in 1783, in Saint Petersburg at the age of 76. This year marked the end of one of the greatest mathematician of all time. There is no field of mathematics in which he made no major contributions.

After reading all of these papers, it should be clear that there is one trick that Euler used extensively and more than any others. This trick is simply the fact that if two power series are equal, then the coefficients in front of the powers of the variable must all be equal. This is simply another way of stating that a function has a unique representation as a power series. This property is very powerful since it helps obtaining infinitely many equations from one simpler equation. Let's take a quick look at a proof from Euler that uses this trick. First, define the function

$$P(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\dots$$

which can be expanded as the following power series:

$$P(x)=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\dots$$

If we expand the product in the definition of $P(x)$, we get that α is the number of ways we can write 1 as a sum of distinct powers of 2, β is the number of ways we can write 2 as a sum of distinct powers of 2, and so on. From the definition of $P(x)$, we have that

$$P(x^2)=(1+x^2)(1+x^4)(1+x^8)(1+x^{16})...=\frac{P(x)}{1+x}$$

and so

$$\begin{aligned}1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\dots&=(1+x)P(x^2)\\&=(1+x)(1+\alpha x^2+\beta x^4+\gamma x^6+\delta x^8+\dots)\\&=1+x+\alpha x^2+\alpha x^3+\beta x^4+\beta x^5+...\end{aligned}$$

Equating both sides of the equation leads to an equality between two series that gives us

$$\alpha=1,\qquad \beta=\alpha,\qquad \beta=\gamma,\qquad etc...$$

and so all the coefficients in the series expansion of $P(x)$ are equal to one. Therefore, every positive integer can be written as a sum of distinct powers of 2 in a unique way.

This is Euler's way of proving that every integer has a unique base-2 representation. It is clear now that Euler understood really well the full power of this simple property of power series. When we see how Euler treated power series as one of the most fundamental object in mathematics, we understand why Joseph Louis Lagrange (1736 - 1813), tried to put series as the foundations of analysis (instead of limits introduced by Cauchy).

Concerning the sum of the reciprocals of the cubes (and any other odd power), it is still an open problem. No one today has found a closed formula for this series. In trying to attack this problem, Euler discovered some of the most curious and surprising formulas. For more informations about the sum of the reciprocals of the cubes, I recommend the book *In Pursuit of Zeta-3* [22] which focuses on this series, and how different mathematician tried to attack the problem.

As it was said earlier, Euler's contributions in mathematics are countless, and this chapter only focuses on a very tiny bit of Euler's full work. To get a broader image of Euler's contributions in mathematics, in strongly recommend the excellent book *Euler: The Master of Us All* [3] by William Dunham.

Exercises

Exercise 1.5.1 (Eulerian Polynomials)**.** In the rational function representation of the series $1^m - 2^m x + 3^m x^2 - 4^m x^3 + \ldots$, denote by $P_n(x)$ the sequence of polynomials found in the numerators. Find a formula for $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_n'(x)$.

Exercise 1.5.2. Prove that $c_n = (-a)^{n-1}\alpha_n$ holds for all $n \geq 0$ by finding the generating function of the c_n's.

Exercise 1.5.3. This exercise outlines a little proof that Euler could have given of a general formula for the series of the mth powers.

(a) Rewrite equation (1.5.4) in terms of the B_n's instead of the α_n's.

(b) Euler only considered alternating divergent series. By ignoring the divergence of the series and using equation (1.5.1), find a general formula for the sum $1^m + 2^m + 3^m + 4^m + \ldots$

(c) According to this formula, what is the value of $1 + 2 + 3 + 4 + 5 + \ldots$?

Exercise 1.5.4. Prove that equation (1.5.5) holds for $n = 0$ using the same method as Euler for the case $n = 1$. [Hint: rewrite $1 \cdot 2 \cdots (n-1)$ as $\frac{1}{n} \cdot 1 \cdot 2 \cdots n$].

Exercise 1.5.5. Prove equation (1.5.7) for all integers $n \geq 1$ in the same way as Euler did when he proved equation (1.5.5).

Exercise 1.5.6. Prove that if two power series are equal, then all of the coefficients are equal using induction.

Chapter 2

Dirichlet's Theorem

2.1 Number Theory in the Beginning of the 19th Century

TODO

1. Lagrange's Quadratic Forms
2. Legendre's Theorem and tentative proof
3. Gauss' Disquisitiones Arithmeticae
   (a) Modular Arithmetic
   (b) Quadratic Forms
   (c) Quadratic Reciprocity

TODO

2.2 Fourier's Theorem

TODO

1. Petit rappel historique sur les séries trigonométriques
2. Travaux de Fourier et publication de son livre
3. Dirichlet 1829, démonstration du théorème pour une grande classe de fonction
4. Dirichlet 1835, utilisation du théorème de Fourier dans le contexte de la théorie des nombres

TODO

2.3 Dirichlet's Theorem

TODO

2.4 Proof of the General Case

TODO

2.5 The Class Number Formula

TODO

Chapter 3

Riemann's 1859 Paper

This part is not written yet.

TODO

3.1 Cauchy's Complex Analysis

3.2 The Riemann ζ Function

3.3 The Functional Equation of the ζ Function

3.4 The Prime Counting Function $\pi(x)$

3.5 The Riemann Hypothesis

Appendix A

The Factorial and the Gamma Functions

This part is not written yet.

TODO

1. Motivate the definition using Understanding Analysis Chapter 8 and Calculus Gallery, chapters on Euler.
2. State and prove basic properties of the function.
3. State and prove that
$$\frac{\pi z}{\Pi(z)\Pi(-z)} = \sin(\pi z)$$

TODO

Definition. The Factorial Function $\Pi : \mathbb{C} \setminus \{-1, -2, -3, ...\} \to \mathbb{C}$ is defined by

$$\Pi(z) = \int_0^\infty e^{-t} t^z dt.$$

Definition. The Gamma Function $\Gamma : \mathbb{C} \setminus \{-1, -2, -3, ...\} \to \mathbb{C}$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Appendix B

The Big-O Notation

This part is not written yet.

TODO

Definition. If f and g are real-valued functions such that

$$|f(x)| \leq C|g(x)|$$

for some positive constants C and for all $x > k$ where $k > 0$, then we say that f is $O(g)$.

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