Solutions to Linear Algebra Done Right (4th Ed) - Sheldon Axler

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Preface

The goal of this document is to share my personal solutions to the exercises in the Fourth Edition of Linear Algebra Done Right by Sheldon Axler during my reading. As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Chapter 1

Vector Spaces

1A $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1
Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib \quad \text{and} \quad \beta = c + id$$

where $a, b, c, d \in \mathbf{R}$, then

$$\begin{aligned}
\alpha + \beta &= (a + ib) + (c + id) \\
&= (a + c) + i(b + d) \\
&= (c + a) + i(d + b) \\
&= (c + id) + (a + ib) \\
&= \beta + \alpha
\end{aligned}$$

which proves that addition is commutative in $\mathbf{C}$ using the fact that it is commutative in $\mathbf{R}$.

Exercise 2
Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
(\alpha + \beta) + \lambda &= [(a + ib) + (c + id)] + (e + if) \\
&= [(a + c) + i(b + d)] + (e + if) \\
&= ([a + c] + e) + i([b + d] + f) \\
&= (a + [c + e]) + i(b + [d + f]) \\
&= (a + ib) + [(c + e) + i(d + f)] \\
&= (a + ib) + [(c + id) + (e + if)] \\
&= \alpha + (\beta + \lambda)
\end{aligned}$$

which proves that addition is associative in $\mathbf{C}$ using the fact that it is associative in $\mathbf{R}$.

Exercise 3
Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
(\alpha\beta)\lambda &= [(a+ib)(c+id)](e+if) \\
&= [(ac-bd)+i(ad+bc)](e+if) \\
&= ([ac-bd]e-[ad+bc]f)+i([ac-bd]f+[ad+bc]e) \\
&= (ace-bde-adf-bcf)+i(acf-bdf+ade+bce) \\
&= (a[ce-fd]-b[cf+de))+i(a[cf+de]+b[ce-fd]) \\
&= (a+ib)[(ce-fd)+i(cf+de)] \\
&= (a+ib)[(c+id)(e+if)] \\
&= \alpha(\beta\lambda)
\end{aligned}$$

which proves that multiplication is associative in $\mathbf{C}$ using the fact that multiplication is associative and addition is commutative in $\mathbf{R}$.

Exercise 4
Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
\lambda(\alpha+\beta) &= (e+if)[(a+ib)+(c+id)] \\
&= (e+if)[(a+c)+i(b+d)] \\
&= [e(a+c)-f(b+d)]+i[e[b+d]+f[a+c]] \\
&= (ea+ec-fb-fd)+i(eb+ed+fa+fc) \\
&= [(ea-fb)+i(eb+fa)]+[(ec-fd)+i(ed+fc)] \\
&= [(e+if)(a+ib)]+[(e+if)(c+id)] \\
&= \lambda\alpha+\lambda\beta
\end{aligned}$$

which proves the distributivity in $\mathbf{C}$ using the distributivity in $\mathbf{R}$.

Exercise 5
Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution
Let $\alpha = a + ib$ and consider $\beta = (-a) + i(-b)$, then we get

$$\begin{aligned}\alpha + \beta &= (a + ib) + ([-a] + i[-b]) \\ &= (a + [-a]) + i(b + [-b]) \\ &= 0 + i0 \\ &= 0\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$, this implies that $\alpha + \beta_1 = \alpha + \beta_2$. If we add β_1 on both sides, we get

$$\begin{aligned}\beta_1 + (\alpha + \beta_1) = \beta_1 + (\alpha + \beta_2) &\implies (\beta_1 + \alpha) + \beta_1 = (\beta_1 + \alpha) + \beta_2 \\ &\implies (\alpha + \beta_1) + \beta_1 = (\alpha + \beta_1) + \beta_2 \\ &\implies 0 + \beta_1 = 0 + \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 6
Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution
Let $\alpha = a + ib \neq 0$, then notice that we must have $a^2 + b^2 \neq 0$. Hence, consider

$$\beta = \left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)$$

Thus, we get

$$\begin{aligned}\alpha\beta &= (a + ib)\left[\left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)\right] \\ &= \left(a\left(\frac{a}{a^2 + b^2}\right) - b\left(-\frac{b}{a^2 + b^2}\right)\right) + i\left(a\left(-\frac{b}{a^2 + b^2}\right) + b\left(\frac{a}{a^2 + b^2}\right)\right) \\ &= \frac{a^2 + b^2}{a^2 + b^2} + i\frac{-ab + ba}{a^2 + b^2} \\ &= 1 + i0 \\ &= 1\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$, this implies that $\alpha\beta_1 = \alpha\beta_2$. If we multiply by β_1 on both sides, we get

$$\begin{aligned}\beta_1(\alpha\beta_1) = \beta_1(\alpha\beta_2) &\implies (\beta_1\alpha)\beta_1 = (\beta_1\alpha)\beta_2 \\ &\implies (\alpha\beta_1)\beta_1 = (\alpha\beta_1)\beta_2 \\ &\implies 1 \cdot \beta_1 = 1 \cdot \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 7
Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution
This is pretty straightforward:

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{(-1+\sqrt{3}i)^3}{2^3}\\ &= \frac{(-1)^3+3(-1)^2(\sqrt{3}i)+3(-1)^1(\sqrt{3}i)^2+(\sqrt{3}i)^3}{8}\\ &= \frac{-1+3\sqrt{3}i+3\cdot 3-3(\sqrt{3}i)}{8}\\ &= \frac{8}{8}\\ &= 1\end{aligned}$$

Exercise 8
Find two distinct square roots of i.

Solution
Consider $\alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\beta = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Hence,

$$\begin{aligned}\alpha^2 &= \left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)^2\\ &= \left(\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\frac{\sqrt{2}}{2}\cdot i\frac{\sqrt{2}}{2} + \left(i\frac{\sqrt{2}}{2}\right)^2\\ &= \frac{2}{4}+i-\frac{2}{4}\\ &= i\end{aligned}$$

and

$$\begin{aligned}\beta^2 &= \left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)^2\\ &= \left(-\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\left(-\frac{\sqrt{2}}{2}\right)\cdot\left(-i\frac{\sqrt{2}}{2}\right) + \left(-i\frac{\sqrt{2}}{2}\right)^2\\ &= \frac{2}{4}+i-\frac{2}{4}\\ &= i\end{aligned}$$

Therefore, α and β are two distinct square roots of i.

Exercise 9
Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution
First, suppose that such an element x exists, then there exist $a, b, c, d \in \mathbf{R}$ such that $x = (a, b, c, d)$ and

$$(4 + 2a, -3 + 2b, 1 + 2c, 7 + 2d) = (5, 9, -6, 8)$$

But notice that this is equivalent to the following system of equations:

$$\begin{cases} 4 + 2a = 5 \\ -3 + 2b = 9 \\ 1 + 2c = -6 \\ 7 + 2d = 8 \end{cases}$$

which implies that

$$\begin{cases} a = \frac{1}{2} \\ b = 6 \\ c = \frac{7}{2} \\ d = \frac{1}{2} \end{cases}$$

Therefore, we get that $x = (\frac{1}{2}, 6, \frac{7}{2}, \frac{1}{2}) \in \mathbf{R}^4$ is indeed a solution to our original equation.

Exercise 10
Explain why there is does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution
By contradiction, suppose there exists a complex number $\lambda = a + ib$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

Then, we would get the following system of equation:

$$\begin{cases} \lambda(2 - 3i) = 12 - 5i \\ \lambda(5 + 4i) = 7 + 22i \\ \lambda(-6 + 7i) = -32 - 9i \end{cases}$$

which is equivalent to

$$\begin{cases} \lambda = 3 + 2i \\ \lambda = 3 + 2i \\ \lambda = \frac{129}{85} + i\frac{278}{85} \end{cases}$$

We clearly have a contradiction since $3 + 2i \neq \frac{129}{85} + i\frac{278}{85}$. Therefore, there doesn't exist such a complex number λ.

Exercise 11
Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution
First, write

$$x = (x_1, ..., x_n), \quad y = (y_1, ..., y_n) \quad \text{and} \quad z = (z_1, ..., z_n)$$

Since addition is commutative in $\mathbf{F}$, we get

$$\begin{aligned}
(x + y) + z &= [(x_1, ..., x_n) + (y_1, ..., y_n)] + (z_1, ..., z_n) \\
&= (x_1 + y_1, ..., x_n + y_n) + (z_1, ..., z_n) \\
&= ([x_1 + y_1] + z_1, ..., [x_n + y_n] + z_n) \\
&= (x_1 + [y_1 + z_1], ..., x_n + [y_n + z_n]) \\
&= (x_1, ..., x_n) + (y_1 + z_1, ..., y_n + z_n) \\
&= (x_1, ..., x_n) + [(y_1, ..., y_n) + (z_1, ..., z_n)] \\
&= x + (y + z)
\end{aligned}$$

which proves that addition is associative in $\mathbf{F}^n$.

Exercise 12
Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution
First, write $x = (x_1, ..., x_n)$. Using associativity of multiplication in $\mathbf{F}$, we get

$$\begin{aligned}
(ab)x &= (ab)(x_1, ..., x_n) \\
&= ((ab)x_1, ..., (ab)x_n) \\
&= (a(bx_1), ..., a(bx_n)) \\
&= a(bx_1, ..., bx_n) \\
&= a[b(x_1, ..., x_n)] \\
&= a(bx)
\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Exercise 13
Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution
Let $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Hence,

$$\begin{aligned}
1x &= 1(x_1, ..., x_n) \\
&= (1 \cdot x_1, ..., 1 \cdot x_n) \\
&= (x_1, ..., x_n) \\
&= x
\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$.

Exercise 14
Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$.

Solution
Let $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$ with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
\lambda(x+y) &= \lambda[(x_1, ..., x_n) + (y_1, ..., y_n)] \\
&= \lambda(x_1 + y_1, ..., x_n + y_n) \\
&= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) \\
&= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) \\
&= \lambda x + \lambda y
\end{aligned}$$

which proves the desired formula.

Exercise 15
Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution
Let $a, b \in \mathbf{F}$ and $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
(a+b)x &= (a+b)(x_1, ..., x_n) & &= ((a+b)x_1, ..., (a+b)x_n) \\
&= (ax_1 + bx_1, ..., ax_n + bx_n) \\
&= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) \\
&= a(x_1, ..., x_n) + b(x_1, ..., x_n) \\
&= ax + bx
\end{aligned}$$

which proves the desired formula.

1B Definition of Vector Space

Exercise 1
Prove that $-(-v) = v$ for every $v \in V$.

Solution
Let $v \in V$, by definition, we know that by definition, $-v$ is defined as the only vector in V satisfying

$$v + (-v) = 0$$

which is equivalent to

$$(-v) + v = 0$$

by commutativity of addition in V. However, notice that by definition, $-(-v)$ is the unique vector satisfying

$$(-v) + [-(-v)] = 0$$

But since v itself also satisfies this equation, we get $-(-v) = v$ by uniqueness.

Exercise 2
Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution
Suppose that $a \neq 0$, then by properties of $\mathbf{F}$, the inverse a^{-1} exists. Hence, if we multiply by a^{-1} on both sides, we get

$$\begin{aligned} av = 0 &\implies a^{-1}(av) = a^{-1}0 \\ &\implies (a^{-1}a)v = 0 \\ &\implies 1v = 0 \\ &\implies v = 0 \end{aligned}$$

Therefore, we either have $a = 0$ or $v = 0$.

Exercise 3
Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution
By properties of vector spaces, since $v \in V$, then $-v \in V$. Similarly, since w and $-v$ are in V, then $w + (-v) \in V$. Finally, since $w + (-v) \in V$, then $3^{-1}(w + (-v)) \in V$. Thus, define x_0 as the vector $3^{-1}(w + (-v))$ in V. Notice that

$$\begin{aligned} v + 3x_0 &= v + 3[3^{-1}(w + (-v))] \\ &= v + (3 \cdot 3^{-1})(w + (-v)) \\ &= v + 1(w + (-v)) \\ &= v + (w + (-v)) \\ &= v + ((-v) + w) \\ &= (v + (-v)) + w \\ &= 0 + w \\ &= w \end{aligned}$$

which shows that the equation has at least one solution. To prove uniqueness, let $x_1 \in V$ be an arbitrary solution to the equation, then we get

$$\begin{aligned}
v + 3x_1 = w &\implies (-v) + (v + 3x_1) = (-v) + w \\
&\implies ((-v) + v) + 3x_1 = w + (-v) \\
&\implies 0 + 3x_1 = w + (-v) \\
&\implies 3x_1 = w + (-v) \\
&\implies 3^{-1}(3x_1) = 3^{-1}(w + (-v)) \\
&\implies (3^{-1}3)x_1 = x_0 \\
&\implies 1x_1 = x_0 \\
&\implies x_1 = x_0
\end{aligned}$$

which proves that x_0 is the unique solution to the equation.

Exercise 4
The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space. Which one?

Solution
The empty set doesn't satisfy the axiom that states that there must be an additive identity since the empty set is empty by definition.

Exercise 5
Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here, the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution
We already know that the axioms of a vector space imply that $0v = 0$ for all $v \in V$. Hence, it suffices to prove that if we assume the axioms of a vector space without the additive inverse condition, then we can prove the additive inverse condition if we also assume the property that $0v = 0$ for all $v \in V$. Let $v \in V$, the by the distributive condition, we get

$$\begin{aligned}
0v = 0 &\implies (1 + (-1))v = 0 \\
&\implies 1v + (-1)v = 0 \\
&\implies v + (-1)v = 0
\end{aligned}$$

which proves that v has an additive inverse for all $v \in V$.

Exercise 6
Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the

notation. Specifically, the sum and product of two reals numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty \\ \infty + (-\infty) = (-\infty) + \infty &= 0 \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution
With these operations of addition and scalar multiplication, $\mathbf{R} \cup \{\infty, -\infty\}$ cannot be a vector space since

$$((-\infty) + \infty) + \infty = 0 + \infty = \infty$$

and

$$(-\infty) + (\infty + \infty) = (-\infty) + \infty = 0$$

which proves that addition isn't associative under this operation.

Exercise 7
Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution
For any f and g in V^S, define $f + g : S \to V$ by $s \mapsto f(s) + g(s)$ for all $s \in S$. Similarly, for all $\alpha \in \mathbf{F}$ and $f \in V^S$, define $\alpha f : S \to V$ by $s \mapsto \lambda f(s)$ for all $s \in S$. With these definitions, let's prove that V^S is a vector space.

- (**commutativity**) Let $f, g \in V^S$, let's show that $f + g = g + f$. Let $s \in S$, then by commutativity in V, we obviously have

$$(f + g)(s) = f(s) + g(s) = g(s) + f(s) = (g + f)(s)$$

  Since it holds for all s, then $f + g = g + f$.

- (**associativity**) Let $f, g, h \in V^S$ and $s \in S$, then by associativity in V, we have

$$\begin{aligned} [(f + g) + h](s) &= (f + g)(s) + h(s) \\ &= [f(s) + g(s)] + h(s) \\ &= f(s) + [g(s) + h(s)] \\ &= f(s) + (g + h)(s) \\ &= [f + (g + h)](s) \end{aligned}$$

Since it holds for all $s \in S$, then $(f+g)+h = f+(g+h)$.
Let now $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, then by associativity in V, we get:

$$\begin{aligned}
[(ab)f](s) &= (ab)f(s) \\
&= a(bf(s)) \\
&= a(bf)(s) \\
&= [a(bf)](s)
\end{aligned}$$

Since it holds for all $s \in S$, then $(ab)f = a(bf)$.

- (**additive identity**) Let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$ and $s \in S$, we have

$$(f + 0_{V^S})(s) = f(s) + 0_{V^S}(s) = f(s) + 0 = f(s)$$

Since it holds for all $s \in S$, then $f + 0_{V^S} = f$ for all $f \in V^S$.

- (**additive inverse**) Again, let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$, we can define the function $g = (-1)f \in V^S$. Hence, for all $s \in S$, we get

$$\begin{aligned}
(f+g)(s) &= f(s) + g(s) \\
&= f(s) + (-1)f(s) \\
&= f(s) + (-f(s)) \\
&= 0 \\
&= 0_{V^S}(s)
\end{aligned}$$

Since it holds for all $s \in S$, then $f + g = 0_{V^S}$.

- (**multiplicative identity**) Let $f \in V^S$, then for all $s \in S$, we have

$$(1f)(s) = 1f(s) = f(s)$$

Since it holds for all $s \in S$, then $1f = f$.

- (**distributive property**) Let $f, g \in V^S$, $a \in \mathbf{F}$ and $s \in S$, then

$$\begin{aligned}
[a(f+g)](s) &= a(f+g)(s) \\
&= a(f(s) + g(s)) \\
&= af(s) + ag(s) \\
&= (af)(s) + (ag)(s) \\
&= (af + ag)(s)
\end{aligned}$$

Since it holds for all $s \in S$, then $a(f+g) = af + ag$. Similarly, for all $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, we have

$$\begin{aligned}
[(a+b)f](s) &= (a+b)f(s) \\
&= af(s) + bf(s) \\
&= (af)(s) + (bf)(s) \\
&= (af + bf)(s)
\end{aligned}$$

Since it holds for all $s \in S$, then $(a+b)f = af + bf$.

Therefore, V^S is a vector space under these definitions.

Exercise 8
Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
$$(a + ib)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with these definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution

- (**commutativity**) Let $u_1, v_1, u_2, v_2 \in V$, then by commutativity in V, we have
$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) &= (u_1 + u_2) + i(v_1 + v_2)\\ &= (u_2 + u_1) + i(v_2 + v_1)\\ &= (u_2 + iv_2) + (u_1 + iv_1)\end{aligned}$$
which proves that addition is commutative.
- (**associativity**) Let $u_1, v_1, u_2, v_2, u_3, v_3 \in V$, then by associativity in V, we have
$$\begin{aligned}[(u_1 + iv_1) + (u_2 + iv_2)] + (u_3 + iv_3) &= [(u_1 + u_2) + i(v_1 + v_2)] + (u_3 + iv_3)\\ &= ([u_1 + u_2] + u_3) + i([v_1 + v_2] + v_3)\\ &= (u_1 + [u_2 + u_3]) + i(v_1 + [v_2 + v_3])\\ &= (u_1 + iv_1) + [(u_2 + u_3) + i(v_2 + v_3)]\\ &= (u_1 + iv_1) + [(u_2 + iv_2) + (u_3 + iv_3)]\end{aligned}$$
Let now $a, b, c, d \in \mathbf{R}$ and $u, v \in V$, then we get:
$$\begin{aligned}&[(a + bi)(c + di)](u + iv)\\ &\quad= [(ac - bd) + i(ad + bc)](u + iv)\\ &\quad= [(ac - bd)u - (ad + bc)v] + i[(ac - bd)v + (ad + bc)u]\\ &\quad= [acu - bdu - adv - bcv] + i[acv - bdv + adu + bcu]\\ &\quad= [a(cu - dv) - b(cv + du)] + i[a(cv + du) + b(cu - dv)]\\ &\quad= (a + ib)[(cu - dv) + i(cv + du)]\\ &\quad= (a + ib)[(c + id)(u + iv)]\end{aligned}$$
which proves the associativity condition.

- (**additive identity**) For all $u, v \in V$,

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv$$

which proves that $0 + i0$ is an additive identity.

- (**additive inverse**) Let $u, v \in V$, then since $(-u), (-v) \in V$, we get

$$(u + iv) + ([-u] + i[-v]) = (u + [-u]) + i(v + [-v]) = 0 + i0$$

which proves that every element has an additive inverse.

- (**multiplicative identity**) Let $u, v \in V$, then

$$(1 + i0)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv$$

which proves that $1 = 1 + i0$ is a multiplicative identity.

- (**distributive property**) Let $a, b \in \mathbf{R}$ and $u_1, v_1, u_2, v_2 \in V$, then

$$\begin{aligned}
&(a + ib)[(u_1 + iv_1) + (u_2 + iv_2)] \\
&\qquad = (a + ib)([u_1 + u_2] + i[v_1 + v_2]) \\
&\qquad = (a[u_1 + u_2] - b[v_1 + v_2]) + i(a[v_1 + v_2] + b[u_1 + u_2]) \\
&\qquad = (au_1 + au_2 - bv_1 - bv_2) + i(av_1 + av_2 + v_1 + bv_2) \\
&\qquad = ([au_1 - bv_1] + [au_2 - bv_2]) + i([av_1 + bu_1] + [av_2 + bu_1]) \\
&\qquad = [(au_1 - bv_1) + i(av_1 + bu_1)] + [(au_2 - bv_2) + i(av_2 + bu_2)] \\
&\qquad = [(a + ib)(u_1 + iv_1)] + [(a + ib)(u_2 + iv_2)]
\end{aligned}$$

Similarly, for all $a, b, c, d \in \mathbf{R}$, and $u, v \in \mathbf{R}$, we have

$$\begin{aligned}
&[(a + ib) + (c{+}id)](u + iv) \\
&\qquad = ([a + c] + i[b + d])(u + iv) \\
&\qquad = ([a + c]u - [b + d]v) + i([a + c]v + [b + d]u) \\
&\qquad = (au + cu - bv - dv) + i(av + cv + bu + du) \\
&\qquad = ([au - bv] + [cu - dv]) + i([av + bu] + [cv + du]) \\
&\qquad = [(au - bv) + i(av + bu)] + [(cu - dv) + i(cv + du)] \\
&\qquad = (a + ib)(u + iv) + (c + id)(u + iv)
\end{aligned}$$

which proves the distributive property.

Therefore, $V_{\mathbf{C}}$ is a vector space under these definitions.

1C Subspaces

Exercise 1
For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution
Heyyy