Solutions to Elementary Number Theory (Second Edition) by David M. Burton

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July 16, 2025

Preface

The goal of this document is to share my personal solutions to the exercises in the Second Edition of Elementary Number Theory by David M. Burton during my reading. To make my solutions clear, for each exercise, I will assume nothing more than the content of the book and the results proved in the preceding exercises.

As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address:

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Contents

Chapter 1

Some Preliminary Considerations

1.1 Mathematical Induction

1. Establish the formulas below by mathematical induction:

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \geq 1$;

(b) $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \geq 1$;

(c) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$ for all $n \geq 1$;

(d) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$ for all $n \geq 1$;

(e) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$ for all $n \geq 1$;

Solution

(a) First, when $n = 1$, we have that both sides of the equation are equal to 1, so the basis for the induction is verified. Suppose now that the equation holds for a natural number k, then adding $k+1$ on both sides gives us

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

But since

$$\frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2},$$

then

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

which implies that the equation holds for $k+1$. Therefore, by induction, it holds for all $n \geq 1$.

(b) First, when $n = 1$, we have that both sides of the equation are equal to 1, so the basis for the induction is verified. Suppose now that the equation holds for a natural number k, then adding $2k + 1$ on both sides gives us

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

which implies that the equation holds for $k + 1$. Therefore, by induction, it holds for all $n \geq 1$.

(c) First, when $n = 1$, we have that both sides of the equation are equal to 2, so the basis for the induction is verified. Suppose now that the equation holds for a natural number k, then adding $(k + 1)(k + 2)$ on both sides gives us

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2).$$

But since

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{(k+1)(k+2)(k+3)}{3},$$

then

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

which implies that the equation holds for $k + 1$. Therefore, by induction, it holds for all $n \geq 1$.

(d) First, when $n = 1$, we have that both sides of the equation are equal to 1, so the basis for the induction is verified. Suppose now that the equation holds for a natural number k, then adding $(2k + 1)^2$ on both sides gives us

$$1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2 + (2k + 1)^2 = \frac{k(4k^2 - 1)}{3} + (2k + 1)^2.$$

But since

$$\begin{aligned} \frac{k(4k^2 - 1)}{3} + (2k + 1)^2 &= \frac{4k^3 - k + 3(2k + 1)^2}{3} \\ &= \frac{4k^3 - k + 12k^2 + 12k + 3}{3} \\ &= \frac{4k^3 + 12k^2 + 11k + 3}{3} \\ &= \frac{(k + 1)(4k^2 + 8k + 3)}{3} \\ &= \frac{(k + 1)(4(k + 1)^2 - 1)}{3}, \end{aligned}$$

then

$$1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2 + (2k + 1)^2 = \frac{(k + 1)(4(k + 1)^2 - 1)}{3}$$

which implies that the equation holds for $k + 1$. Therefore, by induction, it holds for all $n \geq 1$.

(e) First, when $n = 1$, we have that both sides of the equation are equal to 1, so the basis for the induction is verified. Suppose now that the equation holds for a natural number k, then adding $(k+1)^3$ on both sides gives us

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3.$$

But since

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2,$$

then

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

which implies that the equation holds for $k+1$. Therefore, by induction, it holds for all $n \geq 1$.

2. If $r \neq 1$, show that

$$a + ar + ar^2 + \ldots ar^n = \frac{a(r^{n+1} - 1)}{r - 1}$$

for any positive integer n.

Solution
When $n = 1$, both sides of the equation are equal to $a(r+1)$ so the basis for induction is verified. Suppose now that the equation holds for a positive integer k, then adding ar^{k+1} on both sides of the equation gives us

$$a + ar + ar^2 + \ldots ar^k + ar^{k+1} = \frac{a(r^{k+1} - 1)}{r - 1} + ar^{k+1}.$$

But since

$$\frac{a(r^{k+1} - 1)}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r - 1} = \frac{a(r^{k+2} - 1)}{r - 1},$$

then

$$a + ar + ar^2 + \ldots ar^k + ar^{k+1} = \frac{a(r^{k+2} - 1)}{r - 1}$$

and so the equation holds for all $k+1$. Therefore, it holds for all $n \geq 1$.

3. Use the Second Principle of Finite Induction to establish that

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1)$$

for all $n \geq 1$.

Solution
When $n = 1$, both sides of the equation are equal to $a - 1$, so the basis for the

induction is verified. Suppose now that there exists a positive integer k such that the equation holds for all $n = 1, ..., k$. From the identity

$$a^{n+1} - 1 = (a+1)(a^n - 1) - a(a^{n-1} - 1),$$

and by the inductive hypothesis for $n = k$ and $n = k - 1$, we obtain:

$$\begin{aligned}
a^{n+1} - 1 &= (a+1)(a-1)(a^{n-1} + \cdots + 1) - a(a-1)(a^{n-2} + \cdots + 1) \\
&= (a-1)[(a+1)(a^{n-1} + \cdots + 1) - a(a^{n-2} + \cdots + 1)] \\
&= (a-1)[(a+1)(a^{n-1} + \cdots + 1) - (a^{n-1} + \cdots + 1 - 1)] \\
&= (a-1)[(a+1)(a^{n-1} + \cdots + 1) - (a^{n-1} + \cdots + 1) + 1] \\
&= (a-1)[a(a^{n-1} + \cdots + 1) + 1] \\
&= (a-1)(a^n + a^{n-1} + \cdots + a + 1)
\end{aligned}$$

which proves that the equation holds for $n = k+1$. Therefore, by induction, it holds for all $n \geq 1$.

4. Prove that the cube of any integer can be written as the difference of two squares.

Solution
Using part (e) of exercice 1, we get

$$\begin{aligned}
n^3 &= (1^3 + 2^3 + \cdots + n^3) - (1^3 + 2^3 + \ldots (n-1)^3) \\
&= \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{n(n-1)}{2}\right]^2
\end{aligned}$$

which proves that any cube can be written as the difference of two squares.

5.

(a) Find the values of $n \leq 7$ for which $n! + 1$ is a perfect square (it is unknown whether $n! + 1$ is a square for any $n > 7$).

(b) True or false? For positive integers m and n, $(mn)! = m!n!$ and $(m+n)! = m! + n!$.

Solution

(a) For $n = 0, 1$, we have $n! + 1 = 2$ which is not a square. For $n = 2$, we have $2! + 1 = 3$ which is not a square. For $n = 3$, we have $3! + 1 = 7$ which is not a square. When $n = 4$ and $n = 5$, we obtain $4! + 1 = 5^2$ and $5! + 1 = 11^2$. For $n = 6$, we get $6! + 1 = 721$ which is strictly between $26^2 = 676$ and $27^2 = 729$ so it cannot be a square. Finally, for $n = 7$, we obtain $7! + 1 = 71^2$.

(b) In both cases, $m = n = 2$ is a counterexample since $(m+n)! = (mn)! = 24$ and $m!n! = m! + n! = 4$.

6. Prove that $n! > n^2$ for every integer $n \geq 4$, while $n! > n^3$ for every integer $n \geq 6$.

Solution
When $n = 4$, then $n! = 24$ and $n^2 = 16$ so the strict inequality is satisfied. Now that the basis for the induction is verified, suppose that the inequality is satisfied for a positive integer k, then multiplying on both sides by $k+1$ gives the inequality

$$(k+1)! > k^2(k+1) \geq (k+1)(k+1) = (k+1)^2$$

using the fact that $k^2 \geq k+1$ for all $k \geq 2$. Thus, since the inequality is also satisfied by $k+1$, then it is for all $n \geq 4$ by induction.

When $n = 6$, then $n! = 720$ and $n^3 = 216$ so the strict inequality is satisfied. Now that the basis for the induction is verified, suppose that the inequality is satisfied for a positive integer k, then multiplying on both sides by $k+1$ gives the inequality

$$(k+1)! > k^3(k+1) \geq (k+1)^2(k+1) = (k+1)^3$$

using the fact that $k^3 \geq (k+1)^2$ for all $k \geq 4$. Thus, since the inequality is also satisfied by $k+1$, then it is for all $n \geq 6$ by induction.

7. Use mathematical induction to derive the formula

$$1 \cdot (1!) + 2 \cdot (2!) + 3 \cdot (3!) + \cdots + n \cdot (n!) = (n+1)! - 1$$

for all $n \geq 1$.

Solution
If $n = 1$, then both expressions on the two side of the desired equation are equal to 1; so the basis for the induction is verified. Next, if we suppose that the equation holds for a positive integer k, then adding $(k+1) \cdot (k+1)!$ on both sides gives us

$$\begin{aligned} 1 \cdot (1!) + 2 \cdot (2!) + 3 \cdot (3!) + \cdots + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+2) \cdot (k+1)! - 1 \\ &= (k+2)! - 1 \end{aligned}$$

which shows that the equation also holds for $n = k+2$. Therefore, by induction, it holds for all $n \geq 1$.

8.

(a) Verify that

$$2 \cdot 6 \cdot 10 \cdot 14 \cdot \ldots \cdot (4n-2) = \frac{(2n)!}{n!}$$

for all $n \geq 1$.

(b) Use part (a) to to obtain the inequality $2^n(n!)^2 \leq (2n)!$ for all $n \geq 1$.

Solution

(a) Let's prove it by induction on n. When $n = 1$, then both expressions on the two sides of the equation are equal to 2, so the basis for the induction is verified. Now, if we suppose that the equation holds for a positive integer k, then by multiplying both sides by $(4k + 2)$ gives us

$$2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n+2) = \frac{(2n)!}{n!}(4n+2) = \frac{(2n)!}{n!} \cdot \frac{(2n+1)(2n+2)}{n+1} = \frac{(2(n+1)!)}{(n+1)!}.$$

Thus, the equation also holds for $n = k + 1$. Therefore, by induction, it holds for all $n \geq 1$.

(b) First fix a $n \geq 1$ and notice that

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \leq 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1).$$

Next, multiplying both sides by 2^n and using part (a) gives us

$$2^n \cdot n! \leq 2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n - 2) = \frac{(2n)!}{n!}.$$

Finally, multiplying both sides by $n!$ gives us the desired inequality.

9. Establish the Bernoulli inequality: if $1 + a > 0$, then

$$(1 + a)^n \leq 1 + na$$

for all $n \geq 1$.

Solution
Let's prove it by induction on n. When $n = 1$, then $(1+a)^n = 1+a \geq 1+a = 1+na$, and so the basis for induction is verified. Next, suppose that the inequality holds for a positive integer k, then multiplying both sides by $(1 + a)$ preserves the inequality since it is positive. Hence, we obtain:

$$\begin{aligned}(1 + a)^{k+1} &= (1 + a)(1 + a)^k \\ &\geq (1 + a)(1 + ka) \\ &= 1 + (k + 1)a + ka^2 \\ &\geq 1 + (k + 1)a\end{aligned}$$

which shows that the inequality must also hold for $n = k + 1$. Therefore, by induction, it holds for all $n \geq 1$.

10. Prove by mathematical induction that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

for all $n \geq 1$.

Solution
When $n = 1$, both sides of the inequality are equal to 1, so the inequality holds and

so the basis for the induction is verified. Next, if we suppose that the inequality holds for a positive integer k, then adding $\frac{1}{(k+1)^2}$ on both sides gives us

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2}\leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}.$$

But notice that

$$\begin{aligned}0\leq 1 &\implies 2k+k^2\leq 1+2k+k^2\\ &\implies 2k+k^2\leq (k+1)^2\\ &\implies 1-\frac{(k+1)^2}{k}\leq -(k+1)\\ &\implies \frac{1}{(k+1)^2}-\frac{1}{k}\leq -\frac{1}{(k+1)}\end{aligned}$$

and so

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2}\leq 2+\frac{1}{(k+1)^2}-\frac{1}{k}\leq 2-\frac{1}{k+1}.$$

Thus, the inequality holds for $n=k+1$. Therefore, by induction, the inequality holds for all $n\geq 1$.

1.2 The Binomial Theorem

1. Prove that for $n \geq 1$:

(a) $\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!} 2^n.$

(b) $\binom{4n}{2n} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (4n-1)}{[1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)]^2} \binom{2n}{n}.$

Solution

(a) Let's prove it by induction. When $n = 1$, we have

$$\binom{2n}{n} = 2$$

and

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!} 2^n = 2$$

and so it holds in that case. If we now suppose that it holds when $n = k$ for some integer $k \geq 1$, then it follows that

$$\frac{(2k)!}{(k!)^2} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{k!} 2^k.$$

Multiplying both sides by $\frac{(2k+1)(2k+2)}{(k+1)^2}$ gives us

$$\begin{aligned} \binom{2(k+1)}{k+1} &= \frac{(2k+1)(2k+2)}{(k+1)^2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{k!} 2^k \\ &= 2 \frac{(2k+1)}{k+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{k!} 2^k \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1)}{(k+1)!} 2^{k+1} \end{aligned}$$

which shows that the equation also holds for $n = k+1$. Therefore, by induction, it holds for all integers $n \geq 1$.

(b) First, notice that by part (a), it suffices to prove that

$$\binom{4n}{2n} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (4n-1)}{n! \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} 2^n$$

holds for all $n \geq 1$. Let's prove it by induction on n. When $n = 1$, then both sides are equal to 6 and so the statement holds in that case. Suppose now that it holds for some integer $n = k \geq 1$, then

$$\frac{(4k)!}{(2k!)^2} = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (4k-1)}{k! \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)} 2^k.$$

Multiplying both sides by $\frac{(4k+1)(4k+2)(4k+3)(4k+4)}{(2k+1)^2(2k+2)^2}$ gives us

$$\begin{aligned}\binom{4(k+1)}{2(k+1)} &= \frac{(4k+1)(4k+2)(4k+3)(4k+4)}{(2k+1)^2(2k+2)^2}\cdot\frac{1\cdot 3\cdot 5\cdot\dots\cdot(4k-1)}{k!\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2k-1)}2^k\\ &= \frac{(4k+2)(4k+4)}{(2k+1)(2k+2)^2}\cdot\frac{1\cdot 3\cdot 5\cdot\dots\cdot(4k+1)}{k!\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)}2^k\\ &= \frac{4k+4}{(2k+2)(2k+2)}\cdot\frac{1\cdot 3\cdot 5\cdot\dots\cdot(4k+1)}{k!\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)}2^{k+1}\\ &= \frac{1}{k+1}\cdot\frac{1\cdot 3\cdot 5\cdot\dots\cdot(4k+1)}{k!\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)}2^{k+1}\\ &= \frac{1\cdot 3\cdot 5\cdot\dots\cdot(4k+1)}{(k+1)!\cdot 1\cdot 3\cdot 5\cdot\dots\cdot(2k+1)}2^{k+1}\end{aligned}$$

which shows that the equation also holds for $n = k+1$. Therefore, by induction, it holds for all integers $n \ge 1$.

2. If $2 \le k \le n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}, \qquad n \ge 4.$$

Solution
This simply follows from Pascal's Rule:

$$\begin{aligned}\binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} &= \left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right] + \left[\binom{n-2}{k-1} + \binom{n-2}{k}\right]\\ &= \binom{n-1}{k-1} + \binom{n-1}{k}\\ &= \binom{n}{k}.\end{aligned}$$

3. For $n \ge 1$, derive each of the identities below:

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$; [*Hint:* Let $a = b = 1$ in the binomial theorem.]

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$;

(c) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$; [*Hint:* After expanding $n(1+b)^{n-1}$ by the binomial theorem, let $b = 1$: note also that

$$n\binom{n-1}{k} = (k+1)\binom{n}{k+1}.]$$

(d) $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n$;

(e) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \dots$

$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}$; [*Hint:* Use parts (a) and (b).]

(f) $\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \dots + \frac{(-1)^n}{n+1}\binom{n}{n} = \frac{1}{n+1}$; [*Hint:* the left-hand side equals

$\frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \dots + (-1)^n\binom{n+1}{n+1}\right]$.]

Solution

(a) Taking $a = b = 1$ in the Binomial Theorem gives us

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}.$$

(b) Taking $a = 1$ and $b = -1$ in the Binomial Theorem gives us

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots (-1)^n \binom{n}{n}.$$

(c) From the hint, it follows that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} = n\binom{n-1}{0} + n\binom{n-1}{1} + \dots + n\binom{n-1}{n-1} = n2^{n-1}$$

where the last equality follows from part (a).

(d) Taking $a = 1$ and $b = 2$ in the Binomial Theorem gives us

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \dots 2^n \binom{n}{n}.$$

(e) From part (b), we have that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$$

Thus, using part (a), we get

$$\begin{aligned}\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots &= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots \\ &= \frac{1}{2}\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}\right] \\ &= 2^{n-1}\end{aligned}$$

(f) Using the hint, we easily get

$$\begin{aligned}
&\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \cdots + \frac{(-1)^n}{n+1}\binom{n}{n} \\
=&\frac{1}{n+1}\left[\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \cdots + (-1)^n\binom{n+1}{n+1}\right] \\
=&\frac{1}{n+1}\left(1 - \left[\binom{n}{0} - \binom{n+1}{1} + \binom{n+1}{2} - \binom{n+1}{3} + \cdots + (-1)^{n+1}\binom{n+1}{n+1}\right]\right) \\
=&\frac{1}{n+1}(1-0) \\
=&\frac{1}{n+1}
\end{aligned}$$

4. Prove that for $n \geq 1$:

(a) $\binom{n}{r} < \binom{n}{r+1}$ if and only if $0 \leq r < \frac{1}{2}(n-1)$.

(b) $\binom{n}{r} > \binom{n}{r+1}$ if and only if $n-1 \geq r > \frac{1}{2}(n-1)$.

(c) $\binom{n}{r} = \binom{n}{r+1}$ if and only if n is an odd integer, and $r = \frac{1}{2}(n-1)$.

Solution

(a) Let $0 \leq r \leq n-1$ be an integer, then

$$\begin{aligned}
\binom{n}{r} < \binom{n}{r+1} &\iff \frac{n!}{(n-r)!r!} < \frac{n!}{(n-r-1)!(r+1)!} \\
&\iff (n-r-1)!(r+1)! < (n-r)!r! \\
&\iff r+1 < n-r \\
&\iff r < \frac{1}{2}(n-1).
\end{aligned}$$

(b) Let $0 \leq r \leq n-1$ be an integer, then

$$\begin{aligned}
\binom{n}{r} > \binom{n}{r+1} &\iff \frac{n!}{(n-r)!r!} > \frac{n!}{(n-r-1)!(r+1)!} \\
&\iff (n-r-1)!(r+1)! > (n-r)!r! \\
&\iff r+1 > n-r \\
&\iff r > \frac{1}{2}(n-1).
\end{aligned}$$

(c) Let $0 \leq r \leq n-1$ be an integer, then

$$\begin{aligned}
\binom{n}{r} = \binom{n}{r+1} &\iff \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r-1)!(r+1)!} \\
&\iff (n-r-1)!(r+1)! = (n-r)!r! \\
&\iff r+1 = n-r \\
&\iff r = \frac{1}{2}(n-1) \\
&\iff n = 2r+1.
\end{aligned}$$

5. For $n \geq 1$, show that the expressions $\dfrac{(2n)!}{n!(n+1)!}$ and $\dfrac{(3n)!}{6^n n!}$ are both integers.

Solution
For the first expression, it suffices to notice that

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n+1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \frac{(2n)!(n+1) - (2n)!n}{n!(n+1)!} \\ &= \frac{(2n)!}{n!(n+1)!}.\end{aligned}$$

Since the binomial coefficients are integers, then it follows that the expression $\frac{(2n)!}{n!(n+1)!}$ is also an integer. For the second expression, let's prove it by induction. When $n = 1$, we have

$$\frac{(3n)!}{6^n n!} = \frac{3!}{6 \cdot 1} = 1$$

which proves that it holds for $n = 1$. Suppose now that the expression is an integer for some $n = k \geq 1$, then

$$\begin{aligned}\frac{(3(k+1))!}{6^{k+1}(k+1)!} &= \frac{(3k+1)(3k+2)(3k+3)}{6(k+1)} \cdot \frac{(3k)!}{6^k k!} \\ &= \frac{(3k+1)(3k+2)}{2} \cdot \frac{(3k)!}{6^k k!}\end{aligned}$$

where $\frac{(3k)!}{6^k k!}$ is an integer by the inductive hypothesis. Moreover, notice that $3k+1$ and $3k+2$ are two consecutive numbers and so one of them must be divisible by two. Thus, $\frac{(3k+1)(3k+2)}{2}$ is also an integer. Therefore, the case $n = k+1$ also holds since $\frac{(3(k+1))!}{6^{k+1}(k+1)!}$ can be written as the product of two integers.

6.

(a) For $n \geq 2$, prove that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}.$$

[*Hint* Use induction and Pascal's rule.]

(b) From part (a) and the fact that $\dbinom{m}{2} + \dbinom{m+1}{2} = m^2$ for $m \geq 2$, deduce the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

(a) Let's prove it by induction on n. When $n = 2$, we have

$$\binom{2}{2} + \cdots + \binom{n}{2} = \binom{2}{2} = 1 = \binom{3}{3} = \binom{n+1}{3}$$

and so the proposition holds in that case. Suppose now that the proposition holds for $n = k \geq 2$, then

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k}{2} = \binom{k+1}{3}.$$

Adding $\binom{k+1}{2}$ on both sides gives

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k+1}{2} = \binom{k+1}{2} + \binom{k+1}{3} = \binom{k+2}{3}$$

and so the proposition holds for $n = k + 1$. Therefore, by induction, it holds for all $n \geq 2$.

(b) Using the fact that $\binom{m}{2} + \binom{m+1}{2} = m^2$, we can write

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + n^2 &= 1 + \left[\binom{2}{2} + \binom{3}{2}\right] + \left[\binom{3}{2} + \binom{4}{2}\right] + \cdots + \left[\binom{n}{2} + \binom{n+1}{2}\right] \\
&= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2} \\
&= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= \frac{2(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\
&= \frac{2(n+1)n(n-1) + 3(n+1)n}{6} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

which proves the desired formula.

7. For $n \geq 1$, verify that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \binom{2n+1}{3}.$$

Solution

Let's prove it by induction on n. When $n = 1$, we have

$$1^2 + \cdots + (2n-1)^2 = 1 = \binom{3}{3} = \binom{2n+1}{3}.$$

Thus, the proposition holds for $n = 1$. Suppose now that it holds for $n = k \geq 1$, then

$$\begin{aligned}
1^2+3^2+5^2+\cdots+(2k+1)^2 &= (1^2+3^2+5^2+\cdots+(2k-1)^2)+(2k+1)^2\\
&= \binom{2k+1}{3}+(2k+1)^2\\
&= \frac{(2k+1)(2k)(2k-1)}{6}+(2k+1)(2k+1)\\
&= \frac{(2k+1)[2k(2k-1)+6(2k+1)]}{6}\\
&= \frac{(2k+1)(4k^2+10k+6)}{6}\\
&= \frac{(2k+3)(2k+2)(2k+1)}{6}\\
&= \binom{2(k+1)+1}{3}
\end{aligned}$$

which shows that it holds for $n = k+1$. Therefore, by induction, the proposition holds for all $n \geq 1$.

8. Establish the inequality $2^n < \binom{2n}{n} < 2^{2n}$ for $n > 1$.

Solution

Let's prove it by induction on n. When $n = 2$, we have

$$2^n = 4 < 6 = \binom{2n}{2} < 16 = 2^{2n}$$

and so it holds for this case. Suppose now that holds for an integer $n = k \geq 2$, then

$$2^k < \binom{2k}{k} < 2^{2k}.$$

Multiplying both sides by $\frac{(2k+2)(2k+1)}{(n+1)^2} = 2\frac{2k+1}{k+1}$ gives us

$$2^{k+1} \leq 2^k \cdot 2\frac{2k+1}{k+1} < \binom{2(k+1)}{k+1} < 2^{2k} \cdot 2\frac{2k+1}{k+1} \leq 2^{2(k+1)}$$

which shows that it holds for $n = k+1$. Therefore, by induction, the proposition holds for all integers $n > 1$.

1.3 Early Number Theory

1.

(a) A number is triangular if and only if it is of the form $n(n+1)/2$ for some $n \geq 1$.

(b) The integer n is a triangular number if and only if $8n+1$ is a perfect square.

(c) The sum of any two consecutive triangular number is a perfect square.

(d) If n is a triangular number, then so are $9n+1$, $25n+3$ and $49n+6$.

Solution

(a) We already proved in the previous sections that

$$1+2+3\cdots+n = \frac{n(n+1)}{2}$$

so it directly follows that a number of the form of one of the side of the equation can be equivalently written in the form of the other side of the equation.

(b) First, let n be a triangular number, then there is an integer k for which $n = k(k+1)/2$. It follows that

$$8n+1 = 4k(k+1)+1 = 4k^2+4k+1 = (2k+1)^2$$

which shows that $8n+1$ is a perfect square. Suppose now that $8n+1$ is a perfect square for a given integer n. Since $8n+1$ is odd, then it must be the square of an odd number: $8n+1 = (2k+1)^2$. Thus:

$$\begin{aligned} 8n+1 = (2k+1)^2 &\implies 8n+1 = 4k^2+4k+1 \\ &\implies n = \frac{1}{2}k^2 + \frac{1}{2}k \\ &\implies n = \frac{k(k+1)}{2}. \end{aligned}$$

Since n can be written as $k(k+1)/2$, then it is a triangular number.

(c) Let a and b be triangular numbers, then a can be written as $n(n+1)/2$. Since b must have the same form while being the direct successor of a, then b must be equal to $(n+1)(n+2)/2$. Hence:

$$\begin{aligned} a+b &= \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\ &= \frac{n^2+n+n^2+3n+2}{2} \\ &= \frac{2n^2+4n+2}{2} \\ &= n^2+2n+1 \\ &= (n+1)^2 \end{aligned}$$

and so the sum of two consecutive triangular numbers is a perfect square.

(d) Let n be a triangular number, then n can be written as $k(k+1)/2$. It follows that

$$\begin{aligned} 9n+1 &= 9 \cdot \frac{k(k+1)}{2} + 1 \\ &= \frac{1}{2}(9k(k+1)+2) \\ &= \frac{1}{2}(9k^2+9k+2) \\ &= \frac{1}{2}((3k+1)^2+(3k+1)) \\ &= \frac{(3k+1)((3k+1)+1)}{2}, \end{aligned}$$

$$\begin{aligned} 25n+3 &= 25 \cdot \frac{k(k+1)}{2} + 3 \\ &= \frac{1}{2}(25k(k+1)+6) \\ &= \frac{1}{2}(25k^2+25k+6) \\ &= \frac{1}{2}((5k+2)^2+(5k+2)) \\ &= \frac{(5k+2)((5k+2)+1)}{2} \end{aligned}$$

$$\begin{aligned} 49n+6 &= 49 \cdot \frac{k(k+1)}{2} + 6 \\ &= \frac{1}{2}(49k(k+1)+12) \\ &= \frac{1}{2}(49k^2+49k+12) \\ &= \frac{1}{2}((7k+3)^2+(7k+3)) \\ &= \frac{(7k+3)((7k+3)+1)}{2} \end{aligned}$$

and so $9n+1$, $25n+3$ and $49n+6$ are all triangular numbers.

2. If t_n denotes the nth triangular number, prove that in terms of the binomial coefficients

$$t_n = \binom{n+1}{2}, \qquad n \geq 1.$$

Solution

Let $n \geq 1$. We already proved that we can write t_n as $n(n+1)/2$, so using the definition of the binomial coefficients, we have that

$$\binom{n+1}{2} = \frac{(n+1)!}{(n-1)!2!} = \frac{(n+1)n}{2} = t_n$$

which proves the desired formula.

3. Derive the following formula for the sum of triangular numbers, attributed to the Hindu mathematician Aryabhatta (circa 500 A.D.):

$$t_1 + t_2 + t_3 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}, \qquad n \geq 1.$$

[*Hint:* Group the terms on the left-hand side in pairs, noting the identity $t_{k-1} + t_k = k^2$.]

Solution
Let's prove it by cases. If $n = 2k$, then

$$\begin{aligned}
t_1 + t_2 + \cdots + t_{2k-1} + t_{2k} &= (t_1 + t_2) + \cdots + (t_{2k-1} + t_{2k}) \\
&= 2^2 + 4^2 + \cdots + (2k)^2 \\
&= 4(1^2 + 2^2 + \cdots + k^2) \\
&= 4 \cdot \frac{k(k+1)(2k+1)}{6} \\
&= \frac{2k(2k+2)(2k+1)}{6} \\
&= \frac{n(n+2)(n+1)}{6}.
\end{aligned}$$

Suppose now that $n = 2k + 1$, then using the previous result:

$$\begin{aligned}
t_1 + t_2 + \cdots + t_{n-1} + t_n &= \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(n-1+3)}{6} \\
&= \frac{n(n+1)(n+2)}{6}.
\end{aligned}$$

Therefore, the formula is true for all $n \geq 1$.

4. Prove that the square of any odd multiple of 3 is the difference of two triangular numbers; specifically that

$$9(2n+1)^2 = t_{9n+4} - t_{3n+1}.$$

Solution
By direct calculation:

$$\begin{aligned}
t_{9n+4} - t_{3n+1} &= \frac{(9n+4)(9n+5)}{2} - \frac{(3n+1)(3n+2)}{2} \\
&= \frac{81n^2 + 81n + 20 - 9n^2 - 9n - 2}{2} \\
&= \frac{72n^2 + 72n + 18}{2} \\
&= 36n^2 + 36n + 9 \\
&= 9(4n^2 + 4n + 1) \\
&= 9(2n+1)^2.
\end{aligned}$$

5. In the sequence of triangular numbers, find

(a) two triangular numbers whose sum and difference are also triangular numbers;

(b) three successive triangular numbers whose product is a perfect square;

(c) three successive triangular numbers whose sum is a perfect square.

Solution

(a) Take $15 = 1+2+3+4+5$ and $21 = 1+2+3+4+5+6$ since their sum is $36 = 1+2+3+4+5+6+7+8$ and their difference is $6 = 1+2+3$.

(b) Take

$$300 = 1+2+3+\cdots+24,$$
$$325 = 1+2+3+\cdots+25,$$
$$351 = 1+2+3+\cdots+26$$

since their product is

$$300 \cdot 325 \cdot 351 = (5850)^2.$$

(c) Take $15 = 1+2+3+4+5$, $21 = 1+2+3+4+5+6$ and $28 = 1+2+3+4+5+6+7$ since their sum is

$$15 + 21 + 28 = 64 = 8^2.$$

6.

(a) If the triangular number t_n is a perfect square, prove that $t_{4n(n+1)}$ is also a square.

(b) Use part (a) to find three examples of squares which are also triangular numbers.

Solution

(a) Suppose that t_n is a perfect square, then there exists a k such that $k^2 = n(n+1)/2$. It follows that

$$\begin{aligned} t_{4n(n+1)} &= \frac{4n(n+1)[4n(n+1)+1]}{2} \\ &= 2^2 \cdot \frac{n(n+1)}{2} \cdot (4n^2+4n+1) \\ &= (2k(2n+1))^2 \end{aligned}$$

which shows that $t_{4n(n+1)}$ is a square.

(b) Using part (a), it suffices to find one such number to deduce infinitely many others. Since $6^2 = 1+2+3+4+5+6+7+8 = t_8$, then t_{288} and t_{332928} must also be squares.

7. Show that the difference between squares of two consecutive triangular numbers is always a cube.

Solution
Let t_n and t_{n+1} be two consecutive triangular numbers, then

$$\begin{aligned} t_{n+1}^2 - t_n^2 &= \frac{(n+1)^2(n+2)^2}{4} - \frac{n^2(n+1)^2}{4} \\ &= \frac{(n+1)^2}{4}\left[(n+2)^2 - n^2\right] \\ &= \frac{(n+1)^2}{4}(4n+4) \\ &= (n+1)^3. \end{aligned}$$

8. Prove that the sum of the reciprocals of the first n triangular numbers is less than 2; that is,

$$1/1 + 1/3 + 1/6 + 1/10 + \cdots + 1/t_n < 2.$$

[*Hint:* Observe that $\dfrac{2}{n(n+1)} = 2\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$.]

Solution
By direct calculation:

$$\begin{aligned} \frac{1}{1} + \frac{1}{3} + \frac{1}{10} + \cdots + \frac{1}{t_n} &= \frac{2}{1\cdot 2} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \cdots + \frac{2}{n(n+1)} \\ &= 2\left(\frac{1}{1} - \frac{1}{2}\right) + 2\left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + 2\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 2\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}\right) \\ &= 2\left(1 - \frac{1}{n+1}\right) \\ &< 2. \end{aligned}$$

9.

(a) Establish the identity $t_x = t_y + t_z$, where

$$x = 1/2\; n(n+3) + 1, \quad y = n+1, \quad z = 1/2\; n(n+3),$$

and $n \geq 1$, thereby proving that there are infinitely many triangular numbers which are the sum of two other such numbers.

(b) Find three examples of triangular numbers which are sums of two other triangular numbers.

Solution

(a) By direct calculation:

$$\begin{aligned}
t_y + t_z &= \frac{y(y+1)}{2} + \frac{z(z+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} + \frac{\frac{n(n+3)}{2}\left(\frac{n(n+3)}{2}+1\right)}{2} \\
&= \frac{(n+1)(n+2)}{2} + \frac{n(n+3)(n(n+3)+2)}{8} \\
&= \frac{[n(n+3)]^2 + 2n(n+3) + 4(n+1)(n+2)}{8} \\
&= \frac{[n(n+3)]^2 + 2n(n+3) + 4n^2 + 4\cdot 3n + 8}{8} \\
&= \frac{[n(n+3)]^2 + 2n(n+3) + 4n(n+3) + 8}{8} \\
&= \frac{[n(n+3)]^2 + 6n(n+3) + 8}{8} \\
&= \frac{[n(n+3)+2][n(n+3)+4]}{8} \\
&= \frac{\left(\frac{n(n+3)}{2}+1\right)\left(\frac{n(n+3)}{2}+2\right)}{2} \\
&= \frac{x(x+1)}{2} \\
&= t_x.
\end{aligned}$$

(b) By taking plugging $n = 1$, $n = 2$ and $n = 3$ in the previous equation, we obtain that $t_3 = t_2 + t_2$, $t_6 = t_5 + t_3$ and $t_{10} = t_9 + t_4$.