Algebraic Geometry : Homework 5

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Exercise 3: For a homogeneous polynomial $f \in k[x_1, ..., x_{n+1}]$ of degree $d \geq 1$, let $U_f \subset \mathbb{P}^n(k)$ be the open subset

$$U_f := \{p \in \mathbb{P}^n(k) \mid f(p) \neq 0\} = \mathbb{P}^n(k) \setminus V_p(f).$$

In the last lecture, we have seen that U_f is an affine variety when $f = a_1x_1 + \cdots + a_{n+1}x_{n+1}$ is a linear polynomial. Use it to show that U_f is affine for any f.
Hint: use the Veronese embedding.

Solution : If we let m_1, m_2, ..., m_N denote the set of monomials of degree d (here, $N = \binom{n+d}{n}$)), then we can write $f = \sum_i a_i m_i$ where $a_i \in k$. From that, we can define $g = \sum_i a_i x_i \in k[x_1, ..., x_N]$ a linear map. If we consider the Veronomese map ν_d : $\mathbb{P}^n(k) \to \mathbb{P}^{N-1}(k)$, we have that $\nu_d^*(g) = f$ by construction of g. Hence, we can write $U_f = \nu_d^{-1}(U_g) = V \cap U_g$ where V is the image of ν_d. Since U_g is affine by the content of the last lecture, we have that $V \cap U_g$ must be affine as well (V is a projective variety). Therefore, U_f is affine.