Higher Algebra 1 : Assignment 6

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Exercise 39: Let G be a finite group of automorphisms of a commutative ring A, and let A^G denote the subring of G-invariants, that is the collection of all $a \in A$ such that $\sigma(a) = a$ for all $\sigma \in G$. Prove that A is integral over A^G.

Solution : Let $a \in A$ and $a_1, ..., a_n$ be the elements in the orbit of A. There are finitely many elements in the orbit since G is finite. Notice that G acts as a permutation group on the orbit of a because each element in the orbit can be written as $\sigma(a)$ with $\sigma \in G$. Consider the polynomial

$$p(x) = (x - a_1)(x - a_2) \cdots (x - a_n) \in A[x].$$

Clearly, $p(x)$ is monic and $p(a) = 0$ since a is in its orbit. Next, if we expand the product form of $p(x)$, we get an expression

$$p(x) = x^n + s_{n-1}x^{n-1} + \cdots + s_0$$

where $s_{n-1} = -(a_1 + \cdots + a_n)$, $s_{n-2} = a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n$, ..., $s_0 = (-1)^n$. Since $\sigma(-1) = -1$ for all $\sigma \in G$, and G acts as a permutation group on the orbit of A, then s_i is G-invariant by symmetry. Thus, $p(a) = 0$ where $p(x) \in A^G[x]$ is a monic polynomial. Therefore, A is integral over A^G.

Exercise 40: In the situation of the previous exercise, let $\mathfrak{p}$ be a prime ideal of A^G, and let $\mathscr{P}$ be the set of prime ideals P of A such that $P \cap A^G = \mathfrak{p}$. Show that G acts transitively on $\mathscr{P}$. In particular, $\mathscr{P}$ is finite.

Solution : First, let's make sure that G acts on $\mathscr{P}$. Clearly, the identity element σ_0 in G is the identity function on A, hence, for all $P \in \mathscr{P}$, we have $\sigma_0 P = \sigma_0(P) = P$.

Next, let's show that $\sigma(P) \in \mathscr{P}$ for all $\sigma \in G$ and $P \in \mathscr{P}$. Let $Q = \sigma(P)$, since P is prime, then P is nonempty which implies that Q is nonempty. If a and b are in Q, then $\sigma^{-1}(a)$ and $\sigma^{-1}(b)$ are in P, and hence, $\sigma^{-1}(a+b) \in P$. Thus, $a+b \in Q$ and so Q is closed under addition. Similarly, Q is closed under multiplication by elements of A. Therefore, Q is an ideal of A. Let $ab \in Q$, then $\sigma^{-1}(a)\sigma^{-1}(b) \in P$, and hence, $\sigma^{-1}(a) \in P$ or $\sigma^{-1}(b) \in P$ by primality of P. It follows that $a \in Q$ or $b \in Q$, and hence, Q is a prime ideal. Finally, if $a \in Q \cap A^G$, then $\varphi(a) = a$ for all $\varphi \in G$, and $\sigma^{-1}(a) \in a$. Since $\sigma^{-1} \in G$, then $a = \sigma^{-1}(a) \in P$ and so $a \in P \cap A^G = \mathfrak{p}$. Conversely, if $a \in \mathfrak{p} = P \cap A^g$, then $a = \sigma(a) \in Q$ and $a \in A^G$, and so $a \in Q \cap A^G$. Thus, $Q \cap A^G$. Therefore, $\sigma(P) \in \mathscr{P}$.

To finish the proof that G acts on $\mathscr{P}$, we have to show that $g(hP) = (gh)P$ for all $g, h \in G$ and $P \in \mathscr{P}$. But since, σP is defined as $\sigma(P)$ for all $\sigma \in G$, then it follows directly from the definition. Therefore, G acts on $\mathscr{P}$.

To show that G acts transitively on $\mathscr{P}$, let $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathscr{P}$ and let's show that $\mathfrak{p}_1 = \sigma\mathfrak{p}_2$ for some $\sigma \in G$. Let $x \in \mathfrak{p}_1$, then $\prod_{\sigma \in G} \sigma(x) \in \mathfrak{p}_1 \cap A^G = \mathfrak{p} \subset \mathfrak{p}_2$, hence $\sigma(x) \in \mathfrak{p}_2$ for some $\sigma \in G$ by primality of $\mathfrak{p}_2$. Equivalently, this implies that

$$x \in (\sigma)^{-1}(\mathfrak{p}_2) = \sigma^{-1}(\mathfrak{p}_2) \subset \bigcup_{\sigma \in G} \sigma(\mathfrak{p}_2).$$

Since it holds for all $x \in \mathfrak{p}_1$, then $\mathfrak{p}_1 \subset \bigcup_{\sigma \in G} \sigma(\mathfrak{p}_2)$. Since the $\mathfrak{p}_1$ and $\sigma(\mathfrak{p}_2)$ are prime ideals for all i, then the contrapositive of the hint

$$\forall i, \mathfrak{a} \not\subseteq \mathfrak{q}_i \implies \mathfrak{a} \not\subseteq \bigcup_{i=1}^{n} \mathfrak{q}_i$$

gives us that $\mathfrak{p}_1 \subset \sigma(\mathfrak{p}_2)$ for some $\sigma \in G$. Since $\mathfrak{p}_1 \subset \sigma\mathfrak{p}_2$ are prime ideals of A (which is integral over A^G) that satisfy $\mathfrak{p}_1 \cap A^G = \mathfrak{p}$ and $\sigma\mathfrak{p}_2 \cap A^G = \mathfrak{p}$, then $\mathfrak{p}_1 \cap A^G = \sigma\mathfrak{p}_2 \cap A^G$. It follows that $\mathfrak{p}_1 = \sigma\mathfrak{p}_2$ by Proposition 6.4.3. Therefore, G acts transitively on $\mathscr{P}$.

Exercise 41: Let $A \subseteq B$ be an integral extension and $\varphi : A \to k$ a surjective ring homomorphism of A onto an algebraically closed field k. Prove that φ can be extended to a ring homomorphism $B \to k$. Further, give an example showing that the assumption that k is algebraically closed is necessary.

Solution : TODO

Consider the rings $A = \mathbb{Z}$, $B = \mathbb{Z}[i]$ and the field $k = \mathbb{Z}/3\mathbb{Z}$ with the natural surjective homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. Notice that $\mathbb{Z}[i]$ is an integral extension of $\mathbb{Z}$ since i is integral over $\mathbb{Z}$ (i is a root of $x^2 + 1 \in \mathbb{Z}[x]$). However, there is no homomorphism $\chi : \mathbb{Z}[i] \to \mathbb{Z}/3\mathbb{Z}$. To prove this, suppose that there is a homomorphism $\chi : \mathbb{Z}[i] \to \mathbb{Z}/3\mathbb{Z}$. Since $i^2 + 1 = 0$, then $\chi(i)^2 + 1 = 0$ which is impossible in $\mathbb{Z}/3\mathbb{Z}$. This shows that the algebraically closed condition is necessary.

Exercise 46: Prove or disprove: A subring of a Noetherian ring is a Noetherian ring.

Solution : Let R be a non-noetherian ring and k be a field containing R (such a field k may not always exist for an arbitrary ring R but this is not important here since we assume R to have this property). Clearly, k is noetherian since k only has one ideal, the zero ideal. Hence, R is a non-noetherian subring of a noetherian ring. For an actual example, consider the subring $\mathbb{C}[x_1, x_2, ...]$ of $\mathbb{C}(x_1, x_2, ...)$, then $\mathbb{C}[x_1, x_2, ...]$ is non-noetherian while $\mathbb{C}(x_1, x_2, ...)$ is since it is a field.

Exercise 47: Let M be an R-module and let N_1, N_2 be submodules of M. Prove that if M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$.

Solution : Consider the map $f : M \to M/N_1 + M/N_2 : x \mapsto (x + N_1, x + N_2)$. The kernel of f is precisely $N_1 \cap N_2$ since $f(x) = 0$ iff $x + N_1 = 0$ and $x + N_2 = 0$ iff $x \in N_1 \cap N_2$. Hence, by the First Isomorphism Theorem, we have an induced map $f_0 : M/(N_1 \cap N_2) \to M/N_1 + M/N_2$. In other words, $M/(N_1 \cap N_2)$ can be viewed as a submodule of $M/N_1 + M/N_2$. By definition of a module being noetherian, it is clear that any submodule of a noetherian is noetherian, hence, it suffices to prove that $M/N_1 + M/N_2$ is noetherian. Let $S_1 \subseteq S_2 \subseteq \dots$ be an increasing chain of submodules of $M/N_1 + M/N_2$. If we denote by $S_n^{(i)}$ the projection of S_n on M/N_i, we get an increasing chain $S_1^{(i)} \subseteq S_2^{(i)} \subseteq \dots$ that stabilizes. If $S_N^{(i)} = S_{N+1}^{(i)} = \dots$ for some N large enough and for $i = 1, 2$, then $S_N = S_{N+1} = \dots$ for the same N. Therefore, $M/N_1 + M/N_2$ is noetherian, and hence, $M/(N_1 \cap N_2)$ is noetherian.

Exercise 48: Let R be a noetherian commutative ring and let $f : R \to R$ be a surjective ring homomorphism. Prove that f is an isomorphism.

Solution : Let $I_n := \ker f^n$ defines a chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ of R. Since R is noetherian, then there is a natural number N such that $I_N = I_{N+1}$. In other words, we have that $f(f^N(x)) = 0$ if and only if $f^N(x) = 0$ for all $x \in R$. Hence, if y is in the image of f^N, then $f(y) = 0$ implies that $y = 0$. But the image of f^N is R since f is surjective. Therefore, f is also injective, and hence, it is an isomorphism.