MATH 518 Notes : Algebraic Geometry

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These notes are based on lectures given by Professor Eyal Goren at McGill University in Fall 2025. The subject of these lectures is **TODO**. As a disclaimer, it is more than possible that I made some mistakes. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Let k be an algebraically closed field (ex: $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}_p}$, ...)

Definition. The affine n-space $\mathbb{A}_k^n$ (or $\mathbb{A}^n$) is the set of all n-tuples in k (k^n without the ector space structure).

Definition. The ring $k[X_1, X_2, ..., X_n]$ is the ring of polynomials. It has a basis of monomials $X_1^{i_1}...X_n^{i_n}$ on which define $\deg(X_1^{i_1}...X_n^{i_n})$ as $i_1 + \cdots + i_n$. More generally, we define $\deg(f)$ where $f \in k[X_1, ..., X_n]$ as the maximum degree of the monomials that compose f.

Definition. An algebraic subset of $\mathbb{A}_k^n$ is a set of the form

$$V(S) = \{p \in \mathbb{A}_k^n : f(p) = 0 \quad \forall f \in S\}$$

where S is a possibly infinite subset of $k[X_1, ..., X_n]$.

Example:

- When $k = \mathbb{R}$, we get that $V(x^2 + y^2 - 1)$ is the unit circle which is an algebraic subset of $\mathbb{A}_{\mathbb{R}}^2$. Similarly, $V(x^2 + y^2 - z^2)$ is the an algebraic subset of $\mathbb{A}_{\mathbb{R}}^3$ which can be visualized as a double infinite cone.
- As a subset of $\mathbb{A}_{\mathbb{R}}^2$, the set $V(xy)$ is simply the the two axis and so it can be written as $V(x) \cup V(y)$. In this case, we say that $V(xy)$ is reducible.
- $\mathbb{A}_k^n = V(0)$.
- The subset $V(x - a, y - b)$ is simply the point (a, b) in $\mathbb{A}_k^2$.
- Elliptic curves
- cubic curve with singularity: $V(y^2 - x^3 - x)$.

Proposition 0.0.1. *1. When $S_1 \subset S_2$, then $V(S_1) \supset V(S_2)$.*

2. Given a set S, we can consider the ideal $I(S) =$