Higher Algebra 2 : Assignment 1

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Exercise 69: For a finite group G define

$$G^* = \mathrm{Hom}(G, \mathbb{C}^\times),$$

the **character group** of G. Prove that G^* is indeed a group under multiplication of functions. Prove:

(a) $(A \times B)^* = A^* \times B^*$.

(b) If G is a finite abelian group then $G \cong G^*$. (Reduce to the case of cyclic groups.)

Solution : To prove that G^* is a group, we need to prove that pointwise multiplication is indeed an operation on G^* that satisfies the group axioms. Let $f_1, f_2 \in G^*$, to prove that $f_1 f_2 \in G^*$, we need to show that $f_1 f_2 : G \to \mathbb{C}^\times$ is a group homomorphism. This comes form the fact that for $g, h \in G$, we have

$$(f_1 f_2)(gh) = f_1(g) f_1(h) f_2(g) f_2(h) = (f_1 f_2)(g)(f_1 f_2)(h).$$

Finally, the group axioms are satisfies because the identity is the constant function 1, then inverse of a function is literally the inverse of a function, and the associativity follows from the associativity in $\mathbb{C}$. More genrally, all these properties follows from the fact that the operation is pointwise, and that these properties hold pointwise.

(a) Let $\phi : A \times B \to \mathbb{C}^\times$ be a homomorphism, define $\phi_A : A \to \mathbb{C}^\times : a \mapsto \phi(a, 1)$ and $\phi_B : B \to \mathbb{C}^\times : b \mapsto \phi(1, b)$. Notice that ϕ_A is an element of A^* and ϕ_b is an element of B^*. It follows that the function $\phi \mapsto (\phi_A, \phi_B)$ is a function from $(A \times B)^*$ to $A^* \times B^*$. It is easy to prove that this is indeed a homomorphism. To prove that it is an isomorphism, notice that its inverse is the homomorphism $(\psi_1, \psi_2) = \psi_1 \cdot \psi_2$. Therefore, $(A \times B)^* = A^* \times B^*$.

(b) First, let's prove it for cyclic groups. Let $G = \mathbb{Z}/n\mathbb{Z}$, then any homomorphism from G to $\mathbb{C}^\times$ is uniquely determined by where 1 is mapped. Moreover, we can see that 1 must be mapped to an nth root of unity, and that each nth root of unity induces a homomorphism. Thus, G^* is composed of the functions $\phi_k : \overline{m} \mapsto e^{2km\pi i/n}$ where $k \in [\![0, n-1]\!]$. The map $\overline{k} \mapsto \phi_k$ is an isomorphism between G and G^*. Therefore, $G \cong G^*$.

Next, let G be a finite abelian group, then it can be decomposed as a product of cyclic groups. Since it holds for cyclic groups and the claim is preserved under products of groups (part (a)), then it holds for finite abelian groups.