Solutions to Measure, Integration & Real Analysis - Sheldon Axler

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Preface

The goal of this document is to share my personal solutions to the exercises in Measure, Integration & Real Analysis by Sheldon Axler during my reading.
What results will I assume and what results am I going to prove in this document? Most of the time, I will try to state precisely some results that I am going to use without proof. More generaly, I will assume that the reader of this document is already familiar with classical analysis such as the results that can be found in the first chapters of Understanding Analysis by Stephen Abbott or any first class introduction to analysis. For example, I will use without proof the following properties of the infimimum and supremum:

1. $\sup(A+B) = \sup\{a+b : a \in A,\ b \in B\} = \sup A + \sup B$
2. $\inf(A+B) = \inf\{a+b : a \in A,\ b \in B\} = \inf A + \inf B$
3. $\sup A \leq \sup B$ if $A \subset B$
4. $\inf A \geq \inf B$ if $A \subset B$
5. $-\sup A = \inf(-A)$

where A and B are arbitrary bounded subsets of $\mathbf{R}$.
As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Chapter 1

Riemann Integration

1A Review : Riemann Integral

Exercise 1
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function such that

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

for some partition P of $[a, b]$. Prove that f is a constant function on $[a, b]$.

Solution
Let's prove this on the number of subintervals of $[a, b]$ of the partition $P = \{x_0 < x_1 < ... < x_n\}$. For our base case, let $a < b \in \mathbf{R}$, $f : [a, b] \to \mathbf{R}$ be an arbitrary bounded function and $P = \{a, b\}$ be the trivial partition. Suppose that

$$L(f, P, [a, b]) = U(f, P, [a, b])$$

Notice that it is equivalent to

$$\inf_{[a,b]} f = \sup_{[a,b]} f$$

If we let $c := \sup_{[a,b]} f$, then for all $x \in [a, b]$, we have

$$c = \inf_{[a,b]} f \leq f(x) \leq \sup_{[a,b]} f = c$$

Hence, $f \equiv c$ on $[a, b]$ which proves the base case.
For the inductive step, suppose that there is a natural number k such that for all $a < b \in \mathbf{R}$ and for all bounded $f : [a, b] \to \mathbf{R}$, then f is constant on $[a, b]$ whenever $L(f, P, [a, b]) = U(f, P, [a, b])$ where P is a partition splitting $[a, b]$ into k subintervals. Let $a < b \in \mathbf{R}$ be real numbers, f be an arbitrary bounded function on $[a, b]$ and $P = \{a = x_0 < x_1 < ... < x_{k+1} = b\}$ be an arbitrary partition splitting $[a, b]$ into $k+1$ subintervals. Suppose that $L(f, P, [a, b]) = U(f, P, [a, b])$ holds. Let's show that f is constant on $[a, b]$.
First, consider the functions $f_1 := f|_{[a,x_k]}$ and $f_2 := f|_{[x_k,b]}$ and the partitions $P_1 := \{a = x_0 < x_1 < ... < x_k\}$ and $P_2 := \{x_k < x_{k+1} = b\}$ partitioning $[a, x_k]$ and $[x_k, b]$ respectiviely. Notice that $L(f, P, [a, b]) = U(f, P, [a, b])$ is actually equivalent to $L(f_1, P_1, [a, x_k]) = U(f_1, P_1, [a, x_k])$ and $L(f_2, P_2, [x_k, b]) = U(f_2, P_2, [x_k, b])$.
It follows by our induction hypothesis that there exist constants c_1 and c_2 in $\mathbf{R}$ such

that $f_1 \equiv c_1$ and $f_2 \equiv c_2$ on there respetive domains. By definition of f_1 and f_2, we get that $f(x) = c_1$ for all $x \in [a, x_k]$ and $f(x) = c_2$ for all $x \in [x_k, b]$. By plugging-in $x = x_k$, we get that $c_1 = c_2$. It follows that f is constant on $[a, b]$.

Exercise 2
Suppose $a \leq s < t \leq b$. Define $f : [a, b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } s < x < t, \\ 0 & \text{otherwise} \end{cases}$$

Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f = t - s$.

Solution
Let $\epsilon > 0$ and consider the partition $P_\epsilon = \{a < t - \frac{\epsilon}{2} < t + \frac{\epsilon}{2} < s - \frac{\epsilon}{2} < s + \frac{\epsilon}{2} < b\}$. To make sure that P_ϵ is well defined, take ϵ small enough so that $a < t - \frac{\epsilon}{2}$, $t + \frac{\epsilon}{2} < s - \frac{\epsilon}{2}$ and $s + \frac{\epsilon}{2} < b$, i.e., consider ϵ to be stricly smaller than $\min(2(t - a),\ s - t,\ 2(b - s))$. Hence:

$$\begin{aligned} U(f, [a, b]) &\leq U(f, P_\epsilon, [a, b]) \\ &= (t - \frac{\epsilon}{2} - a) \sup_{[a, t - \frac{\epsilon}{2}]} f + (t + \frac{\epsilon}{2} - t + \frac{\epsilon}{2}) \sup_{[t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2}]} f \\ &+ (s - \frac{\epsilon}{2} - t - \frac{\epsilon}{2}) \sup_{[t + \frac{\epsilon}{2}, s - \frac{\epsilon}{2}]} f + (s + \frac{\epsilon}{2} - s + \frac{\epsilon}{2}) \sup_{[s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2}]} f \\ &+ (b - s - \frac{\epsilon}{2}) \sup_{[s + \frac{\epsilon}{2}, b]} f \\ &= (t - \frac{\epsilon}{2} - a) \cdot 0 + \epsilon \cdot 1 + (s - t - \epsilon) \cdot 1 + \epsilon \cdot 1 + (b - s - \frac{\epsilon}{2}) \cdot 0 \\ &= s - t + \epsilon \end{aligned}$$

But $U(f, [a, b])$ don't depend on ϵ so it follows that $U(f, [a, b]) \leq s - t$. Similarly, by construction of P_ϵ, we can prove that $L(f, [a, b]) \geq s - t$ which gives us

$$s - t \leq L(f, [a, b]) \leq U(f, [a, b]) \leq s - t$$

which gives us

$$U(f, [a, b]) = L(f, [a, b]) = s - t$$

Therefore, f is Riemann integrable and $\int_a^b f = s - t$.

Exercise 3
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

Solution
($\implies$) Suppose that f is Riemann integrable, then by definition, $U(f, [a, b]) = L(f, [a, b])$. Let $\epsilon > 0$, then by properties of the infimimum and the supremum, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1, [a, b]) < U(f, [a, b]) + \frac{\epsilon}{2}$$

and

$$L(f,[a,b]) - \frac{\epsilon}{2} < L(f,P_2,[a,b])$$

consider $P = P_1 \cup P_2$, then:

$$\begin{aligned}
U(f,P,[a,b]) - L(f,P,[a,b]) &\leq U(f,P_1,[a,b]) - L(f,P_2,[a,b]) \\
&< U(f,[a,b]) + \frac{\epsilon}{2} - L(f,[a,b]) + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

which proves the first direction of the equivalence.
($\Longleftarrow$) Suppose that for all ϵ, there exists a partition P of $[a,b]$ such that

$$U(f,P,[a,b]) - L(f,P,[a,b]) < \epsilon$$

Then, since for all partitions P of $[a,b]$ we have $U(f,[a,b]) \leq U(f,P,[a,b])$ and $L(f,P,[a,b]) \leq L(f,[a,b])$, then it follows that for all ϵ, we have

$$U(f,[a,b]) - L(f,[a,b]) \leq U(f,P,[a,b]) - L(f,P,[a,b]) < \epsilon$$

for some partition P by our assumption. Since it holds for all $\epsilon > 0$ and since $U(f,[a,b]) - L(f,[a,b])$ is positive, then it follows that $U(f,[a,b]) = L(f,[a,b])$. By definition, this means that f is Riemann integrable.

Exercise 4
Suppose, $f,g : [a,b] \to \mathbf{R}$ are Riemann integrable. Prove that $f+g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Solution
First, consider the following properties of the upper and lower Riemann sums tat we will prove as follows

$$\begin{aligned}
\sup_{[x_i,x_{i+1}]}(f+g) &= \sup\{f(x)+g(x) : x \in [x_i,x_{i+1}]\} \\
&\leq \sup\{f(x)+g(y) : x,y \in [x_i,x_{i+1}]\} \\
&= \sup(\{f(x) : x \in [x_i,x_{i+1}]\} + \{g(x) : x \in [x_i,x_{i+1}]\}) \\
&= \sup\{f(x) : x \in [x_i,x_{i+1}]\} + \sup\{g(x) : x \in [x_i,x_{i+1}]\} \\
&= \sup_{[x_i,x_{i+1}]} f + \sup_{[x_i,x_{i+1}]} g
\end{aligned}$$

where $[x_i,x_{i+1}]$ is an arbitrary closed interval inside $[a,b]$. Similarly, we also have the following property for the infinimum:

$$\inf_{[x_i,x_{i+1}]}(f+g) \geq \inf_{[x_i,x_{i+1}]} f + \inf_{[x_i,x_{i+1}]} g$$

Thus, given a partition P of $[a,b]$, we have

$$\begin{aligned}
U(f+g,P,[a,b]) &= \sum_{i=1}^{n}(x_{i+1}-x_i)\sup_{[x_i,x_{i+1}]}(f+g)\\
&\leq \sum_{i=1}^{n}(x_{i+1}-x_i)(\sup_{[x_i,x_{i+1}]}f+\sup_{[x_i,x_{i+1}]}g)\\
&= \sum_{i=1}^{n}(x_{i+1}-x_i)\sup_{[x_i,x_{i+1}]}f+\sum_{i=1}^{n}(x_{i+1}-x_i)\sup_{[x_i,x_{i+1}]}g\\
&= U(f,P,[a,b])+U(g,P,[a,b])
\end{aligned}$$

and similarly:

$$L(f+g,P,[a,b]) \geq L(f,P,[a,b])+L(g,P,[a,b])$$

These are the main inequalities we will use to prove the additivity of the Riemann integral.

Let's now prove that $f+g$ is Riemann integrable on $[a,b]$ using the criterion proved in the previous exercise. Let $\epsilon>0$, then by the criterion, there exist partitions P_f and P_g of $[a,b]$ such that

$$U(f,P_f,[a,b])-L(f,P_f,[a,b])<\frac{\epsilon}{2}$$

$$U(g,P_g,[a,b])-L(g,P_g,[a,b])<\frac{\epsilon}{2}$$

Consider now P to be the merging of P_f and P_g, i.e., let $P=P_f\cup P_g$, then we get

$$U(f,P,[a,b])-L(f,P,[a,b])<\frac{\epsilon}{2}$$

$$U(g,P,[a,b])-L(g,P,[a,b])<\frac{\epsilon}{2}$$

Thus, by the previous inequalities:

$$\begin{aligned}
U(f+g,P,[a,b])-L(f+g,P,[a,b]) &\leq U(f,P,[a,b])+U(g,P,[a,b])\\
&\quad -L(f,P,[a,b])-L(g,P,[a,b])\\
&= [U(f,P,[a,b])-L(f,P,[a,b])]\\
&\quad +[U(g,P,[a,b])-L(g,P,[a,b])]\\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2}\\
&= \epsilon
\end{aligned}$$

which proves the Riemann integrability of $f+g$.

Now, let's proves equality between $\int_a^b(f+g)$ and $\int_a^b f+\int_a^b g$. To do so, let $\epsilon>0$, then there exist partitions P_1 and P_2 of $[a,b]$ satisfying

$$U(f,[a,b])+\frac{\epsilon}{2}>U(f,P_1,[a,b])$$

and

$$U(g,[a,b])+\frac{\epsilon}{2}>U(g,P_2,[a,b])$$

If we consider $P = P_1 \cup P_2$, we get

$$\begin{aligned}\int_a^b (f+g) &= U(f+g,[a,b]) \\ &\leq U(f+g,P,[a,b]) \\ &\leq U(f,P,[a,b]) + U(g,P,[a,b]) \\ &\leq U(f,P_1,[a,b]) + U(g,P_2,[a,b]) \\ &< U(f,[a,b]) + \frac{\epsilon}{2} + U(g,[a,b]) + \frac{\epsilon}{2} \\ &= \int_a^b f + \int_a^b g + \epsilon\end{aligned}$$

But ϵ is arbitrary and nothing depends on it so by letting $\epsilon \to 0$, we get

$$\int_a^b (f+g) \leq \int_a^b f + \int_a^b g \tag{1}$$

For the reverse inequality, again, let $\epsilon > 0$, then there are paritions P_1 and P_2 of $[a,b]$ satisfying

$$L(f,[a,b]) < L(f,P_1,[a,b]) + \frac{\epsilon}{2}$$

and

$$L(g,[a,b]) < L(g,P_2,[a,b]) + \frac{\epsilon}{2}$$

Thus, by letting $P = P_1 \cup P_2$, we get

$$\begin{aligned}\int_a^b f + \int_a^b g &= L(f,[a,b]) + L(g,[a,b]) \\ &< L(f,P_1,[a,b]) + \frac{\epsilon}{2} + L(g,P_2,[a,b]) + \frac{\epsilon}{2} \\ &= L(f,P,[a,b]) + L(g,P,[a,b]) + \epsilon \\ &\leq L(f+g,P,[a,b]) + \epsilon \\ &\leq L(f+g,[a,b]) + \epsilon \\ &= \int_a^b (f+g) + \epsilon\end{aligned}$$

Letting $\epsilon \to 0$ gives us

$$\int_a^b f + \int_a^b g \leq \int_a^b (f+g) \tag{2}$$

Therefore, combining (1) and (2) gives us

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Exercise 5

Suppose $f : [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that the function $-f$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (-f) = -\int_a^b f$$

Solution
First, notice that for any partition P of $[a,b]$, we have

$$\begin{aligned}
-U(f,P,[a,b]) &= -\sum_{i=1}^{n}(x_{i+1}-x_i)\sup_{[x_i,x_{i+1}]} f \\
&= \sum_{i=1}^{n}(x_{i+1}-x_i)\left(-\sup_{[x_i,x_{i+1}]} f\right) \\
&= \sum_{i=1}^{n}(x_{i+1}-x_i)\inf_{[x_i,x_{i+1}]}(-f) \\
&= L(-f,P,[a,b])
\end{aligned}$$

Similarly, we also have

$$-L(f,P,[a,b]) = U(-f,P,[a,b])$$

Therefore, we get that

$$\begin{aligned}
f \text{ is Riemann integrable} &\implies U(f,[a,b]) = L(f,[a,b]) \\
&\implies -U(f,[a,b]) = -L(f,[a,b]) \\
&\implies -\inf_P\{U(f,P,[a,b])\} = -\sup_P\{L(f,P,[a,b])\} \\
&\implies \sup_P\{-U(f,P,[a,b])\} = \inf_P\{-L(f,P,[a,b])\} \\
&\implies \sup_P\{L(-f,P,[a,b])\} = \inf_P\{U(-f,P,[a,b])\} \\
&\implies L(-f,[a,b]) = U(-f,[a,b]) \\
&\implies -f \text{ is Riemann integrable}
\end{aligned}$$

Hence, by the previous exercise, we get

$$\int_a^b f + \int_a^b (-f) = \int_a^b (f+(-f)) = \int_a^b 0 = 0$$

which directly implies

$$\int_a^b (-f) = -\int_a^b f$$

Exercise 6
Suppose $f : [a,b] \to \mathbf{R}$ is Riemann integrable. Suppose $g : [a,b] \to \mathbf{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a,b]$. Prove that g is Riemann integrable on $[a,b]$ and

$$\int_a^b g = \int_a^b f$$

Solution
Let's prove this by induction on the number of the number of elements in the set $\{x \in [a,b] : g(x) \neq f(x)\}$. For the base case, let $g : [a,b] \to \mathbf{R}$ be a function which differs from f at exactly one point $x_0 \in [a,b]$. Consider the function $h = f - g$ defined

on $[a,b]$ and notice that h is zero everywhere except for $x = x_0$. Now, consider the following cases, if $x_0 \in (a,b)$, then to prove that h is Riemann integrable, let $\epsilon > 0$, define $\epsilon_0 = \epsilon/4|h_0|$ and consider the partition $P = \{a, x_0 - \epsilon_0, x_0 + \epsilon_0, b\}$. Then, we get

$$\begin{aligned} U(f,P,[a,b]) - L(f,P,[a,b]) &= \left(\sup_{[a,x_0-\epsilon_0]} f - \inf_{[a,x_0-\epsilon_0]} f\right)(x_0 - \epsilon_0 - a) \\ &+ \left(\sup_{[x_0-\epsilon_0,x_0+\epsilon_0]} f - \inf_{[x_0-\epsilon_0,x_0+\epsilon_0]} f\right)(x_0 + \epsilon_0 - x_0 + \epsilon_0) \\ &+ \left(\sup_{[x_0+\epsilon_0,b]} f - \inf_{[x_0+\epsilon_0,b]} f\right)(b - x_0 - \epsilon_0) \\ &= 0 \cdot (x_0 - \epsilon_0 - a) + 2|h(x_0)|\epsilon_0 + 0 \cdot (b - x_0 - \epsilon_0) \\ &= 2|h(x_0)|\frac{\epsilon}{4|h(x_0)|} \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Thus, by the criterion proved in exercise 3, h is Riemann integrable. Since $g = f - h$, then g is Riemann integrable as well by exercises 4 and 5.
Now, suppose without loss of generality that $h(x_0)$ is positive, then $L(f,P,[a,b]) = 0$ for any partition P of $[a,b]$. Hence, if we rewrite the last inequality, we get that

$$U(f,P,[a,b]) < \epsilon$$

for some partition P and for all $\epsilon > 0$. Hence, for all $\epsilon > 0$, there is a partition P such that

$$0 = L(f,P,[a,b]) \le U(f,[a,b]) \le U(f,P,[a,b]) < \epsilon$$

It follows that

$$\int_a^b h = U(f,[a,b]) = 0$$

by letting $\epsilon \to 0$. Thus, by exercise 4 and 5, we get

$$\int_a^b f = \int_a^b (h+g) = \int_a^b h + \int_a^b g = \int_a^b g$$

which proves the base case when $x_0 \in (a,b)$. When $x_0 \in \{a,b\}$, the proof is the same up to a small modification of the partition P given $\epsilon > 0$. If $x_0 = a$, define $P = \{a, a + \frac{\epsilon}{2|h(x_0)|}, b\}$ and if $x_0 = b$, define $P = \{a, b - \frac{\epsilon}{2|h(x_0)|}, b\}$.
For the inductive hypothesis, suppose that there is a $k \in \mathbf{Z}^+$ such that any function that differs from a Riemann integrable function f at precisely k points is still Riemann integrable and has its integral to be equal to $\int_a^b f$. Now, let $g : [a,b] \to \mathbf{R}$ be an arbitrary function that differs from f at precisely k points $x_1, x_2, ..., x_{k+1}$. From this, consider the function $g_0 : [a,b] \to \mathbf{R}$ defined by

$$g_0(x) = \begin{cases} f(x) & x = x_{k+1} \\ g(x) & \text{otherwise} \end{cases}$$

Notice that g_0 differs from f at precisely k points. Hence, by the inductive hypothesis, g_0 is integrable and its integral is the same as f. Moreover, g differs from g_0 at precisely one point, hence, by the base case, since g_0 is Riemann integrable, then g is Riemann integrable as well and

$$\int_a^b g = \int_a^b g_0 = \int_a^b f$$

which proves our claim by induction.

Exercise 7

Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function. For $n \in \mathbf{Z}^+$, let P_n denote the partition that divides $[a, b]$ into 2^n intervals of equal size. Prove that

$$L(f, [a, b]) = \lim_{n\to\infty} L(f, P_n, [a, b]) \text{ and } U(f, [a, b]) = \lim_{n\to\infty} U(f, P_n, [a, b])$$

Solution

Let's prove it for the lower Riemann integral. Since $P_{n+1} \subset P_n$ for all $n \in \mathbf{Z}^+$, then $\{L(f, P_n, [a, b])\}_n$ is an increasing sequence that is bounded by $L(f, [a, b])$, thus, it converges to its supremum. Hence, it suffices to prove that $L(f, [a, b]) = \sup_n L(f, P_n, [a, b])$.

Let $\epsilon > 0$, then by properties of the supremum, there exists a partition $P = \{a = x_0, ..., x_m = b\}$ of $[a, b]$ that satisfies

$$L(f, P, [a, b]) > L(f, [a, b]) - \frac{\epsilon}{2}$$

Let $k \in [\![1, m-1]\!]$, then there are a dyadic numbers $a_k/2^{n_k}$ and $b_k/2^{n_k}$ that satisfies the following properties. First, $a_k/2^{n_k}$ is strictly between x_k and x_k minus half the distance between x_k and x_{k-1}. Similarly, $b_k/2^{n_k}$ is stricly between x_k and x_k plus half the distance between x_k and x_{k+1}. This condition is made to ensure that

$$\frac{b_{k-1}}{2^{n_{k-1}}} < \frac{a_k}{2^{n_k}} < x_k < \frac{b_k}{2^{n_k}} < \frac{a_{k+1}}{2^{n_{k+1}}}$$

Moreover, the dyadic numbers also satisfy

$$x_k - \frac{a_k}{2^{n_k}} < \frac{\epsilon}{4M(m-1)}$$

and

$$\frac{b_k}{2^{n_k}} - x_k < \frac{\epsilon}{4M(m-1)}$$

It directly follows that

$$\frac{b_k}{2^{n_k}} - \frac{a_k}{2^{n_k}} < \frac{\epsilon}{2M(m-1)}$$

From this, define N to be the maximum of the n_k's and notice that we can rewrite

$$\frac{\epsilon}{2} = \sum_{k=1}^{m-1} M \frac{\epsilon}{2M(m-1)}$$

Hence, combining this with the previous inequality gives us

$$\frac{\epsilon}{2} > \sum_{k=1}^{m-1} M\left(\frac{b_k}{2^{n_k}} - \frac{a_k}{2^{n_k}}\right)$$

But notice that right hand side is an upper bound for the lower Riemann sum with the partition $P_N \cup P$ where the subintervals are precisely the ones between the dyadic approximations of the x_i's. Hence, since we can split $L(f, P_N \cup P, [a, b])$ into two sums, one that iterates over the subintervals of P_N that are not contained between the dyadic approximations of some x_i and another sum that iterates over the subintervals of $P_N \cup P$ that are contained between the dyadic approximations of some x_i, then we get the following upper bound:

$$L(f, P_N, [a, b]) + \frac{\epsilon}{2} > L(f, P_N \cup P, [a, b])$$

which implies

$$\begin{aligned} L(f, P_N, [a, b]) + \frac{\epsilon}{2} &> L(f, P_N \cup P, [a, b]) \\ &\geq L(f, P, [a, b]) \\ &> L(f, [a, b]) - \frac{\epsilon}{2} \end{aligned}$$

giving us

$$L(f, P_N, [a, b]) > L(f, [a, b]) - \epsilon$$

Thus, the sequence $\{L(f, P_n, [a, b])\}_n$ gets arbitrarily close to $L(f, [a, b])$. But $L(f, [a, b])$ is an upper bound for this sequence. It follows that $L(f, [a, b]) = \sup_n L(f, P_n, [a, b])$. Therefore,

$$L(f, [a, b]) = \lim_{n\to\infty} L(f, P_n, [a, b])$$

The proof for the upper Riemann integral is the same up to some small readjustments.

Exercise 8
Suppose $f : [a, b] \to \mathbf{R}$ is Riemann integrable. Prove that

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\left(a + \frac{j(b-a)}{n}\right).$$

Solution
In this solution, for all $n \in \mathbf{Z}^+$, I will denote by P_n the partition of $[a, b]$ that divides the interval into n equaly spaced subintervals. Let's use the definition of the limit for sequences to prove the claim.
Let $\epsilon > 0$, then there exist partitions $P^{(1)}$ and $P^{(2)}$ of $[a, b]$ satisfying

$$L(f, P^{(1)}, [a, b]) > L(f, [a, b]) - \frac{\epsilon}{2}$$

$$U(f, [a, b]) + \frac{\epsilon}{2} > U(f, P^{(2)}, [a, b])$$

If we consider the merging of the partitions $P = P^{(1)} \cup P^{(2)} = \{a = x_0, x_1, ..., x_m = b\}$, then the previous inequalities still hold even if we replace $P^{(1)}$ and $P^{(2)}$ by P:

$$L(f, P, [a,b]) > L(f, [a,b]) - \frac{\epsilon}{2}$$

$$U(f, [a,b]) + \frac{\epsilon}{2} > U(f, P, [a,b])$$

By the Archimedean Property in $\mathbf{R}$, there is a $N \in \mathbf{Z}^+$ such that

$$\frac{1}{N} < \frac{1}{b-a} \cdot \frac{\epsilon}{4M(m-1)}$$

Moreover, to make the rest of the proof simpler, make N large enough so that $(b-a)/N$ is stricly less than the maximum size of the subintervals in P. Let $n \geq N$, let's first prove that

$$L(f, P \cup P_n, [a,b]) \leq L(f, P_n, [a,b]) + 2M(m-1)\frac{b-a}{n}$$

To do so, since P_n is a partition of $[a,b]$, then any x_i is going to be in a subinterval of P_n of the form $[y_{i_1}, y_{i_2}]$ where $y_{i_1} = a + j\frac{b-a}{n}$ and $y_{i_2} = y_{i_1} + \frac{b-a}{n}$:

$$y_{i_1} \leq x_i \leq y_{i_2}$$

By our assumption on N, there are no x_j between y_{i_1} and x_i or x_i and y_{i_2}. Hence, the lower Riemann sum corresponding to the partition $P \cup P_n$ contains the following terms:

$$(x_i - y_{i_1}) \inf_{[y_{i_1}, x_i]} f + (y_{i_2} - x_i) \inf_{[x_i, y_{i_2}]} f$$

for all $i \in [\![1, m-2]\!]$. But notice that we can find the following upper bound:

$$\begin{aligned}
(x_i - y_{i_1}) \inf_{[y_{i_1}, x_i]} f + (y_{i_2} - x_i) \inf_{[x_i, y_{i_2}]} f &\leq (x_i - y_{i_1})M + (y_{i_2} - x_i)M \\
&= M(y_{i_2} - y_{i_1}) \\
&= M\frac{b-a}{n}
\end{aligned}$$

Summing over all i's gives us

$$\sum_{i=1}^{m-1} \left[(x_i - y_{i_1}) \inf_{[y_{i_1}, x_i]} f + (y_{i_2} - x_i) \inf_{[x_i, y_{i_2}]} f \right] \leq \sum_{i=1}^{m-1} M\frac{b-a}{n} = M(m-1)\frac{b-a}{n}$$

Thus, from the $n + (m-1)$ terms of the lower Riemann sum associated with the partition $P \cup P_n$, we can bound above $2(m-1)$ of the terms by $M(m-1)\frac{b-a}{n}$. What it means is that $L(f, P \cup P_n, [a,b])$ can be bounded above by $M(m-1)\frac{b-a}{n}$ plus $L(f, P_n, [a,b])$ without the $m-1$ subintervals containing the x_i's. But each subinterval in $L(f, P_n, [a,b])$ is of the form $\inf_{[y_j, y_{j+1}]} f \frac{b-a}{n}$ so is greater than $-M\frac{b-a}{n}$.

Thus, if we denote by m_k the infimum of f on the kth subinterval of P_n, we get:

$$\begin{aligned}
L(f, P \cup P_n, [a,b]) &\leq \sum_{i=1}^{n-(m-1)} \left[m_{k_i} \frac{b-a}{n} \right] + M(m-1)\frac{b-a}{n} \\
&= \sum_{i=1}^{n-(m-1)} \left[m_{k_i} \frac{b-a}{n} \right] + \sum_{j=1}^{m-1} \left[-M \frac{b-a}{n} \right] + 2M(m-1)\frac{b-a}{n} \\
&\leq \sum_{i=1}^{n-(m-1)} \left[m_{k_i} \frac{b-a}{n} \right] + \sum_{j=1}^{m-1} \left[m_{k'_j} \frac{b-a}{n} \right] + 2M(m-1)\frac{b-a}{n} \\
&= \sum_{i=1}^{n} \left[m_k \frac{b-a}{n} \right] + 2M(m-1)\frac{b-a}{n} \\
&= L(f, P_n, [a,b]) + 2M(m-1)\frac{b-a}{n}
\end{aligned}$$

which is the desired inequality. Similarly, we can prove an analoguous inequality for the upper Riemann sum:

$$U(f, P \cup P_n, [a,b]) \geq U(f, P_n, [a,b]) - 2M(m-1)\frac{b-a}{n}$$

From these inequalities, we get the following:

$$\begin{aligned}
\frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) + \frac{\epsilon}{2} &\geq \sum_{i=1}^{n} \left[m_i \frac{b-a}{n} \right] + 2M(m-1)\frac{b-a}{n} \\
&= L(f, P_n, [a,b]) + 2M(m-1)\frac{b-a}{n} \\
&\geq L(f, P \cup P_n, [a,b]) \\
&\geq L(f, P, [a,b]) \\
&> L(f, [a,b]) - \frac{\epsilon}{2}
\end{aligned}$$

which implies

$$\int_a^b f - \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) < \epsilon \tag{1}$$

Similarly, with upper Riemann sums, we get

$$\frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) - \int_a^b f < \epsilon \tag{2}$$

Combining (1) and (2) gives us

$$\left| \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) - \int_a^b f \right| < \epsilon$$

Therefore, by definition of the limit of a sequence, we have

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) = \int_a^b f$$

which proves our claim.

Exercise 9
Suppose $f : [a, b] \to \mathbf{R}$ is Riemann integrable. Prove that if $c, d \in \mathbf{R}$ and $a \leq c < d \leq b$, then f is Riemann integrable on $[c, d]$.
[To say that f is Riemann integrable on $[c, d]$ means that f with its domain restricted to $[c, d]$ is Riemann integrable.]

Solution
In this solution, we will denote by $f|_{[c,d]}$ the restriction of f to $[c, d]$. Let's prove this using the criterion proven in exercise 3. Let $\epsilon > 0$, then by Riemann integrability of f, there exists a partition P such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

Consider now the partition $P' = P \cup \{c, d\}$, then the previous still holds if we replace P by P' since P' is a refinement of P:

$$U(f, P', [a, b]) - L(f, P', [a, b]) < \epsilon$$

If we write P' as $\{a = x_0, x_1, ..., x_n = b\}$, then there must exist integers $i < j \in [\![0, n]\!]$ such that $x_i = c$ and $x_j = d$. Define now the partition $P_0 = \{c = x_i, x_{i+1}, ..., x_j = d\}$ and notice that

$$\begin{aligned}
U(f|_{[c,d]}, P_0, [c, d])-&L(f|_{[c,d]}, P_0, [c, d]) \\
&= \sum_{k=i}^{j-1} \left(\sup_{[x_i, x_{i+1}]} f|_{[c,d]} - \inf_{[x_i, x_{i+1}]} f|_{[c,d]} \right) (x_{i+1} - x_i) \\
&= \sum_{k=i}^{j-1} \left(\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \right) (x_{i+1} - x_i) \\
&\leq \sum_{k=1}^{n-1} \left(\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \right) (x_{i+1} - x_i) \\
&= U(f, P', [a, b]) - L(f, P', [a, b]) \\
&< \epsilon
\end{aligned}$$

which proves that f is Riemann integrable on $[c, d]$.

Exercise 10
Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function and $c \in (a, b)$. Prove that f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and f is Riemann integrable on $[c, b]$. Furthermore, prove that if these conditions hold, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Solution
Before proving this, let's show that

$$U(f, [a, b]) = U(f, [a, c]) + U(f, [c, d])$$

and

$$L(f,[a,b]) = L(f,[a,c]) + L(f,[c,d])$$

hold. To do so, we will use properties of the supremum and infimum. Let $\epsilon > 0$, then there exists a partition P of $[a,b]$ such that $U(f,P,[a,b]) < U(f,[a,b]) + \epsilon$. But if we consider $P \cup \{c\} = \{a = x_0, ..., x_j = c, ..., x_n = b,\}$ instead of P, we can split it into two partitions $P_1 = \{x_0, ..., x_j\}$ and $P_2 = \{x_j, ..., x_n\}$ of $[a,c]$ and $[c,b]$ respectively. Hence:

$$\begin{aligned}
U(f,[a,b]) + \epsilon &> U(f,P,[a,b]) \\
&\geq U(f,P\cup\{c\},[a,b]) \\
&= \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \\
&= \sum_{i=1}^{j}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f + \sum_{i=j+1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \\
&= U(f,P_1,[a,c]) + U(f,P_2,[c,b]) \\
&\geq U(f,[a,c]) + U(f,[c,b])
\end{aligned}$$

In short:

$$U(f,[a,c]) + U(f,[c,b]) \leq U(f,[a,b]) + \epsilon$$

But nothing here depends on ϵ so if just take $\epsilon \to 0$, we get

$$U(f,[a,c]) + U(f,[c,b]) \leq U(f,[a,b])$$

Similarly, for any $\epsilon > 0$, there exist partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively such that $U(f,P_1,[a,c]) < U(f,[a,c]) + \frac{\epsilon}{2}$ and $U(f,P_2,[c,b]) < U(f,[c,b]) + \frac{\epsilon}{2}$. Hence, if we consider the partition $P = P_1 \cup P_2 = \{x_0, ..., x_j = c, ..., x_n\}$ of $[a,b]$, we get

$$\begin{aligned}
U(f,[a,c]) + U(f,[c,b]) + \epsilon &> U(f,P_1,[a,c]) + U(f,P_2,[c,b]) \\
&= \sum_{i=1}^{j}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f + \sum_{i=j+1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \\
&= \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f \\
&= U(f,P,[a,b]) \\
&\geq U(f,[a,b])
\end{aligned}$$

In short:

$$U(f,[a,b]) \leq U(f,[a,c]) + U(f,[c,b]) + \epsilon$$

But nothing here depends on ϵ so if just take $\epsilon \to 0$, we get

$$U(f,[a,b]) \leq U(f,[a,c]) + U(f,[c,b])$$

It follows that

$$U(f,[a,b]) = U(f,[a,c]) + U(f,[c,b])$$

The proof for the lower Riemann integral is the same up to some small modifications. Now that we proved these results, the rest will follow easily.

For the equivalence that we need to prove, notice that the forward implication follows from the previous exercise. For the reverse implication, suppose that f is both Riemann integrable on $[a, c]$ and $[c, b]$, then by definition, we have

$$U(f, [a, c]) = L(f, [a, c])$$

and

$$U(f, [c, b]) = L(f, [c, b])$$

Adding the two equations gives us

$$U(f, [a, c]) + U(f, [c, b]) = L(f, [a, c]) + L(f, [c, b])$$

which is equivalent to

$$U(f, [a, b]) = L(f, [a, b])$$

Thus, f is Riemann integrable on $[a, b]$.
Now, suppose that f is Riemann integrable on $[a, b]$ and consequently, on $[a, c]$ and $[c, b]$ as well, then:

$$\int_a^b f = U(f, [a, b]) = U(f, [a, c]) + U(f, [c, b]) = \int_a^c f + \int_c^b f$$

which proves our claim.

Exercise 11
Suppose $f : [a, b] \to \mathbf{R}$ is Riemann integrable. Define $F : [a, b] \to \mathbf{R}$ by

$$F(t) = \begin{cases} 0 & \text{if } t = a \\ \int_a^t f & \text{if } t \in (a, b] \end{cases}$$

Prove that F is continuous on $[a, b]$.

Solution
First, let m be the infimum of f on $[a, b]$ and M be the supremum of f on $[a, b]$. Define A to be the maximum between $|m|$ and $|M|$. Now, let $x \in [a, b]$ and $(x_n)_n$ a sequence in $[a, b]$ that converges to x. For all $n \in \mathbf{Z}^+$, if $x < x_n$ we have

$$(x_n - x) \inf_{[x, x_n]} f \leq \int_x^{x_n} f \leq (x_n - x) \sup_{[x, x_n]} f$$

But by properties of the infimum and supremum, we have

$$m(x_n - x) \leq (x_n - x) \inf_{[a,b]} f \leq \int_x^{x_n} f \leq (x_n - x) \sup_{[a,b]} f \leq M(x_n - x)$$

By definition of A, we have

$$-A(x_n - x) \leq m(x_n - x) \leq \int_x^{x_n} f \leq M(x_n - x) \leq A(x_n - x)$$

By the previous exercise and by definition of F, we have

$$F(x_n) - F(x) = \int_a^{x_n} f - \int_a^x f = \int_x^{x_n} f$$

Thus, plugging this in our inequality gives us

$$-A(x_n - x) \le F(x_n) - F(x) \le A(x_n - x)$$

which is equivalent to

$$|F(x_n) - F(x)| \le A(x_n - x)$$

We assumed here that $x < x_n$ but we actually get the exact same result if $x = x_n$ or if $x > x_n$. Thus, since our last inequality holds for all $n \in \mathbf{Z}^+$, then by the Squeeze Theorem:

$$\lim_{n\to\infty} F(x_n) = F(x)$$

Since it holds for any sequence $(x_n)_n$ converging to x, then by the Sequential Characterization of Continuity, we get that F is continuous at x. Since it holds for all $x \in [a, b]$, then F is continuous on $[a, b]$.

Exercise 12

Suppose $f : [a, b] \to \mathbf{R}$ is Riemann integrable. Prove that $|f|$ is Riemann integrable and that

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Solution

First, let's prove that $|f|$ is Riemann integrable. To do so, let's use the criterion proven in exercise 3. Let $\epsilon > 0$, then there exists a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon$$

Let $k \in [\![1, n]\!]$, define

$$m_k = \inf_{[x_{k-1},x_k]} f \qquad M_k = \sup_{[x_{k-1},x_k]} f$$

$$m'_k = \inf_{[x_{k-1},x_k]} |f| \qquad M'_k = \sup_{[x_{k-1},x_k]} |f|$$

Let's show that $M'_k - m'_k \le M_k - m_k$.
If $M_k \le 0$ or $m_k \ge 0$, it is trivial. Suppose that $M_K \ge 0$ and $m_k \le 0$, then for all $x \in [x_{k-1}, x_k]$:

$$\begin{aligned} m_k \le f(x) &\implies m_k \le f(x) + M_k \\ &\implies m_k - M_k \le f(x) \\ &\implies -(M_k - m_k) \le f(x) \end{aligned}$$

and

$$\begin{aligned} f(x) \le M_k &\implies f(x) + m_k \le M_k \\ &\implies f(x) \le M_k - m_k \end{aligned}$$

Putting the last two inequalities together gives us

$$-(M_k - m_k) \le f(x) \le M_k - m_k$$

which is equivalent to

$$|f(x)| \leq M_k - m_k$$

But it holds for all $x \in [x_{k-1}, x_k]$, so we get

$$M_k' - m_k' \leq M_k' \leq M_k - m_k$$

which is the desired inequality.
Now, simply notice that

$$\begin{aligned} U(|f|, P, [a,b]) - L(|f|, P, [a,b]) &= \sum_{k=1}^{n} (M_k' - m_k')(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= U(f, P, [a,b]) - L(f, P, [a,b]) \\ &< \epsilon \end{aligned}$$

which proves that $|f|$ is Riemann integrable as well.
To prove the triangle inequality, I find it easier to first prove that the Riemann integral is monotone. To do so, let $g_1, g_2 : [a,b] \to \mathbf{R}$ be two Riemann integrable functions such that $g_1 \leq g_2$, then if we define $h = g_2 - g_1 \geq 0$, by exercises 4 and 5, we know that h is Riemann integrable as well and that

$$\int_a^b h = \int_a^b g_2 - \int_a^b g_1$$

Moreover, since h is positive on $[a,b]$, then $\inf_{[a,b]} h$ must be positive as well. It follows that

$$0 \leq (b-a) \inf_{[a,b]} h \leq \int_a^b h = \int_a^b g_2 - \int_a^b g_1$$

which directly implies

$$\int_a^b g_1 \leq \int_a^b g_2$$

Hence, the Riemann integral is monotone. Therefore:

$$-|f| \leq f \leq |f|$$

implies by monotonicity and by exercise 5 that

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

which is equivalent to

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

This proves the triangle inequality for the Riemann integral.

Exercise 13
Suppose $f : [a,b] \to \mathbf{R}$ is an increasing function, meaning that $c, d \in [a,b]$ with

$c < d$ implies $f(c) \leq f(d)$. Prove that f is Riemann integrable on $[a, b]$.

Solution
Let's prove that f is Riemann integrable using the criterion proven in exercise 3. Let $\epsilon > 0$, then by the Archimedean property in $\mathbf{R}$, there exists a $n \in \mathbf{Z}^+$ such that

$$\frac{(b-a)(f(b)-f(a))}{n} < \epsilon$$

Now, consider $P = \{x_0, ..., x_n\}$ to be the the partition of $[a, b]$ that divides the interval into n subintervals of equal size. For all $k \in [\![1, n]\!]$, if we define

$$m_k = \inf_{[x_{k-1}, x_k]} f \qquad\qquad M_k = \sup_{[x_{k-1}, x_k]} f$$

then we get

$$m_k = f\left(a + (k-1)\frac{b-a}{n}\right) \qquad\qquad M_k = f\left(a + k\frac{b-a}{n}\right)$$

since f is increasing. Hence:

$$\begin{aligned}
U(f, P, [a, b]) - &L(f, P, [a, b]) \\
&= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \\
&= \frac{b-a}{n}\sum_{k=1}^{n}\left[f\left(a + k\frac{b-a}{n}\right) - f\left(a + (k-1)\frac{b-a}{n}\right)\right] \\
&= \frac{b-a}{n}(f(b) - f(a)) \\
&< \epsilon
\end{aligned}$$

Therefore, f is Riemann integrable.

Exercise 14
Suppose $f_1, f_2, ...$ is a sequence of Riemann integrable functions on $[a, b]$ such that $f_1, f_2, ...$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbf{R}$. Prove that f is Riemann integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

Solution
First, let's show that f is Riemann integrable using the criterion proven in exercise 3. Let $\epsilon > 0$, then by uniform convergence, there is a $N \in \mathbf{Z}^+$ such that

$$|f(x) - f_N(x)| < \frac{\epsilon}{4(b-a)}$$

for all $x \in [a, b]$. Since f_N is Riemann integrable, then there is a partition $P = \{x_0, ..., x_n\}$ such that

$$U(f_N, P, [a, b]) - L(f_N, P, [a, b]) < \frac{\epsilon}{2}$$

Let $k \in [\![1, n]\!]$ and define

$$m_k = \inf_{[x_{k-1},x_k]} f \qquad M_k = \sup_{[x_{k-1},x_k]} f$$

$$m_k^N = \inf_{[x_{k-1},x_k]} f_N \qquad M_k^N = \sup_{[x_{k-1},x_k]} f_N$$

Let $x \in [x_{k-1}, x_k]$, then

$$\begin{aligned} |f(x) - f_N(x)| < \frac{\epsilon}{4(b-a)} &\implies f(x) - f_N(x) < \frac{\epsilon}{4(b-a)} \\ &\implies f(x) < \frac{\epsilon}{4(b-a)} + f_N(x) \\ &\implies f(x) \leq \frac{\epsilon}{4(b-a)} + M_k^N \end{aligned}$$

However, since the last inequality holds for all $x \in [x_{k-1}, x_k]$ and only the left hand side depends on x, then it follows that

$$M_k \leq \frac{\epsilon}{4(b-a)} + M_k^N \tag{1}$$

Similarly,

$$\begin{aligned} |f(x) - f_N(x)| < \frac{\epsilon}{4(b-a)} &\implies f_N(x) - f(x) < \frac{\epsilon}{4(b-a)} \\ &\implies f_N(x) < \frac{\epsilon}{4(b-a)} + f(x) \\ &\implies m_k^N \leq \frac{\epsilon}{4(b-a)} + f(x) \\ &\implies m_k^N - \frac{\epsilon}{4(b-a)} \leq f(x) \end{aligned}$$

However, since the last inequality holds for all $x \in [x_{k-1}, x_k]$ and only the right hand side depends on x, then it follows that

$$m_k^N - \frac{\epsilon}{4(b-a)} \leq m_k$$

which implies

$$-m_k \leq -m_k^N + \frac{\epsilon}{4(b-a)} \tag{2}$$

Adding (1) and (2) together gives us

$$M_k - m_k \leq M_k^N - m_k^n + \frac{\epsilon}{2(b-a)}$$

for all $k \in [\![1, n]\!]$. Thus:

$$\begin{aligned}
U(f,P,[a,b]) - L(f,P,[a,b]) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\
&\leq \sum_{k=1}^{n} \left[(M_k^N - m_k^N) + \frac{\epsilon}{2(b-a)} \right] (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} (M_k^N - m_k^N)(x_k - x_{k-1}) + \frac{\epsilon}{2(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1}) \\
&= U(f_N, P, [a,b]) - L(f_N, P, [a,b]) + \frac{\epsilon}{2(b-a)}(b-a) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

which proves that f is Riemann integrable.
Now, let's prove that $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$ using the limit definition. Let $\epsilon > 0$, by uniform convergence, there is a $N \in \mathbf{Z}^+$ such that for all $n \geq N$ and $x \in [a, b]$

$$|f(x) - f_n(x)| < \frac{\epsilon}{2(b-a)}$$

Thus, for any $n \geq N$, using the triangle inequality (exercise 12),

$$\begin{aligned}
\left| \int_a^b f - \int_a^b f_n \right| &= \left| \int_a^b (f - f_n) \right| \\
&\leq \int_a^b |f - f_n| \\
&\leq \int_a^b \frac{\epsilon}{2(b-a)} \\
&= \frac{\epsilon}{2(b-a)}(b-a) \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

Therefore, by definition,

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$$

which proves our claim.

1B Riemann Integral Is Not Good Enough

Exercise 1
Define $f : [0,1] \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer} \\ & \text{ such that } a = \frac{m}{n} \text{ for some integer} m. \end{cases}$$

Show that f is Riemann integrable and compute $\int_0^1 f$.

Solution
First, notice that f can be written as the limit of a sequence $f_0, f_1, ...$ of functions defined recursively by $f_0 \equiv 0$ and $f_{n+1} = f_n$ except for the x's which can be written as $\frac{m}{n+1}$ as an irreducible fraction. In that case, define $f_{n+1}(x)$ to be $\frac{1}{n+1}$. It is to see that the sequence of functions converges uniformly to f.
But notice that for all $n \in \mathbf{Z}^+$, the function f_n only differs from the function zero at finitely many points. Thus, by exercise 6 of section 1A, f_n is Riemann integrable and its integral is equal to zero. Hence, by exercise 14 of section 1A, f is Riemann integrable as well and

$$\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n = 0$$

Exercise 2
Suppose that $f : [a,b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if

$$L(-f, [a,b]) = -L(f, [a,b])$$

Solution We actually proved a very similar result in the solution of exercise 5. Let's prove it again here for completeness. Our goal here will be to show that

$$L(-f, [a,b]) = -U(f, [a,b])$$

To do so, consider first an arbitrary partition $P = \{x_0, ..., x_n\}$ of $[a,b]$. By properties of the infimum, we have

$$\begin{aligned} L(f, P, [a,b]) &= \sum_{k=1}^{n} (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} (-f) \\ &= -\sum_{k=1}^{n} (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f \\ &= -U(f, P, [a,b]) \end{aligned}$$

Hence, by properties of the supremum, we get

$$\begin{aligned} L(-f, [a,b]) &= \sup_P L(-f, P, [a,b]) \\ &= \sup_P (-U(f, P, [a,b])) \\ &= -\inf_P U(f, P, [a,b]) \\ &= -U(f, [a,b]) \end{aligned}$$

Therefore, the equivalence can be proved easily as follows:

$$\begin{aligned} f \text{ is Riemann integrable } &\iff U(f,[a,b]) = L(f,[a,b]) \\ &\iff -U(f,[a,b]) = -L(f,[a,b]) \\ &\iff L(-f,[a,b]) = -L(f,[a,b]) \end{aligned}$$

which is the desired equivalence.

Exercise 3
Suppose $f, g : [a,b] \to \mathbf{R}$ are bounded functions. Prove that

$$L(f,[a,b]) + L(g,[a,b]) \leq L(f+g,[a,b])$$

and

$$U(f+g,[a,b]) \leq U(f,[a,b]) + U(g,[a,b]).$$

Solution
Let's prove it for the lower Riemann integral. To do so, let P_1 and P_2 be two arbitrary partitions of $[a,b]$ and consider the common refinement $P = P_1 \cup P_2 = \{x_0, ..., x_n\}$, then by properties of the infimum:

$$\begin{aligned} L(f,P_1,[a,b]) + L(g,P_2,[a,b]) &\leq L(f,P,[a,b]) + L(g,P,[a,b]) \\ &= \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} f + \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} g \\ &= \sum_{i=1}^{n}(x_i - x_{i-1}) \left[\inf_{[x_{i-1},x_i]} f + \inf_{[x_{i-1},x_i]} g\right] \\ &\leq \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{[x_{i-1},x_i]} (f+g) \\ &= L(f+g,P,[a,b]) \\ &\leq L(f+g,[a,b]) \end{aligned}$$

If we fix P_2 and rewrite the inequality as

$$L(f,P_1,[a,b]) \leq L(f+g,[a,b]) - L(g,P_2,[a,b])$$

Then taking the supremum over the P_1's gives us

$$L(f,[a,b]) \leq L(f+g,[a,b]) - L(g,P_2,[a,b])$$

Rewriting the inequality as

$$L(g,P_2,[a,b]) \leq L(f+g,[a,b]) - L(f,[a,b])$$

and taking the supremum over the P_2's gives us

$$L(g,[a,b]) \leq L(f+g,[a,b]) - L(f,[a,b])$$

which can be rewritten as

$$L(f,[a,b]) + L(g,[a,b]) \leq L(f+g,[a,b])$$

The proof for the upper Riemann integral is the same.

Exercise 4
Give an example of bounded functions $f, g : [0,1] \to \mathbf{R}$ such that

$$L(f,[0,1]) + L(g,[0,1]) < L(f+g,[0,1])$$

and

$$U(f+g,[0,1]) < U(f,[0,1) + U(g,[0,1]).$$

Solution
Let f and g be defined by

$$f(x) = \begin{cases} 2 & x \in \mathbf{Q} \cap [0,1] \\ 1 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 1 & x \in \mathbf{Q} \cap [0,1] \\ 2 & \text{otherwise} \end{cases}$$

on $[0,1]$. Then, $L(f,[0,1]) = L(g,[0,1]) = 1$ but $L(f+g,[0,1]) = 3 \neq 2$. Similarly, $U(f,[0,1]) = U(g,[0,1]) = 2$ but $U(f+g,[0,1]) = 3 \neq 4$.

Exercise 5
Give an example of a sequence of continuous real-valued functions $f_1, f_2, ...$ on $[0,1]$ and a continuous real-valued function f on $[0,1]$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f \neq \lim_{k\to\infty} \int_0^1 f_k$$

Solution
Consider the functions $f_1, f_2, ...$ defined by

$$f_k(x) = \begin{cases} nx & x \in [0, \frac{1}{n}] \\ 2 - nx & x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & x \in (\frac{2}{n}, 1] \end{cases}$$

Then, for all $k \in \mathbf{Z}^+$: $\int_0^1 f_k = 1$. However, the f_k's converge pointwise to the constant zero function on $[0,1]$ so $\int_0^1 f = 0$. It follows that $\int_0^1 f$ and $\lim_{k\to\infty} \int_0^1 f_k$ are two different quantities.

Chapter 2

Measures

2A Outer Measure on R

Exercise 1
Prove that if A and B are subsets of $\mathbf{R}$ and $|B| = 0$, then $|A \cup B| = |A|$.

Solution
By finite subadditivity, we have

$$|A \cup B| \leq |A| + |B| = |A| \tag{1}$$

Since $A \subset A \cup B$, then by monotonicity we have

$$|A| \leq |A \cup B| \tag{2}$$

Combining (1) and (2) gives us

$$|A \cup B| = |A|$$

Exercise 2
Suppose $A \subset \mathbf{R}$ and $t \in \mathbf{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t||A|$.
[*Assume that* $0 \cdot \infty$ *is defined to be 0.*]

Solution
First, notice that the statement is trivial for $t = 0$ so suppose t is nonzero. Secondly, if we let $I = (a, b)$ be an arbitrary open set with $a < b \in \mathbf{R}$, then for $t > 0$:

$$\begin{aligned}\ell(tI) &= \ell((ta, tb)) \\ &= tb - ta \\ &= t(b - a) \\ &= |t|\ell(I)\end{aligned}$$

and for $t < 0$:

$$\begin{aligned}\ell(tI) &= \ell((tb, ta)) \\ &= ta - tb \\ &= -t(b - a) \\ &= |t|\ell(I)\end{aligned}$$

Thus, it works for all $t \neq 0$.
Now, let $\{I_1, I_2, ...\}$ be an arbitrary collection of open intervals covering A. It is easy to see that $\{tI_1, tI_2, ...\}$ covers tA. Hence,

$$|tA| \leq \sum_{n=1}^{\infty} \ell(tI_n) = |t| \sum_{n=1}^{\infty} \ell(I_n)$$

which is equivalent to

$$\frac{1}{|t|}|tA| \leq \sum_{n=1}^{\infty} \ell(I_n)$$

But notice that $\{I_n\}_n$ was an arbitrary cover of A so taking the infimum on both sides over all covers $\{I_n\}_n$ of A gives us

$$|tA| \leq |t||A| \tag{1}$$

Proving the reverse inequality can actually be done using equation (1):

$$|A| = \left|\frac{1}{t}(tA)\right| \leq \left|\frac{1}{t}\right| |tA|$$

which is equivalent to

$$|t||A| \leq |tA| \tag{2}$$

Combining (1) and (2) gives us

$$|tA| = |t||A|$$

which is the desired formula.

Exercise 3
Prove that if $A, B \subset \mathbf{R}$ and $|A| < \infty$, then $|B \setminus A| \geq |B| - |A|$.

Solution
By subadditivity and monotonicity, since $B \subset (B \setminus A) \cup A$, then

$$|B| \leq |(B \setminus A) \cup A| \leq |B \setminus A| + |A|$$

Since $|A| < \infty$, then

$$|(B \setminus A) \cup A| \geq |B| - |A|$$

which is the desired inequality.

Exercise 4
Suppose F is a subset of $\mathbf{R}$ with the property that every open cover of F has a finite subcover. Prove that F is closed and bounded.

Solution
Let's prove first that F is bounded. To do so, notice that $\{(k, k+2)\}_{k \in \mathbf{Z}}$ is certainly an open cover for F since $\cup_{k \in \mathbf{Z}}(k, k+2) = \mathbf{R}$. Hence, by our assumption on F, there exist finitely many open intervals that covers F, i.e., F is a subset of a finite union of open intervals of the form $(k, k+2)$ where $k \in \mathbf{Z}$. Obviously, each of these

intervals is bounded, hence a finite union of such intervals is bounded a well. Thus, F is a subset of a bounded set, so it must be bounded as well.
To show that F is closed, let's prove that F^c is open. To prove it, let x be an arbitrary element in F^c and let's show the existence of an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset F$. Consider the collection $\{(-\infty, x-\frac{1}{n}) \cup (x+\frac{1}{n}, \infty)\}_{n\in\mathbf{Z}^+}$. and notice that its union is $\mathbf{R} \setminus \{x\}$. Since $x \notin F$, then $F \subset \mathbf{R} \setminus \{x\}$ which shows that the collection is actually an open cover for F. Again, by our assumption on F, there exist finitely many natural numbers $n_1, n_2, ..., n_N$ such that

$$F \subset \bigcup_{i=1}^{N} \left(-\infty, x - \frac{1}{n_i}\right) \cup \left(x + \frac{1}{n_i}, \infty\right)$$

If we take $M = \max_{1\leq i\leq N}(n_i)$, then

$$F \subset \left(-\infty, x - \frac{1}{M}\right) \cup \left(x + \frac{1}{M}, \infty\right)$$

It follows that

$$\left[x - \frac{1}{M}, x + \frac{1}{M}\right] \subset F^c$$

If we let $\epsilon = \frac{1}{M+1}$, then we get

$$(x-\epsilon, x+\epsilon) \subset F^c$$

which proves that F^c is open, and therefore that F is closed and bounded.

Exercise 5
Suppose $\mathcal{A}$ is a set of closed subsets of $\mathbf{R}$ such that $\cap_{F\in\mathcal{A}} F = \varnothing$. Prove that if $\mathcal{A}$ contains at least one bounded set, then there exist $n \in \mathbf{Z}^+$ and $F_1, ..., F_n \in \mathcal{A}$ such that $F_1 \cap ... \cap F_n = \varnothing$.

Solution
In this proof, I will use the following theorem proved in Exercise 3.3.6.(c) of Understanding Analysis : If $\{A_n\}_n$ is a countable collection of closed and bounded subsets of $\mathbf{R}$ such that any finite intersection is non empty, then $\cap_{n=1}^{\infty} A_n$ is non empty as well. As a corollary, if a countable intersection of closed and bounded sets is empty, then there must be a finite subcollection such that the intersection is empty as well. Notice that the theorem that I just stated is simply a generalisation of the Nested Interval Property.
Since we have no informations about the cardinality of $\mathcal{A}$, the first step of this proof will be to construct a countable collection of closed and bounded sets that will let us apply the previous theorem in a useful way. To do so, recall that any open set in $\mathbf{R}$ can be written as a countable union of open intervals. Moreover, any open interval can be written as a countable union of open intervals with rational endpoints. Hence, any open set can be written as a countable union of open intervals with rational coefficients.
Consider now the set $B = \{(a,b)^c : a, b \in \mathbf{Q}\}$ which is countable ($a, b \mapsto (a,b)^c$ is a bijection from $\mathbf{Q}^2$ to B and we know that $\mathbf{Q}^2$ is countable) and let F be a closed set. By what we said previously, we have that

$$F^c = \bigcup_{i=1}^{\infty} B_i^c$$

for some $\{B_i\}_i \subset B$. It follows that

$$F = \bigcap_{i=1}^{\infty} B_i$$

Since F was an arbitrary closed set, then any closed set can be written as a countable intersection of elements in B. It follows that for every element F in $\mathcal{A}$, there is a coutable collection $\{I_k^{(F)}\}_k$ such that $F = \cap_{k=1}^{\infty} I_k^{(F)}$.
From this, define the collection $I = \cup_{F\in\mathcal{A}}\{I_k^{(F)}\}_k$ which must be countable since it is a subset of B which is countable. Since it is countable, to make the notation easier, enumerate the elements in I as $\{I_1, I_2, ...\}$. Let's prove that $\cap_{n=1}^{\infty} I_n \subset \cap_{F\in\mathcal{A}} F$:

- Suppose that $x \in \cap_{n=1}^{\infty} I_n$ and let $F_0 \in \mathcal{A}$, then $F_0 = \cap_{k=1}^{\infty} I_k^{(F_0)}$. Since $x \in \cap_{n=1}^{\infty} I_n$, then $x \in I_k^{(F_0)}$ for all $k \in \mathbf{Z}^+$. It follows that

  $$x \in \bigcap_{k=1}^{\infty} I_k^{(F_0)} = F_0$$

  Since F_0 was an arbitrary element of $\mathcal{A}$, then $x \in \cap_{F\in\mathcal{A}} F$. Since x was an arbitrary element of $\cap_{n=1}^{\infty} I_n$, then

  $$\bigcap_{n=1}^{\infty} I_n \subset \bigcap_{F\in\mathcal{A}} F$$

Now if we suppose that $\cap_{F\in\mathcal{A}} F = \varnothing$, we get:

$$\bigcap_{n=1}^{\infty}(F_0 \cap I_n) = F_0 \cap \bigcap_{n=1}^{\infty} I_n = \varnothing$$

But notice that on the left hand side, we have a countable intersection of closed and bounded sets. By the theorem stated at the very beginning, we must have a finite subcollection $\{F_0 \cap I_{n_1}, ..., F_0 \cap I_{n_,}\}$ such that

$$F_0 \cap \bigcap_{i=1}^{m} I_{n_i} = \bigcap_{i=1}^{m}(F_0 \cap I_{n_i}) = \varnothing$$

Now, for each $i \in [\![1, m]\!]$, since $I_{n_i} \in I$ and by definition of I, there must be a set $F_i \in \mathcal{A}$ such that $I_{n_i} \in \{I_k^{(F_i)}\}_k$ which implies that

$$F_i = \bigcap_{k=1}^{\infty} I_k^{(F_i)} \subset I_{n_i}$$

Therefore:

$$F_0 \cap F_1 \cap ... \cap F_m \subset F_0 \cap \bigcap_{i=1}^{m} I_{n_i} = \varnothing$$

which proves our claim.

Exercise 6
Prove that if $a, b \in \mathbf{R}$ and $a < b$, then

$$|(a,b)| = |[a,b)| = |(a,b]| = b - a.$$

Solution
Since the sets $\{a\}$, $\{b\}$ and $\{a,b\}$ are all of outer measure zero, then by exercise 1:

- $|(a,b)| = |(a,b) \cup \{a,b\}| = |[a,b]| = b - a$
- $|[a,b)| = |[a,b) \cup \{b\}| = |[a,b]| = b - a$
- $|(a,b]| = |(a,b] \cup \{a\}| = |[a,b]| = b - a$

which proves our claim.

Exercise 7
Suppose a, b, c, d are real numbers with $a < b$ and $c < d$. Prove that

$$|(a,b) \cup (c,d)| = (b-a) + (d-c) \text{ if and only if } (a,b) \cap (c,d) = \varnothing.$$

Solution
First, suppose that $(a,b) \cap (c,d) = \varnothing$, then we either have $b < c$ or $d < a$. Assume sithout loss of generality that $b < c$. By subadditivity, we have

$$|(a,b) \cup (c,d)| \leq |(a,b)| + |(c,d)| = (b-a) + (d-c)$$

By exercise 3, we also have

$$\begin{aligned} |(a,b) \cup (c,d)| &= |(a,d) \setminus [b,c]| \\ &\geq |(a,d)| - |[b,c]| \\ &= (d-a) - (c-b) \\ &= (b-a) + (d-c) \end{aligned}$$

which shows that

$$|(a,b) \cup (c,d)| = (b-a) + (d-c)$$

Suppose now that$(a,b) \cap (c,d) \neq \varnothing$, then we either have $(a,b) \cup (c,d) = (a,d)$ or $(a,b) \cup (c,d) = (c,b)$. Assume without loss of generality that $(a,b) \cup (c,d) = (a,d)$, then since we must have $c < b$, we get

$$\begin{aligned} |(a,b) \cup (c,d)| &= |(a,d)| \\ &= d - a \\ &< d - a + b - c \\ &= (b-a) + (d-c) \end{aligned}$$

Therefore,

$$|(a,b) \cup (c,d)| \neq (b-a) + (d-c)$$

which proves the equivalence between the two statements.

Exercise 8
Prove that if $A \subset \mathbf{R}$ and $t > 0$, then $|A| = |A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))|$.

Solution
First, by subadditivity, we have

$$\begin{aligned} |A| &= |A \cap [(-t,t) \cup (\mathbf{R} \setminus (-t,t))]| \\ &= |[A \cap (-t,t)] \cup [A \cap (\mathbf{R} \setminus (-t,t))]| \\ &\leq |A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))| \end{aligned}$$

which gives us

$$|A| \leq |A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))| \tag{1}$$

Let's now prove the reverse inequality. Let $\epsilon > 0$, then by properties of the infimum, there exists a collection $\{I_k\}_k$ of open intervals that covers A and such that

$$\sum_{k=1}^{\infty} \ell(I_k) < |A| + \frac{\epsilon}{2}$$

Consider now the subcollection $\{I_{1,k}\}_k$ of $\{I_k\}_k$ only composed of the intervals that are fully contained in $(-t,t)$. Similarly, define the subcollection $\{I_{2,k}\}_k$ of $\{I_k\}_k$ only composed of the intervals that are fully contained in $(-t,t)^c$. Obviously, these two subcollection are disjoint but may not partition $\{I_k\}_k$ since there may be intervals that are neither fully contained in $(-t,t)$ nor in $(-t,t)^c$. Concerning these sets, let's define the collections $\{I_{3,k}\}_k$, $\{I_{4,k}\}_k$ and $\{I_{5,k}\}_k$ that will contain the following intervals. Let $I_k = (a_k, b_k) \in \{I_k\}_k$.

- If both t and $-t$ are contained in I_k, then by the previous definitions, we have $I_k \in \{I_{1,k}\}_k$ or $I_k \in \{I_{2,k}\}_k$.
- If I_k contains t but not $-t$, define

$$I_{3,k} = \left(a_k, t + \frac{\epsilon}{2^{k+2}}\right)$$

$$I_{4,k} = \left(t - \frac{\epsilon}{2^{k+2}}, b_k\right)$$

- If I_k contains $-t$ but not t, define

$$I_{3,k} = \left(a_k, -t + \frac{\epsilon}{2^{k+2}}\right)$$

$$I_{4,k} = \left(-t - \frac{\epsilon}{2^{k+2}}, b_k\right)$$

- If both t and $-t$ are contained in I_k, define

$$I_{3,k} = (-t,t)$$

$$I_{4,k} = \left(a_k, -t + \frac{\epsilon}{2^{k+2}}\right)$$

$$I_{5,k} = \left(t - \frac{\epsilon}{2^{k+2}}, b_k\right)$$

Consider now the collections $A_0 = \{I_{1,k}\}_k \cup \{I_{3,k}\}_k$ and $B_0 = \{I_{2,k}\}_k \cup \{I_{4,k}\}_k \cup \{I_{5,k}\}_k$. By construction, A_0 is a collection of open intervals that covers $A \cap (-t,t)$ and B_0 is a collection of open intervals that covers $A \cap (\mathbf{R} \setminus (-t,t))$. Moreover, even if the collections $A_0 \cup B_0$ and $\{I_k\}_k$, the construction was done so that the total length of all the open intervals in $A_0 \cup B_0$ differs from the total length of all the open intervals in $\{I_k\}_k$ by at most $\sum_{k=1}^{\infty} 2\frac{\epsilon}{2^{k+2}} = \frac{\epsilon}{2}$. This gives us

$$\sum_{I \in A_0} \ell(I) + \sum_{I \in B_0} \ell(I) \leq \sum_{k=1}^{\infty} \ell(I_k) + \frac{\epsilon}{2}$$

which implies

$$\begin{aligned} |A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))| &\leq \sum_{I \in A_0} \ell(I) + \sum_{I \in B_0} \ell(I) \\ &\leq \sum_{k=1}^{\infty} \ell(I_k) + \frac{\epsilon}{2} \\ &< |A| + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= |A| + \epsilon \end{aligned}$$

Taking $\epsilon \to 0$ gives us

$$|A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))| \leq |A| \tag{2}$$

Combining (1) and (2) gives us

$$|A| = |A \cap (-t,t)| + |A \cap (\mathbf{R} \setminus (-t,t))|$$

which is the desired equation.

Exercise 9
Prove that $|A| = \lim\limits_{t \to \infty} |A \cap (-t,t)|$ for all $A \subset \mathbf{R}$.

Solution
For this proof, let's first prove by induction that

$$|A \cap (-n,n)| = \sum_{i=1}^{n} |A \cap ((-i,-i+1] \cup [i-1,i))|$$

for all $n \in \mathbf{Z}^+$.

- (Base Case) For $n = 1$, it can be derived as follows

$$\sum_{i=1}^{1} |A \cap ((-i,-i+1] \cup [i-1,i))| = |A \cap ((-1,0] \cup [0,1))| = |A \cap (-1,1)|$$

- (Inductive Step) Suppose that there is a $k \in \mathbf{Z}^+$ such that

$$|A \cap (-k,k)| = \sum_{i=1}^{k} |A \cap ((-i,-i+1] \cup [i-1,i))|$$

holds. Let's prove it for $k+1$. Notice that it suffices to apply the result of the previous exercise to the set $A \cap (-k-1, k+1)$ with $t = k$:

$$\begin{aligned}
|A \cap (-k-1, k+1)| &= |(A \cap (-k-1, k+1)) \cap (\mathbf{R} \setminus (-k, k))| \\
&\quad + \quad |A \cap (-k-1, k+1) \cap (-k, k)| \\
&= |A \cap ((-k-1, -k] \cup [k, k+1))| + |A \cap (-k, k)| \\
&= |A \cap ((-k-1, -k] \cup [k, k+1))| \\
&\quad + \quad \sum_{i=1}^{k} |A \cap ((-i, -i+1] \cup [i-1, i))| \\
&= \sum_{i=1}^{k+1} |A \cap ((-i, -i+1] \cup [i-1, i))|
\end{aligned}$$

which proves it $k+1$.

Now that we proved the formula, let's prove our claim. By subadditivity,

$$\begin{aligned}
|A| &= \left| \bigcup_{i=1}^{\infty} A \cap ((-i, -i+1] \cup [i-1, i)) \right| \\
&\leq \sum_{i=1}^{\infty} |A \cap ((-i, -i+1] \cup [i-1, i))| \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} |A \cap ((-i, -i+1] \cup [i-1, i))| \\
&= \lim_{n \to \infty} |A \cap (-n, n)|
\end{aligned}$$

Moreover, by monotonicity, for all $n \in \mathbf{Z}^+$, we have

$$|A \cap (-n, n)| \leq |A|$$

It follows that

$$\lim_{n \to \infty} |A \cap (-n, n)| \leq |A|$$

Thus,

$$|A| = \lim_{n \to \infty} |A \cap (-n, n)|$$

But we still need to prove it when the limit is taken over all positive real numbers t and not just for positive integers. However, the desired result follows from the fact that $t \mapsto |A \cap (-t, t)|$ is increasing which shows that

$$\lim_{t \to \infty} |A \cap (-t, t)| = \sup_{t \geq 0} |A \cap (-t, t)| = \sup_{n \in \mathbf{Z}^+} |A \cap (-n, n)| = \lim_{n \to \infty} |A \cap (-n, n)|$$

Therefore,

$$|A| = \lim_{t \to \infty} |A \cap (-t, t)|$$

Exercise 10
Prove that $|[0, 1] \setminus \mathbf{Q}| = 1$.

Solution

Since $\mathbf{Q} \cap [0,1]$ is countable, and hence has measure zero, then by exercise 1 of this section:

$$|[0,1] \setminus \mathbf{Q}| = |([0,1] \setminus \mathbf{Q}) \cup (\mathbf{Q} \cap [0,1])| = |[0,1]| = 1$$

Exercise 11

Prove that if $I_1, I_2, ...$ is a disjoint sequence of open intervals, then

$$\left| \bigcup_{k=1}^{\infty} I_k \right| = \sum_{k=1}^{\infty} \ell(I_k).$$

Solution

Let's first prove it for finitely many disjoint open intervals $I_1, ..., I_n$ where $n \in \mathbf{Z}^+$. By subadditivity, we have

$$\left| \bigcup_{k=1}^{n} I_k \right| \leq \sum_{k=1}^{n} \ell(I_k)$$

Moreover, by if write $I_k = (a_k, b_k)$ and suppose that they are ordered as follows

$$a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$$

Then, by exercise 3, we have

$$\begin{aligned} |I_n \cup (a_1, b_{n-1})| &= |(a_1, b_n) \setminus [b_{n-1}, a_n]| \\ &\geq |(a_1, b_n)| - |[b_{n-1}, a_n]| \\ &= b_n - a_1 - a_n + b_{n-1} \\ &= \ell(I_n) + |(a_1, b_{n-1})| \end{aligned}$$

by induction, it follows that

$$\left| \bigcup_{k=1}^{n} I_k \right| \geq \sum_{k=1}^{n} \ell(I_k)$$

Thus, equality holds in the finite case. Consider now the infinite case with the sequence $I_1, I_2, ...$, then again, by subadditivity:

$$\left| \bigcup_{k=1}^{\infty} I_k \right| \leq \sum_{k=1}^{\infty} \ell(I_k)$$

However, notice that for all $n \in \mathbf{Z}^+$, using the finite case, we have

$$\left| \bigcup_{k=1}^{\infty} I_k \right| \geq \left| \bigcup_{k=1}^{n} I_k \right| = \sum_{k=1}^{n} \ell(I_k)$$

Hence, taking $n \to \infty$ gives us

$$\left| \bigcup_{k=1}^{\infty} I_k \right| \geq \sum_{k=1}^{\infty} \ell(I_k)$$

which finishes the proof.

Exercise 12

Suppose $r_1, r_2, ...$ is a sequence that contains every rational number. Let

$$F = \mathbf{R} \setminus \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right)$$

(a) Show that F is a closed subset of $\mathbf{R}$.

(b) Prove that if I is an interval contained in F, then I contains at most one element.

(c) Prove that $|F| = \infty$

Solution

(a) If we rewrite

$$F = \mathbf{R} \cap \bigcap_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right)^c$$

then we get that F is simply an intersection of closed sets. Hence, F is closed as well.

(b) By definition, F contains no rationals. Let I be an interval contained in F. Suppose that I has two distinct elements a and b such that $a < b$, then, $[a, b] \subset F$. However, by the density of $\mathbf{Q}$ in $\mathbf{R}$, there must be a rational r_0 in $[a, b]$ which would imply that $r_0 \in F$. A contradiction. Thus, I contains at most one element.

(c) The proof is straightforward:

$$\begin{aligned}
|F| &= \left| \mathbf{R} \setminus \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right) \right| \\
&\geq |\mathbf{R}| - \left| \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right) \right| \\
&\geq |\mathbf{R}| - \sum_{k=1}^{\infty} \left| \left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k} \right) \right| \\
&= |\mathbf{R}| - \sum_{k=1}^{\infty} 2 \cdot \frac{1}{2^k} \\
&= \infty - 2 \\
&= \infty
\end{aligned}$$

Therefore, $|F| = \infty$.

Exercise 13

Suppose $\epsilon > 0$. Prove that there exists a subset F of $[0,1]$ such that F is closed, every element in F is an irrational number, and $|F| > 1-\epsilon$.

Solution

Since $\mathbf{Q} \cap [0,1]$ is countable, then it has measure zero. By the properties of the infimum, there is a cover $\{I_k\}_k$ of open intervals of $\mathbf{Q} \cap [0,1]$ that satisfies

$$\sum_{k=1}^{\infty} \ell(I_k) < \epsilon$$

Consider now the set F defined by

$$F = [0,1] \setminus \bigcup_{k=1}^{\infty} I_k$$

Then, $F \subset [0,1]$. To show that F is closed, notice that we can write

$$F = [0,1] \cap \bigcap_{k=1}^{\infty} I_k^c$$

which is an intersection of closed sets, hence, closed. To show that F contains only rational numbers, notice that $\cup_{k=1}^{\infty} I_k$ covers $\mathbf{Q} \cap [0,1]$, hence, contains all the rationals in $[0,1]$. It follows that $[0,1] \setminus \cup_{k=1}^{\infty} I_k$ contains no rationals. Finally:

$$\begin{aligned}
|F| &= \left| [0,1] \setminus \bigcup_{k=1}^{\infty} I_k \right| \\
&\geq |[0,1]| - \left| \bigcup_{k=1}^{\infty} I_k \right| \\
&\geq 1 - \sum_{k=1}^{\infty} |I_k| \\
&= 1 - \sum_{k=1}^{\infty} \ell(I_k) \\
&> 1 - \epsilon
\end{aligned}$$

which proves that F has all the required properties.

Exercise 14

Consider the following figure, which is drawn accurately to scale.

[...]

(a) Show that the right triangle whose vertices are (0,0), (20, 0) and (20, 9) has area 90.
[*We have not defined area yet but just use the elementary formulas for the areas of triangles and rectangles that you learned long ago.*]

(b) Show that the yellow (lower) right triangle has area 27.5.

(c) Show that the red rectangle has area 45.

(d) Show that the blue (upper) right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect parts (a) and (e) to have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ. [*You may be tempted to think that what we have here is a two-dimensional example similar to the result about the nonadditivity of outer measure (2.18). However, genuine examples of nonadditivity require much more complicated sets than in this example.*]

Solution

(a) Area $= \frac{20 \cdot 9}{2} = 90$

(b) $\text{Area}_{\text{yellow}} = \frac{11 \cdot 5}{2} = 27.5$

(c) $\text{Area}_{\text{red}} = (20 - 11) \cdot 5 = 45$

(d) $\text{Area}_{\text{blue}} = \frac{(20-11)(9-5)}{2} = 18$

(e) $\text{Area}_{\text{yellow}} + \text{Area}_{\text{red}} + \text{Area}_{\text{blue}} = 27.5 + 45 + 18 = 90.5$

(f) The big triangle composed of the three coloured shapes is actually not a triangle at all. To verify this, if there was a triangle with vertices (0,0), (20, 0) and (20, 9), then a quick calculation shows that it passes through the point (11, 4.95) and not (11, 5).

2B Measurable Spaces and Functions

Exercise 1
Show that $\mathcal{S} = \{\cup_{n\in K}(n, n+1] : K \subset \mathbf{Z}^+\}$ is a σ-algebra on $\mathbf{R}$.

Solution
As most of the proofs showing that a collection is a σ-algebra, let's split this one into three parts:

- ($\varnothing \in \mathcal{S}$) Since $\varnothing \subset \mathbf{Z}$, then $\cup_{n\in\varnothing}(n, n+1] \in \mathcal{S}$. However, notice that $\cup_{n\in\varnothing}(n, n+1] = \varnothing$. It follows that $\varnothing \in \mathcal{S}$.
- (closed under complements) Let $A \in \mathcal{S}$, then there exists a $K_0 \subset \mathbf{Z}$ such that $A = \cup_{n\in K_0}(n, n+1]$ Consider $K_1 = \mathbf{Z} \setminus K_0$ and its associated element $B = \cup_{n\in K_1}(n, n+1]$ in $\mathcal{S}$. Since $A \cap B = \varnothing$ and $A \cup B = \mathbf{R}$, then $B = \mathbf{R} \setminus A$. Hence, $A^c \in \mathcal{S}$ which proves that $\mathcal{S}$ is closed under complements.
- (closed under countable union) Let $\{A_i\}_i$ be a countable collection of elements in $\mathcal{S}$, then for all $i \in \mathbf{Z}^+$, there is a subset K_i of $\mathbf{Z}$ such that $A_i = \cup_{n\in K_i}(n, n+1]$. Consider $K = \cup_{i=1}^{\infty} K_i \subset \mathbf{Z}$ and $A = \cup_{n\in K}(n, n+1] \in \mathcal{S}$. By consruction, $A = \cup_{i=1}^{\infty} A_i \in \mathcal{S}$. Therefore, $\mathcal{S}$ is closed under countable union.

Therefore, $\mathcal{S}$ is a σ-algebra on $\mathbf{R}$.

Exercise 2
Verify both bullet points in Example 2.28.

Solution

- Suppose X is a set and $\mathcal{A}$ is the set of subsets of X that consist of exactly one element:
$$\mathcal{A} = \{\{x\} : x \in \mathcal{A}\}$$
Define $\mathcal{S}$ to be the smallest σ-algebra on X generated by $\mathcal{A}$. Let's prove that $\mathcal{S}$ is precisely the collection of subsets of X that are countable or co-countable. To make it easier, denote by $\mathcal{M}$ the collection of subsets of X that are countable or co-countable.
Hence, we need to prove that $\mathcal{S} = \mathcal{M}$. We already know from example 2.24 that $\mathcal{M}$ is a σ-algebra on X. Moreover, it is easy to see that $\mathcal{A} \subset \mathcal{M}$. It follows that $\mathcal{S} \subset \mathcal{M}$.
To prove the reverse inclusion, let $E \in \mathcal{M}$, then one of E or E^c is countable. If E is countable, then we can simply write E as the countable union of the singletons of its elements, hence, a countable union of elements in $\mathcal{A} \subset \mathcal{S}$. This would imply that $E \in \mathcal{S}$. Similarly, if E^c is countable, then with the same argument, $E^c \in \mathcal{S}$ which also implies that $E \in \mathcal{S}$. Thus, $\mathcal{M} \subset \mathcal{S}$. It follows that $\mathcal{S} = \mathcal{M}$.
- Let $\mathcal{A} = \{(0,1), (0,\infty)\}$ and denote by $\mathcal{S}$ the smallest σ-algebra containing $\mathcal{A}$. Define the collection
$$E = \{\varnothing,\ (0,1),\ (0,\infty),\ (-\infty,0] \cup [1,\infty),\ (-\infty,0],\ [1,\infty),\ (-\infty,1),\ \mathbf{R}\}$$
Let's show that $\mathcal{S} = E$. First, let's prove that E is a σ-algebra:

- ($\varnothing \in E$) By definition of E.
- (closed under complement)
  * $\mathbf{R} \setminus \varnothing = \mathbf{R} \in E$
  * $\mathbf{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty) \in E$
  * $\mathbf{R} \setminus (0, \infty) = (-\infty, 0] \in E$
  * $\mathbf{R} \setminus ((-\infty, 0] \cup [1, \infty)) = (0, 1) \in E$
  * $\mathbf{R} \setminus [1, \infty) = (-\infty, 1) \in E$
  * $\mathbf{R} \setminus (-\infty, 0] = (0, \infty) \in E$
  * $\mathbf{R} \setminus (-\infty, 1) = [1, \infty) \in E$
  * $\mathbf{R} \setminus \mathbf{R} = \varnothing \in E$
- (closed under countable union) Since E is finite, then it suffices to check that E is closed under the regular union between two sets. To be faster, I skipped the trivial unions that involve $\mathbf{R}$ or $\varnothing$.
  * $(0, 1) \cup (0, \infty) = (0, \infty) \in E$
  * $(0, 1) \cup ((-\infty, 0] \cup [1, \infty)) = \mathbf{R} \in E$
  * $(0, 1) \cup [1, \infty) = (0, \infty) \in E$
  * $(0, 1) \cup (-\infty, 0] = (-\infty, 1) \in E$
  * $(0, 1) \cup (-\infty, 1) = (-\infty, 1) \in E$
  * $(0, \infty) \cup ((-\infty, 0] \cup [1, \infty)) = \mathbf{R} \in E$
  * $(0, \infty) \cup [1, \infty) = (0, \infty) \in E$
  * $(0, \infty) \cup (-\infty, 0] = \mathbf{R} \in E$
  * $(0, \infty) \cup (-\infty, 1) = \mathbf{R} \in E$
  * $((-\infty, 0] \cup [1, \infty)) \cup [1, \infty) = ((-\infty, 0] \cup [1, \infty)) \in E$
  * $((-\infty, 0] \cup [1, \infty)) \cup (-\infty, 0] = ((-\infty, 0] \cup [1, \infty)) \in E$
  * $((-\infty, 0] \cup [1, \infty)) \cup (-\infty, 1) = \mathbf{R} \in E$
  * $[1, \infty) \cup (-\infty, 0] = ((-\infty, 0] \cup [1, \infty)) \in E$
  * $[1, \infty) \cup (-\infty, 1) = \mathbf{R} \in E$
  * $(-\infty, 0] \cup (-\infty, 1) = (-\infty, 1) \in E$

Therefore, E is a σ-algebra on $\mathbf{R}$ that contains $\mathcal{A}$. It follows that $\mathcal{S} \subset E$. For the reverse inclusion, let's prove that any element in E can be constructed from elements in $\mathcal{A}$ using the operations of σ-algebras, i.e., complements and unions:

- $\varnothing$ is in $\mathcal{S}$ because $\mathcal{S}$ is a σ-algebra.
- (0,1) is in $\mathcal{S}$ because it is in $\mathcal{A}$.
- $(0, \infty)$ is in $\mathcal{S}$ because it is in $\mathcal{A}$.
- $(-\infty, 0] \cup [1, \infty)$ is in $\mathcal{S}$ because it is the complement of (0,1) which is in $\mathcal{S}$.
- $[1, \infty)$ is in $\mathcal{S}$ because it can be written as $(0, \infty) \setminus (0, 1)$ and we already know that $(0, \infty)$ and $(0, 1)$ are in $\mathcal{S}$.
- $(-\infty, 0]$ is in $\mathcal{S}$ because it can be written as $((-\infty, 0] \cup [1, \infty)) \setminus [1, \infty)$ and we already know that both sets are in $\mathcal{S}$.

- $(-\infty, 1)$ is in $\mathcal{S}$ because it is the complement of $[1, \infty)$ which is in $\mathcal{S}$.
- $\mathbf{R}$ is in $\mathcal{S}$ because $\mathcal{S}$ is a σ-algebra.

Therefore, $\mathcal{S} = E$.

Exercise 3
Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, s] : r, s \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

Solution
To make things easier, let's denote by $\mathcal{B}$ the collection of Borel subsets of $\mathbf{R}$. We need to prove that $\mathcal{S} = \mathcal{B}$. First, notice that for all rationals $r, s \in \mathbf{Q}$ with $r < s$, the set $(r, s] \in \mathcal{B}$ since it can be written as $(r, \infty) \setminus (s, \infty)$ and both are open (so Borel) sets. It follows that $\mathcal{S} \subset \mathcal{B}$.
For the reverse inclusion, let's show that $\mathcal{S}$ contains every open sets. Let's prove first that $\mathcal{S}$ contains open interval Let (a, b) be an open interval with $a < b \in \mathbf{R}$. By density of $\mathbf{Q}$ in $\mathbf{R}$, there exist two sequence $\{q_n\}_n$ and $\{s_n\}$ of rationals that satisfy the following properties : $\{q_n\}_n$ is decreasing and converges to a, $\{s_n\}_n$ is increasing and converges to b. Since all of the terms are rationals, then $(r_n, s_n] \in \mathcal{S}$ for all $n \in \mathbf{Z}^+$. But $\mathcal{S}$ is a σ-algebra so

$$(a, b) = \bigcup_{n=1}^{\infty} (r_n, s_n] \in \mathcal{S}$$

Hence, $\mathcal{S}$ contains every open interval. Now, using the fact that any open set can be written as a countable union of open intervals, it easily follows that $\mathcal{S}$ actually contains every open set. Therefore, $\mathcal{B} \subset \mathcal{S}$ since $\mathcal{B}$ is generated by the open sets so $\mathcal{S} = \mathcal{B}$.

Exercise 4
Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}^+\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

Solution
For this proof, I will use the result of the previous exercise. Hence, define $E = \{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}^+\}$, $E_0 = \{(r, s] : r, s \in \mathbf{Q}\}$ and consider $\mathcal{S}_0$ to be the smallest σ-algebra generated by E_0. Let's denote by $\mathcal{B}$ the collection of Borel subsets of $\mathbf{R}$. Obviously, since $E \subset E_0 \subset \mathcal{S}_0$, then $\mathcal{S} \subset \mathcal{S}_0$.
For the reverse inclusion, let's show that $E_0 \subset \mathcal{S}$. Let $(r, s] \in E_0$ with $r, s \in \mathbf{Q}$. Notice that for all integers $n \geq r$, $(r, n] \in E \subset \mathcal{S}$, hence, taking their union gives us

$$(r, \infty) = \bigcup_{n \geq r} (r, n] \in \mathcal{S}$$

Similarly, $(s, \infty) \in \mathcal{S}$ for the same reasons. Hence, $(r, s] = (r, \infty) \setminus (s, \infty) \in \mathcal{S}$. It follows that $E_0 \subset \mathcal{S}$ which implies that $\mathcal{S}_0 \subset \mathcal{S}$. Therefore

$$\mathcal{S} = \mathcal{S}_0 = \mathcal{B}$$

Exercise 5
Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, r+1) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

Solution
Let $E = \{(r, r+1) : r \in \mathbf{Q}\}$ and denote by $\mathcal{B}$ the collection of Borel subsets of $\mathbf{R}$. Moreover, let $E_0 = \{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}^+\}$ and define $\mathcal{S}_0$ as the smallest σ-algebra containing E_0. By exercise 4, we know that

$$\mathcal{S}_0 = \mathcal{B}$$

Let's show that $\mathcal{S} = \mathcal{B}$. Obviously, since every element in E is a Borel set, then $E \subset \mathcal{B}$ which implies that $\mathcal{S} \subset \mathcal{B}$.
For the reverse inclusion, Let's show that $E_0 \subset \mathcal{S}$. Let $(r, n]$ be an arbitrary element of E_0 with $r \in \mathbf{Q}$ and $n \in \mathbf{Z}^+$. By definition of E, we know that for all $k \geq 0$, the set $(r + \frac{1}{2}k, r + \frac{1}{2}k + 1) \in E \subset \mathcal{S}$. Thus, since $\mathcal{S}$ is closed under countable unions,

$$(r, \infty) = \bigcup_{k=1}^{\infty} \left(r + \frac{1}{2}k, r + \frac{1}{2}k + 1 \right) \in \mathcal{S}$$

Similarly, $(n, \infty) \in \mathcal{S}$ for the same reasons. Hence, $(r, s] = (r, \infty) \setminus (n, \infty) \in \mathcal{S}$. It follows that $E_0 \subset \mathcal{S}$ which implies $\mathcal{B} = \mathcal{S}_0 \subset \mathcal{S}$. Therefore, $\mathcal{S} = \mathcal{B}$.

Exercise 6
Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{[r, \infty) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ is the collection of Borel subsets of $\mathbf{R}$.

Solution
Let $E = \{[r, \infty) : r \in \mathbf{Q}\}$ and denote by $\mathcal{B}$ the collection of Borel subsets of $\mathbf{R}$. Moreover, let $E_0 = \{(r, s] : r, s \in \mathbf{Q}\}$ and define $\mathcal{S}_0$ as the smallest σ-algebra containing E_0. By exercise 3, we know that

$$\mathcal{S}_0 = \mathcal{B}$$

Let's show that $\mathcal{S} = \mathcal{B}$. Since every element of E is closed, then $E \subset \mathcal{B}$ (closed sets are Borel sets). It follows that $\mathcal{S} \subset \mathcal{B}$.
For the reverse inclusion, let's prove that $E_0 \subset \mathcal{S}$. Let $(r, s]$ be an arbitrary set in E_0 with $r < s \in \mathbf{Q}$, then by definition of E, both $[r + \frac{1}{n}, \infty)$ and $[s + \frac{1}{n}, \infty)$ are contained in E and hence in $\mathcal{S}$. Since $\mathcal{S}$ is a σ-algebra, then

$$(r, \infty) = \bigcup_{n=1}^{\infty} \left[r + \frac{1}{n}, \infty \right) \in \mathcal{S}$$

and

$$(s, \infty) = \bigcup_{n=1}^{\infty} \left[s + \frac{1}{n}, \infty \right) \in \mathcal{S}$$

It follows that $(r, s] = (r, \infty) \setminus (s, \infty) \in \mathcal{S}$. Hence, $E_0 \subset \mathcal{S}$ which implies $\mathcal{B} = \mathcal{S}_0 \subset \mathcal{S}$. Therefore, $\mathcal{S} = \mathcal{B}$.

Exercise 7
Prove that the collection of Borel subsets of $\mathbf{R}$ is translation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then $t + B$ is a Borel set.

Solution
Let $B \subset \mathbf{R}$ be a Borel set and t be an arbitrary real number. Let's show that $t+B$ is a Borel set. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $x \mapsto x - t$. Since f is continuous, then f is Borel measurable. It follows that $f^{-1}(B)$ is a Borel set. However, notice that for all $x \in \mathbf{R}$:

$$\begin{aligned} x \in f^{-1}(B) &\iff f(x) \in B \\ &\iff x - t \in B \\ &\iff x \in t + B \end{aligned}$$

Hence, $t + B = f^{-1}(B)$. Therefore, $t + B$ is a Borel set which proves that the collection of Borel sets is translation invariant.

Exercise 8
Prove that the collection of Borel subsets of $\mathbf{R}$ is dilation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then tB (which is defined to be $\{tb : b \in B\}$) is a Borel set.

Solution
Let $B \subset \mathbf{R}$ be a Borel set and t be an arbitrary real number. Let's show that tB is a Borel set. Notice that the case $t = 0$ is trivial since $tB = \{0\}$ in that case and $\{0\}$ is a Borel set. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $x \mapsto \frac{1}{t}x$. Since f is continuous, then f is Borel measurable. It follows that $f^{-1}(B)$ is a Borel set. However, notice that for all $x \in \mathbf{R}$:

$$\begin{aligned} x \in f^{-1}(B) &\iff f(x) \in B \\ &\iff \frac{1}{t}x \in B \\ &\iff x \in tB \end{aligned}$$

Hence, $tB = f^{-1}(B)$. Therefore, tB is a Borel set which proves that the collection of Borel sets is dilation invariant.

Exercise 9
Give an example of a measurable space $(X, \mathcal{S})$ and a function $f : X \to \mathbf{R}$ such that $|f|$ is $\mathcal{S}$-measurable but f is not $\mathcal{S}$-measurable.

Solution
Consider $(X, \mathcal{S}) = (\mathbf{R}, \{\varnothing, \mathbf{R}\})$ and the function $f : X \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Notice that $|f| = \chi_{\mathbf{R}}$ and hence $\mathcal{S}$-measurable since $\mathbf{R} \in \mathcal{S}$. However,

$$f^{-1}(\{1\}) = [0, \infty) \notin \mathcal{S}$$

even if $\{1\}$ is a Borel set. Therefore, $|f|$ is $\mathcal{S}$-measurable but not f.

Exercise 10
Show that the set of real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.

Solution
This proof will have three steps but the idea is the following :

1. Construct the set of reals that contains no digit 5 in their decimal part in a process similar to the construction of the Cantor set. By construction, show that this set is a Borel set.

2. Construct, using the previous set, the set of reals with finitely many 5's in their decimal expansion. By construction, show that this set is a Borel set.

3. Simply take the complement of the previous set. It follows that the desired set is Borel.

(Step 1) Let's construct the set of reals that contains no 5 in their decimal part. Let's construct recursively a sequence of sets that converges to the desired set. To do so, define the Borel set $M_0 = \mathbf{R}$ which simply represents the reals. Define M_1 which represents the set of reals that contains no 5 in their first decimal and M_2 as the set representing the reals with no digit 5 in their first two decimals. To generalize this process, suppose that M_n is a Borel which represents the reals which contains no 5 in their first n decimals, to construct M_{n+1}, simply remove from M_n the reals with a 5 in their $(n+1)$st decimal. Notice that the set of of reals with a 5 in their $(n+1)$st decimal can be written as follows : $\frac{1}{10^{n+1}}(10\mathbf{Z}+5)$. Hence,

$$M_{n+1} = M_n \setminus \frac{1}{10^{n+1}}(10\mathbf{Z}+5)$$

By properties of σ-algebras and exercise 7 and 8, M_{n+1} is also a Borel set. Thus, if we define

$$M = \bigcap_{n=0}^{\infty} M_n$$

by construction of the M_n's, we have that M is precisely the (Borel) set of reals that contains no 5's in their decimal part (such reals can contain a 5 in their decimal representation but only in the integer part).

(Step 2) To construct the set of reals with finitely many 5's in their decimal expansion, notice that if $x \in \mathbf{R}$ has finitely many 5's in their decimal expansion, then there is a natural number n such that $10^n x$ has no 5's in its decimal part. From this observation, we get that the set

$$N = \bigcup_{n=1}^{\infty} \frac{1}{10^n} M$$

is precisely the set of reals with finitely 5's in their decimal expansion. Moreover, by exercise 8 and by properties of σ-algebras, N is a Borel set.

(Step 3) By construction, N^c must be the set of reals with infinitely many 5's in their decimal expansion. Since the collection of Borel sets is closed under complements, then N^c is a Borel set. We could also have shown that it has outer measure 0 by the construction on the interval $[0,1]$ and then extending to the reals but the proof would have been longer.

Exercise 11

Suppose $\mathcal{T}$ is a σ-algebra on a set $\mathcal{Y}$ and $X \in \mathcal{T}$. Let $\mathcal{S} = \{E \in \mathcal{T} : E \subset X\}$.

(a) Show that $\mathcal{S} = \{F \cap X : F \in \mathcal{T}\}$.

(b) Show that $\mathcal{S}$ is a σ-algebra on X.

Solution

(a) Let $E \in \mathcal{S}$, then $E \in \mathcal{T}$ and $E \subset X$. It follows that $E = E \cap X \in \{F \cap X : F \in \mathcal{T}\}$. Hence, $\mathcal{S} \subset \{F \cap X : F \in \mathcal{T}\}$. For the reverse inclusion, let $F \cap X$ be an arbitrary element of $\{F \cap X : F \in \mathcal{T}\}$, then $F \in \mathcal{T}$ which implies that $F \cap X \in \mathcal{T}$. Moreover, $F \cap X \subset X$ so $F \cap X \in \mathcal{S}$. Therefore, $\mathcal{S} = \{F \cap X : F \in \mathcal{T}\}$.

(b) First, since $\varnothing \in \mathcal{T}$ and $\varnothing \subset X$, then $\varnothing \in \mathcal{S}$. Now, if E is an arbitrary element of $\mathcal{S}$, then its complement, $X \setminus E$ is still in $\mathcal{T}$ and obviously is a subset of X. Hence, $E^c \in \mathcal{S}$. Thus, $\mathcal{S}$ is closed under complements. Suppose that $\{E_n\}_n$ is a countable collection of elements in $\mathcal{S}$, then they all are in $\mathcal{T}$ and all are subsets of X. It follows that their union is still in $\mathcal{T}$ and still a subset of X. Hence, their union is in $\mathcal{S}$. Therefore, $\mathcal{S}$ is a σ-algebra.

Exercise 12

Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function.

(a) For $k \in \mathbf{Z}^+$, let

$$\begin{aligned} G_k = \{a \in \mathbf{R} : \text{there exists } \delta > 0 \text{ such that } |f(b) - f(c)| < \tfrac{1}{k} \\ \text{for all } b, c \in (a - \delta, a + \delta)\}. \end{aligned}$$

Prove that G_k is an open subset of $\mathbf{R}$ for each $k \in \mathbf{Z}^+$.

(b) Prove that the set of points at which f is continuous equals $\cap_{k=1}^{\infty} G_k$.

(c) Conclude that the set of points at which f is continuous is a Borel set.

Solution

(a) Let $k \in \mathbf{Z}^+$ and let's prove that G_k is open by proving that every point is an interior point of the set. Let $x \in G_k$, then by definition, there is a $\delta > 0$ such that

$$|f(y) - f(z)| < \frac{1}{k}$$

for all $y, z \in (x-\delta, x+\delta)$. Let's show that $(x-\delta, x+\delta) \subset G_k$. Let $x_0 \in (x-\delta, x+\delta)$ and define $\delta_0 = \min(x_0 - x + \delta, x + \delta - x_0)$. It follows that

$$(x_0 - \delta_0, x_0 + \delta_0) \subset (x - \delta, x + \delta)$$

Hence, for all $y, z \in (x_0 - \delta_0, x_0 + \delta_0)$, we have $y, z \in (x-\delta, x+\delta)$ which implies

$$|f(y) - f(z)| < \frac{1}{k}$$

Thus, $x_0 \in G_k$. Since it holds for all $x_0 \in (x-\delta, x+\delta)$, then $(x-\delta, x+\delta) \subset G_k$. Since it holds for all $x \in G_k$, then G_k is open.

(b) Let's show that the elements in $\cap_{k=1}^{\infty} G_k$ are precisely the points on which f is continuous. Let x be a real number such that f is continuous at x. Let $k \in \mathbf{Z}^+$, then by continuity of f at x, there is a $\delta > 0$ such that

$$|f(y) - f(x)| < \frac{1}{2k}$$

whenever $y \in (x-\delta, x+\delta)$. Hence, for all $b, c \in (x-\delta, x+\delta)$, by the triangle inequality:

$$|f(b) - f(c)| < \frac{1}{k}$$

Thus, $x \in G_k$. Since it holds for all $k \in \mathbf{Z}^+$, then $x \in \cap_{k=1}^{\infty} G_k$. It follows that $C_f \subset \cap_{k=1}^{\infty} G_k$.
For the reverse inclusion, let x be an arbitrary element of $\cap_{k=1}^{\infty} G_k$, let's show that f is continuous at x using the ϵ-δ definition. Let $\epsilon > 0$, then by the Archimedean Property of $\mathbf{R}$, there is a $n \in \mathbf{Z}^+$ such that $\frac{1}{n} < \epsilon$. But recall that $x \in \cap_{k=1}^{\infty} G_k \subset G_n$, hence, there is a $\delta > 0$ such that

$$|f(b) - f(c)| < \frac{1}{n}$$

whenever $b, c \in (x-\delta, x+\delta)$. Let $y \in (x-\delta, x+\delta)$, since x is also in $(x-\delta, x+\delta)$, then

$$|f(x) - f(y)| < \frac{1}{n} < \epsilon$$

Thus, by definition, f is continuous at x. Therefore, $\cap_{k=1}^{\infty} G_k$ is precisely the set of points on which f is continuous.

(c) The set of points at which f is continuous can be written as a countable intersection of open sets. Since open sets are Borel sets and Borel sets are closed under countable intersections, then the set of points at which f is continuous is a Borel set.

Exercise 13

Suppose $(X, \mathcal{S})$ is a measurable space, $E_1, ..., E_n$ are disjoint subsets of X, and $c_1, ..., c_n$ are distinct nonzero real numbers. Prove that $c_1\chi_{E_1} + ... + c_n\chi_{E_n}$ is an $\mathcal{S}$-measurable function if and only if $E_1, ..., E_n \in \mathcal{S}$.

Solution

($\implies$) Suppose that $c_1\chi_{E_1} + ... + c_n\chi_{E_n}$ is $\mathcal{S}$-measurable, then for all borel sets B,

$$(c_1\chi_{E_1} + ... + c_n\chi_{E_n})^{-1}(B) \in \mathcal{S}$$

Hence, for all $k \in [\![1, n]\!]$, since $\{c_k\}$ is a Borel set, then

$$(c_1\chi_{E_1} + ... + c_n\chi_{E_n})^{-1}(\{c_k\}) \in \mathcal{S}$$

But notice that

$$(c_1\chi_{E_1} + ... + c_n\chi_{E_n})^{-1}(\{c_k\}) = E_k$$

since the E_i's are disjoint and the c_i's are distinct. It follows that $E_1, ..., E_n \in \mathcal{S}$.

($\impliedby$) Suppose that $E_1, ..., E_n \in \mathcal{S}$, then for all $k \in [\![1, n]\!]$, the function χ_{E_k} is $\mathcal{S}$-measurable. Moreover, for all $k \in [\![1, n]\!]$, since $g_k : x \mapsto c_k x$ is continuous, then it is Borel measurable. It follows that $c_k\chi_{E_k} = g_k \circ \chi_{E_k}$ is $\mathcal{S}$-measurable. Since measurable functions are closed under addition, then $c_1\chi_{E_1} + ... + c_n\chi_{E_n}$ is $\mathcal{S}$-measurable.

Exercise 14

(a) Suppose $f_1, f_2, ...$ is a sequence of functions from a set X to $\mathbf{R}$. Explain why

$$\begin{aligned}&\{x \in X : \text{ the sequence } f_1(x), f_2(x), ... \text{ has a limit in } \mathbf{R}\}\\ &= \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n})).\end{aligned}$$

(b) Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, ...$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Prove that

$$\{x \in X : \text{ the sequence } f_1(x), f_2(x), ... \text{ has a limit in } \mathbf{R}\}$$

is an $\mathcal{S}$-measurable subset of X.

Solution

(a) First, to make it easier to read, denote by E the set

$$\{x \in X : \text{ the sequence } f_1(x), f_2(x), ... \text{ has a limit in } \mathbf{R}\}$$

Let $x \in E$ be arbitrary, then by definition, the sequence $\{f_n(x)\}_n$ is a convergent sequence in $\mathbf{R}$. It follows that $\{f_n(x)\}_n$ is a Cauchy sequence. Let $n \in \mathbf{Z}^+$, since $\frac{1}{n} > 0$, then there is a $j \in \mathbf{Z}^+$ such that

$$|f_a(x) - f_b(x)| < \frac{1}{n}$$

for all $a, b \geq j$. In particular, for all $k \geq j$, we have

$$|f_j(x) - f_k(x)| < \frac{1}{n}$$

Notice that this can be written as

$$\frac{1}{n} < (f_j - f_k)(x) < \frac{1}{n}$$

which again can be written as

$$x \in (f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

Since it holds for all $j \geq k$, then

$$x \in \bigcap_{j=k}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

Since there is a $k \in \mathbf{Z}^+$ such that it holds, then

$$x \in \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

Since it holds for all $n \in \mathbf{Z}^+$, then

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

It follows that $E \subset \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n}))$.
Now, for the reverse inclusion, suppose that

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

Let's prove that $\{f_n(x)\}_n$ is a Cauchy sequence. Let $\epsilon > 0$, then by the Archimedean Property, there is an integer $N \in \mathbf{Z}^+$ such that $\frac{1}{N} < \epsilon$. By our assumption on x, it follows that

$$x \in \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{2N}, \tfrac{1}{2N}))$$

But it means that there is a $j \in \mathbf{Z}^+$ such that

$$x \in \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\tfrac{1}{2N}, \tfrac{1}{2N}))$$

Let $r, s \geq j$, then the previous statement about x, it implies that

$$x \in (f_j - f_r)^{-1}((-\tfrac{1}{2N}, \tfrac{1}{2N}))$$

and

$$x \in (f_j - f_s)^{-1}((-\tfrac{1}{2N}, \tfrac{1}{2N}))$$

which are both equivalent to

$$|f_j(x) - f_r(x)| < \frac{1}{2N}$$

$$|f_j(x) - f_s(x)| < \frac{1}{2N}$$

By the triangle inequality, this gives us

$$|f_r(x) - f_s(x)| < \frac{1}{N} < \epsilon$$

Thus, $\{f_n(x)\}_n$ is a Cauchy sequence and by completeness of $\mathbf{R}$, we get that the sequence $\{f_n(x)\}_n$ converges in $\mathbf{R}$. Therefore,

$$E = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n}))$$

(b) Since for all $n, j \in \mathbf{Z}^+$ and $k \geq j$, the function $f_j - f_k$ is $\mathcal{S}$-measurable, then

$$(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n})) \in \mathcal{S}$$

since $(-\frac{1}{n}, \frac{1}{n})$ is a Borel set. Since it holds for all $k \geq j$, then

$$\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n})) \in \mathcal{S}$$

Similarly, since it holds for all $j \in \mathbf{Z}^+$, then

$$\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n})) \in \mathcal{S}$$

Again, since it holds for all $n \in \mathbf{Z}^+$, then

$$\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}(f_j - f_k)^{-1}((-\tfrac{1}{n}, \tfrac{1}{n})) \in \mathcal{S}$$

which proves our claim.

Exercise 15

Suppose X is a set and $E_1, E_2, ...$ is a disjoint sequence of subsets of X such that $\cup_{i=1}^{\infty} E_i = X$. Let $\mathcal{S} = \{\cup_{k \in K} E_k : K \subset \mathbf{Z}^+\}$.

(a) Show that $\mathcal{S}$ is a σ-algebra on X.

(b) Prove that a function from X to $\mathbf{R}$ is $\mathcal{S}$-measurable if and only if the function is constant on E_k for every $k \in \mathbf{Z}^+$.

Solution

(a) Since this statement is a generalization of Exercise 1, then the proof will be very similar. As most of the proofs showing that a collection is a σ-algebra, let's split this one into three parts:

- ($\varnothing \in \mathcal{S}$) Since $\varnothing \subset \mathbf{Z}$, then $\cup_{k\in\varnothing} E_k \in \mathcal{S}$. However, notice that $\cup_{k\in\varnothing} E_k = \varnothing$. It follows that $\varnothing \in \mathcal{S}$.
- (closed under complements) Let $A \in \mathcal{S}$, then there exists a $K_0 \subset \mathbf{Z}$ such that $A = \cup_{k\in K_0} E_k$ Consider $K_1 = \mathbf{Z} \setminus K_0$ and its associated element $B = \cup_{k\in K_1} E_k$ in $\mathcal{S}$. Since $A \cap B = \varnothing$ and $A \cup B = X$, then $B = X \setminus A$. Hence, $A^c \in \mathcal{S}$ which proves that $\mathcal{S}$ is closed under complements.
- (closed under countable union) Let $\{A_i\}_i$ be a countable collection of elements in $\mathcal{S}$, then for all $i \in \mathbf{Z}^+$, there is a subset K_i of $\mathbf{Z}$ such that $A_i = \cup_{k\in K_i} E_k$. Consider $K = \cup_{i=1}^{\infty} K_i \subset \mathbf{Z}$ and $A = \cup_{k\in K} E_k \in \mathcal{S}$. By consruction, $A = \cup_{i=1}^{\infty} A_i \in \mathcal{S}$. Therefore, $\mathcal{S}$ is closed under countable union.

Therefore, $\mathcal{S}$ is a σ-algebra on X.

(b) Let $f : X \to \mathbf{R}$, suppose first that f is constant on E_k for every $k \in \mathbf{Z}^+$. Call c_k the constant value of f on E_k. To show that f is $\mathcal{S}$-measurable, let $B \subset \mathbf{R}$ be a Borel set. Notice that

$$f^{-1}(B) = f^{-1}(B \cap \{c_k\}_k) = \bigcup_{k;c_k\in B} E_k \in \mathcal{S}$$

It follows that f is $\mathcal{S}$-measurable.
Suppose now that f is not constant on all E_k's, then, there is a $k_0 \in \mathbf{Z}^+$ and distinct real noumbers a and b such that both $a, b \in f(E_{k_0})$. What we get is that $f^{-1}(\{a\}) \subsetneq E_{k_0}$. Hence, since that E_k's are disjoint, then we cannot write $f^{-1}(\{a\})$ as a union of E_k's. Thus, $f^{-1}(\{a\}) \notin \mathcal{S}$ even if $\{a\}$ is a Borel set. It follows that f is not $\mathcal{S}$-measurable.

Exercise 16
Suppose $\mathcal{S}$ is a σ-algebra on a set X and $A \subset X$. Let

$$\mathcal{S}_A = \{E \in \mathcal{S} : A \subset E \text{ or } A \cap E = \varnothing\}$$

(a) Prove that $\mathcal{S}_A$ is a σ-algebra on X.

(b) Suppose $f : X \to \mathbf{R}$ is a function. Prove that f is $\mathcal{S}$-measurable if and only if f is measurable with respect to $\mathcal{S}$ and f is constant on A.

Solution

(a) First, since $\varnothing \in \mathcal{S}$ and $\varnothing \subset A$, then $\varnothing \in \mathcal{S}_A$. Now, take an arbitrary set E in $\mathcal{S}_A$, then we either have $A \subset E$ or $A \cap E = \varnothing$:

- If $A \subset E$, then $X \setminus E \subset X \setminus A$. It follows that $(X \setminus E) \cap A = \varnothing$. But since $X \setminus E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}_A$.
- If $A \cap E = \varnothing$, then for all $x \in A$, having $x \in E$ would lead to $x \in A \cap E \neq \varnothing$ which is a contradiction. Hence, $x \in X \setminus E$. Hence, $A \subset X \setminus E$. But since $X \setminus E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}_A$.

In all cases, we get that $E^c \in \mathcal{S}_A$. Now, let $\{E_i\}_i$ be a countable collection of elements in $\mathcal{S}_A$, let's show that $\cup_{i=1}^{\infty} E_i \in \mathcal{S}_A$ by cases. If each E_i satisfy $A \cap E_i = \varnothing$, then we must have $A \cap \cup_{i=1}^{\infty} E_i = \varnothing$. Since $\cup_{i=1}^{\infty} E_i \in \mathcal{S}$, then we get that $\cup_{i=1}^{\infty} E_i \in \mathcal{S}_A$. However, if one of the E_i's satisfies $A \subset E_i$, then we get

$$A \subset E_i \subset \bigcup_{i=1}^{\infty} E_i$$

Thus, in both cases, it follows that $\cup_{i=1}^{\infty} E_i \in \mathcal{S}_A$. Therefore, $\mathcal{S}_A$ is a σ-algebra.

(b) ($\Longrightarrow$) Suppose that f is $\mathcal{S}_A$-measurable and let B be a Borel subset of $\mathbf{R}$, then

$$f^{-1}(B) \in \mathcal{S}_A \subset \mathcal{S}$$

which proves that f is $\mathcal{S}$-measurable. Suppose that f is nonconstant on A, then there exist $x_0, x_1 \in A$ such that $f(x_0) \neq f(x_1)$. Consider $f^{-1}(\{f(x_0)\})$, then by our assumption, it is contained in $\mathcal{S}_A$ (since $\{f(x_0)\} isaBorelset)$. Hence, it means that one of $A \subset f^{-1}(\{f(x_0)\})$ or $A \cap f^{-1}(\{f(x_0)\}) = \varnothing$ holds. But $A \subset f^{-1}(\{f(x_0)\})$ cannot hold since $x_1 \in A$ and $x_1 \notin f^{-1}(\{f(x_0)\})$. Similarly, for the same reason, $A \cap f^{-1}(\{f(x_0)\}) = \varnothing$ cannot hold as well. From this contradiction, we get that f must be constant on A.

($\Longleftarrow$) Suppose that f is $\mathcal{S}$-measurable and $f \equiv c$ on A for some $c \in \mathbf{R}$. Let $B \subset \mathbf{R}$ be a Borel set, then, by our assumption, $f^{-1}(B) \in \mathcal{S}$. Now, notice that we either have $c \in B$ or $c \notin B$. If $c \in B$, then it follows that $A \subset f^{-1}(B)$. In that case, $f^{-1}(B) \in \mathcal{S}_A$. If $c \notin B$, then no elements of A are in $f^{-1}(B)$. Hence, $A \cap f^{-1}(B) = \varnothing$. Again, in that case, $f^{-1}(B) \in \mathcal{S}_A$. Therefore, f is $\mathcal{S}_A$-measurable.

Exercise 17

Suppose X is a Borel subset of $\mathbf{R}$ and $f : X \to \mathbf{R}$ is a function such that $\{x \in X : f \text{ not continuous at } x\}$ is a countable set. Prove that f is a Borel measurable function.

Solution

Fix $a \in \mathbf{R}$ and let's show that $f^{-1}((a, \infty))$ is a Borel set. Let $x \in X$, then $x \in f^{-1}((a, \infty))$ if and only if $f(x) > a$. If f is continuous at x, then there is a $\delta_x > 0$ such that

$$f((x - \delta_x, x + \delta_x) \cap X) \subset (a, \infty)$$

which can also be written as

$$(x - \delta_x, x + \delta_x) \cap X \subset f^{-1}((a, \infty))$$

It follows that

$$f^{-1}((a, \infty)) = \left[X \cap \bigcup_{\substack{x \,:\, f \text{ is conti-} \\ \text{nuous at } x}} (x - \delta_x, x + \delta_x) \right] \cup \{x \in X : f \text{ not continuous at } x\}$$

But notice that $\cup_x (x - \delta_x, x + \delta_x)$ is an open set so it is also a Borel set. Since X is Borel as well, then $X \cap \cup_x (x - \delta_x, x + \delta_x)$ is a Borel set. It follows that $f^{-1}((a, \infty))$

is a Borel since any countable set is a Borel set. Therefore, f is Borel measurable.

Exercise 18
Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable at every element of $\mathbf{R}$. Prove that f' is a Borel measurable function from $\mathbf{R}$ to $\mathbf{R}$.

Solution
Consider the sequence $f_1, f_2, ...$ of functions defined as follows:

$$f_n(x) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}}$$

for all $x \in \mathbf{R}$ and $n \in \mathbf{Z}^+$. Since f is differentiable, then f is continuous. It follows that each f_n is continuous as well and consequently, Borel measurable. Now, notice that for all $x \in \mathbf{R}$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} = f'(x)$$

Thus, f' must be Borel measurable as well.

Exercise 19
Suppose X is a nonempty set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Give a characterization of the $\mathcal{S}$-measurable real-valued functions on X.

Solution
In this proof, I will show that that the $\mathcal{S}$-measurable functions are precisely the functions that are constant except on a countable set. Notice that the case where X is finite or countable is easy to prove since in that case, $\mathcal{S} = 2^X$ and hence, every function is $\mathcal{S}$-measurable. Moreover, every function is constant except on a countable set. Hence, the characterization is proved in that case.
Suppose that X is uncountable. Let $f : X \to \mathbf{R}$ be a function such that f is constant except on a countable set. It follows that there is a $x_0 \in X$ such that $f(x) = f(x_0)$ for all $x \in X$ except for countably many x. Let $a \in \mathbf{R}$ and consider the set $f^{-1}((a,\infty))$. If $a \geq f(x_0)$, then for all $x \in X$,

$$\begin{aligned} x \in f^{-1}((a,\infty)) &\implies f(x) > a \\ &\implies f(x) > f(x_0) \\ &\implies f(x) \neq f(x_0) \\ &\implies x \in \{x \in X : f(x) \neq f(x_0)\} \end{aligned}$$

Hence,

$$f^{-1}((a,\infty)) \subset \{x \in X : f(x) \neq f(x_0)\}$$

which is countable. It follows that $f^{-1}((a,\infty)) \in \mathcal{S}$. If $a < f(x_0)$, then similarly, for all $x \in X$,

$$\begin{aligned} x \in f^{-1}((a,\infty))^c &\implies f(x) \leq a \\ &\implies f(x) < f(x_0) \\ &\implies x \in \{x \in X : f(x) \neq f(x_0)\} \end{aligned}$$

which implies

$$f^{-1}((a,\infty))^c \subset \{x \in X : f(x) \neq f(x_0)\}$$

Hence, $f^{-1}((a,\infty))^c$ is countable so $f^{-1}((a,\infty)) \in \mathcal{S}$. Therefore, since it holds for all cases and for all $a \in \mathbf{R}$, it follows that f is $\mathcal{S}$-measurable.
For the converse, consider an $\mathcal{S}$-measurable function f and let's show that it is constant on a countable set. Define the sets

$$A = \{a \in \mathbf{R} : f^{-1}((a,\infty)) \text{ is countable}\}$$

$$B = \{b \in \mathbf{R} : f^{-1}((-\infty,b]) \text{ is countable}\}$$

By the assumption that f is measurable, we must have $A \cup B = \mathbf{R}$. Moreover, if $x \in A \cap B$, then both $f^{-1}((x,\infty))$ and $f^{-1}((-\infty,x])$ are countable. However,

$$X = f^{-1}((x,\infty)) \cup f^{-1}((-\infty,x])$$

which would imply that X is countable. A contradiction since we assumed that X is uncountable. Hence, $A \cap B = \varnothing$. Now, notice that both A and B are nonempty. By contradiction, if $B = \varnothing$, then $A = \mathbf{R}$. Hence, for all $n \in \mathbf{Z}^+$, we get that $f^{-1}((-n,\infty))$ is countable. However, since

$$X = \bigcup_{n=1}^{\infty} f^{-1}((-n,\infty))$$

then it would imply that X is countable. A contradiction that shows that B is nonempty. The proof for A is the same. The last two important properties of A and B are the following, if $a \in A$ and a' is a really number greater than a, then $a' \in A$. Similarly, if $b \in B$ and b' is a real number smaller than b, then $b' \in B$. Let's prove it for A only (the proof is the same for B). Let $a \in A$ and $a' \geq a$, then

$$(a',\infty) \subset (a,\infty)$$

which implies

$$f^{-1}((a',\infty)) \subset f^{-1}((a,\infty))$$

But $f^{-1}((a,\infty))$ is countable so $f^{-1}((a',\infty))$ is countable as well. It follows that $a' \in A$.
All of these properties of A and B show that A is nonempty and bounded below by any element of B and B is nonempty and bounded above by any element of A. Moreover, since $A \cup B = \mathbf{R}$, then $\sup B = \inf A$. Define $c = \sup B$, then we either have

$$B = (-\infty,c) \qquad A = [c,\infty)$$

or

$$B = (-\infty,c] \qquad A = (c,\infty)$$

In both cases, we have

$$(-\infty,c) \subset B \quad \text{and} \quad (c,\infty) \subset A$$

Thus, for all $n \in \mathbf{Z}^+$, we have $c - \frac{1}{n} \in B$ and $c + \frac{1}{n} \in A$. It follows that

$$\begin{aligned} f^{-1}(\mathbf{R} \setminus c) &= f^{-1}((-\infty, c) \cup (c, \infty)) \\ &= f^{-1}((-\infty, c)) \cup f^{-1}((c, \infty)) \\ &= f^{-1}\left(\bigcup_{n=1}^{\infty}\left((-\infty, c - \frac{1}{n}\right]\right) \cup f^{-1}\left(\bigcup_{n=1}^{\infty}\left(c + \frac{1}{n}, \infty\right)\right) \\ &= \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty, c - \frac{1}{n}\right]\right) \cup \bigcup_{n=1}^{\infty} f^{-1}\left(\left(c + \frac{1}{n}, \infty\right)\right) \end{aligned}$$

which shows that $f^{-1}(\mathbf{R} \setminus c)$ is countable since it the union of two countable unions of countable sets. Thus, it means that $f(x) \neq c$ only for countably many $x \in X$. Therefore, f is constant except on a countable set.

Exercise 20
Suppose $(X, \mathcal{S})$ is a measurable space and $f, g : X \to \mathbf{R}$ are $\mathcal{S}$-measurable functions. Prove that if $f(x) > 0$ for all $x \in X$, then f^g (which is the function whose value at $x \in X$ equals $f(x)^{g(x)}$) is an $\mathcal{S}$-measurable function.

Solution
First, recall that both functions

$$\ln : (0, \infty) \to \mathbf{R}$$

$$\exp : \mathbf{R} \to [0, \infty)$$

are continuous on their respective domains (which are Borel sets). Hence, both function of Borel measurable. Since $\operatorname{Im}(f) \subset (0, \infty)$, then $\ln \circ f$ is $\mathcal{S}$-measurable. Since g and $\ln \circ f$ are both $\mathcal{S}$-measurable functions from X to $\mathbf{R}$, then $g \cdot (\ln \circ f)$ is $\mathcal{S}$-measurable. Again, since $\operatorname{Im}(g \cdot (\ln \circ f)) \subset \mathbf{R}$ and exp is $\mathcal{S}$-measurable, then $f^g = \exp \circ (g \cdot (\ln \circ f))$ is $\mathcal{S}$-measurable.

Exercise 21
Prove 2.52.

Solution
Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to [-\infty, \infty]$ is a function such that

$$f^{-1}((a, \infty]) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Let's show that f is $\mathcal{S}$-measurable. To do so, define the collection

$$T = \{A \subset [-\infty, \infty] : f^{-1}(A) \in \mathcal{S}\}$$

By properties of the inverse image of f, the collection T is a σ-algebra. Moreover, by our assumption on f, $\{(a, \infty] : a \in \mathbf{R}\} \subset T$. Let (a, b) be an arbitrary open interval, since $(a, b) = (a, \infty] \setminus (b, \infty]$ and T is closed under set differences, then $(a, b) \in T$. Since it holds for all open intervals, then T contains every open interval. Since every open set can be written as a countable union of open intervals, then T

contains every open set. Since T is a σ-algebra that contains every open set, then it must contain every Borel subsets of $\mathbf{R}$. Now, since

$$\{\infty\} = \bigcap_{n=1}^{\infty}(n, \infty] \in T$$

and

$$\{-\infty\} = \bigcap_{n=1}^{\infty}[-\infty, n] \in T$$

Then every Borel subset of $[-\infty, \infty]$ is contained in T. Therefore, for all Borel subset B of $[-\infty, \infty]$,

$$f^{-1}(B) \in \mathcal{S}$$

It follows that f is $\mathcal{S}$-measurable.

Exercise 22

Suppose $B \subset \mathbf{R}$ and $f : B \to \mathbf{R}$ is an increasing function. Prove that f is continuous at every element of B except for a countable subset of B.

Solution

First, let's prove that every discontinuity is a jump discontinuity by showing that the left and right limits at a point always exist. Let $x_0 \in B$ and notice that if there is no $x \in B$ such that $x < x_0$, then it follows that $\lim_{x \to x_0^-} f(x) = f(x_0)$. Hence, suppose that there some points $x \in B$ such that $x < x_0$ and consider the set $E_x^- = \{f(x) : x < x_0\}$. By our assumptions, the set E_x^- is nonempty and bounded above by $f(x_0)$. Hence, by completeness of $\mathbf{R}$, we can define $s = \sup E_x^-$. Moreover, by properties of the supremum, it is easy to see that $s = \lim_{x \to x_0^-} f(x)$. Thus, for all $x_0 \in B$, its left limit exists. Similarly, if we define the set $E_x^+ = \{f(x) : x > x_0\}$ and take its infimimum, we get that $\lim_{x \to x_0^+} f(x)$ exists as well. Hence, for all $x \in B$, to make the notation lighter,

$$f(x-) = \lim_{x \to x_0^-} f(x)$$

$$f(x+) = \lim_{x \to x_0^+} f(x)$$

Since f is increasing, then for all $x \in B$, we have that $f(x)$ is an upper bound for E_x^- and a lower bound for E_x^+. It follows that $f(x-) \leq f(x+)$ for all $x \in B$.
Consider now the set D_f of discontinuity points of f. Notice that for all $d \in D_f$, we can define the sets

$$F_d^- = \{x \in B : x < d\} = (-\infty, d) \cap B$$

$$F_d^+ = \{x \in B : x > d\} = (d, \infty) \cap B$$

Notice that there are at most two points such that one of F_d^- and F_d^+ is empty. Let A denote the set of such points, then our observation can be translated by $\text{card}(A) \leq 2$. Now define $B = \{d \in D_f : \sup F_d^- = \inf F_d^+ = d\}$ and $C = D_f \setminus (A \cup B)$. By construction, we have $A \cup B \cup C = D_f$. Since A is finite, our goal now is to prove that both B and C are countable.

For all $d \in B$, we must have $\sup F_d^- < d$ or $\inf F_d^+ > d$. If $\sup F_d^- < d$, then define q_d to be any rational inside the interval (m_d, d) where m_d is the midpoint between $\sup F_d^-$ and d (i.e. $m_d = \frac{\sup F_d^- + d}{2}$). Otherwise, if $\inf F_d^+ > d$, define q_d to be any rational inside the interval (d, m_d') where m_d' is the midpoint between $\inf F_d^+$ and d (i.e. $m_d' = \frac{\inf F_d^+ + d}{2}$). Now, let's prove that the function $g : B \to \mathbf{Q}$ defined by $d \mapsto q_d$ is injective. Let $d_1, d_2 \in B$ such that $d_1 < d_2$, let's show that $q_{d_1} \neq q_{d_2}$ by cases.

- If $\sup F_{d_1}^- < d_1$ and $\sup F_{d_2}^- < d_2$, then q_{d_1} and q_{d_2} are rationals between (m_{d_1}, d_1) and (m_{d_2}, d_2) respectively. It follows that $q_{d_1} < d_1$. Moreover, $\sup F_{d_2}^- < m_{d_2} < q_{d_2}$ and $d_1 \in F_{d_2}^-$ so
$$q_{d_1} < d_1 \leq \sup F_{d_2}^- < d_2$$
which shows that $q_{d_1} \neq q_{d_2}$.

- If $\sup F_{d_1}^- < d_1$ and $\inf F_{d_2}^+ > d_2$, then q_{d_1} is a rational inside the interval (m_{d_1}, d_1) and q_{d_2} is a rational inside the interval (d_2, m_{d_1}'). Hence, $q_{d_1} < d_1$ and $d_2 < q_{d_2}$. It follows that
$$q_{d_1} < d_1 < d_2 < q_{d_2}$$
which shows that $q_{d_1} \neq q_{d_2}$.

- If $\inf F_{d_1}^+ > d_1$ and $\sup F_{d_2}^- < d_2$, then q_{d_1} is a rational inside the interval (d_1, m_{d_1}') and q_{d_2} is a rational inside the interval (m_{d_2}, d_2). Hence, $q_{d_1} < m_{d_1}'$ and $m_{d_2} < d_2$. Since $d_1 \in F_{d_2}^-$, then $d_1 \leq \sup F_{d_2}^-$, similarly, $\inf F_{d_1}^+ \leq d_2$. Hence:
$$q_{d_1} < m_{d_1}' = \frac{\inf F_{d_1}^+ + d_1}{2} \leq \frac{d_2 + d_1}{2} \leq \frac{\sup F_{d_2}^- + d_2}{2} = m_{d_2} < q_{d_2}$$
which shows that $q_{d_1} \neq q_{d_2}$.

- If $\inf F_{d_1}^+ > d_1$ and $\inf F_{d_2}^+ > d_2$, then q_{d_1} is a rational inside the interval (d_1, m_{d_1}') and q_{d_2} is a rational inside the interval (d_2, m_{d_2}'). It follows that $q_{d_1} < m_{d_1}'$ and $d_2 < q_{d_2}$. Since $d_2 \in F_{d_1}^+$, then
$$q_{d_2} < m_{d_1}' < \inf F_{d_1}^+ \leq d_2 < q_{d_2}$$
which shows that $q_{d_1} \neq q_{d_2}$.

Therefore, g is injective so $\mathrm{card} B \leq \mathrm{card} \mathbf{Q}$ which implies that B is countable.
Let's now prove that C is countable. Let $d \in C$. Since f is continuous at d if and only if $f(d-) = f(d+)$, then f is discontinuous at d if and only if $f(d-) < f(d+)$. It follows that $f(d-) < f(d+)$. Hence, define q_d to be any rational in the nonempty interval $(f(d-), f(d+))$. Let's prove that the function $h : C \to \mathbf{Q}$ defined by $d \mapsto q_d$ is injective. Let $d_1, d_2 \in C$ such that $d_1 < d_2$. Hence, by definition of C, we must have $d_1 = \inf F_{d_1}^+$. It follows that there must be a $x \in B$ such that $d_1 < x < d_2$. Hence, by monotonicity of f, we have $f(d_1) \leq f(x) \leq f(d_2)$. By definition of $E_{d_1}^+$ and $E_{d_2}^-$, we have
$$f(d_1+) = \inf E_{d_1}^+ \leq f(x) \leq \sup E_{d_2}^- = f(d_2-)$$

Therefore,

$$q_{d_1} < f(d_1+) \leq f(d_2-) < q_{d_2}$$

which shows that $q_{d_1} \neq q_{d_2}$. Hence, h is injective so card$C \leq$ card$\mathbf{Q}$. Thus, C is countable. Since $D_f = A \cup B \cup C$ and A, B, and C are countable, then D_f must be countable as well. Therefore, f is continuous except on the countable set D_f.

Exercise 23

Suppose $f : \mathbf{R} \to \mathbf{R}$ is a strictly increasing function. Prove that the inverse function $f^{-1} : f(\mathbf{R}) \to \mathbf{R}$ is a continuous function.

[*Note that this exercise does not have as a hypothesis that f is continuous.*]

Solution

Let $x_0 \in f(\mathbf{R})$ and let's prove that

$$\lim_{x \to x_0} f^{-1}(x) = f^{-1}(x_0)$$

Let $\epsilon > 0$ and define $x_1 = f^{-1}(x_0) - \epsilon$ and $x_2 = f^{-1}(x_0) + \epsilon$ which are both in the domain of f^{-1}. Notice that $x_1 < x_0 < x_2$. Let $\delta = \min(x_0 - x_1, x_2 - x_0)$ and let $x \in f(\mathbf{R})$ such that $0 < |x - x_0| < \delta$, hence:

$$\begin{aligned} x_1 < x < x_2 &\implies f(f^{-1}(x_0) - \epsilon) < x < f(f^{-1}(x_0) + \epsilon) \\ &\implies f^{-1}(x_0) - \epsilon < f^{-1}(x) < f^{-1}(x_0) + \epsilon \\ &\implies |f^{-1}(x) - f^{-1}(x_0)|\epsilon \end{aligned}$$

So $\lim_{x \to x_0} f^{-1}(x) = f^{-1}(x_0)$ which implies that f^{-1} is continuous at x_0. Therefore, f^{-1} is continuous on $f(\mathbf{R})$.

Exercise 24

Suppose $B \subset \mathbf{R}$ is a Borel set and $f : \mathbf{R} \to \mathbf{R}$ is a strictly increasing function. Prove that $f(B)$ is a Borel set.

Solution

First, we show that $f(\mathbf{R})$ is a Borel set. Since f has countably many discontinuities, then we can enumerate them as $d_1, d_2, \ldots$. Let $i \in \mathbf{Z}^+$, define

$$D_i = [f(d_i-), f(d_i+)] \setminus \{f(d_i)\}$$

which represents the set of points not attained by f because of d_i. In this proof, I denote by $f(x-)$ the limit of f as $z \to x$ from the left and $f(x+)$ the limit of f as $z \to x$ from the right. Notice that each D_i is a Borel set so $\cup_{i=1}^{\infty} D_i$ is a Borel set as well. Define now

$$I_m = (-\infty, \inf_{\mathbf{R}} f]$$

$$I_M = [\sup_{\mathbf{R}} f, \infty)$$

and consider that $I_m = \varnothing$ if f is unbounded below and $I_M = \varnothing$ if f is unbounded above. Again, in all cases, both I_m and I_M are Borel sets. It follows that the set

$$I = \mathbf{R} \setminus \left[I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i \right]$$

is a Borel set. The main goal of this proof is to show that $f(\mathbf{R}) = I$.

- ($\implies$) Let $y \in I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i$. Let's prove by cases that $y \in \mathbf{R} \setminus f(\mathbf{R})$.
  - If $y \in I_m$, then $y \leq \inf_{\mathbf{R}} f$. If $y = f(x_0)$ for some $x_0 \in \mathbf{R}$, then strict monotonicity of f, we have:

    $$\inf_{\mathbf{R}} f \leq f(x_0 - 1) < y \leq \inf_{\mathbf{R}} f$$

    A contradiction that shows that $y \in \mathbf{R} \setminus f(\mathbf{R})$.
  - If $y \in I_M$, the proof is the same as in the previous case.
  - If $\cup_{i=1}^{\infty} D_i$, then there is a $i \in \mathbf{Z}^+$ such that $y \in D_i = [f(d_i-), f(d_i+)] \setminus \{f(d_i)\}$. Suppose by contradiction that $y = f(x_0)$ for some $x_0 \in \mathbf{R}$. Since $y \neq f(d_i)$, then we either have $x_0 < d_i$ or $d_i < x_0$. Suppose without loss of generality that $x_0 < d_i$, then $y \in \{f(x) : x < d_i\}$ which implies that

    $$y \leq \sup\{f(x) : x < d_i\}$$

    But $y \in [f(d_i-), f(d_i+)] \setminus \{f(d_i)\}$ so $y \geq f(d_i-) = \sup\{f(x) : x < d_i\}$. It follows that $y = \sup\{f(x) : x < d_i\}$. However, since $x_0 < d_i$, then there exists a $x_1 \in (x_0, d_i)$. Since $x_0 < x_1$ and $x_1 < d_i$, then

    $$\sup\{f(x) : x < d_i\} = y = f(x_0) < f(x_1) \leq \sup\{f(x) : x < d_i\}$$

    A contradiction that shows that $y \in \mathbf{R} \setminus f(\mathbf{R})$.

  Therefore, $I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i \subset \mathbf{R} \setminus f(\mathbf{R})$ which is equivalent to $f(\mathbf{R}) \subset I$.

- ($\impliedby$) Let's now prove the reverse inclusion. Let $y \in \mathbf{R} \setminus f(\mathbf{R})$, if $y < f(x)$ for all $x \in \mathbf{R}$, then $y \in I_m \subset I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i$. If $y > f(x)$ for all $x \in \mathbf{R}$, then $y \in I_M \subset I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i$. Hence, we can suppose that there exist real numbers x_1 and x_2 such that $f(x_1) < y < f(x_2)$. Consider now the sets

  $$A_1 = \{x \in \mathbf{R} : f(x) < y\}$$

  $$A_2 = \{x \in \mathbf{R} : f(x) > y\}$$

  Since $x_1 \in A_1$ and $x_2 \in A_2$, then the sets are nonempty. Moreover, since $y \neq f(x)$ for all $x \in \mathbf{R}$, then $A_1 \cup A_2 = \mathbf{R}$. Since f is strictly increasing, then any element of A_2 is an upper bound for A_2, it follows that there is a $c \in \mathbf{R}$ such that either

  $$A_1 = (-\infty, c] \qquad A_2 = (c, \infty)$$

  or

  $$A_1 = (-\infty, c) \qquad A_2 = [c, \infty)$$

  Suppose without loss of generality that $f(c) < y$, then

  $$A_1 = (-\infty, c] \qquad A_2 = (c, \infty)$$

  Let's show that f is discontinuous at c. Since y is a lower bound for the set $f(A_2)$, then $y \leq \inf f(A_2)$. However, since f is increasing, notice that

  $$f(c+) = \inf\{f(x) : x > c\} = \inf f((c, \infty)) = \inf f(A_2)$$

Thus,

$$f(c) < y \leq \inf f(A_2) = f(c+)$$

which shows that f is discontinuous at c (otherwise, we would have $f(c) = f(c+)$). Thus, there is a $i \in \mathbf{Z}^+$ such that $c = d_i$. Hence,

$$y \in (f(d_i), f(d_i+)] \subset D_i \subset I_M \subset I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i$$

Therefore, $\mathbf{R} \backslash f(\mathbf{R}) \subset I_M \subset I_m \cup I_M \cup \bigcup_{i=1}^{\infty} D_i$ which is equivalent to $I \subset f(\mathbf{R})$.

Now that we showed that $f(\mathbf{R}) = I$, it follows that $f(\mathbf{R})$ is a Borel set. Using the previous exercice, we get that $f^{-1} : f(\mathbf{R}) \to \mathbf{R}$ is a continuous function, hence, Borel measurable. It follows that $(f^{-1})^{-1}(B)$ is a Borel set. However, since

$$(f^{-1})^{-1}(B) = \{y \in f(\mathbf{R}) : f^{-1}(y) \in B\} = \{y \in f(\mathbf{R}) : y \in f(B)\} = f(B)$$

then $f(B)$ is a Borel set.

Exercise 25
Suppose $B \subset \mathbf{R}$ and $f : B \to \mathbf{R}$ is an increasing function. Prove that there exists a sequence $f_1, f_2, ...$ of strictly increasing functions from B to $\mathbf{R}$ such that

$$f(x) = \lim_{k \to \infty} f_k(x)$$

for every $x \in B$.

Solution
Consider the sequence of functions $f_1, f_2, ...$ defined by

$$f_n(x) = f(x) + \frac{1}{n}x$$

for all $x \in B$ and $n \in \mathbf{Z}^+$. Since f is increasing and $\frac{1}{n}x$ is strictly increasing for all $n \in \mathbf{Z}^+$, then every function in the sequence is strictly increasing. Moreover, for all $x \in B$:

$$\lim_{k \to \infty} f_k(x) = f(x) + x \lim_{k \to \infty} \frac{1}{k} = f(x)$$

which proves our claim.

Exercise 26
Suppose $B \subset \mathbf{R}$ and $f : B \to \mathbf{R}$ is a bounded increasing function. Prove that there exists an increasing function $g : \mathbf{R} \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in B$.

Solution
First, assume that B is non empty. For all $x \in \mathbf{R}$, define the set

$$E_x = \{f(y) : y \leq x,\ y \in B\}$$

Consider the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} \sup E_x & \text{if } E_x \neq \varnothing \\ \inf_{\mathbf{R}} f & \text{if } E_x = \varnothing \end{cases}$$

Notice that g is well defined since f is bounded. Let's show that $g|_B = f$. Take $x \in B$ and notice that E_x is non empty since it contains $f(x)$. Moreover, since f is increasing, then $f(x)$ is an upperbound for E_x. It follows that $g(x) = \sup E_x = f(x)$. Let's now show that g is increasing. If we take $a, b \in \mathbf{R}$ such that $a < b$, then we can proceed by cases.

- If both a and b are in B, then $g(a) \leq g(b)$ follows from the fact that f is increasing :
$$g(a) = f(a) \leq f(b) = g(b)$$
- If $a \in B$ but $b \notin B$, then $f(a) \in E_b \neq \varnothing$ which implies
$$g(a) = f(a) \leq \sup E_b = g(b)$$
- If $a \notin B$ and $b \in B$, then either E_a is empty and we get
$$g(a) = \inf_{\mathbf{R}} f \leq f(b) = g(b)$$
either E_a is nonempty and we get that $f(b)$ is an upperbound for E_a. Hence:
$$g(a) = \sup E_a \leq f(b) = g(b)$$
- If both a and b are not in B, then we, again, have different possible cases.
  - If $E_b = \varnothing$, then we must have $E_a = \varnothing$ as well since $E_a \subset E_b$. Hence,
  $$g(a) = g(b) = \inf_{\mathbf{R}} f$$
  - If $E_b \neq \varnothing$ and $E_a = \varnothing$, then there exists a $x \in B$ such that $f(x) \in E_b$. It follows that
  $$g(a) = \inf_{\mathbf{R}} f \leq f(x) \leq \sup E_b = g(b)$$
  - If $E_b \neq \varnothing$ and $E_a \neq \varnothing$, then we get $E_a \subset E_b$ which implies that $\sup E_a \leq \sup E_b$. Thus:
  $$g(a) = \sup E_a \leq \sup E_b = g(b)$$

After this tedious proof by cases, we now have shown that g is an increasing function that extends f on $\mathbf{R}$.

Exercise 27
Prove or give a counterexample: If $(X, \mathcal{S})$ is a measurable space and

$$f : X \to [-\infty, \infty]$$

is a function such that $f^{-1}((a, \infty)) \in \mathcal{S}$ for every $a \in \mathbf{R}$, then f is an $\mathcal{S}$-measurable function.

Solution
Let $X = \mathbf{R}$ and $\mathcal{S}$ be the σ-algebra of subsets of $\mathbf{R}$ that are either countable or have a countable complement. Define $f : X \to [-\infty, \infty]$ as

$$f(x) = \begin{cases} +\infty & \text{if } x \geq 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

Notice that for all $a \in \mathbf{R}$, we have $f^{-1}((a,\infty)) = \varnothing \in \mathcal{S}$. However, we cannot conclude that f is $\mathcal{S}$-measurable because $\{\infty\}$ is a Borel subset of $[-\infty,\infty]$ but $f^{-1}(\{\infty\}) = [0,\infty) \notin \mathcal{S}$. Therefore, f is not $\mathcal{S}$-measurable.

Exercise 28
Suppose $f : B \to \mathbf{R}$ is a Borel measurable function. Define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B \\ 0 & \text{if } x \in \mathbf{R} \setminus B \end{cases}$$

Prove that g is a Borel measurable function.

Solution
Fix $a \in \mathbf{R}$ and let's show that $g^{-1}((a,\infty))$ is a Borel set. First, notice that B is a Borel set since $B = f^{-1}(\mathbf{R})$ and f is Borel measurable. If $a \geq 0$, then we can show that $g^{-1}((a,\infty)) = f^{-1}((a,\infty))$: if $g(x) > a \geq 0$, then it must be that $x \in B$ which implies that $f(x) = g(x) > a$; and if $f(x) > a$, then it directly follows that $g(x) = f(x) > a$. Thus, $g^{-1}((a,\infty))$ is a Borel set. Now, if $a < 0$, then for the same reasons as above, $g^{-1}((a,\infty)) = f^{-1}((a,\infty)) \cup (\mathbf{R} \setminus B)$ which is again a Borel set. Therefore, g is Borel measurable.

Exercise 29
Give an example of a measurable space $(X, \mathcal{S})$ and a family $\{f_t\}_{t\in\mathbf{R}}$ such that each f_t is an $\mathcal{S}$-measurable function from X to $[0,1]$, but the function $f : X \to [0,1]$ defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\}$$

is not $\mathcal{S}$-measurable.
[*Compare this exercise to 2.53, where the index set is* $\mathbf{Z}^+$ *rather than* $\mathbf{R}$.]

Solution
Let $X = \mathbf{R}$ and $\mathcal{S}$ be the σ-algebra of subsets of $\mathbf{R}$ that are countable or have a countable complement. For all $t < 0$, define f_t to be the constant function zero. If $t \geq 0$, define $f_t = \chi_{\{t\}}$. For each $t \in \mathbf{R}$, the function f_t is $\mathcal{S}$-measurable by construction. However, notice that the function defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\}$$

for all $x \in \mathbf{R}$ is simply the characteristic function of the interval $[0,\infty)$, i.e. $f = \chi_{[0,\infty)}$. But we know that χ_E is $\mathcal{S}$-measurable if and only if $E \in \mathcal{S}$. However, $[0,\infty) \notin \mathcal{S}$ so f is not $\mathcal{S}$-measurable.

Exercise 30
Show that

$$\lim_{j\to\infty}\left(\lim_{k\to\infty}(\cos(j!\pi x))^{2k}\right) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

for every $x \in \mathbf{R}$.
[*This example is due to Henri Lebesgue.*]

Solution
Let $x \in \mathbf{R}$ and consider the case where x is a rational number, then, there exist $a \in \mathbf{Z}$ and $b \in \mathbf{Z}^+$ such that $x = a/b$. For all $j \geq b$, we have $j!x \in \mathbf{Z}$ since $j!$ is a multiple of b which cancels out with the denominator of x. But we know that the cosine of an integer multiple of π is either 1 or -1, hence, for all $k \in \mathbf{Z}^+$, we have $\cos(j!\pi x)^{2k} = 1$. Since it holds for all $k \in \mathbf{Z}^+$, then

$$\lim_{k\to\infty}(\cos(j!\pi x))^{2k} = 1$$

Since it holds for all j large enough, i.e. $j \geq b$, then

$$\lim_{j\to\infty}\left(\lim_{k\to\infty}(\cos(j!\pi x))^{2k}\right) = 1$$

Now, consider the case where x is irrational and let $j \in \mathbf{Z}^+$. Since the cosine of $y\pi$ is 1 or -1 if and only if y is an integer, then $\cos(j!\pi x) \in (-1, 1)$ since $j!x$ is an irrational number. It follows that $(\cos(j!\pi x))^2$ is also in the interval $(-1, 1)$. Hence, if we think of $\{(\cos(j!\pi x))^{2k}\}_k$ as a geometric series with $q = (\cos(j!\pi x))^2$, then

$$\lim_{j\to\infty}\left(\lim_{k\to\infty}(\cos(j!\pi x))^{2k}\right) = 0$$

Since it holds for all $j \in \mathbf{Z}^+$, then

$$\lim_{j\to\infty}\left(\lim_{k\to\infty}(\cos(j!\pi x))^{2k}\right) = 0$$

which proves our claim.

2C Measures and Their Properties

Exercise 1
Explain why there does not exist a measure space $(X, \mathcal{S}, \mu)$ with the property that $\{\mu(E) : E \in \mathcal{S}\} = [0, 1)$.

Solution
By contradiction, let $(X, \mathcal{S}, \mu)$ be such a measure space. Hence, by our assumptions, $\mu(X) < 1$. However, if we let α be any real number in $(\mu(X), 1)$, then $\alpha \in \{\mu(E) : E \in \mathcal{S}\}$ which implies that there is a set $F \in \mathcal{S}$ such that $\mu(F) = \alpha$. But $F \subset X$, so by monotonicity:

$$\mu(X) < \alpha = \mu(F) \leq \mu(X)$$

A contradiction. Therefore, such a measure space cannot exist.

Exercise 2
Suppose μ is a measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$. Prove that there is a sequence $w_1, w_2, ...$ in $[0, \infty]$ such that

$$\mu(E) = \sum_{k \in E} w_k$$

for every set $E \subset \mathbf{Z}^+$.

Solution
First, define the sequence $\{w_k\}_k$ as follows

$$w_k = \mu(\{k\})$$

for all $k \in \mathbf{Z}^+$. Hence, for all $E \in 2^{\mathbf{Z}^+}$, we can write E as the disjoint union of the singletons of its elements. This disjoint union is either finite or countable since $E \subset \mathbf{Z}^+$. Therefore, by finite or countable additivity:

$$\begin{aligned} \mu(E) &= \mu\left(\bigcup_{k \in E} \{k\}\right) \\ &= \sum_{k \in E} \mu(\{k\}) \\ &= \sum_{k \in E} w_k \end{aligned}$$

which proves our claim.

Exercise 3
Give an example of a measure μ on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ such that

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = [0, 1].$$

Solution
Define a measure μ on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ by:

$$\mu(E) = \sum_{k \in E} \frac{1}{2^k}$$

To prove that $\{\mu(E) : E \subset \mathbf{Z}^+\} = [0,1]$, let $c \in [0,1]$ and let's show that there is a $E \subset \mathbf{Z}^+$ such that $\mu(E) = c$. Notice that c has a binary representation of the form:

$$c = \sum_{k=1}^{\infty} a_k \frac{1}{2^k}$$

where the a_k's are either 0 or 1. Hence, if we define $E = \{k : a_k = 1\}$, we get

$$\mu(E) = \sum_{k \in E} \frac{1}{2^k} = \sum_{k=1}^{\infty} a_k \frac{1}{2^k} = c$$

which proves our claim.

Exercise 4

Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1].$$

Solution

Let $X = \mathbf{Z}$, $\mathcal{S} = 2^{\mathbf{Z}}$ and define μ by

$$\mu(\{k\}) = \begin{cases} \frac{1}{2^k} & \text{if } x \geq 1, \\ 3 & \text{if } x \leq 0 \end{cases}$$

Simply use the countable additivity of μ to extend μ on all of $2^{\mathbf{Z}}$ and not just the singletons. Let's show that for all $c \in \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1]$, there is a $E \subset \mathbf{Z}$ such that $\mu(E) = c$. First, consider the case $c = \infty$, then defining $E = \{-k : k \geq 0\}$ gives us

$$\mu(E) = \sum_{k \in E} \mu(\{k\}) = \sum_{k=0}^{\infty} \mu(\{-k\}) = \sum_{k=0}^{\infty} 3 = \infty$$

Now, consider the case $c \in [3k, 3k+1]$, then there exists an integer $k \geq 0$ and a real number $\alpha \in [0,1]$ such that $c = 3k + \alpha$. Since we can write α in binary form as follows:

$$c = \sum_{n=1}^{\infty} a_n \frac{1}{2^n}$$

where the a_n's are either 0 or 1, then we can define the set $E = \{-n : n \in [\![1,k]\!]\} \cup \{n \in \mathbf{Z}^+ : a_n = 1\}$:

$$\begin{aligned}
\mu(E) &= \mu(\{-n : n \in [\![1,k]\!]\} \cup \{n \in \mathbf{Z}^+ : a_n = 1\}) \\
&= \mu(\{-n : n \in [\![1,k]\!]\}) + \mu(\{n \in \mathbf{Z}^+ : a_n = 1\}) \\
&= \sum_{n=1}^{k} \mu(\{-n\}) + \sum_{n=1}^{\infty} a_n \mu(\{n\}) \\
&= \sum_{n=1}^{k} 3 + \sum_{n=1}^{\infty} a_n \frac{1}{2^n} \\
&= 3k + \alpha \\
&= c
\end{aligned}$$

Since we covered all cases, we have,

$$\{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1] \subset \{\mu(E) : E \subset \mathbf{Z}^+\}$$

For the reverse inclusion, consider $E \subset \mathbf{Z}$ and let's show that $\mu(E) \in \{\infty\} \cup \bigcup_{k=0}^{\infty}[3k, 3k+1]$. To do so, notice that by definition of μ, we get

$$\begin{aligned}
\mu(E) &= \mu(\{k \in E : k \leq 0\}) + \mu(\{k \in E : k \geq 1\}) \\
&= \sum_{\substack{k \in E \\ k \leq 0}} \mu(\{k\}) + \sum_{k=1}^{\infty} a_k \mu(\{k\}) \\
&= \sum_{\substack{k \in E \\ k \leq 0}} 3 + \sum_{k=1}^{\infty} a_k \cdot \frac{1}{2^k} \\
&= 3\text{card}\{k \in E : k \leq 0\} + \sum_{k=1}^{\infty} a_k \cdot \frac{1}{2^k}
\end{aligned}$$

where $a_k = 1$ when $k \in E$, otherwise, $a_k = 0$. If the set $\{k \in E : k \leq 0\}$ is infinite, then $\mu(E) = \infty \in \{\infty\} \cup \bigcup_{k=0}^{\infty}[3k, 3k+1]$. If the set is finite, define $n = \text{card}\{k \in E : k \leq 0\}$ and notice that the term $\alpha = \sum_{k=1}^{\infty} a_k \cdot \frac{1}{2^k}$ is simply the binary representation of a number in $[0, 1]$. Hence, $\mu(E) = 3n + \alpha \in [3n, 3n+1] \subset \{\infty\} \cup \bigcup_{k=0}^{\infty}[3k, 3k+1]$. Thus,

$$\{\mu(E) : E \subset \mathbf{Z}^+\} \subset \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1]$$

Therefore,

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = \{\infty\} \cup \bigcup_{k=0}^{\infty} [3k, 3k+1]$$

Exercise 5

Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Prove that if $\mathcal{A}$ is a set of disjoint sets in $\mathcal{S}$ such that $\mu(A) > 0$ for every $A \in \mathcal{A}$, then $\mathcal{A}$ is a countable set.

Solution

Let $n \in \mathbf{Z}^+$ and define the collection

$$\mathcal{A}_n = \{A \in \mathcal{A} : \mu(A) \geq \tfrac{1}{n}\}$$

Suppose that $\mathcal{A}_n$ is infinite, then we can extract a countable sequence $\{A_n\}_n$ of sets in $\mathcal{A}_n$. Since the A_n's are disjoint and in $\mathcal{S}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$. Moreover, by monotonicity, we get

$$\begin{aligned}\mu(X) &\geq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)\\ &= \sum_{n=1}^{\infty} \mu(A_n)\\ &\geq \sum_{n=1}^{\infty} \frac{1}{n}\\ &= \infty\end{aligned}$$

A contradiction. Thus, for all $n \in \mathbf{Z}^+$, the collection $\mathcal{A}_n$ is finite. But since

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

is a countable union of finite sets, then $\mathcal{A}$ is countable.

Exercise 6
Find all $c \in [3, \infty)$ such that there exists a measure space $(X, \mathcal{S}, \mu)$ with

$$\{\mu(E) : E \in \mathcal{S}\} = [0, 1] \cup [3, c].$$

Solution
Let $c \in [3, \infty)$ and suppose that there exists a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = [0, 1] \cup [3, c]$$

Since $\mu(X)$ is both in the set $\{\mu(E) : E \in \mathcal{S}\}$ and an upperbound for the set $\{\mu(E) : E \in \mathcal{S}\}$, then

$$\mu(X) = \sup\{\mu(E) : E \in \mathcal{S}\} = \sup([0, 1] \cup [3, c]) = c$$

Since there is a $E \in \mathcal{S}$ such that $\mu(E) = 1$, then

$$\mu(X \setminus E) = c - 1$$

It follows that c cannot be in the interval $[3, 4)$ (otherwise, we would get a set of measure in the interval [2, 3) which would contradict our assumption on the measure space). Hence, we must have $c \geq 4$. Suppose that $c > 4$, then $3 < c - 1$ which implies that there must be a $\epsilon \in (0, 1)$ such that $3 \leq c - 1 - \epsilon$. Hence, there must be a set $E \in \mathcal{S}$ such that $\mu(E) = c - 1 - \epsilon$. Thus:

$$\begin{aligned}\mu(X \setminus E) &= \mu(X) - \mu(E)\\ &= c - (c - 1 - \epsilon)\\ &= 1 + \epsilon\\ &\in (1, 2)\end{aligned}$$

which is a contradiction. Therefore, the only possible value for c is 4. Let's now prove that there actually is a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,4]$$

Consider the set $X = \mathbf{Z}^+ \cup \{0\}$, $\mathcal{S} = 2^X$ and define the measure μ by

$$\mu(\{n\}) = \begin{cases} 3 & \text{if } n = 0, \\ \frac{1}{2^n} & \text{if } n \geq 1 \end{cases}$$

We can easily extend the definition of μ to any subset of X by countable additivity. Let's now show that this measure space has the right property. Let $c \in [0,1] \cup [3,4]$ and let's show that there is a set $E \subset X$ such that $\mu(E) = c$. If $c \in [0,1]$, then write c in its binary form:

$$c = \sum_{n=1}^{\infty} a_n \frac{1}{2^n}$$

where the a_n's are in the set $\{0,1\}$. Define the set $E = \{n \in \mathbf{Z}^+ : a_n = 1\} \subset X$. Hence, by construction:

$$\mu(E) = \sum_{n \in E} \mu(\{n\}) = \sum_{n=1}^{\infty} a_n \frac{1}{2^n} = c$$

Similarly, if $c \in [3,4]$, consider the binary expansion of $c - 3$, construct the set E as previously and add the element 0 to the set, we would get: $\mu(E) = 3 + \sum_{n=1}^{\infty} b_n \frac{1}{2^n} = 3 + c - 3 = c$ Thus,

$$[0,1] \cup [3,4] \subset \{\mu(E) : E \in \mathcal{S}\}$$

To prove the reverse inclusion, let $E \subset X$. If $0 \in E$, then

$$\mu(E) = 3 + \sum_{n=1}^{\infty} a_n \frac{1}{2^n} \in [3,4] \subset [0,1] \cup [3,4]$$

Similarly, for the same reasons, if $0 \notin E$, then $\mu(E) \in [0,1] \subset [0,1] \cup [3,4]$. Therefore,

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,4]$$

which proves that $c = 4$ is the only possible value.

Exercise 7
Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,\infty)$$

Solution
Consider the measure space defined by $X = \mathbf{Z}$, $\mathcal{S} = 2^{\mathbf{Z}}$ and μ defined by

$$\mu(\{k\}) = \begin{cases} \frac{1}{2^k} & \text{if } k \geq 1, \\ 3 - k & \text{if } k \leq 0 \end{cases}$$

From this, it is easy to extend the definition of μ to any subset of $\mathbf{Z}$. Let's show that

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,\infty)$$

First, let $E \subset \mathbf{Z}$ and split it into $E_1 = \{k \in E : k \geq 1\}$ and $E_2 = \{k \in E : k \leq 0\}$ which are disjoint. Notice that by countable additivity, $\mu(E_2)$ can be written as $\sum_{k=1}^{\infty} a_n \frac{1}{2^n}$ where a_n is 0 when $k \notin E$ and 1 when $k \in E$. But notice that this is simply a base 2 representation of a number in $[0,1]$. Hence, $\mu(E_2) \in [0,1]$. Now, for E_1, notice that $\mu(E_1)$ is a sum of integers greater than or equal to 3 by countable additivity. It follows that $\mu(E_1)$ is either 0 (if it is empty) or an integer greater than or equal to 4. Therefore:

$$\mu(E) = \mu(E_1) + \mu(E_2) \in [0,1] \cup [3,\infty)$$

which shows that

$$\{\mu(E) : E \in \mathcal{S}\} \subset [0,1] \cup [3,\infty)$$

For the reverse inclusion, Let $c \in [0,1] \cup [3,\infty)$ and let's show that there is a subset E of $\mathbf{Z}$ such that $\mu(E) = c$. If $c \in [0,1]$, write it in binary form as $\sum_{n=1}^{\infty} a_n \frac{1}{2^n}$ and define the set $E = \{n : a_n = 1\}$, it follows that

$$\mu(E) = \sum_{k \in E} \mu(\{k\}) = \sum_{n=1}^{\infty} a_n \frac{1}{2^n} = c$$

Similarly, if $c \in [3,\infty)$, then there is an integer $c_0 \geq 3$ and a real $\alpha \in [0,1]$ such that $c = c_0 + \alpha$. As previously, write α in base 2 and define the set E in the same way. Moreover, add to the set E the integer $3 - c_0$, it will follow that $\mu(E) = c_0 + \alpha = c$ for the same reasons as above. Therefore,

$$\{\mu(E) : E \in \mathcal{S}\} = [0,1] \cup [3,\infty)$$

which proves our claim.

Exercise 8

Give an example of a set X, a σ-algebra $\mathcal{S}$ of subsets of X, a set $\mathcal{A}$ of subsets of X such that the smallest σ-algebra on X containing $\mathcal{A}$ is $\mathcal{S}$, and two measures μ and ν on $(X,\mathcal{S})$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$ and $\mu(X) = \nu(X) < \infty$, but $\mu \neq \nu$.

Solution

Let $X = [0,8]$, $\mathcal{A} = \{[0,4],[2,6]\}$, $\mathcal{S}$ the σ-algebra generated by $\mathcal{A}$ and the two follwing measures on $(X,\mathcal{S})$:

$$\mu = \delta_1 + \delta_5$$
$$\nu = \delta_3 + \delta_7$$

where δ_i is the Dirac delta meaasure at i. We can easily prove that both μ and ν are indeed measures on $(X,\mathcal{S})$ (it will be proved in the following exercise). First, notice that

$$\mu([0,4]) = \delta_1([0,4]) + \delta_5([0,4]) = 1 + 0 = 1$$
$$\nu([0,4]) = \delta_3([0,4]) + \delta_7([0,4]) = 1 + 0 = 1$$

and

$$\mu([2,6]) = \delta_1([2,6]) + \delta_5([2,6]) = 0 + 1 = 1$$
$$\nu([2,6]) = \delta_3([2,6]) + \delta_7([2,6]) = 1 + 0 = 1$$

which implies that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$. Moreover,

$$\mu(X) = \delta_1(X) + \delta_5(X) = 1 + 1 = 2$$

$$\nu(X) = \delta_3(X) + \delta_7(X) = 1 + 1 = 2$$

so $\mu(X) = \nu(X) < \infty$. Consider now the set $[2,4] = [0,4] \cap [2,6] \in \mathcal{S}$:

$$\mu([2,4]) = \delta_1([2,4]) + \delta_5([2,4]) = 0 + 0 = 0$$

$$\nu([2,4]) = \delta_3([2,4]) + \delta_7([2,4]) = 1 + 0 = 1$$

so $\mu \neq \nu$. Therefore, X, $\mathcal{S}$, $\mathcal{A}$, μ and ν satisfy all the desired properties.

Exercise 9
Suppose μ and ν are measures on a measurable space $(X, \mathcal{S})$. Prove that $\mu + \nu$ is a measure on $(X, \mathcal{S})$. [Here, $\mu + \nu$ is the usual sum of two functions: if $E \in \mathcal{S}$, then $(\mu + \nu)(E) = \mu(E) + \nu(E)$.]

Solution
We only have two properties to prove, that the empty set is mapped to zero and the countable additivity. First,

$$(\mu + \nu)(\varnothing) = \mu(\varnothing) + \nu(\varnothing) = 0 + 0 = 0$$

Moreover, if $\{E_i\}_i$ is a countable collection of pairwise disjoint sets in $\mathcal{S}$, then

$$\begin{aligned}
(\mu + \nu)\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} E_i\right) + \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \\
&= \sum_{i=1}^{\infty} \mu(E_i) + \sum_{i=1}^{\infty} \nu(E_i) \\
&= \sum_{i=1}^{\infty} [\mu(E_i) + \nu(E_i)] \\
&= \sum_{i=1}^{\infty} (\mu + \nu)(E_i)
\end{aligned}$$

Therefore, $\mu + \nu$ is a measure on $(X, \mathcal{S})$.

Exercise 10
Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset$ ... of sets in $\mathcal{S}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k \to \infty} \mu(E_k).$$

Solution

Let $X = \mathbf{R}$, $\mathcal{S} = 2^{\mathbf{R}}$ and μ be the counting measure on $(\mathbf{R}, 2^{\mathbf{R}})$. Consider the sequence of sets defined by $E_k = [k, \infty)$ for all $k \in \mathbf{Z}^+$. Notice that for all $k \in \mathbf{Z}^+$, the set E_k is infinite so it has infinite measure. Since it holds for all $k \in \mathbf{Z}^+$, then

$$\lim_{k\to\infty} \mu(E_k) = \infty$$

Moreover, since $\cap_{k=1}^{\infty} E_k = \varnothing$, then

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = 0$$

Therefore,

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k\to\infty} \mu(E_k)$$

Exercise 11

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $C, D, E \in \mathcal{S}$ are such that

$$\mu(C \cap D) < \infty, \mu(C \cap E) < \infty, \text{ and } \mu(D \cap E) < \infty.$$

Find an prove a formula for $\mu(C \cup D \cup E)$ in terms of $\mu(C)$, $\mu(D)$, $\mu(E)$, $\mu(C \cap D)$, $\mu(C \cap E)$, $\mu(D \cap E)$, and $\mu(C \cap D \cap E)$.

Solution

First, by 2.61, since $\mu(D \cap E) < \infty$, then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E) \tag{1}$$

Moreover, by 2.61, since $\mu(C \cap D \cap E) \leq \mu(D \cap E) < \infty$ (by monotonicity), then

$$\mu((C \cap D) \cup (C \cap E)) = \mu(C \cap D) + \mu(C \cap E) - \mu(C \cap D \cap E) \tag{2}$$

Lastly, combining (1) and (2) and applying 2.61 with the fact that $\mu((C \cap D) \cup (C \cap E)) \leq \mu(C \cap D) + \mu(C \cap E) < \infty$ gives us

$$\begin{aligned}
\mu(C \cup D \cup E) &= \mu(C \cup (D \cup E)) \\
&= \mu(C) + \mu(D \cup E) - \mu(C \cap (D \cup E)) \\
&= \mu(C) + \mu(D \cup E) - \mu((C \cap D) \cup (C \cap E)) \\
&= \mu(C) + \mu(D) + \mu(E) - \mu(D \cap E) \\
&\quad - [\mu(C \cap D) + \mu(C \cap E) - \mu(C \cap D \cap E)] \\
&= \mu(C) + \mu(D) + \mu(E) - \mu(D \cap E) \\
&\quad - \mu(C \cap D) - \mu(C \cap E) + \mu(C \cap D \cap E)
\end{aligned}$$

which is a satsfying formula for what was asked.

Exercise 12
Suppose X is a set and $\mathcal{S}$ is the σ-algebra of all subsets E of X such that E is countable or $X \setminus E$ is countable. Give a complete description of the set of all measures on $(X, \mathcal{S})$.

Solution
In this proof, the goal will be to show that the measures on $(X, \mathcal{S})$ are precisely the functions of the form

$$\mu(E) = \begin{cases} \sum_{x \in E} w(x) & \text{if } E \text{ is countable} \\ \alpha + \sum_{x \in E} w(x) & \text{if } E \text{ is uncountable} \end{cases}$$

where $\alpha \in [0, \infty]$ and $w : X \to [0, \infty]$ is a function. More precisely, the goal is to show that any measure on $(X, \mathcal{S})$ is of this form and that any function of this form is a measure on $(X, \mathcal{S})$.

- Let $\mu : \mathcal{S} \to [0, \infty]$ be a measure on $(X, \mathcal{S})$. Define $w : X \to [0, \infty]$ by $w : x \mapsto \mu(\{x\})$. This function is well-defined since all singletons are in $\mathcal{S}$ since they are countable. It follows that for all countable sets $E = \{e_1, e_2, ...\}$,

  $$\mu(E) = \sum_{i=1}^{\infty} \mu(\{e_i\}) = \sum_{x \in E} w(x)$$

  Now, suppose that $\sum_{x \in E} w(x) = \infty$, then define $\alpha = \infty$ which makes the following equation true

  $$\mu(E) = \alpha + \sum_{i=1}^{\infty} \mu(\{e_i\}) = \sum_{x \in E} w(x)$$

  for all uncountable set $E \in \mathcal{S}$. In that case, we have shown that μ is of the desired form.
  Suppose now that there is an uncountable set $E_0 \in \mathcal{S}$ such that $\sum_{x \in E_0} w(x) < \infty$, define

  $$\alpha = \mu(E_0) - \sum_{x \in E_0} w(x) \geq 0$$

  [Notice that α is positive since $\mu(E_0)$ is an upper bound for the set $\{\sum_{i=1}^{n} w(x_i) : x_1, ..., x_n \in E_0\}$ and that $\sum_{x \in E_0} w(x) := \sup\{\sum_{i=1}^{n} w(x_i) : x_1, ..., x_n \in E_0\}$.]
  Let's show that $\mu(E) = \alpha + \sum_{x \in E} w(x)$ for all uncountable sets $E \in \mathcal{S}$. If $\sum_{x \in E} w(x) = \infty$, then

  $$\mu(E) = \infty = \alpha + \sum_{x \in E} w(x)$$

If $\sum_{x\in E} w(x) < \infty$, then

$$\begin{aligned}
\mu(E) &= \mu(E_0) - \sum_{x\in E_0\setminus E} w(x) + \sum_{x\in E\setminus E_0} w(x) \\
&= \mu(E_0) - \sum_{x\in E_0\setminus E} w(x) - \sum_{x\in E\cap E_0} w(x) + \sum_{x\in E\cap E_0} w(x) + \sum_{x\in E\setminus E_0} w(x) \\
&= \mu(E_0) - \sum_{x\in E_0} w(x) + \sum_{x\in E} w(x) \\
&= \alpha + \sum_{x\in E} w(x)
\end{aligned}$$

which shows that in any case, μ is of the desired form.

- Consider now a function $\mu : \mathcal{S} \to [0,\infty]$ such that

$$\mu(E) = \begin{cases} \sum_{x\in E} w(x) & \text{if } E \text{ is countable} \\ \alpha + \sum_{x\in E} w(x) & \text{if } E \text{ is uncountable} \end{cases}$$

where $\alpha \in [0,\infty]$ and $w : X \to [0,\infty]$ is a function. Let's prove that μ is a measure on $(X,\mathcal{S})$. First,

$$\mu(\varnothing) = \sum_{x\in\varnothing} w(x) = 0$$

Now, let $\{E_i\}_i$ be a countable pairwise disjoint collection of sets in $\mathcal{S}$. Since they are disjoint, then there is at most one uncountable E_i. Otherwise, if E_{i_1} and E_{i_2} are both uncountable, then E_{i_2} is contained in the complement of E_{i_1} which implies that $E_{i_1}^c$ is also uncountable. A contradiction with the definition of $\mathcal{S}$. Thus, there is at most one uncountable E_i. If none of the E_i's are uncountable, then

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \sum_{x\in\cup_{i=1}^{\infty} E_i} w(x) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} w(e_{i,j}) \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}$$

If (wlog) E_1 is uncountable, then

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \alpha + \sum_{x \in \cup_{i=1}^{\infty} E_i} w(x) \\
&= \alpha + \sum_{x \in E_1} w(x) + \sum_{x \in \cup_{i=2}^{\infty} E_i} w(x) \\
&= \mu(E_1) + \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} w(e_{i,j}) \\
&= \mu(E_1) + \sum_{i=2}^{\infty} \mu(E_i) \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}$$

Therefore, μ is a measure on $(X, \mathcal{S})$.

Therefore, we have a complete description of the measures on the measurable space $(X, \mathcal{S})$.

2D Lebesgue Measure

Hii

2E Convergence of Measurable Functions

hi..