Higher Algebra 1 : Assignment 7

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Exercise 55: Let $A = \mathbb{Q}[x, y, w, z]/(xy - wz)$. Prove that the extension $\mathbb{Q}[x, y, w] \subseteq A$ is not an integral extension. Following the proof of Noether's normalization lemma, exhibit A as an integral extension of $\mathbb{Q}[a, b, c]$ for some algebraically independent elements $a, b, c \in A$. (You may assume that x, y, w are algebraically independent over $\mathbb{Q}$, as this is a bit hard to prove without additional tools.)

Solution : First, let's show that A is not an integral extension of $\mathbb{Q}[x, y, w]$. To do so, we can explicitly show that z is not integral over $\mathbb{Q}[x, y, w]$. By contradiction, suppose that

$$z^n + a_{n-1}(x, y, w)z^{n-1} + \cdots + a_0(x, y, w) = 0$$

for some $a_i \in \mathbb{Q}[x, y, w]$, then multiplying both sides by w^n gives us

$$x^n y^n + a_{n-1}(x, y, w)x^{n-1}y^{n-1}w + \cdots + a_0(x, y, w)w^n = 0.$$

Next, if we let $p(t_1, t_2, t_3) \in \mathbb{Q}[t_1, t_2, t_3]$ be the polynomial

$$p(t_1, t_2, t_3) = t_1^n t_2^n + a_{n-1}(t_1, t_2, t_3)t_1^{n-1}t_2^{n-1}t_3 + \cdots + a_0(t_1, t_2, t_3)t_3^n$$

where t_1, t_2, t_3 are variables, then clearly $p(x, y, w) = 0$. But since x, y, z are algebraically independent, then p must be the zero polynomial:

$$t_1^n t_2^n + a_{n-1}(t_1, t_2, t_3)t_1^{n-1}t_2^{n-1}t_3 + \cdots + a_0(t_1, t_2, t_3)t_3^n = 0.$$

Evaluating both sides at $t_3 = 0$ implies that $t_1^n t_2^n = 0$, a contradiction. Therefore, z is not integral over $\mathbb{Q}[x, y, w]$, so A is not integral over $\mathbb{Q}[x, y, w]$.

Hence, writing $A = \mathbb{Q}[x, y, w, z]$ (here, x, y, and z are to be seen as elements satisfying the relation $xy = wz$, not variables) where x, y, w are algebraically independent and z is algebraic over $\mathbb{Q}[x, y, z]$, we consider the polynomial $f \in \mathbb{Q}[t_1, t_2, t_3, t_4]$ defined by $f(t_1, t_2, t_3, t_4) = t_1t_2 - t_3t_4$. This polynomial satisfies $f(x, y, w, z) = 0$ and $F = f$. If we let $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -1$, we get that

$$F(\lambda_1, \lambda_2, \lambda_3, 1) = 0 \cdot 0 - (-1) \cdot 1 = 1 \neq 0.$$

If we define $w' = w + z$, then

$$0 = f(x, y, w, z) = f(x, y, w' - z, z) = z^2 - w'z + xy$$

which proves that z is integral over $\mathbb{Q}[x, y, w]$. Therefore, A is integral over $\mathbb{Q}[x, y, w + z]$.

Finally, let's prove that $x, y, w + z$ are algebraically independent. Unfortunately, I was not able to prove that x, y, and $w + z$ are algebraically independent, even if we suppose that x, y, w are algebraically independent. I am sure that they are algebraically independent since I was not able to find any algebraic relation between them. However, my proof attemps lead to nothing. Before ending this exercise, here is one of these proof atttempts:

Suppose that there exists a polynomial $p(t_1, t_2, t_3) \in \mathbb{Q}[t_1, t_2, t_3]$ such that $p(x, y, w+z) = 0$, then we can define the polynomial $q(t_1, t_2, t_3, t_4) = p(t_1, t_2, t_3+t_4) \in \mathbb{Q}[t_1, t_2, t_3, t_4]$. If we write

$$q(t_1, t_2, t_3, t_4) = \alpha_n(t_1, t_2, t_3)t_4^n + \alpha_{n-1}(t_1, t_2, t_3)t_4^{n-1} + \cdots + \alpha_0(t_1, t_2, t_3),$$

then $w^n q(x, y, w, z) = h(x, y, w)$ where $h(t_1, t_2, t_3) \in \mathbb{Q}[t_1, t_2, t_3]$ is the polynomial

$$\alpha_n t_1^n t_2^n + \alpha_{n-1} t_1^{n-1} t_2^{n-1} t_3 + \cdots + \alpha_0 t_3^n.$$

Since $h(x, y, w) = w^n q(x, y, w, z) = w^n p(x, y, w+z) = 0$, then h must be the zero polynomial. I am not able to conclude that p is the zero polynomial from the fact that h is the zero polynomial. I defined h because I wanted to reduce the proof to the fact that x, y, and w are algebraically independent (which is what I used to show that $h = 0$).

Exercise 57: Let $f_0, f_1, f_2, f_3, \ldots$ be polynomials in $k[x_1, \ldots, x_n]$, where k is a field. Let I be the ideal of $k[x_1, ..., x_n]$ generated by all of them: $I = \langle f_i : i \in \mathbb{N} \rangle$. Prove that there is a N such that

$$I = \langle f_0, f_1, \ldots, f_N \rangle$$

Solution : Define the following sequence of ideals: $I_n = \langle f_0, f_1, \ldots, f_n \rangle$. Clearly, this is an increasing sequence of ideals of $k[x_1, ..., x_n]$. By Hilbert's Basis Theorem, there must be a natural number N such that $I_N = I_{N+1} = I_{N+2} = \ldots$. Let's prove that $I_N = I$. By definition of I_N and I, we have that $I_N \subset I$ so it suffices to show that $I \subset I_N$. Since the ideal I is generated by the f_i's, then every element f of I can be written as a finite $k[x_1, ..., x_n]$-linear combination of f_i's. If we let m be the maximal index such that f_m is one of the f_i composing f (it may not be unique, but it works as long as we have one representation of f as a combination of f_i's), then we have that $f \in I_m$. If $m \leq N$, then $f \in I_N$ be the increasing property of the sequence of ideals. If $m > N$, then $I_N = I_m$ by the consequence of Hilbert's Basis Theorem. Thus, $f \in I_N$, and hence, $I = I_N$. Therefore:

$$I = \langle f_0, f_1, \ldots, f_N \rangle.$$

Exercise 59: Define a function F on the space of real 3×3 matrices:

$$F : M_3(\mathbb{R}) \to \mathbb{R}, \quad F(A) = \mathrm{trace}(A) + \det(A).$$

Let A be a real 3×3 matrix. Prove that there is an N such that if $F(A) = F(A^2) = \cdots = F(A^N) = 0$, then for all $n > N$, $F(A^n) = 0$.

Solution : Write $A = PTP^{-1}$ where P is an invertible matrix and T is of the form

$$T = \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{C}$, then $F(A) = \mathrm{trace}(T) + \det(T) = a + b + c + abc$. More generaly, we have that $F(A^n) = a^n + b^n + c^n + (abc)^n$ (it suffices to observe the matrix multiplication of matrix of the form of T). Define $f_n \in \mathbb{C}[x, y, z]$ by $f_n = x^n + y^n + z^n + x^n y^n z^n$, then $F(A^n) = f_n(a, b, c)$ for all n. Let I be the ideal generated by the f_i's, then by the previous exercise (exercise 57), we have that $I = \langle f_0, f_1, \ldots, f_N \rangle$ for some natural number N. This means that every f_i can be written as a $k[x, y, z]$-linear combination of $f_0, f_1, ..., f_N$. It follows that if the $f_0, f_1, ..., f_N$ vanish at (a, b, c), then any f_i vanishes at (a, b, c). Therefore, we can rephrase this by saying that $F(A) = F(A^2) = \cdots = F(A^N)$ implies that $F(A^n) = 0$ for all n.

Exercise 61: For the following rings R and left modules M, determine if M is a simple R-module, a semisimple R module, or neither.

(a) $R = \mathbb{Z}$, $M = \mathbb{Q}$.

(b) $R = \mathbb{Z}, M = \mathbb{Q}/\mathbb{Z}$.

(c) $R = M_2(\mathbb{R}), M = \{N \in M_2(\mathbb{R}) : N^t(1,3) = {}^t(0,0)\}$.

(d) R is the ring of upper traingular matrices in $M_3(\mathbb{R})$, M is the ideal comprising upper traingular matrices with zero diagonal.

Solution :

(a) $\mathbb{Q}$ is not a simple $\mathbb{Z}$-module because otherwise, $\mathbb{Q}$ would be isomorphic to $\mathbb{Z}/I$ where I is maximal, which is finite while $\mathbb{Q}$ is infinite. Since $\mathbb{Z}$ is not semisimple, then $\mathbb{Q}$ cannot be a semisimple $\mathbb{Z}$-module.

(b) $\mathbb{Q}/\mathbb{Z}$ is not a simple $\mathbb{Z}$-module because otherwise, $\mathbb{Q}/\mathbb{Z}$ would be isomorphic to $\mathbb{Z}/I$ where I is maximal, which is finite while $\mathbb{Q}/\mathbb{Z}$ is infinite. Since $\mathbb{Z}$ is not semisimple, then $\mathbb{Q}/\mathbb{Z}$ cannot be a semisimple $\mathbb{Z}$-module.

(c) If we solve the equation $N^t(1,3) = {}^t(0,0)$, we get that every element in M can be described as matrix where the first column is a vector in $\mathbb{R}^2$, and the second column is the first column multiplied by a scalar. Hence, M is isomorphic to $\mathbb{R}^2$ as $M_2(\mathbb{R})$-modules. It follows that M is a simple module by Example 24.1.2. Finally, M is semisimple because it can be written as a direct sum of only one simple module, itself.

(d) **TODO**