

Algebraic Geometry : Homework 3

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Exercise 2: Let $V = V(f_1, f_2) \subset k^3$, where $f_1, f_2 \in k[x, y, z]$ are the polynomials given by $f_1 = y - x^2$ and $f_2 = z - x^3$. Let $I(V)^* \subset k[x, y, z, w]$ be the homogenization of the ideal $I(V) = (f_1, f_2)$. Show that $zw - xy \in I(V)^*$ but $zw - xy \notin (f_1^*, f_2^*)$.

Solution : By properties of ideals, the polynomial $z - xy$ is in $I(V)$ since it can be written as $f_2 - xf_1$. It follows that $zw - xy = (z - xy)^* \in I(V)^*$.

Next, we have that $f_1^* = wy - x^2$ and $f_2^* = w^2z - x^3$. By contradiction, if $zw - xy \in (f_1^*, f_2^*)$, then

$$zw - xy = (wy - x^2)P + (w^2z - x^3)Q$$

for some polynomials $P, Q \in k[x, y, z, w]$. If we compare the powers of w on both sides, we get that Q must be zero since otherwise, there would be a w^k for some $k \geq 2$ on one side but not the other. Hence:

$$zw - xy = (wy - x^2)P.$$

Similarly, if we compare the powers of x , we get that P must be zero since otherwise, the powers of x would not match. It follows that $zw - xy = 0$ which is a clear contradiction. Therefore, $zw - xy \notin (f_1^*, f_2^*)$.