Algebraic Geometry : Homework 3

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Exercise 3: Let $V \subset W \subset \mathbb{P}^n(k)$ be two (non-empty) projective varieties, where $V = V(f)$ is a hypersurface defined by a homogeneous irreducible polynomial f. Show that $W = V$ or $W = \mathbb{P}^n(k)$. Deduce that the projective varieties $W \subset \mathbb{P}^n(k)$ that contain H_∞ are $W = \mathbb{P}^n(k)$ and $W = H_\infty$. Determine W_* and $(W_*)^*$ in the two cases.

Solution : By the Projective Nullstellensatz, $I(W) \subset I(V(f)) = (f)$. Since f is irreducible, then $I(W)$ is either the zero ideal of $I(V)$. It follows that W is either V or $\mathbb{P}^n(k)$. Since we can write H_∞ as $V(X_{n+1})$, and X_{n+1} is homogeneous and irreducible, then applying the first part of the exercise with $V = H_\infty$ implies that the projective varieties $W \subset \mathbb{P}^n(k)$ that conatin H_∞ are $W = \mathbb{P}^n(k)$ and $W = H_\infty$.

If $W = H_\infty$, then

$$W_* = V(I(H_\infty)_*) = V(I(V(X_{n+1}))_*) = V((X_{n+1})_*) = V(1) = \varnothing.$$

Hence, $(W_*)^* = \varnothing^* = V(I(\varnothing)^*) = V(k[X_1, \ldots, X_n]^*) = \varnothing$. Similarly, if $W = \mathbb{P}^n(k)$, then

$$W_* = V(I(\mathbb{P}^n(k))_*) = V(\{0\}_*) = V(0) = k^n,$$

and hence, $(W_*)^* = (k^n)^* = V(I(k^n)^*) = V(\{0\}^*) = V(0) = \mathbb{P}^n(k)$.