Solutions to Linear Algebra Done Right (4th Ed) - Sheldon Axler

Samy Lahlou

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Preface

The goal of this document is to share my personal solutions to the exercises in the Fourth Edition of Linear Algebra Done Right by Sheldon Axler during my reading. As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Chapter 1

Vector Spaces

1A $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1
Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that
$$\alpha = a + ib \quad \text{and} \quad \beta = c + id$$
where $a, b, c, d \in \mathbf{R}$, then
$$\begin{aligned} \alpha + \beta &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \\ &= (c + a) + i(d + b) \\ &= (c + id) + (a + ib) \\ &= \beta + \alpha \end{aligned}$$
which proves that addition is commutative in $\mathbf{C}$ using the fact that it is commutative in $\mathbf{R}$.

Exercise 2
Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that
$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$
where $a, b, c, d, e, f \in \mathbf{R}$, then
$$\begin{aligned} (\alpha + \beta) + \lambda &= [(a + ib) + (c + id)] + (e + if) \\ &= [(a + c) + i(b + d)] + (e + if) \\ &= ([a + c] + e) + i([b + d] + f) \\ &= (a + [c + e]) + i(b + [d + f]) \\ &= (a + ib) + [(c + e) + i(d + f)] \\ &= (a + ib) + [(c + id) + (e + if)] \\ &= \alpha + (\beta + \lambda) \end{aligned}$$

which proves that addition is associative in $\mathbf{C}$ using the fact that it is associative in $\mathbf{R}$.

Exercise 3
Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
(\alpha\beta)\lambda &= [(a + ib)(c + id)](e + if) \\
&= [(ac - bd) + i(ad + bc)](e + if) \\
&= ([ac - bd]e - [ad + bc]f) + i([ac - bd]f + [ad + bc]e) \\
&= (ace - bde - adf - bcf) + i(acf - bdf + ade + bce) \\
&= (a[ce - fd] - b[cf + de)) + i(a[cf + de] + b[ce - fd]) \\
&= (a + ib)[(ce - fd) + i(cf + de)] \\
&= (a + ib)[(c + id)(e + if)] \\
&= \alpha(\beta\lambda)
\end{aligned}$$

which proves that multiplication is associative in $\mathbf{C}$ using the fact that multiplication is associative and addition is commutative in $\mathbf{R}$.

Exercise 4
Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Solution
First, suppose that

$$\alpha = a + ib, \quad \beta = c + id \quad \text{and} \quad \lambda = e + if$$

where $a, b, c, d, e, f \in \mathbf{R}$, then

$$\begin{aligned}
\lambda(\alpha + \beta) &= (e + if)[(a + ib) + (c + id)] \\
&= (e + if)[(a + c) + i(b + d)] \\
&= [e(a + c) - f(b + d)] + i[e[b + d] + f[a + c]] \\
&= (ea + ec - fb - fd) + i(eb + ed + fa + fc) \\
&= [(ea - fb) + i(eb + fa)] + [(ec - fd) + i(ed + fc)] \\
&= [(e + if)(a + ib)] + [(e + if)(c + id)] \\
&= \lambda\alpha + \lambda\beta
\end{aligned}$$

which proves the distributivity in $\mathbf{C}$ using the distributivity in $\mathbf{R}$.

Exercise 5
Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Solution
Let $\alpha = a + ib$ and consider $\beta = (-a) + i(-b)$, then we get

$$\begin{aligned}\alpha + \beta &= (a + ib) + ([-a] + i[-b]) \\ &= (a + [-a]) + i(b + [-b]) \\ &= 0 + i0 \\ &= 0\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha+\beta_1 = 0$ and $\alpha + \beta_2 = 0$, this implies that $\alpha + \beta_1 = \alpha + \beta_2$. If we add β_1 on both sides, we get

$$\begin{aligned}\beta_1 + (\alpha + \beta_1) = \beta_1 + (\alpha + \beta_2) &\implies (\beta_1 + \alpha) + \beta_1 = (\beta_1 + \alpha) + \beta_2 \\ &\implies (\alpha + \beta_1) + \beta_1 = (\alpha + \beta_1) + \beta_2 \\ &\implies 0 + \beta_1 = 0 + \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 6
Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Solution
Let $\alpha = a + ib \neq 0$, then notice that we must have $a^2 + b^2 \neq 0$. Hence, consider

$$\beta = \left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)$$

Thus, we get

$$\begin{aligned}\alpha\beta &= (a + ib)\left[\left(\frac{a}{a^2 + b^2}\right) + i\left(-\frac{b}{a^2 + b^2}\right)\right] \\ &= \left(a\left(\frac{a}{a^2 + b^2}\right) - b\left(-\frac{b}{a^2 + b^2}\right)\right) + i\left(a\left(-\frac{b}{a^2 + b^2}\right) + b\left(\frac{a}{a^2 + b^2}\right)\right) \\ &= \frac{a^2 + b^2}{a^2 + b^2} + i\frac{-ab + ba}{a^2 + b^2} \\ &= 1 + i0 \\ &= 1\end{aligned}$$

which proves the existence of such a complex number β. To prove the uniqueness of such a complex number, let β_1 and β_2 be two complex numbers satisfying $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$, this implies that $\alpha\beta_1 = \alpha\beta_2$. If we multiply by β_1 on both sides, we get

$$\begin{aligned}\beta_1(\alpha\beta_1) = \beta_1(\alpha\beta_2) &\implies (\beta_1\alpha)\beta_1 = (\beta_1\alpha)\beta_2 \\ &\implies (\alpha\beta_1)\beta_1 = (\alpha\beta_1)\beta_2 \\ &\implies 1 \cdot \beta_1 = 1 \cdot \beta_2 \\ &\implies \beta_1 = \beta_2\end{aligned}$$

which proves that such a complex number is unique.

Exercise 7
Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution
This is pretty straightforward:

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= \frac{(-1+\sqrt{3}i)^3}{2^3} \\ &= \frac{(-1)^3+3(-1)^2(\sqrt{3}i)+3(-1)^1(\sqrt{3}i)^2+(\sqrt{3}i)^3}{8} \\ &= \frac{-1+3\sqrt{3}i+3\cdot 3-3(\sqrt{3}i)}{8} \\ &= \frac{8}{8} \\ &= 1\end{aligned}$$

Exercise 8
Find two distinct square roots of i.

Solution
Consider $\alpha = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\beta = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. Hence,

$$\begin{aligned}\alpha^2 &= \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 \\ &= \left(\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\frac{\sqrt{2}}{2}\cdot i\frac{\sqrt{2}}{2} + \left(i\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4} + i - \frac{2}{4} \\ &= i\end{aligned}$$

and

$$\begin{aligned}\beta^2 &= \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^2 \\ &= \left(-\frac{\sqrt{2}}{2}\right)^2 + 2\cdot\left(-\frac{\sqrt{2}}{2}\right)\cdot\left(-i\frac{\sqrt{2}}{2}\right) + \left(-i\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4} + i - \frac{2}{4} \\ &= i\end{aligned}$$

Therefore, α and β are two distinct square roots of i.

Exercise 9
Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution
First, suppose that such an element x exists, then there exist $a, b, c, d \in \mathbf{R}$ such that $x = (a, b, c, d)$ and

$$(4 + 2a, -3 + 2b, 1 + 2c, 7 + 2d) = (5, 9, -6, 8)$$

But notice that this is equivalent to the following system of equations:

$$\begin{cases} 4 + 2a = 5 \\ -3 + 2b = 9 \\ 1 + 2c = -6 \\ 7 + 2d = 8 \end{cases}$$

which implies that

$$\begin{cases} a = \frac{1}{2} \\ b = 6 \\ c = \frac{7}{2} \\ d = \frac{1}{2} \end{cases}$$

Therefore, we get that $x = (\frac{1}{2}, 6, \frac{7}{2}, \frac{1}{2}) \in \mathbf{R}^4$ is indeed a solution to our original equation.

Exercise 10
Explain why there is does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution
By contradiction, suppose there exists a complex number $\lambda = a + ib$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i)$$

Then, we would get the following system of equation:

$$\begin{cases} \lambda(2 - 3i) = 12 - 5i \\ \lambda(5 + 4i) = 7 + 22i \\ \lambda(-6 + 7i) = -32 - 9i \end{cases}$$

which is equivalent to

$$\begin{cases} \lambda = 3 + 2i \\ \lambda = 3 + 2i \\ \lambda = \frac{129}{85} + i\frac{278}{85} \end{cases}$$

We clearly have a contradiction since $3 + 2i \neq \frac{129}{85} + i\frac{278}{85}$. Therefore, there doesn't exist such a complex number λ.

Exercise 11
Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution
First, write

$$x = (x_1, ..., x_n), \quad y = (y_1, ..., y_n) \quad \text{and} \quad z = (z_1, ..., z_n)$$

Since addition is commutative in $\mathbf{F}$, we get

$$\begin{aligned}(x + y) + z &= [(x_1, ..., x_n) + (y_1, ..., y_n)] + (z_1, ..., z_n) \\ &= (x_1 + y_1, ..., x_n + y_n) + (z_1, ..., z_n) \\ &= ([x_1 + y_1] + z_1, ..., [x_n + y_n] + z_n) \\ &= (x_1 + [y_1 + z_1], ..., x_n + [y_n + z_n]) \\ &= (x_1, ..., x_n) + (y_1 + z_1, ..., y_n + z_n) \\ &= (x_1, ..., x_n) + [(y_1, ..., y_n) + (z_1, ..., z_n)] \\ &= x + (y + z)\end{aligned}$$

which proves that addition is associative in $\mathbf{F}^n$.

Exercise 12
Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution
First, write $x = (x_1, ..., x_n)$. Using associativity of multiplication in $\mathbf{F}$, we get

$$\begin{aligned}(ab)x &= (ab)(x_1, ..., x_n) \\ &= ((ab)x_1, ..., (ab)x_n) \\ &= (a(bx_1), ..., a(bx_n)) \\ &= a(bx_1, ..., bx_n) \\ &= a[b(x_1, ..., x_n)] \\ &= a(bx)\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Exercise 13
Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution
Let $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Hence,

$$\begin{aligned}1x &= 1(x_1, ..., x_n) \\ &= (1 \cdot x_1, ..., 1 \cdot x_n) \\ &= (x_1, ..., x_n) \\ &= x\end{aligned}$$

which proves the desired formula for all $x \in \mathbf{F}^n$.

Exercise 14
Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$.

Solution
Let $\lambda \in \mathbf{F}$ and $x, y \in \mathbf{F}^n$ with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
\lambda(x+y) &= \lambda[(x_1, ..., x_n) + (y_1, ..., y_n)] \\
&= \lambda(x_1 + y_1, ..., x_n + y_n) \\
&= (\lambda(x_1 + y_1), ..., \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, ..., \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, ..., \lambda x_n) + (\lambda y_1, ..., \lambda y_n) \\
&= \lambda(x_1, ..., x_n) + \lambda(y_1, ..., y_n) \\
&= \lambda x + \lambda y
\end{aligned}$$

which proves the desired formula.

Exercise 15
Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution
Let $a, b \in \mathbf{F}$ and $x = (x_1, ..., x_n) \in \mathbf{F}^n$. Using distributivity in $\mathbf{F}$, we get

$$\begin{aligned}
(a+b)x &= (a+b)(x_1, ..., x_n) && = ((a+b)x_1, ..., (a+b)x_n) \\
&= (ax_1 + bx_1, ..., ax_n + bx_n) \\
&= (ax_1, ..., ax_n) + (bx_1, ..., bx_n) \\
&= a(x_1, ..., x_n) + b(x_1, ..., x_n) \\
&= ax + bx
\end{aligned}$$

which proves the desired formula.

1B Definition of Vector Space

Exercise 1
Prove that $-(-v) = v$ for every $v \in V$.

Solution
Let $v \in V$, by definition, we know that by definition, $-v$ is defined as the only vector in V satisfying

$$v + (-v) = 0$$

which is equivalent to

$$(-v) + v = 0$$

by commutativity of addition in V. However, notice that by definition, $-(-v)$ is the unique vector satisfying

$$(-v) + [-(-v)] = 0$$

But since v itself also satisfies this equation, we get $-(-v) = v$ by uniqueness.

Exercise 2
Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution
Suppose that $a \neq 0$, then by properties of $\mathbf{F}$, the inverse a^{-1} exists. Hence, if we multiply by a^{-1} on both sides, we get

$$\begin{aligned} av = 0 &\implies a^{-1}(av) = a^{-1}0 \\ &\implies (a^{-1}a)v = 0 \\ &\implies 1v = 0 \\ &\implies v = 0 \end{aligned}$$

Therefore, we either have $a = 0$ or $v = 0$.

Exercise 3
Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution
By properties of vector spaces, since $v \in V$, then $-v \in V$. Similarly, since w and $-v$ are in V, then $w + (-v) \in V$. Finally, since $w + (-v) \in V$, then $3^{-1}(w + (-v)) \in V$. Thus, define x_0 as the vector $3^{-1}(w + (-v))$ in V. Notice that

$$\begin{aligned} v + 3x_0 &= v + 3[3^{-1}(w + (-v))] \\ &= v + (3 \cdot 3^{-1})(w + (-v)) \\ &= v + 1(w + (-v)) \\ &= v + (w + (-v)) \\ &= v + ((-v) + w) \\ &= (v + (-v)) + w \\ &= 0 + w \\ &= w \end{aligned}$$

which shows that the equation has at least one solution. To prove uniqueness, let $x_1 \in V$ be an arbitrary solution to the equation, then we get

$$\begin{aligned}
v + 3x_1 = w &\implies (-v) + (v + 3x_1) = (-v) + w \\
&\implies ((-v) + v) + 3x_1 = w + (-v) \\
&\implies 0 + 3x_1 = w + (-v) \\
&\implies 3x_1 = w + (-v) \\
&\implies 3^{-1}(3x_1) = 3^{-1}(w + (-v)) \\
&\implies (3^{-1}3)x_1 = x_0 \\
&\implies 1x_1 = x_0 \\
&\implies x_1 = x_0
\end{aligned}$$

which proves that x_0 is the unique solution to the equation.

Exercise 4
The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space. Which one?

Solution
The empty set doesn't satisfy the axiom that states that there must be an additive identity since the empty set is empty by definition.

Exercise 5
Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here, the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution
We already know that the axioms of a vector space imply that $0v = 0$ for all $v \in V$. Hence, it suffices to prove that if we assume the axioms of a vector space without the additive inverse condition, then we can prove the additive inverse condition if we also assume the property that $0v = 0$ for all $v \in V$. Let $v \in V$, the by the distributive condition, we get

$$\begin{aligned}
0v = 0 &\implies (1 + (-1))v = 0 \\
&\implies 1v + (-1)v = 0 \\
&\implies v + (-1)v = 0
\end{aligned}$$

which proves that v has an additive inverse for all $v \in V$.

Exercise 6
Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the

notation. Specifically, the sum and product of two reals numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty \\ \infty + (-\infty) = (-\infty) + \infty &= 0 \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution
With these operations of addition and scalar multiplication, $\mathbf{R} \cup \{\infty, -\infty\}$ cannot be a vector space since

$$((-\infty) + \infty) + \infty = 0 + \infty = \infty$$

and

$$(-\infty) + (\infty + \infty) = (-\infty) + \infty = 0$$

which proves that addition isn't associative under this operation.

Exercise 7
Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Solution
For any f and g in V^S, define $f + g : S \to V$ by $s \mapsto f(s) + g(s)$ for all $s \in S$. Similarly, for all $\alpha \in \mathbf{F}$ and $f \in V^S$, define $\alpha f : S \to V$ by $s \mapsto \lambda f(s)$ for all $s \in S$. With these definitions, let's prove that V^S is a vector space.

- (**commutativity**) Let $f, g \in V^S$, let's show that $f + g = g + f$. Let $s \in S$, then by commutativity in V, we obviously have

  $$(f + g)(s) = f(s) + g(s) = g(s) + f(s) = (g + f)(s)$$

  Since it holds for all s, then $f + g = g + f$.

- (**associativity**) Let $f, g, h \in V^S$ and $s \in S$, then by associativity in V, we have

  $$\begin{aligned} [(f + g) + h](s) &= (f + g)(s) + h(s) \\ &= [f(s) + g(s)] + h(s) \\ &= f(s) + [g(s) + h(s)] \\ &= f(s) + (g + h)(s) \\ &= [f + (g + h)](s) \end{aligned}$$

Since it holds for all $s \in S$, then $(f+g)+h = f+(g+h)$.
Let now $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, then by associativity in V, we get:

$$\begin{aligned}[(ab)f](s) &= (ab)f(s) \\ &= a(bf(s)) \\ &= a(bf)(s) \\ &= [a(bf)](s)\end{aligned}$$

Since it holds for all $s \in S$, then $(ab)f = a(bf)$.

- (**additive identity**) Let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$ and $s \in S$, we have

$$(f + 0_{V^S})(s) = f(s) + 0_{V^S}(s) = f(s) + 0 = f(s)$$

Since it holds for all $s \in S$, then $f + 0_{V^S} = f$ for all $f \in V^S$.

- (**additive inverse**) Again, let's denote by 0_{V^S} the zero function in V^S, then for all $f \in V^S$, we can define the function $g = (-1)f \in V^S$. Hence, for all $s \in S$, we get

$$\begin{aligned}(f+g)(s) &= f(s) + g(s) \\ &= f(s) + (-1)f(s) \\ &= f(s) + (-f(s)) \\ &= 0 \\ &= 0_{V^S}(s)\end{aligned}$$

Since it holds for all $s \in S$, then $f + g = 0_{V^S}$.

- (**multiplicative identity**) Let $f \in V^S$, then for all $s \in S$, we have

$$(1f)(s) = 1f(s) = f(s)$$

Since it holds for all $s \in S$, then $1f = f$.

- (**distributive property**) Let $f, g \in V^S$, $a \in \mathbf{F}$ and $s \in S$, then

$$\begin{aligned}[a(f+g)](s) &= a(f+g)(s) \\ &= a(f(s) + g(s)) \\ &= af(s) + ag(s) \\ &= (af)(s) + (ag)(s) \\ &= (af + ag)(s)\end{aligned}$$

Since it holds for all $s \in S$, then $a(f+g) = af + ag$. Similarly, for all $f \in V^S$, $a, b \in \mathbf{F}$ and $s \in S$, we have

$$\begin{aligned}[(a+b)f](s) &= (a+b)f(s) \\ &= af(s) + bf(s) \\ &= (af)(s) + (bf)(s) \\ &= (af + bf)(s)\end{aligned}$$

Since it holds for all $s \in S$, then $(a+b)f = af + bf$.

Therefore, V^S is a vector space under these definitions.

Exercise 8
Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
$$(a + ib)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with these definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Solution

- (**commutativity**) Let $u_1, v_1, u_2, v_2 \in V$, then by commutativity in V, we have
$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) &= (u_1 + u_2) + i(v_1 + v_2)\\ &= (u_2 + u_1) + i(v_2 + v_1)\\ &= (u_2 + iv_2) + (u_1 + iv_1)\end{aligned}$$
which proves that addition is commutative.
- (**associativity**) Let $u_1, v_1, u_2, v_2, u_3, v_3 \in V$, then by associativity in V, we have
$$\begin{aligned}[(u_1 + iv_1) + (u_2 + iv_2)] + (u_3 + iv_3) &= [(u_1 + u_2) + i(v_1 + v_2)] + (u_3 + iv_3)\\ &= ([u_1 + u_2] + u_3) + i([v_1 + v_2] + v_3)\\ &= (u_1 + [u_2 + u_3]) + i(v_1 + [v_2 + v_3])\\ &= (u_1 + iv_1) + [(u_2 + u_3) + i(v_2 + v_3)]\\ &= (u_1 + iv_1) + [(u_2 + iv_2) + (u_3 + iv_3)]\end{aligned}$$
Let now $a, b, c, d \in \mathbf{R}$ and $u, v \in V$, then we get:
$$\begin{aligned}&[(a + bi)(c + di)](u + iv)\\ &\quad= [(ac - bd) + i(ad + bc)](u + iv)\\ &\quad= [(ac - bd)u - (ad + bc)v] + i[(ac - bd)v + (ad + bc)u]\\ &\quad= [acu - bdu - adv - bcv] + i[acv - bdv + adu + bcu]\\ &\quad= [a(cu - dv) - b(cv + du)] + i[a(cv + du) + b(cu - dv)]\\ &\quad= (a + ib)[(cu - dv) + i(cv + du)]\\ &\quad= (a + ib)[(c + id)(u + iv)]\end{aligned}$$
which proves the associativity condition.

- (**additive identity**) For all $u, v \in V$,

$$(u + iv) + (0 + i0) = (u + 0) + i(v + 0) = u + iv$$

which proves that $0 + i0$ is an additive identity.

- (**additive inverse**) Let $u, v \in V$, then since $(-u), (-v) \in V$, we get

$$(u + iv) + ([-u] + i[-v]) = (u + [-u]) + i(v + [-v]) = 0 + i0$$

which proves that every element has an additive inverse.

- (**multiplicative identity**) Let $u, v \in V$, then

$$(1 + i0)(u + iv) = (1u - 0v) + i(1v + 0u) = u + iv$$

which proves that $1 = 1 + i0$ is a multiplicative identity.

- (**distributive property**) Let $a, b \in \mathbf{R}$ and $u_1, v_1, u_2, v_2 \in V$, then

$$\begin{aligned}
&(a + ib)[(u_1 + iv_1) + (u_2 + iv_2)] \\
&\qquad = (a + ib)([u_1 + u_2] + i[v_1 + v_2]) \\
&\qquad = (a[u_1 + u_2] - b[v_1 + v_2]) + i(a[v_1 + v_2] + b[u_1 + u_2]) \\
&\qquad = (au_1 + au_2 - bv_1 - bv_2) + i(av_1 + av_2 + v_1 + bv_2) \\
&\qquad = ([au_1 - bv_1] + [au_2 - bv_2]) + i([av_1 + bu_1] + [av_2 + bu_1]) \\
&\qquad = [(au_1 - bv_1) + i(av_1 + bu_1)] + [(au_2 - bv_2) + i(av_2 + bu_2)] \\
&\qquad = [(a + ib)(u_1 + iv_1)] + [(a + ib)(u_2 + iv_2)]
\end{aligned}$$

Similarly, for all $a, b, c, d \in \mathbf{R}$, and $u, v \in \mathbf{R}$, we have

$$\begin{aligned}
&[(a + ib) + (c{+}id)](u + iv) \\
&\qquad = ([a + c] + i[b + d])(u + iv) \\
&\qquad = ([a + c]u - [b + d]v) + i([a + c]v + [b + d]u) \\
&\qquad = (au + cu - bv - dv) + i(av + cv + bu + du) \\
&\qquad = ([au - bv] + [cu - dv]) + i([av + bu] + [cv + du]) \\
&\qquad = [(au - bv) + i(av + bu)] + [(cu - dv) + i(cv + du)] \\
&\qquad = (a + ib)(u + iv) + (c + id)(u + iv)
\end{aligned}$$

which proves the distributive property.

Therefore, $V_{\mathbf{C}}$ is a vector space under these definitions.

1C Subspaces

Exercise 1
For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution

(a) First, define
$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$$
Let's prove that it is indeed a subspace of $\mathbf{F}^3$. Since $0 + 2 \cdot 0 + 3 \cdot 0 = 0$, then $0 = (0, 0, 0) \in U$. Now, let $x, y \in U$ be two arbitrary elements where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then by definition:
$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 0 \\ y_1 + 2y_2 + 3y_3 & = 0 \end{cases}$$
Adding the two equations gives us
$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0 + 0 = 0$$
which proves that $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U$. Similarly, let $x = (x_1, x_2, x_3)$ be an arbitrary element in U and α an arbitrary scalar in $\mathbf{F}$, then by definition of U:
$$x_1 + 2x_2 + 3x_3 = 0$$
Multiplying by α on both sides gives us
$$(\alpha x_1) + 2(\alpha x_2) + 3(\alpha x_3) = \alpha \cdot 0 = 0$$
which proves that $\alpha x \in U$. Therefore, U is a subspace of $\mathbf{F}^3$.

(b) Since $0 = (0, 0, 0)$ doesn't satisfy $x_1 + 2x_2 + 3x_3 = 4$, then the set of such vectors cannot be a subspace since it doesn't contain the zero vector.

(c) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$ and notice that that both $x = (1, 1, 0)$ and $y = (0, 0, 1)$ are in U. However, $x + y$ is obviously not in U since $x + y = (1, 1, 1)$ and $1 \cdot 1 \cdot 1 = 1$. Therefore, U is not a subspace of $\mathbf{F}^3$.

(d) Define $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$ and let's show that it is a subspace of $\mathbf{F}^3$. First, since $0 = 5 \cdot 0$, then $0 = (0, 0, 0) \in U$. To prove that U is closed under addition, let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two arbitrary elements of U, then by definition:
$$\begin{cases} x_1 = 5x_3 \\ y_1 = 5y_3 \end{cases}$$

By adding the two equations together, we get

$$x_1 + y_1 = 5(x_3 + y_3)$$

Thus, $x+y = (x_1+y_1, x_2+y_2, x_3+y_3) \in U$. Finally, to prove that U is closed under scalar multiplication, let $x = (x_1, x_2, x_3)$ be an element of U and $\alpha \in \mathbf{F}$, then

$$\begin{aligned} x_1 = 5x_3 &\implies \alpha x_1 = \alpha(5x_3) \\ &\implies \alpha x_1 = 5(\alpha x_3) \end{aligned}$$

Thus, $\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \in U$. Therefore, U is a subspace of $\mathbf{F}^3$.

Exercise 2

Verify all assertions about subspaces in Example 1.35:

(a) If $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval [0,1] is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The set of differentiable real-valued functions f on the interval (0,3) such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

Solution

(a) Define $U_b = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$ for all $b \in \mathbf{F}$ and suppose first that U is a subspace of $\mathbf{F}^4$, then it must contain the zero vector. Hence, since $(0, 0, 0, 0) \in U$, then by definition:

$$0 = 5 \cdot 0 + b$$

which is equivalent to $b = 0$.
For the converse, let's show that U_0 is a subspace of $\mathbf{F}^4$. Since $0 = 5 \cdot 0$, then $0 = (0, 0, 0, 0) \in U_0$. If $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are arbitrary elements of U_0, then $x_3 = 5x_4$ and $y_3 = 5y_4$. By adding these two equations and by distributivity, we get

$$x_3 + y_3 = 5(x_4 + y_4)$$

which implies that $x+y \in U_0$. Similarly, if $x = (x_1, x_2, x_3, x_4) \in U_0$ and $\alpha \in \mathbf{F}$, then we get

$$x_3 = 5x_4 \implies \alpha x_3 = 5(\alpha x_4)$$

which implies that $\alpha x \in U_0$. Thus, U_0 is a subspace of $\mathbf{F}^4$. Therefore, U_b is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) Let C denote the set of real-valued continuous functions on the interval [0,1] and $0_{\mathbf{R}^{[0,1]}}$ the zero function which acts as the additive identity in $\mathbf{R}^{[0,1]}$. Since the constant zero function is continuous, then $0_{\mathbf{R}^{[0,1]}} \in C$. Similarly, since the sum of two continuous functions is continuous and the multiplication of a continuous function with a scalar is still continuous, then C is closed under addition and scalar multiplication. Therefore, C is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The proof is similar to part (b). The constant zero function is differentiable on $\mathbf{R}$. Moreover, differentiable functions are closed under addition and scalar multiplication. Therefore, the set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) Define $U_b = \{f : (0,3) \to \mathbf{R} \text{ differentiable } : f'(2) = b\}$ for all $b \in \mathbf{R}$. Suppose that U_b is a subspace of $\mathbf{R}^{(0,3)}$, then we must have $0_{(0,3)} \in U_b$ where $0_{(0,3)}$ denotes the constant zero function on (0,3). By definition of U_b, it implies that $0'_{(0,3)}(2) = b$. However, we know that $0'_{(0,3)}(2) = 0$. Thus, $b = 0$.

Conversely, let's show that U_0 is a subspace of $\mathbf{R}^{(0,3)}$. First, the constant zero function $0_{\mathbf{R}^{(0,3)}}$ on (0,3) which acts as the additive identity in $\mathbf{R}^{(0,3)}$, is differentiable on (0,3) and its derivative at 2 is 0. Hence, $0_{\mathbf{R}^{(0,3)}} \in U_0$. Now, let $f, g \in U_0$, then $f + g$ is differentiable on (0,3) and

$$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0$$

so $f + g \in U_0$. Similarly, for any $f \in U_0$ and $\alpha \in \mathbf{F}$, the function αf is still differentiable on (0,3) and

$$(\alpha f)'(2) = \alpha f'(2) = \alpha \cdot 0 = 0$$

so $\alpha f \in U_0$. Thus, U_0 is a subspace of $\mathbf{R}^{(0,3)}$. Therefore, U_b is a subspace if and only if $b = 0$.

(e) Let S be the set of sequences of complex numbers with limit 0. Since the additive identity $(0, 0, ...)$ of C^∞ converges to 0, then it is in S. Let $(a_n)_n, (b_n)_n \in S$, then

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0$$

which implies

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0 + 0 = 0$$

Thus, $(a_n)_n + (b_n)_n \in S$. Similarly, for all $(a_n)_n \in S$ and $\alpha \in \mathbf{C}$, we have

$$\lim_{n\to\infty} \alpha a_n = \alpha \lim_{n\to\infty} a_n = \alpha \cdot 0 = 0$$

so $\alpha(a_n)_n \in S$. Therefore, S is a subspace of $\mathbf{C}^\infty$.

Exercise 3

Show that the set of differentiable real-valued functions f on the interval (-4, 4) such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Solution

Define the set $U = \{f : (-4, 4) \to \mathbf{R} \text{ differentiable } : f'(-1) = 3f(2)\}$ and let's show that it is a subspace of $\mathbf{R}^{(-4,4)}$. First, denote by f_0 to constant zero function on $(-4, 4)$ which is also the additive identity in $\mathbf{R}^{(-4,4)}$. We know that f_0 is differentiable on $(-4, 4)$ with $f_0' = f_0$. Hence, $f_0'(-1) = 0 = 3f_0(2)$ which proves that $f_0 \in U$.
To show that it is closed under addition, let $f, g \in U$, then by definition, f and g are differentiable on (-4, 4) and

$$\begin{cases} f'(-1) = 3f(2) \\ g'(-1) = 3g(2) \end{cases}$$

If we add these two equations, we get

$$(f + g)'(-1) = 3(f + g)(2)$$

which proves that $f + g \in U$ since $f + g$ is differentiable on (-4, 4).
To prove that it is closed under scalar multiplication, let $f \in U$ and $\alpha \in \mathbf{R}$, then

$$\begin{aligned} f'(-1) = 3f(2) &\implies \alpha f'(-1) = \alpha \cdot 3f(2) \\ &\implies (\alpha f)'(-1) = 3(\alpha f)(2) \end{aligned}$$

which proves that $\alpha f \in U$ since αf is differentiable on (-4, 4). Therefore, U is a subspace of $\mathbf{R}^{(-4,4)}$.

Exercise 4

Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

Solution

Let $b \in \mathbf{R}$ and define $I = \{f : [0, 1] \to \mathbf{R} \text{ continuous } : \int_0^1 f = b\}$. Suppose that I is a subspace of $\mathbf{R}^{[0,1]}$, then the additive identity $0 : x \mapsto 0$ must be in I so $\int_0^1 0 = b$. But we know that $\int_0^1 0 = 0$ so it follows that $b = 0$.
Conversly, let's show that I is a subspace of $\mathbf{R}^{[0,1]}$ when $b = 0$. First, the additive identity 0 is obviously continuous with $\int_0^1 0 = 0$ so $0 \in I$. Now, let $f, g \in I$, then f and g are continuous and

$$\int_0^1 f = \int_0^1 g = 0$$

It follows that $f + g$ is a continuous function that satisfies

$$\int_0^1 (f + g) = \int_0^1 f + \int_0^1 g = 0$$

Hence, $f + g \in I$. Similarly, if $f \in I$ and $\alpha \in \mathbf{R}$, then f is continuous and

$$\int_0^1 f = 0$$

which implies that αf is also continuous and

$$\int_0^1 (\alpha f) = \alpha \int_0^1 f = \alpha \cdot 0 = 0$$

Hence, $\alpha f \in I$. Therefore, I is a subspace of $\mathbf{R}^{[0,1]}$.

Exercise 5
Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution
No, it isn't because it is not closed under scalar multiplication since the scalars are complex numbers. For example, $(1,1) \in \mathbf{R}^2$ but $i(1,1) = (i,i) \notin \mathbf{R}^2$. Therefore, $\mathbf{R}^2$ is not a subspace of the complex vector space $\mathbf{C}^2$.

Exercise 6

(a) Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$ a subspace of $\mathbf{R}^3$?

(b) Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = c^3\}$ a subspace of $\mathbf{C}^3$?

Solution

(a) In $\mathbf{R}$, the function $x \mapsto x^3$ is bijective so if we define $I = \{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$, then we actually have $I = \{(a,b,c) \in \mathbf{R}^3 : a = c\}$. Hence, it is easier now to show that I is a subspace of $\mathbf{R}^3$. Obviously, $(0,0,0) \in I$ since $0 = 0$. Moreover, if (x_1,x_2,x_3) and (y_1,y_2,y_3) are in I, then $x_1 = x_3$ and $y_1 = y_3$ which implies that $x_1 + y_1 = x_3 + y_3$. Hence, $(x_1+y_1, x_2+y_2, x_3+y_3)$ in in I. Similarly, for $(x_1,x_2,x_3) \in I$ and $\alpha \in \mathbf{R}$, we must have $x_1 = x_3$ which implies that $\alpha x_1 = \alpha x_3$. Thus, $(\alpha x_1, \alpha x_2, \alpha x_3) \in I$. Therefore, I is a subspace of $\mathbf{R}^3$.

(b) If we let $I = \{(a,b,c) \in \mathbf{R}^3 : a^3 = c^3\}$, notice that $(\frac{-1+\sqrt{3}i}{2}, 0, 1)$ and $(\frac{-1-\sqrt{3}i}{2}, 0, 1)$ are both elements of I. However, their sum is not in I since

$$\left(\frac{-1+\sqrt{3}i}{2}, 0, 1\right) + \left(\frac{-1-\sqrt{3}i}{2}, 0, 1\right) = (-1,0,2) \notin I$$

Therefore, it is not a subspace of $\mathbf{C}^3$ since it is not closed under addition.

Exercise 7
Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

Solution
Consider the set $U = \{(k,k) : k \in \mathbf{Z}\}$ which is obviously closed under addition and taking inverses. Notice that U is not a subspace because it is not closed under scalar multiplication: $(1,1) \in U$ and $\pi \in \mathbf{R}$ but $\pi(1,1) = (\pi,\pi) \notin U$.

Exercise 8
Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar

multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution
Consider the set $U = \{(x, y) \in \mathbf{R}^2 : xy \geq 0\}$, let's first show that it is closed under scalar multiplication. Given $(x, y) \in U$ and $\alpha \in \mathbf{R}$, we know by definition of U tht $xy \geq 0$. Moreover, since α is a real number, then $\alpha^2 \geq 0$. Hence,

$$(\alpha x)(\alpha y) = \alpha^2 xy \geq 0$$

Thus, $(\alpha x, \alpha y) \in U$ so U is indeed closed under scalar multiplication. To show that U is not a subspace, consider the elements $(-1, 0)$ and $(0, 1)$ in U and notice that their addition cannot be in U since $(-1) \cdot 1 \ngeq 0$. Thus, U is not closed under addition which proves that it is not a subspace.

Exercise 9
A function $f : \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x + p) = f(x)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution
Let's prove that this set is not a subspace of $\mathbf{R}^{\mathbf{R}}$ by showing that it is not closed under addition. To do so, consider the functions $x \mapsto \cos(x)$ and $x \mapsto \cos(\pi x)$ defined on $\mathbf{R}$. Obviously, both are periodic since the first one has period 2π and the second one has period 2. Consider their sum $f : \cos(x) + \cos(\pi x)$ and suppose by contradiction that there exists a $p > 0$ such that

$$f(x) = f(x + p) \tag{1}$$

for all $x \in \mathbf{R}$. Notice that

$$\begin{aligned} f(x) = 2 &\implies \cos(x) + \cos(\pi x) = 2 \\ &\implies \cos(x) = 1 \quad \text{and} \quad \cos(\pi x) = 1 \\ &\implies x \in 2\pi\mathbf{Z} \quad \text{and} \quad x \in 2\mathbf{Z} \\ &\implies x = 0 \end{aligned}$$

Hence, f is equal to 2 if and only if $x = 0$. Thus, if we plug-in $x = 0$ in equation (1), we get

$$f(p) = f(0) = 2$$

which implies that $p = 0$, a contradiction since $p > 0$. Therefore, f is not periodic which proves that periodic functions are not closed under addition. With a similar argument, periodic functions are not closed under multiplication either.

Exercise 10
Suppose V_1 and V_2 are subspaces of V. Prove that $V_1 \cap V_2$ is a subspace of V.

Solution
Let's show that $V_1 \cap V_2$ satisfies the three subspace conditions:

- (**additive identity**) Since V_1 and V_2 are subspaces, then they both contain the additive identity 0 of V. It follows that $0 \in V_1 \cap V_2$ since it is contained in both sets.

- (**closed under addition**) Let u and v be two vectors in $V_1 \cap V_2$, then u and v must be contained in V_1. Since V_1 is a subspace, then it is closed under addition so $u+v$ must also be an element of V_1. Similarly, u and v are contained in V_2 so for the same reasons, $u+v$ must be an element of V_2. Thus, $u+v \in V_1 \cap V_2$ since $u+v \in V_1$ and $u+v \in V_2$.

- (**closed under scalar multiplication**) Let $a \in \mathbf{F}$ and $u \in V_1 \cap V_2$, then u must be contained in V_1. Since V_1 is a subspace, then it is closed under scalar multiplication so au must also be an element of V_1. Similarly, u is contained in V_2 so for the same reasons, au must be an element of V_2. Thus, $au \in V_1 \cap V_2$ since $au \in V_1$ and $au \in V_2$.

Therefore, $V_1 \cap V_2$ is a subspace of V.

Exercise 11
Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution
Let $\{V_i\}_{i \in I}$ be an arbitrary collection of subspaces of V, let's show that $\cap_{i \in I} V_i$ is also a subspace of V by proving the three subspace conditions:

- (**additive identity**) Since V_i is a subspace of V, then $0 \in V_i$ for all $i \in I$. It follows that $0 \in \cap_{i \in I} V_i$.

- (**closed under addition**) Let u and v be two vectors in $\cap_{i \in I} V_i$, then u and v must be contained in V_i for all $i \in I$. For any $i \in I$, V_i is a subspace so it is closed under addition, hence $u+v \in V_i$. It follows that $u+v \in \cap_{i \in I} V_i$.

- (**closed under scalar multiplication**) Let $a \in \mathbf{F}$ and $v \in \cap_{i \in I}$. For all $i \in I$, since $u \in V_i$ and V_i is a subspace, then $au \in V_i$. It follows that $au \in \cap_{i \in I} V_i$ since $au \in V_i$ for all $i \in I$.

Therefore, $\cap_{i \in I} V_i$ is a subspace of V.

Exercise 12
Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution
Let V_1 and V_2 be subspaces of V. If $V_1 \subset V_2$ or $V_2 \subset V_1$, then $V_1 \cup V_2$ must be a subspace of V as well. To show the converse, suppose now that $V_1 \cup V_2$ is a subspace of V and that $V_1 \not\subset V_2$. Then there exists a vector $u_1 \in V_1$ such that $u_1 \notin V_2$. Let's prove that $V_2 \subset V_1$ in that case. Let $v \in V_2$ be arbitrary, since u_1 and v are both vectors in $V_1 \cup V_2$, then $u_1 + v \in V_1 \cup V_2$ since it is a subspace. But this implies that $u_1 + v$ is either in V_1 or in V_2. If $u_1 + v \in V_2$, then we must have

$$u_1 = (u_1 + v) - v \in V_2$$

since $v \in V_2$ and V_2 is a subspace. A contradiction since $u_1 \notin V_2$. It follows that $u_1 + v \in V_1$. But again, since V_1 is a subspace and $u_1 \in V_1$, then

$$v = (u_1 + v) - u_1 \in V_1$$

which proves that $V_2 \subset V_1$. Therefore, if $V_1 \cup V_2$ is a subspace, then we either have $V_1 \subset V_2$ or $V_2 \subset V_1$.

Exercise 13
Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution
Let V_1, V_2 and V_3 be three subspaces of V. Obviously, if one contains the other two, then $V_1 \cup V_2 \cup V_3$ is also a subspace of V since it is either equal to V_1, V_2 or V_3.
To show the converse, suppose that $V_1 \cup V_2 \cup V_3$ is a subspace of V. To prove that one subspace contains the other two, suppose that $V_1 \not\supset V_2 \cup V_3$ and $V_2 \not\supset V_1 \cup V_3$, then it suffices to show that V_3 contains $V_1 \cup V_2$. Notice that $V_1 \not\supset V_2 \cup V_3$ and $V_2 \not\supset V_1 \cup V_3$ implies the existence of vectors $v_1, v_2 \in V$ such that

$$\begin{cases} v_1 \in V_1 \cup V_3 \\ v_1 \notin V_2 \end{cases} \quad \text{and} \quad \begin{cases} v_2 \in V_2 \cup V_3 \\ v_2 \notin V_1 \end{cases}$$

To show that $V_3 \supset V_1 \cup V_2$, let's proceed by cases:

- Suppose that $V_1 \cup V_2 \leq V$, then by the previous exercise, we must have $V_1 \cup V_2 \subset V_3$ or $V_3 \leq V_1 \cup V_2$. By contradiction, suppose that $V_3 \subset V_1 \cup V_2$. Since $V_1 \cup V_2 \leq V$, then again, by the previous exercise, $V_1 \subset V_2$ or $V_2 \subset V_1$. If $V_1 \subset V_2$, then $v_1 \in V_1 \cup V_3 \subset V_2 \cup V_3$. But, $v_1 \notin V_2$ so $v_1 \in V_3$. However, $V_3 \subset V_1 \cup V_2 = V_2$ so $v_1 \in V_2$. A contradiction that shows that $V_1 \not\subset V_2$. Similarly, we can prove in the same way using v_2 that $V_2 \subset V_1$. Thus, by contradiction, we get that V_3 is not a subset of $V_1 \cup V_2$. It follows that $V_1 \cup V_2 \subset V_3$.

- Suppose now that $V_1 \cup V_2$ is not a subspace of V, then by the previous exercise, $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$. It follows that there exist vectors $u_1, u_2 \in V$ such that

$$\begin{cases} u_1 \in V_1 \\ u_1 \notin V_2 \end{cases} \quad \text{and} \quad \begin{cases} u_2 \in V_2 \\ u_2 \notin V_1 \end{cases}$$

  Consider now the vector $\alpha_1 u_1 + \alpha_2 u_2$ where α_1 and α_2 are non-zero scalars. Since $\alpha_1 u_1 \in V_1 \subset V_1 \cup V_2 \cup V_3$, $\alpha_2 u_2 \in V_2 \subset V_1 \cup V_2 \cup V_3$ and $V_1 \cup V_2 \cup V_3$ is a subspace of V, then $\alpha_1 u_1 + \alpha_2 u_2 \in V_1 \cup V_2 \cup V_3$. If $\alpha_1 u_1 + \alpha_2 u_2 \in V_1$, then using the fact that $\alpha_1 u_1 \in V_1$ and that V_1 is a subspace gives us

$$\alpha_2 u_2 = (\alpha_1 u_1 + \alpha_2 u_2) - \alpha_1 u_1 \in V_1$$

  Multiplying by α_2^{-1} implies that $u_2 \in V_1$, a contradiction. Similarly, if $\alpha_1 u_1 + \alpha_2 u_2 \in V_2$, then we can show in the same way that $u_1 \in V_2$, another contradiction. Thus, we must have $\alpha_1 u_1 + \alpha_2 u_2 \in V_2$.
  Using the fact that $2 \neq 0$, we get that $u_1 + u_2$, $2u_1 + u_2$ and $u_1 + 2u_2$ are in V_3. Thus,

$$u_1 = (2u_1 + u_2) - (u_1 + u_2) \in V_3$$

  and

$$u_2 = (u_1 + 2u_2) - (u_1 + u_2) \in V_3$$

  We are now ready to show that $V_1 \cup V_2 \subset V_3$ in that case. To do so, let $w \in V_1 \cup V_2$ and proceed again by cases:

- Suppose that $w \in V_1 \setminus V_2$ and consider the vector $w + u_2 \in V_1 \cup V_2 \cup V_3$. If $w + u_2 \in V_1$, then $u_2 \in V_1$, a contradiction. Similarly, if $w + u_2 \in V_2$, then $w \in V_2$, a contradiction. Hence, $w + u_2 \in V_3$. But since $u_2 \in V_3$, then $w \in V_3$.
- Suppose that $w \in V_2 \setminus V_1$ and consider the vector $w + u_1 \in V_1 \cup V_2 \cup V_3$. If $w + u_1 \in V_1$, then $w \in V_1$, a contradiction. Similarly, if $w + u_1 \in V_2$, then $u_1 \in V_2$, a contradiction. Hence, $w + u_1 \in V_3$. But since $u_1 \in V_3$, then $w \in V_3$.
- Suppose that $w \in V_1 \cap V_2$ and consider the vector $w+u_1+u_2 \in V_1 \cup V_2 \cup V_3$. If $w+u_1+u_2 \in V_1$, then $u_2 \in V_1$, a contradiction. Similarly, if $w+u_1+u_2 \in V_2$, then $u_1 \in V_2$, a contradiction. Hence, $w + u_1 + u_2 \in V_3$. But since $u_1 + u_2 \in V_3$, then $w \in V_3$.

Thus, $V_1 \cup V_2 \subset V_3$.

Hence, in all possible cases, we get that $V_1 \cup V_2 \subset V_3$. Therefore, the union of three subspaces is a subspace if and only if one of the subspaces contains the other two.

Exercise 14
Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Solution
By definition, we have

$$\begin{aligned} U + W &= \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} + \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \\ &= \{(x + y, -x + y, 2x + 2y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \\ &= \{(x + y, -x + y, 2(x + y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \end{aligned}$$

From this expression, let's prove that

$$U + W = \{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$$

Obviously, $U + W \subset \{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$ because for any vector $(x + y, -x + y, 2(x + y) \in \mathbf{F}^3$, if we let $a = x + y$ and $b = -x + y$, we can rewrite this vector as $(a, b, 2a)$ which is in $\{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$. Similarly, given an arbitrary vector $(a, b, 2a) \in \mathbf{F}^3$, if we let

$$x = \frac{a - b}{2} \qquad \text{and} \qquad y = \frac{x + y}{2}$$

then we can rewrite the vector as $(x + y, -x + y, 2(x + y))$ which is obviously in $U + W$. It follows that the sets are equal. Without symbols, this just means that $U + W$ is precisely the set of vectors in V such that the third component is twice the first component.

Exercise 15
Suppose U is a subspace of V. What is $U + U$?

Solution
Let's show that $U = U + U$. An arbitrary element in $U + U$ is of the form $x + y$ where x and y are in U. Since U is a subspace, then it is closed under addition which implies that $x + y \in U$. It follows that $U + U \subset U$.
For the reverse inclusion, take an arbitrary $u \in U$ and notice that we can write $u = u + 0$. Again, since U is a subspace of V, then $0 \in V$. Thus, in the expression $u+0$, both vectors are in U. It follows that $u = u+0 \in U+U$. Therefore, $U = U+U$.

Exercise 16
Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution
Let U and W be subspaces of V. Then by commutativity of addition in V, we get

$$\begin{aligned} U + W &= \{u + w : u \in U \text{ and } w \in W\} \\ &= \{w + u : w \in W \text{ and } u \in U\} \\ &= W + U \end{aligned}$$

Therefore, the operation of addition on subspaces of V is commutative.

Exercise 17
Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2 and V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Solution
Let V_1, V_2 and V_3 are subspaces of V and let's show that

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$$

First, take an arbitrary $x + y \in (V_1 + V_2) + V_3$ where $x \in V_1 + V_2$ and $y \in V_3$. Since $x \in V_1 + V_2$, then there exist vectors $a \in V_1$ and $b \in V_2$ such that $x = a + b$. It follows from the associativity of addition in V that

$$x + y = (a + b) + y = a + (b + y)$$

Since $b \in V_2$ and $y \in V_3$, then $b+y \in V_2+V_3$. Hence, $a+(b+y) \in V_1+(V_2+V_3)$ using the fact that $a \in V_1$. Thus, the arbitrary $x + y \in (V_1 + V_2) + V_3$ is in $V_1 + (V_2 + V_3)$ as well so

$$(V_1 + V_2) + V_3 \subset V_1 + (V_2 + V_3)$$

The reverse inclusion has the same proof. The desired equality follows.

Exercise 18
Does the operation of addition on subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution
First, let's show that indeed, the operation of addition on subspaces of V has an additive identity. Define $I = \{0\}$, the subspace of V containing the zero vector only. Take an arbitrary subspace U of V and notice that

$$\begin{aligned} U + I &= \{u + i : u \in U \text{ and } i \in I\} \\ &= \{u + 0 : u \in U \text{ and } i \in I\} \\ &= \{u : u \in U\} \\ &= U \end{aligned}$$

By commutativity of addition of subspaces of V, we also have $I + U = U$. Therefore, I is an additive identity for the addition on subspaces of V.
Concerning additive inverses, let's determine which subspaces of V have an additive inverse by taking an arbitrary subspace U of V and supposing that there is a subspace W of V such that $V + W = I$. Since W is a subspace of V, then $0 \in W$. It follows that for all $u \in U$,

$$u = u + 0 \in U + W = I = \{0\}$$

In other words, $U = \{0\} = I$. Since I obviously has an additive inverse (itself), then the unique subspace having an additive inverse is I.

Exercise 19
Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Solution
Consider the following counterexample. Let $V_1 = U = V$ and $V_2 = \{0\}$. We know from Exercise 15 of this section that

$$V_1 + U = V + V = V$$

Moreover, from Exercise 19, we also have

$$V_2 + U = \{0\} + V = V$$

Thus,

$$V_1 + U = V_2 + U$$

but $V_1 \neq V_2$.

Exercise 20
Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution
Consider the subspace

$$W = \{(0, a, 0, b) \in \mathbf{F}^4 : a, b \in \mathbf{F}\}$$

and the sum $U+W$. First, let's show that the sum is direct by proving that $(0,0,0,0)$ has a unique representation in this sum. Suppose $(x, x, y, y) \in U$ and $(0, a, 0, b) \in W$ satisfy

$$(0,0,0,0) = (x, x, y, y) + (0, a, 0, b)$$

This is equivalent to the system of equation

$$\begin{cases} x = 0 \\ a + x = 0 \\ y = 0 \\ b + y = 0 \end{cases}$$

which clearly has the following unique solution

$$\begin{cases} x = 0 \\ a = 0 \\ y = 0 \\ b = 0 \end{cases}$$

Therefore, in $U + W$, the zero vector can only be written as the sum of two zero vectors. It follows that the sum is direct.
Let's now show that $U \oplus W = \mathbf{F}^4$ by taking an arbitrary vector (x_1, x_2, x_3, x_4). Consider the vectors

$$u = (x_1, x_1, x_3, x_3) \in U \quad \text{and} \quad w = (0, x_2 - x_1, 0, x_4 - x_3) \in W$$

and notice that

$$\begin{aligned} u + w &= (x_1, x_1, x_3, x_3) + (0, x_2 - x_1, 0, x_4 - x_3) \\ &= (x_1, x_1 + x_2 - x_1, x_3, x_3 + x_4 - x_3) \\ &= (x_1, x_2, x_3, x_4) \end{aligned}$$

which shows that $(x_1, x_2, x_3, x_4) \in U \oplus W$. Thus, $\mathbf{F}^4 \subset U \oplus W$. Since U and W are subspaces of $\mathbf{F}^4$, then $U \oplus W$ must also be a subspace of $\mathbf{F}^4$: $U \oplus W \subset \mathbf{F}^4$. Therefore, we get $U \oplus W = \mathbf{F}^4$.

Exercise 21
Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution
Consider the subspace

$$W = \{(0, 0, a, b, c) \in \mathbf{F}^5 : a, b, c \in \mathbf{F}\}$$

and consider the sum $U + W$. Let's first prove that it is actually a direct sum by focusing on the vector zero. Let $(x, y, x + y, x - y, 2x) \in U$ and $(0, 0, a, b, c) \in W$ be two vectors such that

$$(0, 0, 0, 0) = (x, y, x + y, x - y, 2x) + (0, 0, a, b, c)$$

This translates to the following system of equation:

$$\begin{cases} x = 0 \\ y = 0 \\ x + y + a = 0 \\ x - y + b = 0 \\ 2x + c = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} x = 0 \\ y = 0 \\ a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore, since the zero vector can only be written as the sum of two zero vectors, the sum is direct.
Let's now show that $U \oplus W = \mathbf{F}^5$. Obviously, since $U \oplus W$ is a subspace of $\mathbf{F}^5$, we have $U \oplus W \subset \mathbf{F}^5$. Moreover, for any $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$\begin{aligned} &(x_1, x_2, x_3, x_4, x_5) \\ &\quad = (x_1, x_2, [x_1 + x_2] + [x_3 - x_2 - x_1], [x_1 - x_2] + [x_4 + x_2 - x_1], 2x_1 + [x_5 - 2x_1]) \\ &\quad = (x_1, x_2, x_1 + x_2, x_1 - x_2, 2x_1) + (0, 0, x_3 - x_2 - x_1, x_4 + x_2 - x_1, x_5 - 2x_1) \\ &\quad = (x, y, x + y, x - y, 2x) + (0, 0, a, b, c) \\ &\quad \in U \oplus W \end{aligned}$$

where $x = x_1$, $y = x_2$, $a = x_3 - x_2 - x_1$, $b = x_4 + x_2 - x_1$ and $c = x_5 - 2x_1$. Thus, $\mathbf{F}^5 \subset U \oplus W$. Therefore, $U \oplus W = \mathbf{F}^5$.

Exercise 22
Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution

Consider the subspaces

$$\begin{aligned}
W_1 &= \{(0,0,a,0,0) \in \mathbf{F}^5 : a \in \mathbf{F}\} \\
W_2 &= \{(0,0,0,b,0) \in \mathbf{F}^5 : b \in \mathbf{F}\} \\
W_3 &= \{(0,0,0,0,c) \in \mathbf{F}^5 : c \in \mathbf{F}\}
\end{aligned}$$

and their sum $U + W_1 + W_2 + W_3$. Let's first prove that it is actually a direct sum by focusing on the zero vector. Let

$$\begin{aligned}
u &= (x, y, x+y, x-y, 2x) \in U \\
w_1 &= (0,0,a,0,0) \in W_1 \\
w_2 &= (0,0,0,b,0) \in W_2 \\
w_3 &= (0,0,0,0,c) \in W_3
\end{aligned}$$

be arbitrary vectors in their respective sets such that

$$(0,0,0,0,0) = u + w_1 + w_2 + w_3$$

This can be rewritten into the following system of equation:

$$\begin{cases} x = 0 \\ y = 0 \\ x + y + a = 0 \\ x - y + b = 0 \\ 2x + c = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} x = 0 \\ y = 0 \\ a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Hence, $u = w_1 = w_2 = w_3 = (0,0,0,0,0)$. Therefore, since the zero vector can only be written as the sum of zero vectors, the sum is direct.

Let's now show that $U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbf{F}^5$. Obviously, since $U \oplus W_1 \oplus W_2 \oplus W_3$ is a subspace of $\mathbf{F}^5$, we have $U \oplus W_1 \oplus W_2 \oplus W_3 \subset \mathbf{F}^5$. Moreover, for any $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$\begin{aligned}
&(x_1, x_2, x_3, x_4, x_5) \\
&\quad = (x_1, x_2, [x_1 + x_2] + [x_3 - x_2 - x_1], [x_1 - x_2] + [x_4 + x_2 - x_1], 2x_1 + [x_5 - 2x_1]) \\
&\quad = (x_1, x_2, x_1 + x_2, x_1 - x_2, 2x_1) + (0, 0, x_3 - x_2 - x_1, x_4 + x_2 - x_1, x_5 - 2x_1) \\
&\quad = (x, y, x+y, x-y, 2x) + (0,0,a,b,c) \\
&\quad = (x, y, x+y, x-y, 2x) + (0,0,a,0,0) + (0,0,0,b,0) + (0,0,0,0,c) \\
&\quad \in U \oplus W
\end{aligned}$$

where $x = x_1$, $y = x_2$, $a = x_3 - x_2 - x_1$, $b = x_4 + x_2 - x_1$ and $c = x_5 - 2x_1$. Thus, $\mathbf{F}^5 \subset U \oplus W_1 \oplus W_2 \oplus W_3$. Therefore, $U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbf{F}^5$.

Exercise 23
Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Solution
Consider the following counterexample:

$$\begin{aligned} V &= \mathbf{R}^2 \\ V_1 &= \{(0, x) : x \in \mathbf{R}\} \\ V_2 &= \{(x, x) : x \in \mathbf{R}\} \\ U &= \{(x, 0) : x \in \mathbf{R}\} \end{aligned}$$

I will not prove that V_1, V_2 and U are subspaces of V because it is not goal of this exercice. Notice that

$$V_1 + U = \{(x, y) : x, y \in \mathbf{R}\} = \mathbf{R}^2 = V$$

Moreover, for any arbitrary $u = (0, y) \in V_1$ and $y = (x, 0) \in U$, if

$$(0, 0) = u + v = (x, y)$$

then it follows that $u = v = (0, 0)$ since $x = y = 0$. Hence, $V_1 \oplus U = V$. Similarly, let's show that $V_2 \oplus U = V$. To do so, let's first prove that $V_2 + U = V$. Since $V_2 + U \subset V$, it suffices to prove that $V \subset V_2 + U$. Let (a, b) be an arbitrary element in V, then we have

$$(x, y) = (x - y, 0) + (y, y) \in V_2 + U$$

Hence, $V_2 + U = V$. To prove that the sum is direct, let $(x, x) \in V_2$ and $(y, 0) \in U$ such that

$$(x + y, x) = (x, x) + (y, 0) = (0, 0)$$

Since it follows that $x = 0$ and $y = 0$, then it follows that the zero vector can only be written as a sum of two zero vectors in $V_2 + U$. Thus, $V_2 \oplus U = V$. However, notice that $V_1 \neq V_2$ since $(1, 1) \in V_2$ but $(1, 1) \notin V_1$.

Exercise 24
A function $f : \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Solution
First, let's show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$. Since $V_e \oplus V_o \subset \mathbf{R}^{\mathbf{R}}$, then it suffices to prove that $\mathbf{R}^{\mathbf{R}} \subset V_e \oplus V_o$. Given an arbitrary function $f \in \mathbf{R}^{\mathbf{R}}$, define

$$f_e(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in \mathbf{R}$. Notice that for all $x \in \mathbf{R}$, we have

$$\begin{aligned} f_e(-x) &= \frac{f((-x)) + f(-(-x))}{2} \\ &= \frac{f(-x) + f(x)}{2} \\ &= f_e(x) \end{aligned}$$

and

$$\begin{aligned} f_o(-x) &= \frac{f((-x)) - f(-(-x))}{2} \\ &= \frac{f(-x) - f(x)}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= f_o(x) \end{aligned}$$

which proves that $f_e \in V_e$ and $f_o \in V_o$. Moreover, for all $x \in \mathbf{R}$

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$

so $f = f_e + f_o \in V_e + V_o$. Therefore, $\mathbf{R}^{\mathbf{R}} = V_e + V_o$ since we just proved that $\mathbf{R}^{\mathbf{R}} \subset V_e + V_o$. Let's now show that the sum is direct by proving that the zero function can be represented as $f_e + f_o$ where $f_e \in V_e$ and $f_o \in V_o$ only when $f_e = f_o \equiv 0$. To prove this, consider two arbitray functions $f_e \in V_e$ and $f_o \in V_o$ such that

$$f_e(x) + f_o(x) = 0$$

for all $x \in \mathbf{R}$. Then, given any $y \in \mathbf{R}$, we have

$$f_e(y) + f_o(y) = 0$$

and

$$f_e(-y) + f_o(-y) = 0 \implies f_e(y) - f_o(y) = 0$$

by plugging-in $x = y$ and $x = -y$ into our previous equation. Adding the two equations gives us

$$\begin{aligned} [f_e(y) + f_o(y)] + [f_e(y) - f_o(y)] = 0 &\implies 2f_e(y) = 0 \\ &\implies f_e(y) = 0 \end{aligned}$$

It follows that $f_o(y) = 0$ as well since $f_e(y) + f_o(y) = 0$. Thus, since it holds for all $y \in \mathbf{R}$, then $f_e = f_o \equiv 0$. Therefore, $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Chapter 2

Finite-Dimensional Vector Spaces

2A Span and Linear Independence

Exercise 1
Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

Solution
Consider the following list of vectors: $(-1, 0, 1)$, $(0, -1, 1)$, $(1, 1, -2)$ and $(-1, 1, 0)$. To prove that it spans the given set, take an arbitrary $(x, y, z) \in \mathbf{F}^3$ such that $x + y + z = 0$ and notice that

$$(x, y, z) = (-x)(-1, 0, 1) + (-y)(0, -1, 1) + 0(1, 1, -2) + 0(-1, 1, 0)$$

Hence, the given set is in the span of the four vectors. Moreover, any element is the span of the four vectors is in the given set since the four vectors are in the set and the set is closed under linear combinations.

Exercise 2
Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution
Let's prove it. Define the vectors

$$\begin{aligned} u_1 &= v_1 - v_2 \\ u_2 &= v_2 - v_3 \\ u_3 &= v_3 - v_4 \\ u_4 &= v_4 \end{aligned}$$

and B as the set containing these four vectors. To show that B spans V, we need to prove that for any element $v \in V$, there exists a linear combination of the elements in B equal to v. To do so, let $v \in V$, since v_1, v_2, v_3, v_4 spans V, then there exist coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{F}$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

Now, notice that we can write v_1 as $u_1 + u_2 + u_3 + u_4$, v_2 as $u_2 + u_3 + u_4$, v_3 as $u_3 + u_4$ and v_4 simply as u_4. Hence:

$$\begin{aligned} v &= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 \\ &= \alpha_1(u_1 + u_2 + u_3 + u_4) + \alpha_2(u_2 + u_3 + u_4) + \alpha_3(u_3 + u_4) + \alpha_4 u_4 \\ &= \alpha_1 u_1 + (\alpha_1 + \alpha_2)u_2 + (\alpha_1 + \alpha_2 + \alpha_3)u_3 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)u_4 \end{aligned}$$

Thus, we get that v can be written as a linear combination of the vectors in B. Therefore, B spans V.

Exercise 3
Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + ... + v_k.$$

Show that $\text{span}(v_1, ..., v_m) = \text{span}(w_1, ..., w_m)$.

Solution
First, notice that for all $k \in \{2, ..., m\}$, we have

$$v_k = w_k - w_{k-1} \in \text{span}(w_1, ..., w_m)$$

(for $k = 1$, $v_1 = w_1 \in \text{span}(w_1, ..., w_m)$). Hence, since $\text{span}(v_1, ..., v_m)$ is the smallest subspace containing the vectors $v_1, ..., v_m$, and $\text{span}(w_1, ..., w_m)$ is a subspace that contains the vectors $v_1, ..., v_m$, then $\text{span}(v_1, ..., v_m) \subset \text{span}(w_1, ..., w_m)$.
Similarly, by definition, for all $k \in \{1, ..., m\}$, we have

$$w_k = v_1 + ... + v_k \in \text{span}(v_1, ..., v_m)$$

Hence, since $\text{span}(w_1, ..., w_m)$ is the smallest subspace containing the vectors $w_1, ..., w_m$, and $\text{span}(v_1, ..., v_m)$ is a subspace that contains the vectors $w_1, ..., w_m$, then $\text{span}(w_1, ..., w_m) \subset \text{span}(v_1, ..., v_m)$. Therefore, $\text{span}(v_1, ..., v_m) = \text{span}(w_1, ..., w_m)$.

Exercise 4

(a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Solution

(a) Consider the list containing the single vector v_0. By definition, the list is linearly independent if and only if the only choice of scalars in a linear combination of the vectors in the list that is equal to zero is all of the scalars being equal to zero. In our case, this is equivalent to saying that $\alpha v_0 = 0$ only when $\alpha = 0$. However, if it holds, then v_0 cannot be the zero vector since otherwise, $\alpha v_0 = 0$ even when $\alpha \neq 0$. Similarly, if v_0 is not the zero vector then $\alpha v_0 = 0$ can only happen when $\alpha = 0$. Thus, $\alpha v_0 = 0$ only when $\alpha = 0$ is equivalent to $v_0 \neq 0$. Therefore, the list is linearly independent if and only if v_0 is not the zero vector.

(b) For this one, let's prove the converse equivalence: the list is linearly dependent if and only if one vector is a scalar multiple of the other. To do so, suppose that the list is linearly dependent, then there exist scalars $\alpha, \beta \in \mathbf{F}$ such that

$$\alpha v_1 + \beta v_2 = 0$$

but not all scalars are zero. Suppose without loss of generality that $\alpha \neq 0$, then we can rewrite the previous equation as

$$v_1 = -\frac{\beta}{\alpha} v_2$$

Thus, v_1 is a scalar multiple of v_2.
For the reverse implication, suppose that one of the vectors is a multiple of the other. Without loss of generality, suppose that v_1 is a scalar multiple of v_2, then there exists a scalar $\alpha \in \mathbf{F}$ such that $v_1 = \alpha v_2$. But notice that we can rewrite the previous equation as follows:

$$(1)v_1 + (-\alpha)v_2 = 0$$

Since $1 \neq 0$, then there exists a non-trivial linear combination of the vectors in that list that is equal to zero. Thus, the list is linearly dependent.

Exercise 5
Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

Solution
If we take $t = 55/2$, then we can write the vector $(5, 9, t)$ as $\alpha(3, 1, 4) + \beta(2, -3, 5)$ where $\alpha = 15/2$ and $\beta = -1/2$. Hence,

$$\alpha(3, 1, 4) + \beta(2, -3, 5) + (-1)(5, 9, t) = 0$$

is a non-trivial linear combination. Thus, with such a value of t, the three vectors are not linearly independent.

Exercise 6
Show that the list (2,3,1), (1,-1,2),(7,3,c) is linearly dependent in $\mathbf{F}^3$ if and only if

$c = 8$.

Solution

($\Longrightarrow$) Suppose that the list (2,3,1), (1,-1,2),(7,3,c) is linearly dependent in $\mathbf{F}^3$, then there exist scalars $\alpha, \beta, \gamma \in \mathbf{F}$ not all zero such that

$$\alpha(2,3,1) + \beta(1,-1,2) + \gamma(7,3,c) = 0$$

If $\gamma = 0$, then we get that a (2,3,1) and (1,-1,2) are linearly dependent. However, this is impossible since (2,3,1) is not a scalar multiple of (1,-1,2) and vice versa. Thus, γ must be non-zero. Thus:

$$\begin{aligned} \alpha(2,3,1) + \beta(1,-1,2) + \gamma(7,3,c) = 0 &\implies \gamma(7,3,c) = -\alpha(2,3,1) - \beta(1,-1,2) \\ &\implies (7,3,c) = -\frac{\alpha}{\gamma}(2,3,1) - \frac{\beta}{\gamma}(1,-1,2) \end{aligned}$$

Hence, if we let $a = -\alpha/\gamma$ and $b = -\beta/\gamma$, then

$$(7,3,c) = a(2,3,1) + b(1,-1,2)$$

which can be written as the following system of equation:

$$\begin{cases} 2a + b = 7 \\ 3a - b = 3 \\ a + 2b = c \end{cases}$$

To solve the system, we can add equation 1 and 2 to get $5a = 10$. It follows that $a = 2$. If we plug-in $a = 2$ is equation 2, we get $6 - b = 3$ so $b = 3$. Thus, we get that $c = a + 2b = 2 + 2 \cdot 3 = 8$.

($\Longleftarrow$) Suppose that $c = 8$. Using our work from the previous implication gives us

$$(7,3,c) = 2(2,3,1) + 3(1,-1,2)$$

which can be rearranged as

$$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,c) = 0$$

Thus, since there exists a non-trivial linear combination that is equal to zero, then the three vectors are linearly dependent.

Exercise 7

(a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1+i, 1-i$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1+i, 1-i$ is linearly dependent.

Solution

(a) By contradiction, suppose that $1+i, 1-i$ is linearly dependent, then using Exercise 4.(b), we know that there is a scalar $\alpha \in \mathbf{R}$ such that

$$1 + i = \alpha(1 - i) = \alpha + (-\alpha)i$$

But by the unique representation of complex numbers in the form $a + ib$, we get that $\alpha = 1$ and $-\alpha = 1$, a contradiction. Thus, the list $1+i, 1-i$ is linearly independent.

(b) Simply notice that

$$(1 + i) + (-i)(1 - i) = 0$$

even if the scalars are not all zero. Therefore, the list $1+i, 1-i$ is linearly dependent.

Exercise 8

Suppose $v_1, v_2, v_3; v_4$ is linearly independent in V. Prove that

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Solution

To prove that $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent, take arbitrary scalars $a, b, c, d \in \mathbf{F}$ such that

$$a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = 0$$

and prove that $a = b = c = d = 0$. First, notice that we can rearrange the previous equation as follows:

$$\begin{aligned} & a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = 0 \\ \implies & av_1 - av_2 + bv_2 - bv_3 + cv_3 - cv_4 + dv_4 = 0 \\ \implies & av_1 + (b - a)v_2 + (c - b)v_3 + (d - c)v_4 = 0 \end{aligned}$$

But since the list v_1, v_2, v_3, v_4 is linearly independent, then all coefficents in the last equation must be zero. Hence, we get the following system of equation:

$$\begin{cases} a = 0 \\ b - a = 0 \\ c - b = 0 \\ d - c = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} a = 0 \\ a = b = c = d \end{cases}$$

It follows that $a = b = c = d = 0$. Therefore, the given list is linearly independent.

Exercise 9
Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V, then
$$5v_1 - 4v_2, v_2, v_3, ..., v_m$$
is linearly independent.

Solution
To prove that $5v_1 - 4v_2, v_2, v_3, ..., v_m$ is linearly independent, take arbitrary scalars $a_1, ..., a_m \in \mathbf{F}$ such that

$$a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + ... + a_mv_m = 0$$

and prove that $a_1 = a_2 = ... = a_m = 0$. First, notice that we can rearrange the previous equation as follows:

$$\begin{aligned} & a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + ... + a_mv_m = 0 \\ \implies & 5a_1v_1 - 4a_1v_2 + a_2v_2 + a_3v_3 + ... + a_mv_m = 0 \\ \implies & 5a_1v_1 + (a_2 - 4a_1)v_2 + a_3v_3 + ... + a_mv_m = 0 \end{aligned}$$

But since the list $v_1, v_2, ..., v_m$ is linearly independent, then all coefficents in the last equation must be zero. Hence, we get the following system of equation:

$$\begin{cases} 5a_1 = 0 \\ a_2 - 4a_1 = 0 \\ a_3 = 0 \\ \vdots \\ a_m = 0 \end{cases}$$

which is equivalent to $a_1 = a_2 = ... = a_m = 0$ by solving the system. Therefore, the given list is linearly independent.

Exercise 10
Prove or give a counterexample: If $v_1, v_2, ..., v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, ..., \lambda v_m$ is linearly independent.

Solution
To prove that $\lambda v_1, \lambda v_2, ..., \lambda v_m$ is linearly independent, take arbitrary scalars $a_1, ..., a_m \in \mathbf{F}$ such that
$$a_1\lambda v_1 + a_2\lambda v_2 + ... + a_m\lambda v_m = 0$$
and prove that $a_1 = a_2 = ... = a_m = 0$. But since the list $v_1, v_2, ..., v_m$ is linearly independent, then all coefficents in front of the a_i's the last equation must be zero. Hence, we get the following system of equation:

$$\begin{cases} \lambda a_1 = 0 \\ \lambda a_2 = 0 \\ \vdots \\ \lambda a_m = 0 \end{cases}$$

Using the fact that $\lambda \neq 0$, we can divide each equation in the system to get $a_1 = a_2 = ... = a_m = 0$. Therefore, the given list is linearly independent.

Exercise 11
Prove or give a counterexample: If $v_1, ..., v_m$ and $w_1, ..., w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, ..., v_m + w_m$ is linearly independent.

Solution
Consider the following counterexample: take any list $v_1, v_2, ..., v_m$ of linearly independent vectors. We know from Exercise 10 that if we take $\lambda = -1$, then $-v_1, -v_2..., -v_m$ is also linearly independent. However, the list $v_1 + (-v_1), ..., v_m + (-v_m)$ is precisely equal to the list containing m zero vectors. Thus, $v_1+(-v_1), ..., v_m+(-v_m)$ is not linearly independent since any linear combination of the vectors in that list gives the zero vector, even if some scalars are non-zero.

Exercise 12
Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.

Solution
Suppose that $v_1+w, ..., v_m+w$ is linearly dependent, then there must be some scalars $a_1, ..., a_m \in \mathbf{F}$ not all zero such that

$$a_1(v_1 + w) + a_2(v_2 + w) + ... + a_m(v_m + w) = 0$$

But notice that we can rewrite the previous equation as follows:

$$(a_1 + a_2 + ... + a_m)w = a_1(-v_1) + a_2(-v_2) + ... + a_m(-v_m)$$

Define $\alpha = a_1 + a_2 + ... + a_m$ so we can rewrite again the equation as

$$\alpha w = a_1(-v_1) + a_2(-v_2) + ... + a_m(-v_m)$$

and suppose by contradiction that $\alpha = 0$. If $\alpha = 0$, then the previous equation becomes:

$$a_1(-v_1) + a_2(-v_2) + ... + a_m(-v_m) = 0$$

Using Exercise 10, by linear independence of $v_1, ..., v_m$ and by taking $\lambda = -1$, we get that the list $-v_1, ..., -v_m$ is linearly independent as well. Hence, the previous equation implies that $a_1 = a_2 = ... = a_m = 0$. But this is a contradiction with the fact that the scalars $a_1, ..., a_m$ are not all zero. Thus, by contradiction, $\alpha \neq 0$. Hence:

$$\begin{aligned} \alpha w &= a_1(-v_1) + a_2(-v_2) + ... + a_m(-v_m) \\ \implies w &= \left(-\frac{a_1}{\alpha}\right) v_1 + \left(-\frac{a_2}{\alpha}\right) v_2 + ... + \left(-\frac{a_m}{\alpha}\right) v_m \end{aligned}$$

which proves that $w \in \text{span}(v_1, ..., v_m)$.

Exercise 13
Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, ..., v_m, w \text{ is linearly independent} \iff w \notin \text{span}(v_1, ..., v_m).$$

Solution
($\implies$) By contrapositive, suppose that $w \in \text{span}(v_1, ..., v_m)$, then there exist coefficients coefficents $a_1, ..., a_m \in \mathbf{F}$ such that

$$w = a_1v_1 + ... + a_mv_m$$

which can be rewritten as

$$a_1v_1 + ... + a_mv_m + (-1)w = 0$$

Notice that not all scalars in this linear combination are zero. It follows that the list $v_1, ..., v_m, w$ is linearly dependent.

($\impliedby$) Again, by contrapositive, suppose that $v_1, ..., v_m, w$ is linearly dependent, then there exist coefficients $a_1, ..., a_m, \alpha \in \mathbf{F}$ not all zero such that

$$a_1v_1 + ... + a_mv_m + \alpha w = 0$$

By contradiction, if $\alpha = 0$, then we get

$$a_1v_1 + ... + a_mv_m = 0$$

which implies, by linear independence of $v_1, ..., v_m$ that all coefficents are zero. A contradiction since we know that at least one of them is non-zero. Thus, by contradiction, we have that $\alpha \neq 0$:

$$\begin{aligned} & a_1v_1 + ... + a_mv_m + \alpha w = 0 \\ \implies & -\alpha w = a_1v_1 + ... + a_mv_m \\ \implies & w = \left(-\frac{a_1}{\alpha}\right)v_1 + \left(-\frac{a_2}{\alpha}\right)v_2 + ... + \left(-\frac{a_m}{\alpha}\right)v_m \end{aligned}$$

which proves that $w \in \text{span}(v_1, ..., v_m)$.

Exercise 14
Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + ... + v_k.$$

Show that the list $v_1, ...v_m$ is linearly independent if and only if the list $w_1, ..., w_m$ is linearly independent.

Solution
($\implies$) Suppose that $v_1, ...v_m$ is linearly independent, then for any scalars $a_1, ..., a_m \in \mathbf{F}$ such that

$$a_1w_1 + ... + a_mw_m = 0,$$

we can rewrite the previous equation as

$$\begin{aligned} & a_1w_1 + a_1w_2 + ... + a_mw_m = 0 \\ \implies & a_1v_1 + a_2(v_1 + v_2)... + a_m(v_1 + ... + v_m) = 0 \\ \implies & a_1v_1 + a_2v_1 + a_2v_2... + a_mv_1 + ... + a_mv_m = 0 \\ \implies & (a_1 + ... + a_m)v_1 + ... + a_mv_m = 0 \end{aligned}$$

By linear independence of $v_1, ...v_m$, this implies

$$\begin{cases} a_1 + ... + a_m = 0 \\ a_2 + ... + a_m = 0 \\ \vdots \\ a_{m-1} + a_m = 0 \\ a_m = 0 \end{cases}$$

We can easily solve this system of equation and get $a_1 = a_2 = ... = a_m = 0$. Thus, $w_1, ..., w_m$ is linearly independent.

($\Longleftarrow$) Suppose that $w_1, ...w_m$ is linearly independent, then for any scalars $a_1, ..., a_m \in \mathbf{F}$ such that

$$a_1v_1 + ... + a_mv_m = 0,$$

we can rewrite the previous equation as

$$\begin{aligned} & a_1v_1 + a_1v_2 + ... + a_mv_m = 0 \\ \Longrightarrow\ & a_1w_1 + a_2(w_2 - w_1)... + a_m(w_m - w_{m-1}) = 0 \\ \Longrightarrow\ & a_1w_1 + a_2w_2 - a_2w_1... + a_mw_m - a_mw_{m-1} = 0 \\ \Longrightarrow\ & (a_1 - a_2)w_1 + (a_2 - a_3)w_2 + ... + a_mw_m = 0 \end{aligned}$$

By linear independence of $v_1, ...v_m$, this implies

$$\begin{cases} a_1 - a_2 = 0 \\ a_2 - a_3 = 0 \\ \vdots \\ a_m = 0 \end{cases}$$

We can easily solve this system of equation and get $a_1 = a_2 = ... = a_m = 0$. Thus, $v_1, ..., v_m$ is linearly independent.

Exercise 15
Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Solution
We already know a list of size 5 that spans $\mathcal{P}_4(\mathbf{F})$: $1,\ x,\ x^2,\ x^3,\ x^4$. Hence, any list of linearly independent polynomials must have a length smaller than 5. In particular, a linearly independent list of 6 polynomials cannot exist in $\mathcal{P}_4(\mathbf{F})$.

Exercise 16
Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Solution
We already know a linearly independent list of size 5 in $\mathcal{P}_4(\mathbf{F})$: $1,\ x,\ x^2,\ x^3,\ x^4$. Hence, any list of polynomials that spans $\mathcal{P}_4(\mathbf{F})$ must have a length greater or equal

to 5. In particular, a spanning list of 4 polynomials cannot exist in $\mathcal{P}_4(\mathbf{F})$.

Exercise 17
Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, ...$ of vectors in V such that $v_1, ..., v_m$ is linearly independent for every positive integer m.

Solution
($\implies$) Suppose that V is infinite-dimensional, let's define a sequence $v_1, v_2, ...$ of vectors recursively as follows. First, V cannot be the trivial vector space $\{0\}$ because the trivial vector space has a list of vectors that spans it: the list containing the zero vector only. Hence, $\{0\}$ is finite-dimensional so it cannot be equal to V. Thus, define the vector $v_1 \in V$ as any non-zero vector. Obviously, the list v_1 of length 1 is linearly independent by Exercise 4.(a).

Recursively, suppose that we have a linearly independent list $v_1, v_2, ..., v_k$ of vectors in V for some natural number k. Since V is infinite-dimensional, then the given list don't span V. It follows that there exists a vector v_{k+1} such that

$$v_{k+1} \notin \text{span}(v_1, ..., v_k)$$

Therefore, by Exercise 13, the list $v_1, v_2, ..., v_k, v_{k+1}$ is linearly independent as well. Now that we defined our sequence recursively, notice that by construction, for all positive integer m, the list $v_1, ..., v_m$ is linearly independent.

($\impliedby$) Suppose there is a sequence $v_1, v_2, ...$ of vectors in V such that $v_1, ..., v_m$ is linearly independent for every positive integer m. By contradiction, suppose that V is finite-dimensional, then there is a list $w_1, ..., w_N$ that spans V for some positive integer N. However, if we let $m = N + 1$, then our assumption implies that there is a linearly independent list of length $N + 1$ in V. This is in contradiction with Theorem 2.22 so V must be infinite-dimensional.

Exercise 18
Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Solution
Consider the sequence $v_1, v_2, ...$ of sequences in $\mathbf{F}^\infty$ defined as follows: for all positive integer k, define $v_k \in \mathbf{F}^\infty$ as the sequence with all terms equal to 0, except the kth term which is equal to 1. For all positive integer m, consider the list $v_1, ..., v_m$. Notice that by construction, for all scalars $a_1, ..., a_m \in \mathbf{F}$, the linear combination

$$a_1 v_1 + a_2 v_2 + ... + a_m v_m$$

is simply the sequence with terms $(a_1, a_2, ..., a_{m-1}, a_m, 0, 0, 0, ...)$. It follows that this linear combination is equal to the zero sequence if and only if all the coefficients a_i are zero. Thus, the list $v_1, ..., v_m$ is linearly independent. Therefore, by Exercise 17, $\mathbf{F}^\infty$ is infinite-dimensional.

Exercise 19
Prove that the real vector space of all continuous real-valued functions on the interval [0,1] is infinite-dimensional.

Solution

Consider the sequence $v_1, v_2, ...$ of continuous functions on the interval [0,1] defined by $v_k : x \mapsto x^k$ for all positive integer k. Let m be a positive integer and consider the list $v_1, ..., v_m$. To show that this list is linearly independent, take arbitrary scalars $a_1, ..., a_m \in \mathbf{R}$ such that

$$a_1 v_1 + a_2 v_2 + ... + a_m v_m = 0$$

By definition of the v_i's, the previous equation implies that

$$a_1 x + a_2 x^2 + ... + a_m x^m = 0$$

for all $x \in [0, 1]$. If we consider the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = a_1 x + a_2 x^2 + ... + a_m x^m$, then f is differentiable on [0,1]. Thus, differentiating on both sides gives us

$$a_1 + a_2 x + ... + a_m x^{m-1} = 0$$

for all $x \in [0, 1]$. By plugging-in $x = 0$, we get $a_1 = 0$. If we repeat this process m times, we get $a_1 = a_2 = ... = a_m = 0$. Thus, the list is linearly independent. Therefore, by Exercise 17, the real vector space of all continuous real-valued functions on the interval [0,1] is infinite-dimensional.

Exercise 20

Suppose $p_0, p_1, ..., p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, ..., m\}$. Prove that $p_0, p_1, ..., p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Solution

We already know a list of size $m+1$ that spans $\mathcal{P}_m(\mathbf{F})$: 1, x, x^2, ..., x^m. Hence, any list of linearly independent polynomials must have a length smaller than $m+1$. By contradiction, suppose that the given list $p_0, p_1, ..., p_m$ is linearly independent and consider the constant polynomial $p \equiv 1$. Notice that for all scalars $a_0, ..., a_m \in \mathbf{F}$, the new polynomial

$$a_0 p_0 + a_1 p_1 + ... + a_m p_m$$

also vanishes at $x = 2$. Hence, for any linear combination of the list $p_0, p_1, ..., p_m$, the polynomial p must be different from this linear combination they don't evaluate to the same number at $x = 2$. It follows that

$$p \notin \operatorname{span}(p_0, p_1, ..., p_m)$$

Thus, by Exercise 13, the list $p_0, ..., p_m, p$ is linearly independent. However, this list has length $m + 2$ so we get a contradiction. Therefore, $p_0, p_1, ..., p_m$ cannot be linearly independent in $\mathcal{P}_m(\mathbf{F})$.

2B Bases

Exercise 1
Find all vector spaces that have exactly one basis.

Solution
Let V be a vector space over the field $\mathbf{F}$ that have exactly one basis. First, let's show that the basis must contain only one vector. To do so, suppose that the basis is the following list: $b_1, b_2, ..., b_n$ with $n \geq 2$. Consider the new list $b_1, b_1 + b_2, b_3, ..., b_n$ where b_2 is replaced by $b_1 + b_2$. Notice that this new list is also linearly independent because for all scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$:

$$\begin{aligned} & \alpha_1 b_1 + \alpha_2(b_1 + b_2) + \alpha_3 b_3 + ... + \alpha_n b_n = 0 \\ \implies & (\alpha_1 + \alpha_2)b_1 + \alpha_2 b_2 + \alpha_3 b_3 + ... + \alpha_n b_n = 0 \\ \implies & \alpha_1 + \alpha_2 = 0 \quad \text{and} \quad \alpha_i = 0,\ i = 2, ..., n \\ \implies & \alpha_i = 0,\ i = 1, ..., n \end{aligned}$$

Moreover, notice that for all $u \in V$, since $b_1, b_2, ..., b_n$ is a basis for V, then there exist scalars $a_1, ..., a_n \in \mathbf{F}$ such that

$$u = a_1 b_1 + a_2 b_2 + ... a_n b_n$$

Hence,

$$u = (a_1 - a_2)b_1 + a_2(b_1 + b_2) + ... + a_n b_n \in \operatorname{span}(b_1, b_1 + b_2, ..., b_n)$$

It follows that the new list also spans V. Therefore, the new list is a basis. But since V has a unique basis, then the new list must be equal to the first one. In particular, the vector $b_1 + b_2$ must be in the list $b_1, b_2, ..., b_n$ as well. Hence, there is a $i \in \{1, ..., n\}$ such that $b_1 + b_2 = b_i$. If $i = 1$, then we get $b_2 = 0$, a contradiction with the fact that $b_1, ..., b_n$ is linearly independent. Similarly, $i = 2$ would lead to the same contradiction. Now, if $i > 2$, then we can rearrange the equation to $b_1 + b_2 - b_i = 0$. Again, this is impossible since the b_j's are linearly independent. Therefore, by contradiction, the unique basis for V must contain at most 1 vector. Now, we have two cases, either the basis is a list of length 0 or a list of length 1. If the list has length 0, then V must be the trivial vector space $\{0\}$. Indeed, the trivial vector space over any field has a unique basis, the list of length 0.
Suppose now that the basis for V is a list of length 1, call $v_0 \neq 0$ the unique vector in the basis. Let α be a non-zero scalar and consider the list containg the vector αv_0. Since nor α or v_0 is zero, then αv_0 is non-zero. Thus, this new list is linearly independent. Moreover, for any $u \in V$, since the list v_0 spans V, then there is a $\lambda \in \mathbf{F}$ such that $u = \lambda v_0$. Thus:

$$u = \frac{\lambda}{\alpha}(\alpha v_0) \in \operatorname{span}(\alpha v_0)$$

It follows that the list αv_0 is also a basis for V. By uniqueness, we must have $\alpha v_0 = v_0$. But since $v_0 \neq 0$:

$$\begin{aligned} \alpha v_0 = v_0 & \implies \alpha v_0 - v_0 \\ & \implies (\alpha - 1)v_0 = 0 \\ & \implies \alpha = 1 \end{aligned}$$

In conclusion, the only non-zero element in $\mathbf{F}$ is 1 so $\mathbf{F}$ must be the field containing two elements.
From these results, we get that the only vector spaces with exactly one basis are either the trivial vector spaces on any field, or the vectors spaces over the field with two elements with a basis containing only one element.

Exercise 2
Verifiy all assertions in Example 2.27.

Solution

(a) Denote by $e_i \in \mathbf{F}^n$ the vector with all entries equal to zero except the ith entry which is equal to 1. Let's show that the list $e_1, ..., e_n$ is a basis for $\mathbf{F}^n$. First, let's show that it spans $\mathbf{F}^n$. Take an arbitrary $(\alpha_1, ..., \alpha_n) \in \mathbf{F}^n$ and notice that

$$\begin{aligned}(\alpha_1, ..., \alpha_n) &= \alpha_1(1, ..., 0) + ... + \alpha_n(0, ..., 1)\\ &= \alpha_1 e_1 + ... + \alpha_n e_n\\ &\in \operatorname{span}(e_1, ..., e_n)\end{aligned}$$

Hence, the list spans F^n. Moreover, for any scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$, we get

$$\begin{aligned}&\alpha_1 e_1 + ... + \alpha_n e_n = (0, ..., 0)\\ \implies &\alpha_1(1, ..., 0) + ... + \alpha_n(0, ..., 1) = (0, ..., 0)\\ \implies &(\alpha_1, ..., \alpha_n) = (0, ..., 0)\\ \implies &\alpha_i = 0 \text{ for all } i = 1, ..., n\end{aligned}$$

Therefore, the list $e_1, ..., e_n$ is a basis for $\mathbf{F}^n$.

(b) Let's show that the list $(1, 2), (3, 5)$ is a basis of $\mathbf{F}^2$. To do so, notice that for all $(a, b) \in \mathbf{F}^2$, we have

$$(a, b) = (3b - 5a)(1, 2) + (2a - b)(3, 5) \in \operatorname{span}((1, 2), (3, 5))$$

so the list spans $\mathbf{F}^2$. Moreover, for all scalars $\alpha, \beta \in \mathbf{F}$,

$$\begin{aligned}\alpha(1, 2) + \beta(3, 5) = (0, 0) &\implies (\alpha + 3\beta, 2\alpha + 5\beta) = (0, 0)\\ &\implies \begin{cases}\alpha + 3\beta = 0\\ 2\alpha + 5\beta = 0\end{cases}\\ &\implies \alpha = \beta = 0\end{aligned}$$

Thus, the list is linearly independent. Therefore, it is a basis for $\mathbf{F}^2$.

(c) Let's first prove that the list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$.

Notice that for all scalars $\alpha, \beta \in \mathbf{F}$:

$$\begin{aligned}
\alpha(1,2,-4)+\beta(7,-5,6)=0 &\implies (\alpha+7\beta, 2\alpha-5\beta, -4\alpha+6\beta)=0 \\
&\implies \begin{cases} \alpha+7\beta=0 \\ 2\alpha-5\beta=0 \\ -4\alpha+6\beta=0 \end{cases} \\
&\implies \begin{cases} \alpha+7\beta=0 \\ 4\alpha-10\beta=0 \\ -4\alpha+6\beta=0 \end{cases} \\
&\implies \begin{cases} \alpha+7\beta=0 \\ 4\beta=0 \end{cases} \\
&\implies \begin{cases} \alpha+7\beta=0 \\ \beta=0 \end{cases} \\
&\implies \alpha=\beta=0
\end{aligned}$$

It follows that the list is linearly independent. Now, to show that it doesn't span $\mathbf{F}^3$, notice that the standard basis of $\mathbf{F}^3$ is linearly independent list of length 3. Hence, any spanning list of $\mathbf{F}^3$ must have length bigger than or equal to 3. Since the given list has length 2, then it cannot span $\mathbf{F}^3$.

(d) First, recall from part (b) of this exercise that the list $(1,2),(3,5)$ spans $\mathbf{F}^2$. Hence, for all $(a,b) \in \mathbf{F}^2$, there exist scalars $\alpha, \beta \in \mathbf{F}$ such that

$$\begin{aligned}
(a,b) &= \alpha(1,2)+\beta(3,5) \\
&= \alpha(1,2)+\beta(3,5)+0(4,13) \\
&\in \operatorname{span}((1,2),(3,5),(4,13))
\end{aligned}$$

It follows that the list $(1,2),(3,5),(4,13)$ spans $\mathbf{F}^2$. However, it is not linearly independent because the standard basis of $\mathbf{F}^2$ is a spanning list of $\mathbf{F}^2$ of length 2. Hence, any linearly independent list in $\mathbf{F}^2$ must have length lesser than 2. Since our given set has length 3, then it cannot be linearly independent.

(e) Define the set

$$S = \{(x,x,y) \in \mathbf{F}^3 : x, y \in F\}$$

and notice that it is a subspace of F^3. Let's show that the list $(1,1,0),(0,0,1)$ is a basis of S. First, linear independence follows from the fact that for all $\alpha, \beta \in \mathbf{F}$:

$$\begin{aligned}
\alpha(1,1,0)+\beta(0,0,1)=0 &\implies (\alpha,\alpha,\beta)=0 \\
&\implies \begin{cases} \alpha=0 \\ \alpha=0 \\ \beta=0 \end{cases} \qquad \implies \alpha=\beta=0
\end{aligned}$$

To show that the list spans S, let (x,x,y) be an arbitrary element of S where $x, y \in F$, then:

$$(x,x,y) = x(1,1,0)+y(0,0,1) \in \operatorname{span}((1,1,0),(0,0,1))$$

Therefore, the given list is a basis of S.

(f) Define the set
$$S = \{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$$
and notice that it is a subspace of F^3. Let's show that the list $(1, -1, 0), (1, 0, -1)$ is a basis of S. First, linear independence follows from the fact that for all $\alpha, \beta \in \mathbf{F}$:
$$\begin{aligned} \alpha(1, -1, 0) + \beta(1, 0, -1) = 0 &\implies (\alpha + \beta, -\alpha, -\beta) = 0 \\ &\implies \begin{cases} \alpha + \beta = 0 \\ -\alpha = 0 \\ -\beta = 0 \end{cases} \implies \alpha = \beta = 0 \end{aligned}$$
To show that the list spans S, let (x, y, z) be an arbitrary element of S, then we know that $x + y + z = 0$ which implies that $z = -x - y$. Hence:
$$(x, y, z) = (-y)(1, -1, 0) + (x + y)(1, 0, -1) \in \operatorname{span}((1, -1, 0), (1, 0, -1))$$
Therefore, the given list is a basis of S.

(g) Consider the list $1, z, ..., z^m$ as elements of $\mathcal{P}_m(\mathbf{F})$. Notice that for all scalars $\alpha_0, \alpha_1, ..., \alpha_m \in \mathbf{F}$, we have
$$\alpha_0 + \alpha_1 z + ... + \alpha_m z^m = 0 \implies \alpha_i = 0,\ i = 0, 1, ..., m$$
Moreover, given a polynomial $\alpha_0 + \alpha_1 z + ... + \alpha_m z^m \in \mathcal{P}_m(\mathbf{F})$, it directly follows by the way it is written that it is a linear combination of the given list. Therefore, it is a basis of $\mathcal{P}_m(\mathbf{F})$.

Exercise 3

(a) Let U be the subspace of $\mathbf{R}^5$ defined by
$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$
Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

Solution

(a) Consider the list $u_1 = (3, 1, 0, 0, 0), u_2 = (0, 0, 7, 1, 0,), u_3 = (0, 0, 0, 0, 1)$ and let's prove that it is a basis for U. First, for any scalars $\alpha, \beta, \gamma \in \mathbf{R}$:
$$\begin{aligned} \alpha u_1 + \beta u_2 + \gamma u_3 = 0 &\implies (3\alpha, \alpha, 7\beta, \beta, \gamma) = 0 \\ &\implies \begin{cases} 3\alpha = 0 \\ \alpha = 0 \\ 7\beta = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \\ &\implies \alpha = \beta = \gamma = 0 \end{aligned}$$

Moreover, for any $(x_1, x_2, x_3, x_4, x_5) \in U$, we now that $x_1 = 3x_2$ and $x_3 = 7x_4$. Hence:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (3x_2, x_2, 7x_4, x_4, x_5) \\ &= x_2u_1 + x_4u_2 + x_5u_3 \\ &\in \operatorname{span}(u_1, u_2, u_3)\end{aligned}$$

Therefore, it is a basis of U.

(b) As in the proof of 2.32, consider the list $(3, 1, 0, 0, 0)$, $(0, 0, 7, 1, 0,)$, $(0, 0, 0, 0, 1)$, e_1, e_2, e_3, e_4, e_5 where e_1, e_2, e_3, e_4, e_5 is the standard basis for $\mathbf{R}^5$. This list is spanning $\mathbf{R}^5$ but it is not linearly independent. Hence, we can reduce it to a basis as follows. First, remove e_5 because it is already in the list. Keep the first three vectors since we know that they are linearly independent. The vector e_1 cannot be written as a linear combination of the first three vectors so keep it in the list. The vector e_2 can be written as

$$e_2 = (3, 1, 0, 0, 0) - 3e_1$$

so we don't keep it in the list. Similarly, e_3 is not in the span of the first four vectors in the list so we keep it, and e_4 can be written as

$$e_4 = (0, 0, 7, 1, 0) - 7e_3$$

so we remove it. Hence, our original list can be extended to a basis by adding the vectors e_1 and e_3.

(c) As in the proof of 2.33, take $W = \operatorname{span}(e_1, e_3)$, then it follows that $\mathbf{R}^5 = U \oplus W$.

Exercise 4

(a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

Solution

(a) Consider the list $u_1 = (1, 6, 0, 0, 0)$, $u_2 = (0, 0, -2, 1, 0,)$, $u_3 = (0, 0, -3, 0, 1)$ and let's prove that it is a basis for U. First, for any scalars $\alpha, \beta, \gamma \in \mathbf{R}$:

$$\begin{aligned}\alpha u_1 + \beta u_2 + \gamma u_3 = 0 &\implies (\alpha, 6\alpha, -2\beta - 3\gamma, \beta, \gamma) = 0 \\ &\implies \begin{cases}\alpha = 0 \\ 6\alpha = 0 \\ -2\beta - 3\gamma = 0 \\ \beta = 0 \\ \gamma = 0\end{cases} \\ &\implies \alpha = \beta = \gamma = 0\end{aligned}$$

Moreover, for any $(z_1, z_2, z_3, z_4, z_5) \in U$, we now that $6z_1 = z_2$ and $z_3 + 2z_4 + 3z_5 = 0$. Hence:

$$\begin{aligned}(z_1, z_2, z_3, z_4, z_5) &= (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5)\\ &= z_1u_1 + z_4u_2 + z_5u_3\\ &\in \text{span}(u_1, u_2, u_3)\end{aligned}$$

Therefore, it is a basis of U.

(b) As in the proof of 2.32, consider the list u_1, u_2, u_3, e_1, e_2, e_3, e_4, e_5 where e_1, e_2, e_3, e_4, e_5 is the standard basis for $\mathbf{C}^5$. This list is spanning $\mathbf{C}^5$ but it is not linearly independent. Hence, we can reduce it to a basis as follows. First, keep the first three vectors since we know that they are linearly independent. The vector e_1 cannot be written as a linear combination of the first three vectors so keep it in the list. The vector e_2 can be written as

$$e_2 = \frac{1}{6}(u_1 - e_1)$$

so we don't keep it in the list. Similarly, e_3 is not in the span of the first four vectors in the list so we keep it, and e_4 can be written as

$$e_4 = u_2 + 2e_3$$

so we remove it. Again, we also remove e_5 because

$$e_5 = u_3 + 3e_3$$

Hence, our original list can be extended to a basis by adding the vectors e_1 and e_3.

(c) As in the proof of 2.33, take $W = \text{span}(e_1, e_3)$, then it follows that $\mathbf{C}^5 = U \oplus W$.

Exercise 5

Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Solution

Since V is finite-dimensional, then U and W must be finite-dimensional as well by Proposition 2.25. Hence, let $u_1, ..., u_n$ be a list of vectors spanning U, and $w_1, ..., w_m$ be a list of vectors spanning W. Consider the list $u_1, ..., u_n, w_1, ..., w_m$ of vectors in $U \cup W$. Notice that for all $v \in V = U + W$, there exist vectors $u \in U$ and $w \in W$ such that $v = u + w$. Moreover, since the lists $u_1, ..., u_n$ and $w_1, ..., w_m$ are spanning their respective subspaces, then there exist scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$ and $\beta_1, ..., \beta_m \in \mathbf{F}$ such that

$$u = \alpha_1 u_1 + ... + \alpha_n u_n$$

and

$$w = \beta_1 w_1 + ... + \beta_m w_m$$

It follows that

$$v = \alpha_1 u_1 + ... + \alpha_n u_n + \beta_1 w_1 + ... + \beta_m w_m$$

so the list $u_1, ..., u_n, w_1, ..., w_m$ spans V. By Proposition 2.30, this list must contain a sublist that is a basis of V. Thus, such a sublist is indeed a basis of V consisting of vectors in $U \cup W$ by construction.

Exercise 6
Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbf{F})$.

Solution
Consider the following list of polynomials in $\mathcal{P}_3(\mathbf{F})$:

$$\begin{aligned} p_0(x) &= 1 & p_2(x) &= 1 + x + x^2 + x^3 \\ p_2(x) &= x & p_3(x) &= x^3 \end{aligned}$$

and notice that it contains no polynomials of degree 2. Let's show that it is a basis for $\mathcal{P}_3(\mathbf{F})$. To do so, consider the fact that $1, x, x^3 \in \operatorname{span}(p_0, p_1, p_2, p_3)$. Moreover,

$$x^2 = p_2 - p_0 - p_1 - p3 \in \operatorname{span}(p_0, p_1, p_2, p_3)$$

Hence, $\operatorname{span}(p_0, p_1, p_2, p_3)$ is a subspace that contains the standard basis of $\mathcal{P}_3(\mathbf{F})$. It follows that p_0, p_1, p_2, p_3 spans $\mathcal{P}_3(\mathbf{F})$. To prove the linear independence, simply take scalars $a_0, a_1, a_2, a_3 \in \mathbf{F}$ and notice that

$$\begin{aligned} a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 = 0 &\implies a_0 + a_1x + a_2(1 + x + x^2 + x^3) + a_3x^3 = 0 \\ &\implies (a_0 + a_2) + (a_1 + a_2)x + a_2x^2 + (a_3 + a_2)x^3 = 0 \\ &\implies \begin{cases} a_0 + a_2 = 0 \\ a_1 + a_2 = 0 \\ a_2 = 0 \\ a_3 + a_2 = 0 \end{cases} \\ &\implies a_0 = a_1 = a_2 = a_3 = 0 \end{aligned}$$

Therefore, p_0, p_1, p_2, p_3 is a basis of $\mathcal{P}_3(\mathbf{F})$ even if it contains no polynomials of degree 2.

Exercise 7
Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution
First, let's prove that it is linearly independent. Take scalars $a_1, a_2, a_3, a_4 \in \mathbf{F}$, by

linear independence of v_1, v_2, v_3, v_4:

$$\begin{aligned}
&a_1(v_1+v_2)+a_2(v_2+v_3)+a_3(v_3+v_4)+a_4v_4=0\\
\implies &a_1v_1+(a_1+a_2)v_2+(a_2+a_3)v_3+(a_3+a_4)v_4=0\\
\implies &\begin{cases} a_1=0\\ a_1+a_2=0\\ a_2+a_3=0\\ a_3+a_4=0\end{cases}\\
\implies &a_1=a_2=a_3=a_4=0
\end{aligned}$$

To prove that it spans V, take $v \in V$. Since v_1, v_2, v_3, v_4 spans V, then there exist scalars $a_1, a_2, a_3, a_4 \in \mathbf{F}$ such that

$$u = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$$

But since

$$\begin{aligned}
v_1 &= (v_1+v_2)-(v_2+v_3)+(v_3+v_4)-v_4\\
v_2 &= (v_2+v_3)-(v_3+v_4)+v_4
\end{aligned}$$

and

$$v_3 = (v_3+v_4)-v_4,$$

then

$$\begin{aligned}
u &= a_1v_1+a_2v_2+a_3v_3+a_4v_4\\
&= a_1[(v_1+v_2)-(v_2+v_3)+(v_3+v_4)-v_4]\\
&\quad + a_2[(v_2+v_3)-(v_3+v_4)+v_4]\\
&\quad + a_3[(v_3+v_4)-v_4]+a_4v_4\\
&= a_1(v_1+v_2)+(a_2-a_1)(v_2+v_3)\\
&\quad + (a_3-a_2+a_1)(v_3+v_4)+(a_4-a_3+a_2-a_1)v_4
\end{aligned}$$

which proves that it spans V. Therefore, the new list of vectors is also a basis of V.

Exercise 8
Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution
Consider the following counterexample: Take $V = \mathbf{R}^4$, v_1, v_2, v_3, v_4 be the standard basis and define

$$U = \operatorname{span}(v_1, v_2, (0,0,1,1))$$

Obviously, U is a subspace of V that contains v_1 and v_2. Moreover, if $v_3 \in U$, then there would be scalars $a, b, c \in \mathbf{R}$ such that

$$(0,0,1,0) = (a,b,c,c) \implies c = 1 \text{ and } c = 0,$$

a contradiction that shows that $v_3 \notin U$. Using a similar argument, $v_4 \notin U$ as well. However, v_1, v_2 is not a basis of U because it doesn't span it. Take for example

$(0, 0, 1, 1) \in U$ which is not in the span of v_1, v_2.

Exercise 9
Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + ... + v_k.$$

Show that $v_1, ..., v_m$ is a basis of V if and only if $w_1, ..., w_m$ is a basis of V.

Solution
Suppose that $v_1, ..., v_m$ is a basis, then it is linearly independent. By Section 2A Exercise 14, $w_1, ..., w_m$ must be linearly independent as well. Moreover, since $v_1, ..., v_m$ is a basis, then $\text{span}(v_1, ..., v_m) = V$. Again, using Section 2A Exercise 3, we have that $\text{span}(w_1, ..., w_m) = \text{span}(v_1, ..., v_m) = V$. It follows that $w_1, ..., w_m$ spans V. Therefore, $w_1, ..., w_m$ is a basis of V. All the arguments presented here prove the reverse implication as well.

Exercise 10
Suppose U and W are subspace of V such that $V = U \oplus W$. Suppose also that $u_1, ..., u_m$ is a basis of U and $w_1, ..., w_n$ is basis of W. Prove that

$$u_1, ..., u_m, w_1, ..., w_m$$

is a basis of V.

Solution
First, let's prove that $u_1, ..., u_m, w_1, ..., w_m$ spans V. Take an arbitrary $v \in V$, then there exist vectors $u \in U$ and $w \in W$ such that $v = u + w$. Since $u_1, ..., u_m$ is a basis of U and $w_1, ..., w_n$ is basis of W, then there exist scalars $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n \in \mathbf{F}$ such that

$$u = \alpha_1 u_1 + ... + \alpha_m u_m$$

and

$$w = \beta_1 w_1 + ... + \beta_n w_n$$

It follows that

$$v = \alpha_1 u_1 + ... + \alpha_m u_m + \beta_1 w_1 + ... + \beta_n w_n \in \text{span}(u_1, ..., u_m, w_1, ..., w_n)$$

which proves that the list spans V. Now, to prove the inear independence, take arbitrary scalars $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n \in \mathbf{F}$ and recall that $u + w = 0 \implies u = w = 0$ for all $u \in U$ and $w \in W$. Hence,

$$\alpha_1 u_1 + ... + \alpha_m u_m + \beta_1 w_1 + ... + \beta_n w_n = 0$$

implies

$$\begin{cases} \alpha_1 u_1 + ... + \alpha_m u_m = 0 \\ \beta_1 w_1 + ... + \beta_n w_n = 0 \end{cases}$$

But since the lists $u_1, ..., u_m$ and $w_1, ..., w_n$ are linearly independent, then we get

$$\alpha_1 = ... = \alpha_m = \beta_1 = ... = \beta_n = 0$$

Therefore, the list $u_1, ..., u_m, w_1, ..., w_m$ is a basis of V.

Exercise 11
Suppose V is a real vector space. Show that if $v_1, ..., v_n$ is a basis of V (as a real vector space), then $v_1, ..., v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

Solution
First, let's show that $v_1, ..., v_n$ spans $V_{\mathbf{C}}$. To do so, let $u + iv \in V_{\mathbf{C}}$ be an arbitrary vector. Since $u, v \in V$, then there exist scalars $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in \mathbf{R}$ such that

$$u = \alpha_1 v_1 + ... + \alpha_n v_n$$

and

$$v = \beta_1 v_1 + ... + \beta_n v_n.$$

It follows that

$$\begin{aligned} u + iv &= \alpha_1 v_1 + ... + \alpha_n v_n + i\beta_1 v_1 + ... + i\beta_n v_n \\ &\in \operatorname{span}(v_1, ..., v_n) \end{aligned}$$

Thus, $v_1, ..., v_n$ spans $V_{\mathbf{C}}$. To prove the linear independence, let $\alpha_1 + i\beta_1, ..., \alpha_n + i\beta_n \in \mathbf{C}$ be complex scalars and notice that

$$\begin{aligned} & (\alpha_1 + i\beta_1)v_1 + ... + (\alpha_n + i\beta_n)v_n = 0 \\ \implies & [\alpha_1 v_1 + ... + \alpha_n v_n] + i[\beta_1 v_1 + ... + \beta_n v_n] = 0 \\ \implies & \begin{cases} \alpha_1 v_1 + ... + \alpha_n v_n = 0 \\ \beta_1 v_1 + ... + \beta_n v_n = 0 \end{cases} \\ \implies & \alpha_1 = ... = \alpha_n = \beta_1 = ... = \beta_n = 0 \\ \implies & \alpha_1 + i\beta_1 = ... = \alpha_n + i\beta_n = 0 \end{aligned}$$

Thus, the vectors $v_1, ..., v_n$ are linearly independent in $V_{\mathbf{C}}$. Therefore, it is a basis of $V_{\mathbf{C}}$.

2C Dimension

Exercise 1
Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

Solution
Let V be a subspace of $\mathbf{R}^2$. Since $\mathbf{R}^2$ has dimension 2, then $0 \leq \dim V \leq 2$ by Proposition 2.37. If V has dimension 0, then its basis must be the empty set. In that case, $V = \operatorname{span}(\varnothing) = \{0\}$. If V has dimension 1, then its basis must contain a single non-zero vector. It follows that

$$V = \operatorname{span}(u) = \{\lambda u : \lambda \in \mathbf{R}\}.$$

But notice that λu with $\lambda \in \mathbf{R}$ is simply the equation of a line with direction vector u passing through the origin (take $\lambda = 0$), hence, V is a line passing through the origin. Finaly, if V has dimension 2, then by Proposition 2.39, $V = \mathbf{R}^2$. Therefore, since V was an arbitrary subspace, then subspaces of $\mathbf{R}^2$ are $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.
To prove that the subspaces of $\mathbf{R}^2$ are precisely these subsets, let's show that any of these subsets are subspaces. Trivialy, $\{0\}$ and $\mathbf{R}^2$ are indeed subspaces of $\mathbf{R}^2$. Now, let L be a line in $\mathbf{R}^2$ passing through the origin, then L must have a direction vector u. Moreover, if L passes through the point P, then we can write

$$L = \{P + \lambda u : \lambda\}$$

Since L contains the origin, then we can take $P = 0$ which implies that

$$L = \{\lambda u : \lambda\} = \operatorname{span}(u)$$

which is a subspace of $\mathbf{R}^2$. Therefore, the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

Exercise 2
Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$.

Solution
Let V be a subspace of $\mathbf{R}^3$. Since $\mathbf{R}^3$ has dimension 3, then $0 \leq \dim V \leq 3$ by Proposition 2.37. If V has dimension 0, then its basis must be the empty set. In that case, $V = \operatorname{span}(\varnothing) = \{0\}$. If V has dimension 1, then its basis must contain a single non-zero vector. It follows that

$$V = \operatorname{span}(u) = \{\lambda u : \lambda \in \mathbf{R}\}.$$

But notice that λu with $\lambda \in \mathbf{R}$ is simply the equation of a line with direction vector u passing through the origin (take $\lambda = 0$), hence, V is a line passing through the origin. If V has dimension 2, then there are two linearly independent vectors u_1, u_2 such that

$$V = \operatorname{span}(u_1, u_2) = \{\alpha u_1 + \beta u_2 : \alpha, \beta \in \mathbf{R}\}$$

But notice that $\alpha u_1 + \beta u_2$ is simply the equation of a plane containing the origin described by the two vectors u_1 and u_2. Hence, V is a plane containing the origin. Finaly, if V has dimension 3, then by Proposition 2.39, $V = \mathbf{R}^3$. Thus, subspaces of $\mathbf{R}^3$ are $\{0\}$, lines and planes containing the origin and $\mathbf{R}^3$.
To prove that the subspaces of $\mathbf{R}^3$ are precisely these subsets, let's show that any of these subsets are subspaces. Trivialy, $\{0\}$ and $\mathbf{R}^3$ are indeed subspaces of $\mathbf{R}^3$. Now, let L be a line in $\mathbf{R}^3$ passing through the origin, then L must have a direction vector u. Moreover, if L passes through the point P, then we can write

$$L = \{P + \lambda u : \lambda\}$$

Since L contains the origin, then we can take $P = 0$ which implies that

$$L = \{\lambda u : \lambda\} = \operatorname{span}(u)$$

which is a subspace of $\mathbf{R}^3$. Similarly, if P is a plane in $\mathbf{R}^3$ containing the origin, then it must contain two linearly independent vectors u_1, u_2 that describe the orientation of the plane. Moreover, if A is a point on the plan P, then the vectors in the P are described by the equation

$$A + \alpha u_1 + \beta u_2$$

Since P contains the origin, then take $A = 0$ to get:

$$P = \{\alpha u_1 + \beta u_2 : \alpha, \beta \in \mathbf{R}\} = \operatorname{span}(u_1, u_2)$$

It follows that P is a subspace of $\mathbf{R}^3$. Therefore, the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

Exercise 3

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

(a) Consider the list p_1, p_2, p_3, p_4 defined by

$$\begin{aligned} p_1(x) &= x - 6 & p_2(x) &= x^2 - 6x \\ p_3(x) &= x^3 - 6x^2 & p_4(x) &= x^4 - 6x^3 \end{aligned}$$

This list spans U because given any $p \in U$, then p is a polynomial of degree four that has 6 as a root. Hence, we can factorize $x - 6$ such that $p(x) = (x-6)(ax^3 + bx^2 + cx + d)$ for some $a, b, c, d \in \mathbf{F}$. Thus:

$$\begin{aligned} p(x) &= (x-6)(ax^3 + bx^2 + cx + d) \\ &= a(x^4 - 6x^3) + b(x^3 - 6x^2) + c(x^2 - 6x) + d(x - 6) \\ &= ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) \end{aligned}$$

which shows that the list spans U. To show that it is linearly independent, take scalars $a, b, c, d \in \mathbf{F}$ and notice that

$$\begin{aligned}
& ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) = 0 \\
\implies & a(x^4 - 6x^3) + b(x^3 - 6x^2) + c(x^2 - 6x) + d(x - 6) = 0 \\
\implies & ax^4 + (b - 6a)x^3 + (c - 6b)x^2 + (d - 6c)x + (-6d) = 0 \\
\implies & \begin{cases} a = 0 \\ b - 6a = 0 \\ c - 6b = 0 \\ d - 6c = 0 \\ -6d = 0 \end{cases} \\
\implies & a = b = c = d = 0
\end{aligned}$$

Therefore, p_1, p_2, p_3, p_4 is a basis of U.

(b) Since the list p_1, p_2, p_3, p_4 is linearly independent in U, then it must be linearly independent in $\mathcal{P}_4(\mathbf{F})$. Hence, we can extend it to a basis of $\mathcal{P}_4(\mathbf{F})$. To do so, we only need to add one single polynomial to our list because we already know a basis of $\mathcal{P}_4(\mathbf{F})$ of size 5: 1, x, x^2, x^3, x^4. It is easy to notice that the constant polynomial 1 cannot be written as a linear combination of p_1, p_2, p_3, p_4 because for any scalars $a, b, c, d \in \mathbf{F}$:

$$\begin{aligned}
& ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) = 1 \\
\implies & a(x^4 - 6x^3) + b(x^3 - 6x^2) + c(x^2 - 6x) + d(x - 6) = 1 \\
\implies & ax^4 + (b - 6a)x^3 + (c - 6b)x^2 + (d - 6c)x + (-6d) = 1 \\
\implies & \begin{cases} a = 0 \\ b - 6a = 0 \\ c - 6b = 0 \\ d - 6c = 0 \\ -6d = 1 \end{cases} \\
\implies & a = b = c = d = 0 \text{ and } d - \frac{1}{6}
\end{aligned}$$

A contradiction. Therefore, by Section 2A Exercise 13, the list $1, p_1, p_2, p_3, p_4$ is linearly independent. Since the list has length 5, then it must be a basis by Proposition 2.38.

(c) Since $1, p_1, p_2, p_3, p_4$ is a basis of $\mathcal{P}_4(\mathbf{F})$, then we can easily get

$$\mathcal{P}_4(\mathbf{F}) = U \oplus \mathbf{F}$$

where $\mathbf{F}$ denotes the set of constant polynomials.

Exercise 4

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

(a) Consider the list p_1, p_2, p_3, p_4 defined by

$$p_1(x) = 1 \qquad p_2(x) = x$$
$$p_3(x) = \frac{1}{6}x^3 - 3x^2 \qquad p_4(x) = \frac{1}{12}x^4 - x^3$$

To show that it is linearly independent, take scalars $a, b, c, d \in \mathbf{F}$ and notice that

$$\begin{aligned}
& ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) = 0 \\
\implies & a\left(\frac{1}{12}x^4 - x^3\right) + b\left(\frac{1}{6}x^3 - 3x^2\right) + cx + d = 0 \\
\implies & \frac{a}{12}x^4 + \left(\frac{b}{6} - a\right)x^3 + (-3b)x^2 + cx + d = 0 \\
\implies & \begin{cases} \frac{a}{12} = 0 \\ \frac{b}{6} - a = 0 \\ -3b = 0 \\ c = 0 \\ d = 0 \end{cases} \\
\implies & a = b = c = d = 0
\end{aligned}$$

Therefore, p_1, p_2, p_3, p_4 is linearly independent in U. To prove that it is a basis, consider its span. If p_1, p_2, p_3, p_4 don't span U, then we must be able to extend it to a basis of U. However, since $\dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$, then we can add only one polynomial. In this case, U has a basis of length 5 which implies that $U = \mathcal{P}_4(\mathbf{F})$ by Proposition 2.39, a contradiction since $x^2 \notin U$. It follows that the list p_1, p_2, p_3, p_4 must span U. Therefore, it is a basis of U.

(b) Since the list p_1, p_2, p_3, p_4 is linearly independent in U, then it must be linearly independent in $\mathcal{P}_4(\mathbf{F})$. Hence, we can extend it to a basis of $\mathcal{P}_4(\mathbf{F})$. To do so, we only need to add one single polynomial to our list because we already know a basis of $\mathcal{P}_4(\mathbf{F})$ of size 5: 1, x, x^2, x^3, x^4. It is easy to notice that the polynomial x^2 cannot be written as a linear combination of p_1, p_2, p_3, p_4 since $x^2 \notin U$. Therefore, by Section 2A Exercise 13, the list x^2, p_1, p_2, p_3, p_4 is linearly independent. Since the list has length 5, then it must be a basis by Proposition 2.38.

(c) Since x^2, p_1, p_2, p_3, p_4 is a basis of $\mathcal{P}_4(\mathbf{F})$, then we can easily get

$$\mathcal{P}_4(\mathbf{F}) = U \oplus \mathbf{F}x^2$$

where $\mathbf{F}x^2$ denotes the span of the polynomial x^2.

Exercise 5

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

(a) Consider the list p_1, p_2, p_3, p_4 defined by

$$\begin{aligned} p_1(x) &= 1 & p_2(x) &= (x-2)(x-5) \\ p_3(x) &= x(x-2)(x-5) & p_4(x) &= x^2(x-2)(x-5) \end{aligned}$$

To show that it is linearly independent, take scalars $a, b, c, d \in \mathbf{F}$ and notice that

$$\begin{aligned} & ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) = 0 \\ \implies\ & a(x^4 - 7x^3 + 10x^2) + b(x^3 - 7x^2 + 10x) + c(x^2 - 7x + 10) + d = 0 \\ \implies\ & ax^4 + (b - 7a)x^3 + (c - 7b + 10a)x^2 + (10b - 7c)x + (10c + d) = 0 \\ \implies\ & \begin{cases} a = 0 \\ b - 7a = 0 \\ c - 7b + 10a = 0 \\ 10b - 7c = 0 \\ 10c + d = 0 \end{cases} \\ \implies\ & a = b = c = d = 0 \end{aligned}$$

Therefore, p_1, p_2, p_3, p_4 is linearly independent in U. To prove that it is a basis, consider its span. If p_1, p_2, p_3, p_4 don't span U, then we must be able to extend it to a basis of U. However, since $\dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$, then we can add only one polynomial. In this case, U has a basis of length 5 which implies that $U = \mathcal{P}_4(\mathbf{F})$ by Proposition 2.39, a contradiction since $x^2 \notin U$. It follows that the list p_1, p_2, p_3, p_4 must span U. Therefore, it is a basis of U.

(b) Since the list p_1, p_2, p_3, p_4 is linearly independent in U, then it must be linearly independent in $\mathcal{P}_4(\mathbf{F})$. Hence, we can extend it to a basis of $\mathcal{P}_4(\mathbf{F})$. To do so, we only need to add one single polynomial to our list because we already know a basis of $\mathcal{P}_4(\mathbf{F})$ of size 5: 1, x, x^2, x^3, x^4. It is easy to notice that the polynomial x^2 cannot be written as a linear combination of p_1, p_2, p_3, p_4 since $x^2 \notin U$. Therefore, by Section 2A Exercise 13, the list x^2, p_1, p_2, p_3, p_4 is linearly independent. Since the list has length 5, then it must be a basis by Proposition 2.38.

(c) Since x^2, p_1, p_2, p_3, p_4 is a basis of $\mathcal{P}_4(\mathbf{F})$, then we can easily get

$$\mathcal{P}_4(\mathbf{F}) = U \oplus \mathbf{F}x^2$$

where $\mathbf{F}x^2$ denotes the span of the polynomial x^2.

Exercise 6

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

(a) Consider the list p_1, p_2, p_3 defined by

$$\begin{aligned}
p_1(x) &= 1\\
p_2(x) &= (x-2)(x-5)(x-6)\\
p_3(x) &= x(x-2)(x-5)(x-6)
\end{aligned}$$

To show that it is linearly independent, take scalars $a, b, c \in \mathbf{F}$ and notice that

$$\begin{aligned}
& ap_3(x) + bp_2(x) + cp_1(x) = 0\\
\implies\ & a(x^4 - 13x^3 + 52x^2 - 60x) + b(x^3 - 13x^2 + 52x - 60) + c = 0\\
\implies\ & ax^4 + (b - 13a)x^3 + (52a - 13b)x^2 + (52b - 60a)x + (c - 60b) = 0\\
\implies\ & \begin{cases} a = 0\\ b - 13a = 0\\ 52a - 13b = 0\\ 52b - 60a = 0\\ c - 60b = 0 \end{cases}\\
\implies\ & a = b = c = 0
\end{aligned}$$

Therefore, p_1, p_2, p_3 is linearly independent in U. Let's now show that it spans U. To do so, let p be an arbitrary polynomial in U, then $p - p(2)$ must have roots at $x = 2, 5, 6$ which means that

$$p(x) - p(2) = (x-2)(x-5)(x-6)q(x)$$

where q is a polynomial of degree 1. Thus, there exist scalars $a, b \in \mathbf{F}$ such that $q(x) = ax + b$. Thus,

$$\begin{aligned}
p(x) &= (x-2)(x-5)(x-6)q(x) + p(2)\\
&= (x-2)(x-5)(x-6)(ax+b) + p(2)\\
&= ax(x-2)(x-5)(x-6) + b(x-2)(x-5)(x-6) + p(2)\\
&= ap_3(x) + bp_2(x) + p(2)p_1(x)\\
&\in \operatorname{span}(p_1, p_2, p_3)
\end{aligned}$$

Therefore, p_1, p_2, p_3 is a basis for U.

(b) Since the list p_1, p_2, p_3 is linearly independent in U, then it must be linearly independent in $\mathcal{P}_4(\mathbf{F})$. Hence, we can extend it to a basis of $\mathcal{P}_4(\mathbf{F})$. To do so, we need to add two polynomials to our list because $\dim \mathcal{P}_4(\mathbf{F}) = 5$. It is easy to see that the polynomial x cannot be written as a linear combination of p_1, p_2, p_3 since $x \notin U$. Therefore, by Section 2A Exercise 13, the list x, p_1, p_2, p_3 is linearly independent. Let's add one last polynomial to our list to make it a basis of U. Suppose that $x^2 \in \operatorname{span}(x, p_1, p_2, p_3)$, then there exist scalars $a, b, c, d \in \mathbf{F}$ such that

$$x^2 = ax + bp_1(x) + cp_2(x) + dp_3(x)$$

But notice that $bp_1 + cp_2 + dp_3 \in U$ so just define it as p_U, then we have

$$x^2 = ax + p_U(x)$$

where $p_U(2) = p_U(5) = p_U(6)$. This, if we plug-in $x = 2, 5, 6$, we get the following system of equations:

$$\begin{cases} 4 = 2a + p_U(2) \\ 25 = 5a + p_U(2) \\ 36 = 6a + p_U(2) \end{cases}$$

$$\implies \begin{cases} 21 = 3a \\ 11 = a \end{cases}$$

A contradiction that shows that x^2 is not in the span of x, p_1, p_2, p_3. Therefore, by Section 2A Exercise 13, the list x, x^2, p_1, p_2, p_3 is linearly independent. Since the list has length 5, then it must be a basis by Proposition 2.38.

(c) Since x, x^2, p_1, p_2, p_3 is a basis of $\mathcal{P}_4(\mathbf{F})$, then we can easily get

$$\mathcal{P}_4(\mathbf{F}) = U \oplus \operatorname{span}(x, x^2)$$

Exercise 7

(a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : \int_{-1}^{1} p = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Solution

(a) Consider the list p_1, p_2, p_3, p_4 defined by

$$p_1(x) = x \qquad p_2(x) = x^2 - \frac{1}{3}$$
$$p_3(x) = x^3 \qquad p_4(x) = x^4 - \frac{1}{5}$$

To show that it is linearly independent, take scalars $a, b, c, d \in \mathbf{F}$ and notice that

$$\begin{aligned} & ap_4(x) + bp_3(x) + cp_2(x) + dp_1(x) = 0 \\ \implies & a\left(x^4 - \frac{1}{5}\right) + bx^3 + c\left(x^2 - \frac{1}{3}\right) + dx = 0 \\ \implies & ax^4 + bx^3 + cx^2 + dx - \left(\frac{a}{5} + \frac{c}{3}\right) = 0 \\ \implies & \begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = 0 \\ \frac{a}{5} + \frac{c}{3} = 0 \end{cases} \\ \implies & a = b = c = d = 0 \end{aligned}$$

Therefore, p_1, p_2, p_3, p_4 is linearly independent in U. To prove that it is a basis, consider its span. If p_1, p_2, p_3, p_4 don't span U, then we must be able to extend it to a basis of U. However, since $\dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$, then we can add only one polynomial. In this case, U has a basis of length 5 which implies that $U = \mathcal{P}_4(\mathbf{F})$ by Proposition 2.39, a contradiction since $1 \notin U$. It follows that the list p_1, p_2, p_3, p_4 must span U. Therefore, it is a basis of U.

(b) Since the list p_1, p_2, p_3, p_4 is linearly independent in U, then it must be linearly independent in $\mathcal{P}_4(\mathbf{F})$. Hence, we can extend it to a basis of $\mathcal{P}_4(\mathbf{F})$. To do so, we only need to add one single polynomial to our list because $\dim \mathcal{P}_4(\mathbf{F}) = 5$. It is easy to notice that the polynomial 1 cannot be written as a linear combination of p_1, p_2, p_3, p_4 since $1 \notin U$. Therefore, by Section 2A Exercise 13, the list $1, p_1, p_2, p_3, p_4$ is linearly independent. Since the list has length 5, then it must be a basis by Proposition 2.38.

(c) Since $1, p_1, p_2, p_3, p_4$ is a basis of $\mathcal{P}_4(\mathbf{F})$, then we can easily get

$$\mathcal{P}_4(\mathbf{F}) = U \oplus \mathbf{F}$$

where $\mathbf{F}$ denotes the subspace of constant polynomials.

Exercise 8

Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, ..., v_m + w) \geq m - 1.$$

Solution

Consider the list $v_1 + w, ..., v_m + w$ of length m and suppose by contradiction that

$$\dim \operatorname{span}(v_1 + w, ..., v_m + w) = d \leq m - 2,$$

then there exists a linearly independent list of vectors $u_1, ..., u_d$ in V such that

$$\operatorname{span}(v_1 + w, ..., v_m + w) = \operatorname{span}(u_1, ..., u_d).$$

For each $i \in \{1, ..., n\}$, the previous equation implies that

$$v_i + w = \alpha_1 u_1 + ... + \alpha_d u_d,$$

which itself implies that

$$v_i = \alpha_1 u_1 + ... + \alpha_d u_d - w \in \operatorname{span}(u_1, ..., u_d, w).$$

Since it holds for all $i \in \{1, ..., n\}$, then

$$\{v_1, ..., v_m\} \subset \operatorname{span}(u_1, ..., u_d, w) \implies \operatorname{span}(v_1, ..., v_m) \leq \operatorname{span}(u_1, ..., u_d, w).$$

Since subspaces have a dimensions less than the vector space they are contained in (Proposition 2.37), then

$$\dim \operatorname{span}(v_1, ..., v_m) \leq \dim \operatorname{span}(u_1, ..., u_d, w). \tag{1}$$

The v_i's are linearly independent so they form a basis for their span. It follows that

$$\dim \operatorname{span}(v_1, ..., v_m) = m. \tag{2}$$

Moreover, the list $u_1, ..., u_d, w$ is spanning its span (obviously), so it must contain a basis. Since the list has length $d + 1$, then the dimension of the span must be less than $d + 1$. But since $d \leq m - 2$, then

$$\dim \operatorname{span}(u_1, ..., u_d, w) \leq m - 1. \tag{3}$$

Combining equations (1), (2) and (3) gives us

$$m \leq m - 1$$

which is clearly a contradiction. Therefore,

$$\dim \operatorname{span}(v_1 + w, ..., v_m + w) \geq m - 1.$$

Exercise 9

Suppose m is a positive integer and $p_0, p_1, ..., p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, ..., p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solution

Since the list $p_0, p_1, ..., p_m$ has length $m+1$ and we already know that $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$, then it suffices to show that the list is linearly independent by Proposition 2.38. Let's prove by induction on m that any list $p_0, p_1, ..., p_m$ such that each p_k has degree k must be linearly independent.

For the base case, take $m = 0$ and consider the list p_0 where p_0 is a polynomial of degree 0. Hence p_0 must be a nonzero constant polynomial (since the zero polynomial has degree $-\infty$). But we know from Section 2A Exercise 4(a) that the list containing p_0 only must be linearly independent since it is nonzero. This proves that the statement holds for $m = 0$.

Suppose now that it holds for an integer $m \geq 0$ and consider the list $p_0, p_1, ..., p_{m+1}$

such that each p_k has degree k. By our assumption, we know that the list $p_0, p_1, ..., p_m$ is linearly independent. If we take arbitrary scalars $\alpha_0, ..., \alpha_{m+1}$ satisfying

$$\alpha_0 p_0 + \alpha_1 p_1 + ...\alpha_{m+1} p_{m+1} = 0,$$

then notice that α_{m+1} must be equal to zero since p_{m+1} is only polynomial in the linear combination containing a x^{m+1} term. Thus, we get that

$$\alpha_0 p_0 + \alpha_1 p_1 + ...\alpha_m p_m = 0.$$

But by linear independence of the p_i's, we know that $\alpha_0 = ... = \alpha_m = 0$. It follows that the new list is linearly independent as well. Therefore, by induction, all such lists must be linearly independent and hence, a basis for $\mathcal{P}_m(\mathbf{F})$.

Exercise 10

Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k(1-x)^{m-k}.$$

Show that $p_0, ..., p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Solution

Since the list $p_0, p_1, ..., p_m$ has length $m+1$ and we already know that $\dim \mathcal{P}_m(\mathbf{F}) = m+1$, then it suffices to show that the list is linearly independent by Proposition 2.38.

Let $\alpha_0, \alpha_1, ..., \alpha_m \in \mathbf{F}$ be scalars such that

$$\alpha_0 p_0 + \alpha_1 p_1 + ... + \alpha_m p_m = 0.$$

The polynomial on the left hand side is equal to zero, this implies that the coefficients in front of each monomial of the form x^k are zero. Given a k between 0 and m, notice that the polynomial p_k can be written as

$$p_k(x) = x^k \sum_{i=0}^{m-k} \binom{m-k}{i} (-1)^i x^i = \sum_{i=k}^{m} \binom{m-k}{i} (-1)^{i-k} x^i$$

using the Binomial Formula. It follows that the lowest degree term in p_k is x^k. Therefore, in the list $p_0, ..., p_m$, p_0 is the only polynomial containing a constant term. Hence, the constant term in the polynomial $\alpha_0 p_0 + \alpha_1 p_1 + ... + \alpha_m p_m$ is α_0. It follows that $\alpha_0 = 0$. Thus, we now have the equation

$$\alpha_1 p_1 + ... + \alpha_m p_m = 0.$$

In the list $p_1, ..., p_m$, the polynomial p_1 is the only polynomial containing the term x^1. Thus, the coefficient in front of the term x^1 in the polynomial $\alpha_1 p_1 + ... + \alpha_m p_m$ is α_1. It follows that $\alpha_1 = 0$. If we continue in this manner, we can prove by induction that all the α_i's are zero. Therefore, the list is linearly independent and hence, a basis of $\mathcal{P}_m(\mathbf{F})$.

Exercise 11

Suppose U and V are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

Solution
By Proposition 2.43, we know that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) \tag{1}$$

Since $U + W \leq V$, then by Proposition 2.37, $\dim(U + W) \leq \dim V = 6$. Thus, if we substitute this inequality and the known values into equation (1), we get:

$$6 \geq 4 + 4 - \dim(U \cap W),$$

which can be rearranged into

$$\dim(U \cap W) \geq 2.$$

Thus, if we denote by d the dimension of $U \cap W$, then there exists a basis $v_1, ..., v_d$ of $U \cap W$. Since it is a basis and $d \geq 2$, then we can take the vectors $v_1, v_2 \in U \cap W$ and assert that they are linearly independent. Therefore, there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other (by linear independence).

Exercise 12
Suppose that U and V are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

Solution
Since we already know that $U + W = \mathbf{R}^8$, it suffices to prove that $U \cap W = \{0\}$. To do so, notice that $U + W = \mathbf{R}^8$ implies $\dim(U + W) = 8$. Using the formula in Proposition 2.43, we get

$$\dim U + \dim V - \dim(U \cap W) = 8.$$

If we plug-in the known values, we get

$$4 + 4 - \dim(U \cap W) = 8,$$

which can be rearranged into

$$\dim(U \cap W) = 0.$$

But the only zero-dimensional vector space is the trivial vector space $\{0\}$. Hence, $U \cap W = \{0\}$. Therefore, $\mathbf{R}^8 = U \oplus W$.

Exercise 13
Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

Solution
Since $U + W \leq \mathbf{R}^9$, then by Proposition 2.37, $\dim(U + W) \leq \dim \mathbf{R}^9 = 9$. Using the formula in Proposition 2.43, we get

$$\dim U + \dim V - \dim(U \cap W) \leq 9.$$

If we plug-in the known values, we get

$$5 + 5 - \dim(U \cap W) \leq 9,$$

which can be rearranged into

$$\dim(U \cap W) \geq 1.$$

Therefore, $U \cap W$ cannot be $\{0\}$ since otherwise, its dimension would be 0.

Exercise 14
Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Solution
First, consider the subspace $V_1 \cap V_2$. Since $V_1 + V_2 \leq V$, then $\dim(V_1 + V_2) \leq 10$. It follows that

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2) \geq 7 + 7 - 10 = 4.$$

Now, consider the subspace $V_1 \cap V_2 \cap V_3$ as the intersection between $V_1 \cap V_2$ and V_3. Since $(V_1 \cap V_2) + V_3 \leq V$, then $\dim((V_1 \cap V_2) + V_3) \leq 10$. It follows that

$$\begin{aligned}
\dim(V_1 \cap V_2 \cap V_3) &= \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\
&\geq 4 + 7 - 10 \\
&= 1
\end{aligned}$$

Thus, $V_1 \cap V_2 \cap V_3$ cannot be $\{0\}$ since otherwise, its dimension would be 0.

Exercise 15
Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Solution
First, consider the subspace $V_1 \cap V_2$. Since $V_1 + V_2 \leq V$, then $\dim(V_1 + V_2) \leq \dim V$. It follows that

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2) \geq \dim V_1 + \dim V_2 - \dim V.$$

Now, consider the subspace $V_1 \cap V_2 \cap V_3$ as the intersection between $V_1 \cap V_2$ and V_3. Since $(V_1 \cap V_2) + V_3 \leq V$, then $\dim((V_1 \cap V_2) + V_3) \leq \dim V$. It follows that

$$\begin{aligned}
\dim(V_1 \cap V_2 \cap V_3) &= \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\
&\geq \dim V_1 + \dim V_2 - \dim(V_1 + V_2) + \dim V_3 - \dim V \\
&\geq \dim V_1 + \dim V_2 + \dim V_3 - \dim V - \dim V \\
&> 2 \dim V - 2 \dim V \\
&= 0
\end{aligned}$$

Thus, $V_1 \cap V_2 \cap V_3$ cannot be $\{0\}$ since otherwise, its dimension would be 0.

Exercise 16
Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

Solution
Let $u_1, ..., u_m$ be a basis of U and extend it to a basis

$$u_1, ..., u_m, v_1, ..., v_{n-m}$$

of V. For each $i \in \{1, ..., n - m\}$, define the subspace V_i of V as the span of the list $u_1, ..., u_m, v_1, ..., v_{n-m}$ except the vector v_i. Hence, V_i is the span of $n - 1$ linearly independent vectors so $\dim V_i = n-1$. Consider now the intersection $V_1 \cap ... \cap V_{n-m}$. Since each V_i is the span of a list containing a basis of U, then U is a subspace of all the V_i's. It follows that

$$U \leq \bigcap_{i=1}^{n-m} V_i.$$

Let v be an arbitrary vector in $\bigcap_{i=1}^{n-m} V_i$, since $\bigcap_{i=1}^{n-m} V_i \leq V$, then

$$v = \alpha_1 u_1 + ... + \alpha_m u_m + \beta v_1 + ... + \beta_{n-m} v_{n_m}$$

for some scalars $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_{n-m} \in \mathbf{F}$. Let $j \in \{1, ..., n\}$, since $v \in \bigcap_{i=1}^{n-m} V_i$, then $v \in V_j$ in particular. Since V_j is the span of the u_i's and v_i's except v_j, then V_j contains

$$v_0 = \alpha_1 u_1 + ... + \alpha_m u_m + \beta v_1 + ... + \beta_{n-m} v_{n_m}$$

where $\beta_j = 0$. Hence, V_j is a subspace that contains both v and v_0, so it follows that

$$\beta_j v_j = v - v_0 \in V_j.$$

If β_j is non-zero, then $v_j \in V_j$. But since V_j already contains all the vectors in the basis of V except v_j, then V_j contains a basis of V. It follows that $V \leq V_j$ so $n \leq n-1$, a contradiction. Therefore, $\beta_j = 0$. Since it holds for all $j \in \{1, ..., n-m\}$, then all the possibly non-zero coefficients in the linear combination of v are the coefficients in front of the u_i's. Hence:

$$v = \alpha_1 u_1 + ... + \alpha_m u_m + 0 + ... + 0 \in \operatorname{span}(u_1, ..., u_m) = U.$$

Since it holds for all $v \in \cap_{i=1}^{n-m} V_i$, then $\cap_{i=1}^{n-m} V_i \leq U$. Therefore, $U = \cap_{i=1}^{n-m} V_i$, which proves that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

Exercise 17
Suppose that $V_1, ..., V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

Solution
Let $i \in \{1, ..., m\}$ and define d_i as the dimension of V_i. Since V_i is finite-dimensional,

then it has a finite basis $v_1^{(i)}, ..., v_{d_i}^{(i)}$. Consider now the list which merges all of these bases:

$$v_1^{(1)}, ..., v_{d_1}^{(1)}, v_1^{(2)}, ..., v_{d_2}^{(2)}, ..., v_1^{(m)}, ..., v_{d_m}^{(m)}$$

This new list has length $d_1 + d_2 + ... + d_m < \infty$. Moreover, it is easy to see that it is spanning $V_1 + \ldots V_m$ since for all $u \in V_1 + \ldots V_m$, we can write u as $u_1 + ... + u_m$ where $u_i \in V_i$ for all $i \in \{1, ..., m\}$. Hence, for all $i \in \{1, ..., n\}$, since $v_1^{(i)}, ..., v_{d_i}^{(i)}$ is a basis of V_i, then there exist scalars $\alpha_1^{(i)}, ..., \alpha_{d_i}^{(i)} \in \mathbf{F}$ such that

$$u_i = \alpha_1^{(i)} v_1^{(i)} + ... + \alpha_{d_i}^{(i)} v_{d_i}^{(i)}.$$

It follows that

$$u = \alpha_1^{(1)} v_1^{(1)} + ... + \alpha_{d_1}^{(1)} v_{d_1}^{(1)} + ... + \alpha_1^{(m)} v_1^{(m)} + ... + \alpha_{d_m}^{(m)} v_{d_m}^{(m)}$$

which proves that the list

$$v_1^{(1)}, ..., v_{d_1}^{(1)}, v_1^{(2)}, ..., v_{d_2}^{(2)}, ..., v_1^{(m)}, ..., v_{d_m}^{(m)}$$

spans $V_1 + \cdots + V_m$. Hence, $V_1 + \cdots + V_m$ is finite dimensional since it contains a finite spanning list. Moreover, any spanning list must contain a basis. It follows that there exists a sublist of the one presented that it a basis of $V_1 + \cdots + V_m$. This list has a length less than or equal to the length of the presented list (since it is a sublist) which is equal to $d_1 + ... + d_m$. But since it is a basis, then it has length $\dim(V_1 + \cdots + V_m)$. Therefore:

$$\dim(V_1 + \cdots + V_m) \leq d_1 + \cdots + d_m = \dim V_1 + \cdots + \dim V_m.$$

Exercise 18

Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, ..., V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

Solution

Let $v_1, ..., v_n$ be a basis of V and for each $i \in \{1, ..., n\}$, define the subspace V_i as the span of the vector v_i. Let's prove that $V = V_1 \oplus \cdots \oplus V_n$. First, it is clear that $V_1 + \cdots + V_n \leq V$. Moreover, for all $v \in V$, since $v_1, ..., v_n$ is a basis of V, there exist scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$ such that

$$v = \alpha_1 v_1 + ... + \alpha_n v_n.$$

For all $i \in \{1, ..., n\}$, the term $\alpha_i v_i \in V_i$. It follows that $v \in V_1 + \cdots + V_n$. Therefore, $V = V_1 + \cdots + V_n$. To prove that the sum is direct, it suffices to show that zero has a unique representation as a sum of elements in the V_i's (Proposition 1.45). But this follows from the fact the v_i's are linearly independent since it is a basis. Therefore,

$$V = V_1 \oplus \cdots \oplus V_n.$$

Exercise 19

Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1+V_2+V_3) = \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &\quad + \dim(V_1 \cap V_2 \cap V_3).\end{aligned}$$

Then either prove the formula above or give a counterexample.

Solution

Given three finite sets S_1, S_2, S_3, we can easily derive the following formula for the cardinality of the union of the three sets:

$$\begin{aligned}\#(S_1 \cup S_2 \cup S_3) &= \#(S_1 \cup S_2) + \#S_3 - \#((S_1 \cup S_2) \cap S_3) \\ &= \#S_1 + \#S_2 - \#(S_1 \cap S_2) + \#S_3 - \#((S_1 \cap S_3) \cup (S_1 \cap S_3)) \\ &= \#S_1 + \#S_2 + \#S_3 - \#(S_1 \cap S_2) \\ &\quad - \#(S_1 \cap S_3) - \#(S_2 \cap S_3) + \#(S_1 \cap S_2 \cap S_3)\end{aligned}$$

Now, using the correspondence between finite sets and finite-dimensional vector spaces, cardinality and dimension, unions and sums, we could guess that the analoguous fomula for finite-dimensional vector spaces is

$$\begin{aligned}\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) \\ - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3)\end{aligned}$$

However, this formula is false. To see why, consider the following counterexample: Take V_1 to be the span of the vector $(1, 0) \in \mathbf{R}^2$, V_2 to be the span of $(0, 1) \in \mathbf{R}^2$ and V_3 to be the span of $(1, 1) \in \mathbf{R}^2$. Since the three vectors span $\mathbf{R}^2$, then

$$V_1 + V_2 + V_3 = \operatorname{span}((1, 0), (0, 1), (1, 1)) = \mathbf{R}^2.$$

This implies that $\dim(V_1 + V_2 + V_3) = 2$. Moreover, we also have

$$\dim V_1 = \dim V_2 = \dim V_3 = 1$$

and

$$V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = V_1 \cap V_2 \cap V_3 = \{0\}.$$

Thus, if the formula for the dimension of the sum of three subspaces was true, we would have:

$$2 = 1 + 1 + 1 - 0 - 0 - 0 + 0$$

which is clearly a contradiction. Therefore, the formula is false.

Exercise 20
Prove that if V_1, V_2 and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3) = \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.\end{aligned}$$

Solution
First, recall that $V_1+V_2+V_3 = (V_1+V_2)+V_3$. Thus, using the formula in Proposition 2.43, we get

$$\dim(V_1 + V_2 + V_3) = \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3).$$

Now, applying the same formula to $\dim(V_1 + V_2)$ gives us

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3).\end{aligned} \tag{1}$$

We can repeat this process by writing $V_1 + V_2 + V_3$ as $(V_1 + V_3) + V_2$ to get

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_3) - \dim((V_1 + V_3) \cap V_2).\end{aligned} \tag{2}$$

Again, by writing $V_1 + V_2 + V_3$ as $(V_2 + V_3) + V_1$, we get

$$\begin{aligned}\dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_2 \cap V_3) - \dim((V_2 + V_3) \cap V_1).\end{aligned} \tag{3}$$

If we add equations (1), (2) and (3) together, we get

$$\begin{aligned}3\dim(V_1 + V_2 + V_3) = \\ &= 3\dim V_1 + 3\dim V_2 + 3\dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &\quad - \dim((V_1 + V_2) \cap V_3) - \dim((V_1 + V_3) \cap V_2) - \dim((V_2 + V_3) \cap V_1).\end{aligned}$$

By dividing by 3 on both sides, we obtain

$$\begin{aligned}\dim(V_1 + V_2 + V_3) = \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\ &\quad - \frac{\dim((V_1 + V_2) \cap V_3) + \dim((V_1 + V_3) \cap V_2) + \dim((V_2 + V_3) \cap V_1)}{3}.\end{aligned}$$

which is the desired formula.

Chapter 3

Linear Maps

3A Vector Space of Linear Maps

Exercise 1
Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Solution
$(\implies)$ Suppose that T is linear, then we know from Proposition 3.10 that $T0 = 0$. Thus, it follows that

$$T(0, 0, 0) = (b, 0) = (0, 0)$$

which implies that $b = 0$. To prove that $c = 0$, notice that by linearity of T, we have

$$T(2, 2, 2) = 2T(1, 1, 1).$$

If we plug-in the values into the definition of T, we get

$$(4 - 8 + 6 + 0, 12 + 8c) = 2(2 - 4 + 3 + 0, 6 + c)$$

which is equivalent to

$$(2, 12 + 8c) = (2, 12 + 2c).$$

It follows that $12+8c = 12+2c$ which can only be true when $c = 0$. Thus, $b = c = 0$.
$(\impliedby)$ Suppose now that $b = c = 0$, then $T(x, y, z)$ becomes

$$T(x, y, z) = (2x - 4y + 3z, 6x)$$

for all $x, y, z \in \mathbf{R}$. Let's show that T is linear. First, take $(x, y, z), (x', y', z') \in \mathbf{R}^3$ and notice that

$$\begin{aligned}
T((x, y, z) + (x', y', z')) &= T(x + x', y + y', z + z') \\
&= (2(x + x') - 4(y + y') + 3(z + z'), 6(x + x')) \\
&= (2x - 4y + 3z + 2x' - 4y' + 3z', 6x + 6x') \\
&= (2x - 4y + 3z, 6x) + (2x' - 4y' + 3z', 6x') \\
&= T(x, y, z) + T(x', y', z')
\end{aligned}$$

Moreover, given any $\lambda \in \mathbf{R}$ and $(x, y, z) \in \mathbf{R}^3$, we have

$$\begin{aligned} T(\lambda(x,y,z)) &= T(\lambda x, \lambda y, \lambda z) \\ &= (2(\lambda x) - 4(\lambda y) + 3(\lambda z), 6(\lambda x)) \\ &= (\lambda(2x - 4y + 3z), \lambda(6x)) \\ &= \lambda(2x - 4y + 3z, 6x) \\ &= \lambda T(x,y,z) \end{aligned}$$

Therefore, T is linear.

Exercise 2
Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

Solution
($\Longrightarrow$) Suppose that T is linear, then if we let p be the constant polynomial equal to $\pi/2$, we get that T must satisfy

$$T(2p) = 2Tp.$$

If we rewrite this using the definition of T and p, we obtain

$$\left(3\pi + b\pi^2, \pi \int_{-1}^{2} x^3 dx + c\sin(\pi)\right) = 2\left(3\frac{\pi}{2} + b\frac{\pi^2}{4}, \frac{\pi}{2}\int_{-1}^{2} x^3 dx + c\sin\left(\frac{\pi}{2}\right)\right)$$

which can be simplified to

$$\left(3\pi + b\pi^2, \pi \int_{-1}^{2} x^3 dx\right) = \left(3\pi + b\frac{\pi^2}{2}, \pi \int_{-1}^{2} x^3 dx + c\right).$$

This gives us the following system of equations:

$$\begin{cases} 3\pi + b\pi^2 = 3\pi + b\frac{\pi^2}{2} \\ \pi \int_{-1}^{2} x^3 dx = \pi \int_{-1}^{2} x^3 dx + c \end{cases} \implies \begin{cases} b = \frac{1}{2}b \\ c = 0 \end{cases} \implies b = c = 0.$$

($\Longleftarrow$) Suppose that $b = c = 0$, then for all $p \in \mathcal{P}(\mathbf{R})$, we have

$$Tp = \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x)dx\right).$$

Thus, for any $p_1, p_2 \in \mathcal{P}(\mathbf{R})$, we get

$$\begin{aligned} T(p_1 + p_2) &= \left(3(p_1 + p_2)(4) + 5(p_1 + p_2)'(6), \int_{-1}^{2} x^3 (p_1 + p_2)(x)dx\right) \\ &= \left(3(p_1(4) + p_2(4)) + 5(p_1'(6) + p_2'(6)), \int_{-1}^{2} x^3 (p_1(x) + p_2(x))dx\right) \\ &= \left(3p_1(4) + 3p_2(4) + 5p_1'(6) + 5p_2'(6), \int_{-1}^{2} x^3 p_1(x)dx + \int_{-1}^{2} x^3 p_2(x)dx\right) \\ &= \left(3p_1(4) + 5p_1'(6), \int_{-1}^{2} x^3 p_1(x)dx\right) + \left(3p_2(4) + 5p_2'(6), \int_{-1}^{2} x^3 p_2(x)dx\right) \\ &= Tp_1 + Tp_2. \end{aligned}$$

Similarly, for all $\lambda \in \mathbf{R}$ and $p \in \mathcal{P}(\mathbf{R})$, we have

$$\begin{aligned} T(\lambda p) &= \left(3(\lambda p)(4) + 5(\lambda p)'(6), \int_{-1}^{2} x^3 (\lambda p)(x) dx\right) \\ &= \left(3\lambda p(4) + 5\lambda p'(6), \int_{-1}^{2} x^3 \lambda p(x) dx\right) \\ &= \left(\lambda(3p(4) + 5p'(6)), \lambda \int_{-1}^{2} x^3 p(x) dx\right) \\ &= \lambda \left(3p(4) + 5p'(6), \int_{-1}^{2} x^3 p(x) dx\right) \\ &= \lambda T p. \end{aligned}$$

Therefore, T is linear.

Exercise 3
Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, ..., m$ and $k = 1, ..., n$ such that

$$T(x_1, ..., x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, ..., A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, ..., x_n) \in \mathbf{F}^n$.

Solution
Denote by $e_1, ..., e_n$ the standard basis of $\mathbf{F}^n$ and by $f_1, ..., f_m$ the standard basis for $\mathbf{F}^m$, then for all $k \in \{1, ..., n\}$, there exist scalars $A_{1,k}, ..., A_{m,k} \in \mathbf{F}$ such that

$$Te_k = A_{1,k}f_1 + \cdots + A_{m,k}f_m.$$

Therefore, by linearity, for all $(x_1, ..., x_n) \in \mathbf{F}^n$:

$$\begin{aligned} T(x_1, ..., x_n) &= x_1 Te_1 + \cdots + x_n Te_n \\ &= x_1(A_{1,1}f_1 + \cdots + A_{m,1}f_m) + \cdots + x_n(A_{1,n}f_1 + \cdots + A_{m,n}f_m) \\ &= (A_{1,1}x_1 + \cdots + A_{1,n}x_n)f_1 + \cdots + (A_{m,1}x_1 + \cdots + A_{m,n}x_n)f_m \\ &= (A_{1,1}x_1 + \cdots + A_{1,n}x_n, ..., A_{m,1}x_1 + \cdots + A_{m,n}x_n). \end{aligned}$$

Therefore, any linear transformation has this form.

Exercise 4
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $Tv_1, ..., Tv_m$ is a linearly independent list in W. Prove that $v_1, ..., v_m$ is linearly independent.

Solution
To prove that $v_1, ..., v_m$ is linearly independent, take arbitrary scalars $\alpha_1, ..., \alpha_m \in \mathbf{F}$ such that

$$\alpha_1 v_1 + ... + \alpha_m v_m = 0.$$

By evaluating on both sides by T, we get by linearity of T the following equation:

$$\alpha_1 Tv_1 + ... + \alpha_m Tv_m = 0.$$

But since the list $Tv_1, ..., Tv_m$ is a linearly independent in W, then

$$\alpha_1 = ... = \alpha_m = 0$$

which proves that $v_1, ..., v_m$ is linearly independent.

Exercise 5
Prove that $\mathcal{L}(V, W)$ is a vector space, as was asserted in 3.6.

Solution
We already proved in Section 1B Exercise 7 that for any nonempty set S and vector space U, the set U^S equipped with the usual addition and scalar multiplication is a vector space. Hence, if we let $S = V$ and $U = W$, we already know that the set of functions from V tp W is a vector space. Since $\mathcal{L}(V, W) \subset W^V$, then it suffices to show that $\mathcal{L}(V, W)$ is a subspace.
First, notice that $\mathcal{L}(V, W)$ is non-empty since it contains the additive identity map: the constant zero map is linear. Given two linear maps $T_1, T_2 \in \mathcal{L}(V, W)$, we can show that $T_1 + T_2 \in \mathcal{L}(V, W)$ by proving that it is a linear map from V to W. Hence, take arbitrary $x, y \in V$ and $\lambda \in \mathbf{F}$ to get:

$$\begin{aligned}(T_1 + T_2)(x + y) &= T_1(x + y) + T_2(x + y)\\ &= T_1(x) + T_1(y) + T_2(x) + T_2(y)\\ &= (T_1 + T_2)(x) + (T_1 + T_2)(y),\end{aligned}$$

and

$$\begin{aligned}(T_1 + T_2)(\lambda x) &= T_1(\lambda x) + T_2(\lambda x)\\ &= \lambda T_1(x) + \lambda T_2(x)\\ &= \lambda(T_1(x) + T_2(x))\\ &= \lambda(T_1 + T_2)(x).\end{aligned}$$

Thus, $\mathcal{L}(V, W)$ is closed under addition. Similarly, given a linear map $T \in \mathcal{L}(V, W)$ and $\alpha \in \mathbf{F}$, we get that $\alpha T \in \mathcal{L}(V, W)$ because for all $x, y \in V$ and $\lambda \in \mathbf{F}$, we have the following:

$$\begin{aligned}(\alpha T)(x + y) &= \alpha T(x + y)\\ &= \alpha(T(x) + T(y))\\ &= \alpha T(x) + \alpha T(y)\\ &= (\alpha T)(x) + (\alpha T)(y)\end{aligned}$$

and

$$\begin{aligned}(\alpha T)(\lambda x) &= \alpha T(\lambda x)\\ &= \alpha\lambda T(x)\\ &= \lambda\alpha T(x)\\ &= \lambda(\alpha T)(x).\end{aligned}$$

Therefore, $\mathcal{L}(V, W)$ is a vector space since it is a subspace of W^V.

Exercise 6
Prove that the multiplication of linear maps has the associative, identity and distributive properties asserted in 3.8.

Solution

- (Associativity) Let V_1, V_2, V_3, V_4 be vector spaces and $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$ and $T_3 : V_3 \to V_4$ be linear maps. Associativity follows from the fact that for all $x \in V_1$:
$$\begin{aligned}((T_1T_2)T_3)(x) &= (T_1T_2)(T_3(x))\\ &= T_1(T_2(T_3(x)))\\ &= T_1(T_2T_3(x))\\ &= (T_1(T_2T_3))(x).\end{aligned}$$
Since it holds for all $x \in X$, then $(T_1T_2)T_3 = T_1(T_2T_3)$.

- (Identity) Let V and W be vector space. Consider the identity map $I_V : V \to W$ and let's show that it is indeed linear. For all $x, y \in V$:
$$I_V(x+y) = x+y = I_V(x) + I_V(y)$$
and for any $\lambda \in \mathbf{F}$ and $x \in V$:
$$I_V(\lambda x) = \lambda x = \lambda I_V(x).$$
Therefore, I_V is linear. To prove that it is the multiplicative identity in $\mathcal{L}(V, W)$, let $T : V \to W$ be a linear map and $x \in V$, then
$$(I_VT)(x) = I_V(Tx) = Tx$$
and
$$(TI_V)(x) = T(I_Vx) = Tx$$
so $I_VT = TI_V = T$ for all linear maps $T \in \mathcal{L}(V, W)$.

- (Distributivity 1) Let U, V, W be vector spaces, $S_1, S_2 \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, then for all $x \in V$, we have
$$\begin{aligned}[(S_1+S_2)T](x) &= (S_1+S_2)(Tx)\\ &= S_1(Tx) + S_2(Tx)\\ &= (S_1T)(x) + (S_2T)(x)\\ &= [S_1T + S_2T](x).\end{aligned}$$
Since it holds for all $x \in U$, then $(S_1+S_2)T = S_1T + S_2T$.

- (Distributivity 2) Let U, V, W be vector spaces, $S \in \mathcal{L}(V, W)$ and $T_1, T_2 \in \mathcal{L}(U, V)$, then for all $x \in V$ and by linearity of S, we have
$$\begin{aligned}[S(T_1+T_2)](x) &= S((T_1+T_2)(x))\\ &= S(T_1(x) + T_2(x))\\ &= S(T_1(x)) + S(T_2(x))\\ &= (ST_1)(x) + (ST_2)(x)\\ &= [ST_1 + ST_2](x).\end{aligned}$$

Since it holds for all $x \in U$, then $S(T_1 + T_2) = ST_1 + ST_2$.

Exercise 7
Show that every linear map from a one-dimensional vector space to itself is multiplicative by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution
Since $\dim V = 1$, then there is a $v_0 \in V$ such that $V = \operatorname{span}(v_0)$. We have $Tv_0 \in \operatorname{span}(v_0)$ so there is a $\lambda \in \mathbf{F}$ satisfying $Tv_0 = \lambda v_0$. Take $v \in V$, since $v \in \operatorname{span}(v_0)$, then there is an $\alpha \in \mathbf{F}$ such that $v = \alpha v_0$. Thus:

$$Tv = T\alpha v_0 = \alpha T v_0 = \alpha\lambda v_0 = \lambda v.$$

Exercise 8
Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

Solution
Consider the function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ defined by $\varphi(x, y) = \sqrt[3]{(x+y)^3}$, then for all $x, y \in \mathbf{R}$ and $a \in \mathbf{R}$, we have

$$\begin{aligned}\varphi(ax, ay) &= \sqrt[3]{(ax+ay)^3} \\ &= \sqrt[3]{a^3(x+y)^3} \\ &= a\sqrt[3]{(x+y)^3} \\ &= a\varphi(x, y).\end{aligned}$$

However, notice that $\varphi(1, 0) = \varphi(0, 1) = 1$ but $\varphi(1, 1) = \sqrt[3]{2}$ so $\varphi(1, 1) \neq \varphi(1, 0) + \varphi(0, 1)$ so φ is not linear.

Exercise 9
Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here, $\mathbf{C}$ is thought of as a complex vector space.)

Solution
Consider the function $\varphi : \mathbf{C} \to \mathbf{C}$ defined by $\varphi(z) = \operatorname{Re}(z)$, then for all $w, z \in \mathbf{C}$, we know that

$$\operatorname{Re}(w + z) = \operatorname{Re}(w) + \operatorname{Re}(z).$$

However, $\operatorname{Re}(i) = 0$ and $i\operatorname{Re}(1) = i$ so $\operatorname{Re}(i \cdot 1) \neq i\operatorname{Re}(1)$. Therefore, φ is not linear.

Exercise 10
Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by

$Tp = q \circ p$, then T is a linear map.

Solution
Consider the following counterexample: Take $q = x^2$ and define the map $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by
$$Tp = q \circ p = p^2$$
for all $p \in \mathcal{P}(\mathbf{R})$. Notice that $T(x+1) = x^2 + 2x + 1$ but $T(x) + T(1) = x^2 + 1$. Thus, $T(x+1) \neq T(x) + T(1)$ so T is not a linear map.

Exercise 11
Suppose V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

Solution
First, let T be a scalar multiple of the identity, then there is a $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$. Let S be an arbitrary linear map from V to V, then for all $v \in V$:
$$(ST)v = S(Tv) = S(\lambda v) = \lambda Sv = T(Sv) = (TS)v.$$
Since it holds for all $v \in V$, then $ST = TS$.
To prove that the converse holds, fix a basis $v_1, ..., v_n$ of V and for all i between 1 and n, define S_i as the linear map satisfying $S_i v_1 = v_i$ and $S_i v_k = 0$ for all $k \neq 1$ (such a linear map is well-defined and unique by Lemma 3.4). Let $T \in \mathcal{L}(V)$, then for all i between 1 and n, there exist scalars $A_{i,1}, ..., A_{i,n} \in \mathbf{F}$ such that
$$Tv_i = A_{i,1}v_1 + ... + A_{i,n}v_n.$$
Suppose that T satisfies $ST = TS$ for all $S \in \mathcal{L}(V)$, then in particular, for all fixed i between 1 and n, we have $(S_iT)v_1 = (TS_i)v_1$. Using the definitions and properties of S_i and T, we get that
$$\begin{aligned}(S_iT)v_1 = (TS_i)v_1 &\implies S_i(A_{1,1}v_1 + ... + A_{1,n}v_n) = Tv_i \\ &\implies A_{1,1}v_i = A_{i,1}v_1 + ... + A_{i,n}v_n\end{aligned}$$
and by uniqueness of representations of vectors in V as linear combinations of the basis, we get that $A_{i,j} = 0$ for all $i \neq j$ and $A_{1,1} = A_{i,i}$. Thus, if we let $\lambda = A_{1,1}$, we obtain that for all i,
$$Tv_i = A_{i,1}v_1 + ... + A_{i,n}v_n = \lambda v_i.$$
Therefore, it follows that T is equal to λ times the identity map, i.e., a scalar multiple of the identity.

Exercise 12
Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by
$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$
Prove that T is not a linear map on V.

Solution
Since $U \neq V$, then there is a $v_0 \in V \setminus U$. The fact that S is not the zero transformation implies that there is a vector $u \in U$ such that $Su \neq 0$. Moreover, since U is a subspace and $u \in U$, then $v_0 + u \in U$ implies that $v_0 \in U$. A contradiction that shows that $u + v_0 \in V \setminus U$. Thus, by definition of T, we have $Tv_0 = 0$ and $T(v_0 + u) = 0$. If T is linear, then we would get

$$0 = T(v_0 + u) = Tv_0 + Tu = Su.$$

But this is a contradiction since we defined u such that $Su \neq 0$. Thus, no such linear transformation T exists.

Exercise 13
Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution
Let $u_1, ..., u_n$ be a basis of U, then it can be extended to a basis $u_1, ..., u_m$ of V where $m \geq n$. Define T on this basis as follows: $Tu_i = Su_i$ if $i \leq n$ and $Tu_i = 0$ otherwise. By Lemma 3.4, T is a well-defined linear map from V to W. Let's now prove that T extends S. Let $u \in U$, then there exist scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$ such that

$$u = \alpha_1 u_1 + ... + \alpha_n u_n.$$

Applying T on both sides an using the linearity of T, we get

$$Tu = \alpha_1 Tu_1 + ... + \alpha_n Tu_n.$$

By construction of T, we know that $Tu_i = Su_i$ for all i between 1 and n:

$$Tu = \alpha_1 Su_1 + ... + \alpha_n Su_n.$$

Finally, by linearity of S:

$$Tu = S(\alpha_1 u_1 + ... + \alpha_n u_n) = Su.$$

It follows that T is linear map that extends S on V.

Exercise 14
Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Solution
Let $v_1, ..., v_n$ be a basis of V. From Section 2A Exercise 17, we know that there exists a sequence $w_1, w_2, ...$ in W such that the list $w_1, ..., w_m$ is linearly independent for all m. For all k, define the map $T_k : V \to W$ to be the unique linear map such that $T_k v_1 = w_k$ and $T_k v_i = 0$ for all i between 2 and n. Let's show that for all m, the list $T_1, ..., T_m$ is linearly independent in $\mathcal{L}(V, W)$. Let $\alpha_1, ..., \alpha_n \in \mathbf{F}$ be scalars such that

$$\alpha_1 T_1 + ... + \alpha_m T_m = 0,$$

then in particular, if we plug-in v_1, we get

$$\alpha_1 w_1 + ... + \alpha_m w_m = 0.$$

By our assumption on the sequence $w_1, w_2, ...$, we know that it implies that $\alpha_1 = ... = \alpha_m = 0$. Thus, the list $T_1, ..., T_m$ is linearly independent. Since it holds for all m, then by Section 2A Exercise 17, $\mathcal{L}(V, W)$ is infinite-dimensional.

Exercise 15
Suppose $v_1, ..., v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, ..., w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, ..., m$.

Solution
If the list has length 1, then v_1 must be zero vector so it suffices to take $w_1 \in W \setminus \{0\}$. Hence, every linear map T would map v_1 to zero which is different than w_1.
Assume that $m > 1$, since the list $v_1, ..., v_m$ is linearly independent, then without loss of generality, we can assume that v_m can be written as a linear combination of the other vectors. Thus, let $w_1 = ... = w_{m-1} = 0$ and $w_m \in W \setminus \{0\}$ (which must exists since $W \neq \{0\}$). Let T be a linear map and suppose that $Tv_k = w_k$ for each $k = 1, ..., m$. However, since there exist scalars $\alpha_1, ..., \alpha_{m-1}$ such that

$$v_m = \alpha_1 v_1 + ... + \alpha_{m-1} v_{m-1},$$

then by applying T on both sides, we get

$$Tv_m = \alpha_1 Tv_1 + ... + \alpha_{m-1} Tv_{m-1} = 0 \neq w_m.$$

Therefore, no linear map T satisfies $Tv_k = w_k$ for each $k = 1, ..., m$.

Exercise 16
Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

Solution
We know from Exercise 11 that the linear maps that commute with every other linear map are precisely the scalar multiples of the identity map. Hence, it suffices to show that there exists a linear map that is not a scalar multiple of the identity. Let $v_1, ..., v_n$ be a basis of V (so $n \geq 2$) and define $T : V \to V$ to be the unique linear map such that $Tu_1 = u_1$ and $Tu_i = 0$ for i between 2 and n. Such a transformation exists by Lemma 3.4. If T was a scalar multiple of the identity, then $Tu_1 = u_1$ would imply that T is the identity since u_1. However, $Tu_2 = 0$ even if $u_2 \neq 0$. Thus, by contradiction, T is not a scalar multiple of the identity. Therefore, there must be a linear map S such that $ST \neq TS$.

Exercise 17
Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

Solution
Let $\mathcal{E}$ be a two-sided ideal of $\mathcal{L}(V)$, if $\mathcal{E} = \{0\}$, then we are done. Assume that $\mathcal{E} \neq \{0\}$, then there must be a non-zero linear map T in $\mathcal{E}$ and scalars $\{A_{i,j}\} \subset \mathbf{F}$ such that

$$Tv_j = A_{1,j}v_1 + ... + A_{n,j}v_n$$

for all j between 1 and n. Since T is non-zero, then it follows that there exist i_0 and j_0 between 1 and n such that $A_{i_0,j_0} \neq 0$. Moreover, for all i and j between 1 and n, define the linear map $S_{i,j} \in \mathcal{L}(V)$ by

$$S_{i,j}v_k = \begin{cases} v_i & k = j, \\ 0 & k \neq j. \end{cases}$$

Consider the map $\frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T S_{j_0,j_0}$, since $\mathcal{E}$ is a two-sided ideal, then this map belongs to $\mathcal{E}$. Let k be an integer between 1 and n, if $k \neq j_0$, then

$$\frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T S_{j_0,j_0} v_k = \frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T(0) = 0,$$

and if $k = j_0$, then

$$\begin{aligned} \frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T S_{j_0,j_0} v_k &= \frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T v_{j_0} \\ &= \frac{1}{A_{i_0,j_0}} S_{i_0,i_0} (A_{1,j_0}v_1 + ... + A_{n,j_0}v_n) \\ &= \frac{1}{A_{i_0,j_0}} A_{i_0,j_0} v_{i_0} \\ &= v_{i_0}. \end{aligned}$$

Thus, by definition of the maps $S_{i,j}$'s and by uniqueness part of Lemma 3.4, we get that

$$\frac{1}{A_{i_0,j_0}} S_{i_0,i_0} T S_{j_0,j_0} = S_{i_0,j_0}.$$

Hence, the map S_{i_0,j_0} is in $\mathcal{E}$. From this, we get that for all i and j between 1 and n, the map $S_{i,i_0} S_{i_0,j_0} S_{j_0,j}$ is in $\mathcal{E}$ as well. But notice that for all k between 1 and n, if $k \neq j$, then

$$S_{i,i_0} S_{i_0,j_0} S_{j_0,j} v_k = 0,$$

and $k = j$, then

$$S_{i,i_0} S_{i_0,j_0} S_{j_0,j} v_k = S_{i,i_0} S_{i_0,j_0} v_{j_0} = S_{i,i_0} v_{i_0} = v_i.$$

Thus, again, by the uniqueness part of Lemma 3.4 and since it holds for all i, j, then $S_{i,j} \in \mathcal{E}$ for all i, j. We are now ready to show that $\mathcal{E} = \mathcal{L}(V)$. Since $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$, then it suffices to prove that $\mathcal{L}(V) \subset \mathcal{E}$. Let $S \in \mathcal{L}(V)$, then there exist scalars $\{B_{i,j}\}_{i,j} \subset \mathbf{F}$ such that

$$Sv_j = B_{1,j}v_1 + ... + B_{n,j}v_n,$$

for all j. Consider the map $\tilde{S}$ defined by

$$\tilde{S} = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{i,k} S_{i,k}.$$

Since $\mathcal{E}$ is a subspace that contains all the $S_{i,j}$'s, then $\tilde{S} \in \mathcal{E}$. Moreover, notice that for all j,

$$\tilde{S}v_j = \sum_{i=1}^{n} \sum_{k=1}^{n} B_{i,k} S_{i,k} v_j = \sum_{i=1}^{n} B_{i,j} v_i = S v_j.$$

Since it holds for all j, then by Lemma 3.4, we have that $S = \tilde{S} \in \mathcal{E}$. Since it holds for all $S \in \mathcal{L}(V)$, then $\mathcal{L}(V) = \mathcal{E}$. Therefore, the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

3B Null Spaces and Ranges

Exercise 1
Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution
Consider the map $T \in \mathcal{L}(\mathbf{R}^5)$ defined by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0, 0).$$

The range of T is equal to

$$\{(x_1, x_2, 0, 0, 0) : x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}\} = \{(x_1, x_2, 0, 0, 0) : x_1, x_2 \in \mathbf{R}\}.$$

It follows that $\dim \operatorname{range} T = 2$. Concerning the null space, notice that

$$\begin{aligned} \operatorname{null} T &= \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : (x_1, x_2, 0, 0, 0) = 0\} \\ &= \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = x_2 = 0\} \\ &= \{(0, 0, x_3, x_4, x_5) \in \mathbf{R}^5 : x_3, x_4, x_5 \in \mathbf{R}\}. \end{aligned}$$

It follows that $\dim \operatorname{null} T = 2$.

Exercise 2
Suppoe $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subset \operatorname{null} T$. Prove that $(ST)^2 = 0$.

Solution
To prove this, let's show that $(ST)^2$ maps every vector in V to 0. Let $v \in V$, then $Tv \in V$ so $STv \in \operatorname{range} S \subset \operatorname{null} T$. It follows that $T(STv) = 0$. Thus, we obtain that

$$(ST)^2 v = STSTv = ST(Stv) = S0 = 0.$$

Therefore, $(ST)^2 = 0$.

Exercise 3
Suppose $v_1, ..., v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, ..., z_m) = z_1 v_1 + ... + z_m v_m.$$

(a) What property of T corresponds to $v_1, ..., v_m$ spanning V?

(b) What property of T corresponds to the list $v_1, ..., v_m$ being linearly independent?

Solution

(a) Let's prove that T is surjective if and only if $v_1, ..., v_m$ spans V. If T is surjective, then for all $v \in V$, there exists a vector $(z_1, ..., z_m) \in \mathbf{F}^m$ such that $T(z_1, ..., z_m) = v$. But this just means that for all $v \in V$, there exist scalars $z_1, ..., z_m \in \mathbf{F}$ such that $z_1 v_1 + ... + z_m v_m = v$. Hence, by definition, the list is spanning.
To prove the converse, suppose that $v_1, ..., v_m$ spans V, then for all $v \in V$, there exist scalars, $z_1, ..., z_m$ such that $z_1 v_1 + ... + z_m v_m = v$. It follows that $T(z_1, ..., z_m) = v$. Therefore, by definition, T is surjective.

(b) Let's prove that T is injective if and only if $v_1, ..., v_m$ is linearly independent. If T is injective, then for all scalars $\alpha_1, ..., \alpha_m \in \mathbf{F}$ satisfying $\alpha_1 v_1 + ... + \alpha_m v_m = 0$, we can rewrite the equation as

$$T(\alpha_1, ..., \alpha_m) = 0.$$

But notice that $T(0, ..., 0) = 0$ as well so have

$$T(\alpha_1, ..., \alpha_m) = T(0, ..., 0).$$

Thus, by injectivity, we get that $\alpha_1 = ... = \alpha_m = 0$. Therefore, the list is linearly independent.
Suppose now that the list is linearly independent and let $(z_1, ..., z_m) \in \mathbf{F}^m$ be an element in the null space of T, then

$$T(z_1, ..., z_m) = 0,$$

which can be rewritten as

$$z_1 v_1 + ... + z_m v_m = 0.$$

By linear independence of $v_1, ..., v_m$, we obtain that $z_1 = ... = z_m = 0$. Thus, the null space of T is precisely $\{0\}$. Therefore, by Proposition 3.15, we get that T is injective.

Exercise 4
Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Solution
Consider the maps $T_1, T_2 \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ defined by

$$T_1(x_1, x_2, x_3, x_4, x_5) = (x_1, 0, 0, 0)$$

$$T_2(x_1, x_2, x_3, x_4, x_5) = (0, x_2, 0, 0)$$

for all $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5$. Both T_1, T_2 have a 1-dimensional range so by the Fundamental Theorem of Linear Maps, both T_1 and T_2 have 3-dimensional null space. It follows that both T_1 and T_2 are in $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$. However, notice that

$$(T_1 + T_2)(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$$

so it has a 2-dimensional range. Thus, by the Fundamental Theorem of Linear Maps, $\dim \operatorname{null}(T_1 + T_2) = 2$ so $T_1 + T_2 \notin \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$. Therefore, the set is not closed under addition which proves that it cannot be a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Exercise 5
Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution
Consider the map $T \in \mathcal{L}(\mathbf{R}^4)$ defined by

$$T(a, b, c, d) = (c, d, 0, 0)$$

for all $a, b, c, d \in \mathbf{R}$. To prove that $\operatorname{range} T = \operatorname{null} T$, let $(a, b, c, d) \in \operatorname{range} T$, then there exist real number x_1, x_2, x_3, x_4 such that

$$(a, b, c, d) = T(x_1, x_2, x_3, x_4).$$

By definition of T, we get that

$$(a, b, c, d) = (x_3, x_4, 0, 0)$$

which implies that $c = d = 0$. Hence,

$$T(a, b, c, d) = (c, d, 0, 0) = 0$$

so $(a, b, c, d) \in \operatorname{null} T$. To prove the reverse inclusion, let $(a, b, c, d) \in \operatorname{null} T$, then

$$T(a, b, c, d) = 0,$$

so by definition of T:

$$(c, d, 0, 0) = 0.$$

It follows that $(a, b, c, d) = (a, b, 0, 0)$ which shows that we can write (a, b, c, d) as the image of $(0, 0, a, b)$, i.e., (a, b, c, d) is in the range of T. Therefore, both sets are equal.

Exercise 6
Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

Solution
By contradiction, suppose that such a linear map T exists, then by the Fundamental Theorem of Linear maps, we get that

$$\dim \mathbf{R}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T$$

which we can rewrite as

$$5 = 2 \dim \operatorname{range} T.$$

But this is a contradiction since 5 is odd. Therefore, no such transformation exists.

Exercise 7
Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution
Let $n = \dim V$, $m = \dim W$, let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ be a basis of W. Define the linear maps $T_1, T_2 \in \mathcal{L}(V, W)$ by

$$T_1 v_i = \begin{cases} w_1 & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

and

$$T_2 v_i = \begin{cases} 0 & \text{if } i = 1, \\ w_i & \text{if } i \geq 2. \end{cases}$$

Notice that both maps are not injective since for each of them, you can find a non-zero vector such that the map evaluated at this vector is 0 (for example, $T_1 v_2 = 0$ and $T_2 v_1 = 0$). It follows that both maps are in $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$. Consider now the map $T = T_1 + T_2 \in \mathcal{L}(V, W)$, then T satisfies

$$T v_i = w_i$$

for all i between 1 and n. Since the list $w_1, ..., w_n$ is linearly independent, then for all $v = \alpha_1 v_1 + ... + \alpha_n v_n \in V$:

$$\begin{aligned} Tv = 0 &\iff T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0 \\ &\iff \alpha_1 w_1 + ... + \alpha_n w_n = 0 \\ &\iff \alpha_1 = ... = \alpha_n = 0 \\ &\iff v = 0. \end{aligned}$$

It follows from Proposition 3.15 that T is injective so $T \notin \{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$. Therefore, it is not a subspace of $\mathcal{L}(V, W)$ since it is not closed under addition.

Exercise 8
Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution
Let $n = \dim V$, $m = \dim W$, let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ be a basis of W. Define the linear maps $T_1, ..., T_m \in \mathcal{L}(V, W)$ by

$$T_i v_k = \begin{cases} w_i & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

For each i, the range of T_i is equal to the span of w_i which is 1-dimensional. Since $\dim W \geq 2$, then the range of T_i must be a proper subset of W which implies that all T_i's are not surjective. It follows that $T_i \in \{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ for all i between 1 and m. Consider now the map $\tilde{T} = T_1 + ... + T_m$. For all $w \in W$, there exist scalars $\alpha_1, ..., \alpha_m$ such that $w = \alpha_1 w_1 + ... + \alpha_m w_m$. Thus, by linearity and definition of $\tilde{T}$, we get that

$$\begin{aligned} \tilde{T}(\alpha_1 v_1 + ... + \alpha_m v_m) &= T_1(\alpha_1 v_1 + ... + \alpha_m v_m) + ... + T_m(\alpha_1 v_1 + ... + \alpha_m v_m). \\ &= [\alpha_1 T_1 v_1 + ... + \alpha_m T_1 v_m] + ... + [\alpha_1 T_m v_1 + ... + \alpha_m T_m v_m] \\ &= [\alpha_1 w_1 + 0 + ... + 0] + ... + [0 + ... + 0 + \alpha_m w_m] \\ &= \alpha_1 w_1 + ... + \alpha_m w_m \\ &= w \end{aligned}$$

Hence, $w \in \operatorname{range} \tilde{T}$. Since it holds for all $w \in W$, then $\tilde{T}$ is surjective so $\tilde{T} \notin \{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$. Therefore, it is not a subspace of $\mathcal{L}(V, W)$ since it is not closed under addition.

Exercise 9
Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, ..., v_n$ is linearly independent in V. Prove that $Tv_1, ..., Tv_n$ is linearly independent in W.

Solution
Let $\alpha_1, ..., \alpha_n$ be scalars such that

$$\alpha_1 Tv_1 + ... + \alpha_n Tv_n = 0,$$

then by linearity,

$$T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0,$$

by injectivity,

$$\alpha_1 v_1 + ... + \alpha_n v_n = 0,$$

and by linear independence of the list $v_1, ..., v_n$, we obtain that $\alpha_1 = ... = \alpha_n = 0$. Therefore, the list $Tv_1, ..., Tv_n$ is linearly independent.

Exercise 10
Suppose $v_1, ..., v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, ..., Tv_n$ spans $\operatorname{range} T$.

Solution
Let $w \in \operatorname{range} T$, then there exists a vector $v \in V$ such that $w = Tv$. Since $v_1, ..., v_n$ spans V, there exist scalars $\alpha_1, ..., \alpha_n \in \mathbf{F}$ such that

$$v = \alpha_1 v_1 + ... + \alpha_n v_n.$$

If we apply T on both sides, we get that

$$Tv = T(\alpha_1 v_1 + ... + \alpha_n v_n).$$

By linearity of T and definition of v, we get that

$$w = \alpha_1 Tv_1 + ... + \alpha_n Tv_n.$$

Since it holds for all $w \in \operatorname{range} T$, then the list $Tv_1, ..., Tv_n$ spans $\operatorname{range} T$.

Exercise 11
Suppose that V is finite-dimensional and thet $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

Solution
Since $\operatorname{null} T$ is a subspace of V, then by Proposition 2.33, there exists a subspace U of V such that $V = U \oplus \operatorname{null} T$. It follows that $U \cap \operatorname{null} T = \{0\}$. Moreover, since $V = U + \operatorname{null} T$, then

$$\begin{aligned}
\operatorname{range} T &= \{Tv : v \in V\} \\
&= \{T(u+n) : u \in U \text{ and } n \in \operatorname{null} T\} \\
&= \{Tu + Tn : u \in U \text{ and } n \in \operatorname{null} T\} \\
&= \{Tu : u \in U \text{ and } n \in \operatorname{null} T\} \\
&= \{Tu : u \in U\}
\end{aligned}$$

Therefore, U satisfies all the required properties.

Exercise 12
Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution
First, let's prove that $\operatorname{null} T$ has dimension 2. Consider the vectors $u_1 = (5, 1, 0, 0)$ and $u_2 = (0, 0, 7, 1)$. Obviously, the list u_1, u_2 is linearly independent and is contained in $\operatorname{null} T$. Moreover, for all $u \in U$, we know that there exist $x_2, x_4 \in \mathbf{F}$ such that $u = (5x_2, x_2, 7x_4, x_4)$. It follows that $u = x_2 u_1 + x_4 u_2$. Therefore, u_1, u_2 spans $\operatorname{null} T$ which proves that it is a basis and that $\operatorname{null} T$ has dimension 2. Thus, by the Fundamental Theorem of Linear Maps, we have

$$\dim \mathbf{F}^4 = \dim \operatorname{null} T + \dim \operatorname{range} T$$

which we can now write as

$$4 = 2 + \dim \operatorname{range} T.$$

Therefore, $\dim \operatorname{range} T = 2 = \dim \mathbf{F}^2$. Since $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$ of dimension 2, then $\operatorname{range} T = \mathbf{F}^2$. It follows that T is surjective.

Exercise 13
Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

Solution
By the Fundamental Theorem of Linear Maps, we know that

$$\dim \mathbf{R}^8 = \dim U + \dim \operatorname{range} T.$$

By plugging-in the known values, we get

$$8 = 3 + \dim \operatorname{range} T.$$

Thus, $\dim \operatorname{range} T = 5 = \dim \mathbf{R}^5$. Since $\operatorname{range} T$ is a subspace of $\mathbf{R}^5$ of dimension 5, then $\operatorname{range} T = \mathbf{R}^5$. Therefore, T is surjective.

Exercise 14
Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Solution
Define $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$ and let's determine its dimension. Consider the vectors $u_1 = (3, 1, 0, 0, 0)$ and $u_2 = (0, 0, 1, 1, 1)$ and notice that the list u_1, u_2 is linearly independent and contained in U. Moreover, for all $u = (x_1, x_2, x_3, x_4, x_5) \in U$, we have $x_1 = 3x_2$ and $x_3 = x_4 = x_5$. It follows that

$$u = (3x_2, x_2, x_3, x_3, x_3) = x_2 u_1 + x_3 u_2.$$

Therefore, the list u_1, u_2 spans U so it is a basis of U. Hence, $\dim U = 2$. By contradiction, suppose that there is a map $T \in \mathcal{L}(\mathbf{F}^5, \mathbf{F}^2)$ such that $\operatorname{null} T = U$, then by the Fundamental Theorem of Linear Maps, we have

$$\dim \mathbf{F}^5 = \dim U + \dim \operatorname{range} T.$$

If we plug-in the known values, we get that $\dim \operatorname{range} T = 3$. However, since $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$, then

$$3 = \dim \operatorname{range} T \leq \dim \mathbf{F}^2 = 2,$$

a contradiction. Therefore, there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals U.

Exercise 15
Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution
Notice that we cannot use the Fundamental Theorem of Linear Maps since one of the assumptions of the Theorem is that V is finite-dimensional. Let $u_1, ..., u_n$ be a basis of $\operatorname{null} T$ and $v_1, ..., v_m$ be a basis of $\operatorname{range} T$. For each i between 1 and m, define w_i as a vector in V satisfying $Tw_i = v_i$. Now, let $v \in V$ and let $\alpha_1, ..., \alpha_m \in \mathbf{F}$ be scalars such that

$$Tv = \alpha_1 v_1 + ... + \alpha_m v_m,$$

then by definition of the w_i's and by linearity of T, we get that

$$Tv = T(\alpha_1 w_1 + ... + \alpha_m w_m).$$

By rearranging the terms, we get that

$$T(v - [\alpha_1 w_1 + ... + \alpha_m w_m]) = 0$$

so $v - (\alpha_1 w_1 + ... + \alpha_m w_m) \in \operatorname{null} T$. It follows that there exist scalars $\beta_1, ..., \beta_n$ such that

$$v - (\alpha_1 w_1 + ... + \alpha_m w_m) = \beta_1 u_1 + ... + \beta_n u_n.$$

Thus,

$$v = \beta_1 u_1 + ... + \beta_n u_n + \alpha_1 w_1 + ... + \alpha_m w_m$$

which shows that v is in the span of the list $u_1, ..., u_n, w_1, ..., w_m$. Since it holds for all $v \in V$, then the list $u_1, ..., u_n, w_1, ..., w_m$ spans V. Therefore, by definition, V is finite-dimensional.

Exercise 16
Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution
($\Longrightarrow$) This direction is simply the converse of Proposition 3.22.

($\Longleftarrow$) Suppose that $\dim V \leq \dim W$, let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ a basis of W. Define the map $T \in \mathcal{L}(V, W)$ by

$$Tv_i = w_i$$

for all i between 1 and n (which can be done since $n \leq m$). Notice that by linear independence of $w_1, ..., w_n$, we get that for all $v = \alpha_1 v_1 + ... + \alpha_n v_n \in V$,

$$\begin{aligned} Tv = 0 &\iff T(\alpha_1 v_1 + ... + \alpha_n v_n) = 0 \\ &\iff \alpha_1 w_1 + ... + \alpha_n w_n = 0 \\ &\iff \alpha_1 = ... = \alpha_n = 0 \\ &\iff v = 0. \end{aligned}$$

Therefore, $\operatorname{null} T = \{0\}$ which proves that T is injective.

Exercise 17
Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V to W if and only if $\dim V \geq \dim W$.

Solution
($\Longrightarrow$) This direction is simply the converse of Proposition 3.24.
($\Longleftarrow$) Suppose that $\dim V \geq \dim W$, let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$. Consider the linear transformation $T \in \mathcal{L}(V, W)$ defined by

$$Tv_i = \begin{cases} w_i & \text{if } 1 \leq i \leq m, \\ 0 & \text{if } m+1 \leq i \leq n. \end{cases}$$

Let $w = \alpha_1 w_1 + ... + \alpha_m w_m \in V$, then

$$T(\alpha_1 v_1 + ... + \alpha_m v_m) = \alpha_1 w_1 + ... + \alpha_m w_m = w$$

which proves that $w \in \operatorname{range} T$. Therefore, T is surjective.

Exercise 18
Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution
($\Longrightarrow$) Suppose that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Since $\operatorname{range} T$ is a subspace of W, then $\dim \operatorname{range} T \leq \dim W$. It follows by the Fundamental Theorem of Linear maps that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \leq \dim U + \dim W$$

which we can rearrange into

$$\dim U \geq \dim V - \dim W.$$

($\Longleftarrow$) Suppose now that $\dim U \geq \dim V - \dim W$, let $u_1, ..., u_n$ be a basis of U, extend it to a basis $u_1, ..., u_n, v_1, ..., v_m$ of V and let $w_1, ..., w_k$ be a basis of W. Consider the map $T \in \mathcal{L}(V, W)$ defined by

$$Tu_i = 0 \quad \text{and} \quad Tv_j = w_j$$

for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. This can be done because by our assumptions, $k \geq m$. Let's find the null space of T. First, by construction, $U \subset \operatorname{null} T$. Moreover, for all $v \in V$, there exist scalars $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m \in \mathbf{F}$ such that

$$v = \alpha_1 u_1 + ... + \alpha_n u_n + \beta_1 v_1 + ... + \beta_m v_m.$$

Thus,

$$\begin{aligned} Tv = 0 &\implies T(\alpha_1 u_1 + ... + \alpha_n u_n + \beta_1 v_1 + ... + \beta_m v_m) = 0 \\ &\implies \alpha_1 Tu_1 + ... + \alpha_n Tu_n + \beta_1 Tv_1 + ... + \beta_m Tv_m = 0 \\ &\implies 0 + ... + 0 + \beta_1 w_1 + ... + \beta_m w_m = 0 \\ &\implies \beta_1 = ... = \beta_m = 0 \\ &\implies v = \alpha_1 u_1 + ... + \alpha_n u_n. \\ &\implies v \in U. \end{aligned}$$

Therefore, $\operatorname{null} T = U$.

Exercise 19
Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

Solution
$(\implies)$ Suppose that T is injective, then $\dim \operatorname{null} T = \dim\{0\} = 0$ and $\dim \operatorname{range} T \leq \dim W < \infty$. Thus, by Exercise 15, V is finite-dimensional. Let $v_1, ..., v_n$ be a basis of V, then by Exercise 8 and 9, $Tv_1, ..., Tv_n$ is a basis of $\operatorname{range} T$. It can be extended to a basis $Tv_1, ..., Tv_n, w_1, ..., w_m$ of W. Define $S \in \mathcal{L}(W, V)$ by $S(Tv_i) = v_i$ for all $i \in \{1, ..., n\}$ and $Sw_i = 0$ for all $i \in \{1, ..., n\}$, then it follows that the operator $ST \in \mathcal{L}(V)$ satisfies $(ST)v_i = v_i$ for all $i \in \{1, ..., n\}$. By uniqueness in Lemma 3.4, it follows that ST is the identity operator on V.
$(\impliedby)$ Suppose there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V, then for all $x, y \in V$, we have

$$Tx = Ty \implies S(Tx) = S(Ty) \implies (ST)x = (ST)y \implies x = y.$$

Therefore, T is injective.

Exercise 20
Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

Solution
$(\implies)$ Suppose that T is surjective and let $w_1, ..., w_n$ be a basis of W. Define $S \in \mathcal{L}(W, V)$ on the basis as follows: by surjectivity, let Sw_i be equal to a vector v_i that satisfies $Tv_i = w_i$, then for all i between 1 and n, we have that

$$(TS)w_i = T(Sw_i) = Tv_i = w_i.$$

By uniqueness in Lemma 3.4, we get that TS must be the identity operator on W.
$(\impliedby)$ Suppose there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on

W, then for all $w \in W$, we have that $T(Sw) = (TS)w = w$ which proves that w is in the range of T. Since it holds for all $w \in W$, then T is surjective.

Exercise 21
Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

Solution
Let $T^{-1}U$ be the subset of V containing the $v \in V$ satisfying $Tv \in U$. Let's prove that it is a subspace of V. It is non-empty because $T0 = 0 \in U$. Moreover, given $u, v \in T^{-1}U$, we have that $T(u+v) = Tu+Tv \in U$ since U is closed under addition. Similarly, given $v \in T^{-1}U$ and $\alpha \in \mathbf{F}$, we have $T(\alpha v) = \alpha Tv \in U$. Therefore, it is a subspace of V. If we consider the restriction of T on $T^{-1}U$, we can apply the Fundamental Theorem of Linear Maps to get

$$\dim T^{-1}U = \dim \operatorname{null} T|_{T^{-1}U} + \dim \operatorname{range} T|_{T^{-1}U}. \tag{1}$$

But notice that the set of $v \in V$ such that $T|_{T^{-1}U}v = 0$ is the same as the set of $v \in V$ such that $Tv = 0$ since $0 \in U$. Thus, $\dim \operatorname{null} T|_{T^{-1}U} = \dim \operatorname{null} T$. Moreover, if $w \in \operatorname{range} T|_{T^{-1}U}$, then there is a $v \in T^{-1}U \subset V$ such that $T|_{T^{-1}U}v = Tv = w$ so $w \in \operatorname{range} T$. Moreover, since $v \in T^{-1}U$, then $w = Tv \in U$ so $w \in U \cap \operatorname{range} T$. It follows that $\operatorname{range} T|_{T^{-1}U} \subset U \cap \operatorname{range} T$. Conversely, if $w \in U \cap \operatorname{range} T$, then there is a $v \in V$ such that $Tv = w \in U$. But since $Tv \in U$, then by definition, $v \in T^{-1}U$, so it follows that $w = Tv = T|_{T^{-1}U}v$ which implies that $w \in \operatorname{range} T|_{T^{-1}U}$. Thus, $\operatorname{range} T|_{T^{-1}U} = U \cap \operatorname{range} T$. Plugging this into equation (1) gives us

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

Exercise 22
Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Solution
First, notice that

$$u \in \operatorname{null} ST \iff STu = 0 \iff Tu \in \operatorname{null} S \iff u \in \{v \in U : Tv \in \operatorname{null} S\}$$

which implies that $\operatorname{null} ST = \{v \in U : Tv \in \operatorname{null} S\}$. Therefore, by Exercise 21, we get that

$$\begin{aligned}\dim \operatorname{null} ST &= \dim\{v \in U : Tv \in \operatorname{null} S\} \\ &= \dim \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T).\end{aligned}$$

But since $\operatorname{null} S \cap \operatorname{range} T$ is a subspace of $\operatorname{null} S$, then its dimension is less than or equal to $\dim \operatorname{null} S$. Therefore,

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} T + \dim \operatorname{null} S$$

which is the desired inequality.

Exercise 23
Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

Solution
It suffices to show that $\dim \operatorname{range} ST$ is less than both $\dim \operatorname{range} S$ and $\dim \operatorname{range} T$. First, notice that for all $w \in \operatorname{range} ST$, there exists a $u \in U$ such that $w = STu = S(Tu)$. It follows that $w \in \operatorname{range} S$ so $\operatorname{range} ST \subset \operatorname{range} S$. Since both are vector spaces, then $\operatorname{range} ST$ is a subspace of $\operatorname{range} S$ which proves that $\dim \operatorname{range} ST \leq \dim \operatorname{range} S$. Consider now the restriction S_0 of S on the subspace $\operatorname{range} T$ of V, then by the Fundamental Theorem of Linear Maps, we get that

$$\dim \operatorname{range} T = \dim \operatorname{null} S_0 + \dim \operatorname{range} S_0 \geq \dim \operatorname{range} S_0.$$

But notice that if $w \in \operatorname{range} ST$, then there exists a $u \in U$ such that $w = STu = S(Tu)$. Since $Tu \in \operatorname{range} T$, then $w = S(Tu) \in \operatorname{range} S_0$ so we get that $\operatorname{range} ST \subset \operatorname{range} S_0$. Since both are vector spaces, we have that

$$\dim \operatorname{range} ST \leq \dim \operatorname{range} S_0 \leq \dim \operatorname{range} T.$$

Therefore, $\dim \operatorname{range} ST$ is less than both $\dim \operatorname{range} S$ and $\dim \operatorname{range} T$.

Exercise 24

(a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \leq 2$.

(b) Given an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \operatorname{range} TS = 2$.

Solution

(a) First, notice that if $v \in \operatorname{range} T$, then there exists a $w \in V$ such that $v = Tw$. Thus, since $ST = 0$, then $Sv = STw = 0$ so $v \in \operatorname{null} S$. It follows that $\operatorname{range} T \subset \operatorname{null} S$. Since both are vector spaces, then $\operatorname{range} T$ is a subspace of $\operatorname{null} S$ which implies that $\dim \operatorname{range} T \leq \dim \operatorname{null} T$. Now, using the previous exercise, we get that

$$\dim \operatorname{range} TS \leq \dim \operatorname{range} T \leq \dim \operatorname{null} S$$

and

$$\dim \operatorname{range} TS \leq \dim \operatorname{range} S.$$

If we suppose by contradiction that $\dim \operatorname{range} TS \geq 3$, then we get that

$$3 \leq \dim \operatorname{null} S \quad \text{and} \quad 3 \leq \dim \operatorname{range} S.$$

By adding both inequalities and using the Fundamental Theorem of Linear Maps, we conclude that

$$6 \leq \dim \operatorname{null} S + \dim \operatorname{range} S = \dim V = 5$$

which is a contradiction. Therefore, $\dim \operatorname{range} TS \leq 2$.

(b) Consider the maps $S, T \in \mathcal{L}(\mathbf{F}^5)$ defined by

$$S(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, 0, 0)$$

$$T(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, x_1, x_2).$$

for all $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$. Notice that for all $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$ST(x_1, x_2, x_3, x_4, x_5) = S(0, 0, 0, x_1, x_2) = 0$$

which shows that $ST = 0$. Moreover, for all $(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5$, we have

$$TS(x_1, x_2, x_3, x_4, x_5) = T(x_1, x_2, x_3, 0, 0) = (0, 0, 0, x_1, x_2)$$

which implies that

$$\operatorname{range} TS = \{(0, 0, 0, x_1, x_2) : x_1, x_2 \in \mathbf{F}^5\}.$$

Obviously, $(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$ is a basis of $\operatorname{range} TS$ so $\dim \operatorname{range} TS = 2$.

Exercise 25
Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S \subset \operatorname{null} T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

Solution
$(\implies)$ Suppose that $\operatorname{null} S \subset \operatorname{null} T$, let $u_1, ..., u_n$ be a basis of $\operatorname{range} S$ and extend it to a basis $u_1, ..., u_n, w_1, ..., w_m$ of W. Define $E \in \mathcal{L}(W)$ as follows: if $i \in \{1, ..., n\}$, then $u_i = Sv_i$ for some $v_i \in V$, define $Eu_i = Tv_i$, otherwise, define $Ew_j = 0$ for all $j \in \{1, ..., m\}$. To prove that $T = ES$, let $v \in V$ be arbitrary, then $Sv \in \operatorname{range} S$ which proves that there exist coefficiets $\alpha_1, ..., \alpha_n \in \mathbf{F}$ such that $Sv = \alpha_1 u_1 + ... + \alpha_n u_n$. It follows that

$$ESv = \alpha_1 Eu_1 + ... + \alpha_n Eu_n = T(\alpha_1 v_1 + ... + \alpha_n v_n). \tag{1}$$

Moreover, notice that

$$S(v - (\alpha_1 v_1 + ... + \alpha_n v_n)) = (\alpha_1 u_1 + ... + \alpha_n u_n) - (\alpha_1 Sv_1 + ... + \alpha_n Sv_n) = 0$$

which shows that $v - (\alpha_1 v_1 + ... + \alpha_n v_n) \in \operatorname{null} S$. Since $\operatorname{null} S \subset \operatorname{null} T$, then $v - (\alpha_1 v_1 + ... + \alpha_n v_n) \in \operatorname{null} T$ which implies that $Tv = T(\alpha_1 v_1 + ... + \alpha_n v_n)$. Thus, using equation (1), we get that $ESv = Tv$. Therefore, $T = ES$.
$(\impliedby)$ Suppose now that $T = ES$ for some $E \in \mathcal{L}(W)$, then given a $v \in \operatorname{null} S$, we have that $Tv = ESv = E(0) = 0$ so $v \in \operatorname{null} T$. Therefore, $\operatorname{null} S \subset \operatorname{null} T$.

Exercise 26
Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S \subset \operatorname{range} T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

Solution
$(\implies)$ Suppose $\operatorname{range} S \subset \operatorname{range} T$ and let $v_1, ..., v_n$ be a basis of V. For all $i \in \{1, ..., n\}$, we have that $Sv_i \in \operatorname{range} S \subset \operatorname{range} T$ so there exists a $w_i \in V$

such that $Sv_i = Tw_i$. Define the linear map $E \in \mathcal{L}(V)$ as the unique map satisfying $Ev_i = w_i$ for all $i \in \{1, ..., n\}$, then for all $i \in \{1, ..., n\}$, we have that $TEv_i = Tw_i = Sv_i$ which proves that $S = TE$ by uniqueness.
($\Longleftarrow$) Suppose there exists a $E \in \mathcal{L}(V)$ such that $S = TE$ and let $w \in \operatorname{range} S$, then there is a $v \in V$ such that $w = Sv = T(Ev) \in \operatorname{range} T$. Therefore, $\operatorname{range} S \subset \operatorname{range} T$.

Exercise 27
Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Solution
Let $v \in V$ and notice that

$$P(v - Pv) = Pv - P^2v = Pv - Pv = 0$$

so $v - Pv \in \operatorname{null} P$. Hence, we have that

$$v = (v - Pv) + Pv \in \operatorname{null} P + \operatorname{range} P.$$

Since it holds for all $v \in V$, then $V = \operatorname{null} P + \operatorname{range} P$. To prove that the sum is direct, let $v \in \operatorname{null} P \ \cap \ \operatorname{range} P$, then $Pv = 0$ and $v = Pv_0$ for some $v_0 \in V$. Using the fact that $P^2 = P$, we obtain

$$v = Pv_0 = P^2v_0 = Pv = 0$$

which proves that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Exercise 28
Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every non-constant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

Solution
To prove that D is surjective, let's show that $\operatorname{range} D$ contains $\mathcal{P}_n(\mathbf{F})$ for all natural numbers n. Fix a natural number n and define the list $p_0, p_1, ..., p_n$ by $p_i = D(x^{i+1})$ for all $i \in \{0, ..., n\}$. By construction, each p_i is in the range of D and by our assumption on D, each of the p_i has degree i. Thus, by Section 2C Exercise 9, the list $p_0, ..., p_n$ is a basis of $\mathcal{P}_n(\mathbf{F})$. Thus, since $\operatorname{range} D$ is a vector space that contains a basis of $\mathcal{P}_n(\mathbf{F})$, then it must contain $\mathcal{P}_n(\mathbf{F})$. Since it holds for all natural numbers n, then $\operatorname{range} D$ contains $\mathcal{P}(\mathbf{F})$. Therefore, D is surjective.

Exercise 29
Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

Solution
Let $n = \deg p$, if $q \in \mathcal{P}(\mathbf{R})$ has degree m, then $5q'' + 3q'$ has degree $m - 1$. It follows that if q satisfies $5q'' + 3q' = p$, then q must have degree $n + 1$. Thus, consider the linear map $T : \mathcal{P}_{n+1}(\mathbf{R}) \to \mathcal{P}_n(\mathbf{R})$ defined by $q \mapsto 5q'' + 3q'$, then by the Fundamental Theorem of Linear Maps, we have

$$\dim \mathcal{P}_{n+1}(\mathbf{R}) = \dim \operatorname{null} T + \dim \operatorname{range} T$$

which implies that

$$\dim \operatorname{range} T = n + 2 - \dim \operatorname{null} T.$$

Let's determine the dimension of the null space of T. If q satisfies $5q'' + 3q' = 0$, then q cannot have degree strictly greater than 0 because otherwise, $\deg 0 = \deg(5q'' + 3q') \geq 0$ which is a contradiction. Therefore, the elements in the null space are precisely the constant polynomials. It follows that $\dim T = 1$ so

$$\dim \operatorname{range} T = n + 1 = \dim \mathcal{P}_n(\mathbf{R}).$$

Thus, $\operatorname{range} T = \mathcal{P}_n(\mathbf{R})$ which proves that T is surjective. Since $p \in \mathcal{P}_n(\mathbf{R})$, then there exists a $q \in \mathcal{P}_{n+1}(\mathbf{R}) \subset \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

Exercise 30

Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

Solution

First, denote φu by $\alpha \in \mathbf{F} \setminus \{0\}$. Now, for all $v \in V$, notice that

$$\varphi\left(\frac{\varphi(v)}{\alpha} u\right) = \frac{\varphi(v)}{\alpha} \varphi(u) = \varphi(v).$$

It follows that $(v - \frac{\varphi(v)}{\alpha} u) \in \operatorname{null} \varphi$ since $\varphi(v) - \varphi(\frac{\varphi(v)}{\alpha} u) = 0$. Thus,

$$v = \left(v - \frac{\varphi(v)}{\alpha} u\right) + \frac{\varphi(v)}{\alpha} u \in \operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$$

which proves that $V = \operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$. To prove that the sum is direct, notice that for $v \in \operatorname{null} \varphi \ \cap \ \{au : a \in \mathbf{F}\}$, we have that $v = au$ with $\varphi(v) = 0$. But this implies that $a\varphi(u) = 0$ so $a = 0$ (since $\varphi(u)$ is non-zero). Thus, $v = 0$. Therefore, $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.

Exercise 31

Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

Solution

($\Longrightarrow$) Suppose that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$, then by the Fundamental Theorem of Linear Maps, we have that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim X + \dim Y$.

($\Longleftarrow$) Let $n = \dim X$, $m = \dim Y$ and suppose that $\dim V = +m$. Let $w_1, ..., w_m$ be a basis of Y, $u_1, ..., u_n$ be a basis of X and extend it to a basis $u_1, ..., u_n, v_1, ..., v_m$ of V. Define the linear map $T \in \mathcal{L}(V, W)$ on this basis by

$$Tu_i = 0 \ \ \forall i \in \{1, ..., n\} \qquad \text{and} \qquad Tv_i = w_i \ \ \forall i \in \{1, ..., m\}.$$

It follows by construction that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$.

Exercise 32
Suppose V is finite-dimensional with $\dim V > 1$. Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

Solution
Consider the set $U = \operatorname{null} \varphi$. We already know that U is a subspace of V. Moreover, given any $S \in U$ and $T \in \mathcal{L}(V)$, we have that $\varphi(ST) = \varphi(S)\varphi(T) = 0$ so $ST \in U$. Similarly, $TS \in U$ as well. Therefore, U is a two-sided ideal of $\mathcal{L}(V)$ which implies by Section 3A Exercise 17 that $U = \{0\}$ or $U = \mathcal{L}(V)$. By contradiction, if $U = \{0\}$, then $\operatorname{range} \varphi \neq \{0\}$ which implies that $\dim \operatorname{range} \varphi = \dim \mathbf{F} = 1$. Hence, both $\operatorname{null} \varphi$ and $\operatorname{range} \varphi$ are finite-dimensional, so by Exercise 15, V is finite-dimensional. Thus, by the Fundamental Theorem of Linear Maps, we get that

$$\dim V = \dim \operatorname{null} \varphi + \dim \operatorname{range} \varphi = 1.$$

But this contradicts the fact that $\dim V > 1$ so $U = \mathcal{L}(V)$. Therefore, $\varphi = 0$.

Exercise 33
Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}} : V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

for all $u, v \in V$.

(a) Show that $T_{\mathbf{C}}$ is a (complex) linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.

(b) Show that $T_{\mathbf{C}}$ is injective if and only if T is injective.

(c) Show that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ if and only if $\operatorname{range} T = W$.

Solution

(a) To prove that $T_{\mathbf{C}}$ is linear, let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ be elements of $V_{\mathbf{C}}$ with $u_1, u_2, v_1, v_2 \in V$, then

$$\begin{aligned} T_{\mathbf{C}}(u + v) &= T_{\mathbf{C}}((u_1 + v_1) + i(u_2 + v_2)) \\ &= T(u_1 + v_1) + iT(u_2 + v_2) \\ &= Tu_1 + Tv_1 + iTu_2 + iTv_2 \\ &= (Tu_1 + iTu_2) + (Tv_1 + iTv_2) \\ &= T_{\mathbf{C}}(u_1 + iu_2) + T_{\mathbf{C}}(v_1 + iv_2) \\ &= T_{\mathbf{C}}u + T_{\mathbf{C}}v \end{aligned}$$

and given any $\alpha = \alpha_1 + i\alpha_2 \in \mathbf{C}$ and $u = u_1 + iu_2 \in V_{\mathbf{C}}$ with $u_1, u_2 \in V$, then

$$\begin{aligned} T_{\mathbf{C}}(\alpha u) &= T_{\mathbf{C}}((\alpha_1 + i\alpha_2)(u_1 + iu_2)) \\ &= T_{\mathbf{C}}((\alpha_1 u_1 - \alpha_2 u_2) + i(\alpha_1 u_2 + \alpha_2 u_1)) \\ &= T(\alpha_1 u_1 - \alpha_2 u_2) + iT(\alpha_1 u_2 + \alpha_2 u_1) \\ &= (\alpha_1 Tu_1 - \alpha_2 Tu_2) + i(\alpha_1 Tu_2 + \alpha_2 Tu_1) \\ &= (\alpha_1 + i\alpha_2)(Tu_1 + iTu_2) \\ &= \alpha T_{\mathbf{C}}u. \end{aligned}$$

Therefore, $T_{\mathbf{C}}$ is a linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.

(b) ($\Longrightarrow$) Suppose that $T_{\mathbf{C}}$ is injective, and let $u, v \in V$ satisfying $Tu = Tv$, then we can consider the vectors $u_{\mathbf{C}} = u + i0$ and $v_{\mathbf{C}} = v + i0$ in $V_{\mathbf{C}}$ and notice that

$$T_{\mathbf{C}} u_{\mathbf{C}} = Tu + iT(0) = Tu = Tv = Tv + iT(0) = T_{\mathbf{C}} v_{\mathbf{C}},$$

then by injectivity of $T_{\mathbf{C}}$, we get that $u_{\mathbf{C}} = v_{\mathbf{C}}$ which directly implies that $u = v$. Therefore, T is injective.
($\Longleftarrow$) Suppose T is injective and let $u = u_1 + iv_1$ and $v = v_1 + iv_2$ be elements of $V_{\mathbf{C}}$ satisfying $T_C u = T_{\mathbf{C}} v$ with $u_1, u_2, v_1, v_2 \in \mathbf{C}$, then by definition of $T_{\mathbf{C}}$, we have

$$T_C u = T_{\mathbf{C}} v \implies Tu_1 + iTu_2 = Tv_1 + iTv_2.$$

By construction of $W_{\mathbf{C}}$, we get that $Tu_1 = Tv_1$ and $Tu_2 = Tu_2$. By injectivity of T, we have that $u_1 = v_1$ and $u_2 = v_2$. Again, by construction of $V_{\mathbf{C}}$, we get that $u = v$. Therefore, $T_{\mathbf{C}}$ is injective.

(c) ($\Longrightarrow$) Suppose that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ and let $w \in W$. If we define $w_{\mathbf{C}} = w + i0 \in W_{\mathbf{C}} = \operatorname{range} T_{\mathbf{C}}$, then there exists a $u = u_1 + iu_2 \in V_{\mathbf{C}}$ such that $T_{\mathbf{C}} u = w_{\mathbf{C}}$. But notice that

$$T_{\mathbf{C}} u = w_{\mathbf{C}} \implies Tu_1 + iTu_2 = w + i0 \implies Tu_1 = w.$$

Thus, $u_1 \in V$ and $Tu_1 = w$ so $w \in \operatorname{range} T$. Therefore, $\operatorname{range} T = W$.
($\Longleftarrow$) Suppose that $\operatorname{range} T = W$ and let $w = w_1 + iw_2 \in W_{\mathbf{C}}$, then $w_1, w_2 \in W = \operatorname{range} T$. It follows that there exist $u_1, u_2 \in V$ such that $Tu_1 = w_1$ and $Tu_2 = w_2$. Thus, if we define $u = u_1 + iu_2 \in V_{\mathbf{C}}$, we obtain

$$T_{\mathbf{C}} u = Tu_1 + iTu_2 = w_1 + iw_2 = w.$$

Hence, $w \in \operatorname{range} T_{\mathbf{C}}$. It follows that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$.

3C Matrices

Exercise 1
TODO

Solution
TODO

3D Invertibility and Isomorphisms

Exercise 1
TODO

Solution
TODO

3E Products and Quotients of Vector Spaces

Exercise 1
TODO

Solution
TODO

3F Duality

Exercise 1
TODO

Solution
TODO