Data Structures & Algorithms

Adil M. Khan

Professor of Computer Science

Innopolis University

a.khan@innopolis.ru

Disclaimer!!

- This is your first lecture, so somethings might be difficult to understand – don't worry and be patient

- You will be required to do some reading at home to better understand the presented topics

- Somethings from the lecture, you should keep in memory as their answers and explanations will become clear later in the course

Which Phone Will You Buy, and Why?

Which Car Will You Buy, and Why?

Which Sorting Algorithm Will You Choose and Why?

- Insertion Sort
- Selection Sort
- Merge Sort
- Quick Sort

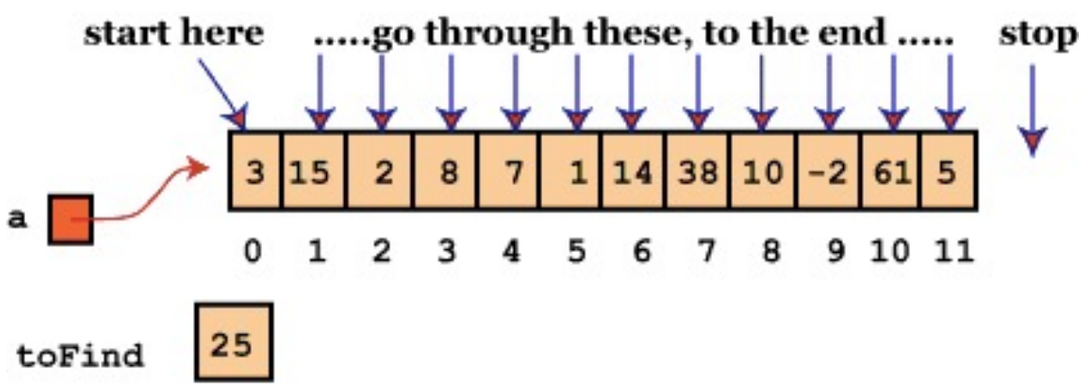
Objectives of This Lecture

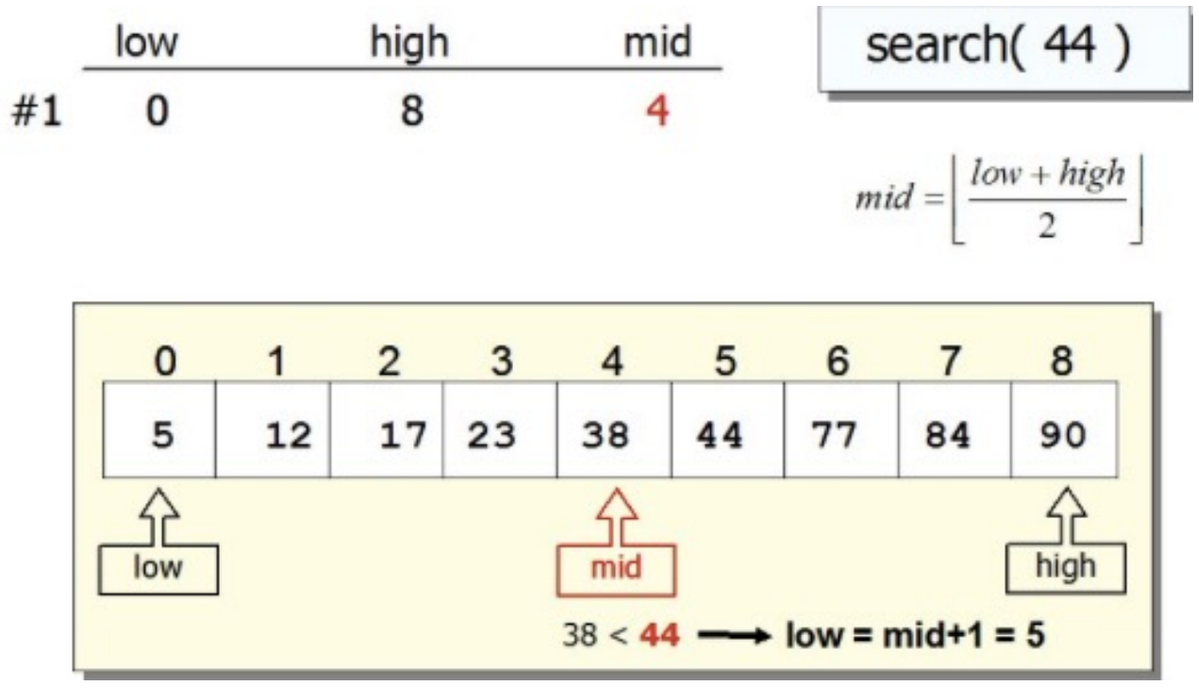
- Learn and be able to describe what is "Algorithm Analysis"
- Learn and be able to describe the importance of analyzing algorithms
- Learn and practice the method to analyze algorithms

Why analyze algorithms?

Algorithm Analysis

- Allows us to:
  - Compare the merits of two alternative approaches to a problem we need to solve
  - Determine whether a proposed solution will meet required resource constraints before we invest money and time coding





Performed before coding!

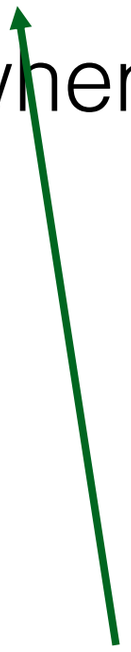
Two Questions

- How to analyze algorithms?

- What to analyze?

Algorithm Analysis

- We analyze how the resource requirements of an algorithm will scale when increasing the input size



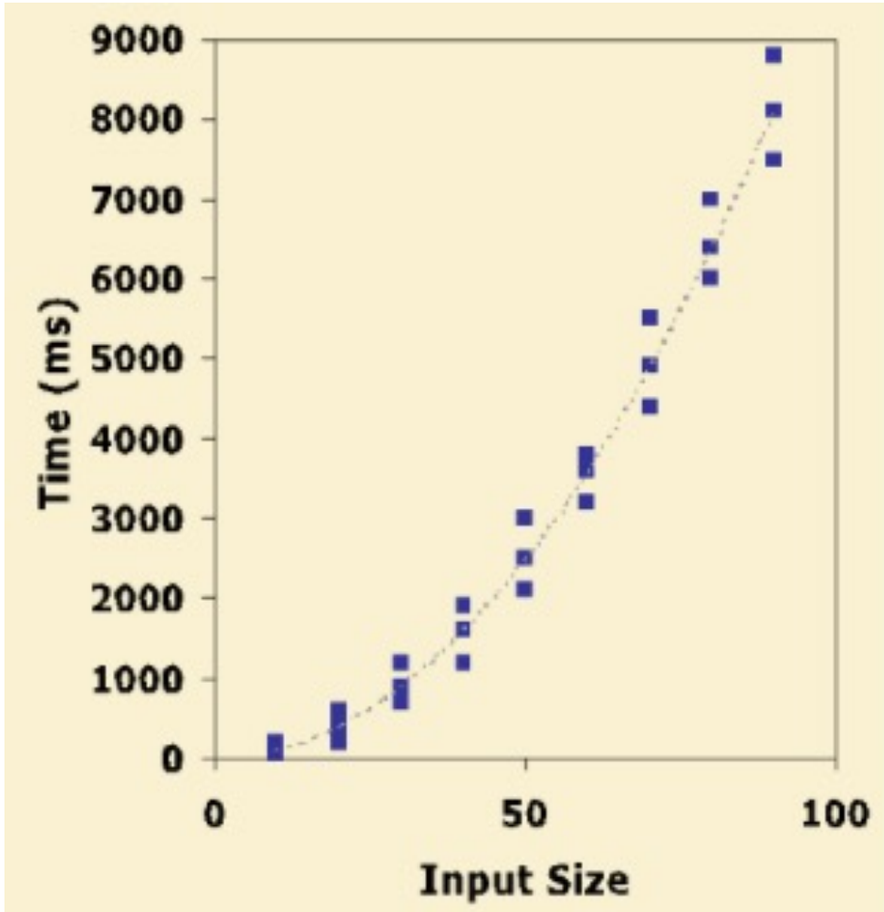
The question is, which resource should we consider?

Time as the Main Resources

- We will focus on time complexity
- That is, we will analyze how much time does an algorithm take to run to its completion
- There are ***Two ways*** with which we can do this!

Experimental Analysis

- Write a program that implements the algorithm
- Run it with inputs of varying size and composition
- Measure the actual running time
- Plot the results



What is wrong with this approach?

Thus!

- We will do *Theoretical Analysis* instead

Pseudocode

- A high-level description of an algorithm
- More structured than English
- Less detailed than a program

Example: find max element of an array

```
Algorithm arrayMax(A, n)
  Input: array A of n integers
  Output: maximum element of A

  currentMax ← A[0]
  for i ← 1 to n – 1 do
    if A[i] > currentMax then
      currentMax ← A[i]
  return currentMax
```

Theoretical Analysis

- Given the Pseudocode description of the algorithm instead of its implementation

  - We use it to characterize running time as a function of the input size, $T(n)$

Input Size (n)

- The n could be
  - The number of items in a data structure
  - The length of a string or file
  - The number of digits (or bits) in an integer
  - The degree of a polynomial

Wait a Second, Please!!

- Even for inputs of the same size, the time consumed can be very different

Example: an algorithm that finds the first prime number in an array by scanning it left to right

Can different situations affect the running time of this algorithm?

3	2	5	6	8	10

12	4	6	8	9	11

Measuring Time Complexity

- Analyze running time for the
  - ***best case:*** usually useless
  - ***average case:*** very difficult to determine
  - ***worst case:*** a safer choice

Why is the worst case a safer choice?

Choosing a Roommate

You can see their past (to judge their behavior) but you have to choose

(1) best case: a day when everything was great
(2) average case: An average day (but it is hard to determine such a day)
(3) worst case: a day when everything was horrible

Thus, in algorithms analysis, we analyze the worst case Time Complexity T(n) of algorithms

How to Measure $T(n)$?

- Consider this statement in your algorithm

$$x = x + 1;$$

- What we want to measure is
  - ***Execution time:*** The time a single execution of this statement would take
  - ***Frequency count:*** The number of times it is executed

Measuring Time Complexity

- Total time taken by each statement is **approximately**

= execution time (represented as constants) x frequency count

Execution Time

- Tied to the underline machine and compiler
- To simplify this, we use the *RAM* model

1. Each simple operation (+, *, -, =, if, call) takes exactly one step
2. Loops and subroutines are not considered simple operations
3. Each memory access takes exactly one time stamp

Read more about RAM model from Cormen's Section 2.2

Thus, in algorithms analysis we analyze the worst case Time Complexity T(n) of algorithms using the RAM model

Example Insertion sort

```
INSERTION-SORT(A)
1  for j = 2 to A.length
2      key = A[j]
3      // Insert A[j] into the sorted
           sequence A[1 .. j − 1].
4      i = j − 1
5      while i > 0 and A[i] > key
6          A[i + 1] = A[i]
7          i = i − 1
8      A[i + 1] = key
```

Let's Compute T(n)

INSERTION-SORT(A)	*cost*	*times*
1 **for** $j = 2$ **to** $A.length$		
2 $key = A[j]$		
3 // Insert $A[j]$ into the sorted sequence $A[1 \ldots j-1]$.		
4 $i = j - 1$		
5 **while** $i > 0$ and $A[i] > key$		
6 $A[i+1] = A[i]$		
7 $i = i - 1$		
8 $A[i+1] = key$		

Let's Compute T(n)

INSERTION-SORT(A)	*cost*	*times*
1 **for** $j = 2$ **to** $A.length$	c_1	n
2 $key = A[j]$	c_2	$n-1$
3 // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.		
4 $i = j - 1$		
5 **while** $i > 0$ and $A[i] > key$		
6 $A[i+1] = A[i]$		
7 $i = i - 1$		
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Let's Compute T(n)

INSERTION-SORT(A)	*cost*	*times*
1 **for** $j = 2$ **to** $A.length$	c_1	n
2 $key = A[j]$	c_2	$n-1$
3 // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.	0	$n-1$
4 $i = j-1$	c_4	$n-1$
5 **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
6 $A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7 $i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8 $A[i+1] = key$	c_8	$n-1$

To understand the execution time for statements 5, 6 and 7, read sections 2.1 and 2.2 from Cormen's book. I will ask you question about it in the next lecture.

Example (cont.)

You already know

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$$

and

$$\sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

Example (cont.)

Thus

$$
\begin{aligned}
T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\
&\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\
&= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\
&\quad - (c_2 + c_4 + c_5 + c_8) \; .
\end{aligned}
$$

Example (cont.)

This can be expressed as

$$T(n) = an^2 + bn + c$$

But what is it telling us?

Time Complexity

- Time complexities of algorithms when expressed in the form of numerical functions over the size of the input are difficult to work with:

- Thus, to make analysis easier, we talk about upper and lower bounds of these functions – Big O Analysis

- This helps us to ignore details that do not impact our comparison of algorithms

Thus, in algorithms analysis we asymptotically analyze the worst case Time Complexity T(n) of algorithms using the RAM model

Russian Winters

- Knowing that
  - the whole world knows that how cold Russia can be in the winters,
  - allows us to express severity of winters of any place in terms of Russian winters

Growth Rates of Common Functions

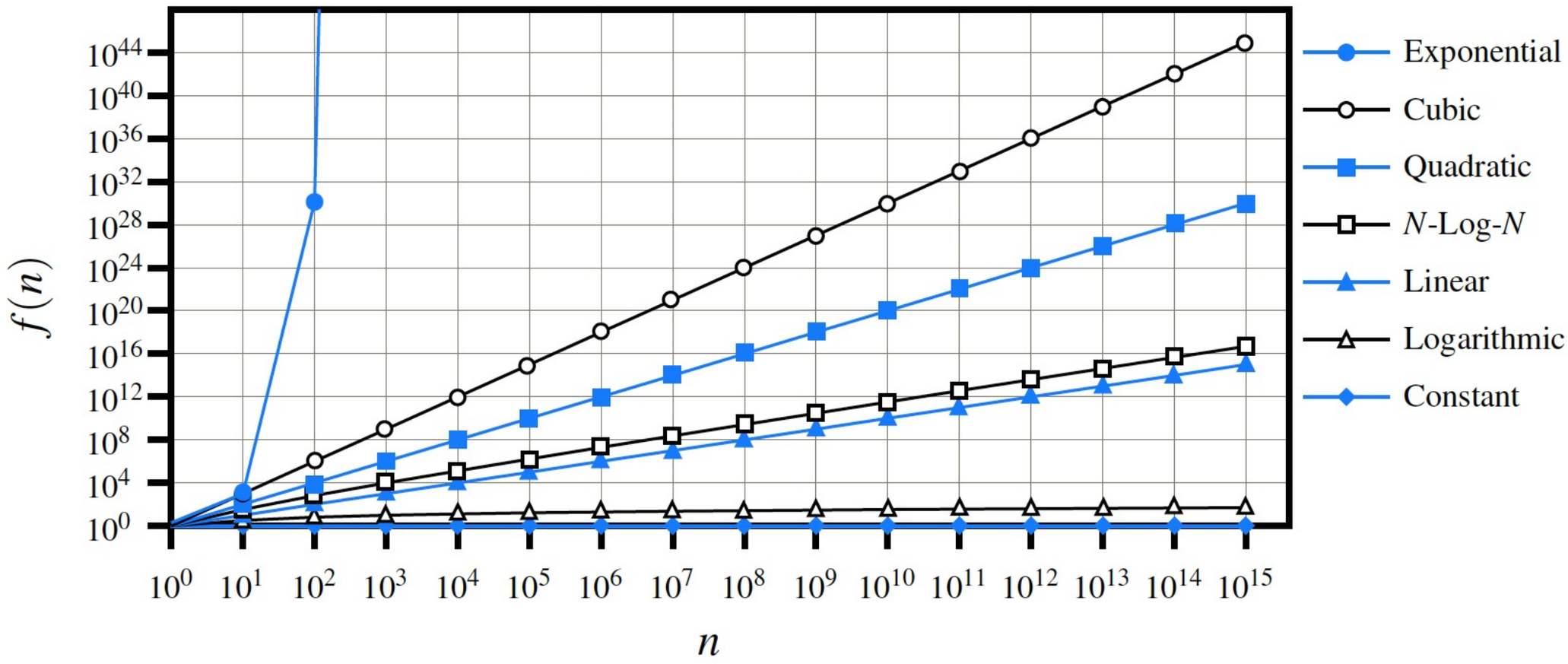


Figure 4.4: Growth rates for the seven fundamental functions used in algorithm analysis. We use base $a = 2$ for the exponential function. The functions are plotted on a log-log chart to compare the growth rates primarily as slopes. Even so, the exponential function grows too fast to display all its values on the chart.

Big O Analysis

- A good thing about Big O, we can ignore
  - ❖ constant factors
  - ❖ lower-order terms
- Examples:
  - $10^2 n + 10^5$ is a *linear function – O(n)*
  - $10^2 n^2 + 10^5 n$ is a *quadratic function – O(n^2)*

Big Oh Analysis

- Why is it not affected by the constant factors and the lower order terms?

❖ $6n$ ***vs.*** $3n$ — getting a computer twice as fast makes the former same as the latter

❖ $2n$ ***vs.*** $2n + 8$ — difference becomes insignificant when n becomes larger and larger

❖ n^3 ***vs.*** kn^2 — the former will always eventually overtake the latter no matter how big you make k

Big Oh Analysis

- So for our example: $T(n) = an^2 + bn + c$
- But we just learned that constant terms and lower order terms don't matter
- Thus, under Big Oh analysis, we can express it as

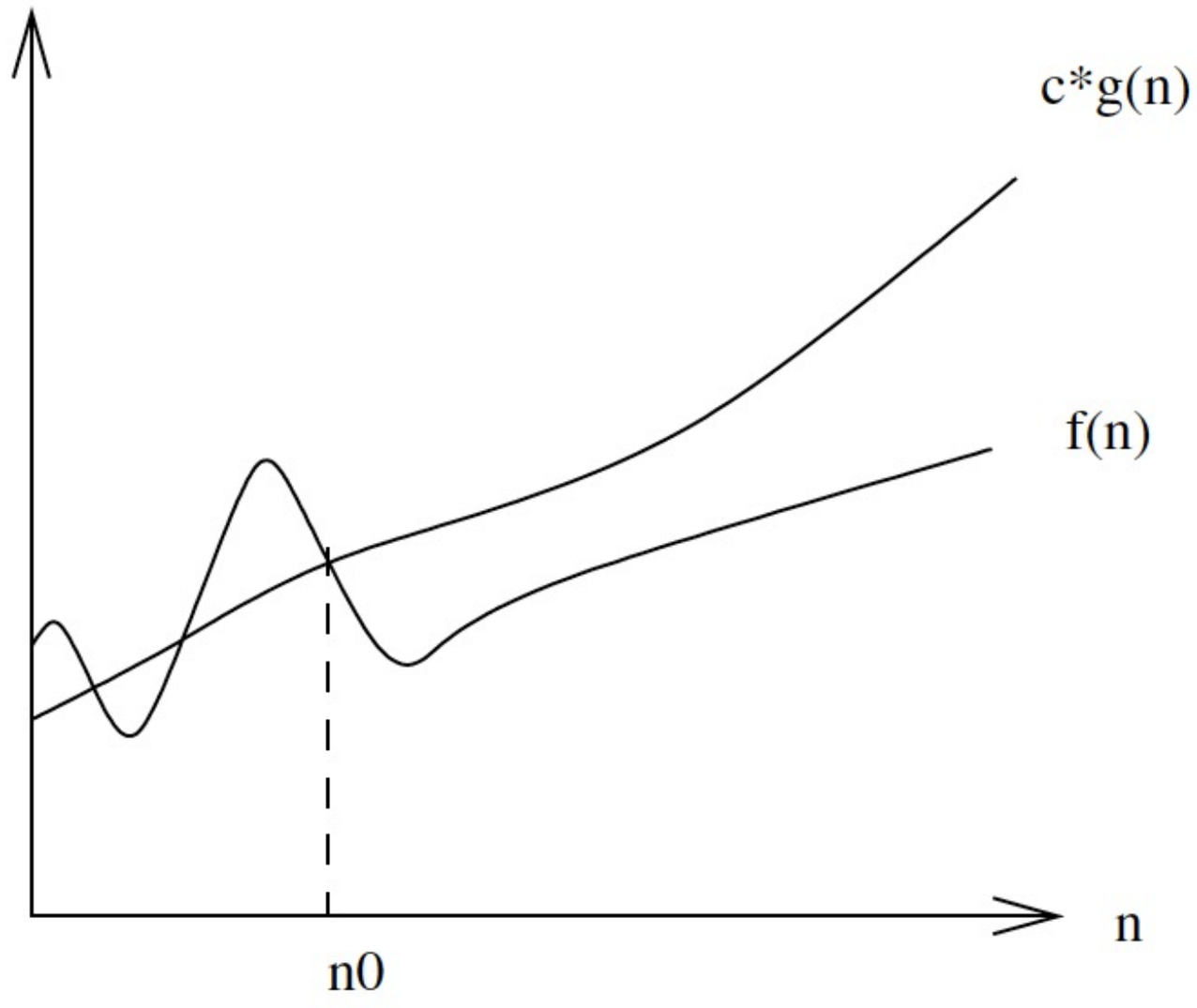
$$T(n) = O(n^2)$$

Big Oh Notations

- $f(n) = O(g(n))$ means $c \cdot g(n)$ is an *upper bound* on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\leq c \cdot g(n)$, for large enough n (i.e. , $n \geq n_0$ for some constant n_0).
- $f(n) = \Omega(g(n))$ means $c \cdot g(n)$ is a *lower bound* on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\geq c \cdot g(n)$, for all $n \geq n_0$.
- $f(n) = \Theta(g(n))$ means $c_1 \cdot g(n)$ is an upper bound on $f(n)$ and $c_2 \cdot g(n)$ is a lower bound on $f(n)$, for all $n \geq n_0$. Thus there exist constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ and $f(n) \geq c_2 \cdot g(n)$. This means that $g(n)$ provides a nice, tight bound on $f(n)$.

You will cover them in more detail in your Tutorial!

Big Oh - O



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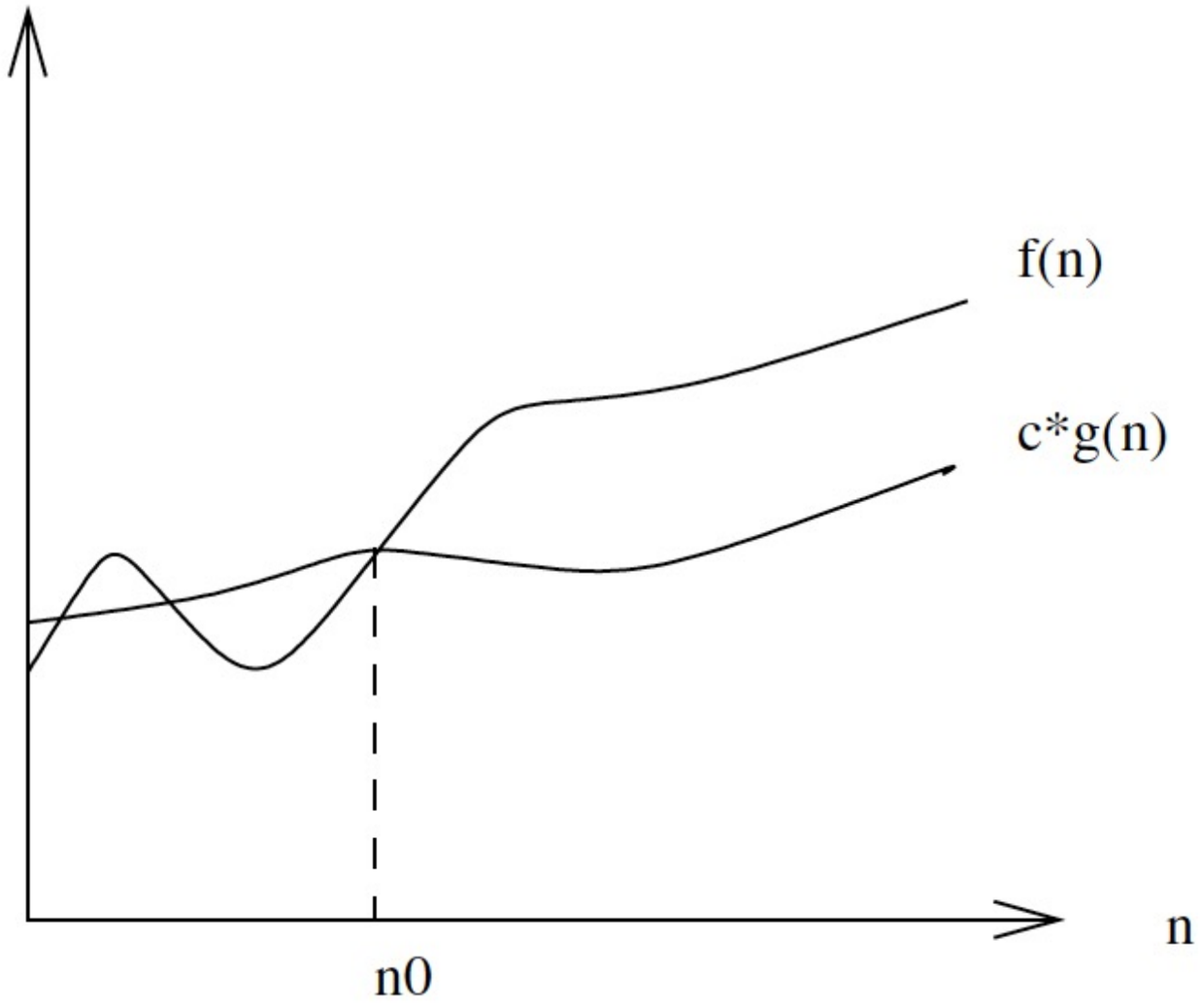
Big Omega - Ω



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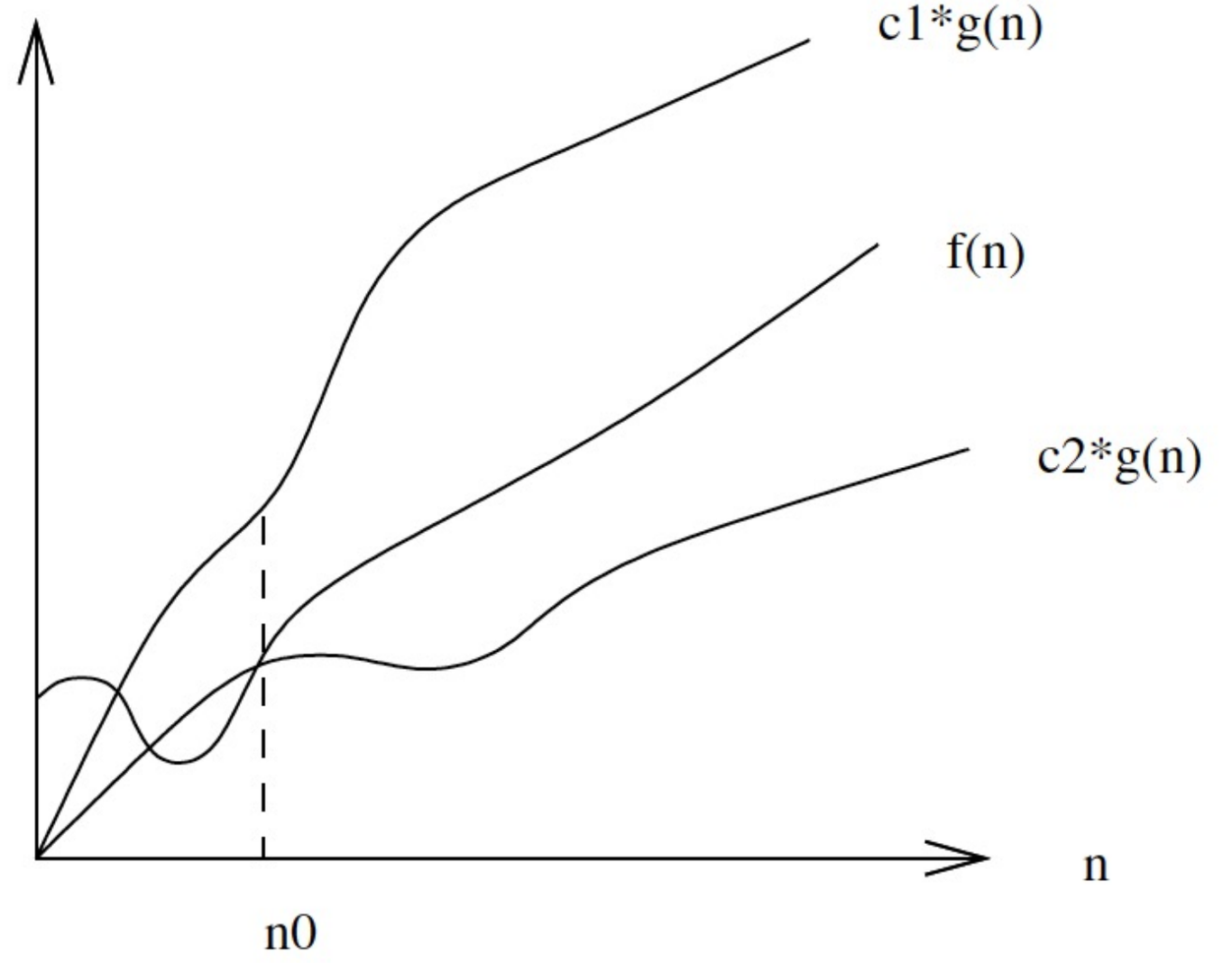
Big Theta - Θ



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Growth Rates of Common Functions

n $f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000	0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000	0.023 μs	0.01 sec	0.23 sec	1.16 days		
100,000,000	0.027 μs	0.10 sec	2.66 sec	115.7 days		
1,000,000,000	0.030 μs	1 sec	29.90 sec	31.7 years		

Space Complexity

- Determine how much space an algorithm requires by analyzing its storage requirements as a function of the input size
- ***Example:***
  - Let's say, our algorithm reads a stream of ***n*** characters
  - But always stores a constant number of them
  - then, its space complexity is ***O(1)***

Space Complexity

- ***Another Example:***

  - Let's say, our algorithm reads a stream of ***n*** characters

  - and stores all of them

  - then, its space complexity is ***O(n)***

Space Complexity

- ***Exercise:***

  - Let's say, our algorithm reads a stream of ***n*** characters

  - and stores all of them, and each record results in the creation of a constant number of other records

  - then, its space complexity is ?

Space Complexity

- ***Another Exercise:***

  - Let's say, our algorithm reads a stream of ***n*** characters

  - and stores all of them, and each record results in the creation of a number of new records — the number is proportional to the size of the data

  - then, its space complexity is ?

Time-Space Tradeoff

- Generally, decreasing the time complexity of an algorithm results in increasing its space complexity — and vice versa
- This is called the time-space tradeoff

- ***Example:*** Storing a sparse matrix as a two-dimensional linked list vs. a two-dimensional array

Did we Achieve Today's Objectives?

- Learn and be able to describe what is "Algorithm Analysis"

- Learn and be able to describe the importance of analyzing algorithms

- Learn and practice the method to analyze algorithms