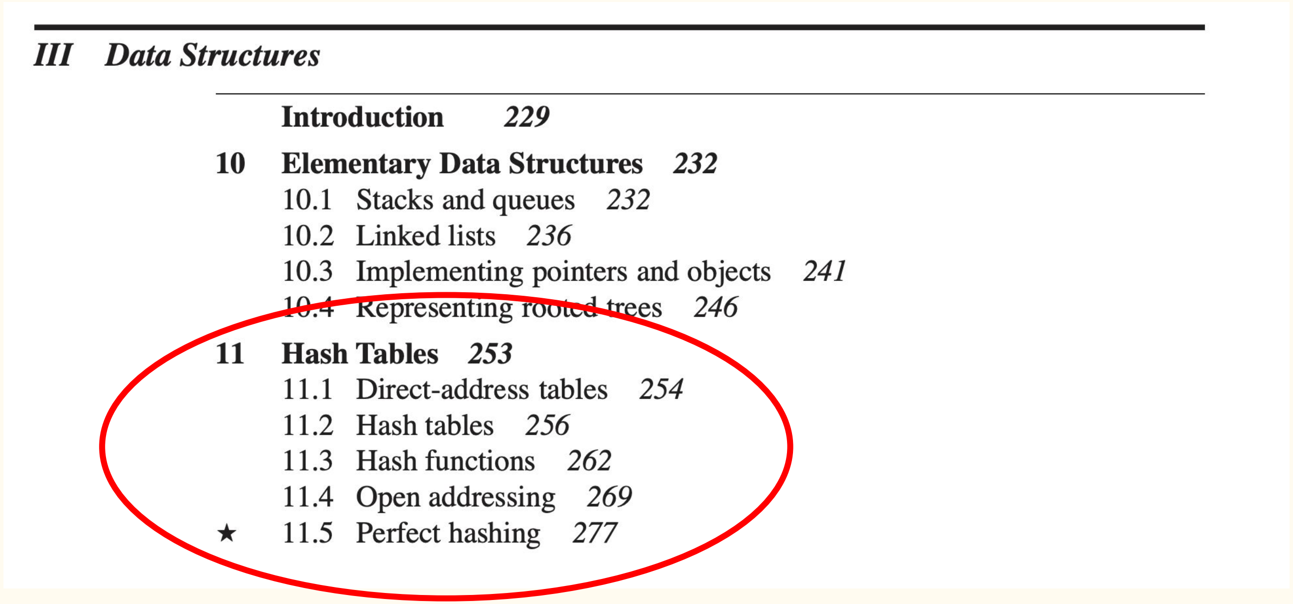
Data Structures and Algorithms

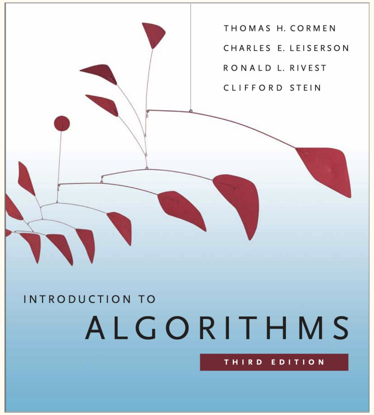
Tutorial 3. Analysis of runtime in hashing

Today's topic is covered in detail in

T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. **Introduction to Algorithms.** The MIT Press 2009.



III Data Structures



Objectives

- Recap: Hashing & Hashtables
- Separate chaining & its analysis
- Open addressing & its analysis

Recap: Hashtables

1. Implementation of a Map/Dictionary ADT:
    - **`get`** value by key
    - **`insert`** a key-value pair
    - **`delete`** a key with an associated value

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2. Uses array to store key-value pairs
    - Directly (using open addressing) or
    - Storing a list of key-value pairs (separate chaining)

3. Relies on a **hash function** to convert key to array index

Recap: Ideal* hash function

1. Avoid collisions entirely

* Ideal for general purpose HashTable implementation. Not necessarily for other purposes, such as cryptography, document integrity checks, unique identifier generators, etc.

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3. Works for arbitrary data types
4. [Cryptography] Tiny change to input changes output significantly (avalanche effect)

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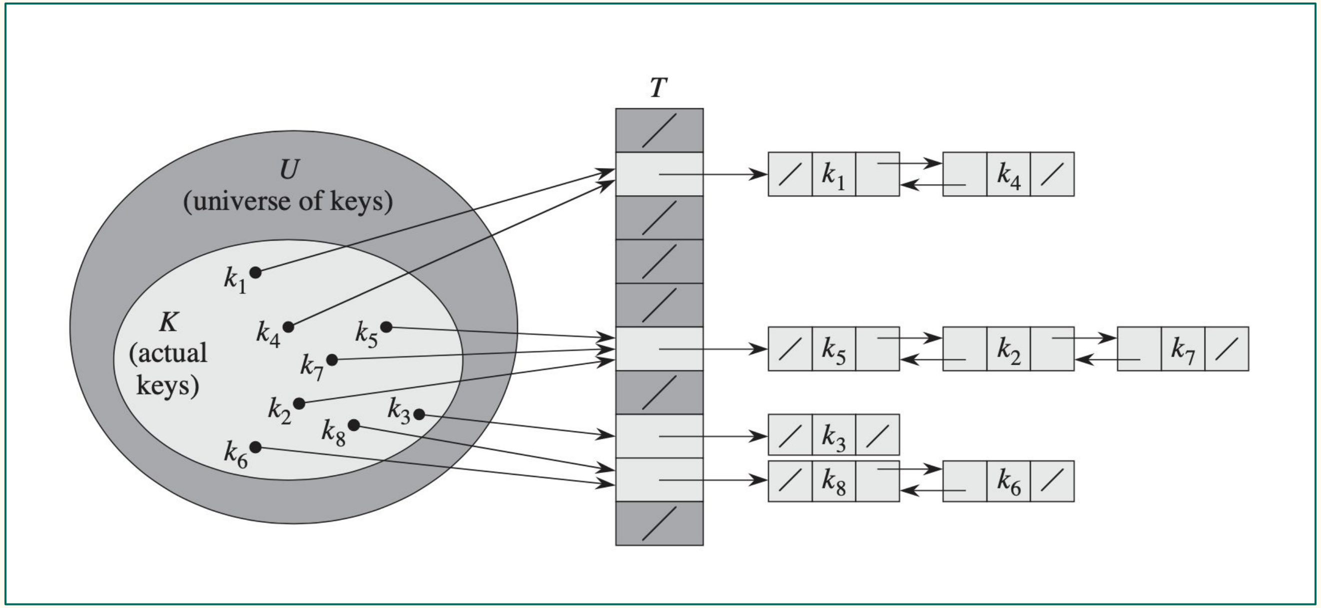
Separate chaining

- T[j] contains a list of key-value pairs (k_i, v_i) such that $h(k_i) = j$



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- To `delete(k)` we delete pair with key **k** from the list **T[h(k)]**
  O(length of T[h(k)]) if we scan the list

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- To **`delete(item)`** we delete pair **item** from the list **T[h(k)]**
  O(1) if we use doubly linked lists (and delete by reference to the node)

Exercise: Separate Chaining

Exercise 3.1. Demonstrate what happens when we insert the keys

$$5;\ 28;\ 19;\ 15;\ 20;\ 33;\ 12;\ 17;\ 10$$

into a hash table with collisions resolved by chaining.
Let the table have 9 slots, and let the hash function be $\mathbf{h(k) = k \bmod 9}$.

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- Hash table T has **m** slots, and stores **n** key-value pairs

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- We assume **simple uniform hashing**:
  «Any given element is equally likely to hash into any of the m slots.»

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- Worst case for chaining — all keys hash to the same slot — O(n).
- So instead we consider **average case!**
- We assume **simple uniform hashing**:
  «Any given element is equally likely to hash into any of the m slots.»
- We also assume it suffices to compute the hash value in O(1) time.

Analysis of separate chaining (unsuccessful search)

- Hash table T has **m** slots, and stores **n** key-value pairs
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Theorem. With separate chaining, an unsuccessful search takes average-case time $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

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Proof. Under simple uniform hashing, any key **k** not already stored in the table is equally likely to hash to any of the **m** slots. The expected time to search unsuccessfully for a key **k** is the expected length of **T[h(k)]**, which is exactly **α**.
Thus, the total time required (including time to compute h(k)) is $\Theta(1+\alpha)$.

Analysis of separate chaining (successful search)

- Hash table T has **m** slots, and stores **n** key-value pairs
- Load factor **α** of T is defined as **n / m**.

Theorem. With separate chaining, a successful search takes average-case time $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

Proof. We assume that the key being searched for is equally likely to be any of the n elements stored in the table. The number of elements examined is equal to the position of the element in the corresponding chain. Intuitively, the expected position is approximately $\alpha/2$, so overall we get $\Theta(1+\alpha)$.
See precise proof in Theorem 11.2 of Cormen et al.

Exercise: Analysis of Separate Chaining

Exercise 3.2. Consider keeping each chain in the hashtable as a sorted list. How would that affect running time for (un)successful searches, inserts, and deletes?

Attendance

https://baam.duckdns.org

Open addressing

- All elements (key-value pairs) occupy the hash table itself
- Corollary: load factor **α** cannot be more that 1.
- To insert we successively **probe** the hash table,
  until we find an empty slot
- The **probe sequence** depends on the key
- Essentially, we now have a hash function with two arguments:
  h(k, 0), h(k, 1), h(k, 2), ...
- In our analysis we assume uniform hashing:
  «The probe sequence of each key is equally likely to be
  any of the **m!** permutations of $\boldsymbol{<0, 1, \ldots, m-1>}$.»

Open addressing: `insert`

1. Let $\mathbf{i = 0}$

2. Try index $\mathbf{h(k, i)}$
    a. If $\mathbf{T[h(k, i)]}$ is empty, use it
    b. Otherwise, increment $\mathbf{i}$ and repeat until $\mathbf{i \geqslant m}$

3. If there was no empty slot, then hash table is full

Open addressing: `search`

- Let **i = 0**

- Try index **h(k, i)**
  - If **T[h(k, i)]** has key **k**, then return associated value and exit;
  - If **T[h(k, i)]** is empty, then exit: there is no key **k** in **T**
  - Otherwise, increment **i** and repeat until $\mathbf{i} \geqslant \mathbf{m}$

- At this point we know that hashtable is full and there is no key **k** in **T**

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  If we remove (mark as EMPTY) an element in the middle of a probing sequence, then some elements will become invisible for search! And if we introduce DELETED mark, then running time of search/insert no longer depends on the load factor.

- So typically, if many deletes are expected, separate chaining is used.

Open addressing: strategies

- **Linear probing**

  $h(k, i) = (h'(k) + i) \bmod m$

  $h(k),\ h(k)+1,\ h(k)+2,\ h(k)+3,\ h(k)+4,\ h(k)+5, \ldots$

Open addressing: strategies

- **Linear probing**

  Primary clustering

  $h(k, i) = (h'(k) + i) \bmod m$

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Primary clustering

- **Quadratic probing**
  $h(k, i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$
  $h(k),\ h(k)+1,\ h(k)+4,\ h(k)+9,\ h(k)+25,\ h(k)+36, \ldots$

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Secondary clustering

- **Double hashing**

  $h(k, i) = (h_1(k) + i\, h_2(k)) \bmod m$

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Secondary clustering

- **Double hashing**
  $h(k, i) = (h_1(k) + i\, h_2(k)) \bmod m$

Typically has nice random-like properties

Exercise: Open addressing

Exercise 3.4. Consider inserting the keys

$$10;\ 22;\ 31;\ 4;\ 15;\ 28;\ 17;\ 88;\ 59$$

into a hash table of length $\mathbf{m = 11}$ using open addressing with the auxiliary hash function $\mathbf{h'(k) = k}$. Illustrate the result of inserting these keys

1. using linear probing,
2. using quadratic probing with $\mathbf{c_1 = 1}$ and $\mathbf{c_2 = 3}$, and
3. using double hashing with $\mathbf{h_1(k) = k}$ and $\mathbf{h_2(k) = 1 + (k \bmod (m - 1))}$.

Analysis of open addressing (unsuccessful search)

- Hash table T has **m** slots, and stores **n** key-value pairs
- Load factor $\boldsymbol{\alpha} = \mathbf{n} / \mathbf{m} < \mathbf{1}$.

Theorem. With open addressing, an unsuccessful search takes average-case time $O(1/(1-\alpha))$ under the assumption of simple uniform hashing.

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Proof. $$\#\text{ of probes} = 1 + \underset{\uparrow}{\frac{n}{m}}\left(1 + \frac{n-1}{m-1}(\ldots)\right)$$

Probability of hitting an occupied slot for the 1st probe

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Proof. $$\#\text{ of probes} = 1 + \frac{n}{m}\Big(1 + \underset{\Big\uparrow}{\frac{n-1}{m-1}}(\ldots)\Big)$$

Probability of hitting an occupied slot for the 2nd probe

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Analysis of open addressing (insertion)

- Hash table T has **m** slots, and stores **n** key-value pairs
- Load factor $\boldsymbol{\alpha} = \mathbf{n} / \mathbf{m} < \mathbf{1}$.

Corollary. With open addressing, an insertion takes average-case time $O(1/(1-\alpha))$ under the assumption of simple uniform hashing.

Analysis of open addressing (successful search)

- Hash table T has **m** slots, and stores **n** key-value pairs
- Load factor $\boldsymbol{\alpha} = \mathbf{n} / \mathbf{m} < \mathbf{1}$.

Theorem. With open addressing, the expected number of probes in a successful search is at most

$$\frac{1}{\alpha} \log \frac{1}{1-\alpha}$$

Proof. See Theorem 11.8 in Cormen et al.

Exercise: Analysis of open addressing

Exercise 3.4. Consider an open-address hash table with uniform hashing. Give upper bounds on the expected number of probes in an unsuccessful search and on the expected number of probes in a successful search when the load factor is $3/4$ and when it is $7/8$.

Remark on open addressing

1. More difficult to implement (compared to separate chaining)

2. Typically requires more fine-tuning

3. Works best if you know the frequency and distribution of keys, as a somewhat specialized solution

There exist other, more complicated open addressing approaches, such as the Cuckoo Hashing, which introduce worst-case O(1) search and delete operations, and amortized O(1) insert.
However, we will not cover those in this course.

A note on terminology

- «Open Hashing» = «Separate chaining»
- «Closed Hashing» = «Open addressing»

Summary

- Hash tables & Hashing
- Separate chaining supports $\Theta(1)$ and/or $\Theta(1+\alpha)$ operations
- Open addressing supports $O(1/(1-\alpha))$ search

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See you next week!