ECS132 **Atharva Chalke**
Prof. Matloff Due Date: 10/14/19

Homework 1

PROBLEM A:

This will be a variant of the bus ridership problem, Sec. 2.11.

In addition to individual riders boarding the bus, there may be pairs, e.g. parent and child. At any stop, either 0 pairs or 1 pair will board, with probability 0.4 and 0.6, respectively. Similarly, any pair already on board will alight at a stop, with probability 0.2. Pairs act independently from individuals and from other pairs.

1. Find P(L2 = 0).

2. A newspaper photograph of the bus arriving at the second stop shows a passenger alighting. Use simulation to find the approximate probability that this passenger was part of a pair. Extra Credit: Do this problem mathematically.

Solution

Part 1
Definitions
Let L_i denote the number of passengers on the bus as it leaves the i^{th} stop, i = 1,2,3......
Let B_i denote the number of individual new passengers who board the bus at the i^{th} stop.
Let R_i denote the number of paired new passengers who board the bus at the i^{th} stop.

Assumptions
The pairs who board together, also leave together.
Passengers who board at a stop do not get off at the same stop.

$$
\begin{aligned}
P(L_2 = 0) = P(B_1 = 0 \; and \; R_1 = 0 \; and \; L_2 = 0) &+ P(B_1 = 0 \; and \; R_1 = 2 \; and \; L_2 = 0) \\
+ P(B_1 = 1 \; and \; R_1 = 0 \; and \; L_2 = 0) &+ P(B_1 = 1 \; and \; R_1 = 2 \; and \; L_2 = 0) \\
+ P(B_1 = 2 \; and \; R_1 = 0 \; and \; L_2 = 0) &+ P(B_1 = 2 \; and \; R_1 = 0 \; and \; L_2 = 0) \\
= \sum_{i=0}^{2} P(B_1 = i \; and \; R_1 = 0 \; and \; L_2 = 0) &+ P(B_1 = i \; and \; R_1 = 2 \; and \; L_2 = 0)
\end{aligned}
$$

Lets solve this for each i

i = 0

$$\begin{aligned}
&= P(B_1 = 0 \; and \; R_1 = 0 \; and \; L_2 = 0) \; + P(B_1 = 0 \; and \; R_1 = 2 \; and \; L_2 = 0) \\
&= P(B_1 = 0 \; and \; R_1 = 0 \;) * P(L_2 = 0 | B_1 = 0 \; and \; R_1 = 0 \;) \\
&\quad\; + P(B_1 = 0 \; and \; R_1 = 2 \;) * P(L_2 = 0 | B_1 = 0 \; and \; R_1 = 2 \;) \quad [Using P(A and B) = P(A) * P(B|A)] \\
&= 0.5 * 0.4 * 0.5 * 0.4 \; + 0.5 * 0.6 * 0.5 * 0.4 * 0.2 \\
&= 0.052
\end{aligned}$$

i = 1

$$\begin{aligned}
&= P(B_1 = 1 \; and \; R_1 = 0 \; and \; L_2 = 0) \; + P(B_1 = 1 \; and \; R_1 = 2 \; and \; L_2 = 0) \\
&= P(B_1 = 1 \; and \; R_1 = 0 \;) * P(L_2 = 0 | B_1 = 1 \; and \; R_1 = 0 \;) \\
&\quad\; + P(B_1 = 1 \; and \; R_1 = 2 \;) * P(L_2 = 0 | B_1 = 1 \; and \; R_1 = 2 \;) \quad [Using P(A and B) = P(A) * P(B|A)] \\
&= 0.4 * 0.4 * 0.5 * 0.4 * 0.2 \; + 0.4 * 0.6 * 0.5 * 0.4 * 0.2 * 0.2 \\
&= 0.00832
\end{aligned}$$

i = 2

$$\begin{aligned}
&= P(B_1 = 2 \; and \; R_1 = 0 \; and \; L_2 = 0) \; + P(B_1 = 2 \; and \; R_1 = 2 \; and \; L_2 = 0) \\
&= P(B_1 = 2 \; and \; R_1 = 0 \;) * P(L_2 = 0 | B_1 = 2 \; and \; R_1 = 0 \;) \\
&\quad\; + P(B_1 = 2 \; and \; R_1 = 2 \;) * P(L_2 = 0 | B_1 = 2 \; and \; R_1 = 2 \;) \quad [Using P(A and B) = P(A) * P(B|A)] \\
&= 0.1 * 0.4 * 0.4 * 0.5 * 0.2^2 \; + 0.6 * 0.1 * 0.5 * 0.4 * 0.2^3 \\
&= 0.000416
\end{aligned}$$

Final Answer
Summing up these values we get

$$\begin{aligned}
&= 0.052 + 0.00832 + 0.000416 \\
&= 0.060736
\end{aligned}$$

Part 2
Assumptions
1. If the pair does get down, then the photo of the pair is taken.

Definitions
Let X_i be the number of pairs getting down at stop i.

Let B_i be the number of pairs getting on the bus at stop i.
Let I_i be the number of individuals getting down at stop i.
Let G_i be the number of individuals getting on at stop i.
Let Y_i be 1 if someone gets down at stop i, 0 otherwise.
Let M_i be 1 if a pair gets down at stop i, 0 otherwise.

$$P(M_2 = 1|Y_2 = 1) = \frac{P(M_2 = 1andY_2 = 1)}{P(Y_2 = 1)}$$
$$[UsingP(A|B) = \frac{P(AandB)}{P(B)}]$$

Lets solve each of the these.
Solving for the numerator, we get:

$$\begin{aligned} P(X_2 = 1andY_2 = 1) &= P(B_1 = 1andX_2 = 1) \\ &= 0.6 * 0.2 \\ &= 0.12 \end{aligned}$$

Solving for the denominator, we get:

$$\begin{aligned} P(Y_2 = 1) = P(&Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 1 \textit{ or } Y_2 = 1 \textit{ and } B_1 = 0 \textit{ and } G_1 = 1 \\ &\textit{or } Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 0 \textit{ or } Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 2 \\ &\textit{or } Y_2 = 1 \textit{ and } B_1 = 0 \textit{ and } G_1 = 2\ ) \\ = P(&Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 1) + P(Y_2 = 1 \textit{ and } B_1 = 0 \textit{ and } G_1 = 1) \\ &+ P(Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 0) + P(Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 2) \\ &+ P(Y_2 = 1 \textit{ and } B_1 = 0 \textit{ and } G_1 = 2) \\ &[Using\ P(AorB) = P(A) + P(B)] \end{aligned}$$

Lets solve each one of these

$$\begin{aligned} P(Y_2 = 1 \textit{ and } B_1 = 1 \textit{ and } G_1 = 1) &= P(B_1 = 1 \textit{ and } G_1 = 1) * P(Y_2 = 1|B_1 = 1 \textit{ and } G_1 = 1) \\ &= 0.6 * 0.4 * P(X_2 = 1 \textit{ and } I_2 = 1 \textit{ or} \\ &\quad X_2 = 1andI_2 = 0orX_2 = 0andI_2 = 1) \\ &= 0.6 * 0.4 * (0.2^2 + 0.2 * 0.8 + 0.2 * 0.8) \\ &= 0.0864 \end{aligned}$$

$$
\begin{aligned}
P(Y_2 = 1 \; and \; B_1 = 0 \; and \; G_1 = 1) &= P(B_1 = 0 \; and \; G_1 = 1) * P(Y_2 = 1 | B_1 = 0 \; and \; G_1 = 1) \\
&= 0.4 * 0.4 * 0.2 \\
&= 0.032
\end{aligned}
$$

$$
\begin{aligned}
P(Y_2 = 1 \; and \; B_1 = 1 \; and \; G_1 = 0) &= P(B_1 = 1 \; and \; G_1 = 0) * P(Y_2 = 1 | B_1 = 1 \; and \; G_1 = 0) \\
&= 0.6 * 0.5 * P(X_2 = 1) \\
&= 0.6 * 0.5 * 0.2 \\
&= 0.06
\end{aligned}
$$

$$
\begin{aligned}
P(Y_2 = 1 \; and \; B_1 = 1 \; and \; G_1 = 2) &= P(B_1 = 1 \; and \; G_1 = 2) * P(Y_2 = 1 | B_1 = 1 \; and \; G_1 = 2) \\
&= 0.6 * 0.1 * P(X_2 = 1 \; and \; I_2 = 1 or X_2 = 1 \; and \; I_2 = 2 \\
&\quad\; or \; X_2 = 0 \; and \; I_2 = 1 or X_2 = 0 \; and \; I_2 = 2 or X_2 = 1 \; and I_2 = 0) \\
&= 0.6 * 0.1 * (0.2^2 + 0.2^3 + 0.2 + 0.2^2 + 0.2) \\
&= 0.02928
\end{aligned}
$$

$$
\begin{aligned}
P(Y_2 = 1 \; and \; B_1 = 0 \; and \; G_1 = 2) &= P(B_1 = 0 \; and \; G_1 = 2) * P(Y_2 = 1 | B_1 = 0 \; and \; G_1 = 2) \\
&= 0.4 * 0.1 * P(X_1 = 1 or X_1 = 1 or X_1 = 2) \\
&= 0.4 * 0.1 * (0.2 + 0.2 + 0.2^2) \\
&= 0.0176
\end{aligned}
$$

Summing up everything,

$$
\begin{aligned}
P(Y_2 = 1) &= 0.0864 + 0.032 + 0.06 + 0.02928 + 0.0176 \\
&= 0.22528
\end{aligned}
$$

Final Answer = $\frac{0.12}{0.22528}$
= 0.5326705

PROBLEM B:

Consider a communications line in which the true bits are 1 or 0, independently with probability of a 1 being p each. At any given bit, the line will fail, independently with probability q. Once the line has failed, it stays failed until it is repaired, reporting each bit as 0 regardless of the bit's true value. Of course, a long string of 0s should make us suspicious and cause us to inspect the line. Let Bi denote the actual value of the ith bit , and Ri the reported value, i = 1,2,3,...

Your answers to Parts 1 and 2 must be in closed form, i.e. no and the like; you may need to use Properties of Geometric Series, pp.73-74. Part 4 also has a closed-form answer.

1. Find P(Bi = Ri) for i = 1,2,3,...

2. Say you have software monitoring the line, which will flag a possible problem whenever it observes k consecutive 0s after a 1 (i.e. a 1 followed by k 0s, the last of which is the most recent bit). Find the probability that a flag is raised at bit r, r = k+1,k+2,k+3,... This is a single expression in p, q , k and r.

3. Write a function with call form simline(nreps,p,q,k,r) that finds via simulation the probability that a flag is raised at bit r but not before that time.

4. The B_i are independent, but intuitively, the R_i are not. Show that to be the case by calculating P(R_1 = 0 and R_2 = 0), P(R_1 = 0) and P(R_2 = 0), and noting that the product of the latter two probabilities is not equal to the first one.
Your answer must be consist of general expressions in p and q. With those equal to 0.6 and 0.2, respectively, one gets the answers 0.36640 and 0.32032.

Solution

Part 1
Definitions
1. Let Z_i be True if line didn't fail at bit i.

$$\begin{aligned} P(B_i = R_i) &= P(B_i = 0 \; and \; R_i = 0 \; or \; B_i = 1 \; and \; R_i = 1) \\ &= P(B_i = 0 \; and \; R_i = 0) + P(B_i = 1 \; and \; R_i = 1) \end{aligned}$$

Solving each one of them we get

$$\begin{aligned} P(B_i = 1 \; and \; R_i = 1) &= P(B_i = 1) * P(R_i = 1 | B_i = 1) \\ &= p * (1-q)^i \end{aligned}$$

$$\begin{aligned} P(B_i = 0 \; and \; R_i = 0) &= P(B_i = 0 \; and \; R_i = 0 \; and \; Z_i = True) + P(B_i = 0 \; and \; R_i = 0 \; and \; Z_i = False) \\ &= P(B_i = 0) * P(R_i = 0 \; and \; Z_i = True | B_i = 0) + P(B_i = 0) * \\ &\quad (R_i = 0 \; and \; Z_i = False | B_i = 0) \\ &= (1-p) * (1-q)^i + (1-p) * q^i \end{aligned}$$

Therefore,
$P(B_i = R_i) = (1-p) * (1-q)^i + (1-p) * q^i + p * (1-q)^i$
Part 2
Unsolved
Part 3
Code: Code1.R
NOTE: numbering starts from 1, i.e., the first bit is bit 1 and not bit 0.

Part 4
Subsection 1: $P(R_1 = 0 \; and \; R_2 = 0)$
Definitions
Let Z_i be false if the line failed for bit i and true if it didn't fail.
Let B_i denote the actual value of the ith bit
Let R_i the reported value, i = 1,2,3,...

$$\begin{aligned} P(R_1 = 0 \; and \; R_2 = 0) &= P(R_2 = 0 \; and \; Z_1 = False \; or R_2 = 0 \; and \; Z_1 = True \\ &\quad and B_1 = 0 \; and \; R_1 = 0) \\ &= P(R_2 = 0 \; and \; Z_1 = False) + P(R_2 = 0 \; and \; Z_1 = True \\ &\quad and \; B_1 = 0 \; and \; R_1 = 0) \\ &\quad [using \; P(A or B) = P(A) + P(B)] \end{aligned}$$

Solving $P(R_2 = 0 \; and \; Z_1 = False)$

$$\begin{aligned} P(R_2 = 0 \; and \; Z_1 = False) &= P(Z_1 = False) * P(R_2 = 0 | Z = False) \\ &\quad [using \; P(A and B) = P(A) * P(B|A)] \\ &= q * 1 \end{aligned}$$

Solving $P(R_2 = 0 \; and \; Z_1 = True \; and \; B_1 = 0 \; and \; R_1 = 0)$

$$\begin{aligned} P(R_2 = 0 \; and \; Z_1 = True \; and \; B_1 = 0 \; and \; R_1 = 0) = & \; P(Z_1 = True \; and \; B_1 = 0and \; R_1 = 0)* \\ & \; P(R_2 = 0|Z_1 = True \; and \; B_1 = 0 \; and \; R_1 = 0) \\ & \; [using \; P(AandB) = P(A) * P(B|A)] \end{aligned}$$

Lets solve the first sub-part of this equation

$$\begin{aligned} P(Z_1 = True \; and \; B_1 = 0and \; R_1 = 0) &= P(B_1 = 0) * P(R_1 = 0 \; and \; Z_1 = True|B_1 = 0) \\ & \quad [Using \; P(AandB) = P(A) * P(B|A)] \\ &= (1 - p) * (1 - q) \end{aligned}$$

Lets solve the second sub-part of this equation
$P(R_2 = 0|Z_1 = True \; and \; B_1 = 0 \; and \; R_1 = 0)$ is the same as $P(R_1 = 0)$ that is solved for in the next part for $P(R_1 = 0)$. It comes out to be $(1 - p) * 1 + p * q$.
Final answer
$P(R_1 andR_2) = q * 1 + ((1 - p) * (1 - q) * (1 - p + p * q))$ For p = 0.6 and q = 0.2 we get 0.3664.

Subsection 2: $P(R_1 = 0) * P(R_2 = 0)$
Definitions
Let Z_i be false if the line failed for bit i and true if it didn't fail.
Let B_i denote the actual value of the ith bit
Let R_i the reported value, i = 1,2,3,...

Lets solve each one of the parts of this equation.

$$\begin{aligned} P(R_1 = 0) &= P(B_1 = 0 \; andR_1 = 0 \; or \; B_1 = 1 \; and \; R_1 = 0) \\ &= P(B_1 = 0 \; andR_1 = 0) + P(B_1 = 1 \; andR_1 = 0) \\ & \quad [UsingP(AorB) = P(A) + P(B)] \\ &= P(B_1 = 0) * P(R_1 = 0|B_1 = 0) + P(B_1) * P(R_1 = 0|B_1 = 1) \\ & \quad [UsingP(AandB) = P(A) * P(B|A)] \\ &= (1 - p) * 1 + p * q \end{aligned}$$

$$\begin{aligned}P(R_2=0) = P(&R_2 = 0 \, and \, B_2 = 0 \, and \, Z_1 = False \, or \, R_2 = 0 \, and \, B_2 = 0 \, and \\ &Z_1 = True \, or \, R_2 = 0 \, and \, B_2 = 1 \, and \, Z_1 = False \, or R_2 = 0 \, and \\ &B_2 = 1 \, and \, Z_1 = True \,) \\ = P(&R_2 = 0 \, and \, B_2 = 0 \, and \, Z_1 = False) + P(R_2 = 0 \, and \, B_2 = 0 \\ &and \, Z_1 = True) + P(R_2 = 0 \, and \, B_2 = 1 \, and \, Z_1 = False) + \\ &P(R_2 = 0 \, and \, B_2 = 1 \, and \, Z_1 = True) \\ &[Using P(A or B) = P(A) + P(B)]\end{aligned}$$

Lets solve each of these. We use P(A and B) = P(A) * P(B—A) for each one of them.

$$\begin{aligned}P(R_2 = 0 \, and \, B_2 = 0 \, and \, Z_1 = False) &= P(B_2 = 0 \, and \, Z_1 = False) * P(R_2 = 0 | B_2 = 0 \\ &\quad and Z_1 = False) \\ &= (1-p) * (1-q)\end{aligned}$$

$$\begin{aligned}P(R_2 = 0 \, and \, B_2 = 0 \, and \, Z_1 = True) &= P(B_2 = 0 \, and \, Z_1 = True) * P(R_2 = 0 | B_2 = 0 \\ &\quad and Z_1 = True) \\ &= (1-p) * q * 1\end{aligned}$$

$$\begin{aligned}P(R_2 = 0 \, and \, B_2 = 1 \, and \, Z_1 = False) &= P(B_2 = 1 \, and \, Z_1 = False) * P(R_2 = 0 | B_2 = 1 \\ &\quad and Z_1 = False) \\ &= p * (1-q) * q\end{aligned}$$

$$\begin{aligned}P(R_2 = 0 \, and \, B_2 = 1 \, and \, Z_1 = True) &= P(B_2 = 1 \, and \, Z_1 = True) * P(R_2 = 0 | B_2 = 1 \\ &\quad and Z_1 = True) \\ &= p * q * 1\end{aligned}$$

Therefore,

$$P(R_2 = 0) = (1-p) * (1-q) + (1-p) * q * 1 + p * (1-q) * q + p * q * 1$$

Substituting p = 0.6 and q = 0.2, we get the following:
$P(R_1 = 0)$ = (1-p) + p*q = 0.52
$P(R_2 = 0)$ = (1-p) * (1-q) + (1-p)*q*1 + p*(1-q)*q + p*q*1 = 0.616
$P(R_1 = 0) * P(R_2 = 0)$ = 0.52 * 0.616 = 0.32032

PROBLEM C:

This problem will be similar to the broken rod example, Sec. 2.14.10.

Say we have a square plate, length 1.0 on each side. The plate is dropped, and breaks into two pieces, as follows: A break point occurs at a random point in the square (call runif() twice), and then along a random angle between 0 and .

Write a function with call form

simplate(nreps,p) that finds by simulation the probability that the smaller piece has area less than p.

Solution

Code: Code1.R

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