Analysis of $MAP/G/1$ queue with inventory and the model of the node of wireless sensor network with energy harvesting*

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Abstract. Queueing systems where certain inventory items are required to provide service to a customer become popular in the literature. Such systems are similar to analysed in the literature models with paired customers, assembly-like queues, passenger-taxi models, etc. During the last few years they are considered in context of modelling operations of the node of wireless sensor network with energy harvesting. Distinguishing feature of the model considered in this paper, besides the suggestion that arrival flow of customers is described by the Markovian Arrival Process, is the assumption about a general distribution of the service time while only exponential or phase-type distribution was previously assumed in the existing literature. We apply the well-known technique of $M/G/1$ type Markov chains to obtain the ergodicity criterion in a transparent form and stationary distribution of the system under study. This creates an opportunity to formulated and solve various optimization problems.
Energy harvesting and queueing and inventory.

1 Introduction

2 Mathematical Model

We consider the single server queueing system of $MAP/G/1$ type. Arrival flow is described by the Markovian Arrival Process (MAP), see [1],[2],[3]. This process assumes that the customers can arrive at the moments of jumps of the underlying Markov chain $\nu_t,\ t \geq 0$, having a finite state space $\{0, 1, \ldots, W\}$ and the generator $D(1) = D_0 + D_1$. The entries of the matrix D_1 of size $\bar{W} = W + 1$ define the intensities of transitions of the chain ν_t that are accompanied by customers arrival. The non-diagonal entries of the matrix D_0 define the intensities of transitions of the chain ν_t that are not accompanied by customer arrival.

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The vector $\boldsymbol{\theta}$ of the stationary distribution of the Markov chain ν_t is the unique solution to the system $\boldsymbol{\theta}D(1) = \mathbf{0},\ \boldsymbol{\theta}\mathbf{e} = 1$. Here and throughout this paper $\mathbf{e}$ is a column vector of appropriate size consisting of 1's, and $\mathbf{0}$ is a row vector of appropriate size consisting of zeroes. The average intensity of customers arrival λ is defined by the formula $\lambda = \boldsymbol{\theta}D_1\mathbf{e}$.

Service of arriving customer is possible only if so-called energy unit is available. Customers arriving when the server is busy or the server is idle but energy units are not available are stored in the buffer of an infinite capacity. Customers are picked up from the buffer according to First-In First-Out discipline. Service time of an arbitrary customer has distribution function $B(t)$ and finite initial moments $b_k = \int\limits_0^\infty t^k dB(t),\ k \geq 1$.

Energy units arrive according to the stationary Poisson process with the intensity γ and are stored in the stock of a finite capacity K. If the stock is full at the energy unit arrival, this unit is lost. One energy unit disappears from the stock at the instant of starting the service of a customer.

Our goal is to analyse the stationary behavior of the described queueing model.

3 Distribution of the number of customers and energy units in the system

We are interested in analysis of the stationary behavior of two-dimensional stochastic process $\zeta_t = \{i_t, k_t\},\ t \geq 0$, where i_t is the number of customers and k_t is the number of energy units in the system at the moment $t,\ i_t \geq 0,\ k_t = \overline{0, K}$, where the notation $k_t = \overline{0, K}$ means that the variable k_t admits the integer values from the set $\{0, 1, \ldots, K\}$. The process ζ_t is non-Markovian. Therefore, to analyse this process we will first consider the embedded Markov chain.

3.1 Embedded Markov Chain

Let t_n be the nth service completion moment, $n \geq 1$. Let i_n be the number of customers in the system at the moment $t_n + 0,\ i_n \geq 0$, and k_n be the number of energy units in the system at the moment $t_n - 0,\ k_n = \overline{0, K}$. Let us consider the three-dimensional process

$$\xi_n = \{i_n, k_n, \nu_n\},\ n \geq 1,$$

where ν_n is the state of the underlying process of the MAP at the moment $t_n,\ \nu_n = \overline{0, W}$.

It is easy to see that the process ξ_n is a discrete-time Markov chain. The prove this formally, we have to present expressions for one-step transition probabilities of this chain. Let us call the set of the states of the process ξ_n having the value (i, k) of the first two components as the level (i, k). Each level consists of $W + 1$ states $(i, k, \nu),\ \nu = \overline{0, W}$.

Let $P\{(i,k) \to (j,k')\}$, be the matrix of transitions probabilities from the level (i,k) to the level (j,k'), i.e., the matrix whose $(\nu,\nu')$the entry is the one-step transition probability

$$P\{(i,k,\nu) \to (j,k',\nu')\} = P\left\{i_{n+1} = j, k_{n+1} = k', \nu_{n+1} = \nu' | i_n = i, k_n = k, \nu_n = \nu\right\}.$$

To present the expressions for the probabilities $P\{(i,k) \to (j,k')\}$, we need the following notation:

- The probability of arrival of k energy units during time t
  $\varphi_k(t) = \frac{(\gamma t)^k}{k!} e^{-\gamma t},\ k \geq 0.$
- The probability of arrival of at least k energy units during time t
  $\hat{\varphi}_k(t) = \sum\limits_{i=k}^{\infty} \varphi_i(t),\ k \geq 0.$
- The probability of arrival of k energy units during service time
  $$\varphi_k = \int\limits_0^{\infty} \varphi_k(t) dB(t),\ k \geq 0.$$
- The probability of arrival of at least k energy units during service time
  $\hat{\varphi}_k = \sum\limits_{i=k}^{\infty} \varphi_i,\ k \geq 0.$
- The matrix entries of which define the probabilities of arrival of i customers and k energy units and the corresponding transitions of the underlying process ν_t of the MAP during service time
  $\Phi(i,k) = \int\limits_0^{\infty} P(i,t)\varphi_k(t) dB(t),\ i \geq 0,\ k \geq 0.$
- The matrix whose entries define the probabilities of arrival of i customers and at least k energy units and transitions of the underlying process ν_t of the MAP during service time
  $\hat{\Phi}(i,k) = \int\limits_0^{\infty} P(i,t)\hat{\varphi}_k(t) dB(t),\ i \geq 0,\ k \geq 0.$
- The matrix whose entries define the probabilities of arrival of m customers and the corresponding transitions of the underlying process of the MAP during the interval between energy units arrival
  $N(m) = \int\limits_0^{\infty} P(m,t)\gamma e^{-\gamma t} dt,\ m \geq 0.$
- The matrix whose entries define the probabilities of arrival of r energy units and transitions of the underlying process of the MAP during the interval between successive customers arrival
  $M(r) = \int\limits_0^{\infty} e^{D_0 t}\varphi_r(t) D_1 dt = \int\limits_0^{\infty} e^{D_0 t}\frac{(\gamma t)^r}{r!} e^{-\gamma I t} D_1 dt = \gamma^r(-D_0 + \gamma I)^{-(r+1)} D_1.$
- The matrix whose entries define the probabilities of arrival of at least r energy units and transitions of the underlying process of the MAP during the interval between successive customers arrival
  $\hat{M}(r) = \sum\limits_{l=r}^{\infty} M(l),\ r \geq 0.$

Lemma 1. *Transition probability matrices $P\{(i,k)\to(j,k')\}$ are defined as follows:*

$$P\{(0,0)\to(j,k')\} = M(0)\sum_{n=0}^{j} N(n)\Phi(j-n,k') + N(0)\sum_{m=0}^{k'} M(m)\Phi(j,k'-m),$$
$$j\geq 0,\ k'=\overline{0,K-2};$$

$$P\{(0,0)\to(j,K-1)\} = M(0)\sum_{n=0}^{j} N(n)\Phi(j-n,K-1)+N(0)[\sum_{m=0}^{K-1} M(m)\Phi(j,K-1-m)+\hat{M}(K)\Phi(j,0)],$$
$$j\geq 0;$$

$$P\{(0,0)\to(j,K)\} = M(0)\sum_{n=0}^{j} N(n)\hat{\Phi}(j-n,K)$$
$$+N(0)(\sum_{m=0}^{K-1} M(m)\hat{\Phi}(j,K-m)+\hat{M}(K)\hat{\Phi}(j,1)),\ j\geq 0;$$

$$P\{(0,k)\to(j,k')\} = \sum_{m=0}^{k'-k+1} M(m)\Phi(j,k'-k+1-m),\ j\geq 0,\ k=\overline{1,K}, k'=\overline{k-1,K-2};$$

$$P\{(0,k)\to(j,K-1)\} = \sum_{m=0}^{K-k} M(m)\Phi(j,K-k-m)+\hat{M}(K-k+1)\Phi(j,0),\ j\geq 0,\ k=\overline{1,K};$$

$$P\{(0,k)\to(j,K)\} = \sum_{m=0}^{K-k} M(m)\hat{\Phi}(j,K-k+1-m)+\hat{M}(K-k+1)\hat{\Phi}(j,1),\ j\geq 0,\ k=\overline{1,K};$$

$$P\{(i,0)\to(j,k')\} = \sum_{m=0}^{j-i+1} N(m)\Phi(j-i+1-m,k'),\ i\geq 1,\ j\geq i-1,\ k'=\overline{0,K-1};$$

$$P\{(i,0)\to(j,K)\} = \sum_{m=0}^{j-i+1} N(m)\hat{\Phi}(j-i+1-m,K),\ i\geq 1,\ j\geq i-1;$$

$$P\{(i,k)\to(j,k')\} = \Phi(j-i+1,k'-k+1),\ i\geq 1,\ j\geq i-1,\ k'=\overline{k-1,K-1},\ k=\overline{1,K};$$

$$P\{(i,k)\to(j,K)\} = \hat{\Phi}(j-i+1,K-k+1),\ i\geq 1,\ j\geq i-1,\ k=\overline{1,K}.$$

Proof. Proof is easily implemented by means of analysis of possible transitions of the components of the Markov chain ξ_n between two successive service completion moments and taking into account the probabilistic meaning of the denotations explained above.

3.2 Stationary distribution at an arbitrary time

Denote by $\tilde{V}_j$ the matrix of transition probabilities of the Markov renewal process $\{i_t, k_t, \nu_t\}, t \geq 0$, from the moment of renewal when the process was in a state with the value 0 of the denumerable component to an arbitrary moment preceding the next renewal when the process is in a state with the value j of the denumerable component.

Let also $\tilde{Y}_{i,j}$ be the matrix of transition probabilities of the Markov renewal process $\{i_t, k_t, \nu_t\}, t \geq 0$, from the moment of the renewal when the process was in a state with the value $i > 0$ of the denumerable component to an arbitrary moment preceding the next renewal when the process is in a state with the value j of the denumerable component.

Denote as $\mathbf{p}_j, j \geq 0$, the stationary distribution of the system at an arbitrary time.

Lemma 2. *The matrices $\tilde{V}_j, j \geq 0$, are calculated as follows.*

$$\tilde{V}_0 = \begin{pmatrix} \tilde{V}^0_{0,0} & \tilde{V}^0_{0,1} & \tilde{V}^0_{0,2} & \cdots & \tilde{V}^0_{0,K-1} & \tilde{V}^0_{0,K} \\ O & \tilde{V}^0_{1,1} & \tilde{V}^0_{1,2} & \cdots & \tilde{V}^0_{1,K-1} & \tilde{V}^0_{1,K} \\ O & O & \tilde{V}^0_{2,2} & \cdots & \tilde{V}^0_{2,K-1} & \tilde{V}^0_{2,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \ldots & O & \tilde{V}^0_{K,K} \end{pmatrix},$$

where

$$\tilde{V}^0_{k,k'} = \gamma^{k'-k}(\gamma I - D_0)^{-(k'-k+1)}, \ 0 \leq k \leq k' \leq K-1; \tag{1}$$

$$\tilde{V}^0_{k,K} = (-D_0)^{-1} - \sum_{l=0}^{K-k-1} \gamma^l (\gamma I - D_0)^{-(l+1)} \tag{2}$$

In the case $j > 0$ the matrix $\tilde{V}_j$ has the following block structure:

$$\tilde{V}_j = \begin{pmatrix} \tilde{V}^j_{0,0} & \tilde{V}^j_{0,1} & \tilde{V}^j_{0,2} & \cdots & \tilde{V}^j_{0,K-2} & \tilde{V}^j_{0,K-1} & \tilde{V}^j_{0,K} \\ \tilde{V}^j_{1,0} & \tilde{V}^j_{1,1} & \tilde{V}^j_{1,2} & \cdots & \tilde{V}^j_{1,K-2} & \tilde{V}^j_{1,K-1} & \tilde{V}^j_{1,K} \\ O & \tilde{V}^j_{2,1} & \tilde{V}^j_{2,2} & \cdots & \tilde{V}^j_{2,K-2} & \tilde{V}^j_{2,K-1} & \tilde{V}^j_{2,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ O & O & O & \ldots & O & \tilde{V}^j_{K,K-1} & \tilde{V}^j_{K,K} \end{pmatrix},$$

where

$$\tilde{V}^{j}_{0,k'} = \delta_{0,k'}\gamma N(j) + \Big[N(0)\sum_{l=0}^{k'} M(l)\tilde{\Phi}(j-1,k'-l) + M(0)\sum_{l=0}^{j-1} N(l)\tilde{\Phi}(j-l-1,k'), \tag{3}$$

$$j \geq 1, 0 \leq k' \leq K-2,$$

$$\tilde{V}^{j}_{0,K-1}\delta_{0,K-1}\gamma N(j)+$$

$$+N(0)[\sum_{l=0}^{K-1} M(l)\tilde{\Phi}(j-1,K-1-l) + \hat{M}(K)\tilde{\Phi}(j-1,0)] + M(0)\sum_{l=0}^{j-1} N(l)\tilde{\Phi}(j-l-1,K-1), \tag{4}$$

$$j \geq 1, 0 \leq k' \leq K-2,$$

$$\tilde{V}^{j}_{0,K} = N(0)[\sum_{l=0}^{K-1} M(l)\tilde{\hat{\Phi}}(j-1,K-l))] + \hat{M}(K)\tilde{\hat{\Phi}}(j-1,1)$$

$$+M(0)\sum_{l=0}^{j-1} N(l)\tilde{\hat{\Phi}}(j-l-1,K), \tag{5}$$

$$j \geq 1;$$

$$\tilde{V}^{j}_{k,k'} = \sum_{l=0}^{k'-k+1} M(l)\tilde{\Phi}(j-1,k'-k-l+1), \tag{6}$$

$$k = \overline{1,K}, k-1 \leq k' \leq K-2, j \geq 1.$$

$$V^{j}_{k,K-1} = \sum_{l=0}^{K-k} M(l)\tilde{\Phi}(j-1,K-k-l) + \hat{M}(K-k+1)\tilde{\Phi}(j-1,0), \tag{7}$$

$$k = \overline{1,K}, j \geq 1.$$

$$\tilde{V}^{j}_{k,K} = \sum_{l=0}^{K-k} M(l)\tilde{\hat{\Phi}}(j-1,K-k-l+1) + \hat{M}(K-k+1)\tilde{\hat{\Phi}}(j-1,1), \tag{8}$$

$$j \geq 0, k = \overline{1,K},$$

where

$$\tilde{\Phi}(i,k) = \int_{0}^{\infty} P(i,t)\varphi_k(t)(1-B(t))dt \; i \geq 0, \; k \geq 0, \tag{9}$$

$$\tilde{\hat{\Phi}}(i,k) = \int\limits_0^\infty P(i,t)\hat{\varphi}_k(t)(1-B(t))dt,\ i \geq 0,\ k \geq 0. \tag{10}$$

Now we will focus on finding the matrices $\tilde{Y}_{i,j}$. As it will be seen, the matrix $\tilde{Y}_{i,j}$ depends on the values i,j only via the difference $j-i$. Denote $r = j-i$ and $\tilde{Y}_r = (\tilde{Y}^r_{k,k'})_{k,k'=\overline{0,K}}$.

Then the following statement takes place.

Lemma 3. *The matrices $\tilde{Y}_r$ have the following form:*

$$\tilde{Y}_r = \begin{pmatrix} \tilde{Y}^r_{0,0} & \tilde{Y}^r_{0,1} & \tilde{Y}^r_{0,2} & \dots & \tilde{Y}^r_{0,K-2} & \tilde{Y}^r_{0,K-1} & \tilde{Y}^r_{0,K} \\ \tilde{Y}^r_{1,0} & \tilde{Y}^r_{1,1} & \tilde{Y}^r_{1,2} & \dots & \tilde{Y}^r_{1,K-2} & \tilde{Y}^r_{1,K-1} & \tilde{Y}^r_{1,K} \\ O & \tilde{Y}^r_{2,1} & \tilde{Y}^r_{2,2} & \dots & \tilde{Y}^r_{2,K-2} & \tilde{Y}^r_{2,K-1} & \tilde{Y}^r_{2,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ O & O & O & \dots & O & \tilde{Y}^r_{K,K-1} & \tilde{Y}^r_{K,K} \end{pmatrix}, r \geq 0,$$

where

$$\tilde{Y}^r_{0,k'} = \sum_{l=0}^{r} N(l)\tilde{\Phi}(r-l,k'), k' = \overline{0,K-1}, r \geq 0, \tag{11}$$

$$\tilde{Y}^r_{0,K} = \sum_{l=0}^{r} N(l)\tilde{\hat{\Phi}}(r-l,K), r \geq 0, \tag{12}$$

$$\tilde{Y}^r_{k,k'} = \tilde{\Phi}(r,k'-k+1), k = \overline{1,K}, k' = \overline{k-1,K-1}, r \geq 0, \tag{13}$$

$$\tilde{Y}^r_{k,K} = \tilde{\hat{\Phi}}(r,K-k+1), k = \overline{1,K}, r \geq 0. \tag{14}$$

Theorem 1. *The vectors $\mathbf{p}_j, j \geq 0$, are calculated via the vectors $\boldsymbol{\pi}_i, i \geq 0$, by the formula*

$$\mathbf{p}_j = \boldsymbol{\pi}_0\tilde{V}_j + \sum_{r=0}^{j-1}\boldsymbol{\pi}_{j-r}\tilde{Y}_r,\ j \geq 0, \tag{15}$$

Thus, the steady state vectors $\mathbf{p}_j, j \geq 0$, are calculated by formula (15). The only question that arises is how to calculate the matrices $\tilde{\Phi}(n,k)$ given by integrals (9). Consider the case of **deterministic distribution** $B(t)$, i.e.,

$$B(t) = \begin{cases} 0, t \leq b_1, \\ 1, t > b_1. \end{cases}$$

Calculating the integral in (9), we rewrite formula (9) as

$$\tilde{\Phi}(n,k) = \int\limits_0^\infty P(n,t)\varphi_k(t)(1-B(t))dt = \int\limits_0^{b_1} P(n,t)\varphi_k(t)dt, \tag{16}$$

To calculate the integral in (16), we use the uniformization procedure. Denote $h = \max\limits_{i=\overline{0,W}} (-D_0)_{ii}$.

Then the matrix $P(n,t),\ n \geq 1$, can be represented in the following form:

$$P(n,t) = \sum_{j=0}^{\infty} e^{-ht} \frac{(ht)^j}{j!} K_n^{(j)},\ n \geq 0, \tag{17}$$

where the matrices $K_n^{(j)},\ n \geq 1, j \geq 0$, are calculated by recursion

$$\begin{aligned} K_0^{(0)} &= I,\ K_n^{(0)} = O,\ n \geq 1, \\ K_0^{(j+1)} &= K_0^{(j)}(I + h^{-1}D_0), \\ K_n^{(j+1)} &= h^{-1}K_{n-1}^{(j)}D_1 + K_n^{(j)}(I + h^{-1}D_0),\ n \geq 1,\ j \geq 0. \end{aligned} \tag{18}$$

Substituting expression (17) for $P(n,t)$ and expression for $\varphi_k(t)$ in (16) and simplifying, we obtain the following relations:

$$\tilde{\Phi}(n,k) = \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j!k!} \int\limits_0^{b_1} e^{-(h+\gamma)t} t^{j+k} dt =$$

$$= \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j!k!} \left[\frac{(j+k)!}{(h+\gamma)^{j+k+1}} - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{(j+k)!}{l!} \frac{b_1^l}{(h+\gamma)^{j+k-l+1}} \right] =$$

$$= \sum_{j=0}^{\infty} K_n^{(j)} \frac{h^j \gamma^k}{j!k!} \frac{(j+k)!}{(h+\gamma)^{j+k+1}} \left[1 - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h+\gamma)]^l}{l!} \right] =$$

$$= \frac{1}{(h+\gamma)k!} \left(\frac{\gamma}{h+\gamma} \right)^k \sum_{j=0}^{\infty} K_n^{(j)} \frac{(j+k)!}{j!} \left(\frac{h}{h+\gamma} \right)^j \left[1 - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h+\gamma)]^l}{l!} \right].$$

Thus, the matrix $\tilde{\Phi}(n,k)$ is calculated by the following formula:

$$\begin{gathered} \tilde{\Phi}(n,k) = \\ = \frac{1}{(h+\gamma)k!} \left(\frac{\gamma}{h+\gamma} \right)^k \sum_{j=0}^{\infty} K_n^{(j)} \frac{(j+k)!}{j!} \left(\frac{h}{h+\gamma} \right)^j \left[1 - e^{-b_1(h+\gamma)} \sum_{l=0}^{j+k} \frac{[b_1(h+\gamma)]^l}{l!} \right], \\ n \geq 0,\ k = \overline{0,K}. \end{gathered} \tag{19}$$

3.3 A Subsection Sample

Please note that the first paragraph of a section or subsection is not indented. The first paragraph that follows a table, figure, equation etc. does not need an indent, either.

Subsequent paragraphs, however, are indented.

Sample Heading (Third Level) Only two levels of headings should be numbered. Lower level headings remain unnumbered; they are formatted as run-in headings.

Theorem 1. *This is a sample theorem. The run-in heading is set in bold, while the following text appears in italics. Definitions, lemmas, propositions, and corollaries are styled the same way.*

Proof. Proofs, examples, and remarks have the initial word in italics, while the following text appears in normal font.

For citations of references, we prefer the use of square brackets and consecutive numbers. Citations using labels or the author/year convention are also acceptable. The following bibliography provides a sample reference list with entries for journal articles [4], an LNCS chapter [5], a book [6], proceedings without editors [7], and a homepage [**?**]. Multiple citations are grouped [4–6], [4, 6, 7, **?**].

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