A. Dudin et al. Friendship University of Russia (RUDN University), Belarusian State University, 4, Nezavisimosti Ave., Minsk, 220030, Belarus dudin-alexander@mail.ru, dudin85@mail.ru, dudina@bsu.by , vklimenok@yandex.ru

Abstract

Queueing systems where certain inventory items are required to provide service to a customer become popular in the literature. Such systems are similar to the analysed in the literature models with paired customers, assembly-like queues, passenger-taxi models, etc. During the last few years they are considered in context of modelling operation of the nodes of a wireless sensor network with energy harvesting. Distinguishing feature of the model considered in this paper, besides the suggestion that arrival flow of customers is described by the Markovian Arrival Process, is the assumption about a general distribution of the service time while only exponential or phase-type distribution was previously assumed in the existing literature. We apply the well-known technique of $M/G/1$ type Markov chains to obtain the ergodicity criterion in a transparent form and stationary distribution of the system under study and sojourn time distribution. This creates an opportunity to formulated and solve various optimization problems.

Energy harvesting and queueing and inventory.

1 Introduction

2 Mathematical Model

We consider the single server queueing system of $MAP/G/1$ type. Arrival flow is described by the Markovian Arrival Process (MAP), see [1],[5],[6]. This process assumes that the customers can arrive at the moments of jumps of the underlying Markov chain $\nu_t,\ t \geq 0$, having a finite state space $\{0, 1, \ldots, W\}$ and the generator $D(1) = D_0 + D_1$. The entries of the matrix D_1 of size $\bar{W} = W + 1$ define the intensities of transitions of the chain ν_t that are accompanied by customers arrival. The non-diagonal entries of the matrix D_0 define the intensities of transitions of the chain ν_t that are not accompanied by customer arrival.

The vector $\boldsymbol{\theta}$ of the stationary distribution of the Markov chain ν_t is the unique solution to the system $\boldsymbol{\theta} D(1) = \mathbf{0},\ \boldsymbol{\theta}\mathbf{e} = 1$. Here and throughout this paper $\mathbf{e}$ is a column vector of appropriate size consisting of 1's, and $\mathbf{0}$ is a row vector of appropriate size consisting of zeroes. The average intensity of customers arrival λ is defined by the formula $\lambda = \boldsymbol{\theta} D_1 \mathbf{e}$.

Service of arriving customer is possible only if so-called energy unit is available. Customers arriving when the server is busy or the server is idle but energy units are not available are stored in the buffer of an infinite capacity. Customers are picked up from the buffer according to First-In First-Out discipline. Service time of an arbitrary customer has distribution function $B(t)$ with Laplace-Stieltjes transform $\beta(s) = \int\limits_0^\infty e^{-st} dB(t),\ Re\ s > 0$, and finite initial moments $b_k = \int\limits_0^\infty t^k dB(t),\ k \geq 1$.

Energy units arrive according to the stationary Poisson process with the intensity γ and are stored in the stock of a finite capacity K. If the stock is full at the energy unit arrival, this unit is lost. One energy unit disappears from the stock at the instant of starting the service of a customer.

Our goal is to analyse the stationary behavior of the described queueing model.

3 Distribution of the number of customers and energy units in the system

We are interested in analysis of the stationary behavior of two-dimensional stochastic process $\zeta_t = \{i_t, k_t\},\ t \geq 0$, where i_t is the number of customers and k_t is the number of energy units in the system at the moment $t,\ i_t \geq 0,\ k_t = \overline{0, K}$, where the notation $k_t = \overline{0, K}$ means that the variable k_t admits the integer values from the set $\{0, 1, \ldots, K\}$. The process ζ_t is non-Markovian. Therefore, to analyse this process we will first consider the embedded Markov chain.

3.1 Embedded Markov Chain. Transition probabilities

Let t_n be the nth service completion moment, $n \geq 1$. Let i_n be the number of customers in the system at the moment $t_n + 0,\ i_n \geq 0$, and k_n be the number of energy units in the system at the moment $t_n - 0,\ k_n = \overline{0, K}$. Let us consider the three-dimensional process

$$\xi_n = \{i_n, k_n, \nu_n\},\ n \geq 1,$$

where ν_n is the state of the underlying process of the MAP at the moment $t_n,\ \nu_n = \overline{0, W}$.

It is easy to see that the process ξ_n is a discrete-time Markov chain. The prove this formally, we have to present expressions for one-step transition probabilities of this chain. Let us call the set of the states of the process ξ_n having the value (i, k) of the first two components as the level (i, k). Each level consists of $W + 1$ states $(i, k, \nu),\ \nu = \overline{0, W}$.

Let $P\{(i, k) \to (j, k')\}$, be the matrix of transitions probabilities from the level (i, k) to the level (j, k'), i.e., the matrix whose $(\nu, \nu')$the entry is the one-step transition probability

$$P\{(i, k, \nu) \to (j, k', \nu')\} = P\left\{i_{n+1} = j, k_{n+1} = k', \nu_{n+1} = \nu' | i_n = i, k_n = k, \nu_n = \nu\right\}.$$

To present the expressions for the probabilities $P\{(i, k) \to (j, k')\}$, we need the following notation:

- The probability of k energy units arrival during time t

  $\varphi_k(t) = \frac{(\gamma t)^k}{k!} e^{-\gamma t},\ k \geq 0.$

- The probability of at least k energy units arrival of during time t

  $\hat{\varphi}_k(t) = \sum\limits_{i=k}^{\infty} \varphi_i(t),\ k \geq 0.$

- The probability of k energy units arrival during a service time

$$\varphi_k = \int\limits_0^{\infty} \varphi_k(t) dB(t),\ k \geq 0.$$

- The probability of arrival of at least k energy units during service time

  $\hat{\varphi}_k = \sum\limits_{i=k}^{\infty} \varphi_i, \ k \geq 0.$

- The matrix entries of which define the probabilities of arrival of i customers and k energy units and the corresponding transitions of the underlying process ν_t of the MAP during service time

  $\Phi(i,k) = \int\limits_0^{\infty} P(i,t)\varphi_k(t)dB(t), \ i \geq 0, \ k \geq 0.$

- The matrix whose entries define the probabilities of arrival of i customers and at least k energy units and transitions of the underlying process ν_t of the MAP during service time

  $\hat{\Phi}(i,k) = \int\limits_0^{\infty} P(i,t)\hat{\varphi}_k(t)dB(t), \ i \geq 0, \ k \geq 0.$

- The matrix whose entries define the probabilities of arrival of m customers and the corresponding transitions of the underlying process of the MAP during the interval between energy units arrival

  $N(m) = \int\limits_0^{\infty} P(m,t)\gamma e^{-\gamma t}dt, \ m \geq 0.$

- The matrix whose entries define the probabilities of arrival of r energy units and transitions of the underlying process of the MAP during the interval between successive customers arrival

  $M(r) = \int\limits_0^{\infty} e^{D_0 t}\varphi_r(t)D_1 dt = \int\limits_0^{\infty} e^{D_0 t}\frac{(\gamma t)^r}{r!}e^{-\gamma I t}D_1 dt = \gamma^r(-D_0+\gamma I)^{-(r+1)}D_1.$

- The matrix whose entries define the probabilities of arrival of at least r energy units and transitions of the underlying process of the MAP during the interval between successive customers arrival

$$\hat{M}(r) = \sum_{l=r}^{\infty} M(l) = \gamma^r(-D_0+\gamma I)^{-r}(-D_0)^{-1}D_1, \ r \geq 0.$$

Transition probability matrices $P\{(i,k) \to (j,k')\}, j \geq \max\{0, i-1\}$, are defined as follows:

$$P\{(0,0) \to (j,k')\} = M(0)\sum_{n=0}^{j} N(n)\Phi(j-n,k') + N(0)\sum_{m=0}^{k'} M(m)\Phi(j,k'-m),$$
$$k' = \overline{0, K-2};$$

$$P\{(0,0) \to (j,K-1)\} = M(0)\sum_{n=0}^{j} N(n)\Phi(j-n,K-1) + N(0)\Big[\sum_{m=0}^{K-1} M(m)\Phi(j,K-1-m)$$
$$+\hat{M}(K)\Phi(j,0)\Big];$$

$$P\{(0,0)\to(j,K)\} = M(0)\sum_{n=0}^{j} N(n)\hat{\Phi}(j-n,K)+N(0)\Big[\sum_{m=0}^{K-1} M(m)\hat{\Phi}(j,K-m)$$

$$+\hat{M}(K)\hat{\Phi}(j,1)\Big];$$

$$P\{(0,k)\to(j,k')\} = \sum_{m=0}^{k'-k+1} M(m)\Phi(j,k'-k+1-m),\ k=\overline{1,K}, k'=\overline{k-1,K-2};$$

$$P\{(0,k)\to(j,K-1)\} = \sum_{m=0}^{K-k} M(m)\Phi(j,K-k-m)+\hat{M}(K-k+1)\Phi(j,0),\ k=\overline{1,K};$$

$$P\{(0,k)\to(j,K)\} = \sum_{m=0}^{K-k} M(m)\hat{\Phi}(j,K-k+1-m)+\hat{M}(K-k+1)\hat{\Phi}(j,1),\ k=\overline{1,K};$$

$$P\{(i,0)\to(j,k')\} = \sum_{m=0}^{j-i+1} N(m)\Phi(j-i+1-m,k'),\ i\ge 1,\ k'=\overline{0,K-1};$$

$$P\{(i,0)\to(j,K)\} = \sum_{m=0}^{j-i+1} N(m)\hat{\Phi}(j-i+1-m,K),\ i\ge 1;$$

$$P\{(i,k)\to(j,k')\} = \Phi(j-i+1,k'-k+1),\ i\ge 1,\ k'=\overline{k-1,K-1},\ k=\overline{1,K};$$

$$P\{(i,k)\to(j,K)\} = \hat{\Phi}(j-i+1,K-k+1),\ i\ge 1,\ k=\overline{1,K}.$$

Proof is easily implemented by means of analysis of possible transitions of the components of the Markov chain ξ_n between two successive service completion moments and taking into account the probabilistic meaning of the denotations explained above.

Let

$$\Psi_1(t) = \begin{pmatrix} O & O & \dots & O & O \\ \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ O & \varphi_0(t) & \dots & \varphi_{K-2}(t) & \hat{\varphi}_{K-1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & \varphi_0(t) & \hat{\varphi}_1(t) \end{pmatrix},$$

$$\Psi_0(t) = \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ O & O & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & O & O \end{pmatrix}.$$

Then, the block matrices $P_{i,j}$ composed by the blocks $P\{(i,k) \to (j,k')\}$, $k' = \overline{\max\{0,k-1\},K}$, $k = \overline{0,K}$, for $i \geq 1$ are defined by

$$P_{i,j} = Y_{j-i+1}, \ j \geq i-1,$$

where the matrices Y_l are defined by formula

$$Y_l = \int_0^\infty \Psi_1(t) \otimes P(l,t)dB(t) + \sum_{m=0}^{l} \int_0^\infty \Psi_0(t) \otimes N(m)P(l-m,t)dB(t), \ l \geq 0.$$

The matrix generating function $Y(z) = \sum_{l=0}^{\infty} Y_l z^l$, $|z| < 1$, of matrices Y_l, $l \geq 0$, has the following form:

$$Y(z) = \int_0^\infty \Psi_1(t) \otimes e^{D(z)t}dB(t) + \gamma \int_0^\infty \Psi_0(t) \otimes (\gamma I - D(z))^{-1} e^{D(z)t}dB(t). \quad (1)$$

In turn, the block matrices $P_{0,j}$ composed by the blocks $P\{(0,k) \to (j,k')\}$, $k' = \overline{\max\{0,k-1\},K}$, $k = \overline{0,K}$, will be denoted by $P_{0,j} = V_j$, $j \geq 0$. Also denote $V(z) = \sum_{l=0}^{\infty} V_l z^l$, $|z| < 1$.

Here $\otimes$ is the symbol of Kronecker product of matrices, see [4].

3.2 Embedded Markov Chain. Ergodicity condition

It follows from Corollary 1 that the Markov chain ξ_n belongs to the class of $M/G/1$ type Markov chains, see [7]. It follows from [7] that the criterion of ergodicity for the Markov chain ξ_n is fulfillment of inequality

$$\mathbf{y} \sum_{k=1}^{\infty} kY_k \mathbf{e} = \mathbf{y}Y'(1)\mathbf{e} < 1 \quad (2)$$

where the vector $\mathbf{y}$ is the unique solution to the system

$$\mathbf{y} \sum_{k=0}^{\infty} Y_k = \mathbf{y}Y(1) = \mathbf{y}, \ \ \mathbf{ye} = 1. \quad (3)$$

By direct substitution into (3), it is possible to verify that solution of system (3) has the form

$$\mathbf{y} = \mathbf{u} \otimes \boldsymbol{\theta} \quad (4)$$

where $\boldsymbol{\theta}$ is the vector of stationary distribution of the underlying process ν_t of the MAP and $\mathbf{u} = (u_0, u_1, \dots, u_K)$ is the unique solution to the system

$$\mathbf{u} \int_0^\infty \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t) \\ O & \varphi_0(t) & \dots & \varphi_{K-2}(t) & \hat{\varphi}_{K-1}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & \varphi_0(t) & \hat{\varphi}_1(t) \end{pmatrix} dB(t) = \mathbf{u}, \ \mathbf{ue} = 1. \quad (5)$$

It is easy to check that the vector $\mathbf{u}$ defines the stationary distribution of the embedded Markov chain for $M/G/1/K$ type queueing model with the stationary arrival process with intensity γ and service time distribution function $B(t)$.

The system (5) can be rewritten in the form

$$u_k = u_0\varphi_k + \sum_{l=1}^{k+1} u_l\varphi_{k+1-l}, k = \overline{0, K-1}, u_K = u_0\hat{\varphi}_K + \sum_{l=1}^{K} u_l\varphi_{K+1-l},$$

and its solution is given by

$$u_k = u_0\psi_k,\ k = \overline{1, K}, \tag{6}$$

and

$$u_0 = (\sum_{k=0}^{K}\psi_k)^{-1}, \tag{7}$$

where the numbers ψ_k can be recursively computed, e.g., from the following recursion

$$\psi_0 = 1,\ \psi_{k+1} \ = \ (\psi_k - \varphi_k - \sum_{i=1}^{k}\psi_i\varphi_{k+1-i})\varphi_0^{-1},\ k = \overline{0, K-1}. \tag{8}$$

By substitution of the vector $\mathbf{y}$ in form (4) into inequality (2), we obtain inequality

$$\lambda b_1 + u_0\frac{\lambda}{\gamma} \ < \ 1. \tag{9}$$

Therefore, we proved the following statement. The embedded Markov chain ξ_n for the queueing model under study is ergodic if and only if the inequality (9) is fulfilled where the probability u_0 is given by (7), (8).

Condition (9) is easily tractable if it is rewritten in the form

$$\lambda(b_1 + u_0\frac{1}{\gamma}) \ < \ 1. \tag{10}$$

Left hand side of (10) defines the mean number of customers arriving into the system between the successive service completion moments when the system is overloaded. Indeed, such a time includes the service time and the time until an energy unit arrival when the buffer of energy is empty. Mean value of this time is $b_1 + u_0\frac{1}{\gamma}$ and the mean number of arriving customers during this time is $\lambda(b_1 + u_0\frac{1}{\gamma})$. It is well known that the system is ergodic if this mean number is less than 1 (average number of arriving customers is less than the number of departing customers).

In what follows we assume that condition (9) is fulfilled.

3.3 Computation of the stationary distribution of embedded Markov chain

Let us denote

$$\pi(i, k, \nu) = \lim_{n\to\infty} P\{i_n = i, k_n = k, \nu_n = \nu\},$$

$$\boldsymbol{\pi}(i,k) = (\pi(i,k,0),\dots,\pi(i,k,W),$$

$$\boldsymbol{\pi}_i = (\boldsymbol{\pi}(i,0),\boldsymbol{\pi}(i,1),\dots,\boldsymbol{\pi}(i,K)),\ i \geq 0.$$

It is well-known that the vectors $\boldsymbol{\pi}_i,\ i \geq 0$, satisfy the following system of linear algebraic equations:

$$\boldsymbol{\pi}_j = \boldsymbol{\pi}_0 V_j + \sum_{i=1}^{j+1} \boldsymbol{\pi}_i Y_{j-i+1},\ \ j \geq 0. \tag{11}$$

There are several possibilities to compute the vectors $\boldsymbol{\pi}_j\ j \geq 0$. One of them is based on the use of the vector generating function $\boldsymbol{\pi}(z) = \sum\limits_{j=0}^{\infty} \boldsymbol{\pi}_j z^j,\ |z| < 1$. It is easy to derive from the system (11) the following functional equation for the vector generating function $\boldsymbol{\pi}(z)$:

$$\boldsymbol{\pi}(z)(Y(z) - zI) = \boldsymbol{\pi}_0(Y(z) - zV(z). \tag{12}$$

This equation can be solved using the reasonings of the analyticity of the generating function $\boldsymbol{\pi}(z)$ in the unit disc of the complex plane and existence of the required number of zeroes of the determinant of the matrix $Y(z) - zI$ in this unit disc. For more details see, e.g., [3].

Another possibility uses consideration of the series of censored Markov chains for the Markov chain ξ_n, for derivation of an alternative system of of linear algebraic equations for the vectors $\boldsymbol{\pi}_j\ j \geq 0$, and its solving based of extension of M. Neuts method with the use of the matrix G. Again, for more details see, e.g., [3].

In what follows, we consider the vectors of stationary probabilities of the embedded Markov chain $\boldsymbol{\pi}_j\ j \geq 0$, be known and compute the vectors of stationary probabilities of the system states at an arbitrary moment as well as the stationary distribution of the virtual and real sojourn time of a customer in the system.

3.4 Stationary distribution of sojourn time

First, we analyze the stationary distribution of the virtual sojourn time, i.e., sojourn time of a customer if it will arrive at an arbitrary moment. The virtual sojourn time consists of (i) the *residual service time* from an arbitrary time instant (associated with virtual customer arrival) to the next service completion epoch; (ii) the *generalized service times* of customers staying in the queue at an arbitrary time and (iii) the *generalized service time* of the virtual customer.

First, we define the *generalized service time*. Let $\hat{B}(x)$ be the matrix distribution function of *generalized service time*. More specifically, let $\bar{B}(x) = (\bar{B}(x)_{k,k'})_{k,k'=\overline{0,K}}$, where $\bar{B}(x)_{k,k'} = P\{t_{n+1} - t_n < x, k_{n+1} = k' \mid k_n = k, i_n \neq 0\}$. Denote $\mathcal{B}(s) = \int\limits_0^{\infty} e^{-st} d\bar{B}(t)$.

The matrix Laplace-Stieltjes transform of the generalized service time distribution is calculated as

$$\mathcal{B}(s)=\int\limits_0^\infty e^{-st}\begin{pmatrix}\gamma(s)\varphi_0(t) & \gamma(s)\varphi_1(t) & \dots & \gamma(s)\varphi_{K-1}(t) & \gamma(s)\hat{\varphi}_K(t)\\ \varphi_0(t) & \varphi_1(t) & \dots & \varphi_{K-1}(t) & \hat{\varphi}_K(t)\\ O & \varphi_0(t) & \dots & \varphi_{K-2}(t) & \hat{\varphi}_{K-1}(t)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ O & O & \dots & \varphi_0(t) & \hat{\varphi}_1(t)\end{pmatrix}dB(t), \tag{12}$$

where $\gamma(s)=\int\limits_0^\infty e^{-st}\gamma e^{-\gamma t}dt$.

Proof. The generalized service time of a tagged customer is just the service time of the customer if there is energy in the system. In this case the Laplace-Stieltjes transform $(\mathcal{B}(s))_{k,k'}, k>0$, of the generalized service time distribution is calculated by $\int\limits_0^\infty e^{-st}\varphi_{k'-k+1}(t)dB(t)$, if $k'<K$ and by $\int\limits_0^\infty e^{-st}\hat{\varphi}_K(t)dB(t)$, if $k'=K$.

However, when there is no energy in the system, the generalized service time consists of the time till a unit of energy arrives and the service time of the tagged customer. The corresponding Laplace-Stieltjes transform is defined by $(\mathcal{B}(s))_{0,k'}=\gamma(s)\int\limits_0^\infty e^{-st}\varphi_{k'}(t)dB(t)$, if $k'<K$ and by $\gamma(s)\int\limits_0^\infty e^{-st}\hat{\varphi}_K(t)dB(t)$, if $k'=K$. The stated result (12) follows immediately.

□

Now we study the *residual time*. To this end, we consider the process $\chi_t=\{j_t,\, k_t,\, \nu_t,\, \tilde{v}_t\}$, $t\geq 0$, whose components are defined as follows: j_t is the number of customers in the system at time t, k_t is the number of energy units in the system at the time immediately after the next service completion epoch, ν_t is the state of the MAP, $\tilde{v}_t$ is the residual time from t to the next service completion epoch.

Using the definition of semi-regenerative processes given in [**?**], it can be verified that the process χ_t is a semi-regenerative one with the embedded Markov renewal process $\{\xi_n, t_n\}$, $n\geq 1$.

Let

$$\tilde{V}(j,k,\nu,x)=\lim_{t\to\infty}P\{i_t=j,k_t=k,\nu_t=\nu,\tilde{v}_t<x\}, \tag{13}$$

$$j\geq 0,\ r=\overline{0,K},\ \nu=\overline{0,W},\ x\geq 0,$$

be the stationary distribution of the process χ_t, $t\geq 0$.

From [**?**], the limits in (13) exist if the process $\{\xi_n,t_n\}$, $n\geq 1$, is irreducible aperiodic recurrent and the value τ of the mean inter-departure time is finite. All these conditions hold if inequality (?) is satisfied.

Let $\tilde{\boldsymbol{V}}(j,x)$ be the row vector of the steady state probabilities $\boldsymbol{V}(j,k,\nu,x)$ arranged according to the lexicographic order of the components (k,ν), and let $\tilde{\boldsymbol{v}}(i,s)$ be the corresponding vector of the Laplace-Stieltjes transforms, that is, let $\tilde{\boldsymbol{v}}(j,s)=\int\limits_0^\infty e^{-sx}d\tilde{\boldsymbol{V}}(j,x)$, $j\geq 0$.

To find the vectors $\tilde{\boldsymbol{v}}(j,s)$ we will try to express this vectors via the stationary distribution $\boldsymbol{\pi}_i$, $i\geq 0$, of the Markov chain ξ_n, $n\geq 1$.

To this end, we first calculate the probabilities

$$p_{i,k,\nu}(j,k',\nu',x) = \lim_{t\to\infty} P\{j_t = j, k_t = k', \nu_t = \nu', \tilde{v}_t < x | j_n = i, k_n = k, \nu_n = \nu\}$$

that the process χ_t is in the state (j,k',ν',x) at an arbitrary time conditional the components j_t, k_t, ν_t were in the states i,k,ν, respectively, at the last service completion epoch t_n.

Define the matrix $C^j_{k,k'}(x) = (p_{0,k,\nu,t}(j,k',\nu',x))_{\nu,\nu'=\overline{0,W}}$ and the block matrix $C_j(x) = (C^j_{k,k'}(x))_{k,k'=\overline{0,K}}$. Let $C_j(s)$ be the corresponding matrix of the Laplace-Stieltjes transforms,that is, $C_j(s) = \int\limits_0^\infty e^{-sx}dC_j(x)$.

Define the matrix $\Omega^r_{k,k'}(x) = (p_{i,k,\nu,t}(i+r,k',\nu',x))_{\nu,\nu'=\overline{0,W}}$ and the block matrix $\Omega_r(x) = (\Omega^r_{k,k'}(x))_{k,k'=\overline{0,K}}$. Let $\Omega_r(s)$ be the corresponding matrix of the Laplace-Stieltjes transforms,that is, $\Omega_r(s) = \int\limits_0^\infty e^{-sx}d\Omega_r(x)$.

The matrix $C_0(s)$ is calculated by

$$C_0(s) = \begin{pmatrix} C^0_{0,0} & C^0_{0,1} & C^0_{0,2} & \dots & C^0_{0,K-1} & C^0_{0,K} \\ O & C^0_{1,1} & C^0_{1,2} & \dots & C^0_{1,K-1} & C^0_{1,K} \\ O & O & C^0_{2,2} & \dots & C^0_{2,K-1} & C^0_{2,K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \dots & O & C^0_{K,K} \end{pmatrix} (\mathcal{B}(s) \otimes I_{\bar{W}}), \qquad (14)$$

where

$$C^0_{k,k'} = \gamma^{k'-k}(\gamma I - D_0)^{k'-k+1},\ 0 \le k \le k' \le K-1; \qquad (15)$$

$$C^0_{k,K} = (-D_0)^{-1} - \sum_{l=0}^{K-k-1} \gamma^l(\gamma I - D_0)^{l+1}. \qquad (16)$$

The matrices $C_j(s)$, $j \ge 1$, have the following block structure:

$$C_j(s) = \begin{pmatrix} C^j_{0,0}(s) & C^j_{0,1}(s) & C^j_{0,2}(s) & \dots & C^j_{0,K-1}(s) & C^j_{0,K}(s) & C^j_{0,K+1}(s) \\ O & C^j_{1,1}(s) & C^j_{1,2}(s) & \dots & C^j_{1,K-1}(s) & C^j_{1,K}(s) & C^j_{1,K+1}(s) \\ O & O & C^j_{2,2}(s) & \dots & C^j_{2,K-1}(s) & C^j_{2,K}(s) & C^j_{2,K+1}(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ O & O & O & \dots & O & C^j_{K,K}(s) & C^j_{K,K+1}(s) \end{pmatrix}, j \ge 1,$$

where

$$C^j_{0,k'}(s) = \gamma N(j)(\mathcal{B}(s))_{0,k'} + N(0)\sum_{l=0}^{k'} M(l)\tilde{\Phi}_{k'-l}(j-1,s) + M(0)\sum_{l=0}^{j-1} N(l)\tilde{\Phi}_{k'}(j-l-1,s), \qquad (17)$$

$$0 \le k' \le K-2,$$

$$C^j_{0,K-1}(s) = \gamma N(j)(\mathcal{B}(s))_{0,K-1} +$$

$$+N(0)[\sum_{l=0}^{K-1} M(l)\tilde{\Phi}_k - l - 1(j-1,s) + \hat{M}(K)\tilde{\Phi}_0(j-1,s)] + M(0)\sum_{l=0}^{j-1} N(l)\tilde{\Phi}_{K-1}(j-l-1,s), \tag{18}$$

$$C^j_{0,K}(s) = N(0)[\sum_{l=0}^{K-1} M(l)\tilde{\hat{\Phi}}_{K-l}(j-1,s))] + \hat{M}(K)\tilde{\hat{\Phi}}_1(j-1,s) + M(0)\sum_{l=0}^{j-1} N(l)\tilde{\hat{\Phi}}_K(j-l-1,s), \tag{19}$$

$$C^j_{k,k'}(s) = \sum_{l=0}^{k'-k+1} M(l)\tilde{\Phi}_{k'-k-l+1}(j-1,s), k = \overline{1,K}, k-1 \leq k' \leq K-2, \tag{20}$$

$$V^j_{k,K-1}(s) = \sum_{l=0}^{K-k} M(l)\tilde{\Phi}_{K-k-l}(j-1,s) + \hat{M}(K-k+1)\tilde{\Phi}_0(j-1,s),\ k = \overline{1,K}, \tag{21}$$

$$C^j_{k,K}(s) = \sum_{l=0}^{K-k} M(l)\tilde{\hat{\Phi}}_{K-k-l+1}(j-1,s) + \hat{M}(K-k+1)\tilde{\hat{\Phi}}_1(j-1,s),\ k = \overline{1,K}, \tag{22}$$

The matrix $\Omega_r(s)$ has the following form:

$$\Omega_r(s) = \begin{pmatrix} \Omega^r_{0,0}(s) & \Omega^r_{0,1}(s) & \Omega^r_{0,2}(s) & \dots & \Omega^r_{0,K-2}(s) & \Omega^r_{0,K-1}(s) & \Omega^r_{0,K}(s) \\ \Omega^r_{1,0}(s) & \Omega^r_{1,1}(s) & \Omega^r_{1,2}(s) & \dots & \Omega^r_{1,K-2}(s) & \Omega^r_{1,K-1}(s) & \Omega^r_{1,K}(s) \\ O & \Omega^r_{2,1}(s) & \Omega^r_{2,2}(s) & \dots & \Omega^r_{2,K-2}(s) & \Omega^r_{2,K-1}(s) & \Omega^r_{2,K}(s) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ O & O & O & \dots & O & \Omega^r_{K,K-1}(s) & \Omega^r_{K,K}(s) \end{pmatrix},$$

$$\Omega^r_{0,k'}(s) = \sum_{l=0}^{r} N(l)\tilde{\Phi_{k'}}(r-l,s), k' = \overline{0,K-1}, r \geq 0, \tag{23}$$

$$\Omega^r_{0,K}(s) = \sum_{l=0}^{r} N(l)\tilde{\hat{\Phi}}_K(r-l,s), r \geq 0, \tag{24}$$

$$\Omega^r_{k,k'}(s) = \tilde{\Phi}_{k'-k+1}(r,s), k = \overline{1,K}, k' = \overline{k-1,K-1}, r \geq 0, \tag{25}$$

$$\Omega^r_{k,K}(s) = \tilde{\hat{\Phi}}_{K-k+1}(r,s), k = \overline{1,K}, r \geq 0. \tag{26}$$

where

$$\tilde{\Phi}_k(i,s)(s) = \int_0^\infty \varphi_k(t) \int_0^t P(i,x)e^{-s(t-x)}dxdB(t)\ i \geq 0,\ k \geq 0, \tag{27}$$

$$\tilde{\Phi}_k(i,s)(s) = \int\limits_0^\infty \hat{\varphi}_k(t) \int\limits_0^t P(i,x)e^{-s(t-x)}dxdB(t)\ i \geq 0,\ k \geq 0. \tag{28}$$

Proof of the theorem is presented in Appendix A.

Using Lemma 3, we express the vectors $\tilde{\boldsymbol{v}}(j,s)$ via the stationary distribution $\boldsymbol{\pi}_i$, $i \geq 0$, of the Markov chain ξ_n, $n \geq 1$.

The vectors $\tilde{\boldsymbol{v}}(j,s)$ are related to the steady state vectors $\boldsymbol{\pi}_i$, $i \geq 0$, of the Markov chain ξ_n, $n \geq 1$, as follows:

$$\tilde{\boldsymbol{v}}(j,s) = \tau^{-1}[\boldsymbol{\pi}_0 C_j(s) + \sum_{r=0}^{j-1} \boldsymbol{\pi}_{j-r}\Omega_r(s)],\ j \geq 0, \tag{29}$$

where the mean inter-departure time τ is defined by

$$\tau = b_1 + \gamma^{-1}\sum_{i=1}^{\infty} \boldsymbol{\pi}(i,0) + \sum_{k=1}^{K} \boldsymbol{\pi}(0,k)(-D_0)^{-1}\mathbf{e} + \boldsymbol{\pi}(0,0)(-D_0+\gamma I)^{-1}$$

$$\times(D_1\gamma^{-1} + \gamma(-D_0)^{-1})\mathbf{e}.$$

Let $p_j(k,\nu)$ be the steady state probability that at an arbitrary time j customers stay in the system and at the time immediately after the next service completion epoch there are k energy units in the system. Denote by $\mathbf{p}_j(k)$ the vector of probabilities $p_j(k,\nu)$ listed in lexicographic order of component $\nu, \nu = \overline{0,W}$.

The vectors $\mathbf{p}_j(k), j \geq 0, k = \overline{0,K}$, are calculated as

$$\mathbf{p}_j(k) = \tilde{\boldsymbol{v}}(j,0)(I_{K+1} \otimes \mathbf{e}_{\bar{W}}),\ j \geq 0, k = \overline{0,K}.$$

Now we are ready to derive the equation for the vector Laplace-Stieltjes transform $\boldsymbol{v}(s)$ of the distribution of the virtual sojourn time in the system. Let $v(k,\nu,x)$ be the probability that, at an arbitrary time, the number of energy units just after the end of the virtual sojourn time is k, MAP is in the state ν and the virtual sojourn time in the system is less than x. Then $\boldsymbol{v}(s)$ is defined as a vector of Laplace-Stieltjes transforms $v(k,\nu,s) = \int\limits_0^\infty e^{-sx}dv(k,\nu,x)$ written in lexicographic order.

As mentioned above, the virtual sojourn time in the system consists of the residual time from an arbitrary time t to the next service completion epoch, the generalized service times of customers that await for a service at time t, and the generalized service time of the virtual customer. Taking into account the structure of the virtual sojourn time and using the law of total probability we express the vector Laplace-Stieltjes transform $\boldsymbol{v}(s)$ in the following theorem

The vector LST of virtual sojourn time in the system is calculated by

$$\boldsymbol{v}(s) = \tilde{\boldsymbol{v}}(0,s) + \sum_{j=1}^{\infty} \tilde{\boldsymbol{v}}(j,s)[\mathcal{B}^j(s) \otimes I_{\bar{W}}]. \tag{30}$$

Having calculated the LST of the virtual sojourn time, we are able to calculate the LST of the distribution of the actual sojourn time in the system.

The Laplace-Stieltjes transform of the actual sojourn time distribution in the system is calculated as follows:

$$v^{(a)}(s) = \lambda^{-1}\boldsymbol{v}(s)(I_{K+1} \otimes D_1)\mathbf{e}.$$

The mean sojourn time in the system is calculated by

$$\bar{v}^{(a)} = -\lambda^{-1}\frac{d\boldsymbol{v}(s)}{ds}|_{s=0}(I_{K+1} \otimes D_1)\mathbf{e}.$$

References

[1] Chakravarthy S.R. The batch Markovian arrival process: a review and future work, in: A. Krishnamoorthy, N. Raju, V. Ramaswami (Eds.), Advances in Probability Theory and Stochastic Processes, Notable Publications Inc., New Jersey, 2001, pp. 21-29.

[2] Cinlar E. Introduction to stochastic processes. Courier Corporation, 2013.

[3] Dudin A. N., Klimenok V.I., Vishnevsky V.M. The theory of queuing systems with correlated flows. Springer. 2019.

[4] Graham, A. Kronecker products and matrix calculus with applications, Courier Dover Publications, 2018.

[5] Lucantoni, D.M. New results on the single server queue with a batch Markovian arrival process, Communications in Statistics-Stochastic Models. 7 (1991) 1-46.

[6] Vishnevskii V. M., Dudin A. N. Queueing Systems with Correlated Arrival Flows and Their Applications to Modeling Telecommunication Networks, Automation and Remote Control, 2017, Vol. 78, No. 8, pp. 13611403.

[7] Neuts M.F. Structured Stochastic Matrices of $M/G/1$ Type and Their Applications. Marcel Dekker. New York. 1989.

Appendix A. Proof of Lemma 2.

We obtain formulas (14)-(26) using the notations introduced in Lemma 1 and the following evident formulas:

$$C^0_{k,k'} = \int\limits_0^\infty e^{D_0 t}\phi_{k'-k}(t)dt, \ 0 \le k \le k' \le K-1,$$

$$C^0_{k,K} = \int\limits_0^\infty e^{D_0 t}\hat{\phi}_{K-k}(t)dt, \ 0 \le k \le K.$$

$$C^j_{0,k'}(s) = \int\limits_0^\infty P(j,t)\phi_0(t)dt(\mathcal{B}(s))_{0,k'}+$$

$$+\int\limits_0^\infty\int\limits_0^t e^{D_0 x}\gamma e^{-\gamma x}dx e^{D_0(t-x)}\sum_{l=0}^{k'}\phi_l(t-x)D_1 dt\tilde{\Phi}_{k'-l}(j-1,s)+$$

$$+\int_0^\infty\int_0^t e^{D_0x}\phi_0(x)D_1dx\sum_{l=0}^{j-1}P(l,t-x)\gamma e^{-\gamma(t-x)}dt\tilde{\Phi}_{k'}(j-l-1,s),$$

$$j\geq 1, 0\leq k'\leq K-2,$$

$$C_{0,K-1}^j(s)=\int_0^\infty P(j,t)\phi_0(t)dt(\mathcal{B}(s))_{0,K-1}+$$

$$+\int_0^\infty\int_0^t e^{D_0x}\gamma e^{-\gamma x}dxe^{D_0(t-x)}[\sum_{l=0}^{K-1}\phi_l(t-x)D_1dt\tilde{\Phi}_k-l-1(j-1,s)dt+$$

$$+\sum_{l=K}^{\infty}\phi_l(t-x)D_1dt\tilde{\Phi}_0(j-1,s)]+$$

$$+\int_0^\infty\int_0^t e^{D_0x}\phi_0(x)D_1dx\sum_{l=0}^{j-1}P(l,t-x)\gamma e^{-\gamma(t-x)}dt\tilde{\Phi}_{K-1}(j-l-1,s), j\geq 1,$$

$$C_{0,K}^j(s)=\int_0^\infty\int_0^t e^{D_0x}\gamma e^{-\gamma x}dxe^{D_0(t-x)}[\sum_{l=0}^{K-1}\phi_l(t-x)D_1dt)\tilde{\tilde{\Phi}}_{K-l}(j-1,s))dt+$$

$$+\sum_{l=K}^{\infty}\phi_l(t-x)D_1dt\sum_{l=1}^{\infty}\tilde{\tilde{\Phi}}_1(j-1,s)]dt]+$$

$$+\int_0^\infty\int_0^t e^{D_0x}\phi_0(x)D_1dx\sum_{l=0}^{j-1}P(l,t-x)\gamma e^{-\gamma(t-x)}dt\tilde{\tilde{\Phi}}_K(j-l-1,s)]dt, j\geq 1;$$

$$C_{k,k'}^j(s)=\int_0^\infty e^{D_0t}\sum_{l=0}^{k'-k+1}\phi_l(t)D_1dt\tilde{\Phi}_{k'-k-l+1}(j-1,s),\ k\geq 1, k-1\leq k'\leq K-2, j\geq 1.$$

$$C_{k,K-1}^j(s)=\int_0^\infty e^{D_0t}[\sum_{l=0}^{K-k}\phi_l(t)D_1dt\tilde{\Phi}_{K-k-l}(j-1,s)+\sum_{l=K-k+1}^{\infty}\phi_l(t)D_1dt\tilde{\Phi}_0(j-1,s)],$$

$$k\geq 1, j\geq 1.$$

$$C_{k,K}^j(s)=\sum_{l=0}^{K-k}\int_0^\infty e^{D_0t}\phi_l(t)D_1dt\sum_{n=K-k-l+1}^{\infty}\tilde{\tilde{\Phi}}_{K-k-l+1}(j-1,s)+$$

$$+\sum_{l=K-k+1}^{\infty}\int_0^\infty e^{D_0t}\phi_l(t)D_1dt\sum_{n=1}^{\infty}\tilde{\tilde{\Phi}}_1(j-1,s),\ j>0, k=\overline{1,K},$$

$$\Omega_{0,k'}^{r}(s)=\int_{0}^{\infty}\sum_{l=0}^{r}P(l,t)\gamma e^{-\gamma t}dt\tilde{\Phi}_{k'}(r,s),k'=\overline{0,K-1},r\geq 0,$$

$$\Omega_{0,K}^{r}(s)=\int_{0}^{\infty}\sum_{l=0}^{r}P(l,t)\gamma e^{-\gamma t}dt\tilde{\tilde{\Phi}}_{K}(r,s),k'=\overline{K-2,K-1},r\geq 0,$$

$$\Omega_{k,k'}^{r}(s)=\tilde{\Phi}_{k'-k+1}(r,s),k=\overline{1,K},k'=\overline{k-1,K-1},r\geq 0,$$

$$\Omega_{k,K}^{r}(s)=\tilde{\tilde{\Phi}}_{K-k+1}(r,s),k=\overline{k-1,K},r\geq 0.$$