Homework #1
Linear Algebra & Probability Theory
Statistical Pattern Recognition

Due Mehr 25th,1395
Format of homework file: Archive all files in a folder named as your student number and send it to MohammadHME@gmail.com. Send your emails with a subject of PRF95_xxxxxx (replace xxxxxx by your student number)

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1. A random variable X has $E(X) = 2$ and $E(X^2) = 2$. Let $Y = -2X - 5$. Compute:
   a) $V(X)$. b) $V(Y)$. c) $E(Y^2)$.

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2. Generate 100 samples from normal distribution specified by $\mu = 1,\ \sigma = 0.5, 1, 2$. Plot histogram of generated samples and compare the results.

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3. Generate and plot samples from normal distribution specified by:
   a. $N = 100, M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$
   b. $N = 100, M = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$
   c. $N = 100, M = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}$

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4. Compute the sample mean and sample covariance matrix of problem 3. (Using MATLAB).

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5. Use the normal distribution to approximate the binomial distribution and find the probability of getting 12 to 15 heads out of 20 flips. Compare this to what you get when you calculate the probability using the binomial distribution.

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6. a) Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$ and compare your results with Matlab outputs.

   b) a 2×2 matrix A matrix has $\lambda_1 = 2$ and $\lambda_2 = 5$, with corresponding eigenvectors $v_1 = [1\ 0]^T$ and $v_2 = [1\ 1]^T$. Find A.

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7. Let

$$f(x,y) = \begin{cases} c(x + y^2) & if\, 0 < x < 1; 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Find appropriate constant c, marginal density $f_X(x)$, conditional density $f_{X|Y}(y)$. Are X and Y independent? What is the probability Pr(X<1/2 | Y=1/2)?

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8. Gaussians: Recall the multivariate Gaussian for a vector $x \in R^n$.

$$p(x|\mu,\Sigma) = \frac{1}{(2\pi)^{(\frac{n}{2})}|\Sigma|^{\frac{1}{2}}} \exp -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$$

A) Let $y = v_i^T x$, where v_i is an eigenvector of Σ with an eigenvalue of λ_i. Find the probability density of p(y).

B) Let x be a zero mean guassian random vector with a isotropic covariance ($\Sigma = I$). Let y=Ax+b. Compute the mean and variance of y.

C) (MATLAB) Generate 500 random samples from a 2 dimensional Gaussian with an isotropic Σ using matlab command **randn**. Transform the data as above with $b = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$, and $A = \begin{bmatrix} -5 & 5 \\ 1 & 1 \end{bmatrix}$.plot the original and transformed points.

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9. If X and Y are independent and identically distributed with mean λ and variance σ^2, find $E[(X - Y)^2]$.

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10. Diagonalize the following two matrices simultaneously:

$$\Sigma_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 1+\sqrt{3/4} & 0.5 \\ 0.5 & 1-\sqrt{3/4} \end{bmatrix}$$

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11. Let p(X) be $N_x(M,\Sigma)$ with

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad M = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Show that:

$p(x_1) = N_{x_1}(m_1, \sigma_1^2)$ (a marginal density)

$p(x_1|x_2) = N_{x_1}(m_1 + \rho\sigma_1(x_2 - m_2)/\sigma_2 \;,\; \sigma_1^2(1-\rho^2))$

(a conditional density)