Graph Theory

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CHAPTER-1. [Fundamental concepts of Graph Theory]

1: Draw and explain a multigraph? Prove that the number of odd vertices in any graph G is always even.

→ If more than one line joining two vertices are allowed, the resulting object is called a multigraph. line joining the same points are called multilines

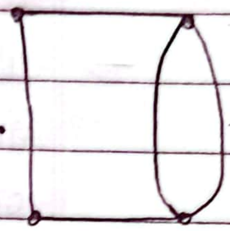

fig-Multigraph

Proof:

Consider a graph G. If G contains no odd vertices, then the result is obvious. Let the graph G contains K number of odd vertices $v_1, v_2, v_3, ---, K$. If G also contains even vertices, then we write them as $u_1, u_2, u_3, ---, u_n$.

Now, by theorem, I,

$$d(v_1)+d(v_2)+d(v_3)+----+d(v_k)+d(u_1)+d(u_2)+d(u_3)+---+d(u_k) = 2ne \quad \text{——} (1)$$

where ne denote the total number of edges in G.

As the numbers $d(u_1), d(u_2), d(u_3), ---, d(u_n)$ are all even, their sum $d(u_1)+d(u_2)+d(u_3)+----+d(u_n)$ is also even.

So, $d(v_1)+d(v_2)+d(v_3)+--+d(v_k) = 2ne - \{d(u_1)+d(u_2)+d(u_3)+---+d(u_n)\}$ is even from (i).

But each of the numbers $d(v_1), d(v_2), --d(v_k)$ is odd.

Hence, K must be even, i.e, G has an even number of odd vertices.

Again, if G has no even vertices, then

$$d(v_1)+d(v_2)+d(v_3)+----+d(v_k) = 2ne.$$

Thus, we again conclude that K is even. Hence the theorem is proved.

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2: Define degree of vertex of a graph? Prove that the sum of the degrees of the vertices of a graph is equal to twice the number of edges.

→ The degree of a point v_i in a graph G is the number of lines incident with v_i. The degree of v_i is denoted by $d_G(v_i)$ or $d(v_i)$ or deg v_i. A point v of degree 0 is called an isolated point. A point v of degree 1 is called an end point.

In other words, the degree of a vertex v in a graph G is the number of edges whose one of their ends is the vertex v. The vertex v is said to be even or odd according as d(v) is even or odd.

Example: Consider the graph G(V, E), where $V(G) = \{a, b, c, d\}$ and $E(G) = \{\{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}\}$, then

$d(a) = 1, d(b) = 3, d(c) = 2, d(d) = 2$.

So, the vertices c and d are even and a and b are odd.

Proof:

Let us consider a graph G having n vertices $v_1, v_2, v_3, \ldots v_n$ and n_e edges. It is clear that each edge contributes degree of 2. Therefore, the sum of the degree of all vertices in G is twice the number of edges in G.

Hence, $\sum_{i=1}^{n} d(v_i) = 2n_e$

Here, $\sum_{i=1}^{n}$ is the sum of all the ~~edges~~ degrees of the vertices of graph G, and n_e denote the total number of edges in G. Hence, it is proved.

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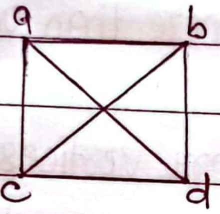
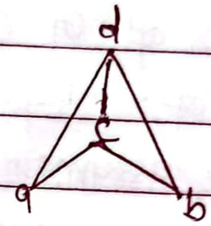
3. Define graph and multi graph with suitable example.

→ A graph is a pair $G = (V, E)$, where $V = \{v_1, v_2, --, v_n\}$ is a set of vertices, and $E \subset = V_2$, where $V_2 = \{\{v_1, v_2\}, \{v_1, v_3\}, ---, \{v_1, v_n\}, ---- \{v_{n-1}, v_n\}$ is an unordered product of V, i.e a set of all two element subsets of V defining the set of all possible edges of the graph, G.

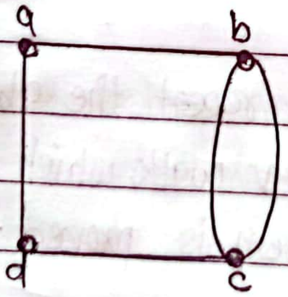
Examples: $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = V_2$.

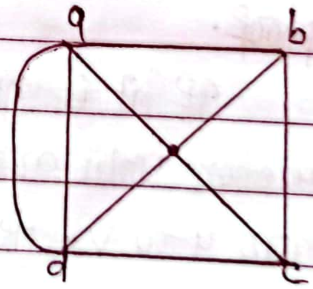




If more than one line joining two vertices are allowed, the resulting object is called a multigraph. Line joining the same points are called multilines.

Examples :





Extra.

4\. If there is a walk from any two vertices u and v of a graph G, then there is a path from these vertices after the deletion, if necessary, of some vertices and edges.

Proof:

Let us define a walk W with the help of vertices u and v of the graph G as $u, e_1, v_1, \ldots v_{k-1}, e_k, v$.

Here, the walk W is a path from u to v if none of the vertices of G occurs in W more than once. It completes the proof.

So, we assume that there are vertices of G that occur in W twice or more. If so, there are distinct, i, j, with $i<j$, such that $v_i = v_j$. If the vertices $v_i, v_{i+1}, \ldots v_{j-1}$ are removed from W, the we get a u-v walk (W_1) having less number of vertices than in W. If there is no repetition of vertices in W_1, then W_1 is a path from u to v. It completes the proof.

If it is not so, then we repeat the above procedure of deletion until arriving at a u-v walk, which is a required path from u to v. Hence, the theorem is proved.

Define cycle.

→ A cycle is defined as a circuit which doesnot repeat any vertices except the initial and final vertices.

Note: 1. A cycle is a non-intersecting circuit.

2. A cycle is of length three or more.

3. If a cycle is of length K, it is called a K-cycle.

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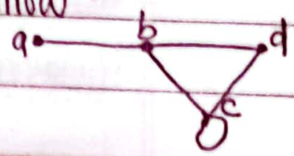
5. Define matrix representation of graph.

→ The matrix representation of graph are:

a. Adjacency matrix system: Let there are n vertices in a graph G and then the adjacency matrix of G is denoted by $A(G) = (e_{ij})_{n \times n}$ is defined as

$$e_{ij} = \begin{cases} (0) \text{ if } V_i \text{ and } V_j \text{ are disconnected.} \\ (1) \text{ if there is a single edge joinip } V_i \text{ and } V_j. \\ (m) \text{ if } m \text{ is the number of edges joining } V_i \text{ and } V_j \text{ where } m > 1. \end{cases}$$

Example: Find the adjacency matrix of G in as follow.



→ The adjacency matrix of G is as follows

$$A(G) = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 2 \\ c & 0 & 1 & 1 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$$

Properties:

i. All the entries belong to principal diagonal of A(G) are all 0's for a graph G. An entry of positive integer on the principal diagonal of A(G) corresponds to the number of self-loops at the ith vertex of multi-graph.

ii. The degree of a vertex equals the number of 1's in the corresponding row or column of A(G).

iii. A graph (G) is disconnected and is with two components G_1 and G_2 if and only if its adjacency matrix A(G) is of the block diagram as given below:

$$A(G) = \left[\begin{array}{c|c} A(G_1) & 0 \\ \hline 0 & A(G_2) \end{array}\right]$$

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Here, $A(G_1)$ and $A(G_2)$ are the adjacency matrices of the components G_{11} and G_{12} of G. The matrix $A(G)$ clearly imply that there exists no edge joining any vertex of G_1 to G_2.

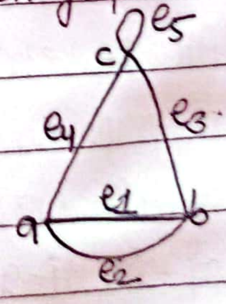
iv. If there is a square symmetric matrix M with order n, then it is possible to construct a multi-graph G with n vertices such that M is the adjacency matrix of G.

b. Incidence matrix system: Let there are $V_1, V_2, ----, V_m$ vertices and $e_1, e_2, ----, e_n$ edges in a graph (or multigraph) G then the incidence matrix of G denoted by $I(G) = (m_{ij})_{m \times n}$ is defined as.

$$m_{ij} = \begin{cases} (0) \text{ if } V_i \text{ is not linked with } e_j \\ (1) \text{ if } V_i \text{ is linked with } e_j \\ (2) \text{ if } V_i \text{ has one loop } e_j \end{cases}$$

Here, m_{ij} denote the number of times as the vertex V_i is linked with edge e_j.

Example: Find the incidence matrix of given graph.



→ The incidence matrix of given graph is

$$I(G) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left| \begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right| \end{array}$$

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Properties :

i. If every edge is linked on exactly two vertices, each column of M(G) has exactly two 1's.

ii. The number of 1's in each row is equal to the degree of the corresponding vertex.

iii. If there is a row with all 0's, it represents as isolated vertex.

iv. Multiple edges in the multi-graph produce identical columns in its incidence matrix.

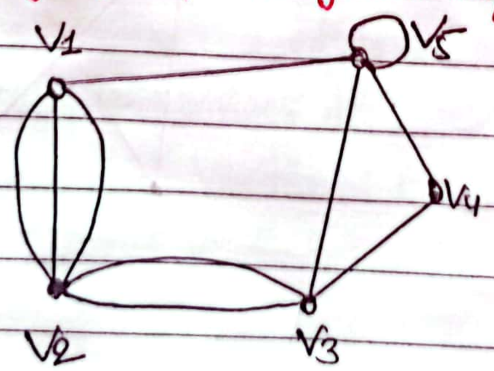
v. If G is disconnected has two components G_1, and G_2, the incidence matrix M(G) of graph G is in the following form.

$$M(G) = \left[\begin{array}{c|c} M(G_1) & 0 \\ \hline 0 & M(G_2) \end{array}\right]$$

Here, $M(G_1)$ and $M(G_2)$ are the incidence matrices of components G_1 and G_2 of the given graph G.

6. **Draw graph of the given adjacency matrix given below.**

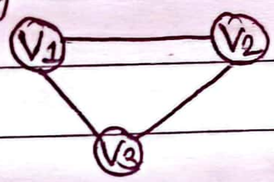
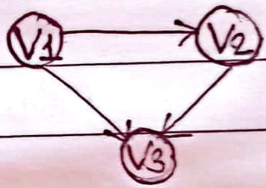
→



	V1	V2	V3	V4	V5
V1	0	3	0	0	1
V2	3	0	2	0	0
V3	0	2	0	1	1
V4	0	0	1	0	1
V5	1	0	1	1	1

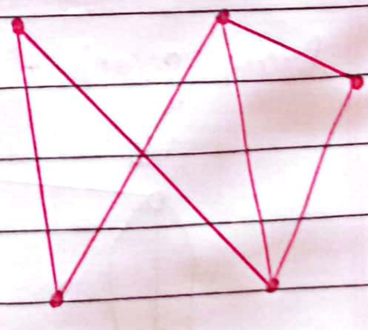
7\. What is difference between graph and directed graph ?

→ The difference between graph and directed graph are :

Graph (Undirected graph)	Directed graph
1. A type of graph that contains unordered pairs of vertices.	1. A type of graph that contains ordered pairs of vertices.
2. Edges donot represent the direction of vertices.	2. Edges represent the direction of vertices.
3. Undirected arcs represent the edges.	3. An arrow represents the edges.
4. Eg: 	4. Eg: 

8\. Write incidence matrix of given graph.

→



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CHAPTER-1

1: What are the difference between path and walk? when the walk will be trial and path?

→ The difference between path and walk are :

Path	Walk
1. A path is a walk which does not repeat verticas.	1. A walk is a series of adjacent verticas that can go backwards as long as there is a line segment.
2. ~~The~~ In a path, each vertex and edge is visited at most once.	2. The vertices in a walk can be repeated, and the walk can revisit edges multiple times.
3. Paths are typically considered to be simple and donot contain any cycles or repetitions.	3. A walk can contain cycles & backtrack on previously visited vertices and edges.

When a walk is referred to as a trial, it usually means that the walk is being used for exploration or experimentation ~~process~~ purposes. A trial walk can be performed to investigate various properties of the graph, such as connectivity, reachability, or the existence of certain paths. It helps in analyzing the structure and characteristics of the graph. A path doesnot typically have any specific connotation of being a trial. It is a fundamental concept used to study the connectivity and routers within a graph. Paths are often used to find the shortest path between two verticas, determine if a path exists between two verticas, or analyze the structure of a graph.

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CHAPTER-2 [Isomorphism and Operations]

1. Define identical and isomorphic graphs? Define bipartite graph and complete bipartite graph with examples.

→ **Identical graphs:** Two graphs are considered identical if they have the same number of vertices and edges, and the vertices and edges are labelled in the same way. In other words, all the structural properties of the two graphs are exactly the same. The arrangement or order of the vertices and edges may differ but the overall structure is identical. For example, let's say we have to graphs:

Graph 1 :→ vertices : A, B, C and edges : (A-B), (B-C).

Graph 2 :→ vertices : C, B, A and edges : (B-C), (A-B)

Despite the different vertex and edge order, Graph 1 and Graph 2 are identical because they have the same number of vertices & edges, and the vertices and edges are labeled the same way.

Isomorphic graphs: Two graphs are considered isomorphic if they have the same underlying structure, but the vertices and edges may be labeled differently. In other words, isomorphic graphs have the same number of vertices and edges and the same connectivity patterns, but the vertex and edge labels may differ.

For example, consider the following two graphs:

Graph 1 :→ vertices : A, B, C and edges : (A-B), (B-C).

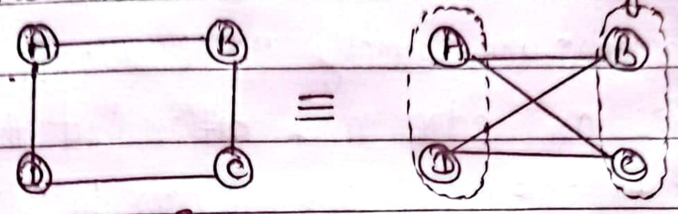
Graph 2 :→ Vertices : X, Y, Z and edges : (X-Y), (Y-Z).

Although the vertices and edges are labeled differently, Graph 1 and Graph 2 are isomorphic because they have the same structure and connectivity patterns. The vertex A in Graph 1 corresponds to the vertex X in Graph 2, vertex B corresponds to Y, & vertex C corresponds to Z. The same correspondence applies to the edges (A-B) and (X-Y), and (B-C) and (Y-Z).

Bipartite graph: A bipartite graph is a special kind of graph with the following properties -

- It consists of two sets of vertices X and Y.
- The vertices of set X join only with the vertices of set Y.
- The vertices within the same set donot join.

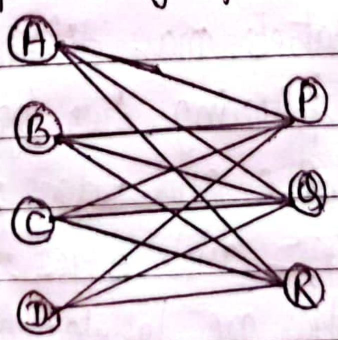
Example:



Here,
- The vertices of the graph can be decomposed into two sets.
- The two sets are X = {a, c} and Y = {B, D}.
- The vertices of set X join only with the vertices of set Y and vice-versa.
- The vertices within the same set donot join.
- Therefore, it is a bipartite graph.

Complete Bipartite graph: A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph.

Example:



Here,
- This graph is a bipartite graph as well as a complete graph.
- Therefore, it is a complete bipartite graph.
- This graph is called as $K_{4,3}$.

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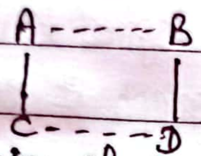
2. Define union and intersection of two graphs with suitable examples.

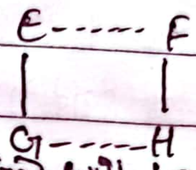
→ The union of two graphs, denoted by $G_1 \cup G_2$, creates a new graph that contains all the vertices and edges from both input graphs. The resulting graph includes all the elements present in either G_1 or G_2.

Example: let's consider two graphs : G_1 and G_2.

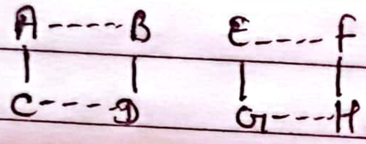
G_1 :



G_2 :



The union of G_1 and G_2 ($G_1 \cup G_2$) will be

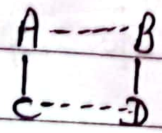


The resulting graph contains all the vertices (A, B, C, D, E, F, G, H) and edges from both G_1 and G_2.

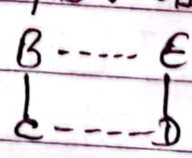
The intersection of two graphs, denoted by $G_1 \cap G_2$, creates a new graph that consists of the vertices and edges that are common to both input graphs. The resulting graph contains only the elements that are present in both G_1 and G_2.

Example: let's consider two graphs : G_1 and G_2.

G_1 :



G_2 :



The Intersection of G_1 and G_2 ($G_1 \cap G_2$) will be.

The resulting graph contains only the common vertices (B, C, D) and edges between them, which are present in both G_1 and G_2.

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3. Define a connected graph. Explain the isomorphic graphs. If G_1 and G_2 be the two isomorphic graphs, with isomorphism ϕ then prove that,

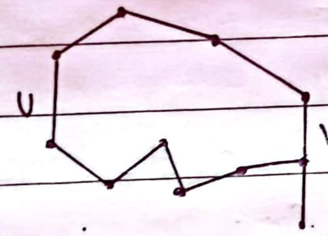
(i) G_1 and G_2 have the same number of vertices.

(ii) G_1 and G_2 have the same number of edges.

(iii) If u is adjacent to v is G_1, then $\phi(u)$ is adjacent to $\phi(v)$ is G_2.

(iv) If u has degree k is G_1, then $\phi(u)$ has degree k is G_2.

→ A graph G is said to be connected graph if every two vertices to have connected otherwise it is disconnected graph.



Proof:

(i) Since ϕ is an one-to-one correspondence, the proof is obvious.

(ii) Here, each edge $\{u,v\}$ in E_1 is corresponded with a unique edge $\{\phi(u), \phi(v)\}$ is E_2 and vice-versa. It implies that $|E_1| = |E_2|$.

(iii) If u is adjacent to v, then $\{u,v\}$ belongs to G_1. Then, the vertex $\{\phi(u), \phi(v)\}$ belongs to G_2. Therefore, $\phi(u)$ is adjacent to the vertex $\phi(v)$ in G_2.

(iv) Let us take $(u) = k$, the graph G_1 has vertices $u_1, u_2, ---, u_k$ which are adjacent to u. Now, from (iii), the vertex $\phi(u)$ in G_2 is adjacent to the vertices $\phi(u_1), \phi(u_2), --- \phi(u_k)$ in G_2. This implies that $d(\phi(u)) = k$. Thus, $\phi(u)$ has degree k in G_2. Hence, it is proved.

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5. Let G_1 be a (p_1, q_1) and G_2 be a (p_2, q_2) graph. Then prove that

(i) $G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph.

(ii) $G_1 + G_2$ is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph.

(iii) $G_1 \times G_2$ is a $(p_1 p_2, q_1p_2+q_2p_1)$ graph.

→ Proof:

(i) Let G_1 be a (p_1, q_1) and G_2 be a (p_2, q_2) graph. Then $G_1 \cup G_2$ is a subgraph having $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$. So, $G_1 \cup G_2$ has a p_1+p_2 vertices and q_1+q_2 edges. So $G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph.

(ii) Number of lines is G_1+G_2 = number of lines in G_1 + number of lines in G_2 + number of lines joining points of V_1 with points of V_2 $= q_1+q_2+p_1p_2$. So, G_1+G_2 is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph.

(iii) Number of points in $G_1 \times G_2$ is p_1p_2. Now, let $(u_1, u_2) \in V_1 \times V_2$. The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 (v_1, v_2) where adjacent to u_1.

$\therefore \deg(u_1, u_2) = \deg(u_1) + \deg(u_2)$.

The total number of lines is $G_1 \times G_2$.

$$= \frac{1}{2}\left[\sum_{i,j} \deg(u_i) + \deg(v_j)\right]$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\deg u_i + \deg v_j), \text{ where } u_i \in V_1 \ \& \ v_j \in V_2.$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1} (p_2 \deg u_i) + \sum_{j=1}^{p_2} (\deg v_j)\right]$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} (p_2 \deg u_i + 2q_2)$$

$$= \frac{1}{2}(2p_2q_1 + 2p_1q_2)$$

$$= p_2q_1 + p_1q_2 \quad \text{Ans}$$

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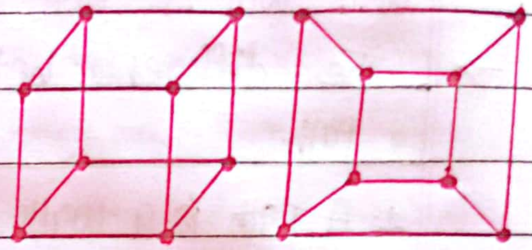
CHAPTER-2

1. Define isomorphic graphs. Check the given graphs are isomorphic or not.



→ Isomorphic graphs are two graphs that have the same underlying structure, even though they may look different in terms of vertex labels or edge placements. Formally, two graphs G and H are said to be isomorphic if there exists a bijection (a one-to-one correspondence) between the vertices of G and the vertices of H, such that for any two vertices u and v in G, u and v are adjacent in G if and only if their corresponding vertices in H are adjacent in H.

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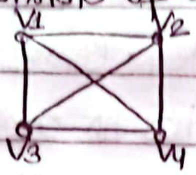
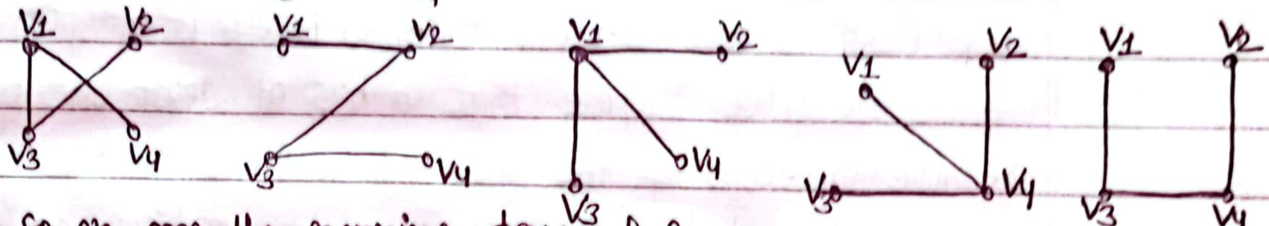
CHAPTER-3 [Eulerian Tours and Hamilton Cycles]

1: What is a spanning tree? If G is tree with n vertices, then prove that it has (n-1) edges.

→ Let G be a graph. A spanning tree G is a subgraph of G which is a tree that consists of all the vertices of G.

Example: If G =  then



and so on, are the spanning trees of G.

Proof:

Let us proceed by the method of induction. When n=1, then G has only one vertex, then, since it has no loops, G cannot have any edges, so, it has no edges. Therefore, the theorem holds for n=1. So, it goes on and the theorem is true for n=k, where k is any positive integer. Now, we will show that the theorem holds for n=k+1. Let G be a tree with k+1 vertices and let y be a vertex of degree 1. If we remove such a vertex and the edge incident on it, the subgraph G−{y} is still connected and has no cycles. Thus, G−{y} is a tree. Now, G−{y} has k vertices because G had k+1 vertices. By our induction hypothesis, the tree G−{y} has k-1 edges. Since the number of edges is less than vertices by 1. It follows that G has k edges, as required. Therefore, if the theorem is true for n=k, then it is true for n=k+1.

In this way, the theorem is true for all positive integers k. It completes the proof.

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2. Prove that a graph is connected if and only if it has subgraph that is a spanning tree.

→ Proof:

First, let us consider that a graph G has a spanning tree. Then, by definition, it is connected. Conversely, let assume that G is a connected graph. If G has no cycles, we are done because G is itself a tree. In this case, G itself is a spanning tree.

Here, we suppose that G has at least one cycle. If we remove an edge of the cycle of G, the resulting graph G_1 is connected and has the same set of vertices as G. If G_1 has no cycles, then G_1 is a spanning tree of G. If G_1, has one cycle, we remove an edge of a cycle of G_1; and so on. In this way, we finally arrive at a connected graph H. The graph H has no cycles and has the same set of vertices as of G. By definition, H is a spanning tree of G. Hence, it is proved.

3. Every Hamilton graph is 2-connected.

→ Proof:

Let G be a Hamilton graph and let c be a Hamilton cycle in G. For any vertex v of G, $c-v$ is connected and hence $G-v$ is also connected. Hence, G has no cut points and therefore, G is 2-connected.

Note: The bipartite graph $K_{m,n}$ is non-Hamiltonian.

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4. Prove that the connected graph G is Eulerian if and only if each vertex has even degree.

→ Proof:

Let us assume that G is an Eulerian graph. we need to show that each of its vertex has even degree. Since G is Eulerian, it contains an Eulerian circuit α, which begins and ends, say, the vertex v. Now, Consider a vertex u different from v. As u is neither the first nor the last vertex of α, each time when u is encountered, it is entered through any edge and exited through another edge. Hence, each occurrence of u in α increases the degree of u by two. Thus, u have even degree. In the case of the vertex v, each occurrence of v except the first and the last gives two to its degree, while the initial and final occurrence of v in α give one each to the degree of v. In this way, every vertex of G bears even degree.

Conversely, let us assume that G is connected graph with each even vertex. Then we prove, G is Eulerian. For, we construct an Eulerian circuit. we begin a trial α_1, starting from any arbitrary vertex v. Here, along any edge e incident on the vertex we let no edge is repeated, we continue tracing the trial α_1, as far as possible. Since every vertex is of even degree, we can let out the trail from any vertex once it started the journey.

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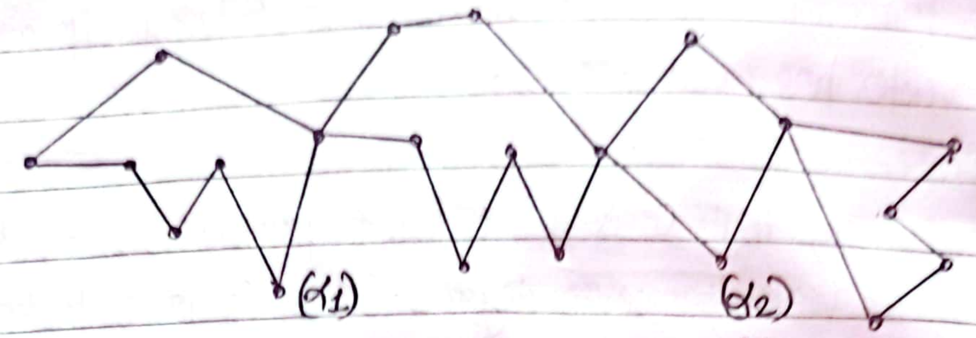

Thus, we keep on going through α_1 until it gets back to its initial vertex v, it means until the trial α_1 is closed it includes all the edges of G, then G is an Eulerian graph.

Let α_1 does not include all edges of G. Consider the graph H formed by excluding all edges of α_1 from G. Here, the subgraph H many not be connected, but each vertex of H has even degree since α_1 contains an even number of edges incident on any vertex. Since G is connected there is an edge of H which has an end point u_1 in α_1. If we build a trial α_2 is H starting from u_1 along e_1 and continue this trial as long as possible. Then as before, α_2 must return to u_1. Here, α_2 must be a circuit as all the vertices of H have even degree. Now, we can put α_1 and α_2 together to form a larger closed trial (circuit) in G. We continue this process until and unless all the edges of G are covered. As a result, we obtain an Eulerian circuit of G. Thus G is Eulerian. Hence, it is proved.

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5\. Define tree. Prove that in a tree G, there are at least two vertices of degree 1.

→ A connected graph G without any cycles is said to be a tree. It is noted that a graph without cycles is an acyclic graph. Thus, a connected graph G is said to be a tree if G is acyclic.

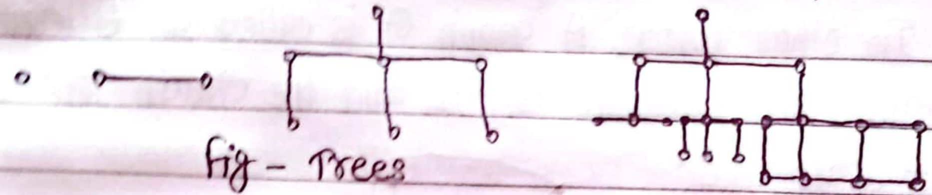


fig - Trees

Proof:

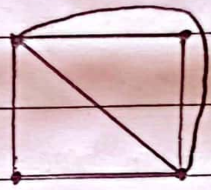
We consider G be a tree. let V_1 be any vertex of G which has degree 1. It is one of the desired vertices. let there is a vertex V_2 adjacent to V_1. If it is of degree 1, it is done. If not, we can choose V_3 adjacent to V_2. By continuing in this way, we get a sequence of vertices $V_1, V_2, ---, V_m$. As a tree has no cycle, this sequence of vertices must terminate for some m, i.e there is a vertex that has only one adjacent vertex. Such a vertex has degree 1. Hence, V_1 and V_m are our desired vertices each with degree 1.

Here, we take, V_1 doesnot have degree 1. Then, we can select two distinct adjacent vertices V_2 and V_2'. By a similar process as above, we will have two sequences of vertices, say, $V_1, V_2, --, V_i$ and $V_1, V_2', ----, V_j'$. Other than V_1 all the elements of these two sequences must be distinct, otherwise the sequences form a cycle. Consequently, each sequence must terminate and give two desired vertices of degree 1. Hence, it is proved.

6. Define Eulerian graphs. If G is a graph in which the degree of every vertex is at least two, then G contains a cycle.

→ Let G be graph. A closed trial or circuit consisting all the edges is called an Eulerian trial. A graph having an Eulerian trial is called an Eulerian graph.

In other words, A graph G is called an Eulerian graph if it contains an Eulerian circuit and the circuit intersects each vertex at least once.



Proof:

First we construct a sequence of the vertices V_1, V_2, V_3 ---- as follows. choose any vertex adjoint to V_1 other than V_1 at any state. If the vertex V_i, $i \geq 2$ is already chosen then choose V_{i+1} to be any vertex adjacent to V_i other than V_{i+1}. Since degree of each vertex is atleast two, the existence of V_{i+1} is always guaranteed. G has only finite number of vertices, at some state, we have to choose a vertex which has been chosen before. let V_k be the first such vertex and let $V_k = V_i$ where $i < k$, then $V_i, V_{i+1}, ---V_k$ is a cycle. Hence, the theorem is proved.

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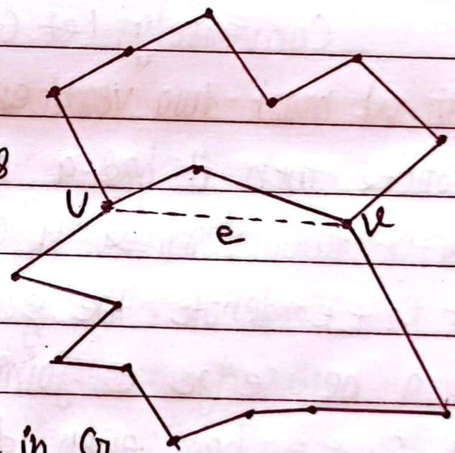
7\. Define Eulerian trial. A multi graph G is traversable if and only if G is connected and has exactly two odd vertices.

→ A trial in a graph G is said to be an Eulerian trial if it covers every edge of G. Then G is called the traversable graph. Thus, the Eulerian trial includes each and every edge of G exactly once.

Proof:

We assume that multigraph G is a traversable graph. Then, there is an Eulerian trial in G. Thus, G has a trial containing all the vertices and edges of G. It follows that G is connected and it has exactly two odd vertices.

Conversely, we consider a connected multigraph G has exactly two odd vertices, say u and v. Then it needs to prove that G is a traversable graph. We insert an edge e in G so that $G' = G \cup \{e\}$ as shown in figure.



As a result all the vertices of the graph G' become even. So, there is a closed trial (circuit) α' of G'. Since α is closed, we can take that α' begins with e. Let β' be the path of α' except its edge e. Then, β' is an Eulerian trial of G beginning at u and ending at v. Thus, G is a traversable graph and hence the multigraph G is traversable.

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8. Define Hamiltonian path. Prove that A connected graph has an Euler trial if and only if it has at most two vertices of odd degree.

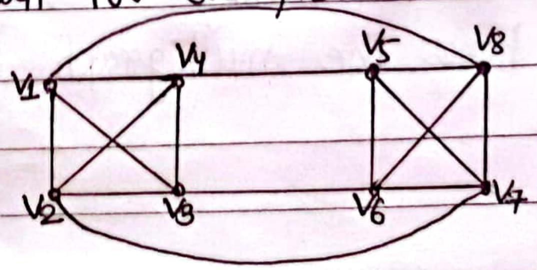
→ A path which includes every vertex of a graph G is said to be the Hamiltonian path. It can be seen in the an adjoining figure marked by bold path.

Proof:

If G has an Euler trial then each vertex other than the origin and terminus of this trial has even degree in G. Hence, G has at most two vertices of odd degree.

Conversely, let G is a nontrivial connected graph with at most two vertices of odd degree. If G has no such vertices then it has a closed Euler trial. Otherwise, G has exactly two vertices u and v of odd degree. In this case, let G+e denote the graph obtained from G by the addition of a new edge e joining u and v. clearly, each vertex of G+e has even degree so G+e has an Euler tour $T = v_0 e_1 v_1 e_2 \dots e_{i+1} v_{i+1}$ where $e_1 = e$. Then the trial $v_1 e_2 v_2 \dots e_{i+1} v_{i+1}$ is an Euler trial of G. It completes the proof.

Hamiltonian cycle: A cycle containing every vertex of a graph G is said to be a Hamiltonian cycle. It is also called a Hamiltonian circuit. For example.



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9. Define forest. Prove that G be a graph without any loops. If for every pair of distinct vertices u and v of G there is a precisely one path from u to v, then G is a tree.

→ A graph whose components are trees is called forest. The forest has also no cycles.

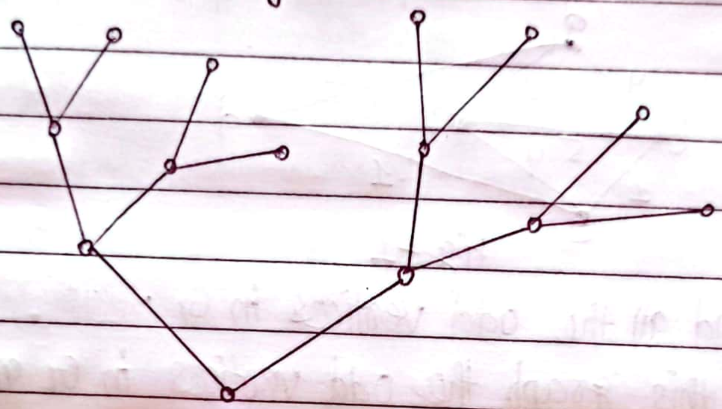

Fig - Forest

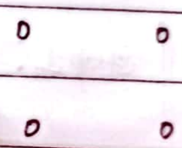

Fig - Trivial forest

Proof:

As each pair of vertices u and v is connected by a path; G must be connected. Now, we have to show that G is a tree. So, it remains to show that G has no cycles.

G has no cycles of length one since it has no loops. If possible, suppose, G has a cycle of length greater than one, say.

$$\alpha = (v_1, v_2, \ldots, v_n, v_1), \text{ for } n \geq 2.$$

If so, any two distinct vertices of the cycle α are joined by two paths which contradict our assumption. Thus, G has no cycles. So, it is a tree.

10- Write an algorithm for chinese post-man problem.

→ To find an Eulerian pseudograph of minimum weight by duplicating edges of a weighted connected graph. The following steps of intersection is necessary to solve the chinese postman problem for the weighted graph.

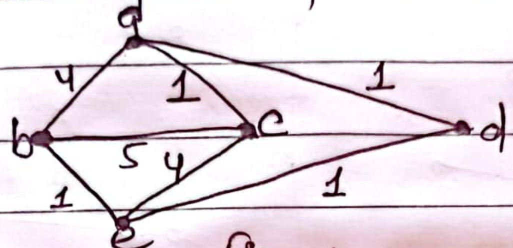

fig-1

step 1: find all the odd vertices in G.

In this graph the odd vertices in G are a, b, c and e.

step 2: for each partition of odd vertices into pair of vertices $\{v_1, w_1\} \{v_2, w_2\} \ldots \{v_m, w_m\}$.

find the length of the shortest path between each v_i and w_i and add these lengths.

Consider all the possible partitions of these vertices into pairs.

Partition into pairs	Sum of the length of shortest paths
{a,b} {c,e}	(a-d-e-b) 1+1+1+1+1+1 (c-a-d-e) = 6.
{a,c} {b,e}	(a-c) = 1+1 (b-e) = 2.
{a,e},{b,c}	(a-d-e) = 1+1+1+1+1+1 (b-e-d-a-c) = 6

step 3: The minimum length in ~~sum~~ step 2 & duplicate the path.
Here we get minimum length 2, so we duplicate the paths (a-c) and (b-e) which is shown below.

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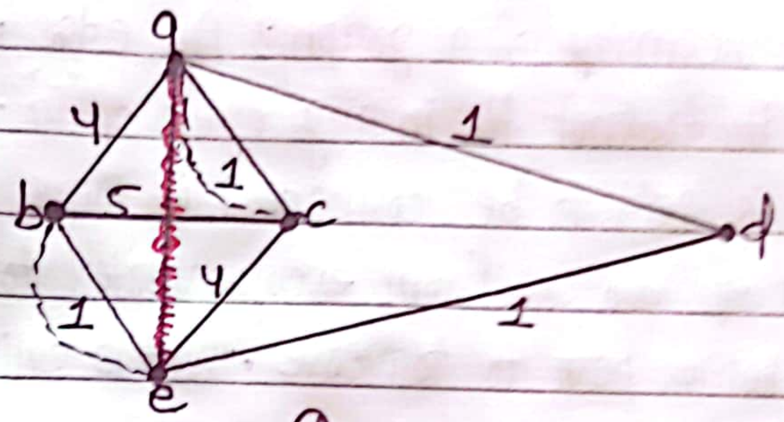


Fig-2

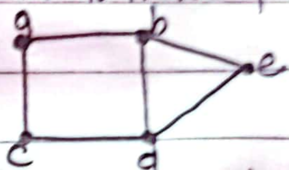
The chinese postman can choose the shortest path.

a-b-e-d-a-c-b-e-c-a.

11. What is connectivity of a graph? An edge 'e' of a graph 'G' is a bridge if it doesnot lie in any cycle of 'G'. prove it.

→ A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph.

Example: In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Proof:

Direction 1: If 'e' is a bridge, then it doesnot lie in any cycle of 'G'.
Assume 'e' is a bridge in 'G'. This means the removing 'e' from 'G' would increase the number of connected components in 'G'. Let's denote 'G-e' as the graph obtained by removing 'e' from 'G'. Since 'e' is a bridge, there must be at least two connected components in 'G-e'.
Now, suppose 'e' lies in a cycle of 'G'. Let's consider the vertices connected by 'e' as 'u' and 'v'. Removing 'e' from the cycle would disconnect 'u' and 'v' and split the cycle into two separate cycles. However, this contradicts the fact that removing 'e' from 'G' results in at least two connected components. Therefore, if 'e' is a bridge, it cannot lie in any cycle of 'G'.

Direction 2: If 'e' doesnot lie in any cycle of 'G', then it is a bridge.
Assume 'e' doesnot lie in any cycle of 'G'. We will prove that removing 'e' from 'G' increases the number of connected components.

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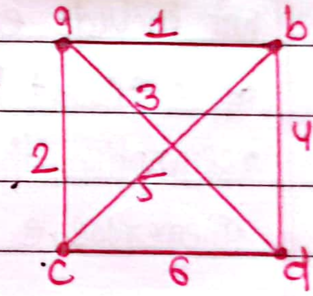
Let's consider 'G-e', the graph obtained by removing 'e' from 'G'. If 'e' doesnot lie in any cycle, it means that there is no alternative path between the vertices connected by 'e'. Therefore, removing 'e' would disconnect these vertices and split 'G' into two separate connected components.

Since 'G-e' has more connected components than 'G', we can conclude that 'e' is a bridge.

By providing both directions, we have show that an edge 'e' of a graph 'G' is a bridge if and only if it doesnt lie in any cycle of 'G'.

12. Find three distinct Hamiltonian cycles in the following graph. Also find their weight.



→ The Hamiltonian cycles and their weight are as follows.

(a) a – b – d – c – a

Weight = 1+4+6+2 = 13

(b) a – b – c – d – a

Weight = 1+5+6+3 = 15

(c) a – d – c – b – a

Weight = 3+6+5+1 = 15

(d) a – c – b – d – a

Weight = 2+5+4+3 = 14

(e) a – d – c – b – a

Weight = 3+6+5+1 = 15

(f) a – c – d – b – a

Weight = 2+6+4+1 = 13

13. Prove that a connected graph G remains connected after removing an edge e from G if and only if e belongs to some circuit in G.

→ Proof:

To prove the statement, we'll need to show both directions:

let's start with the first direction:

a. If a connected graph G remains connected after removing an edge e from G, then e belongs to some circuit in G.

Assume that G is a connected graph and remains connected after removing an edge e from G. We need to show that e belongs to some circuit in G. Since G remains connected after removing e, there must exist a path P between the vertices u and v in G that does not include e. If e doesnot belong to any circuit, then the removal of e should have disconnected G. However, this contradicts our assumption that G remains connected after removing e.

Therefore, e must belong to some circuit in G.

Now, let's move on to the second direction:

b. If e belongs to some circuit in a connected graph G, then G remains connected after removing edge e from G.

Assume that e belongs to some circuit in a connected graph G. We need to show that G remains connected after removing e. Let's denote the end points of e as u and v. Since e belongs to a circuit, there exists a path P in G that starts at u, includes e, and ends at v. If we remove e from G, the path P will still exist in G without e, connecting u and v.

~~Since 'G-e' has more connected components than 'G', we can conclude tha~~

Since G is connected, there must be a path Q in G that connects any two vertices x and y. If x or y is not equal to u or v, then the path Q remains the same after removing e, as it doesn't involve e. If x or y is equal to u to v, we can use the path P to connect x and y, which also remains intact after removing e.

Hence, in both cases, G remains connected after removing e. By providing both directions, we have shown that a connected graph G remains connected after removing an edge e from G if and only if e belongs to some circuit in G.

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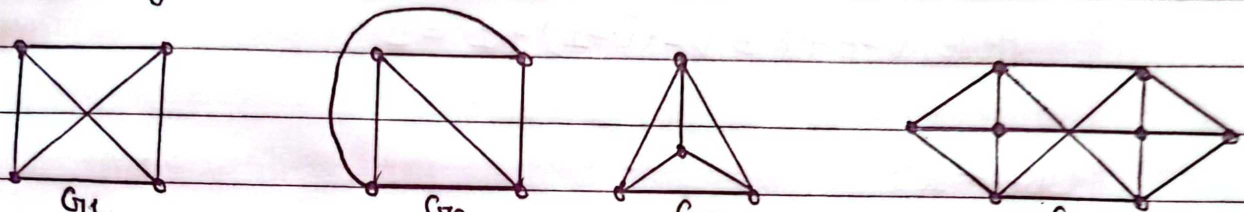
1: Define planar graph. If G is a connected planar graph with $|V| = v$, $|E| = e$ and r be the regions then $v - e + r = 2$.

→ A graph is called a planar graph if it can be represented in the plane in such a way that the edges intersect only at the vertices on which they are incident.



G_1 G_2 G_3 G_4

Among the graphs G_1, G_2, G_3, and G_4, the graphs G_2 and G_3 are planar but G_1 and G_4 are not planar.

~~Now, it is a matter~~

Proof:

Let us prove the theorem by means of induction. First, suppose that $e = 0$, i.e., it has no edges.

So, $v - e + r = 1 - 0 + 1 = 2$

[A trivial graph G.
Here, $e = 0$ and open 1 region, so $r = 1$]

Now, let $e > 0$. Suppose that, for any graph with v vertices, r regions and $e' < e$ edges,

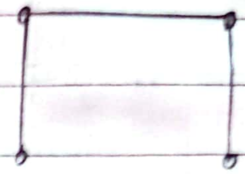
$$v - e' + r = 2$$

It follows two cases

(i) G has no cycles

(ii) G has at least one cycle.

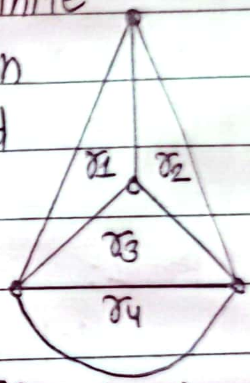
Case – I :



Let G has no cycles. Then, it has no closed regions, as shown below. It has only an open infinite region and thus $r = 1$. Since it is connected, $e = v - 1$.

Here, $v - e + r = v - (v-1) + 1 = 2$

Case – II :



(Four finite regions)

Now, we let G has cycles. It means, it has finite regions. Let us choose a region that has a common edge with the infinite region (outer region) and then delete such an edge. Thus, the new graph has one less edge and one less region. Now, we can apply the induction hypothesis to the new graph as :

$$v - (e-1) + (r-1) = 2$$

Hence, $v - e + r = 2$

It completes the proof.

2. Define connected planar graph with v as vertices and e as edges with $v \geq 3$ then prove that $e \leq 3v - 6$.

→ A connected graph G is said to be 2-connected when it has no bridges. In the graph shown below, there are three components. The component on the left is a 2-connected graph. So, is the component on the right. But the component in the middle is not 2-connected.

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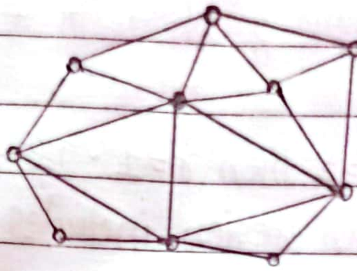 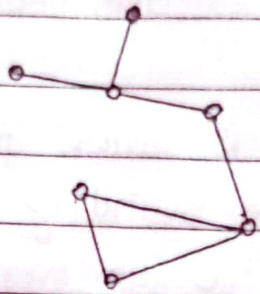 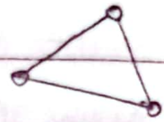

Proof:

For $v = 3$, $e \leq 3$. It is happened because a connected graph with three vertices has at most three edges. We now assume that $v \geq 4$. Let a planar graph G with r regions. For each region r of G, fix the number of edges lying on the boundary of r. Then, determine the number n. It is a sum of all the edges over all the regions of G. As there are at least three edges belonging to the boundary of every region, it gives $n \geq 3r$.

Thus, $3r \leq n \leq 2e$

or, $3r \leq 2e$

or, $-r \geq -\frac{2e}{3}$

Again, we have

$$v - e + r = 2$$

or, $v = e - r + 2$

Thus, $v = e - r + 2 \geq e - \frac{2e}{3} + 2$

or, $v \geq \frac{e}{3} + 2$

Hence, $e \leq 3v - 6$

It completes the proof.

3. If G is a connected, planar, simple graph with $e \geq 2$, then $3r \leq 2e$.

→ Proof:

The result is obvious for $e = 2$, then $r = 1$. If $e > 2$, we know that the boundary of each region includes at least three edges. If so, we sum over every region of the graph where there are at least $3r$ edges. But, while computing this sum, each edge in the graph is found to be counted twice. Therefore, the sum must be $2e$. It completes the proof.

4. A complete graph K_n is planar if and only if $n \leq 4$.

→ Proof:

It is clear that the graph K_n is planar for $n = 1, 2, 3, 4$. Now, we have only to show that K_n is non-planar if $n \geq 5$. For this, we show that K_5 is not planar.

Clearly, K_5 has 5 vertices and 10 edges, so that

$$3V - 6 = 9$$

That is $e > 3V - 6$. So, the graph K_5 cannot be planar. It completes the proof.

5. Find the maximum number of edges in the planar graph with 9 vertices?

→ Here, $V = 9$

Now, $e \leq 3V - 6$

$\Rightarrow e \leq 27 - 6$

$\Rightarrow e \leq 21$

Hence, the maximum number of edges possible in the planar graph with nine vertices is 21.

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6. Find the minimum number of vertices necessary for a graph with 21 edges to be planer ?

→ Here, we have,

$$v \geq \frac{e}{3} + 2$$

For $e = 21$

$$v \geq \frac{21}{3} + 2 = 9$$

$$\Rightarrow v \geq 9$$

Hence, the minimum number of vertices required is 9.

7. If G is a plane graph, then $\sum_{f \in F(G)} d(f) = 2E(G)$.

→ Let G^* be the dual of G. Then,

$$\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f)^*$$

$$= 2E(G^*)$$

$$= 2E(G)$$

$$\Rightarrow \sum_{f \in F(G)} d(f) = 2E(G)$$

This completes the proof of the theorem.

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8. Let G be a connected graph that is not an odd cycle. Then, G has a 2-edge colouring in which both colors are represented at each vertex of degree at least two.

→ Proof:

We may clearly assume that G is nontrivial. First, suppose that G is Eulerian. If G is an even cycle, the proper 2-edge colouring of G has the required property. Otherwise, G has a vertex v_0 of degree at least four. Let, $v_0 e_1 v_1 \ldots e_\epsilon v_0$ be an Euler tour of G, and set

$$E_1 = \{e_i \mid i \text{ odd}\} \text{ and } E_2 = \{e_i \mid i \text{ even}\}$$

Then, the two edge colouring (E_1, E_2) of G has the required property, since each vertex of G is an internal vertex of $v_0 e_1 v_1 \ldots e_\epsilon v_0$.

If G is not Eulerian construct a new graph G^* by adding a new vertex v_0 and joining it to each vertex of odd degree in G. Clearly, G^* is Eulerian. Let $v_0, e_1, v_1, \ldots e_{\epsilon^*} v_0$ be an Euler tour of G^* and define E_1 and E_2 as above. It is then easily verified that the two edge colouring $(E_1 \cap E, E_2 \cap E)$ of G has the required property.

9. If G is bipartite, then $\chi' = \Delta$.

→ Proof:

Let G be a graph with $\chi' > \Delta$, let $\mathcal{C}' = (E_1, E_2, \ldots E_\Delta)$ be an optimal Δ-edge colouring of G and let u be a vertex such that $c(u) < d(u)$. Clearly, u satisfies the hypothesis of lemma 2. Therefore, G contains an odd cycle and so is not bipartite, which is a contradiction. Thus $\chi' = \Delta$.

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10. If G is K-critical, then $\delta \geq K-1$.

→ Proof:

If possible, let G be a K-critical graph $\delta < K-1$, and let v be a vertex of degree δ in G. Since G is K-critical, $G-v$ is $(K-1)$ colourable. Let $(V_1, V_2, ---, V_{K-1})$ be a $(K-1)$-colouring of $G-v$. By definition, v is adjacent in G to $\delta < K-1$ vertices and therefore v must be nonadjacent in G to every vertex of some V_j. But, then $(V_1, V_2, ---, V_j \cup \{v\}, ---, V_{K-1})$ is a $(K-1)$-colouring of G, a contradiction. Thus, $\delta \geq K-1$.

This completes the proof of the theorem.

11. Welch-Powell Algorithm for colouring

→ To colour the vertices of a given graph G, we need to proceed to follow the following algorithm given by Welch-Powell.

Step-1: List down the vertices of G in decreasing order of degrees in a row and then write the degree of the corresponding vertex in the next row.

Step-2: Select a colour to paint the first vertex on the list of step 1. Then, paint each of the other vertices on the list (not adjacent to the vertex previously painted) by the same colour and write down on the third row.

Step-3: Deal the above list of remaining vertices with highest degree to colour the unpainted vertices using a second colour.

Step-4: Keep continue the process of colouring the unpainted vertices with additional colours until all the vertices get painted.

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12\. Explain chromatic number of a graph. Let G be a graph, then prove that chromatic number of G is 2 if and only if G is bipartite.

→ The chromatic number, denoted by $\chi(G)$ of a graph G is the smallest number of colours needed to colour the vertices of G.

Proof:

As given G(V,E) is a bipartite graph so that $V = M \cup N$. Let us paint all vertices of M by a colour C_1 and that of all vertices of N by another color C_2. This implies that $\chi(G) = 2$.

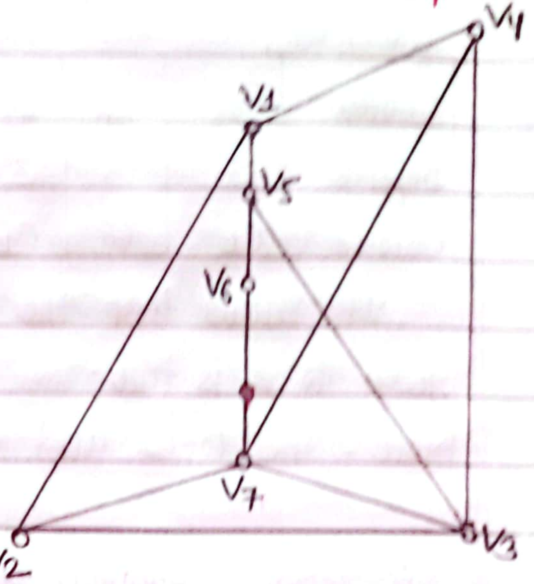
In order to prove it conversely, we take $\chi(G) = 2$ which means G has used two colors. Let M denotes a set of all those vertices painted by color C_1 and N denotes a set of all the vertices painted by next color C_2. Here, no two vertices in M are adjacent. The case is similar to N as well. Therefore, any edge, in the graph G, must join a vertex m and to a vertex in N. Hence, G is a bipartite graph where $V = M \cup N$.

13\. Show that the chromatic number of the bipartite graph $K_{2,3}$ is two.

→ As we know the vertices of a bipartite graph $K_{2,3}$ can be divided into two subsets M and N (say). Here, the two vertices of one Subset M are connected to three vertices of the other subset N so that there is no connection among own vertices of each subset. Thus two vertices of the subset M can be painted by the same color, say C_1. Similarly, three vertices of subset N can be painted by any other color, say C_2.

In this way, the bipartite graph $K_{2,3}$ needs only two colors. Thus, its chromatic number is two.

14. In the given figure, find the chromatic number of the graph G.

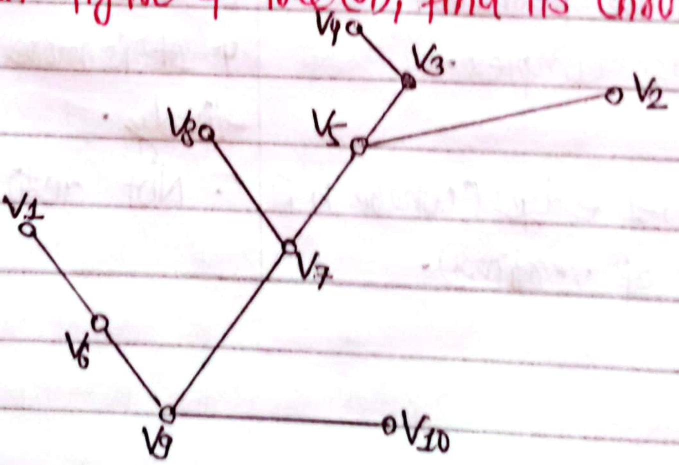


→ Following the algorithm, we have,

Vertex	V_7	V_3	V_2	V_4	V_1	V_5	V_6
Degree	4	4	3	3	3	3	2
Colour	C_1	C_2	C_3	C_3	C_2	C_1	C_3

The vertices V_2, V_3 and V_7 are connected to each other. So, they must be colored by different colours. Here, at least three colours are required to paint the graph G. Thus, $\chi(G) = 3$.

15. In the given figure of tree (G), find its chromatic number.



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→ From the given figure,

we have,

Vertex	V_5	V_7	V_9	V_3	V_6	V_1	V_2	V_4	V_8	V_{10}
Degree	3	3	3	2	2	1	1	1	1	1
Colour	C_1	C_2	C_1	C_2	C_2	C_1	C_2	C_1	C_1	C_2

It shows that the number of colours needed to colour the given tree is two.

Note: The trees and so the forests are 2-Colourable.

16. Difference between tree and graph.

→ The difference between tree and graph are :

Tree	Graph
1. It has only one path between two vertices.	1. It has more than one path is allowed.
2. It has exactly one root vertex.	2. Graph doesn't have a root vertex.
3. No loops are permitted.	3. Graph can have loops.
4. It is less complex.	4. It is more complex comparatively.
5. It has n-1 edges (where n is the number of vertices).	5. Not defined.

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Graph Theory

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CHAPTER-4

1: If $\Delta(G)$ be the largest degree of a vertex in a (P,q)-graph G, then $\chi(G) \leq 1+\Delta(G)$.

Proof: we proof this statement by induction method.
If $P=1$, then there is no edge in G, therefore, $\Delta(G)=0$ and $\chi(G)=1$ Hence, $1 \leq 1+0$ which is true.
Suppose the statement is true for the graph having less than P. let v be any vertex of G and let $\bar{G}=G-v$ be a subgraph of G such that $|\bar{G}|=P-1$. Since $\Delta(\bar{G}) \leq \Delta(G)$ and $\bar{G}$ can be coloured with $\chi(G)$ colours. Since $|V(\bar{G})|=P-1$, then by induction hypothesis, $\chi(\bar{G}) \leq 1+\Delta(\bar{G}) \leq 1+\Delta(G)$. Since there are at most $\Delta(G)$ vertices adjacent to v in G which can be coloured by $\Delta(G)$ colours in $G-v$, the remaining one colour can be used for the vertex v in G. Thus, G can be coloured with at most $1+\Delta(G)$ colours. Hence, $\chi(G) \leq 1+\Delta(G)$. This completes the proof.

2: Determine the chromatic number of K_n.

Proof:

Since, $\Delta(K_n)=n-1$

Then, $\chi(K_n) \leq 1+\Delta(K_n) \leq 1+n-1 \leq n$.

So, every vertex is adjacent to every another vertex of complete graph K_n. Hence $\chi(K_n)=n$.

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3. Lemma: Let $G = (E_1, E_2, ---, E_k)$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that is not represented at u and j represented at least twice at u, then the component of $G(E_i \cup E_j)$ that contains u is an odd cycle.

Proof: Let u be a vertex that satisfies the hypothesis of lemma and denoted by H the component $G(E_i \cup E_j)$ containing u. Suppose that H is not an odd cycle. Then we know that H has a 2-edge colouring in which both colours are represented at each vertex of degree at least two in H. When we recolour the edges of H with colour i and j in this way, we obtain a new-edge colouring $\mathcal{C}' = (E_1', E_2', \ldots, E_k')$ of G. Denoting by $c'(v)$ the number of distinct colours at v in the colouring $\mathcal{C}'$, we have

$$c'(V) = C(u) + 1 .$$

Since, now both i and j are represented at u and also

$c'(V) \geq c(u)$ for $V \neq u$.

Thus, $\Sigma_{v \in V} c'(V) > \Sigma_{v \in V} c(V)$, contradicting the choice of $\mathcal{C}'$.

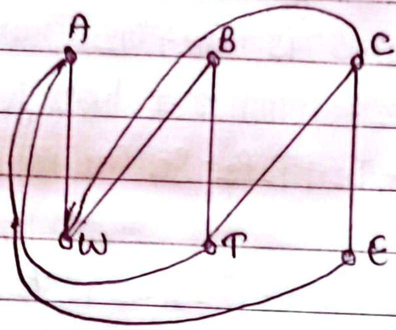
It follows that H is indeed an odd cycle. Hence proved.

4 Prove that a complete bipartite graph $K_{m,n}$ is planner if and only if either $m \leq 2$ or $n \leq 2$. [CHAPTER-4, planar graphs and coloring]

→ Proof:

For $m \leq 2$ or $n \leq 2$, the graph $K_{m,n}$ is planar. Now, we suppose that $m \geq 3$ and $n \geq 3$. In order to prove that the graph $K_{m,n}$ is not planar, it is sufficient to show that $K_{3,3}$ is not planar.

If possible, let $K_{3,3}$ is planar. let a plane graph is $K_{3,3}$ as $K_{3,3}$ has 6 vertices and 9 edges, it follows that it divides the plane into 5 regions.



On the other hand, each region of $K_{3,3}$ is bounded by at least four edges. Thus, if we apply the same argument, we have;

$$2e \geq 4r$$

Now, we have

$$r = 2 - v + e$$

$$\text{or, } 2e \geq 4(2 - v + e) \quad \left[\because r \leq \frac{2e}{4}\right]$$

$$\text{or, } e \geq 4 - 2v + 2e$$

$$\Rightarrow e \leq 2v - 4$$

But $v = 6$ and $e = 9$, which violates this inequality. Hence, $K_{3,3}$ is not planer.

Unit - 5
Digraph and Traversability

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1: What is in-degree and out-degree of a vertex in a digraph? Let G be a digraph with $V_1, V_2, ---, V_n$ and q be the number of directed arcs in G, then prove that:

$$\sum_{i=1}^{n} id(V_i) = \sum_{i=1}^{n} od(V_i) = q.$$

→ The number of edges incident out from the given vertex is called out-degree of it. Similarly, the number of edges incident into the given vertex is called in-degree of it. The out-degree and in-degree are denoted by $od(V_i)$ and $id(V_i)$ respectively.

Proof:

Each arc is counted exactly once when it enters into a vertex. Thus, the total number of in-degrees is equal to total number of directed arcs; that is

$$\sum_{i=1}^{n} id(V_i) = q \text{ —— (i)}$$

Similarly, each arc is counted exactly once when it goes out from each vertex. Thus, the total number of out-degrees is equal to total number of directed arcs; that is,

$$\sum_{i=1}^{n} od(V_i) = q \text{ —— (ii)}$$

Hence, from (i) and (ii),

$$\sum_{i=1}^{n} id(V_i) = \sum_{i=1}^{n} od(V_i) = q \quad \text{proved}$$

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2: Prove that a digraph D is strongly connected if any only if it has closed directed spanning tree(path).

→ Proof:

Let us consider that a digraph D is strongly connected. Let $P = \{v_1, v_2, v_3, \ldots, v_1\}$ be a closed path containing a maximum number of vertices of digraph D.

If possible we take P is not a spanning path. If so, there is a vertex, say v', not involved in the closed path P. But D is a strongly connected digraph, so there are paths $P' = \{v', \ldots, v_1\}$ and $Q' = \{v_1, \ldots, v'\}$. It implies that the path $(P' P Q')$ is a closed path involving the vertices of closed path P and the vertex v'. It contradicts that the closed path P consists of maximum number of vertices of D. It means P is a closed spanning path.

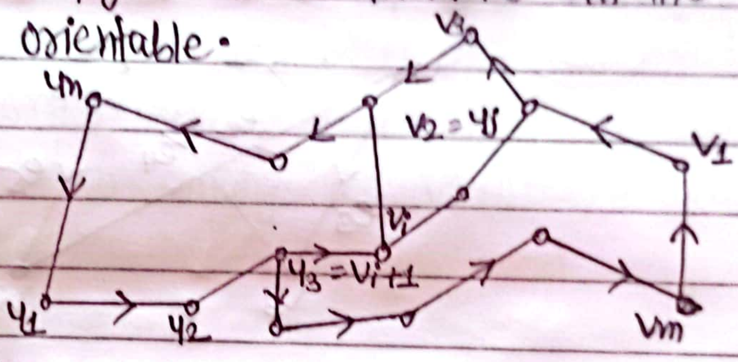
Conversely, let P be a closed spanning path in the digraph D. Then every vertex of D can be reached from any vertex of D through the path P. It implies that D is strongly connected.

3. Prove that a connected graph G is orientable if and only if it contains no bridge.

→ Proof:

Let the graph G is orientable. By contradiction, if possible let G contains a bridge $b = \{b_1, b_2\}$. As we assumed G is orientable, we are allowed to assign a direction to each edge of G and hence we have a strongly connected digraph D. Thus, it is possible to reach from b_1 to b_2 and vice versa in G. But $b = \{b_1, b_2\}$ is a bridge so either one directed path b_1 to b_2 or b_2 to b_1 is possible but not both. Thus, G is not orientable. It contradicts our supposition. So, we conclude that G has no any bridge.

Conversely, let the graph G is connected having no bridges then we have to prove that G is orientable. As G contains no bridges, we have every edge of G lies on a cycle such as $C = \{u_1, u_2, ---, u_n, u_1\}$. Now, we can assign the direction as shown in the figure. As C includes all the vertices of G, G is obviously orientable.



If possible let the vertices of G don't lie on C. Let there is a vertex v_1 which doesn't belong to C. But v_1 is adjacent to C since G is connected. Then $\{v_1, v_j\}$ is an edge of G $(1 \le j \le n)$ Then there exists a cycle, say, $C' = \{v_1, u_i (= v_2), v_3, ---, v_m, v_1\}$ lies on G.

Now, we give the direction arbitrarily as shown in the figure. Now the digraph say D' developed by following this process is connected. So, if D' includes all the vertices of G, it completes the proof of this theorem. Otherwise, we continue this process unless the strongly connected digraph D' includes all the vertices of G. In this way, it establishes that G is orientable. Hence, the theorem is proved.

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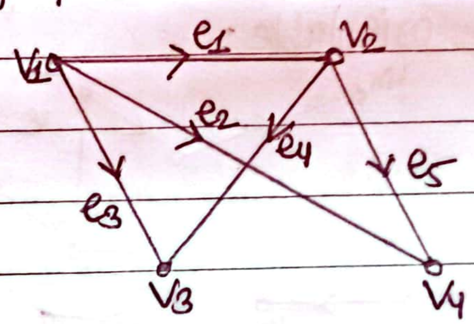
4. Construct a digraph of given adjacency matrix and then find its incidence matrix.

$$M_D = \begin{array}{cc} & \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

→ The required digraph is as drawn below.

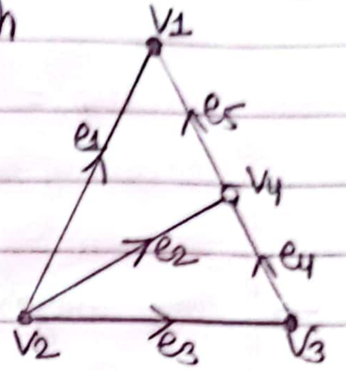


The incidence matrix is given below.

$$M_I = \begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

5. Find the incidence matrix of the given digraph.

→ The incidence matrix for the given digraph is as follows :



$$M_1 = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

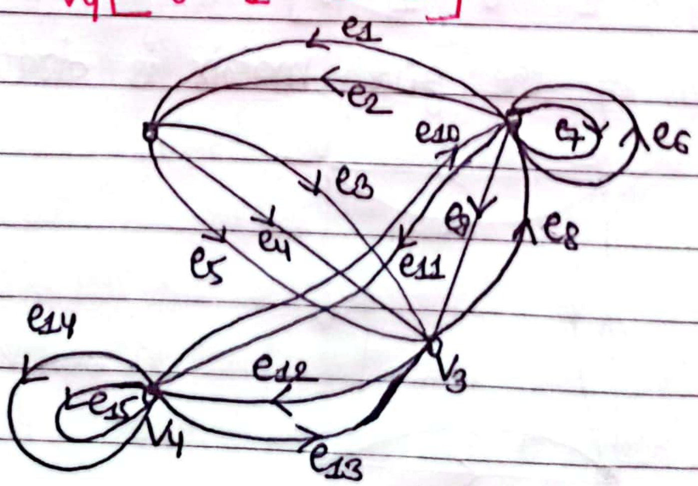
Note : Here, the matrix has been prepared by indicating the relation of vertices with edges. But, if it were formed by taking the relation among the vertices, it would be a square matrix.

6. Construct a digraph of the given matrices.

a.

$$M_2 = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 3 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

→

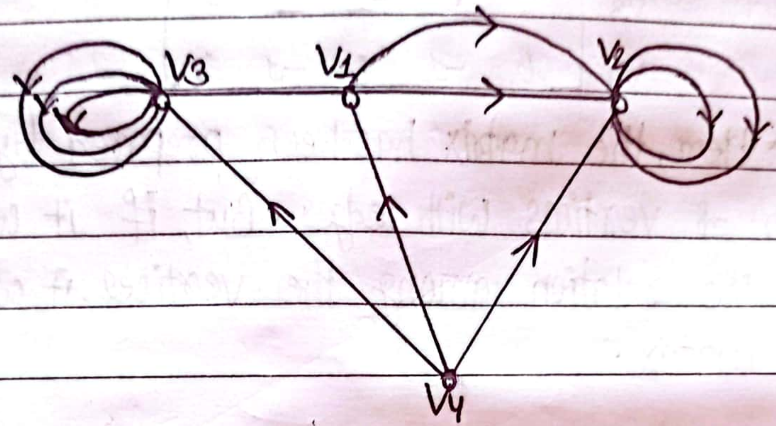


Which is the required digraph.

b.

$$M_D = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

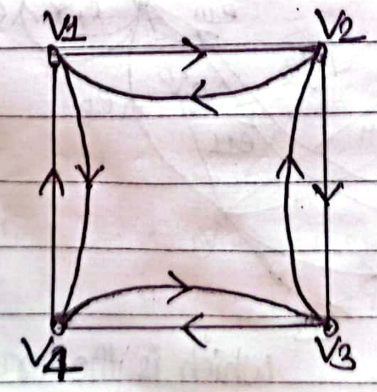
→ The diagraph of the given matrix is given below.



c.

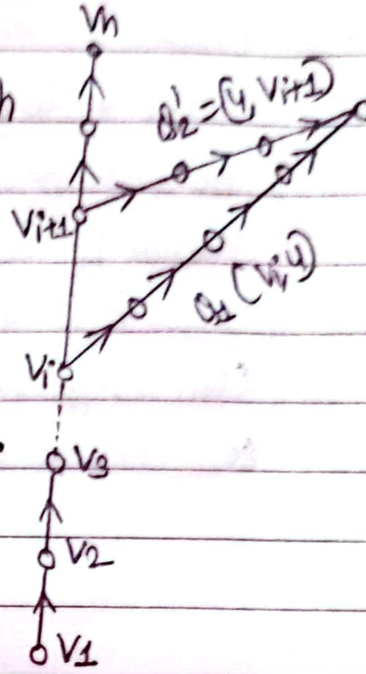
$$M_D = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

→ The diagraph of the given matrix is drawn below.



7. A digraph D is unilaterally connected digraph if and only if there is a directed spanning path in D.

→ Proof:



Let D be an unilaterally connected digraph with a directed path $P = (v_i, v_{i+1})$ or $P' = (v_{i+1}, v_i)$, where $(1 \leq i \leq j \leq n)$. Let $P = (v_i, v_j)$ consists of maximum number of vertices of D. Here, we have to show that D and P have the same number of vertices. If possible, let a vertex $y \in D$ but $y \notin P$. We show that no path is possible from v_i to y or y to v_i.

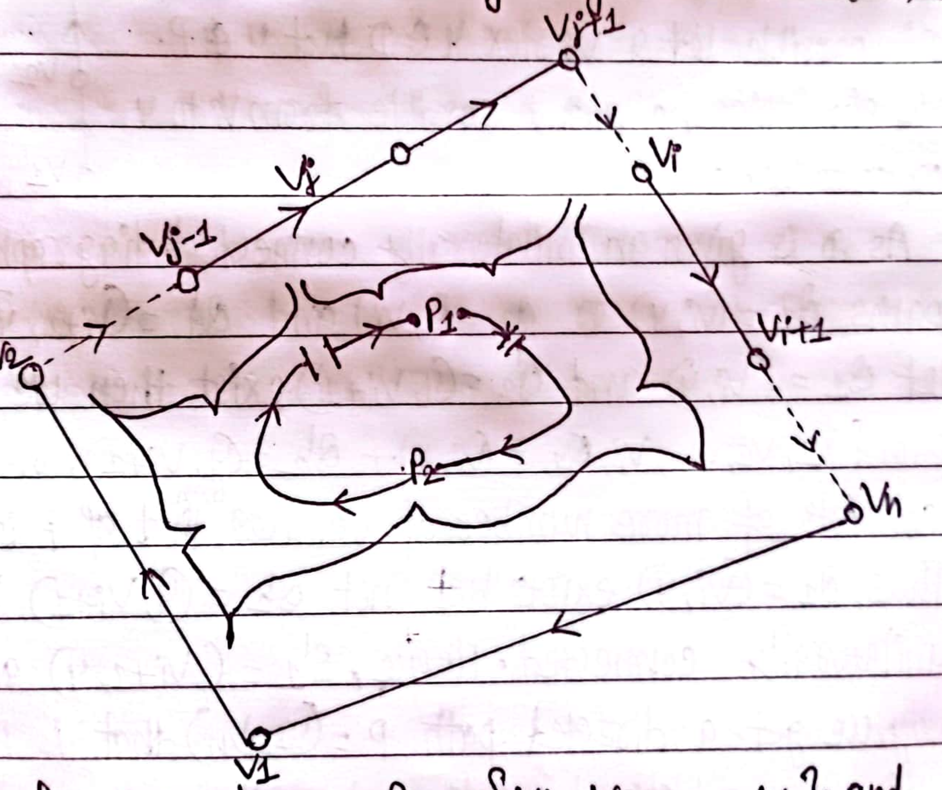
As D is given an unilaterally connected digraph, there can be paths $Q_1 = (v_i, y)$ or $Q_2 = (y, v_i)$ and $Q'_1 = (v_{i+1}, y)$ or $Q'_2 = (y, v_{i+1})$. Let $Q_1 = (v_i, y)$ and $Q'_2 = (y, v_{i+1})$ exist, then we have a directed path $\{v_1, v_2, ---, v_i, Q_1 = (v_i, y) + Q'_2 = (y, v_{i+1}), v_{i+1}, ---, v_n\}$ consists of more number of vertices than that of P is a contraction. Thus, $Q_1 = (v_i, y)$ exists but not $Q'_2 = (y, v_{i+1})$ because D is unilaterally connected. Hence, $Q'_1 = (v_{i+1}, y)$ also exists. In this way, we get a directed path $P = (v_1, v_n)$ that is $P = \{v_i, v_{i+1}\}$ with some additional vertices along the path y, which is a contradiction as above. Therefore, $P = \{v_i, v_{i+1}\}$ is a directed spanning path.

Conversely, let $P = \{v_i, v_{i+1}\}$ is a directed spanning path. Thus, the numbers of vertices in the directed spanning path & the digraph D have the same. Therefore, every $\{v_i, v_{i+1}\}$ are directly connected by any small paths of P. It completes the theorem.

8. A tournament T is Hamiltonian if and only if it is strongly connected.

→ Proof:

Let a tournament T is strongly connected with n vertices. The last theorem ensures that T must have a directed cycle of length n. Since T consisting all the vertices are used by this cycle, it is a Hamiltonian cycle. It implies that T is Hamiltonian. Conversely, we assume that the tournament T is Hamiltonian. If so, it consists of Hamiltonian cycle given as $C = \{v_1, v_2, \ldots, v_n, v_1\}$



From figure we have: $P_1 = \{v_j, v_{j+1}, \ldots, v_i\}$ and

$P_2 = \{v_i, v_{i+1}, \ldots, v_n, v_1, v_2, \ldots, v_{j-1}, v_j\}$

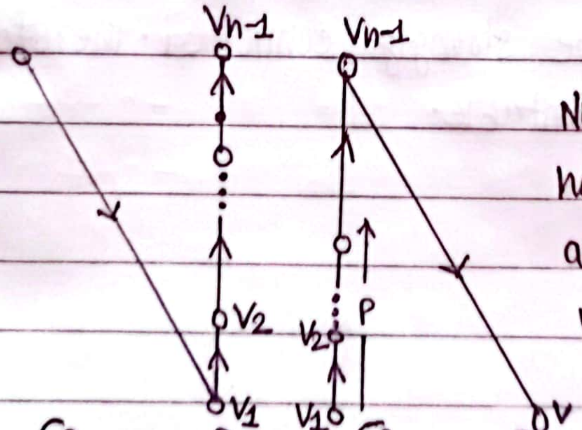
Both P_1 and P_2 are directed paths and they show that each and every vertex is reachable from either vertex. Thus, T is strongly connected. Hence, the theorem.

9. Every tournament T has a directed Hamiltonian path.

→ Proof:

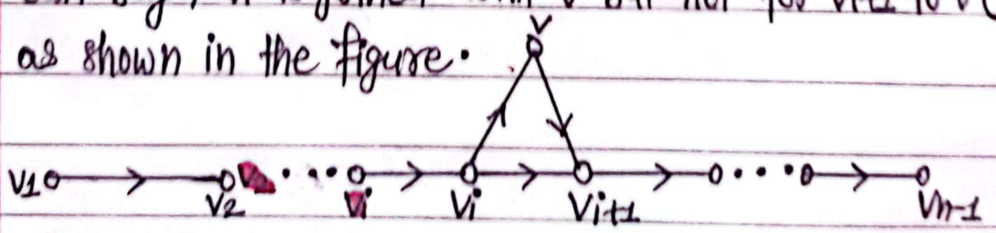
we assume that T is a tournament with n vertices. Then we have to prove that there is a directed Hamiltonian path in T. When n takes the values 1, 2 and 3, it can be verified easily which is obvious too. Hence, we use mathematical induction method for $n \geq 4$.

Let T−v is a tournament with (n−1) vertices and there is a directed Hamiltonian path $P = \{v_1, v_2, --- v_{n-1}\}$ in it. If an arc joins v with v_1 then $P_1 = \{v, v_1, v_2, --- v_{n-1}\}$ is a directed Hamiltonian path in T. But if an arc joins v_{n-1} with, then $P_2 = \{v_1, v_2, ---, v_{n-1}, v\}$ is a directed Hamiltonian path in T. In both the cases, the result is true.



(For case I) (For case II)

Now, we see the cases if there is no arc to join v and v_1 or v_{n-1} and v. Then there is at least one vertex v' on P having an arc joining v' to v where v' is different than v_{n-1}. Now we can say, v_i is joined with v but not for v_{i+1} to v (i.e v to v_{i+1}) as shown in the figure.



In this way, the path $P_3 = \{v_1, v_2, ---, v_i, v, v_{i+1}, ---, v_{n-1}\}$ provides a directed Hamiltonian path in the digraph D. Hence, the theorem.

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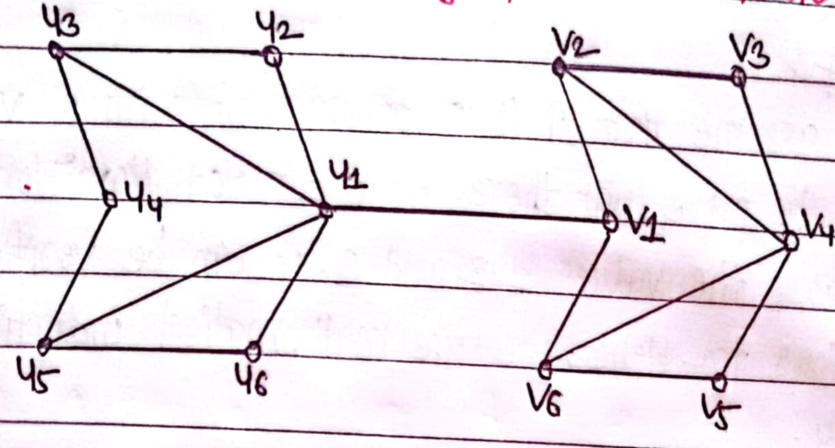
10. Examine whether the given graph is orientable or not.



→ In the given graph, the possible orientation shows that the edge $\{u_1, v_1\}$ receives either u_1 to V_1 or V_1 to u_1. There is no directed walk from u_1 to V_1 in the orientation of V_1 to u_1. Similarly, there is no directed walk from V_1 to u_1 in the orientation of u_1 to V_1. It shows that either V_1 is not reachable from u_1 or u_1 is not reachable from V_1. As a result, the orientation as assigned in the graph G cannot be strongly connected. Therefore, the given graph is not orientable.

Chapter-5

1: A weakly connected digraph D with at least two vertices has a directed open Eulerian trial if any only if D has two vertices u and v such that $od(u) = id(u)+1$ and $id(v) = od(v)+1$, for all other vertices w of D, $id(w) = od(w)$. Furthermore, in this case, the trial begins at u and ends at v.

→ Let an Euler trial T in D such that T begins at u and ends at v. Different than u and v, let us consider w be any arbitrary vertex in D. While enrouting T, every time the vertex w is encountered on T when it entered by another arc and left by another arc. Thus, each occurrence on w in T, it adds 1 to the in-degree and 1 to the out-degree of w. Since incident in and out on w are equal, we have $id(w) = od(w)$.

Moreover, the first arc T adds 1 to the out-degree of u. But, rest of other occurrence of u in T provides 1 to each out-degree and in-degree of u.

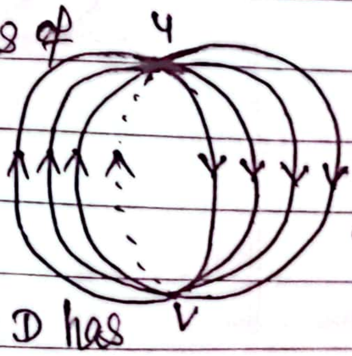
Thus, we get $od(u) = id(u)+1$.

Similarly, the last arc of T adds 1 to the in-degree of v. But rest of other occurrence of v in T provides 1 to each out-degree and in-degree of v. Thus we get $id(v) = od(v)+1$.

Conversely, let u and v be the vertices of a weakly connected digraph D, we have,

$od(u) = id(u)+1$ and $id(v) = od(v)+1$.

So, that, $od(w) = id(w)$ for all other vertices $w \in D$. Here, we have to prove that D has a directed Euler trial that it begins at u and ends at v.

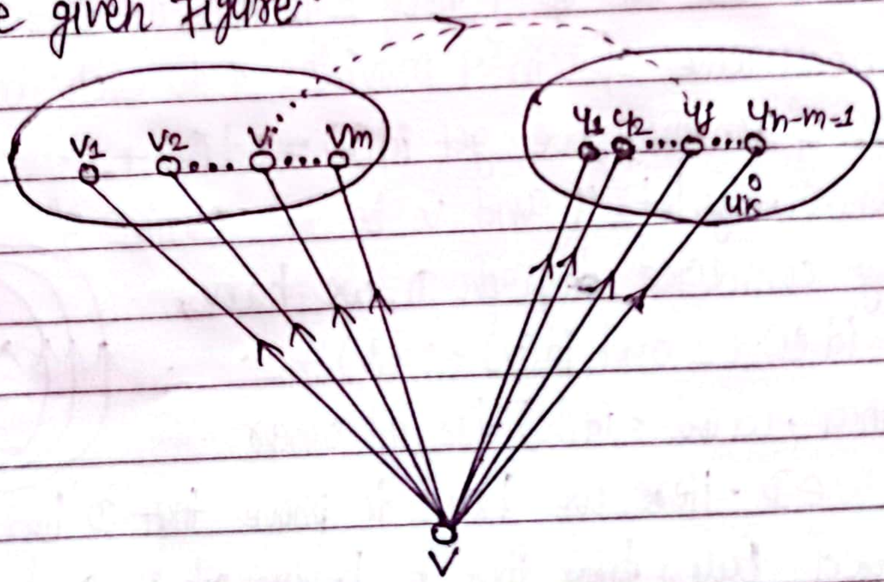


Let $D' = D + e$ where e be the directed arc from v to u. Then, the digraph D' satisfies the condition of above theorem CD with at least one arc is Eulerian iff $od(v) = id(v), v \in D$ to be an Eulerian. Thus, it has a directed Euler tour containing all arcs ~~in~~ of D together with new one arc e. If an arc e is removed, a directed Euler trial is formed in D which begins at u and ends at v. Hence, the theorem is proved.

2: Let an every tournament T having a vertex (v) from which every other vertex is reachable by a path of length at most 2.

→ Let v be the given vertex which is connected with $v_1, v_2, \ldots, v_i, \ldots, v_m$ so that $od(v) = m$.

But, if T has n vertices, we let $u_1, u_2, \ldots u_j, \ldots u_{n-m-1}$ are the remaining vertices adjacent to v. T begin a tournament, there are arcs which join u_j $(1 \le j \le n-m-1)$ to v as shown in the given figure.



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There are directed paths each of length 1 which join v to v_i $(1 \le i \le m)$. Thus, we show that there are directed paths each of length 2 which join v to y_j.

If there is an arc which joins v_i to y_j then we have (v, v_i, v_j) which provides a directed path of length 2. Now if possible, we take y_k $(1 \le k \le n-m-1)$ having no arc from v_i to it. But T being a tournament, there must be an arc which joins y_k with each of v_i. In the meantime, y_k is joined with v. Therefore,:

$$od(y_k) \ge m+1 \text{ (as } v_i = v_1, \ldots v_i, \ldots v_m)$$

It provides an evidence to contradict the fact that $od(v) = m$. so, y_k has an arc from v_i. Thus, it shows that there exists a directed path (v, v_i, v_j) of length 2. It completes the proof.
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Unit-6
Matchings

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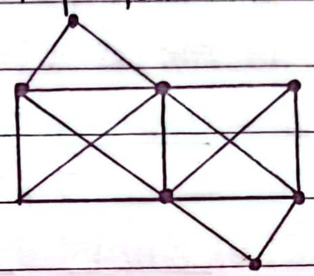
1. Define perfect matching of a graph G. Explain the condition for a matching M to be a covering of the graph G?

→ Let G be a graph and M be a matching in G. M is called a perfect matching if M saturates all vertices of G. For example, we can minutely observe the matching M in the given graph G in given figure where M has saturated all eight vertices of G. So, the matching M is a perfect matching graph s.



Proof:

A matching in a graph is a set of edges such that no two edges share a common vertex. A matching is said to be a covering of a graph if every vertex in the graph is incident to at least one edge in the matching. In other words, a matching M is a covering of a graph G if every vertex in G either an endpoint of an edge in M or adjacent to an endpoint of an edge in M.

To check whether a given matching M is a covering of a graph G, we can follow these steps:

a. Iterate through each vertex v in G.
b. Check if v is an endpoint of any edge in M. If it is, continue to the next vertex.
c. If v is not an endpoint of any edge in M, check if v is adjacent to an endpoint of any edge in M. If it is, continue to the next vertex.
d. If v is not an endpoint of any edge in M & not adjacent to an endpoint of any edge in M, then the matching M is not a covering of the graph G.
e. Repeat steps 2-4 for all vertices in G.
f. If all vertices satisfy either step 2 or step 3, then the matching M is a covering of the graph G.

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2. Find the number of perfect matching in the complete graph K_{2n}.

→ Let us take $V(K_{2n}) = \{v_1, v_2, ---, v_{2n}\}$. The vertex v_1 can be saturated in $2n-1$ way by choosing any edge e_1 incident at v_1. In general having chosen $e_1, e_2, ---, e_k$ can be saturated in $2n-(2k+1)$ ways. We obtain a perfect matching after the choice of n edges in the above process. Hence, the number of perfect matching in K_{2n} is equal to $1.3.5 \dots (2n-1)$. Then

$$1.3.5 --- (2n-1) = \frac{1.2.3.4.5 ---- (2n-1)(2n)}{2.4.6 --- 2n}$$

$$= \frac{(2n)!}{2^n n!}. \blacksquare$$

3. Find the number of perfect matching in the complete bipartite graph $K_{n,n}$.

→ Let $X = \{x_1, x_2, ---, x_n\}$ and $Y = \{y_1, y_2, ---, y_n\}$ be a bi-partition of $K_{n,n}$. It can be seen that any matching of $K_{n,n}$ that saturates every vertex of X is a perfect matching. The vertex x_1 can be saturated in n ways by choosing any of the edges $x_1y_1, x_2y_2, ---, x_1y_n$. As saturated x_1, the vertex x_2 can be saturated in $n-1$ ways. In general having saturated $x_1, x_2, ---, x_i$ the next vertex x_{i+1} can be saturated in $(n-1)$ ways. Hence, the number of perfect matchings in $K_{n,n}$ is $n \cdot (n-1) \dots 2 \cdot 1 = n!$. $\blacksquare$

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4. For what values of n does the complete graph K_n have perfect matching ?

→ It is obvious that any graph having n number of odd vertices has no perfect matching. Also, the complete graph K_n has a perfect matching when n is even. It can be illustrated as when $V(K_n) = \{1, 2, \ldots, n\}$ then $\{12, 34, \ldots, (n-1)n\}$ is a perfect matching of K_n. Thus, K_n has a perfect matching if and only if the given number n is even. ■

5. Show that a tree has at most one perfect matching.

→ Let a tree T has two perfect matchings say M_1 and M_2. Then the degree of every vertex in $H = T[M_1 \Delta M_2]$ is 2. Hence, every component of H is an even cycle with edges alternately in M_1 and M_2 which is a contradiction as T has no cycles. Thus, T has at most one perfect matching. ■

6. Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. [Hall's Marriage Theorem]

→ Proof :

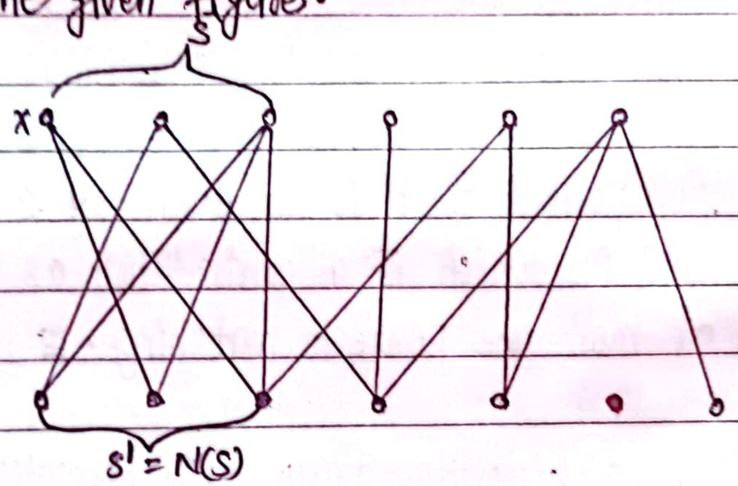
Let us assume that the graph G contains a matching M, and M saturates every vertex in X. We take S be a subset of X. Because of the vertices of S are matched under the matching M with different vertices in $N(S)$, we have $|N(S)| \geq |S|$.

Conversely, let a graph G is a bipartite with $|N(S)| \geq |S|$ for all $S \subseteq X$. We prove this part by contradiction. So

If possible we let G contains no matching that saturates every vertex in X.

Let us assume that a maximum matching denoted by M' is in the graph G where M' doesn't saturate every vertex in X. Let us take x be an M'-unsaturated vertex in X. Let V denote the set of all vertices connected to x by M'-alternating paths (in which both end-vertices are unsaturated by M').

Since M' is a maximum matching, x is the only M'-unsaturated vertex in V. The set $S = V \cap X$ and $S' = V \cap Y$ as shown in the given figure.



Here, the vertices in $S \setminus \{x\}$ are matched under M' with the vertices in S'. Thus, $|S'| = |S| - 1$ —— (i)

Also, $N(S) \supseteq S'$. So, we have, $N(S) = S'$ —— (ii)

because every vertex in $N(S)$ is connected to x by an M'-alternating path. From (i) and (ii), we get $|N(S)| = |S| - 1$.

But, it is obvious that $\{|S| - 1\} < |S|$. Thus, $|N(S)| < |S|$ which is a contradiction.

It completes the proof.

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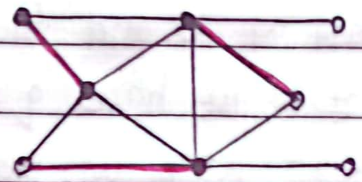
7. Let G be a graph with size of matching $e(M)$ and size of covering $v(C)$. Discuss on $e(M) \leq v(C)$ with required illustrations.

→ If we get a matching M and a covering C of the same size then we have $e(M) = v(C) = 3$ as show in given figure.



But, some graph don't have same sizes of a matching and a covering. For this, we take the following two graphs.

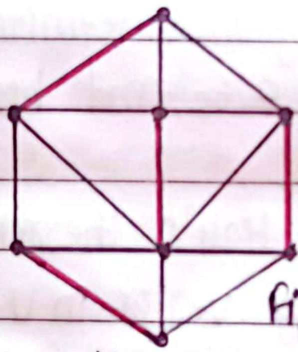


Fig-(i)

[All vertices are saturated]

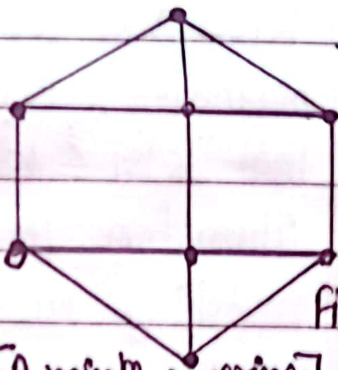


Fig-(ii)

[A min^m covering]

We see that there is a maximum matching of size 4 in figure (i) but a minimum covering of size 5 in the same type of figure (i) as illustrated in given figure (ii). These two figures (i) and (ii) show us that $e(M) < v(C)$.

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8\. Let G be a k-regular bipartite graph with $k > 0$. Then G has a perfect matching.

→ Proof:

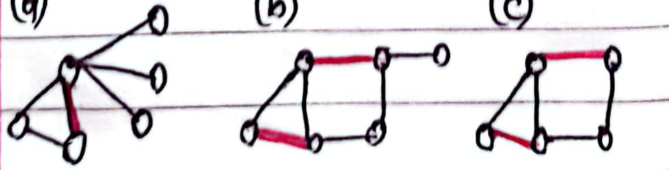
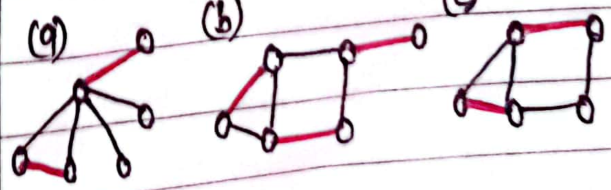
we assume that (U, V) be a bi-partition of G. As each edge of G has one end at U and there are k edges incident with each vertex of U, we have $q = k|U|$. Similarly, $q = k|V|$, so that $k|U| = k|V|$. As $k > 0$, we get then $|U| = |V|$.

Here, we let $S \subseteq U$. Let E_1 denote the set of all edges incident with vertices in $N(S)$. Since G is k-regular, $|E_1| = k|E_2|$ and $|E_2| = k|N(S)|$. Also, by the definition of neighbour set $N(S)$, we have $E_1 \subseteq E_2$; and hence it follows that $k|S| \leq k|N(S)|$.

Therefore, $|N(S)| \geq |S|$. Thus, by Hall's theorem, G has a matching M that saturates every vertex in U. Since $|U| = |V|$, M also saturates all the vertices of V. Therefore, M is a perfect matching. It completes the proof. ■

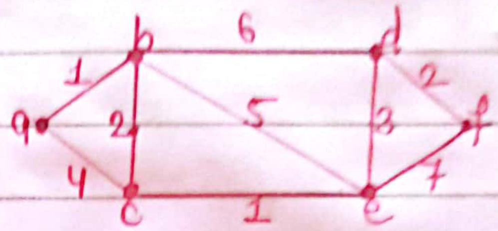
13. Define and differentiate maximal matching and maximum matching with suitable example of graph. [CHAPTER-6]

→

Maximal matching	Maximum matching
1. A maximal matching is a matching M of a graph G that is not a subset of any other matching. A matching M of a graph G is maximal if every edge in G has a non-empty intersection with at least one edge in M.	1. A maximum matching is a matching that contains the largest possible number of edges. There may be many maximum matching. The matching number $\nu(G)$ of a graph G is the size of a maximum matching. Every maximum matching is maximal, but not every maximal matching is a maximum matching.
2. Definition: Maximal matching is the collection of minimum possible collection of non-adjacent edges.	2. Definition: Maximum matching is the collection of maximum non-adjacent edges.
3. Eg: The following figure shows examples of maximal matching (red) in three graphs. (a) (b) (c) 	3. The following figure shows examples of maximum matchings in the same three graphs. (a) (b) (c) 

12: Find the shortest path and its weight from vertex a to vertex f in given graph. [CHAPTER-1]

→ The shortest path from a to f is as follows :-



vertex	a	b	c	d	e	f
level a	0	∞	∞	∞	∞	∞
level b		$\boxed{1}$	4	∞	∞	∞
level c			$\boxed{3}$	7	6	∞
level d				7	$\boxed{4}$	∞
level e				$\boxed{7}$		11
level f						$\boxed{9}$

∴ The shortest distance is 9(weight).

∴ The shortest path is (a) a-b-c-e-d-f

(b) a-b-d-f.