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Discrete Mathematics

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Unit

1

Relations and Digraphs

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- Define Cartesian product of two and three sets.
- Define different types of relations and find the relations between different sets.
- Represent the given relations in different methods.
- Verify the different properties of relations
- Define Boolean matrix with examples.
- Perform the operations on Boolean matrices.
- Find the product of two Boolean matrices.
- Find the composition of two relations.
- Find the operation on relations with transitive closure and Warshall's algorithm

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Introduction

In this chapter I, we first define what we mean by Cartesian product of sets and then use this concept to define a relation from one set to another set and a (binary) relation in a set. We also discuss various types of relations, including an equivalence relation and the partitions

1.1 Product set (Cartesian product)

Consider two arbitrary set A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the product set or Cartesian product of A and B. It is denoted by $A \times B$. read as "A cross B" and written as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$ then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b) (2, c)\}$$

$$B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c,b), (c, c)\}$$

Note: $A \times B \neq B \times A$

The Cartesian product deals with ordered pairs, so naturally the order in which the set are considered is important. If n(S) is the number of elements in a set S. we have $n(A \times B) = n(A).n(B)$

Note: This can be written as $|A \times B| = |A| . |B|$

i.e., $n(A \times B) = n(A) . n(B)$

Theorem : 1

For any two finite non-empty sets A and B, $|A \times B| = |A|.|B|$

Proof:

Suppose that $|A| = m$ and $|B| = n$. To form an ordered pair (a, b), $a \in A$ and $b \in B$. Consider task 1 is to bechoosen a first element from A, and task 2 is to be chosen a second element from B. These are 'm' ways to perform task 1 and there are 'n' ways to perform task 2, so, by the basic principle of multiplication there are $m \times n$ ways to form an ordered pair (a, b) i.e. $|A \times B| = m. n = |A|.|B|$. Hence the theorem is proved.

Note: (i) $A_1 \times A_2 \times A_3 \times A_4 \ldots \times A_n = \prod_{i=1}^{n} A$

$$A \times A = A^2$$

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1.1.1 Partitions

A partition of a nonempty set is a collection $\mathcal{P}$ of nonempty subsets of A such that

(i) Each element of A belong to one of the sets in $\mathcal{P}$ i.e. $A_1 \cup A_2 \cup \ldots \ldots \cup A_n = A$

(ii) If A_1 and A_2 are distinct elements of $\mathcal{P}$ then $A_1 \cap A_2 = \phi$

The set in $\mathcal{P}$ are called the blocks or cells of the partitions.

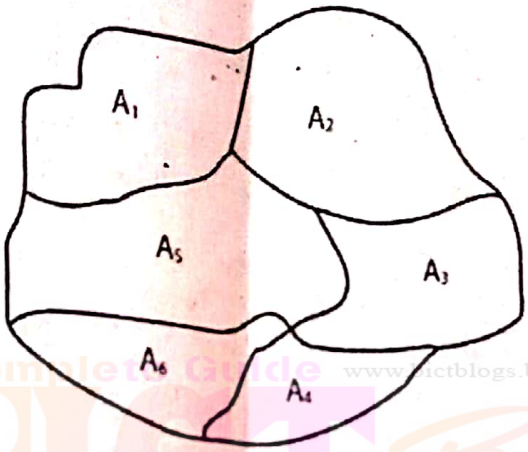

Fig (I)

In the fig (I)

$P = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ into six blocks.

Example – 2

Let $A = \{a, b, c, d, e, f, g, h\}$

$A_1 = \{a, b, c, d\}$

$A_2 = \{a, c, e, f, g, h\}$

$A_3 = \{a, c, e, g\}$

$A_4 = \{b, d\}$

$A_5 = \{f, h\}$

Then $\{A_1, A_2\}$ is not a partition since $A_1 \cap A_2 \neq \phi$. Also, $\{A_1, A_5\}$ is not a partition since $e \notin A_1$ and $e \notin A_5$.

$\therefore$ The collection $\mathcal{P} = \{A_3, A_4, A_5\}$ is a partition of A.

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1.2 Binary Relations and it's Types

Let A and B be two nonempty sets. A binary relation R is defined to be a subset of $A \times B$ from a set A to B. If $(a, b) \in R$ and $R \subseteq A \times B$, then a is related to b, by R i.e. a R b of sets A and B are equal, then we say $R \subseteq A \times A$ is a relation on A. e.g.

(i) Let $A = \{a, b, c\}$, $B = \{r, s, t\}$ then $R = \{(a, r), (b, r), (b, t), (c, s)\}$ is a relation from A to B, which is a subset of $A \times B$.

(ii) Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : a = b\}$

$R = \{(1, 1), (2, 2), (3, 3)\}$ is a relation (equal) on A.

Example - 1

If a set A has n elements, how many relations are there from A to A.

Solution:

If a set A has n elements then $A \times A$ has n^2 elements. So there are 2^{n^2} relations from A to A.

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Example - 2

If a set A has m elements and B has n elements. How many relations are there from A to B and vice versa?

Solution:

There are $m \times n$ elements in $A \times B$; hence there are $2^{m \times n}$ relation from A to B.

Example - 3

Let $A = \{1, 2, 3, 4, 5\}$. Define the following relation R(less than) on A. a R b if and only if $a < b$.

Solution:

$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Example - 4

Let $A = Z^+$, the set of all positive integers define the following relation R on A. a R b if and only if a divides b. Since $4|12$ & $5 + 7$.

So 4 is related to 12, but 5 is not related to 7.

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Domain of binary relation

Consider a binary relation from A to B, the domain of a relation R is the set of all first element of the ordered pairs which belong to R, it is denoted by DOM(R).

Range of binary relation

The range of relation R is the set of all elements in B which are related to some element in A or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN(R).

Example – 5

a) Let A = {1, 2, 3, 4}, B = {a, b, c, d} and R = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (2, d)}

∴ Domain (R) = {1, 2}

Range (R) = {a, b, c, d}

b) A familiar relation on the set Z of integers is "m divides n". A common notation for this relation is to write m| n when m divides n. i.e. 5 | 20 but 3 does not divides 20. So, 3 + 20

c) Consider the set L of lines in the plane perpendicularity, written ⊥ in a relation on L. That is given any pair of lines a and b, either a ⊥ b or a is not perpendicular to b. Similarly, "is parallel to," written as ||, is a relation on L since either a≠ b or a is not parallel to b.

d) Let A be any set. A relation on A is that of equality. {(a, a): a∈ A} which is usually denoted by ' = '. This relation is also called the **identity** or **diagonal relation** on A and it will also be denoted by Δ_A or simply Δ.

e) Let A be any set. Then A×A and ϕ are subsets of A×A and hence are relations on A called the **Universal** and **empty relation** respectively.

1.2.1 Types of Binary Relations

1.2.1.1 Inverse relation

If R ⊆ A × B be a relation from A to B then the inverse relation of R denoted by R^{-1} is a relation from B to A defined by R^{-1} = {(b, a): (a, b) ∈ R}. For e.g.,

If A = {2, 3}, B = {3, 4, 5}, A × B = {(2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)}

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Now, If R is a relation such that R = {(2, 3), (2, 4), (3, 5)}, Domain (R) = {2,3},

Range of R, RAN (R) = {3, 4, 5}

Then, R^{-1} = {(3, 2), (4, 2), (5, 3)}

Domain (R^{-1}) = {3, 4, 5}

Range (R^{-1}) = {2, 3}

Note: *The domain and range of R^{-1} are respectively range and domain of R.*

Example – 1

i) Consider the following relation R on the set of positive integers. Find its inverse if R = {(1,1), (1,2), (1, 3), (2, 1), (3, 1), (3, 2), (2, 3)}

Solution:

R^{-1} = {(1, 1), (2, 1), (3, 1), (1, 2), (1, 3), (2, 3), (3, 2)}

ii) Let A = {1, 2, 3}, B = {x, y, z}, R is the relation from A to B. Where R = {(1, y), (1, z), (3, y)} then find R^{-1}.

Solution:

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R^{-1} = {(y, 1), (z, 1), (y, 3)}

1.2.1.2 Identity relation

A relation R in a set A is said to be identity relation denoted by I_A.

If I_A = {(x, y): x, y ∈ A and x = y} i.e.

I_A = is the set of all those ordered pair (a, b) ∈ A × A in which a = b.

For example,

If A = {1, 2, 3}

I_A = {(1, 1), (2, 2), (3, 3)} is an identity relation in A.

1.2.1.3 Universal relation

A relation R in a set A is said to be universal relation if R coincide with A × A. i.e. R is universal relation if R = A × A. For example,

A = {a, b, c}

R = A × A = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)} is a universal relation.

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1.2.1.4 Void relation (Empty Relation)

A relation R on a set A is said to be void relation if $R = \phi$. For example,

Let $A = \{1, 2, 3\}$ and R is a relation defined by aRb. If $R = \{(x, y) : x + y < 2\}$ b then $R = \phi \subset A \times A$ is a void relation.

1.2.1.5 Total order relation

Consider the relation R on the set A. If it is also the case that for all $a, b \in R$. We have either $(a, b) \in R$ or $(b, a) \in R$ or $a = b$ then the relation R is called total order relation on set A.

1.2.1.6 Circular relation

Consider a binary relation R on a set A. Relation R is called circular if $(a, b) \in R$ and $(b, c) \in R$ implies $(c, a) \in R$.

Example – 1

Consider R is an equivalence relation. Show that R is reflexive and circular.

Solution:

Reflexive : As, the relation R is an equivalence relation. So, reflexivity is the property of an equivalence relation. Hence R is reflexive.

Circular : Let $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ [$\because$ R is transitive]

Since R is equivalence relation so, it is also satisfies symmetric property so,

$\Rightarrow (c, a) \in R$ [$\because$ R is symmetric]

Thus, R is circular.

1.2.2 Some Special Types of Relations

1.2.2.1 Ternary Relation

A ternary relation is a set of ordered triples. In particular, if S is a set, then a subset of S^3 is called a ternary relation on S. For example, if L is a line, then "betweenness" is a ternary relation among points of L.

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Example – 1

Let A = {1, 2, 3, 4, …, 14, 15}. Let R be the ternary relation on A defined by the equation $x^2 + 5y = z$. Write R as a set of ordered triples.

Solution:

Since $x^2 > 15$ for $x > 3$, we need to find solutions for y and z where x = 1, 2, 3. Thus

R = {(1, 1, 6), (1, 2, 11), (2, 1, 9), (2, 2, 14), (3, 1, 14)}

1.2.2.2 n- ary Relations

All the relation, discussed before were binary relations. By an n- ary relation, we mean a set of ordered n- tuples. For any set S_r a subset of the product set S^n is called an n-ary relation on S. In particular, a subset of S^3 is called a ternary relation on S.

In n- ary relation is a set of n- tuples. In particular, if S is a set, then a subset of S^n is called an n-ary relation on S. For example, the set W of solutions of an equation, say $x_1 + 2x_2 + 3x_3 + \dots \dots \dots + nx_n = 0$ may be viewed as an n- ary relation on R.

Example – 1

(a) Let L be a line in the plane. Then "betweenness" is a ternary relation R on the points of L; i.e. (a, b, c) ∈ R if b lies between a and c on L.

(b) The equation $x^2 + y^2 + z^2 = 1$ determines a ternary relation T on the set R of real numbers. That is, a triple (x, y, z) belongs to T if (x, y, z) satisfies the equation. Which means (x, y, z) is the co-ordinates of a point in R^3 on the sphere S with radius 1 and centre at the origin 0 = (0, 0, 0)

Example – 2

Let A = {1, 2, 3, … …, 14, 15}. Let R be the 4- ary relation on A defined by

R = {(x, y, z, t): $4x + 3y + z^2 = t$} write R as a set of 4– tuples.

Note: *we can only have x = 1, 2, 3 thus*

R = {(1, 1, 1, 8), (1, 1, 2, 11), (1, 2, 1, 11), (1, 2, 2, 14), (1, 3, 1, 14), (2, 1, 1, 12), (2, 1, 2, 15),(2, 2, 1, 15)}

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1.3 Different Methods of Representing Relations

1.3.1 Relation as an Order Pairs

Let A and B be two sets. A relation from A to B is a subset of the Cartesian product A×B. Suppose R is a relation form A to B. Then R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$. Every such order pairs is written as a R b and read as 'a is related to b by R or $(a, b) \in R$. If $(a,b) \notin R$, then'a is not related to b' by R.

Example 1 : If A= {1,2,5} and B = {2,4} then Find the binary relation R= {(x,y): x< y}on set A to B. Also, write the relation in order pair form.

Solution: Given two sets are A= {1, 2, 5} and B = {2 , 4}. Now

$A \times B = \{(1,2), (1,4), (2,2),(2,4),(5,2),(5,4)\}$

$R = \{(x,y): x< y\} = \{(1,2), (1,4), (2,4)\}$

1.3.2 Relation as a Matrix

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Let $A = \{a_1, a_2, a_3, \ldots \ldots \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots\ldots \ldots, b_n\}$ are finite sets, containing m and n number of elements respectively. R is a relation from A to B. The relation R can be represented by m × n matrix.

$M_R = [m_{ij}]$ defined as $\quad M_R = [m_{ij}] = \begin{cases} 0 & \text{if } (a_i, b_i) \notin R \\ 1 & \text{if } (a_i, b_i) \in R \end{cases}$

Example: Let A = {1, 2, 3, 4}, B = {a, b, c, d} and R = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (2, d)}

The matrix of relation R is given by

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

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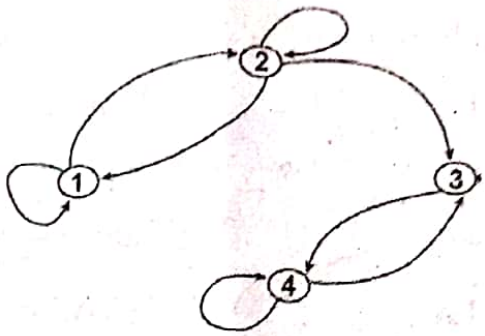
1.3.3 Relation as directed graph

Let P = {1, 2, 3, 4} and

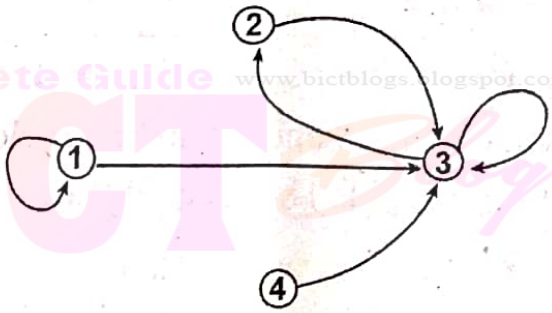
R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 4), (4, 3), (4, 4)} which is relation on P.

If P is a finite set and R is a relation on P, R can be represented as a directed graph asshown in



Example – 1

Find the relations determine by the given figure.



Solution:

From the given figure, R = {(1, 1), (1, 3), (2, 3), (3, 3), (3, 2), (4, 3)}

Example – 2

Let A = {a, b, c, d} and let R be the relation on A that has the matrix

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Construct the diagraph (directed graph) of R and list indegree and outdegree of all vertices.

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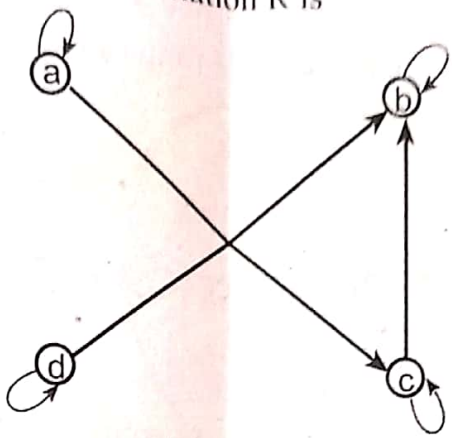
Solution:

The directed graph for the relation R is



The digraph of R is shown in figure. The following table gives the in-degrees and out-degrees of all vertices

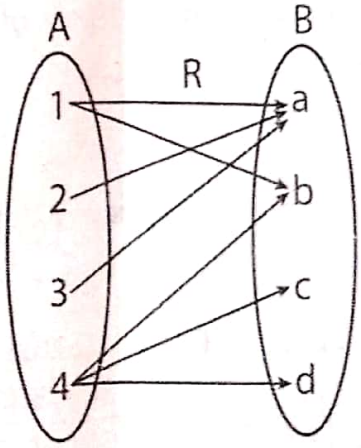
	a	b	c	d
Indegree	1	3	2	1
Outdegree	2	1	2	2

Conclusion: Sum of indegree of vertices is equal to sum of out degree.

Note: A loop is contributed indegree 1 outdegree 1 in digraph.

1.3.4 Relations as Arrow diagram

Let A and B are finite sets and R is a relation from A to B. Relation R can be represented as an arrow diagram as follows. Draw two ellipse for the set A and B. Write down the element of A and element of B, column wise in these ellipse a is related to b and a $\in$A and b $\in$B.



Here, A = {1, 2, 3, 4}, B = {a, b, c, d}

and R = {(1, a), (2, a), (3, a), (1, b), (4, b), (4, c), (4, d)}

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1.3.5 Relation as a table

If A and B are two finite sets and R be a relation from A to B. Relation R can be represented in table form as follows.

	x	y	z	k
1	x	x		
2			x	
3			x	
4				x

$R = \{(1, x), (1, y), (2, z), (3, z), (4, k)\}$

Make the table which contains rows equivalent to elements of A and columns equivalent to element of B. Then place cross (×) in the boxes which represents relation of elements on set A to set B.

1.3.6 Relation as a Graph

First we consider a relation S on the set R of real numbers; that is, S is a subset of $R^2 = R \times R$ since R^2 can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is called the graph of the relation. Frequently, the relation S consists of all ordered pairs of real numbers which satisfy some given equation.

$R = \{(x, y) : f(x, y) = 0\}$

Usually we identify the relation with the equation $f(x, y) = 0$, similarly we can plot the ordered pair of R in graph (Rectangular plane or Cartesian plane).

Example – 1

Let A = R, the set of real numbers, we define the following relation R on A.

or $(x, y) \in R$ if and only if x and y satisfy the equation

$\frac{x^2}{4} + \frac{y^2}{9} = 1$, The set R consists of all points on the ellipse as shown in figure.

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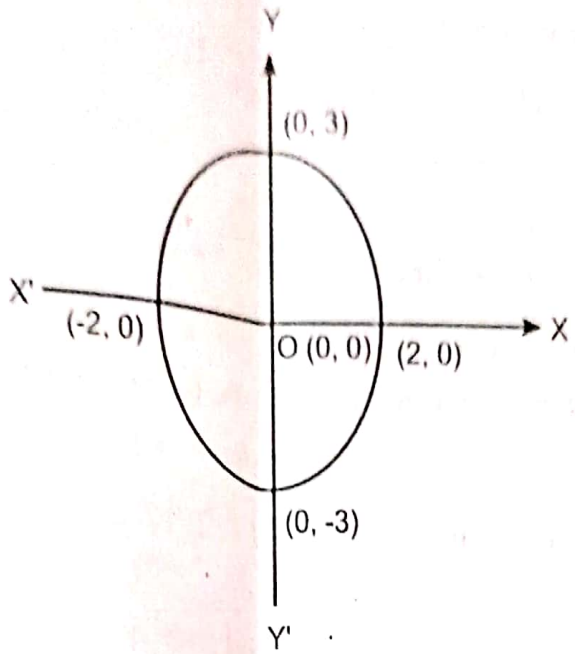
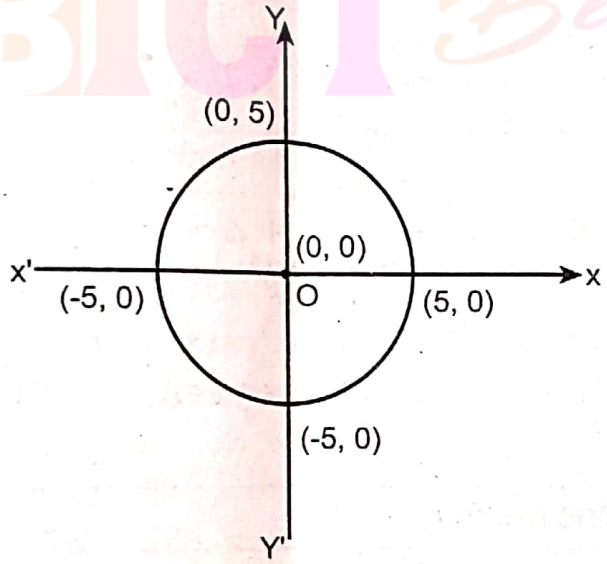


Example – 2

Consider the relation S defined by the equation $x^2 + y^2 = 25$ i.e. S consists of all ordered pairs (x, y) which satisfy the given equation. The graph of the equation is a circle having its centre at the origin and radius 5.



$x^2 + y^2 = 25$

1.4 Properties of relations

Properties of the relation tells the nature or type of the relation. There are many properties of relation i.e. reflexive, irreflexive, symmetric, asymmetric antisymmetric, transitive and equivalence relation.

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1.4.1 Reflexive relation

Consider a binary relation R on a set A. Relation R is called a reflexive relation if for every ain set A, R = {(a, a) ;∀ a ∈ A}

1.4.2 Irreflexive relation

Consider a binary relation R on a set A. Relation R is called irreflexive relation if for every ain set A, (a, a) ∉R, i.e. (a, a) ∉R, ∀ a∈A.

Example – 1

Let A = {1, 2} and R = {(1, 1), (1, 2) (2, 2)} is the relation reflexive or irreflexive?

Solution:

The relation is reflexive as for every a∈A, (a, a)∈ R.

Here, (1, 1), (2, 2)∈ R, So, relation R is reflexive. for 1, 2 ∈ A.

The relation R is not irreflexive.

Example – 2

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Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 1), (3, 1), (4, 1)} is the relation reflexive or irreflexive?

Solution:

The relation R is not reflexive i.e. (1, 1), (2, 2) (3, 3) (4, 4) ∉ R ∀ 1, 2, 3 ∈ A.

The relation is irreflexive as for every a∈A, (a, a) ∈ R.

1.4.3 Symmetric relation

Consider a binary relation R on a set A. Relation R is said to be a symmetric relation if for every (a, b) ∈ R, (b, a) ∈ R for all a, b ∈ A.

1.4.4 Asymmetric relation

Consider a binary relation R on a set A. Relation R is called an asymmetric relation if for every (a, b) ∈ R implies (b, a) ∉ R.

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Example - 1

Let A = {1, 2, 3} and R = {(1, 2), (2, 1), (3, 2), (2, 3)}. Is the relation symmetric or asymmetric?

Solution:

The relation is symmetric as for every (a, b)∈ R, we have (b, a) ∈ R. Here,

$(1, 2) \in R \Rightarrow (2, 1) \in R$.

$(3, 2) \in R \Rightarrow (2, 3) \in R$

∴ The given relation is symmetric. The relation is not asymmetric as we have (b, a) ∈ R for some (a, b) ∈ R.

Example - 2

Let A = {4, 5, 6} and R= {(4, 3), (5, 6), (6, 4)} is the relation symmetric or asymmetric?

Solution:

The relation is not symmetric as for every (a, b)∈ R we have (b, a) ∉ R i.e. (4, 3) ∈ R but (3, 4) ∉ R.

But, The relation is asymmetric as for every (a, b) ∈ R, implies (b, a)∉ R.

1.4.5 Anti–symmetric relations

Consider a binary relation R on a set A. Relation R is called antisymmetric relation if (a, b) ∈ R implies that (b, a)∉R unless a = b. In other words, we can say if (a, b) and (b, a) ∈R then a = b.

Example - 1

Let A = {4, 5, 6} and R = {(4, 4), (4,5), (5, 4), (5, 6), (4, 6)}. Is the relation R antisymmetric?

Solution:

The relation R is not antisymmetric as 4 ≠ 5, but, (4,5) ∈ R and (5, 4) ∈ R.

Example - 2

Let A = {1, 2, 3} and R = {(1, 1), (2, 2)}. Is the relation R antisymmetric?

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Solution:

The relation R is antisymmetric as a = b where (a, b) ∈ R and (b, a) ∈ R

i.e. 1 = 1, 2 = 2, (1, 1)∈ R, (2, 2)∈ R.

1.4.6 Transitive relation

A relation R in a set S is said to be transitive if (a, b) ∈R and (b, c) ∈ R ⇒ (a, c) ∈R

or, aRb and bRc⇒aRc

i.e. If R is transitive if a is related to b and b is related to c imply that a is related to c.

Example – 1

Let A= {1, 2, 3}, R = {(1, 2), (2, 1), (1, 1), (2, 2)} is the relation is transitive?

Solution:

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Yes, the relation R is transitive as for every (a, b), (b, c) ∈ R ⇒ (a, c) ∈ R.

i.e. (1, 2), (2, 1) ∈ R ⇒ (1, 1) ∈ R.

Example – 2

Let A = {1, 2, 3} and R = {(1, 2), (3, 2), (2, 1), (1, 3)} is the relation R transitive?

Solution:

No, the relation R is not transitive because for (3, 2)∈R and (2, 1) ∈ R, but (3, 1) ∉ R.

1.4.7 Equivalence relation (RST relation) and partition

a. Equivalence relation (RST relation)

A relation R in a set S is said to be an equivalence relation if

(i) R is reflexive i.e. aRa∀ a ∈ S **or** (a, a) ∈ R, ∀ a ∈ S

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(ii) R is symmetric i.e. aRb, bRa, $\forall a, b \in S$ or, $(a, b) \in R, (b, a) \in R \ \forall a, b \in S$

or $(a, b) \in R, (b, a) \in R, \forall a, b \in S$

(iii) R is transitive i.e. aRb and bRc$\Rightarrow$aRc$\forall$ a, b, c $\in$ S.

or$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R \ \forall a, b, c \in S$

Example – 1

Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4,2), (4, 4)} Show that R is an equivalence relation:

Solution:

(i) Relation R is reflexive as (1, 1), (2, 2), (3, 3) and (4, 4) $\in$ R $\forall$ 1, 2, 3, 4 $\in$ A

(ii) Relation R is symmetric because when ever $(a, b) \in R, (b, a) \in R$

i.e. (1, 3) and (3, 2) $\in$ R

(2, 4) and (4, 2) $\in$ R

(iii) Transitive: Relation R is transitive because (a, b) and (b, c)$\in$R $\Rightarrow$(a, c)$\in$ R.

i.e. (3, 1) $\in$ R and (1, 3) $\in$ (3, 3) $\in$ R $\Rightarrow$ (3, 3) $\in$ R.

(2, 4) $\in$R and (4, 2) $\in$ R $\Rightarrow$ (2, 2) $\in$ R.

Since, $(1, 3) \in R, (3, 1) \in R \Rightarrow (1, 1) \in R$

$(2, 4) \in R, (4, 2) \in R \Rightarrow (2, 2) \in R$

$(3, 1) \in R, (1, 3) \in R \Rightarrow (3, 3) \in R$

$(4, 1) \in R, (2, 4) \in R \Rightarrow (4, 4) \in R \ \forall 1, 2, 3, 4 \in R$

So, the given relation r satisfies each of the properties of equivalence relation. So, the given relation R is equivalence.

1.4.8 Partial order relation

A relation R on a set A is called a partial order relation if it satisfies the following three properties.

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(i) Relation R is reflexive i.e $aRa \forall a \in A$.

or$(a, a) \in R \ \forall a \in A$]

(ii) Relation R is anti–symmetric aRb and bRa$\Rightarrow a = b$

or$[(a, b) \in R, (b, a) \in R \Rightarrow a = b]$

(iii) Relation R is transitive aRb and bRc$\Rightarrow$aRc.

or $[(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R]$

Example - 1

Show whether the relation $R = \{ (x, y) : x \geq y\}$ defined on the set of positive integers is a partial order relation.

Solution:

Consider the set $A = \{1, 2, 3, 4\}$ containing four positive integers. Find the relation for this set such as $R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$

Reflexive: The relation is reflexive as for every $a \in A$, $(a, a) \in R$. i.e. (1, 1), (2, 2), (3, 3) and $(4, 4) \in R$

Anti–symmetric: The relation is anti–symmetric as whenever (a, b) and $(b, a) \in R$, we have $a = b$, Since the relation (1, 1), (2, 2), (3, 3), (4, 4) $\in R$ & 1=1, 2= 2 , 3= 3 and 4=4.

Transitive: The relation is transitive whenever (a, b) and $(b, c) \in R$, we have $(a, c) \in R$. e.g. $(4, 2) \in R$ and $(2, 1) \in R$ implies $(4, 1) \in R$

$\therefore$ Since the relation is Reflexive, Anti–symmetric and Transitive. Therefore, The given relation is a partial order relation.

1.5 Boolean Matrix Representation of Relations

A matrix whose elements are either zero or one (i.e. 0 or 1) is called Boolean matrix.

Example – 1

$$\text{If } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$\therefore$ Matrix A is Boolean Matrix.

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1.5.1 Boolean Matrix operation

A Boolean matrix is an $m \times n$ matrix whose elements are either zero or one. There are three operations in Boolean matrix.

(i) $A \vee B$ (ii) $A \wedge B$ (iii) $A \odot B$

(i) Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ Boolean matrices.

Then $A \vee B = C = (C_{ij})$ then join of A and B written as

$$C = [C_{ij}] = \begin{cases} 1 & \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if } a_{ij} \text{ and } b_{ij} \text{ both zero} \end{cases}$$

Example – 1

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{4\times3}$$

$$A \vee B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \vee \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{4\times3}$$

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{4\times3}$$

(ii) $A \wedge B$: Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ Boolean matrices then

$A \wedge B = (d_{ij}) = D$, the meet of A and B then

$$D = d_{ij} = \begin{cases} 1, \text{ if } a_{ij} \text{ and } b_{ij} \text{ are both } 1 \\ 0, \text{ if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

Example – 2

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{4\times3}$$

$$\therefore \quad A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3} \wedge \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{4\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{4\times3}$$

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1.9.3 Boolean product

Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix then the Boolean product of A & B denoted by $A \odot B$.

The Boolean product of A and B, denoted by $A \odot B$ is the $m \times n$ matrix with $(i, j)^{th}$ entry $[c_{ij}]$ where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \dots \vee (a_{ik} \wedge b_{kj})$

$$\therefore \quad C = [c_{ij}] = \begin{cases} 1 \text{ if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for } 1 \le k \le p \\ 0 \text{ otherwise} \end{cases}$$

Example - 1

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 3}$$

Compute: $A \odot B$

$$A \odot B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3} \odot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}_{3\times 4}$$

$$= \begin{bmatrix} (1\wedge1)\vee(1\wedge0)\vee(0\wedge1) & (1\wedge0)\vee(1\wedge1)\vee(0\wedge0) & (1\wedge0)\vee(1\wedge1)\vee(0\wedge1) & (1\wedge0)\vee(1\wedge0)\vee(0\wedge1) \\ (0\wedge1)\vee(1\wedge0)\vee(0\wedge1) & (0\wedge0)\vee(1\wedge1)\vee(0\wedge0) & (0\wedge0)\vee(1\wedge1)\vee(0\wedge1) & (0\wedge0)\vee(1\wedge0)\vee(0\wedge1) \\ (1\wedge1)\vee(1\wedge0)\vee(0\wedge1) & (1\wedge0)\vee(1\wedge1)\vee(0\wedge0) & (1\wedge0)\vee(1\wedge1)\vee(0\wedge1) & (1\wedge0)\vee(1\wedge0)\vee(0\wedge1) \\ (0\wedge1)\vee(0\wedge0)\vee(1\wedge1) & (0\wedge0)\vee(0\wedge1)\vee(1\wedge0) & (0\wedge0)\vee(0\wedge1)\vee(1\wedge1) & (0\wedge0)\vee(0\wedge0)\vee(1\wedge1) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee0\vee0 & 0\vee1\vee0 & 0\vee1\vee1 & 0\vee0\vee0 \\ 0\vee0\vee0 & 0\vee1\vee0 & 0\vee1\vee0 & 0\vee0\vee0 \\ 1\vee0\vee0 & 0\vee1\vee0 & 0\vee1\vee0 & 0\vee0\vee0 \\ 0\vee0\vee1 & 0\vee0\vee0 & 0\vee0\vee1 & 0\vee0\vee1 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Example – 2

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$M_R^2 = M_R \odot M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.5.3 Complement of Boolean matrix

The complement of Boolean matrix A, denoted by A', obtained by taking the one's complement of each element A i.e. by replacing 0 with 1 and 1's with zero.

Example : if $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ then Find A'.

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Solution: Given matrix is $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ then the complement of A is denoted by

$$A' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

1.6 Composition of two relations

Let R_1 and R_2 be the relations from the sets A to B and B to C respectively then the composite of R_1 and R_2 is a relation from A to C denoted by R_2oR_1, $R_2oR_1 = \{(a, c) : (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$.

1.6.1 Matrix of composition relations

If we represent composition of relation in matrix form it is called matrix of composition relation. For example"

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Let, $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$

$C = \{8, 9)$ and R_1 be a relation from A to B.

R_2 be a relation from B to C then R_2oR_1 be relation from A to C.

Example – 1

Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4) (3, 2)\}$ and $S = \{(1, 4), (2, 3), (3, 1), (4, 1)\}$.

Since, $(1, 2) \in R, (2, 3) \in S \Rightarrow (1, 3) \in SoR$.

Since, $(1, 1) \in R, (1, 4) \in S \Rightarrow (1, 4) \in SoR$

Since, $(1, 3) \in R, (3, 1) \in S \Rightarrow (1, 1) \in SoR$

Since, $(2, 4) \in R, (4, 1) \in S \Rightarrow (2, 1) \in SoR$

Since, $(3, 2) \in R, (2, 3) \in S \Rightarrow (3, 3) \in SoR$

$\therefore$ $SoR = \{(1,4), (1,3), (1,1), (2,1), (3,3)\}$

$$M_{SOR} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

1.6.2 Properties of composition of relations

Theorem: 1 **If R is a relation from A to B and S be a relation from B to C, then**

$$M_{SOR} = M_R \odot M_S$$

Solution:

Let, A, B and C finite sets with n, p and m elements respectively, let R be a relation from A to B and S be a relation from B to C. Then R and S have Boolean matrices M_R and M_S with respective sizes $n \times p$ and $p \times m$. Thus $M_R \odot M_S$ can be computed and it equals to M_{SOR}. For this Let $A = \{a_1, a_2 \ldots \ldots \ldots \ldots a_n\}$, $B = \{b_1, b_2 \ldots \ldots \ldots \ldots b_P\}$ and $C = \{c_1 \ldots \ldots \ldots \ldots c_m\}$ also, suppose that $M_R = [r_{ij}]$, $M_S = [s_{ij}]$ and $M_{SOR} = [t_{ij}]$ then $t_{ij} = 1$ if and only of $(a_i, c_j) \in SoR$. This condition is identical to the condition needed for $M_R \odot M_S$ to have a 1 in position i & j and thus $M_{SoR} = M_R \odot M_S$ Hence, proved.

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Example – 1

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$M_R \odot M_S =$

$$\begin{bmatrix} (1\wedge0)\vee(1\wedge0)\vee(1\wedge1)\vee(0\wedge1) & (1\wedge0)\vee(1\wedge0)\vee(1\wedge0)\vee(0\wedge0) & (1\wedge0)\vee(1\wedge1)\vee(1\wedge0)\vee(0\wedge0) & (1\wedge1)\vee(1\wedge0)\vee(1\wedge0)\vee(0\wedge0) \\ (0\wedge0)\vee(0\wedge0)\vee(0\wedge1)\vee(1\wedge1) & (0\wedge0)\vee(0\wedge0)\vee(0\wedge0)\vee(1\wedge0) & (0\wedge0)\vee(0\wedge1)\vee(0\wedge0)\vee(0\wedge0) & (0\wedge1)\vee(0\wedge0)\vee(0\wedge0)\vee(0\wedge0) \\ (0\wedge0)\vee(1\wedge0)\vee(0\wedge1)\vee(0\wedge1) & (0\wedge0)\vee(0\wedge0)\vee(0\wedge0)\vee(0\wedge0) & (0\wedge0)\vee(1\wedge1)\vee(0\wedge0)\vee(0\wedge0) & (0\wedge1)\vee(1\wedge0)\vee(0\wedge0)\vee(1\wedge0) \\ (0\wedge0)\vee(0\wedge0)\vee(0\wedge1)\vee(0\wedge1) & (0\wedge0)\vee(0\wedge0)\vee(0\wedge1)\vee(0\wedge1) & (0\wedge0)\vee(0\wedge1)\vee(0\wedge0)\vee(0\wedge0) & (0\wedge1)\vee(0\wedge1)\vee(0\wedge0)\vee(1\wedge0) \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee0\vee1\vee0 & 0\vee0\vee0\vee0 & 0\vee1\vee0\vee0 & 1\vee0\vee0\vee0 \\ 0\vee0\vee0\vee1 & 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 \\ 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 & 0\vee1\vee0\vee0 & 0\vee0\vee0\vee0 \\ 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 & 0\vee0\vee0\vee0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad M_{SOR} = S_R \odot M_S$

Example – 2 If two relations R_1 and R_2 are such that

$R_1 = \{(1, 3), (2, 1), (3, 4)\}$ & $R_2 = \{(1, 2), (2, 3), (3, 4)\}$

Then

i) find $R_2 o R_1$.

ii) Domain $(R_2 o R_1)$ = domain (R_1)

$(1, 3) \in R_1$ and $(3, 4) \in R_2 \Rightarrow (1, 4) \in R_2 o R_1$

$(2, 1) \in R_1$ and $(1, 2) \in R_2 \Rightarrow (2, 2) \in R_2 \text{ o } R_1$

$(3, 4) \in R_1$ and $(4, 3) \in R_2 \Rightarrow (3, 3) \in R_2 \text{ o } R_1$

Hence $R_2 o R_1 = \{(1, 4), (2, 2), (3, 3)\}$

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Theorem: 2 **If R is a relation from A to B, S a relation from B to C, and T a relation from C to D. Then prove that**

(SoR) = (ToS) oR.

Proof: The relation R, S, T are determined by Boolean matrices M_R, M_S and M_T respectively. Then, the matrix of the composition is the Boolean matrix product (i.e. $M_{SOR} = M_R \odot M_S$.)

$$M_{To(SoR)} = M_{SoR} \odot M_T = (M_R \odot M_S) \odot M_T$$

$$M_{(TOS)oR} = M_R \odot M_{TOS} = M_R \odot (M_S \odot M_T)$$

Since, the Boolean matrix operation is associative. So, $(M_R \odot M_S) \odot M_T = M_R \odot (M_S \odot M_T)$

$\therefore$ $M_{To(SoR)} = M_{(ToS)oR}$

Therefore, To (SoR) = (ToS) oR.

Theorem : 3 Let A, B and C be sets, R a relation from A to B and S a relation from B to C then $(SoR)^{-1} = R^{-1}oS^{-1}$.

Proof: Let $c \in C$ and $a \in A$, then $(c, a) \in (SOR)^{-1}$ if and only if $(a, c) \in (SOR)$ that is if and only if there is $a, b \in B$ with $(a, b) \in R$ and $(b, c) \in S$. Finally this is equivalent to $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$. i.e. $(c, a) \in R^{-1}oS^{-1}$.

1.7 Operation on relations on set

1.7.1 Partial order set

A relation R on a set A is called a partial order if R is reflexive, anti-symmetric and transitive. The set A together with the partial order R is called a partially ordered set or simply a Poset and we will denote this Poset by (A, R)

a. **Dual of the Poset:** The Poset (A, R^{-1}) is called the dual of the Poset(A, R) and the partial order R^{-1} is called the dual of the partial order R.

b. **Comparable:** If $(A, \leq)$ is a Poset, the element a and b of A are said to be comparable if $a \leq b$, or $b \leq a$.

c. **Linearly ordered set:** If every pair of element in a Poset A is comparable, then A is linearly ordered set and the partial order is called a linear order. It is also, called A is a chain.

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1.7.2 Closure properties

In the following section, different types of closure properties will be defined with the help of different examples.

1.7.2.1 Reflexive closure (R_F)

Consider a relation R on A. Relation R_F is called reflexive closure of R if R_F is the smallest relation containing R, having the reflexive property. i.e. $R_F = R \cup \Delta$ where Δ is a diagonal relation.

Example - 1

Consider the relation R on A = {7, 8, 9} defined by R = {(7, 8), (7, 9), (8, 8), (9, 7)}. Find the reflexive closure of R.

Solution:

The smallest relation containing R having the reflexive property is

$R \cup \Delta = \{(7, 7), (7, 8), (7, 9), (8, 8), (9, 7), (9, 9)\}$

$$M_{4\times A} = \begin{array}{c} \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccc} 7 & 8 & 9 \end{array} \\ \begin{bmatrix} (7,7) & (7,8) & (7,9) \\ (8,7) & (8,8) & (8,9) \\ (9,7) & (9,8) & (9,9) \end{bmatrix} \end{array}$$

$\Delta = \{(7, 7), (8, 8), (9, 9)\}$

$R = \{(7, 8), (7, 9), (8, 8), (9, 7)\}$

$R \cup \Delta = \{(7, 7), (7, 8), (7, 9), (8, 8), (9, 7), (9, 9)\}$

Example - 2

Let A = {k, l, m} let R is a relation on A defined by

R = {(k, k), (k, l), (l, m), (m, k)}. Find the reflexive closure of R.

Solution:

$R \cup \Delta$ is the smallest relation having reflexive property.

Hence $R_k = R \cup \Delta = \{(k, k), (k, l), (l, l), (l, m), (m, m), (m, k)\}$

Where, $\Delta = \{(k, k), (l, l), (m, m)\}$

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1.7.2.2 Symmetric closure

Consider a relation R on set A. Relation R_S is called the symmetric closure of R, if R_S is the smallest relation containing R_I having the symmetric property. The relation $R_S = R \cup R^{-1}$ is the smallest symmetric relation containing R.

Example – 3

Consider the relation R on A = {4, 5, 6, 7} defined by R = {(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)} find the symmetric closure of R.

Solution:

The smallest relation R having the symmetric property is $R \cup R^{-1}$.

Here, $R = \{(4, 5), (5, 5), (5, 6), (6, 7), (7, 4), (7, 7)\}$

$R^{-1} = \{(5, 4), (5, 5), (6, 5), (7, 6), (4, 7), (7, 7)\}$

i.e. $R_S = R \cup R^{-1}$

$= \{(4, 5), (5, 4), (5, 5), (5, 6), (6, 5), (6, 7), (7, 6), (7, 4), (4, 7), (7, 7)\}$

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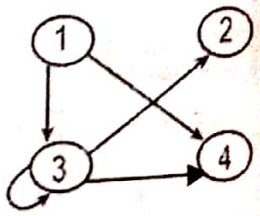
1.8 Transitive Closure and Warshall's Algorithm

1.8.1 Transitive closure

Consider a relation R on a set A. The transitive closure R of a relation R is the smallest transitive relation containing R.

Example – 4

Find the transitive closure R^{∞} of the relation R on A = {1, 2, 3, 4} defined by the directed graph.



Solution:

There is a path from 1 to points 2, 3, and 4 there is no path from 2 to any point. There is a path from 3 to the point 2, 3, and 4. There is no path from 4 to any point. Thus transitive closure.

$R^* = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$

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1.8.2 Warshall's Algorithm

Let R be a relation on a set $A = \{a_1, a_2, \ldots \ldots \ldots \ldots, a_n\}$. If $\{x_1, x_2, \ldots \ldots \ldots \ldots x_m\}$ is a path in R. Then any vertices other than x_1 and x_n are called the interior vertices of the path. Now, for $1 \le k \le n$, we define a Boolean matrix W_k as follows: W_k has a 1 in position i, j if and only if there is a path from a_i to a_j in R whose interior vertices if any, come from the set $\{a_1, a_2, \ldots \ldots \ldots \ldots, a_k\}$.

Since, any vertex must come from the set $\{a_1, a_2, \ldots \ldots \ldots \ldots \ldots, a_n\}$ it follows that the matrix W_n has a 1 in position i, j if and only if some path in R connects a_i with a_j. In other words $W_n = M_R^{\infty}$. If we define w_0 to be M_R, then we will have a sequence $w_0, w_1 \ldots \ldots \ldots \ldots \ldots, w_n$ whose first term is M_R and whose last term is $M_R\infty$. If we define w_0 to be M_R, then we will have a sequence $w_0, w_1, \ldots \ldots \ldots \ldots \ldots w_n$ whose first term is M_R and whose last term is $M_R\infty$. We will show how to compute each matrix W_k from the previous matrix W_{k-i}. Then we can begin with the matrix of R and proceed one step at a time until, in a steps. we reach the matrix of R^{∞}. This procedure is called Warshall's algorithm. The matrices W_k are different from the powers of the matrix M_R and this different result in a considerable savings of steps in the computation of the transitive closure of R.

Example – 5

Consider the relation R defined by the Matrix M_R

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$$

Find the transitive closure of R by Warshall's algorithm.

Solution:

Let $W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times4}$ and $n = 4$ (since it is 4×4 matrix) so, first we find w_1 so that $k = 1$, w_0 has 1's in location 2 of

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column 1 and location 2 of row 1. Thus w_1 is just w_0 a new 1 in position (2, 2).

$$w_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

Now, we compute w_2 so that $k = 2$, we consult column 2 and row 2 of w_1. Matrix w_1 has 1's in location 1 and 2 of column 2 and location 1, 2, and 3 of row 2. Thus to obtain w_2 we write 1's in position (1, 1) (1, 2) (1, 3) (2, 1) (2, 2) and (2, 3) of matrix w_1. (If 1's are n of already there) we get, $w_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Similarly, by proceeding continuously, we see that column 3 of w_2 has 1's in location 1 and 2, and row 3 of w_2 has a 1 in location 4. To obtain w_3, must put, 1's in positions (1, 4) and (2, 4) of w_2. So, $w_3 =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, w_3 has 1 in location 1, and 3 of column 4 and no 1's in row 4. So, no new 1 is added and $M_{R\infty} = w_4 = w_3$. Then we get,

$$w_4 = w_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Required transitive closure of R i.e, $M_{R\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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Exercise – 1

1. If a set A has 'm' elements and the set B has' n' elements then how many relations are there from A to B.
2. Consider the relation R from A to B. If A = {1, 2, 3}, B = {7 ,8} and R = {(1, 7) , (2, 7) ,(1, 8) , (3, 8)}. Find i) R^{-1} ii) R^c
3. The relation R on the set {1, 2, 3, 4, 5} is defined by the rule $(x, y) \in R$ if $3|(x - y)|$. Find the element of R ii) the element of R^{-1} iii) the domain of R iv) the range of R v) the domain of R^{-1} vi) the range of R^{-1}
4. The relation of R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if $x + y \leq 6$. i) Find the element of R ii) the element of R^{-1} iii) the domain of R iv) the range of R v) the domain of R^{-1} vi) the range of R^{-1}
5. What is binary relation R on A if no element of A is related to itself?
6. When is a relation R on a set A is not anti-symmetric?
7. If A = { a, b } find the value of $n(A \times A)$.
8. If A = { a, b, c} then find the value of P(A)
9. What is Cartesian product of two nonempty sets and how is it denoted?
10. Distinguish the equivalence relation and partial order relation.
11. If A = {1, 2} and B = {x, y, z}, C = {3, 4} find the value of $A \times B \times C$
12. Show that the relation of divisibility is a partial order on the set of positive integers.
13. Consider the relation $\perp$ of perpendicularity on the set L of lines in the Euclidean plane. Is $\perp$ and equivalence relation? Give reasons.
14. Let A = {1, 2, 3,... ... 14, 15} Let R be the ternary relation on A defined by the equation $x^2 + 5y = z$ write R as a set of ordered triples
15. The relation R on a set A = {1, 2, 3, 4} is defined by R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4}. Draw the directed graph of R and R^{-1}.

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16. Prove that the relation R defined on the set of positive integers $(x, y) \in R$ if $x - y$ is divisible by 5 is an equivalence relation.

17. Let R is an equivalence relation on the set $A = \{a, b, c, d\}$ defined by partitions $P = \{\{a, d\}, \{\{b, c\}\}$ Determine the elements of equivalence relation and also find the equivalence classes of R.

18. Let A $\{1, 2, 3, 6\}$ if for $x, y \in A$ and $R = \{(x, y) : x \leq y\}$ and $S = \{(x, y)$: x divides y$\}$ write R and S are sets and find $R \cup S$ and $R \cap S$.

19. Let $A = \{0, 1, 2, 3, 4\}$. Show that the relation $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)$ is an equivalence relation.

20. If R and S be the relations on $\{1, 2, 3\}$ defined by $R = \{(1, 1), (1, 2), (3, 4), (4, 2)\}$ and $S = \{(1,1), (2,1), (3, 1), (4, 4), (2, 2)\}$. Find R o S and S o R.

21. Let $A = \{0, 1, 2, 3\}$, $R = \{(x, y) : x + y = 3\}$, $S = \{(x, y) : 3 | (x + y)\}$, $T = \{(x, y) : \max(x, y) = 3\}$ Compute (a) Ro T (b) T o R (c) S o S.

22. R and S be the following relations on $A = \{a, b, c, d\}$ defined by $R = \{(a, a), (a, c), (c, b), (c, d), (d, b)\}$ and $S = \{(b, a), (c, c), (c, d), (d, a)\}$ Find (a) R o S b) S o R (c) R o R d) S o S.

23. Let $A = \{1, 2, 3\}$ and let R and S be the relations on A, Such that

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } M_s = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) $M_{R'}$ b) M_R-1 c) $M_{R \cup S}$ d) $M_{R \cap S}$

24. Given relational matrices M_R and M_S find M_{RoS}, M_R-1, M_S-1, $M_{(RoS)}^{-1}$ and Show that:

$$M_{(RoS)}^{-1} = M_S^{-1} o R^{-1} = M_s^{-1} o M_R^{-1}$$

25. Use Warshall's algorithm to find the transitive closure of the following relations on $A = \{1, 2, 3, 4\}$

i) $\{(1,2), (2,1), (2,3), (3,4), (4, 1)\}$

ii) $\{(1,2), (2,1), (2,3), (3,4)\}$

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26. Draw the diagraph for the relation ' Congruent modulo 3' on the set {2, 3, 4, 6, 7, 9} and hence show that the relation is symmetric, and transitive.

27. Define binary relation form a set A to a set B. Let R = {(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)} be relation on a set A = {a, b, c, d}. Then determine whether the relation is reflexive, irreflexive, symmetric, asymmetric, anti–symmetric or transitive.

28. Let A = {1, 2, 3, 4 … … 14, 15}. Let R be the 4 –ary relation on A defined by R = {(x, y, z, t): $4x + 3y + z^2 = t$} Find R.

29. Suppose R is any relation on A. Show that $R \cup R^{-1}$ is symmetric.

30. If R is a relation on A = {1, 2, 3, 4, 6} defined by ${}_aR_b$ i.e (a, b) ∈ R if and only if a is multiple of b, find the domain, the range, and the inverse of the relation R. Also verify $(M_R)^T = M_{R^{-1}}$.

31. If {{a, c, e}, {b, d, f}} is a partition of the set A = {a, b, c, d, e, f} determine the corresponding equivalence relation R.

32. Show that the set of odd positive integers is countable.

33. Define partial order relation on a set. If P(S) is the set of all subsets of a given set S, show that the inclusion relation '⊆ ' is a partial ordering on the power set P(S).

34. Define composition of relations. Let A = {1, 2, 3}, B = {a, b, c, d} and C = {x, y, z, w} be three sets. If R = {(1,a), (1,b), (3, c), (3, d)} and S = {(a, x), (b, x), (b, z), (d, w)} are two relations from A to B and B to C respectively, verify that i) $M_{SoR} = M_R o M_S$.

35. Define Cross Product and also prove that: $A \times (B \cap C) = (A \times B) \cap (A \times C)$

36. Prove: $(A \cup B) \times C = (A \times C) \cup (B \times C)$

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Answers

1. 2^{mn}

2. i) $R^{-1} = \{(7, 1), (7, 2), (8, 1), (8, 3)\}$

   ii) $R^c = \{(3,7), (2,8)\}$

3. i) $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

   ii) $R^{-1} = \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

   iii) Dom (R) = {1, 2, 3, 4, 5} iv) Ran (R) = {1, 2, 3, 4, 5} v) Dom (R^{-1}) = Dom (R)

   vi) Ran (R) = Ran (R^{-1})

4. $R = R^{-1} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)\}$. Dom (R) = Ran (R) = Dom (R^{-1}) = Ran (R^{-1}) = {1, 2, 3, 4, 5}

6. $(a, b) \in R$, $(b, a) \in R$ but $a \neq b$

7. 4

8. $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b\}\}$ **9.** $A \times B$

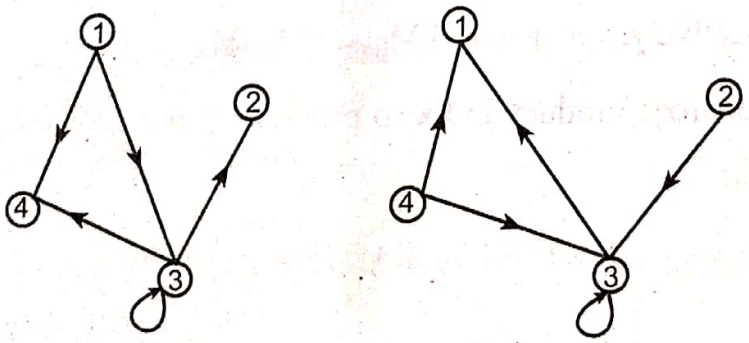
11. {(1, x, 3), (1, x, 4), (1, y, 3), (1, y, 4), (1, z, 3), (1, z, 4), (2, x, 3), (2, x, 4), (2, y, 3), (2, y, 4), (2, z, 3), (2, z, 4)}

13. No

14. R = { (1, 1, 6), (1, 2, 11), (2, 1, 9), (2, 2, 14), (3, 1, 14)}

15. R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)}

    R^{-1} = {(3, 1), (4, 1), (2, 3), (3, 3), (4, 3)}



(R) (R^{-1})

17. R = {(a, a), (d, d), (a, d), (d, a), (b, b), (c, c), (b, c), (c, b)}, [a] = [d] = {a, d}, [b] = [c] = {b, c}

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18. $R = \{(1,1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 3), (2, 6), (3, 3), (3, 6), (6, 6)\}$,

$S = \{(1, 2), (1, 3), (1, 6), (2, 6), (3,6)\}$,

$R \cup S = \{ (1,1), (1,2), (1, 3), (1, 6), (2, 2), (2, 3), (2,6), (3, 3), (3, 6), (6, 6) \}$,

$R \cap S = \{(1, 2), (1, 3), (1, 6), (2, 6), (3, 6)\}$

21. i) $R \circ T = \{(0, 0), (0, 1), (0, 2), (0, +3), (1, 3), (2, 3), (3, 3)\}$

ii) $T \circ S = \{(0,0), (1,0), (2,0), (3,0), (3,1), (3,2), (3, 3)$

iii) $S \circ S = \{(0, 0), (0, 3), (1, 1), (2, 2), (3, 0), (3, 3)\}$

22. $R \circ S = \{(a, c), (a, d), (c, a), (d, a)\}$

$S \circ R = \{(b, a), (b, c), (c, b), (c, d), (d, a), (d, c)\}$

$R \circ R = \{(a, a), (a, b), (a, c), (a, d), (c, b)\}$

$S \circ S = \{(c, c), (c, a), (c, d)$

23. i) $M_R' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ ii) $M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

iii) $M_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ iv) $M_{R \cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

25. i) $R^{\infty} = \{(1, 1), (1, 2), (1, 3), (1,4), (2,1), (2, 2), (2, 3), (3, 4), (3,1), (3,2), (3, 3), (3, 4), (4,1), (4, 2), (4, 3), (4, 4)\}$

ii) $R^{\infty} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$

27. Irreflexive, asymmetric, transitive.

28. $R = \{(1,1,1,8), (1,1, 2, 11), (1, 2, 1, 11), (1, 2, 2, 14), (2, 1, 1, 12), (1,3,1, 14), (2,1,1,12), (2,1,2,15), (2, 2, 1, 15)\}$

31. $R = \{(a, a) (a, c), (a, e), (c, a), (c, c), (c, e), (e, a), (e, c), (e, e), (b, b), (b, f), (b, d), (d, b), (d, d), (d, f), (f, b), (f, d), (f, f)\}$

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Chapter Review

Binary relation from A to B	:	A subset of $A \times B$.
Relation on A	:	A binary relation from A to itself (i.e., a is subset of $(A \times A)$
SoR	:	Composition of R and S.
R^{-1}	:	Inverse of R
R^n	:	n^{th} power of R
Reflexive	:	A relation R on A is reflexive if $(a, a) \in R, \forall a \in A$.
Symmetric	:	A relation R on A is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$
Anti-symmetric	:	A relation R on A is anti–symmetric if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$.
Transitive relation	:	A relations R on A is transitive if $(a, b) \in R\ (b,c) \in R \Rightarrow (a, c) \in R$.
n–ary relation on A_i	:	A subset of $A_1 \times A_2 \times A_3 \ldots\ldots \times A_n$.
Directed graph (Digraph)	:	A set of element called vertices and ordered pair of these elements, called edges.
Loop	:	An edge from (a, a)
Set	:	A collection of distinct & we defined object.
Axiom	:	A basic assumption of a theory.
Paradox	:	A logical inconsistency.
Empty/null set/ void set	:	The set with no members.

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Unit

2

Counting and Combinatories

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- Define basic principle of counting
- Use sum and product rule principle to find the total number of arrangement of objects.
- Explain the method of Permutation and find the permutation of n different object.
- Define combinations and find the combination of different objects.
- Prove Pigeonhole principle and find the number of pigeons and pigeonholes.
- Prove extended pigeonhole principle and find the number of pigeons and pigeonholes.

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2.1 Introduction

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The technique needed to answer this question and a wide variety of other counting problem will be introduced in this section.

2.1 Basic Principle of Counting

If an event can occur in r different steps and

Step 1 can occur in n_1 ways.

Step 2 can occur in n_2 ways.

… … … … … … … … … … …

… … … … … … … … … … …

Step r can occur in n_r ways.

Then the number of possible event that can occur is $n_1.n_2.n_3.n_4 \ldots n_r$ ways. This is the *fundamental principle of counting*.

2.1.1 Sum rule principle

Suppose some event E can occur in m ways and a second event F can occur in n ways. And both events cannot occur simultaneously, Then E or F can occur in m + n ways.

Example: If there are 15 boys and 13 girls in a class, find the number of ways of selecting one student as class captain.

Solution: Since there are 15 boys and 13 girls in a class. Using sum rule there are 15 + 13 = 18 ways of selecting one student (either a boy or a girl) as a class captain.

2.1.2 Product rule principle

In an event A can occur in m ways and event B can occur in n ways, then the two events A and B succession can occur in m .n ways. Similarly, if there are m ways between A and B and n ways between B and C then their simultaneous occurrence will be m .n ways. This rule is known as product rule *principle of counting*. This rule can be extended to three or more events. This is implies *multiplication*.

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Example – 1

Four persons enter into a car, where there are 5 seats. In how many ways can they take up their seats?

Solution:

Since there are four persons and 5 seats. They can have choices to choose the seat. To choose it 1st person can choose in 5 ways, 2nd person can chose in 4 ways 3rd person can chose in 3 ways and 4th person can chose his seat in 2 ways.

$\therefore$ The required number of ways $= 5\times 4 \times 3 \times 2 = 120$ ways.

Example – 2

How many different bit strings of length six are there?

Solution:

There are six bits. These six bits can be chosen in two ways because each bit is either 0 or 1. Therefore, 1st bit can be chosen in 2 ways, 2nd bit can be chosen in 2 ways, 3rd in 2 ways, 4th in 2 ways, 5th in 2 ways, 6th in 2 ways. Therefore, the product rule shows, there are 2.2.2.2.2.2 ways. $= 2^6 = 64$ different bit strings of length six.

2.3 Permutation of n different objects

The arrangement of objects taken some or all of them at a time in a definite order is known as permutation. In permutation, as order changes, its value will change. The arrangement of 2 and 3 taken both of them at a time are 23 and 32 which are different.

Factorial Notation

If n is the natural number, then the factorial n denoted by n! or $\lfloor n$ is the continued product from 1 to n natural numbers. i.e.

$n! = 1 .2 .3 . 4 \ldots (n-1) .n$

$1! = 1, 2! = 2.1, 3! = 3.2.1$

$0! = 1$

$5! = 5 .4 . 3!$

n! Can be written in the form n.(n-1).(n-2)! etc.

If we need to change 5! in to 6! we have to multiply 5! by 6. If we need to change 6! into 5! then we can write 6.5! .

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Theorem : 1

Let A be a set with n elements and $1 \le r \le n$. Then the number of sequences of length r that can be formed from elements of A, allowing repetition is n^r.

Example – 1

How many three letter 'words' can be formed from letters in the set {a, b, y, z} if repeated letters are allowed?

Here, $n = 4, r = 3$

So number of such words $= 4^3 = 64$.

Theorem: 2

The total number of permutation of a set of n objects taken r at a time is given by

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1) \qquad (n \ge r)$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Proof:

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The total number of permutation of a set of n object taken r at a time is equivalent to the number of ways in which r positions can be filled up by those n objects. Now there are n choices to fill up 1st position. When the first position has been filled up, there will be left n-1 objects to fill up 2nd position .i.e. there are n-1 choices to fill up the second position. Similarly, there are n-2 choices to fill up the third position and so on.

At least to fill up the rth position there are n-(r-1) = n-r+1 choices then by the basic principle of counting the number of ways in which r position can be filled up is

$$n(n-1)(n-2)\ldots(n-r+1)$$

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)[(n-r)\ldots 3.2.1]}{[(n-r)\ldots 3.2.1]}$$

$$= \frac{n!}{(n-r)!}$$

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Example – 2

How many 'words' of three distinct letters can be formed from the letters of the word "MATH"?

Solution:

Total number of letters (n) = 4

$\therefore$ Total number of words = ${}^4p_3 = \dfrac{4!}{(4-3)!} = 24$

Theorem : 3

The number of permutation of n objects, of which n_1 objects are of one kind and n_2 object of another kind, when all are taken at a time is $\dfrac{n!}{n_1!n_2!}$.

Proof:

Let us assume that the number of required permutation be K. Now consider a single particular permutation of these K permutations, in which n_1 objects of one kind is followed by n_2 objects of other kind.

Also, assume that all n_1 objects taken all at a time is = ${}^{n_1}P_{n_1} = n_1!$

Also, the number of permutations of n_2 objects taken all at a time = ${}^{n_1}P_{n_2} = n_2!$

By the fundamental principle of counting, these K permutations will give rise to $n_1!\ n_2!$ Permutations by arranging the objects of one kind within the places occupied by them.

Therefore, K permutations will give rise to K. $n_1!\ n_2!$ Permutations.

For n distinct objects, the number of permutations = ${}^nP_n = n!$

Therefore .K.$n_1!\ n_2! = n!$

$$K = \frac{n!}{n_1!\ n_2!}.$$

Hence the theorem is proved.

If n_1 objects are of one kind , n_2 objects are of second kind, n_3 objects are of third kind, and so on up to n_1 objects are of kth type is given by. $\dfrac{n!}{n_1!\ n_2!\ n_3!\ \ldots\ n_k!}$ [Here $n_1 + n_2 + n_3 + \ldots + n_k = n$]

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Theorem : 4

Prove that the number of different permutations of n distinct object taken r at a time when every object is allowed to repeat any of times is given by n^r

Proof:

Assume that with n objects we have to fill r place when repetition of object is allowed.

Therefore, the number of ways of filling the first place = n

The number of ways of filling second place = n

… … … … … … … … … … … … …

… … … … … … … … … … … … …

The number of ways of filling r^{th} place = n

Thus the total number of ways of filling r places with n elements = n .n .n .n….r times = n^r

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Theorem: 5

Prove that the number of circular permutations of n different objects is (n – 1)!

Proof:

Let us consider that, K is number of permutations required. For each such circular permutation of K, there are n corresponding linear permutations. We start from every n objects in the circular permutation. Thus for K circular permutations we have K. n linear permutations.

Therefore, K. n = n!

$$\text{or, } K = \frac{n!}{n}$$

$$\text{or, } K = \frac{n(n-1)!}{n}$$

or, K = (n – 1)! Hence Proved

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Example – 3

Show that $0! = 1$

Solution:

We have $^nP_r = \frac{n!}{(n-r)!}$

Now, put $r = n$ in equation (i)

$^nP_n = \frac{n!}{(n-n)!}$

or, $n! = \frac{n!}{0!}$

or, $0! = \frac{n!}{n!}$ $= 1$ Proved.

Example – 4

How many numbers of 3 digits can be formed with digits 1, 2, 3, 4, 5 when the repetition of digits is allowed?

Solution:

Here, total digits are 1, 2, 3, 4, 5. To fill up three digits number by these digits.

Units digits can be filled in 5 ways

Tens place digit can be filled in 5 ways

Hundred's place digit can be filled in 5 ways.

Therefore, the required number of three digits $= 5 . 5 . 5 = 5^3 = 125$

Example – 5

How many numbers of 3 different digit can be formed from the digits 2, 3, 4, 5, 6?

Solution:

Here, the given digits are 2, 3, 4, 5, 6

So, $n = 5, r = 3$

$\therefore$ The total number of 3 digits numbers $= 5p_3$

$$= \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

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Example – 6

Find the value of following:

a) P(5, 3) b) P(2, 3) c) P(n, q)

Solution:

a. $P(5, 3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$

b. $P(3, 3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$

c. $P(n, q) = \frac{n!}{(n-q)!}$

Example – 7

How many numbers of 4 digits can be formed from the digits 3, 4, 5, 6, 7? How many of these numbers are divisible by 5. How many of these are not divisible by 5?

Solution:

When n = 5, r = 4

The total no. of 4 digits number P(5, 4)

$= \frac{5!}{(5-4)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1} = 120$

To find the number divisible by 5, we need to fix the digits 5 on unit place.

So, $n = 5 - 1 = 4, r = 4 - 1 = 3$

$\therefore$ The required number divisible by 5 = $P(4, 3) = \frac{4!}{(4-3)!} = 24$

$\therefore$ The required number that are not divisible by = 120 – 24 = 96.

Example – 8

In how many ways can the letters of the word 'PERMUTATION' are arranged?

Solution:

There are 11 letters in the word PERMUTATION. So, they are not all different. There are 2T's So, the no. of arrangements =

$\frac{n!}{n_1!} = \frac{11!}{2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 1} = 19958400$

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Example – 9

In how many ways can 5 boys be seated in a round table of 5 seats so, that two particular boys always come together?

Solution:

There are 5 boys and 2 boys are always together. So, these two boys are considered as one.

∴ 4 boys are sitting in a round table of 4 seats.

∴ Arrangement of 4 boys in a round table = $(4-1)! = 3! = 3 \times 2 \times 1$ ways = 6 ways

But 2 boys can take their seats in P(2, 2) ways i.e, $\frac{2!}{(2-2)!} = 2 \times 1$ ways.

Hence, by fundamental principle of multiplication, The total no. of arrangement of 5 boys in a round table so that 2 boys always come together = 6×2 ways = 12 ways.

Example – 10

Find the number of ways in which 5 boys and 3 girls can be seats in a row so that no two girls are together.

Solution:

Here, Total number of boys = 5

Total number of girls = 3

5 boys can occupy 5 places in $5p_5 = 120$ ways

Since, no two girls are to sit together; we may arrange the 3 girls at the 6 places.

Now, the 3 girls at 6 places may be seated to, $6p_3$ ways = $6 \times 5 \times 4$ ways = 120 ways.

∴ The required number of ways = $120 \times 120 = 14400$ ways.

Example – 11

Find the number of ways in which the letters of the word "MACHINE' can be arranged such that the vowel may only odd position.

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Solution:

There are 7 letters in the word 'MACHINE' out of which there are 3 vowels and 4 consonants. Now, the 3 vowels can be placed at any of the three places out of the four marked 1, 3, 5, 7.

So, that 4 consonants at theremaining 4 positions may be arranged in $4p_4 = 4! = 24$ ways.

$\therefore$ The required numbers of ways $= 24 \times 24 = 576$

Example – 12

How many words can be formed from the letters of the word DAUGHTER so that

i. The vowel always come together

ii. The vowel are never come together

Solution:

The given word contains 8 different letters.

i. When the vowel A, U, E are always together, they can supposed to form, an entity treated as one letter.

Now, the letter to be arranged are D, G, H, T, R (AUE). These 6 letters can arranged in $6p_6 = 6!$ ways. But corresponding to each of these arrangements, the vowels in the group (A U E) can be arranged amongst themselves in $3p_3 = 3! = 6$ ways.

$\therefore$ The required number of words $= 720 \times 6 = 4320$.

ii. Similarly, the total number of words formed by using all the 8 letters of the given word is

$8p_8 = 8! = 40320$

$\therefore$ The number of words in which vowels are never together

= Total no. of words –No. of words in which vowels are always together

$= 40320 - 4320 = 36000$

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Example – 13

In how many ways can 6 rings of different type be had in 4 fingers?

Solution:

The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it.

Similarly, each one of the other rings may be worn in 4 ways.

$\therefore$ Required number of ways $= 4^6 = 4096$

Example – 14

In how many ways can 7 people be arranged in

a) a line b) a circle

Solution:

a) The number of ways in which 7 people can be arranged in a line in $7\times6\times5\times4\times3\times2\times1 = 5040$

b) The number of ways in which 7 people can arranged in a circle is $(7-1)! = 6!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways

Example – 15

Find the number of ways in which 7 different beads can be arranged to form a necklace.

Solution:

Fixing the position of one beads, the remaining 6 beads can be arranged in 6! ways but there is no distinction between the clockwise and anticlockwise arrangements.

So, the required number of arrangements $= \frac{1}{2} \times 6!$

$= \frac{1}{2} \times 720 = 360$

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Exercise - 2.1

Short Answers Type Questions.

1. Suppose the license plate contains two letters followed by three digits with the first digit not zero. How many license plates can be printed?
2. Find the number n of license plates that can be made where each plate contains two distinct letters followed by three different digits.
3. Simplify:

   a) $\frac{(n+1)!}{n!}$ b) $\frac{(n+1)!}{(n-1)!}$ c) $\frac{(n-1)!}{(n+2)!}$
4. Find the value of n if i) $P(n, 5) = 2. P(n, 3)$
5. How many permutation of the letters of the word 'MATHEMATICS' are there?
6. Find the number of different permutations of the letters in the following words:

   a) PASCAL b) REQUIREMENTS c) MAST
7. Three person enter into a car, where there are 5 seats. In how many ways can they take up their seats?
8. Define r- permutation of n distinct objects with example.
9. Find the number of ways in which 7 different beads can be arranged to form a necklace.
10. In how many ways can 5 letters be posted in 4 letter boxes?
11. In how many different relatives 4 men and 4 women be seated around a table with men and women alternating.
12. How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed?
13. How many different name can be formed using the letters a, a, a, b, b, b, c, c, c?
14. How many 7 digits numbers can be formed by using digits 1, 7, 2, 7, 6, 7, 6?
15. How many different bit strings of length seven are there?
16. Find the possible value of n if $(2n!) = (n!)!$

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17. In how many ways can a cricket team of eleven be chosen out of batch of 15 player's? How many of them will:

    a) Include a particular player. b) Exclude a particular player?

18. How many ways are these to select a first prize winner, a second prize winner and a third prize winner from 100 different people who have entered a contest?

19. How many permutation of the letters ABCDEFGH contains the string ABC?

20. In how many ways can 7 person form a ring? In how many ways can 7 gentlemen and 7 ladies sit down at a round table, no two ladies being together?

21. How many permutation can be made out of the letters of the word TRIANGLE? How many of these will begin with T and end with E?

22. How many words beginning and ending with a constant can be formed by using the letters of the word 'EQUATION'?

23. Find the numbers of ways in which 7 different beads can be arranged to form a neck lace.

24. Find the number of permutations that can be formed from the letters of the word' ELEVEN'?

    a) How many of them begin and end with E?

    b) How many of them have the three Es together?

    c) How many of them begin with E and end with N?

25. In how many ways can three boys and two girls sit in a row?

26. In how many ways can they sit in a row if the boys and girls are each to sit together?

27. In how many ways can they sit to a row if just the girls are to sit together?

28. How many bit strings of length ten both begin and end with a 1?

29. How many bit strings with length not exceeding n, where n is a positive integer, consists entirely of 1s?

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Answers

1. 608400
2. n = 468000
3. a) n+1 b) n^2+n c) $\frac{1}{n^3+3n^2+2n}$
4. 120
5. 4989600
6. a) 360 b) 39916800 c) 24
7. 60 ways
9. 360
10. 1024
11. 144
12. 256
13. 4200
14. 420
15. 128
16. 0, 3
17. 1365, 1001, 364
18. 970, 200
19. 6!=720
20. 6! × 7! ways
21. 40320, 720
22. 4320
23. $\frac{6!}{2} = 360$
24. 120, 24, 24, 12
25. 120
26. 24
27. 48
28. 2^8
29. n + 1 with empty string

2.4 Combination

Combination of object means just their collection without any regard to order or arrangement. The number of 'n' thing taken 'r' at a time is denoted by ${}^nC_r$. Or C (n, r)

Theorem: 1

The total number of combination of n object taken r at a time C(n, r) is given by $C(n, r) = {}^nC_r = \frac{n!}{(n-r)!r!}$.

Proof:

The number of permutation of n different thing, taken r at a time is given by ${}^nP_r = \frac{n!}{(n-r)!r!}$

As there is no matter about the order or arrangement of the objects. Therefore, to every combination of r things, there are r! Arrangements i.e.

${}^nP_r = r!\ {}^nC_r$

or, ${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\ r!}, n \geq r$

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Hence the theorem is proved.

Cor I: The number of combinations of n things taken r at a time is equal to the number of combinations of n things taken n-r at a time, symbolically. $C(n, r) = C(n, n-r)$

Proof:

$$C(n, n-r) = \frac{n!}{(n-n+r)!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = C(n, r)$$

$$C(n, r) = C(n, n-r)$$

Cor II: If $C(n, r) = C(n, r')$, then either $r = r'$ or $r + r' = n$

Proof:

$C(n, r) = C(n, r')$

or $C(n, r) = C(n, n-r')$ or, $r = n - r'$

$r + r' = n$. Hence proved.

Cor III: ${}^{n}C_r + {}^{n}c_{r-1} = {}^{n+1}C_r$

Proof:

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$$C(n, r) + C(n, r-1) = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\,\frac{n+1}{r(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!} = C(n+1, r)$$

Example – 1

Evaluate the followings:

a) $C(11, 5)$

Solution:

$$C(11, 5) = \frac{11!}{5!(11-5)!} = \frac{11!}{5!6!} = 462$$

b) $C(10, 4)$

Solution:

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$$C(10, 4) = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\,6!} = 210$$

c) $C(9, 9)$

Solution:

$$C(9, 9) = \frac{9!}{9!(9-9)!} = \frac{9!}{9!\,0!} = 1$$

Example – 2

How many committee of five with a given chairperson can be selected from 12 persons?

Solution:

A chairperson can be chosen in 12 ways and the other four on the committee can be chosen from the remaining eleven person in C(11, 4) ways.

Hence, there are $12 \times C(11, 4) = 12 \times 330 = 3960$ such committee.

Example – 3

Show that $C(2n, 2) = 2C(n, 2) + n^2$

LHS = $C(2n, 2)$

$$= \frac{2n!}{2!(2n-2)!}$$

$$= \frac{2n\,.\,(2n-1)\,(2n-2)!}{2 \times 1 \times (2n-2)!}$$

$$= n\,(2n-1) = 2n^2 - n.$$

$$= n^2 + n^2 - n$$

$$= n^2 + n(n-1)$$

$$= n^2 + \frac{n(n-1) \times 2(n-2)!}{2(n-2)!} \qquad \left[\text{Multiplying by } \frac{2(n-2)!}{2(n-2)!}\right]$$

$$= n^2 + \frac{2[n(n-1)\,(n-2)!]}{2(n-2)!}$$

$$= n^2 + \frac{2\,.\,n!}{2\,.\,(n-2)!}$$

$$= n^2 + \frac{2\,.\,n!}{(2 \times 1)\,(n-2)!}$$

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$$= n^2 + 2 \cdot \left[\frac{n!}{2!\,(n-2)!}\right] \qquad = n^2 + 2C(n, 2)$$

RHS proved

Example – 4

If n and r are positive integers such that $1 \le r \le n$ then $\frac{C(n, r)}{C(n, r-1)} = \frac{n-r+1}{r}$

Solution:

$$\text{LHS} = \frac{C(n, r)}{C(n, r-1)}$$

$$= \frac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r-1)!\,[n-(r-1)]!}}$$

$$= \frac{n!}{r!\,(n-r)!} \times \frac{(r-1)!\,[n-(r-1)]!}{n!}$$

$$= \frac{1}{r(r-1)!\,(n-r)!} \times \frac{(r-1)!\,[n-(r-1)]!}{1}$$

$$= \frac{[n-(r-1)]!}{r\,(n-r)!} = \frac{(n-r+1)\,(n-r)!}{r\,(n-r)!} = \frac{n-r+1}{r}$$

RHS proved.

Example - 5

Prove that the product of r consecutive positive integers is divisible by r!

Solution:

Let the r consecutive positive integers be $(n+1), (n+2), (n+3), \ldots \ldots \ldots (n+r)$

$\therefore$ Their product $= (n+1)(n+2)(n+3) \ldots \ldots \ldots (n+r)$

$$= \frac{n!\,(n+1)(n+2)(n+3) \ldots \ldots \ldots (n+r)}{n!}$$

$$= \frac{(n+r)!}{n!} = \frac{r!\,(n+r)!}{r!\,n!} = r!\,\frac{(n+r)!}{r!\,[(n+r)-r]!}$$

$$= r!\left[\frac{(n+r)!}{r!\,[(n+r)-r]!}\right] = r!\,C(n+r, r)$$

Which is divisible by r!. Since $C(n+r, r)$ is a positive integer.

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Example - 6

How many ways are there to select five players from a 10 member tennis team to make a trip to match at another school?

Solution:

The answer is given by the number of 5 combination of a set with 10 elements the number of such combination is $C(10, 5) = \frac{10!}{5!\,(10-5)!}$

$= 252.$

Example - 7

There are 12 points A, B ... ... is a given plane, no three on the same line,

a) How many triangles are determined by the points?

b) How many of these triangles contain the point A as a vertex?

Solution:

a) Since, three points determine a triangle there are $C(12, 3) = 220$ triangles.

b) There are $C(11, 3) = \frac{11!}{3!(11-3)!} = 165$ (Fixing a vertex A)

Example - 8

How many diagonals has a regular polygon with n sides?

Solution:

The regular polygon with n sides has n vertices. Any two vertices determine either a side or a diagonal. Thus there are $C(n, 2) = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2 \times 1 \times (n-2)!} = \frac{n(n-1)}{2}$ sides plus diagonals. But there are n sides, hence there are $\frac{n(n-1)}{2} - n$

$= \frac{n^2 - n}{2} - \frac{2n}{2} = \frac{n^2 - 3n}{2} = \frac{n(n-3)}{2}$ diagonals.

Example - 9

In how many ways can a six-card hand be dealt from a deck of 52 cards?

Solution:

Total number of cards = 52, No. of cards hand = 6

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So, six card hand can be selected from 52 cards in $C(52, 6) = \frac{52!}{6!(52-6)!} = 20358520$ ways.

Example - 10

Suppose that an urn contains 15 balls of which 8 are red and 7 are black. In how many ways can five balls be chosen so that

a) Two are red and three are black?

b) Three are red and two are black?

Solution:

Total number of balls = 15, No. of red balls = 8

No. of black balls = 7

So,

a) 5 balls be chosen from 8 red & 7 black balls in $8C_2 \times 7C_3$ ways = 28 × 35 ways = 980 ways

b) 3 red & 2 black balls can be chosen in $C(8, 3) \times C(7, 2) = 56 \times 21$ ways = 1176 ways

Example - 11

If $C(n, r-1) : C(n, r) = 3 : 4$ and $C(n, r) : C(n, r+1) = 4 : 5$. Find n & r.

Solution:

$$\frac{C(n, r-1)}{C(n, r)} = \frac{3}{4}$$

i.e. $$\frac{\frac{n!}{(r-1)!\,(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{3}{4}$$

or, $$\frac{1}{(r-1)!\,(n-r+1)!} \times \frac{r!\,(n-r)!}{1} = \frac{3}{4}$$

or, $$\frac{r(r-1)!\,(n-r)!}{(r-1)!\,(n-r+1)\,(n-r)!} = \frac{3}{4}$$

or, $4r = 3n - 3r + 3$

or, $7r = 3n + 3$

or, $3n - 7r = -3$ ... ... (i)

Again, $$\frac{C(n, r)}{C(n, r+1)} = \frac{4}{5}$$

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or, $\dfrac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r+1)!\,(n-r-1)!}} = \dfrac{4}{5}$

or, $\dfrac{1}{r!(n-r)!} \times \dfrac{(r+1)\,r!(n-r-1)!}{1} = \dfrac{4}{5}$

or, $\dfrac{1}{(n-r)\,(n-r-1)!} \times \dfrac{(r+1)\,(n-r-1)}{1} = \dfrac{4}{5}$

or, $5r + 5 = 4n - 4r$

$4n - 9r = 5$ ... ... (ii)

Multiplying (i) by 4 & (ii) by 3 then subtracting (ii) from (i) then we get,

$12n - 28r = -12$

$12n - 27r = 15$

$- \quad + \quad -$

$-r = \quad -27$

$\therefore \quad r = 27$

Substituting the value of r in equation

$3n - 7r = -3$

or, $3n - 7 \times 27 = -3$

or, $3n - 189 - 3$

or, $3n - 186$

or, $n = \dfrac{186}{3} = 62$

$\therefore \quad n = 62, r = 27$

Example - 12

From 6 gentleman and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least 3 ladies?

Solution:

The selection of the members in the committee can be made as follows:

Ladies (4)	Gents (6)	Selection
3	2	$C(4, 3) \times C(6, 2)$
4	1	$C(4, 4) \times C(6, 1)$

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The required no. of committee

$\therefore$ $= C(4, 3) \times C(6, 2) + C(4, 4) \times C(6, 1)$

$= 4 \times 15 + 1 \times 6$

$= 60 + 6 = 66$

Example - 13

A student is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions and that the student is not permitted to attempt more than 4 from any group. In how many different ways can students make up choice?

Solution:

The selection of question can be made as follows from group A & B as follows:

	Group A (5)	Group B (5)	Selection
i)	4	2	$5C_4 \times 5C_2$
ii)	3	3	$5C_3 \times 5C_3$
iii)	2	4	$5C_2 \times 5C_4$

$\therefore$ The total number of different ways in which the students can make a selection is

$= C(5, 4) \times C(5, 2) + C(5, 3) \times C(5, 3) + C(5, 2) \times C(5, 4)$

$= 5 \times 10 + 10 \times 10 + 10 \times 5 = 200$

Example - 14

Find the number of ways that six students can be partitioned into two teams containing three students each.

Solution:

Number of Students (n) = 6 number of students containing in each team (r) = 3

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There are ${}^6C_3 = \frac{6!}{3!(6-3)!} = 20$ ordered partitions into two cells containing three students each. Since each unordered partition determines 2! = 2 ordered partitions, there are 20/2 =10 unordered portioned.

Exercise – 2.2

1. Find the values of: a) C(10, 6) b) C(50 , 45)
2. Determine the value of n:
   a) C(n, 4) = C (n,3) b) C (n, n – 2) = 10
   c) C (20, n + 2) = C (20, 2n – 1) d) C(n + 2, 4) = 6. C(n, 2)
3. Show that C (n + 1, r) = C(n, r – 1) + C(n, r).
4. Prove: C (12, 7) = C(11,6) + C(11,7)
5. How many committees of five with a given chairperson can be selected form 12 persons?
6. A school has 6 badminton players. A team of 4 has to be sent to a tournament, in how many ways can the team be selected?
7. A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many committees can be formed?
8. If C(18, r) = C (18, r + 2) then find r and C (r, 3)
9. A man has 11 close friends
   a) In how many ways can she invite five of them to dinner?
   b) In how many ways she can invite if two of the friends are married and will not attend separately?
   c) In how many ways if two of them are not on speaking terms and will not attend together?
10. In how many ways can 14 people be partitioned into 6 committees where two of the committee contains three members and the other two?

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11. In the Discrete Mathematics paper there are 8 questions. In how many ways can an examiner select five questions in all if first question is compulsory.

12. Determine the number of triangles that are formed by selecting points from a set of 15 points out of which 8 are collinear.

13. From 10 persons in how many ways can a selection of 4 be made?

    a) When a one particular person is always included?

    b) When two particular person are always excluded.

14. An examination paper containing 12 questions consists of two parts, A and B, Part A contains 7 questions and part B contains 5 questions. A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?

15. How many diagonals are there in a polygon of n sides?

Answers

1. a) 210	b) 2118760	2. a) 7	b) 5 c) 3 d) 7
5. 3960	6. 15	7. 6160	8. 8,56
9. a) 462	b) 210	c) 252	10. 3153150
11. 35	12. 854	13. a) 84	b) 70
14. 420	15. $\frac{1}{2}n(n-3)$		

2.5 The Pigeonholes principle (Dirichlet Drawer Principle or Shoe Box Principle)

Theorem:

If n pigeons are assigned to m pigeonholes, and m<n, then at least one pigeonhole contain two or more pigeons.

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Proof:

Let each pigeonhole contains at most 1 pigeon then at most m pigeon have been assigned. But m < n, not all pigeons have been assigned pigeonholes. Which is contradict, At least one pigeonhole contains two or more pigeons.

Example - 1

In a group of 27 English words there must be at least two words beginning with the same letter as there are 26 letters in English alphabet.

Example - 2

In a group of 367 people there must be at least two people with the same birthday as there are 365 or 366 days in a year.

Example - 3

In a class of 51 students there must be at least two students scoring same marks in a paper of maximum marks 50.

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Example - 4

Show that among any group of five integers (not necessarily consecutive) there are two integers with the same remainder when divided by 4.

Solution:

There are 4 possible remainders when an integer is divided by 4 i.e. 0,1,2,3. Therefore the pigeonhole principle implies that given five integers at least two have the same remainder.

Example - 5

Show that at least in any set of seven classes there must be two that meet on the same4 day assuming that no class holds on Sunday.

Solution:

There are seven classes and six days. Therefore by the pigeonhole principle here must be at least two classes holding on the same day.

Example - 6

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Show that 13 persons there are at least two persons who were born in the same month.

Solution:

If we consider 13 persons as pigeons and 12 months (Jan... Dec) As pigeonholes then by the pigeonhole principle these are at least two persons having the birth in the same month.

2.6 Extended Pigeonhole Principle

Theorem:

If n pigeons are assigned to m pigeonholes, then one of the pigeonhole must contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

Proof:

(By contradiction) If each pigeonhole contains no more than $\left\lfloor \frac{n-1}{m} \right\rfloor$ pigeons then there are at most m. $\left\lfloor \frac{n-1}{m} \right\rfloor \le m\frac{(n-1)}{m} = n-1$ pigeons in all. Which is contradicts our hypothesis. So one of the pigeonholes must contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ *pigeons.*

Example – 7

How many bit string of length eight start with a 1 bit or end with the two bits 00?

Solution:

Number of bit string of length eight which start with a 1 bit or end with the two bits 00 is $2^7 + 2^6 - 2^5 = 160$

Example – 8

How many different bit strings of length seven are there?

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Solution:

Each of the seven bit can be chose in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit string of length seven.

Exercise – 2.3

1. Show that if any five numbers from 1 to 8 are chosen, and then two of them will add to 9.
2. Show that if any 30 people are selected, then one may choose a subset of five so that all five were born on the same day of the week.
3. How many friends must you have to guarantee at least five of them will have birthday in the same months.
4. Prove that if any 14 numbers from 1 to 25 are chosen then one of them is a multiple of other.
5. Show that if 7 colours used to paint 50 bicycles at least eight bicycles will be the same colour.
6. Show that if seven numbers from 1 to 12 are chosen, then two of them will add up to 13.
7. Find the minimum number of students needed to guarantee that five of them belong to the same class.
8. Find the minimum number n of integers to be selected from S = {1, 2, … 9} so that
   a) The sum of two of the n integers is even
   b) The difference of two of the n integers is 5.
9. What is n, if n is number of integers to be selected from I = {1, 2, 3, … 9} such that the sum of two of the n integers is even.
10. Find the minimum number of balls needed to guarantee that five of them are of same color.

    (Blue, green, red, white)
11. How many bit string are there of length eight?

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12. How many bit string of length ten both begin and end with a 1?
13. How many bit string with length not exceeding n, where n is a positive integer, consist entirely of 1s?
14. Find the minimum number of students in a class to be sure that four out of them are born in the same month.
15. If 9 books are to be kept in 4 shelves, show that there must be at least one shelf which contains at least 3 books.
16. A drawer contains ten black and ten white shocks. What is the least number of socks one must pull out to be sure to get a matched pair.
17. Show that if seven colors are used to paint 50 bicycles, at least eight bicycles will be the same color.

Answers

3.	49 friends	**7.**	17	**8.**	a) $n = 3$	b)	$n = 6$
9.	$n = 3$	**10.**	17	**11.**	2^8	**12.**	2^8
13.	$n + 1$ by counting the empty string			**14.**	37	**16.**	3 shocks

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Chapter Review

The product rule	: A basic counting technique which states that the number of ways to do a procedures that consists of two subtasks in the product of the number of ways to do the second task after the first task has been done.
Permutation	: An ordered arrangement of the elements of a set.
r-permutation	: An ordered arrangement of r elements of a set.
P(n, r)	: The number of n object arrange in r places.
Combinatories	: The study of arrangement of objects.
Enumeration	: The counting of arrangement of objects.
r–combination	: An ordered selection of r elements of a set.
C(n, r)	: The combination of n object taken r at a time.
The pigeonhole principle	: If n pigeons are assigned to m pigeonholes, and m < n then at least one pigeonhole contains two or more pigeons.
Extended pigeonhole principle	: If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contains at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

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Unit 3

The Fundamentals: Algorithms & Matrices

- Define fundamental of different algorithm on the integers and matrices
- Define different types of complexity.
- Use binary and linear search to find the numbers in the list.
- Use Bubble sort and insertion sort to find the element in the list.
- Define matrices and its types
- Find transpose and power of matrices

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3.1 Algorithms

3.1.1 Introduction

An algorithm present the basic frame which is transformed into the program using data structures as the building blocks. Although ordinary language is sometimes adequate to specify an algorithm, many computer scientists prefer pseudo code because of its precision, structure and universality. Pseudo code provides an intermediate step between an English language description of an algorithm and an implementation of this algorithm in a programming language. Pseudo code is so named because it resembles the actual code (programs) of language such as Pascal and C++. There are many versions of pseudo code, any version of pseudo code is acceptable as long as its instructions are unambiguous and it resembles inform.

A computer is needed precise and well defined instruction for finding solutions of problems. An algorithm is a step by step method of solving some problems. An algorithm may have the following characteristics.

1. **Precision:** The steps of an algorithm must be defined precisely.
2. **Uniqueness:** The intermediate results of each step of execution are uniquely defined and depend only on the inputs and the results of the preceding steps.
3. **Finiteness:** An algorithm should produce the desired output after a finite number of steps for any input in the set.
4. **Input:** The algorithm receives input.
5. **Output:** The algorithm produces output.
6. **Generality:** The algorithm applies to a set of inputs.

3.2 Complexity of algorithm

We need to measure the complexity of a problem, algorithm or structure, and to obtain bounds and quantitative relations for complexity. It arises in more and more sciences. Be-sides computer science, the traditional branches of mathematics, statistical physics, biology,

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medicine, social sciences and engineering are also confronted more and more frequently with this problem. In the approach taken by computer science, complexity is measured by the quantity of computational resources (time, storage, program, communication) used up by a particular task.

Computation theory can basically be divided into three parts of different character. First, the exact notions of algorithm, time, storage capacity, etc. must be introduced. For this, different mathematical machine models must be defined, and the time and storage needs of the computations performed on these need to be clarified (this is generally measured as a function of the size of input). Second, one must determine the resource need of the most important algorithms in various areas of mathematics, and give efficient algorithms to prove the certain important problems belong to certain complexity classes. Third, one must find methods to prove "negative results", i.e. for the proof of some problems are actually unsolvable under certain resources restrictions.

Complexity theory, a major branch of the theory of algorithms, investigates the solvability of individual problems under certain resource restrictions. The most important resources are *time* and *space* (storage).

3.2.1 Time complexity

The amount of time required to execute the algorithm and the amount of memory space when it is run. For example, a computer requires a certain amount of time to carry out an arithmetic operation like additions, multiplications, division or any other basic operation.

Time complexity is described in terms of the number of operation required instead of actual computer time. An analysis of the time required to execute an algorithm of a particular size is referred to as the time complexity. An analysis of the computer memory required involves the space complexity of the algorithm.

Time complexity may be classified in following types:

i. **Best case time complexity:** The minimum time needed to execute the algorithm among all inputs of that size. This time is called the best-case time complexity.

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ii. **Worst–case time complexity:** The maximum time needed to execute the algorithm among all input of that size. This time is called the worst–case time complexity.

iii. **Average case time complexity:** The average time needed to execute the algorithm over some finite set of input of given size. Then the complexity is known as the average case time complexity.

3.2.2 Understanding the complexity of algorithm

Some of the most common terminology used to describe the time complexity of algorithm. For example, an algorithm whose complexity function has the same order as n is said to be linear or to run in linear time. An algorithm with complexity function the order of n^2 is called quadratic. An algorithms with complexity function of order n^a for some a > 0 are polynomial or said to run in polynomial time. Algorithms with complexity function of order log n are logarithmic and algorithms with complexity of order a^n for some a > 1 are exponential.

Terminology used for time complexity algorithms

Complexity	Terminology
$f(n) = O(1)$	Constant complexity
$f(n) = O(\log_2 n)$	Logarithmic complexity
$f(n) = O(n)$	Linear complexity
$f(n) = O \log_2 (n)$	n log n complexity
$f(n) = O(n^2)$	Quadratic complexity
$f(n) = O(n^3)$	Cubic complexity
$f(n) = O(n^a)$	Polynomial complexity
$f(n) = O(a^n), a > 1$	Exponential complexity
$f(n) = O(n!)$	Factorial complexity

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3.3 Searching algorithms

In computer science, a search algorithm is an algorithm for finding an item with specified properties among a collection of items. The items may be stored individually as records in a database or may be elements of a search space defined by mathematical formula or procedure, such as the roots of an equation with integer variables.

In this section, we discuss several algorithms for searching and sorting.

3.3.1 Linear Search

In a linear search x is compared to the first element in the list, then the second, then the third and so on. The search is stopped if a match is found at any stage. On the other hand, if the entire lists is processed without finding a match, the x is not in the list.

The linear search begins by comparing x and a_1. When $x = a_1$, the solution is the location of a_1. Namely 1. When $x \neq a_1$ then compare x with a_2 .the solution is the location of a_2. Namely, 2. When $x \neq a_2$ then compare with a_3. Continue the process comparing x successively with each of the list until a match is found. Where the solution is the location of that term, unless no match occurs. If the entire list has been searched without locating x , the solution is 0,

The linear search algorithm

To search a list $a_1, a_2, \ldots\ldots\ldots\ldots a_n$ for the element x

for i = 1 to n.

If $x = a_i$, output "true" and set i = 2n

If $i \neq 2n$, output "false".

The algorithm proceeds in an obvious way setting i = 2n is a little trick which stops the loop as soon as x has been found and, if x is not in the list, ensures that "false" is output at the end.

Example – 1

For x = – 2 and input list

6	0	– 2	1
a_1	a_2	a_3	a_4

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The algorithm sets $i = 1$, determines that $x \neq a_1 = 6$, and sets $i = 2$. Since $x \neq a_2 = 0$, it sets $i = 3$. Since $x = a_3 = -2$, it outputs 'true' and sets $i = 2n = 8$. Since, i is no longer in the range from 1 to n, the loop stops.

3.3.2 Binary Search

A binary search can only be performed on a list whose elements are arranged in ascending (or descending) order. Given a list $a_1, a_2, \ldots\ldots, a_n$ of distinct elements arranged in ascending order, consider the problem of trying to find a particular element x in the list, to use binary search, algorithm determines which half of an ordered list would have to contain the sought–after element, then which half of that half, and so on. At each stage, the list to be searched is one half the length of the previous list.

Binary search algorithm

To search for an element x in an ordered list $a_1 \leq a_2 \leq \ldots\ldots \leq a_n$, proceed as follows,

{n> 0

If n = 1 then

If $x = a_1$ output 'true' and set n = 0 else output 'false' and set n = 0

else set $m = \left\lfloor \frac{n}{2} \right\rfloor$

If $x = a_m$, output 'true' and set n = 0

else if $x < a_m$

set n = m and replace the current list with $a_i \ldots\ldots\ldots\ldots a_m$.

else replace n with n – m and the current list with $a_{m+1} \ldots\ldots\ldots\ldots a_n$.

end}

Example – 2

To search for 12 in the list

5 6 7 10 11 12 15 17 19 20

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Solution:

For x = 12 and the input

5	6	7	10	11	12	15	17	19	20
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}

of length n = 10 > 0, the algorithm $m = \left\lfloor \frac{10}{2} \right\rfloor = 5$. Since $x = a_5 = 11$ and $x < a_m$ are both false, the algorithm replaces n by n – m = 5 (10 – 5 = 5) and replace the original list with

12	15	17	19	20
a_1	a_2	a_3	a_4	a_5

The integer n is still positive so, the algorithm executes the statements inside the {.....}, again, since n ≠ 1, it sets $m = \left\lfloor \frac{5}{2} \right\rfloor =$ 2 and, because $x < a_2 = 15$, if sets n = 2 and replace the current list with

12	15
a_1	a_2

Again, since n ≠ 1, the algorithm set $m = \left\lfloor \frac{2}{2} \right\rfloor = 1$. Since, $x = a_1 =$ 12. The algorithm outputs "true" and set n = 0. Hence, the statement is true. So the algorithm terminates.

3.4 Sorting

To order the elements of a list is a problem in our context. For example it is difficult to explore the name and address and phone number of the person from a telephone directory. The name of person and areas are kept in alphabetical order to solve these type of problem. Therefore the notion of ordering elements of sets in different method will be discuss in the following section.

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3.4.1 Bubble Sort

A sorting algorithm puts a list of numbers in to increasing order or a list of words into alphabetical order. Perhaps the most popular through not efficient method of sorting numeric data is a technique called the bubble sort.

The algorithm compares the first entry a_1 in the given list with its immediate successor a_2. If $a_1 > a_2$ then interchange the value stored in a_1 and a_2. Then a_2, (new value stored) is compared with its immediate successor a_3, the process is continued. After $n - 1$ such comparison, the biggest number in the list is stored in a_n. The process is repeated for the $n - 1$ numbers now stored in the list $a_1, a_2 \ldots\ldots\ldots\ldots a_{n-1}$. In this way, each time this process is carried out, the largest element bubble to the end in order.

Step-by-step example

Let us take the array of numbers "5 1 4 2 8", and sort the array from lowest number to greatest number using bubble sort. In each step, elements written in bold are being compared. Three passes will be required.

First Pass:

(5 1 4 2 8)→ (1 5 4 2 8), Here, algorithm compares the first two elements, and swaps since 5 > 1.

(1 **5** **4** 2 8) →(1 **4** **5** 2 8), Swap since 5 > 4

(1 4 **5** **2** 8) →(1 4 **2** **5** 8), Swap since 5 > 2

(1 4 2 **5** **8**) → (1 4 2 **5** **8**), Since these elements are already in order (8 > 5), algorithm does not swap them.

Second Pass: (**14** 2 5 8) → (**14** 2 5 8)

(1 **4** **2** 5 8) → (1 **2** **4** 5 8), Swap since 4 > 2

(1 2 **4** **5** 8) →(1 2 **4** **5** 8)

(1 2 4 **5** **8**) →(1 2 4 **5** **8**)

Now, the array is already sorted, but our algorithm does not know if it is completed. The algorithm needs one **whole** pass without **any** swap to know it is sorted.

Third Pass:

(**1** **2** 4 5 8) → (**1** **2** 4 5 8)

(1 **2** **4** 5 8) →(1 **2** **4** 5 8)

(1 2 **4** **5** 8) →(1 2 **4** **5** 8)

(1 2 4 **5** **8**) →(1 2 4 **5** **8**)

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Example – 1

Sort the following list of elements using bubble sort: 6, 3, 1, 4, 9, 2.

Solution:

Initial (i = 1) the numbers being compared each stage are underlined
6, 3, 1, 4, 9, 2 → 3, 6, 1, 4, 9, 2 → 3, 1, 6, 4, 9, 2 → 3, 1, 4, 6, 9, 2 → 3, 1, 4, 6, 9, 2 → 3, 1, 4, 6, 2, 9

The sort requires five comparisons and four interchanges.

for, i = 2

3, 1, 4, 6, 2, 9 → 1, 3, 4, 6, 2, 9 → 1, 3, 4, 6, 2, 9 → 1, 3, 4, 6, 2, 9 → 1, 3, 4, 2, 6, 9

Here, four comparisons and two interchanges.

for i = 3

1, 3, 4, 2, 6, 9 → 1, 3, 4, 2, 6, 9 → 1, 3, 4, 2, 6, 9 → 1, 3, 2, 4, 6, 9

Here three comparisons and one interchange.

for i = 4

1, 3, 2, 4, 6, 9 → 1, 3, 2, 4, 6, 9 → 1, 2, 3, 4, 6, 9

Here, two comparisons and one interchanges.

for i = 5

1, 2, 3, 4, 6, 9 → 1, 2, 3, 4, 6, 9

It requires one comparison and no interchanges.

A bubble sort algorithm

Procedure bubble sort (a_1, a_2, ... a_n)

for i : = 1 to n – 1

for j : = 1 to n – 1

If $a_j > a_{j+1}$ then interchange a_j and a_{j+1} {a_1, a_2, ... a_n is an increasing order}

3.4.2 Insertion Sort

A bubble sort relies on a number of small swaps, insertion sort relies on inserting a single element in the right for a given iteration. Every iteration through the collection leaves a greater segment sorted. The insertion sort compare the second element with the first element and insert it before the first element if it does not exceed the first element and after the first element if it exceeds the first element. At this stage, the first

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two elements are in the correct order. The third element is then compared with the first element and if it is larger than the first element, it is compared with the second element, it is inserted into the correct position among the first three elements. This process continues until the last element is placed in the correct order.

The insertion sort algorithm

Procedure: insertion sort ($a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$)

for j = 2 to n

{i = 1

While $a_j > a_i$

i : = i + 1

m : = a_j

for k: 0 to j – i – 1

$a_{j-k} := a_{j-k-1}$

a_i : = m

} || finished

Example – 2

Sort the following list of the element, using the insertion sort.

7, 4, 5, 2, 9, 1

Solution:

Here, sort the list 7, 4, 5, 2, 9, 1 in increasing order.

Step	Comparison	Correct order
I	Compare 4 and 7, 7 > 4	4, 7, 5, 2, 9, 1
II	compare the third element 5 into the 1st two elements i.e., 4 < 5, 7 > 5	4, 5, 7, 2, 9, 1
III	Compare the 4th element 2 into the 1st three elements i.e. 4 > 2, 5 > 2, 7 > 2	2, 4, 5, 7, 9, 1
IV	Compare the 5th element 9 into the 2, 4, 5, 7 since, 9 > 2, 4, 5, 7	2, 4, 5, 7, 9, 1
V	Compare the 6th element 1 into 2, 4, 5, 7, 9 since 1 < 2	1, 2, 4, 5, 7, 9

Hence, 1, 2, 4, 5, 7, 9 is the correct order of the given list.

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Exercise 3.1

1) Define an algorithm. What are the characteristics of an algorithm.
2) List binary search to find 13 in the list 5,6,7,10,11,13,15,18,20 showing each steps
3) List the sequence of steps in using a binary search to find 3 in the list 1,3,5,7,9,11,13,15,17. How many times is 3 compared with an element in the list?
4) Use a bubble sort (i) 3,2,4,1,6 ii) 3,1,5,9,4 showing the lists obtained at each step.
5) Show the sequences of steps in using a bubble sort algorithm to arrange the numbers in ascending order in the list 3,1,5,7 and 4.

Answers

2) 13,15,17,18,20 →13.15, 17 →13
3) 1,3,5,7,9,11,13,15,17→1,3,5,7 →3 comparison $x = a_4$, $x<a_4$ and $x = a_2$ linear search 2 comparison
4) i) 2,3,4,1,6 → 2,1,3,4,6→1,2,3,4,6

   ii) 1,3,5,4,9 →1,3,4,5,9 →1,3,4,5,9
5) 1,3,5,4,7 →1,3,4,5,7→1,3,4,5,7→1,3,4,5,7

3.5 Matrices

Introduction

Matrices play a very important role in various branches of mathematics and in other subjects. It has wide application in many fields such as Operations Research, Statistics, Economics, Engineering and Sociology etc. Here, we use the matrix algebra to find the solution of system of linear equations.

Definition and Types of Matrix

(a) Definition: 1

A matrix is an arrangement of numbers in a rectangular form consisting of one or more rows and columns.

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A matrix containing m rows and n columns, we call it a matrix of **order** m × n. The order of a matrix is also known as size or dimension of the matrix.Matrices are usually denoted by the capital letters A, B, C..... .

(a) Definition: 2

An element containing i^{th} row and j^{th} column of a matrix A. Then it is called the $(i, j)^{th}$element of A, it is denoted by a_{ij}.

In general, m × n matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & & \cdots \\ a_{m_1} & a_{m_2} & a_{m_3} & \cdots & a_{m\times n} \end{bmatrix} = [a_{ij}]_{m\times n}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

Here, A is a 2 × 2 matrix

B is a 3 × 2 matrix

C is a 2 × 3 matrix.

Types of Matrices

(i) Row Matrix

A matrix having only one row is called a row matrix.

Examples: $[0 \;\; 1 \;\; 2]_{1\times 3}$, $[a_{11} \;\; a_{12} \;\; a_{13} \;\ldots\ldots\; a_{1n}]_{1\times n}$

(ii) Column Matrix

A matrix having only one column is said to be column matrix.

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \\ a_{m_1} \end{bmatrix}_{m\times 1}$

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(iii) Null (Zero) Matrix

A matrix having all the elements are zero is said to be zero or null matrix. It is denoted by 0.

Example: $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

(iv) Square Matrix

A matrix having the same number of rows and columns, is called a square matrix.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \\ 2 & 0 & 5 \end{bmatrix}_{3\times 3}$

(v) Diagonal Matrix

A square matrix having all the non-leading diagonal elements are zero is called diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is called a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2}$

Note: The diagonal of a square matrix (a_{ij}) containing the elements a_{11}, a_{22}, a_{33} ..... is called the leading diagonal (or principal diagonal) of the matrix.

(vi) Scalar Matrix

A diagonal matrix is said to be scalar matrix. If the leading diagonal elements are equal (same).

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2}$

(vii) Identity Matrix or Unit Matrix

A diagonal matrix is said to be identity matrix or unit matrix, if the diagonal elements are equal to unity. It is simply denoted by I.

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Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

(viii) Triangular Matrix

(a) **Lower Triangular Matrix:** A square matrix having all the elements above the leading diagonal are zero is called a lower triangular matrix.

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 2 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

(b) **Upper Triangular:** A square matrix having all the elements below the leading diagonal are zero is called an upper triangular matrix.

Example: $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

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(ix) Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i and j.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 3 & 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

(x) Skew-symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a skew symmetric matrix if $a_{ij} = -a_{ji}$ for all i and j.

Example: $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Note: All the leading diagonal elements in a skew-symmetric matrix must be zero.

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(xi) Equal Matrices

Two matrices are said to be equal if they have the same order and their corresponding elements are equal.

Example:

Let $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, A = B if a = w, b = x, c = y, d = z.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Then A = B.

3.5.1 Matrix Arithmetic

(a) Addition of Matrices

Let A and B be two m × n matrices. The m × n matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B is denoted by A + B.

Example:

Let $A = \begin{bmatrix} 0 & 2 \\ 3 & 2 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -3 \\ 4 & 2 \\ 5 & 9 \end{bmatrix}$

Then, $A + B = \begin{bmatrix} 0+6 & 2-3 \\ 3+4 & 2+2 \\ 4+5 & -1+9 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 7 & 4 \\ 9 & 8 \end{bmatrix}$

(b) Difference of Two Matrices

Let A and B be two m × n matrices. Then the difference of matrices A and B is denoted by A – B and is defined as A – B = A + (–B), it is also be a m × n matrix.

In order to find A – B, the element of B must be subtracted from the corresponding elements of A.

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Example:

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$

Then, $A - B = \begin{bmatrix} 1-2 & 3+1 \\ 2-5 & -1-0 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -3 & -1 \end{bmatrix}$

Note: $A \pm B$ is defined only when they are matrices of same order.

(c) Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and K be any scalar, then the scalar multiple of matrix A by K is denoted by KA and is defined as the $m \times n$ matrix obtained by multiplying each element of A by K.

Example:

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 4 \end{bmatrix}$

Then, $kA = \begin{bmatrix} k & 3k & 5k \\ 2k & 0 & 4k \end{bmatrix}$

Where, k is any number.

(d) Matrix Multiplication

Let $A = [a_{ij}]_{m\times p}$ and $B = [b_{jk}]_{p\times n}$ be two matrices such that the number of column in A is equal to the number of rows in B. Then, the product of matrix A and B is denoted by AB is a matrix of order $m \times n$.

i.e. $AB = C = [c_{ij}]_{m\times n}$

***Note*:** Product AB is defined only when the number of columns in A is equal to the number of rows in B.

Example:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$

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$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}_{3\times 2}$$

Same

Order

Here, the number of columns of matrix A is equal to the number of rows of matrix B, so the product AB exist. After multiplication, the order of the product matrix AB will be 2×2.

The process of getting the element of AB is shown below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1\times 0 + 2\times 2 + 3\times 4 \ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots & 1\times 1 + 2\times 3 + 3\times 5 \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ 2\times 0 + 1\times 2 + 0\times 4 \ldots\ldots\ldots & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots & 2\times 1 + 1\times 3 + 0\times 5 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} 1\times 0 + 2\times 2 + 3\times 4 & 1\times 1 + 2\times 3 + 3\times 5 \\ 2\times 0 + 1\times 2 + 0\times 4 & 2\times 1 + 1\times 3 + 0\times 5 \end{bmatrix} = \begin{bmatrix} 16 & 22 \\ 2 & 5 \end{bmatrix}_{2\times 2}$$

(e) Properties of Matrix Multiplication

The multiplication of matrices satisfy the following properties.

(i) $AB \neq BA$ (Not commutative in general)

(ii) $A(BC) = (AB)C$ (Associative property)

(iii) $A(B + C) = AB + AC$ (Distributive)

(iv) $AI = IA = A$ (Multiplicative identity)

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3.5.2 Transpose of a Matrix

Let A be any m × n order matrix then the matrix obtained by interchanging its rows and column is called the transpose of A and is denoted by A′ or A^T. If A is a m × n matrix then A′ will be a n × m matrix.

Example

If $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}$

Properties of Transpose of Matrices

The transpose of a matrix satisfies the following properties:

(a) $(A + B)' = A' + B'$ (b) $(A')' = A$

(c) $(KA)' = KA5$ (d) $(AB)' = B'A'$

3.5.3 Power of Matrices

Given a square matrix A, for n being a nonnegative integer, A^n is called as the product of matrix taking A and multiplying it by itself n times gives the power of matrix. If A is invertible, then $A^{-n}=(A^{-1})^n$, or the product matrix taking $A-1$ and multiplying it by itself n-times.Multiplying a square matrix by an appropriately sized identity matrix does not change this matrix.

If I is a unit matrix of order n , it has the property that it can commute with every square matrix of order n, i.e. If A is a square matrix of order n then A = IA = AI.

Power of square matrices can be defined. When A is an n×n matrix, we have $A^0 = I_n$, andA^r = A.A.A….r times A.

Theorem 1: If A is a square matrix and let r and s be integers and let $k \in R$ be a scalar. Then:

a) $A^rA^s=A^{r+s}$ **b)** $(A^r)^s=A^{rs}$

c) If An is invertible, then $A^{-n}=(A^{-1})^n$ for $n>0$.

d) If $k\neq 0$ and the matrix A is invertible, then $(kA)^{-1}=\frac{1}{k}A^{-1}$.

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Worked Out Examples

Example: 1

Construct a 2×3 matrix whose elements are given by $a_{ij} = 2i - 3j$.

Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ be a required 2×3 matrix, whose elements are given by:

$a_{ij} = 2i - 3j$ then

$a_{11} = 2 \times 1 - 3 \times 1 = -1$

$a_{12} = 2 \times 1 - 3 \times 2 = -4$

$a_{13} = 2 \times 1 - 3 \times 3 = -7$

$a_{21} = 2 \times 2 - 3 \times 1 = 1$

$a_{22} = 2 \times 2 - 3 \times 2 = -2$

$A_{23} = 2 \times 2 - 3 \times 3 = -5$

$$\therefore A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \end{bmatrix}.$$

Example : 2

If $X = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

Find (i) $X + Y$ (ii) $2X + 3Y$

Solution:

(i) $$X + Y = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 3 & 5 & 6 \end{bmatrix}$$

(ii) $$2X + 3Y = 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 6 & 8 & 10 \end{bmatrix} + \begin{bmatrix} 6 & -3 & -6 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 0 \\ 6 & 11 & 13 \end{bmatrix}$$

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Example : 3

If $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2A + B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

Find A.

Solution:

$$2A + B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow 2A + \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\therefore A = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

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Example : 4

Find x, y and z for which

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 6 \end{bmatrix}$$

Solution:

Given, $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 6 \end{bmatrix}$

By definition of equal matrix, then

$x + 3 = 0 \Rightarrow x = -3$

$2y + x = -7$

$2y - 3 = -7$

$2y = -4 \Rightarrow y = -2$

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$z - 1 = 3$

$\Rightarrow z = 4$

$\therefore x = -3, y = -2, z = 4.$

Example : 5

Find AB, when $A = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $B = [3 \;\; 2 \;\; 4]$.

Solution:

$$AB = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}_{3\times1} [3 \;\; 2 \;\; 4]_{1\times3}$$

$$= \begin{bmatrix} 2\times3 & 2\times2 & 2\times4 \\ 3\times3 & 3\times2 & 3\times4 \\ 4\times3 & 4\times2 & 4\times4 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 8 \\ 9 & 6 & 12 \\ 12 & 8 & 16 \end{bmatrix}$$

Example : 6

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Find AB and BC, also verify that A(BC) = (AB) C.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1+4 & 0-6 \\ 3+8 & 0-12 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+0 \\ 2-0 & -2-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix}$$

Again, $A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1+4 & -1-10 \\ 3+8 & -3-20 \end{bmatrix} = \begin{bmatrix} 5 & -11 \\ 11 & -23 \end{bmatrix}$

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$$(AB)C = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5+0 & -5-6 \\ 11-0 & -11-12 \end{bmatrix} = \begin{bmatrix} 5 & -11 \\ 11 & -23 \end{bmatrix}$$

$$\therefore \quad A(BC) = (AB)C = \begin{bmatrix} 5 & -11 \\ 11 & -23 \end{bmatrix}$$

Example: 7

If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$

Verify that $(AB)' = B'A'$.

Solution:

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+15 & -2+2+0 \\ 4+0+0 & -4+2+0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$$

Again, $B'A' = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2+0+15 & 4+0+0 \\ -2+2+0 & -4+2+0 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$$

Hence, $(AB)' = B'A' = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$.

Example: 8

If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = 0$

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Solution:

We have, $A^2 = A.A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

$$-5A = -5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix}$$

$$-14\,I = -14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$\therefore\ A^2 - 5A - 14I = A^2 + (-5A) + (-14\,I)$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 29-15-14 & -25+25+0 \\ -20+20+0 & 24-10-14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A^2 - 5A - 14I = 0$

Example: 9

If $A = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$ then compute A^3.

Given matrix is $A = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$ then $A^2 = A.A = \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9+24 & 18+6 \\ 12+4 & 24+1 \end{bmatrix} = \begin{bmatrix} 33 & 24 \\ 16 & 25 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 33 & 24 \\ 16 & 25 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 99+96 & 198+24 \\ 48+100 & 96+16 \end{bmatrix}$$

$$= \begin{bmatrix} 195 & 222 \\ 148 & 112 \end{bmatrix}$$

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Exercise 3.2

1. If $A = \begin{bmatrix} 5 & -2 & 5 \\ 7 & 0 & 8 \\ 2 & 3 & 6 \end{bmatrix}$

   Find: (a) The number of rows and columns.

   (b) The size of the matrix and.

   (c) value of a_{12}, a_{23}, a_{31}.

2. (a) Construct a 2×3 matrix whose elements are given by $a_{ij} = i + 2j$,

   (b) Construct a 3×3 matrix whose elements are given by

   (i) $a_{ij} = i - j$

   (ii) $a_{ij} = i \times j$.

3. Find A + B and A – B when $A = \begin{bmatrix} 4 & 3 & 3 \\ 2 & 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 & 6 \\ 2 & 3 & 2 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

   Find (i) A + 2B

   (ii) B – 4C

   (iii) A – 2B + 3C.

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$

   Verify that A + (B + C) = (A + B) + C.

6. (i) From the following equation, find the values of x and y.

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$$

   (ii) Find x, y and z satisfying the equation.

$$2\begin{bmatrix} x & z \\ y & 6 \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

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7. (i) Find the matrices X and Y, if

$$X + Y = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & -6 \\ 7 & 3 & 8 \end{bmatrix} \text{ and } X - Y = \begin{bmatrix} -5 & -4 & 8 \\ 11 & 2 & 0 \\ -1 & 7 & 4 \end{bmatrix}$$

(ii) If $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$, find X and Y.

8. Evaluate the following:

(i) $[2\ 3\ 5]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}[1\ 2\ 3]$

(iv) $\begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & 2 \\ 1 & 6 & 3 \end{bmatrix}$

9. (i) If $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}$, find AB and BA, is AB = BA?

(ii) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, show that $AB \neq BA$.

(iii) If $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$

Verify that $A(B + C) = (AB + AC)$.

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(iv) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, verify that $A^2 - 7A - 2I = 0$ where I and O is unit and null matrix respectively.

(v) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 7 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$

Then verify that (AB)C = A (BC).

10. (i) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$

Is $(A + B)^2 = A^2 + 2AB + B^2$?

(ii) Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

Find a matrix D such that AB = CD.

(iii) Find x, if $[2x \quad 3] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = 0$

(iv) If $f(x) = x^2 - 5x + 7$ and $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find f(A).

(v) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find K such that $A^2 - 8A + KI = 0$

Where I and 0 are the unit and null matrices respectively.

11. If $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and K = 3

Verity the following:

(i) $(A + B)' = A' + B'$ (ii) $(AB)' = B'A'$ (iii) $(KB)' = KB'$

(iv) $(A')' = A$ (v) $(A - B)' = A' - B'$.

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Answers

1. (a) 3,3 (b) 3×3 (c) $-2, 8, 2$

2. (a) $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ (b) (i) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 7 & 9 \\ 4 & 9 & 4 \end{bmatrix}, \begin{bmatrix} 7 & -1 & -3 \\ 0 & 3 & 0 \end{bmatrix}$.

4. (i) $\begin{bmatrix} 4 & 10 \\ -1 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & -17 \\ 14 & -11 \end{bmatrix}$ (iii) $\begin{bmatrix} -6 & 13 \\ 16 & 4 \end{bmatrix}$

6. (i) $x = 2, y = 9$ (ii) $x = 3, y = 6, z = 9$.

7. (i) $X = \begin{bmatrix} -2 & -2 & 5 \\ 8 & 3 & -3 \\ 3 & 5 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & 2 & -3 \\ -3 & 1 & -3 \\ 4 & -2 & 2 \end{bmatrix}$

   (ii) $X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$, and $Y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

8. (i) [23] (ii) $\begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$. (iii) $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{bmatrix}$ (iv) $\begin{bmatrix} 9 & -7 & -4 \\ 8 & 40 & 21 \end{bmatrix}$

9. (i) $AB = \begin{bmatrix} 18 & -11 & 10 \\ -16 & 47 & 10 \\ 62 & -23 & 42 \end{bmatrix}$, $BA = \begin{bmatrix} 49 & 24 \\ -7 & 58 \end{bmatrix}$

10. (i) No. (ii) $D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$ (iii) $0, -\frac{3}{2}$

    (iv) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (v) 7

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Chapter Review

Algorithm	:	a finite set of precise instructions for performing a computationorsolving a problem.
Searching algorithm	:	the problem of locating an element in a list.
Sorting	:	the reordering of the elements of a list into non decreasing order.
Time complexity	:	the amount of time required for an algorithm to solve a problem. .
Worst-case time complexity	:	the greatest amount of time required for an algorithm to solve a problem of a given size.
Average –case time complexity	:	the average amount of time required for an algorithm to solve a problem of a given size.
Zero-one matrix	:	A matrix with entry equal to either 0 or 1.
$A \vee B$	:	The join of A and B
$A \wedge B$	:	The meet of A and B

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Unit 4

Recursion on sequence and Series

- Find the nth term of sequence by using recursive formula.
- Use summation notation for the given sequences.
- Define recurrence relations with examples.
- Solve the recursive relations.
- Identify the recursively defined functions and use it to find the term of sequences.

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4.1 Introduction

Meaning of word 'sequence' in ordinary English is a collection of object each following the one before, such that it has known first number, second number, third number and so on. The word sequence in mathematics has same meaning as in ordinary English.

4.2 Sequences and summations

The term sequence develops in the process of counting in a very natural way. It is begin with the number 1. The first number is 1, followed it by number 2, the number 2 by 3 and so on. Then 1, 2, 3, . . . are counting numbers. But students will come across a sequence of numbers which are having a common difference. The difference between two consecutive pairs are the same. Also the ratio of two consecutive pairs are the same. After complete this chapter, student can understand that these two special type of sequences are called arithmetic and geometric sequence respectively

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Sequence

Let us consider the following set of numbers.

i. 27, 1, 25, 28, 11, . . .

ii. 3, 7, 11, 15, . . .

iii. 1, 2, 4, 8, . . .

iv. 2, 7, 17, 32, 52, . . .

In (i) the numbers are not arranged in a particular order. It is, therefore, not possible to indicate the number next to 11. So they are not in sequence. But in (ii), (iii), (iv), the numbers are arranged in a particular order. Then, there are possible to indicate the number next to 15 is 19; next to 8 is 16 and next to 52 is 77. Hence they are in sequences.

Thus, if a set of numbers is arranged by a some mathematical rule is known as sequence. The terms of the sequences are separated by commas.

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Examples:

i. 2, 4, 6, 8, 10, . . . Arithmetic sequence.

ii. 3, 9, 27, 81, . . . Geometric sequence.

iii. 1, 3, 7, 15, 31, . . . A type of sequence.

a. **Finite sequence:** A sequence having finite number of terms is known as finite sequence. Example (i) 3, 6, 9, 12 . . . to 6 terms. (ii) 2, 4, 6, 8, 10, ... 20.

b. **Infinite sequence:** A sequence having infinite number of terms is known as infinite sequence. Example (i) 2, 4, 6, 8, 10, ...

Series

If the terms of the sequences are connected by plus or minus sign, then the resulting expression is called series.

a. **Finite Series:**(i)2 + 4 + 6 + 8 + . . . up to 10 terms.(ii) 3 + 6 + 9+ . . . + 21.

b. **Infinite series**: 5 + 10 + 15 + 20 + ... ∞.

Progression

A set of numbers is called progression if the difference (or ratio) of any term and its preceding term is constant. So,the progression are sequences but all the sequences are not progression.

Examples:

i. 5, 10, 15, 20, . . .

ii. 2, 4, 8, 16, . . .

iii. 16, 14, 12, . . ., 0

iv. 3, 9, 27, . . ., 243

A sequence is defined as a function $f : N \rightarrow S$, where $N = \{1, 2, 3, \ldots \ldots \ldots n\}$ and $S = \{a_1, a_2, a_3 \ldots \ldots \ldots a_n\}$ is any set. If S = R (real number), then the sequence is called a real sequence.

We can list the integer multiples of 2 by assigning each multiple of a position.

Domain	:	1	2	3	4	... ...	n	... ...
Range	:	2	4	6	8	... ...	2n	... ...

A sequence $\{a_n\}$ is known when its nth term a_n is given. For example, if $a_n = 2n$, then the sequence is 2, 4, 6, 8 ... ...

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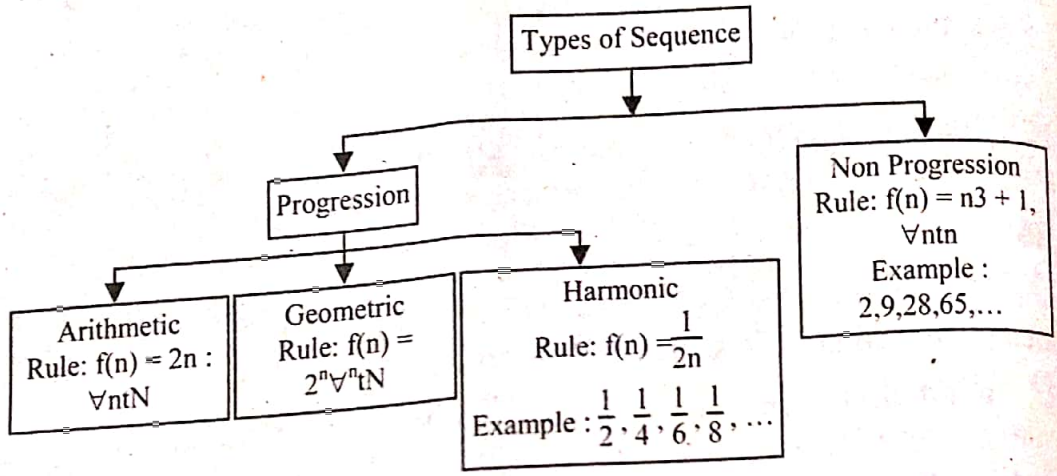
Progression:

If the terms of a sequences are written under specific conditions, then the sequence is called a progression here we will study three special types of progression:



Types of progression

i. Arithmetic progression ii. Geometric progression

iii. Harmonic progression

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4.2.1 Arithmetic Progression

A certain number can be added to (or subtracted from) any term to get the next term is called arithmetic progression (sequence). The fixed (certain) number is called common difference. It is denoted by 'd'. Arithmetic progression is denoted by A.P.

Now, $t_1 = a =$ first term.

$t_2 = a + d = a + (2 - 1)$ = second term

$t_3 = a + 2d = a + (3 - 1)d$ = third term

$t_4 = a + 3d = a + (4 - 1)d$ = Fourth term.

:

$t_n = a + (n - 1)d = n^{th}$ term.

Hence, $b = l =$ last term.

$\therefore \quad b = l = t_n = a + (n - 1)d$

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Arithmetic mean (AM)

A term between first and last term is called an arithmetic means.

i. **Arithmetic mean between a and b is A.**

Then, a, A and b are in AP

by the definition, $A - a = b - A$

or, $A + A = a + b$

or, $2A = a + b$

$\therefore \quad A = \frac{a+b}{2}$

ii. **To insert 'n' arithmetic mean between a and b:**

Let, $A_1, A_2, A_3, \ldots, A_n$ be 'n' arithmetic mean between a and b.

i.e., $a, A_1, A_2, A_3, \ldots A_n, b$ are A.P.

We know that, $t_n = a + (n-1)d$

Where, $n = n + 2, a = a, t_n = b$

Then, $b = a + (n + 2 - 1)d.$

or, $b - a = (n + 1)d.$

Therefore, $d = \frac{b-a}{n+1}$, where n be the number of AMS.

Sum of a series in arithmetic progression:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

OR

$$S_n = \frac{n}{2}[a + l]$$

Where, S_n = Sum to n terms of an A.P.

n = Number of terms.

a = First term

d = common difference

l = last term.

Note: $S_n = \frac{n}{2}[2a + (n-1)d]$ *be the sum of the* n^{th} *term of the arithmetic series.*

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Worked Out Examples

Example: 1

Examine whether the following set of numbers forms an AP.

i. 4, 7, 10, 13, 16, 19

ii. 3, 7, 10, 12 17, 25.

Solution:

i. **Given, 4, 7, 10, 13, 16, 19**

Here, $t_2 - t_1 = 7 - 4 = 3$

$t_3 - t_2 = 10 - 7 = 3$

$t_4 - t_3 = 13 - 10 = 3$

$t_5 - t_4 = 16 - 13 = 3$

$t_6 - t_5 = 19 - 16 = 3$

From above, in all, the difference between two consecutive terms is equal. Hence, the set of numbers 4, 7, 10, 13, 16, 19 form an of A P.

ii. **Given, 3, 7, 10, 12, 17, 25**

Here, $t_2 - t_1 = 7 - 3 = 4$

$t_3 - t_2 = 10 - 7 = 3$

$t_4 - t_3 = 12 - 10 = 2$

$t_5 - t_4 = 17 - 12 = 5$

$t_6 = t_5 = 25 - 17 = 8$

From above, the difference between two consecutive terms is not equal. Hence, the set of numbers are not in sequence.

Example: 2

Find out the fifth term of the series

Solution: $10 + 17 + 24 + \ldots$

Here, $10 + 17 + 24 + \ldots$

$a = 10, d = 17 - 10 = 7, t_5 = ?, \quad n = 5$

We have,

$t_n = a + (n - 1)d$

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$t_5 = 10 + (5 - 1)7$

or $= 10 + 4 \times 7$

$= 10 + 28$

$= 38$

Therefore, Required $t_5 = 38$

Example: 3

Find the first term and common difference, whose 4th and 7th terms are 11 and 17 respectively.

Solution:

For t_4, $t_n = a + (n - 1)d$.

or, $t_4 = a + (4 - 1)d$.

or, $11 = a + 3d$

or, $a = 11 - 3d$ . . . (i)

For t_7, $t_n = a + (n - 1)\,d$.

or, $t_7 = a + (7 - 1)d$.

or, $17 = a + 6d$

or, $17 = 11 - 3d + 6d$ (from (i) a=11-3d)

or, $17 - 11 = 3d$

or, $6 = 3d$

or, $d = \frac{6}{3} = 2$

Form (i), $a = 11 - 3d$

or, $a = 11 - 2 \times 3 = 11 - 6 = 5$

Hence, First term (a) = 5

Common difference (d) = 2

Example: 4

Find the value of $\sum_{n=1}^{6} (4n + 5)$

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Solution: Given

$$\sum_{n=1}^{6}(4n+5) = (4\times1+5)+(4\times2+5)+(4\times3+5)+(4\times4+5)$$
$$+(4\times5+5)+(4\times6+5)$$
$$= 9+13+17+21+25+29$$
$$= 114$$

Example: 5

How many numbers of two digits are divisible by 7?

Solution:

Two digit numbers divisible by 7 are

14, 21, 28, . . . 98.

So, $a = 14, d = 21 - 14 = 7, l = 98.$

Then, $l = a + (n-1)d$

or, $98 = 14 + (n-1)\times 7$

or, $98 - 14 = (n-1)\times 7$

or, $84 = 7(n-1)$

or, $\frac{84}{7} = n - 1$

or, $12 = n - 1$

or, $n = 12 + 1 = 13.$

Hence, there are 13 numbers of two digits divisible by 7.

Example: 6

Find the first four term of $t_n = \frac{2n-1}{n+1}$

Solution:

We have, $t_n = \frac{2n-1}{n+1}$

When = 1, $t_1 = \frac{2\times1-1}{1+1} = \frac{2-1}{2} = \frac{1}{2}$

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When $n = 2$, $t_2 = \frac{2 \times 2 - 1}{2 + 1} = \frac{4 - 1}{3} = \frac{3}{3} = 1$

When $n = 3$, $t_3 = \frac{2 \times 3 - 1}{3 + 1} = \frac{6 - 1}{4} = \frac{5}{4}$

When $x = 4$, $t_4 = \frac{2x4 - 1}{4 + 1} = \frac{8 - 1}{5} = \frac{7}{5}$

Therefore, First four numbers are $\frac{1}{2}$, 1, $\frac{5}{4}$ and $\frac{7}{5}$

Example: 7

In an AP the first term is 3, the last term 45, the sum 235 find the common difference.

Solution:

Given,

$a = 3$

$l = 45$

$s_n = 235$

$d = ?$

We know that, $S_n = \frac{n}{2}(a + l)$

or, $235 = \frac{n}{2}[2 + 45]$

or, $470 = n \times 47$

or, $n = \frac{470}{47} = 10$

Again, $l = a + (n - 1)d$

or, $45 = 2 + (10 - 1)d.$

or, $45 - 2 = 9d$

or, $\frac{43}{9} = d$

Therefore, common difference $(d) = \frac{43}{9}$

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Example: 8

Find the 13th terms of the series $7 + 2 - 3 - 8 - 13, \ldots$

Solution:

Given

$$7 + 2 - 3 - 8 - 13 \ldots$$

Here, $a = 7$, $d = 2 - 7 = -5$, $t_{13} = ?$

We know that,

$$t_n = a + (n - 1)\, d.$$

or, $t_{13} = t + (13 - 1)(-5)$

or, $t_{13} = 7 + 12 \times (-5) = 7 - 60 = -53$

Therefore, $t_{13} = -53$

Example: 9

4, 7, 10, 13, . . . are in Arithmetic sequence, find

i. t_{14}

ii. Which term is 301?

iii. S_{15}

Solution:

Given,

4, 7, 10, 13, . . .

Here, $a = 4$, $d = 7 - 4 = 3$,

i. We have

$$t_n = a + (n - 1)d.$$

or, $t_{14} = 4 + (14 - 1) \times 3$

$= 4 + 13 \times 3$

$= 4 + 39$

$= 43$

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ii. Let t_n be 301

Then, $t_n = a + (n-1)d$

or, $301 = 4 + (n-1) \times 3$

or, $301 - 4 = 3(n-1)$

or, $\frac{297}{3} = n - 1$

or, $99 = n - 1$

or, $n = 100$

Therefore, 301 is the 100th term.

iii. We have, $S_n = \frac{n}{2}[2a + (n-1)d]$

or, $S_{15} = \frac{15}{2}[2 \times 4 + (15-1) \times 3]$

$= \frac{15}{2}[8 + 14 \times 3]$

$= \frac{15}{2} \times 50$

$= 15 \times 25$

$= 375$

Example: 10

Find three numbers in AP such that their sum is 30 and sum of their square is 350.

Solution: Let, $a - d$, a and $a + d$ are three numbers in AP.

Case I: Sum of three numbers is 30

i.e. $a - d + a + a + d = 30$

or, $3a = 30$

or, $a = 10$

Case II: Sum of three number's square is 350.

i.e. $(a-d)^2 + a^2 + (a+d)^2 = 350$.

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or, $(10-d)^2 + (10)^2 + (10+d)^2 = 350$,

or, $100 - 20d + d^2 + 100 + 100 + 20d + d^2 = 350$

or, $300 + 2d^2 = 350$

or, $2d^2 = 350 - 300$

or, $2d^2 = 50$, $d^2 = 5$

or, $d = \pm 5$

When $d = 5, a = 10$

$t_1 = a - d = 10 - 5 = 5$

$t_2 = a = 10$

$t_3 = a + d = 10 + 5 = 15$

When $d = -5, a = 10$

$t_1 = a - d = 10 + 5 = 15$

$t_2 = a = 10$

$t_3 = a + d = 10 - 5 = 5$

Hence, the required three numbers are (5, 10, 15) or (15, 10, 5)

Example: 11

Find 4 arithmetic means between $\frac{1}{2}$ and $4\frac{1}{4}$.

Solution:

Let, A_1, A_2, A_3, A_4 are required four arithmetic mean between $\frac{1}{2}$ and $4\frac{1}{4}$.

So, $n = 4$ (no. of AMS)

$a = \frac{1}{2}, b = 4\frac{1}{4} = \frac{17}{4}$

Then, $d = \frac{b-a}{n+1} = \frac{\frac{17}{4} - \frac{1}{2}}{4+1}$

$= \frac{\frac{17-2}{4}}{5}$

$= \frac{\frac{15}{4}}{5}$

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$$= \frac{15}{5 \times 4} = \frac{3}{4}$$

Therefore,

$$A_1 = a + d = \frac{1}{2} + \frac{3}{4}$$

$$= \frac{2+3}{4} = \frac{5}{4}$$

$$A_2 = a + 2d = \frac{1}{2} + 2 \times \frac{3}{4} = \frac{1}{3} = \frac{3}{2}$$

$$= \frac{1+3}{2} = \frac{4}{2} = 2$$

$$A_3 = a + 3d = \frac{1}{2} + 3 \times \frac{3}{4} = \frac{2+9}{4} = \frac{11}{4}$$

$$A_4 = a + 4d = \frac{1}{2} + 4 \times \frac{3}{4} = \frac{1}{2} + 3$$

$$= \frac{1+6}{2} = \frac{7}{2}$$

Therefore, Required AMS between $\frac{1}{2}$ and $4\frac{1}{4}$

are, $\frac{5}{4}, 2, \frac{11}{4}$ and $\frac{7}{2}$

4.2.2 Geometric Progression

If the ratio of any term (except the first) of the progression to its preceding term is always the same then the progression is form a geometric progression. This ratio is called the common ratio and generally it is denoted by 'r'.

A geometric progression having first term 'a' and common ratio r can be written as a, ar, ar^2, ar^3 ... ... ar^{n-1}

Now, $t_1 = a$ = first term.

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t_2 = second term = $ar = ar^{2-1}$

t_3 = 3rd term = $ar^2 = ar^{3-1}$

.....

.....

.....

$$\boxed{t_n = ar^{n-1}}$$

or

$$\boxed{l = \text{last term} = ar^{n-1}}$$

Where $t_n = ar^{n-1}$ be the general (n^{th}) term of the geometric series.

Note: $S_n = \dfrac{a(r^n - 1)}{r - 1}$, $|r| > 1$ *be the sum of the* n^{th} *term of geometric series.*

Sequence of the form $t_1, t_2, t_3 \ldots \ldots \ldots, t_n$ *are often used in computer science. These finite sequences are also called* ***strings****. The* ***length of the string*** *is number of terms in this string.*

The geometric progression holds the following properties.

a) If each term of G.P is multiplied or divided by a constant number, the resulting progression is also a G.P.

b) If each term of a G.S is raised to a constant power , the resulting progression is again in G.P.

For example: if a, b, c, d … … .. be in G.P and k a constant number not equal to zero then

i) ka, kb, kc, kd, …. …. Are also in G.P

ii) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k} \ldots \ldots \ldots$ is a G.P.

iii) $A^k, b^k, c^k, d^k, \ldots \ldots \ldots$ is in a G.P

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Worked Out Examples

Example – 1

Examine whether the following sequences form a G.P.?

i. 2, 4, 8, 16 . . .

ii. 3, 15, 21, 27, . . .

Solution:

i. Given, 2, 4, 8, 16, . . .

Now, $\frac{t_2}{t_1} = \frac{4}{2} = 2$

$\frac{t_3}{t_2} = \frac{8}{4} = 2$

$\frac{t_4}{t_3} = \frac{16}{8} = 2$

In above, the ratio of any term and the term preceding to it's same. So, the sequence 2, 4, 8,1 6,. . . form a G.P.

ii. Given

3, 15, 21, 27, . . .

Now, $\frac{t_2}{t_1} = \frac{15}{3} = 5$

$\frac{t_3}{t_2} = \frac{21}{15} = \frac{7}{5}$

$\frac{t_4}{t_3} = \frac{27}{21} = \frac{9}{7}$

In above, the ratio of any term and the term preceding to it is not constant so, the given sequence 3, 15, 21, 27, . . . does not form a G.P.

Example: 2

Find the fifth term of the series 6 + 12 + 24 + . . .

Solution: Here,

6 + 12 + 24 + . . .

Now, $a = 6, r = \frac{t_2}{t_1} = \frac{12}{6} = 2$

$t_5 = ?$

$n = 5$

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We have, $t_n = ar^{n-1}$

or, $t_5 = ar^4$

$= 6(2)^4$

$= 6 \times 16 = 96$

Example: 3

Find the number of term of the G.P. $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{128}$.

Solution:

Here, $1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{128}$

So, $a = 1, r = \frac{t_2}{t_1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}, l = \frac{1}{128}$

Now,

$l = ar^{n-1}$

or, $\frac{1}{128} = (1)\left(\frac{1}{2}\right)^{n-1}$

or, $\left(\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^{n-1}$

or, $7 = n - 1$

or, $n = 7 + 1 = 8$

Therefore, Required no. of term (n) = 8.

Example: 4

The second and eight term of a G.P. are 9 and $\frac{1}{81}$ respectively.

i. Find first term

ii. Find common ratio

iii. Find 15th term

iv. Find the series

Solution: Given

$t_2 = 9$ and $t_8 = \frac{1}{81}$

Now, $t_2 = 9$

or, $ar = 9$

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$$a = \frac{9}{r} \quad \ldots (i)$$

or,

Also, $t_8 = \frac{1}{81}$

$$ar^7 = \frac{1}{81}$$

or,

$$\frac{9}{r} \times r^7 = \frac{1}{81} \text{ [From (i)]}$$

or,

$$r^6 = \frac{1}{9 \times 81}$$

or,

$$r^6 = \left(\frac{1}{3}\right)^6$$

or,

$$\therefore \quad r = \frac{1}{3}$$

Again from (i) $a = \frac{9}{\frac{1}{3}} = 9 \times 3 = 27$

Hence, (i) $a = 27$, (ii) $r = \frac{1}{3}$

iii. We have, $t_{15} = ar^{14}$

$$= 27\left(\frac{1}{3}\right)^{14}$$

$$= \left(\frac{1}{3}\right)^{11} = \frac{1}{177147}$$

iv. The geometric series is

$$t_1 + t_2 + t_3 + \ldots$$

$$= ar + ar^2 + ar^3 + \ldots$$

$$= 27 \times \frac{1}{3} + 27 \times \left(\frac{1}{3}\right)^2 + 27\left(\frac{1}{3}\right)^3$$

$= 9 + 3 + 1 + \ldots$ which is the required G.S.

Example: 5

Find the sum of the following geometric series.

i. $1 - 3 + 9 - 27 + \ldots$ to terms.

ii. $3 + 6 + 12 + \ldots + 384$.

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Solution:

i. Given, $1 - 3 + 9 - 27 + \ldots$ to 6 terms

Here, $t_1 = 1, t_2 = -3, n = 6$

$$r = \frac{t_2}{t_1} = \frac{-3}{1} = -3 < 1.$$

We have, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$= \frac{1[1-(-3)^6]}{1-(-3)}$$

$$= \frac{1-729}{1+3} = \frac{-728}{4} = -182$$

ii. Given, $3 + 6 + 12 + \ldots + 384$.

Here,

$a = 3$

$r = \dfrac{t_2}{t_1} = \dfrac{6}{3} = 2$

$l = 384.$

We know that, $S_n = \dfrac{lr - a}{r - 1}$

$$= \frac{384 \times 2 - 3}{2 - 1} = 765$$

Example: 6

The sum of a series in GP, whose common ratio is 3, is 728 and the last term is 486. Find the first term.

Solution: Given,

Common ratio (r) = 3

$S_n = 728$

$l = 486.$

We have,

$$S_n = \frac{lr - a}{r - 1}$$

or, $$728 = \frac{486 \times 3 - a}{3 - 1}$$

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or, $728 = \frac{1458 - a}{2}$

or, $1456 = 1458 - a$

or, $a = 1458 - 1456 = 2$

Therefore, Required first term (a) = 2

Example: 7

Insert 3 geometric means between $2\frac{1}{4}$ and 36.

Solution: Let, G_1, G_2 and G_3 be required three GMs between $2\frac{1}{4}$ and 36.

So, $a = 2\frac{1}{4} = \frac{9}{4}$

$b = 36$

$n = 3$(No. of GMs)

We have, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

or, $r = \left(\frac{36}{\frac{9}{4}}\right)^{\frac{1}{3+1}}$

$= \left[\frac{36 \times 4}{9}\right]^{\frac{1}{4}} = [4 \times 4]^{\frac{1}{4}}$

$= (2^4)^{\frac{1}{4}} = 2$

Then, $G_1 = ar = \frac{9}{4} \times 2 = \frac{9}{2}$

$G_2 = ar^2 = \frac{9}{4}(2)^2 = 9$

$G_3 = ar^3 = \frac{9}{4}(2)^3 = 9 \times 2 = 18$

Hence, Required GMs. are $\frac{9}{2}$, and 18 between $\frac{9}{4}$ and 36.

Example: 8

The sum of three number in G.P. is 14 and their product is 64. Find three numbers.

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Solution: Let, $\frac{a}{r}$, a and ar be three numbers in G.P.

Case I: Sum of three number is 14.

or, $\frac{a}{r} + a + ar = 14 \ldots$ (i)

Case II: Product of three numbers is 64.

or, $\frac{a}{r} . a . ar = 64$

or, $a^3 = 64$

or, $a = 4$

From (i), $\frac{a}{r} + a + ar = 14$

or, $\frac{4}{r} + 4 + 4r = 14$

or, $4 + 4r + 4r^2 = 14r$

or, $4 - 10r + 4r^2 = 0$

or, $2 - 5r + 2r^2 = 0$

or, $2 - 4r - r + 2r^2 = 0$

or, $2(1 - 2r) - r(1 - 2r) = 0.$

or, $(1 - 2r)(2 - r) = 0$

Either, $1 - 2r = 0$ or, $2 - r = 0$

or, $1 = 2r$ or, $r = 2$

or, $\frac{1}{2} = r$

$\therefore$ $r = 2$ or $\frac{1}{2}$

When, $a = 4, r = 2$

$t_1 = \frac{a}{r} = \frac{4}{2} = 2$

$t_2 = a = 4$

$t_3 = ar = 4 \times 2 = 8$

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Also, when $a = 4, r = \frac{1}{2}$

$$t_1 = \frac{a}{r} = \frac{4}{\frac{1}{2}} = 4 \times 2 = 8$$

$$t_2 = a = 4$$

$$t_3 = ar = 4 \times \frac{1}{2} = 2$$

Therefore, Required three number are

(2, 4, 8) or (8, 4, 2)

4.2.3 Harmonic Progression

A progression of numbers is harmonic sequence if its reciprocal is in the form of arithmetic progression.

For example: If 2, 4, 6, 8,… … are in arithmetic progression then

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots \ldots$ are in harmonic progression. Since to every harmonic progression there corresponds an A.P. the term of an H.P are determined with reference to the corresponding A.P.

Properties of Harmonic progression

a) If each term of an H.P be multiplied or divided by a constant quantity the sequence of the resulting numbers is also an H.P. so if a, b, c, d, … … be an H.P and k a constant not equal to 0 then,

i) ka, kb, kc, kd,… … … are in H.P

ii) $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k} \ldots \ldots \ldots$ is again in H.P

Example: If x^2, y^2, z^2 are in A.P then prove that y+z, z+x , x+ y are in H.P.

Solution

Here, y+z, z+x, x+y will be in H.P. if $\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$ are in A.P.

i.e if $\frac{1}{z+x} - \frac{1}{y+z} = \frac{1}{x+y} - \frac{1}{z+x}$

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or, $\frac{y+z-z-x}{(z+x)(y+z)} = \frac{z+x-x-y}{(x+y)(z+x)}$

or, $\frac{y-x}{(z+x)(y+z)} = \frac{z-y}{(x+y)(z+x)}$

or, $y^2-x^2 = z^2-y^2$

i.e if x^2, y^2, z^2 are in A. P which is given

4.2.4 Recurrence relations

In this section, we discussed how relation can be defined recursively. Recurrence relations are useful in certain counting problems. A recurrence relation relates the nth element of a sequence to its predecessors.

Consider the arithmetic progression is an infinite sequence of numbers. Such as 5, 8, 11, 14, 17 ... ... where the difference of each term, other than the first, by its immediate predecessor is a constant, called the common difference. For our sequence this common difference is 8 – 3 = 5, 11 – 8 = 3 and so on. If a_1, a_2, a_3 ... ... is a arithmetic progression. Then, $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 =$ ... ... $= a_n - a_{n-1} = d$ the common difference. In our particular arithmetic progression we have $a_n = a_{n-1} + 3, n \geq 2$. The recurrence relation $a_n = a_{n-1} + 3$, does not define a unique arithmetic progression. The sequence 4, 7, 10, 13 ... ... also satisfies the relation. To pinpoint a particular sequence described by $a_{n+1} = 3a_n$, we need to know one of the terms of the sequence.

Hence, $a_n = a_{n-1} + 3, n \geq 2, a_1 = 5$ ... ... (i)

uniquely defines the sequence, 5, 8, 11, 14, 17 ... ...

The equation (i) is an example of a recurrence relation. A recurrence relation defines a sequence by giving the nth value in terms of certain of its predecessors, the nth value is give in terms of the immediately preceding value. In order for a recurrence relation to define a sequence, a "start–up" (or base) value, such as $a_1 = 5$ must be given. These 'base' values are called initial condition.

Definition:

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more preceding terms of the sequence, namely a_0, a_1 ... ... a_{n-1}.

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Example – 1

The Fibonacci sequence is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, and initial conditions: $f_1 = 1$, $f_2 = 2$.

Example – 2

Find the recurrence relation and initial conditions of the sequence 3, 7, 11, 15 ... ...

Solution:

Here, $a_1 = 3$

$a_2 = a_1 + 4$

$a_3 = a_2 + 4$

... ...

... ...

$a_n = a_{n-1} + 4$

Hence, The recurrence relation is $a_n = a_{n-1} + 4$, $n \geq 2$, $a_1 = 3$

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Example – 3

Find the recurrence relation of the sequence 1, 3, 7, 15, 31, 63 ... ...

Solution:

Here, $a_1 = 1$

$a_2 = 2 . 1 + 1 = 2a_1 + 1$

$a_3 = 2 . 3 + 1 = 2 . 2a_2 + 1$

... ...

... ...

$a_n = 2a_{n-1} + 1$

Hence, the recurrence relation is $a_n = 2a_{n-1} + 1$, $n \geq 2$, $a_1 = 1$

4.2.5 Use of Summation Notation in Series

The symbol Σ(sigma) stands for summation, the notation 'Σ' used to express the sum of the terms. t_1, t_2, t_3 ... ..., t_n from the sequence $\{t_n\}$.

We use the notation $\sum_{i=1}^{n} t_i = t_1 + t_2 + t_3 + \dots \dots \dots + t_n$.

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Here, summation runs through all integers starting with its lower limit '1' and ending with its upper limit n.

Some useful summation formulae

(i) $\sum_{k=0}^{n} ar^{k-1} \ (r \neq 0)$ $\qquad \frac{ar^n - a}{r-1}, r \neq 1$

(ii) $\sum_{k=1}^{n} k$ $\qquad \frac{n(n+1)}{2}$

(iii) $\sum_{k=1}^{n} k^2$ $\qquad \frac{n(n+1)(2n+1)}{6}$

(iv) $\sum_{k=1}^{n} k^3$ $\qquad \left[\frac{n(n+1)}{2}\right]^2$

(v) $\sum_{k=0}^{\infty} x^k, |x| < 1$ $\qquad \frac{1}{1-x}$

(vi) $\sum_{k=1}^{n} (2k-1)$ $\qquad n^2$

(vii) $\sum_{k=1}^{n} 2k$ $\qquad n(n+1)$

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Example – 1

i. $\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5$

ii. $\sum_{1}^{3} (3r + 2) = (3 . 1 + 2) + (3 . 2 + 2) + (3 . 3 + 2)$

Worked out examples:

Example – 2

Find the 5th term of the sequence $\{a_n\}$, where $a_n = 2 . (-3)^n + 5^n$

Solution:

Here, $a_n = 2 . (-3)^n + 5^n$

When, n = 5 then,

$a_5 = 2 . (-3)^5 + 5^5 = 2639$

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Example – 3

List the first 5 terms of the sequence that begins with 2 and in which each successive term is 3 more than the preceding term.

Solution:

Here, the first term (a) = 2

and common difference (d) = 3

Then the first five terms are

$2, 2+3, 2+2.3, 2+3.3, 2+4.3$

$= 2, 5, 8, 11, 14$

Example – 4

Find the formula for the sequence with the first five terms are 3, 6, 11, 18, 27.s

Solution:

Here, the first five terms, are 3, 6, 11, 18, 27

Let, $S_n = 3 + 6 + 11 + 18 + 27 + \ldots\ldots\ldots + t_n$

Also $\underline{S_n = \quad 3 + 6 + 11 + 18 \ldots\ldots\ldots + t_{n-1} + t_n}$

or, Subtraction, $0 = 3 + [3 + 5 + 7 + 9 + \ldots\ldots\ldots + (n-1) \text{ terms}] - t_n$

or, $t_n = 3 + \frac{n-1}{2}[2.3 + (n-1-1)2]$

or, $t_n = 3 + \frac{n-1}{2}[6 + 2n - 4]$

$= 3 + \frac{n-1}{2}[2n + 2]$

$= 3 + (n-1)(n+1)$

$= 3 + n^2 - 1$

$= n^2 + 2$

$\therefore t_n = n^2 + 2$

Example – 5

Find the value of $\sum_{J=0}^{8} (2^{1+J} - 2^J)$

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Solution:

Here, $\sum_{J=0}^{5} (2^{1+J} - 2^J)$

$= (2^{1+0} - 2^0) + (2^{1+1} - 2^1) + (2^{1+2} - 2^2) + (2^{1+3} - 2^3) + (2^{1+4} - 2^4) + (2^{1+5} - 2^5)$

$= (2-1) + (4-2) + (8-4) + (16-8) + (32-16) + (64-32)$

$= 1 + 2 + 4 + 8 + 16 + 32$

$= 63$

Example – 6

What is the value of $\sum_{n \in S} n$ where $S = \{1, 3, 5, 7\}$.

Solution:

Here, $S = \{1, 3, 5, 7\}$

$\sum_{n \in S} n = 1 + 3 + 5 + 7 = 16$

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4.3 Solution of recurrence relation

To solve a recurrence relation involving the sequence $a_0, a_1, a_2 \ldots \ldots \ldots$ is to find an explicit formula for the general term a_n. In many computations, many a times it is easier to obtain a formula by iteration. We use the recurrence relation to write the nth term a_n in terms of certain of its predecessors $a_{n-1}, a_{n-2} \ldots \ldots \ldots a_0$. We then successively use the recurrence relation to replace each of $a_{n-1} \ldots \ldots \ldots$ by certain of their predecessors. We continue until an explicit formula is obtained.

Example – 7

Solve the recurrence relation $a_{n+1} = 3a_n$, $n \geq 0$ and $a_0 = 5$

Solution:

The first four terms of the sequence are:

$a_0 = 5$

$a_1 = 3a_0 = 3(5)$

$a_2 = 3a_1 = 3(3a_0) = 3^2(5)$

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$a_3 = 3a_2 = 3(3^2(5)) = 3^3(5)$

$\therefore$ $a_n = 5(3^n)$ is the general solution of the given recurrence relation.

Example – 8

Solve the recurrence relation $a_n = a_{n-1} + 3$ subject to the initial condition $a_1 = 2$

Solution:

Here, $a_n = a_{n-1} + 3$, $a_1 = 2$ the first five terms of the sequence are

$a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3$

$a_3 = a_2 + 3 = 2 + 3 + 3 = 2 + 6 = 2 + 2.3$

$a_4 = a_3 + 3 = (2 + 2 . 3) + 3 = 2 + 3.3$

$a_5 = a_4 + 3 = (2 + 3 . 3) + 3 = 2 + 4.3$

Hence, $a_n = 2 + 3(n - 1)$ is the general solution of the given recurrence relation.

Example – 9

Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if $a_n = 3(-1)^n + 2^n - n + 2$.

Solution:

Here, $a_n = 3(-1)^n + 2^n - n + 2$, for $n \geq 2$

We have,

$$a_{n-1} + 2a_{n-2} + 2n - 9$$

$$= [3(-1)^{n-1} + 2^{n-1} - (n-1) + 2] + 2[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2] + 2n - 9$$

$$= [3(-1)^n(-1)^{-1} + 2^n . 2^{-1} - n + 1 + 2] + 2[3(-1)^n(-1)^{-2} + 2^n . 2^{-2} - n + 2 + 2] + 2n - 9$$

$$= -3(-1)^n + \frac{2^n}{2} - n + 3 + 6(-1)^n + \frac{2^n}{2} - 2n + 8 + 2n - 9$$

$$= 3(-1)^n + 2^n - n + 2 = a_n$$

Hence, $\{a_n\}$, where $a_n = 3(-1)^n + 2^n - n + 2$, is a solution of the recurrence relation.

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Exercise - 4

Exercise 4.1

1. Find the indicated terms in each of the following sequence whose n^{th} terms are:

   i. $t_n = \frac{n^2}{2^n}$, t_4, t_6

   ii. $t_n = (-1)^{n-1} 5^{n-1}$, t_3

   iii. $t_n = (-1)^{n-1} n^3$, t_9

2. Write the next three terms of the following sequences

   i. $t_2 = 2$, $t_n = t_{n-1} + 1$, $(n \geq 3)$

   ii. $t_1 = 3$, $t_n = 3t_{n-1} + 2$ for all $n > 1$

   iii. $t_1 = 1$, $t_n = \frac{t_{n-1}}{n}$, $(n \geq 2)$

3. List the first 5 terms of the sequence that begins with 3, where each succeeding term is twice the preceding term.

4. For each of these lists of integers provide a simple formula that generates the term of an integer sequence that begins with the given list.

   i. 1, 0, 2, 0, 4, 0, 8, 0, ...

   ii. 3, 6, 12, 24, 48, 96, ...

   iii. 3, 5, 8, 12, 17, 23, ...

5. What is the value of each of these sums of terms of a geometric progression?

   i. $\sum_{n=0}^{8} 3 \cdot 2^n$

   ii. $\sum_{n=1}^{8} 2^n$

   iii. $\sum_{n=2}^{8} (-3)^n$

   iv. $\sum_{n=0}^{8} 2 \cdot (-3)^n$

6. Solve the relation $a_n = n \cdot a_{n-1}$, where $n \geq 1$ and $a_0 = 1$.

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7. Find a recurrence relation, with initial condition, that uniquely determines each of the following sequences.

   i. 2, 10, 50, 250, ...

   ii. 6, – 18, 54, – 162, ...

   iii. 3, 7, 11, 15, ...

8. Solve the relation $a_n = n.\ a_{n-1}$, where $n \geq 1$ and $a_0 = 1$.

9. Find the solution of the recurrence relation $a_n = a_{n-1} + n$, where $a_0 = 1$

10. Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 0$, $a_1 = 1$

Answers

1. i) $1, \frac{9}{16}$ ii) 25 iii) 729

2. i) 3, 4, 5 ii) 11, 35, 107 iii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{24}$

3. 3, 6, 12, 24, 48

4. i) The term in odd numbered location are the successive power of 2, the term in even numbered locations are all 0.

   ii) $t_n = 3 . 2^{n-1}$

   iii) $t_n, \frac{(n^2 + n + 4)}{2}$

5. i) 1533 ii) 510 iii) 4923 iv) 9842

6. $a_n = n!$

7. i) $a_n = 5a_{n-1}, n \geq 1, a_0 = 2$

   ii) $a_n = -3a_{n-1}, n \geq 1, a_0 = 6$

   iii) $a_n = a_{n-1} + 4, n \geq 2, a_1 = 3$

8. $a_n = n!$

9. $a_n = 1 + \frac{n(n+2)}{2}$

10. $a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n$

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4.4 Recursive algorithm, recursion and iteration, the merge sort

Recursive relation, recursive algorithms and mathematical induction are closely related. In all three, prior instance of the current case are assumed known. A recurrence relation uses prior values in a sequence to compute the current value. A recursive algorithm uses smaller instance of the current input to process the current input. The inductive step in a proof by mathematical induction assumes the truth of prior instances of the statement to prove the truth of the current statement.

a. Recursive algorithm

An algorithm is called the **recursive algorithm** if it solves a problem by reducing it to an instance of the same problem with smaller input.

Algorithm1: A recursive algorithm for computing n!

Input: n, an integer greater than or equal to 0.

Output: n!

1. **Procedure** factorial (n)
2. **If** n = 0 then
3. **Return** (1)
4. **Return** (n* factorial (n – 1))
5. **End** factorial

Algorithm 2: A recursive algorithm for computing a^n.

Procedure power (a : non zero real number, n: non negative integer.

if n = 0 **then** power (a, n): = 1

else power (a, n) : a * power (a, n – 1)

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Recursive and iteration

Recursion and iteration are equally expressive: recursion can be replaced by iteration with an explicit stack, while iteration can be replaced with tail recursion. Which approach is preferable depends on the problem under consideration and the language used. In imperative programming, iteration is preferred, particularly for simple recursion, as it avoids the overhead of function calls and call stack management, but recursion is generally used for multiple recursions. By contrast, in functional languages recursion is preferred, with tail recursion optimization leading to little overhead, and sometimes explicit iteration is not available.

c. Merge sort

In this section we shall study a second method for sorting a given list of n items into ascending order. The method is called the merge sort.

In general, a list of n items to sort into ascending order, the merge sort recursively splits the given list and all subsequent sublist in half (or as close as possible to half) until each sublist contains a single element. Then the procedure merges these sublists in ascending order until the original n items have been so sorted. The splitting and merging processes can best be described by a balanced binary trees, as in the next example.

Example – 1

Sorts the list 6, 2, 3, 5, 9, 1, 7, 4, 8 using merge sort.

Solution:

The succession of merged lists in increasing order is represented by the balanced binary tree of height 4 shown at the bottom of the figure below:

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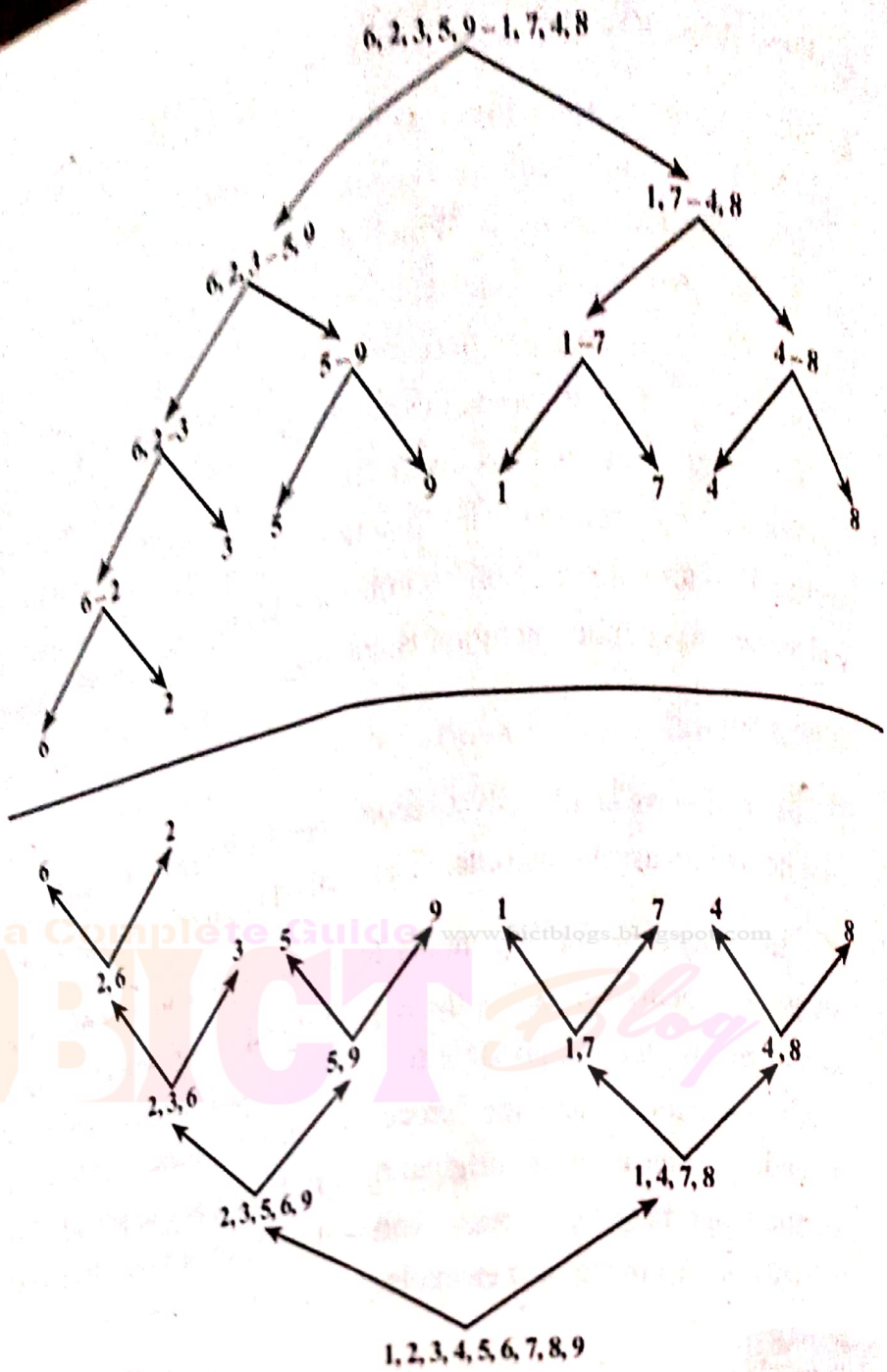

The merge sort of 6, 2, 3, 5, 9, 1, 7, 4, 8

Example – 2

How to merge the two lists 2, 3, 5, 6 and 1, 4

Solution:

First, we compare the smallest elements of each lists. Since $1 \leq 2$, we start out third list with 1, removing 1 from the second list. This leaves us with

2, 3, 5, 6, 4 and 1.

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We compare the first elements of the first two lists. Since $2 \leq 4$, we place 2 after 1 in the third lists and remove 2 from the first list. At this stage, we have the three lists

3, 5, 6, 4 and 1, 2

We continue to compare the first elements in the first two lists, since $3 \leq 4$ we, place 3 after 2 in the third lists and remove 3 from the first list. At this stage, we have the three lists

5, 6, 4 and 1, 2, 3.

By comparing 5 and 4, since 4 is the smaller of these two elements, add it to the third list and remove it from the second list. At this stage, we have the lists 5, 6 and 1, 2, 3, 4.

At this stage the second list is empty, so we simply append the remaining elements 5, 6 of the first list to the third, giving the final list 1, 2, 3, 4, 5, 6.

Example – 3

Merge the two lists 1, 3, 5, 7, 10, 11 and 2, 5, 6, 9.

Solution:

Here the eight steps in our merging the given two lists as shown in table

First list	Second list	Merged list	Comparison
1, 3, 5, 7, 10, 11	2, 5, 6, 9	1	$1 < 2$
3, 5, 7, 10, 11	2, 5, 6, 9	1, 2	$2 < 3$
3, 5, 7, 10, 11	5, 6, 9	1, 2, 3	$3 < 5$
5, 7, 10, 11	5, 6, 9	1, 2, 3, 5	$5 \leq 5$
7, 10, 11	5, 6, 9	1, 2, 3, 5, 5	$5 < 7$
7, 10, 11	6, 9	1, 2, 3, 5, 5, 6	$6 < 7$
7, 10, 11	9	1, 2, 3, 5, 5, 6, 9	$7 < 9$
10, 11		1, 2, 3, 5, 5, 6, 9, 10, 11	

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4.5 Recursively defined functions

A function is said to be recursively defined. If the function must have the following two properties.

i. Specify the base values.

ii. Give a rule for finding its value at an integer from its closer to a base value.

Example – 1

Suppose that f is defined recursively by $f(0) = -1$, $f(1) = 2$, $f(n+1) = f(n) + 3f(n-1)$. Find f(2), f(3), f(4) and f(5)

Solution:

By recursive definition, we have

$f(2) = f(1) + 3\,f(0) = 2 + 3\,(-1) = -1$

$f(3) = f(2) + 3f(1) = -1 + 3(2) = 5$

$f(4) = f(3) + 3f(2) = 5 + 3\,(-1) = 2$

$f(5) = f(4) + 3f(3) = 2 + 3(5) = 17$

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Example – 2

Let a and b be integers and suppose Q(a, b) is defined recursively by

$$Q(a, b) = \begin{cases} 5 \text{ if } a < b \\ Q(a-b, b+2) + a \text{ if } a \geq b \end{cases}$$. Find Q(2, 7), Q(5, 3)

Solution:

$$\text{Here, } Q(a, b) = \begin{cases} 5 \text{ if } a < b \\ Q(a-b, b+2) + a \text{ if } a \geq b \end{cases}$$

$Q(2, 7) = 5\ (2 < 7)$

$Q(5, 3) = Q(5-3, +2) + 5 \qquad (5 \geq 3)$

$= Q(2, 5) + 5$

$= 5 + 5 \qquad (2 < 5)$

$= 10$

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Exercise 4.2

1. Let x and y be positive integers, suppose f is defined recursively as follows:

$$f(x, y) = \begin{cases} 0 & \text{if } x < y \\ f(x - y, y) + 1 & \text{if } y \le x \end{cases}$$

Find f(2, 5) and (12, 5)

2. Let n denote a positive integer, suppose a function L is defined recursively as follows:

$$L(n) = \begin{cases} 0 & \text{if } n = 1 \\ L\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 & \text{if } n > 1 \end{cases}$$

Find L (25)

3. Let x and y be two integers and suppose g(x, y) is defined recursively by

$$g(x, y) = \begin{cases} 5 & \text{if } x < y \\ g(x - y, y + 2) + x & \text{if } x \ge y \end{cases}$$

Find g(15, 2)

4. Sort the list 1, 3, 5, 7, 9, 2, 4, 6, 8, 10 using merge sort.
5. Sort the list 3, 5, 7, 8, 1, 9, 2, 4, 6 using the merge sort.
6. Merge the two list 1, 5, 6, 7, 8 and 2, 3, 4, 9, 10

Answers

1) 0 , 2
2) 4
3) 42

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Unit 5

Special Types of Functions

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- Define floor and ceiling functions with examples.
- Define characteristics function, recursive function, integer value functions remainder functions, factorial function with examples.
- Find the value of factorial function

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5.1 Floor and Ceiling functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x specifically, $\lfloor x \rfloor$, called the floor of x, denotes the greatest integer that does not exceed x.

$\lceil x \rceil$, called the ceiling of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$ otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$

Example – 1

Find the value of

a) (i) $\lfloor 7.5 \rfloor$ (ii) $\lfloor -7.5 \rfloor$ (iii) $\lfloor -18 \rfloor$ (iv) $\lfloor \sqrt{5} \rfloor$

b) (i) $\lceil 7.5 \rceil$ (ii) $\lceil -7.5 \rceil$ (iii) $\lceil -18 \rceil$ (iv) $\lceil \sqrt{5} \rceil$

Solution:

a) By definition, $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x.

Hence (i) $\lfloor 7.5 \rfloor = 7$ (ii) $\lfloor -7.5 \rfloor = -8$

(iii) $\lfloor -18 \rfloor = -18$ (iv) $\lfloor \sqrt{5} \rfloor = 2$

(b) By definition $\lceil x \rceil$ denotes the least integer that is not less than x hence,

(i) $\lceil 7.5 \rceil = 8$

(ii) $\lceil -7.5 \rceil = -7$

(iii) $\lceil -18 \rceil = -18$

(iv) $\lceil \sqrt{5} \rceil = 3$

5.2 Characteristic function

Let A be a subset of the universal set $U = \{u_1, u_2, u_3, \ldots \ldots, u_n\}$. The characteristic function of A is defined as a function from U to {0, 1}. It is defined as:

$$f_A(u_i) = \begin{cases} 1 & if \quad u_i \in A \\ 0 & if \quad u_i \notin A \end{cases}$$

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Example – 1

If A = {4, 7, 9} and U = {1, 2, 3, ... 10} then find the value of:

(i) $f_A(2)$ (ii) $f_A(4)$ (iii) $f_A(7)$ (iv) $f_A(12)$ and is f_A is onto ?

Solution:

$f_A(2) = 0$ since, $2 \notin A$

$f_A(4) = 1$ since, $4 \in A$

$f_A(7) = 1$ since, $7 \in A$

$f_A(12) =$ is undefined since, $12 \notin U$.

$\therefore$ f_A is everywhere defined and onto but is not one to one.

Then we say that, y or f(x) is the logarithm of x to the base a, and is denoted by $\log_a x$.

Thus, the exponential equation $x = a^y$ or $a^{f(x)}$ carries the same meaning as the logarithmic equation, $y = \log_a x$.

Note: *(i) Logarithm to the base 10 is known as common logarithm.*

(ii) Logarithm to the base e is known as natural logarithm is written as $\log_e x = \log x = \ln x$

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5.3 Integer value functions

Let x be any real number. The integer value of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus,

Example – 1

Find the value of:

(i) INT(4.16) (ii) INT(– 7.5) (iii) INT(9) (iv) INT($\sqrt{5}$)

Solution:

(i) INT(4.16) = 4 (ii) INT(-7.5) = –7

(iii) INT(9) = 9 (iv) INT($\sqrt{5}$) = 2

Note: *INT(x) = $\lfloor x \rfloor$ or INT(x) = $\lceil x \rceil$ according to whether x is positive or negative*

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5.4 Remainder function: Modular Arithmetic

Let K be any integer and let M be a positive integer then K (mod M). (Read as K modulo M) will denote the integer remainder when k is divided by M more specifically, K (mod M) is the unique integer r such that, $K = Mq + r$ where $0 \le r < M$. when k is positive, simply divide k by M to obtain the remainder r.

Example – 1

Find:

(a) 26(mod 7) (b) 34 (mod 8) (c) 2345 (mod 6)

(d) – 26 (mod 7) (e) – 371 (mod 8) (f) – 39 (mod 3)

Solutions:

Since K is positive in (a) (b) & (c) so, simply divide K by M to obtain remainder r. Then r = K(mod M) & so,

(a) 26 (Mod 7) = 5 (b) 34 (mod 8) = 2 (c) 2345(mod 6) = 5

Since K is negative in (d) (e) & (f) then divide |K| by the modulus to obtain the remainder r'. Then K(Mod M) = M – r' when $r' \neq 0$.

Thus, (a) – 26(mod 7) = 7 – 5 = 2

(b) – 371(mod 8) = 8 – 3 = 5

(c) – 39(mod 3) = 3 – 0 = 3

Note: *The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows: $a \equiv b$ (mod M) and $a \pm M \equiv a$(mod M)*

Example – 2

Let *f* be mod – 10 function, compute

a. f(81) b. f(316) c. f(1057)

Solution:

a. f(81) = 1 [Since, 1 is remainder when 81 is divided by 10]

b. f(316) = 6 [Since, 6 is remainder when 316 is divided by 10]

c. f(1057) = 7 [Since, 7 is remainder when 1057 is divided by 10]

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5.4.1 Arithmetic modulo M"

Arithmetic modulo M refers to the arithmetic operations of addition, multiplication, and subtraction where the arithmetic value is replaced by its equivalent value in the set. {0, 1, 2, 3, … …, M – 1} or in the set {1, 2, 3, … …, M}. In arithmetic modulo 12, called " clock" arithmetic. For e.g,

$6 + 9 \equiv 3, \quad 7 \times 5 \equiv 11, 2 + 10 \equiv 0, \equiv 12$

(Since 15 = 12 + 3), since (35 = 24 + 11)

(The use of O or M depends on the application.)

5.5 Factorial function

The product of the positive integers from 1 to n, inclusive, is denoted by n! (read as n factorial) i.e. n! = 1. 2. 3. 4. …(n – 2).(n – 1).n.

or, $n! = n(n-1)(n-2)(n-3) \ldots\ldots 3.2.1.$

In factorial function [n! is also symbolized as $\lfloor n$]

a. If n = 0, then n! = 1

b. If n > 0, then n! = n(n – 1)! This definition is recursive & that the function is well-defined.

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Example – 1

Let A = {a, b, c, d} then n(A) = |A| = 4 is cardinality of set A. For eg., Calculate 4! Using the recursive definition.

Solution:

$4! = 4 \times 3!$

$3! = 3 \times 2!$

$2! = 2 \times 1!$

$1! = 1 \times 0!$

$0! = 1$

$1! = 1 \times 1 = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 = 6$

$4! = 4 \times 3 \times 2 = 24$

That is 4! = 24

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Step 1: This defines 4! In terms of 3! So we must postpone evaluating 4! Until we evaluate 3! This postponement is indicated by identifying the next step.

Step 2 : Here 3! is defined in terms of 2! So we must postpone evaluating 3! Until we evaluate 2!

Step 3: This defines 2! In term of 1!

Step 4: This defines 1! In terms of 0!

Step 5: This step can explicitly evaluate 0!, since 0 is the base value of the recursive definition.

Step 6: We backtrack, using 0! To find 1; using 1! To find 2!. Using 2! To find 3! & finally using 3! To find 4!. This back tracking is indicated by the reverse indention.

Recursive function

Let a and b be positive integer and suppose Q is defined recursively as follows:

$$Q(a, b) = \begin{cases} 0 & \text{if } a < b \\ Q(a-b, b) + 1 & \text{if } b \le a \end{cases}$$

Example

a) **Evaluate**

i) Q (2, 5) ii) Q (12, 5)

Solution:

i) Q (2, 5) = 0 (Since 1 < 5)

ii) Q (12 , 5) = Q (7, 5) + 1

= [Q (2, 5) + 1] + 1 [Since, 7 > 5]

= Q (2, 5) + 2

= 0 + 2 [Since, 2 < 5]

= 2

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Exercise 5

1. Compute the following floor and ceiling functions:
   a) $\lfloor 2.78 \rfloor$ b) $\lfloor -2.78 \rfloor$ c) $\lfloor 14 \rfloor$ d) $\lfloor -17.3 \rfloor$
   e) $\lfloor 20.5 \rfloor$ f) $\lceil 2.78 \rceil$ g) $\lceil 21.5 \rceil$
2. For which real numbers x is it true that $\lceil x \rceil - 1 = \lfloor x \rfloor$
3. Let f be the mod-10 function then, compute:
   a) f(417) b) f(38) c) f(253)
4. Find the value of:
   a) 26 (mod 7) b) 2345 (mod 6) c) 495 (mod 11)
   d) – 26 (mod 7) e) – 2345 (mod 6) f) – 371 (mod 6)
   g) – 39 (mod 3)
5. Find the value of the following factorial function
   a) 5! b) 4! c) 7! d) 0!
6. Solve each of the following linear congruence equations:
   a) $3x \equiv 2 \pmod 8$ b) $6x \equiv 5 \pmod 9$
   c) $4x \equiv 6 \pmod{10}$
7. Let a and b denote the positive integers. Suppose a function Q is defined recursively then find the value of:
   a) Q (2, 3) b) Q (14, 3) c) Q (5861, 7)

Answers

1. a) 2 b) – 3 c) 14 d) –18 e) 20 f) 3 g) 22
2. x is non integer real numbers
3. a) 7 b) 8 c) 3
4. a) 5 b) 5 c) 0 d) 2 e) 1 f) 5 g) 0
5. a) 120 b) 24 c) 5040 d) 1
6. a) 6 b) no solution c) 4 and 9
7. a) 0 b) 4 c) 837

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Unit

6

Geometric Transformation

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- Define isometric and Non- Isometric transformation with examples
- Apply the isometric and non isometric transformation to solve numerical examples.
- Represent Isometric and Non-isometric transformation in matrices.

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Introduction

A transformation is a mapping f of set A onto set B such that each element of B is the image of exactly one element of A. A transformation in geometry plays an important role of a function in algebra.

There is a great significance of the concept of transformation in the study of geometrics. The properties of figures and their images are studied under a set of transformation.

6.1 Kinds of Transformation

There are various ways of transforming one geometric figure to another. According to the nature of the transformation. They are classified as:

a. Isometric transformation or isometry.

b. Non isometric transformation.

c. Inversion transformations.

d. Symmetric transformations.

6.2 Isometric Transformation

An isometric transformation is a mapping, which does not distort figure after transformation. The distance between any two points of the figure is same as the distance between the corresponding image points under an isometry. Thus isometry is distance preserving transformation. The following types of isometry are discussed in this chapter. They are:

i. Reflection ii. Translation

iii. Half turn iv. Rotation

v. Glide reflection

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6.2.1 Reflection

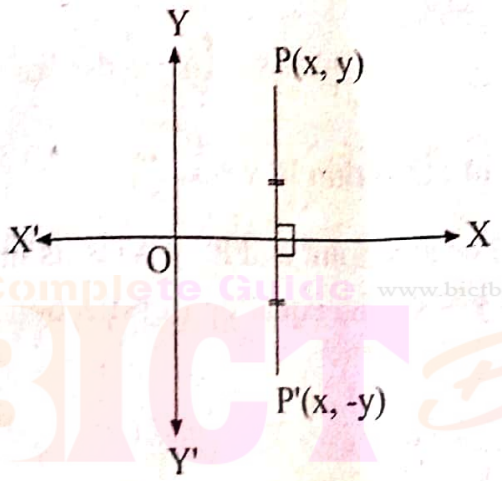
A reflection in a line m, denoted by σ_m is a transformation defined by

$$\sigma_m(P) = \begin{cases} P, \text{ if point P lies on line m} \\ P' \text{ if p is not on the line m} \\ \text{line m is the perpendicular bisector of PP'} \end{cases}$$

(i) **Equation of reflection**

Various equations of reflection are presented in the following sections.

a. **Reflection in the X-axis or in the line y = 0.**



Let P (x, y) be any point and P'(x, –y) be its image after reflection in the X-axis, then the equations of reflection of the point P(x, y) on the X-axis are

$$x' = x$$

$$y' = -y$$

Where, P' (x', y') is the image point of P(x, y). Hence, a point P(x, y) becomes P' (x, –y) when it is reflected through X-axis.

b. **Reflection in the Y-axis or in the line x = 0.**

Let P(x, y) be any point and P'(–x, y) is its image after reflection in the y-axis. Then equation. of reflection of the point P(x, y) in the y-axis are:

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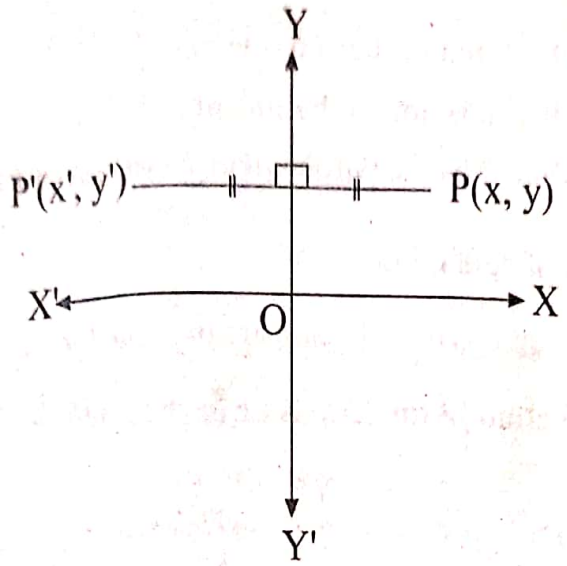
$x' = -x, y' = y$

Where, P' (x', y') is the image of (x, y) under reflection in y-axis.



c. Equation of reflection in y = x.

Let P(x, y) be any point and P' (y, x) is its image after reflection in the line y = x. The equation of reflection of the point P(x, y) in y = x is

$x' = y, \qquad y' = x$

Where, $P^1(x', y')$ is the image of (x, y) under reflection in y = x.

d. Equation of reflection in y = -x

Similarly, reflection eq^n. is the reflection in y = –x is

$x' = -y, \qquad y' = -x$

e. Reflection of a point in the line aX + bY + c = 0.

In the equation of any line is aX + bY + c = 0. Let P' (x', y') be the image of P(x, y) under the reflection by the line aX + bY + c = 0. Then, the line ℓ is perpendicular bisector of PP^1.

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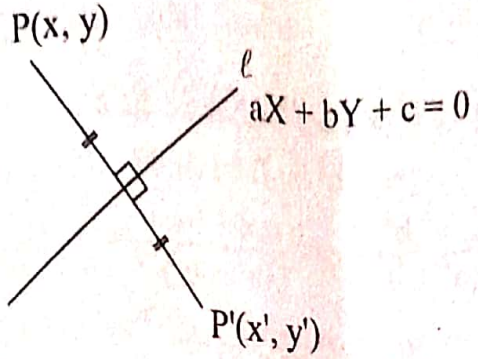

So, Mid point of PP^1 is

$\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right)$ which lies on the line aX +.bY + c = 0.

$$a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + C = 0$$

or, $ax + ax' + by + by' + 2c = 0$

or, $ax' + by' = -ax - by - 2c$ ..... (i)] × b

Again, we have, $\ell \perp PP^1$ so,

(Slope of PP^1) . (Slope of ℓ) = –1

$$\left(\frac{y'-y}{x'-x}\right) \cdot \left(\frac{-a}{b}\right) = 1$$

or, $ay' - ay = bx' - bx$

or, $bx' - ay' = bx - ay$ ..... (ii)] × a

Multiplying eq^n. (i) by b and eq^n. (ii_ by a then subtracting (ii) from (i).

$$abx' + b^2y' = -abx - b^2y - 2bc$$

$$abx' - a^2y' = abx - a^2y$$

$$\underline{-\quad +\qquad -\quad +}$$

$$(a^2 + b^2)y' = -2abx + a^2y - b^2y - 2bc$$

or, $(a^2 + b^2)y' = -2abx + a^2y + b^2y - b^2y - b^2y - 2bc$

[adding and subtracting b^2y]

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or, $(a^2 + b^2)y' = (a^2 + b^2)y - 2abx - 2b^2y - 2bc$

or, $y' = \dfrac{(a^2 + b^2)y - 2b\,(ax + by + c)}{(a^2 + b^2)}$

$= \dfrac{(a^2 + b^2)y}{(a^2 + b^2)} - \dfrac{2b\,(ax + by + c)}{(a^2 + b^2)}$

$y' = y - \dfrac{2b\,(ax + by + c)}{(a^2 + b^2)}$

Similarly, we get,

$x' = x - \dfrac{2a\,(ax + by + c)}{(a^2 + b^2)}$

Example – 1

Find the image of the point (2, 1) under a reflection about the line $3x - y = 5 = 0$.

Solution:

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Since, $a = 2$, $b = -1$, and $c = -5$, the co-ordinates (x', y') of the image of the point (2, 1) are given by

$x' = 2 - \dfrac{2.2\,[2.(-2) + (-1).1 + (-5)]}{2^2 + (-1)^2}$

$= 2 - \dfrac{4(-4 - 1 - 5)}{5}$

$= 2 - 4.(-2)$

$= 2 + 8$

$= 10$

$y' = 1 - \dfrac{2(-1).(-4 - 1 - 5)}{4 + 1}$

$= 1 - 4$

$= -3$

Hence, the co-ordinate of the image point are (10, -3).

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Example - 2

The line $y = 2x - 4$ is reflected in the line $y = x$ followed by reflection in the $y = -x$. Find the final images.

Solution:

Give equation of line is $y = 2x - 4$. Any point on line (i) is $(x, 2x - 4)$ … (i)

Now, Reflection of $(x, 2x - 4)$ in $y = x$ is $(2x - 4, x)$. Again this point is reflected by the line $y = -x$ is $(-x, -(2x - 4))$

So,

$x' = -x$, $\quad y' = -(2x - 4)$

$\therefore x = -x'$ …… (i) $\quad y' = -2x + 4$ …….. (ii)

Subtracting the value of x in (ii) we get,

$y' = -2(-x') + 4$

$y' = 2x' + 4$

$\therefore$ Locus of (x', y') is

$y = 2x + 4$ is the required image of $y = 2x + 4$ in the reflection in $y = x$ followed by $y = -x$.

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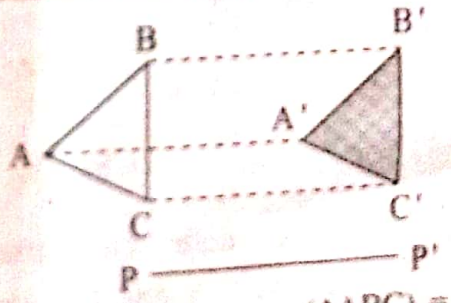
6.2.2 Translation

A translation is defined as a correspondence between points and their image points such that each image point is at the same distance in the same direction from the respective original points.



Here, $\Delta A'B'C'$ is a under translation image of ΔABC in the direction of PP'. It is written as $\tau_{PP'}(\Delta ABC) = \Delta A'B'C'$.

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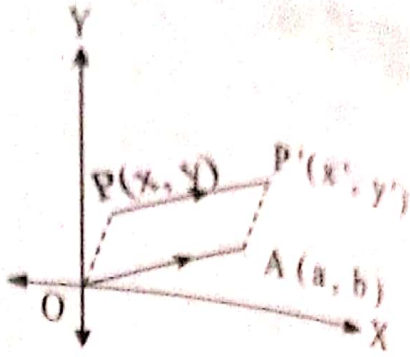
Equation of translation

Let, the co-ordinates of a point A be (a, b). The equation of translation of an arbitrary point P(x, y) to the point P'(x', y') in the direction of OA are:

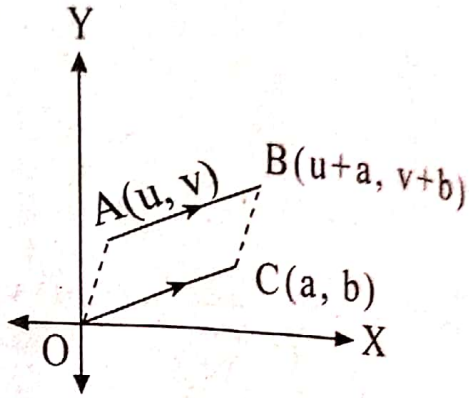


$x' = x + a$

$y' = y + b$

Theorem

Suppose that A, B, C are non-collinear points. Then $\tau_{A,B} = \tau_{O,C}$ if and only if OABC is a parallelogram.



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Proof

Let, the co-ordinates of the point C be (a, b) and that of the point A be (u, v). Then, $\tau_{O,C}$ in the translation of the point O(0, 0) to the point C(a, b) whose equations are

$x' = x + a, \quad y' = y + b$

Therefore,

$\tau_{O,C}(x, y) = (x + a, y + b)$

$\tau_{O,C}(A) = B$

Then, $\tau_{O,C} = \tau_{A,B}$

and $B = (u + a, v + b)$

From the definition of translation OC is equal and parallel to AB. Likewise OA and CB are equal and parallel. Thus, OABC is a parallelogram.

The converse part follows easily by taking OABC is a parallelogram.

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Example 3

Find the co-ordinate of the images points P(5, 7), Q(7, 5), R(3, 1) and S(1, 3) under the translation determined by the vector $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and draw the quadrilateral ABCD and its image.

Solution:

The given vertices of quadrilateral PQRS are (5, 7), (7, 5), (3, 1) and (1, 3) respectively. Let P'Q'R'S' be the images of PQRS under translation by $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

i.e.

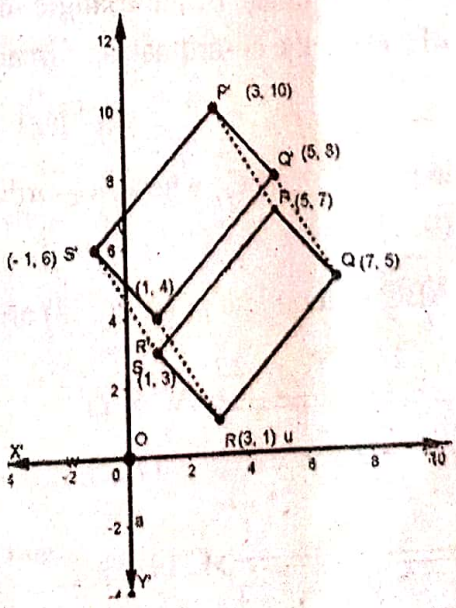
$$P(5, 7) \xrightarrow{\begin{bmatrix} -2 \\ 3 \end{bmatrix}} P'(5 - 2, 7 + 3) \text{ i.e. } P'(3, 10)$$

$$Q(7, 5) \Rightarrow Q'(7 - 2, 5 + 3) \text{ i.e. } Q'(5, 8)$$

$$R(3, 1) \Rightarrow R'(3 - 2, 1 + 3) \text{ i.e. } R'(1, 4)$$

$$S(1; 3) \Rightarrow S'(1 - 2, 3 + 3) \text{ i.e. } S'(-1, 6)$$

The diagram of quadrilateral is given below:



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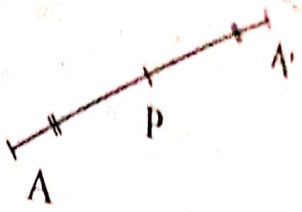
6.2.3 Half-turn

A half-turn denoted by σ_P of a point A about the point P is defined as the point A' such that P is the midpoint of the segment AA'.

i.e. PA = PA'



Equation of Half-turn

Let the co-ordinates of a point A be (x, y). Let (x', y') be the co-ordinates of the half-turn A' of the point A about the point P with co-ordinates (a, b). By the definition of half-turn, the point P is the midpoint of the points A and A'. So,

$$\frac{x+x'}{2} = a, \quad \frac{y+y'}{2} = b$$

Hence, the half-turn σ_P about the point P is the transformation with equation.

$x' = -x + 2a$

$y' = -y + 2b$

The matrix in half-turn is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x+2a \\ -y+2b \end{bmatrix}$

Example – 4

Draw a triangle ABC where A is the point (-1, 2), B is (-3, 1) and C is (0, 1). Draw also the image of the triangle under a half turn about the point P(2, 2) stating the co-ordinates of images of A, B and C.

Solution:

Given co-ordinate of ΔABC whose co-ordinates are A(-1, 2), B(-3, 1) and C(0, 1).

Since half turn of ΔABC is about the point P(2, 2) are

$$A(-1, 2) \xrightarrow{x' = -x + 2.2,\ y' = -y + 2.2} A'(5, 2)$$

$$B(-3, 1) \xrightarrow[x' = -x + 2.2,\ y' = -y + 2.2]{} B'(7, 3)$$

$$C(0, 1) \xrightarrow[x' = -x + 2.2,\ y' = -y + 2.2]{} C'(4, 3)$$

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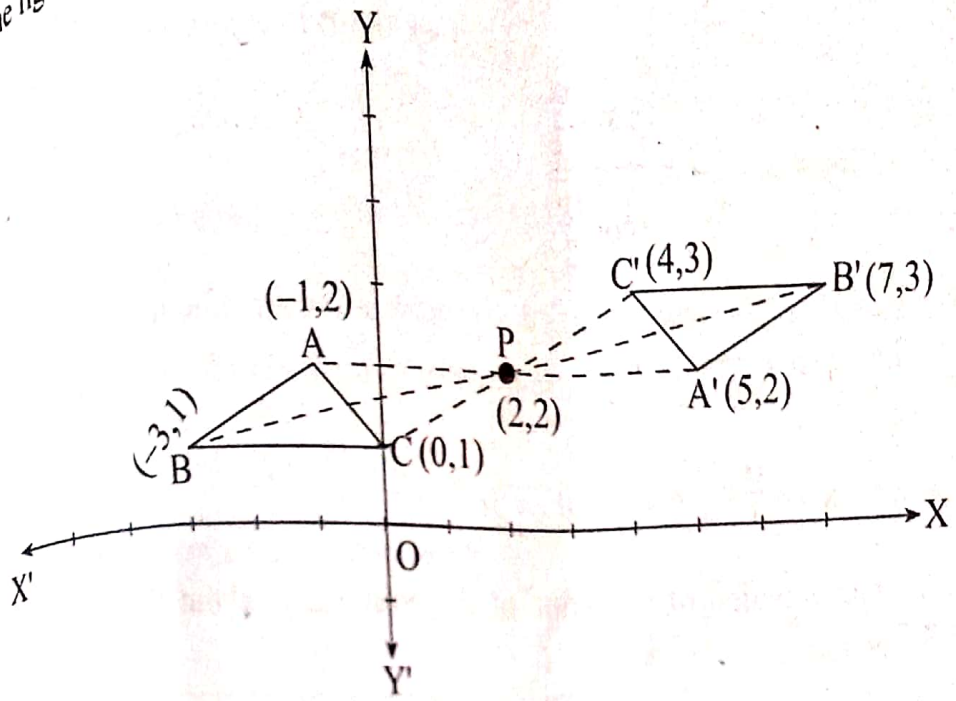
The figure under half turn is given below:



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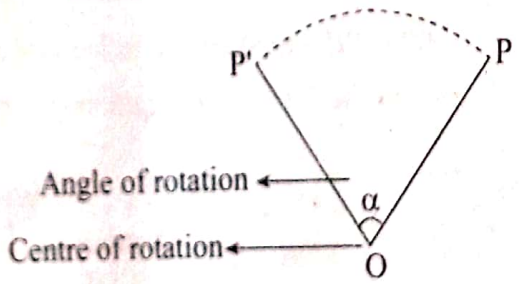
6.2.4 Rotation

Consider a point O in a plane. The rotation of the plane about the point O through an angle α is a transformation of the plane into itself under which to every point P there exists a print P' such that:



$OP = OP'$ and $\measuredangle POP' = \alpha$

It is denoted by $\rho_{O,\alpha}$

Equation of rotation

Solution:

Let the lines m and n be intersect at the origin at an angle $\frac{\alpha}{2}$ in a given direction then,

$\rho_{o,\alpha} = \sigma_n \sigma_m$

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If the line m is x-axis, then equation of line n is $y = \left(\tan\frac{\alpha}{2}\right)x$

$$y = \tan\frac{\alpha}{2}x = \frac{x\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

or, $y\cos\frac{\alpha}{2} = x\sin\frac{\alpha}{2}$

or, $x\sin\frac{\alpha}{2} - y\cos\frac{\alpha}{2} = 0$ ..... (i)

The equation of reflection of the point (x, y) about the line n with equation

$$x\sin\frac{\alpha}{2} - y\cos\frac{\alpha}{2} = 0$$

Now, the equation of reflection σ_m of the point (x, y) are

$$x_1 = x, y_1 = -y$$

Next, the equation of reflection of σ_n of the image point (x, – y) about the line n with equation $x\sin\frac{\alpha}{2} - y\cos\frac{\alpha}{2} = 0$ are

$$x' = x\frac{2\sin\frac{\alpha}{2}\left(x\sin\frac{\alpha}{2} + y\cos\frac{\alpha}{2}\right)}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}$$

$$x' = x - 2x\sin^2\frac{\alpha}{2} - y\,.\,2\sin\frac{\alpha}{2}\,.\,\cos\frac{\alpha}{2}$$

$$= x\left(1 - 2\sin^2\frac{\alpha}{2}\right) - y\sin\alpha$$

$\therefore x' = x\cos\alpha - y\sin\alpha$

and $$y' = y + \frac{2\cos\frac{\alpha}{2}\left(x\sin\frac{\alpha}{2} + y\cos\frac{\alpha}{2}\right)}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}}$$

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$\therefore\ y' = x \sin\alpha + y \cos\alpha$

Hence, the equations of rotations are

$x' = x\cos\alpha - y\sin\alpha$

$y' = x\sin\alpha + y\cos\alpha$

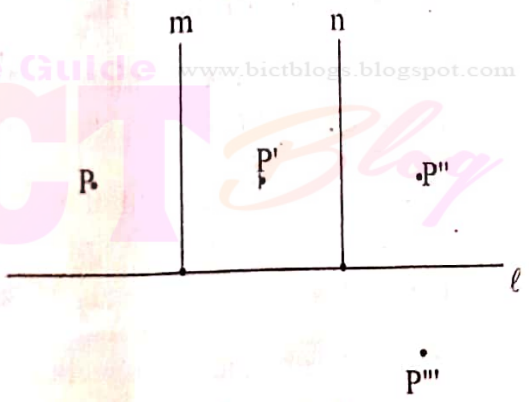
If the rotation is about the point (h, k) other than origin through an angle α, then the equation of rotation are:

$x' = (x - h)\cos\alpha - (y - k)\sin\alpha + h$

$y' = (x - h)\sin\alpha + (y - k)\cos\alpha + k$

6.2.5 Glide reflection

If the lines m and n are perpendicular to the line ℓ, the product of three reflections $\sigma_l\sigma_n\sigma_m$ is called a glide reflection with the line ℓ as axis.



Properties of glide reflection

Glide reflection possesses the following properties:

i. Glide reflection preserves collinearly

ii. There is no invariant points and lines under glide reflection

iii. There is no identity and inverse transformation under glide reflection

iv. Glide reflection fixes no points.

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6.3 Non Isometric Transformation

A transformation in which a geometric figure before transformation changes its size after transformation to a new position is called a non isometric transformation. Thus in this transformation, the image of figure has no same size but has the same shape as the original figure.

6.3.1 Dilation

Let O be a given point and let k be a real number. A dilation is a transformation that maps each point P of plane to an image point P' such that:

a. The point O, P, P' are collinear

b. OP' = K.OP

Special cases

1. If K < -1 then the dilation is called an enlargement but on the opposite side of the centre O.
2. If K > 1, then the dilation is called an enlargement on the same side of the centre O. The image becomes larger than the original figure.
3. If 0 < K < 1 then dilation is called a reduction, contradiction or shrink on the same side of the centre O. The image will be smaller than the original figure and closer to 0.
4. If -1 < K < 0 then a reduction, contraction or shrink on the opposite side of the centre O.
5. If K = 1 the dilation is an identify transformation.

6.3.2 Stretch

A figure is said to be stretched if it is enlarged in one direction only stretch transformation changes the shape of the original figure.

(a) X-stretch

In X-stretch, only X-co-ordinates are altered. If a dilation by a scale factor k is parallel to X-axis the equation of X-stretch are:

$$x' = k.x + 0.y, \qquad y' = 0.x + y$$

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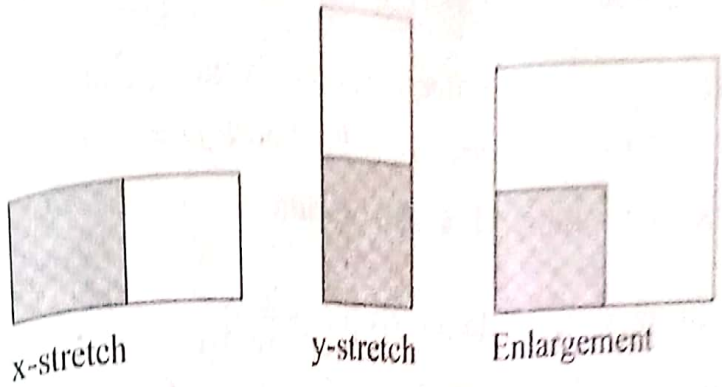
i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



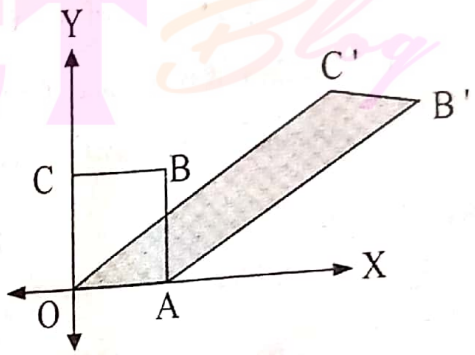
(b) Y-stretch

In y-stretch only y-coordinates are altered. If a dilation by a scale factor K is parallel to y-axis, the equation of y-stretch are

$x' = 1.x + 0.y,$ $\qquad$ $y' = 0.x + k.y$

6.3.3 Shear

A shear is a transformation in which all the points in one line or plane remain fixed and all other points move parallel to the fixed line or plane by a distance proportional to their distance from the fixed line or plane.



(a) Shear parallel to X-axis

Shear parallel to X-axis is given by

$x' = 1.x + a.y$

$y' = 0.x + 1.y$

In matrix for, shear parallel to x-axis is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(b) Shear parallel to y-axis

Shear parallel to y-axis is given by

$x' = 1.x + 0.y$

$y' = b.x + 1.y$

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6.4 Matrix representation of isometric and non-isometric transformations

This section deals the different types of isometric and non-isometric matrices in matrix form.

1) Matrix for Isometric Transformation

a) Reflection matrix in X-axis (i.e. y = 0)

We have, reflection in X-axis is

$x' = x$ $\qquad$ $y' = -y$

$x' = 1.x + 0.y$ $\qquad$ $= 0.x + (-1) - y$

$\therefore$ Required reflection matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Now, equation of reflection can be converted in the following form: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

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b) Equation of reflection in Y-axis

We have equation of reflection in y-axis is

$x' = -x,\ y' = y$

$x' = (-1).\ x + 0.y = y' = 0.x + 1.y$

$\therefore$ Reflection matrix in Y-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then, the line m is the perpendicular bisector of PP' of reflection can be converted in the following form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c) Equation and matrix in reflection by y = x

The reflection by above line is

$x' = y,$ $\qquad$ $y' = x$

$x' = 0.x + 1.y$ $\quad$ $y' = 1.x + 0.y$

$\therefore$ Required reflection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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The required equation of reflection is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

d) **Equation and matrix of reflection in y = -x**

$x' = 0.x + (-1).y,$

$y' = (-1).x + 0.y$

Required matrix for this reflection in y = - x is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

e) **Equation and translation matrix**

The translation of an equation determined by a vector.

If the point P'(x', y') is the translation of point P(x, y) as determined by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ then we have,

$x' = x + a$

$y' = y + b$

These equations are justified from the adjoining figure. These can be written as a matrix equation as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \end{bmatrix}$$

Thus, the translation of point (x, y) to the image point (x + a, y + b) and the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is called the translation matrix.

Particular cases

The translation of P(x, y) to the point P'(x', y') in the direction of x-axis by a units as determined by the vector $\begin{bmatrix} a \\ 0 \end{bmatrix}$ is given by

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$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$$

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + a \\ y \end{bmatrix}$

Similarly, the translation of P(x, y) to the point P'(x', y') in the direction of y-axis by b units as determined by the vector $\begin{bmatrix} 0 \\ b \end{bmatrix}$ is given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y + b \end{bmatrix}$

f) Equation and matrix for half turn about (a,b)

- $x' = -x + 2a$

  $y' = -y + 2b$

  The matrix in half-turn is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2a \\ -y + 2b \end{bmatrix}$

g) Rotation matrices by the angle α

a. Rotation matrix about origin in clockwise direction is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

b. Rotation matrix about origin at 90° anticlockwise direction in

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c. Rotation about origin at 90° in clockwise direction as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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d. Rotation about origin at 180°.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

e. Rotation about origin at 360° is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 360^\circ & -\sin 360^\circ \\ \sin 360^\circ & \cos 360^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2) Matrix for Non Isometric Transformation

a) Equation of dilation

The equations of dilation can be re-written as

$x' = k.x + 0.y$

$y' = 0.x + k.y$

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ describe the equation of dilation in matrix form

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b) Matrix of X-Stretch

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

c) Matrix in Y-Stretch is

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d) Matrix for, Shear transformation parallel to x-axis is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

e) Matrix for Shear transformation parallel to y-axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Worked Out Examples

Example – 1

Find the image of point P(4, -2) under reflection in the live (i) y = x (ii) y = -x.

Solution:

a. Let P' (x', y') is the image point P(4, -2) under reflection in the line y = x then $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 4 \\ -2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 & -2 \\ 4 & -0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}_{2\times 1}$

$\therefore \quad x' = -2, \qquad y' = 4$

Hence, the required point P' (-2, 4)

b. Let P'(x', y') is the image point of point P (4, -2) under reflection in the line y = – x.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Hence, the required point is P'(2, -4)

Example – 2

Let P(1, 1), Q(3, 1) and R(1, 4) respectively be the vertices of the ΔABC. Find, using matrix method, the image of each point (a) in the line y = x and (b) in the line y = -x.

Solution:

a. Let the points A, B and C are transformed into A', B' and C' under reflection in the line y = x, then

$$\begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} P' & Q' & R' \end{array} \\ \begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 1 \end{bmatrix} \end{array}$$

Hence, the required image points of P(1, 1), Q(3, 1) and R(1, 4) are P'(1, 1), Q'(1, 3) and R'(4, 1).

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Under reflection in the line y = -x the point P, Q and R are transformed into P', Q' and R' as

$$\begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} P' & Q' & R' \end{array} \\ \begin{bmatrix} -1 & -1 & -4 \\ -1 & -3 & -1 \end{bmatrix} \end{array}$$

Hence, the image of A(1, 1), B(3, 1) and C(1, 4) are A'(-1, -1), B'(-1, -3), C'(-4, -1).

Example – 3

Draw a triangle ABC on a graph where, A is the point (0, 1), B is (-3, 1) and C(-1, 2). Find the co-ordinates of vertices of ΔA'B'C' when rotated through an angle of 90° about the origin.

Solution:

Given vertices of ΔABC are A(0, 1), B(-3, 1) and C(-1, 2).

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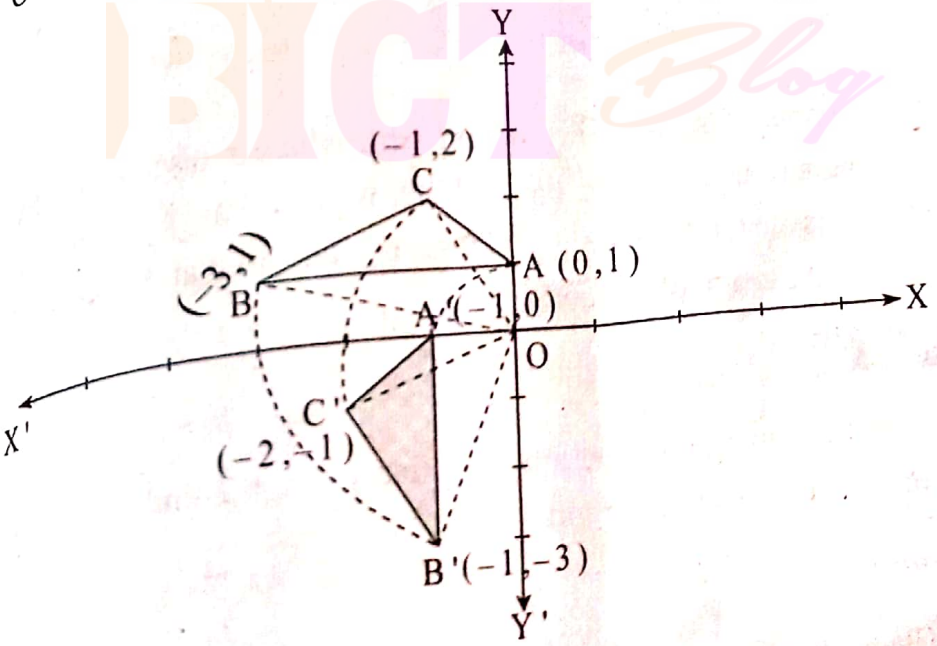

Rotation with centre (0, 0) and angle 90° is

$$P(x, y) \xrightarrow[\rho_{0,+90^\circ}]{} P'(-y, x)$$

$$A(0, 1) \xrightarrow[\rho_{0,+90^\circ}]{} A'(-1, 0)$$

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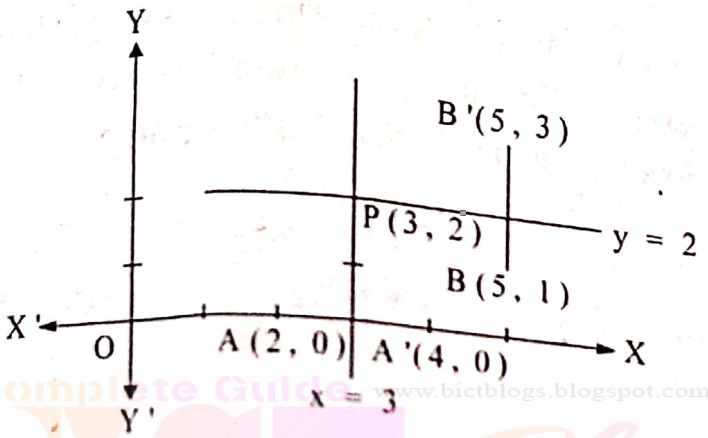
$$B(-3, 1) \xrightarrow[\rho_{0,+90^\circ}]{} B'(-1, -3)$$

$$C(-2, 1) \xrightarrow[\rho_{0,+90^\circ}]{} C'(-1, -2)$$

The corresponding image triangle is drawn in the given graph.

Example – 4

Under rotation about a certain point A(2, 0) → A'(4, 0) and B(5, 1) → B'(5, 3) find the centre of rotation.



The centre of rotation lies on the perpendicular bisector of the line joining any point and its image. Here, we can draw the perpendicular bisectors are the lines x = 3, and y =2 and they meet at the point P(3, 2). This is the centre of rotation.

Example – 5

Find the equation of the image of line y = x + 2 under a clockwise rotation of 90° about origin.

Solution:

Given equation of the line is y = x + 2 ..... (i)

Any point on line (i) is (x, x + 2).

Rotation of point (x, x +2) in clockwise direction about origin can be calculated as follows:

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$$P(x, y) \xrightarrow{\rho\, 0, -90^\circ} (y, -x)$$

$$(x, x+2) \xrightarrow{\rho\, 0, -90^\circ} (x+2, -x)$$

$\therefore$ $x' = x + 2,$ $\quad y' = -x$

$x = x' - 2$ $\quad \therefore$ $y' = -(x' - 2)$

or, $y' = x' + 2$

$\therefore$ $x' + y' = 2$

$\therefore$ The locus of (x', y') is $x + y = 2$

Example – 6

Find the image of the point (5, 2) under the half turn about the point (3, 4)

Solution:

The equation of half turn of the point A(x, y) about the point P(a, b) are,

$x' = -x + 2a,$ $\quad y' = -y + 2b$

Hence, the co-ordinate of the image point $A'(x', y')$ of the point (5, 2) under the half turn about a point (3, 4) is

$x' = -x + 2a,$ $\quad y' = -y + 2b$

or, $x' = -5 + 2.3 = -5 + 6 = 1$

or, $y' = -2 + 2.4 = -2 + 8 = 6$

$\therefore$ The image of (5, 2) under half turn about (3, 4) is (1, 6).

Example – 7

Find the images of a point A(4, 2), B(0, 4), C(2, 3) under the translation (2, –4).

Here given points are A(4, 2), B(0, 4), C(2, 3) under the translation (2, –4).

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The equation of translation is

$x' = x + 2a,$ $\quad y' = y + 2b$

or, $x' = x + 2.2$ $\quad y' = y + 2.(-4)$

or, $x' = x + 4$ $\quad y' = y - 8$

Now,

$A(4, 2) \xrightarrow{x' = x + 4,\ y' = y - 8} A'(4 + 4, 2 - 8)$

$A'(8, -6)$

$B(0, 4) \xrightarrow{x' = x + 4,\ y' = y - 8} B'(0 + 4, 4 - 8)$

$A'(4, -4)$

$C(2, 3) \xrightarrow{x' = x + 4,\ y' = y - 8} C'(2 + 4, 3 - 8)$

$C'(6, -5)$

Example – 8

ΔABC is enlarged into $\Delta A'B'C'$ with origin as the centre of enlargement and scale factor -2. If the co-ordinate of $\Delta A'B'C'$ are $A'(-2, -2)$, $B'(-10, -4)$, $C'(-4, -6)$. Find the co-ordinates of ΔABC.

Solution:

Given centre of enlargement is (0, 0) scale factor (k) =-2,

Image points are $A'(-2, -2)$, $B'(-10, 4)$, $C'(-4, -6)$

We know that,

If A', B', C' be the image of A, B, C under enlargement with scale factor -2 then co-ordinate of ABC can be obtained as,

$$A = \frac{1}{2}(-2, -2) = (-1, -1)$$

$$B = \frac{1}{2}(-10, 4) = (-5, 2)$$

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$C = \frac{1}{2}(-4, -6) = (-2, -3)$

$\therefore$ The co-ordinate of A, B, C are (-1, -1), (-5, 2), (-2, -3).

Example – 9

Enlarge the following triangles with centre of enlargement E and scale factor k as given,

A(1, 3), B(2, 5) and C(6, 1), E(0, 0), k = 2.

Solution:

Given points are A(1, 3), B(2, 5), C(6, 1) and centre of enlargement is E(0, 0) and scale factor k = 2. Now,

$$A(1, 3) \xrightarrow[E(0,0),\ k=2]{} A'(2, 6)$$

$$B(2, 5) \xrightarrow[E(0,0),\ k=2]{} B'(4, 10)$$

$$C(6, 1) \xrightarrow[E(0,0),\ k=2]{} C'(12, 2)$$

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Example – 10

Find the rotation of a point P(h, k) through an angle of 60° in the anticlockwise direction.

Solution

Let P'(x', y') be the position of P(h, k) under a rotation of 60° in the anticlockwise direction then $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -\frac{h}{2} & -\frac{\sqrt{3}}{2}k \\ \frac{\sqrt{3}}{2}h & +\frac{k}{2} \end{bmatrix}$$

$$\therefore x' = \frac{1}{2}(h - k\sqrt{3}),\ y' = \frac{1}{2}(h\sqrt{3} + k)$$

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Exercise – 6

1. Find the image of the following points after reflection in line (a) y = 0, (b) x = 0 whose co-ordinates are:

   i. (3, 2)

   ii. (-4, 7)

   iii. (-4, -3)

   iv. (4, -8)

   v. (6, 2)

2. Find the image of the following points under reflection in the line x = – 1.

   i. (-1, 0)

   ii. (-2, 2)

   iii. (3, 4)

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3. Draw on graph paper the rectangle ABCD, where A(2, -1), B(1, -2), C(3, 4), D(4, -3). Draw also the image of rectangle under reflection in line.

   i. x = 0

   ii. y = 0

   iii. x = 5

   Stating the co-ordinates of the image point in each case.

4. Draw on graph paper the triangle ABC, where A is the point A(0, 2), B(3, 4), C(-1, 3). Draw also the image of the triangle under reflection in the line (i) x = 0, (ii) y = 0, (iii) y = x stating the co-ordinates of the image points in each cases.

5. Find the reflection of each of the following lines in each axis separately,

   i. y = 3x

   ii. $y = \frac{5}{4}x$

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7. The point A(1, 2) is reflected in the line $x + y = 6$ followed by another reflection in the line $x = 4$, find the co-ordinates of the final image of A. If the point A(1, 2) is reflected in the line $x = 4$ and then in the line $x + y = 6$ what is the final image of A? Is the reflection commutative in this case? State the co-ordinates of the invariant point under these two reflections.

6. Draw an graph paper the triangle formed by the points A(-2, 3), B(3, 3), C(0, 5). Draw also the images of triangle under the translation.

   i. $x' = x, y' = y - 4$

   ii. $x' = x - 2, y' = y - 1$ stating the co-ordinates in each cases.

8. Find the image of the line $x + y = 6$ under the translation $x' = x + 2$, $y' = y + 4$.

9. Find the images of the following points under half-turn about the origin.

   i. (0, 3)

   ii. (7, 0)

   iii. (-3, 5)

   iv. (-2, -3)

10. Find the image of the following points under half-turns about the point

   i. (0, 3)

   ii. (5, 1)

   iii. (-3, 5)

   iv. (-2, -3)

11. Find the images of the lines a) $y = 3x + 5$; b) $3x + 7y = 28$ under half-turn σ_P about the point P(– 3, 2).

12. Draw an graph paper ΔABC whose A(3, 1), B(4, 1), C(4, 3) are points. Draw also the image of the triangle under $\rho_{O, 90°}$ stating the co-ordinates of the image of A, B and C.

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13. If A(0, 0), B(2, 0), C(2, 2) and D(0, 2) is a square, find the co-ordinates of the image points A' B' C' and D' under a rotation about origin through an angle 45°.

14. Let A(2, 0), B(4, 4), C(2, 5) and D(0, 4) is a kite. Find the co-ordinates of the image point A' B' C' and D' under the rotation about origin through an angle of i) 45° ii) -90°.

15. Find the image of ΔABC under the scale factor k = 3 where the vertices are A(0, 0), B(4, 1) and C(3, 2)

16. Find a matrix which rotate through 45° and then reflect in y-axis. Find the image of (1, 0) in this reflection.

17. Write the composition ST of the transformations S and T given by S: reflection in the y-axis and T: rotation through 120°.

18. Show that the reflection in the x-axis followed by another reflection in the y-axis is given by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Answers

1. (a) (3, -2), (-4, -7), (-4, 3), (4, 8), (6, -2)

   (b) (-3, 2), (4, 7), (4, -3), (-4, -8), (-6, 2)

2. (i) (-1, 0);

   (ii) (0, 2);

   (iii) (-5, 4)

3. (i) (-2, -1), (-1, -2), (-3, -4), (-4, -3)

   (ii) (2, 1), (1, 2), (3, 4), (4, 3)

   (iii) (8, -1), (9, -2), (7, -4), (6, -3)

4. (i) (0, 2), (-3, 4), (1, 3)

   (ii) (0, -2), (3, -4), (-1, -3)

   (iii) (2, 0), (4, 3), (3, -1)

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