Calculus Notes

ICT 3rd Semester

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Unit-1

Limits and Continuity

* ε-δ definition of limit:

Let $f(x)$ be a function defined in the neighbourhood of 'a', then $f(x)$ approaches to limit I as 'x' approaches to 'a'.

i.e. $\lim_{x \to a} f(x) = I$

And in ε-δ form for each $\varepsilon > 0$ there exists $\delta > 0$ such that,

$|f(x) - I| < \varepsilon \Rightarrow |x-a| < \delta$

Then the function $f(x)$ is said to attain the limit I.

* Left hand limit and right hand limit:

$\lim_{x \to a^-} f(x) =$ left hand limit

$\lim_{x \to a^+} f(x) =$ right hand limit.

$\frac{0}{0}, \infty, -\infty, \frac{\infty}{\infty}$ are undefined form i.e. Indeterminate form.

Find the limits of following:

1) $\lim_{x \to a} \frac{x^3 - a^3}{x-a}$

$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^2+ax+a^2)}{\cancel{(x-a)}}$

$= a^2 + a \cdot a + a^2$

$= 3a^2$

2) $\lim_{x \to 2} \frac{4x^2+4x+1}{x+2}$

$= \frac{4 \cdot 2^2 + 4 \cdot 2 + 1}{2+2}$

$= \frac{16+8+1}{4}$

$= \frac{25}{4}$

3) $\lim_{x\to a} \dfrac{\sqrt{3x}-\sqrt{2x+a}}{2(x-a)}$

$= \lim_{x\to a} \dfrac{\sqrt{3x}-\sqrt{2x+a}}{2(x-a)} \times \dfrac{\sqrt{3x}+\sqrt{2x+a}}{\sqrt{3x}+\sqrt{2x+a}}$

$= \lim_{x\to a} \dfrac{(\sqrt{3x})^2-(\sqrt{2x+a})^2}{2(x-a)(\sqrt{3x}+\sqrt{2x+a})}$

$= \lim_{x\to a} \dfrac{3x-2x-a}{2(x-a)(\sqrt{3x}+\sqrt{2x+a})}$

$= \lim_{x\to a} \dfrac{\cancel{x-a}}{2(\cancel{x-a})(\sqrt{3x}+\sqrt{2x+a})}$

$= \dfrac{1}{2(\sqrt{3a}+\sqrt{2a+a})}$

$= \dfrac{1}{2\cdot 2\sqrt{3a}}$

$= \dfrac{1}{4\sqrt{3a}}$ Ans

4) $\lim_{x\to 64} \dfrac{\sqrt[6]{x}-2}{\sqrt[3]{x}-4}$

$= \lim_{x\to 64} \dfrac{x^{1/6}-2}{x^{1/3}-4}$

$= \lim_{x\to 64} \dfrac{x^{1/6}-2}{x^{1/3}-4} \times \dfrac{x^{1/6}+2}{x^{1/6}+2}$

$= \lim_{x\to 64} \dfrac{(x^{1/6})^2-(2)^2}{(x^{1/3}-4)(x^{1/6}+2)}$

$= \lim_{x\to 64} \dfrac{\cancel{(x^{1/3}-4)}}{\cancel{(x^{1/3}-4)}(x^{1/6}+2)}$

$= \lim_{x\to 64} \dfrac{1}{x^{1/6}+2} = \dfrac{1}{64^{1/6}+2} = \dfrac{1}{2^{6\times\frac{1}{6}}+2} = \dfrac{1}{4}$ Ans

5) $\lim_{x\to\infty} \frac{2x^2}{3x^2+2}$

Let $\infty = \frac{1}{x}$

$$= \lim_{x\to\infty} \frac{2\times\frac{1}{x^2}}{3\times\frac{1}{x^2}+2}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^2}}{\frac{3+2x^2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2}{3+2x^2}$$

$$= \frac{2}{3+\infty} = \frac{2}{3+\frac{1}{0}}$$

$$= \frac{2}{\frac{1}{0}}$$

$$= \frac{0}{1}$$

$$= 0 \text{ Ans}$$

6) $\lim_{x\to\infty} \frac{4x^2+3x+2}{5x^2+4x-3}$

$$= \lim_{x\to\infty} \frac{\frac{4x^2+3x+2}{x^2}}{\frac{5x^2+4x-3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{4x^2}{x^2}+\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{5x^2}{x^2}+\frac{4x}{x^2}-\frac{3}{x^2}} = \lim_{x\to\infty} \frac{4+\frac{3}{x}+\frac{2}{x^2}}{5+\frac{4}{x}-\frac{3}{x^2}}$$

$$= \cancel{\lim_{x\to\infty}} \frac{4+3\times 0+2\times 0}{5+4\times 0-3\times 0} = \frac{4}{5} \text{ Ans}$$

7) $$\lim_{x\to 2} \frac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4}$$

$$= \lim_{x\to 2} \frac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4} \times \frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4} \times \frac{x+\sqrt{8-x^2}}{x+\sqrt{8-x^2}}$$

$$= \lim_{x\to 2} \frac{x^2-(\sqrt{8-x^2})^2}{x+\sqrt{8-x^2}} \times \frac{\sqrt{x^2+12}+4}{(\sqrt{x^2+12})^2-(4)^2}$$

$$= \lim_{x\to 2} \frac{x^2-8+x^2}{x+\sqrt{8-x^2}} \times \frac{\sqrt{x^2+12}+4}{x^2+12-16}$$

$$= \lim_{x\to 2} \frac{2x^2-8}{x+\sqrt{8-x^2}} \times \frac{\sqrt{x^2+12}+4}{x^2-4}$$

$$= \lim_{x\to 2} \frac{2\cancel{(x^2-4)}}{x+\sqrt{8-x^2}} \times \frac{\sqrt{x^2+12}+4}{\cancel{x^2-4}}$$

$$= \frac{2(\sqrt{2^2+12}+4)}{2+\sqrt{8-2^2}}$$

$$= \frac{2(\sqrt{16}+4)}{2+\sqrt{4}}$$

$$= \frac{2(4+4)}{2+2}$$

$$= \frac{16}{4} = 4 \text{ Ans}$$

8) TCT 2074 $$\lim_{x\to\infty} \frac{3x^2+2x+1}{4x^2+x+5}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(3+\frac{2}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(4+\frac{1}{x}+\frac{5}{x^2}\right)}$$ [Taking x^2 common in both]

$$= \frac{3+\frac{2}{\infty}+\frac{1}{\infty^2}}{4+\frac{1}{\infty}+\frac{5}{\infty^2}}$$

$$= \frac{3+0+0}{4+0+0} = \frac{3}{4} \text{ Ans}$$

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Left hand limit & Right hand limit:

Q. Find the limits of:

1) $f(x) = \begin{cases} x+2 & \text{for } x \geq 0 \\ 4x+2 & \text{for } x < 0 \end{cases}$ at $x=0$

Soln:

Left hand limit $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (4x+2) = 4\times0+2 = 2$

Right hand limit $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+2) = 0+2 = 2$

Since L.H.L = R.H.L, so the limit exists.

2) $\lim_{x \to 2} \frac{x-2}{|x-2|}$

Soln Here,

$|x-2| = \begin{cases} -(x-2) & \text{if } x < 2 \\ (x-2) & \text{if } x > 2 \end{cases}$

Left hand limit $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x-2}{-(x-2)} = -1$

Right hand limit $= \lim_{x \to 2^+} f(x) = \frac{x-2}{x-2} = 1$

$\therefore \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, so the limit does not exist.

Continuity

A function $f(x)$ is said to be continuous at $x=a$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Continuity

Q. Discuss the continuity of the following functions.

1) $f(x) = \begin{cases} 2x+1 & \text{for } x<1 \\ 2 & \text{for } x=1 \\ 3x & \text{for } x>1 \end{cases}$ at $x=1$

Soln:

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 2x+1 = 2\times1+1 = 3$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 3x = 3\times1 = 3$

at, $x=1$ $f(x) = f(1) = 2$

$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) \neq f(1)$

Hence, the function is not continuous.

2) A function is defined by

$f(x) = \begin{cases} x^2+2 & \text{for } x<5 \\ 20 & \text{for } x=5 \\ 3x+12 & \text{for } x>5 \end{cases}$

Show that $f(x)$ has removable discontinuity at $x=5$

Soln:

$\lim_{x\to5^-} f(x) = \lim_{x\to5^-} (x^2+2) = 5^2+2 = 27$

$\lim_{x\to5^+} f(x) = \lim_{x\to5^+} (3x+12) = 3\times5+12 = 27$

At, $x=5$ $f(5) = 20$

Here, $\lim_{x\to5^-} f(x) = \lim_{x\to5^+} f(x) \neq f(5)$ which is not continuous.

If the value of $f(x) = 27$ at $x=5$, then the function will be continuous. Hence at $x=5$, the function has removable discontinuity.

3) A function is defined as follows:

$$f(x) = \begin{cases} \frac{2x^2-18}{x-3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{cases}$$

Find the value of k so that f(x) is continuous at $x = 3$.

Soln:

$$\lim_{x \to 3^-} f(x) = \frac{2x^2-18}{x-3} = \frac{0}{0} = \frac{2(x+3)\cancel{(x-3)}}{\cancel{x-3}} = \cancel{2\times3=6}\ 2\times(3+3)$$
$$= 12$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} k = k$$

Functional value. $f(x) = k$

or, $\frac{2x^2-18}{x-3} = k$

or, $\frac{2(x+3)\cancel{(x-3)}}{\cancel{x-3}} = k$

or, $2x+6 = k$

$\therefore k = 2x+6 = 2\times3+6 = 12$

Unit-2

Derivatives

Derivative $\frac{d}{dx}$ $\frac{dy}{dx}$ $\Big| \; y = f(x) = x^2 + 5x$

Formulas

- Power rule : $\frac{d(x^n)}{dx} = nx^{n-1}$
- Sum rule (±) : $y = u \pm v$

  $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
- Product rule : $y = u.v$

  $\frac{dy}{dx} = u.\frac{dv}{dx} + v\frac{du}{dx}$
- Quotient rule : $y = u/v$

  $\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$

Trignometric, logarithmic and exponential famulas:

- $\frac{d(\sin x)}{dx} = \cos x$
- $\frac{d(\cos x)}{dx} = -\sin x$
- $\frac{d(\tan x)}{dx} = \sec^2 x$
- $\frac{d(\cot x)}{dx} = -\mathrm{cosec}^2 x$
- $\frac{d(\sec x)}{dx} = \sec x.\tan x$
- $\frac{d(\mathrm{cosec}\, x)}{dx} = -\mathrm{cosec}\, x.\cot x$
- $\frac{d(\log x)}{dx} = \frac{1}{x}$
- $\frac{d(e^x)}{dx} = e^x$

Types of functions:

1. Explicit → $y = \frac{x^2}{x^3 - 2}$
2. Implicit → $x^2 + y^2 = a^2$
3. Trignometric → $\sin x, \cos x, \tan x$
4. Exponential → $e^x, e^{\sin x}$
5. Logarithmic → $\log x, \log(ax$

A. Find

1)

2)

3)

A. Find the derivative of Explicit functions

1) $y = x^2 + 5x + 6$

Soln:

$$y = x^2 + 5x + 6$$

$$\frac{dy}{dx} = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$$

$$= 2x + 5 \cdot \frac{dx}{dx} + 0$$

$$= 2x + 5$$

2) $y = 125x^3 + 45x^2 + 9x + 10$

$$\frac{dy}{dx} = 125\frac{d(x^3)}{dx} + 45\frac{d(x^2)}{dx} + 9\left(\frac{dx}{dx}\right) + 0$$

$$= \cancel{3.5} \; 125 \cdot 3x^2 + 45 \cdot 2x + 9\cancel{x}$$

$$= 375x^2 + 90x + 9.$$

3) $y = x^2 - \frac{1}{x} + 5$

$$\frac{dy}{dx} = d\left(\frac{x^2}{dx}\right) - \frac{d(1/x)}{dx} + \frac{d(5)}{dx}$$

$$= 2x - (-1x^{-1-1}) + 0$$

$$= 2x + x^{-2}$$

$$= 2x + \frac{1}{x^2}$$

4) $y = \frac{x+2}{x+3}$ $\left(= \frac{u}{v}\right)$

$$\frac{dy}{dx} = \frac{(x+3) \cdot \frac{d(x+2)}{dx} - (x+2)\frac{d(x+3)}{dx}}{(x+3)^2}$$

$$= \frac{(x+3) \cdot \frac{dx}{dx} - (x+2)\frac{dx}{dx}}{(x+3)^2} = \frac{(x+3) - (x+2)}{(x+3)^2}$$

$$= \frac{\cancel{x} + 3 - \cancel{x} - 2}{(x+3)^2}$$

$$= \frac{1}{(x+3)^2}$$

Find from first principle:

1) $y = x^3 - 10$ — (i)

Let Δx & Δy be small increments in x & y respectively.

$\therefore y + \Delta y = (x+\Delta x)^3 - 10$ — (ii)

Subtracting (i) from (ii)

$\Delta y = (x+\Delta x)^3 - \cancel{10} - x^3 + \cancel{10}$

or, $\Delta y = \cancel{x^3} + 3.x^2.\Delta x + 3.x.(\Delta x)^2 + (\Delta x)^3 - \cancel{x^3}$

or, $\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{3x^2\cancel{\Delta x}}{\cancel{\Delta x}} + \frac{3x(\Delta x)^{\cancel{2}}}{\cancel{\Delta x}} + \frac{(\Delta x)^3}{\Delta x}$$

or $\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$

Taking $\lim_{\Delta x \to 0}$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right)$$

$$= 3x^2 + 3x.0 + 0$$

$$= 3x^2 \text{ Ans}$$

Find the derivatives of implicit function:

1) $x^2+y^2=a^2$

Soln:

Differentiating w.r.t. x on both sides

$\frac{d(x^2+y^2)}{dx} = \frac{d(a^2)}{dx}$

or, $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$

or, $2x + \frac{dy^2}{dy} \times \frac{dy}{dx} = 0$

or, $2x + 2y \cdot \frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y}$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$ Ans

2) $x^3+y^3 = 3axy$

Soln:

$\frac{d(x^3+y^3)}{dx} = \frac{d(3axy)}{dx}$

or, $\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx} = 3a\frac{d(xy)}{dx}$

or, $3x^2 + \frac{dy^3}{d\cancel{x}y} \times \frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\frac{dx}{dx}\right]$

or, $3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$

or, $3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$

or, $\frac{dy}{dx} = \frac{3ay-3x^2}{3y^2-3ax}$ Ans

3) $x = y\log(xy)$

soln

$$\frac{dx}{dx} = \frac{d(y\log(xy))}{dx}$$

or $1 = y\frac{d(\log(xy))}{dx}$

or $1 = y.\frac{d(\log(xy))}{d(xy)} \times \frac{d(xy)}{dx} + \log xy . \frac{dy}{dx}$

or $1 = \frac{\cancel{y}}{x\cancel{y}}\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + \log(xy).\frac{dy}{dx}$

or $1 = \frac{1}{\cancel{x}} \times \cancel{x}\frac{dy}{dx} + \frac{1}{x}.y + \log(xy).\frac{dy}{dx}$

or $1 = \frac{dy}{dx} + \frac{y}{x} + \log(xy).\frac{dy}{dx}$

or $\frac{dy}{dx}(1 + \log(xy)) = 1 - \frac{y}{x}$

or $\frac{dy}{dx} = \frac{x-y}{x(1+\log(xy))}$ Ans

4.) $(\cos x)^y = (\sin y)^x$

Taking log on both sides

$\log(\cos x)^y = \log(\sin y)^x$

or $y\log(\cos x) = x\log(\sin y)$

Differentiating w.r.t. x on both sides

$$\frac{d(y\log(\cos x))}{dx} = \frac{d(x\log(\sin y))}{dx}$$

or $y.\frac{d(\log(\cos x))}{dx} + \log(\cos x).\frac{dy}{dx} = x.\frac{d(\log(\sin y))}{dx} + \log\sin y.\frac{dx}{dx}$

or $y.\frac{d(\log(\cos x))}{d(\cos x)} \times \frac{d(\cos x)}{dx} + \log(\cos x).\frac{dy}{dx}$

$= x.\frac{d(\log(\sin y))}{d(\sin y)} \times \frac{d(\sin y)}{dy} \times \frac{dy}{dx} + \log(\sin y).1$

or, $y\frac{1}{\cos x}.(-\sin x) + \frac{1}{\cancel{\cos x}}\log(\cos x)\frac{dy}{dx} = x.\frac{1}{\sin y}.\cos y.\frac{dy}{dx} + \log(\sin y)$

or $\log(\cos x)\frac{dy}{dx} - x.\cot y\frac{dy}{dx} = y\tan x + \log(\sin y)$

or $\frac{dy}{dx}\left[\log(\cos x) - x\cot y\right] = y\tan x + \log(\sin y)$

$\therefore \frac{dy}{dx} = \frac{y\tan x + \log(\sin y)}{\log\cos x - x\cot y}$ Ans

5? 5) $x^y = y^x$

Taking log on both sides

$\log(x^y) = \log(y^x)$

or $y\log x = x\log y$

or Differentiating w.r.t x

$\frac{d(y\log x)}{dx} = \frac{d(x\log y)}{dx}$

or $y.\frac{d(\log x)}{dx} + \log x.\frac{dy}{dx} = x.\frac{d(\log y)}{dx} + \log y.\frac{dx}{dx}$

or $y.\frac{1}{x} + \log x.\frac{dy}{dx} = \cancel{\frac{x.1}{y} + \log y}\; x.\frac{d(\log y)}{dy} \times \frac{dy}{dx} + \log y$

or, $\cancel{\log x\frac{dy}{dx} - \log y = \frac{x}{y} - \frac{y}{x}}$

or $\cancel{\log x}\frac{y}{x} + \log x\frac{dy}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y$

or $\log x\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} = \log y - \frac{y}{x}$

or $\frac{dy}{dx} = \frac{\log y - y/x}{\log x - x/y}$

Find the derivative of trignometric functions:

1) $\sin(4x-5)$

Soln:

$y = \sin(4x-5)$

$$\frac{dy}{dx} = \frac{d[\sin(4x-5)]}{dx}$$

$$= \frac{d[\sin(4x-5)]}{d(4x-5)} \times \frac{d(4x-5)}{dx}$$

$$= \cos(4x-5) \cdot 4$$

$$= 4\cos(4x-5).$$

2) $y = \sec^5\left(\frac{ax+b}{c}\right)$

$$\frac{dy}{dx} = \frac{d\left[\sec^5\left(\frac{ax+b}{c}\right)\right]}{d\left(\frac{ax+b}{c}\right)} \times \frac{d\left(\frac{ax+b}{c}\right)}{dx}$$

$$= 5\sec^4\left(\frac{ax+b}{c}\right) \cdot \tan^5 x\left(\frac{ax+b}{c}\right) \cdot \frac{a}{c}$$

$$= \frac{5a}{c} \cdot \sec^4\left(\frac{ax+b}{c}\right) \cdot \tan^5 x\left(\frac{ax+b}{c}\right)$$

3) $y = \tan^5[\sin(px-q)]$

$$\frac{dy}{dx} = \frac{d[\tan^5(\sin(px-q))]}{d[\sin(px-q)]} \times \frac{d[\sin(px-q)]}{d(px-q)} \times \frac{d(px-q)}{dx}$$

$$= 5\tan^4\{\sin(px-q)\} \cdot \sec^2\{\sin(px-q)\} \cdot \cos(px-q) \times p$$

$$= 5p\tan^4[\sin(px-q)] \sec^2[\sin(px-q)] \cdot \cos(px-q) \text{ Ans}$$

4) $y = \cot\sqrt{\tan 3x}$

$$\frac{dy}{dx} = \frac{d[\cot\sqrt{\tan 3x}]}{d(\sqrt{\tan 3x})} \times \frac{d(\sqrt{\tan 3x})}{d(\tan 3x)} \times \frac{d(\tan 3x)}{d(3x)} \times \frac{d(3x)}{dx}$$

$$= -\operatorname{cosec}^2\sqrt{\tan 3x} \times \frac{1}{2}\frac{1}{\sqrt{\tan 3x}} \times \sec^2 3x \cdot 3$$

$$= -\frac{3}{2}\operatorname{cosec}^2(\tan 3x) \cdot \frac{\sec^2 3x}{\sqrt{\tan 3x}} \text{ Ans}$$

Find from the first principle:

1) $\sin x$

$f(x) = \sin x$

or, $y = \sin x$ — (i)

Let Δx & Δy be small increments in x and y respectively

Then

$y + \Delta y = \sin(x + \Delta x)$ — (ii)

Subtracting (i) from (ii)

$\Delta y = \sin(x+\Delta x) - \sin x$

or $\Delta y = 2\sin\frac{x+\Delta x - x}{2} \cdot \cos\frac{x+\Delta x+x}{2}$

or, $\Delta y = 2\sin\frac{\Delta x}{2} \cdot \cos\frac{2x+\Delta x}{2}$

or, $\dfrac{\Delta y}{\Delta x} = \dfrac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\dfrac{2x+\Delta x}{2}$

or $\dfrac{dy}{dx} = \lim_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\dfrac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\dfrac{2x+\Delta x}{2}\right)$

$= 1 \cdot \cos\dfrac{2x+0}{2} = \cos\dfrac{2x}{2} = \cos x$

OR

$f(x) = \sin x$ —①

Then by formula.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos\frac{x+\Delta x+x}{2} \cdot \sin\frac{x+\Delta x-x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\cos\frac{2x+\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2\cos\frac{2x+\Delta x}{2} \cdot \frac{\sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \cos\frac{2x+\Delta x}{2} \cdot \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= \cos\frac{2x+0}{2} \cdot 1 = \cos\frac{2x}{2}$$

$= \cos x$ Ans.

2) $f(x) = \cos x$ —①

Then by formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\frac{x+\Delta x+x}{2} \cdot \sin\frac{x+\Delta x-x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\sin\frac{2x+\Delta x}{2} \cdot \sin\frac{\Delta x}{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -\sin\frac{2x+\Delta x}{2} \cdot \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= -\sin\frac{2x+0}{2} \cdot 1$$

$$= -\sin\frac{\cancel{2}x}{\cancel{2}}$$

$$= -\sin x \text{ Ans}$$

3) $f(x) = \tan x$

$\therefore f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\sin(x+\Delta x)\cdot\cos x - \cos(x+\Delta x)\cdot\sin x}{\cos(x+\Delta x)\cdot\cos x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\sin(\cancel{x}+\Delta x-\cancel{x})}{\cos(x+\Delta x)\cdot\cos x} = \lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x} \cdot \frac{1}{\cos(x+\Delta x)\cdot\cos x}$$

$$= 1 \cdot \frac{1}{\cos(x+0)\cdot\cos x} = \frac{1}{\cos x \cdot \cos x} = \frac{1}{\cos^2 x} = \sec^2 x \text{ Ans}$$

Find the derivative of logarithmic & exponential functions

1) $y = \log(ax^2+bx+c)$

$$\frac{dy}{dx} = \frac{d[\log(ax^2+bx+c)]}{d(ax^2+bx+c)} \times \frac{d(ax^2+bx+c)}{dx}$$

$$= \frac{1}{ax^2+bx+c} \times \left[a\frac{d(x^2)}{dx} + b\frac{d(x)}{dx} + \frac{d(c)}{dx}\right]$$

$$= \frac{1}{ax^2+bx+c} \times (2ax+b)$$

$$= \frac{2ax+b}{ax^2+bx+c}$$

2) $y = 3x.\log(ax+b)$

Using product rule

$$\frac{dy}{dx} = 3x.\frac{d[\log(ax+b)]}{d(ax+b)} \times \frac{d(ax+b)}{d(x)} + \log(ax+b).\frac{d(3x)}{dx}$$

$$= 3x.\frac{1}{ax+b}.a + \log(ax+b).3$$

$$= \frac{3ax}{ax+b} + 3\log(ax+b) \text{ Ans}$$

3) $y = \log(x+\tan x)$

$$\frac{dy}{dx} = \frac{d[\log(x+\tan x)]}{d(x+\tan x)} \times \frac{d(x+\tan x)}{dx}$$

$$= \frac{1}{(x+\tan x)} \times \left[\frac{dx}{dx} + \frac{d(\tan x)}{dx}\right]$$

$$= \frac{1}{x+\tan x}(1+\sec^2 x)$$

$$= \frac{1+\sec^2 x}{x+\tan x} \text{ Ans}$$

4) $y = \log(e^{ax} + e^{-ax})$

$$\frac{dy}{dx} = \frac{d[\log(e^{ax}+e^{-ax})]}{d(e^{ax}+e^{-ax})} \times \frac{d(e^{ax}+e^{-ax})}{dx}$$

$$= \frac{1}{e^{ax}+e^{-ax}} \times \left[\frac{d(e^{ax})}{dx} + \frac{d(e^{-ax})}{dx}\right]$$

$$= \frac{1}{e^{ax}+e^{-ax}} \times (e^{ax} + e^{-ax})$$

$$= \frac{e^{ax}+e^{-ax}}{e^{ax}+e^{-ax}}$$

$$= 1 \text{ Ans}$$

5) $y = \sec(\log \tan x)$

$$\frac{dy}{dx} = \frac{d[\sec(\log\tan x)]}{d(\log\tan x)} \times \frac{d(\log\tan x)}{d(\tan x)} \times \frac{d(\tan x)}{dx}$$

$$= \sec(\log\tan x)\cdot\tan(\log\tan x) \times \frac{1}{\tan x} \times \sec^2 x$$

$$= \frac{\sec^2 x\cdot\sec(\log\tan x)\cdot\tan(\log\tan x)}{\tan x} \,//$$

6) $y = e^{x^2+5x+6}$

$$\frac{dy}{dx} = \frac{d(e^{x^2+5x+6})}{d(x^2+5x+6)} \times \frac{d(x^2+5x+6)}{dx}$$

$$= e^{x^2+5x+6} \times (2x+5)$$

$$= e^{x^2+5x+6}(2x+5) \text{ Ans}$$

7) $y = e^{5x}\cdot\sin 6x$

$$\frac{dy}{dx} = \frac{d(e^{5x}\cdot\sin 6x)}{dx} = e^{5x}\cdot\frac{d(\sin 6x)}{dx} + \sin 6x\cdot\frac{d(e^{5x})}{d(5x)} \times \frac{d(5x)}{dx}$$

$$= e^{5x}\cdot\cos 6x + \sin 6x\cdot e^{5x}\cdot 5$$

$$= e^{5x}\cos 6x + 5e^{5x}\sin 6x \text{ Ans}$$

Parametric Function

Find $\frac{dy}{dx}$ of parametric functions:

1) $x = a\cos t$, $y = b\sin t$

Soln:

$$\frac{dx}{dt} = \frac{d(a\cos t)}{dt} = \frac{a \cdot d(\cos t)}{dt} = -a\sin t$$

$$\frac{dy}{dt} = \frac{d(b\sin t)}{dt} = b \cdot \frac{d(\sin t)}{dt} = b\cos t$$

Then.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = b\cos t \cdot \frac{1}{-a\sin t} = -\frac{b}{a}\cot t$$

2) $x = \log(\cos\theta)$, $y = e^{\tan\theta}$

Soln:

$$\frac{dx}{d\theta} = \frac{d[\log(\cos\theta)]}{d(\cos\theta)} \times \frac{d(\cos\theta)}{d\theta}$$

$$= \frac{1}{\cos\theta} \times -\sin\theta = -\tan\theta$$

$$\frac{dy}{d\theta} = \frac{d[e^{\tan\theta}]}{d(\tan\theta)} \times \frac{d(\tan\theta)}{d\theta} = e^{\tan\theta} \cdot \sec^2\theta$$

now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}} = e^{\tan\theta} \cdot \sec^2\theta \times \frac{1}{-\tan\theta} = -\frac{e^{\tan\theta}\sec^2\theta}{\tan\theta}$$

3. $x = e^t \cos t,\ y = a \sin t$

Soln:

$$\frac{dx}{dt} = \frac{d(e^t \cos t)}{d(t)} = e^t \cdot \frac{d(\cos t)}{dt} + \cos t \cdot \frac{d(e^t)}{dt}$$

$$= e^t \cdot -\sin t + \cos t \cdot e^t$$

$$= e^t(\cos t - \sin t)$$

$$\frac{dy}{dt} = \frac{d(a \sin t)}{dt} = a \cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = a \cos t \times \frac{1}{e^t(\cos t - \sin t)}$$

$$= \frac{a \cos t}{e^t(\cos t - \sin t)}$$

Q. Find up to the 4th order derivatives:

1) $f(x) = y = 10x^4 + 5x^3 + 6x^2 + 100$

Soln:

1st order $\rightarrow \frac{dy}{dx} = 40x^3 + 15x^2 + \cancel{6}12x$

2nd order $\rightarrow \frac{d^2y}{dx^2} = 120x^2 + 30x + 12$

3rd order $\rightarrow \frac{d^3y}{dx^3} = \cancel{36}240x + 30$

4th order $\rightarrow \frac{d^4y}{dx^4} = 240$

Partial derivative of the function of type $u = f(x, y)$

a) $u = x^2 + y^2 - 2xy$

Soln:

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2+y^2-2xy)}{\partial x} = \frac{\partial(x^2)}{\partial x} + \frac{\partial(y^2)}{\partial x} - 2y \cdot \frac{\partial x}{\partial x}$$

$$= 2x + 0 - 2y$$

$$= 2x - 2y$$

$$\frac{\partial u}{\partial y} = \frac{\partial(x^2+y^2-2xy)}{\partial y} = \frac{\partial(x^2)}{\partial y} + \frac{\partial(y^2)}{\partial y} - \frac{\partial(2xy)}{\partial y}$$

$$= 0 + 2y - 2x \cdot \frac{\partial y}{\partial y}$$

$$= 2y - 2x$$

b) $f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$. find f_x, f_y, f_{xx}, f_{xy}, f_{yx}

Soln:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial(x^3+3x^2y+3xy^2+y^3)}{\partial x}$$

$$= \frac{\partial(x^3)}{\partial x} + 3\frac{\partial(x^2y)}{\partial x} + 3\frac{\partial(xy^2)}{\partial x} + \frac{\partial(y^3)}{\partial x}$$

$$= 3x^2 + 3y \cdot 2x + 3y^2 + 0$$

$$= 3x^2 + 6xy + 3y^2$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial(x^3+3x^2y+3xy^2+y^3)}{\partial y}$$

$$= \frac{\partial(x^3)}{\partial y} + \frac{3\,\partial(x^2y)}{\partial y} + 3\frac{\partial(xy^2)}{\partial y} + \frac{\partial(y^3)}{\partial y}$$

$$= 0 + 3x^2 + 6xy + 3y^2$$

$$= 3x^2 + 6xy + 3y^2$$

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial(3x^2+6xy+3y^2)}{\partial x}$$

$$= \frac{3\,\partial(x^2)}{\partial x} + \frac{6\,\partial(xy)}{\partial x} + \frac{3\,\partial(y^2)}{\partial x}$$

$$= 6x + 6y + 0$$

$$= 6x+6y.$$

$$f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial(3x^2+6xy+3y^2)}{\partial y}$$

$$= 3\cdot\frac{\partial(x^2)}{\partial y} + 6\cdot\frac{\partial(xy)}{\partial y} + 3\cdot\frac{\partial(y^2)}{\partial y}$$

$$= 0 + 6x + 6y$$

$$= 6x+6y$$

$$f_{yx} = \frac{\partial f_y}{\partial x} = \frac{\partial(3x^2+6xy+3y^2)}{\partial x}$$

$$= \frac{3\,\partial(x^2)}{\partial x} + \frac{6\,\partial(xy)}{\partial x} + 3\cdot\frac{\partial(y^2)}{\partial x}$$

$$= 6x + 6y + 0$$

$$= 6x+6y$$

$\because u = (x^2+y^2+z^2)^{1/2}$

$\frac{\partial y}{\partial x}$

Q① If $u = x^2 + y^2 + z^2$, then $x u_x + y u_y + z u_z = 2u$.

Soln:

$u = x^2 + y^2 + z^2$

$u_x = 2x + 0 + 0 = 2x$

$u_y = 0 + 2y + 0 = 2y$

$u_z = 0 + 0 + 2z = 2z$

$\therefore x u_x + y u_y + z u_z = x.2x + y.2y + z.2z$

$= 2x^2 + 2y^2 + 2z^2$

$= 2(x^2 + y^2 + z^2)$

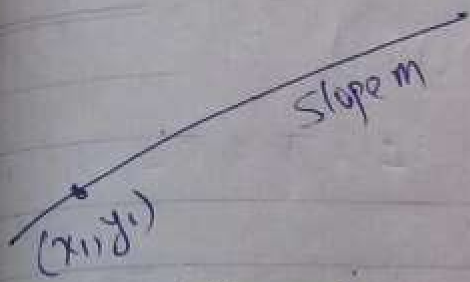
$= 2u$

$=$ RHS proved.

Unit-3

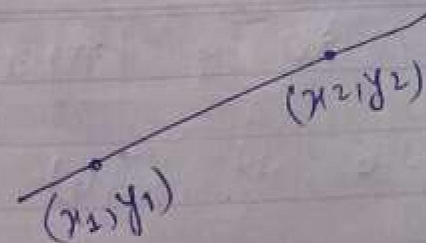
Tangent & Normal

स्पर्शरेखा



$y-y_1=m(x-x_1)$

one point & Slope



$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Two point formula

Equation of straight line & formulas:

① $y=mx+c$

② $\frac{x}{a}+\frac{y}{b}=1$

③ $x\cos\alpha+y\cos\alpha=p$

④ $y-y_1=m(x-x_1)$ → Slope-point formula.

⑤ $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ → Two-point formula.

⑥ $m=\frac{y_2-y_1}{x_2-x_1}$ → Slope formula

Q. Find the equation of tangent and normal of:

① $y = x^2 + 5x + 6$ at $(2, 4)$

Soln:

$$\frac{dy}{dx} = \frac{d(x^2+5x+6)}{dx} = 2x+5$$

$$\therefore \left(\frac{dy}{dx}\right)_{at\,(2,4)} = 2\times2+5 = 9 = m_1$$

$\therefore$ Slope of tangent $(m_1) = 9$

$\therefore$ Slope of normal $(m_1) = \frac{-1}{m} = -\frac{1}{9}$

Equation of tangent at $(2,4)$ is

$y-4 = 9(x-2)$

or, $y-4 = 9x-18$

or, $9x - y = 14.$

Equation of normal is

$y-4 = -\frac{1}{9}(x-2)$

or, $9y - 36 = -x+2$

$\therefore x + 9y = 38.$

② $x^2+y^2 = 4$ at $(3,-1)$

Soln:

Differentiating both sides w.r.t. x

$$\frac{d(x^2+y^2)}{dx} = \frac{d(4)}{dx}$$

or, $\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy}\times\frac{dy}{dx} = 0$

or, $2x + 2y.\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

② rem

$\therefore \frac{dy}{dx}$ at $(3,-1) = \frac{-3}{-1} = 3/1 = 3$

$\therefore$ Slope of tangent $(m_1) = 3$

$\therefore$ Slope of normal $(m_2) = -\frac{1}{m} = -\frac{1}{3}$

Equation of tangent is

$y+1 = 3(x-3)$

or, $y+1 = 3x-9$

$\therefore 3x-y = 10.$

Equation of normal is

$y+1 = -\frac{1}{3}(x-3)$

or $3y+\cancel{3} = -x+\cancel{3}$

$\therefore x+3y = 0.$

③ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(1,7)$

Soln:

Differentiating both sides w.r.t. x

$$\frac{d\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}{dx} = \frac{d(1)}{dx}$$

or, $\frac{1}{a^2}\frac{dx^2}{dx} + \frac{1}{b^2}\frac{dy^2}{dy} \times \frac{dy}{dx} = 0$

or, $\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = \frac{-\cancel{2}x}{a^2} \times \frac{b^2}{\cancel{2}y} = \frac{-b^2x}{a^2y}$

$\therefore \frac{dy}{dx}$ at $(1,7)$ $= -\frac{1}{7}\frac{b^2}{a^2} = \frac{-b^2}{7a^2}$

$\therefore$ Slope of tangent $(m_1) = -\frac{b^2}{7a^2}$

$\therefore$ Slope of normal $(m_2) = -\frac{1}{m} = -\frac{1}{\frac{-b^2}{7a^2}} = \frac{7a^2}{b^2}$

Equation of tangent is

$y-7 = -\frac{b^2}{7a^2}(x-1)$

or, $7a^2y - 49a^2 = -b^2x + b^2$

or, $xb^2 + 7a^2y - 49a^2 - b^2 = 0.$

Equation of normal is

$y-7 = \frac{7a^2}{b^2}(x-1)$

or $yb^2 - 7b^2 = 7xa^2 - \cancel{49a^2}\ 7a^2$

or $7a^2x - b^2y + 7b^2 - \cancel{49a^2} = 0$

Angle of intersection of two curves:

① Find the points on the curve $y = 2x^3 - 15x^2 + 34x - 20$, where the tangents are parallel to $y + 2x = 0$.

Soln:

$\frac{dy}{dx} = 6x^2 - 30x + 34 = m = $ slope of tangent

And slope of given line is $-\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

$= -\frac{2}{1} = -2 = m_2$

According to question, tangent is parallel to the given line

So, $6x^2 - 30x + 34 = -2$

or, $6x^2-30x+36=0$

or, $6(x^2-5x+6)=0$

or, $x^2-2x-3x+6=0$

or, $x(x-2)-3(x-2)=0$

or, $(x-2)(x-3)$

$\therefore x=2$ & $x=3$

if $x=2$, $y=2\times2^3-15\times2^2+34\times2-20$

$=16-60+68-20$

$=84-80$

$=4$

If $x=3$, $y=2\times3^3-15\times3^2+34\times3-20$

$=54-135+102-20$

$=156-155$

$=1$

Hence, the required point are $(2,4)$ & $(3,1)$

(2) Find the points on the curve $y=2x^2-4x+7$, where the tangent is perpendicular to $x+4y=0$.

Soln:

$y=2x^2-4x+7$

Then,

$\frac{dy}{dx}=4x-4=m_1=$ slope of tangent

Also slope of given line $x+4y=0$

is $-\frac{\text{Coeff of } x}{\text{Coeff of } y}=\cancel{\frac{x}{4}}\ -\frac{1}{4}=m_2$

According to question, tangent is perpendicular to the given line

So, $m_1\times m_2=-1$

or, $(4x-4)\times-\frac{1}{4}=-1$

or, $-\frac{1}{\cancel{4}}\times\cancel{4}x+\cancel{4}\times\frac{1}{\cancel{4}}=-1$

or, $-x+1=-1$

$\therefore x=2$

now,

$y = 2\times2^2 - 4\times2 + 7$
$= 8 - 8 + 7$
$= 7$

$\therefore$ The required point is $(2,7)$

3 ③ Find the angle of intersection of:

(i) $x^2 - y^2 = a^2$, $x^2 + y^2 = 2a^2$

Soln:

$x^2 - \cancel{y^2} = a^2$ — (i)
$x^2 + \cancel{y^2} = 2a^2$ — (ii)

Adding (i) & (ii), we get

$2x^2 = 3a^2$

or, $x^2 = \frac{3}{2}a^2$

$\therefore x = \pm\sqrt{\frac{3}{2}}\,a$

Putting value of x in (i)

$\left(\pm\sqrt{\frac{3}{2}}\,a\right)^2 - y^2 = a^2$

or, $\frac{3}{2}a^2 - y^2 = \cancel{y^2}\, a^2$

or, $3a^2$ ~~$- 2y^2 = 2y^2$~~ $3a^2 - 2y^2 = 2a^2$

or, ~~$3a^2 = 4y^2$~~ or, $a^2 = 2y^2$

or, $y^2 = a^2/2$

$\therefore y = \pm \frac{a}{\sqrt{2}}$

Hence, intersecting points are $\left(\sqrt{\frac{3}{2}}\,a, \frac{a}{\sqrt{2}}\right)$

and $\left(-\sqrt{\frac{3}{2}}\,a, -\frac{a}{\sqrt{2}}\right)$

From ①, $\frac{d(x^2-y^2)}{dx} = \frac{d(a^2)}{dx}$

or, $2x - 2y \cdot \frac{dy}{dx} = 0$

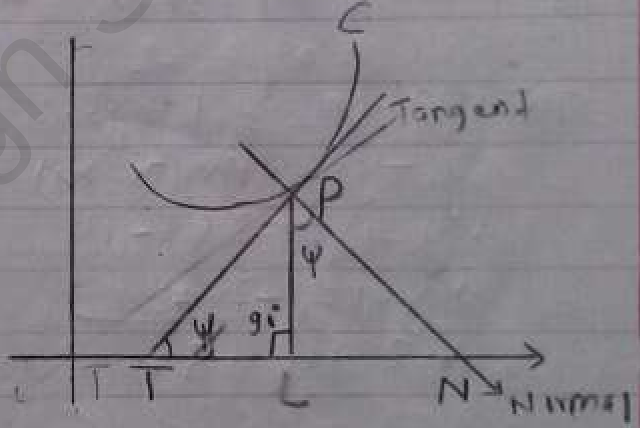
or, $\frac{dy}{dx} = \frac{-2x}{-2y}$

$\therefore \frac{dy}{dx} = x/y$

Q. Draw the diagram of sub-tangent and sub-normal for a curve and find the equation of the tangents to the curve $y = x^2$ and $y = 2 - x^2$ at $(1,1)$.

TCT 2024

Let C be a curve, 'P' is a point on the curve. PT is a tangent and PN is normal to the curve.



Here, TL is subtangent of a curve at P, LN is subnormal of curve P.

2nd part:

Curve $y = x^2$

$\frac{dy}{dx} = \frac{dx^2}{dx}$

$= 2x$

$\left(\frac{dy}{dx}\right)_{at\ (1,1)} = 2 \times 1 = 2$

$\therefore$ slope $(m_1) = 2$

Eq

Equation of tangent at (1,1) is

$y-y_1 = m(x-x_1)$

or $y-1 = 2(x-1)$

or $y-1 = 2x-2$

or $2x-y = 1.$

Again the curve.

$y = 2-x^2$

$\frac{dy}{dx} = -2x$

$\left(\frac{dy}{dx}\right)$ at $(1,1) = -2$

$\therefore$ Slope $(m_2) = -2$

Equation of tangent is

$y-y_1 = m(x-x_1)$

or $y-1 = -2(x-1)$

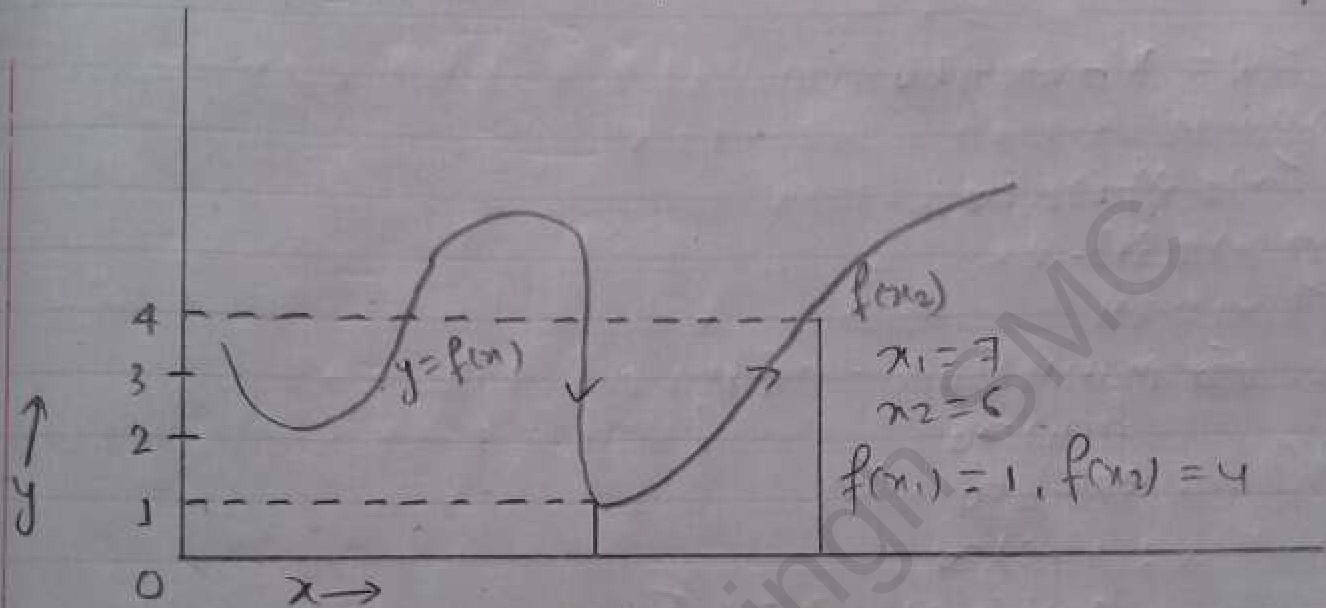
or $y-1 = -2x+2$

or $2x+y = 3.$

Unit-4

Maxima & Minima

Maxima and minima of a function:



Increasing function	Decreasing function
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$	$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

Definition

(a) A function $f(x)$ is called increasing if
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

(b) A function $f(x)$ is called decreasing if
$x_2 < x_1 \Rightarrow f(x_2) > f(x_1)$

A. Find the maximum & minimum value of:
(local maxima & minima).

1. $f(x) = x^3 - 6x + 10$

Soln:

$f(x) = x^3 - 6x + 10$

$f'(x) = 3x^2 - 6$

$f''(x) = 6x$

For maxima and minima.

$f'(x) = 0$

or, $3x^2 - 6 = 0$

or, $3(x^2 - 2) = 0$

or, $\cancel{(x+2)(x-2)=0}$ $x = \pm\sqrt{2}$

$\therefore x = \cancel{2}, -2$

The value of ~~f(x)~~ $f''(x)$ at $x = \sqrt{2}$

$f''(x) = 6x$

$= 6\sqrt{2}$ (positive)

So $f(x)$ has ~~maximum~~ minimum value at $x = \sqrt{2}$

So, minimum value is

$f(\sqrt{2}) = (\sqrt{2})^2 - 6\sqrt{2} + 10$

$= 2\sqrt{2} - 6\sqrt{2} + 10$

$= 10 - 4\sqrt{2}$

$= 4.35$

At $x = -\sqrt{2}$

$f''(x) = 6 \times (-\sqrt{2})$

$= -6\sqrt{2}$ (negative)

So, $f(x)$ has maximum value at $x = -\sqrt{2}$

The maximum value is $f(-\sqrt{2}) = (-\sqrt{2})^3 - 6 \times (-\sqrt{2}) + 10$

$= -2\sqrt{2} + 6\sqrt{2} + 10$

$= 10 + 4\sqrt{2}$

$= 15.65$

2. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

Soln:

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$f''(x) = 12x^2 - 48x + 44.$

For maxima and minima,

$f'(x) = 0$

or, $4x^3 - 24x^2 + 44x - 24 = 0$

or $4(x^3 - 6x^2 + 11x - 6) = 0$

or, $x^3 - 6x^2 + 11x - 6 = 0$

or, $x^3 - x^2 - 5x^2 + 5x + 6x - 6 = 0$

or, $x^2(x-1) - 5x(x-1) + 6(x-1) = 0$

or, $(x-1)(x^2 - 5x + 6) = 0$

or, $(x-1)[x^2 - 2x - 3x + 6] = 0$

or $(x-1)[x(x-2) - 3(x-2)] = 0$

or, $(x-1)(x-2)(x-3) = 0$

$\therefore x = 1, 2, 3$

At $x = 1$.

$f''(x) = 12 - 48 + 44$

$= 8$ (+ve)

So, $f(x)$ has minimum value at $x = 1$

The minimum value is, $f(1) = 1 - 8 + 22 - 24 + 5 = -4$

At $x = 2$.

$f''(x) = 12 \times 2^2 - 48 \times 2 + 44$

$= 48 - 96 + 44$

$= -4$ (negative)

So, $f(x)$ has maximum value at $x = 2$ and given by $f(2) = 2^4 - 8 \times 2^3 + 22 \times 2^2 - 24 \times 2 + 5$

$= 16 - 64 + 88 - 48 + 5 = -3$

At $x=3$, $f''(x) = 12\times3^2 - 48\times3 + 44$
$= 108 - 144 + 44$
$= 8$ (+ve)

So f(x) has minimum value at $x=3$ which is given by
$f(3) = 3^4 - 8\times3^3 + 22\times3^2 - 24\times3 + 5$
$= 81 - 216 + 198 - 72 + 5$
$= 284 - 288$
$= -4.$

So minimum value is -4 & maximum value is -3.

3. $f(x) = x^3 - 4x^2 + 4x - 2$

Soln:
$f(x) = x^3 - 4x^2 + 4x - 2$
$f'(x) = 3x^2 - 8x + 4$
$f''(x) = 6x - 8$

B.

For maxima & minima
$f'(x) = 0$
or, $3x^2 - 8x + 4 = 0$
or, $3x^2 - 6x - 2x + 4 = 0$
or, $3x(x-2) - 2(x-2) = 0$
or, $(x-2)(3x-2) = 0$
$\therefore x = 2, 2/3$

At $x=2$
$f''(x) = 6\times2 - 8 = 4$ (+ve)

So f(x) has minimum value and given by
$f(2) = 2^3 - 4\times2^2 + 4\times2 - 2$
$= 8 - 16 + 8 - 2$
$= -2$

At $x = 2/3$
$f''(x) = 6\times\frac{2}{3} - 8 = -4$ (-ve)

So, $f(x)$ has maximum value given by

$$f(2/3) = (2/3)^3 - 4\times(2/3)^2 + 4\times 2/3 - 2$$

$$= \frac{8}{27} - \frac{16}{9} + \frac{8}{3} - 2$$

$$= \frac{8 - 48 + 72 - 54}{27}$$

$$= \frac{80 - 102}{27}$$

$$= \frac{-22}{27} = -3.14$$

8. Find the global maximum and minimum value of: $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$ at $[0, 2]$

Soln:

$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$f'(x) = 12x^3 - 6x^2 - 12x + 6$

For maximum & minimum value:

$f'(x) = 0$

or $12x^3 - 6x^2 - 12x + 6 = 0$

or, $6(2x^3 - x^2 - 2x + 1) = 0$

or, $2x^3 - 2x - x^2 + 1 = 0$

or $2x(x^2 - 1) - 1(x^2 - 1) = 0$

or, $(x+1)(x-1)(2x-1) = 0$

$\therefore x = 1, -1, 1/2$

Here, $-1 \notin [0, 2]$ so, we search for $x = 1, 1/2, 0$ and 2

Then

$f(1) = 2$, $f(1/2) = 3\times(\frac{1}{2})^4 - 2\times(\frac{1}{2})^3 - 6\times(\frac{1}{2})^2 + 6\times\frac{1}{2} + 1$

$$= \frac{3}{16} - \frac{2}{8} - \frac{6}{4} + 3 + 1$$

$$= \frac{3 - 4 - 24 + 48 + 16}{16} = \frac{39}{16}$$

$\therefore f(0) = 1$

$f(2) = 21$

$\therefore$ Global maxima is 21 and global minima is 1.

c. Find the extreme value of each of the following and show that maximum value is less than the minimum value

ⓐ $f(x) = x + \frac{100}{x} - 25$

[max & min value
Extreme value
Optimum value]

Soln:

$f(x) = x + \frac{100}{x} - 25$

$f'(x) = \frac{d\left(x + \frac{100}{x} - 25\right)}{dx}$

$= \frac{dx}{dx} + 100 \cdot \frac{d(x^{-1})}{dx} - 0$

$= 1 + 100 \cdot (-1) x^{-2}$

$= 1 - 100 \times \frac{1}{x^2}$

$= 1 - \frac{100}{x^2}$

$f''(x) = d\left(1 - \frac{100}{x^2}\right)$

$= 0 - 100 \cdot \frac{d(x^{-2})}{dx}$

$= -100 \cdot (-2) x^{-3}$

$= \frac{200}{x^3}$

For maxima & minima

$f'(x) = 0$

or, $1 - \frac{100}{x^2} = 0$

or $x^2 - 100 = 0$

or $(x)^2 - (10)^2 = 0$

or $(x+10)(x-10) = 0$

$\therefore x = -10, 10$

At $x = -10$

$f''(x) = \frac{200}{(-10)^3} = \frac{200}{-1000} = -\frac{1}{5}$ (negative)

So $f(x)$ has maximum value at $x = -10$

i.e $f(-10) = -10 + \frac{100}{(-10)} - 25$

$= -10 - 10 - 25 = -45$

At $x = 10$

$f''(x) = \frac{200}{(10)^3} = \frac{200}{1000} = \frac{1}{5}$ (Positive)

So $f(x)$ has minimum value at $x = 10$

i.e. $f(10) = 10 + \frac{100}{10} - 25$

$= 10 + 10 - 25 = -5$

$\therefore$ The maximum value is less than the minimum value

Unit-5
Indefinite Integral

Here

if $\frac{dF(x)}{dx} = f(x),\ x \in (a,b)$

As derivative of constant C is zero, $F(x)+c$ is also an antiderivative of f, whenever the function F is so.

if F is an antiderivative of f, we have

$\int f(x)\,dx = F(x) + c$

Formulas

- $\int x^{n-1}\,dx = \frac{x^n}{n} + c \quad \left(\because \int f(x)\,dx = F(x)+c\right)$
- $\int \cos ax\,dx = \frac{\sin ax}{a} + c$
- $\int \sin ax\,dx = -\frac{\cos ax}{a} + c$
- $\int \sec ax \tan ax\,dx = \frac{\sec ax}{a} + c$
- $\int \sec^2 ax\,dx = \frac{\tan ax}{a} + c$
- $\int \operatorname{cosec} ax \cdot \cot ax\,dx = -\frac{\operatorname{cosec} ax}{a} + c$
- $\int \operatorname{cosec}^2 ax\,dx = -\frac{\cot ax}{a} + c$
- $\int \frac{1}{x}\,dx = \log x + c$
- $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$
- $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$
- $\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} + c$

Q. Integrate the following:

① $\int x^2 dx$

Soln.

$I = \int x^2 dx$

$= \frac{x^{2+1}}{2+1} + c$ where c is integration constant

$= \frac{x^3}{3} + c$

② $\int 5\,dx$

$= 5\int dx$

$= 5x + c$

③ $I = \int x^m dx$

$= \frac{x^{m+1}}{m+1} + c$

④ $I = \int (x^6 - x^5 + 4x^2)\,dx$

$= \frac{x^{6+1}}{6+1} - \frac{x^{5+1}}{5+1} + 4 \cdot \frac{x^{2+1}}{2+1} + c$

$= \frac{x^7}{7} - \frac{x^6}{6} + 4 \cdot \frac{x^3}{3} + c$

Q2. Evaluate:

① $I = \int (x+3)^8 dx$

Let $x+3 = y$

$\therefore \frac{d(x+3)}{dx} = \frac{dy}{dx}$

or, $\frac{dx}{dx} + 0 = \frac{dy}{dx}$

or, $1 = \frac{dy}{dx}$

$\therefore dx = dy$

$\therefore I = \int y^8 \, dx$

$= \frac{y^{8+1}}{8+1} + c$

$= \frac{(x+3)^9}{9} + c.$

(2) $I = \int (2x+3)(4x+5)^4 \, dx$

soln:

Let $4x+5 = y$

or, $4x = y-5$

$\therefore x = \frac{y-5}{4}$

Then,

$\frac{d(4x+5)}{dx} = \frac{dy}{dx}$

or, $4 = \frac{dy}{dx}$

$\therefore dx = \frac{dy}{4}$

$\therefore I = \int \left\{ \cancel{2} \cdot \frac{(y-5)}{\cancel{4}_2} + 3 \right\} y^4 \cancel{dx} \frac{dy}{4}$

$= \frac{1}{4} \int \left(\frac{y-5}{2} + 3 \right) y^4 \, dy$

$= \frac{1}{4} \int \frac{y-5+6}{2} y^4 \, dy$

$= \frac{1}{8} \int (y+1) \cdot y^4 \, dy$

$$= \frac{1}{8}\left[\int y^5 dy + \int y^4 dy\right]$$

$$= \frac{1}{8}\left[\frac{y^6}{6} + \frac{y^5}{5}\right] + c$$

$$= \frac{1}{48}(4x+5)^6 + \frac{1}{40}(4x+5)^5 + c \quad \underline{\text{Ans.}}$$

(3) $I = \int \frac{1}{\sqrt{x}}\, dx$

$$= \int x^{-1/2} dx$$

$$= \frac{x^{-1/2+1}}{-1/2+1} + c$$

$$= \frac{x^{1/2}}{1/2} + c = 2\sqrt{x} + c$$

(4) $\int (x^{1/3} + x^{3/4} - 4x^{-5/6})\, dx$

$$= \int x^{1/3} dx + \int x^{3/4} dx - 4\int x^{-5/6} dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + \frac{x^{3/4+1}}{3/4+1} - 4\cdot\frac{x^{-5/6+1}}{-5/6+1}$$

$$= \frac{x^{4/3}}{4/3} + \frac{x^{7/4}}{7/4} - \frac{4\cdot x^{1/6}}{1/6} + c$$

$$= \frac{3x^{4/3}}{4} + \frac{4\cdot x^{7/4}}{7} - 24x^{1/6} + c$$

(5) $\int \frac{x^4-1}{x+1}\, dx$

$$= \int \frac{(x^2)^2 - (1)^2}{x+1}\, dx$$

$$= \int \frac{(x^2+1)\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}\,dx$$

$$= \int (x^2+1)(x-1)\,dx$$

$$= \int (x^3 - x^2 + x - 1)\,dx$$

$$= \frac{x^{3+1}}{3+1} - \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} - x$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x$$

<u>Integrate</u>

① $I = \int \frac{x\,dx}{3x^2-4}$

Let $y = 3x^2-4$

$\therefore dy = 6x\,dx$

$\therefore x\,dx = \frac{1}{6}\,dy$

$\therefore I = \frac{1}{6}\int \frac{dy}{y}$

$= \frac{1}{6}\log y + c$

$= \frac{1}{6}\log(3x^2-4) + c$

② $I = \int \frac{x^{n-1}}{\sqrt{a^n+x^n}}\,dx$

Let, $y = a^n + x^n$

$\frac{dy}{dx} = 0 + nx^{n-1}$ or, $dy = nx^{n-1}\,dx$

$\Rightarrow x^{n-1}\,dx = \frac{1}{n}\,dy$

$$\therefore I = \int \frac{\frac{1}{n}\,dy}{\sqrt{y}}$$

$$= \frac{1}{n} \int \frac{dy}{y^{1/2}}$$

$$= \frac{1}{n} \int y^{-1/2}\,dy$$

$$= \frac{1}{n} \cdot \frac{y^{-1/2+1}}{-1/2+1} + c$$

$$= \frac{1}{n} \frac{y^{1/2}}{1/2}$$

$$= \frac{2}{n} \sqrt{y} + c$$

$$= \frac{2}{n} \sqrt{a^n + x^n} + c$$

(3) $I = \int \frac{(3x+2)}{(3x^2+4x+1)^3}\,dx$

Let, $y = 3x^2+4x+1$

or, $\frac{dy}{dx} = 6x+4$

or, $dy = (6x+4)dx$

or, $dy = 2(3x+2)\,dx$

or, $(3x+2)dx = \frac{1}{2}\,dy$

$$\therefore I = \int \frac{\frac{1}{2}\,dy}{y^3}$$

$$= \frac{1}{2} \int y^{-3}\,dy$$

$$= \frac{1}{2} \cdot \frac{y^{-3+1}}{-3+1} + c$$

$$= \frac{1}{2}\frac{y^{-2}}{-2} + c$$

$$= \frac{y^{-2}}{-4} + c$$

$$= -\frac{1}{4y^2} + c$$

$$= -\frac{1}{4(3x^2+4x+1)^2} + c$$

(4) $\int \sin^2 x \, dx$

$$I = \frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\left[\int 1 \cdot dx - \int \cos 2x \, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$= \frac{1}{2}\left[\frac{2x - \sin 2x}{2}\right] + c$$

$$= \frac{1}{4}\left[2x - \sin 2x\right] + c$$

(5) $\int \tan^2 ax \, dx$

$$= \int (\sec^2 ax - 1)\, dx$$

$$= \int \sec^2 ax \, dx - \int 1 \cdot dx$$

$$= \frac{\tan ax}{a} - x + c$$

(6) $\int \sqrt{1-\sin 2x}\, dx$

$$= \int \sqrt{(\sin^2 x + \cos^2 x - 2\sin x \cdot \cos x)}\, dx$$

$$= \int \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= \int (\sin x - \cos x)\, dx$$

$$= -\cos x - \sin x + c$$

⑦ $\int \frac{dx}{1-\sin x}$

$= \int \frac{(1+\sin x)\,dx}{(1+\sin x)(1-\sin x)}$

$= \int \frac{1+\sin x}{1-\sin^2 x}\,dx$

$= \int \frac{1+\sin x}{\cos^2 x} = \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\,dx$

$= \int \sec^2 x\,dx + \int \tan x\cdot\sec x\,dx$

$= \tan x + \sec x + c.$

⑧ $\int \sin 6x\cdot\cos 3x\,dx$

$= \frac{1}{2}\int 2\sin 6x\cdot\cos 3x\,dx$

$= \frac{1}{2}\int (\sin 9x + \sin 3x)\,dx$

$= \frac{1}{2}\left[-\frac{\cos 9x}{9} - \frac{\cos 3x}{3}\right] + c$

$= -\frac{1}{18}\left[\cos 9x + 3\cos 3x\right] + c$

⑨ $\int \frac{1-e^{3x}}{e^{5x}}\,dx$

$= \int \left(e^{-5x} - e^{-2x}\right) dx$

$= \frac{e^{-5x}}{-5} - \frac{e^{-2x}}{-2} + c$

$= -\frac{1}{5e^{5x}} + \frac{1}{2e^{2x}} + c$

(10) $\int \operatorname{cosec} x \, dx$

$= \int \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

Put $y = \operatorname{cosec} x - \cot x$

$\frac{dy}{dx} = -\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x$

or $\frac{dy}{dx} = \operatorname{cosec} x (-\cot x + \operatorname{cosec} x)$

or $\frac{dy}{dx} = \operatorname{cosec} x (\operatorname{cosec} x - \cot x)$

or, $dy = \operatorname{cosec} x (\operatorname{cosec} x - \cot x) dx$

$\therefore \int \frac{dy}{y} = \log y + c$

$= \log (\operatorname{cosec} x - \cot x) + c$ Ans

(11) $\int x \cos x \, dx$

Take x as first function and $\cos x$ as second.

Then

$\int x \cos x \, dx = x \int \cos x \, dx - \int \frac{d}{dx}(x) \int \cos x \, dx$

$= x \sin x - \int 1 \cdot \sin x \, dx$

$= x \sin x - (-\cos x) + c$

$= x \sin x + \cos x + c.$

Unit - 6

Definite Integral

Question

① $\int_0^3 x^5\,dx$

$= \left[\frac{x^6}{6}\right]_0^3$

$= \frac{1}{6}\left[3^6 - 0^6\right]$

$= \frac{1}{6} \times 729$

$= \frac{243}{2}$

② $I = \int_{-1}^{2} (x^2 + x + 1)\,dx$

$= \int_{-1}^{2} x^2\,dx + \int_{-1}^{2} x\,dx + \int_{-1}^{2} dx$

$= \left[\frac{x^3}{3}\right]_{-1}^{2} + \left[\frac{x^2}{2}\right]_{-1}^{2} + [x]_{-1}^{2}$

$= \frac{1}{3}\left[2^3 - (-1)^3\right] + \frac{1}{2}\left[2^2 - (-1)^2\right] + [2+1]$

$= \frac{1}{3}[8+1] + \frac{1}{2}[4-1] + 3$

$= \frac{1}{3} \times 9 + \frac{1}{2} \times 3 + 3$

$= 3 + 3 + \frac{3}{2}$

$= \frac{15}{2}$

③ $I = \int_0^2 \frac{x\,dx}{\sqrt{x^2+4}}$

Let $y = x^2+4$

or, $\frac{dy}{dx} = 2x$

or $dy = 2x\,dx$

$\therefore x\,dx = \frac{1}{2}dy$

$\therefore I = \int_0^2 \frac{\frac{1}{2}dy}{\sqrt{y}}$

$= \frac{1}{2}\int_0^2 y^{-1/2}\,dy$

$= \frac{1}{2}\left[\frac{y^{-1/2+1}}{-1/2+1}\right]_0^2$

$= \frac{1}{2}\left[\frac{y^{1/2}}{1/2}\right]_0^2$

$= \cancel{\frac{1}{2}} \times \cancel{2}\left[(x^2+4)^{1/2}\right]_0^2$

$= \left[(2^2+4)-(0^2+4)\right]^{1/2}$

$= (8-4)^{1/2}$

$= 4^{1/2}$

$= 2$

④ $I = \int_0^{\pi/6} 5\sqrt{3}\tan^4 x \cdot \sec^2 x\,dx$

$= 5\sqrt{3}\int_0^{\pi/6} \tan^4 x \cdot \sec^2 x\,dx$

Let. $\tan x = y$

$\therefore \sec^2 x dx = dy$

$\therefore I = 5\sqrt{3} \int_0^{\pi/6} y^4 dy$

$= 5\sqrt{3} \left[\frac{y^5}{5}\right]_0^{\pi/6}$

$= \frac{\cancel{5}\sqrt{3}}{\cancel{5}} \left[\tan^5 x\right]_0^{\pi/6}$

$= \cancel{5}\sqrt{3} \left[\tan^5 \pi/6 - \tan^5 0\right]$

$= \sqrt{3} \left[(\tan \pi/6)^5 - (\tan 0)^5\right]$

$= \sqrt{3} \left[\left(\frac{1}{\sqrt{3}}\right)^5 - (0)^5\right]$

$= \sqrt{3} \times \frac{1}{(\sqrt{3})^5}$

$= \frac{1}{(\sqrt{3})^4} = \frac{1}{3^{1/2 \times 4}} = \frac{1}{3^2} = \frac{1}{9}$ "

⑤ $I = \int_0^{\pi/2} (1+\cos x)^2 \cdot \sin x \, dx$

Let $y = 1 + \cos x$

or, $dy = -\sin x \, dx$

or $-dy = \sin x \, dx$

$\therefore I = \int_0^{\pi/2} y^2 \cdot (-dy)$

$$= -\int_0^{\pi/2} y^2 dy$$

$$= -\left[\frac{y^3}{3}\right]_0^{\pi/2}$$

$$= -\frac{1}{3}\left[(1+\cos x)^3\right]_0^{\pi/2}$$

$$= -\frac{1}{3}\left[(1+\cos \pi/2)^3 - (1+\cos 0^\circ)^3\right]$$

$$= -\frac{1}{3}\left[(1+0)^3 - (1+1)^3\right]$$

$$= -\frac{1}{3}[1-8]$$

$$= +\frac{1}{3} \times +7$$

$$= \frac{7}{3}.$$

⑥ $\int_0^{\pi/2} x \sin x \, dx$

$$I = \int x \sin x \, dx = x \int \sin x \, dx - \int\left(\frac{dx}{dx} \cdot \int \sin x \, dx\right) dx$$

$$= x(-\cos x) - \int (1(-\cos x)) dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x$$

$$\therefore I = \int_0^{\pi/2} x \sin x \, dx = \left[(-x\cos x + \sin x)\right]_0^{\pi/2}$$

$$= -\left(\frac{\pi}{2}\cos \pi/2 - 0 \cdot \cos 0\right) + \left(\sin \pi/2 - \sin 0\right)$$

$$= -(\pi/2 \times 0 - 0 \cdot 1) + (1-0)$$

$$= 0 + 1 = 1.$$

Area of Plane regions

① Find the area bounded by the curve $y = 2x^2 - 3$, x-axis, $x=0$, $x=a$. (Use the limit of sum)

Soln:

$f(x) = 2x^2 - 3$

Then, using $\int_0^a f(x)\,dx = \lim_{h\to 0} h[f(h) + f(2h) + f(3h) + \dots + f(nh)]$

We have

$$\int_0^a (2x^2 - 3)\,dx = \lim_{h\to 0} [(2h^2-3) + (2\cdot 2^2h^2 - 3) + (2\cdot 3^2 h\, h^2 - 3) + \dots + (2n^2h^2 - 3)]$$

where, $nh = a - 0 = a$

$$= \lim_{h\to 0} h[2h^2(1^2 + 2^2 + 3^2 + \dots + n^2) - 3n]$$

$$= \lim_{h\to 0} \left\{ \frac{1}{3} \cdot 2h^3 \cdot \frac{n(n+1)(2n+1)}{6} - 3nh \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{1}{3} \cdot nh(nh+h)(2nh+h) - 3nh \right\}$$

$$= \frac{1}{3} \cdot a(a+0)(2a+0) - 3a$$

$$= \frac{2}{3}a^2 - 3a \quad \text{Ans}$$

② Find the area enclosed by $x = 3x$, the x-axis and ordinates $x=0$, $x=4$.

Soln:

The required area $= \int_0^4 y\,dx = \int_0^4 3x\,dx$

$$= 3\int_0^4 \frac{x^2}{2}\,dx = \frac{3}{2}\int_0^4 x^2 = \frac{3}{2}[4^2 - 0^2]$$

$$= \frac{3}{2} \times 16 = 24.$$

③ Find the area bounded by the curve $y^2 = 4ax$, the x-axis and the ordinate which cuts the curve at the point $(a, 2a)$.

Soln:

We have to find the area bounded by the curve $y^2 = 4ax$, the x-axis and the ordinates $x = 0$ & $x = a$.

$$\therefore \text{Area} = \int_0^a y \, dx$$

$$= \int_0^a 2\sqrt{ax}\,dx$$

$$= 2\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{4}{3}\sqrt{a}\left[a^{3/2} - 0^{3/2}\right]$$

$$= \frac{4}{3}a^{1/2}.a^{3/2}$$

$$= \frac{4}{3}a^2$$

④ Find the area enclosed by the axis of x and the curve $y = x^2 - 4x - 3$.

Soln:

The given equation of the curve is $y = x^2 - 4x - 3$. The curve meets the x-axis at the point where

$y = 0$

$\therefore x^2 - 4x - 3 = 0$

or, $(x-1)(x-3) = 0$

$\therefore x = 1, 3$

The required area $= \int_1^3 y\,dx = \int_1^3 (x^2 - 4x + 3)\,dx$

$= \int_1^3 x^2\,dx - 4\int_1^3 x\,dx + 3\int_1^3 dx$

$= \left[\frac{x^3}{3}\right]_1^3 - \frac{4}{2}\left[x^2\right]_1^3 + 3\left[x\right]_1^3$

$= \frac{1}{3}[27 - 1] - \frac{4}{2}[9 - 1] + 3[3 - 1]$

$= \frac{1}{3} \times 26 - \frac{4}{2} \times 8 + 3 \times 2$

$= \frac{26}{3} - 16 + 6$

$= \frac{26 - 48 + 18}{3} = -\frac{4}{3}$

(5) Find the area bounded by the curve $y = x^2$ and $y = 2x$.

Soln:

$y = x^2$ — (I)

$y = 2x$ — (II)

from (I) & (II)

$x^2 = 2x$

or $x^2 - 2x = 0$

or $x(x - 2) = 0$

$\therefore x = 0, 2$

$\therefore$ Area $= \int_0^2 (y_1 - y_2)\,dx$ (Consider $y_1 = 2x$ & $y_2 = x^2$)

$$= \int_0^2 (2x - x^2)\, dx$$

$$= 2\int_0^2 x\, dx - \int_0^2 x^2\, dx$$

$$= \cancel{2}\cdot\left[\frac{x^2}{\cancel{2}}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= [2^2 - 0] - \frac{1}{3}[2^3 - 0]$$

$$= 4 - \frac{1}{3} \times 8$$

$$= \frac{12-8}{3} = \frac{4}{3}$$

⑥ $y^2 - x - 4 = 0$, $x = 2, x = 5$

Soln:

$y^2 - x - 4 = 0$

or $y^2 = x + 4$

$\therefore y = \sqrt{x+4} = (x+4)^{1/2}$

$$\therefore \text{Area}(A) = \int_2^5 y\, dx$$

$$= \int_2^5 (x+4)^{1/2}\, dx$$

$$= \left[\frac{(x+4)^{3/2}}{3/2}\right]_2^5$$

$$= \frac{2}{3}\left[(5+4)^{3/2} - (2+4)^{3/2}\right]$$

$$= \frac{2}{3} \times \left[9^{3/2} - 6^{3/2}\right]$$

$$= \frac{2}{3} \times \left[3^{\cancel{2} \times 3/\cancel{2}} - 6\sqrt{6}\right]$$

$$= \frac{2}{\cancel{3}} \times \cancel{27}^{9} - \frac{2}{\cancel{3}} \times \cancel{6}^{2}\sqrt{6}$$

$$= 18 - 4\sqrt{6}$$

Unit-7

Quadrature, Rectification and Volume

Area

① Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Soln:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or, $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$$=\frac{a^2-x^2}{a^2}$$

or, $y^2=\frac{b^2}{a^2}(a^2-x^2)$

$\therefore y=\sqrt{\frac{b^2}{a^2}(a^2-x^2)}=\frac{b}{a}\sqrt{a^2-x^2}$

Now

Area of OAB $=\int_0^a y\,dx$

$$=b/a\int_0^a\sqrt{a^2-x^2}\,dx$$

$\therefore A=b/a\int_0^a\sqrt{a^2-x^2}$

Put $x=a\sin\theta$

$\frac{dx}{d\theta}=a\cos\theta$

$\therefore dx=a\cos\theta\,d\theta$

$\therefore A=b/a\int_0^a\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$$=b/a\int_0^a\sqrt{a^2(1-\sin^2\theta)}\cdot a\cos\theta\,d\theta$$

$$=\frac{ba^2}{a}\int_0^a\cos\theta\cdot\cos\theta\,d\theta$$

$= ab \int_0^{\pi/2} \cos^2\theta \, d\theta$

$= \cancel{ab \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}}$

$= ab \int_0^{\pi/2} \frac{1}{2} \times 2\cos^2\theta \, d\theta$

$= \frac{ab}{2} \int_0^{\pi/2} (1+\cos 2\theta) \, d\theta$

$= \frac{ab}{2} \int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta \, d\theta$

$= \frac{ab}{2} [\theta]_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin 2\theta}{2}$

$= \frac{ab}{2} [\theta]_0^{\pi/2} + \frac{1}{2} [\sin \cancel{2} \tfrac{\pi}{\cancel{2}} - \sin\theta]$

$= \frac{ab}{2} \left\{ [\pi/2 - 0] + \frac{1}{2} \right\}$

$= \frac{ab}{2} \left[\frac{\pi}{2} + \frac{1}{2} \right]$

$= \frac{ab}{4} (\pi + 1)$ Ans

② Find the area bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Soln:

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

or $\frac{y^2}{16} = 1 - \frac{x^2}{9}$

or, $y = \sqrt{16 \left(\frac{9 - x^2}{9} \right)}$

$\therefore$ ....

or, $y = \frac{4}{3}\sqrt{9-x^2}$

$\therefore$ Area of AOB $= \int_0^3 \frac{4}{3}\sqrt{9-x^2}\,dx$

Put $x = 3\sin\theta$

$\therefore dx = 3\cos\theta\,d\theta$

$I = \frac{4}{3}\int_0^3 \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$

$= \frac{4}{3}\int_0^3 3\sqrt{1-\sin^2\theta}\cdot 3\cos\theta\,d\theta$

$= \frac{4}{3}\int_0^3 3\cos\theta\cdot 3\cos\theta\,d\theta$

$= \frac{4}{3}\int_0^3 9\cos^2\theta\,d\theta$

$= \frac{4}{\cancel{3}}\times \cancel{9}^{3}\int_0^3 \cos^2\theta\,d\theta$

$= 12\int_0^3 \frac{1}{2}\cdot 2\cos^2\theta\,d\theta$

$= \frac{12}{2}\int_0^3 (1+\cos2\theta)\,d\theta$

$= 6\left[\int_0^3 d\theta + \int_0^3 \cos2\theta\,d\theta\right]$

$= 6\left[\theta\right]_0^3 + \cancel{6}^{3}\left[\frac{\sin2\theta}{\cancel{2}}\right]_0^3$

$= 6\left[3-0\right] + 3\left[\sin 6\theta - \sin 0\right]$

$=$

③ find the area bounded by the curve $x^2+y^2=a^2$

Soln:

$x^2+y^2=a^2$

or $y^2=a^2-x^2$

or $y=\sqrt{a^2-x^2}$

$\therefore A=\int_0^a \sqrt{a^2-x^2}\,dx$

Put $x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

If $x=0$, $0=a\sin\theta \Rightarrow \sin\theta=0$

or $\sin\theta=\sin 0$

$\therefore \theta=0^\circ$

If $x=a$, $a=a\sin\theta \Rightarrow \sin\theta=1=\sin\pi/2$

$\therefore \theta=\pi/2$

$\therefore \text{Area}=\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$

$=\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\,d\theta$

$=a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$

$=a^2\int_0^{\pi/2}\frac{1}{2}\cdot 2\cos^2\theta\,d\theta$

$=\frac{a^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

$=\frac{a^2}{2}\left[\int_0^{\pi/2}d\theta+\int_0^{\pi/2}\cos 2\theta\,d\theta\right]$

$$= \frac{a^2}{2}\left[\theta\right]_0^{\pi/2} + \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} - 0\right] + \frac{a^2}{4}\left[\sin 2\times \pi/2 - \sin 0\right]$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} + \frac{a^2}{4}\left[0 - 0\right]$$

$$= \frac{a^2\pi}{4} + \frac{a^2}{8}\, 0$$

$$= \frac{\pi a^2}{4} \text{ sq. unit.}$$

$\therefore$ The total area of the circle $= 4 \times \frac{\pi a^2}{4}$

$= \pi a^2$ sq. units.

(4) Find the area bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$

Soln:

$y^2 = 4ax$ —(i) or $x = y^2/4a$ — y_1

$x^2 = 4ay$ —(ii) or $y = x^2/4a$ — y_2

Solving (i) & (ii)

$$\left(\frac{y^2}{4a}\right)^2 = 4ay$$

or $\frac{y^4}{16a^2} = 4ay$

or $y^4 - 64a^3 y = 0$

or $y(y^3 - 64a^3) = 0$

Either $y = 0$ or $y^3 - 64a^3 = 0$

or $y^3 = 64a^3$

$\therefore y = 4a$

If $y=0, x=0$
if $y=4a, x=4a$
Then
Area bounded by curve $= \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$

$= \sqrt{4a} \int_0^{4a} \sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$

$= \sqrt{4a}\left[\frac{x^{1/2+1}}{1/2+1}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$

$= \sqrt{4a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\cdot\frac{1}{3}\left[x^3\right]_0^{4a}$

$= \sqrt{4a}\times\frac{2}{3}\left[(4a)^{3/2}-0^{3/2}\right] - \frac{1}{12a}\left[(4a)^3-0^3\right]$

$= \frac{2\sqrt{4a}}{3}\left[4^{3/2}\cdot a^{3/2}\right] - \frac{1}{12a}\times 64a^{3}$

$= \frac{16\times 2}{3}\times a^{1/2}\cdot a^{3/2} - \frac{16a^2}{3}$

$= \frac{32}{3}a^2 - \frac{16}{3}a^2$

$= \frac{16}{3}a^2$ sq. units.

$\cancel{= \frac{8a}{3}\times 2^{2\times 3/2}\times a^{3/2} - \frac{64a^2}{12}}$

$\cancel{= \frac{64}{3}a\cdot a^{3/2} - \frac{64a^2}{12}}$

$\cancel{= \frac{64}{3}a^2\cdot\sqrt{a} - \frac{64a^2}{12}}$

$\cancel{= 64a^2\left(\frac{\sqrt{a}}{3}-\frac{1}{12}\right)}$ ~~sq. units.~~

Length

For Cartesian form

Arc length $= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ For curve $y = f(x)$, x is independent variable.

Arc length $= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ for curve $x = f(y)$ where y is independent variable

Questions

① Find the perimeter of the circle, $x^2+y^2=a^2$

Soln:

A $x^2+y^2=a^2$

or, $y^2 = a^2-x^2$

Arc length AB $= \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^a \sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx$

$= \int_0^a \sqrt{\frac{x^2+y^2}{y^2}}\,dx$

$= \int_0^a \sqrt{\frac{y^2+x^2}{y^2}}\,dx$

$= \int_0^a \sqrt{\frac{a^2}{y^2}}\,dx$

$= a\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx$

Put $x = a\sin\theta$

$\therefore dx = a\cos\theta\, d\theta$

If $x=0$, $a\sin\theta = 0 \Rightarrow \sin\theta = \sin 0 \therefore \theta = 0^\circ$

If $x=a$, $a = a\sin\theta \Rightarrow \sin\theta = 1 \therefore \theta = \pi/2$

$$\therefore I = a\int_0^{\pi/2} \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= a\int_0^{\pi/2} \frac{1}{\cancel{a\cos\theta}} \cdot \cancel{a\cos\theta}\, d\theta$$

$$= a\int_0^{\pi/2} d\theta = a[\theta]_0^{\pi/2} = a[\pi/2 - 0] = a\pi/2$$

$$\therefore \text{Total perimeter} = \overset{2}{\cancel{4}} \times \frac{a\pi}{\cancel{2}} = 2a\pi \text{ units.}$$

② Find the length of the arc of parabola $y^2 = 4ax$ cut off by the line $y = 2x$.

Soln:

Solving $y^2 = 4ax$ and $y = 2x$

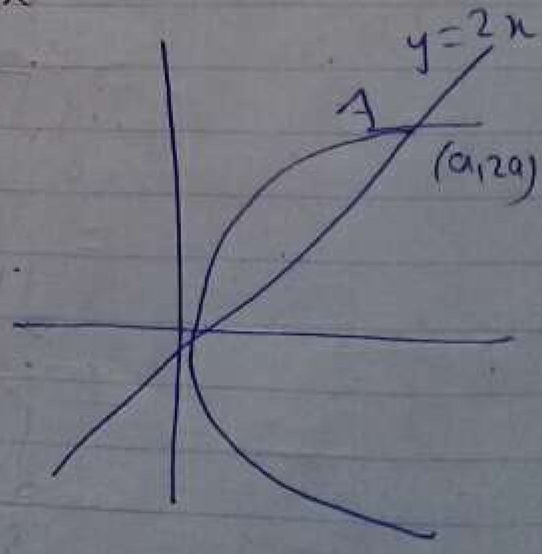
i.e. $(2x)^2 = 4ax$

$\Rightarrow 4x^2 = 4ax$

$\therefore x = a$

Then $y = 2a$

Intersection points are $(0,0)$ & $(a, 2a)$



Now

$$\text{Arc length} = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \left(\frac{2a}{y}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \left(\frac{2a}{\sqrt{4ax}}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \frac{a}{x}}\, dx$$

$$= \int_0^a \sqrt{\frac{a+x}{x}}\, dx$$

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③ Find the length of the arc of parabola $4y = x^2$ from $(-2, 1)$ to $(4, 4)$.

Soln:

$y = \frac{1}{4} x^2$

$\therefore \frac{dy}{dx} = \frac{1}{4} \cdot 2x$

Arc length $= \int_{-2}^{4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_{-2}^{4} \sqrt{1 + \left(\frac{x}{2}\right)^2}\, dx$

$= \int_{-2}^{4} \sqrt{\frac{x^2+4}{4}}\, dx$

$= \frac{1}{2} \int_{-2}^{4} \sqrt{x^2+4}\, dx$

$= \frac{1}{2} \int_{-2}^{4} \sqrt{x^2+2^2}\, dx$

$= \frac{1}{2} \left[\frac{x\sqrt{x^2+2^2}}{2} + \frac{2^2}{2} \log\left(x + \sqrt{x^2+2^2}\right) \right]_{-2}^{4}$

$= \frac{1}{2} \left[\frac{4\sqrt{4^2+2^2}}{2} + 2\log\left(4 + \sqrt{4^2+2^2}\right) \right.$

$\left. - \frac{2\sqrt{(-2)^2+2^2}}{2} + 2\log\left(-2 + \sqrt{(-2)^2+2^2}\right) \right]$

$= \frac{1}{2} \left[2\sqrt{20} + 2\log(4 + \sqrt{20}) + \sqrt{8} - 2\log(-2 + \sqrt{8}) \right)$ units.

Unit-8

Differential Equations

Formula

① $\int e^x dx = e^x$ ② $\int e^{ax} dx = \frac{e^{ax}}{a}$

③ $\frac{d(e^{ax})}{dx} = ae^{ax}$ ④ $\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y$

⑤ $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$

Homogeneous equation:

~~Solve~~

$\cancel{① \frac{dy}{dx} = \frac{y-2x}{x}}$

A differential equation, $Mdx + Ndy = 0$, is said to be homogeneous differential equation if it can be expressed in the form of $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

For example $\frac{dy}{dx} = \frac{3yx + y^2}{x^2}$ is homogeneous form.

Since, $\frac{dy}{dx} = 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$

Note → Every eq^n of above type can be solved by putting $y = vx$.

Solve:

① $\frac{dy}{dx} = \frac{y-2x}{x}$

Soln:

$\frac{dy}{dx} = \frac{y-2x}{x}$ — ①

This is homogeneous differential equation. So let us put, $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence, eqn ① will be

$v + x\frac{dv}{dx} = \frac{vx - 2x}{x}$

or, $v + x\frac{dv}{dx} = \frac{\cancel{x}(v-2)}{\cancel{x}}$

or, $x\frac{dv}{dx} = \cancel{v} - 2 - \cancel{v}$

or, $x\frac{dv}{dx} = -2$

or, $dv = -2\frac{dx}{x}$

Integrating, we get

$\int dv = \int -2\frac{dx}{x}$

or, $v = -2\int\frac{dx}{x}$

or, $v = -2\log x + c$

or, $y/x = -2\log x + c$

or, $y + 2x\log x + cx$ Ans.

② $xy\frac{dy}{dx} = x^2 + y^2$

Soln:

$\frac{dy}{dx} = \frac{x^2}{xy} + \frac{y^2}{xy}$

or, $\frac{dy}{dx} = \left(\frac{x}{y}\right) + \left(\frac{y}{x}\right)$ — ①

This is homogeneous equation. So let $y = vx$

$\therefore \frac{dy}{dx} = v + \frac{dv}{dx}$

Hence, equation ① becomes

$v + \frac{dv}{dx} = \frac{x}{y} + \frac{y}{x}$

or $v + x\frac{dv}{dx} = \frac{x}{vx} + \frac{vx}{x}$

or $\cancel{v} + x\frac{dv}{dx} = \frac{1}{v} + \cancel{v}$

or $vdv = \frac{dx}{x}$

Integrating, we get

$\int vdv = \int \frac{dx}{x}$

or, $\frac{v^2}{2} = \log x + c$

or $v^2 = 2(\log x + c)$

or, $\frac{y^2}{x^2} = 2(\log x + c)$

$\therefore y^2 = 2x^2(\log x + c)$ Ans

③ $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Soln.

$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ —①

This is homogeneous differential equation, so let $y = vx$

$\therefore \frac{dy}{dx} = v + \frac{xdv}{dx}$

or Hence eqn (i) becomes

$$v + \frac{x\,dv}{dx} = \frac{y}{x} + \tan\frac{y}{x}$$

$$\text{or } v + x\frac{dv}{dx} = \frac{v\cancel{x}}{\cancel{x}} + \tan\frac{v\cancel{x}}{\cancel{x}}$$

$$\text{or } \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan v$$

$$\text{or, } x\frac{dv}{dx} = \tan v$$

$$\text{or, } x\frac{dv}{dx} = \frac{1}{\cot v}$$

$$\text{or } \cot v\, dv = \frac{dx}{x}$$

$$\text{or } \frac{\cos v}{\sin v}\, dv = \frac{dx}{x}$$

Integrating, we get

$$\int \frac{\cos v}{\sin v}\, dv = \int \frac{dx}{x}$$

$$\text{or, } \log \sin v = \log x + \log c$$

$$\text{or, } \log \sin v = \log cx$$

$$\therefore \sin v = cx \quad \text{--- (ii)}$$

Restoring the value $v = y/x$ in (ii)

$$\sin\frac{y}{x} = cx \quad \text{Ans}$$

Linear differential equation:

A differential equation in the form $\frac{dy}{dx} + Py = Q$, where both P & Q are functions of x alone or constant is called linear equation of first order.

To solve such equation, we multiply both sides by $e^{\int P dx}$, called integrating factor (I.F)

Solve

① $\frac{dy}{dx} + \frac{1}{x}y = x^2$

Soln:

$$\frac{dy}{dx} + \frac{1}{x}y = x^2 \quad -①$$

Comparing ① with $\frac{dy}{dx} + Py = Q$

$$P = 1/x, \; Q = x^2$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Now multiplying ① by x, $x\frac{dy}{dx} + \cancel{x}\cdot\frac{1}{\cancel{x}}\cdot y = x^2 \cdot x$

$$x\frac{dy}{dx} + y = x^3$$

or, $\frac{x dy + y dx}{dx} = x^3$

or $x dy + y dx = x^3 dx$

or, $d(x \cdot y) = x^3 dx$

On integrating.

$$\int d(x \cdot y) = \int x^3 dx + C$$

or $x \cdot y = \frac{x^4}{4} + C$

(2) $\frac{dy}{dx} + 3y = e^{-x}$

Soln:

$\frac{dy}{dx} + 3y = e^{-x}$ — (1)

Comparing (1) with $\frac{dy}{dx} + Py = Q$

$P = 3,\ Q = e^{-x}$

$I.F = e^{\int P dx} = e^{\int 3 dx} = e^{3\int dx} = e^{3x} = e^{3x}$

Multiplying both sides of (1) by e^{3x}, we get

$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x} \cdot e^{-x}$

or $e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{2x}$

or, $\frac{d}{dx}(y \times e^{3x}) = e^{2x}$

or $d(ye^{3x}) = e^{2x}dx$

Integrating both sides, we get

$\int d(ye^{2x}) = \int e^{2x} dx$

or $ye^{3x} = \frac{e^{2x}}{2} + c \quad \left[\because \int e^{ax}dx = \frac{e^{ax}}{a} + c\right]$

③ $\sin x \frac{dy}{dx} + \cos x \cdot y = \sin x \cos x$

Soln:

$\sin x \frac{dy}{dx} + \cos x \cdot y = \sin x \cdot \cos x$

Dividing both sides by $\sin x$

$\frac{\cancel{\sin x}\, dy}{\cancel{\sin x}\, dx} + \frac{\cos x}{\sin x} y = \frac{\cancel{\sin x} \cdot \cos x}{\cancel{\sin x}}$

or $\frac{dy}{dx} + \cot x y = \cos x$ — ①

Comparing with $\frac{dy}{dx} + Py = Q$

$P = \cot x$, $Q = \cos x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \cot x dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\int \log \sin x} = \sin x$

Multiplying both sides by $\sin x$, we get

$\sin x \frac{dy}{dx} + \cancel{\cos}\, \sin x \cdot \cot x\, y = \sin x \cdot \cos x$

or $\sin x \frac{dy}{dx} + \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} y = \sin x \cos x$

or, $\cancel{\sin}\, \frac{d}{dx}(\sin x \cdot y) = \sin x \cdot \cos x$

or, $d(\sin x \cdot y) = \sin x \cdot \cos x\, dx$

or $d(\sin x \cdot y) = \frac{1}{2} \times 2 \sin x \cdot \cos x\, dx$

or $d(\sin x \cdot y) = \frac{1}{2} \sin 2x\, dx$

Integrating, we get

$\cancel{\sin x \cdot y = \frac{1}{4}}$ $\int d(\sin x \cdot y) = \frac{1}{2}\int \sin 2x\,dx$

or, $\sin x \cdot y = \frac{1}{2} \cdot \frac{-\cos 2x}{2} + c$

or, $\sin x \cdot y = \frac{1}{4}(-\cos 2x) + c$

or, $4y\sin x = -\cos 2x + c$ Ans.

(4) $\tan x \frac{dy}{dx} + y = \sec x$

Soln:

It can be written as

$\frac{dy}{dx} + \frac{1}{\tan x} \cdot y = \frac{\sec x}{\tan x}$

or $\frac{dy}{dx} + \cot x \cdot y = \operatorname{cosec} x$ — (1)

Comparing with $\frac{dy}{dx} + Py = Q$, we get

$P = \cot x$, $Q = \operatorname{cosec} x$

$\therefore$ I.F $= e^{\int P dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$

Multiplying both sides by $\sin x$, we get

$\sin x \frac{dy}{dx} + \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} y = \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}}$

or $\sin x \frac{dy}{dx} + \cos x = 1$

or $\frac{d}{dx}(y \cdot \sin x) = 1$

or $d(y\sin x) = dx$

Integrating, we get

$$\int d(y\sin x) = \int dx$$

$\Rightarrow y\sin x = x + c$ Ans.

Exact differential equation:

A differential equation of the form $Mdx + Ndy = 0$ where both M and N are functions of x and y is said to be exact if there is a function $u(x,y)$ such that $Mdx + Ndy = du$.

i.e. when $Mdx + Ndy = 0$, becomes perfect differential.

Example

① $xdy + ydx$ is exact, since, $xdy + ydx = d(xy)$

② $xdx + ydy$ is exact, since $xdx + ydy = \frac{1}{2}d(x^2 + y^2)$

Solve

① $ydx - xdy = xydy$

Soln:

$ydx - xdy = xydy$

or, $\frac{\cancel{y}dx}{x\cancel{y}} - \frac{\cancel{x}dy}{\cancel{x}y} = \frac{\cancel{xy}\,dy}{\cancel{xy}}$

or, $\frac{dx}{x} - \frac{dy}{y} = dy$

On integrating

$\int \frac{dx}{x} - \int \frac{dy}{y} = \int dy$

or $\log x - \log y = y + c$. Ans

(2) $2xy\,dy - y^2\,dx = 0$

Soln

This equation can be written as

$x \cdot 2y\,dy - y^2\,dx = 0$

or, $x\,d(y^2) - y^2\,dx = 0$

Dividing by x^2

$$\frac{x\,d(y^2) - y^2\,dx}{x^2} = 0$$

or, $\frac{x\,d(y^2)}{x^2} - \frac{y^2\,dx}{x^2} = 0$

or, $x\,d(y^2) - y^2\,dx = 0$

or, $d\left(\frac{y^2}{x}\right) = 0$

Integrating, we get

$\frac{y^2}{x} = c$

or $y^2 = cx$.

(3) $(x+y)\,dy + (y-x)\,dx = 0$

Soln.

$(x+y)\,dy + (y-x)\,dx = 0$

or $x\,dy + y\,dy + y\,dx - x\,dx = 0$

or, $(x\,dy + y\,dx) + y\,dy - x\,dx = 0$

or, $d(xy) + y\,dy - x\,dx = 0$

On integrating.

$\int d(xy) + \int y\,dy - \int x\,dx = c$

or, $xy + \frac{y^2}{2} - \frac{x^2}{2} = \frac{c}{2}$

$\therefore 2xy + y^2 - x^2 = c$ Ans

(4) $\dfrac{dy}{dx} = \dfrac{y-x+1}{y-x+5}$

Soln:

It can be written as

$(y-x+5)dy = (y-x+1)dx$

or $y\,dy - x\,dy + 5dy = y\,dx - x\,dx + dx$

or $x\,dx + y\,dy - (x\,dy + y\,dx) - dx + 5dy = 0$

or $x\,dx + y\,dy - d(xy) - dx + 5dy = 0$

On integrating

$\int x\,dx + \int y\,dy - \int d(xy) - \int dx + 5\int dy = c$

or $\dfrac{x^2}{2} + \dfrac{y^2}{2} - xy - x + 5y = \dfrac{c}{2}$

or, $x^2 + y^2 - 2xy - 2x + 10y = c$ Ans.

(5) $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$

Soln:

It can be written as $\dfrac{dy}{\cos^2 y} = \dfrac{dx}{\sin^2 x}$

or, $-\dfrac{dx}{\sin^2 x} + \dfrac{dy}{\cos^2 y} = 0$

or $-\operatorname{cosec}^2 x\, dx + \sec^2 y\, dy = 0$

or, $d(\cot x) + d(\tan y) = 0$

Integrating, we get

$\int d(\cot x) + \int d(\tan y) = 0$

or $\cot x + \tan y = c$ Ans

⑥ $\sin x \cdot \cos x\,dx + \sin y \cdot \cos y\,dy = 0$

Soln:

The equation can be written as

$\sin x \cdot d(\sin x) + \sin y \cdot d(\sin y) = 0$

or, $2\sin x\,d(\sin x) + 2\sin y \cdot d(\sin y) = 0$

or, $d(\sin^2 x) + d(\sin^2 y) = 0$

integrating,

$\int d(\sin^2 x) + \int d(\sin^2 y) = C$

or $\sin^2 x + \sin^2 y = C$ Ans

Q. Explain the meaning of successive differentiation and find y_2 and y_3 of the following.

i. $y = \sin^3 x$
ii. $y = \log x$

(i) ~~soln~~ $y = \sin^3 x$

$$\frac{dy}{dx} = y_1 = \frac{d(\sin^3 x)}{dx}$$

$$= \frac{d(3\sin x - \sin 3x)}{4\,dx}$$

$$= \frac{1}{4}\left[3\frac{d\sin x}{dx} - \frac{d\sin 3x}{dx}\right]$$

$$= \frac{1}{4}\left[3\cos x - \frac{d\sin 3x}{d(3x)} \times \frac{d\,3x}{dx}\right]$$

$$= \frac{1}{4}\left[3\cos x - \cos 3x \cdot 3\right]$$

$$= \frac{1}{4}\left[3\cos x - 3\cos 3x\right]$$

Again differentiating.

$$y_2 = \frac{dy_1}{dx} = \frac{d\left[\frac{1}{4}(3\cos x - 3\cos 3x)\right]}{dx}$$

$$= \frac{1}{4}\,\frac{d[3\cos x - 3\cos 3x]}{dx}$$

$$= \frac{1}{4}\left[3\frac{d\cos x}{dx} - 3\frac{d\cos 3x}{d3x} \times \frac{d\,3x}{dx}\right]$$

$$= \frac{1}{4}\left[3(-\sin x) - 3(-\sin 3x)\cdot 3\right]$$

$$= \frac{1}{4}\left[-3\sin x + 9\sin 3x\right].$$

(ii) $y = \log x$

$y_1 = \frac{dy}{dx} = \frac{d(\log x)}{dx} = 1/x$

Again
$y_2 = \frac{dy_1}{dx} = \frac{d(1/x)}{dx} = \frac{dx^{-1}}{dx}$

$= -1x^{-2}$

$= \frac{-1}{x^2}$

$\therefore y_2 = -1/x^2$ "