Numerical Analysis Notes

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Unit-1

Computations & Errors

Errors

The gap between exact number and approximate number is called error.

Types of Errors

1. Inherit Error → Error which is already present in the statement or given question before its solution is called inherit error.
2. Rounding off error → Error caused due to rounding off a number is called rounding off error.
3. Truncating error → Error caused due to change of infinite value by finite value.
4. Relative error or absolute error → Error caused due to the absolute difference of the given value.

If x is true value and x' is absolute value, then relative error is given by:

$$E_r = \left| \frac{x - x'}{x} \right| = \frac{|\text{Error}|}{\text{True value}}$$

$$\% \text{ Error} = \left| \frac{x - x'}{x} \right| \times 100\% \quad \text{or,} \quad E_p = E_r \times 100\%$$

E_a → Absolute error $= |x - x'|$

E_r → Relative error $= \left| \frac{x - x'}{x} \right|$

E_p → Percentage Error $= \left| \frac{x - x'}{x} \right| \cancel{\times 100\%}$

x → Exact value

1. Determine the absolute error, relative error and relative percentage error of:

(a) 865250 using 4 significant digits

Soln:

Exact value $(X) = 865250$

Approximate value $(x') = 8652$

$\therefore$ Absolute error $(E_a) = |x - x'| = |865250 - 8652|$

$= 50$

$\therefore$ Relative error $(E_r) = \left|\frac{x - x'}{x}\right| = \frac{50}{865250}$

$= 0.00005778676$

$\therefore$ Relative per. error $(E_p) = E_r \times 100\%$

$= 0.000057\ \%$

(b) 37.46236 using 4 significant digits

Soln:

$X = 37.46236$

$x' = 37.46$

$E_a = |x - x'| = |37.46236 - 37.46| = 0.00235$

$E_r = \left|\frac{x - x'}{x}\right| = \frac{0.00235}{37.46236} = 0.00006272964$

$E_p = E_r \times 100\% = 0.0000627\ \%$

(c) $\sqrt{3}$ using 4 significant digits

Soln:

$x = \sqrt{3} = 1.732050808$

$x' = 1.732$

$E_a = |x - x'| = |1.732050808 - 1.732|$
$= 0.000050808$

$E_r = \left|\frac{x - x'}{x}\right| = \frac{0.000050808}{1.732050808} = 0.0000293401$

$E_p = E_r \times 100\% = 0.0029\%$

(d) Evaluate the sum $S = \sqrt{3} + \sqrt{7} + \sqrt{5}$ to 4 significant digits and find its absolute and relative error.

Soln:

$\sqrt{3} = 1.732050808$

$\sqrt{5} = 2.236067977$

$\sqrt{7} = 2.645751311$

$\therefore S = 6.613870096 \quad \therefore x = 6.613870096, x' = 6.614$

~~Ea = 0.~~

$\therefore E_a = |x - x'| = |6.613870096 - 6.614| = 0.0038$

$E_r = \left|\frac{x - x'}{x}\right| = \frac{0.0038}{6.613870096} = 0.0005$

2\. Determine the error of the function:
$u = \frac{5x^2y}{z^3}$; at $x=y=z=1$ and $\Delta x = 0.001$, $\Delta y = 0.001$; $\Delta z = 0.001$

Soln:

$$\frac{du}{dx} = \frac{df}{dx} + \frac{df}{dy} + \frac{df}{dz}$$

$$= \frac{d(5x^2y/z^3)}{dx} + \frac{d(5x^2y/z^3)}{dy} + \frac{d(5x^2y/z^3)}{dz}$$

$$= \frac{10xy}{z^3} + \frac{5x^2}{z^3} - \frac{15x^2y}{z^4}$$

$$= 10 + 5 - 15$$

$$= 0$$

$\therefore$ Error $= 10 \times 0.001 + 5 \times 0.001 - 15 \times 0.001$

$= 0.001 + 0.005 - 0.0015$

$= 0$

$\therefore$ There is no error.

3\. Determine the error of the function $u = 9x^2y$ at $(1,2)$ $\Delta x = 0.001$ and $\Delta y = 0.002$

Soln:

$$\frac{df}{dx} = \frac{d(9x^2y)}{dx} = 2axy = 2 \times 9 \times 1 \times 2 = 49$$

$$\frac{df}{dy} = \frac{d(9x^2y)}{dy} = 9x^2 = 9 \times 1^2 = 9$$

Error $= 49 \times 0.001 + 9 \times 0.002$

$= 0.0049 + 0.0029$

$= 0.024\,9$.

4. Find the absolute error in the sum of numbers 105.6, 27.28, 5.63, 0.1467, 0.000523, 208.5, 0.0235, 0.432 & 0.0467

Soln:

Sum (S) $= 105.6 + 208.5 + 27.28 + 5.63 + 0.15 + 0.00 + 0.43 + 0.05 = 347.64$

Absolute error $= 2(0.05) + 7(0.005)$
$= 0.14$

5. If $z = \frac{1}{8}xy^3$, find the percentage error in z when $x = 3.14 + 0.0016$ and $y = 4.5 + 0.05$

Soln:

We have.

$z = \frac{1}{8}xy^3$

Suppose exact value of $x = 3.14 + 0.0016$
$= 3.1416$

Exact value of $y = 4.5 + 0.5$
$= 4.55$

Approximate value of $x = 3.14$
Approximate value of $y = 4.5$

$\therefore$ Exact value of $z = \frac{1}{8} \times 3.1416 \times (4.55)^3$

$= 36.99091646$

Approximate value of $Z = \frac{1}{8} \times 3.14 \times (4.5)^3$

$= 35.7665625$

Absolute error $= 1.22435396$

Percentage error $= \frac{\text{Absolute error}}{\text{Exact value}} \times \cancel{100\%}$

$= 0.033 \times 100\%$

$= 3.30\%$

$= 3\%$

Unit-2

Solution of Algebraic and Transcendental Equations

2.1. Linear equations

(i) $y = mx + c$ → degree 1 (linear)

(ii) $y = ax^2 + bx + c = 0$ → degree 2 (quadratic)

degree 3 (cubic), degree 4 (biquadratic)

2.2. Graphical solution of equations

(Solution by graphical method

2.3 Bisection Method

This method states that if a function $f(x)$ is continuous between a & b, and $f(a)$ and $f(b)$ are of opposite signs then there exists at least one root between a & b such that $f(c) = 0$.

Algorithm

Step 1: Start

Step 2: Set up initial guess a & b

Step 3: Is $f(a) \times f(b) < 0$?

Yes : go to step 4

No : go to step 2

Step 4: Calculate $x_0 = \frac{a+b}{2}$

Step 5: Check $f(x_0)$

Step 6: If $f(x_0) < 0$, $x_0 = a$ go to step 7

if $f(x_0) > 0$, $x_0 = b$ go to step 7

If $f(x_0) = 0$, go to step 8

Step 7: Is the criteria met ?

Yes: go to step 8

No: go to step 4

Step 8: Print root is x_0

Step 9: Stop

2. Find the root of $\sin x = \frac{1}{x}$ upto four decimal place

Soln:

$\sin x = 1/x$

$\therefore \sin x - 1/x = 0$

Let initial guess be 2 & 3

$f(2) = 0.4092$

$f(3) = -0.1922$

$\therefore f(2) \times f(3) < 0$, so its negative

Calculating root

$\boxed{\text{Calculator formula} \quad x_0 = \frac{a+b}{2} : \sin x - \frac{1}{x} \text{ and press calc}}$

S.N.	Initial guess a	Initial guess b	$x_0 = \frac{a+b}{2}$	$f(x_0)$
1.	2	3	2.5	+ve
2.	2	2.5	2.25	+ve
3.	2	2.25	2.125	+ve
4.	2	2.125	2.0625	+ve
5.	2	2.0625	2.0132	+ve
6.	2	2.0132	2.0156	+ve
7.	2	2.0156	2.0078	+ve
8.	2	2.0078	2.0039	+ve
9.	2	2.0039	2.0019	+ve
10.	2	2.0019	2.0009	+ve
11.	2	2.0009	2.00045	+ve

$\therefore$ The root is 2.000.

2.5 Iteration method

Step 1: Start
Step 2: Set up initial guess a, b
Step 3: If $f(a) \times f(b) < 0$
Yes: go to step 4
No: go to step 2

Step 4: Calculate $f(x_0)$
Step 5: Is the condition met?
Yes: go to step 7
No: go to step 6

Step 6: $a = x_0$ go to step 4
Step 7: The root is x_0
Step 8: Stop

Q1. Solve using iteration method upto 4 decimal place of the equation: $x^3 - 9x + 1 = 0$

Soln:

$x^3 - 9x + 1 = 0$

or, $x = \sqrt[3]{9x-1}$

Let the initial guess be 2 & 3

$f(2) = -9$

$f(3) = 1$

$\therefore f(2) \times f(3) < 0$

Calculation of root	Calculator formula:
	$x : \sqrt[3]{9x-1}$

S.N.	Initial guess	$x = \sqrt[3]{9x-1}$
1.	2	2.571281
2.	2.571281	2.808035
3.	2.808035	2.895367
4.	2.895367	2.926289
5.	2.926289	2.937082
6.	2.937082	2.940830
7.	2.940830	2.942130
8.	2.942130	2.942580

$\therefore$ The root is 2.942

2. Find the negative root of eqⁿ $x^3 - 2x + 5 = 0$ using iteration method upto 4 decimal place.

Soln:

$x^3 - 2x + 5 = 0$

or, $(-x)^3 - 2(-x) - 5 = 0$

or, $-x^3 + 2x - 5 = 0$

or, $2x - 5 = x^3$

or, $x = \sqrt[3]{2x-5}$

Let the initial guess be -2 and -3

$f(-2) = 1$

$f(-3) = -16$

$\therefore f(-2) \times f(-3) < 0$.

Calculation of root:

Calculator formula
$x: \sqrt[3]{2x-5}$

S.N.	Initial guess	$x = \sqrt[3]{2x-5}$
1.	-3	-2.223980
2.	-2.223980	-2.114037
3.	-2.114037	-2.09750
4.	-2.09750	-2.09499
5.	-2.09499	-2.094618

$\therefore$ The root is $x^3-2x+5=0$ is -2.094

2.4 The method of false position (or Regular falsi method or Interpolation method)

Step 1: Start

Step 2: Set up initial guess a, b

Step 3: Is $f(a) \times f(b) < 0$

Yes : gotostep 4

No : go to step 3

Step 4: Root $(x_1) = a - \frac{b-a}{f(b)-f(a)} \times f(a)$

Step 5: Is $x_1 > 0$?

Yes: Set $b = x_1$ goto step 6

No: goto step 6

Step 6: Is the criteria met ?

Yes : goto step 7

No: goto step 4

Step 7: x_1 is the root.

Step 8: Stop

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1. Find the root of x^3-2x-5 upto 3 decimal place.

Soln:

Let the initial guess be $\overset{A}{2}$ & $\overset{B}{3}$

$f(2) = -1$

$f(3) = 16$

$\therefore f(2) \times f(3) < 0$

Calculation of root:

Let initial guess = A, B

Let $f(A) = C$

$f(B) = D$

$x_1 = a - \frac{b-a}{f(b)-f(a)} \times f(a)$ or, $x_1 = A - \frac{B-A}{D-C} \times C$

Calculator formula

$$C = A^3-2A-5 : D = B^3-2B-5 : A - \frac{B-A}{D-C} \times C : x^3-2x-5$$ press Calc

S.N.	Initial guess A (a)	B (b)	$E = \frac{AD-BC}{D-C}$; $A - \frac{B-A}{D-C} \times C$	F(x)
1.	2	3	2.058823	-ve
2.	2.058823	3	2.081263	-ve
3.	2.081263	3	2.089638	-ve
4.	2.089638	3	2.092739	-ve
5.	2.092739	3	2.093883	-ve
6.	2.093883	3	2.094305	-ve
7.	2.094305	3	2.094460	-ve
8.				

$\therefore$ The root of $x^3-2x-5=0$ is 2.094

Calculator

$$C = A^3-2A-5 : D = B^3-2B-5 : E = \frac{AD-BC}{D-C} : F = E^3-2E-5$$

16.

2.6 Newton-Raphson method

Step 1: Start
Step 2: Set up initial guess x_0
Step 3: Calculate $f(x)$
Step 4: Calculate $f'(x)$
Step 5: Calculate $f(x_0)$
Step 6: Calculate $f'(x_0)$
Step 7: Root $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Step 8: Is the condition met ?
Yes: go to step 9
No: set $x_0 = x_1$ goto step 5

Step 9: Print x_1 as root
Step 10: Stop

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1. Solve using Newton-Raphson method of: $3x = \cos x + 1$ upto 3 decimal place.

Soln:

$f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

Let initial guess be 1

$x_1 = x_0 - \frac{f(x)}{f'(x)}$

Calculation of root:

Calculator formula.

~~x = 3x - Cos x - 1 ÷ 3 + sin x~~ $- \frac{3x - \cos x - 1}{3 + \sin x}$

Let initial guess is A,

Let $f(A) = B$

$f'(A) = C$

$$\boxed{X - \frac{3x - \cos x - 1}{3 + \sin x}}$$

S.N.	Initial guess x_0	$x_1 = x_0 - \frac{3x - \cos x - 1}{3 + \sin x}$
1.	1_0	0.620
2.	0.620	0.607
3.	0.607	0.607

$\therefore$ The root is 0.607

Unit-3

Solution of linear Simultaneous Equations

The set of equations of the form:

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$

$a_3x + b_3y + c_3z + d_3 = 0$ are linear simultaneous equations.

3.1. Gauss elimination method

Solve the following equations by Gauss elimination method:

$x + 4y - z = 5 \quad -(i)$

$x + y - 6z = -12 \quad -(ii)$

$3x - y - z = 4 \quad -(iii)$

Soln:

The augmented matrix is:

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1, \; R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -17 \\ -11 \end{bmatrix}$$

$R_3 \to R_3 - \frac{13}{3} R_2$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13-\frac{13}{3}\times -3 & \frac{59}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -17 \\ 188/3 \end{bmatrix}$$

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or, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 59/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -17 \\ 188/3 \end{bmatrix}$

Now.

$x + 4y - z = 5$ — (iv)

$-3y - 5z = -17$ — (v)

$\frac{59}{3} z = \frac{188}{3}$ — (vi)

From eqn (vi)

$\frac{59}{3} z = \frac{188}{3}$

$\therefore z = 3.18$

Put $z = 3.18$ in eqn (v)

$-3y - 5 \times (3.18) = -17$

or, $-3y = -17 + 5 \times 3.18$

$\therefore y = 0.366$

Put $y = 0.366$ in (iv)

$x + 4 \times 0.366 - 3.18 = 5$

or, $x = 5 + 3.18 - 1.464$

$\therefore x = 6.71$

$\therefore (x, y, z) = (6.71, 0.366, 3.188)$

3.2 Gauss elimination method

In this method, we convert the given equation into diagnol matrix.

Solve the following equations by Gauss-Jordan method:

$x+y+z=9$

$2x-3y+4z=13$

$3x+4y+5z=40$

Soln:

The augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$R_2 \to R_2-2R_1$
$R_3 \to R_3-3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

$R_3 \to R_3+\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

$R_2 \to R_2/-5$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/5 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 1 \\ 12 \end{bmatrix}$$

$R_1 \to R_1-R_2$

$$\begin{bmatrix} 1 & 0 & 7/5 \\ 0 & 1 & -2/5 \\ 0 & 0 & 12/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 1 \\ 12 \end{bmatrix}$$

$$R_3 \to R_3 \times \frac{5}{12} \begin{bmatrix} 1 & 0 & 7/5 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + 2/5 R_3 \\ R_1 \to R_1 - 7/5 R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 - 7/\cancel{5} \times \cancel{5} \\ 1 + 2/\cancel{5} \times \cancel{5} \\ 5 \end{bmatrix}$$

or,
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$\therefore (x, y, z) = (1, 3, 5)$

3.3 Jacobi-Iteration method

Solve the following equations by Jacobi-Iteration method

$20x+y-2z=17$ ①

$3x+20y-z=-18$ ②

$2x-3y+20z=25$ ③

Soln:

We write the equations in the form of:

$$x=\frac{1}{20}(17-y+2z)$$

$$y=\frac{1}{20}(-18-3x+z)$$

$$z=\frac{1}{20}(25-2x+3y)$$

First iteration

Take $x_0, y_0, z_0 = 0$

$x_1=\frac{1}{20}(17-y_0+2z_0)=0.85$

$y_1=\frac{1}{20}(-18-3x_0+z_0)=-0.913$

$z_1=\frac{1}{20}(25-2x_0+3y_0)=1.25$

n	x (A)	y (B)	z (C)
0	-	0	0
1	0.85	-0.913	1.25
2	1.02	-0.965	1.1515
3.	1.0134	-0.9954	1.0032
4.	1.0009	-1.0018	0.9993
5.	1.0000	-1.0002	1.0006
6	1.0000	-1.0000	1.0000

2nd iteration

$x_2=\frac{1}{20}(17-y_1+2z_1)=1.02$

$y_2=\frac{1}{20}(18-3x_1+z_1)=-0.965$

$z_2=\frac{1}{20}(25-2x_1+3y_1)=1.1525$

$\therefore$ Values are:

$x=1, y=-1, z=1$

3.4. **Gauss - Seidal Iteration method**

Solve by Gauss Seidal method

$20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

Soln:

Writing the equations in the form:

$x = \frac{1}{20}(17 - y + 2z)$

$y = \frac{1}{20}(-18 - 3x + z)$

$z = \frac{1}{20}(25 - 2x + 3y)$

Let initial guess be $x_0, y_0, z_0 = 0$

First Iteration

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.85$

$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

n	x	y	z
0	0	0	0
1.	0.85	-1.0275	1.0109
2.	1.0025	-1.000	1.000
3.	1.0000	-1.0000	1.0000

2nd Iteration

$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$

$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$
$= -1.000$

$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$
$= 1.000$

$\therefore x = 1, y = -1, z = 1.$

3rd Iteration

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$

$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$

$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$

$\therefore x = 1, y = -1, z = 1$

3.5 Matrix Inversion method

Solve the equations by matrix inversion method

$3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

Soln:

~~Let~~ The augmented matrix is

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

Let A   Let X   Let B

Now,

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 8$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

Cofactor of $3 = \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} = -3+2 = -1$

Cofactor of $1 = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2+1 = 3$

Cofactor of $2 = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 4+3 = 7$

Cofactor of $2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3$

Cofactor of $-3 = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3-2 = 1$

Cofactor of $-1 = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 6-1 = 5$

Cofactor of $1 = \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} = -1+6 = 5$

Cofactor of $2 = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -3-4 = -7$

Cofactor of $1 = \begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = -9-2 = -11$

$\therefore$ Adj of $A = \begin{bmatrix} -1 & 3 & 7 \\ -3 & 1 & 5 \\ 5 & -7 & -11 \end{bmatrix}$

We know

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \times \text{Adj of } A \times B$$

$$= \frac{1}{8} \begin{bmatrix} -1 & 3 & 7 \\ -3 & 1 & 5 \\ 5 & -7 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 16 \\ 8 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore x = 2, y = 1, z = -1$ Ans

Unit - 4
Finite Differences and Central Differences

A finite difference is a mathematical expression of the form $f(x+b) - f(x+a)$

Three forms are commonly considered: forward, backward and finite differences.

① A forward difference is an expression of the form:
$\Delta_h [f](x) = f(x+h) - f(x)$

② A backward difference ~~is an expression of the form~~ uses the functional values at x and $x-h$, instead of the values at $x+h$ and x

$\nabla_h [f](x) = f(x) - f(x-h)$

③ Finally, the central difference is given by
$\delta_h [f](x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$

Forward difference approximation: $f'(x) \approx \frac{f(x+h) - f(x)}{h}$

Backward difference approximation: $f'(x) \approx \frac{f(x) - f(x-h)}{h}$

Central difference approximation: $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$

Q. Construct forward and backward difference table of $f(x) = x^2 + 2x + 3$ on $[0,10]$ with equal interval of length 2

Soln:

$y = f(x) = x^2 + 2x + 3$

x	0	2	4	6	8	10
y	3	11	27	51	83	123

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	3					
		8 (Δy_0)				
2	11		8 $(\Delta^2 y_0)$			
		16 (Δy_1)		0 $(\Delta^3 y_0)$		
4	27		8 $(\Delta^2 y_1)$		0 $(\Delta^4 y_0)$	
		24 (Δy_2)		0 $(\Delta^3 y_1)$		0 $(\Delta^5 y_0)$
6	51		8 $(\Delta^2 y_2)$		0 $(\Delta^4 y_1)$	
		32 (Δy_3)		0 $(\Delta^3 y_2)$		
8	83		8 $(\Delta^2 y_3)$			
		40 (Δy_4)				
10	123					

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Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0	3					
		$8 (\nabla y_1)$				
2	11		$8 (\nabla^2 y_2)$			
		$16 (\nabla y_2)$		$0 (\nabla^3 y_3)$		
4	27		$8 (\nabla^2 y_3)$		$0 (\nabla^4 y_4)$	
		$24 (\nabla y_3)$		$0 (\nabla^3 y_4)$		$0 (\nabla^5 y_5)$
6	51		$8 (\nabla^2 y_4)$		$0 (\nabla^4 y_5)$	
		$32 (\nabla y_4)$		$0 (\nabla^3 y_5)$		
8	83		$8 (\nabla^2 y_5)$			
		$40 (\nabla y_5)$				
10	123					

Central difference table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		δy_1			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		δy_2		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

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Divided difference

Construct divided difference table:

x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
			$\frac{202-97}{10-5}=21$		$\frac{1-1}{11-4}=0$
7	294			$\frac{27-21}{11-5}=1$	
		$\frac{900-294}{10-7}=202$			$\frac{1-1}{13-4}=0$
			$\frac{310-202}{11-7}=27$		
10	900			$\frac{33-27}{13-7}=1$	
		$\frac{1210-900}{11-10}=310$			
			$\frac{409-310}{13-10}=33$		
11	1210				
		$\frac{2028-1210}{13-11}=409$			
13	2028				

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Relation between operators Δ, ∇, E, μ and δ

$\Delta \rightarrow$ delta

$\nabla \rightarrow$ nebla

$E \rightarrow$ Estimator

$\mu \rightarrow$ mew

$\delta \rightarrow$ delta

Prove the formulas given below:

i) $\Delta = E - 1$

ii) $\nabla = 1 - E^{-1}$

iii) $\delta = E^{1/2} - E^{-1/2}$

iv) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

v) $\Delta = E\nabla = \delta E^{1/2}$

vi) $E = e^{hD}$

Soln:

Let $x_0, x_1, x_2 \dots x_n$ be set of data points and $y_0, y_1, y_2 \dots y_n$ be the corresponding value of $y = f(x)$

$\Delta y_0 = y_1 - y_0$

$\nabla y_1 = y_1 - y_0$

$\delta y_{1/2} = y_1 - y_0$

$E(y_0) = y_1$

$$\mu(y_r) = \frac{y_{r+1/2} + y_{r-1/2}}{2}$$

① $\Delta = E - 1$

We have,

$\Delta y_0 = y_1 - y_0$

or, $\Delta y_0 = E(y_0) - y_0$

or, $\Delta \cancel{y_0} = \cancel{y_0}(E-1)$

$\therefore \Delta = E - 1$ proved.

② $\nabla = 1 - E^{-1}$

We have,

$\nabla y_1 = y_1 - y_0$

or, $\nabla y_1 = y_1 - E^{-1}(y_1)$

or, $\nabla \cancel{y_1} = \cancel{y_1}(1 - E^{-1})$

$\therefore \nabla = 1 - E^{-1}$ proved.

③ $\delta = E^{1/2} - E^{-1/2}$

We have,

$\delta y_{1/2} = y_1 - y_0$

$= y_{1/2+1/2} - y_{1/2-1/2}$

or, $\delta y_{1/2} = E^{1/2} y_{1/2} - E^{-1/2} y_{1/2}$

or, $\delta \cancel{y_{1/2}} = \cancel{y_{1/2}}(E^{1/2} - E^{-1/2})$

$\therefore \delta = E^{1/2} - E^{-1/2}$ proved.

④ $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

We have,

$$\mu y_r = \frac{y_{r+1/2} + y_{r-1/2}}{2}$$

$$\text{or, } \mu y_r = \frac{E^{1/2} y_r + E^{-1/2} y_r}{2}$$

प्रश्न नं. उत्तर

or $\mu y_r = \frac{1}{2} y_r [E^{1/2} + E^{-1/2}]$

$\therefore \mu = \frac{1}{2} [E^{1/2} + E^{-1/2}]$

(5) $\Delta = E\nabla = \delta E^{1/2}$

We have,

$\Delta y_0 = y_1 - y_0 = \nabla y_1 = \nabla E(y_0)$

or $\Delta y_0 = \nabla E(y_0)$

$\therefore \Delta = E\nabla$

Again, we have

$\Delta y_0 = y_1 - y_0$

or $= y_{\frac{1}{2}+\frac{1}{2}} - y_{\frac{1}{2}-\frac{1}{2}}$

$= E^{1/2}(y_{1/2}) - E^{-1/2}(y_{1/2})$

$= (E^{1/2} - E^{-1/2}) y_{1/2}$

or $\Delta y_0 = \delta y_{1/2}$

or $\Delta \cancel{y_0} = \delta (E^{1/2}) \cancel{y_0}$

or $\Delta = \delta E^{1/2}$

$\therefore \Delta = E\nabla = \delta E^{1/2}$ proved.

Factorial Polynomial

The continued product of the form $x(x-1)(x-2)\ldots(x-n)(x-n-1)$ is called factorial polynomial of degree n and is denoted by $(x)^n$ or x^n.

Example

$x^1 = x$

$x^2 = x(x-1)$

$x^3 = x(x-1)(x-2)$

$x^n = nx^{n-1}$

Q1. Express $y = f(x) = 2x^3 - 3x^2 + 3x - 10$ in factorial form and show that $\Delta^3 y = 12$.

Soln:

Let $y = Ax^3 + Bx^2 + Cx + D$

$= 2x^3 - 3x^2 + 3x - 10$

$= 2x(x-1)(x-2) - 3x(x-1) + 3x - 10$

Now,

$y = 2x^3 - 3x^2 + 3x - 10$

$\Delta y = 6x^2 - 6x + 3$

$\Delta^2 y = 12x - 6$

$\Delta^3 y = 12$

$\therefore \Delta^3 y = 12$ proved.

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Q2. Express $y = x^4 - 12x^3 + 24x^2 - 30x + 9$ in factorial form and prove that $\Delta^5 y = 0$

Soln:

$y = x^4 - 12x^3 + 24x^2 - 30x + 9$

$= x(x-1)(x-2)(x-3) - 12x(x-1)(x-2) + 24x(x-1) - 30x + 9$

Now

$y = x^4 - 12x^3 + 24x^2 - 30x + 9$

$\Delta y = 4x^3 - 36x^2 + 48x - 30$

$\Delta^2 y = 12x^2 - 72x + 48$

$\Delta^3 y = 24x - 72$

$\Delta^4 y = 24$

$\Delta^5 y = 0$ proved.

Unit-5

Interpolation with equal Intervals

5.1. Newton-Gregory forward interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$
$$+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} \times \Delta^n y_0$$

Q1. Find the cubic polynomial from the set of data points (1,24), (3,120), (5,336), (7,720). Also estimate y8

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1 (x_0)	24 (y_0)	~~96~~		
		96 (Δy_0)		
3	120		120 ($\Delta^2 y_0$)	
		216 (Δy_1)		48 ($\Delta^3 y_0$)
5	336		168 ($\Delta^2 y_1$)	
		384 (Δy_2)		
7	720			

The Newton forward polynomial function is:

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \quad -①$$

Where,

$x = x_0 + ph$

$\therefore p = \frac{x - x_0}{h} = \frac{x-1}{2}$ put the values in ①

$$y = 24 + \left(\frac{x-1}{2}\right)96 + \frac{\frac{x-1}{2}\left(\frac{x-1}{2}-1\right)\times 120}{2!} + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)}{3!}\times 48$$

$$= 24 + (x-1)48 + \frac{\frac{(x-1)(x-3)}{4}\times 120}{2!} + \frac{\frac{(x-1)(x-3)(x-5)}{8}\times 48}{3!}$$

$$= 24 + (x-1)48 + \frac{(x-1)(x-3)\cancel{30}^{15}}{\cancel{2}} + \frac{(x-1)(x-3)(x-5)\cancel{6}}{\cancel{6}}$$

$$= 24 + (x-1)48 + (x-1)(x-3)15 + (x-1)(x-3)(x-5)$$

$$= 24 + 48x - 48 + 15x^2 - 60x + 45 + x^3 - 9x^2 + 23x + 5$$

$$= x^3 + 6x^2 + 11x + 6$$

$\therefore f(x) = x^3 + 6x^2 + 11x + 6$

$\therefore y(8) = f(8) = 512 + 384 + 88 + 6 = 990$

Q2. Construct ~~the~~ Newton's forward difference table of following data and evaluate the value of y at $x = 160$.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.90	21.27

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
100	10.63						
		2.40					
x_0 150	y_0 13.03	Δy_0	-0.39				
		2.01	$\Delta^2 y_0$	0.15			
200	15.04		-0.24	$\Delta^3 y_0$	-0.07		
		1.77		0.08	$\Delta^4 y_0$	0.02	
250	16.81		-0.16		-0.05	$\Delta^5 y_0$	0.02
		1.6		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

We know,
from Newton's forward difference formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 \quad -①$$

Where

$$p = \frac{x - x_0}{h} = \frac{160-150}{50} = 0.2$$

~~Putting p in eqn ①~~

$$y_{160} = 13.03 + (0.2 \times 2.01) + \frac{0.2(0.2-1)}{2!}\times(-0.24) + \frac{0.2(0.2-1)(0.2-2)}{3!}\times 0.08$$

$$y_{160} = 13.03 + (0.2 \times 2.01) + \frac{0.2(0.2-1)}{2!} \times (-0.24) +$$

$$\frac{0.2(0.2-1)(0.2-2)}{3!} \times 0.008 + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!} \times (-0.05)$$

$$+ \frac{0.2(0.2-1)(0.2-2)(0.2-3)(0.2-4)}{5!} \times 0.04$$

$$= 13.03 + 0.4 + 0.02 + 0.001 + 0.001 + 0.001$$

$$= 13.453$$

प्रश्न न उत्तर पृष्ठ

5.2 Newton-Gregory backward interpolation formula

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$\cdots + \frac{p(p+1)(p+2)\cdots(p+n-1)}{n!} \times \nabla^n y_n$$

Q. find $y(3.2)$, $y(3.5)$ by Newton's backward interpolation

x	0	1	2	3	4
y	-12	-8	10	40	60

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	-12				
		4 (∇y_1)			
1	-8		14 $(\nabla^2 y_2)$		
		18 (∇y_2)		-2 $(\nabla^3 y_3)$	
2	10		12 $(\nabla^2 y_3)$		-20 $(\nabla^4 y_4)$
		30 (∇y_3)		-22 $(\nabla^3 y_4)$	
3	40		-10 $(\nabla^2 y_4)$		
x_n	y_n	20 (∇y_4)			
4	60				

i) $y(3.2)$

$$x = 3.2,\ x_n = 4,\ h = 1,\ p = \frac{x - x_n}{h} = \frac{3.2 - 4}{1} = -0.8$$

$$\therefore y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 60 + (-0.8)\times 20 + \frac{(-0.8)(0.2)(-10)}{2} + \frac{(-0.8)(0.2)(1.2)(-22)}{6}$$

$$+ \frac{(-0.8)(0.2)(1.2)(2.2)(-20)}{24}$$

$$= 45.856$$

ii) $y(3.5)$

$x = 3.5,\ x_n = 4,\ h = 1,\ p = \frac{3.5 - 4}{1} = -0.5$

$$y_{3.5} = 60 + (-0.5)20 + \frac{(-0.5)(0.5)(-10)}{2} + \frac{(0.5)(0.4)(1.5)(-22)}{6}$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(-20)}{24}$$

$$= 53.406$$

Q. The population of town in decennial census was as given below. Estimate population for year 1895.

X	Y	ΔY	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

We use Newton's forward interpolation formula

$x_0 = 1891,\ x = 1895,\ y_0 = 46,\ y = ?$

$h = 10,\ p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$

We know,

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 46 + 0.4\times 20 + \frac{0.4(0.4-1)}{2}\times(-5) + \frac{0.4(0.4-1)(0.4-2)}{3\times 2}\cancel{(\quad)}3\times 2$$

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4\times 3\times 2} - 3$$

$= 54.8528$ thousands.

Q. The sales in particular department stores for the best 5 years is given below. Estimate 1979.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
		3			
1976	43		2		
		5		-3	
1978	48		-1		5
		4		2	
1980	52		1		
		5			
1982	57				

We have to use Newton's backward formula

Here

$x_n = 1982,\ x = 1979,\ h = 2$

$\therefore p = \dfrac{1975 - 1982}{2} = -1.5$.

प्रश्न नं. उत्तर

By formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 57 + (-1.5) + (-1.5)(-1.5+1)\tfrac{1}{2} + (-1.5)(-1.5+1)(-1.5+2)\times\frac{2}{6}$$

$$+ (-1.5)(-1.5+1)(-1.5+2)(-1.5+3)\times\frac{5}{24}$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1171$$

$$= 50.1171 \text{ Ans}$$

प्रश्न न. उत्तर परीक्षक

Unit-6

Interpolation with Unequal Intervals

Newton's divided difference interpolation formula For

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + \cdots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)[x_0,x_1,x_2 \cdots x_n]$$

① Using Newton's divided difference evaluate $f(3.5)$

x	y	$\Delta y / f[x_0,x_1]$	$\Delta^2 y / f[x_0,x_1,x_2]$	$\Delta^3 y / f[x_0,x_1,x_2,x_3]$
$0\ (x_0)$	$-3\ (y_0)$			
		$\frac{6+3}{1-0} = 9\ (\Delta y_0)$		
$1\ (x_1)$	$6\ (y_1)$		$\Delta^2 y_0$ $\frac{11.5-9}{3-0} = 0.83$	$\Delta^3 y_0$
		$\frac{29-6}{3-1} = 11.5$		$\frac{-8.5-0.83}{4-0} = -2.33$
$3\ (x_2)$	$29\ (y_2)$		$\frac{-14-11.5}{4-1} = -8.5$	
		$\frac{15-29}{4-3} = -14$		
$4\ (x_3)$	$15\ (y_3)$			

We have

$$y = f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 +$$
$$(x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$

$$f(3.5) = -3 + (3.5-0)9 + (3.5-0)(3.5-1)\times 0.83 + (3.5-0)(3.5-1)(3.5-3) - 2.33$$
$$= 25.57 \text{ Ans}$$

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(2) Using Newton's divided difference evaluate f(8) and f(15)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$4 (x_0)$	$49 (y_0)$					
		52		for y_8		
$5 (x_1)$	$100 (y_1)$		15			
		97		1		
$7 (x_2)$	$294 (y_2)$		21		0	
		202		1		0
$10 (x_3)$	$900 (y_3)$		27		0	
		310		1		
$11 (x_4)$	$120 (y_4)$		33			
		409		for y_{15}		
$13 (x_5)$	$2028 (y_5)$					

We have,

$y = f(x) = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] +$
$(x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)[x_0,x_1,x_2,x_3]$

$f(8) = 49 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)\times 1$
$+ (8-4)(8-5)(8-7)(8-10)\times 0 + (8-4)(8-5)(8-7)(8-10)(8-11)\times 10$
$= 448$

$f(15) = 2028 + (15-13)\times 409 + (15-13)(15-11)\times 33 + (15-13)(15-11)$
$(15-10)\times 1 + (15-13)(15-11)(15-10)(15-7)\times 0 + (15-13)(15-11)(15-10)$
$(15-7)(15-5)\times 0 = 3150.$

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Unit-7

Central Difference Interpolation Formula

7.1 Gauss' Forward interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} +$$

$$--- + \frac{p(p+1)(p-1)(p+2)(p-2) + \cdots + (p+n)(p-n)}{n!}\Delta^n y_{n-2}$$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Q. Find f(30) using Gauss' forward formula

x	n	$y=f(x)$	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
21	-2	18.4708				
			-0.6564			
25	-1	17.8344		-0.0510		$\Delta^4 y_{-2}$
		y_0	-0.7074	$\Delta^2 y_{-1}$	-0.0074	-0.0022
29 →30	0	17.1070	Δy_0	-0.0564	$\Delta^3 y_{-1}$	
			-0.7638		0.0076	
33	1	16.3432		-0.0640		
			-0.8278			
37	2	15.5154				

$$p = \frac{x - x_0}{h} = \frac{30-29}{4} = 0.25$$

We have,

$$y_{30} = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_{-1}}{2!} + \frac{p(p+1)(p-1)\Delta^3 y_{-1}}{3!}$$

$$+ \frac{p(p+1)(p-1)(p-2)\Delta^4 y_{-2}}{4!}$$

$$= 17.1070 + (0.25)(-0.7638) + (0.25)(0.25-1)(-0.0564) +$$
$$0.25(0.25-1)$$
$$= 16.739$$

7.2. Gauss' backward interpolation formula

$$y_p = y_0 + p\nabla y_{-1} + \frac{p(p+1)}{2!}\nabla^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\nabla^3 y_{-2}$$

$$+ \dots + \frac{p(p+1)(p-1)(p+2)}{4!}\nabla^4 y_{-2}$$

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_{-3}	y_{-3}						
		∇y_{-3}					
x_{-2}	y_{-2}		$\nabla^2 y_{-3}$				
		∇y_{-2}		$\nabla^3 y_{-3}$			
x_{-1}	y_{-1}		$\nabla^2 y_{-2}$		$\nabla^4 y_{-3}$		
		∇y_{-1}		$\nabla^3 y_{-2}$		$\nabla^5 y_{-3}$	
x_0	y_0		$\nabla^2 y_{-1}$		$\nabla^4 y_{-2}$		$\nabla^6 y_{-3}$
		∇y_0		$\nabla^3 y_{-1}$		$\nabla^5 y_{-2}$	
x_1	y_1		$\nabla^2 y_0$		$\nabla^4 y_{-1}$		
		∇y_1		$\nabla^3 y_0$			
x_2	y_2		$\nabla^2 y_1$				
		∇y_2					
x_3	y_3						

Q. Apply Gauss backward interpolation formula to find the population of town of year 1972.

x	n	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1939	-3	11					
			3				
1949	-2	15		2			
			5		0		
1959	-1	20	∇y_{-1}	2	$\nabla^3 y_{-2}$	3	
			7	$\nabla^2 y_{-1}$	3		-10
1969	0	27		5		-7	$\nabla^5 y_{-3}$
1972		y_0	12		-4	$\nabla^4 y_{-2}$	
1979	1	39		1			
			13				
1989	2	52					

Soln:

$x = 1972,\ x_0 = 1969$

$p = \frac{x - x_0}{h} = \frac{1972 - 1969}{10} = 0.3$

$$y = y_0 + p\nabla y_{-1} + \frac{p(p+1)}{2!}\nabla^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\nabla^3 y_{-2}$$

$$+ \frac{p(p+1)(p-1)(p+2)}{4!}\nabla^4 y_{-2} + \frac{p(p+1)(p-1)(p+2)(p-2)}{5!}\nabla^5 y_{-3}$$

$$= 27 + 0.3\times 7 + \frac{0.3(0.3+1)}{2!}\times 5 + \frac{0.3(0.3+1)(0.3-1)}{3!}\times 3$$

$$+ \frac{0.3(0.3+1)(0.3-1)(0.3+2)}{4!}\times -7 + \frac{0.3(0.3+1)(0.3-1)(0.3+2)(0.3-2)}{5!}\times -10$$

$= 27 + 2.1 + 0.975 - 0.1365 + 0.183 - 0.088$

$= 30.034$ approx.

प्रश्न नं | उत्तर | प्राप्तांक

7.3 Bessel's formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right)\right.$$

$$\left. - \frac{1}{120}\Delta^5 y_{-2} \cancel{- \frac{1}{240}\left(\Delta^6 y_{-3} + \Delta^6 y_{-2}\right)}\right]$$

Q. Find $\frac{dy}{dx}$ at $x = 1.3$

x	n	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	-3	7.989						
			0.414					
1.1	-2	8.403		-0.036				
			0.378		0.006			
1.2	-1	8.781		-0.036		-0.002		
x_0		y_0	0.348	$\Delta^2 y_{-1}$	0.004	$\Delta^4 y_{-2}$	0.001	
1.3	0	9.129	Δy_0	-0.026	$\Delta^3 y_{-1}$	-0.001	$\Delta^5 y_{-2}$	0.002
			0.322		0.003		0.003	
1.4	1	9.451		-0.023		-0.002		
			0.299	$\Delta^2 y_0$	0.005	$\Delta^4 y_{-1}$		
1.5	2	9.750		-0.018				
			0.281					
1.6	3	10.031						

We have,

$h = 0.1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right)\right.$$

$$\left. - \frac{1}{120}\Delta^5 y_{-2} \cancel{- \frac{1}{240}\left(\Delta^6 y_{-3} + \Delta^6 y_{-2}\right)}\right]$$

$$= \frac{1}{0.1}\left[0.322 - \frac{1}{4}(-0.026 - 0.023) + \frac{1}{12}(0.003) - \frac{1}{24}(-0.001 + 0.002)\right.$$

$$\left. - \frac{1}{120}(0.003)\right]$$

प्रश्न नं.	उत्तर

$$= \frac{1}{0.1}\left[3.22 + 0.01225 + 0.00025 - 0.000041 - 0.000025\right]$$

$$= 3.34434 \quad \text{Ans.}$$

7.4 Stirling's formula

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Q. Using Stirling's formula, find the value of $y_{12.2}$

x	n	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
10	-2	0.23967				
			0.04093			
11	-1	0.28060	Δy_{-1}	-0.00365	$\Delta^3 y_{-2}$	
		y_0	0.03728	$\Delta^2 y_{-1}$	0.00058	$\Delta^4 y_{-2}$
12	0	0.31788		-0.00307		-0.00013
12.2			0.03421		-0.00045	
13	1	0.35209	Δy_0	-0.00062	$\Delta^3 y_{-1}$	
			0.03159			
14	2	0.38368				

here

$$p = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2$$

$$y_{12.2} = 0.3178 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + (0.2)^2\left(\frac{-0.00307}{2!}\right)$$

$$+ 0.2\left(\frac{0.2^2 - 1}{3!}\right) \times \left(\frac{0.00058 - 0.00045}{2}\right)$$

$= 0.3249$

Unit - 8

Numerical Differentiation

8.2. Derivative using ~~Newton's~~ forward difference formula

Newton's forward formula is:

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)}{3!}\Delta^3 y_0 + \cdots$$

Newton's forward formula for differentiation is:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right]$$

Q. find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$

x	n	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	-1	7.989						
			0.414					
1.1	0	$\boxed{8.403}$		-0.036				
			$\boxed{0.378}$		0.006			
1.2	1	8.781		$\boxed{-0.030}$		-0.002		
			0.348		$\boxed{0.004}$		-0.001	
1.3	2	9.129		-0.026		$\boxed{-0.001}$		0.002
			0.332		-0.003		$\boxed{0.003}$	
1.4	3	9.451		-0.023		0.002		
			0.299		0.005			
1.5	4	9.750		-0.018				
			0.281					
1.6	5	10.031						

We have,

$h = 0.1$

By formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.378 - \tfrac{1}{2}(-0.030) + \tfrac{1}{3}(0.004) - \tfrac{1}{4}(-0.001) + \tfrac{1}{5}(0.003)\right)$$

$$= 3.952$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \tfrac{11}{12}\Delta^4 y_0 - \tfrac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right]$$

$$= \frac{1}{(0.1)^2}\left[\{-0.030 - (0.004) + \tfrac{11}{12}(-0.001) - \tfrac{5}{6}(0.003)\}\right]$$

$$= -3.74$$

8.3 Derivative using backward ~~formula~~ difference formula

Newton's backward formula is:

$$y = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p-1)}{3!}\nabla^3 y_0 + \cdots$$

Newton's backward formula for differentiation

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \frac{1}{5}\nabla^5 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \frac{5}{6}\nabla^5 y_0 + \frac{137}{180}\nabla^6 y_0 + \cdots\right]$$

प्रश्न नं उत्तर प्राप्तांक

Q. find $f'(x)$, $f''(x)$ from the following at $x=5$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	0				
		1			
2	1		3		
		4		9	$\nabla^4 y_0$
3	5		12	$\nabla^3 y_0$	-31
		16	$\nabla^2 y_0$	-22	
4	21	∇y_0	-10		
	y_0	6			
5 →	27				

here $h=1$

By Newtons backward difference formula

$$f'(x)=\frac{dy}{dx}=\frac{1}{h}\left[\nabla y_0+\frac{1}{2}\nabla^2 y_0+\frac{1}{3}\nabla^3 y_0+\frac{1}{4}\nabla^4 y_0\right]$$

$$=\frac{1}{1}\left[6-\frac{1}{2}\times 10-\frac{1}{3}\times 22-\frac{1}{4}\times 31\right]$$

$$=-14.083$$

$$f''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_0+\nabla^3 y_0+\frac{11}{12}\nabla^4 y_0\right\}$$

$$=\frac{1}{1}\left[-10-22-31\right]$$

$$=64$$

Unit-9
Numerical Integration

The process of evaluating definite integral from the integrant f(x) is called numerical integration.

9.1 General quadrature formula for equidistant ordinates

$$\int_{y_0}^{y_0+nh} f(x)dx = \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \dots\right]$$

9.2 Trapezoidal Rule

$$\int_{y_0}^{y_0+nh} f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\dots y_{n-1})\right]$$

Q. Evaluate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal rule

Soln:

Let $f(x) = \frac{1}{1+x} = \frac{1}{1+1} = 0.5$

$h = 0.2$

x	0	0.2	0.4	0.6	0.8	1
fx	1	0.833	0.714	0.625	0.556	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

Using Trapezoidal's rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_5) + 2(y_1+y_2+y_3+y_4)\right]$$

$= \frac{0.2}{2}\left[(1+0.5)+2(0.833+0.714+0.625+0.556+0.5)\right]$

$= \frac{0.2}{2}\left[1.5+5.4552\right]$

$= 0.69552$ Ans

9.3. Simpson's One-Third (1/3) rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

Q. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Simpson's 1/3 rule

Soln:

Let $f(x) = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6
fx	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$h = 1$

By Simpson's rule,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$

$= 1.366$ Ans

9.4. Simpson's Three-Eighth (3/8) rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\ldots) + 2(y_3+y_6+\ldots)\right]$$

Q. Evaluate $\int_0^1 \frac{x^2}{1+x^3}$ using Simpson's 3/8 rule

Soln:

Let $f(x) = \frac{x^2}{1+x^3}$

x	0	0.2	0.4	0.6	0.8	1
fx	0	0.0396	0.1503	0.2960	0.423	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$h = 0.2$

By Simpson's 3/8 rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_5) + 3(y_1+y_2+y_4) + 2(y_3)\right]$$

$$= \frac{3\times 0.2}{8}\left[(0+0.5) + 3(0.0396+0.1503+0.423) + 2(0.2960)\right]$$

$$= 0.075\,[0.5 + 1.8387 + 0.592]$$

$$= 0.075 \times 2.9307$$

$$= 0.2198$$

Evaluate using: Trapezoidal rule
Simpson's $\frac{1}{3}$ rule

Simpson's 3/8 rule

i) $\int_{\pi/4}^{\pi} e^{\sin x} dx$

Soln:
Let $f(x) = e^{\sin x}$

	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$	$7\pi/12$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$11\pi/12$	π
x	0.785	0.523	1.308	1.570	1.832	2.094	2.356	2.617	2.879	3.141
fx	2.028	1.648	2.627	2.718	2.628	2.378	2.028	2.288	1.296	1.001
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9

$h = 0.261$

Using Trapezoidal's rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0+y_9) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8)\right]$$

$$= \frac{0.261}{2}\left[(2.028+1.001) + 2(1.648+2.627+2.718+2.628+2.378 + 2.028+2.288+1.296)\right]$$

$$= 0.13\,[3.029 + 35.222]$$

$$= 4.972$$

Using Simpson's 1/3 rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_9) + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6+y_8)\right]$$

प्रश्न नं उत्तर

$$= \frac{0.261}{3}\left[(2.028+1.001) + 4(1.648+2.718+2.378+2.298) + 2(2.627+2.628+2.028+1.296)\right]$$

$$= 0.087\left[3.029 + 36.128 + 17.158\right]$$

$$= 4.90 \text{ Ans}$$

Using Simpson's 3/8 rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_9) + 3(y_1+y_2+y_4+y_5+y_7+y_8) + 2(y_3+y_6)\right]$$

$$= \frac{3\times 0.261}{8}\left[(2.028+1.001) + 3(1.648+2.627+2.628+2.378+2.298+1.296) + 2(2.718+2.028)\right]$$

$$= 0.097\left[2.029 + 38.595 + 9.492\right)$$

$$= 4.861 \text{ Ans}$$

ii) $\int_1^2 \int_1^2 \frac{dx\,dy}{xy}$

Soln:

Let $f(x) = \frac{1}{xy}$

$$I = \int_1^2 \frac{dy}{y} \int_1^2 \frac{dx}{x}$$

$$I_1 = \int_1^2 \frac{dx}{x}$$

So $f_1(x) = \frac{1}{x}$

x	1	1.2	1.4	1.6	1.8	2
$f_1 x$	1	0.833	0.714	0.625	0.555	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

We know, from Trapezoidal's rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1+0.5)+2(0.833+0.714+0.625+0.555)\right]$$

$$= 0.69546$$

$$I = \int_1^2 \frac{dy}{y} \int_1^2 \frac{dx}{x}$$

$$= \int_1^2 \frac{dy}{y}\, 0.69546$$

$$\therefore I = \int_1^2 \frac{0.69546}{y}\, dy \qquad (63.)$$

Let $f(y) = \frac{0.69546}{y}$

y	1	1.2	1.4	1.6	1.8	2
$f(y)$	0.69546	0.57955	0.49675	0.4346	0.3863	0.34773
	y_0	y_1	y_2	y_3	y_4	y_5

From Trapezoidal's rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

or $$\int_1^2 \frac{0.69546}{y}\,dy = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0.69546+0.34773)+2(0.57955+0.49675+0.4346+0.3863+0.34773)\right]$$

$$= 0.482$$

iii) $\displaystyle\int_1^{2.8}\int_2^{3.2}\frac{dx\,dy}{x+y}$

Soln:

Let $I = \displaystyle\int_1^{2.8}\int_2^{3.2}\frac{dx\,dy}{x+y}$

$= \displaystyle\int_1^{2.8} dy \int_2^{3.2}\frac{dx}{x+y}$

Let $f_1(x) = \displaystyle\int_2^{3.2}\frac{dx}{x+y}$

x	2	2.4	2.8	3.2
$f_1(x)$	$2+y$	$2.4+y$	$2.8+y$	$3.2+y$
	y_0	y_1	y_2	y_3

From Trapezoidal's rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\therefore \int_2^{3.2}\frac{dx}{x+y} = \frac{0.4}{2}\left[(2+y+3.2+y)+2(2.4+y+2.8+y)\right]$$

$$= 0.2\left[5.2+2y+4y+10.4\right]$$

$$= 1.2y+3.12$$

$$I = \int_1^{2.8} dy \int_2^{3.2}\frac{dx}{x+y} = \int_1^{2.8}(1.2y+3.12)\,dy$$

Let $f_2(x) = (1.2y + 3.12)$

y	1	1.3	1.6	1.9	2.2	2.5	2.8
$f_2(y)$	4.32	4.68	5.04	5.4	5.4	6.12	6.48
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

From Trapezoidal's rule

$$\int_1^{2.8} (1.2y + 3.12)\,dy = \frac{0.3}{2}\left[(4.32 + 6.48) + 2(4.68 + 5.04 + 5.4 + 5.76 + 6.12)\right]$$

$$= 0.15\,[10.8 + 54.0]$$

$$= 9.72 \text{ Ans.}$$

9.5 Boole's Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{2h}{45}\left[7y_0+32y_1+12y_2+32y_3+14y_4+32y_5+32y_7 + 14y_8\right]$$

9.6 Weddle's Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+12y_6+5y_7+y_8\right]$$

Q. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Boole's rule and Weddle's rule

Soln:

Let $f(x) = \int_0^6 \frac{dx}{1+x^2} = \frac{1}{1+x^2}$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

from Boole's rule

$$\int_{x_0}^{6} \frac{dx}{1+x^2} = \frac{2h}{45}\left[7y_0+32y_1+12y_2+32y_3+14y_4+32y_5+12y_6\right]$$

$$= \frac{2\times1}{45}\left[(7\times1)+(32\times0.5)+12(0.2)+(32\times0.1)+64\times0.588) + 32\times0.0385+12\times0.027)\right]$$

$$= 1.375$$

from Weddle's rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6\right]$$

$$= \frac{3\times1}{10}\left[1+5\times0.5+0.2+6\times0.1+0.0588+5\times0.0385 + 2\times0.027\right]$$

$$= 0.3\left[1+2.5+0.2+0.6+0.0588+0.1925+0.054\right]$$

$= 1.3815$ Ans