Discrete Mathematics Notes

ICT 7th Semester

by

Mukesh Singh

(ICT 5th batch, 2073)

Sukuna Multiple Campus

Sundarharaincha-12, Morang

Units	Chapter name / Contents
Unit-1	Relation and Diagraph (9 hrs)
	1.1 Product set and partitions
	1.2 Binary relations and its types
	1.3 Different methods representing relations
	• Relation as an order pairs
	• Relation as matrix
	• Relation as directed graphs
	• Relation as an arrow diagram
	• Relation as graph
	1.4 Properties of relations:
	• Reflexive
	• Symmetric
	• Asymmetric
	• Transitive
	• Equivalence relation
	• Partial order relations
	1.5 Boolean matrix representation of Relations
	• Boolean matrix operations
	• Boolean products
	1.6 Composition of two relations
	1.7 Operation on relations
	1.8. Transitive closure and Warshall's

Page No.
Date

algorithm

Unit-2 Counting and Combinatories (7 hrs)

2.1 Introduction
2.2 Basic principles of Counting
- Sum rule principle
- Product rule principle

2.3 Permutation of n-different objects

2.4 Combination
2.5 The pigeonhole principle
2.6 The extended pigeonhole principle

Unit-3 The Fundamental Algorithms, and Matrices

3.1 Algorithms
3.2 Complexity of algorithm
- Time Complexity
- Understanding the Complexity of algorithm

3.3 Searching algorithm:
- Linear Search
- Binary Search

3.4. Sorting
- Bubble Sort
- Insertion Sort

3.5 Matrices
- Matrix arithmetic
- Transpose of matrix
- Power of matrices

Unit-4 Recursion on Sequence and Series (8 hrs)

4.1 Introduction
4.2 Sequence and summations
- Arithmetic Progression
- Geometric Progression
- Harmonic Progression
- Relations and properties
- Recurrence relations
- Use of series on summation notation

4.3 Solutions for recursive relations

4.4 Recursive algorithm, recursion and iteration, the merge sort

4.5 Recursively defined functions

Page No.
Date

Unit-5 Special types of Functions (6 hrs)

5.1 Floor and ceiling functions
5.2 Characteristics functions
5.3 Integer value functions
5.4 Remainder function: modular arithmetic
5.5 Factorial function

Unit-6 Geometric Transformation (9 hrs)

6.1 Geometric properties of plane linear transformation
6.2 Isometric transformation
- Reflection
- Translation
- Half turn
- Rotation
- Glide reflection

6.3 Non-isometric transformation
- Dilation
- Stretch
- Shear

6.4 Matrix representation of isometric and non-isometric transformations

Unit-1

Relations and Diagraph

Page No.
Date

Discrete mathematics is the branch of mathematics dealing with objects that can consider only distinct, separated values.

1.1 Product Set and Partitions

A set is defined as collection of distinct objects of the same type or class of objects.

Product Sets

Let A and B be two non-empty sets. The Product set or Cartesian product $A \times B$ is the set of all order pairs (a,b) with $a \in A$ and $b \in B$, that is

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Example

Let $A = \{1,2,3\}$ and $B = \{r,s\}$

$A \times B = \{(1,r), (1,s), (2,r), (2,s), (3,r), (3,s)\}$

$|A \times B| = |A||B|$

A|B	r	s
1	(1,r)	(1,s)
2	(2,r)	(2,s)
3	(3,r)	(3,s)

Partitions

A partition or a quotient set of a non-empty set A is a collection of P of non-empty subsets of A such that:

(1) Each element of A belongs to one of these sets of P.

(2) If A_1 and A_2 are distinct elements of P, then $A_1 \cap A_2 = \emptyset$.

Then sets in P are called the blocks or cells of the partition.

Example

Let $A = \{a, b, c, d, e, f, g, h\}$. Consider subsets of A:
$A_1 = \{a, b, c, d\}$, $A_2 = \{a, c, e, f, g, h\}$, $A_3 = \{a, c, e, g\}$,
$A_4 = \{b, d\}$, $A_5 = \{f, h\}$

Then $\{A_1, A_2\}$ is not the partition because $A_1 \cap A_2 = \emptyset$.

$\{A_1, A_5\}$ is not a partition because $\cancel{A_1 \cap A_2 = \emptyset}$ $e, g \notin A_1$ and A_5.

$\{A_2, A_4\}$ and $\{A_3, A_4, A_5\}$ are partitions of A.

~~Binary relations and its types~~

Some important Sets and Notations

N → the set of all natural numbers = {1, 2, 3, 4 ....}

I / Z → the set of all integers = {... , -3, -2, -1, 0, 1, 2, 3 ....}

I^+ / Z^+ → the set of all ~~integers~~ positive integers.

Q → the set of all rational numbers.

R → the set of all real numbers.

W → the set of all whole numbers.

∅ → Empty set

U → Universal set

Cardinality of a Sets:

The total number of unique elements in the set is called the cardinality of the set. The cardinality of the countably infinite set is countably infinite.

Example

Let P = {k, l, m, n}

The cardinality of the set P is 4.

1.2 Binary Relations and its types

Let A and B be two non-empty sets. A relation R from A to B is a subset of $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

If $R \subseteq A \times B$ and $(a,b) \in R$, then we say that a is related to b by R, denoted as aRb. Other notation: $a =_R b$.

If a is not related to b, then $a \neq_R b$.

If sets A and B are equal, then we say $R \subseteq A \times A$ is a relation ~~from~~ on A.

Eg:

(i) Let $A = \{a, b, c\}$

$B = \{r, s, t\}$

Then $R = \{(a,r), (b,r), (b,t), (c,s)\}$ is a relation from A to B.

(ii) Let $A = \{1,2,3\}$ and $B = A$

$R = \{(1,1), (2,2), (3,3)\}$ is a relation (equal) on A.

Q1. If a set has 'n' elements, how many relations are there from A to A.

Soln:

If set A has 'n' elements, $A \times A$ has n^2 elements. So, there are 2^{n^2} relations from A to A.

Q2. If A has m elements and B has n elements. How many relations are there from A to B and vice versa?

Soln:
There are $m \times n$ elements; hence there are $2^{m \times n}$ relations from A to B.

Q3. If a set $A = \{1,2\}$. Determine all relations from A to A.

Soln:
There are $2^2 = 4$ elements i.e. $\{(1,2),(2,1),(1,1),(2,2)\}$ in $A \times A$. So, there are $2^4 = 16$ relations from A to A i.e.

(1) $\{(1,2),(2,1),(1,1),(2,2)\}$, (2) $\{(1,2),(2,1)\}$ (3) $\{(1,2),(1,1)\}$,
(4) $\{(1,2),(2,2)\}$, (5) $\{(2,1),(1,1)\}$, (6) $\{(2,1),(2,2)\}$
(7) $\{(1,1),(2,2)\}$, (8) ~~$\{(1,2),(2,1)\}$ (9) $\{(1,1)\}$~~
(8) $\{(1,2),(2,1),(1,1)\}$ (9) $\{(1,2),(1,1),(2,2)\}$
(10) $\{(2,1),(1,1),(2,2)\}$ (11) $\{(1,2),(2,1),(2,2)\}$
(12) $\{(1,2),(2,1),(1,1),(2,2)\}$ and ϕ.

Domain and Range of a relation

If there are two sets A and B and Relation from A to B is $R(a,b)$, then:
Domain is defined as the set $\{a \mid (a,b) \in R$ for some b in $B\}$ ~~and~~
Range is defined as the set $\{b \mid (a,b) \in R$ for some a in $A\}$.

Page No.
Date

Types of Binary Relation:

1. Empty Relation →
The empty relation between the sets A and B, or, on E, is the empty set ϕ.

2. Full Relation:
The full relation between the sets A and B is the set $A \times B$.

3. Identity Relation
The identity relation on set X is the set $\{(x,x) \mid x \in X\}$

4. Inverse Relation
The inverse relation R' of a relation R is defined as $R' = \{(b,a) \mid (a,b) \in R\}$
Eg:
if $R = \{(1,2),(2,3)\}$ then R' will be $\{(2,1),(3,2)\}$.

5. Reflexive Relation
A relation R on set A is called reflexive if $(a,a) \in R$ for every $a \in A$.
Example
The relation $R = \{(a,a),(b,b)\}$ on set $X = \{a,b\}$ is reflexive.

Date

6. Irreflexive Relation -
A relation R on set A is said to be irreflexive if $(a, a) \notin R$ for every $a \in A$.

Example
The relation $R = \{(a, b), (b, a)\}$ on set $X = \{a, b\}$ is irreflexive.

7. Symmetric Relation
A relation R is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$.

Example
The relation $R = \{(1,2), (2,1), (3,2), (2,3)\}$ on set $A = \{1, 2, 3\}$ is symmetric.

8. Asymmetric Relation
A relation R is asymmetric if $(b, a) \notin R$ whenever $(a, b) \in R$.

~~Example~~

9. Antisymmetric Relation
A relation R is antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

10. Transitive Relation
A relation R on a set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Example → Relation $R = \{(1,2), (2,3), (1,3)\}$ on a set $A = \{1, 2, 3\}$ is transitive.

11. Equivalence Relation

A relation R is an Equivalence Relation if it is reflexive, symmetric and transitive.

Example

Relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$ on set $A = \{1,2,3\}$ is equivalence relation as it is reflexive, symmetric and transitive.

Complement of a Relation:

Consider a relation from set A to B. The complement of relation R denoted by $\bar{R}$ is a relation from A to B such that:

$\bar{R} = \{(a,b) : (a,b) \notin R\}$

Example

Consider the relation R from X to Y.

$X = \{1,2,3\}$

$Y = \{8,9\}$

$R = \{(1,8),(2,8),(1,9),(3,9)\}$

Find the complement of R.

Soln

$X \times Y = \{(1,8),(2,8),(3,8),(1,9),(2,9),(3,9)\}$

Now we find the complement relation $\bar{R}$ from $X \times Y$.

$\bar{R} = \{(3,8),(2,9)\}$

Page No.
Date

1.3 Different methods of representing relations

Relations can be represented in many ways. Some of which are as follows:

1. Relation as a Matrix

In this zero-one is used to represent the relationship that exists between two sets. If a element is present then it is represented by 1 else it is represented by 0.

Let $P = [a_1, a_2, a_3 \ldots a_m]$ and
$Q = [b_1, b_2, b_3 \ldots b_m]$ are finite sets, containing m and n number of elements respectively. R is a relation from P to Q. The relation R can be represented by m×n matrix $M = [M_{ij}]$, defined as

$$M_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

__Example__

Let $P = \{1,2,3,4\}$, $Q = \{a,b,c,d\}$
and $R = \{(1,a),(1,b),(1,c),(2,b),(2,c),(2,d)\}$
The matrix of relation is shown as:

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2. Relation as a Directed Graph

There is another way of picturing a relation R when R is a relation from a finite set to itself.

It consists of set 'V' of vertices and with the edges 'E'. Here E is represented by ordered pair of vertices.

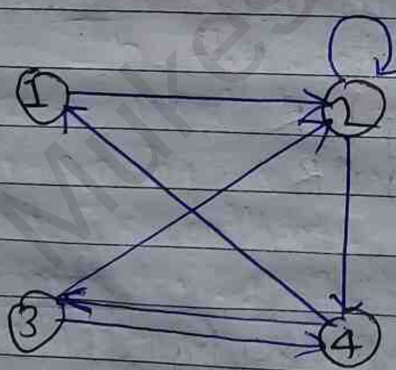
In the edge (a,b), a is the initial vertex and b is the final vertex.

If edge is (a,a), then this is regarded as loop.

Example

A = {1,2,3,4}

R = {(1,2),(2,2),(2,4),(3,2),(3,4),(4,1),(4,3)}



3. Relation as an Arrow Diagram

If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented as an arrow diagram as follows.

Draw two ellipses for the sets P and Q. Write down the elements of P and Q Column wise in three ellipses. Then draw an arrow from the first ellipse to the second ellipse if a is related to b and $a \in P$ and $b \in Q$.

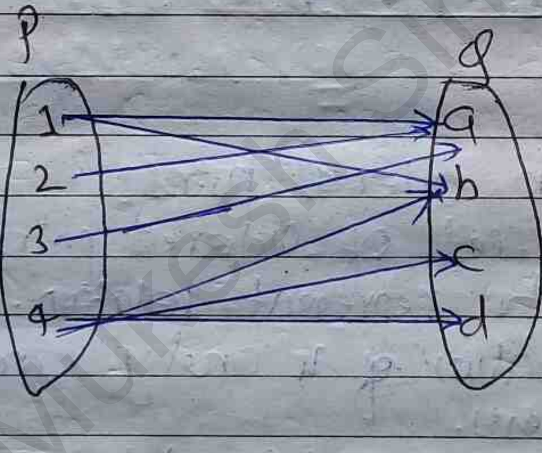
<u>Example</u>

Let $P = \{1, 2, 3, 4\}$

$Q = \{a, b, c, d\}$

$R = \{(1, a), (2, a), (3, a), (1, b), (4, b), (4, c), (4, d)\}$

The arrow diagram is as follows:



4. Relation as a Table:

If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented in tabular form.

Make the table which contains equivalent to an element of P and columns equivalent to the element of Q. Then place a cross (X) in the boxes which represent relations of elements on set P to set Q.

Example:

Let $P = \{1,2,3,4\}$

$Q = \{x, y, z, k\}$

$R = \{(1,x), (1,y), (2,z), (3,z), (4,k)\}$

The tabular form of relation as shown in fig:

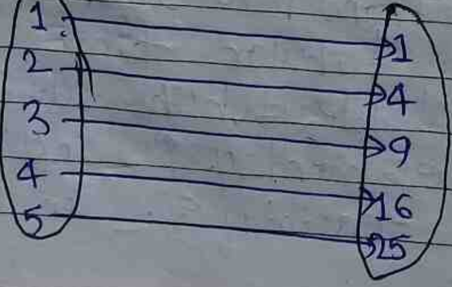
	x	y	z	k
1	X	X		
2			X	
3			X	
4				X

5. Relation as an ordered pairs

In this, set of ordered pairs of x and y are used to represent relation. In this corresponding values of x and y are represented using parenthesis.

Example $\{(1,1), (2,4), (3,9), (4,16), (5,25)\}$

This represent square of a number which means if $x=1$ then $y = x * x = 1$ and so on.



Page No.
Date

1.4 Properties of relations:

- Reflexive
- Symmetric
- Assymetric
- Transitive
- Equivalence
- partial order relations

Partial Order relations

A relation R on a set A is called a partial order relation if it satisfies the following three properties:

1. Relation R is Reflexive i.e. aRa, $a \in A$.
2. Relation R is Antisymmetric i.e., aRb and $bRa \Rightarrow a=b$
3. Relation R is transitive, i.e. aRb and $bRc \Rightarrow aRc$.

Example

Show whether the relation $(x,y) \in R$, if $x \geq y$ defined on the set of +ve integers is a partial order relation.

Soln:

Consider set $A = \{1,2,3,4\}$ containing four +ve integers. Find the relation for this set such as $R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (1,1), (2,2)$ $(3,3), (4,4)\}$

~~Reflexive~~

The relation is reflexive as $(1,1),(2,2),(3,3),(4,4) \in R$.

The relation is ~~transitive as~~ antisymmetric as whenever (a,b) and $(b,a) \in R$, we have $a=b$.

The relation is transitive as whenever (a,b) and $(b,c) \in R$, we have $(a,c) \in R$.

Hence it is partial order relation.

Page No.
Date

1.5 Boolean Matrix representation of Relations

(1) Boolean Matrix Operations

A Boolean matrix is a matrix whose entries are 0 or 1.

Let $A = [a_{ij}]$, and $B = [b_{ij}]$ be $m \times n$ Boolean matrices.

(i) The join of A and B : $C = A \vee B = [c_{ij}]$ where

$$c_{ij} = \begin{cases} 1 \text{ if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 \text{ if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

(ii) the meet of A and B : $D = A \wedge B = [d_{ij}]$ where

$$d_{ij} = \begin{cases} 1 \text{ if } a_{ij} = 1 \text{ and } b_{ij} = 1 \\ 0 \text{ if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

(iii) Let $A = [a_{ij}]$ be $m \times p$, and $B = [b_{ij}]$ be $p \times n$ Boolean matrices.

the Boolean product of A and B : $E = A \odot B = [e_{ij}]$ Where

$$e_{ij} = \begin{cases} 1 \text{ if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k, \; 1 \leq k \leq p \\ 0 \text{ Otherwise} \end{cases}$$

Page No.
Date

If $M_g = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $M_s = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ find

i) $M_g \cup M_s = M_g \vee M_s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

ii) $M_g \cap M_s = M_g \wedge M_s = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

iii) $M_g \odot M_s = M_g \odot M_s = \begin{bmatrix} 1\vee0\vee0 & 0\vee0\vee0 & 1\vee0\vee0 \\ 1\vee0\vee0 & 0\vee1\vee0 & 1\vee0\vee0 \\ 0\vee0\vee0 & 0\vee1\vee0 & 0\vee0\vee0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

iv) $(M_s)^2 = M_s \otimes M_s$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 1+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Boolean Products

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} 1+0+0 & 0+1+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+1+0 & 0+0+0 \\ 1+0+0 & 0+1+0 & 0+1+0 & 0+0+0 \\ 0+0+1 & 0+0+0 & 0+0+1 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

1.6 Composition of Two relations.

Let A, B and C be sets.
Let R is the relation from A to B
i.e. $R \subseteq A \times B$.
S is the relation from B to C
i.e. $S \subseteq B \times C$.

The composition of R and S gives the relation from A to C, indicated by $R \circ S$ and defined by:

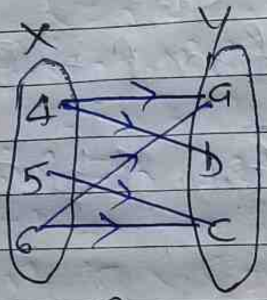
$R \circ S = \{(a,c) \in A \times C :$ for some $b \in B$, $(a,b) \in R$ and $(b,c) \in S\}$

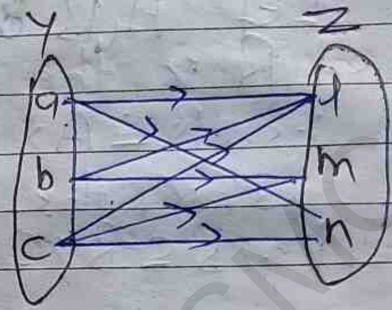
Sometimes, $R \circ S$ is simply denoted by RS.

Let R is a relation on set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself, is always represented. Also, $R \circ R$ is sometimes denoted by R^2.
Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on.
Thus, R^n is defined for all positive n.

Example
Let $X = \{4,5,6\}$, $Y = \{a,b,c\}$ and $Z = \{l,m,n\}$.
Consider the relation R_1 from X to Y and R_2 from Y to Z.

$R_1 = \{(4,a),(4,b),(5,c),(6,a),(6,c)\}$
$R_2 = \{(a,l),(a,n),(b,l),(b,m),(c,l),(c,m),(c,n)\}$





Find the Composition of relation
(i) $R_1 \circ R_2$ (ii) $R_1 \circ R^{-1}$

(i) Soln:
The Composition of $R_1 \circ R_2$ as shown below:

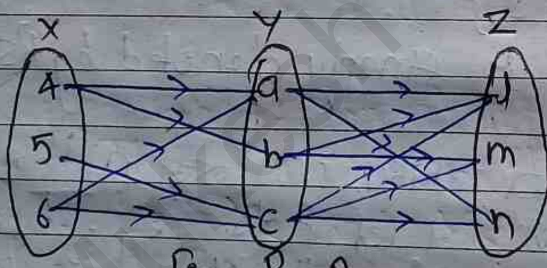


Fig: $R_1 \circ R_2$

$R_1 \circ R_2 = \{(4,l),(4,n),(4,m),(5,l),(5,m),(5,n),$
$(6,l),(6,m),(6,n)\}$

(ii) The composition of relation $R_1 \circ R_1^{-1}$ as shown below:

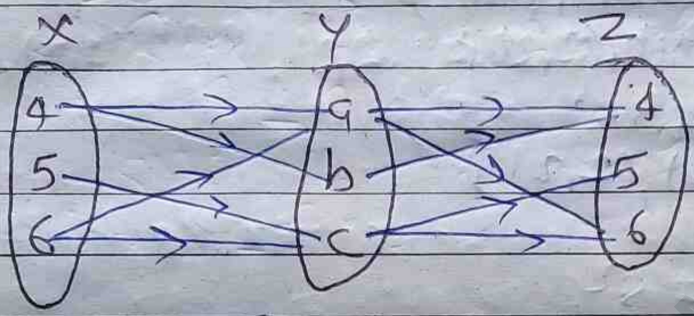


Fig: $R_o R^{-1}$

$R_1 \circ R^{-1} = \{(4,4), (5,5), (5,6), (6,4), (6,5), (4,6), (6,6)\}$

Q2.
Let $R = \{(1,2), (3,4), (2,2)\}$
$S = \{(4,2), (2,5), (3,1), (1,3)\}$
Find R_oS, S_oR, $R_o(S_oR)$, R_oR, S_oS.
Soln:

(i) $R_oS = \{(1,5), (3,2), (2,5)\}$

(ii) $S_oR = \{(4,2), (3,2), (1,4)\}$

(iii) $R = \{(1,2), (3,4), (2,2)\}$
$S_oR = \{(4,2), (3,2), (1,4)\}$

$R_o(S_oR) = \{(3,2)\}$

(iv) $R \circ R = R^2$

$R = \{(1,2),(3,4),(2,2)\}$
$R = \{(1,2),(3,4),(2,2)\}$

$R \circ R = \{(1,2),(2,2)\}$

(v) $S \circ S = S^2$
$S = \{(4,2),(2,5),(3,1),(1,3)\}$
$S = \{(4,2),(2,5),(3,1),(1,3)\}$

$S \circ S = \{(4,5),(3,3),(1,1)\}$

Composition of Relations and Matrices
There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrix.

Example
Let $P = \{2,3,4,5\}$. Consider the relation R and S on P defined by
$R = \{(2,2),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),(5,3)\}$

$S = \{(2,3),(2,5),(3,4),(3,5),(4,2),(4,3),(4,5),(5,2),(5,5)\}$
Find the matrices of the above relations. Use matrices to find the following composition of the relation R and S.

Page No. ______
Date ______

(i) $R \circ S$ (ii) $R \circ R$ (iii) $S \circ R$

Soln:
The matrices of the relation R and S are shown in fig:

$$M_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \text{ and } M_S = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(i) To obtain the composition of relation R and S. First multiply M_R with M_S to obtain the matrix $M_R \times M_S$ as shown in fig:

The non-zero entries in the matrix $M_R \times M_S$ tells the elements related in $R \circ S$. So,

$$M_R \times M_S = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 4 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Hence, the composition of R∘S of the relation R and S is:

$R \circ S = \{(2,2),(2,3),(2,4),(3,2),(3,3),(4,2),(4,5),(5,2),(5,3),(5,4),(5,5)\}$

(ii) First, multiply the matrix M_R by itself, as shown in fig:

$$M_R \times M_R = \begin{array}{c|cccc} & 2 & 3 & 4 & 5 \\ \hline 2 & 1 & 2 & 2 & 3 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 1 \end{array}$$

Hence, the composition of R∘R of the relation R and S is:

$R \circ R = \{(2,2),(3,2),(3,3),(3,4),(4,2),(4,5),(5,2),(5,3),(5,5)\}$

(iii) Multiply the matrix M_S with M_R to obtain the matrix $M_S \times M_R$ as shown in fig:

$$M_S \times M_R = \begin{array}{c|cccc} & 2 & 3 & 4 & 5 \\ \hline 2 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 2 & 2 & 2 \\ 5 & 1 & 1 & 1 & 1 \end{array}$$

Page No.
Date

The non-zero entries in Matrix $M_S \times M_R$ tells the elements related ~~to~~ in $S \circ R$.

Hence, the Composition ~~of~~ $S \circ R$ of the relation S and R is:

$S \circ R = \{(2,4), (2,5), (3,3), (3,4), (3,5), (4,2), (4,4), (4,5), (5,2), (5,3), (5,4), (5,5)\}$.

1.7 Operation on relations

The following operations are performed on the relations:

1) Union
2) Intersection
3) Set difference

} Binary operation

4) Complement
5) Inverse or Converse relation

} Unary operation

1) Union

If R and S are two relations from set A to B, then the union of R and S relations are denoted by $R \cup S$, defined by

$$R \cup S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ or } (a,b) \in S\}$$

2) Intersection

If R and S are two relations from set A to set B, then the intersection of two relations R and S is denoted by $R \cap S$, it can be defined as

$$R \cap S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ and } (a,b) \in S\}$$

3.) Set difference

If R and S are two relations from set A to set B, then set difference of two relations R and S is denoted by $R - S$, defined by

Page No.
Date

$R - S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ and } (a,b) \notin S\}$

$S - R = \{(a,b) \mid a \in A, b \in B, (a,b) \in S \text{ and } (a,b) \notin R\}$.

R R R
S S S $\therefore \boxed{R - S \neq S - R}$
$R \cup S$ $R \cap S$ $R - S$

$R \cup S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ or } (a,b) \in S\}$

$R \cap S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ and } (a,b) \in S\}$

$R - S = \{(a,b) \mid a \in A, b \in B, (a,b) \in R \text{ and } (a,b) \notin S\}$
$S - R = \{(a,b) \mid a \in A, b \in B, (a,b) \in S \text{ and } (a,b) \notin R\}$.

4) Complement Operation
Let R be a relation, then complement of relation R is denoted by R^c or $\overline{R}$ or R', defined by
$R^c = \{(a,b) \mid (a,b) \in A \times B \text{ and } (a,b) \notin R\}$

$R^c = (A \times B) - R$

$S^c = \{(a,b) \mid (a,b) \in A \times B \text{ and } (a,b) \notin S\}$

$S^c = (A \times B) - S$

5) Inverse Operation

Let R be relation, the inverse operation on a relation R is denoted by R^{-1}, defined by

$R^{-1} = \{(a,b) \mid (b,a) \in R^{-1}, a \in A, b \in B\}$

$S^{-1} = \{(a,b) \mid (b,a) \in S^{-1}, a \in A, b \in B\}$

Let R

R^{-1}

$R^{-1} = \{(a,b) \mid (a,b) \in R \text{ and } (b,a) \in R^{-1}\}$

$(a,b) \in R$

Then

$(b,a) \in R^{-1}$

$S^{-1} = \{(a,b) \mid (a,b) \in S \text{ and } (b,a) \in S^{-1}\}$

Page No.
Date

1.8 Transitive closure and Warshall's algorithm

Transitive closure of a relation:

The transitive closure R^+ is the smallest transitive relation that contains R as a subset.

Let a relation R is defined on set A containing 'n' elements, then

$$\boxed{R^+ = R \cup R^2 \cup \ldots \cup R^n}$$

Eg

Let $A = \{1,2,3\}$ and $R = \{(1,2),(2,3),(3,1)\}$. Find the transitive closure.

Soln:

Given

$R = \{(1,2),(2,3),(3,1)\}$

$R^2 = R \circ R = \{(1,2),(2,3),(3,1)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,3),(2,1),(3,2)\}$

$R^3 = R^2 \circ R = \{(1,3),(2,1),(3,2)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,1),(2,2),(3,3)\}$

So, $R^+ = R \cup R^2 \cup R^3$
$= \{(1,2),(2,3),(3,1)\} \cup \{(1,3),(2,1),(3,2)\} \cup \{(1,1),(2,2),(3,3)\}$

$= \{(1,2),(2,3),(3,1),(1,3),(2,1),(3,2),(1,1),(2,2),(3,3)\}$

~~Warshall's Algorithm~~

If relation R for a set is not in transitive, we need to apply closure to make it transitive. This closure is known as transitive closure.

Warshall's Algorithm:

A more efficient method ~~to~~ for computing the transitive closure of a relation.

Eg

Using Warshall's algorithm, find transitive closure of the relation,

$R = \{(1,4),(2,1),(2,3),(3,1),(3,4),(4,3)\}$ on set $A = \{1,2,3,4\}$

Soln:

Given,

$R = \{(1,4),(2,1),(2,3),(3,1),(3,4),(4,3)\}$

Representing the relation in matrix:

$$M_R = P_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

P_0 = initial matrix.

Page No.
Date

1st Iteration: 1st Col and 1st row

Now,

C   R

{2,3}   {4}

$C \times R = \{(2,4),(3,4)\}$

$$P_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

2nd Iteration: 2nd Col. and 2nd row

C   R

{ }   {1,3,4}

$C \times R = \{ \}$

$$P_2 = P_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

3rd Iteration: 3rd Col. and 3rd row

C   R

{2,4}   {1,4}

$C \times R = \{(2,1),(2,4),(4,1),(4,4)\}$

$$P_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

4th Iteration: 4th Col. and 4th row

C   R

{1,2,3,4}   {1,3,4}

$C \times R = \{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)\}$

$$
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 & 1 \\
3 & 1 & 0 & 1 & 1 \\
4 & 1 & 0 & 1 & 1
\end{array}
$$

$P^4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

$\therefore R^+ = \{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),$
$(4,1),(4,3),(4,4)\}$

Page No.
Date

Unit-2

Counting and Combinatories

Page No.
Date

2.1. Introduction

In daily lives, many a times one needs to find out the number of all possible outcomes for a series of events. For instance, in how many ways can a panel judges comprising of 6 men and 4 women be chosen from among 50 men and 38 women? How many different 10 lettered PAN numbers can be generated such that the first five letters are capital letters (alphabets), the next four are digits and the last is again a capital letter.

For solving these problems, Mathematical theory of counting are used. Counting mainly encompasses fundamental counting rule, the permutation rule, and the combination rule.

Page No.
Date

2.2 Basic principles of Counting

1. Sum rule principle

Assume that some event E occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in m+n ways.

In general, if there are n events and no two events can occurs in same time then the event occur in $n_1 + n_2 \cdots n$ ways.

If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

Example

If 8 male professor and 5 female professor teaching DMS then the student can choose professor in $8+5=13$ ways.

2. Product rule principle

Suppose there is an event E which can occur in m ways and independent of this event, there is a second event F which can occur in n ways. Then combination of E and F can occur in mn

Page No.
Date

ways.

In general, if there are 'n' events occurring independently then all events can occur in the order indicated as $n_1 \times n_2 \times n_3 \ldots n$ ways.

Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$.

Example

In a class, there are 4 boys and 10 girls, if a boy and a girl have to be chosen for the class monitor, the students can choose monitor in $4 \times 10 = 40$ ways.

Factorial function:

The product of the first n natural number is called factorial n. It is denoted by n!, read "n factorial."

The factorial n can also be written as:

$$n! = n(n-1)(n-2)(n-3) \cdots 1$$
$$= 1 \text{ and } 0! = 1$$

Example: Find the value of: 5!

Soln:

$$5! = 5 \times (5-1)(5-2)(5-3)(5-4)$$
$$= 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

Page No.
Date

2.5 Permutation of n-different Objects

A permutation is an ordered ~~arrangement~~ combination of elements.

Example

* From a set $S = \{x, y, z\}$ by taking two at a time all permutations are:
  xy, yx, xz, zx, yz, zy.

* We have to form a permutation of three digit numbers from a set of numbers $S = \{1, 2, 3\}$. Different three digit numbers will be formed when we arrange the digits. The permutation will be:
  $= 123, 132, 213, 231, 312, 321$

Number of Permutations:

The number of permutations of 'n' different objects taken 'r' at a time is denoted by:

$$P(n, r) \text{ or } {}^{n}P_{r} = \frac{n!}{(n-r)!}, \quad r \leq n.$$

Where,

$$n! = n(n-1)(n-2)(n-3)\cdots 1$$

Theorem: Prove that the number of permutations of n things taken all of a time is n!

Proof - Let there be 'n' different elements.

There are n number of ways to fill up the first place. After filling the first place (n-1) number of elements left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place (n-2) number of elements left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up the r-th place as $[n-(r-1)]$ $= n-r+1$

So, the total no. of ways to fill up from first place up to r-th place -

$$^nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

$$= [n(n-1)(n-2)\ldots(n-r+1)][(n-r)(n-r-1)\ldots 3.2.1] / [(n-r)(n-r-1)\ldots 3.2.1]$$

Hence,

$$^nP_r = \frac{n!}{(n-r)!}$$

Permutation with Restrictions:

The number of permutations of n different objects taken r at a time in which p particular objects do not occur is

$\cancel{^{n}p_r}$ $^{n-p}P_r$

The number of permutations of n different objects taken r at a time in which p particular objects are present is:

$$^{n-p}P_{r-p} \times {}^{r}P_p$$

~~Question~~ Circular Permutations:

(i) If clockwise and anticlockwise orders are different $= (n-1)!$

(ii) If orders are not different $= \frac{(n-1)!}{2!}$

(iii) (n, r), if orders are different $= {}^{n}P_r / r$

(iv) If orders are not different $= {}^{n}P_r / 2r$

Questions from Permutation

1. How many different signals can be made ~~from~~ by 5 flags from 8 flags of different Colours:

Soln:

$n=8, r=5$

$P_{nr} = {}^{8}P_{5} = \frac{n!}{(n-r)!} = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720.$

2. How many words can be made by using the letters of the word 'SIMPLETON' taken all at a time.

Soln:

$n=9, r=9$

${}^{9}P_{9} = \frac{9!}{0!} = 9! = 362880$

3. How many permutations of the letters ABCDEFGH contains the string ABC?

Soln:

$\underbrace{ABC}_{1}, \underbrace{DEFGH}_{5}] \rightarrow 6! = 720$ permutations.

single letter

Related to Circular

4. In how many ways ten people can sit around a table?

Soln:

$(n-1)! = (10-1)! = 9!$ Ans

5. How many necklace of 12 beads each can be made from 18 beads of different colours?

Soln:

Clockwise and anticlockwise arrangement are same.

$n = 18, r = 12$

$^nP_r/2r = \frac{^{18}P_{12}}{24} = \frac{18!}{6! \times 24}$ Ans

Important permutations formula:

1). No. of permutations of 'n' things taken 'r' at a time when a particular r thing is to be always included in each arrangement. $]\left(r\, ^{n-1}P_{r-1}\right)$

2) $P(n,r)$ when a particular thing is fixed. $\left(^{n-1}P_{r-1}\right)$

3) $P(n,r)$ when 'm' specified things always comes together. $]\left(^{n-1}P_r\right)$

4] p(n, r) When 'm' specified things always come together.] → [$m! \times (n-m+1)!$]

5] No. of permutations of 'n' things, taken all at a time when 'm' specified things always come together. → $n! - [m! \times (n-m+1)!]$

2.4. Combination

A combination is a selection of some or all, objects from a set of given objects, where the order of the objects does not matter. The number of combinations of 'n' objects, taken 'r' at a time represented by ~~the~~ ^nC_r or $C(n,r)$.

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

Proof: The number of permutations of n different things, r at a time is given by:

$$^nP_r = \frac{n!}{(n-r)!}$$

As there is no matter about the order of ~~of~~ arrangements of the objects, therefore, to every combination of r things, there are ! arrangements i.e.

$$^nP_r = r!\,{}^nC_r \quad \text{or} \quad {}^nC_r = \frac{^nP_r}{r!}$$

$$= \frac{n!}{(n-r)!\,r!}, \; n \geq r$$

Thus, $$^nC_r = \frac{n!}{(n-r)!\,r!}$$

Page No. ____
Date ____

Questions:

1. How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Soln:

$n = 52$

$r = \cancel{2}5$

$$C(52,5) = {}^{52}C_5 = \frac{5!}{(52-5)!\,5!}$$

2. Prove that $C(n,r) = C(n,n-r)$

Soln.

$$C(n,r) = \frac{n!}{(n-r)!\,r!} \quad \text{— (I)}$$

$$C(n,n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,(\cancel{n}-\cancel{n}+r)!}$$

$$= \frac{n!}{(n-r)!\,r!} \quad \text{— (II)}$$

Eqn (I) = Eqn (II) Hence proved.

Page No.
Date

3. Find the number of subsets of the set {1, 2, 3, 4, 5, 6} having 3 elements.

Soln:

The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the number of subsets will be $^6C_3 = 20$

4. There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Soln:

The number of ways to choose 3 men from 6 men is $6C_3$ and the number of ways to choose 2 women from 5 women is $^5C_2$.

Hence, the total number of ways is $6C_3 \times {}^5C_2 = 20 \times 10 = 200$.

5. How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Soln:

Let us ~~...~~ number the group as 1, 2, and 3.

For choosing 3 students for 1^{st} group, the number of ways - $^9C_3$.

The number of way for choosing 3 student for 2nd group after choosing 1^{st} group - $^6C_3$

The number of ways for choosing 3 students for 3rd group after choosing 1^{st} and 2^{nd} group - $^3C_3$.

Hence, the total number of ways

$$= {}^9C_3 \times {}^6C_3 \times {}^3C_3 = 84 \times 20 \times 1 = 1680.$$

6. In how many ways can a crick et eleven be chosen out of 15 players? If ~~in a particular~~

(i) a particular player is always chosen?

(ii) a particular player is never chosen?

Soln:

$n = 15,\ r = 11,\ p = 1$

(i) $^{n-p}C_{r-p} = {}^{14}C_{10} = \dfrac{14!}{10! \times 4!} = 1365$

(ii) $^{n-p}C_r = {}^{14}C_{11} = \dfrac{14!}{11! \times 3!} = 364.$

Page No.
Date

Important Combinations formula:

(i) No. of Combinations of 'n' different things taken 'r' at a time, when 'p' particular things are always included.

$\rightarrow {}^{n-p}C_{r-p}$

(ii) No. of Combinations of 'n' different things taken 'r' at time, when 'p' particular things are always to be excluded.

$\rightarrow {}^{n-p}C_{r}$

2.5 The Pigeonhole Principle

In 1834, German mathematician, Peter Gustav Lejeune Dirichlet, stated a principle which he called the drawer principle. Now, it is known as the pigeonhole principle.

Pigeonhole Principle states that, "If there are more pigeons than the pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it."

If 'n' pigeons are put into m pigeonholes where $n \geq m$, there's a hole with more than one pigeon.

OR

If 'n' pigeonholes are occupied by $kn+1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

P	P
P	PP

Example

4 pigeonholes
5 pigeons

* 10 men are in a room and they are taking part in handshakes. If each person shakes hand at least once and no man shakes the same man's hand more than once then two men took part in the same number of handshakes.

Date

* There must be at least two people in a class of 30 whose names start with the same alphabet.

Questions

1. Find the minimum number of students in a class to be sure that three of them are both in the same month.

Soln:

Here, $n=12$ months are pigeonholes

And $k+1=3$

$k=2$

2. Show that at least two people must have their birthday in the same month if 13 people are assigned in a room.

Soln:

We assigned each person the month of the year on which he was born. Since there are 12 months in a year.

So, according to the pigeonhole principle, there must be at least two people assigned to the same month.

Page No.
Date:

Q3. How many students must be in a class to guarantee that at least 2 students receive same score. [0 to 100]

Soln:

0 → 100] 101 scores

There must be 102 students so that 2 students score the same marks.

Q4. Find the minimum number of students in a class so that 2 students were born in the same month, date.

Soln:

Same months, 12 months

↳ 12+1 = 13 students

Same date → 365 days

↳ 365+1 = 366 students.

Page No.
Date

Unit-3

The Fundamental Algorithms, and Matrices

Page No.
Date

3.1 Algorithms

An algorithm is a step-by-step method for solving a problem.

Characteristics of Algorithms:

Algorithms generally have the following characteristics:

i) Input → The algorithm receives input. Zero or more quantities are externally supplied.

ii) Output → The algorithm produces output. At least one quantity is produced.

iii) Precision → The steps are precisely stated. Each instruction is clear and unambiguous.

iv) Feasibility → It must be feasible to execute each instruction.

v) Flexibility → It should be possible to make changes in the algorithm without putting so much effort on it.

vi) Generality → The algorithm applies to a set of inputs.

vi) Finiteness → Algorithm must complete after a finite number of instruction have been executed.

3.2 Complexity of algorithm

The analysis ~~of an~~ (complexity) of an algorithm refers to the process of of deriving the time and space needed to execute the algorithm.

The complexity of an algorithm computes the amount of time and spaces required by an algorithm for an input of size(n). The complexity of an algorithm can be divided into two types. The time complexity and the space complexity.

1. Space Complexity

Space complexity is ~~required~~ defined as the amount of memory space required by the algorithm to solve a particular problem.

Space complexity ~~of any~~ $S(P)$ of any algorithm P is: $S(P) = C + S(I)$.

Where C is fixed a point and S(I) is the variable part of the algorithm, which depends on instance characteristics I.

2. Time Complexity

~~Space co~~ Time Complexity is defined as the amount of time required by the algorithm to solve a particular problem.

The time required to execute an algorithm is a function of the input. Instead of dealing directly with the input, parameters are used to characterize the size of the input e.g. if the input is a set containing n elements, the size of the input n. There are three cases noting about the time complexity of an algorithm since determining the exact time complexity of an algorithm is a difficult task.

- Worst-Case: f(n) represents the maximum number of steps taken on any instance of size n. It is the slowest possible time to solve a problem.

- Best Case: f(n) represents the minimum number of steps taken on any instance of size n. It is the fastest possible time to solve a problem.

- Average Case: f(n) represents the average number of steps taken on any instance of size n. It is the average possible time to solve a problem.

Asymptotic Notations

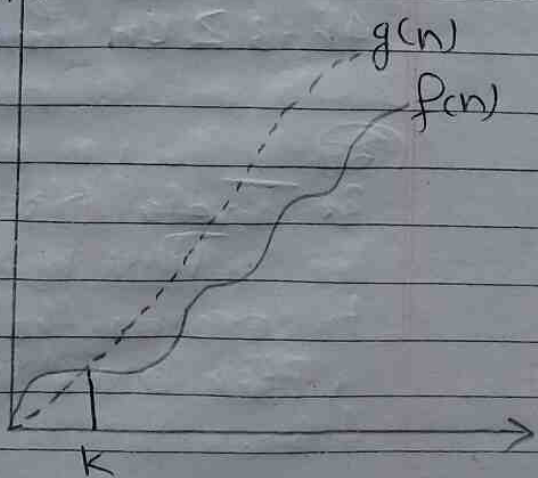
Asymptotic notations are used to describe the execution time of an algorithm. It represents the efficiency and performance of an algorithm in a systematic and meaningful manner. The notations show the order of growth of functions. Here the time taken by an algorithm is mapped regarding ~~the~~ mathematical functions. There are many asymptotic notations like $O, \Theta, \Omega, \omega$ each having its importance.

1. Big Oh Notation (O)

This notation, $O(n)$ is the formal way to express the upper bound of an algorithm's running time. It measures the worst case time complexity or the longest amount of time that an algorithm can possibly take to complete.

Definition

$f(n) = O(g(n))$ if there exists constants n_0 and C such that $f(n) \leq C.g(n)$ for all $n \geq n_0$.



Example:

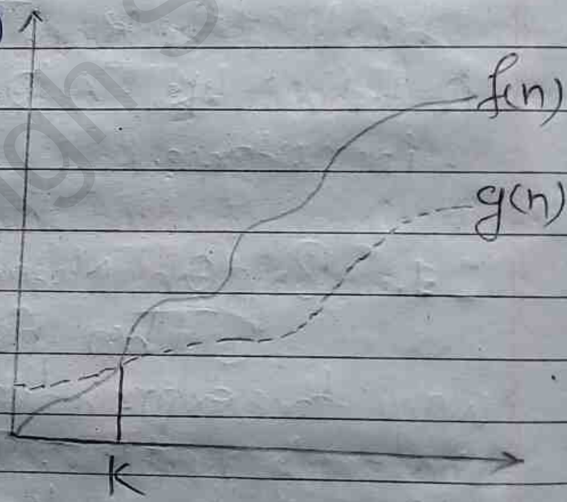
function $4n+3 = O(n)$ as $4n+3 \leq 5$ for all $n \geq n_0$.

2. Omega Notation (Ω)

This notation, $\Omega(n)$ is the formal way to express the lower bound of an algorithm's running time. It ~~means~~ measures the best case time complexity or the best amount of time an algorithm can possibly take to complete.

Definition

~~This notation, $\Omega(n)$ is the format~~ way to express ~~the lower bound~~ ~~of an~~



$f(n) = \Omega(g(n))$ if there exists constants n_0 & C such that $f(n) \geq C \cdot g(n)$ for all $n \geq n_0$.

Example:

The function $4n+3 = \Omega(n)$ as $4n+3 \geq 4n$ for all $n \geq 1$.

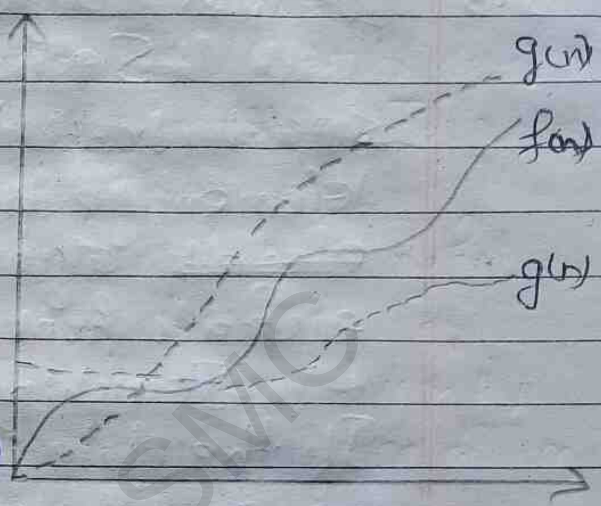
3. Theta Notation (θ)

This notation, $\theta(n)$ is the formal way to express the lower bound and the upper bound of an algorithm's running time.

Page No.
Date

Definition

If $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n) = \Theta(g(n))$.



Example

The function $4n+3 = \Theta(n)$ as $4n+3 \geq 4n$ for all $n \geq 3$ and $4n+3 \leq 5n$ for all $n \geq 3$.

3.3 Search algorithm

Searching algorithms are used to search or find one or more than one element from a dataset. These types of ~~elements~~ algorithms are used to find elements from a specific data structures.

Searching may be sequential or not. If the data in the dataset are random, then we need to use sequential searching. Otherwise we can use other different searching techniques to reduce the complexity.

There are various types of search algorithms. A Two of them are discussed below:

1. Linear Search

Linear search finds a element in a list by checking every element in the list one by one until it is found. A search will be unsuccessful if all the elements are accessed and desired element is not found. Linear search is also known as sequential search. It is named linear because its time complexity is of the order of n O(n).

Complexity of Linear Search:

- Time Complexity – O(n)
- Space Complexity – O(1)

Algorithm

Step 1: Read all search element from the user.

Step 2: Compare the search element with the first element in the list.

Step 3: If both are matching, then display "Given element found!" and terminate the function.

Step 4: If both are not matching, then compare search element with the next element in the list.

Step 5: Repeat step 3 and 4 until the search element is ~~completed~~ compared with the last element in the list.

Step 6: If the last element in the list doesn't match, then display "Element not found!!!" and terminate the function.

Example

Consider the following list of element and search element:

	0	1	2	3	4	5	6	7
List	65	20	10	55	32	12	50	99

Search element 12.

Page No. ______
Date ______

Step 1: Search element (12) is compared with ~~the~~ first element(5) ~~and Search element~~

	0	1	2	3	4	5	6	7
list	65	20	10	55	32	12	50	99
	12							

Both are not matching. So move to next elements.

Step 2: Search element(12) is compared with next element (20).

	0	1	2	3	4	5	6	7
list	65	20	10	55	32	12	50	99
		12						

Both are not matching. So move to next elements.

Step 3: Search element (12) is compared with with next element (10)

	0	1	2	3	4	5	6	7
list	65	20	10	55	32	12	50	99
			12					

Both are not matking. So move to next element.

Step 4: Search element (12) is compared with next element (55).

	0	1	2	3	4	5	6	7
list	65	20	10	55	32	12	50	99
				12				

Both are not matching so move to next element (32)

Step 5: Search element(12) is compared with next element(32).

0	1	2	3	4	5	6	7
65	20	10	55	32	12	50	99
				12			

Both are not matching, so move to next element (12).

Step 6: Search element (12) is compared with next element (12).

0	1	2	3	4	5	6	7
65	20	10	55	32	12	50	99
					12		

Both are matching, so we stop comparing and display element is found at index 5.

Page No.
Date

2. Binary Search

Binary Search is also known as half-interval search or logarithmic search. It is a search algorithm that finds the position of a target value within a sorted array.

Binary search is an efficient algorithm for finding an item from a sorted list of items. It works repeatedly dividing half the portion of the list that could contain the item, until you have narrowed down the possible locations to just one.

Algorithm:

Step 1: Let min = 1 and max = n.

Step 2: Guess the average max and min, rounded down so it is an integer.

Step 3: If you guessed the number, stop. You found it.

Step 4: If the guess was too low, set min to be larger than the guess.

Step 5: If the guess was too high, set

Page No. ______
Date ______

max to be one smaller than the guess.

Step 6: Go back to step 2.

Example 1

If searching element 23 in the 10 element array.

2	5	8	12	16	23	38	56	72	91

23>16, take 2nd half.

L									H
2	5	8	12	16	23	38	56	72	91

23<56 take 1st half

					L				H
2	5	8	12	16	23	38	56	72	91

Found 23, return 5

					L	H			
2	5	8	12	16	23	38	56	72	91

Example 2

Consider the list of element and search element.

	0	1	2	3	4	5	6	7	8
List	10	12	20	32	50	55	65	80	99

Search element 12

Step 1: Search element (12) is compared with the middle element (50).

	0	1	2	3	4	5	6	7	8
List	10	12	20	32	50	55	65	80	99
					12				

Both are not matching and 12 is smaller than 50. So we search only in the left sublist.

Page No.
Date

(i.e 10, 12, 20, 32).

	0	1	2	3	4	5	6	7	8
List	10	12	20	32	50	55	65	80	99

Step 2: Search element (12) is compared with middle element (12).

0	1	2	3	4	5	6	7	8
10	12	20	32	50	55	65	80	99

12

Both are matching, so the result is "Element Found at index 1".

3.4. Sorting

Sorting is the process of arranging the elements of any array or list in ascending or descending order.

For example, consider an array A = {A1, A2, A3, A4 ... An}, the array is called to be in ascending order if element of A are arranged like A1 > A2 > A3 > A4 > A5 ? > An.

Consider a list of values : 2, 4, 6, 8, 9, 1, 22, 4, 77, 8, 9
After sorting the values: 1, 2, 4, 4, 6, 8, 8, 9, 9, 22, 77.

Consider an array;
int [10] = {5, 4, 10, 2, 30, 45, 34, 14, 18, 9}
The array after sorting in ascending order will be:
A[] = {2, 4, 5, 9, 10, 14, 18, 30, 34, 45}

There are lots of sorting algorithms. ~~We~~ Here, we only discuss about 2 sorting algorithms:

1. Bubble Sort
2. Insertion Sort

1. Bubble Sort

Bubble sort is the simplest sorting algorithm that works by repeatedly swapping the adjacent values/elements if they are in wrong order.

Algorithm

Step 1: Starting with the first element (index = 0), Compare the current element with the next element of the array.

Step 2: If the current element is greater than the next element of the array, swap them.

Step 3: If the current element is less than the next element, move to the next element.

Step 4: Repeat Step 1.

Example:

First pass

(5 1 4 2 8) → (1 5 4 2 8), Here algorithm compares first two elements, and swaps since 5>1.

(1 5 4 2 8) → (1 4 5 2 8), swap since 5>4.

(1 4 5 2 8) → (1 4 2 5 8), swap since 5>2

(1 4 2 5 8) → (1 4 2 5 8), Now since these elements are already in order. (8>5), algorithm does not swap them.

Second Pass:

(1 4 2 5 8) → (1 4 2 5 8)

(1 4 2 5 8) → (1 2 4 5 8), Swap since 4>2

Now the array is already sorted, but our algorithm ~~does~~ needs one whole pass without any swap to know it is sorted.

Third Pass:

(1 2 4 5 8) → (1 2 4 5 8)

(1 2 4 5 8) → (1 2 4 5 8)

(1 2 4 5 8) → (1 2 4 5 8)

(1 2 4 5 8) → (1 2 4 5 8)

Example 2.

0	1	2	3	4
15	16	6	8	5

pass 1

15 16 6 8 5   15 < 16 no swap

15 16 6 8 5   16 > 6 swap

15 6 16 8 5   16 > 8 swap

15 6 8 16 5   16 > 5 swap

15 6 8 5 16   Go to next pass.

0	1	2	3	4
15	6	8	5	16

pass 2

6 15 8 5 16   15 > 6 swap

6 8 15 5 16   15 > 8 swap

6 8 5 15 16   15 > 5 swap

6 8 5 15 16   15 < 16 no swap
Go to next pass.

pass 3

6	8	5	15	16

6 8 5 15 16   6 < 8 no Swapping

6 5 8 15 16   8 > 5 Swap

6 5 8 15 16   no swap

6 5 8 15 16   move to next pass

6	5	8	15	16

5 6 8 15 16   6 > 5 swap

5 6 8 15 16   no swap

5 6 8 15 16   no swap

5	6	8	15	16

The elements are in correct order.

Example 3

0	1	2	3	4
16	14	5	6	8

pass 1

14 16 5 6 8   16 > 14 Swap

14 5 16 6 8   16 > 5 Swap

14 5 6 16 8   16 > 6 Swap

14 5 6 8 16   Go to next pass

pass 2

14	5	6	8	16

5 14 6 8 16   14 > 5 Swap

5 6 14 8 16   14 > 6 Swap

5 6 8 $\boxed{14 \ 16}$ Go ~~to next~~ pa

Since, the elements are in sorted order, no more pass is required.

No. of pass = n - 1 (n = no of elements)
But in some cases
no. of pass = n - 1 - i.

Page No.
Date

2. Insertion Sort

Insertion sort is a simple sorting algorithm that works the way we sort playing cards in our hands.

The array is searched sequentially and unsorted items are moved and inserted into the sorted sub-list (in the same array). This algorithm is not suitable for large data sets as its average and worst case complexity are of $O(n^2)$, where n is the number of elements.

Algorithm:

Step 1: If it is the first element, it is already sorted. return 1;

Step 2: Pick the next element.

Step 3:- Compare with all elements in the sorted sub-list.

Step 4: Shift all the elements in the sorted sub-list that is greater than the value to be sorted.

Step 5: Insert the value.

Step 6: Repeat until list is sorted.

Page No. ______
Date ______

Example 1:

5 2 4 6 1 3

2 5 4 6 1 3

2 4 5 6 1 3

2 4 5 6 1 3

1 2 4 5 6 3

1 2 3 4 5 6

Example 2

1st Pass	23	1	10	5	2	⇒	23	1	10	5	2
2nd Pass	23	1	10	5	2	⇒	1	23	10	5	2
3rd Pass	1	23	10	5	2	⇒	1	10	23	5	2
4th Pass	1	10	23	5	2	⇒	1	5	10	23	2
5th Pass	1	5	10	23	2	⇒	1	2	5	10	23

Page No.
Date

3.5 Matrices

A matrix is a rectangular array of numbers or symbols which are generally arranged in rows and columns.

The order of the matrix is defined as the number of rows and columns.

Matrix example

We have a 3x2 matrix that is because the number of rows here is equal to 3 and the number of columns is equal to 2.

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix}$$

The dimension of matrix can be defined as the number of rows and columns of the matrix in that order.

Since the given matrix A has 2 rows and 3 columns, it is known as 2x3 matrix.

Types of Matrices:

1. Null matrix → A matrix have all the elements zero (0) is called a null matrix or zero matrix. It is denoted by O. Thus, $A = [a_{ij}]_{m \times n}$ is a null-matrix if $a_{ij} = 0$ for all i and j.

Examples: ① $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Page No.
Date

2. Row Matrix

A matrix having only one row is called a row matrix. Thus $A = [a_{ij}]_{m \times n}$ is a row matrix if $m = 1$. So, a row matrix can also be represented as $A = [a_{ij}]_{1 \times n}$.

Example

$A = [1\ 2\ 4\ 5]$, $B = [4\ 6\ 9]$

(order 1×4) (order 1×3).

3. Column Matrix

A matrix having only one column is called a column matrix. Thus $A = [a_{ij}]_{m \times n}$ is a column matrix if $n = 1$. The order is $m \times 1$.

Example

$A = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ 8 \\ 2 \end{bmatrix}$ (Order 4×1)

(order 3×1)

4. Singleton Matrix

A matrix having only one element is called a singleton matrix.

Thus, $A = [a_{ij}]_{m \times n}$ is a singleton matrix if $m = n = 1$.

Example:

$[2], [3], [9]$.

5. Horizontal matrix

A matrix of order $m \times n$ is a horizontal matrix if $n > m$ i.e no. of column is greater than number of rows.

Example:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$ Here, $m = 2$, $n = 4$
$\therefore n > m$.

6. Vertical matrix

A matrix of order $m \times n$ is a vertical matrix if $m > n$ i.e. no. of rows is greater than no. of columns.

Example: $$\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$$ Here, $m = 4$, $n = 2$
$\therefore m > n$.

7. Square Matrix

~~If the~~ A matrix having equal number of rows and columns is called a square matrix.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ Here, $m = n$.

Page No. ______
Date ______

8. Triangular Matrix

A matrix having all of its elements above the diagnol or below the diagnol zero is called a triangular matrix.

Example.

① $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$ ② $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

Upper triangular matrix Lower triangular matrix.

9. Diagnol Matrix

If all the elements, except the principle diagnol, in a square matrix ~~is~~, are zero, it is called a diagnol matrix. Thus, a square matrix $A = [a_{ij}]$ is a diagnol matrix if $a_{ij} = 0$, when $i \neq j$.

Example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

10. ~~Di~~ Scalar Matrix

If all the elements in the diagnol of a diagnol matrix are equal, it is called a scalar matrix.

Example

$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Page No.
Date

11. Unit or Identity Matrix

If all the elements of a principal diagnol in a diagnol matrix are 1, then it is called a unit matrix.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12. Symmetric Matrix

A square matrix $A = [a_{ij}]$ is known as symmetric matrix if $a_{ij} = a_{ji}$, for all i,j values.

For example:

$$A = \boxed{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}}$$

13. Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = a_{ji}$, for all values of i,j. Thus in a ~~square~~ skew-symmetric matrix, all diagnol elements are zero.

Example

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Matrix Arithmetic

Matrix arithmetic operations may involve three algebraic operations which are addition of matrices, subtraction of matrices, and multiplication of matrices.

1. Addition of Matrices

The sum of two matrices of the same order/size is obtained by adding elements in the corresponding positions. Matrix of different sizes/order cannot be added.

Consider two matrices A & B of order 2x2. Then the sum is given by:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} \text{ then, } A+B = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Properties of matrix addition:

(a) Commutative Law: $A+B = B+A$

(b) Associative Law: $(A+B)+C = A+(B+C)$

(c) Identity of the matrix: $A+0 = 0+A = A$, where 0 is zero matrix.

(d) Additive Inverse: $A+(-A) = 0 = (-A)+A$, where $(-A)$ is obtained by changing the sign of every element of A which is additive inverse of the matrix.

2. Subtraction of Matrices

The difference of two matrices of same order/size ~~can be~~ is obtained by subtracting elements in the corresponding position.

Consider two matrices ~~of~~ A & B of order 2x2. Then the difference is given by

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} - \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}, B = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}. A - B = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

3. Multiplication of Matrices

If A and B are any two matrices, then their product ~~will be~~ AB will be defined only when the number of column in A is equal to the number of rows in B.

Example

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$$

find $A \times B$.

Date

Soln:

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+12 & -4+1-9 \\ 3-4+4 & -6-2-3 \\ -1+0+4 & 2+0-3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -12 \\ 3 & -11 \\ 3 & -1 \end{bmatrix}$$

$B \times A$ is not possible since number of columns of $B \neq$ number of rows of A.

Example 2

Q. Find the value of x and y if

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Soln:

$$\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

or $$\begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Equating the Corresponding elements

$2+y=5$
$\therefore y=3$

$2x+2=8$
or, $2x=6$
$\therefore x=3$

Hence, $x=3$ and $y=3$.

Q. find the value of a, b, c & d if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

Soln:

$a-b=-1 \quad -(1)$
$2a+c=5 \quad -(2)$
$2a-b=0 \quad -(3)$
$3c+d=13 \quad -(4)$

Sub Eqn (1) from (3)

$$\begin{array}{r} 2a - \cancel{b} = 0 \\ a - \cancel{b} = -1 \\ - \quad + \quad\quad + \\ \hline a \quad = 1 \end{array}$$

$\therefore a=1$

Page No.
Date

Putting $a=1$ in (2)

$2\times1+c=5$

or, $c=5-2$

$\therefore c=3$

Putting $c=3$ in (4)

$3\times3+d=13$

or, $d=13-9$

$\therefore d=4.$

$\therefore a=1, b=2, c=3, d=4.$

~~# Transpose of Matrix:~~

Inverse of matrix:

If A and B are two square matrices of the same order, such that $AB = BA = I$ (I = unit matrix).

Then B is called the inverse of A, i.e. $B = A^{-1}$ and A is the inverse of B. Where A is non-singular, i.e. $|A| \neq 0$.

Q. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ What is the inverse of A?

Solution:

By using the formula $A^{-1} = \dfrac{\text{adj} A}{|A|}$, we can obtain the value of A^{-1}.

We have

$$A_{11} = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} = -28 + 30 = 2$$

$$A_{12} = -\begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix} = -(-21 - 0) = 21$$

$$A_{13} = \begin{bmatrix} 3 & 4 \\ 0 & -6 \end{bmatrix} = -18 - 0 = -18$$

Page No.
Date

$$A_{21} = \begin{bmatrix} 0 & -1 \\ -6 & -7 \end{bmatrix} = -(0-6) = 6$$

$$A_{22} = \begin{bmatrix} 1 & -1 \\ 0 & -7 \end{bmatrix} = -7-0 = -7$$

$$A_{23} = \begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix} = -(-6-0) = 6$$

$$A_{31} = \begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix} = 0+4 = 4$$

$$A_{32} = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = -(5+3) = -8$$

$$A_{33} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = 4-0 = 4$$

$$\therefore \text{Adj of } A = \begin{bmatrix} 2 & 21 & 18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ 18 & 6 & 4 \end{bmatrix}$$

Also,

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4 \times (-7) - (-6) \times 5 - 3 \times (-6)\}$$

$$= -28 + 30 + 18 = 20$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Transpose of Matrix

The number of rows and columns in A is equal to number of columns and rows in B respectively. Thus, the matrix B is known as the transpose of the matrix, A. The transpose of matrix is represented by A' or A^T.

The following statement generalizes ~~the~~ transpose of a matrix:

If $A = (a_{ij})_{m \times n}$ then $A' = (a_{ij})_{n \times m}$

Thus, the transpose of a matrix is defined as ~~the~~ "A matrix which is formed by turning all the rows of a given matrix into columns and vice-versa.

Example:

Find the transpose of the given matrix:

$$M = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & -17 \\ 3 & 6 & 15 \\ 4 & 13 & 1 \end{bmatrix}$$

Soln:

Given a matrix of order 4×3.

$$M^T = \begin{bmatrix} 2 & 13 & 3 & 4 \\ -9 & 11 & 6 & 13 \\ 3 & -17 & 15 & 1 \end{bmatrix}$$

Properties of transpose of matrix:

(i) Transpose of the transpose matrix : $(A')' = A$

(ii) Addition Property of Transpose: $(A+B)' = A' + B'$

(iii) Multiplication by Constant: $(KA)' = KA'$

(iv) Multiplication by property of transpose:
$(AB)' = B'A'$

Page No.
Date

Power of Matrices

For a square matrix A and positive integer k, the k^{th} power of A is defined by multiplying this matrix itself repeatedly; that is,

$$A^k = A \times A \times \dots \times A$$

Where there are k copies of the matrix A.

Example

If $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$ find A^2 and A^3.

Soln:

$$A^2 = A \times A$$

$$= \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & -3-15 \\ 2+10 & -6+25 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -18 \\ 12 & 19 \end{pmatrix}$$

$$A^3 = A \times A^2$$

$$= \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \times \begin{pmatrix} -5 & -18 \\ 12 & 19 \end{pmatrix} = \begin{pmatrix} -41 & -75 \\ 50 & 59 \end{pmatrix}$$

Unit-4

Recursion on Sequence and Series

Page No.
Date

4.1. Introduction

Recursion is the process of starting with an element and performing a specific process to obtain the next term.

Sequence is an arrangement of any object or number in a particular order followed by some rule. If $a_1, a_2, a_3, a_4 \ldots$ denote the terms of the sequence, then $1, 2, 3, 4 \ldots$ denotes the position of the term.

A sequence can be defined based on the number of terms i.e. either finite sequence or infinite sequence.

Series can be highly generalized as the sum of all the terms in a sequence.
If $a_1, a_2, a_3, a_4 \ldots$ is a sequence, then the corresponding series is given by
$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

If we go with the definition of a recursive sequence, then both arithmetic sequences and geometric sequences are also recursive. Why?
In ~~geom~~ arithmetic sequence, each term is obtained by adding a specific number to

the previous term.

In a geometric sequence, each term is obtained by multiplying the previous terms by a specific number.

If a sequence is recursive, we can write recursive equations for the sequence. Recursive equations usually come in pairs: the first equation tells us what the first term is, and the second equation tells us how to get the n^{th} term in relation to the previous term (or terms).

Example 1: Write the recursive equations for the sequence 5, 7, 9, 11 ...

Solution: The first term of the sequence is 5, and each term is 2 more than the previous term, so our equations are:

$a_1 = 5$

$a_n = a_{n-1} + 2$ for $n > 1$.

Example 2: Write the recursive equation for the sequence 2, 4, 8, 16 ...

Solution: The first term is 2, and each term after that is twice the previous term, so the equations are:

$a_1 = 2$

$a_n = 2a_{n-1}$, for $n > 1$.

Page No.
Date

4.2 Types of Sequences and Series

(A) Arithmetic Sequences (Progressions)
(B) Geometric Sequences / Progression
(C) Harmonic Sequences / Progression

A. Arithmetic Sequence / Progression

A sequence in which every term is created by adding or subtracting a definite number to the preceding number is an arithmetic sequence.

An arithmetic progression is a sequence of the form,

$a, a+d, a+2d, \dots a+nd \dots$

Where the initial term 'a' and the common difference 'd' are real numbers.

Questions

1(a) If 4, 7, 10, 13, 16, 19, 22 ... is a sequence, find
i) Common difference
ii) n^{th} term
iii) 21^{st} term

Soln:

Given sequence is: 4, 7, 10, 13, 16, 19, 22 ...

i) Common difference $= 7-4 = 3$

ii) n^{th} term, $T_n = a+(n-1)d$
$= 4+(n-1)3 = 4+3n-3 = 3n+1$

iii) 21^{st} term: $T_{21} = 4+(21-1)\times 3 = 4+60 = 64$

2. Find the 35^{th} term in the arithmetic sequence 3, 9, 15, 21 ...

Soln:

first term $= a_1 = 3$

Common difference $= d = 6$

term position $= n = 35$

We have,

$a_n = a_1 + (n-1)d$

$a_{35} = 3 + (35-1) \times 6$

$= 3 + 34 \times 6$

$= 3 + 204 = 207$

$\therefore$ 35^{th} term is 207.

3. Find the 125^{th} term in the arithmetic sequence 4, -1, -6, -11 ...

Soln:

first term $= a = 4$

difference $= d = -5$

term position $= n = 125$

We have,

$a_n = a + (n-1)d$

$= 4 + (125-1) \times -5$

$= 4 - 124 \times 5$

$= 4 - 620$

$= -616$

$\therefore$ 125^{th} term is -616.

Page No.
Date

4. Given two terms in the arithmetic sequence, $a_5 = -8$ and $a_{25} = 72$; find the 100th term (a_{100}).

Soln:

~~Given~~ We have

$a_5 = -8$ $\qquad\qquad$ $a_{25} = 72$

or, $a_1 + 4d = -8$ — (i) $\qquad$ or, $a_1 + 24d = 72$ — (ii)

Eqn (i) − Eqn (ii)

$$\cancel{a_1} + 4d = -8$$
$$\underline{\cancel{a_1} + 24d = 72}$$
$$-20d = -80$$

$\therefore d = 4$

Common difference $(d) = 4$

~~Now,~~

~~100th term (a_{100}) = $a_1 + 99d$~~

from eqn (i)

$a_1 + 4d = -8$

or, $a_1 + 4 \times 4 = -8$

or, $a_1 = -8 - 16$

$\therefore a_1 = -24$

now,

100th term $= a_{100} = a_1 + 99d$
$= -24 + 99 \times 4$
$= -24 + 396$
$= 372$

$\therefore$ 100th term is 372.

~~B. Geometric Progression / Sequence~~

5. Find the sum of first 30 multiples of 4.

Soln:

Given,

$a = 4$

$n = 30$

$d = 4$

We know

$S = n/2 [2a + (n-1) \times d]$

$= 30/2 [2 \times 4 + (30-1) \times 4]$

$= 15 [8 + 29 \times 4]$

$= 15 \times 124$

$S = 1860$

The arithmetic mean between a and b is $A.M = [a+b]/2$

Series of A.P.: $S_n = [n/2](a+l)$, $S_n = (n/2)[2a + (n-1)d]$, where S_n is the sum of n terms of A.P.

B. Geometric Sequence / Progression

A sequence in which every term is obtained by multiplying or dividing a definite number with preceding numbers is known as a geometric sequence.

A geometric progression is a sequence of the form

$a, ar, ar^2 \ldots ar^n \ldots$

Where the initial term a and the common ratio 'r' are real numbers.

Questions

1. Consider the sequence 1, 4, 16, 64, 256, 1024...
Find the common ratio and 9th term.

Soln: $a = 1$

Common ratio $(r) = 4/1 = 4$

$n = 9$

Now, We have, $T_n = ar^{(n-1)}$

9th term, $T_9 = 1 \times (4)^{9-1} = 4^8 = 65536$

Date

2. Find the n^{th} term and 26^{th} term of the geometric sequence with $a_5 = 5/4$ and $a_{12} = 160$.

Soln:

Given

$a_5 = 5/4$ $a_{12} = 160$

~~or, $ar^{11} = 160$ — (ii)~~

~~or, $ar^4 = 5/4$ — (i)~~

We know,

$a_{12} = a_5 \times r^7$ (We can get from 5th to 12th term by multiplying the fifth term by the common ratio seven times)

or, $160 = \frac{5}{4} \times r^7$

or, $r^7 = 128$

$\therefore r = 2$

Also, we know

$a_5 = ar^4$

or, $\frac{5}{4} = a \times 2^4$

or, $\frac{5}{4} = a \times 16$

or $a = \frac{5}{16 \times 4}$

$\therefore a = \frac{5}{64}$

We have,

n^{th} term $= a_n = a \cdot r^{n-1}$

$= \frac{5}{64} \times 2^{n-1}$

$= \left(\frac{5}{2^6}\right)(2^{n-1})$

$= 5 \times 2^{-6} \times 2^{n-1}$

$= 5 \times 2^{n-7}$

26^{th} term $= a_{26} = 5(2^{26-7})$

$= 5 \times 2^{19}$

$= 2621440.$

3. ~~Sum~~ Find the sum of first 4 terms of the geometric sequence 10, 30, 90, 270, 810, 2430 ....

Soln:

$a = 10$ (first term)

$r = \frac{30}{10} = 3$ (Common ratio)

$n = 4$ (we want to sum the first 4 terms)

So, by formula, we have

$$\sum_{k=0}^{n-1} (ar^k) = a\left(\frac{1-r^n}{1-r}\right)$$

becomes

Date

$$\sum_{k=0}^{4-1} (10 \times 3^k) = 10\left(\frac{1-3^4}{1-3}\right) =$$

$$= 10\left(\frac{1-81}{-2}\right)$$

$$= 10 \times \frac{\cancel{-}80}{\cancel{-}2}$$

$$= 400.$$

We can check it:

$10 + 30 + 90 + 270 = 400.$

The geometric mean between a and b is G.M. $= \pm\sqrt{ab}$

Series of G.P.: $S_n = [a(1-r^n)] / [1-r]$; where S_n is the sum to n terms of G.P.

The sum ~~of~~ 'S' of infinite geometric series is $S = a / 1-r$.

C. Harmonic Progression / Sequence

A series of numbers is said to be in harmonic sequence if the reciprocals of all the elements of the sequence form an arithmetic sequence.

OR, A sequence obtained by the reciprocals of the elements of the arithmetic sequence.

The n^{th} term of harmonic sequence is :

$a_n = 1/[a+(n-1)d]$

The harmonic mean between a and b is

$H.M = 2ab/[a+b]$

Harmonic sequence, in mathematics, a sequence of numbers $a_1, a_2, a_3 \ldots$ such that their reciprocals $1/a_1, 1/a_2, 1/a_3 \cdots$ form an arithmetic sequence (numbers separated by a common difference).

Infinite harmonic progressions are not summable.

Example of harmonic sequence:

(i) $1, 1/2, 1/3, 1/4, 1/5, 1/6 \ldots$

(ii) $30, -30, -10, -6, \ldots$

~~Questions:~~

Harmonic mean is calculated as the reciprocal of the arithmetic mean of the reciprocals. The formula to calculate harmonic mean is given by:

$H.M = n / [(1/a) + (1/b) + (1/c) + (1/d) + \dots]$

Where, a, b, c, d are the values and n is the number of values present.

Questions:

1. Determine the 4th and 8th term of the harmonic progression 6, 4, 3 ...

Soln:

$H.P. = 6, 4, 3 \dots$

Let us take A.P. from the given ~~G~~ H.P.

$A.P = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$

Here, $T_2 - T_1 = T_3 - T_2 = \frac{1}{12} = d$

So, in order to find the 4th term of an A.P. ~~using~~ use the formula.

The nth term of A.P. $= a + (n-1)d$

Here, $a = \frac{1}{6}$, $d = \frac{1}{12}$

Now, we have to find the 4th term
So take, $n = 4$
Now, put the values in the formula.

4th term of A.P $= (1/6) + (4-1)(1/12)$

$= 1/6 + \cancel{3} \times \frac{1}{\cancel{12}_4}$

$= \frac{1}{6} + \frac{1}{4}$

$= \frac{5}{12}$

Similarly, 8th term of A.P $= a + 7d$

$= 1/6 + 7 \times \frac{1}{12}$

$= \frac{1}{6} + \frac{7}{12}$

$= \frac{9}{12}$

Since, H.P. is the reciprocal of A.P., we can write the values as:

4th term of H.P. $= 1/$4th term of an A.P
$= 12/5$

8th term of H.P. $= 1/$8th term of an A.P.
$= 12/9 = 4/3$

Page No.
Date

Date

2. Compute the 16th term of H.P. if the 6th and 11th term of H.P are 10 and 18, respectively.

Soln:

The H.P. is written ~~as~~ ~~A.P~~ in terms of A.P are given below:

6th term of A.P. $= a + 5d = 1/10$ — ①

11th term of A.P. $= a + 10d = 1/18$ — ②

Eqn ① − Eqn ②

$$a + 5d = 1/10$$

$$a + 10d = 1/18$$

$$- \quad - \qquad -$$

$$-5d = \frac{1}{10} - \frac{1}{18}$$

or, $-5d = \frac{9-5}{90}$

or, $-5d = \frac{4}{90}$

or, $d = -\frac{2}{225}$

Putting value of d in eqn ①

$a = 13/90$

Now,

16th term of A.P $= a+15d$

$$= \frac{13}{90} + 15 \times \frac{-2}{225}$$

$$= \frac{13}{90} - \frac{30}{225}$$

$$= \frac{65-60}{450}$$

$$= \frac{5}{450}$$

$$= \frac{1}{90}$$

Thus, 16th term of HP $= 1/$16th term of A.P

$$= 90$$

Relation between AP, GP and HP

For any two numbers, if A.M, G.M and H.M. are the Arithmetic, Geometric and Harmonic mean respectively, then the relationship between these three is given by:

- $G.M.^2 = A.M. \times H.M$, where A.M., G.M., H.M are in G.P.

- $A.M. \geq G.M. \geq H.M$

Recurrance Relations

A recurrance relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

A recurrence relation is a functional relation between the independent variable x, dependent variable $f(x)$ and the differences of various order of $f(x)$. A recurrance relation is also called a difference equation, and we will use these two terms interchangeably.

Example 1: The equation $f(x+3h)+3f(x+2h)+6f(x+h)+9f(x)=0$ is a recurrence relation.

It can also be written as:

$$a_{r+3}+3a_{r+2}+6a_{r+1}+9a_r=0$$

$$y_{k+3}+3y_{k+2}+6y_{k+1}+9y_k=0$$

Example 2: Fibonacci Series:

$$F_n = F_{n-1}+F_{n-2}$$

$a_r = a_{r-2}+a_{r-1}$, $r \geq 2$, with the initial conditions $a_0 = 1$ and $a_1 = 1$.

Tower of Hanoi - $F_n = 2F_{n-1} + 1$

Linear Recurrance Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format:

$$x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \cdots A_k x_{n-k}$$

(A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

There are some examples of linear recurrance relations.

Recurrance relations	Initial Values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

Summations:

Use of Series on Summation notation

A series can be represented in a compact form, called summation or sigma notation. The Greek capital letter, Σ, is used to represent the sum.

The series $4+8+12+16+20+24$ can be expressed as $\sum_{n=1}^{6} 4n$. The expression is read as the sum of $4n$ as n goes from 1 to 6. The variable n is called the index of summation.

Last value of n

$$\sum_{n=1}^{6} 4n$$

formula for the terms

Index of summation

first value of n

To generate the terms of ~~the~~ a series given in sigma notation, successfully replace the index of summation with consecutive integers from the first value to last value of the index.

To generate the terms of the series given in sigma notation above, replace n by 1, 2, 3, 4, 5 & 6.

$$\sum_{n=1}^{6} 4n = 4(1)+4(2)+4(3)+4(4)+4(5)+4(6)$$
$$= 4+8+12+16+20+24$$
$$= 84$$

The sum of the series is 84

Use sigma notation to express each series.

1.) $8+11+14+17+20$

Soln:

This is arithmetic series with five terms whose first term is 8 and whose common difference is 3. Therefore, $a_1 = 8$ and $d = 3$

The n^{th} term of the corresponding sequence is

$$a_n = a_1 + (n-1)d$$
$$= 8 + (n-1)3$$
$$= 8 + 3n - 3$$
$$= 3n + 5$$

Since there are five terms, the given series can be written as:

$$\sum_{n=1}^{5} a_n = \sum_{n=1}^{5} (3n+5).$$

2) $\frac{2}{3} - 1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{8} - \frac{81}{16}$

This is a geometric series with six terms, where $a_1 = \frac{2}{3}$ and $r = -\frac{3}{2}$.

The n^{th} term of the sequence is

$a_n = a_1 r^{n-1}$

$= \frac{2}{3} \times \left(-\frac{3}{2}\right)^{n-1}$

$= \cancel{\frac{2}{3}}$

Since, there are six terms in the given series, the sum can be written as:

$$\sum_{n=1}^{6} a_n = \sum_{n=1}^{6} \frac{2}{3}\left(-\frac{3}{2}\right)^{n-1}$$

Remaining of 4.2

Sum of the terms of an Arithmetic sequence (Arithmetic series)

To find the sum of first n terms of an arithmetic sequence, we the formula:

$S_n = \frac{n(a_1 + a_n)}{2}$ where, S_n is the number of terms, a_1 is the first term and a_n is the last term.

Examples:

1) Find the sum of first 20 terms of the ~~sequence~~ arithmetic series if $a_1 = 5$ and $a_{20} = 62$.

Soln:

$a_1 = 5$
$a_{20} = 62$

$S_n = \frac{n(a_1 + a_n)}{2}$

or $S_{20} = \frac{20(5+62)}{2} = 670.$

2) Find the sum of first 40 terms of the arithmetic sequence. 2, 5, 8, 11, 14...

Soln: $a_1 = 2$, $d = 3$

first find the 40th term

$a_{40} = a_1 + 39d$
$= 2 + 39 \times 3 = 119$

Now the sum is:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$\Rightarrow S_{40} = \frac{40(2+119)}{2} = 2420.$$

3) Find the sum:

$$\sum_{k=1}^{50} (3k+2)$$

Soln:

first find a_1 and a_{50}

$a_1 = 3(1) + 2 = 5$

$a_{50} = 3(50) + 2 = 152$

Then find the sum,

$$S_k = \frac{k(a_1 + a_k)}{2}$$

$$\Rightarrow S_{50} = \frac{50(5+152)}{2}$$

$$= 3925$$

Sum of the terms of a Geometric Sequence (Geometric Series)

To find the sum of the first n terms of a geometric sequence, use the formula:

$$S_n = \frac{a_1(1-r^n)}{1-r}, \quad r \neq 1,$$

Where n is the number of terms, a_1 is the first term and r is the common ratio.

Examples

1) Find the sum of first 8 terms of the geometric series if $a_1 = 1$ and $r = 2$

Soln:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$\text{or } S_8 = \frac{1(1-2^8)}{1-2} = 255.$$

2) Find ~~the~~ S_{10} of the geometric series:

$24 + 12 + 6 + \cdots$

Soln:

$a_1 = 24$

$r = \frac{r_2}{r_1} = \frac{12}{24} = \frac{1}{2}$

Now, the sum is

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Page No.
Date

or $S_{10} = \frac{24\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = \frac{3069}{64}$

3) Evaluate:

$$\sum_{n=1}^{10} 3(-2)^{n-1}$$

Soln:

(You are finding S_{10} of the series $3 - 6 + 12 + -24 + \dots$ where common ratio is -2.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{3\left[1-(-2)^{10}\right]}{1-(-2)} = \frac{3(1-1024)}{3} = -1023$$

4.3 Solutions for recursive relations

How to solve linear recurrence relations

Suppose, a two ordered linear recurrence relation is - $F_n = AF_{n-1} + BF_{n-2}$, where A and B are real numbers.

The characteristic equation for the above recurrence relation is :

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots:

Case 1 → If this equation factors as $(x-x_1)(x-x_1)=0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here a & b are constants].

Case 2 → If this equation factors as $(x-x_1)^2=0$ and it produces single real root x_1, then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 → If this equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then

$F_n = r^n(a\cos(n\theta) + b\sin(n\theta))$ is the solution.

Page No.
Date

Questions:

1) Solve the recurrence relation ~~$F_n = 5F_{n-1}$~~
$F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$.

Soln:

The characteristic equation of the recurrence relation is -

$x^2 - 5x + 6 = 0$,

So, $(x-3)(x-2) = 0$

Hence, the roots are $x_1 = 3$ and $x_2 = 2$

The roots are real and distinct, so, this is in the form of case 1.

Hence, the solution is

$$F_n = a x_1^n + b x_2^n$$

Here,

$F_n = a3^n + b2^n$ (As $x_1 = 3$ and $x_2 = 2$)

Therefore,

$1 = F_0 = a3^0 + b2^0 = a + b$

$4 = F_1 = a3^1 + b2^1 = 3a + 2b$

Solving these two equations, we get

$a = 2$ and $b = -1$

Hence, the final solution is:

$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$

2) Solve the recurrence relation,

$F_n = 10F_{n-1} - 25F_{n-2}$ Where $F_0 = 3$ and $F_1 = 17$.

Soln:

The characteristic equation of the recurrence relation is:

$x^2 - 10x - 25 = 0$

So, $(x-5)^2 = 0$

Hence, the single real root $x_1 = 5$.

As there is single ~~val~~ real valued root, this is in the form of Case 2.

Hence, the Solution is:

$F_n = ax_1^n + bnx_1^n$

$3 = F_0 = a.5^0 + (b)(0 \times 5)^0 = a$

$17 = F_1 = a.5^1 + b.1 \cdot 5^1 = 5a + 5b$

Solving these two equations, we ~~get~~ get $a = 3$ and ~~$b = 2/5$~~

$b = 2/5$.

Hence, the final solution is $F_n = 3.5^n + (2/5).n.2^n$.

3) Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ Where $F_0 = 1$ and $F_1 = 3$

Soln.

The characteristic equation of the recurrence relation is:

$x^2 - 2x - 2 = 0$

Hence, the roots are:

$x_1 = 1+i$ and $x_2 = 1-i$

Date

In polar form,

$x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$,

where $r = \sqrt{2}$ and $\theta = \pi/4$

The roots are imaginary. So this is in the form of case 3.

Hence, the solution is

$$F_n = (\sqrt{2})^n \left(a\cos(n.\pi/4) + b\sin(n.\pi/4)\right)$$

$$1 = F_0 = (\sqrt{2})^0 \left(a\cos(0.\pi/4) + b\sin(0.\pi/4)\right) = a$$

$$3 = F_1 = (\sqrt{2})^1 \left(a\cos(1.\pi/4) + b\sin(1.\pi/4)\right) = \sqrt{2}\left(a/\sqrt{2} + b/\sqrt{2}\right)$$

Solving these two equations, we get

$a = 1$ and $b = 2$.

Hence, the final solution is

$$F_n = (\sqrt{2})^n \left(\cos(n.\pi/4) + 2\sin(n.\pi/4)\right).$$

4.4 Recursive algorithm, recursion and iteration, the merge sort.

Recursive Algorithm

A recursive algorithm is an algorithm which calls itself with "smaller (or simpler)" input values, and which obtains the result for the current input by applying simple operations to the returned value for the smaller (or simpler) input.

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

For the algorithm to terminate, the instance of the problem must ~~be~~ eventually be reduced to some initial case for which the solution is known.

A recursive algorithm is one in which objects are defined in terms of other objects of the same type.

Some recursive algorithms principles are ~~explained~~ given below:

1) A recursion algorithm must have a base case.
2) It is a condition that allows the algorithm to stop recursion.
2) A recursion algorithm must change ~~its~~ state

and move towards the base case.

3) A recursive algorithm must call itself.

Examples of recursive algorithms:

1. Factorial
2. Fibonacci Sequence
3. GCD
4. Tower of Hanoi

1) Recursive factorial Algorithm

Q. Give a recursive algorithm for computing n! where n is a non-negative integer.

Solution:

1) Input a non-negative integer n.

~~2) If n=0, then declare n! = 1~~

2) If n=0, then return 1

3) Else return n * factorial(n-1)

4) Output is n!

5) Stop.

2) Fibonacci Sequence Algorithm

1) Input a non-negative integer n
2) If n=0 then fibo(0)=0
3) ~~Else~~ If n=1 then fibo(1)=1
4) Else fibo(n) = fibo(n-1) + fibo(n-2).
5) ~~Output~~ Stop.

3) GCD recursion algorithm

1) Input non-negative numbers a, b where a<b.
2) if a=0 then return b
3) Else return gcd(b mod a, a)
4) output is gcd(a,b)
5) Stop.

4) Tower of Hanoi using recursion.

Tower of Hanoi is a game played with three poles and a number of different sized disks. Each disk has a hole in the center that makes easy to place into poles.

Initial state

a) There are three poles named: origin, intermediate and destination.

b) 'n' numbers of different sized disks, having

hole at the center is stacked around the origin pole in decreasing order.
c) The disks are numbered by 1,2,3,4...n

Rules

a) Move only one disk at a time.
b) Each disk must always be placed around one of the pole.
c) Never place larger disk on the top of smaller disk.

Algorithm

To move a tower of 'n' disks from origin to destination pole (where 'n' is positive integers)

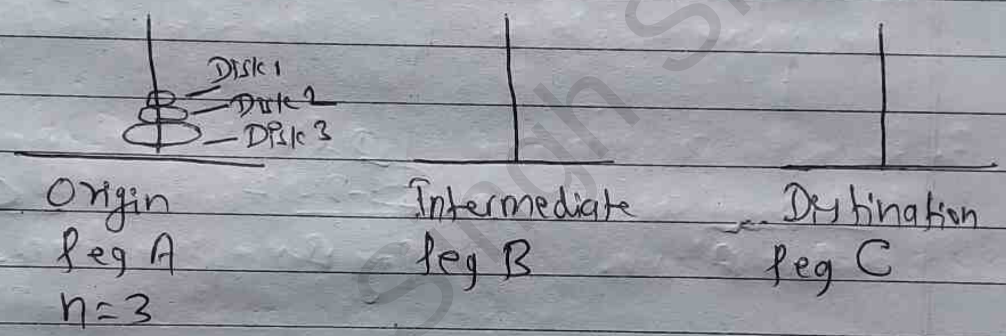
1) If n=1, move a single disk from origin to destination pole.

2) If n>1
a) Let intermediate be the remaining pole other than origin and destination.
b) Move tower of (n-1) disks from origin to intermediate.
c) Move single disk from origin to destination.
d) Move a tower of (n-1) disks from intermediate to destination.

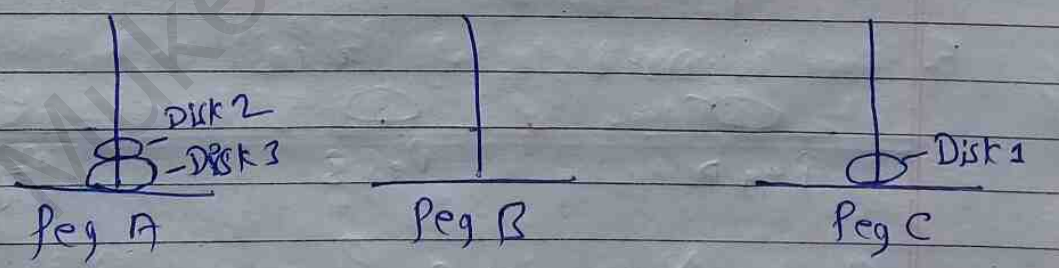
Solution to Tower of Hanoi problem when number of disks 'n' = 3.

Let the origin, intermediate and destination pole or peg be denoted as peg 'A', peb 'B' and peg 'C' respectively. Let disks be denoted as Disk 1, Disk 2, Disk 3.

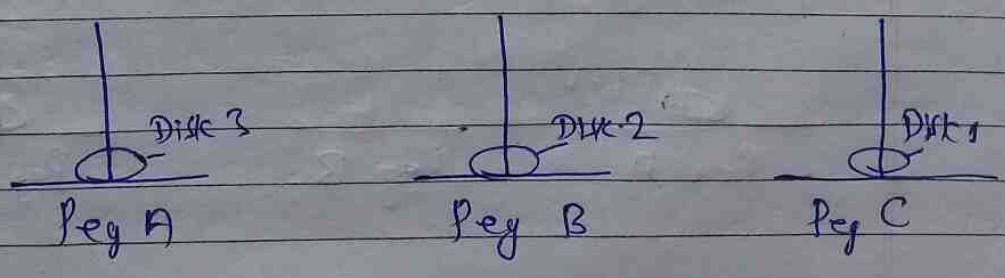
Step 1:



Step 2: Move disk 1 from peg A to peg C.

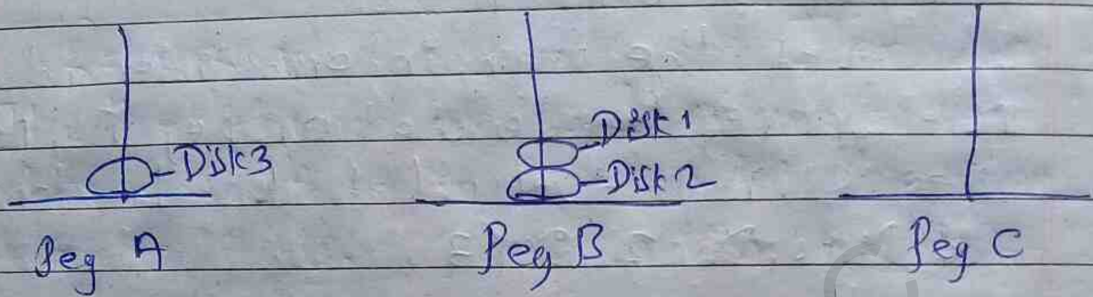


Step 3: Move disk 2 from peg A to peg B.

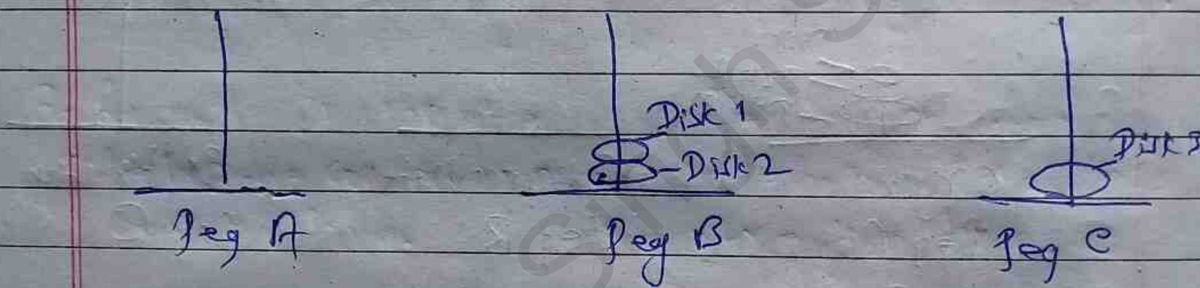


Page No. ______
Date ______

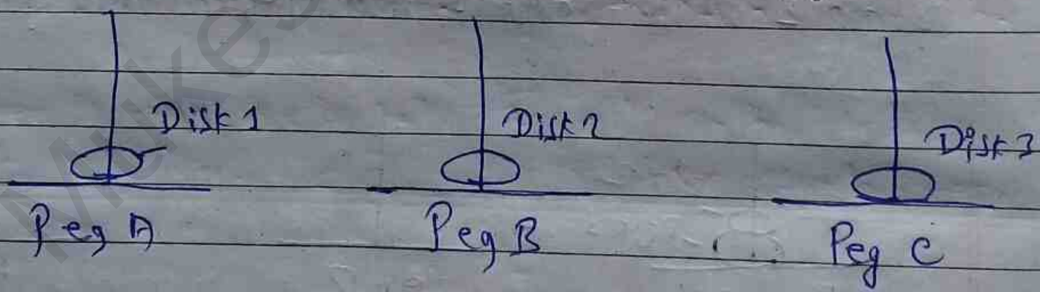
Step 4: Move disk 1 from peg 'C' to peg B.



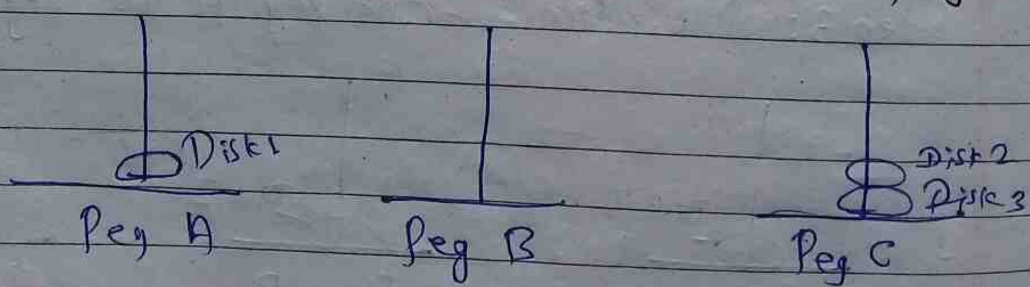
Step 5: Move disk 3 from peg A to peg C.



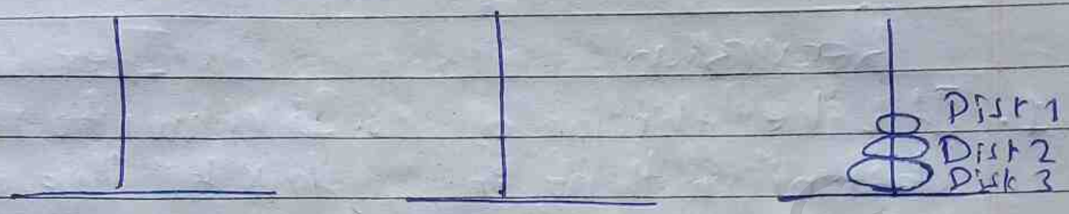
Step 6: Move disk 1 from peg B to peg A.



Step 7: Move disk 2 from peg B to peg C



Step 8: Move disk 1 from Peg A to peg C



Tricks for Tower of Hanoi

$2^n - 1$

for 1 disk, $2^1 - 1 = 1$ move

for 2 disks, $2^2 - 1 = 3$ moves

for 3 disks, $2^3 - 1 = 7$ moves

for 4 disks, $2^4 - 1 = 15$ moves

Date

Recursion and Iteration

Recursion	Iteration
1. It is the technique of defining anything in terms of itself.	1. It allows for repeat the task till the given condition is satisfied.
2. All problems may not have recursive solution.	2. Any recursive problems can be solve iteratively.
3. Recursive programming technique conserves more execution time.	3. Iterative programming techniques takes less execution time.
4. Recursive technique is top-down approach.	4. Iterative technique is bottom-up approach.
5. Given program is divided into modules.	5. It constructs the solution step by step.

The merge sort

It is a sorting algorithm based on divide and conquer technique. Algorithm divides the array into two equal halves, recursively sorts them and finally merges them in sorted order.

Example

[14 | 33 | 27 | 10 | 35 | 19 | 42 | 44]

[14 | 33 | 27 | 10]   [35 | 19 | 42 | 44]

[14 | 33]   [27 | 10]   [35 | 19]   [42 | 44]

[14]   [33]   [27]   [10]   [35]   [19]   [42]   [44]

[14 | 33]   [10 | 27]   [19 | 35]   [42 | 44]

[10 | 14 | 27 | 33]   [19 | 35 | 42 | 44]

[10 | 14 | 19 | 27 | 33 | 35 | 42 | 44]

Recursively defined functions

A recursively defined function is a function whose definition refers back to itself. A classic example of such a function is the factorial;

$n! = 1.2.3 \ldots n,$

Which can be defined recursively as follows:

$0! = 1$

$n! = n(n-1)!$ for all $n \in N$.

A recursive definition has two parts:

① Definition of the smallest argument (usually $f(0)$ or $f(1)$)

② Definition of $f(n)$, given $f(n-1)$, $f(n-2)$ etc.

Example 1

Here is an example of a recursively defined function.

$$\begin{cases} f(0) = 5 \\ f(n) = f(n-1) + 2 \end{cases}$$

We can calculate the values of this function:

$f(0) = 5$

$f(1) = f(0) + 2 = 5+2 = 7$

$f(2) = f(1) + 2 = 7+2 = 9$

$f(3) = f(2) + 2 = 9+2 = 11$

This recursively defined function is equivalent to the explicitly defined function $f(n) = 2n+5$. However, the recursive function is defined only for non-negative integers.

Example 2

Here is one more example of a recursively defined function

$$\begin{cases} f(0) = 1 \\ f(n) = n \cdot f(n-1). \end{cases}$$

The values of this function are:

$$f(0) = 1$$
$$f(1) = 1 \cdot f(0) = 1 f_1 = 1$$
$$f(2) = 2 f(1) = 2 f_1 = 2$$
$$f(3) = 3 f(2) = 3 f_2 = 6$$
$$f(4) = 4 f(3) = 4 f_3 = 24$$
$$f(5) = 5 f(4) = 5 f_4 = 120$$

This is the recursive definition of the factorial function, $f(n) = n!$

Not all recursively defined functions have an explicit definition.

The Fibonacci Numbers

One special recursively defined function which has no simple explicit definition yields the Fibonacci numbers.

$$\begin{cases} f(1) = 1 \\ f(2) = 1 \\ f(n) = f(n-2) + f(n-1) \end{cases}$$

The values of this function are:

$f(1) = 1$
$f(2) = 1$
$f(3) = 1+1 = 2$
$f(4) = 1+2 = 3$
$f(5) = 2+3 = 5$
$f(6) = 3+5 = 8$
$f(7) = 5+8 = 13$
$f(8) = 8+13 = 21$
$f(9) = 13+21 = 34$

Then the sequence of Fibonacci numbers is 1, 1, 2, 3, 5, 8, 13, 21, 34.

Unit-5
Special Types of Functions

5.1 Floor and Ceiling Function

The floor and ceiling function give us the nearest integer up or down.

Let x be a real number.
The floor function of x, denoted by $[x]$ or floor (x), is defined to be the greatest integer that is less than or equal to x.

The ceiling function of x, denoted by $[x]$ or ceil$[x]$, is defined to be the least ~~number~~ integer that is greater than or equal to x.

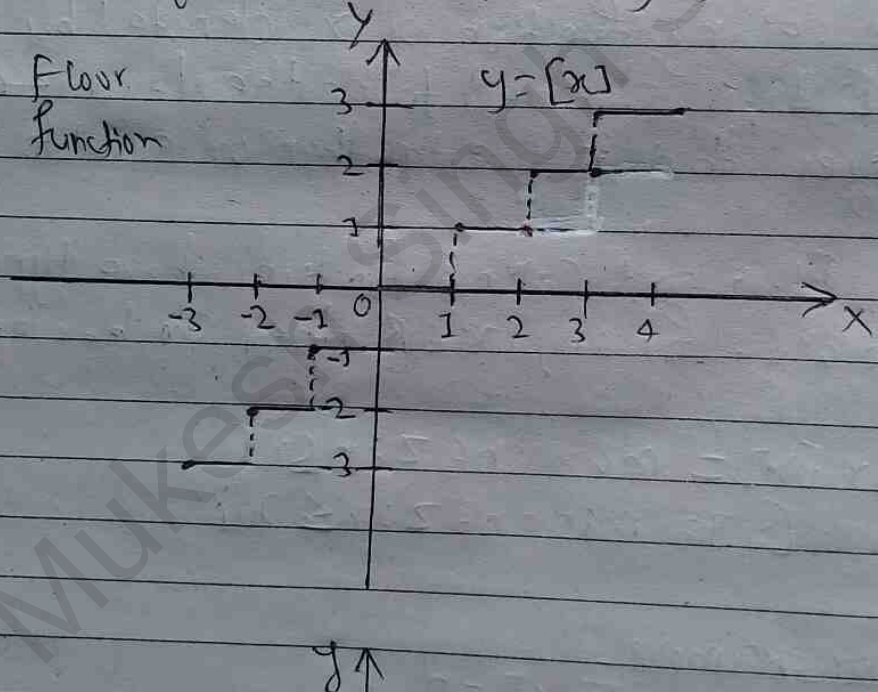
Floor and Ceiling functions have type $R \to Z$.
Formally, for any $x \in R$ they can be defined as:

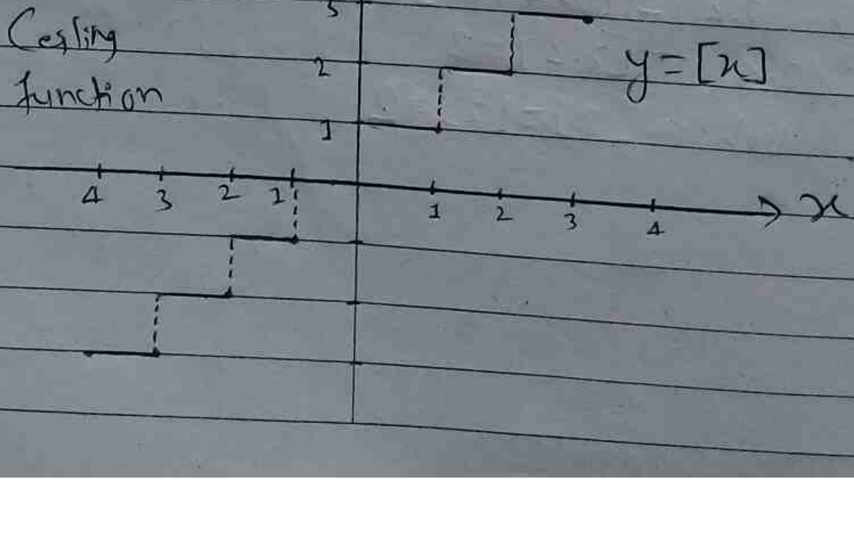
floor: $[x] = \max \{n \in Z : n \leq r\}$
ceiling: $[x] = \min \{n \in Z : n \leq x\}$.

For example
floor $(2.4) = [2.4] = 2$
ceil $(2.4) = [2.4] = 3$
while $[2] = [2] = 2$

x	floor	Ceiling
-1.1	-2	-1
0	0	0
1.01	1	2
2.9	2	3
3	3	3

Graphs of floor and ceiling functions:





Page No.
Date

Properties of floor and ceiling function

1) $[x] = n$ iff $n \le x < n+1$
2) $[x] = n$ iff $n-1 < x \le n$
3) $[x] = n$ iff $x-1 < n \le x$
4) $[x] = n$ iff $x \le n < x+1$
5) $[-x] = -[x]$

~~6) [x] =~~

6) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in Z \\ -1 & \text{if } x \notin Z \end{cases}$

7) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in Z \\ 1 & \text{if } x \notin Z \end{cases}$

8) $[x+n] = [x] + n$

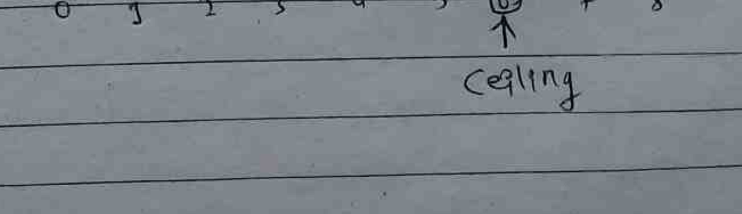
~~9) [x+n] =~~

Q. Evaluate the ceiling function of ~~the graph~~ 5.39 with the help of the graph.

~~Ceil [5.39] =~~

Ceil $(5.39) = [5.39] = 6$

It can be demonstrated on the number line as below:



Page No.
Date

5.2 Characteristics functions

A characteristic function is a function defined on set X that indicates membership of an element in a subset A of X, having the value 1 for all elements of X in A and the value 0 for all elements of X not in A.

Mathematical defn:

(I) Let E be a given set and $A \subseteq E$ then Characteristic function χ_A of set A is a real ~~number~~ valued function defined on E by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

or

Let A be a subset of universal set $U = \{u_1, u_2 \ldots u_n\}$. Then the characteristic function of A is defined as a function from U to $\{0, 1\}$. It is defined as;

$$\chi_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$$

where u_i is the variable which can take any element from U.

[χ_A = kia of A]

Example:

If $A = \{4, 7, 9\}$ and $U = \{1, 2, 3 \ldots 10\}$ then find the value of:

① $\chi_A(2) = 0$ since $2 \notin A$

② $\chi_A(4) = 1$ since $4 \notin A$

③ $\chi_A(7) = 1$ since $7 \in A$

④ $\chi_A(12) =$ undefined since $12 \notin U$.

Page No.
Date

5.3 Integer Value functions

An integer valued function is a function whose value are integers. In other words, it is a function that assigns an integer to each member of its domain.

Let x be any real number. Then, the integer value of x written $INT(x)$ converts x into an integer by ~~detect~~ deleting the factorial part of the number.

Floor and Ceiling functions are examples of integer valued functions.

Example

$x \rightarrow$ real number

Then, $Int(x) = Int(2) = 1.2 \rightarrow 2 \rightarrow 1$

$INT(x) = [x]$ if $x > 0$ (positive) $\rightarrow$ floor

$INT(x) = [x]$ if $x < 0$ (negative) $\rightarrow$ ceiling

$INT(4.10) = 4$

$INT(-7.5) = -7$

5.4 Remainder function ; modular arithmetic

Let a be any integer and b be a positive integer then $a \pmod b$ will denote the integer remainder when a is divided by b.

Specifically, $a \pmod b$ is the unique integer r such that $a = bq + r$ where $0 \le r < b$

When a is positive, simply divide a by b to obtain r.

Examples

1. $26 \pmod 7 = 5$
2. $34 \pmod 8 = 2$
3. $2345 \pmod 6 = 5$
4. $-26 \pmod 7 = 2$
5. $-371 \pmod 8 = 5$
6. $-39 \pmod 3 = 0$

Rough

$a = bq + r$

or, $26 = 7 \times 3 + 5$

$-26+7 = -19+7 = -12+7 = -5$

$-5+7 = 2$

$\rightarrow |26| = 26/7 = 5 \rightarrow 7-5 = 2$

$|371| = 371/8 = 3 \rightarrow 8-3 = 5$

$\rightarrow |39| = 39/3 = 13 \rightarrow 0-0 = 0$

a is positive in 1, 2, 3 so simply divide a by b to obtain remainder r. Then $r = a \pmod b$

Page No.
Date

~~If~~ Since a is negative in 4,5,6 then divide $|a|$ by b to obtain remainder r & $a (\bmod b) = b - r'$ when $r' \neq 0$.

Note → The term "mod" is also used for the mathematical congruence relation, which is denoted and defined as follows:

$a \equiv b \ (\bmod M)$ and

Arithmetic Modulo (M):

Modulo is the remainder or the number that is left after a number is divided by another value.

modulo arithmetic as remainders:

$15 \equiv -9 \ (\bmod 12)$

$15 (\bmod 12) = 3$ | $-9 + 12 = 3 < 12$

$-9 (\bmod 12) = 12 - 9 = 3$

$a \equiv b (\bmod M)$

$a (\bmod M) = b (\bmod M)$

5.5 Factorial function

Factorial function is a mathematical formula represented by an exclamation mark, "!". In the factorial formula you must multiply all the integers and positives that exist between the number that appears in the formula and the number 1.

Example

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040.$

Mathematical defn

The product of the positive integers from 1 to n, inclusive is denoted by n! (read as n factorial)

i.e. $n! = n(n-1)(n-2)(n-3)\ldots 3.2.1$.

Note : if $n=0$ then $n! = 1$

if $n>0$ then $n! = n(n-1)!$

This definition is recursive and that the function is well defined.

Page No.
Date

Example: Let $A = \{a, b, c, d\}$ then $n(A) = |A| = 4$ is cardinality of set A.

for eg. Calculate 4! using recursive definition.

Soln:

(1) $4! = 4.3!$

$3! = 3 \times 2!$

$2! = 2 \times 1!$

$1! = 1 \times 0!$

$0! = 1$

$1! = 1 \times 1 = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 = 6$

$4! = 4 \times 6 = 24$

Directly:

$3! = 3 \times 2 \times 1 = 6$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Unit-6

Geometric Transformation

Page No.
Date

A geometric transformation is any bijection of a set to itself with some salient geometrical underplanning.

Transformation means to change. Hence, a geometric transformation would mean to make some changes in any given geometric shape.

6.1 Geometric properties of a plane linear transformation

Let V and W be vector spaces. The function $T: V \to W$ is called a linear transformation of V into W if the following two properties are true for all u and v in V and for any scalar c.

1) $T(u+v) = T(u) + T(v)$
2) $T(cu) = cT(u)$

A linear transformation is said to be operation reserving. (the operation of addition and scalar multiplication).

Let T be a linear transformation from V into W, where u and v are in V. Then the following properties are true.

1) $T(0) = 0$
2) $T(-v) = -T(v)$
3) $T(u-v) = T(u) - T(v)$
4) If $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$, then
$T(v) = c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n)$.

Page No.
Date

6.2 Isometric Transformation

An isometric transformation (or isometry) is a shape-preserving transformation (movement) in the plane or in space.

If a transformation does not change the figure (no change in shape & size) after transformation, is called an isometric transformation.

The isometric transformations are:

1) Reflection
2) Rotation
3) Translation
4) Half turn
5) Glide reflection

1) Reflection

It is a transformation which produces a mirror image of an object. The mirror image can be either about x-axis or y-axis.

The object is rotated by 180°. We can say the object is flipped.

i) Reflection about the x-axis $(y=0)$ → The value of ~~obj~~ x will remain same and the value of y will be negative.

i.e. $P(x,y) \xrightarrow[\text{reflection}]{\text{X-axis}} P'(x,-y)$

In matrix form:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) Reflection on Y-axis (i.e. $x=0$)
The value of y will remain same and the value of x changes sign.
i.e

$$P(x,y) \xrightarrow[\text{reflection}]{\text{Y-axis}} P'(-x,y)$$

Example

$$P(2,3) \longrightarrow P'(-2,3)$$

In matrix form:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

iii) Reflection about an axis perpendicular to xy plane and passing through the origin (i.e. reflection on $y=-x$)
The value of x and y both will be reversed. This is also called half revolution about the origin.

$$P(x,y) \xrightarrow[\text{reflection}]{y=-x} P(-y,-x)$$

In matrix form:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$P(2,3) \longrightarrow P'(-3,-2)$

iv) Reflection on $y = x$.

First of all the object is rotated at 45° clockwise. After the reflection is done concerning x-axis. The last step is the rotation of $y = x$ back to its original position that is anticlockwise 45°.

$$P(x,y) \xrightarrow[\text{reflection}]{y=x} P'(y,x)$$

Matrix form:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$P(2,3) \longrightarrow P'(3,2)$

2) Rotation

It is a process of changing the angle of the object. Rotation can be clockwise or anticlockwise.

i) Anticlockwise (positive quarter turn, $+90°$)

The object $P(x,y)$ is rotated through $+90°$ (or $-270°$) about the origin and its image is $P'(-y,x)$

i.e.,

$$P(x,y) \xrightarrow[\text{rotation}]{(0,0)\ 90°} P'(-y,x)$$

~~in matrix form~~:

Example

$$P(2,3) \longrightarrow P'(-3,2)$$

ii) Clockwise Rotation (Negative quarter turn, $-90°$)

The object $P(x,y)$ is rotated through $-90°$ about the origin and its image is $P'(y,-x)$.

i.e.,

$$P(x,y) \xrightarrow{(0,0)\ -90°} P'(y,-x).$$

Example

$$P(2,3) \longrightarrow P'(3,-2)$$

Page No.
Date

3) ~~Scaling~~ Translation

It is a straight line movement of an object from one position to another, through a certain distance and direction.

The object $P(x,y)$ is translated using a translation vector $T = \begin{pmatrix} a \\ b \end{pmatrix}$ to get the image $P'(x+a, y+b)$.

i.e. $P(x,y) \xrightarrow[\text{translation}]{T = \begin{bmatrix} a \\ b \end{bmatrix}} P'(x+a, y+b)$

We can also ~~write~~ take (t_x, t_y) as translation vector.

$x_1 = x + t_x$

$y_1 = y + t_y$.

In matrix form:

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$A(2,3) \xrightarrow{T = \binom{3}{4}} A'(2+3, 3+4)$

$= A'(5,7)$.

4) Half turn

A rotation through 180° is a half turn.

The object $P(x,y)$ is rotated through 180° about the origin to get the image $P'(-x,-y)$.

i.e

$$P(x,y) \xrightarrow[\text{rotation}]{(0,0)\ 180^\circ} P'(-x,-y)$$

Example.

$A(2,3) \longrightarrow A'(-2,-3)$

5) Glide reflection

A glide reflection is a symmetric operation/transformation that consists of a reflection over a line and then translation along that line, combined into a single operation.

The effect of a reflection ~~with~~ combined with any translation is a glide reflection.

For example, there is an isometry consisting of the reflection on x-axis, followed by translation of one unit parallel to it. Then it can be represented as:

$$P(x, y) \longrightarrow P'(x+1, -y)$$

6.3 Non-Isometric Transformation

If a transformation ~~does not~~ changes the figure (change in shape & size) after ~~the~~ the transformation, then it is called non-isometric transformation. ~~Non~~

Non-isometric transformation maps a figure to a similar image. It means object figure and image figure are similar under non-isometric transformation.

The non-isometric transformations are:

1. Dilation
2. Stretch
3. Shear

1. Dilation

A dilation is a transformation that produces an image that is same ~~as~~ shap as the original, but is of different size.

A dilation that that creates a larger image is called enlargement.

A dilation that creates a smaller image is called a reduction.

So, dilations are enlargements or reductions

- If the scale factor is greater than 1, the image is enlargement. It expands.
- If the scale factor is between 0 and 1, the image is reduction. It contracts.

If the scale factor is 1, the figure and the image are exact the same size (congruent).

* Dilation with scale factor 2, multiply by 2
  Center at the origin
  $P(x,y) \longrightarrow P'(2x,2y)$

* Dilation with scale factor 1/2, multiply by 1/2.
  Center at the origin.
  $P(x,y) \longrightarrow P'(\frac{1}{2}x, \frac{1}{2}y)$

* For a dilation not at the origin, measure the distances.

* When dilating an image with scale factor k:
  $P(x,y) \xrightarrow{D_k} P'(kx, ky)$.

Example

$A(-4,1) \xrightarrow{D_2} A'(2\times -4, 2\times 1)$
$= A'(-8, 2)$

2) Stretch

A stretch is a transformation in which a figure is enlarged (or reduced) in only one direction. ~~the change is referred to as~~ In a stretch, the figure is distorted, and is not necessarily similar to the original figure.

We know that ~~the~~ a dilation with a center ~~and~~ at the origin and a scale factor of k can be expressed as:

$$P(x, y) \longrightarrow P'(kx, ky)$$

But, a stretch will expand the size of only one of the coordinates as mention below:

* A stretch with stretch factor k and invariant x-axis:

$$P(x, y) \xrightarrow[x\text{-axis}]{S_k} P'(x, ky)$$

* A stretch with stretch factor k and invariant y-axis:

$$P(x, y) \xrightarrow[y\text{-axis}]{S_k} P'(kx, y).$$

Combination of Stretch:

* A stretch with stretch factor a in x-direction and stretch factor b in y-direction:

$$P(x, y) \longrightarrow P'(ax, by)$$

~~Exam~~ Example

$$A(2,3) \xrightarrow[\text{X-axis}]{S_2} A'(2, 2\times 3)$$
$$= A'(2,6)$$

$$A(2,3) \xrightarrow[\text{Y-axis}]{S_2} A'(2\times 2, 3)$$
$$= A'(4,3)$$

Date

3) Shear

A transformation in which all points along a given line L remained fixed while other points are shifted parallel to L by a distance proportional to their perpendicular distance from L is called shear.

Shearing a plane figure does not change its area.

A shear can also be generalized to three dimensions, in which planes are translated instead of lines.

A transformation that slants the shape of an object is called the shear transformation.

There are two shear transformations :- X-shear and Y-shear.

One shifts X-coordinates values and other shifts Y-Coordinate values. However, in both cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as skewing.

X-shear (shearing in X-direction)

The X-shear preserves the Y coordinate and changes are made to X coordinates.

i.e. ~~$Y' = Y + Sh_y \cdot X$~~ ~~$X' = X$~~

$$X' = X + Sh_x \cdot y$$
$$Y' = Y$$

Page No.
Date

or,
$P(x, y) \longrightarrow P'(x + Sh_x \cdot y, y)$
or, $P(x,y) \longrightarrow P'(x + cy, y)$
In matrix form

$$\begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or,} \quad Sh_x = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c = Shearing parameter.
$c = Sh_x$

Y-Shear

It is ~~also~~ a vertical shearing.
The Y-shear preserves the value of X-coordinates and changes are made to Y-coordinates.
i.e

$X' = X$
$Y' = y + Sh_y \cdot X$

i.e. $P(x, y) \longrightarrow P'(x, y + Sh_y \cdot x)$
or, $P(x,y) \longrightarrow P'(x, y + cx)$

In matrix form,

$$\begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad Sh \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c = Sh_y$.

Date

Shearing in X-Y directions

Here, layers will be slided in both x as well as y direction.

Sliding in both horizontal and vertical directions.

i.e.

$x' = x + Sh_x \cdot y$

$y' = y + Sh_y \cdot x$

$P(x, y) \longrightarrow P'(x + Sh_x \cdot y,\ y + Sh_y \cdot x)$

In matrix form:

$$\begin{bmatrix} 1 & Sh_y & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Find the image of the point (3,4) after a shear of ~~4~~ factor 4 parallel to x-axis.

Soln:

$$P(3,4) \xrightarrow{Sh_x = 4} P'(3 + 4 \times 4,\ 4)$$
$$= P'(19, 4)$$