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Uvod u interaktivno dokazivanje teorema

Vežbe 6

Zadatak 1 *Svojstva funkcija*

Pokazati da je slika unije, unija pojedinačnih slika.
Savet: Razmotriti teoreme *image-def* i *vimage-def*.

lemma *image-union*:
  shows $f \text{ ` } (A \cup B) = f \text{ ` } A \cup f \text{ ` } B$
proof
  show $f \text{ ` } (A \cup B) \subseteq f \text{ ` } A \cup f \text{ ` } B$
  proof
    fix y
    assume $y \in f \text{ ` } (A \cup B)$
    then have $\exists\ x.\ x \in A \cup B \wedge f\,x = y$ **by** *auto*
    then obtain x **where** $x \in A \cup B$ $f\,x = y$ **by** *auto*
    then have $x \in A \vee x \in B$ **by** *auto*
    then have $f\,x \in f \text{ ` } A \vee f\,x \in f \text{ ` } B$ **by** *auto*
    with $\langle f\,x = y \rangle$ **show** $y \in f \text{ ` } A \cup f \text{ ` } B$ **by** *auto*
  qed
next
  show $f \text{ ` } A \cup f \text{ ` } B \subseteq f \text{ ` } (A \cup B)$
  proof
    fix y
    assume $y \in f \text{ ` } A \cup f \text{ ` } B$
    then have $y \in f \text{ ` } A \vee y \in f \text{ ` } B$ **by** *simp*
    then show $y \in f \text{ ` } (A \cup B)$
    proof
      assume $y \in f \text{ ` } A$
      then have $\exists\ x.\ x \in A \wedge f\,x = y$ **by** *auto*
      then obtain x **where** $x \in A$ $f\,x = y$ **by** *auto*
      then have $x \in A \cup B$ **by** *simp*
      then have $f\,x \in f \text{ ` } (A \cup B)$ **by** *simp*
      with $\langle f\,x = y \rangle$ **show** $y \in f \text{ ` } (A \cup B)$ **by** *auto*
    next
      assume $y \in f \text{ ` } B$
      then have $\exists\ x.\ x \in B \wedge f\,x = y$ **by** *auto*
      then obtain x **where** $x \in B$ $f\,x = y$ **by** *auto*
      then have $x \in A \cup B$ **by** *simp*
      then have $f\,x \in f \text{ ` } (A \cup B)$ **by** *simp*
      with $\langle f\,x = y \rangle$ **show** $y \in f \text{ ` } (A \cup B)$ **by** *auto*
    qed
  qed
qed

Neka je funkcija f injektivna. Pokazati da je slika preseka, presek pojedinačnih slika.
Savet: Razmotriti teoremu *inj-def*.

lemma *image-inter*:
  assumes $inj\ f$
  shows $f\ `\ (A \cap B) = f\ `\ A \cap f\ `\ B$
proof
  show $f\ `\ (A \cap B) \subseteq f\ `\ A \cap f\ `\ B$
  proof
    fix y
    assume $y \in f\ `\ (A \cap B)$
    then have $\exists\ x \in A \cap B.\ f\ x = y$ **by** *auto*
    then obtain x **where** $x \in A \cap B\ f\ x = y$ **by** *auto*
    then have $x \in A \land x \in B$ **by** *auto*
    then have $f\ x \in f\ `\ A \land f\ x \in f\ `\ B$ **by** *auto*
    with $\langle f\ x = y \rangle$ **show** $y \in f\ `\ A \cap f\ `\ B$ **by** *auto*
  qed
next
  show $f\ `\ A \cap f\ `\ B \subseteq f\ `\ (A \cap B)$
  proof
    fix y
    assume $y \in f\ `\ A \cap f\ `\ B$
    then have $y \in f\ `\ A\ y \in f\ `\ B$ **by** *auto*
    from $\langle y \in f\ `\ A \rangle$ **obtain** xa **where** $xa \in A\ f\ xa = y$ **by** *auto*
    moreover
    from $\langle y \in f\ `\ B \rangle$ **obtain** xb **where** $xb \in B\ f\ xb = y$ **by** *auto*
    ultimately
    have $xa = xb$ **using** *assms* **by** (*simp add*: *inj-def*)
    with $\langle xa \in A \rangle$ **have** $xb \in A$ **by** *auto*
    with $\langle xb \in B \rangle$ **have** $xb \in A \land xb \in B$ **by** *auto*
    then have $xb \in A \cap B$ **by** *auto*
    then have $f\ xb \in f\ `\ (A \cap B)$ **by** *auto*
    with $\langle f\ xb = y \rangle$ **show** $y \in f\ `\ (A \cap B)$ **by** *auto*
  qed
qed

Savet: Razmotriti teoremu *surj-def* i *surjD*.

lemma *surj-image-vimage*:
  assumes $surj\ f$
  shows $f\ `\ (f\ -`\ B) = B$
proof
  show $f\ `\ f\ -`\ B \subseteq B$
  proof
    fix y
    assume $y \in f\ `\ f\ -`\ B$
    then obtain x **where** $x \in f\ -`\ B\ f\ x = y$ **by** *auto*
    then have $f\ x \in B$ **by** *auto*
    with $\langle f\ x = y \rangle$ **show** $y \in B$ **by** *auto*
  qed
next
  show $B \subseteq f\ `\ f\ -`\ B$
  proof
    fix y
    assume $y \in B$

    with *assms* **obtain** x **where** $f\ x = y$ **using** *surjD* **by** *metis*
    with ‹$y \in B$› **have** $x \in f -‘\ B$ **by** *auto*
    then have $f\ x \in f\ ‘\ (f -‘\ B)$ **by** *auto*
    with ‹$f\ x = y$› **show** $y \in f\ ‘\ f -‘\ B$ **by** *auto*
  qed
qed

Pokazati da je kompozicija injektivna ako su pojedinačne funkcije injektivne.
Savet: Razmotrite teoremu *inj-eq*.

lemma *comp-inj*:
  assumes *inj f*
  assumes *inj g*
  shows *inj* $(f \circ g)$
proof
  fix $x\ y$
  assume $(f \circ g)\ x = (f \circ g)\ y$
  then have $f\ (g\ x) = f\ (g\ y)$ **by** *auto*
  with ‹*inj f*› **have** $g\ x = g\ y$ **by** (*simp add*: *inj-eq*)
  with ‹*inj g*› **show** $x = y$ **by** (*simp add*: *inj-eq*)
qed

lemma
  assumes *inj f*
  shows $x \in A \longleftrightarrow f\ x \in f\ ‘\ A$
proof
  assume $x \in A$
  then show $f\ x \in f\ ‘\ A$ **by** *auto*
next
  assume $f\ x \in f\ ‘\ A$
  then obtain x' **where** $x' \in A\ f\ x = f\ x'$ **by** *auto*
  with ‹*inj f*› **have** $x = x'$ **by** (*simp add*: *inj-eq*)
  with ‹$x' \in A$› **show** $x \in A$ **by** *auto*
qed

lemma *inj-vimage-image*:
  assumes *inj f*
  shows $f -‘\ (f\ ‘\ A) = A$
proof
  show $f -‘\ f\ ‘\ A \subseteq A$
  proof
    fix x
    assume $x \in f -‘\ (f\ ‘\ A)$
    then obtain y **where** $y \in f\ ‘\ A\ f\ x = y$ **by** *auto*
    then obtain x' **where** $x' \in A\ f\ x' = y$ **by** *auto*
    with ‹$f\ x = y$› **have** $f\ x = f\ x'$ **by** *auto*
    with *assms* **have** $x = x'$ **by** (*simp add*: *inj-eq*)
    with ‹$x' \in A$› **show** $x \in A$ **by** *auto*
  qed
next
  show $A \subseteq f -‘\ f\ ‘\ A$
  proof

fix x
assume $x \in A$
then have $f\,x \in f\ `\ A$ **by** *auto*
then show $x \in f\ -`\ f\ `\ A$ **by** *auto*
qed
qed

Kompozicija je surjekcija ako su pojedinačne funkcije surjekcije.

lemma *comp-surj*:
assumes *surj f*
assumes *surj g*
shows *surj* $(f \circ g)$
unfolding *surj-def*
proof
fix z
from ‹*surj f*› **obtain** y **where** ‹$z = f\ y$› **by** *auto*
moreover
from ‹*surj g*› **obtain** x **where** ‹$y = g\ x$› **by** *auto*
ultimately
have $z = f\ (g\ x)$ **by** *auto*
then show $\exists\, x.\ z = (f \circ g)\ x$ **by** *auto*
qed

lemma *vimage-compl*:
shows $f\ -`\ (-\ B) = -\ (f\ -`\ B)$
proof
show $f\ -`\ (-\ B) \subseteq -\ f\ -`\ B$
proof
fix x
assume $x \in f\ -`\ (-\ B)$
then obtain y **where** $y \in -\ B\ f\ x = y$ **by** *auto*
then have $y \notin B$ **by** *auto*
with ‹$f\ x = y$› **have** $f\ x \notin B$ **by** *auto*
then have $x \notin f\ -`\ B$ **by** *auto*
then show $x \in -\ f\ -`\ B$ **by** *auto*
qed
next
show $-\ f\ -`\ B \subseteq f\ -`\ (-\ B)$
proof
fix x
assume $x \in -\ f\ -`\ B$
then have $x \notin f\ -`\ B$ **by** *auto*
then have $f\ x \notin B$ **by** *auto*
then have $f\ x \in -B$ **by** *auto*
then show $x \in f\ -`\ (-\ B)$ **by** *auto*
qed
qed