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On Turing degrees of points in computable topology*

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This paper continues our study of computable point-free topological spaces and the metamathematical points in them. For us, a *point* is the intersection of a sequence of basic open sets with compact and nested closures. We call such a sequence a *sharp filter*. A function f_F from points to points is generated by a function F from basic open sets to basic open sets such that sharp filters map to sharp filters. We restrict our study to functions that have at least all computable points in their domains.

We follow Turing's approach in stating that a point is computable if it is the limit of a computable sharp filter; we then define the Turing degree $\mathrm{Deg}(x)$ of a general point x in an analogous way. Because of the vagaries of the definition, a result of J. Miller applies and we note that not all points in all our spaces have Turing degrees; but we also show a certain class of points do. We further show that in $\mathbb{R}^n$ all points have Turing degrees and that these degrees are the same as the classical Turing degrees of points defined by other researchers.

We also prove the following: For a point x that has a Turing degree and lies either on a computable tree $\mathbf{T}$ or in the domain of a computable function f_F, there is a sharp filter on $\mathbf{T}$ or in $\mathrm{dom}(F)$ converging to x and with the same Turing degree as x. Furthermore, all possible Turing degrees occur among the degrees of such points for a given computable function f_F or a complete, computable, binary tree $\mathbf{T}$. For each $x \in \mathrm{dom}(f_F)$ for which x and $f_F(x)$ have Turing degrees, $\mathrm{Deg}(f_F(x)) \leq \mathrm{Deg}(x)$. Finally, the Turing degrees of the sharp filters convergent to a given x are closed upward in the partial order of all Turing degrees.

1 Introduction

The computable numbers were first described by Turing in [13] as the real numbers with computable binary expansions. Various other representations of computable numbers have been considered, including expansions in other base systems, continued fractions, and the like. The set of computable numbers is the same in all of these representations. In fact, if to every real number we assign a degree of unsolvability, commonly called a Turing degree, equal to the Turing degree of its binary expansion, it turns out that the Turing degree of its expansion in any other base system or of its continued fraction expansion is the same; i. e., the Turing degree of a real number is stable with respect to its representation. (Of course, some of the rationals have two binary expansions, but if this occurs, then both expansions are computable. Since no other real number has more than one binary expansion, each real number has a unique Turing degree.)

There is a problem in a binary (or any other base) representation of a real number, though, and that is that given a sequence of computable real numbers converging to a computable real number we cannot make in all cases a finite determination as to the binary representation of the limit. In particular, if a computer program produces a binary output beginning $0.111\ldots$, it may not be clear from the program code whether the 1's continue forever to make the final output equal to 1. If this does not occur, we will see it in the output sooner or later. But if the sequence of 1's produced is infinitely long, there might be no finite way of proving that it is. In fact, infinitely long expansions (in any base) are even more mischievous than this, since they can cause serious interference with an intuitively acceptable definition of computable functions. When using, say, binary expansions to name numbers, "addition" is not a computable function.

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Turing therefore proposed representing a real number as the unique member in the intersection of a sequence of nested finite open intervals which vanish in length. This leads to a point-free system by taking intervals as the principal objects and naming points with sequences of intervals. This approach can be useful for studying computable functions and computable real analysis because in this setting, the basic arithmetic operations (and many others) are computable.

In several of our previous papers we have studied computable topological spaces from a similar point-free perspective, taking basic open sets to be the primitive objects and letting a point be the unique member of the intersection of a (properly formed) nested sequence of basic open sets. In this paper we define and investigate Turing degrees of the points so obtained.

In Section 2 we give the necessary background from our previous work. Section 3 sets forth the definition of our Turing degrees of points. We are able to conclude there that all points of a particular type in any of our spaces have Turing degrees. We expand that result in Section 4 to show that the Turing degree of a point in $\mathbb{R}$, as it is defined by using our point-free approach, is the same as the Turing degree of its binary expansion, so that all points of $\mathbb{R}$, and indeed of $\mathbb{R}^n$, have Turing degrees. We then visit a result of J. Miller to note that not all points in all spaces of our type have Turing degrees. In Section 5 we study the Turing degrees of points in the domain or range of a computable quantum function. Several authors, including Rettinger and Zheng (see [10]), have extensively researched on the computability of real numbers, with particular reference to monotone computability. Their work led them to investigate k-monotone and other reducibilities. In this paper we do not address those issues, but look at some of the more basic aspects of Turing reducibility.

Next, by using trees whose branches are nested sequences of basic open sets that converge to distinct points, we can prove some interesting theorems about our spaces. If we assume our basic open sets to have some of the properties characteristic of the usual computable subbasis of $\mathbb{R}$ that consists of intervals with rational endpoints, we can make still further observations. Sections 6 and 7 introduce the concepts we need related to trees and subbases.

In Section 8 we note that if a point is the limit of a branch of one of our computable trees, then its Turing degree is the same as that of the branch. We also show that all possible Turing degrees are represented among the Turing degrees of the points to which the branches of a complete, computable, binary tree converge, and that the same is true for the points in the domain of a computable function, provided all computable points lie in that domain. Finally, in Section 9 we show that the Turing degrees of all nested sequences of open sets converging to a point are closed upward in the partial ordering of all Turing degrees.

2 Background, definitions, and basic results

In this section, we recall some key definitions and results from our previous works [3] and [5], and fix the basic topological and recursion theoretic properties of the spaces of our study at the end.

2.1 Basic topological properties of our spaces

Throughout this paper, we study $\langle X, \Delta\rangle$, where X is a first countable, connected space containing at least two points, and Δ, its subbasis, is comprised of basic open sets each of which is connected and has compact closure. These assumptions imply that X is regular, second countable, and of second category. Fundamental examples of these spaces are $\mathbb{R}^n$, for every $n \geq 1$, with appropriate subbases. Familiar examples are $\mathbb{R}$ with the subbasis of open intervals with rational endpoints, $\mathbb{R}^2$ with the subbasis of open rectangles whose corners have rational coordinates and whose sides are parallel to the axes, and $\mathbb{R}^2$ with the subbasis of open balls with rational radii and centers with rational coordinates.

2.2 A convenient notation

We use the notation $\alpha \Subset \beta$ to mean $\overline{\alpha} \subseteq \beta$ (α's closure is a subset of β). We do so because often we work with situations where we have a string like $\alpha \Subset \beta \Subset \gamma \Subset \cdots$. Similarly, we write $\alpha \Supset \beta$ to mean $\alpha \supseteq \overline{\beta}$, to allow efficient mention of a string like $\alpha \Supset \beta \Supset \gamma \Supset \cdots$.

2.3 Points, sharp filters, and enumerations

In this subsection, we summarize our machinery for capturing (computable) points through (computable) sharp filters, specify our topological and recursion theoretic settings, and describe an acceptable enumeration that captures all computable sharp filters.

Our use of sharp filters to name points can be viewed as an application of the concept of *representation* encountered in the "Type 2 Theory of Effectivity" which has been developed by Weihrauch and others (see [14]). A sharp filter can also be thought of as an example of what Spreen calls a *strong base* that *converges* to a point (see [12]); but where Spreen assumes the prior existence of points, we take a point-free approach. So we define a sharp filter in a way that makes it a close cousin of Martin-Löf's *maximal approximation* (see [8]). But where a maximal approximation is a maximal filterbase in a topological basis, and hence generates a maximal filter in the topology, a sharp filter is a discrete, nested filterbase which also generates a maximal filter in the topology. The nesting is given by property 1. of Definition 2.1; property 2. is essentially the same as what Martin-Löf uses to distinguish an approximation from a maximal approximation. Further, since the collection of open sets forms a complete lattice under containment, our approach is related to the study of such lattices. In fact, for a compact Hausdorff space, our relation of "$\Subset$" is the same as the relation "way below". Gierz et al. [2] studies these notions in the context of continuous lattices and domain theory, both of which can be related to our point-free approach. The study of more detailed connections will be the subject of a future project.

Definition 2.1 For a topological space X with subbasis $\Delta = \{\delta_n : n \in \omega\}$, a sequence $A = \{\alpha_i : i \in \omega\}$ of basic open sets is a *sharp filter in* Δ if

1. $(\forall i)(\alpha_{i+1} \Subset \alpha_i)$,
2. $(\forall \beta, \gamma \in \Delta)[(\beta \Subset \gamma) \Rightarrow (\exists i)[(\alpha_i \cap \beta = \emptyset) \vee (\alpha_i \subseteq \gamma)]]$.

We say α *resolves* $\langle \beta, \gamma \rangle$ if

$$(\beta \Subset \gamma) \Rightarrow [(\alpha \cap \beta = \emptyset) \vee (\alpha \subseteq \gamma)].$$

We refer to a pair $\langle \beta, \gamma \rangle$ with $\beta, \gamma \in \Delta$ as a *target on* X. (Note that if $\overline{\beta}$ is not a subset γ, then α resolves $\langle \beta, \gamma \rangle$ trivially.) If a sequence $\{\alpha_i : i \in \omega\}$ satisfies clause 2. for fixed β and γ, we say that it *resolves* $\langle \beta, \gamma \rangle$. We will refer to the property in item 2. above as *the resolution property*.

Let $\pi_0, \pi_1 : \omega \longrightarrow \omega$ be recursive functions such that if $n = \langle n_0, n_1 \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard "pairing function", then $\pi_0(n) = n_0$ and $\pi_1(n) = n_1$. We shall say that $\gamma \in \Delta$ *resolves targets* 0 *to* n *on* X if γ resolves $\langle \delta_{\pi_0(i)}, \delta_{\pi_1(i)} \rangle$ for $0 \leq i \leq n$.

We say *a sharp filter* A *converges to* x, or x *is the limit of* A, and write $A \searrow x$, if $\bigcap \alpha_i = \{x\}$.

Definition 2.2 $\langle X, \Delta \rangle$ is *semi-computably presentable* if $\Delta = \{\delta_n : n \in \omega\}$ and for all $\alpha, \beta \in \Delta$, the predicates "$\alpha \subseteq \beta$", "$\overline{\alpha} \subseteq \beta$", and "$\alpha \cap \beta = \emptyset$" are decidable. Let $\langle X, \Delta \rangle$ be a semi-computably presentable space, and let $A = \{\alpha_i : i \in \omega\}$ be a sharp filter in Δ. A is *computable* if there is a computable function $f : \omega \longrightarrow \omega$ such that for every i, $\alpha_i = \delta_{f(i)}$. For $x \in X$, we say x *is computable* if there is a computable sharp filter A such that $A \searrow x$.

2.4 Basic computational properties of our spaces

In addition to the topological properties (mentioned in Subsection 2.1) assumed of the spaces studied in this paper, we require that all our spaces be semi-computably presented. In particular, that means that our subbasis Δ is indexed by the natural numbers, and the decidability of relations among the basic open sets in Δ is dependent on that indexing.

2.5 Correspondences and functions

Here we recall the basic definitions for quantum correspondences and quantum functions and the interrelationship between them. A continuous function from a subset S of a space X to a space Y will be called a *quantum function* if all computable points are in S. Since our context is actually a point-free one, we must define quantum functions via functions from basic open sets to basic open sets that we call *quantum correspondences*.

Definition 2.3 Two sharp filters $A = \{\alpha_i : i \in \omega\}$ and $B = \{\beta_i : i \in \omega\}$ are *equivalent* if they converge to the same point; that is, if $(\forall i)(\alpha_i \cap \beta_i \neq \emptyset)$. In such a case we write $A \equiv B$.

Definition 2.4 Let $\Delta_X = \{\delta_n : n \in \omega\}$ and $\Delta_Y = \{\varepsilon_n : n \in \omega\}$. A partial function $F : \Delta_X \longrightarrow \Delta_Y$ is a *quantum correspondence* if

1. $(\forall \alpha, \beta)[[F(\alpha)\downarrow \wedge F(\beta)\downarrow \wedge \alpha \subseteq \beta] \Rightarrow [F(\alpha) \subseteq F(\beta)]]$;
2. $(\forall \alpha, \beta)[[F(\alpha)\downarrow \wedge F(\beta)\downarrow \wedge \alpha \Subset \beta] \Rightarrow [F(\alpha) \Subset F(\beta)]]$;
3. for every computable sharp filter B in Δ_X, there exists a computable sharp filter $A = \{\alpha_i : i \in \omega\}$ in Δ_X such that

$$(A \equiv B) \wedge ((\forall i)F(\alpha_i)\downarrow \wedge (F(A) \text{ is a sharp filter in } \Delta_Y)).$$

If F is also partially computable, we say it is a *computable quantum correspondence*.

In [3] we thoroughly study computable quantum correspondences and describe the motivation for their concept and the satisfying behavior of such objects despite their anomalies.

Definition 2.5 A correspondence F is said to be *honest* if for every sharp filter A in Δ_X (computable or not) with $A \subseteq \mathrm{dom}(F)$, there is a sharp filter $B \subseteq A$ where $F(B)$ is a sharp filter in Δ_Y.

In [4] we show that while computable dishonest correspondences exist, each computable quantum correspondence is replaceable with an "equivalent" honest computable quantum correspondence.

Definition 2.6 For a computable quantum correspondence F we refer to the function generated by it on the space, f_F, as a *computable quantum function*.

2.6 Trees and points

In this subsection we define what is understood as tree in our setting. It is important to point out that for our study of trees the "addresses" of nodes (strings of 0's and 1's) and the content of the nodes (basic open sets) play crucial roles (see [6]).

Definition 2.7 Let Σ be the set of all finite binary strings on the set $\{0, 1\}$. We usually denote members of Σ by σ or τ. The length of σ is denoted by $\mathrm{lh}(\sigma)$. If σ is an initial segment of τ, we write $\sigma \subseteq_{\mathrm{str}} \tau$. If σ is lexicographically before τ, we write $\sigma \leq_{\mathrm{lex}} \tau$. A *tree of sharp filters* $\mathbf{T}$ is the range of a partial function $\Theta : \Sigma \longrightarrow \Delta$ satisfying the following conditions:

1. $\Theta(\emptyset)\downarrow$;
2. for all $\sigma, \tau \in \Sigma$, if $\Theta(\tau)\downarrow$ and $\sigma \subseteq_{\mathrm{str}} \tau$, then $\Theta(\sigma)\downarrow$ and $\Theta(\tau) \Subset \Theta(\sigma)$;
3. if $\sigma \nsubseteq_{\mathrm{str}} \tau$, $\tau \nsubseteq_{\mathrm{str}} \sigma$, $\Theta(\sigma)\downarrow$, and $\Theta(\tau)\downarrow$, then $\Theta(\sigma) \cap \Theta(\tau) = \emptyset$;
4. if $b : \omega \longrightarrow \Sigma$ is a total function such that $\mathrm{lh}(b(n)) = n$, $\Theta(b(n))\downarrow$, and $\Theta(b(n+1)) \Subset \Theta(b(n))$ for every n (i.e., if $\Theta \circ b$ is a *branch through* $\mathbf{T}$), then $\Theta \circ b$ is a sharp filter. (Note that this condition requires $b(n)$ to be a substring of $b(n+1)$.)

$\mathbf{T}$ is a *complete tree of sharp filters* if Θ is a total function. For an infinite branch $\mathbf{b}$ through $\mathbf{T}$ that is a sharp filter and converges to a point, we denote that point by $x_{\mathbf{b}}$.

For notational convenience, for $\sigma \in \Sigma$ we denote $\Theta(\sigma)$ by θ_σ; thus we have

$$\mathbf{T} = \{\Theta(\sigma) : \sigma \in \Sigma'\} = \{\theta_\sigma : \sigma \in \Sigma'\},$$

where $\Sigma' = \mathrm{dom}(\Theta)$ is a subset of Σ and a tree under the ordering $\subseteq_{\mathrm{str}}$.

For a tree $\mathbf{T}$, let $\mathcal{T} = \{x_{\mathbf{b}} : \mathbf{b} \text{ is a branch in } \mathbf{T}\}$.

Definition 2.8 $\mathbf{T}$ is a Π^0_1 *tree* if there is a computable procedure which determines, for each $n \in \omega$ and each finite sequence $\langle \delta_0, \dots, \delta_n \rangle$ of members of Δ, whether there is $\sigma \in \mathrm{dom}(\Theta)$ with $\mathrm{lh}(\sigma) = n$, such that

$$(\forall \tau \subseteq_{\mathrm{str}} \sigma)(\forall k \leq n)[(\mathrm{lh}(\tau) = k) \Rightarrow (\theta_\tau = \delta_k)],$$

and if so, which computes σ.

3 Computable trees and Turing degrees

Recall that in our setting when in the meta language we refer to a "point", we mean the equivalence class of all of the sharp filters converging to that point. In this section, we define a filter-based Turing degree of a (meta language) point and show that any *pseudo-irrational* point, a point that is not on the boundary of any $\delta \in \Delta$, has a Turing degree.

Definition 3.1 Let $\Delta = \{\delta_n : n \in \omega\}$, and let, for some $f : \omega \longrightarrow \omega$, $A = \{\delta_{f(i)} : i \in \omega\}$ be a sharp filter. Then $\deg(A)$, *the Turing degree of* A, is the Turing degree of the set $\{\langle i, f(i)\rangle : i \in \omega\}$. (Here $\langle \cdot, \cdot \rangle$ is a Cantor pairing function.)

This of course is the classical definition of Turing degree of a set.

The following definition is the one whose consequences we will pursue in the rest of this paper. Because a point in our point-free setting is simply an equivalence class of sharp filters, the degree of a point is best defined in terms of the degrees of those sharp filters, and this definition seems to be natural. In particular, as we shall see in the next section, it agrees with the generally accepted definition of Turing degree for points in $\mathbb{R}$.

Definition 3.2 Let $x \in X$. *The filter-based Turing degree of* x is

$$\mathrm{Deg}(x) = \min\{\deg(A) : (A \text{ is a sharp filter}) \wedge (A \searrow x)\},$$

provided this is well-defined.

One of the theorems of this section shows that in a general space of our type many points do indeed have Turing degrees. As a preliminary definition we give the following.

Definition 3.3 Let $x \in X$. We define $\chi_x : \Delta_X \longrightarrow \{0, 1\}$ by

$$\chi_x(\delta) = \begin{cases} 0 & \text{if } x \notin \delta, \\ 1 & \text{if } x \in \delta. \end{cases}$$

In the first of the next three results, for a given point x, we construct a "canonical" sharp filter converging to it that is Turing reducible to χ_x. Then we show that χ_x is Turing reducible to any sharp filter converging to x if x is pseudo-irrational, and end with concluding that every pseudo-irrational point has a Turing degree in our filter-based sense.

Theorem 3.4 *For any $x \in X$, there is a canonical sharp filter A_x converging to x such that $A_x \leq_{\mathrm{T}} \chi_x$.*

P r o o f. Let $x \in X$ be given. We produce a sharp filter converging to x, and show that its degree is less than or equal to the degree of χ_x.

Let $\Delta = \{\delta_i : i \in \omega\}$. Form a sharp filter A_x as follows.

S t a g e 0: Find the least k such that δ_k resolves target 0 on X, and $\chi_x(\delta_k) = 1$. Let $\alpha_0 = \delta_k$.

S t a g e $s + 1$: Assume $\alpha_0, \ldots, \alpha_s$ have been defined so that $\alpha_{i+1} \Subset \alpha_i$ for each $i < s$, α_i resolves targets 0 to i for every $i \leq s$, and $\chi_x(\alpha_s) = 1$. Since Δ is a subbasis for X, there exists k such that $\delta_k \Subset \alpha_s$, δ_k resolves the $(s + 1)^{\text{st}}$ pair, and $\chi_x(\delta_k) = 1$; let $\alpha_{s+1} = \delta_k$, for the least such k.

Let $A_x = \{\alpha_s : s \in \omega\}$. Then A_x is a sharp filter converging to x, and $A_x \leq_{\mathrm{T}} \chi_x$. □

Definition 3.5 In the space $\langle X, \Delta\rangle$, a point $x \in X$ is Δ-*pseudo-irrational* if it does not belong to the boundary of any basic open set. When the context is clear, we might drop the reference to Δ.

Theorem 3.6 *If x is pseudo-irrational, then x has a Turing degree. Indeed, for such a pseudo-irrational x, we have* $\mathrm{Deg}(x) = \deg(\chi_x)$.

P r o o f. Note that as per Theorem 3.4 we have $A_x \leq_{\mathrm{T}} \chi_x$.

Next, let B be any sharp filter which converges to x. Note that given $\delta \in \Delta$, since here x is pseudo-irrational, we cannot have x on the boundary of δ, so there is $\beta \in B$ such that either $\beta \subseteq \delta$ or $\beta \cap \delta = \emptyset$. Enumerate B until the correct possibility is revealed. If $\beta \subseteq \delta$, then $\chi_x(\delta) = 1$; if $\beta \cap \delta = \emptyset$, then $\chi_x(\delta) = 0$. So $\chi_x \leq_{\mathrm{T}} B$.

Thus we have $A_x \leq_{\mathrm{T}} \chi_x \leq_{\mathrm{T}} B$ for any sharp filter B converging to x. It clearly follows that x has a Turing degree and $\mathrm{Deg}(x) = \deg(A_x) = \deg(\chi_x)$. □

4 On the Turing degree of a point

We can apply the ideas of the previous sections to any computable metric space. In this section we present contrasting results about Turing degrees in such spaces.

At first, we apply our approach to $\mathbb{R}^n$ with the subbasis Δ^n comprised of all n-dimensional rectangles whose corners have rational coordinates and whose sides are parallel to the axes; with this arrangement, a point is pseudo-irrational in $\langle \mathbb{R}^n, \Delta^n \rangle$ if and only if all of its coordinates are irrational. We use our filter-based definition of a Turing degree of a point, as per Definition 3.2, and prove that every x in $\mathbb{R}^n$ has a Turing degree in our sense. This fact becomes even more pleasing when we show that our filter-based Turing degree of x, $\mathrm{Deg}(x)$, is exactly the same as the classical Turing degree based on the binary expansion of x, $\deg(x)$.

Do all points of all spaces of our type have Turing degrees? At the end of this section, we examine a result of J. Miller [9] that demonstrates that there is a point f in the space of all continuous functions on $[0, 1]$ that does not have a Turing degree in our sense.

Theorem 4.1 *Every real number has a Turing degree.*

Proof. Every rational number r is the limit of the computable sharp filter

$$A = \{(r - \frac{1}{n+1}, r + \frac{1}{n+1}) \,:\, n \in \omega\};$$

since $\deg(A) = \mathbf{0}$, it follows that $\mathrm{Deg}(r) = \mathbf{0}$. Furthermore, since every irrational real is pseudo-irrational, Theorem 3.6 applies and thus every irrational real, too, has a Turing degree. $\square$

Corollary 4.2 *For each $n = 1, 2, \ldots$, every point of $\langle \mathbb{R}^n, \Delta^n \rangle$ has a Turing degree.*

Proof. We show this by induction. We have seen in Theorem 4.1 that this is true of $\mathbb{R}$. So suppose it is true of $\mathbb{R}^n$ and consider $\mathbb{R}^{n+1}$. Let $x \in \mathbb{R}^{n+1}$. If all coordinates of x are irrational, x is a pseudo-irrational in $\mathbb{R}^{n+1}$; so x has a Turing degree. Now suppose at least one coordinate of x is rational; say $x = (x_1, \ldots, x_{n+1})$ with x_p rational for some $p \in \{1, \ldots, n+1\}$. Let

$$y = (x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{n+1}),$$

so $y \in \mathbb{R}^n$. If $A \subseteq \Delta^n$ is any sharp filter converging to y, we can use A to find a sharp filter $B \subseteq \Delta^{n+1}$ converging to x in this way: If $A = \{\alpha_i \,:\, i \in \omega\}$ with each $\alpha_i = (a_{i,1}, b_{i,1}) \times (a_{i,2}, b_{i,2}) \times \cdots \times (a_{i,n}, b_{i,n})$, then let

$$\begin{aligned}\beta_i = (a_{i,1}, b_{i,1}) \times \cdots \times (a_{i,p-1}, b_{i,p-1}) \times (x_p - \frac{1}{i+1}, x_p + \frac{1}{i+1})& \\ \times (a_{i,p+1}, b_{i,p+1}) \times \cdots \times (a_{i,n}, b_{i,n})&\end{aligned}$$

and let $B = \{\beta_i \,:\, i \in \omega\}$. Then clearly $B \equiv_{\mathrm{T}} A$.

In a similar way, given any sharp filter $B \searrow x$, we can find $A \searrow y$ by expunging the p^{th} coordinates, and thus $A \leq_{\mathrm{T}} B$.

Now suppose A_0 is a sharp filter that converges to y and such that $\mathrm{Deg}(y) = \deg(A_0)$. Then as above we can find $B_0 \searrow x$ with $B_0 \equiv_{\mathrm{T}} A_0$. We shall show $\deg(B_0) = \mathrm{Deg}(x)$.

As noted above, for *any* B such that $B \searrow x$ we can find $A \searrow y$ such that $A \leq_{\mathrm{T}} B$. Since

$$\deg(A_0) = \mathrm{Deg}(y) \leq \deg(A),$$

we then have $B_0 \equiv_{\mathrm{T}} A_0 \leq_{\mathrm{T}} A \leq_{\mathrm{T}} B$. So $\deg(B_0)$ is minimal among the degrees of sharp filters that converge to x, and therefore $\mathrm{Deg}(x) = \deg(B_0)$. $\square$

Definition 4.3 For any $x \in [0, 1]$, choose $a_1, a_2, \ldots \in \{0, 1\}$ such that

$$x = \textstyle\sum_{i=0}^{\infty} a_i \cdot 2^{-i}$$

and let $f_x : \omega \longrightarrow \{0, 1\}$ be the function such that for each i, $f_x(i) = a_i$.

Define *the degree of the binary expansion of* x to be the Turing degree of f_x: $\deg(x) = \deg(f_x)$.

Remark 4.4 Note that for every $x \in [0,1]$, $\deg(x)$ is well-defined. This is because if x has more than one binary expansion, then it has exactly two binary expansions, both computable, so that $\deg(x) = \mathbf{0}$.

It is natural to ask if $\mathrm{Deg}(x) = \deg(x)$ for every $x \in \mathbb{R}$. In the next theorem, we show that this equality is in fact the case by proving that it holds for $x \in [0,1]$.

Theorem 4.5 *For all* $x \in [0,1]$, $\mathrm{Deg}(x) = \deg(x)$.

Proof. First note that if $x \in \mathbb{Q}$, then $\mathrm{Deg}(x) = \mathbf{0} = \deg(x)$. Now suppose $x \in [0,1]$ is irrational. Then f_x, as defined above, is uniquely determined and

$$x = \textstyle\sum_{i=0}^{\infty} f_x(i) \cdot 2^{-i}.$$

For convenience, for each $n \in \omega$, let $x_{[n]} = \sum_{i=0}^{n} f_x(i) \cdot 2^{-i}$, so that $x_{[n]} < x < x_{[n]} + 2^{-n}$.

First, we show that $\mathrm{Deg}(x) \leq \deg(x)$. We form a sharp filter $A = \{\alpha_i \,:\, i \in \omega\}$ as follows: Let $\alpha_0 = (0,1)$. Now suppose we have formed $\alpha_0, \dots, \alpha_n$ with $\alpha_i \Supset \alpha_{i+1}$ for $i = 0, \dots, n-1$, $|\alpha_i| \leq 2^{-i}$ for $i = 0, \dots, n$, and $x \in \alpha_n$. Since $x \notin \mathbb{Q}$, we can use f_x to find $m > n$ such that $(x_{[m]}, x_{[m]} + 2^{-m}) \Subset \alpha_n$. Choose the least such m. Let $\alpha_{n+1} = (x_{[m]}, x_{[m]} + 2^{-m})$. Then $\alpha_{n+1} \Subset \alpha_n$, $|\alpha^{n+1}| = 2^{-m} \leq 2^{-(n+1)}$, and $x \in \alpha_{n+1}$. So A is a sharp filter converging to x and $A \leq_{\mathrm{T}} f_x$. Thus $\mathrm{Deg}(x) \leq_{\mathrm{T}} \deg(A) \leq_{\mathrm{T}} \deg(f_x) = \deg(x)$.

Next we show that $\deg(x) \leq \mathrm{Deg}(x)$. Let $B = \{\beta_i \,:\, i \in \omega\}$ be a sharp filter converging to x such that

$$\deg(B) = \mathrm{Deg}(x).$$

First note that since $x \in (0,1)$ we have $f_x(0) = 0$. Suppose we know $x_{[n]}$ and wish to find $f_x(n+1)$. We note that $x_{[n]} < x_{[n]} + 2^{-(n+1)} < x_{[n]} + 2^{-n}$ and we remember that $x_{[n]} < x < x_{[n]} + 2^{-n}$.

Furthermore, $x \neq x_{[n]} + 2^{-(n+1)}$ since $x \notin \mathbb{Q}$, so there is some $\beta_i \subseteq (x_{[n]}, x_{[n]} + 2^{-n})$ such that

$$x_{[n]} + 2^{-(n+1)} \notin \beta_i.$$

If $\beta_i \subseteq (x_{[n]}, x_{[n]} + 2^{-(n+1)})$, then $f_x(n+1) = 0$; otherwise

$$\beta_i \subseteq (x_{[n]} + 2^{-(n+1)}, x_{[n]} + 2^{-n}),$$

so $f_x(n+1) = 1$. Thus $f_x \leq_{\mathrm{T}} B$ and so $\deg(x) = \deg(f_x) \leq_{\mathrm{T}} \deg(B) = \mathrm{Deg}(x)$. □

This theorem in essence reveals that our definition of the Turing degree of a point is on the mark for spaces like $\mathbb{R}^n$. In some spaces though, this is inadequate, since J. Miller [9] has shown that there are spaces in which not all points have Turing degrees in our sense. The outline of his counterexample is as follows. Let $\mathcal{M}$ be the metric space $C[0,1]$ (the continuous functions on the interval $[0,1]$) with the metric

$$d(f,g) = \max\nolimits_{x \in [0,1]} |f(x) - g(x)|.$$

Let $\mathcal{Q}^{C[0,1]}$ be the set of all polygonal functions in $\mathcal{M}$ whose line segments have rational coordinates at their endpoints. For a given $f \in \mathcal{Q}^{C[0,1]}$ and a given $r \in \mathbb{Q}^+$ (where $\mathbb{Q}^+$ is the set of positive rational numbers), let

$$B_{f,r} = \{g \in \mathcal{M} \,:\, |f - g| < r\}.$$

Let

$$\Delta_{\mathcal{M}} = \{B_{f,r} \,:\, f \in \mathcal{M} \wedge r \in \mathbb{Q}^+\}.$$

Then $\langle \mathcal{M}, \Delta_{\mathcal{M}} \rangle$ is a semi-computably presented resolvable space. The next theorem shows that not all points in this space have Turing degrees in our sense.

Theorem 4.6 (J. Miller [9]) *There is a function $f \in \langle \mathcal{M}, \Delta_{\mathcal{M}} \rangle$ such that no sharp filter converging to f has least Turing degree among all such sharp filters, so* $\mathrm{Deg}(f)$ *is undefined.*

5 Turing degrees of points and computable quantum functions

Now that we have defined Turing degrees of sharp filters and points, and have seen that pseudo-irrational points in any of our spaces and all points in $\mathbb{R}^n$ have Turing degrees, we shall investigate what can be said about the degrees of points that lie in the domains or ranges of computable quantum functions.

Theorem 5.1 *Let F be a computable quantum correspondence from X to Y and let $x \in \operatorname{dom}(f_F)$ be a point whose Turing degree exists. Then there is a sharp filter $A \subseteq \operatorname{dom}(F)$ such that $A \searrow x$ and $\deg(A) = \operatorname{Deg}(x)$.*

Proof. (Note that here F need not be honest.) Assume a computable enumeration of $\operatorname{dom}(F)$ is given and let $B = \{\beta_i : i \in \omega\}$ be a sharp filter converging to x such that $\deg(B) = \operatorname{Deg}(x)$. We shall construct A in stages using B as an oracle.

Stage 1: Enumerate $\operatorname{dom}(F)$ until some δ appears such that δ resolves $\langle \beta_1, \beta_0 \rangle$ and there is some $\beta_m \subseteq \delta$. Such a δ exists because $x \in \operatorname{dom}(f_F)$. Let α_0 be the first such δ encountered. Let $A_1 = \{\alpha_0\}$.

Stage $n+1$: Assume that $\alpha_0, \ldots, \alpha_n$ have been enumerated into A, that $\alpha_{i+1} \Subset \alpha_i$ for all $i \leq n$, that every α_i resolves $\langle \beta_{i+1}, \beta_i \rangle$, and that there is some $\beta_m \subseteq \alpha_n$. Then

$$x \in \beta_m \cap \beta_{n+1} \subseteq \alpha_n \cap \beta_{n+1},$$

where $\alpha_n \subseteq \beta_n$ as α_n resolves $\langle \beta_{n+1}, \beta_n \rangle$. Fix the least such m. Enumerate $\operatorname{dom}(F)$ until some δ appears such that δ resolves $\langle \beta_{m+2}, \beta_{m+1} \rangle$ and for some k, $\beta_k \subseteq \delta$. There is such δ because $x \in \operatorname{dom}(f_F)$. Let α_{n+1} be the first such δ encountered. Then $\beta_k \subseteq \alpha_{n+1}$, so $x \in \alpha_{n+1} \cap \beta_{m+2}$, and hence we obtain $\alpha_{n+1} \subseteq \beta_{m+1}$. This implies two things: first, since $\beta_m \subseteq \alpha_n \subseteq \beta_n$, we have $\alpha_{n+1} \subseteq \beta_{m+1} \subseteq \beta_{n+1}$, so α_{n+1} resolves $\langle \beta_{n+2}, \beta_{n+1} \rangle$; second, $\overline{\alpha_{n+1}} \subseteq \overline{\beta_{m+1}} \subseteq \beta_m \subseteq \alpha_n$. Let $A_{n+1} = A_n \cup \{\alpha_{n+1}\}$.

Now let $A = \lim_n A_n$. Then A is a sharp filter converging to x. Also A is a total function that has been enumerated computably in B, so $\deg(A) \leq \deg(B)$. Since $\deg(B)$ is minimal among the degrees of all sharp filters converging to x, $\deg(A) = \deg(B) = \operatorname{Deg}(x)$. □

From this theorem we can derive an observation about the Turing degree of a point in the range of a computable quantum correspondence and its relation to the Turing degree (if any) of its preimage.

Corollary 5.2 *Let F be a computable quantum correspondence from X to Y and suppose $x \in X$, $y \in Y$ are such that $\operatorname{Deg}(x)$ and $\operatorname{Deg}(y)$ exist and $f_F(x) = y$. Then $\operatorname{Deg}(y) \leq \operatorname{Deg}(x)$.*

Proof. We may assume F is honest. Let $x \in \operatorname{dom}(f_F)$, and as per the previous theorem, pick $A \searrow x$ such that $\deg(A) = \operatorname{Deg}(x)$ and $A \subseteq \operatorname{dom}(F)$. Then $F(A)$ is a sharp filter converging to y, and clearly

$$\operatorname{Deg}(y) \leq \deg(F(A)) \leq \deg(A) = \operatorname{Deg}(x),$$

since F is partially computable. □

6 Free trees

In our previous papers we have required that a tree of sharp filters $\mathbf{T} = \{\theta_\sigma : \sigma \in \Sigma'\}$ satisfy the following two conditions: each $\sigma \in \Sigma'$ is a string on the alphabet $\{0, 1\}$, and for $\sigma, \tau \in \Sigma'$ such that $\sigma \neq \tau$ and $\operatorname{lh}(\sigma) = \operatorname{lh}(\tau)$, $\theta_\sigma \cap \theta_\tau = 0$. In this paper (and most likely in our next papers too) we will abolish these requirements, which we had adopted only for convenience. In their place, the requirement that our alphabet is called computable will be imposed, and we will occasionally use the following definition.

Definition 6.1 Let $\alpha \in \Delta$. Then $\mathbf{T}$ is called a *free tree of sharp filters in* α if it is the range of a partial function $\Theta : \Sigma \longrightarrow \Delta$ satisfying the following conditions:

1. $\Theta(\emptyset) = \alpha$;
2. for all $\sigma, \tau \in \Sigma$, if $\Theta(\tau)\downarrow$ and $\sigma \subsetneq_{\mathrm{str}} \tau$, then we have $\Theta(\sigma)\downarrow$ and $\Theta(\tau) \Subset \Theta(\sigma)$;
3. if $b : \omega \longrightarrow \Sigma$ is a total function such that for every n, $\operatorname{lh}(b(n)) = n$, $\Theta(b(n))\downarrow$, and $b(n) \subseteq_{\mathrm{str}} b(n+1)$, then $\Theta \circ b = \mathbf{b}$ is a sharp filter.

For such a $\mathbf{b}$ as in clause 3., with $\beta_i = \Theta \circ b(i)$ for $i \in \omega$, we refer to $(\beta_0, \beta_1, \ldots)$ as an *infinite branch of* $\mathbf{T}$, and write $(\beta_0, \beta_1, \ldots) \in \mathbf{T}$ to indicate this fact. Similarly, for any $b : \omega \longrightarrow \Sigma$, total or not, with $\beta_i = \Theta \circ b(i)$ for $i \in \{0, \ldots, n\}$, we write $(\beta_0, \beta_1, \ldots, \beta_n) \in \mathbf{T}$, and refer to $(\beta_0, \beta_1, \ldots, \beta_n)$ as a *finite branch of* $\mathbf{T}$.

The reader is encouraged to compare this with the definition of a tree in Section 2.

Definition 6.2 Let $\mathbf{T} = \{\theta_\sigma : \sigma \in \Sigma'\}$ be a free tree of sharp filters. A partial function $f : \Sigma \longrightarrow \omega$ is a *separation function for* $\mathbf{T}$ if $\Sigma' \subseteq \mathrm{dom}(f)$ and for every $\sigma \in \Sigma'$, $f(\sigma) > \mathrm{lh}(\sigma)$ and there is an open set $\mathcal{G}_\sigma \subseteq X$ such that

1. for every $\tau \in \Sigma'$ with $\mathrm{lh}(\tau) = f(\sigma)$ and $\tau \supseteq \sigma$, $\theta_\tau \subseteq \mathcal{G}_\sigma$;
2. for every $\tau \in \Sigma'$ with $\mathrm{lh}(\tau) = f(\sigma)$ and $\tau \not\supseteq \sigma$, $\theta_\tau \subseteq (X - \mathcal{G}_\sigma)^\circ$.

(Here and elsewhere, for $\mathcal{A} \subseteq X$, $\mathcal{A}^\circ$ indicates the interior of the set $\mathcal{A}$.)

A free tree that has a separation function is *separated*, and a free tree with a computable separation function is *computably separated*.

Definition 6.3 For a branch $\mathbf{b} = \Theta \circ b$ through $\mathbf{T}$, where $b : \omega \longrightarrow \Sigma - \{\emptyset\}$, we write $\mathbf{b} = \{\beta_n : n \in \omega\}$ to mean $\beta_n = \Theta \circ b(n)$ for every $n \in \omega$. If $b(n) = \sigma$ and $\beta_n = \Theta \circ b(n) = \alpha$, we say $\beta_n = \alpha$ *at* σ.

Theorem 6.4 *Let* $\mathbf{b}_0 = \Theta \circ b_0$, $\mathbf{b}_1 = \Theta \circ b$ *be branches through a separated tree* $\mathbf{T}$ *with* $b_0 \neq b_1$. *Then*

$$\bigcap \mathbf{b}_0 \neq \bigcap \mathbf{b}_1.$$

Proof. Let Υ be a separation function for $\mathbf{T}$ and let n be such that $b_0(n) \neq b_1(n)$. Let

$$\sigma_0 = b_0(n) \quad \text{and} \quad \sigma_1 = b_1(n),$$

so $\mathrm{lh}(\sigma_0) = \mathrm{lh}(\sigma_1) = n + 1$. Suppose $\Upsilon(\sigma_0) = m$. Then there is an open set $\mathcal{G}_{\sigma_0} \subseteq X$ such that

$$\mathbf{b}_0(m-1) \subseteq \mathcal{G}_{\sigma_0},$$

but

$$\mathbf{b}_1(m-1) \subseteq X - \mathcal{G}_{\sigma_0}.$$

Hence $\bigcap \mathbf{b}_0 \in \mathcal{G}_{\sigma_0}$ while $\bigcap \mathbf{b}_1 \in X - \mathcal{G}_{\sigma_0}$ □

Corollary 6.5 *For* b_0, b_1, σ_0, *and* m *as in the previous theorem, if*

$$b_0' : \omega \longrightarrow \Sigma - \{\emptyset\}$$

is an extension of $b_0 \upharpoonright (m)$ *and*

$$b_1' : \omega \longrightarrow \Sigma - \{\emptyset\}$$

is an extension of $b_1 \upharpoonright (m)$ *such that* $\mathbf{b}_0' = \Theta \circ b_0'$ *and* $\mathbf{b}_1' = \Theta \circ b_1'$ *are branches through* $\mathbf{T}$, *then*

$$\bigcap \mathbf{b}_0' \in \mathcal{G}_{\sigma_0} \quad \textit{and} \quad \bigcap \mathbf{b}_1' \in X - \mathcal{G}_{\sigma_0}.$$

Proof. $\bigcap \mathbf{b}_0' \in \mathbf{b}_0(m-1)$ and $\bigcap \mathbf{b}_1' \in \mathbf{b}_1(m-1)$. □

7 Dense and unconfined subbases and auxiliaries

Let $\Delta = \{\delta_n : n \in \omega\}$. Because of the computability of the relation "$\alpha \subseteq \beta$", we may, if we wish, assume the enumeration of Δ is one-to-one.

Definition 7.1 Δ is called a *dense subbasis for* X if the enumeration $\{\delta_n : n \in \omega\}$ is one-to-one and Δ has the following property:

$$(\forall \alpha, \beta \in \Delta)[\alpha \Subset \beta \Rightarrow (\exists \gamma \in \Delta)[\alpha \Subset \gamma \wedge \gamma \Subset \beta]].$$

It is clear that Δ^n, the subbasis we adopted for $\mathbb{R}^n$ in Section 4, is a dense subbasis.

Definition 7.2 If Δ is a dense subbasis for X, then let $\mathrm{Ins} : (\omega \times \omega) \times \omega \longrightarrow \omega$ be a total computable *inserting function* with the following properties for all $i, j, r \in \omega$:

1. if $\delta_i \Supset \delta_j$, then $\delta_i \Supset \delta_{\mathrm{Ins}(i,j;r)} \Supset \delta_{\mathrm{Ins}(i,j;r+1)} \Supset \delta_j$;
2. if $\langle i, j, r\rangle < \langle i', j', r'\rangle$ (where $\langle \cdot, \cdot, \cdot\rangle$ is the usual Cantor encoding), then $\mathrm{Ins}(i, j; r) < \mathrm{Ins}(i', j'; r')$.

It should be noted that because of the use of Cantor encoding such an inserting function is a one-to-one function whose range is computable (this will be relevant in our forthcoming argument in Theorem 9.1).

Thus, if $\delta_i \Supset \delta_j$, then

$$\delta_i \Supset \delta_{\mathrm{Ins}(i,j;0)} \Supset \delta_{\mathrm{Ins}(i,j;1)} \Supset \delta_{\mathrm{Ins}(i,j;2)} \Supset \cdots \Supset \delta_{\mathrm{Ins}(i,j;r)} \Supset \cdots \Supset \delta_j .$$

Remark 7.3 We shall use Ins to conduct successive interpolations between adjacent members of a sharp filter, so as to produce another sharp filter which is a supersequence of the given one, in which certain information is encoded by the interpolants.

Finally we give a definition.

Definition 7.4 Δ is an *unconfined subbasis for* X if the enumeration $\{\delta_n : n \in \omega\}$ is one-to-one and Δ has the following properties:

1. $X \notin \Delta$;
2. $(\forall\alpha \in \Delta)(\exists\beta \in \Delta)[\alpha \Subset \beta]$.

It is further clear that the subbasis Δ^n is also unconfined.

8 Turing degrees, trees, and correspondences

In this section we examine the Turing degrees of points on computably separated trees in the first two theorems, and use a result from [7] to strengthen them further. Finally we apply our findings to establish a further result on domains of computable quantum functions.

The first theorem in this section guarantees us that points on a computably separated Π^0_1 tree have Turing degrees, and that the degree of such a point is equal to the degree of the branch converging to it.

Theorem 8.1 *Let* $\mathbf{T}$ *be a computably separated* Π^0_1 *tree of sharp filters and let* $\mathbf{b} = \{\beta_n : n \in \omega\}$ *be an infinite branch through* $\mathbf{T}$ *converging to* $x_{\mathbf{b}}$*. Then* $x_{\mathbf{b}}$ *has a Turing degree, and in fact* $\mathrm{Deg}(x_{\mathbf{b}}) = \deg(\mathbf{b})$.

Proof. Let $T = \{\theta_\sigma : \sigma \in \Sigma'\}$, $A = \{\alpha_n : n \in \omega\}$ be a sharp filter converging to $x_{\mathbf{b}}$, and f be a computable separating function for $\mathbf{T}$. We shall show that $\mathbf{b} \leq_{\mathrm{T}} A$.

Suppose $\beta_0, \dots, \beta_k$ are known. There is some $\sigma \in \Sigma'$ such that $\beta_k = \theta_\sigma$ at σ. Then $\mathrm{lh}(\sigma) = k + 1$, and

$$(\theta_{\sigma\restriction 0}, \dots, \theta_{\sigma\restriction k}, \theta_\sigma)$$

is an initial segment of $\mathbf{b}$. There is thus some $i \in \mathcal{A}$ such that $\beta_{k+1} = \theta_{\sigma^\frown i}$ at $\sigma^\frown i$.

Claim 8.1.1 *There is a unique* $i \in \mathcal{A}$ *with* $\sigma^\frown i \in \Sigma'$ *and such that there exists some* $\tau \in \Sigma'$ *with* $\tau \supseteq \sigma^\frown i$, $\mathrm{lh}(\tau) = f(\sigma^\frown i)$*, and for some* $m \in \omega$, $\alpha_m \Subset \theta_\tau$.

Proof. Take $i \in \mathcal{A}$ such that $\beta_{k+1} = \theta_{\sigma^\frown i}$ at $\sigma^\frown i$. Let $n = f(\sigma^\frown i)$. Pick $\tau \in \Sigma$ with $\theta_\tau = \beta_{n1}$. Then there is $m \in \omega$ such that $\sigma_m \Subset \theta_\tau$. □ (Claim 8.1.1)

Now let $j \in \mathcal{A}$, $j \neq i$, and suppose $\sigma^\frown j \in \Sigma'$. Consider any $\upsilon \in \Sigma'$ with $\upsilon \geq \sigma^\frown j$ and $\mathrm{lh}(\upsilon) = n$. The definition of a separating function guarantees that $\theta_\tau \subseteq \mathcal{G}_{\sigma^\frown i}$, so $x_{\mathbf{b}} \in \mathcal{G}_{\sigma^\frown i}$, and that $\theta_\upsilon \subseteq X - \mathcal{G}_{\sigma^\frown i}$, so $x_{\mathbf{b}} \notin \theta_\upsilon$. Hence there exists no $m \in \omega$ with $\sigma_m \Subset \theta_\upsilon$. We now describe how to find β_{k+1} using A as an oracle, knowing that $\beta_k = \theta_\sigma$ also. Since $\mathbf{T}$ is computable we can computably enumerate all $i \in \mathcal{A}$ such that $\sigma^\frown i \in \Sigma'$, and for every such i we can computably enumerate all the nodes of $\mathbf{T}$ of the form θ_τ, where $\tau \supseteq \sigma^\frown i$, $\mathrm{lh}(\tau) = f(\sigma^\frown i)$. By Claim 8.1.1 we are guaranteed a unique $i \in \mathcal{A}$ such that $\alpha_m \Subset \theta_\tau$ for some such τ and some $m \in \omega$, and of course we can find this i by using A. By the uniqueness of i, $\beta_{k+1} = \theta_{\sigma^\frown i}$ at $\sigma^\frown i$. □

Next, we show that a complete tree of sharp filters is rich in Turing degrees.

Theorem 8.2 *Let* $\mathbf{T}$ *be a complete binary*[1] *computable, computably separated tree of sharp filters. Then for any Turing degree* $\boldsymbol{a}$*, there is a point* x *lying in* $\mathcal{T}$ *with* $\mathrm{Deg}(x) = \boldsymbol{a}$*.*

Proof. Let $\mathbf{T} = \{\theta_\sigma : \sigma \in \Sigma\}$, where Σ is the set of all finite strings on the alphabet $\{0, 1\}$ and let f be a computable separation function for $\mathbf{T}$. Let $A \subseteq \omega$ be such that $\deg(A) = \boldsymbol{a}$. For each $n \in \omega$ let

$$\sigma_n = \langle x_A(0), \dots, x_A(n-1)\rangle$$

and let $\beta_n = \theta_{\sigma_n}$. Then $\mathbf{b} = \{\theta_n : n \in \omega\} = \{\theta_{\sigma_n} : n \in \omega\}$ is a branch through $\mathbf{T}$. Let x be the point of X to which $\mathbf{b}$ converges.

By Theorem 8.1, x has a Turing degree and $\mathrm{Deg}(x) = \deg(\mathbf{b})$. Clearly we have $\mathbf{b} \leq_{\mathrm{T}} A$, since $\mathbf{T}$ is computable. So we need only show that $A \leq_{\mathrm{T}} \mathbf{b}$. Suppose we know $X_A \restriction n$ for some $n \in \omega$. Then of course we also know σ_n. Let $m = f(\sigma_n{}^\frown 0)$. There is an open set $\mathcal{G}$ such that for any τ with $\mathrm{lh}(\tau) = m$, $\theta_\tau \subseteq \mathcal{G}$ if $\tau \supseteq \sigma_n{}^\frown 0$, and $\theta_\tau \subseteq X - \mathcal{G}$ if $\tau \supseteq \sigma_n{}^\frown 1$. Now since $\mathrm{lh}(\sigma_m) = m$ and $\beta_m = \theta_{\sigma_m}$, we have

$$\beta_m \in \{\theta_\tau : \mathrm{lh}(\tau) = m\}.$$

So we can find $\chi_A(n)$ as follows: Find $\tau \in \Sigma$ with $\mathrm{lh}(\tau) = m$ and $\theta_\tau = \beta_m$. If $\tau \supseteq \sigma^\frown 0$, then $\chi_A(n) = 0$; and if $\tau \supseteq \sigma^\frown 1$, then $\chi_A(n) = 1$. Hence $A \leq_{\mathrm{T}} \mathbf{b}$, proving that $\boldsymbol{a} = \deg(\mathbf{b}) = \mathrm{Deg}(x)$. □

For a space with a dense and unconfined subbasis the conclusions of the last two theorems can be established for trees highly computable in $\mathbf{0}'$, strengthening these results. The next two corollaries, corresponding to these theorems, respectively, give the strengthened forms; their proofs are included in [7] and follow from its main result (Theorem 4).

Corollary 8.3 (To Theorem 8.1) *Let* Δ *be dense and unconfined. Let* $\mathbf{T}$ *be a computably separated tree of sharp filters that is highly computable in* $\mathbf{0}'$*. If* $\mathbf{b}$ *is an infinite branch through* $\mathbf{T}$ *converging to* $x_{\mathbf{b}}$*, then* $x_{\mathbf{b}}$ *has a Turing degree, and* $\mathrm{Deg}(x_{\mathbf{b}}) = \deg(\mathbf{b})$.

Corollary 8.4 (To Theorem 8.2) *Let* Δ *be dense and unconfined, and let* $\mathbf{T}$ *be a complete, binary, computably separated tree that is computable in* $\mathbf{0}'$*. Then for any Turing degree* $\boldsymbol{a}$*, there exists a point* x *lying in* $\mathcal{T}$ *and such that* $\mathrm{Deg}(x) = \boldsymbol{a}$*.*

Finally, we use Theorem 8.2 to show that any computable quantum function has a domain rich in Turing degrees as well.

Theorem 8.5 *Let* F *be a computable quantum correspondence from* X *to* Y*. Then for every Turing degree* $\boldsymbol{a}$ *and every* $\alpha \in \Delta$ *there is a point* $x \in \alpha \cap \mathrm{dom}(f_F)$ *such that* $\mathrm{Deg}(x) = \boldsymbol{a}$*.*

Proof. By [5, Theorem 3.3], assume F is an honest computable quantum correspondence. Let $\mathbf{T}$ be a complete tree whose range is a subset of $\{\theta \subseteq \alpha : \theta \in \mathrm{dom}(F)\}$ as per [5, Theorem 8.4]. Then for every branch $\mathbf{b}$ through $\mathbf{T}$, $F(\mathbf{b})$ is a sharp filter, so any such branch converges to a point in the domain of f_F. By Theorem 8.2 for every Turing degree $\boldsymbol{a}$, there is x lying on $\mathbf{T}$ such that $\mathrm{Deg}(x) = \boldsymbol{a}$, and as we have seen, $x \in \mathrm{dom}(f_F)$. □

In a space X containing points without Turing degrees, none of those points can be on a computably separated Π^0_1 tree (as per Theorem 8.1). Therefore the tree-based techniques of this paper (Section 8) are not intricate enough to capture such enigmatic points.

9 Upward closure

In this section, we show that, for spaces with a dense subbasis Δ, the Turing degrees of sharp filters converging to a given point are closed upward. Given the definition of the Turing degree of a point, this result states that if x has a Turing degree, then the degrees of the sharp filters that converge to x form an upper cone in the Turing degrees.

[1] Actually, it is not necessary for the tree to be binary; rather, the alphabet that the tree is based on should have two or more members.

Using the standard definition for "$\otimes$", that is, if $g, h : \omega \longrightarrow \omega$, then $g \otimes h$ is a function from ω to ω, where

$$g \otimes h(2n) = g(n) \quad \text{and} \quad g \otimes h(2n+1) = h(n),$$

we have the following.

Theorem 9.1 *Let Δ be dense. Let $A = \{\delta_{k_n} : n \in \omega\}$ be a sharp filter that converges to a point $x \in X$, and let $f : \omega \longrightarrow \omega$ be a total function. Then there is a sharp filter C converging to x such that A is a subsequence of C and $C \equiv_{\mathrm{T}} A \otimes f$.*

Proof. With the help of the auxiliary function "Ins" defined in Section 7, we define $C = \{\gamma_n : n \in \omega\}$, a function from ω to Δ, as follows:

$$\gamma_{2n} = \delta_{k_n}, \quad \gamma_{2n+1} = \delta_{\mathrm{Ins}(k_n, k_{n+1}; f(n))}.$$

Therefore $A = \{\gamma_{2n} : n \in \omega\}$, and for each $n \in \omega$ we have $\gamma_{2n} \Supset \gamma_{2n+1} \Supset \gamma_{2n+2}$.

Hence C has the monotonicity property; since A is a subsequence of C, this implies that C is a sharp filter converging to x.

To show that $C \leq_{\mathrm{T}} A \otimes f$, first note that to find γ_{2n} we need only know δ_{k_n} since $\gamma_{2n} = \delta_{k_n}$. To find γ_{2n+1}, we first use A to find k_n and k_{n+1}, next use f to find $m = \mathrm{Ins}(k_n, k_{n+1}; f(n))$; then we conclude $\gamma_{2n+1} = \delta_m$.

To see that $A \leq_{\mathrm{T}} C$, we note that $A = \{\gamma_{2n} : n \in \omega\}$, as mentioned above. To observe that $f \leq_{\mathrm{T}} C$, determine $f(n)$ as follows: First, find k_n and k_{n+1} such that

$$\delta_{k_n} = \gamma_{2n} \quad \text{and} \quad \delta_{k_{n+1}} = \gamma_{2n+2}.$$

Next, find m such that $\delta_m = \gamma_{2n+1}$. Now there is a unique p such that $m = \mathrm{Ins}(k_n, k_{n+1}; p)$, and this p can be computably determined. Then $f(n) = p$. $\square$

Corollary 9.2 *Let Δ be dense. Then the Turing degrees of the sharp filters converging to a point $x \in X$ are closed upward.*

Proof. Given $x \in X$, a sharp filter A converging to x, and a set B whose Turing degree is greater than or equal to $\deg(A)$, form a sharp filter C converging to x such that $C \equiv_{\mathrm{T}} A \otimes \chi_B \equiv_{\mathrm{T}} B$. $\square$

10 Epilogue

It appears that the approach and the techniques of this paper can be applied in order to study the Turing degrees of Specker points, cluster points, and points in domain or range of a computable quantum function as investigated in [6].

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