ku-theoretic spectral decompositions for spheres and projective spaces

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Abstract. Ben-Zvi–Sakellaridis–Venkatesh described a conjectural extension of the geometric Satake equivalence to spherical varieties, whose spectral decomposition is described by Hamiltonian varieties. The goal of this article is to study their conjecture in the case of spherical varieties of relative rank 1 using tools from homotopy theory. Our discussion relates their conjecture to classical topics in homotopy theory such as the EHP sequence and Hopf fibrations, as well as more modern topics such as Hochschild (co)homology. We will also study an analogue of the derived geometric Satake equivalence and of the Ben-Zvi–Sakellaridis–Venkatesh conjecture with coefficients in connective complex K-theory. In this generalized setting, the dual *group* (à la Langlands, Gaitsgory–Nadler, Sakellaridis–Venkatesh, Knop–Schalke) remains unchanged, but the specific dual "representation" of the dual group changes. On the spectral/Langlands dual side, we expect that the appropriate replacement of Hamiltonian varieties are given by what we term "ku-Hamiltonian varieties"; this is a notion interpolating between Hamiltonian and quasi-Hamiltonian varieties (à la Alekseev–Malkin–Meinrenken). Finally, we suggest possible generalizations to more exotic cohomology theories such as the sphere spectrum.

Contents

1. Introduction 3
 1.1. Spherical harmonics and its geometrization 3
 1.2. The proof of Theorem 1.1.10 8
 1.3. ku-theoretic aberrations 11
 1.4. Spherical coefficients 16
 1.5. Notation 19
 1.6. Acknowledgements 20
2. Equivariant connective K-theory 21
 2.1. Shearing and the Koszul sign rule 21
 2.2. Equivariant K-theory 24
 2.3. Equivariant connective K-theory 27
 2.4. Equivariant ku-homology 31
3. (Derived) geometric Satake and variants 34
 3.1. Full faithfulness of global sections 34
 3.2. Review of derived geometric Satake 40

Part of this work was done when the author was supported by NSF DGE-2140743.

3.3. Spherical varieties 46
3.4. Whittaker induction and [**BZSV23**, Conjecture 7.5.1] 51
3.5. The G-equivariant ku-(co)homology of $\mathcal{L}(G/H)$ 56
3.6. ku-theoretic derived geometric Satake 68
4. Case-by-case analysis 78
4.1. Homology of loops on a sphere 80
4.2. Type A_n 86
4.3. Type B_n 93
4.4. Type C_n 95
4.5. Type D_n 99
4.6. Type F_4 101
4.7. Type G_2 105
4.8. Type B_3' 107
5. Structures on the spectral side 110
5.1. ku-Hamiltonian varieties 110
5.2. ku-Langlands duals and coisotropicity 115
5.3. Equivariant spectra and graded groups 120
5.4. Langlands duality and the sphere spectrum 124
Appendix A. Sheaves on loop spaces 133
Appendix B. Questions/further directions 135
References 139

1. Introduction

The goal of this article is multifold: we will prove the main conjecture of Ben-Zvi–Sakellaridis–Venkatesh ([**BZSV23**, Conjecture 7.5.1]) in the case of affine homogeneous spherical varieties of relative rank 1 (under Hypothesis 3.5.21); we will also prove through examples that this conjecture admits a generalization with coefficients in connective complex K-theory; and finally, we will propose a broader program in stable homotopy theory which attempts to relate the aforementioned conjecture and the main results of [**Dev23a**]. Each of these steps will be explained in detail later in the introduction. Since the goals of this article involve a few different areas of mathematics, we will give a somewhat long-winding introduction to the results presented here (with advance apologies to experts for the prolonged introduction).

1.1. Spherical harmonics and its geometrization. Broadly speaking, the Langlands program aims to study spectral decompositions of automorphic representations. The historically first example of such a spectral decomposition is the Fourier transform:

Example 1.1.1. The Fourier transform says that any $f \in L^2(S^1; \mathbf{C})$ can be expressed in terms of the spherical functions $\exp(inx)$ (which are eigenvectors for the derivative operator). One can view Fourier analysis as providing a decomposition $L^2(S^1; \mathbf{C}) \cong \ell^2(\mathbf{Z})$ into irreducible representations of S^1 acting on $L^2(S^1; \mathbf{C})$. Note that $\ell^2(\mathbf{Z})$ is a completion of an infinite direct sum of the irreducible representations $\mathbf{C} \cdot \exp(inx)$ of S^1; each appears with multiplicity 1.

The Fourier transform was soon generalized to the theory of spherical harmonics, which studies the decomposition of $L^2(S^{n-1}; \mathbf{C})$ under the action of the group of rotations $\mathrm{O}_n \subseteq \mathrm{GL}_n(\mathbf{R})$.

Example 1.1.2. Let $\mathcal{H}_j(\mathbf{R}^n)$ denote the space of homogeneous harmonic polynomials $\mathbf{R}^n \to \mathbf{C}$ of degree j, and let $\mathcal{H}_j(S^{n-1})$ denote the space of functions on S^{n-1} obtained by restricting elements of $\mathcal{H}_j(\mathbf{R}^n)$ to $S^{n-1} \subseteq \mathbf{R}^n$. Then, $L^2(S^{n-1}; \mathbf{C})$ is isomorphic to a completion of the direct sum $\bigoplus_{j\geq 0} \mathcal{H}_j$, and each $\mathcal{H}_j$ is an irreducible O_n-representation appearing with multiplicity 1.

Observing that $S^{n-1} \simeq \mathrm{O}_n/\mathrm{O}_{n-1}$, the theory of spherical harmonics can be generalized even further: if G is a reductive algebraic group over $\mathbf{C}$, K is a maximal compact subgroup of $G(\mathbf{C})$, and $H \subseteq G$ is a closed subgroup, one can attempt to understand the decomposition of $L^2(G(\mathbf{C})/H(\mathbf{C}); \mathbf{C})$ into irreducible K-representations. One can also state this goal in the p-adic setting, where the maximal compact subgroup $K \subseteq G(\mathbf{C})$ is replaced by the subgroup $G(\mathbf{Z}_p) \subseteq G(\mathbf{Q}_p)$. Namely, if $H \subseteq G$ is a closed subgroup of a reductive algebraic group over $\mathbf{Z}_p$, one can study the decomposition of the space $C_c(G(\mathbf{Q}_p)/H(\mathbf{Q}_p); \mathbf{C})$ of compactly supported functions into irreducible $G(\mathbf{Z}_p)$-representations. This turns out to be especially understandable in the case of multiplicity 1, in which case $H \subseteq G$ is called "spherical". Questions of this form have been placed into the context of the Langlands program by Sakellaridis and Venkatesh (among others) in [**SV17**].

The archimedean and p-adic settings being too elaborate, it is often simpler to *geometrize* such questions by studing their function field variants.

Recollection 1.1.3. There is a standard analogy between p-adic number fields such as $\mathbf{Q}_p$ (along with its ring of integers $\mathbf{Z}_p$) and function fields such as $\overline{\mathbf{F}}_p(\!(t)\!)$

(along with its ring of integers $\overline{\mathbf{F}}_p[\![t]\!]$). There is a further analogy between $\overline{\mathbf{F}}_p(\!(t)\!)$ (along with its ring of integers $\overline{\mathbf{F}}_p[\![t]\!]$) and $\mathbf{C}(\!(t)\!)$ (along with its ring of integers $\mathbf{C}[\![t]\!]$). In this case, $G(\mathbf{C}(\!(t)\!))$ can be regarded as the $\mathbf{C}$-points of the formal loop group of G, and hence acquires a natural topology. Therefore, instead of studying the decomposition of the space $C_c(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C})$ of compactly supported functions into irreducible $G(\mathbf{C}[\![t]\!])$-representations, we can further study a decomposition of the compactly supported cohomology ring $\mathrm{H}^*_c(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C})$ into irreducible $G(\mathbf{C}[\![t]\!])$-representations. (In the function field case, one can similarly consider the étale cohomology of $G(\overline{\mathbf{F}}_p(\!(t)\!))/H(\overline{\mathbf{F}}_p(\!(t)\!))$.)

The "spherical"[1] part of this cohomology ring is given by the *equivariant* cohomology $\mathrm{H}^*_{c,G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C})$. This cohomology ring can be itself be categorified: namely, one can consider the ∞-category $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C})$ of constructible $G(\mathbf{C}[\![t]\!])$-equivariant sheaves of $\mathbf{C}$-vector spaces on $G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!))$. This ∞-category will be the main topic of study in this article: in particular, when G/H is an affine spherical variety of rank 1 (the meaning of which will be explained below), we will provide a spectral decomposition of this ∞-category.

In order to explain the precise sense in which this ∞-category admits a spectral decomposition, let us return to the de-categorified function field setting; in other words, consider the vector space $C_{c,G(\overline{\mathbf{F}}_p[\![t]\!])}(G(\overline{\mathbf{F}}_p(\!(t)\!))/H(\overline{\mathbf{F}}_p(\!(t)\!)); \mathbf{C})$ of $G(\overline{\mathbf{F}}_p[\![t]\!])$-invariant compactly supported functions $G(\overline{\mathbf{F}}_p(\!(t)\!))/H(\overline{\mathbf{F}}_p(\!(t)\!)) \to \mathbf{C}$. This vector space canonically upgrades to an action of the vector space $C_{c,G(\overline{\mathbf{F}}_p[\![t]\!])\times G(\overline{\mathbf{F}}_p[\![t]\!])}(G(\overline{\mathbf{F}}_p(\!(t)\!)); \mathbf{C})$ equipped with the multiplication given by convolution. From now, let us assume (for simplicity) that G is semisimple. One then has the following famous theorem:

Theorem 1.1.4 (Satake isomorphism)**.** *There is an explicit isomorphism (defined by Macdonald)*

$$C_{c,G(\overline{\mathbf{F}}_p[\![t]\!])\times G(\overline{\mathbf{F}}_p[\![t]\!])}(G(\overline{\mathbf{F}}_p(\!(t)\!)); \mathbf{C}) \cong \mathbf{C}[\mathbb{X}_*(T)]^W,$$

where W is the Weyl group and $\mathbb{X}_*(T)$ is the lattice of cocharacters of T.

The right-hand side is not quite the complexification of the representation ring of G, which would instead be isomorphic to $\mathbf{C}[\mathbb{X}^*(T)]^W$ by highest weight theory; instead, it is the complexification of the representation ring of the *Langlands dual* group $\check{G}$, which is defined so that the weights, coweights, roots, and coroots of G are the coweights, weights, coroots, and roots of $\check{G}$ (respectively). In other words, the Satake isomorphism gives an explicit isomorphism

$$C_{c,G(\overline{\mathbf{F}}_p[\![t]\!])\times G(\overline{\mathbf{F}}_p[\![t]\!])}(G(\overline{\mathbf{F}}_p(\!(t)\!)); \mathbf{C}) \cong K_0(\mathrm{Rep}_{\mathbf{C}}(\check{G})) \otimes \mathbf{C}.$$

One therefore obtains an action

$$K_0(\mathrm{Rep}_{\mathbf{C}}(\check{G})) \otimes \mathbf{C} \circlearrowright C_{c,G(\overline{\mathbf{F}}_p[\![t]\!])}(G(\overline{\mathbf{F}}_p(\!(t)\!))/H(\overline{\mathbf{F}}_p(\!(t)\!)); \mathbf{C}),$$

[1]Apologies for the unfortunate, but standard, terminology! The terminology clash is even worse than one might expect: we will study both the sphere spectrum, as well as spectral decomposition for spherical varieties (and even propose the existence of a theory of spectral decomposition over the sphere spectrum for spherical varieties!).

I recently learned that "chromatic aberration" also goes by the name "spherochromatism". Motivated by the title of [**Dev23a**], I was considering changing the title of this article to "Spherochromatic spectral decompositions for spheres and the sphere spectrum", but I'm glad I decided not to go down that route — and anyone else reading this article probably is, too!

and the task of providing a "spectral decomposition" of this vector space can be more precisely phrased as giving an explicit description of this action in terms of the Langlands dual group $\check{G}$.

This interpretation of our task can be categorified, since both sides of the Satake isomorphism admit natural categorifications. A categorification of the Satake isomorphism itself is provided by the famous geometric Satake equivalence of Mirkovic-Vilonen [**MV07**]. To state it, let us switch back to the Laurent series ring $\mathbf{C}(\!(t)\!)$. Let $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})$ denote the ∞-category of $G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])$-equivariant constructible sheaves of $\mathbf{C}$-vector spaces on $G(\mathbf{C}(\!(t)\!))$ equipped with its perverse t-structure. Convolution defines a symmetric monoidal structure on the heart of the perverse t-structure, and then one has:

Theorem 1.1.5 (Mirkovic-Vilonen)**.** *There is a symmetric monoidal equivalence*

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})^\heartsuit \simeq \mathrm{Rep}(\check{G})$$

of abelian categories.

The naïve guess that this equivalence promotes to an equivalence between $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})$ and the derived ∞-category of $\mathrm{Rep}(\check{G})$ is *false*; we will return to it momentarily. For the moment, note that there is a canonical action of $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})^\heartsuit$ on $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!));\mathbf{C})^\heartsuit$, and this discussion suggests that the appropriate categorification of the task of providing a spectral decomposition of $C_{c,G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!));\mathbf{C})$ would be:

Goal 1.1.6. Describe the action

$$\mathrm{Rep}(\check{G}) \circlearrowleft \mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!));\mathbf{C})^\heartsuit,$$

perhaps by proving an equivalence between this category and the abelian category of quasicoherent sheaves on a quotient stack $\check{M}/\check{G}$ for some $\check{G}$-variety $\check{M}$.

Unfortunately, such an equivalence is generally not possible, since taking the heart of a t-structure is a rather severe process. It is therefore natural to ask for a generalization of the Mirkovic-Vilonen equivalence describing the full ∞-category $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})$. Such a description was provided by Bezrukavnikov-Finkelberg in [**BF08**] (following earlier work of Lusztig and Ginzburg; see [**Lus83, Gin95**]), and the answer involves derived algebraic geometry. We will state the *arithmetically sheared* (in the sense of [**BZSV23**, Section 6.7]) version of the derived geometric Satake equivalence.

Theorem 1.1.7 (Bezrukavnikov-Finkelberg; Theorem 3.2.7)**.** *Let $\check{\mathfrak{g}}$ denote the Lie algebra of the Langlands dual group, viewed as a $\mathbf{C}$-vector space, let $2\rho : \mathbf{G}_m \to \check{T}$ denote the sum of the positive coroots of $\check{G}$, and let $\check{\mathfrak{g}}[2-2\rho]$ denote its $(2-2\rho)$-fold shift. Then, there is an $\mathbf{E}_3$-monoidal equivalence*

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C}) \simeq \mathrm{Perf}(\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]). \tag{1}$$

This is often known as the *derived* geometric Satake equivalence, and it has some precedence in homotopy theory.

Remark 1.1.8. Let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{g}$, and let W denote the Weyl group of G. Then, there is an isomorphism[2] $\mathrm{Spec}\,\mathrm{H}^*_G(*;\mathbf{C}) \cong \mathfrak{t}[2]/\!\!/W$. There

[2] Only for the purposes of this introduction, we will purposely conflate grading shifts with homological shifts. For instance, there is in fact only an isomorphism $\mathrm{Spec}\,\mathrm{H}^*_G(*;\mathbf{C}) \cong \mathfrak{t}(2)/\!\!/W$ of

is also an isomorphism $\check{\mathfrak{t}}^* \cong \mathfrak{t}$, so that the Chevalley restriction theorem gives an isomorphism $\mathfrak{t}[2]/\!\!/W \cong \check{\mathfrak{g}}^*[2]/\!\!/\check{G}$, and hence an isomorphism $\operatorname{Spec} \mathrm{H}^*_G(*; \mathbf{C}) \cong \check{\mathfrak{g}}^*[2]/\!\!/\check{G}$. This implies an equivalence

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!]) \times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}[\![t]\!]); \mathbf{C}) \simeq \mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(*; \mathbf{C}) \simeq \mathrm{Perf}(\check{\mathfrak{g}}[2]/\!\!/\check{G}),$$

which is a restriction of the derived geometric Satake equivalence to equivariant sheaves on $G(\mathbf{C}[\![t]\!]) \subseteq G(\mathbf{C}(\!(t)\!))$.

The $\mathbf{E}_3$-monoidal structure on the left-hand side of the derived geometric Satake equivalence comes from the geometry of the affine Grassmannian (and is spelled out in [**Noc20**]), and the $\mathbf{E}_3$-monoidal structure on the right-hand side comes from restricting the natural tensor product of perfect complexes. Unfortunately, the fact that the derived geometric Satake equivalence is $\mathbf{E}_3$-monoidal does not seem to be recorded anywhere in the literature (although it is closely related to forthcoming work of Campbell-Raskin), and we will also not address this point in our discussion. We will give an argument for the above equivalence in Theorem 3.2.7 which is slightly different from that of [**BF08**]; the key step (already accomplished in [**BFM05**], and [**YZ11**] in arbitrary characteristic) is the construction of a homomorphism

$$\operatorname{Spec} \mathrm{H}^G_*(\Omega G; \mathbf{C}) \to \check{G} \times \check{\mathfrak{g}}^*[2]/\!\!/\check{G} \tag{2}$$

of group schemes over $\check{\mathfrak{g}}^*[2]/\!\!/\check{G} \cong \operatorname{Spec} \mathrm{H}^*_G(*; \mathbf{C})$.

We can now formulate the "correct" version of Goal 1.1.6.

Goal 1.1.9. There is a canonical action of the ∞-category $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!]) \times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!)); \mathbf{C})$ on $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C})$, and we can state the task of providing a spectral decomposition of the latter ∞-category as explicitly describing the action

$$\mathrm{Perf}(\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]) \circlearrowleft \mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C}).$$

The work of Ben-Zvi–Sakellaridis–Venkatesh [**BZSV23**] provides numerous conjectures about this description: namely, they conjecture in [**BZSV23**, Conjecture 7.5.1] that if G/H is a spherical G-variety (satisfying some other conditions), there is a graded *Hamiltonian* $\check{G}$-variety $\check{M}$ such that there is an equivalence[3]

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!)); \mathbf{C}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}(-2\rho)), \tag{3}$$

and the action of $\mathrm{Perf}(\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho])$ on the left-hand side is specified by the moment map $\mu : \check{M}/\check{G} \to \check{\mathfrak{g}}/\check{G}$. Here, $\mathrm{sh}^{1/2}$ denotes a shearing, which converts gradings into homological shifts (more precisely, it sends a module in weight $2n$ to the same module shifted homologically by $2n$). Moreover, they give a precise construction of the predicted dual variety $\check{M}$. One of our main goals in this article is to show that [**BZSV23**, Conjecture 7.5.1] is true for the building blocks of spherical varieties:

graded $\mathbf{C}$-schemes. Applying the shearing functor to this equivalence then produces an isomorphism $\operatorname{Spec} \mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{C}) \cong \mathfrak{t}[2]/\!\!/W$ of derived $\mathbf{C}$-schemes.

[3] This is technically a slight lie: the left-hand side is replaced by a certain subcategory defined using the action of $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!]) \times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!)); \mathbf{C})$.

Theorem 1.1.10 (Theorem 3.4.13)**.** *Suppose G/H is an affine spherical variety of rank 1, and assume Hypothesis 3.5.21 holds*[4]*. Then* [**BZSV23**, Conjecture 7.5.1] *is true, i.e., there is an equivalence* (3) *for the dual variety $\check{M}$ constructed in* [**BZSV23**, Section 4].[5]

More precisely, for each line of Table 1, there is an equivalence

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!));\mathbf{C}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{Y}/\check{G}_X \times \text{“Normalization”}),$$

where the normalization term accounts for the parabolic subgroup stabilizing the open B-orbit in G/H, and $\check{G}_X$ (isomorphic to SL_2 in our case) denotes the dual group of [**SV17**] *(see also* [**GN10, KS17**]*).*

Name	$X = G/H$	Dual $\check{Y}$	Topological explanation
A_n	$\mathrm{PGL}_{n+1}/\mathrm{GL}_n$	$T^*(2n)\mathbf{A}^2(2n,0)$	Hopf fibration
B_n	$\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$	$T^*(2n)\mathbf{A}^2(4n-2,0)$	EHP sequence
C_n	$\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$	$T^*(4n-4)\mathbf{A}^2(4n-2,0)$	Hopf fibration
D_n	$\mathrm{SO}_{2n}/\mu_2 \cdot \mathrm{SO}_{2n-1}$	$\mathfrak{sl}_2(2n-2n\rho_{\mathrm{SL}_2})$	James splitting
F_4	F_4/Spin_9	$T^*(16)\mathbf{A}^2(22,0)$	Exceptional Hopf fibration
G_2	G_2/SL_3	$T^*(6)\mathbf{A}^2(10,0)$	EHP sequence
B_3'	SO_7/G_2	$\mathfrak{sl}_2(6-6\rho_{\mathrm{SL}_2})$	James splitting

TABLE 1. Table of dual varieties and topological phenomena corresponding to each of the rank one affine homogeneous spherical varieties with no “roots of type N” (such varieties are excluded by [**SV17, BZSV23**]). For each of these varieties, the dual group is $\check{G}_X = \mathrm{SL}_2$ (which is also equipped with a certain grading that we have omitted in this table). With varied columns, this table will appear again in the present article; see, in particular, Table 4. Here, the notation $\mathbf{A}^2(i,j)$ denotes an affine 2-space with coordinates in weights $-i$ and $-j$; and $T^*(j)(X)$ denotes the cotangent bundle with cotangent fibers placed in weight j.

Remark 1.1.11. In the real and p-adic settings, the analogue of Theorem 1.1.10 was proved by Gan and Gomez as [**GG14**, Theorem 1].

Remark 1.1.12 (Why rank 1?)**.** Most of this article does not restrict attention to affine spherical varieties of rank 1; this assumption is only imposed in Section 4 for doing computations. The restriction to rank 1 here is not for any particularly deep reason: these varieties have very simple equivariant cell structures, which makes them more amenable to calculations. These examples also capture many interesting phenomena expected in [**BZSV23**], and for these examples, the resulting homotopy-theoretic explanations for these phenomena become easier to understand.

[4]If the sheaf theory for the $G(\mathbf{C}[\![t]\!])$-action on $G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!))$ is sufficiently well-behaved, it should be possible to forego this hypothesis; but regardless of the sheaf-theoretic setup and the ultimate correctness of Hypothesis 3.5.21, we believe that the calculations of Section 4 will be the key to proving any sort of Langlands duality.

[5]In fact, we only prove a bare equivalence; namely, we do not check compatibility with the action of the spherical Hecke category $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{C})$. We do not expect this to be an especially difficult task, but it is one we decided to omit.

Remark 1.1.13. The reader might notice a conspicuous absence of loop-rotation equivariance in this article, which, under Langlands duality, conspires to a deformation quantization of the spectral/coherent side of Theorem 1.1.10. We have chosen to separate this topic into a different article **[Dev24]**, in order to give it the detailed treatment it deserves (as well as for the purposes of length).

In the remainder of this introduction, we will:

(a) explain the meaning of the terms in, and the proof of, the above theorem, and illustrate it in the example of the spherical PGL_2-variety $\mathrm{PGL}_2/\mathbf{G}_m$. When applied to the spherical $G \times G$-variety G, this discussion recovers the derived geometric Satake equivalence (even when G is not of rank 1).
(b) explain the generalization of the derived geometric Satake equivalence to coefficients in connective complex K-theory, and some limited analogues of the above theorem on relative rank 1 spherical varieties.
(c) discuss some conjectures and expectations about a further generalization to coefficients in the sphere spectrum.

The homotopically-minded reader is suggested to skip to (b) and return to (a) as needed, and the conjecturally-minded reader is suggested to skip to (c).

1.2. The proof of Theorem 1.1.10.

The basic strategy to prove Theorem 1.1.10 is discussed in Theorem 3.5.23. Let us give a high-level summary of this argument. Assume throughout that $H \subseteq G$ is a connected closed subgroup of a connected compact Lie group whose complexification is a spherical subgroup of $G_{\mathbf{C}}$.

Strategy 1.2.1. (a) One of the first steps is reducing Theorem 1.1.10 to a question in homotopy theory; this reduction relies on a result of Quillen's (see Proposition 3.5.19), which implies that the $G_{\mathbf{C}}(\mathbf{C}[\![t]\!])$-equivariant topology of $G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!))$ can be understood via the G-equivariant topology of the free loop space $\mathcal{L}G/\mathcal{L}H \simeq \mathcal{L}(G/H)$. It is here that the assumption that the quotient $G_{\mathbf{C}}/H_{\mathbf{C}}$ is an affine spherical variety is used. For instance, as shown in **[GN10]**, the set of $G_{\mathbf{C}}(\mathbf{C}[\![t]\!])$-orbits of $G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!))$ is countable if and only if $G_{\mathbf{C}}/H_{\mathbf{C}}$ is spherical (so this could be taken as the definition of a spherical variety). This step does not require assuming that $G_{\mathbf{C}}/H_{\mathbf{C}}$ is rank 1.

(b) On the Langlands dual side, recall (as mentioned before (3)) that Ben-Zvi–Sakellaridis–Venkatesh construct the dual Hamiltonian $\check{G}$-variety $\check{M}$ using the spherical geometry of the quotient G/H. One important observation is that the conjectures of **[BZSV23]** in particular predict that there is a closed immersion $\kappa : \check{\mathfrak{h}}^*[2]/\!\!/\check{H} \to \check{M}$ called the *Kostant section* whose image "generates" $\mathcal{O}_{\check{M}}$ under the $\check{G}$-action; see Remark 3.5.7. In the case of affine homogeneous rank 1 spherical varieties, we construct the Kostant section case-by-case.

(c) Using a compact generation argument, Theorem 1.1.10 is reduced to proving that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^H(\Omega(G/H); \mathbf{C}) \cong \check{\mathfrak{h}}^*[2]/\!\!/\check{H} \times_{\check{M}/\check{G}[-2\rho]} \check{\mathfrak{h}}^*[2]/\!\!/\check{H}$$

of graded group schemes over $\check{\mathfrak{h}}^*[2]/\!\!/\check{H}$. In other words, $\operatorname{Spec} \mathrm{H}_*^H(\Omega(G/H); \mathbf{C})$ is the stabilizer of the image of the Kostant section. The *nonequivariant* homology of $\Omega(G/H)$ has been studied by many authors using Morse theory, at least in the case of symmetric spaces; see, e.g., [**BS58, Zil77**].

Together, the properties (b) and (c) of κ imply that one can recover $\check{M}$ from the $\operatorname{Spec} \mathrm{H}_*^H(\Omega(G/H); \mathbf{C})$-action on $\check{\mathfrak{h}}^*[2]/\!\!/\check{H}$, which ultimately leads to the proof of Theorem 1.1.10.

It is the isomorphism of (c) which we will establish in the rank 1 case through case-by-case analysis, since the spaces S^n, $\mathbf{C}P^n$, $\mathbf{H}P^n$, and $\mathbf{O}P^2$ form a finite list of such affine homogeneous spherical varieties up to finite covers (see [**Akh83**]). Although most of these cases behave quite similarly to each other, each case showcases some interesting basic homotopy-theoretic facts (see Table 4). In fact, we will establish the isomorphism of (c) even for homology with coefficients in $\mathbf{Z}[1/2]$, and in some cases with coefficients in ku (see the next section).

Example 1.2.2 (Geometrized spherical harmonics, i.e., $G/H = \mathrm{PGL}_2/\mathbf{G}_m = \mathrm{SO}_3/\mathrm{SO}_2$)**.** Let us illustrate (c) in the case of the spherical PGL_2-variety $\mathrm{PGL}_2/\mathbf{G}_m$ (so $G = \mathrm{PGL}_2 \cong \mathrm{SO}_3$ and $H = \mathbf{G}_m \cong \mathrm{SO}_2$). The Hopf fibration gives a homotopy equivalence $(\mathrm{PGL}_2/\mathbf{G}_m)(\mathbf{C}) \simeq S^2$, so (c) reduces to computing $\mathrm{H}_*^{S^1}(\Omega S^2; \mathbf{C})$. The Borel-equivariant analogue of this computation is quite simple: there is a homotopy fixed points spectral sequence

$$E_2^{*,*} \cong \mathrm{H}_*(\Omega S^2; \mathbf{C}) \otimes_{\mathbf{C}} \mathrm{H}^*(BS^1; \mathbf{C}) \Rightarrow \pi_* C_*(\Omega S^2; \mathbf{C})^{hS^1},$$

with a single d_2-differential. This spectral sequence degenerates on the E_3-page, and gives an isomorphism

$$\pi_* C_*(\Omega S^2; \mathbf{C})^{hS^1} \cong \mathbf{C}[\![x]\!][b]/bx,$$

where $|b| = 2$ and $|x| = -2$. Replacing the left-hand side by $\mathrm{H}_*^{S^1}(\Omega S^2; \mathbf{C})$ simply has the effect of making x into a polynomial (as opposed to power series) variable.

Ignoring degrees for a moment, write $\mathbf{A}^1 = \operatorname{Spec} \mathrm{H}^*_{S^1}(*; \mathbf{C})$, and let $\kappa : \mathbf{A}^1 \to T^*(\mathbf{A}^2)$ denote the map sending x to the point $(x, 1)$ in the cotangent fiber over $(1, 0) \in \mathbf{A}^2$. If we equip $T^*(\mathbf{A}^2)$ with its natural SL_2-action (coming from the SL_2-action on $\mathbf{A}^2$), one can compute that there is an isomorphism

$$\mathbf{A}^1 \times_{T^*(\mathbf{A}^2)/\mathrm{SL}_2} \mathbf{A}^1 \cong \operatorname{Spec} \mathbf{C}[x, b]/bx,$$

and hence an (ungraded) isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{S^1}(\Omega S^2; \mathbf{C}) \cong \mathbf{A}^1 \times_{T^*(\mathbf{A}^2)/\mathrm{SL}_2} \mathbf{A}^1.$$

The right-hand side can be equipped with a grading such that the above isomorphism is one of graded schemes, which gives (c). In the case of PGL_2, this leads to an equivalence

$$\mathrm{Shv}^c_{\mathrm{PGL}_2}(\mathcal{L}(\mathrm{PGL}_2/\mathbf{G}_m); \mathbf{C}) \simeq \mathrm{Perf}(T^*[2](\mathbf{A}^2[2,0])/\mathrm{SL}_2[-2\rho]).$$

Before proceeding to the ku-theoretic generalization, let us mention that many aspects of the BZSV conjecture can be understood from the perspective of Hochschild (co)homology. For the purposes of the introduction, it will be convenient to Borel-complete, i.e., to work with the ring $\pi_* \mathbf{C}[\Omega(G/H)]^{hH}$.

Proposition (Corollary 3.5.6, Corollary 3.5.8)**.** *The* $\mathbf{C}$*-algebra* $\mathbf{C}[\Omega(G/H)]^{hH}$ *can be identified with the relative Hochschild cohomology* $\mathrm{HC}(\mathbf{C}^{hH}/\mathbf{C}^{hG})$ *of the map*

$\mathbf{C}^{hG} \to \mathbf{C}^{hH}$ *(this was already observed in* [**Dev23a**, Remark A.6]*). In particular, the Hochschild-Kostant-Rosenberg theorem implies that there is an isomorphism*

$$\mathrm{Spf}\, \mathrm{H}^*(\mathcal{L}(G/H)_{hG}; \mathbf{C}) \cong T[-1]((\widehat{\check{\mathfrak{h}}}^*[2]/\!\!/\check{H})/(\widehat{\check{\mathfrak{g}}}^*[2]/\!\!/\check{G}),$$

where the right-hand side is the 1*-shifted relative tangent bundle of the map* $\widehat{\check{\mathfrak{h}}}^*[2]/\!\!/\check{H} \to \widehat{\check{\mathfrak{g}}}^*[2]/\!\!/\check{G}$*, and the hats denote completion at the origin. Moreover, the Deligne conjecture equips* $\mathbf{C}[\Omega(G/H)]^{hH}$ *with the structure of an* $\mathbf{E}_2$*-*$\mathbf{C}^{hG}$*-algebra.*

Remark 1.2.3. In Strategy 1.2.1, we said that Conjecture 3.4.11 often reduces to proving an isomorphism

$$\mathrm{Spec}\, \mathrm{H}_*^H(\Omega(G/H); \mathbf{C}) \cong \check{\mathfrak{h}}^*/\!\!/\check{H} \times_{\check{M}/\check{G}} \check{\mathfrak{h}}^*/\!\!/\check{H},$$

where we are ignoring gradings. The right-hand side is a group scheme over $\check{\mathfrak{h}}^*/\!\!/\check{H}$, which we denote by $\check{J}_X$; it is an analogue for $\check{M}$ of the regular centralizer group scheme. If the left-hand side in the above isomorphism is replaced by $\mathrm{Spf}\, \pi_* C^*_H(\Omega(G/H); \mathbf{C})^*$, i.e., the dual of equivariant *co*homology, the preceding proposition says that the right-hand side must be replaced by $T^*[1]((\check{\mathfrak{h}}^*/\!\!/\check{H})/(\check{\mathfrak{g}}^*/\!\!/\check{G}))$; this is simply the *Lie algebra* of $\check{J}_X$. In other words, the dual of equivariant cohomology allows one to access the Lie algebra of $\check{J}_X$; to understand $\check{J}_X$ itself involves a "decompletion", which in homotopy theory is given by working with equivariant homology itself (since this is a predual of equivariant cohomology).

The above result can be viewed as a relative version of [**BF08**, Theorem 1]. In a sense, most of this article can be regarded as an attempt to understand the decompletion of this Hochschild-Kostant-Rosenberg isomorphism. As a perhaps helpful guide, a general principle about equivalences of the form conjectured in [**BZSV23**] is that the ∞-category $\mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$ is equivalent to an ∞-category $\mathrm{Perf}(\check{M}/\check{G})$ of $\check{G}$-equivariant perfect complexes on $\check{M}$, which is often the twisted cotangent bundle $T^*_\psi(\check{X})$ of an $\check{G}$-variety $\check{X}$. This leads to some analogies between topology and algebra, a limited collection of which we have recorded in Table 2.

A-side/topology	B-side/algebra
Spherical G-variety X	$\check{G}$-variety $\check{X}$
Free loop space $\mathcal{L}X$	(Twisted) cotangent bundle $T^*_\psi(\check{X})$
Based loop space ΩX	Cotangent fiber
Sheaves on $(\mathcal{L}X)/G$ with $\mathbf{Q}$-coefficients	Perfect complexes on $T^*_\psi(\check{X})/\check{G}$

TABLE 2. Analogies between topology and algebra. Passing from the left-hand to the right-hand column is roughly implemented by (rational) cohomology.

The above relationship to Hochschild cohomology allows us to make an observation about the structure of a Hamiltonian $\check{G}$-variety on $\check{M}$. A result of Safronov's from [**Saf16**] can be used to translate the desired Hamiltonian $\check{G}$-structure on $\check{M}$ into the language of shifted symplectic geometry (à la [**PTVV13**]). This translation in turn predicts that equipping $\check{M}$ with the structure of a Hamiltonian $\check{G}$-variety should, in particular, imply that the map

$$\check{\mathfrak{h}}^*[2]/\!\!/\check{H} \times_{\check{M}/\check{G}[-2\rho]} \check{\mathfrak{h}}^*[2]/\!\!/\check{H} \to \check{\mathfrak{g}}^*[2]/\!\!/\check{G} \times_{\check{\mathfrak{g}}^*[2-2\rho]/\check{G}[-2\rho]} \check{\mathfrak{g}}^*[2]/\!\!/\check{G} \tag{4}$$

is coisotropic (in an appropriate derived sense). It turns out that this coisotropicity has a homotopy-theoretic explanation in terms of the ring $\mathrm{H}_*^H(\Omega(G/H); \mathbf{C})$.

Example 1.2.4. In the special case when G is replaced by $G \times G$ and H is replaced by the diagonal embedding of G (so that the associated homogeneous variety is just $(G \times G)/G^{\mathrm{diag}} \cong G$), the Hochschild cohomology $\mathrm{HC}(\mathbf{C}^{hG}/\mathbf{C}^{h(G\times G)})$ can be identified with the $\mathbf{E}_2$-center $\mathfrak{Z}_{\mathbf{E}_2}(\mathbf{C}^{hG}/\mathbf{C})$. Again, the Deligne conjecture equips $\mathbf{C}[\Omega G]^{hG} \simeq \mathfrak{Z}_{\mathbf{E}_2}(\mathbf{C}^{hG}/\mathbf{C})$ with the structure of an $\mathbf{E}_3$-$\mathbf{C}$-algebra; this $\mathbf{E}_3$-algebra structure is closely related to the $\mathbf{E}_3$-monoidality of the derived geometric Satake equivalence.

Note that $\pi_*\mathbf{C}[\Omega G]^{hG}$ is equipped with the structure of a (graded) Poisson algebra whose Poisson bracket has weight 2. This Poisson structure in fact comes from a symplectic form (of weight 2) on $\mathrm{Spf}\, \pi_*\mathbf{C}[\Omega G]^{hG}$. Moreover, Strategy 1.2.1 says, in particular, that there is an isomorphism

$$\mathrm{Spf}\, \pi_*\mathbf{C}[\Omega G]^{hG} \cong \widehat{\check{\mathfrak{g}}}^*[2]/\!\!/\check{G} \times_{\widehat{\check{\mathfrak{g}}}^*[2-2\rho]/\check{G}[-2\rho]} \widehat{\check{\mathfrak{g}}}^*[2]/\!\!/\check{G}.$$

Returning to relative Langlands, one can also show that if $H \subseteq G$ is a subgroup, $\mathbf{C}[\Omega(G/H)]^{hH}$ admits the structure of an $\mathbf{E}_2$-$\mathbf{C}[\Omega G]^{hG}$-algebra. The coisotropicity of (4) translates into the requirement that the natural map

$$\mathrm{Spec}\, \mathrm{H}_*^H(\Omega(G/H); \mathbf{C}) \to \mathrm{Spec}\, \mathrm{H}_*^G(\Omega G; \mathbf{C})$$

is coisotropic. One can directly prove the Borel-completed analogue of this requirement:

Proposition (Observation 5.2.10)**.** *The natural map*

$$\mathrm{Spf}\, \mathrm{H}^*_H(\Omega(G/H); \mathbf{C})^\vee \to \mathrm{Spf}\, \mathrm{H}^*_G(\Omega G; \mathbf{C})^\vee$$

is coisotropic (in an appropriate derived sense).

This result turns out to be a simple consequence of the fact that $\mathbf{C}[\Omega(G/H)]^{hH}$ admits the structure of an $\mathbf{E}_2$-$\mathbf{C}[\Omega G]^{hG}$-algebra, and a general property of $\mathbf{E}_n$-centers as established in [**Fra13**, Theorem 1.1]. We hope that further study of the relative Langlands program from the perspective of Hochschild (co)homology might shed more light into some of the structures predicted in [**BZSV23**].

1.3. ku-theoretic aberrations. In the course of proving Theorem 1.1.10, or even the derived geometric Satake equivalence, the reader will likely observe that many components of the proof do not depend very heavily on the particular choice of coefficient ring for the ∞-category of constructible sheaves. In particular, calculations such as that of $\mathrm{H}_*^G(\Omega G; \mathbf{C})$ (to construct the homomorphism (2)) work equally well with G-equivariant $\mathbf{C}$-(co)homology replaced by any well-behaved equivariant generalized cohomology theory. Motivated by this observation, our second goal in this article is to suggest that the (geometric) Langlands program should admit a generalization to sheaves with coefficients in more "exotic" rings, such as the sphere spectrum or complex cobordism. We will discuss the conceptual role of these coefficients in the next section.

Establishing this generalized form of the Langlands program is rather tricky, and so our focus in this article will be on the simpler example of *connective complex K-theory*. Our focus is on this particular example for at least two reasons: first, it is a general principle in homotopy theory that statements about ordinary rational/integral cohomology which admit analogues for connective K-theory will likely admit generalizations to other complex-oriented spectra; second, it is mostly psychological, in that it seems quite likely that the results of this article will admit analogues for equivariant elliptic cohomology, but this requires further technical setup and distracts from the main features of Langlands duality.[6] An analogue of the derived geometric Satake equivalence with coefficients in periodic complex K-theory and elliptic cohomology was proved in [**Dev23a**].

If G is a compact Lie group, Atiyah and Segal defined G-equivariant complex K-theory KU_G in [**Seg68, AS69**]: this is a generalized cohomology theory, viewed as a spectrum in the sense of homotopy theory, which classifies G-equivariant vector bundles on finite G-spaces. Direct sum and tensor products of G-equivariant vector bundles equips KU_G with the structure of a *ring* spectrum; in fact, it is an $\mathbf{E}_\infty$-ring, meaning (for instance) that the multiplication on cohomology can be refined by Adams operations. Despite its definition, the geometric interpretation of cocycles for equivariant K-theory as equivariant vector bundles will play no role below. Two examples will play an important conceptual role:

Example 1.3.1. When G is the trivial group, KU_G is simply periodic complex K-theory KU, and Bott periodicity gives a graded isomorphism $\pi_*\mathrm{KU} \cong \mathbf{Z}[\beta^{\pm 1}]$ with the Bott class β in weight 2. On the other hand, when G is a connected compact Lie group with complex representation ring $R_{\mathbf{C}}(G)$, the coefficient ring $\pi_*\mathrm{KU}_G$ is the tensor product $R_{\mathbf{C}}(G) \otimes_{\mathbf{Z}} \mathbf{Z}[\beta^{\pm 1}]$. In particular, if G is a torus T, then $\operatorname{Spec} \pi_*\mathrm{KU}_T$ is the corresponding algebraic torus $T_{\mathbf{Z}[\beta^{\pm 1}]}$ over $\mathbf{Z}[\beta^{\pm 1}]$.

Nonequivariant complex K-theory is in some sense the simplest generalized cohomology theory which is not just ordinary integral cohomology. In fact, the oft-cited analogies between them are more than coincidental:

Recollection 1.3.2. There is an $\mathbf{E}_\infty$-ring ku called *connective* complex K-theory such that there is a graded isomorphism $\pi_*\mathrm{ku} \cong \mathbf{Z}[\beta]$. If we set $\beta = 0$, this $\mathbf{E}_\infty$-ring simply degenerates to the Eilenberg-Maclane spectrum $\mathbf{Z}$ representing ordinary integral cohomology; and if we invert β, it recovers periodic complex K-theory. In other words, ku interpolates between $\mathbf{Z}$ and KU, and can be viewed as a one-parameter deformation of the ring $\mathbf{Z}$ in a homotopy-theoretic direction (namely, along the Bott class β).

If G is a compact Lie group, one can also construct an $\mathbf{E}_\infty$-ring ku_G called G-equivariant connective K-theory which interpolates between G-equivariant integral cohomology and G-equivariant (periodic) complex K-theory. In a precise sense (known in the homotopy-theoretic literature as *complex-oriented/abelian descent*), the $\mathbf{E}_\infty$-ring ku_G is determined by the $\mathbf{E}_\infty$-rings ku_T for compact abelian Lie groups T, which are in turn determined by the $\mathbf{E}_\infty$-ring ku_{S^1}.

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[6] A third reason, in keeping with the epigraph of our previous article [**Dev23a**], is yet another quote of ~~J. F. Adams~~ E_8 from [**Ada80**]: "[To] consider the question [of torsion in the cohomology of E_8] at all reveals a certain preoccupation with ordinary cohomology. Any impartial observer must marvel at your obsession with this obscure and unhelpful invariant. The author, like all respectable Lie groups, is much concerned to present a decorous and seemly appearance to the eyes of K-theory..." (It should be said immediately that we do not study E_8 in this article.) I do not have such strong feelings against ordinary cohomology, but the general thrust of this quote still applies: going from ordinary cohomology to K-theory should reveal deeper structures under Langlands duality.

Example 1.3.3. Since ku_{S^1} interpolates between S^1-equivariant integral cohomology and KU_{S^1}, and there are isomorphisms $\operatorname{Spec} \mathrm{H}^*_{S^1}(*; \mathbf{Z}) \cong \mathbf{G}_a(2)$ (with coordinate in weight -2) and $\operatorname{Spec} \pi_* \mathrm{KU}_{S^1} \cong \mathbf{G}_{m,\mathbf{Z}[\beta^{\pm 1}]}$, one expects $\operatorname{Spec} \pi_* \mathrm{ku}_{S^1}$ to interpolate between $\mathbf{G}_m$ and $\mathbf{G}_a$. In fact, equivariant connective K-theory is concocted so that $\operatorname{Spec} \pi_* \mathrm{ku}_{S^1}$ is the canonical degeneration from $\mathbf{G}_m$ to $\mathbf{G}_a$:

$$\operatorname{Spec} \pi_* \mathrm{ku}_{S^1} \cong \operatorname{Spec} \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}],$$

where x is in weight -2. We will denote this group scheme (where the group structure makes $1 + \beta x$ into a grouplike element) by $\mathbf{G}_\beta$. The case of a general compact abelian Lie group T is a straightforward generalization:

$$\operatorname{Spec} \pi_* \mathrm{ku}_T \cong \operatorname{Hom}(\mathbb{X}^*(T), \mathbf{G}_\beta) =: T_\beta.$$

Note that when $\beta = 0$, this group scheme is just $\mathfrak{t}(2)$; and when β is inverted, this group scheme is $T_{\mathbf{Z}[\beta^{\pm 1}]}$. This story is discussed further in Section 2.

Very simply, the effect of studying derived geometric Satake with coefficients in ku (instead of coefficients in $\mathbf{Z}$) is that the dual *group* remains unchanged, and every appearance/consequence of the Cartan subalgebra $\mathfrak{t}$ in the "classical" story is replaced by the group scheme T_β over $\mathbf{Z}[\beta]$. In order to make this more precise, let us explain the ku-theoretic analogue of the derived geometric Satake equivalence as proved in Theorem 3.6.21.

Setup 1.3.4. Write $\mathrm{sh}^{1/2}\mathbf{Z}[\beta]$ to denote the polynomial $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra where β lives in homological degree 2, so that it is obtained as a shearing of the graded ring $\pi_*(\mathrm{ku})$.

- There is a $\mathrm{sh}^{1/2}\mathbf{Z}[\beta]$-linear ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku})$ of $G\times G$-equivariant sheaves of ku-modules on $\mathcal{L}G$. The definition of this ∞-category is given in Construction 3.6.18. However, this definition is extremely unsatisfactory and *ad hoc*, so one might wish to view the notation $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku})$ as an object which we hope will agree with a "correct" definition (see Remark 3.6.20 for a little more on this point).
- Suppose $\check{G}$ is a group scheme defined over $\mathbf{Z}$. Let $\mathbf{G}_\beta^\vee$ denote the Cartier dual of $\mathbf{G}_\beta$, and let $\check{G}_\beta$ denote the group scheme over $\pi_*(\mathrm{ku}) \cong \mathbf{Z}[\beta]$ given by $\operatorname{Hom}(\mathbf{G}_\beta^\vee, G_{\mathbf{Z}[\beta]})$. We will view $\check{G}_\beta$ as a *β-deformation* of $\check{G}$. The quotient stack $\check{G}_\beta/\check{G}$ is related to the Hochschild-Kostant-Rosenberg filtration for the quotient stack $B\check{G}$.

Theorem 1.3.5 (Derived geometric Satake with ku-theoretic coefficients; Theorem 3.6.21)**.** *Let G be a simply-laced simply-connected semisimple Lie group, and invert the order of the Weyl group W (for simplicity). Then there is a $\mathrm{sh}^{1/2}\mathbf{Z}[1/|W|, \beta]$-linear equivalence*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)),$$

where $\check{G}^{\mathrm{sc}}$ is the simply-connected form of the Langlands dual group.

Remark 1.3.6. Upon setting $\beta = 0$, we have

$$\check{G}^{\mathrm{sc}}(-2\rho)_\beta|_{\beta=0} = \operatorname{Hom}(\mathbf{G}_a^\vee, \check{G}^{\mathrm{sc}}(-2\rho)) = \operatorname{Hom}(\hat{\mathbf{G}}_a^\sharp, \check{G}^{\mathrm{sc}}(-2\rho)) = \check{\mathfrak{g}}^{\mathrm{sc}}(2-2\rho);$$

here, $\hat{\mathbf{G}}_a^\sharp$ denotes the divided power completion of the origin in $\mathbf{G}_a$, further completed at the divided power filtration. Since $\check{G}$ is semisimple, its center is finite, and hence $\check{\mathfrak{g}}^{\mathrm{sc}}(2-2\rho) \cong \check{\mathfrak{g}}(2-2\rho)$. It follows that upon setting $\beta = 0$, the left-hand side of Theorem 1.3.5 becomes $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathbf{Z}[1/|W|])$, and the right-hand side becomes $\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]$. In other words, Theorem 1.3.5 just reduces to the derived geometric Satake equivalence (in the simply-laced simply-connected semisimple case).

On the other hand, upon inverting β, Theorem 1.3.5 is related to the KU-theoretic derived geometric Satake equivalence of [**Dev23a**], since

$$\check{G}^{\mathrm{sc}}(-2\rho)_\beta|_{\beta^{-1}} = \mathrm{Hom}(\mathbf{G}_m^\vee, \check{G}^{\mathrm{sc}}) = \mathrm{Hom}(\underline{\mathbf{Z}}, \check{G}^{\mathrm{sc}}) = \check{G}^{\mathrm{sc}}.$$

All objects on the right hand sides of the above displayed isomorphisms are to be understood as base-changed from $\mathbf{Z}$ to $\mathbf{Z}[\beta^{\pm 1}]$ (we omitted this from the notation for readability): since β lives in weight 2 and is invertible, we may ignore the -2ρ-shift. In particular, Theorem 1.3.5 implies that the β-adic filtration on $\mathrm{Shv}^{c,\mathrm{Sat}}_{(G\times G)/Z(G)}(\mathcal{L}G; \mathrm{KU})$ corresponds to the Hochschild-Kostant-Rosenberg filtration on the free loop space of the quotient stack $B\check{G}$.

Example 1.3.7. Let us illustrate Theorem 1.3.5 in the case when G is a torus T; the result still holds despite T not being simply-connected. Identifying $\mathcal{L}T$ with $T\times\Omega T$, we see that the ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku})$ is just $\mathrm{Shv}^c_T(\Omega T; \mathrm{ku})$. However, ΩT is simply the discrete set $\mathbb{X}_*(T)$ of cocharacters of T, so that $\mathrm{Shv}^c_T(\Omega T; \mathrm{ku}) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Shv}^c_T(*; \mathrm{ku})$. Almost by construction, there is an equivalence $\mathrm{Shv}^c_T(*; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} T_\beta)$. On the other hand, there is an equivalence $\bigoplus_{\mathbb{X}_*(T)} \mathrm{Mod}_{\mathrm{sh}^{1/2}\mathbf{Z}[\beta]} \simeq \mathrm{Perf}(B\check{T})$. Together, we obtain an equivalence $\mathrm{Shv}^c_T(\Omega T; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} T_\beta \times B\check{T})$, which is (analogous to) the right-hand side of Theorem 1.3.5.

Since the group scheme $\check{G}^{\mathrm{sc}}_\beta$ may seem somewhat mysterious, let us mention that it has an extremely concrete interpretation if, for instance, $\check{G} = \mathrm{PGL}_n$.

Example 1.3.8. When $\check{G} = \mathrm{PGL}_n$, we have $\check{G}^{\mathrm{sc}} = \mathrm{SL}_n$, and $\check{G}^{\mathrm{sc}}_\beta = \mathrm{SL}_{n,\beta}$ is the group scheme whose R-points (for R being a graded $\mathbf{Z}[\beta]$-algebra) consists of those $n\times n$-matrices A such that

$$\frac{\det(I+\beta A) - 1}{\beta} = 0.$$

Since the derivative of the determinant is the trace, the specialization of this condition to $\beta = 0$ is simply the condition that A is traceless. More conceptually, $\check{G}^{\mathrm{sc}}_\beta$ is a variant of the simply-connected form $\check{G}^{\mathrm{sc}}$ whose Cartan subgroup is replaced by its β-deformation, but whose unipotent parts remain unchanged. For instance, there is an analogue of the Bruhat decomposition for $\check{G}_\beta$ where the big cell is $\check{N}^- \times \check{T}_\beta \times \check{N}$.

Given Theorem 1.3.5, one is naturally led to wonder if there is an analogue of the relative Langlands program, and in particular of [**BZSV23**, Conjecture 7.5.1], in the context of ku-theoretic coefficients. We do not have a conjecture as precise as that of *loc. cit.* in this setting, but we do prove an analogue of Theorem 1.1.10 for affine homogeneous spherical varieties of rank 1 on types A_n, C_n, D_2, and G_2 with ku-theoretic coefficients. For instance, we have the following result describing "ku-theoretic geometrized spherical harmonics" for $\mathrm{PGL}_2/\mathbf{G}_m = \mathrm{SO}_3/\mathrm{SO}_2$:

Example 1.3.9 (Corollary 4.2.19 and Remark 5.1.18)**.** There is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2}(\mathcal{L}(\mathrm{PGL}_2/\mathbf{G}_m); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{V}_\beta/\mathrm{SL}_2(-2\rho)),$$

where $\check{V}_\beta$ is the affine closure of $\mathrm{SL}_2 \times^{\mathbf{G}_a} (\mathbf{G}_\beta \times \mathbf{A}^1) \subseteq T^*(\mathbf{A}^2_{\mathbf{Z}[\beta]} - \{0\})$, with $\mathbf{G}_a$ acting on $\mathbf{G}_\beta \times \mathbf{A}^1 \subseteq \mathbf{A}^2_{\mathbf{Z}[\beta]}$ via $b : (x, y) \mapsto (x, y + bx)$. The scheme $\check{V}_\beta$ can be viewed as a β-deformation of $T^*(\mathbf{A}^2)$; it can be explicitly identified with the open subscheme of $\mathbf{A}^4_{\mathbf{Z}[\beta]} = \mathbf{A}^4 \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[\beta]$ given by the complement of the hypersurface

$$1 + \beta(aD - Bc) = 0,$$

where a is in weight 0, c is in weight -2, B is in weight 0, and D is in weight -2.

The structure that seems to emerge out of these considerations is a β-deformation of the notion of a graded Hamiltonian $\check{G}$-variety, as explained in Section 5. To define this notion, it is convenient to use the language of shifted symplectic geometry as introduced in [**PTVV13**] (see Recollection 5.1.2 for a brief review).

Recollection 1.3.10. It was shown in [**Saf16**] that the quotient stack $\check{\mathfrak{g}}(2)/\check{G}$ admits a 1-shifted symplectic structure, and that a graded Hamiltonian $\check{G}$-variety $\check{M}$ is equivalent to the data of a *Lagrangian morphism* $\check{M}/\check{G} \to \check{\mathfrak{g}}(2)/\check{G}$. In particular, the local geometric story of [**BZSV23**] can be restated as the expectation that for certain affine spherical $G_{\mathbf{C}}$-varieties $G_{\mathbf{C}}/H_{\mathbf{C}}$, there is a dual Lagrangian morphism to $\check{\mathfrak{g}}(2)/\check{G}$ such that [**BZSV23**, Conjecture 7.5.1] holds.

In Proposition 5.1.10, we show that $\check{G}^{\mathrm{sc}}_\beta/\check{G}$ admits a 1-shifted symplectic structure. Moreover, the ku-theoretic calculations in the relative rank one cases of types A_n, C_n, D_2, and G_2 show that there are equivalences of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}_\beta/\check{G}(-2\rho)),$$

where $\check{M}_\beta$ is a graded $\check{G}$-variety over $\mathrm{Spec}\,\mathbf{Z}[\beta]$. The action of $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho))$ on the left-hand side of the above equivalence suggests that there is a Lagrangian morphism $\check{M}_\beta/\check{G} \to \check{G}^{\mathrm{sc}}_\beta/\check{G}$. One might therefore hope that the local geometric story of [**BZSV23**] admits a ku-theoretic analogue, where for certain affine spherical $G_{\mathbf{C}}$-varieties $G_{\mathbf{C}}/H_{\mathbf{C}}$, there is a dual Lagrangian morphism to $\check{G}^{\mathrm{sc}}_\beta/\check{G}$ such that an analogue of [**BZSV23**, Conjecture 7.5.1] holds.

Example 1.3.11. Lagrangian morphisms to $\check{G}^{\mathrm{sc}}_\beta/\check{G}$ give rise to ku-*Hamiltonian $\check{G}$-varieties*, which are roughly graded $\check{G}$-varieties $\check{M}_\beta$ over $\mathbf{Z}[\beta]$ with some additional structure and a moment map $\check{M}_\beta \to \check{G}^{\mathrm{sc}}_\beta$. See Definition 5.1.11. Specializing $\beta = 0$, the aforementioned result from [**Saf16**] says that ku-Hamiltonian $\check{G}$-varieties specialize to graded Hamiltonian $\check{G}$-varieties. Upon inverting β, there is an isomorphism $\check{G}^{\mathrm{sc}}_\beta[\beta^{-1}]/\check{G} \cong \check{G}^{\mathrm{sc}}/\check{G} \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[\beta^{\pm 1}]$, and another result from [**Saf16**] then implies that ku-Hamiltonian $\check{G}$-varieties give rise to *quasi-Hamiltonian $\check{G}$*-varieties in the sense of [**AMM98**].

Unfortunately, the structure theory of quasi-Hamiltonian varieties does not seem to be well-developed in the literature, so it is hard at the current moment to make conjectures as precise as those in [**BZSV23**] regarding the nature of these ku-Hamiltonian $\check{G}$-varieties. We do, however, strongly believe that such a theory will play an important role in understanding ku-theoretic deformations of the geometric Langlands program.

Remark 1.3.12. There is some precedence for ku-Hamiltonian varieties in the setting of integrable systems: namely, just as the phase space of a classical integrable system (such as the Calogero-Moser system and the Toda lattice) forms a

Hamiltonian variety, the phase space of a "relativistic" integrable system and its degeneration to a "nonrelativistic" system (such as the Ruijsenaars-Schneider system degenerating into the Calogero-Moser system, and the relativistic Toda lattice degenerating into the classical Toda lattice) naturally forms a ku-Hamiltonian variety. In the case of the relativistic Toda lattice degenerating into the classical Toda lattice, this essentially follows from [**BFM05**] and the calculations of Theorem 3.6.10. We will explain the example of the Ruijsenaars-Schneider system degenerating into the Calogero-Moser system in future work.

As the reader will likely observe (and as mentioned in the beginning of this subsection), the proofs of these statements for ku ultimately use very little on the specific structure of equivariant connective K-theory, and are almost axiomatic in nature: they only rely on certain basic properties of this equivariant cohomology theory, in particular an analogue of the Goresky-Kottwitz-MacPherson theorem [**GKM98**]. While proving Theorem 1.1.10 and its ku-theoretic variant, we will also describe some calculations of independent interest along the way (such as Remark 4.2.4 and Remark 5.4.11, which identifies the homology of $\Omega^2\mathrm{SU}(n)$ with the cohomology of the classifying stack of a shearing of the group scheme of length $n-1$ Witt vectors).

1.4. Spherical coefficients. One ultimate goal of this project is to understand a version of the (relative) geometric Langlands program with coefficients in the sphere spectrum. We do not resolve this problem in this article (far from it!), but instead formulate some expectations in Section 5.4; I hope that reporting my (meagre) partial progress will motivate further study into this topic.

In order to motivate why this is a natural question, let us begin with some general remarks about the nature of the geometric Satake equivalence. A longstanding expectation has been that Langlands-type equivalences are of a "motivic nature".

Example 1.4.1. In the arithmetic incarnation of the Langlands program, results of this form are often very deep; for instance, V. Lafforgue has conjectured an independence of ℓ result for Langlands parametrization in the case of global function fields (see [**Laf18**, Conjecture 12.12]).

As mentioned earlier in the introduction, it is often simpler to geometrize the archimedean and p-adic settings into the setting of complex curves. In this context, we will treat the word "motivic" as more a "way of life" instead of a precise mathematical word. For instance, the motivic nature of the geometric Langlands equivalence could be interpreted as the expectation that spectral decompositions should exist for sheaves/automorphic forms valued in (modules) over an (essentially) arbitrary base ring.

Example 1.4.2. One reflection of this motivic nature already appears in the geometric Satake equivalence from [**MV07**], which (re)constructs the Chevalley split form of a reductive group scheme. It is also true of the derived geometric Satake equivalence (see, e.g., [**YZ11, Zhu09**]); namely, Theorem 1.1.7 still holds if the coefficients $\mathbf{C}$ are replaced by some localization $\mathbf{Z}'$ of $\mathbf{Z}$, in which case $\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]$ is replaced by a lift to $\mathbf{Z}'$.

The field of motivic homotopy theory, as introduced by Morel-Voevodsky, suggests that the *stable* motivic category is a more refined version of (integral) motives (although one which is perhaps less accessible by the general public). Taking this perspective into account suggests that one can generalize the motivic expectation of geometric Langlands equivalences to also include sheaves with coefficients in ring spectra. The discussion of the preceding section (e.g., Theorem 1.3.5) shows that this expectation is not implausible: namely, although spectral decompositions exist, their nature is modified according to the behaviour of Chern classes for the ring spectrum.

Example 1.4.3. For instance, this relationship to Chern classes is the basic source of the difference between the case of "ordinary" derived geometric Satake (whose spectral side is $\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]$) and KU-theoretic derived geometric Satake (whose spectral side is $\check{G}^{\mathrm{sc}}/\check{G}$): indeed, Chern classes in integral cohomology and complex K-theory are very different from each other![7] In the case of elliptic cohomology with associated elliptic curve E (as studied in [**Dev23a**]), the spectral side is in turn replaced by the moduli stack $\mathrm{Bun}_{\check{G}}^{\mathrm{ss},0}(E^\vee)$ of semistable degree zero $\check{G}$-bundles on the dual elliptic curve $E^\vee$. The importance of Chern classes in geometric Langlands is reflected in our setting in Example 1.3.7, as well as in the classical setting of geometric Satake, where the the Chern class of the determinant line bundle on Gr_G for G semisimple can be identified with a regular nilpotent element for the dual Lie algebra $\check{\mathfrak{g}}$ (see [**Gin95**] and Proposition 5.4.8).

Motivated by this discussion, it is natural to wonder:

Question 1.4.4. Is there is an analogue of the derived geometric Satake equivalence with coefficients in an arbitrary ring spectrum R which admits a theory of Chern classes?

Such ring spectra are called *complex-oriented*. Associated to any complex-oriented ring spectrum R, one can define a graded (1-dimensional) formal group $\hat{\mathbf{G}}_R$ over $\pi_*(R)$ given by $\mathrm{Spf}\, \pi_*(R^{hS^1}) = \mathrm{Spf}\, \pi_* R^{\mathbf{C}P^\infty_+}$.

Observation 1.4.5. Let R be a complex-oriented ring spectrum. A generalization of the derived geometric Satake equivalence along the lines of Theorem 1.3.5 should involve replacing the 1-dimensional group scheme $\mathbf{G}_\beta$ by a 1-dimensional group scheme $\mathbf{G}_R$ which is related to the complex-oriented structure of R. If T is a maximal torus of a reductive algebraic group G, the group scheme $T_{\mathbf{G}} := \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G}_R)$ would play the role of a Cartan subgroup of a "$\mathbf{G}_R$-deformation" of the group scheme G.

However, the existence of a theory of Chern classes only grants us access to the *formal* group $\hat{\mathbf{G}}_R$, as opposed to an honest 1-dimensional algebraic group. (For instance, in the case of periodic complex K-theory KU, we have $\hat{\mathbf{G}}_{\mathrm{KU}} = \hat{\mathbf{G}}_m$, as opposed to the multiplicative group $\mathbf{G}_m$.) As discussed in [**Lur09**] (motivated by the Atiyah-Segal completion theorem), the data of a decompletion $\mathbf{G}_R$ of $\hat{\mathbf{G}}_R$ can be viewed as an algebraic incarnation of a *genuine S^1-equivariant* analogue R_{S^1} of R. Namely, $\mathbf{G}_R$ can be understood as the graded group scheme $\mathrm{Spec}\, \pi_*^{S^1}(R_{S^1})$ over $\pi_*(R)$ underlying $\mathrm{Spec}\, R_{S^1}$. Note that the group structure on $\mathbf{G}_R$ comes from the coproduct

$$R_{S^1} \to R_{S^1 \times S^1} \xleftarrow{\sim} R_{S^1} \otimes_R R_{S^1};$$

[7]More precisely, the first Chern class is additive in integral cohomology, but is (essentially) multiplicative in complex K-theory. This distinction manifests itself in many ways in other (related) parts of mathematics; for instance, the Todd class appearing in the Grothendieck-Riemann-Roch theorem is just the ratio of the first Chern classes in rational cohomology and K-theory.

in particular, it is important that the assignment $T \mapsto R_T$ from tori satisfy the Künneth formula. Therefore, one should require the additional data of a genuine equivariant refinement of R in order to answer Question 1.4.4.

Remark 1.4.6. There is in fact a universal example of a complex-oriented ring spectrum, given by *complex cobordism* MU. This is an $\mathbf{E}_\infty$-ring whose origin is geometric in nature (via cobordism classes of stably almost-complex manifolds), but nevertheless exerts strong control over the ∞-category of spectra (in a sense described below). In similar fashion, there is a universal ring L_* carrying a 1-dimensional formal group equipped with a coordinate; a theorem of Lazard's shows that L_* – called the *Lazard ring* – is isomorphic to a polynomial algebra on infinitely many generators (which encode the coefficients of the group law in the chosen coordinate). In **[Qui69]**, Quillen showed the following profound and deeply influential statement:

Theorem (Quillen)**.** *The map $L_* \to \pi_*(\mathrm{MU})$ classifying the 1-dimensional formal group $\hat{\mathbf{G}}_{\mathrm{MU}}$ is an isomorphism. In other words, the universal 1-dimensional formal group with a coordinate can be identified with the homotopically-defined formal group $\hat{\mathbf{G}}_{\mathrm{MU}}$.*

If G is a compact Lie group, there is also a notion of *G-equivariant* complex cobordism MU_G, defined using equivariant Thom spaces (and not geometrically via equivariant cobordism, thanks to the failure of equivariant transversality); see **[Uri18]** for a survey. However, setting $\mathbf{G}_{\mathrm{MU}} = \operatorname{Spec} \pi_*^{S^1}(\mathrm{MU}_{S^1})$ does *not* produce a 1-dimensional group scheme! The problem is precisely the failure of the Künneth formula for the assignment $T \mapsto \mathrm{MU}_T$. Instead, as explained in **[Hau22]**, the appropriate structure encoded by the assignment $T \mapsto \mathrm{MU}_T$ is that of a *graded group law*. This is a functor $\mathbf{G}$ from abelian compact Lie groups to graded commutative rings satisfying a certain condition which forces $\operatorname{Spec} \mathbf{G}(S^1)$ to behave like a 1-dimensional group scheme over $\mathbf{G}(*)$. (See Definition 5.3.6 for further discussion.) In **[Hau22**, Theorem C], Hausmann showed that the assignment $T \mapsto \pi_*^T \mathrm{MU}_T$ defines the universal graded group law; this can be regarded as an analogue of Quillen's theorem about MU.

A positive answer to Question 1.4.4 in the universal case of MU would therefore suggest that if $\check{G}$ is a (split) reductive algebraic group (over $\mathbf{Z}$, say) with a chosen maximal torus $\check{T}$, then *every* graded group law $\mathbf{G}$ defines a "$\mathbf{G}$-analogue" $\check{G}_{\mathbf{G}}$ of $\check{G}$, where the role of the Cartan subgroup is played by $\check{T}_{\mathbf{G}} := \operatorname{Spec} \mathbf{G}(\mathbb{X}^*(\check{T}))$. Unfortunately, I do not know how to define such a $\mathbf{G}$-analogue. If $\mathbf{G}$ comes from a 1-dimensional algebraic group, Setup 1.3.4 suggests defining $\check{G}_{\mathbf{G}} := \operatorname{Hom}(\mathbf{G}^\vee, \check{G})$, where $\mathbf{G}^\vee$ is the Cartier dual of $\mathbf{G}$ (but this definition is also somewhat lacking).[8] The definition of a graded group law, however, is so general that it is not clear how to define Cartier duals in this context (or even if it should be possible to do so!). Nevertheless, we propose some expectations in Section 5.4 about a putative derived geometric Satake equivalence with coefficients in MU, but (as the reader will see) we could not make it very far before getting stuck.

[8] For instance, it cannot be correct if $\mathbf{G}$ is an elliptic curve, since its Cartier dual is the zero group. However, one could instead define the quotient stack $\check{G}_{\mathbf{G}}/\check{G}$ as $\mathrm{Map}(\operatorname{Hom}(\mathbf{G}, B\mathbf{G}_m), B\check{G})$; this, too, is not quite correct in the case of an elliptic curve, but for more subtle reasons.

Remark 1.4.7. It is quite easy to give a positive answer to Question 1.4.4 in the case of a torus: namely, for any reasonably definition of T-equivariant sheaves of MU-modules on (ind-finite) T-spaces, there will be an equivalence

$$\mathrm{Shv}^c_{T\times T}(\mathcal{L}T;\mathrm{MU}) \simeq \mathrm{Perf}(T_{\mathbf{G}_{\mathrm{MU}}} \times B\check{T}).$$

Remark 1.4.8. The sphere spectrum is *not* complex-oriented, so it is not clear that there should be an analogue of the derived geometric Satake equivalence with coefficients in S^0. Nevertheless, the sphere spectrum admits a "local" complex-orientation, in the sense that the unit map $S^0 \to \mathrm{MU}$ behaves as an fpqc cover. More precisely, work of Quillen and Landweber-Novikov suggests that rather than considering $\mathrm{Spec}\,\pi_*(S^0)$, it is more appropriate to consider the stack $\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}$ of 1-dimensional formal groups equipped with a square root of the dualizing line bundle; the stacky nature of $\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}$ corresponds to the failure of S^0 to be complex-oriented. Using the discussion involving equivariant complex cobordism above, we formulate some expectations in Section 5.4 about a putative derived geometric Satake equivalence with coefficients in the sphere spectrum. However, making these expectations into precise conjectures seems to require fundamental new inputs from (equivariant) homotopy theory.

All the aforementioned difficulties come from attempting to view the effect of genuine equivariance on (spectral) algebraic geometry. However, some aspects of Langlands duality do not require working with equivariant cohomology. For instance, recall from **[Gin95]** (see Proposition 5.4.8) that there is an isomorphism $\mathrm{H}^*(\Omega G;\mathbf{C}) \cong U(\check{\mathfrak{g}}^e)$ of Hopf algebras, with e being the principal nilpotent element corresponding to the Chern class of the determinant line bundle $\Omega G \to \mathbf{C}P^\infty$. In Section 5.4, we prove an analogue of this isomorphism for the sphere spectrum in the case $G = \mathrm{SU}(n)$. This calculation is rather simple, and it would be interesting to prove an analogue for arbitrary compact Lie groups.

1.5. Notation.

In writing this article, I discovered that it is extremely easy to fall into grading hell[9], and escaping it is a painful task; I hope the following list of conventions is helpful to the reader.

Notation 1.5.1. We will always use homological degrees. For instance, if X is a space, a class $x \in \mathrm{H}^n(X;\mathbf{Q})$ in cohomology lives in homological degree $-n$.

Notation 1.5.2. Let V be a finite-dimensional affine space over a (possibly graded) commutative ring R, so that $V = \mathrm{Spec}\,\mathrm{Sym}^*_R(V^*)$. We will denote $V(n)$ to denote the affine space V with weight n. Then, we have

$$V(n) \cong \mathrm{Spec}\,\mathrm{Sym}^*_R(V(n)^*) \cong \mathrm{Spec}\,\mathrm{Sym}^*_R(V^*(-n)).$$

We will also write $\mathbf{A}^n(i_1,\cdots,i_n)$ to denote the product $\prod_{j=1}^n \mathbf{A}^1(i_j)$.

Notation 1.5.3. Let V be a finite-dimensional affine space over a (possibly graded) commutative ring R, so that $V = \mathrm{Spec}\,\mathrm{Sym}^*_R(V^*)$. If $n \in \mathbf{Z}$, we will write $V[n]$ to denote the derived R-scheme which underlies the graded derived R-scheme $V[n](n) = \mathrm{Spec}\,\mathrm{sh}^{1/2}\,\mathrm{Sym}^*_R(V(n)^*)$. Note that by Lemma 2.1.5, this definition

[9] Especially as a graduate student!

may not be well-behaved unless n is even (but this will be the case in all examples of interest). Note that

$$\mathrm{sh}^{1/2}\,\mathrm{Sym}^*_R(V(n)^*) = \bigoplus_{j\geq 0} \mathrm{sh}^{1/2} V^*(-nj) = \bigoplus_{j\geq 0} V^*(-nj)[-nj],$$

so that $V[n](n)$ is the graded derived R-scheme where the coordinate lives in degree n and weight n.

Warning 1.5.4. Note that $V[n]$ is generally *not* equivalent to the derived R-scheme $\mathrm{Spec}\,\mathrm{Sym}^*_R(V^*[-n])$! For example, suppose $n = -2$, and say that $V = \mathbf{A}^1_R$ itself. Then the ring of functions $\mathcal{O}_{V[-2]}$ is the polynomial algebra $R[x]$ with $|x| = 2$, but $\mathrm{Sym}^*_R(V^*[-n]) = \mathrm{Sym}^*_R(R[2])$ is isomorphic to the sheared divided power algebra $\bigoplus_{j\geq 0} \Gamma^j_R(R)[2j]$ by décalage. Of course, if R is a $\mathbf{Q}$-algebra, these two algebras are isomorphic to each other, but it is often (psychologically) safer to not make this assumption.

Notation 1.5.5. If A is a ring spectrum with even homotopy groups, one obtains a graded affine scheme $\mathrm{Spec}\,\pi_* A$. In particular, a class $x \in \pi_n A$ defines a map $\mathrm{Spec}\,\pi_* A \to \mathbf{A}^1(-n)$, i.e., lives in weight n. This is somewhat opposed to standard practice in homotopy theory, where a class in $\pi_{2n} A$ lives in weight n (as opposed to our convention, where it has weight $2n$).

Notation 1.5.6. We will often write $\mathbf{Z}'$ to denote a localization of the ring of integers. This will essentially always mean that the prime 2 has been inverted. In some instances, it will denote the localization $\mathbf{Z}[1/|W|]$ obtained by inverting the order of a Weyl group.

Notation 1.5.7. The symbol HC will denote Hochschild cohomology, and HH will denote Hochschild homology.

Notation 1.5.8. We will always write a double-slash to mean GIT quotients, and not stacky quotients (which will be denoted by a single slash). For instance, $\mathfrak{g}(2)/\!\!/G = \mathrm{Spec}\,\mathrm{Sym}(\mathfrak{g}^*(-2))^G$, while $\mathfrak{g}(2)/G$ is a graded stack over BG.

1.6. Acknowledgements. I am very grateful to Yiannis Sakellaridis for extremely helpful conversations, his encouragement and patience, and showing me [**Sak21**, Table 1] when I visited Johns Hopkins University in February 2023. I am very grateful to David Gepner and Milton Lin for the opportunity to visit. For me, this table made clear the analogy between Langlands dual L-functions for rank 1 spherical varieties, and the James splitting and the EHP + generalized Hopf fibrations; this was the starting point of the present article, so I owe its entire existence to Yiannis. I am also grateful to David Ben-Zvi, Justin Hilburn, Mark Macerato, Kevin Lin, and Tomer Schlank for helpful exchanges and enjoyable conversations; I'm especially grateful to Mark Macerato for giving me very helpful feedback on this writeup. Finally, as always, I am very thankful to my advisors Dennis Gaitsgory and Mike Hopkins for their support and encouragement.

2. Equivariant connective K-theory

2.1. Shearing and the Koszul sign rule. The operation of *shearing* will play a vital role in many of our constructions below. Outside of a few sources like [**Lur15**] and [**Rak20**, Proposition 3.3.4], there does not seem to be much literature developing this notion from a homotopy-coherent perspective. Let us recall the shearing functor $\mathrm{sh} : \mathrm{Sp}^{\mathrm{gr}} \xrightarrow{\sim} \mathrm{Sp}^{\mathrm{gr}}$ as constructed in [**Rak20**, Proposition 3.3.4]. Throughout this section, we will let $\mathbf{Z}^{\mathrm{ds}}$ denote the set of integers viewed as a discrete space.

Construction 2.1.1. Recall from the universal property of Day convolution that a lax $\mathbf{E}_2$-monoidal functor $\mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ is the same data as a lax $\mathbf{E}_2$-monoidal functor $\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}$. Let $f : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathrm{Sp})$ denote the $\mathbf{E}_2$-composite

$$\mathbf{Z}^{\mathrm{ds}} \xrightarrow{\Omega^2(\mathbf{C}P^\infty \to \mathrm{BU})} \mathrm{BU} \times \mathbf{Z} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}).$$

This defines a lax $\mathbf{E}_2$-monoidal functor via the composite

$$\mathbf{Z}^{\mathrm{ds}} \times \mathrm{Sp}^{\mathrm{gr}} = \mathbf{Z}^{\mathrm{ds}} \times \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathrm{Sp}) \xrightarrow{f,\mathrm{ev}} \mathrm{Pic}(\mathrm{Sp}) \times \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp}.$$

It is not difficult to see that the lax $\mathbf{E}_2$-monoidal functor $\mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ constructed above is in fact a strictly $\mathbf{E}_2$-monoidal equivalence. This functor will be denoted by sh, and will be called shearing. Explicitly, it sends $M_\bullet \mapsto M_\bullet[2\bullet]$, with $\bullet$ denoting the weight.

Remark 2.1.2. In [**DHL$^+$23**, Proposition 3.10], it is shown that the functor $\mathrm{sh} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ is in fact a *framed* $\mathbf{E}_2$-monoidal functor. However, it *cannot* be made into an $\mathbf{E}_3$-monoidal functor (see [**DHL$^+$23**, Remark 3.11]).

A first simple observation is the following.

Lemma 2.1.3. *The shearing functor* $\mathrm{sh} : \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}}$ *admits a natural symmetric monoidal structure.*

Proof. The same argument as in Construction 2.1.1 will show that the shearing functor $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}}$ is symmetric monoidal, as long as the map $\mathbf{Z}^{\mathrm{ds}} \xrightarrow{f} \mathrm{Pic}(\mathrm{Sp}) \to \mathrm{Pic}(\mathrm{MU})$ admits an $\mathbf{E}_\infty$-structure. Let $J : \mathrm{BU} \times \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathrm{Sp})$ denote the J-homomorphism, so that J is an $\mathbf{E}_\infty$-map. Since MU is the Thom spectrum of the $\mathbf{E}_\infty$-composite $\mathrm{BU} \to \mathrm{BU} \times \mathbf{Z}^{\mathrm{ds}} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp})$, it can be understood as the initial $\mathbf{E}_\infty$-ring R equipped with a nullhomotopy of the $\mathbf{E}_\infty$-map $\mathrm{BU} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}) \to \mathrm{Pic}(R)$. In particular, there is a commutative diagram of $\mathbf{E}_\infty$-maps:

$$\begin{tikzcd}
\mathrm{BU} \times \mathbf{Z} = \Omega^\infty \mathrm{ku} \arrow[r, "J"] \arrow[d] & \mathrm{Pic}(\mathrm{Sp}) \arrow[d] \\
\mathbf{Z}^{\mathrm{ds}} \simeq \Omega^\infty \tau_{\leq 0}\mathrm{ku} \arrow[r] & \mathrm{Pic}(\mathrm{MU}),
\end{tikzcd}$$

which proves the desired claim. □

Remark 2.1.4. One might wonder whether there is also an $\mathbf{E}_2$-monoidal structure on the functor $\mathrm{sh}^{1/2} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ which sends $M_\bullet \mapsto M_\bullet[\bullet]$. In fact, one can already ask whether there is an $\mathbf{E}_2$-monoidal structure on the functor $\mathrm{sh}^{1/2} : \mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}} \to \mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}}$ which sends $M_\bullet \mapsto M_\bullet[\bullet]$. The essential difficulty is that of the Koszul sign rule. Namely, suppose that there was an $\mathbf{E}_2$-monoidal structure on $\mathrm{sh}^{1/2}$. Applying $\mathrm{sh}^{1/2}$ to the graded $\mathbf{E}_\infty$-algebra $\mathbf{Z}[x]$ with x in degree zero and weight 1 would produce a graded $\mathbf{E}_2$-algebra $\mathbf{Z}[w]$ with w in degree 1 and weight 1. The Koszul sign rule forces $w^2 = -w^2$, i.e., $2w^2 = 0$, which is a contradiction. This is one of the basic topological reasons for why we will work with evenly-graded objects throughout this article.

Let us note the following related result:

Lemma 2.1.5. *There is no $\mathbf{E}_2$-map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathbf{Z})$ sending $1 \mapsto \mathbf{Z}[1]$. However, there is a unique $\mathbf{E}_1$-map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathrm{Sp})$ sending $1 \mapsto S^1$, and hence the method of Construction 2.1.1 produces an $\mathbf{E}_1$-monoidal structure on the functor $\mathrm{sh}^{1/2} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$.*

Proof. Let us first show that there is a unique $\mathbf{E}_1$-map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathrm{Sp})$ sending $1 \mapsto S^1$. This is easy, since $\mathbf{Z}^{\mathrm{ds}}$ is the group completion of the free $\mathbf{E}_1$-space $\mathbf{Z}^{\mathrm{ds}}_{\geq 0}$ on a single class: the choice of $S^1 \in \pi_0 \mathrm{Pic}(\mathrm{Sp})$ defines an $\mathbf{E}_1$-map $\mathbf{Z}^{\mathrm{ds}}_{\geq 0} \to \mathrm{Pic}(\mathrm{Sp})$, which factors through $\mathbf{Z}^{\mathrm{ds}}$ since $\mathrm{Pic}(\mathrm{Sp})$ is group-complete.

To show that there is no $\mathbf{E}_2$-map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathbf{Z})$ sending $1 \mapsto \mathbf{Z}[1]$, let us make the following observation. There is an fiber sequence of $\mathbf{E}_\infty$-spaces

$$\tau_{\geq 1} \mathrm{Pic}(\mathbf{Z}) = \mathrm{BGL}_1(\mathbf{Z}) = \mathbf{R}P^\infty \to \mathrm{Pic}(\mathbf{Z}) \to \pi_0 \mathrm{Pic}(\mathbf{Z}) = \mathbf{Z}^{\mathrm{ds}}, \tag{5}$$

where the final $\mathbf{Z}^{\mathrm{ds}}$ is generated by S^1. An $\mathbf{E}_2$-map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(\mathbf{Z})$ would give an $\mathbf{E}_2$-splitting of this fiber sequence, which we claim is impossible. Indeed, let $\mathfrak{pic}(\mathbf{Z})$ denote the connective spectrum delooping $\mathrm{Pic}(\mathbf{Z})$; then the above fiber sequence deloops to a cofiber sequence

$$\Sigma \mathfrak{gl}_1(\mathbf{Z}) = \mathbf{F}_2[1] \to \mathfrak{pic}(\mathbf{Z}) \to \mathbf{Z}. \tag{6}$$

One can use the J-homomorphism and the $\mathbf{E}_\infty$-map $\mathrm{MSO} \to \mathbf{Z}$ to show that $\mathfrak{pic}(\mathbf{Z}) \simeq \tau_{\leq 1}\mathrm{ko}$. (Applying Ω^∞, this amounts to the identification $\mathrm{Pic}(\mathbf{Z}) \simeq (\mathbf{Z} \times \mathrm{BO})/\mathrm{BSO}$.) Therefore, the boundary map in (6) identifies with the first k-invariant of ko, which is given by the composite

$$\mathbf{Z} \to \mathbf{F}_2 \xrightarrow{\mathrm{Sq}^2} \mathbf{F}_2[2]. \tag{7}$$

In particular, (6) is *not* split as a cofiber sequence of spectra (this can be viewed as a manifestation of the Koszul sign rule). However, the above description of the boundary map also lets us show that (5) does not split as a fiber sequence of $\mathbf{E}_2$-spaces. Namely, if (5) did split as a fiber sequence of $\mathbf{E}_2$-spaces, then the twice-delooped fiber sequence

$$K(\mathbf{Z}/2, 3) \to B^2 \mathrm{Pic}(\mathbf{Z}) \to B^2 \mathbf{Z}^{\mathrm{ds}} = \mathbf{C}P^\infty$$

would also admit a splitting. But the boundary map $\mathbf{C}P^\infty \to K(\mathbf{Z}/2, 4)$ in this fiber sequence represents the generator of $\mathrm{H}^4(\mathbf{C}P^\infty; \mathbf{Z}/2)$, which is certainly nonzero. □

Remark 2.1.6. Lest Lemma 2.1.5 seem like a problem specific to $\mathrm{Pic}(\mathbf{Z})$, we note that the same problem persists for $\mathrm{Pic}(\mathbf{Q})$ (as well as for $\mathrm{Pic}(\mathbf{F}_p)$ with $p > 2$, but not for $\mathrm{Pic}(\mathbf{F}_2) \cong \mathbf{Z}$). Indeed, the Postnikov fiber sequence for $\mathfrak{pic}(\mathbf{Q})$ is given by

$$\Sigma \mathfrak{gl}_1(\mathbf{Q}) = \mathbf{Q}^\times[1] \to \mathfrak{pic}(\mathbf{Q}) \to \pi_0 \mathfrak{pic}(\mathbf{Q}) = \mathbf{Z}^{\mathrm{ds}}.$$

Recall that $\mathbf{Q}^\times \cong \mathbf{Z}/2 \oplus \bigoplus_{\text{primes}} \mathbf{Z}$. Under this identification, the boundary map $\mathbf{Z}^{\text{ds}} \to \mathbf{Q}^\times[2]$ composes to the composite

$$\mathbf{Z} \to \mathbf{Z}/2[2] \oplus \bigoplus_{\text{primes}} \mathbf{Z}[2] \xrightarrow{\text{pr}} \mathbf{Z}/2[2]$$

which identifies with (7). Since this composite is not null as an $\mathbf{E}_2$-map upon applying Ω^∞ by the argument of Lemma 2.1.5, the map $\text{Pic}(\mathbf{Q}) \to \mathbf{Z}^{\text{ds}}$ does not admit an $\mathbf{E}_2$-splitting.

Remark 2.1.7. Consider the fully faithful functor $2 : \text{Sp}^{\text{gr}} \hookrightarrow \text{Sp}^{\text{gr}}$ which doubles the weight. Then the composite

$$\text{Sp}^{\text{gr}} \xrightarrow{2} \text{Sp}^{\text{gr}} \xrightarrow{\text{sh}^{1/2}} \text{Sp}^{\text{gr}}$$

identifies with the usual shearing functor, and hence admits an $\mathbf{E}_2$-monoidal structure. Similarly, if we replace Sp^{gr} by $\text{Mod}^{\text{gr}}_{\text{MU}}$ (and in particular $\text{Mod}^{\text{gr}}_{\mathbf{Z}}$), the analogue of the above composite admits a symmetric monoidal structure. In particular, if $M_\bullet$ is an $\mathbf{E}_n$-algebra in graded spectra (resp. graded MU- or $\mathbf{Z}$-module spectra) which is concentrated in even weights, its "half-shear" $\text{sh}^{1/2}(M_\bullet)$ admits an $\mathbf{E}_{\min(n,2)}$-algebra structure in graded spectra (resp. $\mathbf{E}_n$-algebra structure in MU- or $\mathbf{Z}$-module spectra).

Moreover, $\text{sh}^{1/2} : \text{Mod}^{\text{gr}}_{\mathbf{F}_2} \to \text{Mod}^{\text{gr}}_{\mathbf{F}_2}$ admits a *symmetric* monoidal structure. Indeed, there is an $\mathbf{E}_\infty$-map $\mathbf{Z}^{\text{ds}} \to \text{Pic}(\mathbf{F}_2)$ sending $1 \mapsto \mathbf{F}_2[1]$. There are many ways to see this; for instance, one can argue as in Lemma 2.1.3. Namely let $J_{\mathbf{R}} : \text{BO} \times \mathbf{Z}^{\text{ds}} \to \text{Pic}(\text{Sp})$ denote the real J-homomorphism, so that $J_{\mathbf{R}}$ is an $\mathbf{E}_\infty$-map, and $1 \in \mathbf{Z}^{\text{ds}} \mapsto S^1 \in \text{Pic}(\text{Sp})$. Since MO is the Thom spectrum of the $\mathbf{E}_\infty$-composite $\text{BO} \to \text{BO} \times \mathbf{Z}^{\text{ds}} \xrightarrow{J_{\mathbf{R}}} \text{Pic}(\text{Sp})$, it can be understood as the initial $\mathbf{E}_\infty$-ring R equipped with a nullhomotopy of the $\mathbf{E}_\infty$-map $\text{BO} \xrightarrow{J_{\mathbf{R}}} \text{Pic}(\text{Sp}) \to \text{Pic}(R)$. In particular, there is a commutative diagram of $\mathbf{E}_\infty$-maps:

$$\begin{tikzcd}
\text{BO} \times \mathbf{Z} = \Omega^\infty \text{ko} \arrow[r, "J_{\mathbf{R}}"] \arrow[d] & \text{Pic}(\text{Sp}) \arrow[d] \\
\mathbf{Z}^{\text{ds}} \simeq \Omega^\infty \tau_{\leq 0}\text{ko} \arrow[r] & \text{Pic}(\text{MO}).
\end{tikzcd}$$

There is an $\mathbf{E}_\infty$-orientation $\text{MO} \to \mathbf{F}_2$, so we obtain an $\mathbf{E}_\infty$-map $\mathbf{Z}^{\text{ds}} \to \text{Pic}(\text{MO}) \to \text{Pic}(\mathbf{F}_2)$ sending $1 \mapsto \mathbf{F}_2[1]$, as desired.

The following observation will be useful below; in particular, the final sentence says that have polynomial homotopy on even-degree classes automatically forces "formality", even as an $\mathbf{E}_2$-algebra (over the sphere spectrum).

Lemma 2.1.8. *If R is an $\mathbf{E}_\infty$-$\mathbf{F}_2$-algebra, and A is an $\mathbf{E}_1$-R-algebra such that $\pi_\bullet A$ is a finitely generated polynomial R-algebra (whose generators need not live in even degrees), there is an equivalence $A \simeq \text{sh}^{1/2}\pi_\bullet A$ of $\mathbf{E}_1$-R-algebras.*

If R is an $\mathbf{E}_\infty$-ring, and A is an $\mathbf{E}_1$-R-algebra such that $\pi_\bullet A$ is a finitely generated polynomial R-algebra generated by classes in even degrees, there is an equivalence $A \simeq \text{sh}^{1/2}\pi_\bullet A$ of $\mathbf{E}_1$-R-algebras. If A furthermore admits an $\mathbf{E}_2$-R-algebra structure, the equivalence $A \simeq \text{sh}^{1/2}\pi_\bullet A$ can be upgraded to one of $\mathbf{E}_2$-R-algebras.

PROOF. Let us first prove the claims about $\mathbf{E}_1$-algebra equivalences. Write $\pi_\bullet A \cong R[x_1, \cdots, x_n]$ with x_j in weight $2i_j$. Observe that the shearing $\mathrm{sh}^{1/2} R[x_j]$ is the free $\mathbf{E}_1$-R-algebra on a class in degree $2i_j$. Similarly, if R is an $\mathbf{F}_2$-algebra and y_j lives in weight i_j, the shearing $\mathrm{sh}^{1/2} R[y_j]$ is the free $\mathbf{E}_1$-R-algebra on a class in degree i_j. This implies that there are $\mathbf{E}_1$-R-algebra maps $\mathrm{sh}^{1/2} R[x_j] \to A$ for all j such that x_j is sent to the eponymous class on homotopy groups. Together, these define maps $\bigotimes_{j=1}^n \mathrm{sh}^{1/2} R[x_j] \to A$. By construction, this map induces an isomorphism on homotopy (and hence is an equivalence).

Now assume that A is an $\mathbf{E}_2$-R-algebra such that $\pi_\bullet A$ is a finitely generated polynomial R-algebra generated by classes in even degrees; by induction on the number of polynomial generators, we may assume that $\pi_\bullet A \cong \pi_\bullet(R)[x_{2n}]$ with x_{2n} in weight $2n$. Let $R[x_{0,1}] = R[\mathbf{Z}_{\geq 0}]$ denote the flat *graded* polynomial R-algebra on a class in weight 1 and degree zero, so that $\mathrm{sh}^{\circ n}(R) = R[x_{2n,1}]$ is a graded polynomial R-algebra on a class in weight 1 and degree $2n$. By [**DHL$^+$23**, Corollary 3.12], $R[x_{2n,1}]$ admits the structure of a framed $\mathbf{E}_2$-algebra in $\mathrm{Mod}_R^{\mathrm{gr}}$. Let $\mathrm{und}(R[x_{2n,1}])$ denote the underlying ungraded $\mathbf{E}_2$-R-algebra (in the body of this article, we will often simply omit "und", but we keep it here for clarity), so that $\mathrm{und}(R[x_{2n,1}])$ is the free $\mathbf{E}_1$-R-algebra on a class in degree $2n$. We will construct an equivalence $\mathrm{und}(R[x_{2n,1}]) \xrightarrow{\sim} A$ of $\mathbf{E}_2$-R-algebras.

To do this, it suffices to show that $\mathrm{und}(R[x_{2n,1}])$ admits an *even* cell structure as an $\mathbf{E}_2$-R-algebra. Indeed, the class $x_{2n} \in \pi_{2n}(A)$ defines a map from the bottom $\mathbf{E}_2$-cell into A; all obstructions to extending this map along the higher $\mathbf{E}_2$-cells of $\mathrm{und}(R[x_{2n,1}])$ live in odd degrees, but the odd homotopy of A vanishes, so such an extension $\mathrm{und}(R[x_{2n,1}]) \to A$ exists. By construction, this map is an isomorphism on homotopy, and hence is an equivalence. To construct an $\mathbf{E}_2$-cell structure for $\mathrm{und}(R[x_{2n,1}])$, note that it in fact suffices to construct an $\mathbf{E}_2$-cell structure for $R[x_{0,1}]$ in $\mathrm{Alg}_{\mathbf{E}_2}(\mathrm{Mod}_R^{\mathrm{gr}})$: indeed, the desired $\mathbf{E}_2$-cell structure on $\mathrm{und}(R[x_{2n,1}])$ then follows from shearing and the fact that sh is $\mathbf{E}_2$-monoidal by Construction 2.1.1. Since $R[x_{0,1}]$ is an augmented R-algebra whose augmentation ideal is concentrated in positive weights, an $\mathbf{E}_2$-cell structure for $R[x_{0,1}]$ is specified by the 2-fold bar construction $\mathrm{Bar}^{(2)}(R[x_{0,1}])$. This is a standard calculation: one finds that $\mathrm{Bar}^{(2)}(R[x_{0,1}]) \simeq \bigoplus_{n\geq 0} R[2n](n)$, at least as R-modules. For a reference in slightly different language, see [**Lur15**, Proposition 3.4.5]. □

Remark 2.1.9. In [**DHL$^+$23**, Remark 3.11], we show that $\mathrm{sh} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ *cannot* be made into an $\mathbf{E}_3$-monoidal functor. This implies that, in the setting of Lemma 2.1.8, if A admits an $\mathbf{E}_3$-R-algebra structure, the equivalence $A \simeq \mathrm{sh}^{1/2}\pi_\bullet A$ need not upgrade to an equivalence of $\mathbf{E}_3$-R-algebras. This is closely related to the subtlety of refining the derived geometric Satake equivalence (proved in the present article as Theorem 3.2.7) into an $\mathbf{E}_3$-monoidal equivalence.

2.2. Equivariant K-theory. Let G be a compact Lie group. Atiyah and Segal constructed *G-equivariant K-theory* using the theory of G-equivariant vector bundles. We will review this theory here and describe the spectral algebro-geometric perspective on equivariant K-theory following [**Lur09**].

Definition 2.2.1. A *finite G-space X* is a space with G-action which is constructed from finitely many G-cells of the form $G/H \times D^n$, where $H \subseteq G$ is a closed subgroup.

Let $\mathcal{S}(G)$ denote the ∞-category of finite G-spaces and G-equivariant maps between them.

Definition 2.2.2. Let X be a finite G-space. A *G-equivariant vector bundle* on X is a vector bundle $\mathcal{V}$ over X equipped with a continuous G-action, such that the map $\mathcal{V} \to X$ is G-equivariant. Let $\mathrm{KU}^0_G(X)$ denote the Grothendieck group of the monoid of G-equivariant vector bundles on X.

Atiyah and Segal showed that the assignment $X \mapsto \mathrm{KU}^0_G(X)$ from the (opposite of the) homotopy category of finite G-spaces to groups extends to a cohomology theory which is represented in the homotopy category of G-spectra by a spectrum denoted by KU_G.

In order to see all the structure on equivariant K-theory, it will be convenient to phrase the construction in terms of the ∞-category of orbispaces.

Definition 2.2.3. Let Orb denote the *global orbit ∞-category* as defined in [**GM20**, Definition 2.7]. Heuristically, this is the full subcategory of the ∞-category of topological stacks spanned by objects of the form $*/G$. An *orbispace* is a functor $\mathrm{Orb}^{\mathrm{op}} \to \mathcal{S}$. Let $\mathcal{S}_{\mathrm{Orb}}$ denote the ∞-category of orbispaces.

Let $\mathcal{S}_G$ denote the ∞-category of G-spaces, and let Orb_G denote the full subcategory of $\mathcal{S}_G$ spanned by G-spaces of the form G/H with $H \subseteq G$ being a closed subgroup. By [**GM20**, Proposition 2.16], there is a fully faithful functor $\mathrm{Orb}_G \to \mathrm{Orb}_{/*/G}$, whose essential image is spanned by those maps $*/H \to */G$ which arise via an inclusion of subgroups $H \subseteq G$.

Remark 2.2.4. Note that $\mathcal{S}(G)$ is the full subcategory of $\mathcal{S}_G$ generated by G-spaces of the form G/H (for closed subgroups $H \subseteq G$) under finite colimits.

A more invariant construction of KU_G, along with its $\mathbf{E}_\infty$-ring structure, is as follows; see [**GM20**, Section 4].

Construction 2.2.5. Let $\mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$ denote the functor sending $*/G \mapsto \mathrm{Rep}_{\mathbf{C}}(G)$. Taking connective additive K-theory, we obtain a functor $K : \mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}$. The functor K is a module over the constant functor $\mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}$ sending $*/G \mapsto K(\mathrm{Vect}_{\mathbf{C}}) \simeq \tau_{\geq 0}(\mathrm{KU})$. Therefore, inverting the Bott class $\beta \in \pi_2\mathrm{KU}$ produces a functor $\mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}}$ sending $*/G \mapsto \mathrm{KU}_G$. Right Kan extending along the functor $\mathrm{Orb}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}_{\mathrm{Orb}}$ defines a lax symmetric monoidal functor $\mathcal{S}^{\mathrm{op}}_{\mathrm{Orb}} \to \mathrm{CAlg}_{\mathrm{KU}}$ sending an orbispace $X/G \mapsto \mathrm{KU}_G(X)$.

One important property of equivariant K-theory, which is also satisfied/posited to hold (depending on the construction) for equivariant analogues of other complex-oriented cohomology theories, is that it satisfies *abelian descent*. Let us review this, following [**GM20**, Section 4].

Definition 2.2.6. Let $\mathcal{A}$ denote a family of compact Lie groups (so that $\mathcal{A}$ is closed under isomorphisms, subgroups, and quotients). Define $\mathrm{Orb}^{\mathcal{A}}$ to be the full subcategory of Orb spanned by those $*/G$ with $G \in \mathcal{A}$. For $*/G \in \mathrm{Orb}$, let $\mathrm{Orb}^{\mathcal{A}}_G$ denote the full subcategory of $\mathrm{Orb}^{\mathcal{A}}_{/*/G}$ spanned by those morphisms $*/H \to */G$ which arise via an inclusion of subgroups $H \subseteq G$. Note that by [**GM20**, Proposition 2.16], one can identify $\mathrm{Orb}^{\mathcal{A}}_G$ with the full subcategory of Orb_G spanned by those G/H with $H \in \mathcal{A}$.

Theorem 2.2.7. *Let $\mathcal{A}$ denote the family of abelian compact Lie groups. The functor* $\mathrm{Orb}^{\mathrm{op}} \to \mathrm{CAlg}_{\mathrm{KU}}$ *sending* $*/G \mapsto \mathrm{KU}_G$ *is right-Kan extended along the inclusion* $\mathrm{Orb}^{\mathcal{A},\mathrm{op}} \hookrightarrow \mathrm{Orb}^{\mathrm{op}}$.

Proof. For each $*/G$, one first notes that the inclusion $\mathrm{Orb}_G^{\mathcal{A}} \to \mathrm{Orb}^{\mathcal{A}}_{/*/G}$ is final, so we need to show that the canonical map $\mathrm{KU}_G \to \lim_{*/H \in \mathrm{Orb}_G^{\mathcal{A}}} \mathrm{KU}_H$ is an equivalence. Let $E\mathcal{A} \simeq \mathrm{colim}_{G/H \in \mathrm{Orb}_G^{\mathcal{A}}} G/H$, so that $E\mathcal{A}^K$ is empty if $K \notin \mathcal{A}$, and $E\mathcal{A}^K \simeq *$ if $K \in \mathcal{A}$. In fact, this property characterizes $E\mathcal{A}$ up to weak equivalence. Then $\lim_{*/H \in \mathrm{Orb}_G^{\mathcal{A}}} \mathrm{KU}_H \simeq \mathrm{KU}_G(E\mathcal{A})$, so we only need to show that the canonical map $\mathrm{KU}_G \to \mathrm{KU}_G(E\mathcal{A})$ is an equivalence. But this is [**AHJM88**, Corollary 1.3]. □

Remark 2.2.8. Instead of appealing to [**AHJM88**, Corollary 1.3] in Theorem 2.2.7, one can argue explicitly as follows in the case when G is connected with torsion-free $\pi_1(G)$. Let T be a maximal torus of G, and let G/T be the flag variety. If $H \subseteq G$ is a closed subgroup, H is abelian if and only if some conjugate gHg^{-1} is contained in T, which in turn happens if and only if $(G/T)^H$ is nonempty. This implies that the H-invariants of the geometric realization $|(G/T)^{\times \bullet+1}|$ is nonempty if and only if H is abelian, in which case it is contractible. Therefore, by uniqueness of $E\mathcal{A}$, there is a weak equivalence $E\mathcal{A} \simeq |(G/T)^{\times \bullet+1}|$. This implies that

$$\mathrm{KU}_G(E\mathcal{A}) \simeq \mathrm{Tot}\, \mathrm{KU}_G((G/T)^{\times \bullet+1}) \simeq \mathrm{Tot}\, \mathrm{KU}_G(G/T)^{\otimes_{\mathrm{KU}_G} \bullet+1}.$$

To conclude that this totalization is equivalent to KU_G by the unit map, it therefore suffices to show that the map $\mathrm{KU}_G \to \mathrm{KU}_G(G/T)$ induces a faithfully flat map on homotopy. But $\mathrm{KU}_G(G/T) = \mathrm{KU}_T$, so by 2-periodicity, we only need to show that the map $R_{\mathbf{C}}(G) \to R_{\mathbf{C}}(T)$ on complex representation rings is faithfully flat. In fact, $R_{\mathbf{C}}(T)$ is a free $R_{\mathbf{C}}(G)$-module by the main theorem of [**Pit72**], thanks to our assumption that G is connected with torsion-free π_1.

Observation 2.2.9. The $\mathbf{E}_\infty$-KU-algebra KU_T is 2-periodic, with π_0 given by the complex representation ring $R_{\mathbf{C}}(T)$. In particular, $\mathrm{Spec}\, \pi_0 \mathrm{KU}_T \cong \mathrm{Spec}\, \mathbf{Z}[\mathbb{X}^*(T)]$, where $\mathbb{X}^*(T)$ is the lattice of characters. This is precisely the algebraic group $\mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G}_m)$.

Proposition 2.2.10. *Let* $\mathbf{G}_{m,\mathrm{KU}}$ *denote the* $\mathbf{E}_\infty$*-KU-scheme given by* $\mathrm{Spec}\, \mathrm{KU}[\mathbf{Z}]$. *Let T be an abelian compact Lie group. Then there is an equivalence* $\mathrm{Spec}\, \mathrm{KU}_T \simeq \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G}_{m,\mathrm{KU}})$.

Variant 2.2.11. Let $\mathcal{A}$ denote the family of abelian compact Lie groups. Let $\mathrm{Orb}^{\mathcal{A}} \to \mathrm{Sch}_{/\mathrm{KU}}$ denote the functor to spectral schemes over KU sending $*/T \mapsto \mathrm{Spec}\, \mathrm{KU}_T$. That this is well-defined is essentially [**GM20**, Proposition 4.4]. The left Kan extension of this functor along the inclusion $\mathrm{Orb}^{\mathcal{A}} \hookrightarrow \mathrm{Orb}$ defines a functor $\mathrm{Orb} \to \mathrm{Sch}_{/\mathrm{KU}}$, which, by Theorem 2.2.7, sends $*/G \mapsto \mathrm{Spec}\, \mathrm{KU}_G$. Further left Kan extending along the inclusion $\mathrm{Orb} \to \mathcal{S}_{\mathrm{Orb}}$ defines a functor $\mathcal{S}_{\mathrm{Orb}} \to \mathrm{Sch}_{/\mathrm{KU}}$ sending $X/G \mapsto \mathrm{Spec}\, \mathrm{KU}_G(X)$.

Remark 2.2.12. This construction can be extended further. Namely, consider the composite $\mathrm{Orb}^{\mathcal{A},\mathrm{op}} \to \mathrm{Sch}^{\mathrm{op}}_{/\mathrm{KU}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{KU}})$, where the functor $\mathrm{Sch}^{\mathrm{op}}_{/\mathrm{KU}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{KU}})$ is given by taking quasicoherent sheaves. Right Kan extending this functor along the inclusion $\mathrm{Orb}^{\mathcal{A},\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}_{\mathrm{Orb}}$ defines a functor $\mathcal{S}^{\mathrm{op}}_{\mathrm{Orb}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{KU}})$, which we will denote by $X/G \mapsto \mathrm{Loc}_G(X; \mathrm{KU})$. The ∞-category

$\mathrm{Loc}_G(X; \mathrm{KU})$ could (somewhat abusively) be called the ∞-category of *G-equivariant local systems of* KU*-modules on X*. We will not use this notion below.

Remark 2.2.13. The functor $\mathcal{S}_{\mathrm{Orb}} \to \mathrm{Sch}_{/\mathrm{KU}}$ refines to a functor $(\mathcal{S}_{\mathrm{Orb}})_{/*/G} \to \mathrm{Sch}_{/\mathrm{KU}_G}$. In particular, if X is a space with G-action, and X/G denotes the associated orbispace (so that $X/G \in (\mathcal{S}_{\mathrm{Orb}})_{/*/G}$), there is a canonical map $\mathrm{Spec}\,\mathrm{KU}_G(X) \to \mathrm{Spec}\,\mathrm{KU}_G$. We will often write $\mathcal{F}_G(X)$ to denote the associated $\mathbf{E}_\infty$-KU_G-algebra.

2.3. Equivariant connective K-theory. We will need a good theory of equivariant *connective* K-theory. (This is *not* the functor K of Construction 2.2.5.) It will be most convenient to adopt the spectral algebro-geometric perspective of Variant 2.2.11. To motivate the construction, let us briefly recall the definition of nonequivariant connective K-theory.

Definition 2.3.1. Let ku denote the connective cover of complex K-theory KU. Then $\pi_*\mathrm{ku} \cong \mathbf{Z}[\beta]$ with $|\beta| = 2$, so that $\mathrm{ku}[\beta^{-1}] = \mathrm{KU}$, and $\mathrm{ku}/\beta \simeq \mathbf{Z}$.

Let us suggest some desiderata in the simple case of S^1-equivariance.

Expectation 2.3.2. By construction, there is an isomorphism $\mathrm{Spec}\,\mathrm{KU}_{S^1} \simeq \mathbf{G}_{m,\mathrm{KU}}$ of spectral schemes over KU. Recall that $\mathrm{Spec}\,\mathrm{H}^*(BS^1; \mathbf{Z}) \cong \hat{\mathbf{G}}_a(2)$ as graded $\mathbf{Z}$-schemes, where the coordinate of $\hat{\mathbf{G}}_a$ lives in weight -2. This can also be identified with the graded $\mathbf{Z}$-scheme $\mathrm{Spec}\,\mathrm{H}^*_{S^1}(*; \mathbf{Z}) = \mathbf{G}_a(2)$, since equipping the coordinate on $\mathbf{G}_a$ with the nonzero weight -2 allows us to identify $\hat{\mathbf{G}}_a(2) \cong \mathbf{G}_a(2)$. Therefore, if $\mathbf{Z}_{S^1}$ denotes the $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra representing S^1-equivariant $\mathbf{Z}$-cohomology, one expects the appropriate notion of S^1-equivariant connective K-theory ku_{S^1} to be a sufficiently structured ku-algebra such that there is a diagram where each square is Cartesian:

$$\begin{tikzcd}
\mathbf{G}_a(2) \arrow[r] \arrow[d, equal] & \mathbf{G} \arrow[d, equal] & (\mathbf{G}_m)_{\mathbf{Z}[\beta^{\pm 1}]} \arrow[l] \arrow[d, equal] \\
\mathrm{Spec}\,\pi_*\mathbf{Z}_{S^1} \arrow[r] \arrow[d] & \mathrm{Spec}\,\pi_*\mathrm{ku}_{S^1} \arrow[d] & \mathrm{Spec}\,\pi_*\mathrm{KU}_{S^1} \arrow[l] \arrow[d] \\
\mathrm{Spec}\,\mathbf{Z} \arrow[r] & \mathrm{Spec}\,\mathbf{Z}[\beta] & \mathrm{Spec}\,\mathbf{Z}[\beta^{\pm 1}] \arrow[l]
\end{tikzcd}$$

In particular, one expects that there is an isomorphism of graded $\mathbf{Z}[\beta]$-group schemes

$$\mathrm{Spec}\,\pi_*\mathrm{ku}_{S^1} \cong \mathrm{Spec}\,\mathbf{Z}[\beta][x, \tfrac{1}{1+\beta x}],$$

where x lives in weight -2 and the group structure is given by $x \mapsto x \otimes 1 + 1 \otimes x + \beta x \otimes x$.

Let us recall a construction of the group scheme $\mathrm{Spec}\,\mathbf{Z}[\beta][x, \frac{1}{1+\beta x}]$.

Recollection 2.3.3. Let $\mathbf{Z}[y]$ denote the graded $\mathbf{Z}$-algebra where y has weight 1. The $(t-1)$-adically filtered ring $\mathbf{Z}[t^{\pm 1}]$ defines a commutative algebra object $\mathrm{Mod}^{\mathrm{fil}}_{\mathbf{Z}}$, which, by the Rees construction $\mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}[y]} \simeq \mathrm{Mod}^{\mathrm{fil}}_{\mathbf{Z}}$, defines a commutative algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}[y]}$. This algebra is simply $\mathbf{Z}[y][t^{\pm 1}, \frac{t-1}{y}]$. The shearing autoequivalence of $\mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}}$ sending $M_\bullet \mapsto M_\bullet[2\bullet]$ sends $\mathbf{Z}[y]$ to a graded ring $\mathbf{Z}[\beta']$ with β' in weight 1 and degree 2, so that shearing defines an equivalence

$$\mathrm{Mod}^{\mathrm{fil}}_{\mathbf{Z}} \simeq \mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}[y]} \simeq \mathrm{Mod}^{\mathrm{gr}}_{\mathbf{Z}[\beta']}.$$

Under this equivalence, the $(t-1)$-adically filtered ring $\mathbf{Z}[t^{\pm 1}]$ is sent to the graded $\mathbf{Z}[\beta']$-algebra $\mathbf{Z}[\beta', t^{\pm 1}, \frac{t-1}{\beta'}]$.

We will adapt essentially the same argument to the spectral setting, with some minor homotopical issues complicating the story.

Proposition 2.3.4 ([**Mou19**]). *The $\mathbf{E}_\infty$-MU-algebra $\mathrm{MU}[y]$ (with y in degree zero) admits a grading where y is in weight 1. There is a symmetric monoidal equivalence $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}[y]} \simeq \mathrm{Mod}^{\mathrm{fil}}_{\mathrm{MU}}$. In particular, if $\mathrm{MU}[\beta']$ denotes the graded $\mathbf{E}_\infty$-MU-algebra given by the shearing of $\mathrm{MU}[y]$ (so that β' lives in degree 2 and weight 1), Lemma 2.1.3 defines a symmetric monoidal equivalence $\mathrm{Rees}_{\beta'} : \mathrm{Mod}^{\mathrm{fil}}_{\mathrm{MU}} \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MU}[\beta']}$.*

Lemma 2.3.5. *Let $\mathrm{oblv}(\mathrm{MU}[\beta'])$ denote the ungraded $\mathbf{E}_\infty$-MU-algebra which underlies $\mathrm{MU}[\beta']$. There is an $\mathbf{E}_\infty$-MU-algebra map $\mathrm{oblv}(\mathrm{MU}[\beta']) \to \mathrm{ku}$ sending $\beta' \mapsto \beta$.*

Proof. Note that by construction, $\mathrm{oblv}(\mathrm{MU}[\beta'])$ can be identified with the Thom spectrum of the composite

$$\mathrm{BU} \times \mathbf{Z}_{\geq 0} \to \mathrm{BU} \times \mathbf{Z} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}).$$

Let $\mathrm{MU}_{\mathrm{per}}$ denote the Thom spectrum of the map $J : \mathrm{BU} \times \mathbf{Z} \to \mathrm{Pic}(\mathrm{Sp})$. Then there is a canonical map $\mathrm{oblv}(\mathrm{MU}[\beta']) \to \mathrm{MU}_{\mathrm{per}}$. There is an $\mathbf{E}_\infty$-map $\mathrm{MU}_{\mathrm{per}} \to \mathrm{KU}$ which sends $\mathrm{oblv}(\beta') \in \pi_2 \mathrm{MU}_{\mathrm{per}}$ to the Bott class (see, e.g., [**HJN$^+$21**, Remark 6.3]), and hence an $\mathbf{E}_\infty$-map $\mathrm{oblv}(\mathrm{MU}[\beta']) \to \mathrm{KU}$. which does the same. But $\mathrm{oblv}(\mathrm{MU}[\beta'])$ is connective, so this map factors through an $\mathbf{E}_\infty$-map $\mathrm{oblv}(\mathrm{MU}[\beta']) \to \mathrm{ku}$, as desired. $\square$

Construction 2.3.6. Let I denote the fiber of the $\mathbf{E}_\infty$-map $\mathrm{MU}[t^{\pm 1}] \to \mathrm{MU}$ given by MU-chains of the map $\mathbf{Z} \to *$. Then the functor $\mathbf{Z}_{\geq 0} \to \mathrm{Mod}_{\mathrm{MU}}$ sending $n \mapsto I^{\otimes_{\mathrm{MU}[t^{\pm 1}]} n}$ defines an object $\mathrm{F}^\star_{(t-1)}\mathrm{MU}[t^{\pm 1}] \in \mathrm{CAlg}(\mathrm{Mod}^{\mathrm{fil}}_{\mathrm{MU}})$. Applying the symmetric monoidal equivalence of Proposition 2.3.4, we obtain a graded $\mathbf{E}_\infty$-$\mathrm{MU}[\beta']$-algebra $\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{MU}[t^{\pm 1}])$. Note that this graded $\mathbf{E}_\infty$-$\mathrm{MU}[\beta']$-algebra admits a strictly cocommutative $\mathbf{E}_\infty$-MU-coalgebra structure by the map $t \mapsto t \otimes t$.

Let ku_{S^1} denote the $\mathbf{E}_\infty$-ku-bialgebra given by

$$\mathrm{ku}_{S^1} := \mathrm{oblv}(\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{MU}[t^{\pm 1}])) \otimes_{\mathrm{oblv}(\mathrm{MU}[\beta'])} \mathrm{ku},$$

where the $\mathbf{E}_\infty$-map $\mathrm{oblv}(\mathrm{MU}[\beta']) \to \mathrm{ku}$ is given by Lemma 2.3.5. We will write $\mathrm{Spec}\,\mathrm{ku}_{S^1}$, or sometimes $\mathbf{G}_{\mathrm{ku},\beta}$, to denote the functor $\mathrm{CAlg}_{\mathrm{ku}} \to \mathcal{S}$ which is corepresented by ku_{S^1}. Since ku_{S^1} is a strictly cocommutative $\mathbf{E}_\infty$-ku-coalgebra, this functor in fact lands in the ∞-category $s\mathrm{Ab}$ of simplicial abelian groups. We will write $\mathbf{G}_\beta$ to denote the underlying graded $\pi_*(\mathrm{ku}) = \mathbf{Z}[\beta]$-scheme.

Remark 2.3.7. By comparison to Recollection 2.3.3, it is not difficult to see that

$$\pi_* \mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{MU}[t^{\pm 1}]) \cong \pi_*(\mathrm{MU})[\beta', t^{\pm 1}, \tfrac{t-1}{\beta'}],$$

where β' is in weight 1 and degree 2, and $\frac{t-1}{\beta'}$ is in weight -1 and degree -2. Therefore,

$$\pi_* \mathrm{ku}_{S^1} \cong \mathbf{Z}[\beta][t^{\pm 1}, \tfrac{t-1}{\beta}],$$

as expressed in Expectation 2.3.2.

Remark 2.3.8. Note that if we invert the Bott class in $\pi_2\mathrm{ku}_{S^1}$, we obtain

$$\begin{aligned}\mathrm{ku}_{S^1}[\beta^{-1}] &= \mathrm{oblv}(\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{MU}[t^{\pm 1}])[\beta'^{-1}]) \otimes_{\mathrm{oblv}(\mathrm{MU}[\beta'^{\pm 1}])} \mathrm{KU} \\ &\simeq \mathrm{MU}[\beta'^{\pm 1}][t^{\pm 1}] \otimes_{\mathrm{MU}[\beta'^{\pm 1}]} \mathrm{KU} \simeq \mathrm{KU}[t^{\pm 1}] = \mathrm{KU}_{S^1},\end{aligned}$$

as expressed in Expectation 2.3.2.

Observation 2.3.9. It is not difficult to extend the above construction to arbitary abelian compact Lie groups T. Namely, given a functor $F : \mathrm{CAlg}_{\mathrm{ku}} \to s\mathrm{Ab}$ and an abelian compact Lie group T, one obtains a new functor $F_T : \mathrm{CAlg}_{\mathrm{ku}} \to s\mathrm{Ab}$ given by $\mathrm{Hom}(\mathbb{X}^*(T), F)$. If F is corepresentable, the same is true of F_T. Applied to the functor $\mathbf{G}_{\mathrm{ku},\beta} : \mathrm{CAlg}_{\mathrm{ku}} \to s\mathrm{Ab}$, we obtain an $\mathbf{E}_\infty$-ku-algebra ku_T. This assignment evidently defines a functor from the ∞-category of abelian compact Lie groups to $\mathbf{E}_\infty$-ku-algebras. We will write the underlying graded group $\mathbf{Z}[\beta]$-scheme of $\mathrm{Spec}\,\mathrm{ku}_T$ as T_β.

In order to extend the functoriality of the assignment $T \mapsto \mathrm{ku}_T$, we need the following.

Lemma 2.3.10. *There is an* $\mathbf{E}_\infty$-ku-*algebra map* $\mathrm{ku}_{S^1} \to \mathrm{ku}^{hS^1}$ *which is given on homotopy by the map*

$$\mathbf{Z}[\beta][t^{\pm 1}, \tfrac{t-1}{\beta}] \to \mathbf{Z}[\beta][\![\hbar]\!] \cong \mathbf{Z}[\![t-1]\!][\beta][\![\hbar]\!]/(\beta\hbar = t-1)$$

sending $\frac{t-1}{\beta} \mapsto \hbar$.

Proof. The orientation of the spectral formal multiplicative group $\hat{\mathbf{G}}_m$ over KU defines an equivalence of $\mathbf{E}_\infty$-rings $\mathrm{KU}^{hS^1} \xrightarrow{\sim} \mathrm{KU}[\![t-1]\!]$ (see [**Lur18**, Sections 4.4 and 6.5]), where the map sends $\beta\hbar \mapsto t-1$ on π_0. This defines a canonical $\mathbf{E}_\infty$-map $S[\![t-1]\!] \to \mathrm{KU}^{hS^1}$, which can be interpreted as a BS^1-family of $\mathbf{E}_\infty$-map $S[\![t-1]\!] \to \mathrm{KU}$. Since $S[\![t-1]\!]$ is connective, this is the same as a BS^1-family of $\mathbf{E}_\infty$-maps $S[\![t-1]\!] \to \mathrm{ku}$, i.e., an $\mathbf{E}_\infty$-map $S[\![t-1]\!] \to \mathrm{ku}^{hS^1}$.

This defines an $\mathbf{E}_\infty$-map $\mathrm{ku}[t^{\pm 1}] \to \mathrm{ku}^{hS^1}$ via the ku-linearization of the composite

$$S[t^{\pm 1}] \to S[\![t-1]\!] \to \mathrm{ku}^{hS^1}.$$

Note that the pushout $\mathrm{ku}^{hS^1} \otimes_{S[t^{\pm 1}]} S$ equips $\mathrm{ku}^{hS^1}/\beta\hbar$ with the structure of an $\mathbf{E}_\infty$-ku^{hS^1}-algebra. Let J denote the fiber of the $\mathbf{E}_\infty$-map $\mathrm{ku}^{hS^1} \to \mathrm{ku}^{hS^1}/\beta\hbar$, so that π_*J is the ideal of $\pi_*\mathrm{ku}^{hS^1} = \mathbf{Z}[\beta][\![\hbar]\!]$ generated by $\beta\hbar$. Let $\mathrm{F}^\star_{\beta\hbar}\mathrm{ku}$ denote the filtered $\mathbf{E}_\infty$-ring $J^{\otimes_{\mathrm{ku}^{hS^1}} *}$. The above discussion produces a map $\mathrm{F}^\star_{(t-1)}\mathrm{ku}[t^{\pm 1}] \to \mathrm{F}^\star_{\beta\hbar}\mathrm{ku}$ of filtered $\mathbf{E}_\infty$-ku-algebras, and hence a map

$$\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{ku}[t^{\pm 1}]) \to \mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{\beta\hbar}\mathrm{ku}^{hS^1})$$

of graded $\mathbf{E}_\infty$-$\mathrm{ku}[\beta']$-algebras. In particular, applying oblv and base-changing along the $\mathbf{E}_\infty$-ku-algebra map $\mathrm{oblv}(\mathrm{ku}[\beta']) \to \mathrm{ku}$ from Lemma 2.3.5 defines a map of ungraded $\mathbf{E}_\infty$-ku-algebras

$$\mathrm{oblv}(\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{(t-1)}\mathrm{ku}[t^{\pm 1}])) \otimes_{\mathrm{oblv}(\mathrm{ku}[\beta'])} \mathrm{ku} \to \mathrm{oblv}(\mathrm{Rees}_{\beta'}(\mathrm{F}^\star_{\beta\hbar}\mathrm{ku}^{hS^1})) \otimes_{\mathrm{oblv}(\mathrm{ku}[\beta'])} \mathrm{ku}.$$

But it is not difficult to see that the target is precisely ku^{hS^1}. $\square$

Proposition 2.3.11. *The group scheme $\mathbf{G}_{\mathrm{ku},\beta}$ is preoriented (compatibly with the orientation on $\mathbf{G}_{m,\mathrm{KU}}$), and the construction from Observation 2.3.9 extends to a functor $\mathrm{Orb}^{\mathcal{A}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$ sending $T \mapsto \operatorname{Spec} \mathrm{ku}_T$.*

Proof. Following [**GM20**, Construction 3.13 and Proposition 4.4], the desired functor can be defined as follows. First, note that the $\mathbf{E}_\infty$-map of Lemma 2.3.10 defines a map

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ku}}}(\mathrm{ku}^{hS^1}, \mathrm{ku}) \to \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ku}}}(\mathrm{ku}_{S^1}, \mathrm{ku}) = \mathbf{G}_{\mathrm{ku},\beta}(\mathrm{ku}).$$

There is an obvious map

$$\mathbf{C}P^\infty \to \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ku}}}(\mathrm{ku}^{hS^1}, \mathrm{ku})$$

of simplicial abelian groups. The resulting map $\mathbf{C}P^\infty \to \mathbf{G}_{\mathrm{ku},\beta}(\mathrm{ku})$ defines a preorientation $*/S^1 \to \mathbf{G}_{\mathrm{ku},\beta}$, and hence a functor $\mathrm{Orb}^{\mathcal{A}} \times \mathrm{CAlg}_{\mathrm{ku}} \to \mathcal{S}$ sending

$$(X, R) \mapsto \mathrm{Map}_{s\mathrm{Ab}_{*/S^1/}}(\mathrm{Map}(X, */S^1), \mathbf{G}_{\mathrm{ku},\beta}(R)).$$

This is adjoint to the desired functor $\mathrm{Orb}^{\mathcal{A}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$. It is not difficult to see that this functor sends $T \mapsto \mathrm{ku}_T$. □

Motivated by Variant 2.2.11, we are led to:

Definition 2.3.12. Let $\mathcal{S}_{\mathrm{Orb}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$ denote the functor given by left Kan extending the functor $\mathrm{Orb}^{\mathcal{A}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$ along the inclusion $\mathrm{Orb}^{\mathcal{A}} \hookrightarrow \mathcal{S}_{\mathrm{Orb}}$. It is not hard to see that this functor in fact lands in the full subcategory spanned by the representable functors, so we will denote this functor by $X/G \mapsto \operatorname{Spec} \mathcal{F}_G(X)$. The functor $\mathcal{S}_{\mathrm{Orb}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$ refines to a functor $(\mathcal{S}_{\mathrm{Orb}})_{/*/G} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})$. In particular, if X is a space with G-action, and X/G denotes the associated orbispace (so that $X/G \in (\mathcal{S}_{\mathrm{Orb}})_{/*/G}$), there is a canonical map $\operatorname{Spec} \mathcal{F}_G(X) \to \operatorname{Spec} \mathrm{ku}_G$. We will write $\mathrm{ku}_G^*(X)$ to denote $\pi_{-*}\mathcal{F}_G(X)$.

Remark 2.3.13. This construction can be extended further. Namely, consider the composite $\mathrm{Orb}^{\mathcal{A},\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{ku}})$, where the functor $\mathrm{Fun}(\mathrm{CAlg}_{\mathrm{ku}}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{ku}})$ is given by taking quasicoherent sheaves. Right Kan extending this functor along the inclusion $\mathrm{Orb}^{\mathcal{A},\mathrm{op}} \to \mathcal{S}_{\mathrm{Orb}}^{\mathrm{op}}$ defines a functor $\mathcal{S}_{\mathrm{Orb}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{ku}})$, which we will denote by $X/G \mapsto \mathrm{Loc}_G(X; \mathrm{ku})$. The ∞-category $\mathrm{Loc}_G(X; \mathrm{ku})$ could (somewhat abusively) be called the ∞-category of *G-equivariant local systems of* ku*-modules on X*. We will not use this notion below.

Notation 2.3.14. Let $\mathcal{M}_G$ denote the underlying graded $\mathbf{Z}[\beta]$-scheme of $\operatorname{Spec} \mathrm{ku}_G$, i.e., $\mathcal{M}_G = \operatorname{Spec} \pi_* \mathrm{ku}_G$.

Proposition 2.3.15. *Let G be a connected compact Lie group whose π_1 is torsion-free, and let $T \subseteq G$ be a maximal torus with associated Weyl group W. Let X/G be an orbispace over $*/G$, and let X/T denote the associated orbispace of $*/T$. Upon inverting $|W|$, the natural map $\operatorname{Spec} \mathcal{F}_T(X) \to \operatorname{Spec} \mathcal{F}_G(X)$ exhibits the graded scheme $\operatorname{Spec} \mathrm{ku}_G^*(X)$ as the GIT quotient $\operatorname{Spec} \mathrm{ku}_T^*(X)/\!\!/W$. In particular, $\mathcal{M}_G \cong T_\beta/\!\!/W$.*

Proof. Following Remark 2.2.8, we can identify $\mathcal{F}_G(X)$ with the totalization of the diagram

$$\mathcal{F}_T(X) \rightrightarrows \mathcal{F}_T(X \times G/T) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \mathcal{F}_T(X \times (G \times G)/T) \cdots$$

There is an isomorphism $\mathcal{F}_T(X \times G/T) \simeq \mathcal{F}_T(X) \otimes_{\mathrm{ku}_T} \mathcal{F}_T(G/T)$. After inverting $|W|$, there is an isomorphism $\mathcal{F}_T(G/T) \simeq \mathrm{ku}_T \otimes_{\mathrm{ku}} \mathrm{ku}^{(G/T)_+}$ of ku_T-modules, and the cochains $\mathrm{ku}^{(G/T)_+}$ can be identified with the regular representation of W. This implies that there is an equivalence $\mathcal{F}_G(X) \simeq \mathcal{F}_T(X)^{hW}$; but W is a finite group, and its order is assumed to be inverted, so that $\pi_*(\mathcal{F}_T(X)^{hW}) \cong \pi_*(\mathcal{F}_T(X))^W$. □

2.4. Equivariant ku-homology. The goal of this section is to set up the theory of equivariant ku-homology. Fix a compact Lie group G throughout.

Definition 2.4.1. Let $\mathcal{F}_G(-)^\vee : \mathcal{S}(G) \to \mathrm{Mod}_{\mathrm{ku}_G}$ denote the functor given by sending $X/G \mapsto \mathcal{F}_G(X)^\vee$, where $\mathcal{F}_G(X)^\vee$ denotes the ku_G-linear dual of $\mathcal{F}_G(X)$. We will refer to $\mathcal{F}_G(X)^\vee$ as the *G-equivariant* ku-*homology* of X, and often write the homotopy groups of this spectrum as $\mathrm{ku}^G_*(X)$.

Remark 2.4.2. Note that the functor $\mathcal{F}_G(-)^\vee : \mathcal{S}(G) \to \mathrm{Mod}_{\mathrm{ku}_G}$ is in fact symmetric monoidal. Since every G-space is naturally equipped with a diagonal map, this refines $\mathcal{F}_G(-)^\vee$ to a functor $\mathcal{S}(G) \to \mathrm{coCAlg}_{\mathrm{ku}_G}$.

The above definition is badly behaved if X is not a finite G-space. This is not special to the equivariant setting, as the following example shows.

Example 2.4.3. The integral cohomology of the discrete space $\mathbf{Z}$ is given by the ring $\mathrm{Map}(\mathbf{Z}, \mathbf{Z})$ of all functions $\mathbf{Z} \to \mathbf{Z}$. Given such a function f, one can define a new function $\Delta f : \mathbf{Z} \to \mathbf{Z}$ via $f(x+1) - f(x)$. Then, we formally have $f(x) = \sum_{k\geq 0} (\Delta^k f)(0)\binom{x}{k}$. This series converges in the completion of the ring $\mathbf{Z}[\binom{x}{k}]_{k\geq 0}$ of numerical polynomials. However, the $\mathbf{Z}$-linear dual of $\mathrm{Map}(\mathbf{Z}, \mathbf{Z})$ is *not* isomorphic to the group algebra $\mathbf{Z}[t^{\pm 1}] = \mathbf{Z}[\mathbf{Z}] = \mathrm{H}_*(\mathbf{Z}; \mathbf{Z})$.

The basic issue is the infinitude of $\mathbf{Z}$, which leads to a difference between $\mathbf{Z}^{\mathbf{Z}}$ and $\mathbf{Z}^{\oplus \mathbf{Z}}$. The simplest fix is to observe that $\mathbf{Z}$ admits a filtration by finite subsets $I_n = \{-n, \cdots, n\}$, and that $\mathrm{H}_*(I_n; \mathbf{Z}) = \mathbf{Z}\{t^{-n}, \cdots, t^n\}$ is indeed the $\mathbf{Z}$-linear dual of $\mathrm{H}^*(I_n; \mathbf{Z}) = \mathrm{Map}(I_n, \mathbf{Z})$. Note that this filtration of $\mathbf{Z}$ equips it with the structure of a filtered group: namely, the addition on $\mathbf{Z}$ gives maps $I_n \times I_m \to I_{n+m}$ for each $n, m \in \mathbf{Z}$.

We will therefore rely on the following construction.

Construction 2.4.4. Let $\mathcal{F}_G(-)^\vee : \mathcal{S}_G \to \mathrm{coCAlg}_{\mathrm{ku}_G}$ denote the left Kan extension of the functor $\mathcal{S}(G) \to \mathrm{coCAlg}_{\mathrm{ku}_G}$ along the inclusion $\mathcal{S}(G) \to \mathcal{S}_G$. Explicitly, if $X \in \mathcal{S}_G$ is a G-space equipped with a presentation $X = \mathrm{colim}_{j\in\mathcal{J}} X_j$ as the filtered colimit of a filtered diagram $\mathcal{J} \to \mathcal{S}(G)$ of finite G-spaces, then $\mathcal{F}_G(X)^\vee$ is the filtered colimit $\mathrm{colim}_{j\in\mathcal{J}} \mathcal{F}_G(X_j)^\vee$; we will refer to it as the *G-equivariant* ku-*homology of* X. Note that the forgetful functor $\mathrm{coCAlg}_{\mathrm{ku}_G} \to \mathrm{Mod}_{\mathrm{ku}_G}$ preserves colimits, so this filtered colimit can be computed in $\mathrm{coCAlg}_{\mathrm{ku}_G}$ or $\mathrm{Mod}_{\mathrm{ku}_G}$.

In most examples of interest, there will be a geometrically defined presentation of X.

Remark 2.4.5. Let X be a *finite* G-space equipped with an $\mathbf{E}_n$-algebra structure in G-spaces. Then $\mathcal{F}_G(X)^\vee$ admits an $\mathbf{E}_n$-algebra structure in $\mathrm{coCAlg}_{\mathrm{ku}_G}$. If X is not a finite G-space, but is equipped with an $\mathbf{E}_n$-algebra structure in $\mathcal{S}_G$, the definition of G-equivariant ku-homology via Construction 2.4.4 does not guarantee the existence of an $\mathbf{E}_n$-algebra structure on $\mathcal{F}_G(X)^\vee \in \mathrm{coCAlg}_{\mathrm{ku}_G}$. Rather, if $\mathcal{J}$ is a filtered index category equipped with an $\mathbf{E}_n$-monoidal structure and $\mathcal{J} \to \mathcal{S}_G$ is an

$\mathbf{E}_n$-algebra object in the ∞-category $\mathrm{Fun}(\mathcal{J}, \mathcal{S}_G)$ equipped with the Day convolution monoidal structure, Construction 2.4.4 will define an $\mathbf{E}_n$-algebra structure on $\mathcal{F}_G(X)^\vee \in \mathrm{coCAlg}_{\mathrm{ku}_G}$. We will refer to such a presentation of X as a *multiplicative* presentation.

A basic fact about equivariant connective K-theory is the localization theorem. Although one can make statements about ku_G for arbitrary compact Lie groups G, we will restrict attention only to the case when $G = T$ is an abelian compact Lie group. In this case, we have the following simple observation.

Lemma 2.4.6. *Let T be an abelian compact Lie group, and let X be a finite T-space. Let $T_0 \subseteq T$ be a closed subgroup, and let $\mathcal{U}_{T_0} \subseteq T_\beta$ denote the complement of the union of the closed subschemes T'_β ranging over all closed subgroups $T' \subseteq T$ which do not contain T_0. Then the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^{T_0})$, and hence the map $\mathcal{F}_T(X^{T_0})^\vee \to \mathcal{F}_T(X)^\vee$, is an equivalence upon restriction to $\mathcal{U}_{T_0}$.*

Proof. By induction on the orbit stratification on X, we are reduced to the case when $X = T/T_1$ for some closed subgroup $T_1 \subseteq T$. In this case, the fixed points X^{T_0} is empty if $T_0 \not\subseteq T_1$, and $X^{T_0} = X$ if $T_0 \subseteq T_1$. It therefore suffices to show that $\mathcal{F}_T(X)|_{T_\beta - T_{1,\beta}} = 0$ if $T_0 \not\subseteq T_1$; but this is clear, because $\mathcal{F}_T(X) \cong \mathcal{O}_{T_{1,\beta}}$. $\square$

Remark 2.4.7. One special case of Lemma 2.4.6 which is worth restating (coresponding to $T_0 = T$) is the following. Let T_β° denote the complement of the union of the closed subscheme T'_β ranging over all closed *proper* subgroups $T' \subsetneq T$. Then the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^T)$, and hence the map $\mathcal{F}_T(X^T)^\vee \to \mathcal{F}_T(X)^\vee$, is an equivalence upon restriction to T_β°.

Lemma 2.4.8. *Let T be a torus, and let X be a finite T-space. If $\pi_*\mathcal{F}_T(X)$ is a projective $\pi_*\mathrm{ku}_T$-module, the map $\pi_*\mathcal{F}_T(X) \to \pi_*\mathcal{F}_T(X^T)$ is an injection.*

Proof. Since the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^T) \to \mathcal{F}_T(X^T)_{T_\beta^\circ}$ factors as $\mathcal{F}_T(X) \to \mathcal{F}_T(X)|_{T_\beta^\circ} \to \mathcal{F}_T(X^T)|_{T_\beta^\circ}$, and the map $\mathcal{F}_T(X)|_{T_\beta^\circ} \to \mathcal{F}_T(X^T)|_{T_\beta^\circ}$ is an equivalence by Lemma 2.4.6, it suffices to show that the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X)|_{T_\beta^\circ}$ induces an injection on homotopy groups. But $\pi_*\mathcal{F}_T(X)$ was assumed to be a projective $\pi_*\mathrm{ku}_T$-module, so one is reduced to the case $X = *$, i.e., to showing that the map $\mathrm{ku}_T \to \mathrm{ku}_T|_{T_\beta^\circ}$ induces an injection on homotopy groups. This, however, is clear, since the closed subscheme $T'_\beta \hookrightarrow T_\beta$ defined by each closed subgroup $T' \subseteq T$ is cut out by a regular sequence. $\square$

Definition 2.4.9. Let X be a finite T-space equipped with a chosen presentation in terms of T-cells. Say that X is a *GKM space* if the following conditions are satisfied:

- $\pi_*\mathcal{F}_T(X)$ is a projective $\pi_*\mathrm{ku}_T$-module;
- if $X^{(1)}$ denotes the equivariant 1-skeleton of X, then $X^{(1)}$ consists of a finite number of spheres S^λ meeting only at the fixed points, where λ ranges over characters of T.

Let V denote the set X^T of fixed points, and let E denote the set of characters λ such that $S^\lambda \subseteq X^{(1)}$. There are two maps $E \rightrightarrows V$ sending λ to the points $0, \infty \in S^\lambda \subseteq X^{(1)}$.

Proposition 2.4.10 (Goresky-Kottwitz-MacPherson)**.** *Let X be a finite GKM T-space equipped with a chosen presentation in terms of T-cells. For each character λ :*

$T \to S^1$, let T_λ denote the kernel of T, and let $S(\lambda)$ denote the unit representation sphere, so that $\mathrm{ku}_{T_\lambda} \cong \mathcal{F}_T(S(\lambda))$. Then there is an equalizer diagram

$$\pi_*\mathcal{F}_T(X) \hookrightarrow \pi_*\mathcal{F}_T(X^T) \cong \mathrm{Map}(V, \pi_*\mathrm{ku}_T) \rightrightarrows \prod_{\lambda \in E} \pi_*\mathrm{ku}_{T_\lambda},$$

where the two maps in the equalizer are defined in the evident manner.

Proof. Let us first show that the maps $\mathcal{F}_T(X) \to \mathcal{F}_T(X^T)$ and $\mathcal{F}_T(X^{(1)}) \to \mathcal{F}_T(X^T)$ have the same images on homotopy. There is an evident map from the image of $\mathcal{F}_T(X) \to \mathcal{F}_T(X^T)$ on homotopy to the image of $\mathcal{F}_T(X^{(1)}) \to \mathcal{F}_T(X^T)$ on homotopy, which we will denote by f. The map f is an injection by Lemma 2.4.8. Let T' denote a proper closed subgroup of T of codimension 1, and let $U' \subseteq T'_\beta$ denote the complement of the union of the closed varieties T''_β ranging over the proper closed subgroups $T'' \subseteq T'$. By Lemma 2.4.6, the map f is an isomorphism upon restriction to $U' \subseteq T'_\beta \subseteq T_\beta$ for each proper closed subgroup $T' \subseteq T$ of codimension 1.

Therefore, the locus $Z \subseteq T_\beta$ over which f fails to be an isomorphism is contained in the union of closed subvarieties T'_β for finitely many $T' \subseteq T$ of codimension at least 2. However, the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X)|_{T_\beta - Z}$ is an isomorphism (by Hartogs). Since the same is true of the map $\mathcal{F}_T(X^T) \to \mathcal{F}_T(X^T)|_{T_\beta - Z}$, and the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^T)$ factors through the map $\mathcal{F}_T(X^{(1)}) \to \mathcal{F}_T(X^T)$, the desired result follows.

For the equalizer diagram, an easy induction on the cell structure of X reduces us to the case $X = S^\lambda$ for a character $\lambda : T \to S^1$. In this case, the isomorphism $T/T_\lambda \cong S^\lambda$ defines an isomorphism $\mathrm{ku}_{T_\lambda} \cong \mathcal{F}_T(S(\lambda))$. Since $S^\lambda \cong \Sigma S(\lambda)$, we obtain an equalizer diagram

$$\pi_*\mathrm{ku}_T(S^\lambda) \to \pi_*\mathrm{ku}_T \oplus \pi_*\mathrm{ku}_T \cong \mathrm{Map}(\{0, \infty\}, \mathrm{ku}_T) \rightrightarrows \pi_*\mathrm{ku}_{T_\lambda}.$$

This proves the desired claim. □

Remark 2.4.11. Note that the statement of Proposition 2.4.10 is natural in X, and in particular, one can use Proposition 2.4.10 to describe the $\pi_*\mathrm{ku}_T$-algebra structure on $\pi_*\mathcal{F}_T(X)$. By dualizing Proposition 2.4.10, one can also describe the $\pi_*\mathrm{ku}_T$-*co*algebra structure on $\mathrm{ku}^T_*(X)$. Moreover, suppose that X is a T-space equipped with a presentation $X = \mathrm{colim}_{j \in \mathcal{J}} X_j$ in terms of finite T-spaces, each of which is GKM and equipped with a chosen presentation in terms of T-cells, and such that the transition maps $X_j \to X_{j'}$ are maps of cellular T-spaces. Then Proposition 2.4.10 can be extended to compute $\pi_*\mathcal{F}_T(X)$ and $\mathrm{ku}^T_*(X)$.

3. (Derived) geometric Satake and variants

3.1. Full faithfulness of global sections.
In this section, we prove an analogue of a result of Ginzburg's from [**Gin91**]. We will closely follow [**CMNO22**, Section 4.7] and [**SW18**, Section 8].

Setup 3.1.1. Let G be a compact Lie group, and fix a maximal torus $T \subseteq G$. Let X be a finite G-space whose G-equivariant orbit stratification indexed by a poset P (neccessarily finite). Let X_λ denote the stratum corresponding to $\lambda \in P$, and let $X_{\leq\lambda}$ denote its closure in X. Suppose further that each X_λ is a complex affine space of complex dimension n_λ on which G acts linearly. In particular, this implies that $\mathrm{H}^*_G(X; j_{\lambda,!}\underline{\mathbf{Q}})$ is concentrated in even degrees for each $\lambda \in P$, where $j_\lambda : X_\lambda \hookrightarrow X_{\leq\lambda}$ denotes the inclusion.

Let $X_{<\lambda} = X_{\leq\lambda} - X_\lambda$, and let $i_\lambda : X_{<\lambda} \hookrightarrow X_{\leq\lambda}$ denote the complementary closed embedding. We will also write j_λ to denote the inclusion $X_\lambda \hookrightarrow X$. Let $\mathrm{Shv}^c_G(X; \mathbf{Q})$ denote the ∞-category of G-equivariant sheaves on X which are constructible for the G-equivariant orbit stratification of X. Recall that the cohomology functor $\Gamma : \mathrm{Shv}^c_G(X; \mathbf{Q}) \to \mathrm{Mod}_{\mathbf{Q}}$ is given by $*$-pushforward to a point and then taking G-homotopy fixed points of the resulting $\mathbf{Q}$-module with G-action.

Definition 3.1.2. Let $\mathcal{F} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$. Say that $\mathcal{F}$ is $*$-even if the $*$-pullback $j^*_\lambda\mathcal{F} \in \mathrm{Shv}^c_G(X_\lambda; \mathbf{Q})$ is a direct sum of constant sheaves concentrated in even degrees for all $\lambda \in P$. Similarly, say that $\mathcal{F}$ is !-even if the !-pullback $j^!_\lambda\mathcal{F} \in \mathrm{Shv}^c_G(X_\lambda; \mathbf{Q})$ is a direct sum of constant sheaves concentrated in even degrees for all $\lambda \in P$. Say that $\mathcal{F}$ is even if it is both $*$-even and !-even. Finally, say that $\mathcal{F}$ is (!- or $*$-)odd if $\mathcal{F}[1]$ is (!- or $*$-)even.

The goal of this section is to prove the following result, by inducting on the stratification of X:

Theorem 3.1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be even objects of $\mathrm{Shv}^c_G(X; \mathbf{Q})$. Then the map*

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X;\mathbf{Q})}(\mathrm{H}^*_G(X; \mathcal{F}), \mathrm{H}^*_G(X; \mathcal{G}))$$

of graded $\mathbf{Q}$-vector spaces is a graded isomorphism (the grading denoted by $\bullet$), where the Hom *on the right-hand side is taken in the* 1*-category of graded $\mathrm{H}^*_G(X; \mathbf{Q})$-modules in $\mathrm{Mod}^\heartsuit_{\mathbf{Q}}$.*

Remark 3.1.4. Although Theorem 3.1.3 is stated only for X being a *finite* G-space, it can be extended easily to the situation when X is not necessarily finite. Namely, suppose that X is a G-space equipped with a presentation $X = \mathrm{colim}_{j\in\mathcal{J}} X_j$ in terms of finite G-spaces where each map $X_j \to X_{j'}$ is a closed embedding. In this case, we will write $\mathrm{Shv}^c_G(X; \mathbf{Q})$ to denote the inverse limit $\lim_{j\in\mathcal{J}} \mathrm{Shv}^c_G(X_j; \mathbf{Q})$ taken over !-pullbacks; and the meaning of evenness is exactly as in Definition 3.1.2. With this definition of $\mathrm{Shv}^c_G(X; \mathbf{Q})$, Theorem 3.1.3 continues to hold verbatim.

Remark 3.1.5. The argument for Theorem 3.1.3 below is sufficiently general that if A is an $\mathbf{E}_\infty$-ring with homotopy concentrated in even degrees, given a good theory of G-equivariant constructible sheaves of A-modules on a stratified (finite) G-space (including a six functor formalism), Theorem 3.1.3 will continue to hold as long as $\pi_* A_G$ is concentrated in even degrees. In particular, it continues to hold if $A = \mathbf{F}_2$. However, we will only prove Theorem 3.1.3 for coefficients in $\mathbf{Q}$.

Lemma 3.1.6. *Let R be an $\mathbf{E}_\infty$-ring, and let $M_1 \to M_2 \to M_3$ be a cofiber sequence of R-modules such that each of M_1, M_2, and M_3 have homotopy concentrated in even degrees. Then there is a short exact sequence of graded $\pi_* R$-modules*

$$0 \to \pi_* M_1 \to \pi_* M_2 \to \pi_* M_3 \to 0.$$

Lemma 3.1.7. *Let $F : \mathrm{Shv}^c_G(X; \mathbf{Q}) \to \mathrm{Mod}_{\mathbf{Q}}$ be an exact functor. Then F sends $*$-even sheaves to $\mathbf{Q}$-modules with even homotopy groups if and only if $F(j_{\lambda,!}\underline{\mathbf{Q}})$ has even homotopy groups for each $\lambda \in P$.*

Proof. Suppose that F sends $*$-even sheaves to a $\mathbf{Q}$-module with even homotopy groups. We claim that $F(j_{\lambda,!}\underline{\mathbf{Q}})$ has even homotopy groups for each $\lambda \in P$: for this, it suffices to show that for each $\lambda' \in \Lambda$, the pullback $j^*_{\lambda'} j_{\lambda,!}\underline{\mathbf{Q}}$ is a direct sum of constant sheaves concentrated in even degrees. But this is clear, because this pullback is zero unless $\lambda' = \lambda$, in which case it is just $\underline{\mathbf{Q}}$.

Let us now show the other direction. Let $\mathcal{F} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ be an $*$-even sheaf, and fix $\lambda \in P$ such that X_λ is contained in the support of $\mathcal{F}$. Then there is a recollement

$$j_{\lambda,!} j^!_\lambda \mathcal{F} \to \mathcal{F} \to i_{\lambda,*} i^*_\lambda \mathcal{F}.$$

Since $j_\lambda : X_\lambda \hookrightarrow X$ is open, we can identify $j^!_\lambda = j^*_\lambda$, and so $j^*_\lambda \mathcal{F}$ is a direct sum of constant sheaves concentrated in even degrees by assumption on $\mathcal{F}$. This implies that $j_{\lambda,!} j^!_\lambda \mathcal{F}$ is $*$-even (by the argument in the preceding paragraph), so that $F(j_{\lambda,!} j^!_\lambda \mathcal{F})$ has even homotopy groups by our assumption on F. Similarly, by induction on the strata contained in the support of $\mathcal{F}$, we may assume that $F(i_{\lambda,*} i^*_\lambda \mathcal{F})$ has even homotopy groups. Since $F(\mathcal{F})$ is an extension of $F(i_{\lambda,*} i^*_\lambda \mathcal{F})$ by $F(j_{\lambda,!} j^!_\lambda \mathcal{F})$, this implies that $F(\mathcal{F})$ also has even homotopy groups. □

Lemma 3.1.8. *The functor $\Gamma : \mathrm{Shv}^c_G(X; \mathbf{Q}) \to \mathrm{Mod}_{\mathbf{Q}}$ sends $*$-even sheaves to $\mathbf{Q}$-modules with even homotopy groups.*

Proof. By Lemma 3.1.7, we need to show that if $\lambda \in \Lambda$, the global sections $\Gamma_G(X; j_{\lambda,!}\underline{\mathbf{Q}})$ has homotopy concentrated in even degrees. This is true by our assumption on X_λ. □

Lemma 3.1.9. *Let $\mathcal{G} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ be $!$-even. Then the functor $\mathrm{Shv}^c_G(X; \mathbf{Q}) \to \mathrm{Mod}_{\mathbf{Q}}$ given by $\mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(-, \mathcal{G})$ sends $*$-even sheaves to $\mathbf{Q}$-modules with even homotopy groups.*

Proof. By Lemma 3.1.7, we need to show that if $\lambda \in \Lambda$, the $\mathbf{Q}$-module $\mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(j_{\lambda,!}\underline{\mathbf{Q}}, \mathcal{G})$ has even homotopy. This $\mathbf{Q}$-module can be identified with $\mathrm{Map}_{\mathrm{Shv}^c_G(X_\lambda;\mathbf{Q})}(\underline{\mathbf{Q}}, j^!_\lambda \mathcal{G}) = \Gamma_G(X_\lambda; j^!_\lambda \mathcal{G})$. Since $j^!_\lambda \mathcal{G}$ is a direct sum of constant sheaves concentrated in even degrees (by assumption on $\mathcal{G}$), the desired result again follows from the assumption that $\mathrm{H}^*_G(X_\lambda; \mathbf{Q})$ is concentrated in even degrees. □

Lemma 3.1.10. *Let $\mathcal{F} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ be $*$-even, and let $\mathcal{G} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ be $!$-even. Then for each $\lambda \in P$ such that X_λ is open in the support of $\mathcal{F}$, there is an exact sequence*

$$0 \to \mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X_{<\lambda};\mathbf{Q})}(i^*_\lambda \mathcal{F}, i^!_\lambda \mathcal{F}) \to \mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X_\lambda;\mathbf{Q})}(j^!_\lambda \mathcal{F}, j^*_\lambda \mathcal{G}) \to 0.$$

Proof. Recall that there is a recollement cofiber sequence

$$j_{\lambda,!} j^!_\lambda \mathcal{F} \to \mathcal{F} \to i_{\lambda,*} i^*_\lambda \mathcal{F}.$$

Applying $\mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(-,\mathcal{G})$ produces a cofiber sequence of $\mathbf{Q}$-modules
(8)
$$\mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(i_{\lambda,*}i_\lambda^*\mathcal{F},\mathcal{G}) \to \mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(\mathcal{F},\mathcal{G}) \to \mathrm{Map}_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(j_{\lambda,!}j_\lambda^!\mathcal{F},\mathcal{G}).$$
Observe that $i_{\lambda,*}i_\lambda^*\mathcal{F}$ and $j_{\lambda,!}j_\lambda^!\mathcal{F}$ are both $*$-even, so that Lemma 3.1.9 implies that each term in (8) has even homotopy. In particular, Lemma 3.1.6 implies that (8) induces a split exact sequence on homotopy groups. Note that by adjunction, we can rewrite the first term of (8) as $\mathrm{Map}_{\mathrm{Shv}^c_G(X_{<\lambda};\mathbf{Q})}(i_\lambda^*\mathcal{F}, i_\lambda^!\mathcal{G})$, and the final term of (8) as $\mathrm{Map}_{\mathrm{Shv}^c_G(X_\lambda;\mathbf{Q})}(j_\lambda^!\mathcal{F}, j_\lambda^*\mathcal{G})$. Together with the above discussion, this proves the desired claim. □

Lemma 3.1.11. *Let $\mathcal{F} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ be $*$-even, and let $\mathcal{G} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ be !-even. Then for each $\lambda \in P$ such that X_λ is open in the support of $\mathcal{F}$, there are exact sequences of graded $\mathbf{Q}$-vector spaces*
$$0 \to \mathrm{H}^*_G(X; j_{\lambda,!}j_\lambda^!\mathcal{F}) \to \mathrm{H}^*_G(X;\mathcal{F}) \to \mathrm{H}^*_G(X_{<\lambda}; i_\lambda^*\mathcal{F}) \to 0,$$
$$0 \to \mathrm{H}^*_G(X_{<\lambda}; i_\lambda^!\mathcal{G}) \to \mathrm{H}^*_G(X;\mathcal{G}) \to \mathrm{H}^*_G(X_\lambda; j_\lambda^*\mathcal{G}) \to 0.$$

Proof. We will only prove the first exact sequence; the second follows by an entirely analogous argument. Again, recall that there is a recollement cofiber sequence
$$j_{\lambda,!}j_\lambda^!\mathcal{F} \to \mathcal{F} \to i_{\lambda,*}i_\lambda^*\mathcal{F}.$$
Applying $\Gamma_G(X;-)$ gives a cofiber sequence
$$\Gamma_G(X; j_{\lambda,!}j_\lambda^!\mathcal{F}) \to \Gamma_G(X;\mathcal{F}) \to \Gamma_G(X_{<\lambda}; i_\lambda^*\mathcal{F}).$$
Observe that $i_{\lambda,*}i_\lambda^*\mathcal{F}$ and $j_{\lambda,!}j_\lambda^!\mathcal{F}$ are both $*$-even, so that Lemma 3.1.8 implies that each term in this cofiber sequence has even homotopy. In particular, Lemma 3.1.6 implies that this cofiber sequence induces a split exact sequence on homotopy groups, as desired. □

Lemma 3.1.12. *Let V be a complex affine space on which G acts linearly, and equip V with the trivial stratification. Then:*

(a) The functor $\mathrm{H}^*_G(V;-) : \mathrm{Shv}^c_G(*;\mathbf{Q}) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{Sp})$ is an equivalence, where the right-hand side denotes the ∞-category of $\mathrm{H}^*_G(V;\mathbf{Q})$-modules in spectra (i.e., the derived ∞-category of chain complexes of $\mathrm{H}^*_G(V;\mathbf{Q})$-modules).

Moreover, if $\mathcal{F},\mathcal{G} \in \mathrm{Shv}^c_G(V;\mathbf{Q})$ are sheaves such that $\mathrm{H}^*_G(V;\mathcal{F})$ is a projective $\mathrm{H}^*_G(V;\mathbf{Q})$-module, there is a graded isomorphism
$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(V;\mathbf{Q})}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(V;\mathbf{Q})}(\mathrm{H}^*_G(V;\mathcal{F}), \mathrm{H}^*_G(V;\mathcal{G})).$$
Here, the Hom *on the right-hand side is taken in the* 1*-category of graded $\mathrm{H}^*_G(X;\mathbf{Q})$-modules in $\mathrm{Mod}^\heartsuit_{\mathbf{Q}}$.*

(b) The compactly supported equivariant cohomology $\mathrm{H}^*_{G,c}(V;\mathbf{Q})$ is isomorphic to a free $\mathrm{H}^*_G(*;\mathbf{Q})$-module generated by a single class $[V]$ in degree $\dim_{\mathbf{R}}(V)$.

Proof. Let us first show (a). Since V is equipped with the trivial stratification, $\mathrm{Shv}^c_G(V;\mathbf{Q})$ is equivalent to the ∞-category of G-equivariant local systems on V. Because V is a complex affine space, this is simply equivalent to the ∞-category $\mathrm{Loc}_G(*;\mathbf{Q})$ of G-equivariant local systems of $\mathbf{Q}$-modules on a point. Almost by definition, there is an equivalence $\mathrm{Loc}_G(*;\mathbf{Q}) \simeq \mathrm{Mod}_{C^*_G(*;\mathbf{Q})}(\mathrm{Sp})$. Since G is assumed

to be a compact Lie group, the $\mathbf{Q}$-algebra $\pi_* C^*_G(*;\mathbf{Q}) = \mathrm{H}^{-*}_G(*;\mathbf{Q})$ is isomorphic to a graded polynomial $\mathbf{Q}$-algebra on generators in even negative (homological) degrees. Since the free $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra on a generator x in even degree is isomorphic to the polynomial algebra $\mathbf{Q}[x]$, choosing polynomial generators for $\pi_* C^*_G(*;\mathbf{Q})$ defines an equivalence $\mathrm{H}^*_G(*;\mathbf{Q}) \cong C^*_G(*;\mathbf{Q})$ of $\mathbf{E}_\infty$-$\mathbf{Q}$-algebras. It follows that there is an equivalence $\mathrm{Loc}_G(*;\mathbf{Q}) \cong \mathrm{Mod}_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{Sp})$. Finally, if $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^c_G(V;\mathbf{Q})$ are sheaves such that $\mathrm{H}^*_G(V;\mathcal{F})$ is a projective graded $\mathrm{H}^*_G(V;\mathbf{Q})$-module, the spectral sequence
(9)
$$E_2 = \mathrm{Ext}^\bullet_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{H}^*_G(*;\mathcal{F}), \mathrm{H}^*_G(*;\mathcal{G})) \Rightarrow \pi_{-*} \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{Sp})}(\mathrm{H}^*_G(*;\mathcal{F}), \mathrm{H}^*_G(*;\mathcal{G}))$$
degenerates at the E_2-page, where $\mathrm{Ext}^\bullet$ denotes the *graded* Ext-groups taken internal to the 1-category of graded $\mathrm{H}^*_G(*;\mathbf{Q})$-modules in $\mathrm{Mod}^\heartsuit_\mathbf{Q}$. In fact, the E_2-page is concentrated entirely in the zero line, since $\mathrm{H}^*_G(V;\mathcal{F})$ is a projective graded $\mathrm{H}^*_G(V;\mathbf{Q})$-module (so there are no higher Ext-groups). In particular, we have
$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(V;\mathbf{Q})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(V;\mathbf{Q})}(\mathrm{H}^*_G(V;\mathcal{F}), \mathrm{H}^*_G(V;\mathcal{G})),$$
as desired.

Part (b) is simply the statement of (equivariant) Poincaré duality on an affine space. $\square$

Remark 3.1.13. It is natural to ask whether the statement of Lemma 3.1.12(a) is true for arbitrary $\mathcal{F}$. Unfortunately, this need not be true. For example, suppose that $G = S^1$, and that V is the trivial vector space (without loss of generality). Using the equivalence $\mathrm{Loc}_{S^1}(*;\mathbf{Q}) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{H}^*_{S^1}(*;\mathbf{Q})}(\mathrm{Sp})$ and the fact that $\mathrm{H}^*_{S^1}(*;\mathbf{Q}) = \mathbf{Q}[x]$ with x in $\mathrm{H}^2_{S^1}(*;\mathbf{Q})$, one can define an S^1-local system on the point by the $\mathrm{H}^*_{S^1}(*;\mathbf{Q})$-module $\mathbf{Q}[x]/x = \mathbf{Q}$. The extension class
$$\mathbf{Q}[-2] \to \mathbf{Q}[x]/x^2 \to \mathbf{Q}$$
defines a nontrivial element $\delta \in \pi_1 \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{H}^*_{S^1}(*;\mathbf{Q})}(\mathrm{Sp})}(\mathbf{Q}, \mathbf{Q})$. However, this class cannot be seen from the graded Hom group: indeed,
$$\mathrm{Hom}^\bullet_{\mathrm{H}^*_{S^1}(*;\mathbf{Q})}(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q}$$
in weight zero, generated by multiples of the identity map. The spectral sequence (9) still degenerates at the E_2-page in this case, but there is a nontrivial class in $\mathrm{Ext}^1_{\mathrm{H}^*_{S^1}(*;\mathbf{Q})}(\mathbf{Q}, \mathbf{Q})$ which detects the class δ.

The above example is intended to illustrate the difference between the ∞-category of $\mathrm{H}^*_{S^1}(*;\mathbf{Q})$-modules in spectra, which can be identified with the derived ∞-category of chain complexes of $\mathrm{H}^*_{S^1}(*;\mathbf{Q})$-modules, and the category of graded $\mathrm{H}^*_{S^1}(*;\mathbf{Q})$-modules in $\mathrm{Mod}^\heartsuit_\mathbf{Q}$.

Lemma 3.1.14. *Let R be a graded (discrete) $\mathbf{Q}$-algebra. Fix two exact sequences*
$$0 \to M_1 \to M_2 \to M_3 \to 0,$$
$$0 \to N_1 \to N_2 \to N_3 \to 0$$
of graded R-modules. Then there is a sequence
$$0 \to \mathrm{Hom}^\bullet_R(M_3, N_1) \to \mathrm{Hom}^\bullet_R(M_2, N_2) \to \mathrm{Hom}^\bullet_R(M_1, N_3)$$
which is exact on the left (i.e., the second map is injective). Here, the Homs *are taken in the* 1*-category of graded R-modules in* $\mathrm{Mod}^\heartsuit_\mathbf{Q}$.

If[10] *the maps*

$$\mathrm{Hom}^\bullet_R(M_3, N_1) \hookrightarrow \mathrm{Hom}^\bullet_R(M_2, N_1),$$
$$\mathrm{Hom}^\bullet_R(M_2, N_3) \twoheadrightarrow \mathrm{Hom}^\bullet_R(M_1, N_3)$$

are isomorphisms, the above sequence is also exact in the middle.

Proof. Exactness on the left is clear, since the map α factors as injections

$$\mathrm{Hom}^\bullet_R(M_3, N_1) \hookrightarrow \mathrm{Hom}^\bullet_R(M_3, N_2) \hookrightarrow \mathrm{Hom}^\bullet_R(M_2, N_2).$$

Exactness in the middle given the assumptions follows from noting that the desired sequence can be written as the composite

$$\begin{tikzcd} & \mathrm{Hom}^\bullet_R(M_3, N_1) \arrow[d, hook, "\sim"] & & \mathrm{Hom}^\bullet_R(M_1, N_3) \\ 0 \arrow[r] & \mathrm{Hom}^\bullet_R(M_2, N_1) \arrow[r, hook] & \mathrm{Hom}^\bullet_R(M_2, N_2) \arrow[r] & \mathrm{Hom}^\bullet_R(M_2, N_3), \arrow[u, two heads, "\sim"'] \end{tikzcd}$$

where the bottom row is exact in the middle (by left exactness of Hom). □

Lemma 3.1.15. *Let* $\mathcal{F} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ *be* $*$*-even, and let* $\mathcal{G} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ *be* $!$*-even. Suppose that the map* $\mathrm{H}^*_G(X; \mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda \mathcal{F})$ *is surjective for every* $\lambda \in P$*, and that the map* $\mathrm{H}^*_G(X_\lambda; j^!_\lambda \mathcal{G}) \to \mathrm{H}^*_G(X; \mathcal{G})$ *is injective for every* $\lambda \in P$*. Then there is an exact sequence*

$$0 \to \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X_{<\lambda};\mathbf{Q})}(\mathrm{H}^*_G(X_{<\lambda}; i^*_\lambda \mathcal{F}), \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X;\mathbf{Q})}(\mathrm{H}^*_G(X; \mathcal{F}), \mathrm{H}^*_G(X; \mathcal{G}))$$
$$\to \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X_\lambda;\mathbf{Q})}(\mathrm{H}^*_G(X; j_{\lambda,!} j^!_\lambda \mathcal{F}), \mathrm{H}^*_G(X_\lambda; j^*_\lambda \mathcal{G})).$$

Here, the Hom*s are taken in the* 1*-category of graded* $\mathrm{H}^*_G(X_{<\lambda}; \mathbf{Q})$*-modules (resp.* $\mathrm{H}^*_G(X; \mathbf{Q})$*- and* $\mathrm{H}^*_G(X_\lambda; \mathbf{Q})$*-modules) in* $\mathrm{Mod}^\heartsuit_{\mathbf{Q}}$*.*

Proof. Applied to $R = \mathrm{H}^*(X; \mathbf{Q})$ and the exact sequences of Lemma 3.1.11, we see that the composite of Lemma 3.1.15 is exact on the left. For exactness in the middle, Lemma 3.1.15 says that we need to check:

(a) Every graded $\mathrm{H}^*_G(X; \mathbf{Q})$-linear map $\mathrm{H}^*_G(X; \mathcal{F}) \to \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G})$ factors through the map $\mathrm{H}^*_G(X; \mathcal{F}) \twoheadrightarrow \mathrm{H}^*_G(X_{<\lambda}; i^*_\lambda \mathcal{F})$.
(b) Every graded $\mathrm{H}^*_G(X; \mathbf{Q})$-linear map $\mathrm{H}^*_G(X; j_{\lambda,!} j^!_\lambda \mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda \mathcal{G})$ extends through the map $\mathrm{H}^*_G(X; j_{\lambda,!} j^!_\lambda \mathcal{F}) \hookrightarrow \mathrm{H}^*_G(X; \mathcal{F})$.

The proof of (b) is entirely analogous to that of (a), so we will only show (a). Recall that X_λ was assumed to be a complex affine space on which G acts linearly. Therefore, the compactly supported equivariant cohomology $\mathrm{H}^*_{G,c}(X_\lambda; \mathbf{Q})$ is isomorphic to a free $\mathrm{H}^*_G(*; \mathbf{Q})$-module on a single generator (by Lemma 3.1.12(b)). Let us denote this generator by $[X_\lambda]$.

Suppose we are given a graded $\mathrm{H}^*_G(X; \mathbf{Q})$-linear map $f : \mathrm{H}^*_G(X; \mathcal{F}) \to \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G})$. Then the following composite is zero

$$\mathrm{H}^*_G(X; \mathcal{F}) \xrightarrow{f} \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G}) \xrightarrow{\cdot [X_\lambda]} \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G})[2n_\lambda],$$

[10] In words: every graded R-linear map $M_2 \to N_1$ factors through the surjection $M_2 \twoheadrightarrow M_3$, and every graded R-linear map $M_1 \to N_3$ extends along $M_1 \hookrightarrow M_2$.

because $[X_\lambda]$ is zero in $\mathrm{H}^*_G(X_{<\lambda};\mathbf{Q})$. Since f is $\mathrm{H}^*_G(X;\mathbf{Q})$-linear, this implies that the following composite is also zero:

$$\mathrm{H}^*_G(X;\mathcal{F})[-2n_\lambda] \xrightarrow{\cdot[X_\lambda]} \mathrm{H}^*_G(X;\mathcal{F}) \xrightarrow{f} \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G}).$$

In particular, the map $f : \mathrm{H}^*_G(X;\mathcal{F}) \to \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda\mathcal{G})$ factors through the quotient $\mathrm{H}^*_G(X;\mathcal{F})/\operatorname{im}([X_\lambda])$. In order to show that the map f factors through the map $\mathrm{H}^*_G(X;\mathcal{F}) \twoheadrightarrow \mathrm{H}^*_G(X_{<\lambda}; i^*_\lambda\mathcal{F})$, it suffices to show that there is a dotted injection making the following diagram commute:

$$\begin{tikzcd}
\mathrm{H}^*_G(X;\mathcal{F}) \arrow[d, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & \\
\mathrm{H}^*_G(X_{<\lambda}; i^*_\lambda\mathcal{F}) \arrow[r, hook, dashed] & \mathrm{H}^*_G(X;\mathcal{F})/\operatorname{im}([X_\lambda]).
\end{tikzcd}$$

Equivalently, using Lemma 3.1.11, it suffices to show that $\mathrm{H}^*_G(X; j_{\lambda,!}j^!_\lambda\mathcal{F}) \hookrightarrow \mathrm{H}^*_G(X;\mathcal{F})$ is contained in $\operatorname{im}([X_\lambda])$. By Poincaré duality on the affine space X_λ, multiplication by $[X_\lambda]$ defines an isomorphism

$$\mathrm{H}^*_G(X; j_{\lambda,*}j^*_\lambda\mathcal{F})[-2n_\lambda] \xrightarrow{\cdot[X_\lambda]} \mathrm{H}^*_G(X; j_{\lambda,!}j^!_\lambda\mathcal{F}).$$

Observe that there is a commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^*_G(X;\mathcal{F})[-2n_\lambda] \arrow[r, "\cdot[X_\lambda]"] \arrow[d] & \mathrm{H}^*_G(X;\mathcal{F}) \\
\mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F}) \arrow[r, "\sim"] & \mathrm{H}^*_G(X; j_{\lambda,*}j^*_\lambda\mathcal{F})[-2n_\lambda] \arrow[r, "\sim", "\cdot[X_\lambda]"'] & \mathrm{H}^*_G(X; j_{\lambda,!}j^!_\lambda\mathcal{F}), \arrow[u, hook]
\end{tikzcd}$$

To show that $\mathrm{H}^*_G(X; j_{\lambda,!}j^!_\lambda\mathcal{F}) \hookrightarrow \mathrm{H}^*_G(X;\mathcal{F})$ is contained in $\operatorname{im}([X_\lambda])$, it suffices that the left vertical map be surjective; but this is precisely our assumption on $\mathcal{F}$. $\square$

Lemma 3.1.16. *Let $\mathcal{F} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ be $*$-even, and suppose that the map $\mathrm{H}^*_G(X;\mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F})$ is surjective for every $\lambda \in P$. Then the same is true of $i^*_\mu\mathcal{F}$ for any $\mu \in P$.*

Similarly, let $\mathcal{G} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ be $!$-even, and suppose that the map $\mathrm{H}^*_G(X_\lambda; j^!_\lambda\mathcal{G}) \to \mathrm{H}^*_G(X;\mathcal{G})$ is injective for every $\lambda \in P$. Then the same is true of $i^!_\mu\mathcal{G}$ for any $\mu \in P$.

Proof. The proof for $\mathcal{G}$ is analogous to the proof for $\mathcal{F}$, so we will only prove the latter. First, it is clear that $i^*_\mu\mathcal{F}$ is $*$-even for any $\mu \in P$. To prove the surjectivity claim, it evidently suffices to assume that X_λ is contained in the support of $i^*_\mu\mathcal{F}$ (otherwise $\mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F}) = 0$). Let λ be such that X_λ is contained in $X_{<\mu} = X_{\leq\mu} - X_\mu$, so that $i_\mu : X_{<\mu} \hookrightarrow \mathrm{Supp}(\mathcal{F})$ is the inclusion. Then, we have maps

$$\mathrm{H}^*_G(X;\mathcal{F}) \to \mathrm{H}^*_G(X_{<\mu}; i^*_\mu\mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F}).$$

The composite is surjective, and hence the map $\mathrm{H}^*_G(X_{<\mu}; i^*_\mu\mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F})$ is surjective. This gives the desired claim, since $j^*_\lambda\mathcal{F} \cong j^*_\lambda i^*_\mu\mathcal{F}$. $\square$

Proof of Theorem 3.1.3. Let us begin by showing that if $\mathcal{F} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ is even, the map $\mathrm{H}^*_G(X;\mathcal{F}) \to \mathrm{H}^*_G(X_\lambda; j^*_\lambda\mathcal{F})$ is surjective for every $\lambda \in P$; and that if $\mathcal{G} \in \mathrm{Shv}^c_G(X;\mathbf{Q})$ is even, the map $\mathrm{H}^*_G(X_\lambda; j^!_\lambda\mathcal{G}) \to \mathrm{H}^*_G(X;\mathcal{G})$ is injective for every $\lambda \in P$. The claim for $\mathcal{F}$ follows from the fact that it is even, and hence !-even, and the second exact sequence of Lemma 3.1.11. Similarly, the claim for $\mathcal{G}$ follows from the fact that it is even, and hence $*$-even, and the first exact sequence of Lemma 3.1.11.

Let us now prove Theorem 3.1.3. Assume that $\mathcal{F} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ is even, and that $\mathcal{G} \in \mathrm{Shv}^c_G(X; \mathbf{Q})$ is even. The preceding paragraph implies that the assumptions of Lemma 3.1.15 are satisfied. We will show by induction that the canonical map $\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X;\mathbf{Q})}(\mathrm{H}^*_G(X; \mathcal{F}), \mathrm{H}^*_G(X; \mathcal{G}))$ is an isomorphism. Let $\lambda \in P$ be such that X_λ is open in the union of the supports of $\mathcal{F}$ and $\mathcal{G}$. There is a map of sequences

$$\begin{tikzcd}
\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X_{<\lambda};\mathbf{Q})}(i^*_\lambda \mathcal{F}, i^!_\lambda \mathcal{F}) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X_{<\lambda};\mathbf{Q})}(\mathrm{H}^*_G(X_{<\lambda}; i^*_\lambda \mathcal{F}), \mathrm{H}^*_G(X_{<\lambda}; i^!_\lambda \mathcal{G}) \arrow[d] \\
\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X;\mathbf{Q})}(\mathcal{F}, \mathcal{G}) \arrow[r, "\beta"] \arrow[d] & \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X;\mathbf{Q})}(\mathrm{H}^*_G(X; \mathcal{F}), \mathrm{H}^*_G(X; \mathcal{G})) \arrow[d] \\
\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(X_\lambda;\mathbf{Q})}(j^!_\lambda \mathcal{F}, j^*_\lambda \mathcal{G}) \arrow[r, "\gamma"'] & \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(X_\lambda;\mathbf{Q})}(\mathrm{H}^*_G(X; j_{\lambda,!} j^!_\lambda \mathcal{F}), \mathrm{H}^*_G(X_\lambda; j^*_\lambda \mathcal{G})).
\end{tikzcd}$$

By Lemma 3.1.10, the left vertical composite is a short exact sequence. By Lemma 3.1.15, the right vertical composite is left exact (i.e., the first map is injective, and it is exact in the middle). By Lemma 3.1.16 and the inductive hypothesis, the map denoted α is a graded isomorphism. The map γ is also a graded isomorphism: indeed, since $\mathcal{F}$ is !-even, $j^!_\lambda \mathcal{F}$ is a direct sum of constant sheaves (in even degrees); similarly, since $\mathcal{G}$ is $*$-even, $j^*_\lambda \mathcal{G}$ is also a direct sum of constant sheaves (in even degrees). In particular, $\mathrm{H}^*_G(X_\lambda; j^!_\lambda \mathcal{F})$ is a projective $\mathrm{H}^*_G(X_\lambda; \mathbf{Q})$-module, so Lemma 3.1.12(a) implies that the map denoted γ is also a graded isomorphism. This implies that the map denoted β is also a graded isomorphism, as desired. □

3.2. Review of derived geometric Satake. Using the results of the preceding section, let us review the argument for the derived geometric Satake equivalence of [**BF08**], because it will serve as a model for the arguments appearing later in this article. We will use this as an opportunity to review some facts from geometric representation theory.

Setup 3.2.1. Fix a connected semisimple compact Lie group G throughout this section, and let $G_\mathbf{C}$ denote the associated complex algebraic group over $\mathbf{C}$ (so that $G_\mathbf{C}$ is a connected semisimple group). Let $\check{G}$ denote the Chevalley form over $\mathbf{Z}$ of the split semisimple algebraic group (over $\mathbf{Z}$) whose root datum is Langlands dual to that of $G_\mathbf{C}$. We will simply write $\check{G}$ to denote the base-change of $\check{G}$ to $\mathbf{Q}$; the base over which $\check{G}$ is defined will be clear from context.

Later in this section, we will also fix a Borel subgroup $\check{B} \subseteq \check{G}$, and write $\check{N}$ to denote its unipotent radical. Let Φ denote the set of roots of $\check{G}$ (i.e., coroots of $G_\mathbf{C}$), and Λ will denote the character lattice of $\check{G}$ (i.e., cocharacter lattice of $G_\mathbf{C}$). The choice of $\check{B}$ defines a subset Φ^+ of positive roots, and $\Delta \subseteq \Phi^+$ will denote a base of simple roots. Let Λ^+ denote the subset of dominant weights of $\check{G}$, so that Λ^+ is in (order-preserving) bijection with the set of orbits of the G-action on ΩG by [**Zhu17**, Theorem 1.6.1 and Equation 2.1.1]. We will add checks above each of these symbols to denote coroots, positive coroots, and simple coroots, respectively.

Definition 3.2.2. Let $\mathrm{Shv}^c_G(\Omega G; \mathbf{Q})$ denote the ∞-category of G-equivariant sheaves of $\mathbf{Q}$-modules on ΩG which are constructible for the orbit stratification. More precisely, if we write ΩG as the direct limit $\mathrm{colim}_{\lambda\in\Lambda^+} \mathrm{Gr}_G^{\leq\lambda}(\mathbf{C})$ of the finite-dimensional G-equivariant Schubert strata $\mathrm{Gr}_G^{\leq\lambda}(\mathbf{C})$ for $\lambda \in \Lambda^+$, the ∞-category $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ is defined to be the inverse limit $\lim_{\lambda\in\Lambda^+} \mathrm{Shv}_G^c(\mathrm{Gr}_G^{\leq\lambda}(\mathbf{C}); \mathbf{Q})$ along !-pullbacks.

In addition to Theorem 3.1.3, there are two key results needed to prove the derived Satake theorem. The first of these is the following.

Theorem 3.2.3 (Abelian geometric Satake, [**MV07**])**.** *Let $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q})$ denote the abelian 1-category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr_G, so that $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q})$ admits a symmetric monoidal structure arising from convolution on the affine Grassmannian. Then there is a symmetric monoidal equivalence $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q}) \simeq \mathrm{Rep}(\check{G})$.*

Remark 3.2.4. We will not need to spell out the definition of $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q})$ in the remainder of this article; in fact, all that we will need is the consequence that there is a fully faithful functor $\mathrm{Rep}(\check{G}) \hookrightarrow \mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$.

Definition 3.2.5. The action of $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q})$ on $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ via convolution defines, via Theorem 3.2.3, an action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$. Let $\mathrm{IC}_0 \in \mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ denote the pushforward $i_! \underline{\mathbf{Q}}$ of the constant sheaf along the inclusion $i : \{*\} \hookrightarrow \Omega G$ of the basepoint. Let $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G; \mathbf{Q})$ denote the full subcategory of $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ generated by IC_0 under the action of $\mathrm{Rep}(\check{G})$.

Notation 3.2.6. Let $2\rho = \sum_{\check{\alpha}\in\check{\Phi}^+} \check{\alpha}$ denote the sum of the positive coroots, so that it defines a homomorphism $2\rho : \mathbf{G}_m \to \check{T}$. The adjoint action defines an action of $\mathbf{G}_m$ on $\check{\mathfrak{g}}$ via $(2-2\rho)$, which fixes the element e. The adjoint action of $\check{G}$ on $\check{\mathfrak{g}}$ refines to a *graded* action if $\check{G}$ is equipped with the grading coming from 2ρ. We will *only* view $\check{\mathfrak{g}}$ (resp. $\check{G}$) as a graded scheme via the $(2-2\rho)$-action (resp. -2ρ-action). To emphasize this, we will denote these graded schemes as $\check{\mathfrak{g}}(2-2\rho)$ and $\check{G}(-2\rho)$.

For instance, if $\check{G} = \mathrm{SL}_2$, the grading equips the entries of an element $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2$ with the gradings where a lives in weight -2, b lives in weight 0, and c lives in weight -4; similarly, the entries an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$ have the gradings here a and d live in weight 0, b lives in weight 2, and c lives in weight -2.

Theorem 3.2.7 (Derived geometric Satake, [**BF08**])**.** *There is an $\mathbf{E}_3$-monoidal equivalence $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G; \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{\mathfrak{g}}(2-2\rho)/\check{G}(-2\rho))$ of $\mathbf{Q}$-linear ∞-categories.*

Remark 3.2.8. The shifts appearing in Theorem 3.2.7 are different than those which appear in [**BF08**]; this is because Theorem 3.2.7 is stated using the *arithmetic shearing* of [**BZSV23**, Section 6.7].

To prove Theorem 3.2.7, we need a few more ingredients. First, we need to explain the theory of the Kostant slice.

Recollection 3.2.9. Since $\check{\mathfrak{g}}$ is semisimple, the map $\check{\mathfrak{g}} \to \check{\mathfrak{g}}^*$ from the Killing form is a $\check{G}$-equivariant isomorphism. This implies that $\check{\mathfrak{g}}/\!\!/\check{G} \cong \check{\mathfrak{g}}^*/\!\!/\check{G}$, and the Chevalley restriction theorem gives an isomorphism $\check{\mathfrak{g}}^*/\!\!/\check{G} \cong \mathfrak{t}/\!\!/W$. In particular, there is an isomorphism $\mathfrak{t}/\!\!/W \cong \check{\mathfrak{g}}/\!\!/\check{G}$. There is also an *ungraded* isomorphism $\mathrm{Spec}\,\mathrm{H}_G^*(*; \mathbf{Q}) \cong \mathfrak{t}/\!\!/W$.

This can be upgraded to a graded isomorphism as follows. Let $\check{\mathfrak{g}}(2) = \mathrm{Spec}\,\mathrm{Sym}_{\mathbf{Q}}^*(\check{\mathfrak{g}}^*(-2))$ denote the $\mathbf{Q}$-vector space where the coordinate has weight -2. Similarly, let $\mathfrak{t}(2)$

denote $\operatorname{Spec}\operatorname{Sym}^*_{\mathbf{Q}}(\mathfrak{t}^*(-2))$ denote the $\mathbf{Q}$-vector space where the coordinate has weight -2. Then, there is a graded isomorphism

$$\operatorname{Spec}\mathrm{H}^*_G(*;\mathbf{Q}) \cong \mathfrak{t}(2)/\!\!/W \cong \check{\mathfrak{g}}(2)/\!\!/\check{G}.$$

Definition 3.2.10. Fix a Borel subgroup $\check{B} \subseteq \check{G}$, and let $\check{N}$ denote its unipotent radical. Let $e \in \check{\mathfrak{g}}$ be a principal nilpotent element, i.e., an element $e \in \check{\mathfrak{n}}$ such that for each simple root $\alpha \in \Delta$, the image of e under the following composite is nonzero:

$$\check{\mathfrak{n}} \twoheadrightarrow \check{\mathfrak{n}}/[\check{\mathfrak{n}}, \check{\mathfrak{n}}] \cong \prod_{\alpha\in\Delta} \mathbf{G}_a \xrightarrow{\mathrm{pr}_\alpha} \mathbf{G}_a.$$

Let $i : \mathfrak{sl}_2 \to \check{\mathfrak{g}}$ denote the Lie algebra homomorphism arising from the Jacobson-Morozov theorem, so that i sends $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$ to $e \in \check{\mathfrak{g}}$. Let $h \in \check{\mathfrak{g}}$ denote the image of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}_2$, and let $f \in \check{\mathfrak{g}}$ denote the image of $\left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$, so that $f \in \mathfrak{b}^-$. The adjoint action of h equips e with weight 2 and f with weight -2. In fact, the adjoint action of h on $\check{\mathfrak{g}}$ equips it with a grading where all the weights are *even* integers.

Let $\check{\mathfrak{g}}^f$ denote the centralizer of f in $\check{\mathfrak{g}}$. The *Kostant slice* is defined to be the affine subspace $\kappa : e + \check{\mathfrak{g}}^f \hookrightarrow \check{\mathfrak{g}}$, so that κ is a closed immersion. The grading on $\check{\mathfrak{g}}$ via $(2-2\rho)$ restricts to a grading on $e + \check{\mathfrak{g}}^f \subseteq \check{\mathfrak{g}}$.

Theorem 3.2.11 (Kostant, [**Kos78**, Theorem 1.2])**.** *Fix notation as in Definition 3.2.10, and equip $e + \check{\mathfrak{b}}^-$ with the grading coming from the adjoint action of $2-2\rho$. Then the adjoint action of $\check{N}^-$ on $e + \check{\mathfrak{b}}^-$ is one of graded schemes. The natural maps*

$$e + \check{\mathfrak{g}}^f \to (e + \check{\mathfrak{b}}^-)/\check{N}^- \to \check{\mathfrak{g}}(2)/\!\!/\check{G} \leftarrow \mathfrak{t}(2)/\!\!/W$$

are isomorphisms of graded schemes. Moreover, the closed subscheme $e + \check{\mathfrak{g}}^f \subseteq \check{\mathfrak{g}}(2-2\rho)$ meets every regular orbit exactly once, and transversally so.

Remark 3.2.12. Thanks to Theorem 3.2.11, we will often abusively view the Kostant slice as a graded map $\kappa : \mathfrak{t}/\!\!/W \cong e + \check{\mathfrak{g}}^f \subseteq \check{\mathfrak{g}}$, which provides a section to the quotient map $\check{\mathfrak{g}} \to \check{\mathfrak{g}}/\!\!/\check{G} \cong \mathfrak{t}/\!\!/W$.

Motivated by Theorem 3.2.11, we are led to the following definition.

Definition 3.2.13. Let $\check{C}$ denote the (graded) subgroup scheme of $\check{G}(-2\rho) \times \check{\mathfrak{g}}(2-2\rho)$ consisting of those pairs (g, x) such that x is regular and $\mathrm{Ad}_g(x) = x$. Write $\check{J}$ to denote the fiber product $\check{C} \times_{\check{\mathfrak{g}}} (e + \check{\mathfrak{g}}^f)$. The projection $\check{J} \to (e + \check{\mathfrak{g}}^f)$ equips $\check{J}$ with the structure of a graded group scheme over $(e + \check{\mathfrak{g}}^f) \cong \check{\mathfrak{g}}(2)/\!\!/\check{G}$; we will call $\check{J}$ the *group scheme of regular centralizers*. The fibers of the map $\check{J} \to \check{\mathfrak{g}}(2)/\!\!/\check{G}$ are commutative group schemes, so $\check{J}$ is a commutative group scheme over $\check{\mathfrak{g}}(2)/\!\!/\check{G}$.

Theorem 3.2.14 ([**BFM05, YZ11**])**.** *There is an isomorphism*

$$\operatorname{Spec}\mathrm{H}^G_*(\Omega G;\mathbf{Q}) \cong \check{J}$$

of commutative graded group schemes over $\check{\mathfrak{g}}(2)/\!\!/\check{G}$.

Remark 3.2.15. Suppose G is simply-connected. The Borel-completion of $\mathrm{H}^G_*(\Omega G;\mathbf{Q})$ is simply $\pi_*\mathbf{Q}[\Omega G]^{hG}$, which is Koszul dual to $\mathbf{Q}[G]$ acting on $\mathbf{Q}[\Omega G]$. According to rational homotopy theory, this can in turn be viewed as a deformation of $\pi_*\mathbf{Q}[\Omega G]$ along $\{0\} \hookrightarrow \operatorname{Spec}\pi_*\mathbf{Q}^{hG} \cong \widehat{\mathfrak{t}(2)}/\!\!/W$. But $\pi_*\mathbf{Q}[\Omega G]$ can be identified with the universal enveloping algebra $U(\pi_*(\Omega G)_{\mathbf{Q}})$, which is isomorphic to the dual Lie algebra $(\check{\mathfrak{g}}^e)^*$ by Theorem 3.2.14 (or Proposition 5.4.8). Theorem 3.2.14 implies that the relevant deformation of $\pi_*\mathbf{Q}[\Omega G]$ along $\{0\} \hookrightarrow \widehat{\check{\mathfrak{t}}(2)}/\!\!/W$ is precisely the relative enveloping algebra of Lie algebra of (the completion of) $\check{J}$.

Corollary 3.2.16 ([**Bao06**])**.** *The classifying stack $B_{\check{\mathfrak{g}}(2)/\!\!/\check{G}}\check{J}$ is isomorphic to the stacky quotient $\check{\mathfrak{g}}(2-2\rho)^{\mathrm{reg}}/\check{G}(-2\rho)$. In particular, there is a graded isomorphism*

$$(\check{\mathfrak{g}}(2)/\!\!/\check{G} \times \check{G}(-2\rho))/\check{J} \cong \check{\mathfrak{g}}(2-2\rho)^{\mathrm{reg}}.$$

Proof. For notational simplicity, let us drop the gradings. By definition of $\check{J}$, the classifying stack $B_{\check{\mathfrak{g}}(2)/\!\!/\check{G}}\check{J}$ is a 1-stack whose objects given by the $\check{G}$-orbit of the Kostant slice $e + \check{\mathfrak{g}}^f \subseteq \check{\mathfrak{g}}$, and such that an isomorphism $x \xrightarrow{\sim} y$ is given by an element $g \in \check{G}$ such that $\mathrm{Ad}_g(x) = y$. By Theorem 3.2.11, the $\check{G}$-orbit of the Kostant slice is precisely $\check{\mathfrak{g}}^{\mathrm{reg}}$, so we obtain the desired result. $\square$

We will also need to use the following general result, discussed in [**BF08**, Section 6.5]; this result relies on several results from [**BBD82**, Section 6.1].

Setup 3.2.17. Let F be a number field, and let R be a localization of the ring of integers $\mathcal{O}_F$. Let ℓ be a prime which is invertible in R, and fix an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}_\ell}$. Let X be an R-scheme of finite type equipped with an action of an affine group scheme H over R, such that the set of orbits is finite[11]. If R' is a ring equipped with a ring map $R \to R'$, let $X_{R'}$ denote the base-change of X to R' (and similarly for $H_{R'}$).

Theorem 3.2.18 ([**BF08**, Proposition 5])**.** *In Setup 3.2.17, there is a localization $R \subseteq R'$ such that for any k-point $R' \twoheadrightarrow k$ with k being a finite field, there are equivalences (which are functorial in X)*

$$\begin{aligned}\mathrm{Shv}^{c,\mathrm{et}}_{H_{\overline{\mathbf{F}}_q}}(X_{\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell}) &\xrightarrow{\sim} \mathrm{Shv}^{c,\mathrm{et}}_{H_{\overline{F}}}(X_{\overline{F}}; \overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} \mathrm{Shv}^{c,\mathrm{et}}_{H_{\mathbf{C}}}(X_{\mathbf{C}}; \overline{\mathbf{Q}_\ell}) \\ &\xrightarrow{\sim} \mathrm{Shv}^{c}_{H(\mathbf{C})}(X(\mathbf{C}); \overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} \mathrm{Shv}^{c}_{H(\mathbf{C})}(X(\mathbf{C}); \mathbf{C}).\end{aligned}$$

Remark 3.2.19. We are implicitly stating in Theorem 3.2.18 that the displayed equivalences are of presentable symmetric monoidal stable ∞-categories. However, we will not prove this here.

We will apply Theorem 3.2.18 to the case when X is an ind-finite R-scheme (namely, the affine Grassmannian of G): see Theorem A.4. In order to use Theorem 3.2.18 in this manner, we need the following result.

Theorem 3.2.20 (Quillen, Garland-Raghunathan, [**GR75, Mit88**])**.** *Let G_c be a compact Lie group, and let G denote the associated reductive algebraic group over $\mathbf{C}$. Then there is a homotopy equivalence $G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!]) \simeq \Omega G_c$ which is equivariant for the left-action of $G_c \subseteq G(\mathbf{C}) \subseteq G(\mathbf{C}[\![t]\!])$ on the left-hand side and the action of G_c on the right-hand side given by conjugation.*

The final input we need to prove Theorem 3.2.7 is the following elementary observation.

Observation 3.2.21. Let A be an $\mathbf{E}_n$-$\mathbf{Q}$-algebra spectrum with $n \geq 1$. View $\pi_\bullet A$ as a *graded* $\mathbf{Q}$-algebra spectrum (where $\pi_j A$ is placed in degree zero and weight j). Then, its half-shearing $\mathrm{sh}^{1/2}(\pi_\bullet A)$ has underlying $\mathbf{Q}$-module spectrum $\bigoplus_{j\in\mathbf{Z}} \pi_j(A)[j]$. Note that Lemma 2.1.5 and Remark 2.1.6 say that $\mathrm{sh}^{1/2}(\pi_\bullet A)$ will

[11] In [**BF08**], the group scheme H is assumed to be smooth, but this is not necessary.

a priori only admit an $\mathbf{E}_1$-algebra structure in graded $\mathbf{Q}$-modules; but if A only has *even* homotopy groups, Remark 2.1.7 says that $\mathrm{sh}^{1/2}\pi_\bullet A$ will indeed admit an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra structure in graded $\mathbf{Q}$-modules. If A is formal, then there is an equivalence $A \simeq \mathrm{sh}^{1/2}(\pi_\bullet A)$ of $\mathbf{E}_1$-$\mathbf{Q}$-algebra spectra. (Here, we have implicitly applied the forgetful functor $\mathrm{Mod}_{\mathbf{Q}}^{\mathrm{gr}} \to \mathrm{Mod}_{\mathbf{Q}}$ to the right-hand side.)

Let us now prove Theorem 3.2.7.

PROOF OF THEOREM 3.2.7. For notational simplicity, let us write $\mathcal{C}$ to denote $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G; \mathbf{Q})$. Let $\widetilde{\mathcal{C}}$ denote the base-change $\mathcal{C} \otimes_{\mathrm{Rep}(\check{G})} \mathrm{Mod}_{\mathbf{Q}}$, so that IC_0 is a compact generator of $\widetilde{\mathcal{C}}$ (by definition of $\mathcal{C}$). It follows from the Barr-Beck theorem [**Lur16**, Theorem 4.7.3.5] that there is an equivalence $\Phi : \widetilde{\mathcal{C}} \xrightarrow{\sim} \mathrm{Perf}_{\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)}$, implemented by the functor $\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0, -)$. Recall that $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ is equipped with an $\mathbf{E}_3$-monoidal structure (see, e.g., [**Noc20**]) where IC_0 is the monoidal unit, which equips $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ with an $\mathbf{E}_3$-monoidal structure. Note that the $\mathbf{E}_3$-monoidal structure on $\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})$ induces one on $\mathcal{C}$. With this $\mathbf{E}_3$-monoidal structure, the equivalence Φ is $\mathbf{E}_3$-monoidal. (For simplicity, we will ignore questions of $\mathbf{E}_3$-monoidality in the argument below.) Write $\check{\mathcal{R}} \in \mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G; \mathbf{Q})$ denote the (perverse) sheaf obtained by the action of $\mathcal{O}_{\check{G}} \in \mathrm{Rep}(\check{G})$ on IC_0. By definition of $\widetilde{\mathcal{C}}$, we can identify $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})$.

A key claim proved in [**BF08**] is that the $\mathbf{E}_3$-algebra $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ is in fact formal. In other words, by Observation 3.2.21, there is an isomorphism

$$\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathrm{sh}^{1/2}(\mathrm{Ext}^\bullet_{\mathrm{Shv}_G^c(\Omega G; \mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})). \tag{10}$$

We will only give a sketch of this below. First, note that Theorem 3.2.20 lets us identify $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{Shv}_G^c(\Omega G; \mathbf{C})$. Next, note that the claimed formality of the $\mathbf{Q}$-algebra $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ can be proved after base-changing to $\mathbf{C}$. In this case, we can identify

$$\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathrm{End}_{\widetilde{\mathcal{C}} \otimes_{\mathbf{Q}} \mathbf{C}}(\mathrm{IC}_0).$$

Let us write $\mathcal{D}$ to denote $\mathrm{Shv}^{c,\mathrm{et}}_{G_{\overline{\mathbf{F}}_q}}(\mathrm{Gr}_{G,\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell})$, and $\widetilde{\mathcal{D}}$ to denote the base-change of the $\mathrm{Rep}(\check{G}_{\overline{\mathbf{Q}_\ell}})$-module $\mathcal{D}$ to $\mathrm{Mod}_{\overline{\mathbf{Q}_\ell}}$. Applying Theorem A.4, we can identify

$$\mathrm{End}_{\widetilde{\mathcal{C}} \otimes_{\mathbf{Q}} \mathbf{C}}(\mathrm{IC}_0) \simeq \mathrm{End}_{\widetilde{\mathcal{D}}}(\mathrm{IC}_0) \simeq \mathrm{Hom}_{\mathrm{Shv}^{c,\mathrm{et}}_{G_{\overline{\mathbf{F}}_q}[\![t]\!]}(\mathrm{Gr}_{G,\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}).$$

As discussed in the next paragraph, $\mathrm{IC}_0 \star \check{\mathcal{R}}$ is a direct sum over $\lambda \in \Lambda^+$ of a finite number of copies of $\mathrm{IC}_\lambda[\langle 2\rho, \lambda\rangle](\langle \check{\rho}, \lambda\rangle)$. Using [**BY13**, Example 3.1.4], one finds that both IC_0 and $\mathrm{IC}_0 \star \check{\mathcal{R}}$ are pure of weight zero. Therefore, [**BY13**, Lemma 3.1.5] implies that Frobenius acts on $\pi_{-j} \mathrm{Hom}_{\mathrm{Shv}^{c,\mathrm{et}}_{G_{\overline{\mathbf{F}}_q}[\![t]\!]}(\mathrm{Gr}_{G,\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})$ by multiplication[12] by $q^{j/2}$. Since the action of Frobenius respects the ring structure on $\mathrm{End}_{\widetilde{\mathcal{D}}}(\mathrm{IC}_0)$, it follows that the action of Frobenius splits the Postnikov filtration on $\mathrm{End}_{\widetilde{\mathcal{D}}}(\mathrm{IC}_0)$ as a $\overline{\mathbf{Q}_\ell}$-algebra. In particular, $\mathrm{End}_{\widetilde{\mathcal{D}}}(\mathrm{IC}_0)$ is formal as a $\overline{\mathbf{Q}_\ell}$-algebra, as desired.

[12] *A priori*, one needs to fix a square root $q^{1/2}$ of q to state this result. However, as shown below, $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$, and hence $\mathrm{End}_{\widetilde{\mathcal{D}}}(\mathrm{IC}_0)$, is concentrated entirely in even degrees. Therefore, the integer j can be assumed to be even without any loss of generality, and no choice of a square root of q is needed.

In order to compute the Ext-algebra on the right-hand side of (10), we will apply Theorem 3.1.3. It is easy to see that IC_0 is even (it is supported only on the basepoint of ΩG). The fact that $\mathrm{IC}_0 \star \check{\mathcal{R}}$ is even is a consequence of the Peter-Weyl theorem and the proof of Theorem 3.2.3. Namely, since $\mathcal{O}_{\check{G}} = \bigoplus_{\lambda \in \Lambda^+} \mathrm{End}(V_\lambda) \cong \bigoplus_{\lambda \in \Lambda^+} V_\lambda \otimes V_\lambda^*$, one can identify $\mathrm{IC}_0 \star \check{\mathcal{R}}$ with the direct sum $\bigoplus_{\lambda \in \Lambda^+} \mathrm{IC}_\lambda[\langle 2\check{\rho}, \lambda\rangle] \otimes V_\lambda^*$. However, if $\mu \leq \lambda$ and $j_\mu : \mathrm{Gr}_G^\mu \hookrightarrow \mathrm{Gr}_G^{\leq \lambda}$ is the inclusion, $j_\mu^* \mathrm{IC}_\lambda \cong j_\mu^! \mathrm{IC}_\lambda \cong \underline{\mathbf{Q}}_{\mathrm{Gr}_G^\mu}[2 \dim \mathrm{Gr}_G^\mu]$. In particular, each IC_λ is even in the sense of Theorem 3.1.3.

Applying Theorem 3.1.3, we conclude that there is a graded isomorphism

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\Omega G;\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) &\cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(\Omega G;\mathbf{Q})}(\mathrm{H}^*_G(\Omega G; \mathrm{IC}_0), \mathrm{H}^*_G(\Omega G; \mathrm{IC}_0 \star \check{\mathcal{R}})) \\ &\cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{H}^*_G(\Omega G; \mathrm{IC}_0), \mathrm{H}^*_G(\Omega G; \mathrm{IC}_0 \star \check{\mathcal{R}}))^{\mathrm{Spec}\, \mathrm{H}_*^G(\Omega G;\mathbf{Q})}.\end{aligned}$$

But $\mathrm{H}^*_G(\Omega G; \mathrm{IC}_0) \cong \mathrm{H}^*_G(*; \mathbf{Q}) \cong \mathcal{O}_{\mathfrak{t}(2)/\!\!/W}$, and $\mathrm{H}^*_G(\Omega G; \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathcal{O}_{\mathfrak{t}(2)/\!\!/W \times \check{G}}$. Note that here and below, the symbol $\mathcal{O}$ denotes the *classical* (and not derived) ring of functions; in the two cases above, this distinction does not matter, but it will momentarily. This, along with Theorem 3.2.14, implies that

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\Omega G;\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) &\cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(*;\mathbf{Q})}(\mathrm{H}^*_G(\Omega G; \mathrm{IC}_0), \mathrm{H}^*_G(\Omega G; \mathrm{IC}_0 \star \check{\mathcal{R}}))^{\mathrm{Spec}\, \mathrm{H}_*^G(\Omega G;\mathbf{Q})} \\ &\cong \mathrm{Hom}^\bullet_{\mathcal{O}_{\mathfrak{t}(2)/\!\!/W}}(\mathcal{O}_{\mathfrak{t}(2)/\!\!/W}, \mathcal{O}_{\mathfrak{t}(2)/\!\!/W \times \check{G}(-2\rho)})^{\check{J}} \\ &\cong \mathcal{O}^{\check{J}(2)}_{\mathfrak{t}(2)/\!\!/W \times \check{G}} \cong \mathcal{O}_{(\mathfrak{t}(2)/\!\!/W \times \check{G}(-2\rho))/\check{J}}.\end{aligned}$$

Corollary 3.2.16 and the Chevalley restriction theorem precisely identifies this with $\mathcal{O}_{\check{\mathfrak{g}}^*(2-2\rho)^{\mathrm{reg}}}$. Since this is the classical (and not derived) ring of functions[13], and the complement of $\check{\mathfrak{g}}^*(2-2\rho)^{\mathrm{reg}} \subseteq \check{\mathfrak{g}}^*(2-2\rho)$ has codimension ≥ 2, there is an isomorphism $\mathcal{O}_{\check{\mathfrak{g}}^*(2-2\rho)^{\mathrm{reg}}} \cong \mathcal{O}_{\check{\mathfrak{g}}^*(2-2\rho)}$ by the algebraic Hartogs lemma. We conclude that $\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\Omega G;\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathcal{O}_{\check{\mathfrak{g}}^*(2-2\rho)}$. This implies that there is an isomorphism

$$\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \cong \mathrm{sh}^{1/2}(\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\Omega G;\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})) \cong \mathrm{sh}^{1/2}(\mathcal{O}_{\check{\mathfrak{g}}^*(2-2\rho)}) \cong \mathcal{O}_{\check{\mathfrak{g}}^*[2-2\rho]}.$$

This implies that $\widetilde{\mathcal{C}} \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2-2\rho])$, and so $\mathcal{C} \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2-2\rho]/\check{G}[-2\rho])$, as desired. □

Remark 3.2.22. The proof of (10) using Theorem A.4 can be circumvented using Lemma 2.1.8: namely, $\mathrm{End}^\bullet_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \simeq \mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\Omega G;\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})$ is isomorphic to the finitely generated polynomial algebra $\mathcal{O}_{\check{\mathfrak{g}}^*(2)}$, and hence Lemma 2.1.8 shows that $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ is formal as an $\mathbf{E}_2$-$\mathbf{Q}$-algebra. In fact, using Lemma 2.1.8 instead of Theorem A.4 allows us to replace $\mathbf{Q}$ by a finite localization $\mathbf{Z}'$ of $\mathbf{Z}$ in Theorem 3.2.7. Moreover, this observation can be used to show that the equivalence of Theorem 3.2.7 with coefficients in $\mathbf{Z}'$ is in fact $\mathbf{E}_2$-monoidal. However, lifting this to an $\mathbf{E}_3$-monoidal equivalence is rather nontrivial. Indeed, since ring spectra with polynomial homotopy need *not* be formal as $\mathbf{E}_3$-algebras (as mentioned in Remark 2.1.9), it is a much more subtle task to establish formality of $\mathrm{End}^\bullet_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ as an $\mathbf{E}_3$-*algebra*.

[13]We emphasize that the desired result would be *false* if one was to instead take derived rings of functions. Indeed, it is not true that the inclusion $\mathbf{A}^2 - \{0\} \subseteq \mathbf{A}^2$ induces an isomorphism on derived global sections of the structure sheaf (namely, $\pi_{-1}\mathcal{O}_{\mathbf{A}^2-\{0\}} \cong \mathbf{Q}[x,y]/(x^\infty, y^\infty)$), although it is certainly true at the level of classical rings of functions.

However (at least for $\mathbf{Q}$-coefficients), the argument using Theorem A.4 is more general, since it does not assume that the relevant Ext-algebra is polynomial. This argument will be useful as a model for Theorem 3.5.23 below.

Remark 3.2.23. It is a consequence of the above proof that under the equivalence of Theorem 3.2.7, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G; \mathbf{Q}) \arrow[r, "\sim", "\text{Theorem 3.2.7}"'] \arrow[d, "\Gamma_G(\Omega G; -)"'] & \mathrm{Perf}(\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho]) \arrow[d, "\kappa^*"] \\ \mathrm{Shv}_G^c(*; \mathbf{Q}) \arrow[r, "\sim", "\text{Lemma 3.1.12(a)}"'] & \mathrm{Perf}(\mathfrak{t}[2]/\!\!/W). \end{tikzcd} \tag{11}$$

Here, the map κ is the map of ungraded derived schemes underlying the half-shearing of the Kostant slice $\kappa : \mathfrak{t}(2)/\!\!/W \to \check{\mathfrak{g}}(2-2\rho)^{\mathrm{reg}}/\check{G}(-2\rho)$. Note that since these two schemes have even weights, Remark 2.1.7 ensures that the half-shearing of κ will indeed be a map of $\mathbf{E}_\infty$-$\mathbf{Q}$-schemes.

The proof of the main result of this article will follow the same outline. We will still need Theorem 3.2.3 as input; but the heart of our work lies in proving an analogue of Theorem 3.2.14.

3.3. Spherical varieties. In this section, we will review some of the theory of spherical varieties. Since the examples we will study in this article are rather simple (from the perspective of representation theory), we do not, strictly speaking, need the general theory. However, the recollections of this section will nevertheless be useful in placing basic phenomena that we will observe later into a broader context (see Section 3.4).

We will not give any proofs in this section, but instead refer to [**BLV86, LV83, Tim11, BZSV23**] for details; in particular, this section is *not* intended to be an introduction to the theory of spherical varieties or to the theory of their Hamiltonian duals. (Instead, the reader should see [**Per14**] for a very readable introduction to spherical varieties.) The base field in this section will always be the complex numbers, G will always be a connected reductive algebraic group over $\mathbf{C}$, $B \subseteq G$ will denote a chosen Borel subgroup, and N will be its unipotent radical.

Definition 3.3.1. A subgroup $H \subseteq G$ is called *spherical* if any of the following equivalent conditions are satisfied:

(a) For any G-variety X and any H-fixed point $x \in X$, the closure $\overline{G \cdot x}$ contains finitely many G-orbits.
(b) There are finitely many H-orbits in the flag variety G/B of G.
(c) There is an open H-orbit in G/B.
(d) The action of B on G/H has an open dense orbit.

An irreducible G-variety X is called *spherical* if it is normal and admits a dense open B-orbit $X^\circ \subseteq X$. In this case, X also contains an open G-orbit given by $G \cdot X^\circ$. If $x \in X^\circ$ and H is its stabilizer, there is an isomorphism $X^\circ = G/H$, and H is a spherical subgroup of G.

Before delving into examples, let us mention that the condition of being a spherical G-variety is relevant for our purposes because of the following result:

Theorem 3.3.2 ([**GN10**, Theorem 3.2.1])**.** *Let $H \subseteq G$ be a subgroup. Then the following conditions are equivalent:*

(a) *G/H is a spherical G-variety.*
(b) *The group $H(\mathbf{C}(\!(t)\!))$ acts on $\mathrm{Gr}_G(\mathbf{C}) = G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!])$ with countably many orbits.*
(c) *The group $G(\mathbf{C}[\![t]\!])$ acts on $(G/H)(\mathbf{C}(\!(t)\!))$ with countably many orbits.*

Remark 3.3.3. We refer the reader to **[GN10]** for a proof of Theorem 3.3.2, but since the argument is so short, let us recall why (b) implies (a). Suppose $\lambda : \mathbf{G}_m \to G$ is a subgroup, so that we obtain a point $x_\lambda \in \mathrm{Gr}_G(\mathbf{C})$. Then the G-orbit $X_\lambda = G \cdot x_\lambda \subseteq \mathrm{Gr}_G$ is a flag variety of G, and by (b), the number of $H(\mathbf{C}(\!(t)\!))$-orbits intersecting X_λ is countable. This implies that there is an $H(\mathbf{C}(\!(t)\!))$-orbit which intersects X_λ in an open set. If we choose a point $y \in X_\lambda$ in this open set, this implies that there is a surjection $\mathfrak{h} \twoheadrightarrow T_y X_\lambda$. If $\mathfrak{p}_y$ is the Lie algebra of the parabolic subgroup of G stabilizing y, the tangent space $T_y X_\lambda$ can be identified with $\mathfrak{g}/\mathfrak{p}_y$. In particular, if we choose λ to be regular, $\mathfrak{p}_y$ is isomorphic to a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, and hence there is a surjection $\mathfrak{h} \twoheadrightarrow \mathfrak{g}/\mathfrak{b}$. But this implies that H has an open orbit in G/B, so H is spherical.

There are a lot of examples of spherical varieties: it includes the class of flag varieties, symmetric spaces (essentially by the Iwasawa decomposition), and toric varieties.

Example 3.3.4. The quotient $\mathrm{GL}_n/\mathrm{GL}_{n-1}$ is an affine spherical GL_n-variety; it is isomorphic to the variety $\{(x, V) \in \mathbf{C}^{n+1} \times \mathrm{Gr}_n(\mathbf{C}^{n+1}) | x \notin V\}$. The fact that the $\mathbf{C}$-points of $\mathrm{GL}_n/\mathrm{GL}_{n-1}$ is homotopy equivalent to S^{2n-1} motivates the terminology "spherical".

Example 3.3.5. As mentioned above, any symmetric space is a spherical variety. In particular, since G is the fixed points of the involution on $G \times G$ which swaps the two factors, we see that $G \cong (G \times G)/G^\Delta$ is a spherical $G \times G$-variety. This will often be called the *group case*.

Example 3.3.6. Suppose $G = \mathrm{PGL}_2$. Since the flag variety of G is isomorphic to $\mathbf{P}^1$, a subgroup $H \subseteq \mathrm{PGL}_2$ is spherical if and only if it has an open orbit in $\mathbf{P}^1$. This is equivalent to saying that it is a subgroup of positive dimension. It is not difficult to see that all positive-dimensional subgroups of PGL_2 can be conjugated either to PGL_2 itself, the diagonal torus $\mathbf{G}_m \subseteq \mathrm{PGL}_2$, its normalizer $\mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m) \cong \mathrm{PO}_2 \subseteq \mathrm{PGL}_2$, or $S \cdot N \subseteq \mathrm{PGL}_2$, where N is the strictly upper-triangular matrices and $S \subseteq \mathbf{G}_m$. In general, a spherical subgroup $H \subseteq G$ is called *horospherical* if H contains the unipotent radical of the Borel $B \subseteq G$; the motivation for this term being, of course, that horocycles in $\mathrm{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{Z})$ are orbits of the subgroup of strictly upper-triangular matrices in $\mathrm{SL}_2(\mathbf{R})$. These kinds of spherical varieties are *not* considered in the present article.

Example 3.3.7. Suppose G is a connected semisimple compact Lie group with complexification $G_\mathbf{C}$, and $H \subseteq G$ is a connected closed subgroup with complexification $H_\mathbf{C}$. Then $G_\mathbf{C}/H_\mathbf{C}$ is a spherical $G_\mathbf{C}$-variety if and only if (G, H) is a Gelfand pair, i.e., $L^2(G/H)$ is multiplicity-free as a G-representation. See, e.g., **[Vin01]**f for more discussion.

Warning 3.3.8. If G is a semisimple algebraic group and $T \subseteq G$ is a maximal torus, the quotient G/T is generally *not* a spherical G-variety. Indeed, there generally will not be an open dense T-orbit in G/B, since $|\Phi^-|$ is often larger than $\mathrm{rank}(T)$, where

Φ^- is the set of negative roots of G. For instance, although the quotient $\mathrm{SL}_2/\mathbf{G}_m$ is a spherical SL_2-variety, the quotient SL_3/T is not a spherical SL_3-variety.

Remark 3.3.9. There is a finite list of closed connected spherical subgroups of simple algebraic groups: see [**KR15, Kra79**].

Example 3.3.10. Let G be a torus T. Then a T-variety X is spherical if it is normal and contains a dense orbit, and hence is precisely an affine toric variety. Let Λ denote the monoid of weights of T. Note that $\mathcal{O}_X$ is a T-submodule of $\mathcal{O}_T$, and so $\mathcal{O}_X = \bigoplus_{\lambda \in S_X} \mathbf{C}_\lambda$ for some subset $S_X \subseteq \Lambda$. A standard fact from the theory of affine toric varieties is that a subset $S_X \subseteq \Lambda$ arises from an affine toric variety if and only if $S_X = C \cap \Lambda$ for some convex cone $C \subseteq \Lambda_{\mathbf{R}}$ generated by finitely many elements of Λ which span $\Lambda_{\mathbf{R}}$. Equivalently, if $\check{C} \subseteq \check{\Lambda}$ denotes the dual cone, one observes that C spans $\Lambda_{\mathbf{R}}$ if and only if $\check{C}$ is strictly convex (i.e., contains no line). Therefore, affine toric varieties are classified by strictly convex rational polyhedral cones of $\check{\Lambda}_{\mathbf{R}}$.

Example 3.3.10 is the first indication that certain spherical varieties admit interesting combinatorial data. In particular, this combinatorial data will be useful in defining the *Langlands dual group* to a spherical variety. We will recall some generalities on defining this dual group below, and then explain its manifestation in examples.

To define this dual group following [**SV17**], let us now suppose that X is a homogeneous quasi-affine spherical G-variety. In this case, if $X^\circ \subseteq X$ is the open B-orbit, we will write H to be the stabilizer of a point $X^\circ(\mathbf{C})$, so that $X = G/H$ and $B \cdot H \subseteq G$ is open.

Construction 3.3.11. Let $\mathrm{Frac}(\mathcal{O}_X)$ denote the fraction field of $\mathcal{O}_X$, and let $\mathrm{Frac}(\mathcal{O}_X)^{(B)}$ denote the subset of $\mathrm{Frac}(\mathcal{O}_X) - \{0\}$ consisting of the nonzero rational B-eigenfunctions. Then the lattice $\mathcal{X}_X$ is simply the group of B-eigencharacters, and there is an exact sequence

$$1 \to \mathbf{C}^\times \to \mathrm{Frac}(\mathcal{O}_X)^{(B)} \to \mathcal{X}_X \to 1;$$

in other words, for a fixed $\lambda \in \mathcal{X}_X$, the functions $f \in \mathrm{Frac}(\mathcal{O}_X)^{(B)}$ which are χ-eigenvectors are all proportional by a scalar in $\mathbf{C}^\times$ (this follows from X being spherical). Let Λ_X denote the dual lattice to $\mathcal{X}_X$. Then Λ_X defines a torus T_X, and we will write $\mathfrak{t}_X$ to denote $\Lambda_X \otimes \mathbf{Q}$. The rank of the lattice Λ_X (which is also the rank of $\mathcal{X}_X$) is called the *rank* of X.

Remark 3.3.12. Suppose $X = G/H$ is a homogeneous quasi-affine G-variety, and let $\mathcal{X}_X = \mathrm{Frac}(\mathcal{O}_X)^{(B)}/\mathbf{C}^\times$ as above. It is not difficult to see that X is spherical if and only if $\mathcal{X}_X$ is a lattice of finite rank. If K is a maximal compact subgroup of $G(\mathbf{C})$, [**Akh88**] shows that

$$\mathrm{rank}(X) = \dim(K\backslash X(\mathbf{C})).$$

This is a purely topological description of the rank of X.

Construction 3.3.13. The stabilizer of the open B-orbit $X^\circ \subseteq X$ is a parabolic subgroup $P(X)$. We will write $L(X)$ to denote the Levi quotient of $P(X)$; it will often be viewed as a subgroup of $P(X)$ when convenient. Let T be a maximal torus of $B \cap L(X)$; then the torus T_X from above can be identified with $T/(T \cap B)$. The T_X-orbit of a point in the open B-orbit $X^\circ(\mathbf{C})$ defines an embedding $T_X \hookrightarrow X^\circ(\mathbf{C})$.

In other words, the B-action on X° defines a T-action on $X^\circ /\!\!/ N = \operatorname{Spec} \mathcal{O}_X^N$, and this T-action factors through the quotient $T \twoheadrightarrow T_X$.

Remark 3.3.14. In [**Kno94**, Lemma 3.1], Knop showed that if X is quasi-affine, the set of coroots in the span of $\Delta_{L(X)}$ in Λ is precisely the set of coroots $\check{\alpha} \in \Lambda$ which are perpendicular to Λ_X.

Construction 3.3.15. Suppose $v : \mathrm{Frac}(\mathcal{O}_X)^\times \to \mathbf{Q}$ is a discrete valuation which is trivial on $\mathbf{C}^\times$. Then the restriction of v to $\mathrm{Frac}(\mathcal{O}_X)^{(B)}$ defines a homomorphism $\Lambda_X \to \mathbf{Q}$, i.e., a point of $\mathfrak{t}_X$. It is known that the map from G-invariant valuations to $\mathfrak{t}_X$ is an injection, and so we will write $\mathcal{V} \subseteq \mathfrak{t}_X$ to denote the subspace of G-invariant valuations. Let $\check{\Lambda}_X^+$ denote the intersection $\Lambda_X \cap \mathcal{V}$ of G-invariant $\mathbf{Z}$-valued valuations.

It turns out that the subset $\mathcal{V} \subseteq \mathfrak{t}_X$ is a fundamental domain for the Weyl group W_X of a root system in Λ (where the weight lattice is Λ_X). In other words, the reflections over faces of $\mathcal{V}$ of codimension 1 generate a finite reflection subgroup $W_X \subseteq \mathrm{GL}(\mathfrak{t}_X)$, and this Weyl group W_X is called the *little Weyl group* of X. One can canonically identify W_X with a subgroup of W which normalizes the Weyl group $W_{L(X)}$ of $L(X)$ (with respect to the chosen torus T).

Remark 3.3.16. The definition of the little Weyl group given above does not immediately relate to the microlocal nature of X. In [**Kno94**], Knop gave an alternative construction of W_X using the Hamiltonian G-action on T^*X. Very briefly, let us review this construction. The quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defines an inclusion $(\mathfrak{g}/\mathfrak{h})^* \subseteq \mathfrak{g}^*$, and we will denote this by $\mathfrak{h}^\perp$ (it can be viewed as a subspace of $\mathfrak{g}$ via the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ given by the Killing form). Consider the moment map $\mu : T^*X \cong (G \times \mathfrak{h}^\perp)/H \to \mathfrak{g}^*$ of the Hamiltonian G-action on T^*X. Composing with the characteristic polynomial map $\mathfrak{g}^* \to \mathfrak{g}^* /\!\!/ G \cong \mathfrak{t}^* /\!\!/ W$ defines a map $T^*X \to \mathfrak{t}^* /\!\!/ W$. Observe also that the quotient map $T \twoheadrightarrow T_X$ induces an inclusion $\mathfrak{t}_X^* \hookrightarrow \mathfrak{t}^*$.

Fix a character $\chi : T_X \to \mathbf{G}_m$. Then, there is a $(P(X), \chi)$-eigenfunction $f_\chi \in \mathcal{O}_{X^\circ}$ (unique up to scalar multiplication) defines a section $d\log(f_\chi) : X^\circ \to T^*X^\circ$. This section is independent of the choice of f_χ, since f_χ is unique up to scalar multiplication. Ranging over all characters χ, one obtains a map $\mathfrak{t}_X^* \times X^\circ \to T^*X^\circ$. If $\mathcal{P}$ denotes the set of conjugates of the parabolic subgroup $P(X)$, we further obtain a map $\mathfrak{t}_X^* \times (\mathcal{P} \times X^\circ) \to T^*X$. Knop showed that the image of this map is dense, and that there is an isomorphism $(T^*X) /\!\!/ G \cong \mathfrak{t}_X^* /\!\!/ W_X$.

Said slightly differently, the fiber product $T^*X \times_{\mathfrak{t}^* /\!\!/ W} \mathfrak{t}^*$ generally has multiple irreducible components. If C is an irreducible component which dominates T^*X, we obtain a covering $C \to T^*X$, and W_X is the Galois group of this covering. In particular, note that this construction describes W_X as a subquotient of W. However, there is in fact a canonical embedding $W_X \hookrightarrow W$.

Remark 3.3.17. Continuing Theorem 3.3.2, one can show (see [**LV83**, Proposition 4.10] or [**GN10**, Theorem 3.2.1]) that the $G(\mathbf{C}[\![t]\!])$-orbits on $(G/H)(\mathbf{C}(\!(t)\!))$ are in bijection with $H(\mathbf{C}(\!(t)\!))$-orbits on $\mathrm{Gr}_G(\mathbf{C})$, which in turn are in bijection with $\check{\Lambda}_X/W_X \cong \check{\Lambda}_X^+$. This generalizes the Cartan decomposition, in the sense that when applied to the group case of Example 3.3.5, it recovers the standard parametrization of the $G(\mathbf{C}[\![t]\!])$-orbits on Gr_G. The bijection between $G(\mathbf{C}[\![t]\!])$-orbits on $(G/H)(\mathbf{C}(\!(t)\!))$ and $\check{\Lambda}_X^+$ sends a map $\lambda : \mathcal{O}_{G/H} \to \mathbf{C}(\!(t)\!)$ to the valuation given by the composite

$$\mathcal{O}_{G/H} \to \mathcal{O}_{G/H} \otimes_{\mathbf{C}} \mathcal{O}_G \xrightarrow{\lambda} \mathcal{O}_G((t)) \xrightarrow{v_t} \mathbf{Z}.$$

This is a G-invariant discrete valuation of $\mathcal{O}_{G/H}$,

Construction 3.3.18. Let $\mathcal{V}^\perp$ denote the cone $\{\chi \in \mathfrak{t}_X^* | \langle \chi, v\rangle \leq 0 \text{ for each } v \in \mathcal{V}\}$. Let Σ_X denote the set of generators of intersections of extremal rays of $\mathcal{V}^\perp$ with Λ_X. It turns out that the elements of Σ_X are linearly independent; they are known as the *spherical roots* of X. In fact, they form the set of simple roots of the based root system mentioned in Construction 3.3.15.

Remark 3.3.19. It turns out that for each spherical root $\gamma \in \Sigma_X$, there is some element $n \in \{\frac{1}{2}, 1, 2\}$ such that $\gamma' = n\gamma$ is either a positive root of G, or is the sum $\alpha + \beta$ of two positive roots which are orthogonal to each other and α and β are elements of some system of simple roots. These simple roots need not correspond to the choice of B! Let Δ_X denote the set $\{\gamma' | \gamma \in \Sigma_X\}$; then Δ_X is called the set of *normalized spherical roots.* Moreover, if Φ_X denotes the set of W_X-translates of Δ_X, it is shown in [**SV17**, Proposition 2.2.1] that the pair (Φ_X, W_X) defines a root system (called the *normalized spherical root system* of X) where Δ_X forms a set of simple roots. Let $(\check{\Phi}_X, W_X)$ denote the dual root system, and $\check{\Delta}_X$ the set of simple coroots.

Theorem 3.3.20 ([**SV17**, Proposition 2.2.2], [**KS17**])**.** *Suppose that Σ_X does not contain any elements of the form 2α for α being a root of G. Then, $(\Lambda_X, \Phi_X, \check{\Lambda}_X, \check{\Phi}_X)$ forms a root datum, with associated split complex reductive group G_X.*

Definition 3.3.21. Let $\check{G}_X$ denote the complex reductive group with maximal torus $\check{T}_X$ with root datum given by the dual of that of Theorem 3.3.20. We will refer to $\check{G}_X$ as the *(Langlands) dual group* of X. It can be viewed as a subgroup of $\check{G}$. Also see [**GN10, KS17**].

Example 3.3.22. As in Example 3.3.5, if $X = G$ is viewed as a spherical $G \times G$-variety, the group $\check{G}_X$ is simply the Langlands dual $\check{G}$ of G itself.

Example 3.3.23 (Spherical PGL_2-varieties)**.** Recall the classification of spherical subgroups $H \subseteq \mathrm{PGL}_2$ from Example 3.3.6. Let us describe the root datum of $X = \mathrm{PGL}_2/H$ from Theorem 3.3.20 in each case.

(a) If $H = \mathrm{PGL}_2$, the quotient X is a point, and everything is trivial.
(b) If $H = \mathbf{G}_m$, the orbits of B on X are the same as orbits of $\mathbf{G}_m$ on $\mathbf{P}^1$. There are therefore three orbits, given by $\mathbf{G}_m$ (the open orbit) and the points 0 and ∞. To describe the spherical roots, let us instead consider $\mathrm{SL}_2/\mathbf{G}_m \cong (\mathbf{P}^1 \times \mathbf{P}^1) - \mathbf{P}^1_{\mathrm{diag}}$. Note that $\mathcal{O}_{\mathrm{SL}_2/\mathbf{G}_m} = \mathcal{O}_{\mathrm{SL}_2}^{\mathbf{G}_m} \cong \bigoplus_{n \geq 0} V_{n\alpha}$, where α is the positive root of SL_2 and $V_{n\alpha}$ is the representation with highest weight n. It follows that $\Lambda_X \cong \mathbf{Z}$, generated by α. A little calculation implies that $\mathcal{V} \subseteq \mathfrak{t}_X$ identifies with $\{v \in \mathfrak{t}_X | \langle v, \alpha\rangle \leq 0\}$. This implies that $\Sigma_X = \Delta_X = \{\alpha\}$, and so $\check{G}_X = \mathrm{PGL}_2$. If we worked with $\mathrm{PGL}_2/\mathbf{G}_m$ instead, we would find that $\check{G}_X = \mathrm{SL}_2$.
(c) If $H = \mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m)$, the sublattice $\Lambda_X \subseteq \Lambda_{\mathrm{PGL}_2/\mathbf{G}_m}$ has index two. In particular, by (b) above, we see that $\Lambda_X = \mathbf{Z} \cdot 2\alpha$, and $\Sigma_X = \{2\alpha\}$. In particular, Theorem 3.3.20 does not apply to this particular case.

(d) If $H = S \cdot N \subseteq \mathrm{PGL}_2$, the orbits of B on X are the same as orbits of H on $\mathbf{P}^1$. There are therefore two orbits, given by $\mathbf{A}^1$ (the open orbit) and the point ∞. Let us assume for simplicity that $S = \{1\}$. Again, $\Lambda_X \cong \mathbf{Z}$, and one now calculates that Σ_X is empty. One therefore finds that $\check{G}_X = \check{T}$. In general, the dual group of horospherical varieties is the Cartan subgroup.

The cases (b), (c), and (d) above are known as types T, N, and U. The spherical $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-variety PGL_2 (i.e., the group case of Example 3.3.5) is known as type G.

Remark 3.3.24. If α is a simple root of G (or α and β are two orthogonal simple roots of G) and P_α (or $P_{\alpha\beta}$) is the associated parabolic subgroup, then the spherical variety $X^\circ P_\alpha/U_{P_\alpha}$ is isomorphic to one of $\mathrm{PGL}_2/\mathrm{PGL}_2$, PGL_2/T for T being a torus, $\mathrm{PGL}_2/\mathrm{N}_{\mathrm{PGL}_2}(T)$, or $(\mathrm{PGL}_2 \times \mathrm{PGL}_2)/\mathrm{PGL}_2^{\mathrm{diag}}$. Correspondingly, the unique element of Σ_X is a normalized spherical root, and its type is as defined in Example 3.3.23. In particular, the condition of Theorem 3.3.20 asks that X have no normalized spherical root of type N.

Remark 3.3.25. Assume from now on that X does not have any spherical roots of type N. As in [**SV17**, Section 3.6], the embedding $\check{G}_X \hookrightarrow \check{G}$ commutes with the image of a principal $\mathrm{SL}_2 \to \check{L}(X)$. In particular, there is a map $\iota : \check{G}_X \times \mathrm{SL}_2 \to \check{G}$ such that upon restriction to the diagonal torus $\mathbf{G}_m \subseteq \mathrm{SL}_2$, the map $\mathbf{G}_m \to \check{L}(X)$ is given by $2\rho_{L(X)} = \sum_{\alpha \in \Phi^+_{L(X)}} \alpha$ (regarded as a coweight of $\check{G}$). Since we will mainly deal with spherical varieties of rank 1 below, where $\check{G}_X$ itself will sometimes be SL_2, we will distinguish the SL_2 above with a superscript: namely, we will write it as $\mathrm{SL}_2^{\mathrm{Arth}}$.

3.4. Whittaker induction and [BZSV23, Conjecture 7.5.1]. In this section, we will review the notion of Whittaker induction (following [**BZSV23**, Section 3.4]), and the statement of [**BZSV23**, Conjecture 7.5.1]. This construction takes as input a map $H \times \mathrm{SL}_2^{\mathrm{Arth}} \to G$ and produces a functor from Hamiltonian H-varieties to Hamiltonian G-varieties. We warn the reader that our notation will differ slightly from that of [**BZSV23**, Section 3.4].

Recollection 3.4.1. A Hamiltonian G-variety is a smooth symplectic variety M (with symplectic form ω) equipped with a Hamiltonian G-action (i.e., the map $i : \mathfrak{g} \to T_M$ given by the derivative of the G-action lands in the subspace of Hamiltonian vector fields on M). The moment map $\mu : M \to \mathfrak{g}^*$ is characterized by the property that for each $x \in \mathfrak{g}$, we have $d\langle \mu, x\rangle = \langle i(x), \omega\rangle$. We will often simply specify a Hamiltonian G-variety as the pair (M, ω) along with its moment map. There will frequently be a grading present, which we encode by an action of $\mathbf{G}_{m,\mathrm{rot}}$ on M, G, and ω. We will say that $(M, \omega, \mu : M \to \mathfrak{g}^*)$ is a *graded* Hamiltonian G-variety (for a given $\mathbf{G}_{m,\mathrm{gr}}$-action on G) if M has a $\mathbf{G}_{m,\mathrm{gr}}$-action which acts on ω with weight 2, and the moment map μ is $\mathbf{G}_{m,\mathrm{gr}}$-equivariant.

Let us review the basic example of Whittaker induction.

Example 3.4.2. Let G be a connected reductive group (over $\mathbf{C}$), and let $e \in \mathfrak{g}$ be a principal nilpotent element, so that the Jacobson-Morozov theorem produces a map $\mathrm{SL}_2^{\mathrm{Arth}} \to G$. Let H be the trivial group, and let M denote the trivial Hamiltonian H-variety. Then the Whittaker induction of M along the map $\iota : \{1\} \times \mathrm{SL}_2^{\mathrm{Arth}} \to G$ is given by $\mathrm{WhitInd}_\iota^G(M) = (e + \mathfrak{b}^-) \times^{N^-} G$, where N^- is the unipotent radical of the opposite Borel subgroup B^- corresponding to e, and $\mathfrak{b}^-$ is the Lie algebra of B^-. Note that Theorem 3.2.11 gives isomorphisms

$$\mathrm{WhitInd}_\iota^G(M)/G \cong (e+\mathfrak{b}^-)/N^- \cong \mathfrak{g}/\!\!/G.$$

Let us now describe the construction in general.

Construction 3.4.3. Suppose we are given a map $H \times \mathrm{SL}_2^{\mathrm{Arth}} \to G$ of reductive algebraic groups over $\mathbf{C}$ such that H centralizes the map $\mathrm{SL}_2^{\mathrm{Arth}} \to G$. Let $f \in \mathfrak{g}$ be the image of $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2^{\mathrm{Arth}}$ inside $\mathfrak{g}$. The action of $\mathbf{G}_m^{\mathrm{Arth}} \subseteq \mathrm{SL}_2^{\mathrm{Arth}}$ on $\mathfrak{g}$ defines a decomposition

$$\mathfrak{g} = \mathfrak{z}^{\mathrm{Arth}} \oplus \overline{\mathfrak{n}} \oplus \mathfrak{n}_0 \oplus \mathfrak{n},$$

where $\mathfrak{z}^{\mathrm{Arth}}$ is the centralizer of $\mathfrak{sl}_2^{\mathrm{Arth}} \to \mathfrak{g}$, and $\overline{\mathfrak{n}}$, $\mathfrak{n}_0$, and $\mathfrak{n}$ are the negative, zero, and positive weight spaces. Let N denote the associated unipotent subgroup of G. Note that all the weights of the $\mathbf{G}_m^{\mathrm{Arth}}$-action on $\mathfrak{g}$ are integers, and that $e \in \mathfrak{n}$. Note that the orthogonal complement to $\mathfrak{z}^{\mathrm{Arth}} \subseteq \mathfrak{g}$ is a Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$. Let $L \subseteq G$ denote the associated subgroup.

Let $\mathfrak{n}_+$ denote the subspace of $\mathfrak{n}$ of elements with weight ≥ 2, and let N_+ denote the associated unipotent subgroup. One can then equip $\mathfrak{n}/\mathfrak{n}_+$ with the structure of a Hamiltonian HN-space. There is an H-invariant symplectic form ω on $\mathfrak{n}/\mathfrak{n}_+$, given by $\omega(x,y) = \langle f, [x,y]\rangle$.[14] Since H preserves ω, we obtain a homomorphism $H \to \mathrm{Sp}(\mathfrak{n}/\mathfrak{n}_+)$, and hence a map $\mathfrak{h} \to \mathfrak{sp}_{\mathfrak{n}/\mathfrak{n}_+}$. The group H acts on $\mathfrak{n}/\mathfrak{n}_+$ by the adjoint action. Moreover, the group N acts on $\mathfrak{n}/\mathfrak{n}_+ \cong N/N_+$ via translation. The moment map $\mu : \mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{h}^* \oplus \mathfrak{n}^*$ is defined as follows:

- The map $\mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{h}^*$ is adjoint to the map
$$\mathfrak{n}/\mathfrak{n}_+ \oplus \mathfrak{h} \to \mathfrak{n}/\mathfrak{n}_+ \oplus \mathfrak{sp}_{\mathfrak{n}/\mathfrak{n}_+} \xrightarrow{(x,g)\mapsto \frac{1}{2}\omega(gx,x)} \mathfrak{g}_a.$$
- The map $\mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{n}^*$ is given by the composite
$$\mathfrak{n}/\mathfrak{n}_+ \xrightarrow{\omega} (\mathfrak{n}/\mathfrak{n}_+)^* \xrightarrow{x\mapsto f+x} \mathfrak{n}^*.$$
Here, f is viewed as an element of $\mathfrak{n}^*$ via the identification $\mathfrak{n}^* \cong \mathfrak{n}$. Under this isomorphism, the image of $\mathfrak{n}/\mathfrak{n}_+$ inside $\mathfrak{n}$ is simply $f + \mathfrak{n}_1$, where $\mathfrak{n}_1$ is the weight 1 eigenspace.

Remark 3.4.4. There is a natural grading defined on $\mathfrak{n}/\mathfrak{n}_+$, as well as a natural $\mathbf{G}_{m,\mathrm{gr}}$-action on N via the conjugation action of $\mathbf{G}_m^{\mathrm{Arth}}$. If H is equipped with the trivial $\mathbf{G}_{m,\mathrm{gr}}$-action, the Hamiltonian HN-space $\mathfrak{n}/\mathfrak{n}_+$ from Construction 3.4.3 can be viewed as a graded Hamiltonian HN-space.

Definition 3.4.5. Fix a map $\iota : H \times \mathrm{SL}_2^{\mathrm{Arth}} \to G$ of reductive algebraic groups over $\mathbf{C}$ such that H centralizes the map $\mathrm{SL}_2^{\mathrm{Arth}} \to G$. The conjugation action of $\mathbf{G}_m^{\mathrm{Arth}}$ on G composed with the square character equips G with a grading (which we will think of as a $\mathbf{G}_{m,\mathrm{gr}}$-action). Let M be a graded Hamiltonian H-variety. Then the *Whittaker induction* $\mathrm{WhitInd}_\iota^G(M)$ is defined as

$$\mathrm{WhitInd}_\iota^G(M) = (M \times \mathfrak{n}/\mathfrak{n}_+) \times^{HN}_{\mathfrak{h}^*\oplus\mathfrak{n}^*} (T^*G),$$

[14] Note that this symbol is well-defined: if $x \in \mathfrak{n}_+$, then $[x,y]$ lives in weight ≥ 3, so $\langle f, [x,y]\rangle = 0$ since f has weight -2. Moreover, this form is indeed nondegenerate: if $x \in \mathfrak{n}$ is nonzero of weight 1, then $[f,x]$ is a nonzero element of weight -1. This implies that there is some $y \in \mathfrak{n}$ of weight 1 such that $\langle [f,x], y\rangle = \langle f, [x,y]\rangle$ is nonzero, as desired.

where T^*G is regarded as a Hamiltonian HN-space via restriction along $HN \subseteq G$. There is a natural grading on $\mathrm{WhitInd}_\iota^G(M)$, coming from the grading on M, the grading on $\mathfrak{n}/\mathfrak{n}_+$ from Remark 3.4.4, and the grading on T^*G coming from the $\mathbf{G}_{m,\mathrm{gr}}$-action on G. In particular, note that there is an isomorphism of stacks

$$\mathrm{WhitInd}_\iota^G(M)/G \cong ((M \times \mathfrak{n}/\mathfrak{n}_+) \times_{\mathfrak{h}^* \oplus \mathfrak{n}^*} \mathfrak{g}^*)/HN.$$

The simplest way to understand Whittaker induction in the case when M is a symplectic H-representation is as follows.

Lemma 3.4.6 ([**BZSV23**, Section 3.4.8])**.** *Suppose M is a symplectic H-representation, and fix an isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$. Then there is an isomorphism of stacks*

$$\mathrm{WhitInd}_\iota^G(M)/G \cong (M \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}^e))/H$$

over BG.

Proof. Using [**GG02**, Lemma 2.1], one obtains an inclusion $f + \mathfrak{g}^e \subseteq f + \mathfrak{n}_+^\perp$ which is a slice of the N-action on $f + \mathfrak{n}_+^\perp$. Therefore, there is an isomorphism

$$\begin{aligned} N \times (M \times_{\mathfrak{h}^*} \mathfrak{g}^e) &\to (M \times \mathfrak{n}/\mathfrak{n}_+) \times_{\mathfrak{h}^* \oplus \mathfrak{n}^*} \mathfrak{g}^* \\ &\cong \{(v, x) \in M \times (f + \mathfrak{n}_+^\perp) \text{ such that } \mu(v) = x|_\mathfrak{h}\}, \end{aligned}$$

sending $(n, v, y) \mapsto (v, n \cdot (f + y))$. This isomorphism is H-equivariant, so it follows that $\mathrm{WhitInd}_\iota^G(M)/G$ is isomorphic to $(M \times_{\mathfrak{h}^*} \mathfrak{g}^e)/H$ as stacks over BG. This implies the desired claim, since $M \times_{\mathfrak{h}^*} \mathfrak{g}^e \cong M \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}^e)$. ☐

Let us now recall a statement of [**BZSV23**, Conjecture 7.5.1]; our presentation will follow [**BZSV23**, Section 4.3]. Assume for now that X is an affine spherical G-variety over $\mathbf{C}$ which is the affine closure of its open G-orbit (for instance, this holds if X is affine and homogeneous).

Definition 3.4.7. A *color* of X is an irreducible B-stable divisor which is not G-stable (if X is homogeneous, this is simply an irreducible B-stable divisor). Following [**BZSV23**, Definition 4.3.4], a standard parabolic $P \subseteq G$ is said to be of *even spherical type* if the spherical P/U_P-variety $X^\circ P/U_P$ is isomorphic to either the spherical SO_{2n+1}-variety $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$ or the spherical G_2-variety G_2/SL_3. (Note that there are diffeomorphisms $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n} \cong S^{2n}$ and $G_2/\mathrm{SL}_3 \cong S^6$.) A color D is said to be of *even spherical type* if it meets $X^\circ P$ for a standard parabolic P of even spherical type. Let $\mathcal{C}_X$ denote the set of colors of X of even spherical type.

Suppose that the elements of $\mathcal{C}_X$ freely generate a direct summand of $\check{\Lambda}_X$. Let $\mathcal{D}_X$ denote the set of dominant W_X-translates of $\mathcal{C}_X \subseteq \check{\Lambda}_X$, and let $\mathcal{D}_X^{\max}$ denote the subset of maximal elements of $\mathcal{D}_X$ (with respect to the ordering via coroots of $\check{G}_X$). Let S_X denote the $\check{G}_X$-representation with highest weights $\mathcal{D}_X^{\max}$. It is expected (see [**BZSV23**, Conjecture 4.3.16]) that S_X admits an $\check{G}_X$-invariant symplectic form.

Example 3.4.8 ([**BZSV23**, Example 4.3.9])**.** Consider the example of the spherical GL_2-variety $X = \mathrm{GL}_2/\mathbf{G}_m$ (in which case $\check{G}_X = \check{G} = \mathrm{GL}_2$). Then $U\backslash X^\circ \cong \mathbf{G}_m^2$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (b, d^{-1}\det)$. The colors of X are given by the vanishing loci of b and d, and are both of even spherical type. As explained in [**BZSV23**, Example 4.3.9], this implies that $\mathcal{C}_X$ is the subset $\{\check{\alpha}_1, -\check{\alpha}_2\}$ of $\check{\Lambda}_X = \check{\Lambda}$, which in turn implies that $S_X = \mathbf{A}^2 \oplus (\mathbf{A}^2)^* \cong T^*(\mathbf{A}^2)$ as an $\check{G}_X$-representation. However, as remarked in [**BZSV23**], the condition that the elements of $\mathcal{C}_X$ freely generate a direct summand of $\check{\Lambda}_X$ is *not* true in the example of $\mathrm{PGL}_2/\mathbf{G}_m$ (whose dual group is $\check{G}_X = \check{G} = \mathrm{SL}_2$). Nevertheless, the variant of Definition 3.4.7 discussed in [**BZSV23**, Section 4.4] shows that S_X is the $\check{G}_X$-representation $T^*(\mathbf{A}^2)$.

Example 3.4.9. For $n > 2$, the spherical GL_n-variety $X = \mathrm{GL}_n/\mathrm{GL}_{n-1}$ still has $\check{G}_X = \mathrm{GL}_2$, but the representation S_X is zero. (I am very grateful to Justin Hilburn and Yiannis Sakellaridis for this point.) For instance, when $n = 3$, the Whittaker induction $\mathrm{WhitInd}_\iota^{\mathrm{GL}_3} S_X$ along the map $\iota : \mathrm{GL}_2 \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{GL}_3$ of Remark 3.3.25 can be identified with $T^*(\mathrm{GL}_3/\mathrm{GL}_2)$ using Lemma 3.4.6.

Example 3.4.10. Consider the example of the spherical SO_4/μ_2-variety $\mathrm{SO}_4/\mu_2 \cdot \mathrm{SO}_3$ (in which case $\check{G}_X = \mathrm{SL}_2$). Since $\mathrm{Spin}_4 \cong \mathrm{SL}_2 \times \mathrm{SL}_2$, there is an isomorphism $\mathrm{SO}_4/\mu_2 \cong \mathrm{SO}_3 \times \mathrm{SO}_3$, under which the embedding of SO_3 into SO_4/μ_2 is given by the diagonal. Therefore, there is an isomorphism $\mathrm{SO}_4/\mu_2\mathrm{SO}_3 \cong \mathrm{SO}_3$, and this spherical SO_4/μ_2-variety can be understood as the group case for SO_3. Using this, one can show that $\check{G}\backslash \mathrm{WhitInd}_\iota^{\check{G}} S_X \cong \mathfrak{sl}_2/\check{G}_X$. If we had instead worked with the spherical SO_4-variety $\mathrm{SO}_4/\mathrm{SO}_3 \cong \mathrm{SL}_2$, we would have $\check{G}_X = \mathrm{PGL}_2$, and $\check{G}\backslash \mathrm{WhitInd}_\iota^{\check{G}} S_X \cong \mathfrak{pgl}_2/\check{G}_X$.

The following is a slight variant of [**BZSV23**, Conjecture 7.5.1].

Conjecture 3.4.11. *Suppose X is a smooth affine spherical G-variety over $\mathbf{C}$ which is the affine closure of its open G-orbit, and with no roots of type N. Let $\iota : \check{G}_X \times \mathrm{SL}_2^{\mathrm{Arth}} \to \check{G}$ denote the map of Remark 3.3.25. Suppose that S_X admits an $\check{G}_X$-invariant symplectic form, and let $\check{M}$ denote $\mathrm{WhitInd}_\iota^{\check{G}} S_X$. Then:*

- *There is an equivalence*[15]
$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G(\mathbf{C}[\![t]\!])}(X(\mathbf{C}(\!(t)\!));\mathbf{Q}) \cong \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}(-2\rho)).$$
- *This equivalence is equivariant for the actions of* $\mathrm{Shv}^{c,\mathrm{Sat}}_{G(\mathbf{C}[\![t]\!])\times G(\mathbf{C}[\![t]\!])}(G(\mathbf{C}(\!(t)\!));\mathbf{Q})$ *and* $\mathrm{Perf}(\check{\mathfrak{g}}^*[2-2\rho]/\check{G}[-2\rho])$ *under the equivalence of Theorem 3.2.7.*

Remark 3.4.12. One of the requirements for the equivalence of Conjecture 3.4.11 is the "pointing" of [**BZSV23**, Section 7.5.2]. Namely, the pushforward of the constant sheaf along $i : X(\mathbf{C}[\![t]\!]) \to X(\mathbf{C}(\!(t)\!))$ must be sent under the equivalence of Conjecture 3.4.11 to the structure sheaf of $\mathrm{sh}^{1/2}\check{M}/\check{G}$. This implies, in particular, that
$$\mathrm{End}_{\mathrm{Shv}^{c,\mathrm{Sat}}_{G(\mathbf{C}[\![t]\!])}(X(\mathbf{C}(\!(t)\!));\mathbf{Q})}(i_*\underline{\mathbf{Q}}_{X(\mathbf{C}[\![t]\!])}) \simeq \mathcal{O}^{\check{G}}_{\mathrm{sh}^{1/2}\check{M}}.$$
The left-hand side is simply $C^*_{G(\mathbf{C}[\![t]\!])}(X(\mathbf{C}[\![t]\!]);\mathbf{Q}) \simeq C^*_G(X;\mathbf{Q})$, while the right-hand side is $\mathcal{O}_{\mathrm{sh}^{1/2}\check{M}/\!\!/\check{G}}$. Therefore, the "pointing" requirement can be restated as the existence of an equivalence of $\mathbf{E}_1$-$\mathbf{Q}$-algebras $C^*_G(X;\mathbf{Q}) \simeq \mathcal{O}_{\mathrm{sh}^{1/2}\check{M}/\!\!/\check{G}}$. If $X = G/H$, the left-hand side is exactly $C^*_H(*;\mathbf{Q}) \simeq \mathrm{sh}^{1/2}\mathrm{H}^*_H(*;\mathbf{Q})$, so this equivalence can be rephrased as a graded isomorphism
$$\check{M}/\!\!/\check{G} \cong \mathrm{Spec}\,\mathrm{H}^*_H(*;\mathbf{Q}) \cong \check{\mathfrak{h}}^*(2)/\!\!/\check{H}. \tag{12}$$
Using Lemma 3.4.6, one can identify $\check{M}/\!\!/\check{G} \cong (S_X \oplus (\check{\mathfrak{g}}_X^\perp \cap \check{\mathfrak{g}}^e))/\!\!/\check{G}_X$; it might be possible to prove the resulting identification with $\check{\mathfrak{h}}^*(2)/\!\!/\check{H}$ in a direct manner (without having first established Conjecture 3.4.11).

Name	$X = G/H$	Semisimple part of $L(X)$	Normalized spherical root	Root type	$\check{G}_X \circlearrowleft \check{Y}$
A_n	$\mathrm{GL}_{n+1}/\mathrm{GL}_n$	$[1, n-1, 1]$	$\alpha_1 + \cdots + \alpha_n$	T	$\mathrm{GL}_2 \circlearrowleft T^*\mathbf{A}^2$
B_n	$\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$	SO_{2n-1}	$\alpha_1 + \cdots + \alpha_n$	T	$\mathrm{SL}_2 \circlearrowleft T^*\mathbf{A}^2$
C_n	$\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$	$\mathrm{SL}_2 \times \mathrm{Sp}_{2n-4}$	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$	T	$\mathrm{SL}_2 \circlearrowleft T^*\mathbf{A}^2$
D_n	$\mathrm{SO}_{2n}/\mu_2 \cdot \mathrm{SO}_{2n-1}$	SO_{2n-2}/μ_2	$2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	G	$\mathrm{SL}_2 \circlearrowleft \mathfrak{sl}_2$
F_4	F_4/Spin_9	Spin_7	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	T	$\mathrm{SL}_2 \circlearrowleft T^*\mathbf{A}^2$
G_2	G_2/SL_3	SL_2	$2\alpha_1 + \alpha_2$	T	$\mathrm{SL}_2 \circlearrowleft T^*\mathbf{A}^2$
B_3'	SO_7/G_2	SL_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	G	$\mathrm{SL}_2 \circlearrowleft \mathfrak{sl}_2$

TABLE 3. Table of affine homogeneous rank one spherical varieties with no roots of type N. Note that all the examples above except for types G_2 and B_3' are in fact symmetric varieties; and these are in bijection with compact Riemannian symmetric spaces of rank one. See Table 4 for a more refined version of this table.

Let us end this section with a table of affine homogeneous rank one spherical varieties, as classified by [**Akh83**], and their conjectured dual Hamiltonian varieties. This table is essentially lifted from [**SV17**, Section A.3.6] and [**Sak21**, Table 1]. The main result of this article is the following:

Theorem 3.4.13. *Assuming Hypothesis 3.5.21, the first part of Conjecture 3.4.11 is true for each of the affine homogeneous rank* 1 *spherical varieties in Table 3.*

Using Theorem 3.2.20 and Theorem 3.2.18, we will replace the left-hand side in Conjecture 3.4.11 with $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}X; \mathbf{Q})$.

Remark 3.4.14. For each row in Table 3, there is a graded isomorphism

$$\check{M}/\check{G} \cong \check{Y}/\check{G}_X \times \text{“Normalization”}$$

of stacks. This normalization term can be identified with $\mathfrak{l}_X^\wedge /\!\!/ L_X^\wedge$, where $L_X^\wedge$ is the subgroup of $\check{G}$ from [**KS17**]. (This is *not* quite the Langlands dual of the Levi subgroup $L(X)$.) This essentially follows from Lemma 3.4.6, which identifies $S_X \oplus (\check{\mathfrak{g}}_X^\perp \cap \check{\mathfrak{g}}^e)$ with $\check{Y} \times$ “Normalization”. Note that this is somewhat surprising: for instance, when $X = \mathrm{GL}_{n+1}/\mathrm{GL}_n$ (so $\check{G}_X = \mathrm{GL}_2$), we have $S_X = T^*(\mathbf{A}^2)$ when $n = 1$ by Example 3.4.8; but $S_X = 0$ for $n \geq 2$. Nevertheless, $\check{Y}$ always identifies with $T^*(\mathbf{A}^2)$ as GL_2-schemes.

In any case, Theorem 3.4.13 therefore states that there is a grading on $\check{Y}$ and an equivalence

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}X; \mathbf{Q}) \cong \mathrm{Perf}(\mathrm{sh}^{1/2}\check{Y}/\check{G}_X \times \text{“Normalization”});$$

it is this form of Theorem 3.4.13 which we will prove below. See Table 4.

Remark 3.4.15. As mentioned in the introduction, the analogue of Theorem 3.4.13 in the real and p-adic cases was proved by Gan and Gomez as [**GG14**, Theorem 1].

As mentioned at the end of Section 3.2, the proof of Theorem 3.4.13 is modeled after the proof of Theorem 3.2.7. In fact, in the course of proving Theorem 3.4.13, we will accomplish more than what is stated: namely, our argument also suggests that there is a “chromatic deformation” of Conjecture 3.4.11, which we will explain in greater detail after the proof of Theorem 3.4.13 (which will in turn occupy Section 4).

[15]The ∞-category on the left-hand side is defined in Definition 3.5.17 using the action of $\mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G; \mathbf{Q})$ on $\mathrm{Shv}_G^c(\mathcal{L}X; \mathbf{Q})$.

3.5. The G-equivariant ku-(co)homology of $\mathcal{L}(G/H)$. Fix a compact Lie group G, and let $H \subseteq G$ be a closed subgroup. Throughout this section, we will always assume that H and G are connected. Before proceeding to the proof of Theorem 3.4.13, we will prove some general statements about the G-equivariant ku-(co)homology of $\mathcal{L}(G/H)$. The following basic result is an analogue of the calculation that $G\backslash T^*(G/H) \cong H\backslash(\mathfrak{g}/\mathfrak{h})^*$.

Proposition 3.5.1. *Let H act on G/H, and hence on $\Omega(G/H)$, by conjugation (equivalently, left-translation). Then the G-space $\mathcal{L}(G/H)$ is G-equivariantly homotopy equivalent to $\mathrm{Ind}_H^G \Omega(G/H)$. In particular, there is an equivalence of orbispaces*

$$G\backslash\mathcal{L}(G/H) \simeq H\backslash\Omega(G/H).$$

Proof. It is a classical fact that the map $m : G \times \Omega G \to \mathcal{L}G$ sending (g, γ) to the loop $\gamma_g : t \mapsto g\gamma(t)$ is a homotopy equivalence. The left action of G on $\mathcal{L}G$ is simply given by

$$G \ni g' : \gamma_g(t) \mapsto \gamma_{g'g}(t),$$

which allows us to identify $\mathcal{L}G \simeq \mathrm{Ind}_{\{1\}}^G \Omega G$. Recall that there is a principal fibration

$$H \to G \to G/H,$$

which gives equivalences $\Omega(G/H) \simeq \Omega G/\Omega H$ and $\mathcal{L}(G/H) \simeq \mathcal{L}G/\mathcal{L}H$. Since and the diagram

$$\begin{tikzcd} H \times \Omega H \arrow[r] \arrow[d, "m"'] & G \times \Omega G \arrow[d, "m"] \\ \mathcal{L}H \arrow[r] & \mathcal{L}G \end{tikzcd}$$

commutes, we find that there is an equivalence of G-spaces

$$\mathcal{L}G/\mathcal{L}H \simeq (\mathrm{Ind}_{\{1\}}^G \Omega G)/(\mathrm{Ind}_{\{1\}}^H \Omega H) \simeq \mathrm{Ind}_H^G \Omega(G/H),$$

as desired. $\square$

Remark 3.5.2. Let $X = G_{\mathbf{C}}/H_{\mathbf{C}}$. By Theorem 3.3.2, Remark 3.3.17, and Proposition 3.5.1, the H-orbits on $\Omega(G/H)$ can be parametrized by $\check{\Lambda}_X/W_X$. There is a multiplicative presentation of $\Omega(G/H)$ as $\mathrm{colim}_{\lambda \in \check{\Lambda}_X/W_X} X_\lambda$ via finite H-spaces X_λ, and the induced G-spaces $\mathrm{Ind}_H^G X_\lambda$ defines a presentation of $\mathcal{L}(G/H)$ by finite G-spaces. It follows that there is an equivalence $\mathcal{F}_G(\mathcal{L}(G/H)) \cong \mathcal{F}_H(\Omega(G/H))$ of $\mathbf{E}_\infty$-ku_G-algebras, where the right-hand side is viewed as an $\mathbf{E}_\infty$-ku_G-algebra via its natural ku_H-algebra structure and the canonical map $\mathrm{ku}_G \to \mathrm{ku}_H$.

Warning 3.5.3. Although there is an equivalence $\mathcal{F}_G(\mathcal{L}(G/H)) \cong \mathcal{F}_H(\Omega(G/H))$ of $\mathbf{E}_\infty$-ku_G-algebras, there is *not* an equivalence $\mathcal{F}_G(\mathcal{L}(G/H))^\vee \cong \mathcal{F}_H(\Omega(G/H))^\vee$ of ku_G-modules. Indeed, $\mathcal{F}_G(\mathcal{L}(G/H))^\vee$ denotes the ku_G*-linear* dual of $\mathcal{F}_G(\mathcal{L}(G/H))$, while $\mathcal{F}_H(\Omega(G/H))^\vee$ denotes the ku_H*-linear* dual of $\mathcal{F}_H(\Omega(G/H))^\vee$.

Remark 3.5.4. Proposition 3.5.1 breaks the natural symmetry on $G\backslash\mathcal{L}(G/H)$. Namely, since the action of G on $\mathcal{L}(G/H)$ is defined via the G-action on G/H, the orbispace $G\backslash\mathcal{L}(G/H)$ has an action of the circle S^1_{rot} given by rotating loops. However, this structure is not naturally visible on the orbispace $\Omega(G/H)/H$. Indeed, the proof of Proposition 3.5.1 used the splitting $G \times \Omega G \xrightarrow{\sim} \mathcal{L}G$; but this splitting is *not* S^1_{rot}-equivariant.

A slight variant of Proposition 3.5.1 lets us describe the G-equivariant ku-*co*homology of $\mathcal{L}(G/H)$. The following result is proved nonequivariantly in [**Dev23a**, Remark A.6].

Proposition 3.5.5. *There is an S^1_{rot}-equivariant equivalence of $\mathbf{E}_\infty$-ku_G-algebras*

$$\mathcal{F}_G(\mathcal{L}(G/H)) \simeq \mathrm{HH}(\mathrm{ku}_H/\mathrm{ku}_G),$$

where the right-hand side denotes the relative Hochschild homology of the $\mathbf{E}_\infty$-map $\mathrm{ku}_H \to \mathrm{ku}_G$ (equipped with its natural S^1-action).

Proof. Since G/H is itself the fiber product $* \times_{*/G} */H$ in orbispaces, there is an equivalence

$$G\backslash\mathcal{L}(G/H) \simeq G\backslash(* \times_{\mathcal{L}(*/G)} \mathcal{L}(*/H)) \simeq */G \times_{\mathcal{L}(*/G)} \mathcal{L}(*/H).$$

But $\mathcal{L}(*/G) \simeq */G \times_{*/G\times */G} */G$, where the two maps $*/G \to */G \times */G$ are both given by the diagonal. Therefore, we can identify

$$\begin{aligned} G\backslash\mathcal{L}(G/H) &\simeq */G \times_{*/G\times_{*/G\times */G}*/G} (*/H \times_{*/H\times */H} */H) \\ &\simeq */H \times_{*/H\times_{*/G}*/H} */H. \end{aligned} \tag{13}$$

By construction of equivariant ku, it follows that there is an equivalence of $\mathbf{E}_\infty$-ku_G-algebras

$$\mathcal{F}_G(\mathcal{L}(G/H)) \simeq \mathrm{ku}_H \otimes_{\mathrm{ku}_H \otimes_{\mathrm{ku}_G} \mathrm{ku}_H} \mathrm{ku}_H = \mathrm{HH}(\mathrm{ku}_H/\mathrm{ku}_G).$$

Moreover, the equivalence of (13) is manifestly S^1-equivariant, so we obtain the desired claim. $\square$

Base-changing Proposition 3.5.5 along the map $\mathrm{ku} \to \mathbf{Q}$ and using the Hochschild-Kostant-Rosenberg theorem (in the form proved in [**Rak20, MRT19**]), one finds:

Corollary 3.5.6. *There is an equivalence*

$$C^*_G(\mathcal{L}(G/H); \mathbf{Q}) \simeq L\Omega^*_{\mathrm{sh}^{1/2}\mathrm{H}^*_H(*;\mathbf{Q})/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*;\mathbf{Q})},$$

where the right-hand side is derived Hodge cohomology. This equivalence identifies the loop rotation action of S^1 on the left-hand side with the de Rham differential on the right-hand side.

Remark 3.5.7. Suppose G/H is an affine spherical G-variety, and assume that (12) of Remark 3.4.12 holds for G/H (which would follows from Conjecture 3.4.11). Then Corollary 3.5.6 implies that if $\mu : \check{M} \to \check{\mathfrak{g}}^*$ denotes the moment map, there is an isomorphism (ignoring gradings for simplicity)

$$\mathrm{H}^*_G(\mathcal{L}(G/H); \mathbf{Q}) \simeq L\Omega^*_{\check{M}/\!\!/\check{G}/\check{\mathfrak{g}}^*/\!\!/\check{G}}.$$

In fact, more is true: taking cohomology defines a functor

$$\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}G/\mathcal{L}H; \mathbf{Q}) \xrightarrow{\mathrm{H}^*_G(-;\mathbf{Q})} \mathrm{Mod}(\mathrm{H}^*_G(\mathcal{L}G/\mathcal{L}H; \mathbf{Q})).$$

By Conjecture 3.4.11, the left-hand side can be identified with $\mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G})$. By Corollary 3.5.6, the right-hand side can be identified with the ∞-category of modules over the shearing of $L\Omega^*_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H}/\check{\mathfrak{g}}^*(2)/\!\!/\check{G}}$, i.e., the ∞-category of perfect complexes over the shearing of the (-1)-shifted tangent bundle $T[-1](\check{\mathfrak{h}}^*(2)/\!\!/\check{H}/\check{\mathfrak{g}}^*(2)/\!\!/\check{G})$.

In particular, there is a natural map $\mathrm{H}^*_H(*;\mathbf{Q}) \to \mathrm{H}^*_G(\mathcal{L}G/\mathcal{L}H;\mathbf{Q})$, which gives a functor

$$\mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}) \to \mathrm{Perf}(\mathrm{sh}^{1/2}T[-1](\check{\mathfrak{h}}^*(2)/\!\!/\check{H}/\check{\mathfrak{g}}^*(2)/\!\!/\check{G})) \to \mathrm{Perf}(\mathrm{sh}^{1/2}\check{\mathfrak{h}}^*(2)/\!\!/\check{H}).$$

When $G = H \times H$, so that $\check{M} = T^*\check{H}$, this is precisely the Kostant functor of [**BF08**, Section 2.6].

Let us ignore gradings in the following discussion. It is natural to expect that the above analogue of the Kostant functor is induced by pullback along a certain map

$$\kappa_{\check{M}} : \check{\mathfrak{h}}^*/\!\!/\check{H} \to \check{M}/\check{G}.$$

For instance, when $G = H \times H$, so that $\check{M} = T^*\check{H}$, the map κ is simply the Kostant slice for $\check{H}$. Moreover, in the general case, the compatibility of the equivalence of Conjecture 3.4.11 with the action of the Satake category implies that there is a commutative square

$$\begin{tikzcd} \check{\mathfrak{h}}^*/\!\!/\check{H} \arrow[r, "\kappa_{\check{M}}"] \arrow[d] & \check{M}/\check{G} \arrow[d, "\mu"] \\ \check{\mathfrak{g}}^*/\!\!/\check{G} \arrow[r, "\kappa"] & \check{\mathfrak{g}}^*/\check{G}. \end{tikzcd}$$

Therefore, (12) and Corollary 3.5.6 together make the concrete prediction that on the spectral side of Conjecture 3.4.11, there is an isomorphism $\check{M}/\!\!/\check{G} \cong \check{\mathfrak{h}}^*/\!\!/\check{H}$ and a "Kostant section" $\kappa_{\check{M}}$ which makes the above square commute. Just as the Kostant slice plays a crucial role in the geometric Langlands program, we expect the Kostant section $\kappa_{\check{M}}$ to play a central role in the story of relative geometric Langlands.

Corollary 3.5.8. *The $\mathbf{E}_1$-ku_G-algebra structure obtained via the $\mathbf{E}_\infty$-map $\mathrm{ku}_G \to \mathrm{ku}_H$ on the ku_H-linear dual of $\mathcal{F}_H(\Omega(G/H))$ – which is* not *$\mathcal{F}_H(\Omega(G/H))^\vee$ in the notation of Construction 2.4.4 – refines to an $\mathbf{E}_2$-ku_G-algebra structure.*

PROOF. Taking the ku_H-linear dual of the right-hand side of Proposition 3.5.5 produces the Hochschild *co*homology $\mathrm{HC}(\mathrm{ku}_H/\mathrm{ku}_G)$. By the Deligne conjecture (in the form proved in [**Lur16**, Section 5.3]), this admits the structure of an $\mathbf{E}_2$-ku_G-algebra. On the other hand, by Proposition 3.5.1, the right-hand side of Proposition 3.5.5 can be identified with the equivariant cohomology $\mathcal{F}_H(\Omega(G/H))$. The desired result follows. □

Remark 3.5.9. One can also identify the ku_H-linear dual of $\mathcal{F}_H(\Omega G)$ with the $\mathbf{E}_2$-centralizer of the map $\mathrm{ku}_G \to \mathrm{ku}_H$; see Recollection 5.2.4. The $\mathbf{E}_2$-structure on the ku_H-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is essentially the reason for the $\mathbf{E}_2$-monoidal structure on the automorphic relative Langlands category $\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}G/\mathcal{L}H;\mathbf{Q})$ (to be defined below) from [**BZSV23**, Remark 7.5.12 and Section 16].

In the special case when $G = H \times H$ and H is embedded diagonally, one can identify $\mathrm{HC}(\mathrm{ku}_H/\mathrm{ku}_G)$ with the $\mathbf{E}_2$-*Hochschild cohomology* $\mathrm{HC}_{\mathbf{E}_2}(\mathrm{ku}_H/\mathrm{ku})$. The Deligne conjecture therefore equips the ku_H-linear dual of $\mathcal{F}_H(\Omega H)$ with an $\mathbf{E}_3$-algebra structure, and again this is essentially the source of the folklore $\mathbf{E}_3$-monoidal structure on the spherical Hecke category $\mathrm{Shv}^{c,\mathrm{Sat}}_{H\times H}(\mathcal{L}H;\mathbf{Q})$. We will use the perspective of $\mathbf{E}_n$-centers to establish some coisotropicity results in Section 5.2.

Warning 3.5.10. The reader should keep Warning 3.5.3 in mind: the ku_H-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is *not* equivalent to the ku_G-linear dual of $\mathcal{F}_G(\mathcal{L}(G/H))$. In fact, as mentioned in Corollary 3.5.8, the ku_H-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is also not equivalent to the equivariant homology $\mathcal{F}_H(\Omega(G/H))^\vee$; the former is only a *completion* of the latter.

Example 3.5.11. Let us illustrate Corollary 3.5.8, or rather, the identification of the ku_H-linear dual of $\mathcal{F}_H(\Omega(G/H))$ with Hochschild cohomology, after base-changing along the map $\mathrm{ku}_H \to \mathrm{ku} \to \mathbf{Z}$ (to $\mathbf{Z}[1/2]$ in the second example) in two simple cases:

(a) Let $H = \mathrm{SU}(n-1) \subseteq \mathrm{SU}(n) = G$. Then $G/H \simeq S^{2n-1}$, and so there is an isomorphism

$$\pi_*\mathcal{F}_H(\Omega(G/H))^\vee \otimes_{\mathrm{ku}_H} \mathbf{Z} \cong \mathrm{H}_*(\Omega S^{2n-1}; \mathbf{Z}) \cong \mathbf{Z}[y],$$

where y lives in weight $2n-2$. On the other hand, the map $\mathrm{H}^*_G(*; \mathbf{Z}) \to \mathrm{H}^*_H(*; \mathbf{Z})$ identifies with the map

$$\mathbf{Z}[c_1, \cdots, c_n] \to \mathbf{Z}[c_1, \cdots, c_{n-1}]$$

sending $c_n \mapsto 0$, where the ith Chern class c_i lives in weight $-2i$. Taking Hochschild homology along this map identifies

$$\mathrm{HH}(\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z})/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{Z})) \simeq \mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{HH}(\mathbf{Z}/\mathrm{sh}^{1/2}\mathbf{Z}[c_n]).$$

But $\pi_*\mathrm{HH}(\mathbf{Z}/\mathrm{sh}^{1/2}\mathbf{Z}[c_n])$ is isomorphic to the divided power algebra $\mathbf{Z}\langle\sigma^2(c_n)\rangle$, where σ denotes "suspension", so that $\sigma^2(c_n)$ lives in degree $-2n+2$; it follows that there is an isomorphism

$$\pi_*\mathrm{HH}(\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z})/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{Z})) \cong \mathbf{Z}[c_1, \cdots, c_{n-1}]\langle\sigma^2(c_n)\rangle.$$

This in turn implies that there is an isomorphism

$$\pi_*\mathrm{HC}(\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z})/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{Z})) \cong \mathbf{Z}[c_1, \cdots, c_{n-1}][\![y]\!]$$

where the class y in weight $2n-2$ is dual to $\sigma^2(c_n)$. Killing $c_1, \cdots, c_n$ (i.e., base-changing along $\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z}) \to \mathbf{Z}$) precisely recovers a completion of $\mathrm{H}_*(\Omega S^{2n-1}; \mathbf{Z})$. We will discuss a generalization (and decompletion) of this calculation in Theorem 4.1.13.

(b) Let $H = \mathrm{SO}_{2n} \subseteq \mathrm{SO}_{2n+1} = G$ with $n > 0$, and recall that we are replacing $\mathbf{Z}$ by $\mathbf{Z}' = \mathbf{Z}[1/2]$. Then $G/H \simeq S^{2n}$, and so a standard argument with the Serre spectral sequence shows that there is an isomorphism

$$\pi_*\mathcal{F}_H(\Omega(G/H))^\vee \otimes_{\mathrm{ku}_H} \mathbf{Z}' \cong \mathrm{H}_*(\Omega S^{2n}; \mathbf{Z}') \cong \mathbf{Z}'[y, z]/z^2,$$

where z lives in weight $2n-1$ and y lives in weight $4n-2$. On the other hand, the map $\mathrm{H}^*_G(*; \mathbf{Z}') \to \mathrm{H}^*_H(*; \mathbf{Z}')$ identifies with the map

$$\mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n] \to \mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n^{1/2}]$$

sending $p_n \mapsto (p_n^{1/2})^2$, where the ith Pontryagin class p_i lives in weight $-4i$ and the Euler class $p_n^{1/2}$ lives in weight $-2n$. Taking Hochschild homology along this map identifies

$$\mathrm{HH}(\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z}')/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{Z}')) \simeq \mathrm{sh}^{1/2}\mathbf{Z}'[p_1, \cdots, p_{n-1}] \otimes_{\mathbf{Z}'} \mathrm{HH}(\mathrm{sh}^{1/2}\mathbf{Z}'[p_n^{1/2}]/\mathrm{sh}^{1/2}\mathbf{Z}'[p_n]),$$

and so computing the Hochschild cohomology from Corollary 3.5.8 amounts to computing the Hochschild cohomology $\mathrm{HC}(\mathbf{Z}'[p_n^{1/2}]/\mathbf{Z}'[p_n])$. Lemma 3.5.12 implies that there is an isomorphism

$$\pi_*\mathrm{HC}(\mathrm{sh}^{1/2}\mathrm{H}^*_H(*; \mathbf{Z}')/\mathrm{sh}^{1/2}\mathrm{H}^*_G(*; \mathbf{Z}')) \cong \mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n^{1/2}][\![w]\!]/p_n^{1/2}w,$$

with w in in weight $4n-2$. Upon killing $p_1, \cdots, p_{n-1}, p_n^{1/2}$ (i.e., base-changing along $\mathrm{sh}^{1/2}\mathrm{H}_H^*(*;\mathbf{Z}) \to \mathbf{Z}$), one precisely recovers a completion of $\mathrm{H}_*(\Omega S^{2n};\mathbf{Z}')$. We will discuss a generalization (and decompletion) of this calculation in Theorem 4.1.13.

Lemma 3.5.12. *Let x be a class in homological degree $2n$, and let $j \geq 1$. Then there is an isomorphism*

$$\pi_*\mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[x^j]) \cong \mathbf{Z}[x][\![w]\!]/jx^{j-1}w,$$

where w lives in weight $-2nj-2$.

Proof. Since $\mathbf{Z}[x] = \mathrm{sh}\mathbf{Z}[x_n]$ where x_n lives in weight n and degree 0, it suffices to work in the graded setting and assume that x lives in weight n and degree 0. Let us first work in the ungraded setting; fix a nonconstant polynomial $g(x) \in \mathbf{Z}[x]$, and consider $\mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[g])$. There is an isomorphism

$$\mathbf{Z}[x] \otimes_{\mathbf{Z}[g]} \mathbf{Z}[x] \cong \mathbf{Z}[x,x']/(g(x)-g(x')) \cong \mathbf{Z}[x,z]/zf,$$

where $z = x'-x$ and $f = \frac{g(x)-g(x+z)}{z}$. (If x has weight n and g is homogeneous of degree j, the class z lives in degree 0 and weight n, and f lives in degree 0 and weight $n(j-1)$.) Our goal is to compute $\pi_* \mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, where the map $\mathbf{Z}[x,z]/zf \to \mathbf{Z}[x]$ sends $z \mapsto 0$. There are several ways to compute this: one is to note that there is a presentation

$$\mathbf{Z}[x] \simeq (\mathbf{Z}[x,z,u]\langle v\rangle/(zf,u^2), d(u)=z, d(v)=uf)$$

of $\mathbf{Z}[x]$ as a $\mathbf{Z}[x,z]/zf$-algebra. If x has weight n and g is homogeneous of degree j, the class u is in degree 1 and weight n, and v is a divided power class in degree 2 and weight nj. This implies that there is an equivalence

$$\mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x]) \simeq (\mathbf{Z}[x,u'][\![w]\!]/{u'}^2, d(u') = f(z=0)w),$$

where u' is dual to u and w is dual to v. If x has weight n and g is homogeneous of degree j, the class u' is in degree -1 and weight $-n$, and w is in degree -2 and weight $-nj$. It follows that there is a class $w \in \pi_{-2}\mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$ such that $f(z=0)w = 0 \in \pi_{-2}\mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, which gives an isomorphism

$$\pi_*\mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[g]) = \pi_*\mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x]) \cong \mathbf{Z}[x][\![w]\!]/g'(0)w.$$

If x has weight n and g is homogeneous of degree j, the class w lives in $\pi_{-2,-nj}\mathrm{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, and we obtain a graded isomorphism

$$\pi_*\mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[x^j]) \cong \mathbf{Z}[x][\![w]\!]/jx^{j-1}w,$$

which gives the desired calculation by shearing. $\square$

Remark 3.5.13. As in Lemma 3.5.12, one can also compute $\pi_*\mathrm{HC}(\pi_*\mathrm{ku}_{S^1}/\pi_*\mathrm{ku}_{\mathrm{SU}(2)})$ to obtain the following:

$$\pi_*\mathrm{HC}(\pi_*\mathrm{ku}_{S^1}/\pi_*\mathrm{ku}_{\mathrm{SU}(2)}) \cong \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}][\![w]\!]/w(x-\overline{x}).$$

Here, w lives in degree 0 and weight 2, and $\overline{x} = -\frac{x}{1+\beta x}$ is the negative of x under the group law on $\mathbf{G}_\beta = \mathrm{Spec}\,\pi_*\mathrm{ku}_{S^1}$. When $\beta = 0$, this recovers Lemma 3.5.12 for $j=2$ and $n=-1$. There is a spectral sequence whose E_1-page is $\pi_*\mathrm{HC}(\pi_*\mathrm{ku}_{S^1}/\pi_*\mathrm{ku}_{\mathrm{SU}(2)})$ which converges to $\pi_*\mathrm{HC}(\mathrm{ku}_{S^1}/\mathrm{ku}_{\mathrm{SU}(2)})$; this spectral sequence degenerates. Since the 2-series of x is $[2](x) = (1+\beta x)(x-\overline{x})$, we find that

$$\pi_*\mathrm{HC}(\mathrm{ku}_{S^1}/\mathrm{ku}_{\mathrm{SU}(2)}) \cong \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}][\![w]\!]/w[2](x),$$

where w lives in degree 2.

Remark 3.5.14. The reader might observe that one can analyze $\mathrm{HH}(\pi_*\mathrm{ku}_H/\pi_*\mathrm{ku}_G)$ essentially using the combinatorics of the weight lattices and Weyl groups of H and G. More generally, therefore, let $W_1 \to W_2$ be a homomorphism of finite groups acting on vector spaces $V_1 \to V_2$ over a field k (possibly of nonzero characteristic). Then there is a map $V_1/\!\!/W_1 \to V_2/\!\!/W_2$, and hence one can consider the Hochschild homology $\mathrm{HH}(V_1/\!\!/W_1/V_2/\!\!/W_2)$. This should be an interesting invariant associated to homomorphisms of finite groups, but it is likely only well-behaved if the map $V_1/\!\!/W_1 \to V_2/\!\!/W_2$ is an affine bundle. For instance, Proposition 3.5.5 and Example 3.5.28 below imply that the Hochschild cohomology $\mathrm{HC}(\mathfrak{t}/\mathfrak{t}/\!\!/W)$ is closely related to a completion of the "regular centralizer" $\check{\mathfrak{t}}^* \times_{\check{G}\backslash\overline{T^*(\check{G}/\check{N})}} \check{\mathfrak{t}}^*$ for $\overline{T^*(\check{G}/\check{N})}$.

One example which does not come from Lie-theoretic data is the dihedral group D_{2n} acting on $\mathbf{A}^2 = \operatorname{Spec}\mathbf{C}[x_1, x_2]$ by the matrices $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $r = \left(\begin{smallmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{smallmatrix}\right)$. The algebra of invariants $\mathbf{C}[x_1, x_2]^{D_{2n}}$ is simply $\mathbf{C}[x_1x_2, x_1^n + x_2^n]$. As in Lemma 3.5.12, one finds that

$$\pi_*\mathrm{HC}(\mathbf{A}^2/\mathbf{A}^2/\!\!/D_{2n}) \cong \mathbf{C}[x_1, x_2][\![w_1, w_2]\!]/\left(\begin{smallmatrix} x_1 & x_2 \\ x_2^{n-1} & x_1^{n-1} \end{smallmatrix}\right)\vec{w}.$$

Example 3.5.15. Consider the subgroup $G^{\mathrm{diag}} \subseteq G\times G$, so that $(G\times G)/G^{\mathrm{diag}} \simeq G$ (this is the "group case" of Example 3.3.5). Then Proposition 3.5.5 says that there is an S^1-equivariant equivalence of $\mathbf{E}_\infty$-ku_G-algebras

$$\mathcal{F}_{G\times G}(\mathcal{L}G) \simeq \mathrm{HH}(\mathrm{ku}_G/\mathrm{ku}_{G\times G}).$$

By construction of equivariant ku, there is an equivalence $\mathrm{ku}_{G\times G} \simeq \mathrm{ku}_G \otimes_{\mathrm{ku}} \mathrm{ku}_G$, so that the right-hand side can be identified with the factorization homology

$$\mathrm{HH}(\mathrm{ku}_G/\mathrm{ku}_G \otimes_{\mathrm{ku}} \mathrm{ku}_G) \simeq \int_{S^2} \mathrm{ku}_G/\mathrm{ku}.$$

Note that by Proposition 3.5.1, the left-hand side can be identified with $\mathcal{F}_G(\Omega G)$, so Proposition 3.5.5 describes the G-equivariant ku-*co*homology of the affine Grassmannian:

$$\mathrm{ku}_G^*(\Omega G) \simeq \pi_* \int_{S^2} \mathrm{ku}_G/\mathrm{ku}. \tag{14}$$

Note that there is an equivalence

$$\mathrm{HH}(\mathrm{ku}_G/\mathrm{ku}_G \otimes_{\mathrm{ku}} \mathrm{ku}_G) \simeq \mathrm{ku}_G \otimes_{\mathrm{ku}} \mathrm{HH}(\mathrm{ku}/\mathrm{ku}_G).$$

Upon killing the Bott class β, (14) implies that

$$C_G^*(\Omega G; \mathbf{Z}) \simeq \int_{S^2} C_G^*(*; \mathbf{Z})/\mathbf{Z}.$$

As argued in [**Dev23a**, Example A.8], this recovers [**BF08**, Theorem 1] and [**Gin95**, Section 1.7] upon rationalization.

Remark 3.5.16. Unlike Proposition 3.5.1, Proposition 3.5.5 gives an S^1_{rot}-equivariant equivalence. In particular, it allows us to calculate the S^1_{rot}-equivariant cohomology $\mathrm{ku}^*_{G\times G\times S^1_{\mathrm{rot}}}(\mathcal{L}G) \simeq \mathrm{ku}^*_{G\times S^1_{\mathrm{rot}}}(\Omega G)$. We will discuss this in a future article, since addressing loop rotation in the detail it deserves will take us too far afield.

However, since it is not very difficult to make explicit, let us explicate Proposition 3.5.5 (or rather, its variant for Hochschild cohomology describing $\mathrm{ku}_*^{G\times S^1_{\mathrm{rot}}}(\Omega G)$)

in the case when $G = T$ is a torus. As in [**Dev23a**, Proposition 3.3.4], the associative graded ring $\mathrm{ku}_*^{T \times S^1_{\mathrm{rot}}}(\Omega T)$ can be identified with the algebra of $\mathbf{G}_\beta$-differential operators on the dual torus $\check{T}$. This is an analogue of the algebra of (asymptotic) differential operators. Let us assume for simplicity that T is of rank 1; then the algebra $\mathrm{ku}_*^{T \times S^1_{\mathrm{rot}}}(\Omega T)$ is the F-Weyl algebra $F\mathcal{D}_{\square, \mathbf{G}_m}$ of [**DM23**, Definition 4.4.1] for $F(x, y) = x + y + \beta xy$, at least up to completion. Explicitly, when $T = S^1$, we have

$$\mathrm{ku}_*^{T \times S^1_{\mathrm{rot}}}(\Omega T) \cong \mathbf{Z}[\beta, \hbar, \tfrac{1}{1+\beta\hbar}]\{x, a^{\pm 1}\}[\tfrac{1}{1+\beta x}]/([x, a] = a\hbar(1 + \beta x)).$$

Here, the curly brackets denotes the free associative algebra generated by the elements enclosed within. The classes $\hbar$ and x live in weight -2 (they are the S^1-equivariant Chern classes for ku), β lives in weight 2, and a lives in weight zero. Let us note two specializations of this associative algebra:

(a) If $\beta = 0$, the right-hand side above simply becomes $\mathbf{Z}[\hbar]\{x, a^{\pm 1}\}/([x, a] = \hbar a)$, which is precisely the algebra of asymptotic differential operators on $\check{T} = \operatorname{Spec} \mathbf{Z}[a^{\pm 1}]$ over $\mathbf{Z}$. Namely, $x = \hbar a \partial_a$; see [**DM23**, Example 4.4.2].

(b) If β is inverted, all elements can be pushed to degree zero. Namely, let $q = 1 + \beta\hbar$ and $\Theta = 1 + \beta x$. Then there is an isomorphism

$$\mathbf{Z}[\beta^{\pm 1}, \hbar, \tfrac{1}{1+\beta\hbar}]\{x, a^{\pm 1}\}[\tfrac{1}{1+\beta x}]/([x, a] = a\hbar(1+\beta x)) \cong \mathbf{Z}[\beta^{\pm 1}, q^{\pm 1}]\{\Theta^{\pm 1}, a^{\pm 1}\}/(\Theta a = qa\Theta),$$

so that this algebra can be identified with the q-Weyl algebra of $\check{T} = \operatorname{Spec} \mathbf{Z}[a^{\pm 1}]$. Namely, $\Theta = q^{a\partial_a}$; see [**DM23**, Example 4.4.3].

In general, $\mathrm{ku}_*^{T \times S^1_{\mathrm{rot}}}(\Omega T)$ interpolates between the algebra of asymptotic differential and q-difference operators on $\check{T}$.

Let us end this section with a definition of $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q})$.

Definition 3.5.17. Let $\mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$ denote the ∞-category of G-equivariant sheaves of $\mathbf{Q}$-modules on $\mathcal{L}(G/H)$ which are constructible for the orbit stratification on $\mathcal{L}(G/H)$. Note that since the orbit stratification is countable (by assumption that $H_\mathbf{C} \subseteq G_\mathbf{C}$ is a spherical subgroup and Remark 3.3.17), the ∞-category $\mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$ is well-defined. There is a natural left-action of the $\mathbf{E}_3$-monoidal ∞-category $\mathrm{Shv}_{G \times G}^c(\mathcal{L}G; \mathbf{Q})$ on $\mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$, and in particular, a left-action of $\mathrm{Rep}(\check{G})$ by Theorem 3.2.3. Let $\mathrm{IC}_0 \in \mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$ denote the pushforward $i_! \underline{\mathbf{Q}}$ of the constant sheaf along the inclusion $i : G/H \hookrightarrow \mathcal{L}(G/H)$ of the constant loops. Note that i is the analytic realization of the natural map $(G/H)(\mathbf{C}[\![t]\!]) \to (G/H)(\mathbf{C}(\!(t)\!))$. Let $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q})$ denote the full subcategory of $\mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$ generated by IC_0 under the action of $\mathrm{Rep}(\check{G})$.

Example 3.5.18. In the group case of Example 3.3.5, it is not difficult to see that $\mathrm{Shv}_{G \times G}^{c,\mathrm{Sat}}(\mathcal{L}((G \times G)/G^{\mathrm{diag}}); \mathbf{Q})$ from Definition 3.5.17 agrees with Definition 3.2.5.

In order to understand $\mathrm{Shv}_G^c(\mathcal{L}(G/H); \mathbf{Q})$, we will rely on an algebraic model (in order to use a formality argument as in the proof of Theorem 3.2.7). For this, we need the following analogue of Theorem 3.2.20 for *affine* homogeneous spherical varieties.

Proposition 3.5.19. *Let $G_{\mathbf{C}}$ be a reductive group over $\mathbf{C}$, and let $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a reductive subgroup such that $G_{\mathbf{C}}/H_{\mathbf{C}}$ is an affine spherical variety*[16]. *Then there is a homotopy equivalence of orbifolds*

$$G_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!)) \simeq G\backslash \mathcal{L}(G/H).$$

PROOF. Theorem 3.2.20 gives $G_{\mathbf{C}}(\mathbf{C}[\![t]\!])$-equivariant (resp. $H_{\mathbf{C}}(\mathbf{C}[\![t]\!])$-equivariant) homotopy equivalences

$$G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/G_{\mathbf{C}}(\mathbf{C}[\![t]\!]) \simeq \Omega G, \;\; H_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}[\![t]\!]) \simeq \Omega H.$$

In particular, there is an equivalence of orbifolds

$$G_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}[\![t]\!]) \simeq G\backslash(\mathcal{L}G)/H.$$

We therefore have the following chain of equivalences of orbifolds:

$$\begin{aligned} G\backslash\mathcal{L}(G/H) &\simeq G\backslash\mathcal{L}G/\mathcal{L}H \simeq G\backslash(\mathcal{L}G)/H \times^{H\backslash\mathcal{L}H/H} H\backslash\mathcal{L}H/\mathcal{L}H \\ &\simeq G_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}[\![t]\!]) \times^{H_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash H_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}[\![t]\!])} H_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash H_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!)) \\ &\simeq G_{\mathbf{C}}(\mathbf{C}[\![t]\!])\backslash G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!)), \end{aligned}$$

as desired. □

Remark 3.5.20. Proposition 3.5.19 in the case of symmetric varieties was already proved in [**Mit88**].

We can now state our main criterion for proving equivalences of the form Conjecture 3.4.11.

Hypothesis 3.5.21. Say that a connected reductive spherical subgroup $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is *optimal* if:

- The $G_{\mathbf{C}}$-action on $G_{\mathbf{C}}/H_{\mathbf{C}}$ is defined over a finite type $\mathbf{Z}$-algebra $R \subseteq \mathbf{C}$, and that the $G_R(R[\![t]\!])$-action on $(G_R/H_R)(R(\!(t)\!))$ is "weakly placid" in the sense of Definition A.2.
- Let $\check{\mathcal{R}}\star\mathrm{IC}_0 \in \mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q})$ denote the sheaf obtained by the action of $\mathcal{O}_{\check{G}} \in \mathrm{Rep}(\check{G})$ on IC_0. Then the sheaves $\mathrm{IC}_0, \mathrm{IC}_0\star\check{\mathcal{R}} \in \mathrm{Shv}^{c,\mathrm{et}}_{G(\overline{\mathbf{F}_q}[\![t]\!])}(G(\overline{\mathbf{F}_q}(\!(t)\!))/H(\overline{\mathbf{F}_q}(\!(t)\!)))$ are pure of weight zero for $q \gg 0$.
- For the placid presentation of the $G_R(R[\![t]\!])$-action on $(G_R/H_R)(R(\!(t)\!))$ from Definition A.2, the evenness assumption of Theorem 3.1.3 are satisfied for IC_0 and $\mathrm{IC}_0 \star \check{\mathcal{R}}$.

Remark 3.5.22. Before proceeding to the argument, let us make one comment about Hypothesis 3.5.21: as the terminology suggests, a spherical subgroup being optimal is a rather idealized situation; it is essentially the smallest set of hypotheses needed to make the argument of Theorem 3.2.7 go through. It is, of course, possible that examples of interest (even the ones considered in this article) do not satisfy this condition. Verifying Hypothesis 3.5.21 is likely no easy task, and working out the appropriate subtleties of the sheaf theories involved in Conjecture 3.4.11 will be very important to understanding microlocal aspects of relative geometric Langlands. However, the point of Hypothesis 3.5.21 is to isolate some hard sheaf-theoretic subtleties, assume that they might be resolved in the cases of interest,

[16]Note that in this case, the quotient $G_{\mathbf{C}}/H_{\mathbf{C}}$ is affine. In fact, if $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a closed subgroup such that $G_{\mathbf{C}}/H_{\mathbf{C}}$ is *smooth* and affine, Matsushima's theorem says that $H_{\mathbf{C}}$ is a reductive subgroup of $G_{\mathbf{C}}$.

and to then state Theorem 3.5.23. Its criteria (we believe) extract the key ways in which (relative) Langlands duality is born, and also gives psychologically more manageable conjectures at "category level 0".

Theorem 3.5.23. *Let $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a connected reductive spherical subgroup which is optimal in the sense of Hypothesis 3.5.21. Let $\check{M}$ denote a "dual" affine graded Hamiltonian $\check{G}$-variety (e.g., in rank 1, $\check{M}$ is constructed from Table 3 as in Section 3.4).*

Suppose that there is a "Kostant section" $\kappa_{\check{M}} : \check{\mathfrak{h}}^*(2)/\!\!/\check{H} \hookrightarrow \check{M}$ (see, e.g., Remark 3.5.7) such that:

(a) Let $\check{J}_X$ denote the (possibly non-flat) group scheme $\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times_{\check{M}/\check{G}(-2\rho)} \check{\mathfrak{h}}^*(2)/\!\!/\check{H}$ over $\check{\mathfrak{h}}^*(2)/\!\!/\check{H}$. Then the algebra of regular functions on $(\check{\mathfrak{h}}^*(2)/\!\!/\check{H}\times \check{G}(-2\rho))/\check{J}_X$ is isomorphic to $\mathcal{O}_{\check{M}}$. (For instance, this holds by the algebraic Hartogs lemma if the $\check{G}$-orbit $\check{M}^{\mathrm{reg}}$ of the image of $\kappa_{\check{M}}$ is open with complement of codimension ≥ 2.)

(b) There is an isomorphism of graded group schemes over $\check{\mathfrak{h}}^*(2)/\!\!/\check{H}$:
$$\operatorname{Spec} \mathrm{H}_*^H(\Omega(G/H);\mathbf{Q}) \cong \check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times_{\check{M}/\check{G}(-2\rho)} \check{\mathfrak{h}}^*(2)/\!\!/\check{H} = \check{J}_X.$$

Then there is an equivalence of $\mathbf{Q}$-linear ∞-categories
$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}(-2\rho)). \tag{15}$$

Let $\mu : \check{M}/\check{G}(-2\rho) \to \check{\mathfrak{g}}^*(2\rho-2)/\check{G}(-2\rho)$ denote the moment map, and assume that there is a commutative diagram
$$\begin{tikzcd}
\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \arrow[r, "\kappa_{\check{M}}"] \arrow[d] & \check{M}/\check{G}(-2\rho) \arrow[d, "\mu"] \\
\check{\mathfrak{g}}^*(2)/\!\!/\check{G} \arrow[r, "\kappa"] & \check{\mathfrak{g}}^*(2-2\rho)/\check{G}(-2\rho),
\end{tikzcd}$$
so that there is an induced map
$$\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times_{\check{M}/\check{G}(-2\rho)} \check{\mathfrak{h}}^*(2)/\!\!/\check{H} \cong \check{J}_X \to \check{\mathfrak{g}}^*(2)/\!\!/\check{G} \times_{\check{\mathfrak{g}}^*(2-2\rho)/\check{G}(-2\rho)} \check{\mathfrak{g}}^*(2)/\!\!/\check{G} \cong \check{J}.$$
If the isomorphism of (b) fits into a commutative diagram
$$\begin{tikzcd}
\operatorname{Spec} \mathrm{H}_*^H(\Omega(G/H);\mathbf{Q}) \arrow[r, "\sim", "(b)"'] \arrow[d] & \check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times_{\check{M}/\check{G}(-2\rho)} \check{\mathfrak{h}}^*(2)/\!\!/\check{H} \arrow[d] \\
\operatorname{Spec} \mathrm{H}_*^G(\Omega G;\mathbf{Q}) \arrow[r, "\sim", "\text{Theorem 3.2.7}"'] & \check{\mathfrak{g}}^*(2)/\!\!/\check{G} \times_{\check{\mathfrak{g}}^*(2-2\rho)/\check{G}(-2\rho)} \check{\mathfrak{g}}^*(2)/\!\!/\check{G},
\end{tikzcd}$$
then the equivalence (15) is equivariant for the left-action of $\mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{\mathfrak{g}}^*(2-2\rho)/\check{G}(-2\rho))$ via Theorem 3.2.7.

Proof. The proof will follow the first half of the proof of Theorem 3.2.7. Let $\mathcal{C}$ denote $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q})$, so that $\mathcal{C}$ admits a left-action of $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\Omega G;\mathbf{Q}) \simeq \mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G;\mathbf{Q})$. In particular, Theorem 3.2.3 implies that $\mathcal{C}$ admits a left-action of $\mathrm{Rep}(\check{G})$. Let $\widetilde{\mathcal{C}}$ denote the base-change $\mathcal{C}\otimes_{\mathrm{Rep}(\check{G})}\mathrm{Mod}_{\mathbf{Q}}$, so that IC_0 is a compact generator of $\widetilde{\mathcal{C}}$ (by definition of $\mathcal{C}$). It follows from the Barr-Beck theorem [**Lur16**, Theorem 4.7.3.5] that there is an equivalence $\Phi : \widetilde{\mathcal{C}} \xrightarrow{\sim} \mathrm{Perf}_{\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)}$, implemented by the functor $\mathrm{Hom}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0,-)$. Recall that $\check{\mathcal{R}}\star\mathrm{IC}_0 \in \mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q})$ denotes the sheaf obtained by the action of $\mathcal{O}_{\check{G}} \in \mathrm{Rep}(\check{G})$ on IC_0. By definition of $\widetilde{\mathcal{C}}$, we can identify $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})$.

The same argument as in the proof of Theorem 3.2.7 shows that $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$ is formal. The key input needed is Proposition 3.5.19 and the hypothesis that if q is a sufficiently large prime number, the objects

$$\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}} \in \mathrm{Shv}^{c,\mathrm{et}}_{G(\overline{\mathbf{F}_q}[\![t]\!])}(G(\overline{\mathbf{F}_q}(\!(t)\!))/H(\overline{\mathbf{F}_q}(\!(t)\!)))$$

are pure of weight zero. This allows us to use [**BY13**, Lemma 3.1.5] and Theorem A.4 to obtain the formality of $\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0)$. It follows that

$$\mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0) \simeq \mathrm{sh}^{1/2}(\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\mathcal{L}(G/H);\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}})). \tag{16}$$

To compute this Ext-algebra, we will use Theorem 3.1.3. One can show that the assumptions of Theorem 3.1.3 are satisfied for IC_0 and $\mathrm{IC}_0 \star \check{\mathcal{R}}$, so the cited result gives a graded isomorphism

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\mathcal{L}(G/H);\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_G(\mathcal{L}(G/H);\mathbf{Q})}(\mathrm{H}^*_G(\mathcal{L}(G/H); \mathrm{IC}_0), \mathrm{H}^*_G(\mathcal{L}(G/H); \mathrm{IC}_0 \star \check{\mathcal{R}})).$$

There is an isomorphism

$$\mathrm{H}^*_G(\mathcal{L}(G/H); \mathrm{IC}_0) \cong \mathrm{H}^*_G(G/H; \mathbf{Q}) \cong \mathrm{H}^*_H(*; \mathbf{Q}) \cong \mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H}},$$

and hence an isomorphism

$$\mathrm{H}^*_G(\mathcal{L}(G/H); \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times \check{G}(-2\rho)}.$$

Moreover, there is an isomorphism $\mathrm{H}^*_G(\mathcal{L}(G/H); \mathbf{Q}) \cong \mathrm{H}^*_H(\Omega(G/H); \mathbf{Q})$ by Remark 3.5.2, so we find that there is a graded isomorphism

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\mathcal{L}(G/H);\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) &\cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_H(\Omega(G/H);\mathbf{Q})}(\mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H}}, \mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times \check{G}(-2\rho)}) \\ &\cong \mathrm{Hom}^\bullet_{\mathrm{H}^*_H(*;\mathbf{Q})}(\mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H}}, \mathcal{O}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times \check{G}(-2\rho)})^{\mathrm{Spec}\, \mathrm{H}^H_*(\Omega(G/H);\mathbf{Q})} \\ &\cong \mathcal{O}^{\mathrm{Spec}\, \mathrm{H}^H_*(\Omega(G/H);\mathbf{Q})}_{\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times \check{G}(-2\rho)}.\end{aligned}$$

By (b), there is an isomorphism $\mathrm{Spec}\, \mathrm{H}^H_*(\Omega(G/H); \mathbf{Q}) \cong \check{J}_X$, and hence there is an isomorphism

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c_G(\mathcal{L}(G/H);\mathbf{Q})}(\mathrm{IC}_0, \mathrm{IC}_0 \star \check{\mathcal{R}}) \cong \mathcal{O}_{(\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \times \check{G}(-2\rho))/\check{J}_X}.$$

By (a), this is further isomorphic to $\mathcal{O}_{\check{M}}$. Using (16), it follows that $\widetilde{\mathcal{C}}$ is equivalent to the ∞-category $\mathrm{Perf}_{\mathrm{sh}^{1/2}\mathcal{O}_{\check{M}}} \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{M}))$. This in turn implies that $\mathcal{C}$ is equivalent to the ∞-category $\mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}(-2\rho))$, as desired. □

Before proceeding, let us note the following consequence of the argument of Theorem 3.5.23.

Lemma 3.5.24. *There is a homomorphism of graded group schemes*

$$\mathrm{Spec}\, \mathrm{H}^H_*(\Omega(G/H); \mathbf{Q}) \to \check{G}(-2\rho) \times \check{\mathfrak{h}}^*(2)/\!\!/\check{H}$$

over $\check{\mathfrak{h}}^*(2)/\!\!/\check{H} \cong \mathrm{Spec}\, \mathrm{H}^*_H(*; \mathbf{Q})$.

Remark 3.5.25. One can see Lemma 3.5.24 more directly as follows. Theorem 3.2.14 and Corollary 3.2.16 together imply that there is a homomorphism

$$\mathrm{Spec}\, \mathrm{H}^G_*(\Omega G; \mathbf{Q}) \cong \check{J} \to \check{G}(-2\rho) \times \check{\mathfrak{g}}^*(2)/\!\!/\check{G}$$

of graded group schemes over $\check{\mathfrak{g}}^*(2)/\!\!/\check{G} \cong \operatorname{Spec} \mathrm{H}^*_G(*; \mathbf{Q})$. Base-changing along the map $\mathrm{H}^*_G(*; \mathbf{Q}) \to \mathrm{H}^*_H(*; \mathbf{Q})$, we obtain a homomorphism

$$\operatorname{Spec} \mathrm{H}^H_*(\Omega G; \mathbf{Q}) \to \check{G}(-2\rho) \times \check{\mathfrak{h}}^*(2)/\!\!/\check{H}.$$

Composition with the map $\operatorname{Spec} \mathrm{H}^H_*(\Omega(G/H); \mathbf{Q}) \to \operatorname{Spec} \mathrm{H}^H_*(\Omega G; \mathbf{Q})$ induced by the map $G \to G/H$ produces the desired homomorphism.

Remark 3.5.26. One can compute the *non*equivariant homology $\mathrm{H}_*(\Omega(G/H); \mathbf{Q})$ rather easily (this works even with $\mathbf{Q}$ replaced by the square ℓ_G of the ratio of the lengths of long roots and the short roots of G). Namely, it is shown in [**YZ11**, Theorem 6.1] that if e is a principal nilpotent element of $\check{\mathfrak{g}}$, there is an isomorphism $\operatorname{Spec} \mathrm{H}_*(\Omega G; \mathbf{Q}) \cong Z_e(\check{G})$. There is a fiber sequence of $\mathbf{E}_1$-spaces

$$\Omega H \to \Omega G \to \Omega(G/H),$$

which gives an equivalence

$$C_*(\Omega(G/H); \mathbf{Q}) \simeq C_*(\Omega G; \mathbf{Q}) \otimes_{C_*(\Omega H; \mathbf{Q})} \mathbf{Q}$$

of $\mathbf{E}_1$-$\mathbf{Q}$-algebras. This in turn implies that there is an isomorphism

$$\mathrm{H}_*(\Omega(G/H); \mathbf{Q}) \cong \Gamma(Z_e(\check{G})(-2\rho_G) \times_{Z_{e_H}(\check{H})(-2\rho_H)} \{1\}; \mathcal{O}),$$

where the fiber product is taken in the derived sense. For instance, let $G = \mathrm{SL}_2$ and $H = \mathbf{G}_m$; then, $G/H = S^2$, and so $\mathrm{H}_*(\Omega(G/H); \mathbf{Q}) \cong \mathbf{Q}[b, c]/c^2$. On the other hand, $Z_e(\check{G}) \cong \mathbf{G}_a$ and $Z_{e_H}(\check{H}) = \mathbf{G}_m$. Therefore, there is an isomorphism

$$Z_e(\check{G}) \times_{Z_{e_H}(\check{H})} \{1\} \cong \mathbf{G}_a \times (\{1\} \times_{\mathbf{G}_m} \{1\}),$$

whose global sections is indeed $\mathrm{H}_*(\Omega S^2; \mathbf{Q})$.

Remark 3.5.27. In this article, we will only focus on applying Theorem 3.5.23 to the case when $G_{\mathbf{C}}/H_{\mathbf{C}}$ is a spherical $G_{\mathbf{C}}$-variety of rank 1. However, in [**Dev23b**], we use Theorem 3.5.23 to prove Conjecture 3.4.11 for the spherical subgroups $\mathrm{PGL}_2^{\mathrm{diag}} \subseteq \mathrm{PGL}_2^{\times 3}$ and $G_2 \subseteq \mathrm{SO}_8/\mu_2$ of relative rank 3.

There are some expected equivalences of the form (15) which do not fit into the parameters of Theorem 3.5.23, and are difficult to make precise.

Example 3.5.28. For simplicity, we will ignore gradings in the following discussion. Let $G_{\mathbf{C}}$ be an almost simple algebraic group over $\mathbf{C}$, and let $T_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a maximal torus. Then, one expects that there is a $\mathbf{Q}$-linear equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times T}(\mathcal{L}G; \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\overline{T^*(\check{G}/\check{N})}/(\check{G} \times \check{T})) \tag{17}$$

which is equivariant for the left-action of $\mathrm{Shv}^{c,\mathrm{Sat}}_{(G\times T)\times(G\times T)}(\mathcal{L}(G\times T); \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{\mathfrak{g}}^*(2-2\rho)/\check{G}(-2\rho) \times \check{\mathfrak{t}}^*[2]/\check{T})$ via Theorem 3.2.7. Note that (17) implies that there is a $\mathbf{Q}$-linear equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G}(\mathcal{L}(G/T); \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\overline{T^*(\check{G}/\check{N})}/\check{G}).$$

Moreover, were the equivalence (17) true, the canonical action of the Weyl group $W = N_G(T)/T$ on the left-hand side of (17) should correspond to the $\check{G} \times \check{T}$-equivariant (semi-classical) Gelfand-Graev action on $\overline{T^*(\check{G}/\check{N})}$ as studied in [**GK22**]. These expectations are indeed true in the case when G has semisimple rank 1; see Corollary 4.2.19 and Remark 4.2.20.

The equivalence (17), however, is difficult to make precise outside of the semisimple rank 1 case, since $(G_{\mathbf{C}} \times T_{\mathbf{C}})/T_{\mathbf{C}}^{\mathrm{diag}} \cong G_{\mathbf{C}}$ is generally not a spherical $G_{\mathbf{C}} \times T_{\mathbf{C}}$-variety: otherwise, $G_{\mathbf{C}}/T_{\mathbf{C}}$ would be a spherical $G_{\mathbf{C}}$-variety, which is generally not true (see Warning 3.3.8). Nevertheless, the criteria (a) and (b) of Theorem 3.5.23 can be checked to hold (see below), if we set $\check{M} = \overline{T^*(\check{G}/\check{N})}$; this suggests that (17) might indeed hold, given an appropriate definition of the left-hand side. Note that the verification of the criteria of Theorem 3.5.23 is essentially [**GR15**, Theorem 2.3.1].

First, we will show Theorem 3.5.23(b). The relevant Kostant section can be defined as follows. Fix a nondegenerate character $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$, and define the map

$$\kappa : \check{\mathfrak{t}}^* \to \check{\mathfrak{b}}^* \hookrightarrow \mathrm{Ind}_{\check{N}}^{\check{G}} \check{\mathfrak{b}}^* \hookrightarrow \overline{T^*(\check{G}/\check{N})},$$

where the first map is given by the inclusion $\check{\mathfrak{t}}^* \subseteq \check{\mathfrak{t}}^* \oplus \check{\mathfrak{n}}^*$ sending $x \mapsto (x, \psi)$. Then, there is an isomorphism

$$\begin{aligned} \check{\mathfrak{t}}^* \times_{\check{M}/(\check{G}\times\check{T})} \check{\mathfrak{t}}^* &\cong \check{\mathfrak{t}}^* \times_{T^*(\check{G}/\check{N})/(\check{G}\times\check{T})} \check{\mathfrak{t}}^* \\ &\cong \check{\mathfrak{t}}^* \times_{\check{\mathfrak{b}}^*/\check{B}} \check{\mathfrak{t}}^*. \end{aligned}$$

It follows from the main result of [**ABG04**], or equivalently [**YZ11**, Theorem 6.1] (see also [**Dev23a**, Section 4.1] for a 2-periodified analogue), that there is an isomorphism

$$\check{\mathfrak{t}}^* \times_{\check{\mathfrak{b}}^*/\check{B}} \check{\mathfrak{t}}^* \cong \mathrm{Spec}\, \mathrm{H}_*^T(\Omega G; \mathbf{Q}) \cong \mathrm{Spec}\, \mathrm{H}_*^{G\times T}(\mathcal{L}G; \mathbf{Q})$$

of graded group schemes over $\check{\mathfrak{t}}^*$. This implies Theorem 3.5.23(b).

It remains to check Theorem 3.5.23(a). For this, we need to check that there is an isomorphism

$$\mathcal{O}_{(\check{G}\times\check{T}\times\check{\mathfrak{t}}^*)/(\check{\mathfrak{t}}^*\times_{\check{\mathfrak{b}}^*/\check{B}}\check{\mathfrak{t}}^*)} \cong \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}.$$

It is well-known that the $\check{G} \times \check{T}$-orbit of the image of $\kappa : \check{\mathfrak{t}}^* \to T^*(\check{G}/\check{N})$ is the regular locus $T^*(\check{G}/\check{N})^{\mathrm{reg}}$, so that there is an isomorphism

$$\mathcal{O}_{(\check{G}\times\check{T}\times\check{\mathfrak{t}}^*)/(\check{\mathfrak{t}}^*\times_{\check{\mathfrak{b}}^*/\check{B}}\check{\mathfrak{t}}^*)} \cong \mathcal{O}_{T^*(\check{G}/\check{N})^{\mathrm{reg}}}.$$

The inclusion $T^*(\check{G}/\check{N})^{\mathrm{reg}} \subseteq T^*(\check{G}/\check{N})$ has complement of codimension ≥ 2, since it can be identified with the inclusion $\mathrm{Ind}_{\check{N}}^{\check{G}} \check{\mathfrak{b}}^{*,\mathrm{reg}} \subseteq \mathrm{Ind}_{\check{N}}^{\check{G}} \check{\mathfrak{b}}^*$; the claim follows from Lemma 4.2.10(a) along with the observation that the inclusion $\check{\mathfrak{b}}^{*,\mathrm{reg}} \subseteq \check{\mathfrak{b}}^*$ has complement of codimension ≥ 2. The inclusion $T^*(\check{G}/\check{N}) \subseteq \overline{T^*(\check{G}/\check{N})}$ also has complement of codimension ≥ 2 (e.g., by Lemma 4.2.10(b)), so that there are isomorphisms

$$\mathcal{O}_{T^*(\check{G}/\check{N})^{\mathrm{reg}}} \cong \mathcal{O}_{T^*(\check{G}/\check{N})} \cong \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}$$

by the algebraic Hartogs lemma. This verifies Theorem 3.5.23(a), as desired.

Remark 3.5.29. One can extend Example 3.5.28 to more general Levi subgroups as follows. Let $I \subseteq \check{\Delta}$ denote a subset of the simple roots of G, let P_I denote the associated parabolic subgroup, and let $L_I \subseteq P_I$ denote a fixed Levi factor. The subset I defines a subset of simple roots of $\check{G}$, which we will also denote by I (for simplicity). Following [**Mac23**], let $\psi_I : \check{N} \to \mathbf{G}_a$ denote the additive character given by the composite

$$\check{N} \to \check{N}/[\check{N}, \check{N}] \cong \prod_{\Delta} \mathbf{G}_a \xrightarrow{\mathrm{Proj}_I} \prod_I \mathbf{G}_a \xrightarrow{\sum} \mathbf{G}_a,$$

so that ψ_I defines an element of $\check{\mathfrak{n}}^*$. This allows one to define the Whittaker reduction $T^*(\check{G}/_{\psi_I}\check{N})$, which admits a natural $\check{G}$-action. Let $\overline{T^*(\check{G}/_{\psi_I}\check{N})}$ denote its affine closure. We then expect the following:

Conjecture 3.5.30. *Let L_I denote the above Levi subgroup of G, and assume that it is spherical*[17]. *Then there is an equivalence*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/L_I);\mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\overline{T^*(\check{G}/_{\psi_I}\check{N})}/\check{G})$$

which is equivariant for the left-action of $\mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G;\mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{\mathfrak{g}}^*/\check{G})$ *via Theorem 3.2.7; we have again omitted the grading on the spectral side for simplicity.*

In particular, Conjecture 3.5.30 should be a consequence of [**Mac23**, Theorem 4.4.5]. However, the relationship between the spectral side of Conjecture 3.5.30 and the predicted dual variety of Conjecture 3.4.11 seems to be very nontrivial. The above conjecture in the case of the spherical Levi $\mathrm{GL}_j \times \mathrm{GL}_{n-j} \subseteq \mathrm{GL}_n$ is work-in-progress of Chen-Macerato-Nadler-O'Brien.

Let us remark on one interesting consequence of Conjecture 3.5.30. Write $\mathrm{N}_G(L_I)$ to denote the normalizer of $L_I \subseteq G$. There is a natural action of the relative Weyl group $W_I = \mathrm{N}_G(L_I)/L_I$ on the left-hand side of Conjecture 3.5.30, which defines an action of W_I on the right-hand side. Based on Example 3.5.28, it is natural to hope that there is in fact an action of W_I on $\overline{T^*(\check{G}/_{\psi_I}\check{N})}$ which commutes with its natural $\check{G}$-action. This would be a parabolic variant of the (semi-classical) Gelfand-Graev action, and would imply the following extension of Conjecture 3.5.30:

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/\mathrm{N}_G(L_I));\mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\overline{T^*(\check{G}/_{\psi_I}\check{N})}/W_I)\check{G}).$$

In particular, the Hamiltonian $\check{G}$-"space" which is dual to the spherical G-variety $G/\mathrm{N}_G(L_I)$ (which often has roots of type N) is the *stack* $\overline{T^*(\check{G}/_{\psi_I}\check{N})}/W_I$. In joint work with Macerato, we will address this question and Conjecture 3.5.30 using "Whittaker descent" and the calculations of [**Mac23**] (which in turn rely on the calculations of [**BFM05, YZ11**]).

To generalize the discussion of this section to ku-theoretic coefficients, we need some preliminary results.

3.6. ku-theoretic derived geometric Satake.

Recall from the proof of Theorem 3.2.7 that the key step in the argument, once given Theorem 3.2.3, is Theorem 3.2.14. Our goal in this section is to prove a ku-theoretic analogue of Theorem 3.2.14 in the case when G is assumed to be simply-laced and simply-connected. Throughout, $\mathbf{G}_\beta^\vee$ will denote the Cartier dual of $\mathbf{G}_\beta$.

Definition 3.6.1. Let X be a (possibly graded) scheme over a commutative ring R. The ku-*loop space* $\mathcal{L}_\beta X$ of X is defined to be the graded $R[\beta]$-scheme $\mathrm{Map}(B\mathbf{G}_\beta^\vee, X_{R[\beta]})$.

The following is a slight variant of the main result of [**MRT19**] (see also [**Mou21**, Corollary 6.1]).

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[17]Note that there is a classification of spherical Levi subgroups of simple linear algebraic groups using Krämer's classification [**Kra79**]; see [**Bru98**, Theorem 4.1]. For instance, if G is a classical group, the only possibilities are $\mathrm{GL}_j \times \mathrm{GL}_{n-j} \subseteq \mathrm{GL}_j$, $\mathrm{SO}_2 \times \mathrm{SO}_{2n-1} \subseteq \mathrm{SO}_{2n+1}$, $\mathrm{GL}_n \subseteq \mathrm{SO}_{2n+1}$, $\mathbf{G}_m \times \mathrm{Sp}_{2n-2} \subseteq \mathrm{Sp}_{2n}$, $\mathrm{GL}_n \subseteq \mathrm{Sp}_{2n}$, $\mathrm{SO}_2 \times \mathrm{SO}_{2n-2} \subseteq \mathrm{SO}_{2n}$, and $\mathrm{GL}_n \subseteq \mathrm{SO}_{2n}$.

Lemma 3.6.2. *Let X be a derived scheme over a commutative ring R. Then the pushforward of the structure sheaf along the canonical map $\mathcal{L}_\beta(X)/\mathbf{G}_m \to \mathrm{Spec}(R[\beta])/\mathbf{G}_m$ corresponds (under the equivalence between quasicoherent sheaves on $\mathrm{Spec}(R[\beta])/\mathbf{G}_m$ and filtered R-modules) to the Hochschild-Kostant-Rosenberg filtration on the Hochschild homology $\mathrm{HH}(X/R)$.*

PROOF. In [**MRT19**], it is shown that if R is a $\mathbf{Z}_p$-algebra, W is the ring scheme of p-typical Witt vectors, and F is its Frobenius, the pushforward of the structure sheaf along the canonical map $\mathrm{Map}(W[F = \beta^{p-1}], X)/\mathbf{G}_m \to \mathrm{Spec}(R[\beta])/\mathbf{G}_m$ corresponds to the Hochschild-Kostant-Rosenberg filtration on $\mathrm{HH}(X/R)$. It therefore suffices to identify $\mathbf{G}_\beta^\vee$ with a completion of $W[F = \beta^{p-1}]$. As shown in [**Dev23c**, Proposition C.6], there is an isomorphism

$$\mathbf{G}_\beta^\vee \cong \mathrm{Spf}\, \mathbf{Z}_p \left[\beta, \tfrac{y(y-\beta)\cdots(y-(n-1)\beta)}{n!}\right]^\wedge \tag{18}$$

where the element y is primitive (i.e., the coproduct sends $y \mapsto y \otimes 1 + 1 \otimes y$) and lives in weight 2. Here, the completion is taken with respect to the β-deformed divided power filtration (i.e., with respect to $\frac{1}{n!}\prod_{j=0}^{n-1}(y - j\beta)$ for $n \geq 1$). The desired identification with the completion of $W[F = \beta^{p-1}]$ is now given by [**Dev23c**, Remark C.7]. Using the arithmetic fracture square, it only remains to prove the lemma when R is a $\mathbf{Q}$-algebra. In this case, $\mathbf{G}_\beta^\vee$ is isomorphic to $\mathrm{Spf}\, \mathbf{Q}[\beta][\![y]\!] \cong \hat{\mathbf{G}}_a$, from which the desired result follows since the the Hochschild-Kostant-Rosenberg filtration on $\mathrm{HH}(X/R)$ splits, and the Hodge complex of X over R can be identified with the global sections of the mapping stack $\mathrm{Map}(B\hat{\mathbf{G}}_a, X)$. □

Definition 3.6.3. Let H be a graded algebraic group over a commutative ring R. Let H_β denote the graded group scheme over $R[\beta]$ given by $\mathrm{Hom}(\mathbf{G}_\beta^\vee, H_{R[\beta]})$. Note that there is a canonical action of $H_{R[\beta]}$ on H_β by conjugation, and the quotient stack $H_\beta/H_{R[\beta]}$ is isomorphic to $\mathcal{L}_\beta(BH) = \mathrm{Map}(B\mathbf{G}_\beta^\vee, BH_{R[\beta]})$.

Lemma 3.6.4. *If H is a graded algebraic group, there is an isomorphism $H_\beta[\beta^{-1}]/\mathbf{G}_m \cong H$, and a graded isomorphism $H_\beta/\beta \cong \mathfrak{h}(2)$.*

PROOF. Since $\mathrm{Map}(B\widehat{\mathbf{G}_a^\sharp(-2)}, BH) \cong \mathfrak{h}(2)/H$ by Example 5.1.5, and $\mathrm{Map}(B\mathbf{Z}, BH) \cong H/H$, it suffices to show that $\mathbf{G}_\beta^\vee[\beta^{-1}]/\mathbf{G}_m \cong \mathbf{Z}$, while $\mathbf{G}_\beta^\vee/\beta$ is isomorphic to the completion $\widehat{\mathbf{G}_a^\sharp(-2)}$ of the PD-hull of the origin in $\mathbf{G}_a(-2)$ at the divided power filtration. This in turn follows from the fact that $\mathbf{G}_\beta[\beta^{-1}]/\mathbf{G}_m \cong \mathbf{G}_m$ and $\mathbf{G}_\beta/\beta \cong \mathbf{G}_a(2)$, and that $\mathbf{Z}$ (resp. $\mathbf{G}_a^\sharp(-2)$) is the Cartier dual of $\mathbf{G}_m$ (resp. $\mathbf{G}_a(2)$). □

Remark 3.6.5. In Definition 3.6.1, there was no reason to restrict to considering maps out of $B\mathbf{G}_\beta^\vee$: we could have considered *any* 1-dimensional group scheme over $\mathbf{Z}[\beta]$ in place of $\mathbf{G}_\beta$. (This sort of philosophy fits very naturally into Section 5.4, which more generally suggests that it should be very interesting to study the *universal* case, where $\mathbf{Z}[\beta]$ is replaced with the Lazard ring carrying the universal formal group law itself.) For instance, a particularly important example which arises naturally in chromatic homotopy theory is the following. Fix a prime p, and consider the formal group over $\mathbf{Q}[\beta]$ whose logarithm is given by the invertible "p-typical polylogarithmic" power series

$$\ell_F(x) = \sum_{j\geq 0} \beta^{p^{nj}-1} \tfrac{x^{p^{nj}}}{p^j}.$$

Here, the class x lives in weight -2. That the power series $F(x,y) = \ell_F^{-1}(\ell_F(x) + \ell_F(y))$ has coefficients in $\mathbf{Z}_{(p)}[\beta]$ is a consequence of Hazewinkel's functional equation lemma [**Haz78**, Section I.2]; write $\widehat{\mathbf{G}}_{k_{\mathbf{Z}}(n)}$ to denote the associated formal group law over $\mathbf{Z}_{(p)}[\beta]$. When base-changed to $\mathbf{F}_p[\beta]$, we will denote it by $\widehat{\mathbf{G}}_{k(n)}$. The Cartier dual of $\widehat{\mathbf{G}}_{k(n)}$ was computed in [**DM23**, Example 4.5.14], where it was shown that

$$\widehat{\mathbf{G}}_{k(n)}^\vee \cong \operatorname{Spec} \mathbf{F}_p[\beta][y_0, y_1, \cdots]/(y_{n+j-1}^p - \beta^{p^j(p^n-1)} y_j).$$

Here, the classes y_j live in weight $2p^j$. Observe that for $\beta = 0$, one recovers the Cartier dual $\mathbf{G}_a^\sharp$ of $\widehat{\mathbf{G}}_a$. For $n = 1$, the formal group $\widehat{\mathbf{G}}_{k(1)}$ is isomorphic to the p-typification of $\widehat{\mathbf{G}}_\beta$.

To connect this to chromatic homotopy theory, note that $\ell_F(x)$ only depends on β through β^{p^n-1}; so $\widehat{\mathbf{G}}_{k_{\mathbf{Z}}(n)}$ is in fact defined over $\mathbf{Z}_{(p)}[\beta^{p^n-1}]$. The class β^{p^n-1} is often denoted v_n, and the resulting formal group law over $\mathbf{Z}_{(p)}[v_n]$ is the one associated to the complex orientation of (a form of) integral Morava K-theory of height n. When H is a group scheme over $\mathbf{F}_p$, the group scheme $\operatorname{Hom}(\widehat{\mathbf{G}}_{k(n)}^\vee, H)$ is closely related to the combinatorial constructions of [**DM23**], and we expect it to capture a lot of interesting aspects of the modular representation theory of H (in particular, the semiclassical limit of the "philosophy of generations" from [**LW18**]). For instance, when $n = 1$, it essentially reduces to the group scheme H_β studied in the present section. In general, it is an interesting deformation of the Lie algebra $\mathfrak{h}(2)$ to $\mathbf{F}_p[\beta]$. We will not discuss this generalization of H_β further here, but plan to in [**Dev24**].

Lemma 3.6.6. *Let R be a commutative ring, and let K be a commutative group scheme over R with Cartier dual $K^\vee$. Then there is an isomorphism* $\operatorname{Hom}(K^\vee, \mathbf{G}_a) \cong \operatorname{Lie}(K)$.

Lemma 3.6.7. *Let $B \subseteq \mathrm{GL}_2$ denote the Borel subgroup of upper-triangular matrices, graded by the action of $2n\rho : \mathbf{G}_m \to \mathrm{GL}_2$. If R_* is a graded $\mathbf{Z}[\beta]$-algebra, the group scheme $B_\beta(R_*)$ is isomorphic to the subgroup of $B(R_*)$ consisting of matrices of the form $\left(\begin{smallmatrix} 1+\beta x & \beta y \\ 0 & 1+\beta w \end{smallmatrix}\right)$, where $x, w \in \mathbf{G}_\beta(R) \subseteq R_{-2}$ and $y \in R_{2n-2}$.*

Proof. There is a graded extension

$$\mathbf{G}_a(-2n) \to B \to \mathbf{G}_m^2,$$

which implies that there is a graded extension

$$\mathbf{G}_a(-2n)_\beta \to B_\beta \to (\mathbf{G}_m^2)_\beta.$$

By construction, $\mathbf{G}_{m,\beta}$ is the Cartier dual of $\mathbf{G}_\beta^\vee$, i.e., $\mathbf{G}_{m,\beta} \cong \mathbf{G}_\beta$. Moreover, Lemma 3.6.6 gives an isomorphism $\mathbf{G}_a(-2n)_\beta \cong \operatorname{Lie}(\mathbf{G}_\beta)(-2n) \cong \mathbf{G}_a(2-2n)$. It follows that there is an extension

$$\mathbf{G}_a(2-2n) \to B_\beta \to \mathbf{G}_\beta^2.$$

This extension precisely classifies matrices of the form $\left(\begin{smallmatrix} 1+\beta x & \beta y \\ 0 & 1+\beta w \end{smallmatrix}\right)$ with $x, w \in \mathbf{G}_\beta(R) \subseteq R_{-2}$ and $y \in R_{2n-2}$. □

Lemma 3.6.8. *Let $\check{G}$ be a reductive group over a commutative ring R. Let $\check{G}_\beta^{\mathrm{reg}}$ denote the open subgroup consisting of those elements $x \in \check{G}_\beta$ such that the centralizer $Z_{\check{G}}(x) \subseteq \check{G}$ has minimal dimension (i.e., the rank of $\check{G}$). Similarly, let $\check{B}_\beta^{\mathrm{reg}}$ denote $\check{B}_\beta \cap \check{G}_\beta^{\mathrm{reg}} \subseteq \check{G}_\beta$. Then the morphism $\check{B}_\beta^{\mathrm{reg}}/\check{B} \to \check{G}_\beta^{\mathrm{reg}}/\check{G}$ is a ramified W-Galois cover.*

Definition 3.6.9. Let $e \in \check{G}$ be a principal unipotent element, so that e defines a homomorphism $\mathrm{SL}_2 \to \check{G}$ by the Jacobson-Morozov theorem. This homomorphism is $\mathbf{G}_m$-equivariant if SL_2 (resp. $\check{G}$) is graded by $2\rho : \mathbf{G}_m \to \mathrm{SL}_2$ (resp. $2\rho : \mathbf{G}_m \to \check{G}$). There is an induced homomorphism $B_{\mathrm{SL}_2,\beta} \to \check{G}_\beta$, where $B_{\mathrm{SL}_2} \subseteq \mathrm{SL}_2$ is the Borel subgroup of upper-triangular matrices. Let $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \in B_{\mathrm{SL}_2,\beta}$ denote the element defined by Lemma 3.6.7; the image of this element in $\check{G}_\beta$ will be denoted by e_β.

Let $\widetilde{\kappa} : \check{T}_\beta \to \check{B}_\beta$ denote the map sending $x \mapsto e_\beta x$, so that $\widetilde{\kappa}$ induces a map $\check{T}_\beta \to \check{B}_\beta/\check{B}$ (which will also be denoted by $\widetilde{\kappa}$). It is not difficult to see that $\widetilde{\kappa} : \check{T}_\beta \to \check{B}_\beta/\check{B}$ is W-equivariant, so Lemma 3.6.8 implies that the composite

$$\check{T}_\beta \xrightarrow{\widetilde{\kappa}} \check{B}_\beta/\check{B} \to \check{G}_\beta/\check{G}$$

descends to a morphism $\check{T}_\beta /\!\!/ W \to \check{G}_\beta/\check{G}$, which we will denote by κ. The map $\kappa : \check{T}_\beta /\!\!/ W \to \check{G}_\beta/\check{G}$ will be called the *β-deformed Kostant slice*. It defines a graded map $\check{T}_\beta /\!\!/ W \to \check{G}(-2\rho)_\beta/\check{G}(-2\rho)$.

In the remainder of this section, we will invert the order of the Weyl group W, and write $\mathbf{Z}' = \mathbf{Z}[1/|W|]$; in particular, $\pi_*\mathrm{ku} \cong \mathbf{Z}'[\beta]$.

Theorem 3.6.10. *Let G be a simply-laced and simply-connected compact Lie group with associated reductive group $G_{\mathbf{C}}$ over $\mathbf{C}$, so that $(G \times G)/Z(G)^{\mathrm{diag}}$ acts on $\mathcal{L}G$ by left and right translation. Then there is a graded $\mathbf{Z}'[\beta]$-linear isomorphism*

$$\operatorname{Spec} \mathrm{ku}_*^{(G\times G)/Z(G)^{\mathrm{diag}}}(\mathcal{L}G) \cong \check{T}_\beta /\!\!/ W \times_{\check{G}(-2\rho)_\beta/\check{G}(-2\rho)} \check{T}_\beta /\!\!/ W$$

of group schemes over $\check{T}_\beta /\!\!/ W = \operatorname{Spec} \mathrm{ku}^*_{G/Z(G)}(*)$.

Corollary 3.6.11. *In the setup of Theorem 3.6.10, there is a graded $\mathbf{Z}'[\beta]$-linear isomorphism*

$$\operatorname{Spec} \mathrm{ku}_*^{G\times G}(\mathcal{L}G) \cong T_\beta /\!\!/ W \times_{\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)} T_\beta /\!\!/ W$$

of group schemes over $T_\beta /\!\!/ W = \operatorname{Spec} \mathrm{ku}^*_G(*)$, where $\check{G}^{\mathrm{sc}}$ is the simply-connected cover of $\check{G}$.

Remark 3.6.12. Corollary 3.6.11 is a simultaneous generalization of [**Dev23a**, Proposition 4.1.5 and Theorem 4.2.5] (which is in turn related to [**BFM05**, Theorem 2.12 and Theorem 2.15]).

To prove Theorem 3.6.10, we will first calculate $\mathrm{ku}_*^{G\times G}(\mathcal{L}G)$ for not necessarily simply-laced G.

Notation 3.6.13. Let $(T^*_\beta \check{T})^{\mathrm{bl}}$ denote the affine blowup of $\check{T} \times T_\beta$ given by

$$(T^*_\beta \check{T})^{\mathrm{bl}} \cong \operatorname{Spec} \mathcal{O}_{\check{T} \times T_\beta}[\tfrac{\check{\alpha}-1}{x_\alpha}, \alpha \in \Phi],$$

where $\check{\alpha} - 1 \in \mathcal{O}_{\check{T}}$ and $x_\alpha \in \mathcal{O}_{T_\beta}$ denote the functions associated to $\alpha \in \Phi$. There is a canonical action of the Weyl group W on $(T^*_\beta \check{T})^{\mathrm{bl}}$.

Proposition 3.6.14. *Let G be a connected compact Lie group whose π_1 is torsion-free. There is a graded isomorphism*

$$\operatorname{Spec} \mathrm{ku}_*^{G \times G}(\mathcal{L}G) \cong (T^*_\beta \check{T})^{\mathrm{bl}} /\!\!/ W$$

over $T_\beta /\!\!/ W$.

Proof. There is an isomorphism $(\mathcal{L}G)/(G \times G) \cong (\Omega G)/G$ of orbifolds, and hence an isomorphism $\mathrm{ku}_*^{G\times G}(\mathcal{L}G) \cong \mathrm{ku}_*^G(\Omega G)$. Since $\pi_1 G$ is assumed to be torsionfree, Proposition 2.3.15 implies that there is an isomorphism $\mathrm{ku}_*^G(\Omega G) \cong \mathrm{ku}_*^T(\Omega G)^W$. To compute $\mathrm{ku}_*^T(\Omega G)$, we can now use [**Dev23a**, Theorem 3.2.12] applied to $A = \mathrm{ku}$ and $\mathbf{G} = \mathbf{G}_{\mathrm{ku},\beta}$ to conclude that there is a graded isomorphism

$$\operatorname{Spec} \mathrm{ku}_*^T(\Omega G) \cong (T^*_\beta \check{T})^{\mathrm{bl}}; \tag{19}$$

taking the GIT quotient by W produces the desired result. Note that [**Dev23a**, Theorem 3.2.12] asks that $\mathbf{G}$ be an *oriented* group scheme over A, but this is in fact not necessary: it suffices that $\mathbf{G}$ be preoriented, so that Proposition 2.4.10 continues to hold. In this case, the desired preorientation of $\mathbf{G}_{\mathrm{ku},\beta}$ is given by Proposition 2.3.11. $\square$

The following result is essentially [**YZ11**, Step II of Theorem 6.1].

Lemma 3.6.15. *The scheme $\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta$ is flat over $\check{T}_\beta$ after inverting $|W|$.*

Proof. There is a closed immersion

$$\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \subseteq \check{T}_\beta \times \check{B},$$

which exhibits the left-hand side as the subgroup of those (x, g) such that g stabilizes $\kappa(x)$; in particular, it is cut out by $\dim \check{N}$ equations, so that the fibers of the projection $\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \to \check{T}_\beta$ are at most $\dim \check{B} - \dim \check{N} = \dim \check{T}$-dimensional. To prove that $\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta$ is flat over $\check{T}_\beta$, it suffices to show that all the fibers of the projection

$$\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \to \check{T}_\beta$$

are $\dim \check{T}$-dimensional.

Let $g \in \check{B}$ and $x \in \check{T}_\beta$. Since $\kappa(x) = e_\beta x$, we have $\mathrm{Ad}_g \kappa(x) = \mathrm{Ad}_g(e_\beta)\mathrm{Ad}_g(x) \in \check{B}_\beta \subseteq \check{B}_{\mathbf{Z}[\beta]}$. Since x is semisimple, the same is true of $\mathrm{Ad}_g(x)$. It therefore suffices to show that the subgroup $Z_{\check{B}}(e_\beta) = \{g \in \check{B} | \mathrm{Ad}_g(e_\beta) = e_\beta\} \subseteq \check{B}$ is $\dim \check{T}$-dimensional. If $\mathrm{Ad}_g(e_\beta) = e_\beta$, then $\mathrm{Ad}_g(e) = e$, where $e \in \check{\mathfrak{g}}$ is the associated nilpotent element; this implies that $\dim Z_{\check{B}}(e_\beta) \leq \dim Z_{\check{B}}(e)$. Therefore, it suffices to show that the centralizer $Z_{\check{B}}(e)$ is $\dim \check{T}$-dimensional, which is even true with a smaller set of primes inverted (see [**Ken87**]). $\square$

Proposition 3.6.16. *Upon inverting $|W|$, there is a graded isomorphism*

$$\check{T}_\beta /\!\!/ W \times_{\check{G}(-2\rho)_\beta/\check{G}(-2\rho)} \check{T}_\beta /\!\!/ W \cong ((T^*_\beta \check{T})^{\mathrm{bl}} /\!\!/ W)/Z(G)$$

over $T_\beta /\!\!/ W$, where the map $\kappa : \check{T}_\beta /\!\!/ W \to \check{G}_\beta/\check{G}$ is the β-deformed Kostant slice.

Proof. For simplicity, we will ignore gradings in the following discussion. By definition of κ, it suffices to show that there is a W-equivariant graded isomorphism

$$\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \cong (T^*_\beta \check{T})^{\mathrm{bl}}/Z(G) \tag{20}$$

over T_β. Since G is simply-laced and simply-connected, there is an isomorphism $\check{T} \cong T/Z(G)$, so that $(T^*_\beta \check{T})^{\mathrm{bl}}/Z(G)$ is simply isomorphic to the affine blowup of $\check{T} \times \check{T}_\beta$ given by

$$(T^*_\beta \check{T})^{\mathrm{bl}}/Z(G) \cong \operatorname{Spec} \mathcal{O}_{\check{T}\times\check{T}_\beta}[\tfrac{\check{\alpha}-1}{x_{\check{\alpha}}}, \alpha \in \Phi].$$

Using [**BFM05**, Claim following Lemma 4.1], one finds that the scheme $(T^*_\beta \check{T})^{\mathrm{bl}}/Z(G)$ is flat over $\check{T}_\beta$. The scheme $\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta$ is also flat over $\check{T}_\beta$ by Lemma 3.6.15. Therefore, the argument of [**BFM05**, Section 4.3] reduces us to checking the isomorphism (20) in the case when G (equivalently $\check{G}$) has semisimple rank 1. For this, it in turn suffices to prove (20) when $\check{G} = \mathrm{SL}_2$ and PGL_2; the case when $\check{G}$ is a torus is straightforward.

Let $\check{G} = \mathrm{SL}_2$ (so $G = \mathrm{SO}_3$, and $Z(G)$ is trivial). Recall that there is a closed immersion

$$\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \subseteq \check{T}_\beta \times \check{B},$$

which exhibits the left-hand side as the subgroup of those (x, g) such that g stabilizes $\kappa(x)$. Let $\overline{x}$ denote the inverse of x in the group structure on $\mathbf{G}_\beta$, so that $1 + \beta\overline{x} = (1 + \beta x)^{-1}$. Since $\kappa(x) \in \check{B}_\beta$ is the matrix

$$\kappa(x) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\beta x & 0 \\ 0 & (1+\beta x)^{-1} \end{pmatrix} = \begin{pmatrix} 1+\beta x & \frac{\beta}{1+\beta x} \\ 0 & 1+\beta\overline{x} \end{pmatrix},$$

it follows that that if $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, we have

$$\mathrm{Ad}_g \kappa(x) = \begin{pmatrix} 1+\beta x & a^2\beta(1+\beta\overline{x}) - ab\beta(x-\overline{x}) \\ 0 & 1+\beta\overline{x} \end{pmatrix}.$$

It follows that g fixes $\kappa(x)$ if and only if

$$a^2\beta(1 + \beta\overline{x}) - ab\beta(x - \overline{x}) = \tfrac{\beta}{1+\beta x},$$

i.e., if and only if

$$b = \tfrac{1}{1+\beta x} \tfrac{a-a^{-1}}{x-\overline{x}}.$$

It follows that there is an isomorphism

$$\check{T}_\beta \times_{\check{B}_\beta/\check{B}} \check{T}_\beta \cong \operatorname{Spec} \mathbf{Z}'[\beta, x, \tfrac{1}{1+\beta x}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-\overline{x}}];$$

but this is precisely $(T^*_\beta \check{T})^{\mathrm{bl}}/Z(G) = (T^*_\beta \check{T})^{\mathrm{bl}}$, as desired. A similar calculation proves (20) when $\check{G} = \mathrm{PGL}_2$. $\square$

Proof of Theorem 3.6.10. It follows from Proposition 3.6.14 and Proposition 3.6.16 that there are graded isomorphisms

$$\operatorname{Spec} \mathrm{ku}_*^{(G\times G)/Z(G)}(\mathcal{L}G) \cong ((T^*_\beta \check{T})^{\mathrm{bl}} /\!\!/ W)/Z(G) \cong \check{T}_\beta /\!\!/ W \times_{\check{G}(-2\rho)_\beta/\check{G}(-2\rho)} \check{T}_\beta /\!\!/ W.$$

Checking that this is an isomorphism of group schemes is not difficult; we leave it to the reader. $\square$

Example 3.6.17. Let us describe the calculation of Corollary 3.6.11 when $G = \mathrm{SO}_3$; this example combines both [**Dev23a**, Examples B.3 and B.6]. Namely, there is an isomorphism

$$\begin{aligned}\mathrm{ku}_*^{\mathrm{SO}_3\times\mathrm{SO}_3}(\mathcal{L}\mathrm{SO}_3) &\cong \mathbf{Z}'[\beta, x, \tfrac{1}{1+\beta x}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-\overline{x}}]^{\mathbf{Z}/2} \\ &\cong \mathbf{Z}'[\beta, x\overline{x}, a+a^{-1}, \tfrac{a-a^{-1}}{x-\overline{x}}],\end{aligned}$$

since the invariant $x+\overline{x}$ is $-\beta x\overline{x}$. Let $\Phi = x\overline{x}$, $U = a+a^{-1}$, and $V = \frac{a-a^{-1}}{x-\overline{x}}$. Using that

$$(\beta^2\Phi - 4)\Phi = (x+\overline{x})^2 - 4x\overline{x} = (x-\overline{x})^2,$$

it is not difficult to see that $\operatorname{Spec}\mathrm{ku}_*^{\mathrm{SO}_3\times\mathrm{SO}_3}(\mathcal{L}\mathrm{SO}_3)$ is cut out inside $\mathbf{A}^3_{\mathbf{Z}[\beta]}$ with coordinates Φ, U, V by the equation

$$U^2 - 4 = (\beta^2\Phi - 4)\Phi V^2.$$

Here, Φ lives in weight -4, U lives in weight 0, and V lives in weight 2.

It is also possible to compute the S^1_{rot}-equivariant homology $\mathrm{ku}_*^{\mathrm{SO}_3\times\mathrm{SO}_3\times S^1_{\mathrm{rot}}}(\mathcal{L}\mathrm{SO}_3)$ as in [**Dev23a**, Example B.3 and Example B.6]. Namely, using Remark 3.5.16, one obtains an isomorphism

$$\mathrm{ku}_*^{\mathrm{SO}_3\times\mathrm{SO}_3\times S^1_{\mathrm{rot}}}(\mathcal{L}\mathrm{SO}_3) \cong \Big(\mathbf{Z}'[\beta, \hbar, \tfrac{1}{1+\beta\hbar}]\{x, a^{\pm 1}, \tfrac{1}{x-\overline{x}}(a-a^{-1})\}[\tfrac{1}{1+\beta x}]/I\Big)^{\mathbf{Z}/2},$$

where the ideal I encodes the commutation relation

$$[x, a] = a\hbar(1+\beta x).$$

Here, the curly brackets denotes the free associative algebra generated by the elements inside the brackets. The ring of invariants can be computed explicitly: again, there are three generators, given by $\Phi = x\overline{x}$, $U = a+a^{-1}$, and $V = \frac{1}{x-\overline{x}}(a-a^{-1})$. We will leave the tedious (but straightforward) exercise of computing their commutators to the interested reader.

Construction 3.6.18. Let G be a simply-laced simply-connected almost simple compact Lie group, so that Theorem 3.6.10 gives a homomorphism

$$\operatorname{Spec}\mathrm{ku}_*^G(\Omega G) \hookrightarrow \check{G}(-2\rho) \times T_\beta /\!\!/ W$$

of group schemes over $T_\beta/\!\!/W$. Let $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a connected *reductive* spherical subgroup. Then there is a homomorphism (in fact, closed imersion) of graded group schemes

$$\operatorname{Spec}\mathrm{ku}_*^H(\Omega(G/H)) \to \check{G}(-2\rho) \times T_{H,\beta}/\!\!/W_H$$

over $T_{H,\beta}/\!\!/W \cong \operatorname{Spec}\mathrm{ku}^*_H(*)$ constructed as the following composite:

$$\begin{aligned}\operatorname{Spec}\mathrm{ku}_*^H(\Omega(G/H)) \to \operatorname{Spec}\mathrm{ku}_*^H(\Omega G) &\cong \operatorname{Spec}\mathrm{ku}_*^G(\Omega G) \times_{\operatorname{Spec}\mathrm{ku}_G^*(*)} \operatorname{Spec}\mathrm{ku}_H^*(*) \\ &\hookrightarrow (\check{G}(-2\rho) \times T_\beta/\!\!/W) \times_{T_\beta/\!\!/W} T_{H,\beta}/\!\!/W_H \cong \check{G}(-2\rho) \times T_{H,\beta}/\!\!/W_H.\end{aligned}$$

Let A_β denote the $\mathcal{O}_{T_{H,\beta}/\!\!/W_H}$-algebra of regular functions on the quotient $(\check{G} \times T_{H,\beta}/\!\!/W_H)/\operatorname{Spec}\mathrm{ku}_*^H(\Omega(G/H))$ over $T_{H,\beta}/\!\!/W_H$, so that A_β admits a canonical grading, as well as a canonical action of $\check{G}(-2\rho)$. Define $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$ to be the $\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]$-linear ∞-category $\mathrm{Perf}(\mathrm{sh}^{1/2}\operatorname{Spec} A_\beta/\check{G}(-2\rho))$.

The reason that $H_{\mathbf{C}}$ is assumed to be reductive is precisely thanks to the proof of Theorem 3.5.23, which implies Proposition 3.6.19 below; this result ensures consistency in notation (and showing that Theorem 3.6.21 implies Theorem 3.2.7 at least in the simply-laced and simply-connected case):

Proposition 3.6.19. *Let G be a simply-laced simply-connected almost simple compact Lie group, let $G_{\mathbf{C}}$ be the associated algebraic group over $\mathbf{C}$, and let $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a connected reductive spherical subgroup which is optimal in the sense of Hypothesis 3.5.21. Let $\mathrm{sh}^{1/2}\mathbf{Z}'[\beta] \to \mathbf{Q}$ denote the $\mathbf{E}_\infty$-map given by rationalization and sending $\beta \mapsto 0$. Then there is an equivalence*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \otimes_{\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]} \mathbf{Q} \simeq \mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}).$$

Remark 3.6.20. Construction 3.6.18 is extremely unsatisfactory and *ad hoc*. For instance, it is not defined intrinsically to the G-action on G/H, and instead uses the algebra of functions on $\check{G}$. Its only saving grace is Proposition 3.6.19.

Instead, Construction 3.6.18 is intended to be a replacement of Definition 3.5.17 in the case where one does not have an analogue of Theorem 3.2.3. It might be the case that there is a well-behaved ku_G-linear ∞-category $\widetilde{\mathrm{Shv}}_G^c(\mathcal{L}(G/H); \mathrm{ku})$ of constructible sheaves of G-equivariant ku-modules defined on ind-finite G-spaces such as $\mathcal{L}(G/H)$. In the nonequivariant case, there *is* a well-behaved notion of constructible sheaves of ku-modules defined using exit path categories à la [**Lur16**, Appendix A]. Roughly, then, $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$ would be an "associated graded for the Postnikov filtration" on $\widetilde{\mathrm{Shv}}_G^c(\mathcal{L}(G/H); \mathrm{ku})$.

Namely, suppose that there was a ku_G-linear ∞-category $\widetilde{\mathrm{Shv}}_G^c(\mathcal{L}(G/H); \mathrm{ku})$, and an $\mathbf{E}_\infty$-coalgebra $\widetilde{\check{\mathcal{R}}}$ in $\mathrm{Alg}_{\mathbf{E}_2}(\widetilde{\mathrm{Shv}}_{G\times G}^c(\mathcal{L}G; \mathrm{ku}))$ which lifted the regular sheaf (corresponding to $\mathcal{O}_{\check{G}} \in \mathcal{P}\mathrm{erv}_{G\times G}(\mathcal{L}G; \mathbf{Z})$). Let $\widetilde{\mathcal{C}}$ denote the de-equivariantization of $\widetilde{\mathrm{Shv}}_G^c(\mathcal{L}(G/H); \mathrm{ku})$ with respect to this $\mathbf{E}_2$-Hopf algebra, and let $\mathrm{IC}_0 \in \widetilde{\mathrm{Shv}}_G^c(\mathcal{L}(G/H); \mathrm{ku})$ denote the !-pushforward of the constant sheaf in $\widetilde{\mathrm{Shv}}_G^c(G/H; \mathrm{ku}) \simeq \widetilde{\mathrm{Shv}}_H^c(*; \mathrm{ku})$. Then the full subcategory $\mathcal{C}$ of $\widetilde{\mathcal{C}}$ generated by $\mathrm{IC}_0 \star \widetilde{\check{\mathcal{R}}}$ can be identified with left modules over the ku-algebra $\widetilde{A} := \mathrm{End}_{\widetilde{\mathcal{C}}}(\mathrm{IC}_0 \star \widetilde{\check{\mathcal{R}}})$. Finally, let $\widetilde{\mathrm{Shv}}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$ denote the equivariantization of $\mathcal{C}$ with respect to the $\mathbf{E}_2$-Hopf algebra $\widetilde{\check{\mathcal{R}}}$. Using Theorem 3.1.3 and Remark 3.1.5, we may identify $\pi_*\widetilde{A} = A_\beta$, and the coaction of $\widetilde{\check{\mathcal{R}}}$ on $\widetilde{A}$ induces the $\mathcal{O}_{\check{G}}$-coaction on A_β from Construction 3.6.18. In particular, $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$ is an "associated graded" of $\widetilde{\mathrm{Shv}}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$.

Of course, it would be ideal to work with $\widetilde{\mathrm{Shv}}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku})$ itself instead of the *ad hoc* ∞-category of Construction 3.6.18. However, carrying out the above program has proven to be very challenging (for both technical and conceptually interesting reasons), and Construction 3.6.18 is an attempt to salvage the situation somewhat.

Corollary 3.6.11 implies the following generalization of Theorem 3.2.7:

Theorem 3.6.21. *Let G be a simply-laced simply-connected semisimple compact Lie group. There is an $\mathbf{E}_2$-monoidal equivalence of $\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]$-linear ∞-categories*

$$\mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)).$$

PROOF. By Construction 3.6.18, we need to check that there is a graded $\mathbf{Z}'[\beta]$-linear isomorphism

$$\mathcal{O}_{\check{G}(-2\rho)^{\mathrm{sc}}_\beta} \cong \mathcal{O}_{(\check{G}(-2\rho)\times T_\beta/\!\!/W)/(T_\beta/\!\!/W\times_{\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)} T_\beta/\!\!/W)},$$

for which it in turn suffices to check that there is a graded $\mathbf{Z}'[\beta]$-linear isomorphism

$$\mathcal{O}_{\check{G}(-2\rho)_\beta} \cong \mathcal{O}_{(\check{G}(-2\rho)\times \check{T}_\beta/\!\!/W)/(\check{T}_\beta/\!\!/W\times_{\check{G}(-2\rho)_\beta/\check{G}(-2\rho)} \check{T}_\beta/\!\!/W)}.$$

For simplicity, let us momentarily ignore gradings. We claim that the $\check{G}$-orbit of the image of $\kappa : \check{T}_\beta/\!\!/W \to \check{G}_\beta$ is isomorphic to $\check{G}_\beta^{\mathrm{reg}}$, so that the right-hand side above is $\mathcal{O}_{\check{G}_\beta^{\mathrm{reg}}}$. For this, it suffices to show that the $\check{B}$-orbit of the image of $\widetilde{\kappa} : \check{T}_\beta \to \check{B}_\beta$ is isomorphic to $\check{B}_\beta^{\mathrm{reg}}$ where $\widetilde{\kappa} : \check{T}_\beta \to \check{B}_\beta/\check{B}$ is as in Definition 3.6.9. First, we claim that $\check{B}\cdot \mathrm{im}(\widetilde{\kappa}) \subseteq \check{B}_\beta^{\mathrm{reg}}$. Indeed, if $x \in \check{T}_\beta$, then for a fixed $b \in \check{B}$, there is an isomorphism $Z_{\check{B}}(b\cdot\widetilde{\kappa}(x)) \to Z_{\check{B}}(\widetilde{\kappa}(x))$ sending $g \mapsto bgb^{-1}$. It is not difficult to show that both $\check{B}\cdot \mathrm{im}(\widetilde{\kappa})$ and $\check{B}_\beta^{\mathrm{reg}}$ are flat over $\mathbf{Z}'[\beta]$ (the latter because it is an open subscheme of $\check{B}_\beta$, which is smooth over $\mathbf{Z}'[\beta]$), so it suffices to show that the inclusion $\check{B}\cdot \mathrm{im}(\widetilde{\kappa}) \subseteq \check{B}_\beta^{\mathrm{reg}}$ is an isomorphism after inverting β and setting $\beta = 0$. By Lemma 3.6.4, this is equivalent to the following pair of well-known facts: the inclusions $\check{B}\cdot\mathrm{im}(\widetilde{\kappa} : \check{T} \to \check{B}) \subseteq \check{B}^{\mathrm{reg}}$ and $\check{B}\cdot\mathrm{im}(\widetilde{\kappa} : \check{\mathfrak{t}} \to \check{\mathfrak{b}}) \subseteq \check{\mathfrak{b}}^{\mathrm{reg}}$ are isomorphisms.

Since $\check{G}_\beta$ is normal and irreducible, the desired isomorphism $\mathcal{O}_{\check{G}(-2\rho)_\beta^{\mathrm{reg}}} \cong \mathcal{O}_{\check{G}(-2\rho)_\beta}$ is therefore a consequence of the following claim (and the algebraic Hartogs lemma): the complement of $\check{G}(-2\rho)_\beta^{\mathrm{reg}} \subseteq \check{G}(-2\rho)_\beta$ is of codimension ≥ 2. This complement is flat over $\mathbf{Z}'[\beta]$, and so it suffices to check the claim after inverting β and setting $\beta = 0$. Again, this reduces to the well-known facts that the closed subschemes in $\check{G}$ and $\check{\mathfrak{g}}$ of irregular elements has complement of codimension ≥ 2. □

Remark 3.6.22. Since ku interpolates between $\mathbf{Z}$ and KU, it can be understood as encoding the β-adic filtration on KU. In the same way, the right-hand side of Theorem 3.6.21 interpolates between $\mathrm{Perf}(\check{\mathfrak{g}}[2-2\rho]/\check{G}[-2\rho])$ and $\mathrm{Perf}(\check{G}^{\mathrm{sc}}/\check{G})$. Therefore, Theorem 3.6.21 along with Lemma 3.6.2 say that the β-adic filtration on the topological/A-side corresponds to the Hochschild-Kostant-Rosenberg filtration on the free loop space of $B\check{G}$ (up to the issue of replacing $\check{G}$ with its simply-connected cover, and changing gradings by -2ρ).

The next result follows as in Theorem 3.5.23 (and is in fact simpler than Theorem 3.5.23, thanks to Construction 3.6.18 being so *ad hoc*).

Proposition 3.6.23. *Let G be a simply-laced simply-connected semisimple compact Lie group, and let $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ be a reductive spherical subgroup (so that $G_{\mathbf{C}}/H_{\mathbf{C}}$ is an affine spherical $G_{\mathbf{C}}$-variety). Let $\mathcal{M}_H$ be the graded $\mathbf{Z}[\beta]$-scheme defined in Notation 2.3.14. Let $\check{M}_\beta$ denote a "dual" affine graded $\check{G}$-variety over $\mathbf{Z}[\beta]$ equipped with a closed immersion $\kappa_{\check{M}_\beta} : \mathcal{M}_H \hookrightarrow \check{M}_\beta$ (called the β-deformed Kostant slice) such that:*

- *(a) Let $\check{J}_{X,\beta}$ denote the (possibly non-flat) group scheme $\mathcal{M}_H \times_{\check{M}_\beta/\check{G}(-2\rho)} \mathcal{M}_H$ over $\mathcal{M}_H$. Then the algebra of regular functions on $(\mathcal{M}_H \times \check{G}(-2\rho))/\check{J}_{X,\beta}$ is isomorphic to $\mathcal{O}_{\check{M}_\beta}$. (For instance, this holds by the algebraic Hartogs lemma if the $\check{G}$-orbit $\check{M}_\beta^{\mathrm{reg}}$ of the image of $\kappa_{\check{M}_\beta}$ is open with complement of codimension ≥ 2.)*

(b) There is an isomorphism of graded group schemes over $\mathcal{M}_H$:

$$\operatorname{Spec} \mathrm{ku}_*^H(\Omega(G/H)) \cong \mathcal{M}_H \times_{\check{M}_\beta/\check{G}(-2\rho)} \mathcal{M}_H = \check{J}_{X,\beta}.$$

Then there is an equivalence of $\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]$*-linear* ∞*-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}_\beta/\check{G}(-2\rho)). \tag{21}$$

Suppose, further, that there is a morphism $\mu : \check{M}_\beta/\check{G}(-2\rho) \to \check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)$ *over* $\mathbf{Z}'[\beta]$ *such that there is a commutative diagram*

$$\begin{tikzcd}
T_{H,\beta}/\!\!/W_H \arrow[r, "\kappa_{\check{M}_\beta}"] \arrow[d] & \check{M}_\beta/\check{G}(-2\rho) \arrow[d, "\mu"] \\
T_\beta/\!\!/W \arrow[r, "\kappa"] & \check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho),
\end{tikzcd}$$

so that there is an induced map

$$T_{H,\beta}/\!\!/W_H \times_{\check{M}_\beta/\check{G}(-2\rho)} T_{H,\beta}/\!\!/W_H \cong \check{J}_{X,\beta} \to T_\beta/\!\!/W \times_{\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)} T_\beta/\!\!/W.$$

If the isomorphism of (b) fits into a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathrm{ku}_*^H(\Omega(G/H)) \arrow[r, "\sim", "(b)"'] \arrow[d] & T_{H,\beta}/\!\!/W_H \times_{\check{M}_\beta/\check{G}(-2\rho)} T_{H,\beta}/\!\!/W_H \arrow[d] \\
\operatorname{Spec} \mathrm{ku}_*^G(\Omega G) \arrow[r, "\sim", "\textit{Theorem 3.2.7}"'] & T_\beta/\!\!/W \times_{\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho)} T_\beta/\!\!/W,
\end{tikzcd}$$

then the equivalence (21) *is equivariant for the left-action of* $\mathrm{Shv}_{G\times G}^{c,\mathrm{Sat}}(\mathcal{L}G; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho))$ *via Theorem 3.6.21.*

Remark 3.6.24. It would be interesting to use Proposition 3.6.23 to prove the ku-theoretic analogue of Conjecture 3.5.30 for spherical Levi subgroups of G.

4. Case-by-case analysis

The main results of this section can be summarized below in a table, refining Table 3. This table should be read as follows: there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}X;\mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}(\check{Y}/\check{G}_X) \times \text{“Normalization”}), \tag{22}$$

where $\check{Y}$ is a Hamiltonian $\check{G}_X$-variety, and the normalization term can be identified with $\mathfrak{l}_X^\wedge /\!\!/ L_X^\wedge$, where $L_X^\wedge$ is the subgroup of $\check{G}$ from **[KS17]**. This is *not* the Lie algebra of the dual Levi $\check{L}(X)$. Here, $\check{G}_X = \mathrm{SL}_2$ for each of the varieties in Table 4.

Warning. *Throughout this section, we will assume Hypothesis 3.5.21 holds for these spherical varieties.* As mentioned in Remark 3.5.22, it is possible that this hypothesis simply fails; so it should perhaps be stated at the outset that our actual goal in this section is to explicitly verify the conditions of Theorem 3.5.23, and that Hypothesis 3.5.21 *only* comes in when using Theorem 3.5.23 to prove an equivalence of ∞-categories.

Remark. Below, we will only prove the bare equivalence (22). We have not proved compatibility with the equivalence of Theorem 3.2.7, but we expect it to be possible using the second part of Theorem 3.5.23. One can check the desired compatibility in type A_n. In fact, Conjecture 3.5.30 (essentially) implies Corollary 4.2.19.

During the course of proving the equivalence (22) in types A_n, C_n, D_2, and G_2, we will in fact prove an analogue of the above equivalence for $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}X;\mathrm{ku})$; the conceptual intepretation of the results thus obtained will be discussed in a later section. The reader only interested in Theorem 3.4.13 should simply set $\beta = 0$ everywhere.

Remark. Whenever we work with integral (i.e., non-rational) coefficients, we will always invert the order of the Weyl group of G.

The proof of the equivalence (22) for the spherical varieties of Table 4 relies on the criterion of Theorem 3.5.23 (or, in the ku-theoretic case, Proposition 3.6.23). Namely, in each case, we will:

- compute the H-equivariant homology $\mathrm{H}_*^H(\Omega(G/H);\mathbf{Q})$ of $\Omega(G/H)$; and
- show that $\mathrm{Spec}\,\mathrm{H}_*^H(\Omega(G/H);\mathbf{Q})$ can be identified with the group scheme $\check{J}_X$ from Theorem 3.5.23(a).

The key input into the first part is the calculation of the H-equivariant homology of the based loop space ΩS^V of the representation spheres (i.e., one-point compactifications) of (unitary) H-representations V. This is accomplished in Corollary 4.1.17. The underlying analytic spaces of each of the rank one spherical varieties in Table 4 are either representation spheres themselves, or can be built as a quotient of a representation sphere, so Corollary 4.1.17 lets us describe the H-equivariant (ku-)homology of $\Omega(G/H)$. Once $\mathrm{H}_*^H(\Omega(G/H);\mathbf{Q})$ has been computed, the second part is rather straightforward. Indeed, the group scheme $\check{J}_X = \mathrm{Spec}\,\mathrm{H}_*^H(\Omega(G/H);\mathbf{Q})$ is quite simple in each example, and the main difficulty in checking Theorem 3.5.23(a) is in bookkeeping weights.

Remark. In applying Theorem 3.5.23, we will see that the only piece of $\mathrm{Spec}\,\mathrm{H}_H^*(*;\mathbf{Q}) \cong \check{\mathfrak{t}}_H^*(2)/\!\!/W_H \cong \check{\mathfrak{h}}^*(2)/\!\!/\check{H}$ which “interacts” with $\check{Y}/\check{G}_X$ via the Kostant section of

Name	Citation	$X = G/H$	Type	$\check{Y}$	$\check{G}_X$ grading	"Normalization"	Topological phenomenon
A_n	Corollary 4.2.19	$\mathrm{PGL}_{n+1}/\mathrm{GL}_n$	T	$T^*(2n)\mathbf{A}^2(2n,0)$	$2n\rho_{\check{G}_X}$	$\mathfrak{gl}_{n-1}[2]/\!\!/\mathrm{GL}_{n-1}$	Hopf fibration $S^1 \to \Omega\mathbf{C}P^n \to \Omega S^{2n+1}$
B_n	Theorem 4.3.1	$\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$	T	$T^*(2n)\mathbf{A}^2(4n-2,0)$	$(4n-2)\rho_{\check{G}_X}$	$\mathfrak{sp}_{2n-2}[2]/\!\!/\mathrm{Sp}_{2n-2}$	EHP sequence $S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}$
C_n	Theorem 4.4.1	$\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$	T	$T^*(4n-4)\mathbf{A}^2(4n-2,0)$	$(4n-2)\rho_{\check{G}_X}$	$(\mathfrak{sp}_2 \times \mathfrak{sp}_{2n-4})[2]/\!\!/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-4})$	Hopf fibration $S^3 \to \Omega\mathbf{H}P^{n-1} \to \Omega S^{4n-1}$
D_n	Theorem 4.5.1	$\mathrm{SO}_{2n}/\mu_2 \cdot \mathrm{SO}_{2n-1}$	G	$\mathfrak{sl}_2(2n-2-(2n-2)\rho_{\check{G}_X})$	$2n\rho_{\check{G}_X}$	$\mathfrak{spin}_{2n-3}[2]/\!\!/\mathrm{Spin}_{2n-3}$	James splitting for ΩS^{2n-1}
F_4	Theorem 4.6.1	F_4/Spin_9	T	$T^*(16)\mathbf{A}^2(22,0)$	$22\rho_{\check{G}_X}$	$\mathfrak{sp}_6[2]/\!\!/\mathrm{Sp}_6$	Exceptional Hopf fibration $S^7 \to \Omega\mathbf{O}P^2 \to \Omega S^{23}$
G_2	Theorem 4.7.1	G_2/SL_3	T	$T^*(6)\mathbf{A}^2(10,0)$	$10\rho_{\check{G}_X}$	$\mathfrak{sl}_2[2]/\!\!/\mathrm{SL}_2$	EHP sequence $S^5 \to \Omega S^6 \to \Omega S^{11}$
B_3'	Theorem 4.8.1	SO_7/G_2	G	$\mathfrak{sl}_2(6-6\rho_{\check{G}_X})$	$6\rho_{\check{G}_X}$	$\mathfrak{sp}_2[2]/\!\!/\mathrm{Sp}_2$	James splitting for ΩS^7

TABLE 4. Table of dualities for affine homogeneous rank one spherical varieties with no roots of type N. For each of these varieties, $\check{G}_X = \mathrm{SL}_2$. The column labeled by $\check{Y}$ should be thought of encoding the cotangent bundle to the dual Hamiltonian $\check{G}$-variety to X. The dual group $\check{G}_X$ is equipped with the grading via the cocharacter specified above. The normalization term is of the form $\check{\mathfrak{l}}^*[2]/\!\!/\check{L}$, where $\check{\mathfrak{l}}$ is the Lie algebra of the subgroup $L_X^\wedge$ of $\check{G}$ from [**KS17**]; in particular, see [**KS17**, Final column of Table 3] for the groups $L_X^\wedge$ in the present case.
We have also included some topological phenomena corresponding to each of the rank one affine homogeneous spherical varieties. Most of these are standard, except perhaps for the "exceptional Hopf fibration" for the octonionic projective plane, which is proved as [**DM81**, Theorem 1.2]. The point is that these homotopy-theoretic aspects of ΩX control essentially all of the properties of the Langlands dual of X. For instance, if $S^i \to \Omega X \to \Omega S^j$ is a fibration as in the table with both i and j being odd, the graded scheme $\check{Y}$ can be identified with $T^*(j-i)\mathbf{A}^2(j-1,0)$; and, if $\Omega X = \Omega S^{2j+1}$, the graded scheme $\check{Y}$ can be identified with $\mathfrak{sl}_2(2j-2j\rho_{\check{G}_X})$.

Theorem 3.5.23 is the coordinate corresponding to the highest degree fundamental invariant of W_H acting on $\check{\mathfrak{t}}_H^*(2)$. The remainder of $\check{\mathfrak{t}}_H^*(2)/\!\!/W_H$ makes up the normalization term $\mathfrak{l}_X^\wedge/\!\!/L_X^\wedge$.

Since this section is somewhat technical, let us make an observation about the main qualitative difference between root types T and G which appears in the course of the proof. Let us focus on the prototypical cases of types A_1 (which is of root type T) and D_2 (which is of root type G). In these cases, there is no normalization term; in general, this normalization term comes from cohomology classes in $\mathrm{H}^*(BG;\mathbf{Q})$ which do not "interact" with $\mathrm{H}_*(\Omega(G/H);\mathbf{Q})$.

(a) In type A_1, there are isomorphisms of derived $\mathbf{Q}$-schemes

$$\begin{aligned}\operatorname{Spec}\mathrm{sh}^{1/2}\mathrm{H}_*^{\mathrm{SO}_2}(\Omega(\mathrm{SO}_3/\mathrm{SO}_2);\mathbf{Q}) &\cong \operatorname{Spec}\mathbf{Q}[b,x]/bx \\ &\cong \mathbf{A}^1[2] \times_{T^*[2]\mathbf{A}^2[2,0]/\mathrm{SL}_2[-2\rho_{\mathrm{SL}_2}]} \mathbf{A}^1[2],\end{aligned}$$

where b lives in degree 2, and x lives in degree -2. Let $\check{J}_X$ denote the above group scheme. Then $\check{J}_X$ is *not* flat over $\mathbf{A}^1[2] = \operatorname{Spec}\mathbf{Q}[x]$.

The nonflatness of $\check{J}_X$ over $\check{\mathfrak{h}}^*[2]/\!\!/\check{H}$ is characteristic of the case of roots of type T. Topologically, this corresponds to the observation that the H-invariant subspace $(G/H)^H$ is just S^0. (In general, one can use

Atiyah-Bott localization and Theorem 3.5.23 to see that the phenomenon of $\Omega(G/H)^H$ being rationally contractible is Langlands dual to the triviality of the stabilizer in $\check{G}$ of a generic point in $\check{M}$.)

(b) In type D_2, there is an isomorphism $\mathrm{SO}_4/\mu_2 \cong \mathrm{SO}_3 \times \mathrm{SO}_3$, so that $\mathrm{SO}_4/\mu_2\mathrm{SO}_3 \simeq \mathrm{SO}_3$. Then, there are isomorphisms of derived $\mathbf{Q}$-schemes

$$\begin{aligned}\operatorname{Spec} \mathrm{sh}^{1/2}\mathrm{H}_*^{\mathrm{SO}_3}(\Omega(\mathrm{SO}_4/\mu_2\mathrm{SO}_3); \mathbf{Q}) &\cong \operatorname{Spec} \mathrm{sh}^{1/2}\mathrm{H}_*^{\mathrm{SO}_3}(\Omega\mathrm{SO}_3; \mathbf{Q}) \\ &\cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, \tfrac{a-a^{-1}}{2x}]^{\mathbf{Z}/2} \\ &\cong \mathbf{A}^1[2]/\!\!/(\mathbf{Z}/2) \times_{\mathfrak{sl}_2[2-2\rho_{\mathrm{SL}_2}]/\mathrm{SL}_2[-2\rho_{\mathrm{SL}_2}]} \mathbf{A}^1[2]/\!\!/(\mathbf{Z}/2),\end{aligned}$$

where the map $\mathbf{A}^1[2]/\!\!/(\mathbf{Z}/2) \to \mathfrak{sl}_2[2-2\rho_{\mathrm{SL}_2}]/\mathrm{SL}_2[-2\rho_{\mathrm{SL}_2}]$ is the Kostant slice sending $x^2 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ x^2 & 0 \end{smallmatrix}\right)$. Let $\check{J}_X$ denote the above group scheme. Then $\check{J}_X$ is flat over $\mathbf{A}^1[2]/\!\!/(\mathbf{Z}/2) = \operatorname{Spec} \mathbf{Q}[x^2]$, and can be identified with the group scheme of regular centralizers of $\check{G}_X = \mathrm{SL}_2$.

The identification of $\check{J}_X$ over $\check{\mathfrak{h}}^*[2]/\!\!/\check{H}$ with (the product of the normalization factor with) the group scheme of regular centralizers of SL_2 is characteristic of the case of roots of type G. Topologically, this corresponds to the observation that the T-invariant subspace $(G/H)^T$ is just S^1 (where T is a maximal torus of H).

As the reader will see, the calculations of this section are quite repetitive; it is possible to handle all the type T and type G cases simultaneously, but at the risk of confusing oneself with various gradings (in other words, the repetitive nature of this section is mostly for my purposes, and one can conglomerate these calculations into a more uniform argument if desired).

4.1. Homology of loops on a sphere. The proof of Theorem 3.4.13 rests on a key topological calculation, namely that of the equivariant homology $\mathrm{ku}_*^T(\Omega S^V)$ for S^V being the one-point compactification of a (unitary) representation V of a (compact) torus T. To illustrate this calculation, let us begin with two simple (but exemplifying) examples.

Notation 4.1.1. Recall the group scheme $\mathbf{G}_\beta$ from Construction 2.3.6, i.e., the graded group scheme over $\mathbf{Z}'[\beta]$ given by $\operatorname{Spec} \mathbf{Z}'[\beta, t^{\pm 1}, \frac{t-1}{\beta}]$ where β lives in weight 2, and with coproduct determined by the formula $t \mapsto t \otimes t$. The invertible class t defines a homomorphism $\mathbf{G}_\beta \to (\mathbf{G}_m)_{\mathbf{Z}'[\beta]}$ which exhibits $\mathbf{G}_\beta$ as an affine blowup of $(\mathbf{G}_m)_{\mathbf{Z}'[\beta]}$ at the identity of $\mathbf{G}_m$. The kernel of this map is given by a group scheme $\mathbf{G}_\beta^0$, whose underlying graded $\mathbf{Z}'[\beta]$-scheme is isomorphic to $\mathbf{A}^1(-2) = \operatorname{Spec} \mathbf{Z}'[\beta, \frac{t-1}{\beta}]$, and whose group structure is given by

$$\tfrac{t-1}{\beta} \mapsto \tfrac{t-1}{\beta} \otimes 1 + 1 \otimes \tfrac{t-1}{\beta} + \beta \tfrac{t-1}{\beta} \otimes \tfrac{t-1}{\beta}.$$

A compact torus T defines a group scheme T_β given by $\mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G}_\beta)$, and hence a subgroup scheme $T_\beta^0 = \ker(T_\beta \to T)$. Let $\lambda : T \to S^1$ be a character, let T_λ denote its kernel, and let $n \geq 0$. Define $T_{\lambda,\beta}^{[n]}$ to be the closed subscheme of T_β given by the nth infinitesimal neighborhood of $T_{\lambda,\beta}$; similarly for $T_{\lambda,\beta}^{0,[n]}$. Let $x_\lambda \in \mathcal{O}_{T_\beta}$ denote the function which cuts out $T_{\lambda,\beta}$.

Since $\mathbf{G}_\beta = \operatorname{Spec} \mathbf{Z}'[\beta, t^{\pm 1}, \frac{t-1}{\beta}]$, if we call $x = \frac{t-1}{\beta}$ (so that x lives in weight -2), this ring can be identified with $\operatorname{Spec} \mathbf{Z}'[\beta, x, \frac{1}{1+\beta x}]$. It will often be more convenient to consider this presentation of $\mathbf{G}_\beta$. For $n \in \mathbf{Z}$, we will write $[n](x)$ to denote the n-fold sum of x in the group structure on $\mathbf{G}_\beta$, so that $[n](x) = \frac{(1+\beta x)^n - 1}{\beta}$.

Example 4.1.2. Let std denote the standard 1-dimensional complex representation of S^1, and consider the one-point compactification S^{std} (so that its underlying nonequivariant space is S^2). We will be interested in computing the (for the moment) Borel-equivariant homology $\mathrm{H}_*^{S^1}(\Omega S^{\mathrm{std}}; \mathbf{Z}) = \pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}$. There is a homotopy fixed points spectral sequence

$$E_2^{*,*} \cong \mathrm{H}_*(\Omega S^{\mathrm{std}}; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{H}^*(\mathbf{C}P^\infty; \mathbf{Z}) \Rightarrow \pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}.$$

The E_2-page can be computed easily to be

$$E_2^{*,*} \cong \mathbf{Z}[a, b][\![x]\!]/a^2,$$

where $x \in \mathrm{H}^2(\mathbf{C}P^\infty; \mathbf{Z})$ is the first Chern class, $a \in \mathrm{H}_1(\Omega S^{\mathrm{std}}; \mathbf{Z})$ coming from the inclusion $S^1 \subseteq \Omega S^2$, and $b \in \mathrm{H}_2(\Omega S^{\mathrm{std}}; \mathbf{Z})$ coming from the map $S^2 \to \Omega S^2$ adjoint to the Hopf fibration $S^3 \to S^2$. There is a single differential $d_2(a) = bx$ (if std is replaced by $\mathrm{std}^{\otimes_{\mathbf{C}} n}$ for some $n \geq 1$, this differential is simply replaced by nbx). After running this differential, the spectral sequence is concentrated in even degrees, and we find that

$$\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1} \cong \mathbf{Z}[b][\![x]\!]/bx.$$

Compare to Example 3.5.11(b) with $n = 1$. Exactly the same calculation holds with $\mathbf{Z}$ replaced by ku:

$$\pi_* \mathrm{ku}[\Omega S^{\mathrm{std}^{\otimes_{\mathbf{C}} n}}]^{hS^1} \cong \mathbf{Z}[\beta, b][\![x]\!]/b[n](x).$$

Remark 4.1.3. Note that the quotient $\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}/x$ is precisely the nonequivariant homology $\mathrm{H}_*(\Omega S^{\mathrm{std}}; \mathbf{Z})$. Indeed, Example 4.1.2 says that the class b is x-torsion in $\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}$; therefore, if we kill x, the class bx in degree zero bumps up to a class $\sigma(bx)$ in degree 1.

One interesting observation is that the homotopy quotient $\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]_{hS^1}$ (which would compute what is traditionally called equivariant homology) is *not* concentrated in even degrees: namely, the fact that b is x-torsion implies that the $\mathbf{Z}[\![x]\!]$-linear dual of $\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}$ will have odd homotopy groups.

Let us mention that the fact that $\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}}]^{hS^1}$ is concentrated in even degrees is an absolutely crucial fact (related to the subtleties of Lemma 2.1.5), which has important implications in the Langlands duality of Theorem 3.4.13, and emphasizes the role of equivariance in our discussion.

Example 4.1.4. Again, let std denote the standard 1-dimensional complex representation of S^1, and consider the one-point compactification $S^{\mathrm{std} \oplus \mathbf{R}}$ (so that its underlying nonequivariant space is S^3). We will be interested in computing the (for the moment) Borel-equivariant homology $\mathrm{H}_*^{S^1}(\Omega S^{\mathrm{std} \oplus \mathbf{R}}; \mathbf{Z}) = \pi_* \mathbf{Z}[\Omega S^{\mathrm{std} \oplus \mathbf{R}}]^{hS^1}$. There is a homotopy fixed points spectral sequence

$$E_2^{*,*} \cong \mathrm{H}_*(\Omega S^{\mathrm{std} \oplus \mathbf{R}}; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{H}^*(\mathbf{C}P^\infty; \mathbf{Z}) \Rightarrow \pi_* \mathbf{Z}[\Omega S^{\mathrm{std} \oplus \mathbf{R}}]^{hS^1}.$$

Now, the E_2-page is simply $\mathbf{Z}[b][\![x]\!]$, where again $x \in \mathrm{H}^2(\mathbf{C}P^\infty; \mathbf{Z})$ is the first Chern class, and $b \in \mathrm{H}_2(\Omega S^3; \mathbf{Z})$ is the generator. The entire spectral sequence is concentrated in even degrees, so there can be no differentials, and the spectral sequence degenerates. This implies that

$$\pi_* \mathbf{Z}[\Omega S^{\mathrm{std}\oplus\mathbf{R}}]^{hS^1} \cong \mathbf{Z}[b][\![x]\!].$$

Compare to Example 3.5.11(a) with $n = 1$.

Remark 4.1.5. Note that in Example 4.1.4, the class bx is topologically nilpotent, so that $1+bx$ is invertible. The inclusion $S^{\mathbf{R}} \subseteq S^{\mathrm{std}\oplus\mathbf{R}}$ induces a map $\mathbf{Z} \cong \Omega S^{\mathbf{R}} \to \Omega S^{\mathrm{std}\oplus\mathbf{R}}$; this map is in fact just the inclusion of the S^1-fixed points of $\Omega S^{\mathrm{std}\oplus\mathbf{R}}$. Therefore, there is a map $\mathbf{Z}[\Omega S^{\mathbf{R}}] \cong \mathbf{Z}[a^{\pm 1}] \to \mathbf{Z}[\Omega S^{\mathrm{std}\oplus\mathbf{R}}]^{hS^1}$, and it is not difficult to check that this map sends $a \mapsto 1 + bx$.

In order to describe the main calculation, we need to introduce some notation.

Setup 4.1.6. Let T be a *compact* torus, and let V be a (unitary) representation of T with no nonzero fixed vectors. Let $\Lambda(V)$ denote the set of weights of V, and let $\chi_V : T \to S^1$ to denote the character of V. Moreover, if $\lambda : T \to S^1$ is a character, we will write T_λ to denote the kernel of λ. Note that $\langle \chi_V, \lambda\rangle$ is the dimension of the λ-weight space of V (as a complex vector space).

Setup 4.1.7. Throughout, when we talk about coefficients in a homology theory, we will *invert the prime* 2. This will be implicit in the notation, and we will write $\mathbf{Z}'$ to denote $\mathbf{Z}[1/2]$.

Definition 4.1.8. In Setup 4.1.6, fix an integer $j \in \mathbf{Z}$, and define $\mathcal{C}_V$ to be the graded $\mathbf{Z}'[\beta]$-scheme given by the union

$$\mathcal{C}_V(-j) = (T^0_\beta \times \{0\}) \cup \bigcap_{\lambda\in\Lambda(V)} (T^{0,[\langle \chi_V,\lambda\rangle]}_{\lambda,\beta} \times \mathbf{A}^1(-j)).$$

We will call $\mathcal{C}_V(-j)$ the *V-coordinate axes with weight j*.

Definition 4.1.9. In Setup 4.1.6, consider the blowup

$$X := \mathrm{Bl}\left(\bigcap_{\lambda\in\Lambda(V)} (T^{[\langle \chi_V,\lambda\rangle]}_{\lambda,\beta} \times \{1\}) \subseteq T_\beta \times \mathbf{G}_m\right).$$

Let $\mathcal{B}_V$ denote the complement of the proper preimage of $\{0_{T_\beta}\} \times \mathbf{G}_m$ from X, so that $\mathcal{B}_V$ is an affine blowup of $T_\beta \times \mathbf{G}_m$.

Example 4.1.10. Suppose $T = S^1$, and let V denote the weight n representation of T. Then there is an isomorphism

$$\mathcal{C}_V(-j) \cong \operatorname{Spec} \mathbf{Z}'[\beta, x, b]/bx,$$

where x lives in weight -2 and b lives in weight j. This is the reason for the terminology of Definition 4.1.8: we are more concerned with the weight of the function b on $\mathcal{C}_V(-j)$. Similarly, there is an isomorphism

$$\mathcal{B}_V \cong \operatorname{Spec} \mathbf{Z}'[\beta, t^{\pm 1}, \tfrac{t-1}{\beta}, a^{\pm 1}, \tfrac{(a-1)\beta}{t-1}] \cong \operatorname{Spec} \mathbf{Z}'[\beta, x, \tfrac{1}{1+\beta x}, a^{\pm 1}, \tfrac{a-1}{x}],$$

where t lives in weight 0 (so $\frac{t-1}{\beta}$ lives in weight 2), and a lives in weight 0 (so $\frac{(a-1)\beta}{t-1}$ lives in weight 2).

Example 4.1.11. More generally, let $T = (S^1)^m$, and let V denote the representation $\bigoplus_{i=1}^m d_i \mathrm{std}_i^{\otimes_{\mathbf{C}} n_i}$. Then

$$\mathcal{C}_V(-j) \cong \operatorname{Spec} \mathbf{Z}'[\beta, x_1, \cdots, x_m, \prod_{i=1}^m \tfrac{1}{1+\beta x_i}, b]/b \prod_{i=1}^m [n_i](x_i)^{d_i},$$

with each x_i in weight -1 and b in weight j. Similarly, we have

$$\mathcal{B}_V \cong \operatorname{Spec} \mathbf{Z}'[\beta, x_1, \cdots, x_m, \prod_{i=1}^m \tfrac{1}{1+\beta x_i}, a^{\pm 1}, \prod_{i=1}^m \tfrac{a-1}{[n_j](x_j)^{d_j}}],$$

where each x_i lives in weight -1, and a lives in weight 0 (so $\prod_{i=1}^m \frac{a-1}{[n_j](x_j)^{d_j}}$ lives in weight $2\sum_{i=1}^m d_j = \dim_{\mathbf{R}}(V)$). This immediately implies:

Lemma 4.1.12. *In Setup 4.1.6, the fiber of the projection map $\mathcal{B}_V \to \mathbf{G}_m$ over $\{1\} \in \mathbf{G}_m$ is isomorphic to $\mathcal{C}_V(-\dim_{\mathbf{R}}(V))$.*

Theorem 4.1.13. *In Setup 4.1.6, let S^V denote the one-point compactification of V. Then there are graded isomorphisms of $\pi_* \mathrm{ku}_T$-algebras*

$$\mathrm{ku}_*^T(\Omega S^V) \cong \mathcal{O}_{\mathcal{C}_V(2-2\dim_{\mathbf{R}}(V))},$$
$$\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}}) \cong \mathcal{O}_{\mathcal{B}_V}.$$

In particular, both $\mathrm{ku}_*^T(\Omega S^V)$ and $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}})$ are concentrated in even weights, and are graded commutative $\pi_* \mathrm{ku}_T$-algebras.

Remark 4.1.14. If T acts on a pointed space X (and we are given some multiplicative presentation of ΩX as a T-space), the equivariant homology $\mathrm{ku}_*^T(\Omega X)$ need not be a commutative $\mathbf{Z}'[\beta]$-algebra in general: *a priori*, it is only an associative $\mathbf{Z}'[\beta]$-algebra, since ΩX generally only admits the structure of an $\mathbf{E}_1$-space. Although ΩS^V is still generally only an $\mathbf{E}_1$-space (unless V is isomorphic to $\emptyset$, $\mathbf{R}$, or $\mathbf{R}^3$), Theorem 4.1.13 implies that $\mathrm{ku}_*^T(\Omega S^V)$ and $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}})$ are in fact concentrated in even weights, and generated (as a $\pi_* \mathrm{ku}_T$-algebra) by a single class. In particular, it is necessarily a commutative $\mathbf{Z}'[\beta]$-algebra[18]. We will implicitly use this observation throughout this article, by rewriting Theorem 4.1.13 as a pair of graded isomorphisms of $\mathbf{Z}'[\beta]$-schemes

$$\operatorname{Spec} \mathrm{ku}_*^T(\Omega S^V) \cong \mathcal{C}_V(2 - 2\dim_{\mathbf{R}}(V)),$$
$$\operatorname{Spec} \mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}}) \cong \mathcal{B}_V.$$

Proof of Theorem 4.1.13. Let us first compute the *Borel* T-equivariant ku-homology of ΩS^V. Write $V = \mathbf{C}^n$; then there is a homotopy fixed points spectral sequence

$$E_2^{*,*} \cong \pi_* \mathrm{ku}^{hT} \otimes_{\mathbf{Z}'[\beta]} \mathrm{ku}_*(\Omega S^{\mathbf{C}^n}) \Rightarrow \pi_* \mathrm{ku}[\Omega S^V]^{hT}.$$

The EHP sequence for ΩS^V is the fibration given by

$$S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}.$$

Note that the map $S^{2n-1} \to \Omega S^{2n} \simeq \Omega\Sigma S^{2n-1}$ is *not* defined equivariantly: the unit sphere $S(V)$ does not have T-fixed points (by assumption on V), so there is no T-equivariant basepoint of $S(V)$ with respect to which the reduced suspension

[18]This is a manifestation of the fact that in classical algebra, an associative algebra being commutative is a *property*, whereas in homotopy theory, it is more *structure*.

can be constructed. In any case, the EHP sequence splits after inverting the prime 2 (this is the reason for Setup 4.1.7), which implies that there is an isomorphism

$$\mathrm{ku}_*(\Omega S^{2n}) \cong \mathbf{Z}'[\beta, a, b]/a^2,$$

where $|a| = 2n-1$ and $|b| = 4n-2$. (See, e.g., Example 3.5.11 for a massively overblown derivation of this isomorphism.)

Remark 4.1.15. That the EHP sequence splits after inverting the prime 2 goes all the way back to Serre: on [**Ser53**, p. 281], he showed that there is a equivalence

$$\mathrm{can} \times \Omega[\iota_{2n}, \iota_{2n}] : S^{2n-1} \times \Omega S^{4n-1} \to \Omega S^{2n}$$

after inverting 2. Here, $[\iota_{2n}, \iota_{2n}] \in \pi_{4n-1}(S^{2n})$ denotes the Whitehead product of $\iota_{2n} \in \pi_{2n}(S^{2n})$ with itself.

It follows that the E_2-page can be identified with

$$E_2^{*,*} \cong \mathcal{O}_{T_\beta}^\wedge[a, b]/a^2.$$

Each weight $\lambda \in \Lambda(V)$ defines a function $x_\lambda \in \mathcal{O}_{T_\beta}$, and there is a single differential in this spectral sequence, given by

$$d_{\dim_{\mathbf{R}}(V)}(a) = b \prod_{\lambda \in \Lambda(V)} x_\lambda. \tag{23}$$

One can see this by reducing to the case when V is one-dimensional, in which case (23) follows from Example 4.1.2. The $E_{\dim_{\mathbf{R}}(V)+1}$-page of the spectral sequence is then concentrated entirely in even degrees, and therefore degenerates. This implies that

$$\pi_*\mathrm{ku}[\Omega S^V]^{hT} \cong \mathcal{O}_{T_\beta}^\wedge[b]/b \prod_{\lambda \in \Lambda(V)} x_\lambda. \tag{24}$$

Note that this is nearly $\mathcal{O}_{\mathcal{C}_V(2-2\dim_{\mathbf{R}}(V))}$, except for the completion.

The above calculation of $\pi_*\mathrm{ku}[\Omega S^V]^{hT}$ is enough to imply the desired calculation of $\mathrm{ku}_*^T(\Omega S^V)$. Indeed, let e_j be a given basis vector of $\mathbb{X}^*(T)$, let T_j denote the kernel of $e_j : T \to S^1$, and let V_j denote the fixed locus V^{T_j}. Let $\mathcal{U}_j$ denote the complement of the union of the closed subschemes T'_β ranging over all closed subgroups $T' \subseteq T$ which do not contain T_j. Then Lemma 2.4.6 gives an isomorphism

$$\mathrm{ku}_*^T(\Omega S^V)|_{\mathcal{U}_j} \simeq \mathrm{ku}_*^T(\Omega S^{V_j})|_{\mathcal{U}_j}. \tag{25}$$

Indeed, since $(\Omega S^V)^{T_j} \simeq \Omega(S^V)^{T_j}$, it suffices to note that $(S^V)^{T_j} \simeq S^{V_j}$. Using the fracture square and induction on the dimension of V, one finds that there is a Cartesian square

$$\begin{tikzcd}
\mathrm{ku}_*^T(\Omega S^V) \arrow[r] \arrow[d] & \pi_*\mathrm{ku}[\Omega S^V]^{hT} \arrow[d] \\
\pi_*\mathrm{ku}_T|_{T_\beta - \bigcup_{1\le j\le m} T_{j,\beta}} \arrow[r] & \pi_*\mathrm{ku}^{tT}.
\end{tikzcd}$$

This precisely has the effect of correcting the completion in (24), which recovers $\mathcal{O}_{\mathcal{C}_V(2-2\dim_{\mathbf{R}}(V))}$.

Turning to $\Omega S^{V\oplus\mathbf{R}}$, let us first compute the *Borel* T-equivariant ku-homology of $\Omega S^{V\oplus\mathbf{R}}$. Write $V=\mathbf{C}^n$; then there is a homotopy fixed points spectral sequence

$$E_2^{*,*}\cong \pi_*\mathrm{ku}^{hT}\otimes_{\mathbf{Z}'[\beta]}\mathrm{ku}_*(\Omega S^{\mathbf{C}^n\oplus\mathbf{R}})\Rightarrow \pi_*\mathrm{ku}[\Omega S^{V\oplus\mathbf{R}}]^{hT}.$$

There is an isomorphism $\mathrm{ku}_*(\Omega S^{\mathbf{C}^n\oplus\mathbf{R}})\cong \mathrm{ku}_*[b]$, where $|b|=2n$. This implies that the entire spectral sequence is concentrated in even degrees, so there are no differentials, and we find that

$$\pi_*\mathrm{ku}[\Omega S^{V\oplus\mathbf{R}}]^{hT}\cong \mathbf{Z}'[\beta,b][\![x_1,\cdots,x_n]\!]. \tag{26}$$

To compute $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}})$, we will again use the fracture square. Again, it is not difficult to reduce to the case when $T=S^1$ and $V=\mathrm{std}^{\otimes\mathbf{C}n}$. Then there is a Cartesian square

$$\begin{tikzcd}
\mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}}) \arrow[r] \arrow[d] & \mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}})^\wedge_x \arrow[d] \\
\mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}})[x^{-1}] \arrow[r] & \mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}})^\wedge_x[x^{-1}].
\end{tikzcd}$$

Note that the S^1-fixed point set $(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}})^{S^1}$ is simply $\Omega S^1\cong\mathbf{Z}$, so that Lemma 2.4.6 lets us identify the bottom-left corner with $\pi_*\mathrm{ku}_T[a^{\pm1}]$. By (26), we can identify the above Cartesian square with

$$\begin{tikzcd}
\mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}}) \arrow[r] \arrow[d] & \mathbf{Z}'[\beta,b][\![x]\!] \arrow[d] \\
\mathbf{Z}'[\beta,a^{\pm1},x^{\pm1},\frac{1}{1+\beta x}] \arrow[r] & \mathbf{Z}'[\beta,b](\!(x)\!),
\end{tikzcd}$$

where the bottom map sends $a\mapsto 1+bx$. It follows that

$$\mathrm{ku}_*^{S^1}(\Omega S^{\mathrm{std}^{\otimes\mathbf{C}n}\oplus\mathbf{R}})\cong \mathbf{Z}'[\beta,x,\tfrac{1}{1+\beta x},a^{\pm1},\tfrac{a-1}{x}]\cong \mathcal{O}_{\mathcal{B}_V},$$

where $\frac{a-1}{x}\mapsto b$. □

Remark 4.1.16. The evenness of Theorem 4.1.13 is always true for $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}})$ (in fact, more generally for $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}^{2n+1}})$ for any $n\geq 0$), but it is *not* true for $\mathrm{ku}_*^T(\Omega S^{V\oplus\mathbf{R}^{2n}})$ if $n>0$.

The following result is an immediate consequence of Theorem 4.1.13 and Proposition 2.3.15.

Corollary 4.1.17. *Let G be a connected compact Lie group whose π_1 is torsion-free, and let $T\subseteq G$ be a maximal torus with associated Weyl group W. If V is a (unitary) G-representation with no nonzero T-fixed vectors, there are graded isomorphisms of $\pi_*\mathrm{ku}_G$-schemes*

$$\operatorname{Spec}\mathrm{ku}_*^G(\Omega S^V)\cong \mathcal{C}_V(2-2\dim_{\mathbf{R}}(V))/\!\!/W,$$
$$\operatorname{Spec}\mathrm{ku}_*^G(\Omega S^{V\oplus\mathbf{R}})\cong \mathcal{B}_V/\!\!/W.$$

Remark 4.1.18. If H is a compact Lie group with maximal torus T and V is a (unitary) H-representation with no nonzero T-fixed vectors, then we implicitly view $\mathrm{ku}_*^H(\Omega S^V)$ as a commutative algebra over $\pi_*\mathrm{ku}_H$ as in Remark 4.1.14. In the case-by-case analysis below, this is in fact not as abusive as it might seem:

namely, if G is a compact Lie group and $H \subseteq G$ is a closed inclusion of subgroups such that $S^V \cong G/H$ as H-equivariant spaces, the natural $\mathbf{E}_1$-algebra structure on the ku_H-linear dual of $\mathcal{F}_H(\Omega S^V)$ in fact upgrades to an $\mathbf{E}_2$-algebra structure via Corollary 3.5.8. In particular, the commutative ring structure on the completion of $\mathrm{ku}_*^H(\Omega S^V)$ does indeed have a homotopical origin.

4.2. Type A_n. Our goal in this section is to prove Theorem 3.4.13 in type A_n, i.e., for the spherical GL_{n+1}-variety $\mathrm{GL}_{n+1}/\mathrm{GL}_n$. We will write $G = \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$, so $\check{G} = \mathrm{GL}_{n+1}$. Recall from Table 3 that $\check{G}_X = \mathrm{GL}_2$. Equip GL_2 with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight zero, b lives in weight $2n$, and c lives in weight $-2n$; we will write $\mathrm{GL}_2(-2n\rho_{\check{G}_X})$ to denote this graded group.

Theorem 4.2.1 (Theorem 3.4.13 in type A_n)**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q}) \simeq \mathrm{Perf}(T^*[2n](\mathbf{A}^2[2n,0])/\mathrm{GL}_2[-2n\rho_{\check{G}_X}] \times \mathfrak{gl}_{n-1}[2]/\!\!/\mathrm{GL}_{n-1}).$$

Example 4.2.2. For instance, if $n = 1$, we have $\check{G}_X = \check{G}$, and so Theorem 4.2.1 states that there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_{\mathrm{GL}_2}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{GL}_2/\mathbf{G}_m);\mathbf{Q}) \cong \mathrm{Perf}(T^*[2](\mathbf{A}^2[2,0])/\mathrm{GL}_2[-2\rho_{\check{G}_X}]).$$

The Koszul dual of this statement is also proved as [**BF22**, Theorem 1.8(2)].

Remark 4.2.3. There is a relationship between Theorem 4.2.1 and the mirabolic Satake equivalence of [**BFGT21**], which studies the spectral decomposition of the spherical $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$-variety $(\mathrm{GL}_n \times \mathrm{GL}_{n-1})/\mathrm{GL}_{n-1} \simeq \mathrm{GL}_n$. In our language, their main result states that there is an equivalence

$$\mathrm{Shv}_{\mathrm{U}(n)\times\mathrm{U}(n-1)}^{c,\mathrm{Sat}}(\mathcal{L}\mathrm{U}(n);\mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{GL}_n\backslash T^*(\mathrm{Hom}(\mathbf{A}^n,\mathbf{A}^{n-1}))/\mathrm{GL}_{n-1}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ denotes the ∞-category of perfect complexes on a shearing. (We have omitted the precise gradings for brevity.) If $n = 2$, the above equivalence specializes to

$$\mathrm{Shv}_{\mathrm{U}(2)\times\mathrm{U}(1)}^{c,\mathrm{Sat}}(\mathcal{L}\mathrm{U}(2);\mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{GL}_2\backslash T^*\mathbf{A}^2/\mathbf{G}_m),$$

and forgetting the $\mathbf{G}_m$-quotient on the coherent side is equivalent to extending $\mathrm{U}(2)\times\mathrm{U}(1)$-equivariance to $\mathrm{U}(2)\times\mathcal{L}\mathrm{U}(1)$-equivariance on the left-hand side; this in turn recovers Theorem 4.2.1 for the spherical GL_2-variety $\mathrm{GL}_2/\mathbf{G}_m$.

If $n > 2$, then using the equivalence

$$\mathrm{Shv}_{\mathrm{U}(n-1)}^{c,\mathrm{Sat}}(*;\mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{gl}_{n-1}^*/_\psi N_{n-1})$$

with N_{n-1} being the unipotent radical of a Borel subgroup of $\mathfrak{gl}_{n-1}$ and ψ being a nondegenerate character on its Lie algebra, the mirabolic Satake equivalence implies that

$$\mathrm{Shv}_{\mathrm{U}(n)}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{U}(n)/\mathrm{U}(n-1));\mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{GL}_n\backslash T^*(\mathrm{Hom}(\mathbf{A}^n,\mathbf{A}^{n-1}))/_\psi N_{n-1}).$$

Justin Hilburn has shown using the results of [**NT17**] that there is an isomorphism of stacks

$$\mathrm{GL}_n\backslash T^*(\mathrm{Hom}(\mathbf{A}^n,\mathbf{A}^{n-1}))/_\psi N_{n-1} \cong T^*(\mathbf{A}^2)/\mathrm{GL}_2 \times \mathfrak{gl}_{n-2}/\!\!/\mathrm{GL}_{n-2},$$

which shows that Theorem 4.2.1 is in fact a consequence of the mirabolic Satake equivalence of [**BFGT21**].

Remark 4.2.4. One can use the analogue of Theorem 4.2.1 for sheaves with coefficients in $\mathbf{Z}$ to show that there is an equivalence

$$\mathrm{Loc}(\Omega(\mathrm{U}(n{+}1)/\mathrm{U}(n));\mathbf{Z}) \simeq \mathrm{Perf}(\{0\}\times_{\mathfrak{gl}_{n+1}[2]/\!\!/\mathrm{GL}_{n+1}}(T^*[2n](\mathbf{A}^2[2n,0])/\mathrm{GL}_2[-2n\rho_{\check{G}_X}]\times\mathfrak{gl}_{n-1}[2]/\!\!/\mathrm{GL}_{n-1}))$$

describing (derived) *local systems* on $\Omega(\mathrm{U}(n{+}1)/\mathrm{U}(n))$. Here, the map to $\mathfrak{gl}_{n+1}[2]/\!\!/\mathrm{GL}_{n+1}$ is via the moment map. However, one can compute that there is an isomorphism

$$(T^*(\mathbf{A}^2)/\mathrm{GL}_2 \times \mathfrak{gl}_{n-1}/\!\!/\mathrm{GL}_{n-1}) \times_{\mathfrak{gl}_{n+1}/\!\!/\mathrm{GL}_{n+1}} \{0\} \cong B\mathbf{G}_a,$$

where we have ignored shifts for notational simplicity. Putting shifts back in, we conclude that there is a Fourier equivalence

$$\mathrm{Loc}(\Omega(\mathrm{U}(n+1)/\mathrm{U}(n));\mathbf{Z}) \simeq \mathrm{Perf}(B\mathbf{G}_a[-2n]); \tag{27}$$

this equivalence is not very difficult to prove directly using Koszul duality, but it is satisfying to see the right-hand side fall out of Theorem 4.2.1. This equivalence sends the skyscraper sheaf at the basepoint of $\Omega(\mathrm{U}(n+1)/\mathrm{U}(n))$ to the structure sheaf of $B\mathbf{G}_a[-2n]$.

Using the equivalence (27) to compute endomorphisms of the skyscraper sheaf at the basepoint of $\Omega(\mathrm{U}(n+1)/\mathrm{U}(n))$, we find that there is an isomorphism

$$\pi_*\mathbf{Z}[\Omega^2 S^{2n+1}] \cong \pi_*\mathrm{sh}^{1/2}\Gamma(B\mathbf{G}_a(-2n);\mathcal{O}).$$

More generally, one can show that there is an equivalence of $\mathbf{E}_{n+1}$-$\mathbf{Z}$-algebras

$$\mathbf{Z}[\Omega^{n+1}S^{j+1}] \simeq \mathrm{sh}^{1/2}\Gamma(B^n\mathbf{G}_a(-j);\mathcal{O}); \tag{28}$$

this follows from the fact that $\mathbf{Z}[\Omega^{n+1}S^{j+1}]$ is the free $\mathbf{E}_{n+1}$-$\mathbf{Z}$-algebra on a class in degree $j-n$, hence is the shearing $\mathrm{sh}^{1/2}\mathrm{Free}_{\mathbf{E}_{n+1}}(\Sigma^{-n}\mathbf{Z}(j))$; but there is an equivalence $\mathrm{Free}_{\mathbf{E}_{n+1}}(\Sigma^{-n}\mathbf{Z}(j)) \simeq \mathrm{LSym}_{\mathbf{Z}}(\mathbf{Z}[-n](j)) \simeq \Gamma(B^n\mathbf{G}_a(-j);\mathcal{O})$ of $\mathbf{E}_{n+1}$-$\mathbf{Z}$-algebras.

Returning to the case of $\Omega^2 S^{2n+1}$, let us replace $\mathbf{Z}$ by $\mathbf{F}_p$ for some prime p. Then, one has:

$$\begin{aligned}\pi_*\mathbf{F}_p[\Omega^2 S^{2n+1}] &\cong \pi_*\mathrm{sh}^{1/2}\Gamma(B\mathbf{G}_a(-2n);\mathcal{O})\\ &\cong \begin{cases}\mathbf{F}_p[\tau_0^{[n]},\tau_1^{[n]},\cdots] & p=2\\ \mathbf{F}_p[\zeta_1^{[n]},\zeta_2^{[n]},\cdots]\otimes_{\mathbf{F}_p}\mathbf{F}_p[\tau_0^{[n]},\tau_1^{[n]},\cdots]/(\tau_i^{[n]}|i\geq 0)^2 & p>2.\end{cases}\end{aligned}$$

Here, $\tau_i^{[n]}$ lives in degree $2np^i-1$, and $\zeta_j^{[n]}$ lives in degree $2(np^j-1)$.

Let $j\geq 1$ be an integer, and let $J_{p^j-1}(S^{2n}) \subseteq \Omega S^{2n+1}$ denote the (p^j-1)st partial James construction. Using the EHP fiber sequence

$$J_{p^j-1}(S^{2n}) \to \Omega S^{2n+1} \to \Omega S^{2np^j+1},$$

one can similarly show that there is a Fourier equivalence

$$\mathrm{Loc}(\Omega J_{p^j-1}(S^{2n});\mathbf{F}_p) \simeq \mathrm{Perf}(B\alpha_{p^j}[-2n]), \tag{29}$$

which sends the skyscraper sheaf at the basepoint of $\Omega J_{p^j-1}(S^{2n})$ to the structure sheaf of $B\alpha_{p^j}[-2n]$. Computing endomorphisms of this skyscraper sheaf, we find that there is an isomorphism[19]

$$\pi_*\mathbf{F}_p[\Omega^2 J_{p^j-1}(S^{2n})] \cong \pi_*\mathrm{sh}^{1/2}\Gamma(B\alpha_{p^j}(-2n); \mathcal{O}).$$

The proof of Theorem 4.2.1 will occupy the remainder of this section; in fact, we will prove a ku-theoretic deformation of Theorem 4.2.1 below in Corollary 4.2.17.

Lemma 4.2.5. *There is a homotopy equivalence* $(\mathrm{GL}_{n+1}/\mathrm{GL}_n)(\mathbf{C}) \simeq S^{2n+1}$. *Furthermore, if* $B \subseteq G$ *is the subgroup of upper-triangular matrices, the Levi quotient* $L(X)$ *of the parabolic subgroup stabilizing the open* B*-orbit in* $\mathrm{GL}_{n+1}/\mathrm{GL}_n$ *is given by the partition* $[1, n-1, 1]$ *of* $n+1$.

Lemma 4.2.6. *There is an isomorphism of graded* $\pi_*\mathrm{ku}$*-algebras*

$$\pi_*\mathrm{ku}_{\mathrm{U}(n)} \cong \mathbf{Z}'[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1+\cdots+\beta^n c_n}] \cong \mathcal{O}_{T^n_\beta /\!\!/ \Sigma_n},$$

where c_j *lives in weight* $-2j$.

Proof. Let T^n denote the standard diagonal torus of $\mathrm{U}(n)$, so that

$$\pi_*\mathrm{ku}_{T^n} \cong \mathbf{Z}'[\beta, x_1, \cdots, x_n, \prod_{i=1}^n \tfrac{1}{1+\beta x_i}].$$

Since the Weyl group of T^n inside $\mathrm{U}(n)$ is the symmetric group Σ_n, Proposition 2.3.15 says that $\pi_*\mathrm{ku}_{\mathrm{U}(n)} \cong (\pi_*\mathrm{ku}_{T^n})^{\Sigma_n}$. The action of Σ_n on $\pi_*\mathrm{ku}_{T^n}$ is simply given by permuting the x_j. If we set c_j to denote the jth elementary symmetric polynomial in the variables $x_1, \cdots, x_n$, the lemma follows immediately. $\square$

Proposition 4.2.7. *There is an isomorphism of graded* $\pi_*\mathrm{ku}_{\mathrm{U}(n)}$*-algebras*

$$\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1}) \cong \mathbf{Z}'[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1+\cdots+\beta^n c_n}, a^{\pm 1}, \tfrac{a-1}{c_n}],$$

where c_j *lives in weight* $-2j$ *and* a *lives in weight* 0.

---

[19]One conceptual way to compute the cohomology of $B\mathbf{G}_a(-2)$ and $B\alpha_{p^j}(-2)$ is as follows. Let W denote the p-typical Witt ring scheme, let $V : F_*W \to W$ denote the Verschiebung, let W_n denote the quotient ring scheme of p-typical Witt vectors of length n, and let $W_n[F^j]$ denote the kernel of j-fold Frobenius on W_n (so that $W_1 = \mathbf{G}_a$ and $W_1[F^j] = \alpha_{p^j}$). All of these group schemes admit a natural action of $\mathbf{G}_m$ where the jth ghost coordinate lives in weight $2p^j$ (to compute the cohomology of $B\mathbf{G}_a(-2m)$, say, one simply replaces $2p^j$ by $2mp^j$). Then, there is a graded isomorphism

$$\mathrm{H}^*(BW_n[F^j]; \mathcal{O}) \cong \mathbf{F}_p[\zeta_n, \cdots, \zeta_{n+j-1}] \otimes_{\mathbf{F}_p} \mathbf{F}_p[\tau_0, \cdots, \tau_{j-1}]/(\tau_i^2 | 0 \le i \le j-1),$$

where $\zeta_i \in \mathrm{H}^2(BW_n[F^j]; \mathcal{O})$ and $\tau_i \in \mathrm{H}^1(BW_n[F^j]; \mathcal{O})$ both live in weight $2p^j$.

To see this, first observe that $\mathrm{H}^*(BW; \mathcal{O}) \cong \mathbf{F}_p[\tau_0, \cdots]/(\tau_i^2 | i \ge 0)$ and that $\mathrm{H}^*(B^2 F_*^n W; \mathcal{O}) \cong \mathbf{F}_p[\zeta_n, \cdots]$; this follows, for instance, either from the existence of ghost coordinates, or from the fact that W is an extension of the group schemes $F_*^j \mathbf{G}_a^\sharp$, and that there are isomorphisms

$$\mathrm{H}^*(BF_*^j\mathbf{G}_a^\sharp; \mathcal{O}) \cong \mathbf{F}_p[\tau_j]/(\tau_j^2), \ \mathrm{H}^*(B^2F_*^j\mathbf{G}_a^\sharp; \mathcal{O}) \cong \mathbf{F}_p[\zeta_j].$$

Next, note that there is an exact sequence

$$F_*^n W[F^j] \xrightarrow{V^n} W[F^j] \to W_n[F^j]$$

of flat group schemes. This induces an exact sequence

$$BW[F^j] \to BW_n[F^j] \to B^2 F_*^n W[F^j]$$

of commutative group stacks. However, since $W[F^j]$ is the kernel of $F^j : W \to F_*^j W$, the above calculation of $\mathrm{H}^*(BW; \mathcal{O})$ and $\mathrm{H}^*(B^2F_*^nW; \mathcal{O})$ implies that $\mathrm{H}^*(BW[F^j]; \mathcal{O}) \cong \mathbf{F}_p[\tau_0, \cdots, \tau_{j-1}]/(\tau_i^2 | 0 \le i \le j-1)$ and that $\mathrm{H}^*(B^2F_*^nW[F^j]; \mathcal{O}) \cong \mathbf{F}_p[\zeta_n, \cdots, \zeta_{n+j-1}]$. The desired calculation of $\mathrm{H}^*(BW_n[F^j]; \mathcal{O})$ follows from this.

PROOF. By Corollary 4.1.17, $\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1}) \cong \mathrm{ku}_*^{T^n}(\Omega S^{2n+1})^{\Sigma_n}$. As a T^n-representation, S^{2n+1} is the one-point compactification of $V = \mathbf{R} \oplus \bigoplus_{j=1}^n \mathrm{std}_j$. Theorem 4.1.13 says that

$$\mathrm{ku}_*^{T^n}(\Omega S^{2n+1}) \cong \mathbf{Z}'[\beta, x_1, \cdots, x_n, \prod_{i=1}^n \tfrac{1}{1+\beta x_i}, a^{\pm 1}, \tfrac{a-1}{x_1 \cdots x_n}],$$

so since the action of Σ_n simply permutes the x_j and leaves y invariant, we see from Lemma 4.2.6 (and $c_n = x_1 \cdots x_n$) that

$$\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1}) \cong \mathbf{Z}'[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1 + \cdots + \beta^n c_n}, a^{\pm 1}, \tfrac{a-1}{c_n}],$$

as desired. □

Corollary 4.2.8. *There is an isomorphism of graded $\pi_* \mathrm{ku}_{\mathrm{U}(n)}$-algebras*

$$\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega \mathbf{C}P^n) \cong \mathbf{Z}'[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1 + \cdots + \beta^n c_n}, b]/bc_n,$$

where c_j lives in weight $-2j$ and b lives in weight $2n$.

PROOF. Let V denote the $\mathrm{U}(n)$-representation $\mathbf{C}^n$, so that the $\mathrm{U}(n)$ action on $\mathbf{C}P^n$ is obtained by viewing it as $\mathbf{C}P(V \oplus \mathbf{R})$. We then have the generalized $\mathrm{U}(n)$-equivariant Hopf fibration

$$S^1 \to S^{V \oplus \mathbf{R}} \to \mathbf{C}P(V \oplus \mathbf{R}),$$

which induces a $\mathrm{U}(n)$-equivariant fibration of $\mathbf{E}_1$-spaces

$$\Omega S^1 \cong \mathbf{Z} \to \Omega S^{V \oplus \mathbf{R}} \to \Omega \mathbf{C}P(V \oplus \mathbf{R}).$$

This implies that there is an equivalence of $\mathbf{E}_1$-$\mathrm{ku}_{\mathrm{U}(n)}$-algebras

$$\mathcal{F}_{\mathrm{U}(n)}(\Omega \mathbf{C}P^n)^\vee \simeq \mathcal{F}_{\mathrm{U}(n)}(\Omega S^{2n+1})^\vee \otimes_{\mathrm{ku}[\Omega S^1]} \mathrm{ku}.$$

By Proposition 4.2.7, $\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1})$ is a flat $\mathrm{ku}_*(\Omega S^1) \cong \mathbf{Z}'[\beta, a^{\pm 1}]$-module, so we obtain an isomorphism of graded $\pi_* \mathrm{ku}_{\mathrm{U}(n)}$-algebras

$$\mathrm{ku}_*^{\mathrm{U}(n)}(\Omega \mathbf{C}P^n) \cong \mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1}) \otimes_{\mathbf{Z}'[\beta, a^{\pm 1}]} \mathbf{Z}'[\beta] \cong \mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1})/(a-1).$$

The desired result follows from the calculation of Proposition 4.2.7: the class $\frac{a-1}{c_n} \in \mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1})$ is sent to the class denoted $b \in \mathrm{ku}_{2n}^{\mathrm{U}(n)}(\Omega \mathbf{C}P^n)$ under the generalized Hopf fibration. □

Remark 4.2.9. The generalized Hopf fibration above also shows that

$$\mathrm{ku}_*^{T^n}(\Omega \mathbf{C}P^n) \cong \mathrm{ku}_*^{T^n}(\Omega S^{2n+1})/(a-1) \cong \mathcal{O}_{\mathcal{B}_V \times \mathbf{G}_m \{1\}}.$$

In particular, Lemma 4.1.12 implies that there is an isomorphism $\operatorname{Spec} \mathrm{ku}_*^{T^n}(\Omega \mathbf{C}P^n) \cong \mathcal{C}_V(-\dim_{\mathbf{R}}(V))$, and hence an isomorphism

$$\operatorname{Spec} \mathrm{ku}_*^{\mathrm{U}(n)}(\Omega \mathbf{C}P^n) \cong \mathcal{C}_V(-\dim_{\mathbf{R}}(V)) /\!\!/ \Sigma_n.$$

Before proceeding, we need the following lemma.

Lemma 4.2.10. *The following statements hold:*

(a) *Let $G_2 \subseteq G_1$ be a closed subgroup scheme. Let $Y \to Z$ be an open immersion of schemes with G_2-action whose complement has codimension $\geq d$. Then the induced map $\mathrm{Ind}_{G_2}^{G_1} Y \to \mathrm{Ind}_{G_2}^{G_1} Z$ is an open immersion of schemes with G_1-action whose complement has codimension $\geq d$.*

(b) Let Y be an integral quasi-affine variety such that $\pi_0\Gamma(Y;\mathcal{O}_Y)$ is Noetherian. Then the map $Y \to \overline{Y}$ to its affine closure is an open immersion whose complement has codimension ≥ 2.

PROOF. Part (a) is clear. For part (b), let $Z \subseteq \overline{Y}$ denote a closed subscheme of Y associated to a minimal prime in the complement of $Y \subseteq \overline{Y}$, so that $Y \subseteq \overline{Y} - Z$. Then there are maps

$$\mathrm{H}^0(\mathcal{O}_{\overline{Y}}) \xrightarrow{f} \mathrm{H}^0(\mathcal{O}_{\overline{Y}-Z}) \xrightarrow{g} \mathrm{H}^0(\mathcal{O}_Y).$$

The map g is an isomorphism, and the composite is also an isomorphism (by assumption). Therefore, f is also an isomorphism. We claim that this forces $\mathfrak{q}$ is necessarily of height ≥ 2. Indeed, let $A = \mathrm{H}^0(\mathcal{O}_{\overline{Y}})$, so that A is Noetherian. The desired claim then follows from the more general observation: if A is Noetherian and $\mathfrak{p} \subseteq A$ is a height one prime ideal corresponding to a closed subscheme $Z \subseteq \mathrm{Spec}(A)$ with complement $U \subseteq \mathrm{Spec}(A)$, the inclusion $A \subseteq \Gamma(\mathcal{O}_U)$ is strict. To see this, standard arguments reduce us to the case when A is local. Since A is Noetherian, $\mathfrak{p}$ is the radical of any nonzero $a \in \mathfrak{p}$. Therefore, the inclusion $A \subseteq \Gamma(\mathcal{O}_U)$ corresponds to the inclusion $A \subseteq A[1/a]$, which is evidently strict. □

Construction 4.2.11. Let $\check{V}_\beta = \mathbf{A}^1 \times T^n_\beta /\!\!/ \Sigma_n$; we will denote a point of $\check{X}$ by $(c_0, c_1, \cdots, c_n)$ where c_j lives in weight $-2j$. Write $\check{V}$ to denote $\check{V}_\beta/\beta$.

Let $\kappa : T^n_\beta /\!\!/ \Sigma_n \to \check{V}_\beta$ denote the map $\vec{c} = (c_1, \cdots, c_n) \mapsto (1, \vec{c})$. Equip GL_2 with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight zero, b lives in weight $2n$, and c lives in weight $-2n$. Let $\check{B}(-2n)$ denote the mirabolic subgroup of GL_2 of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, so that $\check{B}(-2n)$ is an extension of $\mathbf{G}_m$ by $\mathbf{G}_a(-2n)$. There is an action of $\check{B}(-2n)$ on $\check{V}_\beta$, where $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \check{B}(-2n)$ acts via

$$\check{V}_\beta \ni (c_0, \cdots, c_n) \to (ac_0 - bc_n, c_1, \cdots, c_n).$$

Proposition 4.2.12. *There is an isomorphism of graded group schemes over $T^n_\beta /\!\!/ \Sigma_n$:*

$$\mathrm{Spec}\, \mathrm{ku}^{\mathrm{U}(n)}_*(\Omega S^{2n+1}) \cong T^n_\beta /\!\!/ \Sigma_n \times_{\check{V}_\beta/\check{B}(-2n)} T^n_\beta /\!\!/ \Sigma_n.$$

Moreover, the $\check{B}(-2n)$-orbit of $\kappa(T^n_\beta /\!\!/ \Sigma_n) \subseteq \check{V}_\beta$ has complement of codimension ≥ 2.

PROOF. By Corollary 4.2.8, it suffices to show that there is an isomorphism of graded schemes over $T^n_\beta /\!\!/ \Sigma_n$:

$$T^n_\beta /\!\!/ \Sigma_n \times_{\check{V}_\beta/\check{B}(-2n)} T^n_\beta /\!\!/ \Sigma_n \cong \mathrm{Spec}\, \mathbf{Z}'[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1 + \cdots + \beta^n c_n}, a^{\pm 1}, \tfrac{a-1}{c_n}].$$

There is a closed immersion

$$T^n_\beta /\!\!/ \Sigma_n \times_{\check{V}_\beta/\check{B}(-2n)} T^n_\beta /\!\!/ \Sigma_n \hookrightarrow T^n_\beta /\!\!/ \Sigma_n \times \check{B}(-2n)$$

which exhibits $T^n_\beta /\!\!/ \Sigma_n \times_{\check{V}_\beta/\check{B}(-2n)} T^n_\beta /\!\!/ \Sigma_n$ as the subscheme of pairs $(\vec{c}, \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right))$ such that b stabilizes $\kappa(\vec{c})$. But $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ sends $\kappa(\vec{c}) \mapsto (a - bc_n, \vec{c})$, so the necessary condition is that $b = \frac{a-1}{c_n}$, as desired. This also shows that the $\check{B}(-2n)$-orbit of $\kappa(T^n_\beta /\!\!/ \Sigma_n)$ is the complement of the closed subscheme $(0, *, \cdots, *, 0) \subseteq \check{V}_\beta$. This closed subscheme has codimension ≥ 2, as desired. □

Observation 4.2.13. Equip $\check{G}_X = \mathrm{GL}_2$ with the grading via the action of $2n\rho_{\check{G}_X}$, so that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$, the elements a and d have weight 0, b has weight $2n$, and c has weight $-2n$. Let V denote the affine space $\mathbf{A}^2(2n, 0)$, so that there is an action of $\check{G}_X = \mathrm{GL}_2$ on V via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x, y) = (ax + cy, bx + dy);$$

here, x lives in degree $-2n$ and y lives in degree zero. There is an isomorphism

$$\check{B}(-2n)\backslash\check{G}_X \cong V - \{0\},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c, d),$$

and the above action of $\check{G}_X$ on V restricts on $V - \{0\}$ to the right-action of $\check{G}_X$ on $\check{B}(-2n)\backslash\check{G}_X$.

There is a $\check{G}_X$-equivariant fibration

$$\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta \to \check{B}(-2n)\backslash\check{G}_X \cong V - \{0\} \tag{30}$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta}$ denote the affine closure of $\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta$, so that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta} \to \overline{\check{B}(-2n)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$. Finally, let $\mathcal{Y}_\beta$ denote the induction

$$\mathcal{Y}_\beta := \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta}.$$

Lemma 4.2.10 implies:

Lemma 4.2.14. *There is an open immersion*

$$\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta} = \mathcal{Y}_\beta$$

which exhibits the target as the affine closure of the source, and whose complement is of codimension ≥ 2.

Proof. By Lemma 4.2.10, it suffices to show that there is an open immersion

$$\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta \hookrightarrow \overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta}$$

which exhibits the target as the affine closure of the source, and whose complement is of codimension ≥ 2. The statement about being the affine closure is true by definition, and the fact that the complement is of codimension ≥ 2 is a consequence of the fact that the open subscheme $\check{B}(-2n)\backslash\check{G}_X \hookrightarrow \overline{\check{B}(-2n)\backslash\check{G}_X}$ has complement of codimension 2. $\square$

The map $\kappa : T_\beta^n /\!\!/ \Sigma_n \to \check{V}_\beta$ defines a locally closed immersion

$$T_\beta^n /\!\!/ \Sigma_n \xrightarrow{\kappa} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\check{B}(-2n)}^{\check{G}} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta} \cong \mathcal{Y}_\beta.$$

We will denote this map by $\kappa_{\mathcal{Y}_\beta}$.

Remark 4.2.15. Here (and in the remaining sections), the stack $\mathcal{Y}_\beta$ and its fiber $\mathcal{Y}$ over $\beta = 0$ only act as crutches. Lemma 3.4.6 implies that $\mathcal{Y}$ is isomorphic to the dual variety $\check{M}$ of Theorem 3.4.13, but we have opted to use different notation since we are not viewing $\mathcal{Y}$ as a Hamiltonian $\check{G}$-variety in the present context.

Lemma 4.2.16. *The $\check{G}$-orbit of the image of $\kappa_{\mathcal{Y}_\beta}$ has complement of codimension ≥ 2. Moreover, there is an isomorphism of graded group schemes over $T^n_\beta /\!\!/ \Sigma_n$:*

$$\operatorname{Spec} \mathrm{ku}_*^{\mathrm{U}(n)}(\Omega S^{2n+1}) \cong T^n_\beta /\!\!/ \Sigma_n \times_{\mathcal{Y}_\beta/\check{G}(-2\rho)} T^n_\beta /\!\!/ \Sigma_n.$$

Proof. For the first statement, Lemma 4.2.14 implies that it suffices to show that the $\check{G}$-orbit of the image of the composite

$$T^n_\beta /\!\!/ \Sigma_n \xrightarrow{\kappa} \check{V}_\beta \hookrightarrow \operatorname{Ind}_{\check{B}(-2n)}^{\check{G}} \check{V}_\beta$$

has complement of codimension ≥ 2. Let $\check{V}_\beta^{\mathrm{reg}}$ denote the $\check{B}(-2n)$-orbit of $\kappa(T^n_\beta /\!\!/ \Sigma_n)$ inside $\check{V}_\beta$. Applying Lemma 4.2.10 to the inclusion $\check{V}_\beta^{\mathrm{reg}} \hookrightarrow \check{V}_\beta$, it suffices to show that $\check{V}_\beta^{\mathrm{reg}} \subseteq \check{V}_\beta$ has complement of codimension ≥ 2; but this is precisely Proposition 4.2.12. $\square$

The above lemma combined with Proposition 3.6.23 and the isomorphism $\mathcal{Y}_\beta/\check{G}(-2\rho) \cong \overline{(\operatorname{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta)}/\check{G}_X(-2n\rho_{\check{G}_X})$ implies:

Corollary 4.2.17. *Recall that $G = \mathrm{GL}_{n+1}$ and $H = \mathrm{GL}_n$, so $\check{G} = \mathrm{GL}_{n+1}$ and $\check{G}_X = \mathrm{GL}_2$. There is an equivalence of $\mathrm{sh}^{1/2}(\mathbf{Z}'[\beta])$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} \overline{\operatorname{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}_\beta}/\check{G}_X(-2n\rho_{\check{G}_X})).$$

The following simple observation is helpful for bookkeeping weights.

Lemma 4.2.18. *There is a graded isomorphism $T^*(j)\mathbf{A}^2(m,n) = \mathbf{A}^2(m,n) \times \mathbf{A}^2(j-m, j-n)$.*

Proof of Theorem 4.2.1. There is an isomorphism $\check{V}_\beta/\beta \cong \mathfrak{t}^{n-1}(2) /\!\!/ \Sigma_{n-1} \times \mathbf{A}^2(0, 2n)$, so Lemma 4.2.18 and the fibration (30) defines an isomorphism

$$\operatorname{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V} \cong \mathfrak{t}^{n-1}(2) /\!\!/ \Sigma_{n-1} \times T^*(2n)(V - \{0\}),$$

and hence

$$\overline{\operatorname{Ind}_{\check{B}(-2n)}^{\check{G}_X} \check{V}} \cong \mathfrak{t}^{n-1}(2) /\!\!/ \Sigma_{n-1} \times T^*(2n)(V).$$

Corollary 4.2.17 now implies Theorem 4.2.1. $\square$

Corollary 4.2.19. *Let $G = \mathrm{PGL}_{n+1}$ and $H = \mathrm{GL}_n$, so $\check{G} = \mathrm{SL}_{n+1}$ and $\check{G}_X = \mathrm{SL}_2$. There is an equivalence of $\mathrm{sh}^{1/2}(\mathbf{Z}'[\beta])$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} \mathcal{Y}_\beta/\mathrm{SL}_{n+1}[-2\rho]).$$

When $\beta = 0$, this specializes to an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(T^*[2n](\mathbf{A}^2[2n,0])/\mathrm{SL}_2[-2n\rho_{\check{G}_X}] \times \mathfrak{gl}_{n-1}[2] /\!\!/ \mathrm{GL}_{n-1}).$$

Proof. Recall that $\mathrm{PU}(n+1) = \mathrm{U}(n+1)/\mathrm{U}(1)^{\mathrm{diag}}$, so that there are equivalences

$$\begin{aligned}
\mathrm{Shv}_{\mathrm{PU}(n+1)}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{PU}(n+1)/\mathrm{U}(n)); \mathrm{ku}) &\simeq \mathrm{Shv}_{\mathrm{U}(n+1)}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{U}(n+1)/\mathrm{U}(n)); \mathrm{ku}) \otimes_{\mathrm{Shv}_{\mathrm{U}(1)}(\mathcal{L}\mathrm{U}(1);\mathrm{ku})} \mathrm{LMod}_{\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]} \\
&\overset{\text{Corollary 4.2.17}}{\simeq} \mathrm{Perf}(\mathrm{sh}^{1/2}\mathcal{Y}_\beta/\mathrm{GL}_{n+1}[-2\rho]) \otimes_{\mathrm{Perf}(B\mathbf{G}_m)} \mathrm{LMod}_{\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]} \\
&\simeq \mathrm{Perf}\left(\mathrm{sh}^{1/2}\mathcal{Y}_\beta/\mathrm{GL}_{n+1}[-2\rho] \times_{B\mathbf{G}_m} \operatorname{Spec} \mathrm{sh}^{1/2}\mathbf{Z}'[\beta]\right)
\end{aligned}$$

However, the displayed fiber product is precisely $\mathrm{sh}^{1/2}\mathcal{Y}_\beta/\mathrm{SL}_{n+1}[-2\rho]$. The claim about identifying its reduction modulo β with $T^*[2n](\mathbf{A}^2[2n,0])/\mathrm{SL}_2[-2\rho_{\mathrm{SL}_2}] \times \mathfrak{gl}_{n-1}[2]/\!\!/\mathrm{GL}_{n-1}$ follows from the construction of $\mathcal{Y}_\beta$. □

Remark 4.2.20. In the case of the spherical variety $\mathrm{PGL}_2/\mathbf{G}_m$, Corollary 4.2.19 states that there is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2}(\mathcal{L}(\mathrm{PGL}_2/\mathbf{G}_m);\mathbf{Q}) \simeq \mathrm{Perf}(T^*[2n](\mathbf{A}^2[2n,0])/\mathrm{SL}_2[-2n\rho_{\check{G}_X}]).$$

The PGL_2-variety $\mathrm{PGL}_2/\mathbf{G}_m$ has a natural action of the Weyl group $\mathbf{Z}/2 = \mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m)/\mathbf{G}_m$ (under the homotopy equivalence $(\mathrm{PGL}_2/\mathbf{G}_m)(\mathbf{C}) \simeq S^2$, this is the antipodal action). This equips the left-hand side of the above equivalence with a natural $\mathbf{Z}/2$-action. One can show that under this equivalence, the resulting $\mathbf{Z}/2$-action on the right-hand side identifies with the natural $\mathbf{Z}/2$-action on $T^*(\mathbf{A}^2)$ via the symplectic form. (See Example 3.5.28 for an expected generalization to arbitrary semisimple algebraic groups.) The normalizer $\mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m)$ can be identified with PO_2, which implies that there is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2}(\mathcal{L}(\mathrm{PGL}_2/\mathrm{PO}_2);\mathbf{Q}) \simeq \mathrm{Perf}(T^*[2n](\mathbf{A}^2[2n,0])/(\mathrm{SL}_2[-2n\rho_{\check{G}_X}] \times \mathbf{Z}/2)).$$

Note that the spherical root of $\mathrm{PGL}_2/\mathrm{PO}_2$ is (by definition) of type N, and so the spherical PGL_2-variety $\mathrm{PGL}_2/\mathrm{PO}_2$ is excluded by **[SV17, BZSV23]**. Nevertheless, the preceding equivalence shows that the Hamiltonian SL_2-"space" which should be dual (in the sense of **[BZSV23]**) to $\mathrm{PGL}_2/\mathrm{PO}_2$ is the *stack* $T^*(\mathbf{A}^2)/(\mathbf{Z}/2)$.

4.3. Type B_n. Our goal in this section is to prove Theorem 3.4.13 in type B_n, i.e., for the spherical SO_{2n+1}-variety $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$. Note that if (V,q) is a quadratic space and $v \in V$ with $q(v)=1$, then $\mathrm{SO}_V/\mathrm{SO}_{v^\perp}$ can be identified with the hyperboloid $\{w \in V | q(w)=1\}$. Write $G = \mathrm{SO}_{2n+1}$ and $H = \mathrm{SO}_{2n}$, so that $\check{G} = \mathrm{Sp}_{2n}$. Recall from Table 3 that $\check{G}_X = \mathrm{SL}_2$. In this section, we will only consider coefficients in $\mathbf{Z}'$ (instead of ku).

Theorem 4.3.1 (Theorem 3.4.13 in type B_n)**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G}(\mathcal{L}(G/H);\mathbf{Q}) \simeq \mathrm{Perf}(T^*[2n](\mathbf{A}^2[4n-2,0])/\mathrm{SL}_2[-(4n-2)\rho_{\check{G}_X}] \times \mathfrak{sp}_{2n-2}[2]/\!\!/\mathrm{Sp}_{2n-2}).$$

The proof of Theorem 4.3.1 will take up the remainder of this section.

Lemma 4.3.2. *There is a homotopy equivalence $(\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n})(\mathbf{C}) \simeq S^{2n}$. Moreover, if $B \subseteq G$ is the Borel subgroup of upper-triangular matrices, the Levi quotient $L(X)$ of the parabolic subgroup stabilizing the open B-orbit in $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$ is given by SO_{2n-1}.*

Lemma 4.3.3. *Let $W = (\mathbf{Z}/2)^{n-1} \rtimes \Sigma_n$ denote the Weyl group of SO_{2n}. Then there is an isomorphism*

$$\mathrm{H}^*_{\mathrm{SO}_{2n}}(*;\mathbf{Z}') \cong \mathbf{Z}'[p_1,\cdots,p_{n-1},c_n],$$

where the injective map $\mathrm{H}^*_{\mathrm{SO}_{2n}}(*;\mathbf{Z}') \to \mathrm{H}^*_{T^n}(*;\mathbf{Z}')$ sends p_j to the jth elementary symmetric polynomial in the variables $x_1^2,\cdots,x_{n-1}^2$ (so p_j lives in weight $-4j$), and $c_n \mapsto x_1\cdots x_n$.

Proposition 4.3.4. *There is a graded isomorphism of $\mathrm{H}^*_{\mathrm{SO}_{2n}}(*;\mathbf{Z}')$-algebras*

$$\mathrm{H}_*^{\mathrm{SO}_{2n}}(\Omega S^{2n};\mathbf{Z}') \cong \mathbf{Z}'[p_1,\cdots,p_{n-1},c_n,b]/bc_n,$$

where b lives in weight $4n-2$.

Proof. The restriction of the SO_{2n}-action on S^{2n} to the maximal torus $T^n \subseteq \mathrm{SO}_{2n}$ exhibits S^{2n} as the one-point compactification of the standard n-dimensional complex representation std. Corollary 4.1.17 implies that there is a graded isomorphism of $\mathrm{H}^*_{\mathrm{SO}_{2n}}(*; \mathbf{Z}')$-algebras

$$\mathrm{H}_*^{\mathrm{SO}_{2n}}(\Omega S^{2n}; \mathbf{Z}') \cong \left(\mathbf{Z}'[x_1, \cdots, x_n, b]/bx_1 \cdots x_n\right)^W,$$

where b lives in weight $4n-2$. The W-invariants on the right-hand side can be computed using Lemma 4.3.3 (note that the action of W on b is trivial), and gives the desired calculation. □

Remark 4.3.5. Note that $\mathrm{H}_*^{\mathrm{SO}_{2n}}(\Omega S^{2n}; \mathbf{Z}')$ is *not* flat over $\mathrm{H}^*_{\mathrm{SO}_{2n}}(*; \mathbf{Z}')$.

Definition 4.3.6. Let $\check{V}$ denote the graded affine space $\mathbf{A}^1(2-2n) \times \mathfrak{t}^n(2)/\!\!/W$. There is an action of $B(2-4n)$ on $\check{V}$, where $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in B(2-4n)$ sends $(z, \vec{p}, c_n) \mapsto (az - bc_n, \vec{p}, c_n)$. Note that b lives in weight $4n-2$. Equip $\check{G}_X = \mathrm{SL}_2$ with the grading coming from $(4n-2)\rho_{\mathrm{SL}_2}$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are equipped with the following weights: a and d have weight 0, b has weight $4n-2$, and c has weight $2-4n$. Let V denote the affine space $\mathbf{A}^2(4n-2, 0)$, so that there is an action of $\check{G}_X = \mathrm{SL}_2$ on V via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x, y) = (ax + cy, bx + dy);$$

here, x lives in degree $2-4n$ and y lives in degree 0. There is an isomorphism

$$\mathbf{G}_a(2-4n)\backslash \check{G}_X \cong V - \{0\}, \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c, d),$$

and the above action of $\check{G}_X$ on V restricts on $V - \{0\}$ to the right-action of $\check{G}_X$ on $\mathbf{G}_a(2-4n)\backslash \check{G}_X$.

Let $\kappa : \mathfrak{t}^n(2)/\!\!/W \to \check{V}$ denote the map sending $(\vec{p}, c_n) \mapsto (0, \vec{p}, c_n)$. Let W' denote the Weyl group of SO_{2n-1}, so Lemma 4.2.18 implies that there is an isomorphism

$$\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V} \cong \mathfrak{t}^{n-1}(2)/\!\!/W' \times T^*(2n)(V - \{0\}).$$

In particular, there is an open immersion

$$\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V} \hookrightarrow \mathfrak{t}^{n-1}(2)/\!\!/W' \times T^*(2n)(V)$$

which exhibits the target as the affine closure of $\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}$. Inducing along the map $\check{G}_X \to \check{G}$ produces an open immersion

$$\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}} \check{V} \hookrightarrow \mathfrak{t}^{n-1}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} T^*(2n)(V),$$

and we will write $\mathcal{Y} = \mathfrak{t}^{n-1}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} T^*(2n)(V)$.

The map κ defines a locally closed immersion

$$\mathfrak{t}^n(2)/\!\!/W \xrightarrow{\kappa} \check{V} \hookrightarrow \mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}} \check{V} \hookrightarrow \mathfrak{t}^{n-1}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} T^*(2n)(V) = \mathcal{Y},$$

which we will denote by $\kappa_{\mathcal{Y}}$.

Remark 4.3.7. In Definition 4.3.6, it does not make sense to ask that $\kappa : \mathfrak{t}^n(2)/\!\!/W \to \check{V}$ instead send $(\vec{p}, c_n) \mapsto (1, \vec{p}, c_n)$. Indeed, the point $1 \in \mathbf{A}^1(2-2n)$ is not well-defined, since it would have to be cut out by the ideal $(z-1)$, which is not homogeneous (i.e., is not a graded ideal).

Lemma 4.3.8. *Let $C = \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}'[c]$, and let $\mathbf{A}^2 = \operatorname{Spec} \mathbf{Z}'[z, c]$. Let $\underline{\mathbf{G}_a}$ denote the constant group scheme over C acting on $\mathbf{A}^2$ by $b \cdot (z, c) = (z + bc, c)$. Let $f : C \to Y$ denote the map $c \mapsto (0, c)$, and let J denote the stabilizer group scheme of the image of f (over C). Then J is not flat over C, and the quotient $\underline{\mathbf{G}_a}/J$ is isomorphic to $\mathbf{A}^2$.*

Proof. It is immediate that $J \cong \operatorname{Spec} \mathbf{Z}'[b, c]/bc$, so is not flat over C. (This implies that the $\underline{\mathbf{G}_a}$-orbit of the image of f is only a constructible subset of $\mathbf{A}^2$: indeed, this $\underline{\mathbf{G}_a}$-orbit is the standard example $(\mathbf{A}^2 - \{z = 0\}) \cup \{(0, 0)\}$ of a constructible subset.) Let $\underline{\mathbf{G}_a} \to \mathbf{A}^2$ denote the map given by the action of $\underline{\mathbf{G}_a}$ on C (so it is given on rings of functions by $\mathbf{Z}'[c, z] \to \mathbf{Z}'[b, c]$ sending $z \mapsto bc$). Then $C \times_{\mathbf{A}^2} \underline{\mathbf{G}_a} \cong J$, from which one can see that $\underline{\mathbf{G}_a}/J \cong \mathbf{A}^2$. □

Proposition 4.3.9. *There is an isomorphism of graded group schemes over $\mathfrak{t}^n(2)/\!\!/W$:*

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{SO}_{2n}}(\Omega S^{2n}; \mathbf{Z}') \cong \mathfrak{t}^n(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^n(2)/\!\!/W.$$

Moreover, if $\check{J}_X$ denotes the above group scheme over $\mathfrak{t}^n(2)/\!\!/W$, the algebra of regular functions on $(\mathfrak{t}^n(2)/\!\!/W \times \check{G})/\check{J}_X$ is isomorphic to $\mathcal{O}_{\mathcal{Y}}$.

Proof. There is an isomorphism

$$\mathfrak{t}^n(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^n(2)/\!\!/W \cong \mathfrak{t}^n(2)/\!\!/W \times_{\check{V}/\mathbf{G}_a(2-4n)} \mathfrak{t}^n(2)/\!\!/W,$$

as well as a closed immersion

$$\mathfrak{t}^n(2)/\!\!/W \times_{\check{V}/\mathbf{G}_a(2-4n)} \mathfrak{t}^n(2)/\!\!/W \hookrightarrow \mathfrak{t}^n(2)/\!\!/W \times \mathbf{G}_a(2-4n),$$

which exhibits $\mathfrak{t}^n(2)/\!\!/W \times_{\check{V}/\mathbf{G}_a(2-4n)} \mathfrak{t}^n(2)/\!\!/W$ as the subscheme of pairs $(\vec{p}, c_n, b)$ such that b stabilizes $\kappa(\vec{p}, c_n)$. But by definition of κ, this happens if and only if $bc_n = 0$. In other words, there is an isomorphism of graded schemes over $\mathfrak{t}^n(2)/\!\!/W$:

$$\mathfrak{t}^n(2)/\!\!/W \times_{\check{V}/\mathbf{G}_a(2-4n)} \mathfrak{t}^n(2)/\!\!/W \cong \operatorname{Spec} \mathbf{Z}'[p_1, \cdots, p_{n-1}, c_n, b]/bc_n.$$

The first part of the proposition therefore follows from Proposition 4.3.4. The second part of the proposition follows from Lemma 4.3.8 (rather, its obvious variant for graded affine spaces). □

Proof of Theorem 4.3.1. This follows from Theorem 3.5.23 and Proposition 4.3.9, along with the isomorphism between $\mathcal{Y}/\check{G}(-2\rho)$ and $T^*(2n)(V)/\check{G}_X(-(4n-2)\rho_{\check{G}_X}) \times \mathfrak{t}^{n-1}(2)/\!\!/W'$. □

4.4. Type C_n. Our goal in this section is to prove Theorem 3.4.13 in type C_n, i.e., for the spherical Sp_{2n}-variety $\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$. Let $G = \mathrm{Sp}_{2n}$ and $H = \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$, so that $\check{G} = \mathrm{SO}_{2n+1}$. Recall from Table 3 that $\check{G}_X = \mathrm{SL}_2$.

Theorem 4.4.1 (Theorem 3.4.13 in type C_n)**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(T^*[4n-4](\mathbf{A}^2[4n-2, 0])/\mathrm{SL}_2[-(4n-2)\rho_{\check{G}_X}] \times \check{\mathfrak{h}}^*[2]/\!\!/\check{H}),$$

where $H = \mathrm{Sp}_2 \times \mathrm{Sp}_{2(n-2)}$.

The proof of Theorem 4.4.1 will take up the remainder of this section; in fact, we will prove a ku-theoretic deformation.

Lemma 4.4.2. *There is a homotopy equivalence* $(\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}))(\mathbf{C}) \simeq \mathbf{H}P^{n-1}$. *Moreover, if* $B \subseteq G$ *is the Borel subgroup of upper-triangular matrices, the Levi quotient* $L(X)$ *of the parabolic subgroup stabilizing the open* B*-orbit in* $\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$ *is given by* $\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-4}$.

Lemma 4.4.3. *Let* W *denote the Weyl group of* Sp_{2n}. *Then there is an isomorphism*

$$\pi_*\mathrm{ku}_{\mathrm{Sp}_{2n}} \cong \mathbf{Z}'[\beta, p_1, \cdots, p_n],$$

where the map $\pi_*\mathrm{ku}_{\mathrm{Sp}_{2n}} \to \pi_*\mathrm{ku}_{T^n}$ *sends* p_j *to the* j*th elementary symmetric polynomial in the variables* $x_1\overline{x}_1, \cdots, x_n\overline{x}_n$.

Although one can give an argument for the following result using Theorem 4.1.13, it is simpler to give an argument "from scratch".

Proposition 4.4.4. *There is an isomorphism of graded* $\pi_*\mathrm{ku}_{\mathrm{Sp}_{2n-2}}$*-algebras*

$$\mathrm{ku}_*^{\mathrm{Sp}_{2n-2}}(\Omega\mathbf{H}P^{n-1}) \cong \mathbf{Z}'[\beta, p_1, \cdots, p_{n-1}, b]/bp_{n-1},$$

where b *lives in weight* $4n-2$.

Proof. Let $T^{n-1} \subseteq \mathrm{Sp}_{2n-2}$ denote the maximal torus. Then the homotopy fixed points spectral sequence for $\pi_*\mathrm{ku}[\Omega\mathbf{H}P^{n-1}]^{hT^{n-1}}$ is given by

$$E_2^{*,*} \cong \mathrm{ku}_*(\Omega\mathbf{H}P^{n-1}) \otimes_{\pi_*\mathrm{ku}} \pi_*\mathrm{ku}^{hT^{n-1}} \Rightarrow \pi_*\mathrm{ku}[\Omega\mathbf{H}P^{n-1}]^{hT^{n-1}}.$$

To compute the E_2-page, observe that the Hopf fibration $S^3 \to S^{4n-1} \to \mathbf{H}P^{n-1}$ implies that there is an equivalence $\Omega\mathbf{H}P^{n-1} \simeq S^3 \times \Omega S^{4n-1}$. This gives an isomorphism

$$\mathrm{ku}_*(\Omega\mathbf{H}P^{n-1}) \cong \mathbf{Z}'[\beta, a, b]/a^2,$$

where a lives in weight 3 and b lives in weight $4n-2$. Therefore,

$$E_2^{*,*} \cong \mathbf{Z}'[\beta, a, b][\![x_1, \cdots, x_{n-1}]\!]/a^2,$$

where each x_j lives in weight -2. Recall that the action of T^{n-1} on $\mathbf{H}P^{n-1}$ is induced by the inclusion $T^{n-1} \subseteq \mathrm{Sp}_{2n-2} \subseteq \mathrm{U}(4n-4)$ given by the representation $\bigoplus_{j=1}^{n-1} \mathrm{std} \oplus \mathrm{std}^{-1}$. This forces a single differential in the above spectral sequence, given by

$$d_2(a) = bx_1\overline{x}_1 \cdots x_{n-1}\overline{x}_{n-1}.$$

After running this differential, the spectral sequence is concentrated in even degrees, and we find that there is an isomorphism

$$\pi_*\mathrm{ku}[\Omega\mathbf{H}P^{n-1}]^{hT^{n-1}} \cong \mathbf{Z}'[\beta, b][\![x_1, \cdots, x_{n-1}]\!]/bx_1\overline{x}_1 \cdots x_{n-1}\overline{x}_{n-1}.$$

To calculate $\mathrm{ku}_*^{T^{n-1}}(\Omega\mathbf{H}P^{n-1})$ itself (and not just its completion $\pi_*\mathrm{ku}[\Omega\mathbf{H}P^{n-1}]^{hT^{n-1}}$), the strategy of Theorem 4.1.13 reduces us to showing that the restriction $\mathrm{ku}_*^{T^{n-1}}(\Omega\mathbf{H}P^{n-1})|_{T_\beta^\circ} = \pi_*\mathrm{ku}_{T^{n-1}}$. By Lemma 2.4.6, there is an isomorphism $\mathrm{ku}_*^{T^{n-1}}(\Omega\mathbf{H}P^{n-1})|_{T_\beta^\circ} \cong \mathrm{ku}_*^{T^{n-1}}(\Omega(\mathbf{H}P^{n-1})^{T^{n-1}})|_{T_\beta^\circ}$. It therefore suffices to show that $\Omega(\mathbf{H}P^{n-1})^{T^{n-1}}$ is contractible, but this is a consequence of the simple observation that $(\mathbf{H}P^{n-1})^{T^{n-1}} \cong S^0$. This discussion gives an isomorphism of graded $\pi_*\mathrm{ku}_{T^{n-1}}$-algebras

$$\mathrm{ku}_*^{T^{n-1}}(\Omega\mathbf{H}P^{n-1}) \cong \mathbf{Z}'[\beta, x_1, \cdots, x_{n-1}, \prod_{j=1}^{n-1} \tfrac{1}{1+\beta x_j}, b]/bx_1\overline{x}_1 \cdots x_{n-1}\overline{x}_{n-1}.$$

This isomorphism is W-equivariant (where W is the Weyl group of Sp_{2n-2}), so Proposition 2.3.15 implies that there is an isomorphism of graded $\pi_*\mathrm{ku}_{\mathrm{Sp}_{2n-2}}$-algebras

$$\mathrm{ku}_*^{\mathrm{Sp}_{2n-2}}(\Omega\mathbf{H}P^{n-1}) \cong \left(\mathbf{Z}'[\beta, x_1, \cdots, x_{n-1}, \prod_{j=1}^{n-1} \tfrac{1}{1+\beta x_j}, b]/bx_1\overline{x}_1 \cdots x_{n-1}\overline{x}_{n-1}\right)^W.$$

Noting that the action of W leaves b invariant, Lemma 4.4.3 computes the right-hand side; the resulting answer is precisely the right-hand side of the proposition. □

Notation 4.4.5. Let W denote the Weyl group of $\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$, so that it is the product of $\mathbf{Z}/2$ with the Weyl group of Sp_{2n-2}. There is a natural action of W on the torus $\mathbf{G}_m \times T^{n-1} \subseteq \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$, and hence an action of W on T^n_β. It is an easy consequence of Lemma 4.4.3 that there is an isomorphism

$$T^n_\beta /\!\!/ W \cong \operatorname{Spec} \mathbf{Z}'[\beta, p_1', p_1, \cdots, p_{n-1}],$$

where p_1' lives in weight -4, and p_j lives in weight $-4j$.

Construction 4.4.6. Let $\check{V}_\beta$ denote the graded affine scheme $\mathbf{A}^1(-2) \times T^n_\beta /\!\!/ W$, and let $\kappa : T^n_\beta /\!\!/ W \to \check{V}_\beta$ denote the map sending $(p_1', \vec{p}) \mapsto (0, p_1', \vec{p})$. There is an action of $\mathbf{G}_a(2-4n)$ on $\check{V}_\beta$, where $b \in \mathbf{G}_a(2-4n)$ sends

$$(z, p_1', \vec{p}) \mapsto (z - bp_{n-1}, p_1', \vec{p}).$$

Equip $\check{G}_X = \mathrm{SL}_2$ with the grading coming from $(4n-2)\rho_{\mathrm{SL}_2}$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are equipped with the following weights: a and d have weight 0, b has weight $4n-2$, and c has weight $2-4n$. Let V denote the affine space $\mathbf{A}^2(4n-2, 0)$, so that there is an action of $\check{G}_X = \mathrm{SL}_2$ on V via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x, y) = (ax + cy, bx + dy);$$

here, x lives in degree $2-4n$ and y lives in degree 0. There is an isomorphism

$$\mathbf{G}_a(2-4n)\backslash \check{G}_X \cong V - \{0\}, \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c, d),$$

and the above action of $\check{G}_X$ on V restricts on $V - \{0\}$ to the right-action of $\check{G}_X$ on $\mathbf{G}_a(2-4n)\backslash \check{G}_X$. There is a fibration

$$\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta \to \mathbf{G}_a(2-4n)\backslash \check{G}_X \cong V - \{0\}$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta}$ denote the affine closure of $\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta$, so that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta} \to \overline{\mathbf{G}_a(2-4n)\backslash \check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\mathcal{Y}_\beta$ denote the induction

$$\mathcal{Y}_\beta = \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta}.$$

Lemma 4.2.10 implies:

Lemma 4.4.7. *There is an open immersion*

$$\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}} \check{V}_\beta \to \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta} = \mathcal{Y}_\beta$$

which exhibits the target as the affine closure of the source, and whose complement is of codimension ≥ 2.

The map $\kappa : T_\beta^n /\!\!/ W \to \check{V}_\beta$ defines a locally closed immersion

$$T_\beta^n /\!\!/ W \xrightarrow{\kappa} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\check{G}_X}^{\check{G}} \overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta},$$

which we will denote by $\kappa_{\mathcal{Y}_\beta}$.

Proposition 4.4.8. *There is an isomorphism of graded group schemes over $T_\beta^n /\!\!/ W$:*

$$\mathrm{Spec}\, \mathrm{ku}_*^{\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}}(\Omega \mathbf{H}P^{n-1}) \cong T_\beta^n /\!\!/ W \times_{\mathcal{Y}_\beta / \check{G}(-2\rho)} T_\beta^n /\!\!/ W.$$

Moreover, if $\check{J}_{X,\beta}$ denotes the above group scheme over $T_\beta^n /\!\!/ W$, the algebra of regular functions on $(T_\beta^n /\!\!/ W \times \check{G})/\check{J}_{X,\beta}$ is isomorphic to $\mathcal{O}_{\mathcal{Y}_\beta}$.

Proof. There is an isomorphism

$$T_\beta^n /\!\!/ W \times_{\mathcal{Y}_\beta / \check{G}(-2\rho)} T_\beta^n /\!\!/ W \cong T_\beta^n /\!\!/ W \times_{\check{V}_\beta / \mathbf{G}_a(2-4n)} T_\beta^n /\!\!/ W,$$

as well as a closed immersion

$$T_\beta^n /\!\!/ W \times_{\check{V}_\beta / \mathbf{G}_a(2-4n)} T_\beta^n /\!\!/ W \hookrightarrow T_\beta^n /\!\!/ W \times \mathbf{G}_a(2-4n),$$

which exhibits $T_\beta^n /\!\!/ W \times_{\check{V}_\beta / \mathbf{G}_a(2-4n)} T_\beta^n /\!\!/ W$ as the subscheme of tuples $(p_1', \vec{p}, b)$ such that b stabilizes $\kappa(p_1', \vec{p})$. By definition of κ, this happens if and only if $bp_{n-1} = 0$, which gives an isomorphism of graded schemes over $T_\beta^n /\!\!/ W$:

$$T_\beta^n /\!\!/ W \times_{\check{V}_\beta / \mathbf{G}_a(2-4n)} T_\beta^n /\!\!/ W \cong \mathrm{Spec}\, \mathbf{Z}'[\beta, p_1', p_1, \cdots, p_{n-1}, b]/bp_{n-1}.$$

On the other hand, Proposition 4.4.4 gives an isomorphism of graded $\pi_* \mathrm{ku}_{\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2}$-algebras

$$\mathrm{ku}_*^{\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}}(\Omega \mathbf{H}P^{n-1}) \cong \mathbf{Z}'[\beta, p_1', p_1, \cdots, p_{n-1}, b]/bp_{n-1},$$

which implies the first part of the proposition. The second part of the proposition follows from Lemma 4.3.8 (rather, its obvious variant for graded affine spaces). □

Proposition 4.4.8 and Proposition 3.6.23 imply:

Corollary 4.4.9. *Let $G = \mathrm{Sp}_{2n}$ and $H = \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$, so $\check{G} = \mathrm{SO}_{2n+1}$ and $\check{G}_X = \mathrm{SL}_2$. There is an equivalence of $\mathrm{sh}^{1/2}(\mathbf{Z}'[\beta])$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} \mathcal{Y}_\beta / \mathrm{SO}_{2n+1}).$$

Proof of Theorem 4.4.1. Combining Corollary 4.4.9 with Proposition 3.6.19, we see that if $\mathcal{Y} = \mathcal{Y}_\beta / \beta$, there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} \mathcal{Y} / \mathrm{SO}_{2n+1}).$$

It suffices to describe $\mathcal{Y}/\mathrm{SO}_{2n+1}$. Let W' denote the Weyl group of Sp_{2n-4}, so that it acts on $\mathfrak{t}^{n-1}(2)$ such that $\mathfrak{t}^{n-1}(2) /\!\!/ W' \cong \mathrm{Spec}\, \mathbf{Z}'[p_1', p_1, \cdots, p_{n-2}]$. Recall that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X} \check{V}_\beta} \to \overline{\mathbf{G}_a(2-4n) \backslash \check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$, which implies (by setting $\beta = 0$) that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X}\check{V}} \to \overline{\mathbf{G}_a(2-4n)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}$. By Lemma 4.2.18, this implies that $\overline{\mathrm{Ind}_{\mathbf{G}_a(2-4n)}^{\check{G}_X}\check{V}} \cong T^*(4n-4)(V)$, and so

$$\mathcal{Y} \cong \mathfrak{t}^{n-1}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} T^*(4n-4)(V);$$

this implies that $\mathcal{Y}/\mathrm{SO}_{2n+1}$ is isomorphic to $\mathfrak{t}^{n-1}(2)/\!\!/W' \times T^*(4n-4)(V)/\check{G}_X(-(4n-2)\rho_{\check{G}_X})$, which implies the desired claim. □

4.5. Type D_n. Our goal in this section is to prove Theorem 3.4.13 in type D_n, i.e., for the spherical SO_{2n}/μ_2-variety $\mathrm{SO}_{2n}/\mu_2 \cdot \mathrm{SO}_{2n-1}$. Write $G = \mathrm{SO}_{2n}/\mu_2$ and $H = \mathrm{SO}_{2n-1}$, so $\check{G} = \mathrm{Spin}_{2n}$ and $\check{G}_X = \mathrm{Spin}_3 = \mathrm{SL}_2$. Throughout this section, we will write $\overline{x} = -\frac{x}{1+\beta x}$ to denote the inverse of $x \in \mathcal{O}_{\mathbf{G}_\beta}$ under the group structure on $\mathbf{G}_\beta$. As a shorthand, we will also write $N = 2n-2$.

Theorem 4.5.1 (Theorem 3.4.13 in type D_n)**.** *Let $N = 2n-2$. There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q}) \cong \mathrm{Perf}(\mathfrak{sl}_2[N-N\rho_{\check{G}_X}]/\mathrm{SL}_2[-N\rho_{\check{G}_X}]\times\mathfrak{spin}_{2n-3}[2]/\!\!/\mathrm{Spin}_{2n-3}).$$

Example 4.5.2. If $n = 2$, Theorem 4.5.1 says that there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_{\mathrm{SO}_4/\mu_2}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{SO}_4/\mu_2\mathrm{SO}_3);\mathbf{Q}) \cong \mathrm{Perf}(\mathfrak{sl}_2[2-2\rho]/\mathrm{SL}_2[-2\rho]).$$

By Example 3.4.10, the left-hand side can therefore be identified with $\mathrm{Shv}_{\mathrm{SO}_3\times\mathrm{SO}_3}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{SO}_3);\mathbf{Q})$, and the above equivalence is simply the derived Satake equivalence of Theorem 3.2.7 for SO_3.

Remark 4.5.3. When $n = 2$, Theorem 4.5.1 admits a ku-theoretic deformation, given by a $\mathrm{sh}^{1/2}\mathbf{Z}'[\beta]$-linear equivalence

$$\mathrm{Shv}_{\mathrm{SO}_4/\mu_2}^{c,\mathrm{Sat}}(\mathcal{L}(\mathrm{SO}_4/\mu_2\mathrm{SO}_3);\mathrm{ku}) \cong \mathrm{Perf}(\mathrm{sh}^{1/2}\mathrm{Spin}_3(-2\rho)_\beta/\mathrm{Spin}_3(-2\rho)).$$

Indeed, as in Example 4.5.2, the left-hand side can be identified with $\mathrm{Shv}_{\mathrm{SO}_3\times\mathrm{SO}_3}^{c,\mathrm{Sat}}(\mathcal{L}\mathrm{SO}_3;\mathrm{ku})$, while the right-hand side can be identified with $\mathrm{Perf}(\mathrm{sh}^{1/2}\mathrm{SL}_{2,\beta}/\mathrm{SL}_2)$. The desired equivalence is then a consequence of Theorem 3.6.21 applied to $G = \mathrm{SO}_3$. (Note that Theorem 3.6.21 as stated only applies to simply-laced and *simply-connected* G; but SO_3 is not simply-connected. Nevertheless, the statement continues to hold with $G = \mathrm{SO}_3$.)

The proof of Theorem 4.5.1 will occupy the remainder of this section.

Lemma 4.5.4. *There is a homotopy equivalence $(\mathrm{SO}_{2n}/\mu_2{\cdot}\mathrm{SO}_{2n-1})(\mathbf{C}) \simeq \mathbf{R}P^{2n-1}$. Furthermore, if $B \subseteq G$ is the Borel subgroup of upper-triangular matrices, the Levi quotient $L(X)$ of the parabolic subgroup stabilizing the open B-orbit in $\mathrm{SO}_{2n}/\mu_2 \cdot \mathrm{SO}_{2n-1}$ is given by SO_{2n-2}/μ_2.*

Lemma 4.5.5. *Let $W = (\mathbf{Z}/2)^{n-1} \rtimes \Sigma_{n-1}$ denote the Weyl group of Spin_{2n-1}. Then there is an isomorphism of graded $\mathbf{Z}'$-algebras*

$$\pi_*\mathrm{ku}_{\mathrm{Spin}_{2n-1}} \cong \mathbf{Z}'[\beta, p_1, \cdots, p_{n-1}] \cong \mathcal{O}_{T_\beta^{n-1}/\!\!/W},$$

where p_j lives in weight $-4j$. The map $\mathrm{ku}_{\mathrm{Spin}_{2n-1}} \to \mathrm{ku}_{T^{n-1}}$ sends p_j to the jth elementary symmetric polynomial in the variables $\frac{x_1^2}{1+\beta x_1}, \cdots, \frac{x_{n-1}^2}{1+\beta x_{n-1}}$.

Proposition 4.5.6. *There is an isomorphism of graded $\pi_*\mathrm{ku}_{\mathrm{SO}_{2n-1}}$-algebras*

$$\mathrm{ku}_*^{\mathrm{SO}_{2n-1}}(\Omega \mathbf{R}P^{2n-1}) \cong \mathbf{Z}'[\beta, x_1, \cdots, x_{n-1}, \prod_{i=1}^{n-1} \tfrac{1}{1+\beta x_i}, a^{\pm 1}, \tfrac{a-a^{-1}}{\prod_{i=1}^{n-1}(x_i - \overline{x}_i)}]^W.$$

Here, the action of the jth $\mathbf{Z}/2 \subseteq (\mathbf{Z}/2)^{n-1} \subseteq W$ sends $x_j \mapsto \overline{x}_j$ and $a \mapsto a^{-1}$, and the symmetric group acts by permuting the variables $\frac{x_1^2}{1+\beta x_1}, \cdots, \frac{x_{n-1}^2}{1+\beta x_{n-1}}$ (and leaves a invariant).

Proof. The restriction of the Spin_{2n-1}-action on $\mathrm{Spin}_{2n}/\mathrm{Spin}_{2n-1}$ (which is homotopy equivalent to S^{2n-1}) to $T^{n-1} \subseteq \mathrm{Spin}_{2n-1}$ exhibits S^{2n-1} as the one-point compactification of $\mathrm{std} \oplus \mathbf{R}$, where std is the standard $(n-1)$-dimensional complex representation of T^{n-1}. Corollary 4.1.17 implies that there is a graded isomorphism of $\pi_*\mathrm{ku}_{\mathrm{Spin}_{2n-1}}$-algebras

$$\mathrm{ku}_*^{\mathrm{Spin}_{2n-1}}(\Omega S^{2n-1}) \cong \mathbf{Z}'[\beta, x_1, \cdots, x_{n-1}, \prod_{i=1}^{n-1} \tfrac{1}{1+\beta x_i}, a^{\pm 1}, \tfrac{a-1}{x_1 \cdots x_{n-1}}]^W.$$

The calculation of $\mathrm{ku}_*^{\mathrm{SO}_{2n-1}}(\Omega \mathbf{R}P^{2n-1})$ is a consequence of the above description of $\mathrm{ku}_*^{\mathrm{Spin}_{2n-1}}(\Omega S^{2n-1})$, the fact that $\mathrm{Spin}_{2n-1}/\mu_2 \cong \mathrm{SO}_{2n-1}$, and the fact that $S^{2n-1}/(\mathbf{Z}/2) \cong \mathbf{R}P^{2n-1}$. $\square$

Let us now study the case of general n, with coefficients in $\mathbf{Z}'$ (instead of ku).

Construction 4.5.7. Let W' denote the Weyl group of Spin_{2n-3}. Equip SL_2 with the grading coming from $-N\rho$, and consider the $\check{G}_X = \mathrm{SL}_2$-scheme $\mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X})$ (where SL_2 acts only on the factor $\mathfrak{sl}_2(N - N\rho_{\check{G}_X})$, via the adjoint action). Define

$$\mathcal{Y} = \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} \mathfrak{sl}_2(N - N\rho_{\check{G}_X}).$$

Let $\kappa : \mathfrak{t}^{n-1}(2)/\!\!/W \to \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X})$ denote the closed immersion sending

$$(p_1, \cdots, p_{n-1}) \mapsto (p_1, \cdots, p_{n-2}), \left(\begin{smallmatrix} 0 & 1 \\ p_{n-1} & 0 \end{smallmatrix}\right).$$

There is a closed immersion

$$\begin{aligned} \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X}) &\cong \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}_X} \mathfrak{sl}_2(N - N\rho_{\check{G}_X}) \\ &\hookrightarrow \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} \mathfrak{sl}_2(N - N\rho_{\check{G}_X}) \cong \mathcal{Y}, \end{aligned}$$

and hence κ defines a closed immersion

$$\mathfrak{t}^{n-1}(2)/\!\!/W \xrightarrow{\kappa} \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X}) \to \mathcal{Y}.$$

We will denote the above map by $\kappa_{\mathcal{Y}}$.

Proposition 4.5.8. *The $\check{G}$-orbit of the image of $\kappa_{\mathcal{Y}}$ has complement of codimension ≥ 2. Moreover, there is an isomorphism of graded group schemes over $\mathfrak{t}^{n-1}(2)/\!\!/W$:*

$$\mathrm{Spec}\, \mathrm{H}_*^{\mathrm{SO}_{2n-1}}(\Omega \mathbf{R}P^{2n-1}; \mathbf{Z}') \cong \mathfrak{t}^{n-1}(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^{n-1}(2)/\!\!/W.$$

In particular, the conditions of Theorem 3.5.23 hold for the spherical SO_{2n}/μ_2-variety $\mathrm{SO}_{2n}/\mu_2\mathrm{SO}_{2n-1}$.

Proof. Let us denote by Y the SL_2-orbit of the image of $\kappa : \mathfrak{t}^{n-1}(2)/\!\!/W \to \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X})$. Then $Y \cong \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2^{\mathrm{reg}}(N - N\rho_{\check{G}_X})$, and it is well-known that the complement of $\mathfrak{sl}_2^{\mathrm{reg}} \subseteq \mathfrak{sl}_2$ has complement of codimension ≥ 2. Applying Lemma 4.2.10 to the SL_2-equivariant inclusion $Y \hookrightarrow \mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X})$ and the map $\mathrm{SL}_2 \to \check{G} = \mathrm{Spin}_{2n}$, we conclude that the $\check{G}$-orbit of the image of $\kappa_{\mathcal{Y}}$ has complement of codimension ≥ 2.

To prove the second part of the proposition, Proposition 4.5.6 reduces us to showing that there is a graded isomorphism

$$\operatorname{Spec} \mathbf{Z}'[x_1, \cdots, x_{n-1}, a^{\pm 1}, \tfrac{a-a^{-1}}{\prod_{i=1}^{n-1}(x_i - \overline{x}_i)}]^W \cong \mathfrak{t}^{n-1}(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^{n-1}(2)/\!\!/W. \tag{31}$$

Since $\overline{x}_i = -x_i$, and 2 is inverted in $\mathbf{Z}'$, we have $\frac{a-a^{-1}}{\prod_{i=1}^{n-1}(x_i-\overline{x}_i)} = 2^{1-n}\frac{a-a^{-1}}{x_1\cdots x_{n-1}}$. The jth copy of $\mathbf{Z}/2 \subseteq (\mathbf{Z}/2)^{n-1} \subseteq W$ sends $x_j \mapsto -x_j$ and $a \mapsto a^{-1}$, so there is an isomorphism

$$\begin{aligned}\mathbf{Z}'[x_1, \cdots, x_{n-1}, a^{\pm 1}, \tfrac{a-a^{-1}}{\prod_{i=1}^{n-1}(x_i - \overline{x}_i)}]^W &\cong \mathbf{Z}'[p_1, \cdots, p_{n-1}, a + a^{-1}, \tfrac{a-a^{-1}}{x_1\cdots x_{n-1}}] \\ &\cong \mathcal{O}_{\mathfrak{t}^{n-2}(2)/\!\!/W'} \otimes_{\mathbf{Z}'} \mathbf{Z}'[p_{n-1}, a + a^{-1}, \tfrac{a-a^{-1}}{x_1\cdots x_{n-1}}],\end{aligned} \tag{32}$$

where we recall that p_j is the jth elementary symmetric polynomial in the variables $x_1^2, \cdots, x_{n-1}^2$. In particular, $p_{n-1} = (x_1 \cdots x_{n-1})^2$.

On the other hand, there is a graded isomorphism

$$\begin{aligned}\mathfrak{t}^{n-1}(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^{n-1}(2)/\!\!/W &\cong \mathfrak{t}^{n-1}(2)/\!\!/W \times_{(\mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X}))/\mathrm{SL}_2(-2n\rho)} \mathfrak{t}^{n-1}(2)/\!\!/W \\ &\cong \mathfrak{t}^{n-2}(2)/\!\!/W' \times (\mathbf{A}^1(2n-2)/\!\!/(\mathbf{Z}/2) \times_{\mathfrak{sl}_2(N-N\rho_{\check{G}_X})/\mathrm{SL}_2(-2n\rho)} \mathbf{A}^1(2n-2)/\!\!/(\mathbf{Z}/2)).\end{aligned}$$

By construction, the map $\mathbf{A}^1(4n-4) \to \mathfrak{sl}_2(N - N\rho_{\check{G}_X})/\mathrm{SL}_2(-2n\rho)$ is precisely a shifted version of the Kostant slice, so the discussion in [**Dev23a**, Remark B.4] implies that if we write p_{n-1} to denote the coordinate on $\mathbf{A}^1(4n-4)$, there is an isomorphism

$$\begin{aligned}\mathbf{A}^1(2n-2)/\!\!/(\mathbf{Z}/2) \times_{\mathfrak{sl}_2(N-N\rho_{\check{G}_X})/\mathrm{SL}_2(-2n\rho)} \mathbf{A}^1(2n-2)/\!\!/(\mathbf{Z}/2) &\cong \operatorname{Spec} \mathbf{Z}'[c_{n-1}, a^{\pm 1}, \tfrac{a-a^{-1}}{c_{n-1}}]^{\mathbf{Z}/2} \\ &\cong \operatorname{Spec} \mathbf{Z}'[p_{n-1}, a + a^{-1}, \tfrac{a-a^{-1}}{c_{n-1}}],\end{aligned}$$

where $c_{n-1}^2 = p_{n-1}$. This, along with (32), implies (31); it is not difficult to observe that (31) is in fact an isomorphism of group schemes. $\square$

Proof of Theorem 4.5.1. This follows from Proposition 4.5.8 and Theorem 3.5.23, along with the identification between $\mathcal{Y}/\check{G}(-2\rho)$ and $\mathfrak{t}^{n-2}(2)/\!\!/W' \times \mathfrak{sl}_2(N - N\rho_{\check{G}_X})/\check{G}_X(-N\rho_{\check{G}_X})$. $\square$

Remark 4.5.9. Note that the normalization term $\mathfrak{t}^{n-2}(2)/\!\!/W'$ identifies with $\mathfrak{so}_{2n-3}(2)/\!\!/\mathrm{SO}_{2n-3}$, which is the group $L_X^\wedge$ from [**KS17**].

4.6. Type F_4. Our goal in this section is to prove Theorem 3.4.13 in type F_4, i.e., for the spherical F_4-variety F_4/Spin_9. Let $G = F_2$ and $H = \mathrm{Spin}_9$, so that $\check{G} = F_4$. Recall from Table 3 that $\check{G}_X = \mathrm{SL}_2$.

Theorem 4.6.1 (Theorem 3.4.13 in type F_4)**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(T^*[16](\mathbf{A}^2[22,0])/\mathrm{SL}_2[-22\rho_{\check{G}_X}] \times \mathfrak{sp}_6^*[2]/\!\!/\mathrm{Sp}_6).$$

The proof of Theorem 4.6.1 will take up the remainder of this section; in fact, we will prove a ku-theoretic deformation.

Lemma 4.6.2. *There is a homotopy equivalence* $(F_4/\mathrm{Spin}_9)(\mathbf{C}) \simeq \mathbf{O}P^2$. *Moreover, if* $B \subseteq G$ *is the Borel subgroup of upper-triangular matrices, the Levi quotient* $L(X)$ *of the parabolic subgroup stabilizing the open* B*-orbit in* F_4/Spin_9 *is given by* Spin_7.

Proposition 4.6.3. *Let* $W = (\mathbf{Z}/2)^4 \rtimes \Sigma_4$ *denote the Weyl group of* Spin_9. *There is an isomorphism of graded* $\pi_*\mathrm{ku}_{\mathrm{Spin}_9}$*-algebras*

$$\mathrm{ku}_*^{\mathrm{Spin}_9}(\Omega\mathbf{O}P^2) \cong \mathbf{Z}'[\beta, p_1, \cdots, p_4, b]/bp_4,$$

where b *lives in weight* 22.

Proof. The argument is essentially the same as that of the preceding subsections. Let $T^4 \subseteq \mathrm{Spin}_9$ denote the maximal torus; we will begin by describing $\mathrm{ku}_*^{T^4}(\Omega\mathbf{O}P^2)$. The homotopy fixed points spectral sequence for $\pi_*\mathrm{ku}[\Omega\mathbf{O}P^2]^{hT^4}$ is given by

$$E_2^{*,*} \cong \mathrm{ku}_*(\Omega\mathbf{O}P^2) \otimes_{\pi_*\mathrm{ku}} \pi_*\mathrm{ku}^{T^4} \Rightarrow \pi_*\mathrm{ku}[\Omega\mathbf{O}P^2]^{hT^4}.$$

To compute the E_2-page, we need to compute $\mathrm{ku}_*(\Omega\mathbf{O}P^2)$. Using the Atiyah-Hirzebruch spectral sequence, we will first calculate $\mathrm{H}_*(\Omega\mathbf{O}P^2;\mathbf{Z}')$. Although there is no Hopf fibration $S^7 \to S^{23} \to \mathbf{O}P^2$ (otherwise, the cofiber of the map $S^{23} \to \mathbf{O}P^2$ would provide a contradiction to the Hopf invariant one problem), we can instead compute $\mathrm{H}_*(\Omega\mathbf{O}P^2;\mathbf{Z}')$ using the Serre spectral sequence for the fibration

$$\Omega\mathbf{O}P^2 \to * \to \mathbf{O}P^2$$

and the fact that $\mathrm{H}_*(\mathbf{O}P^2;\mathbf{Z}')$ is isomorphic to a free graded $\mathbf{Z}'$-module on classes $\{1, x_1, x_2\}$ in weights 0, 8, and 16. This is a standard argument: one finds that $\mathrm{H}_*(\Omega\mathbf{O}P^2;\mathbf{Z}') \cong \mathbf{Z}'[a,b]/a^2$ where a lives in weight 7 and b lives in weight 22; the differentials in the Serre spectral sequence are given by

$$d^8(b^j x_1) = ab^j,\ d^8(b^j x_2) = ab^j x_1,\ d^{23}(ab^j x_2) = b^{j+1}.$$

The Atiyah-Hirzebruch spectral sequence for $\mathrm{ku}_*(\Omega\mathbf{O}P^2)$ degenerates at the E_1-page (with no multiplicative extensions), and we obtain an isomorphism $\mathrm{ku}_*(\Omega\mathbf{O}P^2) \cong \mathbf{Z}'[\beta, a, b]/a^2$. Returning to the homotopy fixed points spectral sequence, the above discussion implies that

$$E_2^{*,*} \cong \mathbf{Z}'[\beta, a, b][\![x_1, \cdots, x_4]\!]/a^2.$$

There is a single differential

$$d_2(a) = bx_1\overline{x_1}\cdots x_4\overline{x_4},$$

and the spectral sequence is concentrated in even degrees after running this differential. It therefore collapses on the E_3-page, and we find that there is an isomorphism

$$\pi_*\mathrm{ku}[\Omega\mathbf{O}P^2]^{hT^4} \cong \mathbf{Z}'[\beta, b][\![x_1, \cdots, x_4]\!]/bx_1\overline{x_1}\cdots x_4\overline{x_4}.$$

To calculate $\mathrm{ku}_*^{T^4}(\Omega\mathbf{O}P^2)$ itself (and not just its completion $\pi_*\mathrm{ku}[\Omega\mathbf{O}P^2]^{hT^4}$), the strategy of Theorem 4.1.13 reduces us to showing that the restriction $\mathrm{ku}_*^{T^4}(\Omega\mathbf{O}P^2)|_{T^\circ_\beta} = \pi_*\mathrm{ku}_{T^4}$. By Lemma 2.4.6, there is an isomorphism $\mathrm{ku}_*^{T^4}(\Omega\mathbf{O}P^2)|_{T^\circ_\beta} \cong \mathrm{ku}_*^{T^4}(\Omega(\mathbf{O}P^2)^{T^4})|_{T^\circ_\beta}$. It therefore suffices to show that $\Omega(\mathbf{O}P^2)^{T^4}$ is contractible, but this is a consequence of the simple observation that $(\mathbf{O}P^2)^{T^4} \cong S^0$. This discussion gives an isomorphism of graded $\pi_*\mathrm{ku}_{T^4}$-algebras

$$\mathrm{ku}_*^{T^4}(\Omega\mathbf{O}P^2) \cong \mathbf{Z}'[\beta, x_1, \cdots, x_4, \prod_{j=1}^{4} \tfrac{1}{1+\beta x_j}, b]/bx_1\overline{x_1}\cdots x_4\overline{x_4}.$$

This isomorphism is W-equivariant (where W is the Weyl group of Spin_9), so Proposition 2.3.15 implies that there is an isomorphism of graded $\pi_*\mathrm{ku}_{\mathrm{Spin}_9}$-algebras

$$\mathrm{ku}_*^{\mathrm{Spin}_9}(\Omega\mathbf{O}P^2) \cong \left(\mathbf{Z}'[\beta, x_1, \cdots, x_4, \prod_{j=1}^{4} \tfrac{1}{1+\beta x_j}, b]/bx_1\overline{x}_1\cdots x_4\overline{x_4}\right)^W.$$

Noting that the action of W leaves b invariant, Lemma 4.5.5 computes the right-hand side; the resulting answer is precisely the right-hand side of the proposition. $\square$

Construction 4.6.4. Let $\check{V}_\beta$ denote the graded affine scheme $\mathbf{A}^1(-6) \times T^4_\beta /\!\!/ W$, and let $\kappa : T^4_\beta/\!\!/W \to \check{V}_\beta$ denote the map sending $\vec{p} \mapsto (0, \vec{p})$. There is an action of $\mathbf{G}_a(-22)$ on $\check{V}_\beta$, where $b \in \mathbf{G}_a(-22)$ sends

$$(x, \vec{p}) \mapsto (x - bp_4, \vec{p}).$$

Equip $\check{G}_X = \mathrm{SL}_2$ with the grading coming from $22\rho_{\mathrm{SL}_2}$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ are equipped with the following weights: a and d have weight 0, b has weight 22, and c has weight -22. Let V denote the affine space $\mathbf{A}^2(22, 0)$, so that so that there is an action of $\check{G}_X = \mathrm{SL}_2$ on V via

$$\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \cdot (x, y) = (ax + cy, bx + dy);$$

here, x lives in degree -22 and y lives in degree 0. There is an isomorphism

$$\mathbf{G}_a(-22)\backslash\check{G}_X \cong V - \{0\},\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \mapsto (c, d),$$

and the above action of $\check{G}_X$ on V restricts on $V - \{0\}$ to the right-action of $\check{G}_X$ on $\mathbf{G}_a(-22)\backslash\check{G}_X$. There is a fibration

$$\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta \to \mathbf{G}_a(-22)\backslash\check{G}_X \cong V - \{0\}$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\overline{\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta}$ denote the affine closure of $\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta$, so that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta} \to \overline{\mathbf{G}_a(-22)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\mathcal{Y}_\beta$ denote the induction

$$\mathcal{Y}_\beta = \mathrm{Ind}_{\check{G}_X}^{\check{G}}\overline{\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta}.$$

Lemma 4.2.10 implies:

Lemma 4.6.5. *There is an open immersion*

$$\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}}\check{V}_\beta \to \mathrm{Ind}_{\check{G}_X}^{\check{G}}\overline{\mathrm{Ind}_{\mathbf{G}_a(-22)}^{\check{G}_X}\check{V}_\beta} = \mathcal{Y}_\beta$$

which exhibits the target as the affine closure of the source, and whose complement is of codimension ≥ 2.

The map $\kappa : T^4_\beta /\!\!/ W \to \check{V}_\beta$ defines a locally closed immersion

$$T^4_\beta /\!\!/ W \xrightarrow{\kappa} \check{V}_\beta \hookrightarrow \mathrm{Ind}^{\check{G}}_{\mathbf{G}_a(-22)} \check{V}_\beta \hookrightarrow \mathrm{Ind}^{\check{G}}_{\check{G}_X} \overline{\mathrm{Ind}^{\check{G}_X}_{\mathbf{G}_a(-22)} \check{V}_\beta},$$

which we will denote by $\kappa_{\mathcal{Y}_\beta}$.

Proposition 4.6.6. *There is an isomorphism of graded group schemes over $T^4_\beta /\!\!/ W$:*

$$\mathrm{Spec}\,\mathrm{ku}_*^{\mathrm{Spin}_9}(\Omega \mathbf{O}P^2) \cong T^4_\beta /\!\!/ W \times_{\mathcal{Y}_\beta/\check{G}(-2\rho)} T^4_\beta /\!\!/ W.$$

Moreover, if $\check{J}_{X,\beta}$ denotes the above group scheme over $T^4_\beta /\!\!/ W$, the algebra of regular functions on $(T^4_\beta /\!\!/ W \times \check{G})/\check{J}_{X,\beta}$ is isomorphic to $\mathcal{O}_{\mathcal{Y}_\beta}$.

Proof. There is an isomorphism

$$T^4_\beta /\!\!/ W \times_{\mathcal{Y}_\beta/\check{G}(-2\rho)} T^4_\beta /\!\!/ W \cong T^4_\beta /\!\!/ W \times_{\check{V}_\beta/\mathbf{G}_a(-22)} T^4_\beta /\!\!/ W,$$

as well as a closed immersion

$$T^4_\beta /\!\!/ W \times_{\check{V}_\beta/\mathbf{G}_a(-22)} T^4_\beta /\!\!/ W \hookrightarrow T^4_\beta /\!\!/ W \times \mathbf{G}_a(-22),$$

which exhibits $T^4_\beta /\!\!/ W \times_{\check{V}_\beta/\mathbf{G}_a(-22)} T^4_\beta /\!\!/ W$ as the subscheme of tuples $(\vec{p}, b)$ such that b stabilizes $\kappa(\vec{p})$. By definition of κ, this happens if and only if $bp_4 = 0$, which gives an isomorphism of graded schemes over $T^4_\beta /\!\!/ W$:

$$T^4_\beta /\!\!/ W \times_{\check{V}_\beta/\mathbf{G}_a(-22)} T^4_\beta /\!\!/ W \cong \mathrm{Spec}\,\mathbf{Z}'[\beta, p_1, \cdots, p_4, b]/bp_4.$$

On the other hand, Proposition 4.6.3 gives an isomorphism of graded $\pi_*\mathrm{ku}_{\mathrm{Spin}_9}$-algebras

$$\mathrm{ku}_*^{\mathrm{Spin}_9}(\Omega \mathbf{O}P^2) \cong \mathbf{Z}'[\beta, p_1, \cdots, p_4, b]/bp_4,$$

which implies the first part of the proposition. The second part of the proposition follows from Lemma 4.3.8 (rather, its obvious variant for graded affine spaces). $\square$

Proposition 4.6.6 and Proposition 3.6.23 imply:

Corollary 4.6.7. *Let $G = F_4$ and $H = \mathrm{Spin}_9$, so $\check{G} = F_4$ and $\check{G}_X = \mathrm{SL}_2$. There is an equivalence of $\mathrm{sh}^{1/2}(\mathbf{Z}'[\beta])$-linear ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\mathcal{Y}_\beta/F_4).$$

Proof of Theorem 4.6.1. Combining Corollary 4.6.7 with Proposition 3.6.19, we see that if $\mathcal{Y} = \mathcal{Y}_\beta/\beta$, there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\mathcal{Y}/F_4).$$

It suffices to compute $\mathcal{Y}/F_4$. Let W' denote the Weyl group of Spin_7, so that $W' \cong (\mathbf{Z}/2)^3 \rtimes \Sigma_3$ acts on $\mathfrak{t}^3(2)$ such that $\mathfrak{t}^3(2) /\!\!/ W' \cong \mathrm{Spec}\,\mathbf{Z}'[p_1, p_2, p_3]$. Recall that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}^{\check{G}_X}_{\mathbf{G}_a(-22)} \check{V}_\beta} \to \overline{\mathbf{G}_a(-22)\backslash \check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$, which implies (by setting $\beta = 0$) that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}^{\check{G}_X}_{\mathbf{G}_a(-22)} \check{V}} \to \overline{\mathbf{G}_a(-22)\backslash \check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}$. By Lemma 4.2.18, this implies that $\overline{\mathrm{Ind}^{\check{G}_X}_{\mathbf{G}_a(-22)} \check{V}} \cong T^*(16)(V)$, and so the isomorphism between $\mathcal{Y}/F_4$ and $\mathfrak{t}^3(2) /\!\!/ W' \times T^*(16)(V)/\check{G}_X(-22\rho_{\check{G}_X})$ gives the desired claim. $\square$

4.7. Type G_2. Our goal in this section is to prove Theorem 3.4.13 in type G_2, i.e., for the spherical G_2-variety G_2/SL_3. Let $G = G_2$ and $H = \mathrm{SL}_3$, so that $\check{G} = G_2$. Recall from Table 3 that $\check{G}_X = \mathrm{SL}_2$.

Theorem 4.7.1 (Theorem 3.4.13 in type G_2)**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H);\mathbf{Q}) \simeq \mathrm{Perf}(T^*[6](\mathbf{A}^2[10,0])/\mathrm{SL}_2(-10\rho_{\check{G}_X}) \times \mathfrak{sl}_2[2]/\!\!/\mathrm{SL}_2).$$

The proof of Theorem 4.7.1 will take up the remainder of this section; in fact, we will prove a ku-theoretic deformation.

Lemma 4.7.2. *There is a homotopy equivalence $(G_2/\mathrm{SL}_3)(\mathbf{C}) \simeq S^6$. Moreover, if $B \subseteq G$ is the Borel subgroup of upper-triangular matrices, the Levi quotient $L(X)$ of the parabolic subgroup stabilizing the open B-orbit in G_2/SL_3 is given by SL_2.*

Lemma 4.7.3. *Let $W = \Sigma_n$ denote the Weyl group of SL_n. Then there is an isomorphism*

$$\pi_*\mathrm{ku}_{\mathrm{SU}(n)} \cong \mathbf{Z}'[\beta, c_2, \cdots, c_n],$$

where the injective map $\pi_*\mathrm{ku}_{\mathrm{SU}(n)} \to \pi_*\mathrm{ku}_{T^{n-1}}$ sends c_j to the jth elementary symmetric polynomial in the variables $x_1, \cdots, x_{n-1}, x_n$, where x_n is the sum of $x_1, \cdots, x_{n-1}$ in the group structure on $\mathbf{G}_\beta$.

Proposition 4.7.4. *There is an isomorphism of graded $\pi_*\mathrm{ku}_{\mathrm{SU}(3)}$-algebras*

$$\mathrm{ku}_*^{\mathrm{SU}(3)}(\Omega S^6) \cong \mathbf{Z}'[\beta, c_2, c_3, b]/bc_3,$$

where b is in weight 10.

Proof. The restriction of the SU(3)-action on S^6 to $T^2 \subseteq \mathrm{SU}(3)$ exhibits S^6 as the one-point compactification of the T^2-representation with weights λ_1, λ_2, and $\lambda_1 + \lambda_2$. Therefore, Corollary 4.1.17 implies that

$$\mathrm{ku}_*^{\mathrm{SU}(3)}(\Omega S^6) \cong \left(\mathbf{Z}'[\beta, x_1, x_2, \tfrac{1}{(1+\beta x_1)(1+\beta x_2)}, b]/bx_1x_2x_3\right)^{\Sigma_3}.$$

The action of Σ_3 permutes x_1, x_2, and x_3, and leaves b invariant. Therefore, Lemma 4.7.3 implies that this ring of invariants can be identified with $\mathbf{Z}'[\beta, c_2, c_3, b]/bc_3$, as desired. □

Definition 4.7.5. Let $\check{V}_\beta$ denote the graded affine scheme $\mathbf{A}^1(-4) \times T^2_\beta/\!\!/\Sigma_3$, and let $\kappa : T^2_\beta/\!\!/\Sigma_3 \to \check{V}_\beta$ denote the map sending $\vec{c} \mapsto (0, \vec{c})$. There is an action of $\mathbf{G}_a(-10)$ on $\check{V}_\beta$, where $b \in \mathbf{G}_a(-10)$ sends

$$(z, \vec{c}) \mapsto (z - bc_3, \vec{c}).$$

Equip $\check{G}_X = \mathrm{SL}_2$ with the grading coming from $10\rho_{\mathrm{SL}_2}$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are equipped with the following weights: a and d have weight 0, b has weight 10, and c has weight -10. Let V denote the affine space $\mathbf{A}^2(10, 0)$, so that there is an action of $\check{G}_X = \mathrm{SL}_2$ on V via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x, y) = (ax + cy, bx + dy);$$

here, x lives in degree -10 and y lives in degree 0. There is an isomorphism

$$\mathbf{G}_a(-10)\backslash\check{G}_X \cong V - \{0\}, \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c, d),$$

and the above action of $\check{G}_X$ on V restricts on $V - \{0\}$ to the right-action of $\check{G}_X$ on $\mathbf{G}_a(-10)\backslash\check{G}_X$. There is a fibration

$$\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta \to \mathbf{G}_a(-10)\backslash\check{G}_X \cong V - \{0\}$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta}$ denote the affine closure of $\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta$, so that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta} \to \overline{\mathbf{G}_a(-10)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$. Let $\mathcal{Y}_\beta$ denote the induction

$$\mathcal{Y}_\beta = \mathrm{Ind}_{\check{G}_X}^{\check{G}}\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta}.$$

Lemma 4.2.10 implies:

Lemma 4.7.6. *There is an open immersion*

$$\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}}\check{V}_\beta \to \mathrm{Ind}_{\check{G}_X}^{\check{G}}\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta} = \mathcal{Y}_\beta$$

which exhibits the target as the affine closure of the source, and whose complement is of codimension ≥ 2.

The map $\kappa : T^2_\beta/\!\!/\Sigma_3 \to \check{V}_\beta$ defines a locally closed immersion

$$T^2_\beta/\!\!/\Sigma_3 \xrightarrow{\kappa} \check{V}_\beta \hookrightarrow \mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}}\check{V}_\beta \hookrightarrow \mathrm{Ind}_{\check{G}_X}^{\check{G}}\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta},$$

which we will denote by $\kappa_{\mathcal{Y}_\beta}$.

Proposition 4.7.7. *There is an isomorphism of graded group schemes over* $T^2_\beta/\!\!/\Sigma_3$*:*

$$\mathrm{Spec}\,\mathrm{ku}_*^{\mathrm{SU}(3)}(\Omega S^6) \cong T^2_\beta/\!\!/\Sigma_3 \times_{\mathcal{Y}_\beta/\check{G}(-2\rho)} T^2_\beta/\!\!/\Sigma_3.$$

Moreover, if $\check{J}_{X,\beta}$ denotes the above group scheme over $T^2_\beta/\!\!/\Sigma_3$, the algebra of regular functions on $(T^2_\beta/\!\!/\Sigma_3 \times \check{G})/\check{J}_{X,\beta}$ is isomorphic to $\mathcal{O}_{\mathcal{Y}_\beta}$.

Proof. There is an isomorphism

$$T^2_\beta/\!\!/\Sigma_3 \times_{\mathcal{Y}_\beta/\check{G}(-2\rho)} T^2_\beta/\!\!/\Sigma_3 \cong T^2_\beta/\!\!/\Sigma_3 \times_{\check{V}_\beta/\mathbf{G}_a(-10)} T^2_\beta/\!\!/\Sigma_3,$$

as well as a closed immersion

$$T^2_\beta/\!\!/\Sigma_3 \times_{\check{V}_\beta/\mathbf{G}_a(-10)} T^2_\beta/\!\!/\Sigma_3 \hookrightarrow T^2_\beta/\!\!/\Sigma_3 \times \mathbf{G}_a(-10),$$

which exhibits $T^2_\beta/\!\!/\Sigma_3 \times_{\check{V}_\beta/\mathbf{G}_a(-10)} T^2_\beta/\!\!/\Sigma_3$ as the subscheme of pairs $(\vec{c}, b)$ such that b stabilizes $\kappa(\vec{c})$. By definition of κ, this happens if and only if $bc_3 = 0$, which gives an isomorphism of graded schemes over $T^2_\beta/\!\!/\Sigma_3$:

$$T^2_\beta/\!\!/\Sigma_3 \times_{\check{V}_\beta/\mathbf{G}_a(-10)} T^2_\beta/\!\!/\Sigma_3 \cong \mathrm{Spec}\,\mathbf{Z}'[\beta, c_2, c_3, b]/bc_3.$$

Proposition 4.7.4 therefore implies the first part of the proposition. The second part of the proposition follows from Lemma 4.3.8 (rather, its obvious variant for graded affine spaces). $\square$

Proposition 4.7.7 and Proposition 3.6.23 imply:

Corollary 4.7.8. *Let $G = G_2$ and $H = \mathrm{SL}_3$, so $\check{G} = G_2$ and $\check{G}_X = \mathrm{SL}_2$. There is an equivalence of $\mathrm{sh}^{1/2}(\mathbf{Z}'[\beta])$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\mathcal{Y}_\beta/G_2(-2\rho)).$$

Proof of Theorem 4.7.1. Combining Corollary 4.7.8 with Proposition 3.6.19, we see that if $\mathcal{Y} = \mathcal{Y}_\beta/\beta$, there is an equivalence of $\mathbf{Q}$-linear ∞-categories

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\mathcal{Y}/G_2(-2\rho)).$$

It suffices to compute $\mathcal{Y}/G_2(-2\rho)$. Let $W' = \mathbf{Z}/2$ denote the Weyl group of SL_2, so that it acts on $\mathfrak{t}^1(2)$ such that $\mathfrak{t}^1(2)/\!\!/W' \cong \mathbf{A}^1(4)$. Recall that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}_\beta} \to \overline{\mathbf{G}_a(-10)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}_\beta$, which implies (by setting $\beta = 0$) that there is a $\check{G}_X$-equivariant fibration

$$\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}} \to \overline{\mathbf{G}_a(-10)\backslash\check{G}_X} \cong V$$

whose fibers are isomorphic to $\check{V}$. By Lemma 4.2.18, this implies that $\overline{\mathrm{Ind}_{\mathbf{G}_a(-10)}^{\check{G}_X}\check{V}} \cong T^*(6)(V)$, and so the isomorphism between $\mathcal{Y}/G_2(-2\rho)$ and $\mathfrak{t}^1(2)/\!\!/W' \times T^*(6)(V)/\check{G}_X(-10\rho_{\check{G}_X})$ implies the desired result. $\square$

4.8. Type B_3'. We will only work with coefficients in $\mathbf{Z}'$ below. Our goal in this section is to prove Theorem 3.4.13 in type B_3', i.e., for the spherical SO_7-variety SO_7/G_2. Write $G = \mathrm{SO}_7$ and $H = G_2$, so that $\check{G} = \mathrm{Sp}_6$. Recall from Table 3 that $\check{G}_X = \mathrm{SL}_2$.

Theorem 4.8.1 (Theorem 3.4.13 in type B_3')**.** *There is an equivalence of $\mathbf{Q}$-linear ∞-categories*

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) \cong \mathrm{Perf}(\mathfrak{sl}_2[6 - 6\rho_{\check{G}_X}]/\mathrm{SL}_2[-6\rho_{\check{G}_X}] \times \mathfrak{sp}_2[2]/\!\!/\mathrm{Sp}_2).$$

Lemma 4.8.2. *There is a homotopy equivalence $(\mathrm{Spin}_7/G_2)(\mathbf{C}) \cong S^7$, which implies that $(\mathrm{SO}_7/G_2)(\mathbf{C}) \cong \mathbf{R}P^7$. Moreover, if $B \subseteq G$ is the Borel subgroup of upper-triangular matrices, the Levi quotient $L(X)$ of the parabolic subgroup stabilizing the open B-orbit in SO_7/G_2 is given by SL_3.*

Lemma 4.8.3. *Let $W = \Sigma_3 \times \mathbf{Z}/2$ denote the Weyl group of G_2. Then there is an isomorphism*

$$\mathrm{H}^*_{G_2}(*; \mathbf{Z}') \cong \mathbf{Z}'[c_2, c_6],$$

where the injective map $\mathrm{H}^*_{G_2}(*; \mathbf{Z}') \to \mathrm{H}^*_{T^2}(*; \mathbf{Z}')$ sends

$$c_2 \mapsto x_1^2 + x_2^2 + x_1x_2,$$
$$c_6 \mapsto x_1^2x_2^2(x_1 + x_2)^2.$$

Proof. The action of $\Sigma_3 \subseteq W$ on $\mathrm{H}^*_{T^2}(*; \mathbf{Z}') = \mathbf{Z}'[x_1, x_2]$ is given by the reduced standard representation (i.e., x_1, x_2, and $-(x_1 + x_2)$ are permuted), and the action of $\mathbf{Z}/2 \subseteq W$ is given by negating the x_i simultaneously. It follows that

$$\mathrm{H}^*_{G_2}(*; \mathbf{Z}') = \mathrm{H}^*_{T^2}(*; \mathbf{Z}')^W \cong (\mathbf{Z}'[x_1, x_2]^{\Sigma_3})^{\mathbf{Z}/2} \cong \mathbf{Z}'[x_1^2+x_2^2+x_1x_2, x_1x_2(x_1+x_2)]^{\mathbf{Z}/2},$$

which is precisely $\mathbf{Z}'[c_2, c_6]$, as desired. $\square$

Proposition 4.8.4. *There is an isomorphism of graded $\pi_*\mathrm{ku}_{G_2}$-algebras*

$$\mathrm{ku}_*^{G_2}(\Omega\mathbf{R}P^7) \cong \mathbf{Z}'[\beta, x_1, x_2, \tfrac{1}{(1+\beta x_1)(1+\beta x_2)}, a^{\pm 1}, \tfrac{a-a^{-1}}{(x_1-\overline{x}_1)(x_2-\overline{x}_2)(x_3-\overline{x}_3)}]^W,$$

where $x_3 = x_1 + x_2 + \beta x_1 x_2$ is the sum of the two weights of x_1 and x_2 in the group structure on $\mathbf{G}_\beta$. Here, the action of $\mathbf{Z}/2 \subseteq W$ sends $x_j \mapsto \overline{x}_j$ and $a \mapsto a^{-1}$, and the symmetric group acts by permuting the variables x_1, x_2, x_3 (and leaves a invariant).

Proof. The restriction of the G_2-action on $(\mathrm{Spin}_7/G_2)(\mathbf{C}) \simeq S^7$ exhibits it as the one-point compactification of $\mathrm{std} \oplus \mathbf{R}$, where std is the standard 3-dimensional complex representation of G_2. In particular, the restriction of the G_2-action on S^7 to $T^2 \subseteq G_2$ exhibits it as the one-point compactification of the 3-dimensional representation of T^2 with weights λ_1, λ_2, and $\lambda_1 + \lambda_2$. Given this observation, the isomorphism of the proposition is a consequence of Corollary 4.1.17 (just as with Proposition 4.5.6). □

Construction 4.8.5. Let W' denote the Weyl group of Sp_2, so that $\mathfrak{t}^1(2)/\!\!/W' \cong \mathbf{A}^1(4)$. Equip $\check{G}_X = \mathrm{SL}_2$ with the grading by -6ρ, and consider the $\check{G}_X = \mathrm{SL}_2$-scheme $\mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X})$ (where SL_2 acts only on the factor $\mathfrak{sl}_2(6-6\rho_{\check{G}_X})$ by the adjoint action). Define

$$\mathcal{Y} = \mathfrak{t}^1(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X}).$$

Let $\kappa : \mathfrak{t}^2(2)/\!\!/W \to \mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X})$ denote the closed immersion sending

$$(c_2, c_6) \mapsto c_2, \left(\begin{smallmatrix} 0 & 1 \\ c_6 & 0 \end{smallmatrix}\right).$$

There is a closed immersion

$$\begin{aligned}\mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X}) &\cong \mathfrak{t}^1(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}_X} \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X}) \\ &\hookrightarrow \mathfrak{t}^1(2)/\!\!/W' \times \mathrm{Ind}_{\check{G}_X}^{\check{G}} \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X}) \cong \mathcal{Y},\end{aligned}$$

and hence κ defines a closed immersion

$$\mathfrak{t}^2(2)/\!\!/W \xrightarrow{\kappa} \mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X}) \to \mathcal{Y}.$$

We will denote the above map by $\kappa_{\mathcal{Y}}$.

Proposition 4.8.6. *The $\check{G}$-orbit of the image of $\kappa_{\mathcal{Y}}$ has complement of codimension ≥ 2. Moreover, there is an isomorphism of graded group schemes over $\mathfrak{t}^2(2)/\!\!/W$:*

$$\mathrm{Spec}\,\mathrm{H}_*^{G_2}(\Omega\mathbf{R}P^7; \mathbf{Z}') \cong \mathfrak{t}^2(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^2(2)/\!\!/W.$$

In particular, the conditions of Theorem 3.5.23 hold for the spherical SO_7-variety SO_7/G_2.

Proof. Let us denote by Y the SL_2-orbit of the image of $\kappa : \mathfrak{t}^2(2)/\!\!/W \to \mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X})$. Then $Y \cong \mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2^{\mathrm{reg}}(6)$, and it is well-known that the complement of $\mathfrak{sl}_2^{\mathrm{reg}} \subseteq \mathfrak{sl}_2$ has complement of codimension ≥ 2. Applying Lemma 4.2.10 to the SL_2-equivariant inclusion $Y \hookrightarrow \mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6 - 6\rho_{\check{G}_X})$ and the map $\mathrm{SL}_2 \to \check{G} = \mathrm{Sp}_6$, we conclude that the $\check{G}$-orbit of the image of $\kappa_{\mathcal{Y}}$ has complement of codimension ≥ 2.

It follows from Proposition 4.8.4 that there is an isomorphism of graded schemes over $\mathfrak{t}^2(2)/\!\!/W$:

$$\mathrm{Spec}\,\mathrm{H}_*^{G_2}(\Omega\mathbf{R}P^7; \mathbf{Z}') \cong \mathbf{Z}'[x_1, x_2, a^{\pm 1}, \tfrac{a-a^{-1}}{x_1 x_2 (x_1+x_2)}]^W.$$

The action of $\Sigma_3 \subseteq W$ on $\mathrm{H}^*_{T^2}(*;\mathbf{Z}') = \mathbf{Z}'[x_1, x_2]$ is given by the reduced standard representation (i.e., x_1, x_2, and $-(x_1+x_2)$ are permuted, and a is fixed), and the action of $\mathbf{Z}/2 \subseteq W$ is given by negating the x_i simultaneously and sending $a \mapsto a^{-1}$. It follows that

$$\mathbf{Z}'[x_1, x_2, a^{\pm 1}, \tfrac{a-a^{-1}}{x_1x_2(x_1+x_2)}]^W \cong \mathbf{Z}'[c_2, c_6, a+a^{-1}, \tfrac{a-a^{-1}}{x_1x_2(x_1+x_2)}].$$

On the other hand, there is a graded isomorphism

$$\begin{aligned}\mathfrak{t}^2(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^2(2)/\!\!/W &\cong \mathfrak{t}^2(2)/\!\!/W \times_{(\mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6-6\rho_{\check{G}_X}))/\mathrm{SL}_2(-6\rho)} \mathfrak{t}^2(2)/\!\!/W \\ &\cong \mathfrak{t}^1(2)/\!\!/W' \times (\mathbf{A}^1(6)/\!\!/(\mathbf{Z}/2) \times_{\mathfrak{sl}_2(6-6\rho_{\check{G}_X})/\mathrm{SL}_2(-6\rho)} \mathbf{A}^1(6)/\!\!/(\mathbf{Z}/2)).\end{aligned}$$

By construction, the map $\mathbf{A}^1(6)/\!\!/(\mathbf{Z}/2) \cong \mathbf{A}^1(12) \to \mathfrak{sl}_2(6-6\rho_{\check{G}_X})/\mathrm{SL}_2(-6\rho)$ is precisely a shifted version of the Kostant slice, so the discussion in [**Dev23a**, Remark B.4] implies that if we write c_6 to denote the coordinate on $\mathbf{A}^1(12)$, there is an isomorphism

$$\begin{aligned}\mathbf{A}^1(6)/\!\!/(\mathbf{Z}/2) \times_{\mathfrak{sl}_2(6-6\rho_{\check{G}_X})/\mathrm{SL}_2(-6\rho)} \mathbf{A}^1(6)/\!\!/(\mathbf{Z}/2) &\cong \operatorname{Spec} \mathbf{Z}'[c_3, a^{\pm 1}, \tfrac{a-a^{-1}}{x_1x_2(x_1+x_2)}]^{\mathbf{Z}/2} \\ &\cong \operatorname{Spec} \mathbf{Z}'[c_6, a+a^{-1}, \tfrac{a-a^{-1}}{c_3}],\end{aligned}$$

where $c_3^2 = c_6$. Therefore, there is an isomorphism

$$\mathfrak{t}^2(2)/\!\!/W \times_{\mathcal{Y}/\check{G}(-2\rho)} \mathfrak{t}^2(2)/\!\!/W \cong \operatorname{Spec} \mathbf{Z}'[c_2, c_6, a+a^{-1}, \tfrac{a-a^{-1}}{x_1x_2(x_1+x_2)}],$$

which gives the desired claim. $\square$

Proof of Theorem 4.8.1. This follows from Proposition 4.8.6 and Theorem 3.5.23, along with the isomorphism between $\mathcal{Y}/\check{G}(-2\rho)$ and $\mathfrak{t}^1(2)/\!\!/W' \times \mathfrak{sl}_2(6-6\rho_{\check{G}_X})/\check{G}_X(-6\rho_{\check{G}_X})$. $\square$

Remark 4.8.7. Note that the normalization term $\mathfrak{t}^{n-2}(2)/\!\!/W'$ identifies with $\mathfrak{so}_3(2)/\!\!/\mathrm{SO}_3$, which is the group $L_X^\wedge$ from [**KS17**].

5. Structures on the spectral side

5.1. ku-Hamiltonian varieties. Our goal in this section is to place the calculations of the preceding section into a broader context. The basic topic of study in this section is the β-deformation of a group scheme introduced in Definition 3.6.3. We will soon focus on the graded quotient stack $\check{G}(-2\rho)_\beta/\check{G}(-2\rho)$ appearing in Theorem 3.6.21.

Remark 5.1.1. Since we will work in the setting of graded schemes, and both $\check{G}(-2\rho)_\beta$ and $\check{G}(-2\rho)$ are shifted by the same cocharacter of $\check{T}$, we can (and will) simply ignore this cocharacter. In other words, we will focus only on the quotient stack $\check{G}_\beta/\check{G}$, instead of $\check{G}(-2\rho)_\beta/\check{G}(-2\rho)$.

We will begin with a brief review of the theory of shifted symplectic stacks; for optimal generality, we will work over a discrete commutative ring R, and all reductive groups will be assumed to be split.

Recollection 5.1.2 ([**PTVV13, Cal15**]). Let X be a derived R-stack which admits a (co)tangent complex which is a perfect $\mathcal{O}_X$-module, and let $\mathrm{F}^\star_{\mathrm{H}}\mathrm{dR}_{X/R}$ denote the Hodge-filtered de Rham complex of X relative to R. A *closed j-form of degree n* on X is a global section ω of $\mathrm{F}^{\geq j}_{\mathrm{H}}\mathrm{dR}_{X/R}[n-j]$; let $\Omega^{j,\mathrm{cl}}_{BG/R,n}$ denote $\mathrm{H}^0(X;\mathrm{F}^{\geq j}_{\mathrm{H}}\mathrm{dR}_{X/R}[n-j])$. A closed 2-form ω of degree n defines an *n-shifted symplectic structure* on X if the section of $(\wedge^2 L_{X/R})[n] \simeq \mathrm{Sym}^2(L_{X/R}[1])[n-2]$ defined by the image of ω under the map

$$\mathrm{F}^{\geq 2}_{\mathrm{H}}\mathrm{dR}_{X/R}[n-2] \to \mathrm{gr}^2_{\mathrm{H}}\mathrm{dR}_{X/R}[n-2] \cong (\wedge^2 L_{X/R})[n]$$

defines an equivalence $T_{X/R} \xrightarrow{\sim} L_{X/R}[n]$. If X is an n-shifted symplectic stack (the closed 2-form will be left implicit in the notation), let $\overline{X}$ denote X equipped with the opposite symplectic structure.

Let X be an n-shifted symplectic stack, and let $f : L \to X$ be a morphism of derived R-stacks, where L and f each admit perfect (co)tangent complexes. An *isotropic structure* on f is a nullhomotopy of the composite

$$T_L \to f^*T_X \overset{\omega}{\simeq} f^*L_X[n] \to L_L[n].$$

An isotropic structure is called *Lagrangian* if the above composite is a cofibe sequence. If X and Y are n-shifted symplectic stacks, a *Lagrangian correspondence* is a Lagrangian morphism $L \to X \times \overline{Y}$. By [**Saf16**, Theorem 1.2], if $L_1 \to X \times \overline{Y}$ and $L_2 \to Y \times \overline{Z}$ are Lagrangian correspondences, the fiber product $L_1 \times_Y L_2 \to X \times \overline{Z}$ is also a Lagrangian correspondence. As a special case, if $L_1, L_2 \to X$ are Lagrangian morphisms to an n-shifted symplectic stack, the fiber product $L_1 \times_X L_2$ admits the structure of an $(n-1)$-shifted symplectic stack.

Proposition 5.1.3 ([**PTVV13**]). *Let G be a reductive group. Then any nondegenerate G-invariant bilinear form on $\mathfrak{g}^*$ defines a 2-shifted symplectic structure on BG.*

Proof. The object $L_{BG/R} \in \mathrm{Perf}(BG)$ can be identified with the coadjoint representation $\mathfrak{g}^*[-1]$. The underlying graded R-algebra of the de Rham complex $\mathrm{dR}_{BG/R}$ can be identified with $\Gamma^*_R(L_{BG/R}[-1]) \cong \Gamma^*_R(\mathfrak{g}^*[-2])$. There is a décalage isomorphism $\Gamma^n_R(M[-2]) \simeq \mathrm{Sym}^n_R(M)[-2n]$ for any bounded-below R-module M, and hence an isomorphism $\Gamma^*_R(\mathfrak{g}^*[-2]) \cong \mathrm{sh}\,\mathrm{Sym}^*_R(\mathfrak{g}^*(-1))$. The de Rham differential is trivial, and so $\Omega^{2,\mathrm{cl}}_{BG/R,n} \cong \mathrm{H}^0(BG; \mathrm{Sym}^2_R(\mathfrak{g}^*)[n-2])$. This is nonzero if and only if $n = 2$, in which case it is $\mathrm{Sym}^2_R(\mathfrak{g}^*)^G$. Therefore, any G-invariant bilinear form on $\mathfrak{g}^*$ defines a closed 2-form of degree 2 on BG. It is easy to see that this closed 2-form defines a 2-shifted symplectic structure if and only if the G-invariant bilinear form on $\mathfrak{g}^*$ is nondegenerate. $\square$

Proposition 5.1.4 ([**PTVV13**, Theorem 2.5], [**Saf16**, Theorem 3.5])**.** *Let X be an d-oriented stack, i.e., a stack equipped with a map $\Gamma(X; \mathcal{O}_X) \to R[-d]$ such that for any animated R-algebra A and any $\mathcal{F} \in \mathrm{Perf}(X \otimes_R A)$, the induced map $\Gamma(X \otimes_R A; \mathcal{F})^\vee \to \Gamma(X \otimes_R A; \mathcal{F}[-d]^\vee)$ is an isomorphism. The data of an n-shifted symplectic structure on Y equips the mapping stack $\mathrm{Map}(X, Y)$ with an $(n-d)$-shifted symplectic structure.*

More generally, if $L \to Y$ is a Lagrangian morphism, the induced map $\mathrm{Map}(X, L) \to \mathrm{Map}(X, Y)$ acquires a natural Lagrangian structure.

Example 5.1.5. Let $\mathbf{G}_a^\sharp(-2)$ denote the divided power hull of the origin in $\mathbf{G}_a$ equipped with a $\mathbf{G}_m$-action of weight -2, and let $\widehat{\mathbf{G}_a^\sharp(-2)}$ denote the completion of $\mathbf{G}_a^\sharp(-2)$ at the divided power filtration. Then $X = B\widehat{\mathbf{G}_a^\sharp(-2)}$ is a 1-oriented stack: the cohomology of its structure sheaf is isomorphic to $R[\epsilon]/\epsilon^2$ with ϵ in homological degree -1 (and weight -2). Moreover, if Y is any derived R-stack which admits a cotangent complex, and $\widehat{T_Y^\sharp(2)}$ denotes the completion of the PD-hull of the zero section of the tangent bundle of Y with respect to the divided power filtration, the mapping stack $\mathrm{Map}(B\widehat{\mathbf{G}_a^\sharp(-2)}, Y)$ can be identified with the stack $\widehat{T_Y^\sharp[-1](2)}$. It follows from Proposition 5.1.4 that the data of an n-shifted symplectic structure on Y equips $\widehat{T_Y^\sharp[-1](2)}$ with an $(n-1)$-shifted symplectic structure.

Let G be a reductive group over R. When $Y = BG$, there is an isomorphism $T_{BG}^\sharp(1) \cong \mathfrak{g}[1](2) \in \mathrm{Perf}(BG)$, and the divided power filtration corresponds to the filtration by infinitesimal thickenings of the origin. Since the coordinate of $\mathfrak{g}[1](2)$ has nonzero weight, completing with respect to this filtration has no effect, and we conclude that $\widehat{T_{BG}^\sharp[-1](2)} \cong \mathfrak{g}(2)/G$. It follows from Proposition 5.1.3 that $\mathfrak{g}(2)/G$ admits a 1-shifted symplectic structure of weight 2; forgetting the grading, the same is true of $\mathfrak{g}/G$.

Example 5.1.6. Let S^1 denote the constant stack $B\mathbf{Z}$. Then S^1 is a 1-oriented stack, since $\Gamma(S^1; \mathcal{O}) \cong C^*(S^1; R)$, and the circle admits a canonical orientation. It follows from Proposition 5.1.4 that the data of an n-shifted symplectic structure on Y equips $\mathrm{Map}(S^1, Y)$ with an $(n-1)$-shifted symplectic structure. In particular, if G is a reductive group over R, then applying Proposition 5.1.3 implies that $\mathrm{Map}(S^1, BG) \cong G/G$ admits a 1-shifted symplectic structure.

Proposition 5.1.7 ([**Saf16**])**.** *Let $R = \mathbf{C}$, and let G be a complex reductive group. A Lagrangian morphism $L \to \mathfrak{g}/G$ is equivalent to the data of a Hamiltonian G-variety. Similarly, a Lagrangian morphism $L \to G/G$ is equivalent to the data of a quasi-Hamiltonian G-variety in the sense of* [**AMM98**].

The reader not familiar with the definition of a quasi-Hamiltonian G-variety can take the second part of Proposition 5.1.7 to be a definition. Motivated by Lemma 3.6.4, we make the following observation.

Lemma 5.1.8. *The stack $B\mathbf{G}_\beta^\vee$ admits a canonical* 1*-orientation of weight* 2*.*

Proof. We will just construct the orientation on $\mathbf{G}_\beta^\vee$, and leave verifying the properties of Proposition 5.1.4 to the reader. To compute the cohomology $\mathrm{H}^*(B\mathbf{G}_\beta^\vee; \mathcal{O})$, let us first compute $\mathbf{G}_\beta^\vee$. As shown in [**Dev23c**, Proposition C.6], the Cartier dual $\mathbf{G}_\beta^\vee$ is isomorphic to $\mathrm{Spf}\, \mathbf{Z}[\beta, \frac{1}{n!}\prod_{j=0}^{n-1}(y-j\beta)]^\wedge$ where the element y is primitive (i.e., the coproduct sends $y \mapsto y\otimes 1 + 1\otimes y$) and lives in weight 2. Here, the completion is taken with respect to the β-deformed divided power filtration (i.e., with respect to $\frac{1}{n!}\prod_{j=0}^{n-1}(y-j\beta)$ for $n \geq 1$). Computing the cohomology of the trivial $\mathbf{G}_\beta^\vee$-representation in the standard manner shows that $\mathrm{H}^0(B\mathbf{G}_\beta^\vee; \mathcal{O}) \cong R$, $\mathrm{H}^1(B\mathbf{G}_\beta^\vee; \mathcal{O})$ is isomorphic to the submodule of primitive elements in $\mathcal{O}_{\mathbf{G}_\beta^\vee}$, and $\mathrm{H}^j(B\mathbf{G}_\beta^\vee; \mathcal{O})$ is zero for $j > 1$. It is not difficult to see that the only primitive elements in $\mathcal{O}_{\mathbf{G}_\beta^\vee}$ are scalar multiples of y, and so the cohomology ring $\mathrm{H}^*(B\mathbf{G}_\beta^\vee; \mathcal{O})$ is exterior on a single class in cohomological degree 1 and weight 2. This generator of $\mathrm{H}^1(B\mathbf{G}_\beta^\vee; \mathcal{O})$ gives the desired 1-orientation of $B\mathbf{G}_\beta^\vee$. □

Remark 5.1.9. The 1-orientation on $B\mathbf{G}_\beta^\vee$ is closely connected to the preorientation on $\mathbf{G}_{\beta,\mathrm{ku}}$ from Proposition 2.3.11. Indeed, the 1-orientation on $B\mathbf{G}_\beta^\vee$ can be viewed as a map $B\mathbf{G}_\beta^\vee \to B\mathbf{G}_a$. Since there is an isomorphism

$$\mathrm{Map}(B\mathbf{G}_\beta^\vee, B\mathbf{G}_a) \cong \mathrm{Hom}(\mathbf{G}_\beta^\vee, \mathbf{G}_a)/\mathbf{G}_a \cong \mathrm{Hom}(\mathbf{G}_\beta^\vee, \mathbf{G}_a) \times B\mathbf{G}_a,$$

we can identify

$$\mathrm{H}^1(B\mathbf{G}_\beta^\vee; \mathcal{O}) \cong \pi_0\mathrm{Map}(B\mathbf{G}_\beta^\vee, B\mathbf{G}_a) \cong \pi_0\,\mathrm{Hom}(\mathbf{G}_\beta^\vee, \mathbf{G}_a),$$

and the 1-orientation on $B\mathbf{G}_\beta^\vee$ can be viewed as a homomorphism $\mathbf{G}_\beta^\vee \to \mathbf{G}_a$. However, $\mathrm{Hom}(\mathbf{G}_\beta^\vee, \mathbf{G}_a)$ is isomorphic to the Lie algebra of $\mathbf{G}_\beta$, and a section of this Lie algebra is precisely the datum of a preorientation on $\mathbf{G}_\beta$.

Proposition 5.1.10. *Let G be a reductive group over R. The choice of a nondegenerate G-invariant bilinear form on $\mathfrak{g}^*$ equips $\mathrm{Map}(B\mathbf{G}_\beta^\vee, BG) \cong G_\beta/G$ with a* 1*-shifted symplectic structure of weight* 2*. If G is semisimple and G^{sc} is its simply-connected form, the quotient stack G_β^{sc}/G admits a* 1*-shifted symplectic structure of weight* 2 *such that the map $G_\beta^{\mathrm{sc}}/G \to G_\beta/G$ is a morphism of* 1*-shifted symplectic stacks.*

Proof. This is a consequence of Proposition 5.1.3, Proposition 5.1.4, and Lemma 5.1.8. □

Definition 5.1.11. Let G be a reductive group over R. Let M be a graded $R[\beta]$-stack which admits a cotangent complex, and suppose M is equipped with a G-action. A ku-*Hamiltonian structure* on M is a Lagrangian morphism $M/G \to G_\beta/G$. The resulting G-equivariant map $M \to G_\beta$ will be called the ku-*moment map.*

Remark 5.1.12. Motivated by Lemma 3.6.2, one alternative name for ku-Hamiltonian structures might be "HKR-filtered quasi-Hamiltonian structures" (where HKR stands for Hochschild-Kostant-Rosenberg).

Remark 5.1.13. As in Remark 3.6.5, it is not really crucial to treat $\mathbf{G}_\beta$ as the fundamental object here: we could have considered any 1-dimensional group scheme over $\mathbf{Z}[\beta]$ in place of $\mathbf{G}_\beta$. Suppose we permit 1-dimensional *formal* group schemes.

Fix a prime p, let $\widehat{\mathbf{G}}_{k_{\mathbf{Z}}(n)}$ be the 1-dimensional formal group over $\mathbf{Z}_{(p)}[\beta]$ from Remark 3.6.5, and let G be a reductive group over $\mathbf{Z}_{(p)}[\beta]$. One can then prove that $\mathrm{Map}(B\widehat{\mathbf{G}}^\vee_{k_{\mathbf{Z}}(n)}, BG)$ admits a 1-shifted symplectic structure over $\mathbf{Z}_{(p)}[\beta]$; Lagrangian morphisms to this stack provide a(n integral) Morava K-theoretic analogue of ku-Hamiltonian varieties. Again, we will not study these objects further in the present article, but we hope to in [**Dev24**].

Remark 5.1.14. In the setting of differential geometry (as studied in [**AMM98**]), where G is replaced by a compact Lie group, [**AMM98**, Theorem 8.3] proves that there is an equivalence between quasi-Hamiltonian G-varieties and Hamiltonian LG-varieties (the latter needing some care to define because of infinite-dimensional analytic issues). It seems likely that the notion of a ku-Hamiltonian structure in the setting of differential geometry might be equivalent to the theory of Hamiltonian spaces for $\mathrm{Map}(S^1_\beta, G)$, with S^1_β being defined as

$$S^1_\beta = \{(\beta, z) | \|z\| = \beta\} \subseteq \mathbf{R} \times \mathbf{C},$$

and the $\mathrm{Map}(S^1_\beta, G)$-Hamiltonian space is equipped with a compatible fibration to $\mathbf{R}$. In other words, the radii of the fibers of $S^1_\beta \to \mathbf{R}$ should be related to the parameter β.

Proposition 5.1.15. *Let $P \subseteq G$ be a parabolic subgroup, let $U_P \subseteq P$ denote its unipotent radical, and let $L = P/U_P$ denote the Levi quotient. Then $\mathrm{Ind}^G_{U_P} P_\beta$ admits the structure of a* ku*-Hamiltonian $G \times L$-variety where the* ku*-moment map $\mathrm{Ind}^G_{U_P} P_\beta \to G_\beta$ is given by conjugation.*

Proof. As shown in [**Saf16**, Lemma 3.4], the maps $BP \to BG$ and $BP \to BL$ define a Lagrangian correspondence $BP \to BG \times BL$, essentially because there is an exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{g} \oplus \mathfrak{l} \to \mathfrak{p}^* \to 0.$$

It follows from Proposition 5.1.4 and Lemma 5.1.8 that there is a Lagrangian correspondence

$$P_\beta/P \cong \mathrm{Map}(B\mathbf{G}^\vee_\beta, BP) \to \mathrm{Map}(B\mathbf{G}^\vee_\beta, BG \times BL) \cong G_\beta/G \times L_\beta/L.$$

Since $P_\beta/P \times_{BG \times BL} \mathrm{Spec}\, \mathbf{Z}[\beta] \cong \mathrm{Ind}^G_{U_P} P_\beta$, this produces the desired ku-Hamiltonian $G \times L$-structure on $\mathrm{Ind}^G_{U_P} P_\beta$. $\square$

Remark 5.1.16. Upon inverting β and quotienting by $\mathbf{G}_m$, Proposition 5.1.15 reduces to [**Boa11**, Theorem 9].

Proposition 5.1.17. *Let $B \subseteq G$ be a Borel subgroup with unipotent radical N. Then $\mathrm{Ind}^G_N B_\beta$ admits the structure of a* ku*-Hamiltonian G-variety, where the* ku*-moment map $\mathrm{Ind}^G_N B_\beta \to G_\beta$ is given by conjugation.*

Proof. Let us first show show that the map $(\mathrm{Ind}^G_N B_\beta)/G \cong B_\beta/N \to G_\beta/G$ admits a Lagrangian structure. Using Proposition 5.1.15, Recollection 5.1.2, and the isomorphism $B_\beta/B \times_{G_\beta/G \times T_\beta/T} (G_\beta/G \times T_\beta) \cong B_\beta/N$, it suffices to show that the map $T_\beta \to T_\beta/T$ is a Lagrangian morphism. For this, note that the tangent complex to H_β/H is given by the complex $\underline{\mathfrak{h}} \to T_{H_\beta}$, where $\underline{\mathfrak{h}}$ denotes $\mathfrak{h} \otimes \mathcal{O}_{H_\beta}$, and the differential $\mathfrak{h} \to \Gamma(H_\beta; \mathcal{O})$ is given by the adjoint action $\xi \mapsto \xi^R - \xi^L$. Here, ξ^L and ξ^R denote the vector fields generating the left and right actions of H on H_β. (Note that there is an isomorphism $T_{H_\beta} \cong \underline{\mathfrak{h}}$.) When H is commutative (such as

$H = T$), $\xi^R = \xi^L$, and so the tangent complex is split. There is an obvious cofiber sequence

$$\underline{\mathfrak{t}} \to \underline{\mathfrak{t}} \oplus \underline{\mathfrak{t}}[1] \to \underline{\mathfrak{t}}[1],$$

which identifies with the cofiber sequence

$$T_{T_\beta} \to T_{T_\beta/T} \cong L_{T_\beta/T} \to L_{T_\beta}[1].$$

This gives the desired Lagrangian structure. □

Remark 5.1.18. Suppose $G = \mathrm{SL}_2$, so that the affine closure of $G/N \cong \mathrm{SL}_2/\mathbf{G}_a \cong \mathbf{A}^2 - \{0\}$ is smooth. Using the algebraic Hartogs lemma, one can show that the ku-Hamiltonian structure on the G-variety $\mathrm{Ind}_N^G B_\beta \cong \mathrm{SL}_2 \times^{\mathbf{G}_a} B_\beta$ from Proposition 5.1.17 extends to a ku-Hamiltonian structure on its affine closure $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B_\beta}$. It seems reasonable to expect that this is true in general, i.e., that the ku-Hamiltonian structure on the G-variety $\mathrm{Ind}_N^G B_\beta$ from Proposition 5.1.17 extends to $\overline{\mathrm{Ind}_N^G B_\beta}$. This is not immediately clear from the perspective of derived algebraic geometry, since neither of the affine closures $\overline{G/N}$ or $\overline{\mathrm{Ind}_N^G B_\beta}$ are smooth outside of the rank one case.

For the sake of concreteness, let us describe the affine closure $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B_\beta}$ explicitly. Fix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$ and $(x, y) \in B_\beta$ with a, d, and y in weight 0, c and x in weight -2, and b in weight 2. Then the action of $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \in \mathbf{G}_a$ (with z in weight 2) sends

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & az+b \\ c & cz+d \end{smallmatrix}\right), \ (x, y) \mapsto (x, y + z[-2](x)),$$

where $[-2](x) = -\frac{\beta x^2 + 2x}{1+\beta x}$ is the (-2)-series of x in the group law on $\mathbf{G}_\beta$. In particular, the $\mathbf{G}_a$-action fixes a, c, and x, as well as $ay - [-2](x)b$ and $cy - [-2](x)d$. If we write $B = ay - [-2](x)b$ (in weight 0) and $D = cy - [-2](x)d$ (in weight -2), the only relation is

$$cB - aD = [-2](x). \tag{33}$$

In other words, $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B_\beta}$ is cut out inside $\mathbf{A}^4_{\mathbf{Z}[\beta]} \times_{\mathrm{Spec}\, \mathbf{Z}[\beta]} \mathbf{G}_\beta$ by the above equation, where the affine space has coordinates a, c, B, D.

If we had instead replaced the $\mathbf{G}_a$-action on B_β by conjugation with the $\mathbf{G}_a$-action on $\check{V}_\beta = \mathbf{A}^1 \times \mathbf{G}_\beta$ (following the notation of Construction 4.2.11 with $n = 1$), the above equations would continue to hold if $[-2](x)$ was replaced by $-x$. The analogue of the equation (33) implies that $x = aD - cB$, so $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} V_\beta}$ is the open subscheme of $\mathbf{A}^4_{\mathbf{Z}[\beta]}$ given by the complement of the hypersurface

$$1 + \beta(aD - cB) = 0.$$

Note that when $\beta = 0$, this is the entirety of $\mathbf{A}^4$. Just as with $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B_\beta}$, the scheme $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} V_\beta}$ also admits a ku-Hamiltonian structure for its natural SL_2-action, and perhaps deserves to be called $T^*_\beta \mathbf{A}^2$.

Remark 5.1.19. More generally, if $H \subseteq G$ is a closed subgroup, the cotangent bundle of the quotient G/H admits the structure of a Hamiltonian G-variety. It is natural to ask whether there is a natural β-deformation of the cotangent bundle of G/H to a ku-Hamiltonian G-variety. Even more simply, if $H \subseteq G$ is a subgroup, it is natural to ask whether there is a "ku-theoretic" cotangent bundle $T^*_\beta(G/H)$. It seems rather difficult to define such an object for arbitrary subgroups H. However, in the case that the annhilator of $\mathfrak{h} \subseteq \mathfrak{g}$ under an invariant bilinear form on $\mathfrak{g}$ is itself a Lie subalgebra, we expect this to be possible, but we will not study this topic here. (For instance, Proposition 5.1.17 fits into this general class of examples.) Upon inverting β, i.e., working with quasi-Hamiltonian G-varieties, this was shown in [**BM22**].

5.2. ku**-Langlands duals and coisotropicity.** Following [**BZSV23**], it is natural to hope:

Expectation 5.2.1. Suppose that G is a simply-laced simply-connected semisimple compact Lie group, $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a closed reductive spherical subgroup, and $\check{M}_\beta$ is a "dual" affine graded $\check{G}$-variety over $\mathbf{Z}[\beta]$ satisfying the hypotheses of Proposition 3.6.23 equipped with a morphism $\check{M}_\beta \to \check{G}^{\mathrm{sc}}_\beta$. Then, there is an equivalence $\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}_\beta/\check{G}(-2\rho))$ which is compatible with the action of $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}G; \mathrm{ku}) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{G}^{\mathrm{sc}}(-2\rho)_\beta/\check{G}(-2\rho))$. Based on Conjecture 3.4.11, we expect that the map $\check{M}_\beta/\check{G} \to \check{G}^{\mathrm{sc}}_\beta/\check{G}$ admits a Lagrangian structure, i.e., that $\check{M}_\beta$ admits the structure of a ku-Hamiltonian $\check{G}$-variety.

In the generality of Proposition 3.6.23, it is not clear how one might prove Expectation 5.2.1. This can be shown in the case of $\mathrm{PGL}_2/\mathbf{G}_m$, but contributions from the Levi factor/Whittaker induction present difficulties in proving Expectation 5.2.1 for types A_n with $n > 1$, C_n, D_2, and G_2 as studied in Section 4. However, we can use the discussion in Section 3.5 to prove some partial results along these lines. The starting point of this discussion is the following.

Observation 5.2.2. Recall that if R is an $\mathbf{E}_\infty$-ring and A is a (nonunital) $\mathbf{E}_n$-R-algebra with $n \geq 2$, the homotopy groups $\pi_*(A)$ admit the structure of a graded (nonunital) Poisson algebra over $\pi_*(R)$, where the graded Poisson bracket has weight $n-1$ (i.e., if f, g are functions in weights i, j respectively, the graded Poisson bracket $\{f, g\}$ is in weight $i + j + n - 1$) and is $\pi_*(R)$-linear. This graded Poisson bracket comes from action of the generator of $\pi_{n-1}\mathrm{Conf}_2(\mathbf{R}^n) \cong \pi_{n-1}S^{n-1}$ on $\pi_*(R)$. (See, e.g., [**Law20**, Example 4.5].)

Let $\widehat{\mathcal{F}_G(\Omega G)}$ denote the ku_G-linear dual of $\mathcal{F}_G(\mathcal{L}G)$. There is a graded Poisson structure of weight 2 on $\mathrm{Spf}\, \pi_*\widehat{\mathcal{F}_G(\Omega G)}$ arising from the $\mathbf{E}_3$-algebra structure on $\widehat{\mathcal{F}_G(\Omega G)}$ (viewed as the $\mathbf{E}_2$-center $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{ku}_G/\mathrm{ku})$ via Corollary 3.5.8). Using Proposition 3.6.16, this implies that (a completion of) the fiber product $T_\beta /\!\!/ W \times_{\check{G}^{\mathrm{sc}}_\beta/\check{G}} T_\beta /\!\!/ W$ admits a graded Poisson structure where the graded Poisson bracket has weight 2.

Remark 5.2.3. The Kostant slice $\kappa : \mathfrak{t}/\!\!/ W \to \check{\mathfrak{g}}/\check{G}$ is Lagrangian for the 1-shifted symplectic structure on $\check{\mathfrak{g}}/\check{G}$ from Example 5.1.5; see, e.g., [**Saf20**, Proposition 4.18]. Recollection 5.1.2 implies that the self-intersection $\mathfrak{t}/\!\!/ W \times_{\check{\mathfrak{g}}/\check{G}} \mathfrak{t}/\!\!/ W$ admits a (0-shifted) symplectic structure; the underlying Poisson structure can be identified with the $\beta = 0$ degeneration of Observation 5.2.2.

If Expectation 5.2.1 holds, the map

$$T_{H,\beta}/\!\!/ W_H \times_{\check{M}_\beta/\check{G}} T_{H,\beta}/\!\!/ W_H \to T_\beta/\!\!/ W \times_{\check{G}^{\mathrm{sc}}_\beta/\check{G}} T_\beta/\!\!/ W \tag{34}$$

from Proposition 3.6.23 will, in particular, be coisotropic (in an appropriate derived sense). Our goal in this section is to show that this consequence of Expectation 5.2.1

is always true; hopefully some variant of our discussion below could imply Expectation 5.2.1 itself. In order to explain this, we need to rephrase the graded Poisson bracket on $T_\beta /\!\!/ W \times_{\check{G}^{\mathrm{sc}}_\beta/\check{G}} T_\beta /\!\!/ W$ in homotopy-theoretic terms. As the reader will observe, the coisotropicity of (34) is a rather general phenomenon. Let us begin by reviewing the notion of an $\mathbf{E}_n$-center.

Recollection 5.2.4 ([**Lur16**, Section 5.3] and [**Fra13**])**.** Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category with unit $\mathbf{1}$, and let $f : A \to B$ be a morphism in $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})$. The centralizer $\mathfrak{Z}_{\mathbf{E}_n}(f)$ is the universal $\mathbf{E}_n$-algebra object of $\mathcal{C}$ equipped with the data of commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & \mathfrak{Z}_{\mathbf{E}_n}(f) \otimes A \arrow[d] \\ & B \end{tikzcd}$$

in $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})$. The existence of centralizers is proved in [**Lur16**, Theorem 5.3.1.14]. One can explicitly identify $\mathfrak{Z}_{\mathbf{E}_n}(f) = \mathrm{Map}_{\mathrm{Mod}_A^{\mathbf{E}_n}}(A, B) = \mathrm{Map}_{\int_{S^{n-1}} A/\mathcal{C}}(A, B)$. If f is the identity map on A, we will simply write $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C})$ to denote $\mathfrak{Z}_{\mathbf{E}_n}(\mathrm{id}_A)$; moreover, if R is an $\mathbf{E}_\infty$-ring, we will write $\mathfrak{Z}_{\mathbf{E}_n}(A/R)$ to denote $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathrm{Mod}_R)$.

Remark 5.2.5. In the setup of Recollection 5.2.4, there is a canonical $\mathbf{E}_n$-algebra map $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \to \mathfrak{Z}_{\mathbf{E}_n}(f)$ defined using the universal property of $\mathfrak{Z}_{\mathbf{E}_n}(f)$ and the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\mathrm{id}_A"'] & \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A \arrow[d] & \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

in $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})$.

Remark 5.2.6. The scenario of Proposition 3.6.23 can be modeled as follows. Under the hypotheses of Proposition 3.6.23, the map (34) can be identified with the composite map

$$\mathrm{Spec}\,\mathrm{ku}_*^H(\Omega(G/H)) \to \mathrm{Spec}\,\mathrm{ku}_*^H(\Omega G) \to \mathrm{Spec}\,\mathrm{ku}_*^G(\Omega G). \tag{35}$$

Let $R = \mathrm{ku}$, $A = \mathrm{ku}_G$, and $B = \mathrm{ku}_H$, so that there is a map $f : A \to B$ of $\mathbf{E}_\infty$-R-algebras. Following Corollary 3.5.8 and Warning 3.5.10, we can identify a completion of $\mathcal{F}_H(\Omega(G/H))^\vee$ with the Hochschild cohomology $\mathfrak{Z}_{\mathbf{E}_1}(B/A)$, and a completion of $\mathcal{F}_G(\Omega G)^\vee$ with the $\mathbf{E}_2$-center $\mathfrak{Z}_{\mathbf{E}_2}(A/R)$. It is also easy to see that a completion of $\mathcal{F}_H(\Omega G)^\vee$ can be identified with the centralizer $\mathfrak{Z}_{\mathbf{E}_2}(f)$. In particular, the above composite can be identified, at least upon completion, with a map

$$\mathrm{Spec}\,\pi_*\mathfrak{Z}_{\mathbf{E}_1}(B/A) \to \mathrm{Spec}\,\pi_*\mathfrak{Z}_{\mathbf{E}_2}(f) \to \mathrm{Spec}\,\pi_*\mathfrak{Z}_{\mathbf{E}_2}(A/R).$$

Lemma 5.2.7. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, let $A \in \mathrm{Alg}_{\mathbf{E}_{n+1}}(\mathcal{C})$, and let $B \in \mathrm{Alg}_{\mathbf{E}_{n-1}}(\mathrm{Mod}_A(\mathcal{C}))$ with unit map $f : A \to B$. Then there is a canonical $\mathbf{E}_n$-$\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C})$-algebra structure on the $\mathbf{E}_n$-A-algebra $\mathfrak{Z}_{\mathbf{E}_{n-1}}(B/\mathrm{Mod}_A(\mathcal{C}))$ such that the unit map factors as a composite*

$$\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \to \mathfrak{Z}_{\mathbf{E}_n}(f) \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A).$$

Proof. Let $g : B \to B'$ be a map of $\mathbf{E}_{n-1}$-A-algebras. Then there is a map $\mu_g : \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g) \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(g)$ of $\mathbf{E}_{n-1}$-A-algebras defined using the universal property of $\mathfrak{Z}_{\mathbf{E}_{n-1}}(g)$ as follows. Recall that there are commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\mathrm{id}_A"'] & \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A \arrow[d] \\ & A \end{tikzcd}, \quad \begin{tikzcd} B \arrow[r] \arrow[dr, "g"'] & \mathfrak{Z}_{\mathbf{E}_{n-1}}(g) \otimes_A B \arrow[d] \\ & B', \end{tikzcd}$$

where the maps in the first diagram are of $\mathbf{E}_n$-algebras in $\mathcal{C}$, and the maps in the second diagram are of $\mathbf{E}_{n-1}$-A-algebras. Since A is an $\mathbf{E}_{n+1}$-algebra, the first diagram can be upgraded to a commutative diagram of $\mathbf{E}_n$-A-algebras. Tensoring these two diagrams over A produces a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "g"'] & (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g)) \otimes_A B \arrow[d] \\ & B' \end{tikzcd}$$

of $\mathbf{E}_{n-1}$-A-algebras, which gives the desired map of $\mathbf{E}_{n-1}$-A-algebras

$$\mu_g : \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g) \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(g).$$

It is not difficult to see that this map is compatible with composition in g, in the sense that if $g' : B' \to B''$ is another morphism and $c : \mathfrak{Z}_{\mathbf{E}_{n-1}}(g) \otimes_A \mathfrak{Z}_{\mathbf{E}_{n-1}}(g') \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(g' \circ g)$ is the composition coming from functoriality of $\mathbf{E}_{n-1}$-centers, there is a commutative digram of $\mathbf{E}_{n-1}$-A-algebras

$$\begin{tikzcd} (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g)) \otimes_A (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g')) \arrow[r, "\mathrm{mult}_{\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C})} \otimes c"] \arrow[d, "\mu_g \otimes \mu_{g'}"'] & \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(g' \circ g) \arrow[d, "\mu_{g' \circ g}"] \\ \mathfrak{Z}_{\mathbf{E}_{n-1}}(g) \otimes_A \mathfrak{Z}_{\mathbf{E}_{n-1}}(g') \arrow[r] & \mathfrak{Z}_{\mathbf{E}_{n-1}}(g' \circ g). \end{tikzcd}$$

Since A is an $\mathbf{E}_{n+1}$-algebra, the ∞-category $\mathrm{Alg}_{\mathbf{E}_{n-1}}(\mathrm{LMod}_A(\mathcal{C}))$ admits an $\mathbf{E}_1$-monoidal structure. Taking $g = g' = \mathrm{id}_B$, we find that the map $\widetilde{\mu} := \mu_{\mathrm{id}_B}$ can be upgraded to a map $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A) \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A)$ of $\mathbf{E}_1$-algebras in $\mathbf{E}_{n-1}$-A-algebras, i.e., of $\mathbf{E}_n$-A-algebras. In particular, this equips $\mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A)$ with the structure of an $\mathbf{E}_n$-$\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C})$-algebra.

The factorization through the centralizer $\mathfrak{Z}_{\mathbf{E}_n}(f)$ is a consequence of the construction of $\widetilde{\mu}$. Namely, the unit map $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A)$ can be regarded as giving a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "\mathrm{id}_B"'] & (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A) \otimes_A B \arrow[d] \\ & B \end{tikzcd}$$

of $\mathbf{E}_{n-1}$-A-algebra maps, which is in turn obtained via a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & \mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A \arrow[d] \\
B \arrow[r] \arrow[dr, "\mathrm{id}_B"'] & (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A) \otimes_A B \arrow[d] \\
& B,
\end{tikzcd}$$

where the square is given by the tensor product in $\mathcal{C}$. The universal property of the centralizer $\mathfrak{Z}_{\mathbf{E}_n}(f)$ implies that there is a factorization

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\mathrm{id}_B"'] & (\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes A) \otimes_A B \arrow[d] \\
B \arrow[r] \arrow[dr, "\mathrm{id}_B"'] & (\mathfrak{Z}_{\mathbf{E}_n}(f) \otimes A) \otimes_A B \arrow[d] \\
& B,
\end{tikzcd}$$

which in turn gives the desired A-linear map $\mathfrak{Z}_{\mathbf{E}_n}(f) \otimes A \to \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A)$ factoring the unit $\mathfrak{Z}_{\mathbf{E}_n}(A/\mathcal{C}) \otimes \mathfrak{Z}_{\mathbf{E}_{n-1}}(B/A)$. $\square$

Proposition 5.2.8. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, let $A \in \mathrm{Alg}_{\mathbf{E}_{n+1}}(\mathcal{C})$, and let $B \in \mathrm{Alg}_{\mathbf{E}_n}(\mathrm{Mod}_A(\mathcal{C}))$ with unit map $f : A \to B$. Then the fiber $\mathrm{fib}(f)$ admits the structure of a nonunital $\mathbf{E}_{n+1}$-algebra in $\mathcal{C}$.*

Proof. The map f factors as a composite

$$A \xrightarrow{g} \mathfrak{Z}_{\mathbf{E}_n}(B/\mathcal{C}) \xrightarrow{h} B,$$

where g is a map of $\mathbf{E}_{n+1}$-algebras in $\mathcal{C}$, and the $\mathbf{E}_n$-map $h : \mathfrak{Z}_{\mathbf{E}_n}(B/\mathcal{C}) \to B$ is the unit. Let $T^{\mathbf{E}_n}_{B/\mathcal{C}}$ denote the $\mathbf{E}_n$-cotangent complex of B (viewed as an object of $\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})$), so that $T^{\mathbf{E}_n}_{B/\mathcal{C}}[-n]$ admits the structure of a nonunital $\mathbf{E}_{n+1}$-algebra in $\mathcal{C}$, and there is a map $T^{\mathbf{E}_n}_{B/\mathcal{C}}[-n] \to \mathfrak{Z}_{\mathbf{E}_n}(B/\mathcal{C})$ of nonunital $\mathbf{E}_{n+1}$-algebras. We claim that there is a Cartesian square

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d] & T^{\mathbf{E}_n}_{B/\mathcal{C}}[-n] \arrow[d] \\
A \arrow[r, "g"'] & \mathfrak{Z}_{\mathbf{E}_n}(B/\mathcal{C}),
\end{tikzcd}$$

so that $\mathrm{fib}(f)$ is canonically equipped with the structure of a nonunital $\mathbf{E}_{n+1}$-algebra in $\mathcal{C}$. To see this, note that the factorization of f as $h \circ g$ implies a fiber sequence

$$\mathrm{fib}(g) \to \mathrm{fib}(f) \to \mathrm{fib}(h).$$

It therefore remains to identify $\mathrm{fib}(h)$ with $T^{\mathbf{E}_n}_{B/\mathcal{C}}[-n]$; but this follows from [**Fra13**, Theorem 1.1] (or equivalently [**Lur16**, Theorem 7.3.5.1]). $\square$

We are now in a position to explain the "coisotropic" property of the map (34) (or more generally of (35)). To motivate the use of this term, let us make the following observation.

Observation 5.2.9. Let P_* be a graded adic algebra with Poisson bracket of weight n, and let Q_* be a graded commutative adic P_*-algebra. The map $\operatorname{Spf} Q_* \to \operatorname{Spf} P_*$ is coisotropic if the unit map $P_* \to Q_*$ is surjective, and its kernel is closed under the Poisson bracket. Suppose that the Poisson structure on P_* arises from an $\mathbf{E}_{n+1}$-ring P with $\pi_* P \cong P_*$, and Q_* arises as the homotopy groups of an $\mathbf{E}_n$-P-algebra Q with unit map $f : P \to Q$. Then, Proposition 5.2.8 implies that $\mathrm{fib}(f)$ admits a nonunital $\mathbf{E}_{n+1}$-algebra structure. We therefore find that the map $\operatorname{Spf} Q_* \to \operatorname{Spf} P_*$ is the inclusion of a coisotropic *subvariety* precisely when the map $\pi_* P \to \pi_* Q$ is surjective (since its kernel is then $\pi_* \mathrm{fib}(f) \subseteq \pi_* P$, which is closed under the Poisson bracket).

Using Observation 5.2.9 with $n = 2$, one can show that the map of (34) is very nearly coisotropic: the only obstructions are given by a completion issue, and that the map (34) need not be a closed immersion (i.e., the unit map is $\mathrm{ku}_*^G(\Omega G) \to \mathrm{ku}_*^H(\Omega(G/H))$ need not be surjective). Keeping these in mind, we are led to the following.

Observation 5.2.10. Let $H \subseteq G$ be a closed subgroup of a compact Lie group such that the ku_G-linear dual $\widehat{\mathcal{F}_G(\Omega G)}^\vee$ of $\mathcal{F}_G(\Omega G)$ and the ku_H-linear dual $\mathcal{F}_H(\widehat{\Omega(G/H)})^\vee$ of $\mathcal{F}_H(\Omega(G/H))$ are concentrated in even degrees[20]. Remark 5.2.6 and Lemma 5.2.7 together imply that the unit map $\widehat{\mathcal{F}_G(\Omega G)}^\vee \to \mathcal{F}_H(\widehat{\Omega(G/H)})^\vee$ exhibits $\mathcal{F}_H(\widehat{\Omega(G/H)})^\vee$ as an $\mathbf{E}_2$-$\widehat{\mathcal{F}_G(\Omega G)}^\vee$-algebra, and so Proposition 5.2.8 implies that the fiber of the unit map admits a nonunital $\mathbf{E}_3$-algebra structure. For instance, this means that if the unit map induces a surjection on homotopy (which is not common!), the map $\operatorname{Spf} \pi_* \mathcal{F}_H(\widehat{\Omega(G/H)})^\vee \to \operatorname{Spf} \pi_* \widehat{\mathcal{F}_G(\Omega G)}^\vee$ will be the inclusion of a coisotropic subvariety.

Remark 5.2.11. Applying Lemma 5.2.7 to Proposition 5.2.8 is of course quite "lossy", in the sense that the $\mathbf{E}_n$-algebra structure from Lemma 5.2.7 is of a very specific kind. The setup of Lemma 5.2.7 hopefully exhibits further special features which allows us to understand Expectation 5.2.1 further.

Remark 5.2.12. The coisotropicity of Observation 5.2.10 can be understood from the perspective of boundary theories for topological quantum field theories (TQFTs for short). Namely, consider an n-dimensional TQFT $Z : \mathrm{Bord}_n \to \mathrm{Cat}_{(\infty,n-1)}$ (valued in, e.g., some (∞, n)-category of "presentable" $\mathbf{C}$-linear $(\infty, n-1)$-categories). Then $Z(S^j)$ is naturally with the structure of an $\mathbf{E}_{j+1}$-algebra object in the $(\infty, n-j)$-category of $\mathbf{C}$-linear $(\infty, n-j-1)$-categories, since S^j is an $\mathbf{E}_{j+1}$-algebra in Bord_n. Indeed, if I is a set of n points on $\mathbf{R}^{j+1}$, the manifold $S^{j+1} - I$ can be viewed as a cobordism $\amalg_{i=0}^n S^j \rightsquigarrow S^j$.

A boundary theory $\mathcal{B}$ for Z is a natural transformation from the trivial n-dimensional TQFT (i.e., whose value on a point is the unit object) to Z, so that $\mathcal{B}$ is an $(n-1)$-dimensional TQFT. In particular, $\mathcal{B}(S^{n-2})$ is an $\mathbf{E}_{n-1}$-algebra

[20] This is a very mild condition, which can be checked to hold in all examples in this article. However, it is somewhat subtle, in the sense that working equivariantly is crucial. See, e.g., Example 4.1.2.

object of the $\mathbf{E}_{n-1}$-monoidal ∞-category $Z(S^{n-2})$. The cell decomposition $S^{n-1} \cong D^{n-1} \amalg_{S^{n-2}} D^{n-1}$ implies that $Z(S^{n-1})$ is the $\mathbf{E}_n$-algebra (in $\mathbf{C}$-modules) given by the $\mathbf{E}_{n-1}$-center of $Z(S^{n-2})$. This implies that $\mathcal{B}(S^{n-2})$ is an $\mathbf{E}_{n-1}$-$Z(S^{n-1})$-algebra.

A special case of the above situation is when Z is determined by an $\mathbf{E}_n$-algebra A in $\mathrm{Mod}_{\mathbf{C}}(\mathrm{Sp})$, in which case $\mathcal{B}$ is determined by an $\mathbf{E}_n$-A-algebra B. The statement that $\mathcal{B}(S^{n-2})$ is an $\mathbf{E}_{n-1}$-$Z(S^{n-1})$-algebra then translates to the statement that $\mathfrak{Z}_{\mathbf{E}_{n-2}}(B/A)$ is an $\mathbf{E}_{n-1}$-$\mathfrak{Z}_{\mathbf{E}_{n-1}}(A/\mathbf{C})$-algebra, which is precisely Lemma 5.2.7.

Returning to the general case, the structure of an $\mathbf{E}_{n-1}$-$Z(S^{n-1})$-algebra on $\mathcal{B}(S^{n-2})$ implies that there is a map $\mathrm{Spec}\,\pi_*\mathcal{B}(S^{n-2}) \to \mathrm{Spec}\,\pi_* Z(S^{n-1})$, at least if $n \geq 3$. Proposition 5.2.8 (and the general discussion in Observation 5.2.10) implies that this map should be coisotropic (in an appropriate sense) for the Poisson bracket of weight $n-1$ on $\pi_* Z(S^{n-1})$. It will be too much of a digression to discuss this here, but this coisotropicity can in fact be deduced from the secondary product structure (as discussed in [**BBB$^+$20**]) on local operators in extended TQFTs arising via topological descent.

5.3. Equivariant spectra and graded groups. Motivated by the discussion of the previous sections, we will propose a generalization of the Langlands duality equivalences of Theorem 3.6.21 and Theorem 3.4.13 to coefficients in the sphere spectrum. Our goal is not to provide a fully fleshed-out story, but instead to suggest some questions that we expect to play an important role in the sequel. In order to motivate our presentation, we will need to recall some background on the Adams-Novikov spectral sequence, as well as some results of Hausmann regarding equivariant complex cobordism.

In previous sections, we implicitly used the construction taking in a commutative ring spectrum R and producing the (graded) affine scheme $\mathrm{Spec}\,\pi_*(R)$. This construction is well-behaved if R is concentrated in even degrees, but generally not so otherwise. One important insight suggested by chromatic homotopy theory is that the algebro-geometric object associated to a ring spectrum which is not even should instead be a (graded) *stack*. This perspective is made precise (for $\mathbf{E}_\infty$-rings) in [**HRW22**] and [**DHRY23**]. Let us briefly recall some of the relevant constructions.

Recollection 5.3.1. An $\mathbf{E}_\infty$-ring A will be called *even* if $\pi_*(A)$ is concentrated in even weights. If R is an $\mathbf{E}_\infty$-ring spectrum, let $\mathcal{M}_R^{\mathrm{fil}}$ denote the filtered stack

$$\mathcal{M}_R^{\mathrm{fil}} := \mathrm{colim}_{R\to A}\,\mathrm{Spec}\,\tau_{\geq\star}(A),$$

where the colimit is taken over all $\mathbf{E}_\infty$-ring maps $R \to A$ with A being even. The structure sheaf of $\mathcal{M}_R^{\mathrm{fil}}$ defines a filtered $\mathbf{E}_\infty$-ring $\mathrm{F}^\star_{\mathrm{ev}}R := \lim_{R\to A}\tau_{\geq\star}A$. If M is an R-module, let $\mathcal{F}_M^{\mathrm{fil}}$ denote the quasicoherent sheaf over $\mathcal{M}_R^{\mathrm{fil}}$ defined by $\lim_{R\to A}\tau_{\geq\star}(A\otimes_R M)$. Of course, if R is already even, $\mathcal{M}_R^{\mathrm{fil}} \cong \mathrm{Spec}\,\tau_{\geq\star}(R)$. The unit map $S^0 \to R$ defines a map $\mathcal{M}_R^{\mathrm{fil}} \to \mathcal{M}_{S^0}^{\mathrm{fil}}$.

A map $R \to A$ is called an *eff cover* (for "evenly faithfully flat") if for every even $\mathbf{E}_\infty$-R-algebra B, the base-change $A\otimes_R B$ is even and the map $\pi_*(A) \to \pi_*(A\otimes_R B)$ is faithfully flat. If R admits an eff cover $R \to A$ by an even $\mathbf{E}_\infty$-ring A, there is an equivalence

$$\mathcal{M}_R^{\mathrm{fil}} \cong \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\,\mathrm{Spec}\,\tau_{\geq\star}(A^{\otimes_R \bullet+1}).$$

If M is an R-module, $\mathcal{F}_M^{\mathrm{fil}}$ can be identified with $\lim_{\mathbf{\Delta}} \tau_{\geq\star}(M \otimes_R A^{\otimes_R \bullet+1})$. Each of these filtered stacks and quasicoherent sheaves defines graded stacks and quasicoherent sheaves, which we will simply denote by $\mathcal{M}_R$ and $\mathcal{F}_M$.

Example 5.3.2. The map $S^0 \to \mathrm{MU}$ is an eff cover: if B is an even $\mathbf{E}_\infty$-ring, it admits a complex orientation, and so $\mathrm{MU} \otimes B \cong B[\mathrm{BU}]$; but BU has even cells, so that $\pi_*(\mathrm{MU} \otimes B)$ is a free $\pi_*(B)$-module on classes in even weights, and hence is itself concentrated in even weights. Furthermore, results of Quillen, Araki, Landweber, and Novikov (see [**Qui69, Ara73, Lan67, Nov67**]) identify $\mathcal{M}_{S^0}$ with the moduli stack $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ of graded (1-dimensional) spin formal groups in the sense of [**Mil19**]. Explicitly, if R is a commutative ring, an R-point of $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ is the data of a line bundle $\mathcal{L}$ over $\mathrm{Spec}(R)$, a formal group $\hat{\mathbf{G}}$ over $\mathrm{Spec}(R)$, and isomorphism $\omega_{\hat{\mathbf{G}}} \cong \mathcal{L}^{\otimes 2}$. We will write $\hat{\mathbf{G}}_{\mathrm{univ}}$ to denote the universal spin formal group over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$, and ω to denote the line bundle over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ given by $\omega_{\hat{\mathbf{G}}_{\mathrm{univ}}}$. Note that there is a canonical square root $\omega^{1/2}$; this plays the role of the "weight 1" line bundle $\mathcal{O}(1)$ which appears throughout this article.

Any spectrum M therefore defines a quasicoherent sheaf $\mathcal{F}_M$ on $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$. If X is a space, let $\mathcal{F}(X; S^0)$ denote the ind-coherent sheaf $\mathcal{F}_{(S^0)^{X_+}}$ associated to the spherical cochains $(S^0)^{X_+}$, so that the pullback of $\mathcal{F}(X; S^0)$ along the map $\mathrm{Spec}\, \pi_*(\mathrm{MU}) \to \mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ can be identified with $\pi_* \mathrm{MU}^{X_+}$. Note that the diagonal on X equips $\mathcal{F}(X; S^0)$ with the structure of an $\mathbf{E}_\infty$-algebra in $\mathrm{IndCoh}(\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}})$. For instance, let $q : \hat{\mathbf{G}}_{\mathrm{univ}} \to \mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ denote the universal spin formal group over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$; then $\mathcal{F}(\mathbf{C}P^\infty; S^0) \cong \mathcal{O}_{\hat{\mathbf{G}}_{\mathrm{univ}}}$. If X is a finite space, let $\mathcal{F}(X; S^0)^\vee$ denote the $\mathcal{O}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}}$-linear dual of $\mathcal{F}(X; S^0)$.

Example 5.3.2 can be categorified, following ideas from [**Pst18, GIKR18, Gre21**].

Construction 5.3.3. Let X be a finite space equipped with a stratification, and let $\mathrm{Shv}^c(X; S^0)$ denote the ∞-category of constructible sheaves of spectra on X. Let $q^*(\mathrm{MU})$ denote the constant sheaf with value MU, and define the *filtered* ∞-category

$$\mathrm{Shv}^c(X; S^0)^{\mathrm{Syn}} := \mathrm{Mod}_{\mathrm{Tot}(\tau_{\geq\star}(q^*(\mathrm{MU})^{\otimes\bullet+1}))}(\mathrm{Shv}^c(X; S^0)^{\mathrm{fil}}).$$

Note that this ∞-category is linear over the ∞-category $\mathrm{Sp}^{\mathrm{Syn}} := \mathrm{Mod}_{\mathrm{Tot}(\tau_{\geq\star}(\mathrm{MU}^{\otimes\bullet+1}))}(\mathrm{Sp}^{\mathrm{fil}})$ of synthetic spectra. This in turn leads to a *graded* ∞-category $\mathrm{Shv}^{c,\mathrm{gr}}(X; S^0)$ which is linear over the associated graded of the ∞-category $\mathrm{Sp}^{\mathrm{Syn}}$, namely $\mathrm{IndCoh}(\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}})$. Note that $\mathrm{Shv}^{c,\mathrm{gr}}(X; S^0)$ is *not* a $\pi_*(S^0)$-linear ∞-category!

In order to state results analogous to Theorem 3.6.21 and Theorem 3.4.13, we need to incorporate genuine S^1-equivariance into this picture. The primary motivation for this discussion is the work of Hausmann (see [**Hau22**]). The theory of genuine S^1-equivariant synthetic spectra does not seem to exist in the literature, so we will instead proceed in a rather *ad hoc* manner.

Notation 5.3.4. In what follows, we will write $\mathrm{Sp}_{\mathrm{glob}}$ to denote the ∞-category of global spectra as defined in [**Sch18**], so that each compact Lie group G defines a symmetric monoidal restriction functor $\mathrm{Sp}_{\mathrm{glob}} \to \mathrm{Sp}_G$ to the ∞-category of genuine G-equivariant global spectra. These functors are jointly conservative over all G. Note that [**Sch18**] works in the model-categorical setting, but we will work with the corresponding ∞-categories; the reader is referred to [**LNP22**] for a discussion of global spectra in ∞-categorical language.

Construction 5.3.5. Let $\mathcal{A}$ denote the family of abelian compact Lie groups, let $\mathrm{Orb}^{\mathcal{A}}$ denote the ∞-category of Definition 2.2.6, and let $\mathrm{Orb}^{\mathcal{A}}_*$ denote the (non-full) subcategory of pointed objects in $\mathrm{Orb}^{\mathcal{A}}$. Then $\mathrm{Orb}^{\mathcal{A}}_*$ is equivalent to the (nerve of the) topological category of abelian compact Lie groups.

Let $\underline{R}$ be an $\mathbf{E}_\infty$-algebra in $\mathrm{Sp}_{\mathrm{glob}}$. Then there is a natural lax symmetric monoidal functor $\underline{\tau}_{\geq\star} : \mathrm{Mod}_{\underline{R}}(\mathrm{Sp}_{\mathrm{glob}}) \to \mathrm{Fun}(\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_*, \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}R})$ sending $\underline{A} \in \mathrm{Sp}_{\mathrm{glob}}$ to the functor $T \mapsto \tau^T_{\geq\star}(A_T)$. This constrution can be slightly modified as follows: let Lat denote the 1-category of lattices, so that Pontryagin duality naturally gives a fully faithful functor $\mathrm{Lat} \subseteq \mathrm{Orb}^{\mathcal{A},\mathrm{op}}_*$. Restricting $\underline{\tau}_{\geq\star}$ along this inclusion defines a functor $\mathrm{Mod}_{\underline{R}}(\mathrm{Sp}_{\mathrm{glob}}) \to \mathrm{Fun}(\mathrm{Lat}, \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}R})$. The composite

$$\mathrm{Mod}_{\underline{R}}(\mathrm{Sp}_{\mathrm{glob}}) \xrightarrow{\underline{\tau}_{\geq\star}} \mathrm{Fun}(\mathrm{Lat}, \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}R}) \xrightarrow{\text{left Kan extension}} \mathrm{Fun}(\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_*, \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}R})$$

will be denoted $\underline{\tau}^{\mathrm{toral}}_{\geq\star}$. Note that there is no reason for $\underline{\tau}^{\mathrm{toral}}_{\geq\star}\underline{E}$ to agree with $\underline{\tau}_{\geq\star}\underline{E}$; some criteria for when this is true are listed in [**Hau22**, Lemma 5.23]. In particular, $\underline{\tau}^{\mathrm{toral}}_{\geq\star}\underline{E} \cong \underline{\tau}_{\geq\star}\underline{E}$ if the latter is concentrated in even weights. We will write $\underline{\pi}_*$ and $\underline{\pi}^{\mathrm{toral}}_*$ to denote the composite functors

$$\mathrm{Mod}_{\underline{R}}(\mathrm{Sp}_{\mathrm{glob}}) \xrightarrow{\underline{\tau}^{\mathrm{toral}}_{\geq\star}, \underline{\tau}_{\geq\star}} \mathrm{Fun}(\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_*, \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}R}) \to \mathrm{Fun}(\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_*, \mathrm{Mod}^{\mathrm{gr}}_{\pi_* R}).$$

Let $\underline{\mathrm{MU}}$ denote the global complex cobordism spectrum, so that $\underline{\tau}_{\geq\star}(\underline{\mathrm{MU}})$ defines a functor $\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_* \to \mathrm{Mod}^{\mathrm{fil}}_{\tau_{\geq\star}\mathrm{MU}}$. In fact, this refines to a functor $\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_* \to \mathrm{CAlg}^{\mathrm{fil}}_{\tau_{\geq\star}\mathrm{MU}}$, which in particular defines a functor $\mathrm{Orb}^{\mathcal{A},\mathrm{op}}_* \to \mathrm{CAlg}^{\mathrm{gr}}_{\pi_*\mathrm{MU}}$. Recall that the graded ring $\pi_*(\mathrm{MU})$ classifies the universal graded (1-dimensional) formal group law. Hausmann proved a similar characterization of $\underline{\pi}_*(\underline{\mathrm{MU}})$, too.

Definition 5.3.6. A *graded (1-dimensional) group law* is a functor $\mathcal{O}_{\mathbf{G}} : \mathrm{Lat} \to \mathrm{CAlg}^{\heartsuit,\mathrm{gr}}$ equipped with an element $x \in \mathcal{O}_{\mathbf{G}}(\mathbf{Z})_{-2}$ in weight -2 such that for every lattice Λ and split injective homomorphism $\chi : \mathbf{Z} \to \Lambda$, the sequence of graded abelian groups

$$0 \to \mathcal{O}_{\mathbf{G}}(\Lambda)_{*+2} \xrightarrow{\cdot\chi^* x} \mathcal{O}_{\mathbf{G}}(\Lambda)_* \to \mathcal{O}_{\mathbf{G}}(\Lambda/\mathbf{Z}\chi)_* \to 0$$

is exact. Such an element x will be called a *coordinate*. Say that an n-tuple $x^{(1)}, \cdots, x^{(n)}$ of coordinates on a graded group law $\mathcal{O}_{\mathbf{G}}$ is *strict* if each $x^{(j)}$ is a multiple of $x^{(1)}$ by a unit $\lambda_j \in \mathcal{O}_{\mathbf{G}}(\mathbf{Z})_0$ whose restriction along the map $\mathcal{O}_{\mathbf{G}}(\mathbf{Z}) \to \mathcal{O}_{\mathbf{G}}(\{0\})$ is 1.

A *graded (1-dimensional) generalized group* is a functor $\mathcal{O}_{\mathbf{G}} : \mathrm{Lat} \to \mathrm{CAlg}^{\heartsuit,\mathrm{gr}}$ such that there are elements $a_1, \cdots, a_n \in \mathcal{O}_{\mathbf{G}}(\{0\})$ with $a_1 + \cdots + a_n = 1$, such that for each $1 \leq j \leq n$, the functor $\mathcal{O}_{\mathbf{G}}[a_i^{-1}] : \mathrm{Lat} \to \mathrm{CAlg}^{\heartsuit,\mathrm{gr}}$ given by

$$\mathcal{O}_{\mathbf{G}}[a_i^{-1}](\Lambda) \cong \mathcal{O}_{\mathbf{G}}(\Lambda) \otimes_{\mathcal{O}_{\mathbf{G}}(\{0\})} \mathcal{O}_{\mathbf{G}}(\{0\})[a_i^{-1}]$$

admits the structure of a graded group law. We will sometimes view graded generalized groups as functors $\mathbf{G} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{Aff}^{\heartsuit,\mathrm{gr}}$ via the identification $\mathrm{Spec} : \mathrm{CAlg}^{\heartsuit,\mathrm{gr},\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}^{\heartsuit,\mathrm{gr}}$.

Example 5.3.7. Graded group laws $\mathcal{O}_{\mathbf{G}} : \mathrm{Lat} \to \mathrm{CAlg}^{\heartsuit,\mathrm{gr}}$ which preserve coproducts are simply specified by $\mathcal{O}_{\mathbf{G}}(\{0\})$, $\mathcal{O}_{\mathbf{G}}(\mathbf{Z})$ viewed as a Hopf algebra over $\mathcal{O}_{\mathbf{G}}(\{0\})$, and a regular element of $\mathcal{O}_{\mathbf{G}}(\mathbf{Z})_{-2}$ which generates the augmentation ideal. In particular, graded generalized groups which preserve coproducts are the same as 1-dimensional linear graded algebraic groups over a graded commutative ring R (namely $\mathcal{O}_{\mathbf{G}}(\{0\})$) which are weight-connected over R.

Theorem 5.3.8 (Hausmann, [**Hau22**, Theorem A]; Comezaña, [**Com96**, XXVIII Theorem 5.3]; Löffler, [**Lof73**])**.** *The ring $\underline{\pi}_*\underline{\mathrm{MU}}$ is concentrated in even weights, and $(\underline{\pi}_*\underline{\mathrm{MU}})(\Lambda)$ is a free $\pi_*(\mathrm{MU})$-module for each lattice Λ.*

Moreover, for any $n \geq 0$, let $x_\tau^{(1)}, \cdots, x_\tau^{(n)} \in \underline{\pi}_*(\underline{\mathrm{MU}}^{\otimes n})$ denote the n different tautological complex orientations of $\underline{\mathrm{MU}}$. If $(\mathcal{O}_{\mathbf{G}}, x^{(1)}, \cdots, x^{(n)})$ is a strict n-tuple of coordinates on a graded group law, there is a unique homomorphism of graded group laws $(\underline{\pi}_*(\underline{\mathrm{MU}}^{\otimes n}), x_\tau^{(1)}, \cdots, x_\tau^{(n)}) \to (\mathcal{O}_{\mathbf{G}}, x^{(1)}, \cdots, x^{(n)})$ which sends $x_\tau^{(j)} \mapsto x^{(j)}$.

Construction 5.3.9. Motivated by Theorem 5.3.8, note that the cosimplicial diagram $\underline{\mathrm{MU}}^{\otimes\bullet+1}$ defines a morphism of simplicial diagrams

$$\underline{\pi}_*(\underline{\mathrm{MU}}^{\otimes\bullet+1}) : \mathrm{Lat} \to \mathrm{CAlg}^{\mathrm{gr}}_{\pi_*(\mathrm{MU}^{\otimes\bullet+1})} \simeq \mathrm{CAlg}(\mathrm{Mod}^{\mathrm{gr}}_{\pi_*(\mathrm{MU}^{\otimes\bullet+1})}),$$

where the source denotes the constant simplicial object. Write $\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}} : \mathrm{Lat} \to \mathrm{CAlg}(\mathrm{IndCoh}(\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}))$ to denote the resulting map on totalizations. This functor can be viewed as the "universal graded generalized group" over $\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}$. In our context, it is the analogue for the sphere spectrum of the graded group scheme $\mathbf{G}_\beta$ associated to ku. A variant of Comezaña's freeness result from [**Com96**] implies that $\underline{\pi}_*(\underline{\mathrm{MU}}^{\otimes n})(\Lambda)$ is flat over $\pi_*(\mathrm{MU}^{\otimes n})$ for each n and lattice Λ. This implies that if Λ is a lattice, $\mathbf{G}_{\mathrm{univ}}(\Lambda) = \mathrm{Spec}\,\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}}(\Lambda)$ is flat over $\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}$. Note that if T is a compact abelian Lie group, there is an fpqc cover $\mathrm{Spec}\,\pi_*^T(\mathrm{MU}_T) \to \mathbf{G}_{\mathrm{univ}}(\mathbb{X}^*(T))$. If M is a finite T-equivariant spectrum, let $\mathcal{F}_{T,M}$ denote the graded quasicoherent sheaf over $\mathbf{G}_{\mathrm{univ}}(\mathbb{X}^*(T))$ specified by $\lim_{\boldsymbol{\Delta}} \pi_*^T(M^\vee \otimes \mathrm{MU}_T^{\otimes\bullet+1})$. If M is the suspension spectrum of a finite T-space X, let $\mathcal{F}_T(X) = \mathcal{F}_{T,M}$. If X is a T-space, let $\mathcal{F}_T(X)^\vee$ denote the $\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}(\mathbb{X}^*(T))}$-linear dual of $\mathcal{F}_T(X)$, and if $X = \mathrm{colim}_{i\in\mathcal{I}} X_i$ is an ind-finite T-space, let $\mathcal{F}_T(X)^\vee = \mathrm{colim}_{i\in\mathcal{I}} \mathcal{F}_T(X_i)^\vee$.

The notation $\mathbf{G}_{\mathrm{univ}}$ is motivated by the following.

Conjecture 5.3.10. *Let $\underline{S}^0$ denote the global sphere spectrum, i.e., the unit for the tensor product on $\mathrm{Sp}_{\mathrm{glob}}$. Then, the unit map $\underline{S}^0 \to \mathrm{Tot}(\underline{\mathrm{MU}}^{\otimes\bullet+1})$ in $\mathrm{Sp}_{\mathrm{glob}}$ is an equivalence.*

Notation 5.3.11. Let G be a compact Lie group. It will be convenient to write $\mathrm{F}^\star_{\mathrm{ev}}(S^0_G)$ to denote the $\mathrm{F}^\star_{\mathrm{ev}}S^0$-algebra in $\mathrm{Sp}_{\mathrm{fil}}$ given by $\mathrm{Tot}(\tau^G_{\geq\star}(\mathrm{MU}_G^{\otimes\bullet+1}))$. If Conjecture 5.3.10 is false, this notation is rather abusive unless T is a compact abelian Lie group, since the underlying unfiltered spectrum need not be S^0_G. Note that if T is a compact abelian Lie group, $\mathrm{gr}^\star_{\mathrm{ev}}(S^0_T)$ is precisely $\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}}(\mathbb{X}^*(T))$ as an object of $\mathrm{CAlg}(\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{gr}^\star_{\mathrm{ev}}S^0}) \simeq \mathrm{CAlg}(\mathrm{IndCoh}(\mathcal{M}^{\mathrm{s}}_{\mathrm{FG}}))$.

Since the arguments establishing Theorem 3.2.14 and Theorem 3.6.10 ultimately reduce to the case of *torus*-equivariant (co)homology, if one is to generalize these results to equivariant MU, it is first natural to ask whether equivariant MU admits abelian descent. Perhaps the most naïve formulation of this question is the following: if G is a connected Lie group whose π_1 is torsion-free, $T \subseteq G$ is a maximal torus, and W is its Weyl group, does restriction induce an isomorphism $\pi_*^G(\mathrm{MU}_G) \xrightarrow{\sim} \pi_*^T(\mathrm{MU}_T)^W$? This turns out to be *false*:

Lemma 5.3.12 ([**Sch23**, Remark 1.2])**.** *The restriction map $\pi_*^{\mathrm{U}(n)}(\mathrm{MU}_{\mathrm{U}(n)}) \to \pi_*^T(\mathrm{MU}_T)$ is not injective if $n > 1$.*

Remark 5.3.13. It is also not known whether $\pi_*^G(\mathrm{MU}_G)$ is concentrated in even degrees for a general compact Lie group G. This is a famous open problem in equivariant algebraic topology; see [**Uri18**].

Remark 5.3.14. Lemma 5.3.12 leads to the issue of whether an appropriate analogue of Theorem 3.2.14 and Theorem 3.6.10 should use G-equivariant MU, or instead a more limited analogue which is built from A-equivariant MU over all compact abelian subgroups A of G. (At a first pass, this can be viewed as the difference betweeen $\underline{\pi}_*\underline{\mathrm{MU}}$ and $\underline{\pi}_*^{\mathrm{toral}}\underline{\mathrm{MU}}$.) The resolution of this issue in the context of Langlands duality is not entirely clear to me, although it seems to be the case that the latter analogue of equivariant MU should be more relevant.

5.4. Langlands duality and the sphere spectrum. Naturally, one is interested in proving the "universal" analogue of Theorem 3.2.7 and Theorem 3.6.21, except for sheaves with coefficients in the sphere spectrum. Let us begin with some speculations about the form of such an equivalence, and then discuss an actual mathematical statement relating to these speculations. There are far too many components of this speculation which do not have well-behaved foundations at the moment, and so we will record this discussion as a series of expectations. It is quite likely that these expectations are too naïve, and that more refinement is needed to make them precise. I apologize in advance for the speculative nature of this section!

Expectation 5.4.1. Let G be a compact Lie group, and suppose that X is a stratified finite space with G-action respecting the stratification. Then, there should be a $\mathrm{F}^\star_{\mathrm{ev}}(S^0_G)$-linear ∞-category $\mathrm{Shv}^c_G(X; S^0)^{\mathrm{Syn}}$ of equivariant "synthetic" constructible sheaves of G-spectra on X. This theory should admit a well-behaved six functor formalism, and should also extend to ind-finite stratified G-spaces. Changing coefficients of the underlying ∞-category $\mathrm{Shv}^c_G(X; S^0)$ along the unit map $S^0 \to \mathbf{Z}$ should produce the ∞-category $\mathrm{Shv}^c_G(X; \mathbf{Z})$.

When G is trivial, $\mathrm{Shv}^c_G(X; S^0)^{\mathrm{Syn}}$ should be the ∞-category of Construction 5.3.3. Thanks to Notation 5.3.11, if T is a compact abelian Lie group, the associated graded ∞-category $\mathrm{Shv}^{c,\mathrm{gr}}_T(X; S^0)$ will be $\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}}(\mathbb{X}^*(T))$-linear. If G is a connected compact Lie group whose π_1 is torsion-free, we expect that the associated graded ∞-category $\mathrm{Shv}^{c,\mathrm{gr}}_G(X; S^0)$ should be $\mathcal{O}_{\mathbf{G}_{\mathrm{univ}}}(\mathbb{X}^*(T))^W$-linear (at least, depending on the resolution of Remark 5.3.14). There should also be an analogue of Theorem 3.1.3 in this context.

For a sufficiently robust theory as in Expectation 5.4.1, it should be possible to define the ∞-category $\mathrm{Shv}^{c,\mathrm{gr}}_{G\times G}(\mathcal{L}G; S^0)$. In order to obtain a spectral decomposition of this ∞-category analogous to Theorem 3.2.7 and Theorem 3.6.21, we need an analogue of the quotient stack G_β/G from Definition 3.6.3.

Expectation 5.4.2. Let $\mathbf{G} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{Aff}^{\heartsuit,\mathrm{gr}}$ be a graded generalized group, and let H be a graded group scheme over $\mathbf{G}(\{0\})$. Then there should be a graded stack $H_{\mathbf{G}}/H$ over the classifying stack $B_{\mathbf{G}(\{0\})}H$ satisfying the following property: if $\mathbf{G}$ preserves products, so that it can be identified with the data of the weight-connected 1-dimensional linear graded algebraic group $\mathbf{G}(\mathbf{Z})$ over $\mathcal{O}_{\mathbf{G}}(\{0\})$ by Example 5.3.7), there should be an isomorphism

$$H_{\mathbf{G}}/H \cong \mathrm{Map}(\underline{\mathrm{Hom}}(\mathbf{G}(\mathbf{Z}), B\mathbf{G}_m), BH). \tag{36}$$

In particular, $H_{\mathbf{G}_\beta}/H \cong H_\beta/H$. Note that $\underline{\mathrm{Hom}}(\mathbf{G}(\mathbf{Z}), B\mathbf{G}_m)$ is the shifted Cartier dual of $\mathbf{G}(\mathbf{Z})$. Let us remark that it is not obvious how one might define the Cartier dual of a graded group $\mathbf{G}$ which does not necessarily preserve products: the $\mathcal{O}_{\mathbf{G}(\{0\})}$-linear dual of $\mathcal{O}_{\mathbf{G}(\mathbf{Z})}$ does *not* necessarily admit a ring structure.

Remark 5.4.3. Suppose $\mathbf{G} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{Aff}^{\heartsuit,\mathrm{gr}}$ is a graded generalized group which preserves products, and let H be a graded group scheme over $\mathbf{G}(\{0\})$. Let $H_{\mathbf{G}}$ denote the fiber product $H_{\mathbf{G}}/H \times_{B_{\mathbf{G}(\{0\})}H} \mathbf{G}(\{0\})$, where $H_{\mathbf{G}}/H$ is defined as in (36). Then $H_{\mathbf{G}}$ can be viewed as a "probing" of H by $\mathbf{G}$. If H is semisimple and base-changed from $\mathbf{Z}$, $H_{\mathbf{G}}$ "replaces" the Cartan T of H by $T_{\mathbf{G}}$, and leaves the unipotent part alone (at least if $\mathbf{G}$ is a graded group *law*). For instance, if $\mathbf{G} = \mathbf{G}_a(2)$, we can identify $H_{\mathbf{G}}$ with $\mathfrak{h}(2)$; and if $\mathbf{G} = \mathbf{G}_m$, we can identify $H_{\mathbf{G}}$ with H.

Expectation 5.4.4 (Derived Satake over the sphere spectrum)**.** Let G be a simply-laced simply-connected semisimple algebraic group over $\mathbf{Z}$, and invert the integer $|W|$. Let $\mathbf{G}_{\mathrm{univ}} : \mathrm{Lat} \to \mathrm{CAlg}(\mathrm{IndCoh}(\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}} \otimes_{\mathbf{Z}} \mathbf{Z}'))$ denote the graded generalized group of Construction 5.3.9, and let $\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G}$ denote the group scheme of Expectation 5.4.2. Then:

- there is an isomorphism
$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathrm{gr}^{\star}_{\mathrm{ev}}(S^0_{G/Z(G)}) \cong \check{G}_{\mathbf{G}_{\mathrm{univ}}}/\!\!/\check{G},$$
as well as a "Kostant slice"
$$\kappa_{\mathrm{univ}} : \check{G}_{\mathbf{G}_{\mathrm{univ}}}/\!\!/\check{G} \to \check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G};$$
- there is a $\mathrm{gr}^{\star}_{\mathrm{ev}}(S^0_{G/Z(G)})$-linear equivalence
$$\mathrm{Shv}^{c,\mathrm{gr}}_{G/Z(G)}(\Omega G; S^0) \simeq \mathrm{Perf}(\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G})$$
such that changing coefficients along the unit map $S^0 \to \mathbf{Z}'$ produces the derived geometric Satake equivalence with coefficients in $\mathbf{Z}'$.
- the above equivalence fits into a commutative diagram
$$\begin{tikzcd}
\mathrm{Shv}^{c,\mathrm{gr}}_{G/Z(G)}(\Omega G; S^0) \arrow[r, "\sim"] \arrow[d, "\mathrm{pushforward}"'] & \mathrm{Perf}(\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G}) \arrow[d, "\kappa^*_{\mathrm{univ}}"] \\
\mathrm{Shv}^{c,\mathrm{gr}}_{G/Z(G)}(*; S^0) \arrow[r, "\sim"'] & \mathrm{Perf}(\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\!\!/\check{G}).
\end{tikzcd}$$

The expected equivalence above can be viewed as describing (a categorification of) the $G/Z(G)$-equivariant stratified stable homotopy type of ΩG in terms of the Langlands dual group, i.e., via the combinatorics of G.

Remark 5.4.5. It would be interesting to generalize Expectation 5.4.4, or even Theorem 3.6.21, to the non-simply-laced case.

Remark 5.4.6. The right-hand side of Expectation 5.4.4 cannot be correct as written. To explain this, observe that if the construction of the left-hand side of Expectation 5.4.4 is sufficiently well-behaved, it should imply that if A is any

2-periodic $\mathbf{E}_\infty$-ring equipped with an oriented group scheme $\mathbf{G}$ over A (with underlying group scheme $\mathbf{G}_0$ over $\pi_0(A)$) and a commutative diagram

$$\begin{tikzcd} \mathbf{G} \arrow[r] \arrow[d] & \mathrm{Spec}(\mathrm{MU}_T^T) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(\mathrm{MU}) \end{tikzcd}$$

of $\mathbf{E}_\infty$-rings, then there is an equivalence of $\pi_0(A_G)$-linear ∞-categories

$$\mathrm{Shv}^{c,\mathrm{gr}}_{G/Z(G)}(\Omega G; A) \simeq \mathrm{Perf}(\check{G}_{\mathbf{G}_0}/\check{G}).$$

Note that the right-hand side is indeed well-defined by (36), since $\mathbf{G}_0$ is an honest 1-dimensional linear algebraic group. Such an equivalence would in particular imply an equivalence between the localizing subcategory of the left-hand side spanned by *locally constant sheaves* (i.e., local systems) and the localization of the right-hand side given by restriction to the regular locus. This more limited equivalence was proved in [**Dev23a**] in the cases of 2-periodic rational cohomology and periodic complex K-theory. However, the case of elliptic cohomology is more subtle.

If $\mathbf{G}_0$ is an elliptic curve E over $\pi_0(A)$, then $\check{G}_{\mathbf{G}_0}/\check{G}$ is isomorphic to the moduli stack $\mathrm{Bun}_{\check{G}}(E^\vee)$ of $\check{G}$-bundles on the dual elliptic curve $E^\vee$. This is *not* the object appearing in [**Dev23a**]: instead, the relevant object is the substack of degree zero semistable $\check{G}$-bundles on $E^\vee$. This subtlety suggests that the correct form of Expectation 5.4.4 should have similar adornments on the dual side, but it is difficult to guess what these should be.

Remark 5.4.7. It should also be possible to extend Conjecture 3.4.11 to the case of coefficients in S^0 (once Expectation 5.4.1 is appropriately resolved), although at this stage of development, there is very little that can be said about the general theory of relative Langlands duality with coefficients in S^0. One might hope that $\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G}$ admits a 1-shifted symplectic structure when viewed as a stack over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$. The analogue of Hamiltonian $\check{G}$-varieties in this new context should then be given by Lagrangian morphisms to $\check{G}_{\mathbf{G}_{\mathrm{univ}}}/\check{G}$.

This expected generalization of Conjecture 3.4.11 is in some ways not entirely unreasonable, since (as mentioned at the end of Expectation 5.4.4) it can be viewed as concerned with the G-equivariant stratified stable homotopy type of $\mathcal{L}(G/H)$, or equivalently with the H-equivariant stratified stable homotopy type of $\Omega(G/H)$. The proof of Theorem 3.4.13, and in particular Table 4, shows that these spectral decompositions already have manifestations at the level of spaces themselves, and not just at the level of (ku-)chains.

Let us end with a calculation in the *non*equivariant setting. One could interpret this calculation as either a step towards Expectation 5.4.4, or as evidence that equivariance is the most interesting part of Expectation 5.4.4. Namely, recall the following simple consequence (and historical antecedent) of Theorem 3.2.7:

Proposition 5.4.8 (Ginzburg, [**Gin95**, Proposition 1.7.2])**.** *Let G be a connected semisimple algebraic group over $\mathbf{C}$; we will abusively also write G to denote the maximal compact subgroup of $G(\mathbf{C})$. Let $\check{\mathfrak{g}}^e(2)$ denote the centralizer of a principal nilpotent element e in $\check{\mathfrak{g}}$. Then there is an isomorphism* $\mathrm{H}^*(\Omega G; \mathbf{C}) \cong U(\check{\mathfrak{g}}^e(2))$ *of*

Hopf algebras over $\mathbf{C}$*, such that the Chern class* $c_1(\det)$ *of the determinant line bundle*[21] *over* ΩG *is sent to the element* $e \in U(\check{\mathfrak{g}}^e(2))$.

Moreover, if λ *is a dominant weight of* $\check{G}$ *with associated highest weight representation* $\check{V}_\lambda$*, and* IC_λ *is the* IC*-sheaf associated to* $\mathrm{Gr}_G^{\leq\lambda} \subseteq \mathrm{Gr}_G$*, there is an isomorphism* $\mathrm{H}^*(\mathrm{Gr}_G; \mathrm{IC}_\lambda) \cong \check{V}_\lambda$ *of* $\mathrm{H}^*(\Omega G; \mathbf{C}) \cong U(\check{\mathfrak{g}}^e(2))$*-comodules.*

Remark 5.4.9. By [**Gin91**, Theorem 1.10.3], if G is simply-connected and λ, μ are dominant coweights of G with associated irreducible representations $\check{V}_\lambda, \check{V}_\mu$ of $\check{G}$, there is an isomorphism

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c(\mathrm{Gr}_G;\mathbf{C})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \cong \mathrm{Hom}^\bullet_{\mathrm{Rep}(Z_e(\check{G}))}(\check{V}_\lambda, \check{V}_\mu). \tag{37}$$

Indeed, using Theorem 3.1.3, taking cohomology defines an isomorphism

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c(\mathrm{Gr}_G;\mathbf{C})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathrm{H}^*(\Omega G;\mathbf{C})}(\mathrm{H}^*(\Omega G; \mathrm{IC}_\lambda), \mathrm{H}^*(\Omega G; \mathrm{IC}_\mu)).$$

The desired isomorphism (37) then follows from the graded isomorphism $\mathrm{H}^*(\Omega G; \mathrm{IC}_\lambda) \cong \check{V}_\lambda$ via Theorem 3.2.3 and first part of Proposition 5.4.8.

In [**Gin91**, Proposition 1.10.4], the isomorphism (37) is rephrased without appeal to a particular choice of regular nilpotent element as follows. Let $\check{\mathcal{N}}$ denote the nilpotent cone of $\check{\mathfrak{g}}$ (equipped with its natural $\mathbf{G}_m$-action), and for any $\check{G}$-representation $\check{V}$, let $\check{\mathcal{V}}$ denote the associated vector bundle $\mathcal{O}_{\check{\mathcal{N}}} \otimes_{\mathbf{C}} \check{V}$. If G is simply-connected and λ, μ are dominant coweights of G, there is an isomorphism

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}^c(\mathrm{Gr}_G;\mathbf{C})}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \cong \mathrm{Hom}^\bullet_{\mathrm{Coh}(\check{\mathcal{N}}/\check{G})}(\check{\mathcal{V}}_\lambda, \check{\mathcal{V}}_\mu).$$

This can be deduced from (37) using the fact that the regular nilpotent orbit $\check{\mathcal{N}}^{\mathrm{reg}} \subseteq \check{\mathcal{N}}$ is isomorphic to $\check{G}/Z_e(\check{G})$, and has complement of codimension ≥ 2.

Remark 5.4.10. The isomorphism $\mathrm{H}^*(\Omega G; \mathbf{C}) \cong U(\check{\mathfrak{g}}^e(2))$ is defined via a map $\mathrm{H}^*(\Omega G; \mathbf{C}) \to U(\check{\mathfrak{g}}(2))$ of Hopf algebras, which is constructed using Theorem 3.2.3. Namely, taking cohomology defines a functor $\mathcal{P}\mathrm{erv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{Rep}(\check{G}) \to \mathrm{Mod}_{\mathrm{H}^*(\Omega G;\mathbf{C})}$, and hence a map of Hopf algebras as desired.

Remark 5.4.11. Let ℓ_G denote the square of the ratio of the lengths of long roots and the short roots of G. If we replace $\mathbf{C}$ by $\mathbf{Z}[1/\ell_G]$, there is an isomorphism of Hopf algebras between $\mathrm{H}^*(\Omega G; \mathbf{Z}[1/\ell_G])$ and the divided power Hopf algebra $U^\sharp(\check{\mathfrak{g}}^e(2))$ (i.e., distributions on $Z_{\check{G}}(e)$). In fact, there is an isomorphism $\mathrm{Spec}\, \mathrm{H}_*(\Omega G; \mathbf{Z}[1/\ell_G]) \cong Z_e(\check{G})$ of group schemes over $\mathbf{Z}[1/\ell_G]$; see [**YZ11**, Theorem 6.1]. After rationalization (or even just inverting $|W|$), this follows from Theorem 3.2.14.

Let us mention how this isomorphism can be deduced after inverting $|W|$ using the analogue of Theorem 3.2.7 for sheaves with coefficients in $\mathbf{Z}[1/|W|]$. Namely, this equivalence states that there is an equivalence $\mathrm{Loc}_G(\Omega G; \mathbf{Z}[1/|W|]) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G})$, so that

$$\mathrm{Loc}(\Omega G; \mathbf{Z}[1/|W|]) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G} \times_{\check{\mathfrak{g}}^*[2]/\!\!/\check{G}} \{0\}).$$

But the fiber product $\check{\mathfrak{g}}^{*,\mathrm{reg}}[2] \times_{\check{\mathfrak{g}}^*[2]/\!\!/\check{G}} \{0\}$ is precisely the regular locus $\check{\mathcal{N}}^{\mathrm{reg}}$ in the (shifted) nilpotent cone $\check{\mathcal{N}}$, so that $\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G} \times_{\check{\mathfrak{g}}^*[2]/\!\!/\check{G}} \{0\} \cong \check{\mathcal{N}}^{\mathrm{reg}}/\check{G}$. By [**Kos63**],

[21] This is simply defined to be $\Omega^2(BG \to K(\mathbf{Z}, 4))$, where the map $BG \to K(\mathbf{Z}, 4)$ detects the Killing form in $\mathrm{H}^4(BG; \mathbf{Z}) \cong \mathrm{Sym}^2(\mathbb{X}^*(T))^W$.

there is a unique regular $\check{G}$-orbit in $\check{\mathcal{N}}$, and so $\check{\mathcal{N}}^{\mathrm{reg}}/\check{G} \cong BZ_e(\check{G})$. This implies that there is an equivalence

$$\mathrm{Loc}(\Omega G; \mathbf{Z}[1/|W|]) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} BZ_e(\check{G})).$$

This equivalence sends the skyscraper sheaf at the basepoint of ΩG to the pushforward of the structure sheaf along the map $\mathrm{Spec}(\mathbf{Z}[1/|W|]) \to BZ_e(\check{G})$, and the constant sheaf on ΩG to the structure sheaf of $BZ_e(\check{G})$. For instance, taking endomorphisms of the constant sheaf on the left-hand side, we get an isomorphism $\mathrm{H}_*(\Omega G; \mathbf{Z}[1/|W|]) \cong \mathcal{O}_{Z_e(\check{G})}$, as expected from [**YZ11**, Theorem 6.1].

Moreover, as in Remark 4.2.4, the above equivalence of ∞-categories implies (by computing endomorphisms of the skyscraper sheaf at the basepoint of ΩG) that there is an isomorphism

$$\mathrm{H}_*(\Omega^2 G; \mathbf{Z}[1/|W|]) \cong \mathrm{H}^*(\mathrm{sh}^{1/2} BZ_e(\check{G}); \mathcal{O}).$$

For instance, if $G = \mathrm{SU}(n)$ (in which case one does not need to invert $n!$), the centralizer $Z_e(\mathrm{PGL}_n)$ is isomorphic to the group scheme $\mathbf{W}_{n-1}$ of length n Witt vectors (see [**Dev23a**, Example 4.1.8]) where the jth ghost coordinate lives in weight $2j$: indeed, both group schemes can be identified with the group of matrices of the form

$$Z_e(\mathrm{PGL}_n) \cong \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ & 1 & x_1 & x_2 & \cdots & x_{n-2} \\ & & \ddots & \vdots & \cdots & \vdots \\ & & & 1 & x_1 & x_2 \\ & & & & 1 & x_1 \\ & & & & & 1 \end{pmatrix} \right\} \cong \mathbf{W}_{n-1}.$$

There is then an isomorphism

$$\mathrm{H}_*(\Omega^2 \mathrm{SU}(n); \mathbf{Z}) \cong \mathrm{H}^*(\mathrm{sh}^{1/2} B\mathbf{W}_{n-1}; \mathcal{O});$$

moreover, this isomorphism is compatible as n varies. After base-changing to $\mathbf{F}_p$, this recovers [**Rav93**, Theorem A] and the main result of [**Yam86**].

Remark 5.4.12. As an aside, let us note that it is natural to wonder if there is an analogue of the determinant line bundle for arbitrary spherical varieties X, which reduces to the determinant line bundle on Gr_G in the group case for G. If Gr^X denotes the relative Grassmannian of [**BZSV23**, Section 8.2], there is a tautological map $\mathrm{Gr}^X \to \mathrm{Gr}_G$, so one can simply consider the pullback of the determinant line bundle to Gr^X. However, this construction in some sense misses the point of the determinant line bundle: namely, it captures the first nontrivial integral cohomology class on Gr_G, and hence the principal nilpotent $e \in \check{\mathfrak{g}}$, but the above line bundle on Gr^X does not capture anything new.

In order to understand the new phenomena which appear in the relative setting, suppose $X = G/H$ with $H \subseteq G$ being a connected reductive subgroup. Throughout this article, we have emphasized that the quotient $\Omega(G/H) = \mathrm{Gr}_G/\mathrm{Gr}_H$ captures the homotopically interesting content of the local relative geometric Langlands conjectures, so it is natural to ask whether $\Omega(G/H)$ carries an analogue of the determinant line bundle. The answer to this particular question is "no" for a very naïve reason: if G/H is 3-connected, as is often the case (e.g., $\mathrm{GL}_n/\mathrm{GL}_{n-1}$ for $n > 2$), then $\mathrm{H}^2(\Omega(G/H); \mathbf{Z}) = 0$, and so all complex line bundles over $\Omega(G/H)$ are trivial.

However, if one relaxes the notion of a line bundle by permitting categorification, the answer seems to be "yes". Consider, for instance, the case of $\mathrm{Spin}_{2n+2}/\mathrm{Spin}_{2n+1} \simeq S^{2n+1}$, so that $\mathrm{H}^*(\Omega S^{2n+1}; \mathbf{Z}) \cong \mathbf{Z}\langle x\rangle$ with x in cohomological degree $2n$. As we have seen in Theorem 4.5.1, the class x plays the role of the principal nilpotent $e \in \check{\mathfrak{g}}_X$ (just as in Proposition 5.4.8). In any case, x may be viewed as a map $\Omega S^{2n+1} \to K(\mathbf{Z}, 2n)$, and hence it classifies a nontrivial $(2n-2)$-$\mathbf{G}_m$-gerbe over ΩS^{2n+1}. When $n = 1$, this is simply the determinant line bundle over $\Omega S^3 = \mathrm{Gr}_{\mathrm{SL}_2}$. For larger n, one can wonder whether this $(2n-2)$-$\mathbf{G}_m$-gerbe arises from a vector bundle of rank n over ΩS^{2n-1}, i.e., whether the map $\Omega S^{2n+1} \to K(\mathbf{Z}, 2n)$ factors as a composite

$$\Omega S^{2n+1} \xrightarrow{f} \mathrm{BU}(n) \to K(\mathbf{Z}, 2n).$$

Unfortunately, this turns out to be impossible for $n \geq 2$; let us just focus on the case $n = 2$. Indeed, in this case, the putative map $f : \Omega S^5 \to \mathrm{BU}(2)$ will necessarily factor through $\mathrm{BSU}(2)$, since all line bundles on ΩS^5 are trivial. By construction, the composite $S^4 \to \Omega S^5 \to \mathrm{BSU}(2) = \mathbf{H}P^\infty$ will induce an isomorphism on $\mathrm{H}^4(-; \mathbf{Z})$, so to get the desired map f, we need the attaching map of the 8-skeleton of ΩS^5 to be null in $\pi_7(\mathbf{H}P^\infty)$. However, this is false; see [**Dev20**]. In general, the geometric/representation-theoretic construction and meaning of such putative determinant $\mathbf{G}_m$-gerbes on $\Omega(G/H)$ seems very interesting.

Our goal is to prove an analogue of Proposition 5.4.8 over the sphere spectrum in the simplest case of $G = \mathrm{SL}_n$, and (a piece) of the second part of *loc. cit.* when $n = 2$. Since our goal is to illustrate certain phenomena, as opposed to proving the most general statement, we will only stick to this simple case.

Construction 5.4.13. Let X be a scheme (or even a stack), let $\mathcal{L}$ be a line bundle over X, and let $\mathbf{V}(\mathcal{L})$ denote its total space. Let $\mathbf{W}$ denote the Witt ring scheme, and let $\mathbf{W}(\mathcal{L}) = \mathrm{Hom}_X(\mathbf{V}(\mathcal{L}), \mathbf{W})$ denote the associated *Witt group scheme* over X. Similarly, one can define the length n Witt group scheme $\mathbf{W}_n(\mathcal{L}) = \mathrm{Hom}_X(\mathbf{V}(\mathcal{L}), \mathbf{W}_n)$. Its Cartier dual will be denoted $\mathbf{W}_n^\vee(\mathcal{L}^{-1})$, so that $\mathbf{W}_n^\vee(\mathcal{L}^{-1}) \cong \mathrm{Hom}_X(\mathbf{V}(\mathcal{L}^{-1}), \mathbf{W}_n^\vee)$.

The following result is a refinement of [**Dev23a**, Example 4.1.8], and can be viewed as an analogue of Proposition 5.4.8 for $G = \mathrm{SL}_n$.

Lemma 5.4.14. *Let $\hat{\mathbf{G}}_{\mathrm{univ}}$ denote the universal spin formal group over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$, and let $\hat{\mathbf{G}}_{\mathrm{univ}}^\vee$ denote its Cartier dual. There is an isomorphism*

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega\mathrm{SU}(n); S^0)^\vee \cong \mathbf{W}_{n-1}(\omega)$$

of group stacks over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$, and which induces an isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega\mathrm{SU}(n); S^0) \cong \mathbf{W}_{n-1}^\vee(\omega^{-1}) \cong \mathrm{Hom}(\hat{\mathbf{G}}_{\mathrm{univ}}^\vee, \mathbf{W}_{n-1}^\vee)$$

of group stacks over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$.

Proof. The natural map $\mathbf{C}P^{n-1} \to \Omega\mathrm{SU}(n)$ is Bott's generating complex from [**Bot58**], so that there is an isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega\mathrm{SU}(n); S^0)^\vee \cong \mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathrm{Sym}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}}(\mathcal{F}(\mathbf{C}P^{n-1}; S^0)^\vee),$$

the latter being $\mathbf{V}(\mathcal{F}(\mathbf{C}P^{n-1}; S^0))$. It therefore suffices to observe that there is an isomorphism of group schemes

$$\mathbf{V}(\mathcal{F}(\mathbf{C}P^{n-1}; S^0)) \cong \mathbf{W}_{n-1}(\omega).$$

Since $\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega\mathrm{SU}(n); S^0)^\vee$ is Cartier dual to $\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega\mathrm{SU}(n); S^0)$, this implies that the latter is isomorphic to $\mathbf{W}_{n-1}^\vee(\omega^{-1})$ (being Cartier dual to $\mathbf{W}_{n-1}(\omega)$). Lemma 3.6.6 implies that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}}(\hat{\mathbf{G}}_{\mathrm{univ}}^\vee, \mathbf{W}_{n-1}) \cong \mathbf{W}_{n-1}(\omega^{-1}),$$

since $\mathrm{Lie}(\hat{\mathbf{G}}_{\mathrm{univ}}) \cong \omega^{-1}$; this in turn implies that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}}(\hat{\mathbf{G}}_{\mathrm{univ}}^\vee, \mathbf{W}_{n-1}^\vee) \cong \mathbf{W}_{n-1}^\vee(\omega^{-1}),$$

as desired. □

Remark 5.4.15. The group scheme $\mathbf{W}_{n-1}(\omega)$ over $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ can be identified with the fiber product $\mathbf{W}_{n-1}(\mathcal{O}(2)) \times_{B\mathbf{G}_m} \mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$, where the map $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}} \to B\mathbf{G}_m$ classifies $\omega^{1/2}$.

Remark 5.4.16. The $\mathbf{E}_2$-map $\Omega S^3 \to \mathbf{C}P^\infty$ classifying the determinant line bundle induces a map

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\mathbf{C}P^\infty; S^0)^\vee \to \mathrm{Spec}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}} \mathcal{F}(\Omega S^3; S^0)^\vee,$$

which, under Lemma 5.4.14 for $n = 2$ can be identified with the canonical map $\hat{\mathbf{G}}^\vee \to \mathbf{V}(\omega)$ classifying the tautological section of $\mathrm{Lie}(\hat{\mathbf{G}}_{\mathrm{univ}}) \otimes \omega \cong \mathcal{O}_{\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}}$. The canonical map

$$\mathbf{W}_1^\vee(\omega^{-1}) \cong \hat{\mathbf{V}}(\omega^{-1})^\sharp \to \hat{\mathbf{G}}_{\mathrm{univ}}$$

classifies the first Chern class of the determinant line bundle, and can be regarded as an analogue of the map $\mathrm{Spec}\, \mathrm{H}^*(\Omega\mathrm{SU}(n); \mathbf{Z}) \to \mathbf{A}^1$ detecting the principal nilpotent element of $\check{\mathfrak{g}} = \mathfrak{pgl}_n$ under Proposition 5.4.8.

According to Lemma 5.4.14, $\mathbf{W}_{n-1}(\omega)$ is to be understood as the analogue for the sphere of the centralizer $Z_{\mathrm{PGL}_n}(e)$.[22] In this setting, the grading on $Z_{\mathrm{PGL}_n}(e)$ translates into tensor powers of the line bundle ω. As a consequence of the above discussion, $\mathbf{W}_{n-1}^\vee(\omega^{-1})$ is to be understood as the analogue for the sphere of the divided power enveloping algebra $U^\sharp(\mathfrak{pgl}_n^e)$. Ultimately, the fact that the *nonequivariant* (co)homology of $\Omega\mathrm{SU}(n)$ is accessible, and looks analogous to the case of integral cohomology, is a Langlands dual manifestation of the fact that the centralizer of a regular nilpotent element is a unipotent group scheme.

Let us now describe an analogue of the second part of Proposition 5.4.8 for $G = \mathrm{SL}_2$.[23] In this case, we have the following (well-known) calculation.

Lemma 5.4.17. *Consider the dominant (co)weight (i,j) of GL_2, so that $i \geq j$. Then there is a cell decomposition*

$$\mathrm{Gr}_{\mathrm{GL}_2}^{\leq(i,j)} \cong \begin{cases} \coprod_{0\leq k\leq (i-j)/2} \mathrm{Gr}_{\mathrm{GL}_2}^{(i-k,j+k)} & i-j \text{ even}, \\ \coprod_{0\leq k\leq (i-j-1)/2} \mathrm{Gr}_{\mathrm{GL}_2}^{(i-k,j+k)} & i-j \text{ odd}. \end{cases}$$

[22] Note that there is an isomorphism $Z_{\mathrm{PGL}_n}(e) \cong \mathbf{W}_{n-1}(\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})})$ of group schemes over $\mathbf{Z}$; see [**Dev23a**, Example 4.1.8]. Therefore, pulling back the isomorphism of Lemma 5.4.14 along the map $\mathrm{Spec}(\mathbf{Z}) \to \mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ classifying the additive formal group precisely recovers the isomorphism $\mathrm{Spec}\, \mathrm{H}_*(\Omega\mathrm{SU}(n); \mathbf{Z}) \cong Z_{\mathrm{PGL}_n}(e)$.

[23] To a representation theorist, this might seem like an especially trivial case, and likewise for a homotopy theorist; but perhaps for somewhat different reasons (either $\check{G} = \mathrm{PGL}_2$ is too simple, or $\Omega G = \Omega S^3$ is too simple). For a geometric representation theorist, this example is trivial for two reasons!

If $i > j$, there is an isomorphism

$$\mathrm{Gr}_{\mathrm{GL}_2}^{(i,j)} \cong \mathrm{Map}(\mathrm{Spec}\,\mathbf{C}[\epsilon]/\epsilon^{i-j}, \mathbf{P}^1),$$

and $\mathrm{Gr}_{\mathrm{GL}_2}^{(i,i)} = \mathrm{Spec}(\mathbf{C})$.

Proof sketch. The only nontrivial claim is the calculation of $\mathrm{Gr}_{\mathrm{GL}_2}^{(i,j)}$ for $i > j$. Since there is an isomorphism $\mathrm{Gr}_{\mathrm{GL}_2}^{(i,j)} \cong \mathrm{Gr}_{\mathrm{GL}_2}^{(i-1,j-1)}$, we may assume that $j = 0$. In this case, the $\mathrm{GL}_2(\mathbf{C}[\![t]\!])$-orbit of the loop $\begin{pmatrix} t^i & 0 \\ 0 & t^j \end{pmatrix}$ consists of equivalence classes of matrices of the form $\begin{pmatrix} t^i a & b \\ t^i c & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{C}[\![t]\!]$. Note that both b and d cannot both be nonconstant, so $\mathrm{Gr}_{\mathrm{GL}_2}^{(i,0)}$ is covered by the open loci where the constant term of b (resp. of d) is invertible. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Over the locus where the constant term of b is invertible, we have the relation

$$\frac{1}{\det(A)}\begin{pmatrix} t^i a & b \\ t^i c & d \end{pmatrix}\begin{pmatrix} b & 0 \\ -at^i & adb^{-1} - c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t^i & db^{-1} \end{pmatrix}.$$

Multiplying on the right by some strictly upper-triangular matrix in $\mathrm{GL}_2(\mathbf{C}[\![t]\!])$, we can further assume that the bottom-right corner lies in $\mathbf{C}[t]/t^i$. This gives an isomorphism between $\{b(0) \neq 0\} \subseteq \mathrm{Gr}_{\mathrm{GL}_2}^{(i,0)}$ and $\mathbf{C}[t]/t^i$. One can similarly obtain an isomorphism between $\{d(0) \neq 0\} \subseteq \mathrm{Gr}_{\mathrm{GL}_2}^{(i,0)}$ and $\mathbf{C}[t]/t^i$ via matrices of the form $\begin{pmatrix} t^i & bd^{-1} \\ 0 & 1 \end{pmatrix}$. The intersection of these two opens is precisely given by sending $db^{-1} \mapsto bd^{-1}$, i.e., is obtained by gluing $\mathbf{C}[t]/t^i$ to itself along the inversion automorphism of $(\mathbf{C}[t]/t^i)^\times$. But this is precisely the mapping scheme $\mathrm{Map}(\mathrm{Spec}\,\mathbf{C}[\epsilon]/\epsilon^{i-j}, \mathbf{P}^1)$. $\square$

Remark 5.4.18. As a special case, Lemma 5.4.17 contains the well-known fact that $\mathrm{Gr}_{\mathrm{SL}_2}^{\leq 1}(\mathbf{C}) \cong \mathrm{Gr}_{\mathrm{GL}_2}^{\leq (1,-1)}(\mathbf{C})$ can be identified with the total space of $\mathcal{O}(2)$ over $\mathbf{P}^1$ (since this is the tangent bundle of $\mathbf{P}^1$).

Corollary 5.4.19. *Let $i \geq 0$ be an integer. If $J_i(S^2) \hookrightarrow \Omega S^3$ denotes the partial James construction (with top cell in dimension $2i$), Lemma 5.4.17 gives a homotopy equivalence*

$$\mathrm{Gr}_{\mathrm{SL}_2}^{\leq i}(\mathbf{C}) \cong \mathrm{Gr}_{\mathrm{GL}_2}^{\leq (i,-i)}(\mathbf{C}) \simeq J_{2i}(S^2)$$

which is compatible with the homotopy equivalence $\mathrm{Gr}_{\mathrm{SL}_2}(\mathbf{C}) \simeq \Omega S^3$ from Theorem 3.2.20.

Remark 5.4.20. It follows from Corollary 5.4.19 and (29) that if p is odd, there is an equivalence

$$\mathrm{Loc}(\mathrm{Gr}_{\mathrm{SL}_2}^{\leq \frac{p^j-1}{2}}(\mathbf{C}); \mathbf{F}_p) \simeq \mathrm{Perf}(\Omega\alpha_{p^j}[-2])$$

of $\mathbf{F}_p$-linear ∞-categories, where $\Omega\alpha_{p^j} = \{0\} \times_{\alpha_{p^j}} \{0\}$.

Importantly, Lemma 5.4.17 implies that the Schubert varieties for SL_2 are rationally smooth, and so IC_λ is simply the pushforward of the constant sheaf on $\mathrm{Gr}_{\mathrm{SL}_2}^{\leq\lambda}$. Therefore, if $i \geq 0$ is an integer (viewed as a dominant coweight of SL_2), Corollary 5.4.19 implies that we may identify

$$\mathrm{H}^*(\Omega\mathrm{SU}(2); \mathrm{IC}_i) \cong \mathrm{H}^*(J_{2i}(S^2); \mathbf{C}),$$

and hence Proposition 5.4.8 says (in particular) that

$$\mathrm{H}^*(J_{2i}(S^2); \mathbf{C}[-2i]) \cong \check{V}_i \tag{38}$$

with $\check{V}_i$ being the $(2i+1)$-dimensional irreducible representation of PGL_2. To prove an analogue of the second half of Proposition 5.4.8, we therefore need to compute $\mathcal{F}(J_{2i}(S^2); S^0)$ as a $\mathcal{F}(\Omega S^3; S^0)$-comodule (i.e., by Lemma 5.4.14, as a $\mathbf{V}(\omega)$-representation). This is quite simple (ha!):

Lemma 5.4.21. *Let $\Gamma^j_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) = (\omega^{\otimes j})^{\Sigma_j}$ denote the jth divided power of ω, and let $\Gamma^{\leq 2i}_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) = \bigoplus_{0\leq j\leq 2i} \Gamma^j_{\mathcal{M}^s_{\mathrm{FG}}}(\omega)$. Then there is an isomorphism*

$$\mathcal{F}(J_{2i}(S^2); S^0[-2i]) \cong \Gamma^{\leq 2i}_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) \otimes \omega^{\otimes -i},$$

and the inclusion $J_{2i}(S^2) \to \Omega S^3$ induces the canonical map

$$\mathcal{O}_{\hat{\mathbf{V}}(\omega^{-1})^\sharp} \cong \Gamma^*_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) \to \Gamma^{\leq 2i}_{\mathcal{M}^s_{\mathrm{FG}}}(\omega).$$

Since the irreducible representation $\check{V}_i$ of PGL_2 (over $\mathbf{Z}$) can be identified with $\Gamma^{\leq 2i}(\mathbf{Z}\cdot e)$, Lemma 5.4.21 can be viewed as an analogue of (38) over the sphere spectrum.

Remark 5.4.22. Note that Lemma 5.4.21 (or even (38)) only describes the action of the centralizer $Z_{\mathrm{PGL}_2}(e)$, which is isomorphic to the unipotent radical of the upper-triangular Borel subgroup of PGL_2, on the representation $\check{V}_i$. The action of the entirety of PGL_2 is encoded in natural structures present in this setting. Namely, in the classical setting of Proposition 5.4.8, the action of the Cartan subgroup is encoded by the natural grading on $\mathrm{H}^*(\Omega\mathrm{SU}(2); \mathrm{IC}_i)$. Moreover, the action of the unipotent radical of the opposite Borel is encoded by the action of $Z_{\mathrm{PGL}_2}(e)$ under "Poincaré duality" on $J_{2i}(S^2)$. Although $J_{2i}(S^2)$ is generally not a smooth manifold (and is often not even a Poincaré duality complex unless $(2i)!$ is inverted), its integral homology and cohomology *groups* are dual to each other, with a shift of $4i$.

In the setting of Lemma 5.4.21, the action of the Cartan subgroup is encoded by the tensor powers of the line bundle ω appearing in $\mathcal{F}(J_{2i}(S^2); S^0[-2i])$. Again, the action of the unipotent radical of the opposite Borel is still encoded by "Poincaré duality" on $J_{2i}(S^2)$, which gives an isomorphism of $\mathcal{O}_{\mathcal{M}^s_{\mathrm{FG}}}$-modules

$$(\Gamma^{\leq 2i}_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) \otimes \omega^{\otimes -i})^\vee \cong \Gamma^{\leq 2i}_{\mathcal{M}^s_{\mathrm{FG}}}(\omega) \otimes \omega^{\otimes i}.$$

As a first step towards Expectation 5.4.4, it would be interesting and important to prove an analogue of Proposition 5.4.8 with coefficients in the sphere spectrum for arbitrary (simply-connected and simply-laced) G, and not just for SL_2.

Appendix A. Sheaves on loop spaces

The proof of Theorem 3.5.23 relied on a comparison to positive characteristic. In the finite-type case, this comparison is provided by Theorem A.4, but the relevant comparison turns out to be more subtle in the infinite-type situation.

Notation A.1. Let R be a commutative ring, and let X be an R-scheme. Write $X(\!(t)\!)$ to denote the prestack sending an R-algebra S to $X(S(\!(t)\!))$. Similarly, write $X[\![t]\!]$ to denote the prestack sending an R-algebra S to $X(S[\![t]\!])$, and let $X[\![t]\!]/t^n$ denote the prestack sending an R-algebra S to $X(S[t]/t^n)$.

Definition A.2. Let X be an affine scheme defined over a commutative ring R equipped with an action of a linear algebraic group G over R. The $G[\![t]\!]$-action on $X(\!(t)\!)$ is called *weakly placid* if:

- there is a presentation $X(\!(t)\!) = \varinjlim_j X^j$, where each X^j is an inverse limit $\varprojlim_n X^j_n$ with each X^j_n being a $G[\![t]\!]$-equivariant scheme of finite type;
- the action of $G[\![t]\!]$ on X^j_n factors through $G[\![t]\!]/t^{m_n}$ for some $m_n \gg 0$ (compatibly in n).

Note that this is weaker than the condition that $G[\![t]\!]$-action on $X(\!(t)\!)$ being placid in the sense of [**BZSV23**, Section 7.3.1]: there, it is required that the transition maps $X^j_n \to X^j_{n-1}$ also be torsors for a unipotent group scheme.

Construction A.3. In the above setup, let $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j_n;\overline{\mathbf{Q}_\ell})$ denote the ∞-category $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]/t^{m_n}}(X^j_n;\overline{\mathbf{Q}_\ell})$. Note that since the kernel of the surjection $G[\![t]\!] \twoheadrightarrow G[\![t]\!]/t^{m_n}$ is unipotent, and the action of $G[\![t]\!]$ factors through this surjection, the ∞-category $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]/t^{m_n}}(X^j_n;\overline{\mathbf{Q}_\ell})$ would be unchanged if we replace m_n by any $m \geq m_n$.

Let $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j;\overline{\mathbf{Q}_\ell})$ denote the direct limit

$$\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j;\overline{\mathbf{Q}_\ell}) = \varinjlim_{f^!_{j,n}} \mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j_n;\overline{\mathbf{Q}_\ell})$$

of the ∞-categories $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j_n;\overline{\mathbf{Q}_\ell})$ along !-pullbacks. Finally, define

$$\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X(\!(t)\!);\overline{\mathbf{Q}_\ell}) = \varinjlim_{g^j_!} \mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j;\overline{\mathbf{Q}_\ell})$$

to be the direct limit of the ∞-categories $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]}(X^j;\overline{\mathbf{Q}_\ell})$ along the !-pushforward functors associated to the maps $g^j : X^j \to X^{j+1}$.

Suppose that there are only countably many $G[\![t]\!]$-orbits on $X(\!(t)\!)$. (If X is affine and G is reductive, Theorem 3.3.2 says that this is the case if and only if X is a spherical G-variety.) Then there are only finitely many $G[\![t]\!]/t^{m_n}$-orbits on X^j_n, and the maps $f_{j,n} : X^j_n \to X^j_{n-1}$ are $G[\![t]\!]$-equivariant and respect the stratifications on X^j_n and X^j_{n-1}. Define $\mathrm{Shv}^{c,\mathrm{et}}_{G[\![t]\!]}(X(\!(t)\!);\overline{\mathbf{Q}_\ell})$ to be the ∞-category obtained via the above procedure, except where $\mathrm{Shv}^{\mathrm{et}}_{G[\![t]\!]/t^{m_n}}(X^j_n;\overline{\mathbf{Q}_\ell})$ is replaced by the ∞-category $\mathrm{Shv}^{c,\mathrm{et}}_{G[\![t]\!]/t^{m_n}}(X^j_n;\overline{\mathbf{Q}_\ell})$ of $G[\![t]\!]/t^{m_n}$-equivariant étale sheaves on X^j_n which are constructible with respect to the orbit stratification on X^j_n.

Theorem A.4. *Let $q \gg 0$ be a large prime power. Fix a prime $\ell \neq p$, and choose an isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$. In Setup 3.2.17, suppose that the $H(R[\![t]\!])$-action on*

$X(\!(t)\!)$ is weakly placid. Then is there is a localization $R \subseteq R'$ such that for any k-point $R' \twoheadrightarrow k$ with k being a finite field, there is a natural equivalence

$$\mathrm{Shv}^{c,\mathrm{et}}_{H_{\overline{\mathbf{F}}_q}[\![t]\!]}(X(\!(t)\!)_{\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} \mathrm{Shv}^{c}_{H(\mathbf{C}[\![t]\!])}(X(\mathbf{C}(\!(t)\!)); \mathbf{C}).$$

PROOF. By definition of the ∞-categories involved, it suffices to show that for each j and n, there are compatible equivalences

$$\mathrm{Shv}^{c,\mathrm{et}}_{H_{\overline{\mathbf{F}}_q}[\![t]\!]/t^{m_n}}(X^j_{n,\overline{\mathbf{F}}_q}; \overline{\mathbf{Q}_\ell}) \xrightarrow{\sim} \mathrm{Shv}^{c}_{H_{\mathbf{C}}[\![t]\!]/t^{m_n}}(X^j_{n,\mathbf{C}}; \mathbf{C}).$$

This in turn is a consequence of Theorem 3.2.18 applied to the group scheme $H[\![t]\!]/t^{m_n}$ over R. □

Remark A.5. Theorem A.4 implicitly uses Remark 3.2.19: it is necessary to treat the equivalence in the proof of Theorem A.4 as those of ∞-categories, so that the (co)limit constructions of Construction A.3 are legal.

We will often use Theorem A.4 in the case when X is an affine homogeneous spherical H-variety. For this, we need:

Conjecture A.6. *If X is an affine homogeneous spherical H-variety, the $H[\![t]\!]$-action on $X(\!(t)\!)$ is weakly placid (so that we can apply Theorem A.4).*

However, not being a specialist in the relevant technical details, I have not been able to verify this. It is quite likely that the assumption of weak placidity is not necessary to prove Theorem 3.5.23, since it is only used to conclude formality of a certain Ext-algebra.

Appendix B. Questions/further directions

The work presented in this article is clearly far from being a complete story. There are numerous questions left open by our discussion, of varying levels of difficulty. I hope that the discussion in this article is compelling enough to motivate further study of these problems.

Let us begin with some broad questions. First, some conceptual questions about the place of this story in mathematics and physics:

(a) Having been raised a topologist, it is inspiring to see the appearance of *unstable* homotopy-theoretic structures in Table 4 (such as EHP sequences and Hopf fibrations). It suggests that there might be an unstable analogue of Expectation 5.4.4 (the latter is already not well-defined!) giving a spectral decomposition for "G-equivariant sheaves of *spaces* on ΩG". There are some indications that this might be possible, and many that it might be impossible — but it is perhaps better to be optimistic!

This question seems exceptionally difficult, even in the simplest case of G being trivial. In this case, one can interpret the question as asking for an analogue of the theory of synthetic spectra à la [**Pst18**] for spaces/homotopy types/anima. Considerations from the theory of power operations suggests that one should replace the fpqc stack $\mathcal{M}_{\mathrm{FG}}^{\mathrm{s}}$ of groupoids classifying spin formal groups and isomorphisms between them by some sort of fpqc stack of categories classifying spin formal groups and *homomorphisms* (in particular, including isogenies) between them.

(b) As mentioned in [**Dev23a**, Heuristic B.5], an unpublished conjecture of Gaiotto (which I learned about from Nakajima) says that the Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory over $\mathbf{R}^3 \times S^1$ with a generic choice of complex structure and gauge group G can be modeled via the periodic G-equivariant complex K-theory of ΩG. The calculations of this paper (see, e.g., Example 3.6.17) suggest that perhaps one can modify this proposal to use *connective* equivariant complex K-theory instead, and that the Bott class is related to the radius of the circle S^1 (see also Remark 5.1.14). The Bott class being sent to zero then might correspond to the degeneration of 4d $\mathcal{N} = 2$ pure gauge theory into 3d $\mathcal{N} = 4$ pure gauge theory. For instance, in Remark 3.5.16, we find that if $T = S^1$ (say), going from integral to ku-homology has the effect of *deforming* the relation $[p, a] = \hbar a$ in the Weyl algebra of $\mathbf{G}_a = \operatorname{Spec} \mathbf{Z}[a]$ over $\mathbf{Z}$ to the relation $[p, a] = \hbar(1 + \beta x)$ over $\mathbf{Z}[\beta]$. Does such a deformation have any precedence in physics?

More optimistically, Kapustin and Witten (among others) proposed viewing the derived geometric Satake equivalence of Theorem 3.2.7 as an equivalence of categories stemming from S-duality for 4d $\mathcal{N} = 4$ gauge theory. Does Theorem 3.6.21 have any relationship to 5d $\mathcal{N} = 2$ gauge theory compactified on a circle of finite radius?

(c) One glaring omission in this article is the study of equivariance for loop rotation on the topological/A-side; as explained in [**NS09, BBB$^+$20**] via the Ω-deformation and [**Dev23a**, Remark B.2], this corresponds to deformation quantizing the spectral/B-side. From one point of view, it can be described as the study of the homology of the operad $\mathbf{E}_3 \rtimes S^1$ associated to rotations about a line inside $\mathbf{R}^3$. Motivated by this, and considerations involving Frobenius-constant quantizations à la Bezrukavnikov-Kaledin [**BK08**], we are planning to discuss the $\mathrm{U}(n)$-equivariant cohomology of the $\mathbf{E}_{2n+1}$-operad with *arbitary* complex-oriented coefficients in a later article [**Dev24**], where we also plan to introduce formal group law variants of quantum groups. This discussion is heavily motivated by [**DM23**] (in particular, [**DM23**, Section 4.4] can be regarded as the simplest case of this story, namely the case of tori). Part of the reason for not discussing loop rotation equivariance in this article is our reliance on Proposition 3.5.1, which inherently breaks the natural S^1-action on the free loop space.

(d) How might one generalize Theorem 3.6.21 to the setting of *global* geometric Langlands duality?

(e) In many calculations, one gets the sense that the condition of sphericity should not be the maximal generality in which relative geometric Langlands should work. For instance, the criteria (a) and (b) of Theorem 3.5.23 do not use sphericity of $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ at all (and as in Example 3.5.28 and Conjecture 3.5.30, there are should be many interesting examples of relative Langlands duality for non-spherical subgroups). One of the immediate difficulties encountered when working with non-spherical subgroups is that $G_{\mathbf{C}}(\mathbf{C}[\![t]\!])$-orbits on $G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!))$ are not parametrized by a discrete set, which makes defining the ∞-category $\mathrm{Shv}^c_{G_{\mathbf{C}}(\mathbf{C}[\![t]\!])}(G_{\mathbf{C}}(\mathbf{C}(\!(t)\!))/H_{\mathbf{C}}(\mathbf{C}(\!(t)\!)); \mathbf{Q})$ rather difficult. Nevertheless, it would be interesting to collect other non-spherical examples satisfying conditions (a) and (b) of Theorem 3.5.23; a sufficient supply of examples might suggest a way to understand this generalization of relative Langlands duality.

(f) In the proof (from [**BF08**]) of Theorem 3.2.7, many calculations can be reduced to the case of semisimple rank 1, essentially by localization on the affine space $\check{\mathfrak{t}}^*(2)/\!\!/W = \operatorname{Spec} \mathrm{H}^*_T(*; \mathbf{Z}')$. Is it possible to tackle Conjecture 3.4.11 for affine homogeneous spherical varieties of higher rank by reduction to the rank 1 case (in which case Theorem 3.4.13 can be applied)? The technique of *localization* of spherical varieties, studied in [**Lun97, Kno14**], should be crucial here.

(g) Let C_2 be the cyclic group of order 2. In the case of symmetric spaces, Conjecture 3.4.11 should be closely related to C_2-equivariant derived algebraic geometry as studied by Mike Hill and his collaborators (see, e.g., [**Hil14**]). Let σ denote the sign representation of C_2 on $\mathbf{R}$, and let $\varrho = 1 + \sigma$ denote the regular representation of C_2 on $\mathbf{C}$.[24] If Y is a C_2-space, let $\mathcal{L}^\sigma Y$ denote the space of maps $\mathrm{Map}(S^\sigma, Y)$ equipped with its natural C_2-action.

Let G be a connected compact Lie group, let θ be a conjugate-linear involution on $G_{\mathbf{C}}$ preserving G, and let G^θ denote the maximal compact

[24] The notation is intended to distinguish this C_2 from the $\mathbf{Z}/2$ appearing elsewhere in this article, as well as ϱ from the half-sum of positive roots (which is usually denoted ρ). On the topic of notation: it would have been great if ϱ was used to denote the *r(h)otation* representation of C_2 (which is frequently denoted by $\lambda = 2\sigma$), instead of the regular representation of C_2.

subgroup of the fixed subgroup $G^\theta_{\mathbf{C}}$. Then, there is a C_2-equivariant equivalence of orbispaces

$$G\backslash\mathcal{L}^\sigma G/G \simeq \mathrm{Bun}_{(G,\theta)}(\mathbf{C}P^1_{\mathbf{R}}),$$

which gives an equivalence

$$(G\backslash\mathcal{L}^\sigma G/G)^{C_2} \simeq G\backslash\mathcal{L}(G/G^\theta).$$

Let R denote an $\mathbf{E}_\infty$-algebra in genuine C_2-spectra. We expect that there is an ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}^\sigma G; R)$ of "C_2-equivariant sheaves of $G\times G$-equivariant R-modules on $\mathcal{L}^\sigma G$", and an analogue of Theorem 3.2.7 which computes $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}^\sigma G; R)$ whose underlying nonequivariant equivalence is Theorem 3.2.7. Taking geometric fixed points (in some appropriate categorical sense) of the putative spectral decomposition of $\mathrm{Shv}^{c,\mathrm{Sat}}_{G\times G}(\mathcal{L}^\sigma G; R)$ should produce an analogue/special case of Conjecture 3.4.11 which describes $\mathrm{Shv}^{c,\mathrm{Sat}}_{G}(\mathcal{L}(G/G^\theta); \Phi^{C_2}R)$. For instance, one can try to access $\mathrm{Shv}^{c,\mathrm{Sat}}_{G}(\mathcal{L}(G/G^\theta); \Phi^{C_2}R)$ via an analogue of Smith theory. We hope to study this question in future work. Developing this theory should lead to important interactions between geometric Langlands for symmetric spaces and C_2-equivariant homotopy theory.

Such a conjecture should incorporate the example of $G = \mathrm{U}(n)$ equipped with the C_2-action given by complex conjugation. Let $\underline{\mathbf{Z}}$ denote the constant Mackey functor; then, there should be a $\underline{\mathbf{Z}}$-linear equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{U}(n)_{\mathbf{R}}\times \mathrm{U}(n)_{\mathbf{R}}}(\mathcal{L}^\sigma \mathrm{U}(n)_{\mathbf{R}}; \underline{\mathbf{Z}}) \simeq \mathrm{Perf}(\mathfrak{gl}_n^*[\varrho]/\mathrm{GL}_n),$$

where $\mathfrak{gl}_n^*[\varrho]$ denotes the "ϱ-shearing" of $\mathrm{Spec}\,\mathrm{Sym}^*(\mathfrak{gl}_n(-2))$ (so that $\mathcal{O}_{\mathfrak{gl}_n^*[\varrho]}$ can be identified with $\bigoplus_{j\geq 0}\mathrm{Sym}^j(\mathfrak{gl}_n)[-j\varrho]$). Moreover, this equivalence should be $\mathbf{E}_{\sigma\oplus\varrho}$-monoidal. Note that the associated symmetric variety $X = \mathrm{GL}_n/\mathrm{O}_n$ has roots of type N.

Some of the most important issues in this article on the topological/A-side come from defining well-behaved sheaf theories.

(h) Verify Hypothesis 3.5.21. More generally, if X is an affine homogeneous spherical G-variety (over a localization of a number ring), when is the $G[\![t]\!]$-action on $X(\!(t)\!)$ weakly placid?

(i) Let G be a compact Lie group, and let X be an (ind-)finite stratified G-space. Is there a good $\mathrm{sh}^{1/2}\mathbf{Z}[\beta]$-linear ∞-category $\mathrm{Shv}^c_G(X; \mathrm{ku})$ which agrees with the *ad hoc* construction of Construction 3.6.18?

(j) Is there a good sheaf theory of synthetic G-equivariant constructible sheaves of spectra as in Expectation 5.4.1? As a first step, it seems important to study G-equivariant analogues of synthetic spectra, as well as nonequivariant synthetic analogues of constructible sheaves of spectra (as in Construction 5.3.3) and a corresponding six-functor formalism.

There are also several broad questions arising from considering the spectral/B-side.

(k) Can one define $\check{G}_{\mathbf{G}_{\mathrm{univ}}}$ as in Expectation 5.4.2?

(l) Is there an *a priori* reason that if $\check{M}_\beta$ is an $\check{G}$-variety over $\mathbf{Z}[\beta]$ as in Proposition 3.6.23, the map $\check{M}_\beta/\check{G} \to \check{G}^{\mathrm{sc}}_\beta/\check{G}$ admits a Lagrangian structure? See Observation 5.2.10 for some progress in this direction.

The discussion in this article also suggests several (less lofty, and presumably more approachable) questions.

(m) As in Remark 3.5.14, let $W_1 \to W_2$ be a homomorphism of reflection groups acting on vector spaces $V_1 \to V_2$ over a field k (possibly of nonzero characteristic), so that there is a map $V_1/\!\!/W_1 \to V_2/\!\!/W_2$. The Hochschild homology $\mathrm{HH}(V_1/\!\!/W_1/V_2/\!\!/W_2)$ should be an interesting invariant associated to homomorphisms of reflection groups; what can one prove about it? In the case that the map of reflection groups comes from an inclusion of root data, this Hochschild homology is the content of Proposition 3.5.5, and therefore plays an important role in the relative Langlands program.

(n) Is there an analogue of Theorem 4.5.1 for ku? The difficulty lies in finding a β-deformation of $\check{M}$.

(o) Can one extend Theorem 3.6.21 to the non-simply-laced case? Similarly, can one extend Proposition 5.4.8 to coefficients with the sphere spectrum? Following Lemma 5.4.14, a first step will be an understanding of $\mathrm{MU}_*(\Omega G)$ as a Hopf algebra. A description of the $\mathrm{MU}_*\mathrm{MU}$-comodule structure will then give the desired generalization of Proposition 5.4.8.

(p) Can the results of this article (even only the ones concerning ordinary coefficients) be extended to the setting of p-compact groups? This question was suggested by Haynes Miller. See [**Gro10**] for a survey of the theory of p-compact groups. In [**Dev23b**, Remark 3.3], we propose a concrete question along these lines: namely, is there a 2-compact group G equipped with an $\mathbf{E}_1$-map $\mathrm{DI}_4 \to G$ from the Dywer-Wilkerson 2-compact group [**DW93**], which is built using some sort of triality for G? For instance, one should have $G/\mathrm{DI}_4 \simeq \mathbf{R}P^{15} \times \mathbf{R}P^{15}$ (just as $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2 \cong \mathbf{R}P^3 \times \mathbf{R}P^3$ and $\mathrm{SO}_8/\mu_2 \cdot G_2 \cong \mathbf{R}P^7 \times \mathbf{R}P^7$). The putative quotient stack $\check{M}/\check{G}$ which is dual to the "subgroup" $\mathrm{DI}_4 \to G$ should be given by $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3} \times \mathbf{A}^2$.

(q) Finding other examples of the ku-theoretic analogue of [**BZSV23**, Conjecture 7.5.1] is an important goal. For instance, what is the ku-theoretic dual of the spherical GL_n-variety $\mathrm{GL}_n/(\mathrm{GL}_j \times \mathrm{GL}_{n-j})$ (which is homotopy equivalent to $\mathrm{Gr}_j(\mathbf{C}^n)$)? In the arithmetic case, this was studied by Jacquet-Rallis in [**JR96**], and is described in Conjecture 3.5.30 in the geometric case.

(r) Does Conjecture 5.3.10 hold? Categorifying this question: is there an analogue of the theory of synthetic spectra for global homotopy theory?

(s) Is there a structure theory for ku-Hamiltonian varieties? Namely, is there a well-behaved theory of "hyperspherical" ku-Hamiltonian varieties which generalizes the notion of hyperspherical Hamiltonian varieties introduced in [**BZSV23**, Section 3], and an analogue of [**BZSV23**, Theorem 3.6.1]?

(t) Related to the preceding point, can one develop a theory of β-deformed cotangent bundles for certain G-varieties? In the case of "coisotropic subgroups" as in Remark 5.1.19, this would be a β-deformation of [**BM22**] (or rather, of the specialization of their results to quasi-Hamiltonian varieties).

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