

# Hodge theory for elliptic curves and the Hopf element $\nu$

Sanath Devalapurkar

ABSTRACT. We show that the vector bundle on the moduli stack  $M_{\text{ell}}$  of elliptic curves associated to the 2-cell complex  $C\nu$  is isomorphic to the de Rham cohomology sheaf  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{ell}})$  of the universal elliptic curve  $\mathcal{E} \rightarrow M_{\text{ell}}$ . We use this to calculate the homotopy groups of the  $\mathbf{E}_1$ -quotient  $\text{tmf}/\nu$  of  $\text{tmf}$  by  $\nu$ , called the spectrum of “topological quasimodular forms”, by relating its Adams–Novikov spectral sequence to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration.

## 1. Introduction

In this paper, we study the relationship between the Hopf invariant one element  $\nu \in \pi_3 \text{tmf}$  and the Hodge filtration for elliptic curves. Namely, we show that the vector bundle on the moduli stack  $M_{\text{ell}}$  of elliptic curves associated to  $C\nu$  is isomorphic to the (middle) de Rham cohomology  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{ell}})$  of the universal elliptic curve  $\mathcal{E} \rightarrow M_{\text{ell}}$ . A version of this relationship had been stated by Hopkins in [Hop02, Section 5]. Using this, we calculate the homotopy groups of the  $\mathbf{E}_1$ -quotient  $\text{tmf}/\nu$  of  $\text{tmf}$  by  $\nu \in \pi_3(\mathbb{S})$  by showing that the  $E_2$ -page of its Adams–Novikov spectral sequence is isomorphic to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration. The  $\mathbf{E}_1$ -ring  $\text{tmf}/\nu$  is called the spectrum of *topological quasimodular forms* (see Remark 4.2). The results of this paper are probably well-known to the experts.

The ring spectrum  $\text{tmf}/\nu$  is interesting for several reasons. One motivation for studying it comes from the Ando–Hopkins–Rezk orientation  $\text{MU}\langle 6 \rangle \rightarrow \text{tmf}$  (see [AHR10]). As is made clear during the course of the proof, a key reason for why this orientation does not factor through the map  $\text{MU}\langle 6 \rangle \rightarrow \text{MSU}$  is because  $\text{tmf}$  detects the element  $\nu \in \pi_3(\mathbb{S})$ ; this in turn is related to the fact that the weight 2 Eisenstein series is not a modular form. Since  $\text{tmf}/\nu$  is the “smallest” coherently structured (i.e.,  $\mathbf{E}_1$ -)  $\text{tmf}$ -algebra with a nullhomotopy of  $\nu$ , one might expect the composite

$$(1) \quad \text{MU}\langle 6 \rangle \rightarrow \text{tmf} \rightarrow \text{tmf}/\nu$$

to factor through  $\text{MSU}$  via an  $\mathbf{E}_1$ -map; see also Remark 4.7. Although we do not prove in this paper that the composite (1) factors through  $\text{MSU}$ , we will use the results of this paper to address this question in future work. The connection between  $\nu$  and the weight 2 Eisenstein series is also discussed in Section 4. The relationship between  $\nu$  and de Rham cohomology is also intrinsically intriguing, because the Hodge–de Rham filtration on the de Rham cohomology of an elliptic curve is related to many deep topics in arithmetic geometry (such as Grothendieck–Messing theory).

We begin in Section 2 by recalling some background on Hodge theory for cubic curves from algebraic geometry. In particular, we give a Hopf algebroid presentation for the moduli stack

$M_{\text{cub}}^{\text{dR}}$  of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence. In Section 3, we prove our main technical result relating the Adams–Novikov spectral sequence of  $\text{tmf}/\nu$  to the cohomology of the moduli stack  $M_{\text{cub}}^{\text{dR}}$ . Finally, in Section 4 we prove Theorem 4.1, which calculates this Adams–Novikov spectral sequence. It degenerates at the  $E_4$ -page, and  $\text{TMF}/\nu$  is found to be 24-periodic. Moreover,  $\text{tmf}/\nu$  is homotopy commutative. These results were discovered independently by Rezk in unpublished work, and we give our own proof of his calculation of  $\pi_*(\text{tmf}/\nu)$ .

**1.1. Acknowledgements.** I would like to thank Charles Rezk: after we proved part of Theorem 4.1, we discovered that he had proved the result independently; I would like to thank him for discussions about the results in this paper, and for letting us write up this result. I am also grateful to Mark Behrens, Robert Burklund, Tyler Lawson, Lennart Meier, and Andrew Senger for helpful discussions, and in particular to Robert Burklund for providing helpful comments on a draft.

## 2. Background on Hodge theory

In this section, we recall some background on Hodge theory for cubic curves over a general base scheme. Multiple sources (such as [Kat73, Appendix A1.2]) discuss Hodge theory for (smooth) elliptic curves.

Let  $f : X \rightarrow Y$  be a morphism of schemes. One then has the  $f^{-1}\mathcal{O}_Y$ -linear relative de Rham complex  $\Omega_{X/Y}^\bullet$ .

**DEFINITION 2.1.** The  $i$ th relative de Rham cohomology  $H_{\text{dR}}^i(X/Y)$  of  $f : X \rightarrow Y$  is defined to be the hypercohomology sheaf  $\mathbf{R}^i f_*(\Omega_{X/Y}^\bullet)$  on  $Y$ .

The hypercohomology spectral sequence defines the Hodge–de Rham spectral sequence of sheaves on  $Y$ :

$$E_1^{s,t} = R^t f_* \Omega_{X/Y}^s \Rightarrow H_{\text{dR}}^{s+t}(X/Y).$$

The following (easy) result is well-known.

**THEOREM 2.2.** *If  $f : X \rightarrow Y$  is a smooth, proper, and surjective morphism of relative dimension 1 with geometrically connected fibers, then the Hodge–de Rham spectral sequence degenerates at the  $E_1$ -page.*

Since  $f$  is of relative dimension 1, the only interesting de Rham cohomology is in the middle dimension, i.e.,  $H_{\text{dR}}^1(X/Y)$ . In particular, for such  $f$ , there is an exact sequence

$$0 \rightarrow f_* \Omega_{X/Y}^1 \rightarrow H_{\text{dR}}^1(X/Y) \rightarrow R^1 f_* \mathcal{O}_X \rightarrow 0$$

of quasicoherent sheaves on  $Y$ ; this is called the *Hodge–de Rham exact sequence*. Moreover, the pairing  $H_{\text{dR}}^1(X/Y) \otimes_{\mathcal{O}_Y} H_{\text{dR}}^1(X/Y) \rightarrow \mathcal{O}_Y$  is determined by the canonical perfect pairing

$$R^1 f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} f_* \Omega_{X/Y}^1 \rightarrow R^1 f_* \Omega_{X/Y}^1 \xrightarrow{\text{trace}} \mathcal{O}_Y.$$

We now specialize to the case when  $f : X \rightarrow Y$  is an elliptic curve  $f : E \rightarrow S$ . Then  $f_* \Omega_{E/S}^1$  is the line bundle  $\omega_{E/S}$  of invariant differentials. The pairing  $\omega_{E/S} \otimes_{\mathcal{O}_S} R^1 f_* \mathcal{O}_E \rightarrow \mathcal{O}_S$  is perfect, and so  $R^1 f_* \mathcal{O}_E \cong \omega_{E/S}^{-1}$ . In particular, the Hodge–de Rham exact sequence for  $E \rightarrow S$  becomes

$$(2) \quad 0 \rightarrow \omega_{E/S} \rightarrow H_{\text{dR}}^1(E/S) \rightarrow \omega_{E/S}^{-1} \rightarrow 0.$$

If  $M_{\text{ell}}$  denotes the moduli stack of elliptic curves, and  $\mathcal{E} \rightarrow M_{\text{ell}}$  is the universal elliptic curve, then (2) exhibits  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{ell}})$  as an element of  $\text{Ext}_{M_{\text{ell}}}^1(\omega^{-1}, \omega)$ .

REMARK 2.3. If  $S$  is a  $p$ -adic scheme, then (2) corresponds to the Hodge filtration of the Dieudonné module  $\mathbf{D}(E[p^\infty]/S)$  of  $E$  under the isomorphism  $H_{\text{dR}}^1(E/S) \cong \mathbf{D}(E[p^\infty]/S)$ ; see, for instance, [Kat81, Section V].

The following is an immediate consequence of [Kat73, Equation A1.2.3] (see also [Poo20]):

PROPOSITION 2.4. *If  $f : E \rightarrow S$  is an elliptic curve, then there is an isomorphism  $H_{\text{dR}}^1(E/S) \otimes \omega_{E/S} \cong f_* \Omega_{E/S}^1(2\infty)$ .*

The map

$$f_* \Omega_{E/S}^1(2\infty) \cong H_{\text{dR}}^1(E/S) \otimes \omega_{E/S} \rightarrow \omega_{E/S}^{-1} \otimes \omega_{E/S} = \mathcal{O}_S$$

induced by the Hodge–de Rham exact sequence sends a section of  $f_* \Omega_{E/S}^1(2\infty)$  to its residue at  $\infty$ .

We can now generalize the above story to the non-smooth setting. First, we recall the definition of a cubic curve.

DEFINITION 2.5. A *cubic curve*  $f : E \rightarrow S$  over a scheme  $S$  is a flat and proper morphism of finite presentation whose fibers are reduced, irreducible curves of arithmetic genus 1, along with a section  $\infty : S \rightarrow E$  whose image is contained in the smooth locus  $E^{\text{sm}}$  of  $f$ . Let  $M_{\text{cub}}$  denote the stack of cubic curves, and let  $f : \mathcal{E} \rightarrow M_{\text{cub}}$  denote the universal cubic curve.

Let  $\omega$  denote the line bundle on  $M_{\text{cub}}$  assigning to a cubic curve  $f : E \rightarrow S$  the cotangent bundle  $\omega_{E/S}$  along the section  $\infty$ . The vector bundle  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{ell}})$  over  $M_{\text{ell}}$  defines a class in  $\text{Ext}_{M_{\text{ell}}}^1(\omega^{-1}, \omega) \cong H^1(M_{\text{ell}}; \omega^{\otimes 2})$ . This class extends to an element in  $\text{Ext}_{M_{\text{cub}}}^1(\omega^{-1}, \omega) \cong H^1(M_{\text{cub}}; \omega^{\otimes 2})$ , and we will, in a terrible abuse of notation, denote the resulting vector bundle on  $M_{\text{cub}}$  by  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{cub}})$ .

DEFINITION 2.6. Let  $M_{\text{cub}}^{\text{dR}}$  denote the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence, and let  $M_{\text{ell}}^{\text{dR}} = M_{\text{cub}}^{\text{dR}} \times_{M_{\text{cub}}} M_{\text{ell}}$  denote the moduli stack of elliptic curves with a chosen splitting of the Hodge–de Rham exact sequence.

In order to do calculations with  $M_{\text{cub}}^{\text{dR}}$ , we would like to obtain a Hopf algebroid presentation of this stack. To do so, we recall a Hopf algebroid presentation of  $M_{\text{cub}}$ . Zariski-locally on any base scheme  $S$ , a cubic curve is described by a Weierstrass equation

$$(3) \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with other choices of coordinates  $(x, y)$  given by the transformations

$$x \mapsto x + r, \quad y \mapsto y + sx + t.$$

The moduli stack  $M_{\text{cub}}$  of cubic curves is presented by the Hopf algebroid

$$(D, \Gamma) = (\mathbf{Z}[a_1, a_2, a_3, a_4, a_6], D[r, s, t]),$$

with gradings  $|a_i| = i$  and  $|r| = 2$ ,  $|s| = 1$ , and  $|t| = 3$ . Studying how the coefficients  $a_i$  transform gives the right unit  $\eta_R : D \rightarrow \Gamma$  of this Hopf algebroid:

$$\begin{aligned} a_1 &\mapsto a_1 + 2s, \\ a_2 &\mapsto a_2 - a_1s + 3r - s^2, \\ a_3 &\mapsto a_3 + a_1r + 2t, \\ a_4 &\mapsto a_4 + a_3s + 2a_2r - a_1t - a_1rs - 2st + 3r^2, \\ a_6 &\mapsto a_6 + a_4r - a_3t + a_2r^2 - a_1rt - t^2 + r^3. \end{aligned}$$

To determine a Hopf algebroid presentation of  $M_{\text{cub}}^{\text{dR}}$ , note that after choosing  $x, y$ , the coordinate  $x$  defines a function on the smooth locus of  $\mathcal{E}$  with a double pole at  $\infty$ , and it is in

fact the only such non-constant function on the smooth locus of  $\mathcal{E}$  (this follows from the usual calculation [KM85, Section 2.2.5] with the Riemann-Roch formula). In particular, Proposition 2.4 implies that the element  $[r]$  in the cobar complex for the Hopf algebroid  $(D, \Gamma)$  detects the extension in  $\text{Ext}_{M_{\text{cub}}}^1(\mathcal{O}, \omega^{\otimes 2}) = \text{Ext}_{\Gamma}^{1,2}(D, D)$  determined by the de Rham cohomology  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{cub}})$ . By the exact sequence (2), a choice of Hodge–de Rham splitting on the universal cubic curve amounts to fixing a choice of  $x$  (although  $y$  is allowed to vary). Consequently:

**PROPOSITION 2.7.** *The moduli stack  $M_{\text{cub}}^{\text{dR}}$  of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence is presented by the Hopf algebroid  $(D, \Sigma) = (\mathbf{Z}[a_1, a_2, a_3, a_4, a_6], D[s, t])$ , with gradings<sup>1</sup>  $|a_i| = i$ ,  $|s| = 1$ , and  $|t| = 3$ . The right unit is the same as in that of the elliptic curve Hopf algebroid, except with  $r = 0$ :*

$$(4) \quad \begin{aligned} a_1 &\mapsto a_1 + 2s, \\ a_2 &\mapsto a_2 - a_1 s - s^2, \\ a_3 &\mapsto a_3 + 2t, \\ a_4 &\mapsto a_4 + a_3 s - a_1 t - 2st, \\ a_6 &\mapsto a_6 - a_3 t - t^2. \end{aligned}$$

### 3. The relationship with $\text{tmf}/\nu$

In this section, we study the  $\mathbf{E}_1$ -quotient  $\text{tmf}/\nu$  of  $\text{tmf}$  by  $\nu$ , and relate its Adams–Novikov spectral sequence to the cohomology of  $M_{\text{cub}}^{\text{dR}}$ . The results of this section are well-known to the experts.

We begin by recalling one construction of the  $\mathbf{E}_1$ -quotient  $\text{tmf}/\nu$ . This satisfies the following universal property: if  $R$  is any  $\mathbf{E}_1$ - $\text{tmf}$ -algebra, then

$$\text{Map}_{\text{Alg}_{\mathbf{E}_1}(\text{Mod}(\text{tmf}))}(\text{tmf}/\nu, R) = \begin{cases} \Omega^{\infty+4}R & \text{if } \nu = 0 \in \pi_3 R \\ \emptyset & \text{else.} \end{cases}$$

One of the main results of [AB19] justifies the following definition:

**DEFINITION 3.1.** The  $\mathbf{E}_1$ -ring  $\text{tmf}/\nu$ , called *topological quasimodular forms* (see Remark 4.2 for a justification for the name), is the Thom spectrum of the dotted extension in the following diagram:

$$\begin{array}{ccc} S^4 & \xrightarrow{\nu} & BGL_1(\text{tmf}) \\ \downarrow & \nearrow \text{dotted} & \\ \Omega S^5 & & \end{array}$$

**REMARK 3.2.** The element  $\nu \in \pi_3(\text{tmf})$  is spherical, and so this diagram factors as

$$\begin{array}{ccccc} S^4 & \xrightarrow{2v_1^2} & B\text{Spin} & \xrightarrow{J} & BGL_1(\mathbb{S}) \\ \downarrow & \nearrow \text{dotted} & & & \downarrow \\ \Omega S^5 & & & & BGL_1(\text{tmf}). \end{array}$$

Following the notation of [Dev19], we will write  $A$  to denote the Thom spectrum of the loop map  $\Omega S^5 \rightarrow BGL_1(\mathbb{S})$ . Then there is a canonical equivalence  $\text{tmf}/\nu \simeq \text{tmf} \otimes A$  of  $\mathbf{E}_1$ - $\text{tmf}$ -algebras, so there is in particular a ring map  $A \rightarrow \text{tmf}/\nu$ .

<sup>1</sup>Recall that the topological grading is *double* the algebraic grading.

The  $\mathbf{E}_1$ -ring  $A$  will be useful below. In [Dev19], it is shown that there is an  $\mathbf{E}_1$ -map  $A \rightarrow \mathrm{BP}$ . Moreover, the BP-homology of  $A$  at the prime 2 is isomorphic to  $\mathrm{BP}_*[y_2]$ , where  $y_2$  is sent to  $t_1^2$  modulo decomposables under the map  $\mathrm{BP}_*(A) \rightarrow \mathrm{BP}_*(\mathrm{BP})$ . In particular,  $H_*(A; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4]$ . One then has:

**PROPOSITION 3.3.** *There is a nontrivial simple 2-torsion element  $\sigma_1 \in \langle \eta, \nu, 1_A \rangle \subseteq \pi_5(A) \cong \pi_5(C\nu)$  specified up to indeterminacy by the relation  $\eta\nu = 0$ . One choice of this element is represented by  $[t_2]$  in the Adams–Novikov spectral sequence for  $A$ , and by  $h_{21}$  in the (mod 2) Adams spectral sequence for  $A$ .*

**REMARK 3.4.** The element  $\sigma_1^4 \in \pi_{20}(\mathbb{S}/\nu)$  is in the Hurewicz image, and is detected by  $\bar{\kappa} \in \pi_{20}(\mathbb{S})$ .

To connect  $\mathrm{tmf}/\nu$  and Hodge theory for cubic curves, we make the following observation. Recall that  $H_{\mathrm{dR}}^1(\mathcal{E}/M_{\mathrm{cub}}) \in \mathrm{Ext}_{M_{\mathrm{cub}}}^1(\omega^{-1}, \omega)$ .

**PROPOSITION 3.5.** *Let  $f : \mathcal{E} \rightarrow M_{\mathrm{cub}}$  denote the universal cubic curve over the moduli stack of cubic curves. Then  $H_{\mathrm{dR}}^1(\mathcal{E}/M_{\mathrm{cub}}) \in \mathrm{Ext}_{M_{\mathrm{cub}}}^1(\omega^{-1}, \omega) \cong H^1(M_{\mathrm{cub}}; \omega^2)$  detects  $\nu$  in the  $E_2$ -page of the descent spectral sequence for  $\mathrm{tmf}$ .*

**PROOF.** This is essentially argued in [Hop02, Section 5.2]. We know that  $H^1(M_{\mathrm{cub}}; \omega^2) = \mathbf{Z}/12$  by the calculations in [Bau08]; the element  $[r]$  in the cobar complex determined by the Hopf algebroid  $(D, \Gamma)$  is a representative for the generator. This element detects  $\nu$  in the Adams–Novikov spectral sequence for  $\mathrm{tmf}$ , and by the discussion before Proposition 2.7, also detects the extension class of the Hodge–de Rham exact sequence.  $\square$

**COROLLARY 3.6.** *The rank two vector bundle  $\mathcal{F}(C\nu)$  on the moduli stack of cubic curves corresponding to  $C\nu$  is isomorphic to  $H_{\mathrm{dR}}^1(\mathcal{E}/M_{\mathrm{cub}})$ .*

**REMARK 3.7.** In [Rez13, Section 11.5], the Hodge–de Rham exact sequence appears in a different but related guise, as a class in the  $E_2$ -page of a spectral sequence  $\{E_r^{s,t}\}$  converging to the homotopy groups of the space of  $\mathbf{E}_\infty$ -maps  $\mathbf{Z}_+ \rightarrow \mathrm{TMF}$ . The element  $H_{\mathrm{dR}}^1(\mathcal{E}/M_{\mathrm{ell}}) \in E_2^{1,4}$  detects a nontrivial class in  $\pi_3 \mathrm{Map}_{\mathbf{E}_\infty}(\mathbf{Z}_+, \mathrm{TMF}) = \pi_3 \mathbb{G}_m(\mathrm{TMF})$ , i.e., an  $\mathbf{E}_\infty$ -map  $K(\mathbf{Z}, 3)_+ \rightarrow \mathrm{TMF}$ . This is related to the  $\mathbf{E}_\infty$ -twisting of  $\mathrm{TMF}$  explored in [ABG10].

Since  $\mathrm{tmf}/\nu$  is the  $\mathbf{E}_1$ -quotient of  $\mathrm{tmf}$  by  $\nu$  by Remark 3.2, it is the universal  $\mathbf{E}_1$ - $\mathrm{tmf}$ -algebra with a nullhomotopy of  $\nu$ . If  $\mathrm{tmf}/\nu$  is a homotopy commutative ring (which we will show is indeed the case in Corollary 4.8), then we would be able to consider the stack associated to  $\mathrm{tmf}/\nu$  (in the sense of [DFHH14, Chapter 9], [Mat16, Section 2.1]), and it would be reasonable to expect that Proposition 3.5 implies that this stack is the moduli of cubic curves with a choice of splitting of the Hodge–de Rham spectral sequence. We have:

**THEOREM 3.8.** *Let  $g : M_{\mathrm{cub}}^{\mathrm{dR}} \rightarrow M_{\mathrm{cub}}$  denote the structure morphism. Then the sheaf on  $M_{\mathrm{cub}}$  associated to  $A$  is isomorphic as an algebra to the pushforward  $g_* \mathcal{O}_{M_{\mathrm{cub}}^{\mathrm{dR}}}$ .*

**PROOF.** Let  $\mathcal{C}$  be a presentable symmetric monoidal  $(\infty)$ -category, and let  $T$  denote the functor  $\mathcal{C}^{\mathrm{unital}} \rightarrow \mathrm{Alg}_{\mathbf{E}_1}(\mathcal{C})$  sending a unital object  $i : \mathbf{1} \rightarrow X$  to the free  $\mathbf{E}_1$ -algebra in  $\mathcal{C}$  whose unit factors through  $i$ : this may be defined via the homotopy pushout

$$\begin{array}{ccc} \mathrm{Free}_{\mathbf{E}_1}(\mathbf{1}) & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathrm{Free}_{\mathbf{E}_1}(X) & \longrightarrow & T(X) \end{array}$$

in  $\text{Alg}_{\mathbf{E}_1}(\mathcal{C})$ . The functor  $\mathcal{F}(-)$  from the homotopy category of  $\text{tmf}$ -modules to  $\text{QCoh}(M_{\text{cub}})$  fits into a commutative diagram

$$\begin{array}{ccc} h\text{Mod}(\text{tmf})^{\text{unital}} & \longrightarrow & h\text{Alg}_{\mathbf{E}_1}(\text{Mod}(\text{tmf})) \\ \downarrow T & & \downarrow T \\ \text{QCoh}(M_{\text{cub}})^{\text{unital}} & \longrightarrow & \text{Alg}_{\text{assoc}}(\text{QCoh}(M_{\text{cub}})). \end{array}$$

It is easy to see by the universal property of  $T$  that  $T(C\nu) \simeq A$ , so it follows from Proposition 3.5 that  $\mathcal{F}(A) \cong T(f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty))$ . It therefore suffices to show that  $T(f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty)) \cong g_*\mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$ . This is not immediate, since the sheaf  $T(f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty))$  satisfies a universal property with respect to associative  $\mathcal{O}_{M_{\text{cub}}}$ -algebras. Nonetheless, its universal property defines an algebra map  $\varphi : T(f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty)) \rightarrow g_*\mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$  of sheaves on  $M_{\text{cub}}$ . To check that this is an isomorphism, it suffices to show by [Mat16, Lemma 4.4] that the map induces an isomorphism on the cuspidal cubic over  $\mathbf{F}_p$  for all primes  $p$ .

In fact, we can check this over  $\mathbf{Z}$ . We know that  $g_*\mathcal{O}_{M_{\text{cub}}^{\text{dR}}}$  is isomorphic to a polynomial ring on a single generator: indeed, the space of all splittings of the Hodge filtration is isomorphic to the projective line  $\mathbf{P}(H_{\text{dR}}^1(\mathcal{E}))$  minus the point corresponding to the line determined by the invariant differentials. Moreover,  $T(f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty))$  is isomorphic to a free associative ring on one generator, which is also a polynomial ring on a single generator. It therefore suffices to show that the generator maps to the generator under the algebra map  $\varphi$ , but this (after unwinding) is precisely the fact (Proposition 2.4) that  $H_{\text{dR}}^1(\mathcal{E}/M_{\text{cub}}) \otimes \omega = f_*\Omega_{\mathcal{E}/M_{\text{cub}}}^1(2\infty)$ .  $\square$

COROLLARY 3.9. *There is a descent spectral sequence*

$$E_2^{s,2t} = H^s(M_{\text{cub}}^{\text{dR}}; g^*\omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\text{tmf}/\nu),$$

and it is isomorphic to the Adams–Novikov spectral sequence for  $\text{tmf}/\nu$ .

PROOF. There is a descent spectral sequence

$$E_2^{s,2t} = H^s(M_{\text{cub}}; \mathcal{F}(A) \otimes_{\mathcal{O}_{M_{\text{cub}}}} \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\text{tmf}/\nu),$$

and this is isomorphic to the Adams–Novikov spectral sequence for  $\text{tmf}/\nu$  by [Mat16, Corollary 5.3] and the main theorem of [Dev18]. Combining Theorem 3.8 with the projection isomorphism shows that  $\mathcal{F}(A) \otimes_{\mathcal{O}_{M_{\text{cub}}}} \omega^{\otimes t} \cong g_*(g^*\omega^{\otimes t})$ . The morphism  $g$  is flat and affine, so  $E_2^{s,2t} \cong H^s(M_{\text{cub}}^{\text{dR}}; g^*\omega^{\otimes t})$ , as desired.  $\square$

REMARK 3.10. Corollary 3.9 says that although  $\text{tmf}/\nu$  is not *a priori* a homotopy commutative ring, there is a descent spectral sequence which would exist if there was a sheaf of structured ring spectra on  $M_{\text{cub}}^{\text{dR}}$  whose global sections was  $\text{tmf}/\nu$ . We pose this as a conjecture:

CONJECTURE 3.11. *For some  $k \geq 2$ , there is a sheaf of even-periodic  $\mathbf{E}_k$ -rings  $\mathcal{O}^{\text{der}}$  on the étale site of  $M_{\text{ell}}^{\text{dR}}$  such that if  $f : \text{Spec } R \rightarrow M_{\text{ell}}^{\text{dR}}$  is an étale map, then  $\mathcal{O}^{\text{der}}(f)$  is the Landweber-exact theory corresponding to the composite  $\text{Spec } R \rightarrow M_{\text{ell}}^{\text{dR}} \rightarrow M_{\text{ell}} \rightarrow M_{FG}$ , and such that the global sections  $\Gamma(M_{\text{ell}}^{\text{dR}}; \mathcal{O}^{\text{der}})$  is equivalent as an  $\mathbf{E}_1$ -ring to  $\text{TMF}/\nu$ . Moreover, the resulting  $\mathbf{E}_k$ -ring structure on  $\text{TMF}/\nu$  extends to an  $\mathbf{E}_k$ -ring structure on  $\text{tmf}/\nu$ .*

#### 4. The descent spectral sequence

Our goal in this section is to calculate the homotopy groups of  $\text{tmf}/\nu$  via the descent spectral sequence of Corollary 3.9. To do this calculation, we will use the Hopf algebroid presentation in Proposition 2.7. The calculation of the descent spectral sequence was done independently by Charles Rezk; although he stated part of the result to us in an email, the argument is ours (so errors are our fault).

THEOREM 4.1. *There is an isomorphism*

$$(5) \quad H^*(M_{\text{cub}}^{\text{dR}}; g^* \omega^{\otimes *}) \cong \mathbf{Z}[b_2, b_4, b_6, b_8, h_1, h_{21}]/I,$$

where  $b_i \in H^0(M_{\text{cub}}^{\text{dR}}; g^* \omega^{\otimes i})$  of total degree  $2i$ ,  $h_1 \in H^1(M_{\text{cub}}^{\text{dR}}; g^* \omega)$  of total degree 1, and  $h_{21} \in H^1(M_{\text{cub}}^{\text{dR}}; g^* \omega^{\otimes 3})$  of total degree 5. If one defines

$$c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6, \\ \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6,$$

then the ideal  $I$  is generated by the relations

$$2h_1 = 0, \quad 2h_{21} = 0, \quad h_1^2b_6 = h_{21}^2b_2, \quad 4b_8 = b_2b_6 - b_4^2, \quad 1728\Delta = c_4^3 - c_6^2.$$

Moreover, the descent/Adams–Novikov spectral sequence of Corollary 3.9 collapses on the  $E_4$ -page, and  $\pi_*(\text{tmf}/\nu)$  is determined by the differentials

$$d_3(b_2) = h_1^3, \quad d_3(b_4) = h_1^2h_{21}, \quad d_3(b_6) = h_1h_{21}^2, \quad d_3(b_8) = h_{21}^3.$$

One has the relations

$$\eta^3 = 0, \quad \eta^2\sigma_1 = 0, \quad \eta\sigma_1^2 = 0, \quad \sigma_1^3 = 0$$

in the homotopy of  $\text{tmf}/\nu$ , in addition to the relations in  $I$ . Here  $\eta$  is represented by  $h_1$ , and  $\sigma_1$  is represented by  $h_{21}$ . All the torsion in  $\text{tmf}/\nu$  is concentrated in dimensions congruent to 1, 2 (mod 4).

Before giving the proof, we discuss some consequences.

REMARK 4.2. By Theorem 4.1, there is a ring isomorphism

$$H^0(M_{\text{cub}}^{\text{dR}}; g^* \omega^{\otimes *}) \cong \mathbf{Z}[b_2, b_4, b_6, b_8]/(4b_8 = b_2b_6 - b_4^2, \quad 1728\Delta = c_4^3 - c_6^2).$$

This ring has been studied before (in characteristic zero, in which case  $b_8 = (b_2b_6 - b_4^2)/4$ ), e.g., in [KZ95, Mov12], where it is referred to as the ring of *quasimodular forms*. Theorem 4.1 provides a calculation of the ring of integral quasimodular forms, and also justifies calling the ring spectrum  $\text{tmf}/\nu$  by the name “topological quasimodular forms”.

The following corollary is a calculation via Theorem 4.1.

COROLLARY 4.3. *In Adams–Novikov filtration zero,  $b_i^2$  and twice any monomial in the  $b_i$ s survive to the  $E_\infty$ -page for  $i = 2, 4, 6, 8$ , as do  $b_2b_6$  and  $\lambda_1b_8b_2^2 + \lambda_2b_2b_4b_6$  for  $\lambda_1 \equiv \lambda_2 \pmod{2}$ . In particular,  $\Delta \in \pi_{24}(\text{tmf}/\nu)$ , so the  $\mathbf{E}_1$ -ring  $\text{TMF}/\nu$  is 24-periodic with periodicity generator  $\Delta$ .*

REMARK 4.4. Note that  $\text{tmf}/\nu$  is complex orientable after inverting 2. This can be seen algebraically by noting that the Hopf algebroid  $(D, \Sigma)$  presenting  $M_{\text{cub}}^{\text{dR}}$  becomes discrete after inverting 2; indeed, the transformation  $y \rightarrow y - a_1x/2 - a_3/2$  transforms the Weierstrass equation (3) into

$$y^2 = x^3 + a_2x^2 + a_4x + a_6,$$

and one cannot make any coordinate changes to  $x$  since it is fixed. We find that  $(D, \Sigma)$  is isomorphic to the discrete Hopf algebroid  $(D' = \mathbf{Z}[1/2][a_2, a_4, a_6], D')$ , and so  $\pi_*(\text{tmf}[1/2]/\nu) \cong \mathbf{Z}[1/2][a_2, a_4, a_6]$ , with  $|a_i| = 2i$ . In light of this, we only need to prove Theorem 4.1 after 2-localization.

REMARK 4.5. The Hurewicz image of  $\text{tmf}$  in  $\text{tmf}/\nu$  can be determined from Theorem 4.1. The subring generated by  $\eta$ ,  $\sigma_1$ ,  $2b_2$ , and  $b_2^2$  is in the image of the map  $\pi_*A \rightarrow \pi_*\text{tmf}/\nu$ . There relationship between  $\pi_*(\text{tmf}/\nu)$  and  $\pi_*(\text{tmf})$ , however, is more interesting than merely the Hurewicz image. The (2-local) calculation in [Bau08] shows that  $\bar{\kappa}\nu$  vanishes in  $\pi_{23}(\text{tmf})$ ; this is detected in the Adams–Novikov spectral sequence by a  $d_5$ -differential  $d_5(\Delta) = \bar{\kappa}\nu$ . This



implies that the element  $8\Delta \in \pi_{24}(\mathrm{tmf})$  can be expressed as an element of the Toda bracket  $\langle 8, \nu, \bar{\kappa} \rangle$ . Equivalently, the map  $\bar{\kappa} : S^{20} \rightarrow \mathrm{tmf}$  extends to a map from  $\Sigma^{20}C\nu$ , and hence from  $\Sigma^{20}\mathrm{tmf} \wedge C\nu$ . Composition with the map  $S^{24} \rightarrow \Sigma^{20}C\nu$  which is degree 8 on the top cell produces the element  $8\Delta \in \pi_{24}\mathrm{tmf}$  (up to indeterminacy). In other words,  $8\Delta$  comes from an element of  $\pi_{24}(\Sigma^{20}\mathrm{tmf} \wedge C\nu) \cong \pi_4(\mathrm{tmf} \wedge C\nu)$ . Under the canonical map  $\mathrm{tmf} \wedge C\nu \rightarrow \mathrm{tmf}/\nu$ , this element corresponds to  $2b_2 \in \pi_4(\mathrm{tmf}/\nu)$ . Similarly, the element  $\Delta\eta \in \pi_{24}(\mathrm{tmf})$  can be related to the element  $\sigma_1 \in \pi_5(\mathrm{tmf}/\nu)$ . This is related to the approach taken in [Dev19] to show that the Ando-Hopkins-Rezk orientation  $\mathrm{MString} \rightarrow \mathrm{tmf}$  from [AHR10] is surjective on homotopy.

REMARK 4.6. One can give explicit  $q$ -expansions for each of the generators of  $H^0(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes*}) \cong \mathbb{Z}[b_2, b_4, b_6, b_8]$ . The element  $b_2$  is related to the weight 2 Eisenstein series via the  $q$ -expansion:

$$b_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \text{ where } \sigma_1(n) = \sum_{d \geq 1, d|n} d.$$

REMARK 4.7. The explicit characteristic power series for the Witten genus (in [HBJ92, Section 6.3], for instance) shows that the Witten genus of a spin  $4k$ -manifold is a modular form if its first Pontryagin class vanishes. This restriction arises essentially because of the appearance of a weight 2 Eisenstein series, which is not a modular form, in the characteristic power series. By Remark 4.6, the weight 2 Eisenstein series is in fact a global section of  $H^0(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes*})$ , and so one might expect to be able to refine the orientation  $\mathrm{MU}\langle 6 \rangle \rightarrow \mathrm{tmf}$  from [AHR10] to an orientation  $\mathrm{MSU} \rightarrow \mathrm{tmf}/\nu$ . We will return to this question in forthcoming work; see also Remark 3.7.

The following application is due to Rezk.

COROLLARY 4.8. *The  $\mathbf{E}_1$ -ring  $\mathrm{tmf}/\nu$  admits the structure of a homotopy commutative ring.*

PROOF. We know that  $\mathrm{tmf}/\nu$  only has  $\mathrm{tmf}$ -module cells in dimensions divisible by 4, so all obstructions to its homotopy commutativity live in these dimensions. We know that these obstructions vanish after inverting 2, since  $A[1/2]$  is homotopy commutative. By Theorem 4.1, all the homotopy groups of  $\mathrm{tmf}/\nu$  in dimensions divisible by 4 are torsion-free, so  $\mathrm{tmf}/\nu$  must indeed be homotopy commutative.  $\square$

An immediate consequence of Theorem 3.8 and Corollary 4.8 is:

COROLLARY 4.9. *The stack  $M_{\mathrm{tmf}/\nu}$  associated to the homotopy commutative ring  $\mathrm{tmf}/\nu$  is isomorphic to  $M_{\mathrm{cub}}^{\mathrm{dR}}$ .*

REMARK 4.10. Corollary 4.9 implies, for instance, that the fact that  $\nu$  is not detected by  $L_{K(1)}\mathrm{tmf}$  is related to the existence of a dotted map

$$\begin{array}{ccc} & & M_{\mathrm{cub}}^{\mathrm{dR}} \\ & \nearrow & \downarrow \\ M_{\mathrm{ell}}^{\mathrm{ord}} & \longrightarrow & M_{\mathrm{ell}}, \end{array}$$

where  $M_{\mathrm{ell}}^{\mathrm{ord}}$  denotes the moduli stack of ordinary elliptic curves. This existence of this dotted map is well-known in arithmetic geometry: it is the statement that the Frobenius (which exists for ordinary elliptic curves via quotienting out by the canonical subgroup) splits the Hodge filtration (see [Kat73, Section A2.3]).

Finally, we give the proof of Theorem 4.1.



PROOF OF THEOREM 4.1. We shall implicitly 2-localize everywhere; this is sufficient by Remark 4.4. We begin by calculating  $H^*(M_{\text{cub}}^{\text{dR}}; g^*\omega^{\otimes*}) = \text{Ext}_{\Sigma}(D, D)$ , where

$$(D, \Sigma) = (\mathbf{Z}[a_1, a_2, a_3, a_4, a_6], D[s, t]).$$

Following [Sil86, Chapter III], define quantities

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6, \quad b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2.$$

Notice that  $b_2b_6 - b_4^2 = 4b_8$  and that  $1728\Delta = c_4^2 - c_6^2$  where  $c_4, c_6$ , and  $\Delta$  are as in the theorem statement.

Let  $I$  denote the ideal  $(2, a_1, a_3, a_4)$ , and define a Hopf algebroid

$$(\overline{D}, \overline{\Sigma}) = (D/I, \Sigma/I) = (\mathbf{F}_2[a_2, a_6], \overline{D}[s, t]).$$

The right unit sends

$$a_2 \mapsto a_2 + s^2, \quad a_6 \mapsto a_6 + t^2.$$

Then there is a Bockstein spectral sequence

$$(6) \quad E_1^{p,q,n} = \text{Ext}_{\overline{\Sigma}}^{p,n}(\overline{D}, \text{Sym}_{\overline{D}}^q(I/I^2)) \Rightarrow \text{Ext}_{\Sigma}^{p,n}(D, D),$$

with  $d_r : E_r^{p,q,n} \rightarrow E_r^{p+1,q+r,n}$ . Now,  $I/I^2 = \overline{D} \otimes_{\mathbf{F}_2} V$ , with  $V = \mathbf{F}_2\{\overline{a}_0, \overline{a}_1, \overline{a}_3, \overline{a}_4\}$  where  $\overline{a}_0, \overline{a}_1, \overline{a}_3$ , and  $\overline{a}_4$  represent 2,  $a_1$ ,  $a_3$ , and  $a_4$  respectively. The comodule structure  $I/I^2 \rightarrow I/I^2 \otimes_{\overline{D}} \overline{\Sigma}$  sends

$$(7) \quad \overline{a}_0 \mapsto \overline{a}_0, \quad \overline{a}_1 \mapsto \overline{a}_1 + \overline{a}_0s, \quad \overline{a}_3 \mapsto \overline{a}_3 + \overline{a}_0t, \quad \overline{a}_4 \mapsto \overline{a}_4 + \overline{a}_3s + \overline{a}_1t + \overline{a}_0st.$$

There is a map  $\overline{D} \rightarrow \mathbf{F}_2$  induced by sending  $a_2$  and  $a_6$  to zero, and so we obtain a Hopf algebroid  $(\mathbf{F}_2, C)$  with

$$C = \mathbf{F}_2 \otimes_{\overline{D}} \overline{\Sigma} \otimes_{\overline{D}} \mathbf{F}_2 \cong \mathbf{F}_2[a_2, a_6, s, t]/(a_2, a_6, \eta_R(a_2), \eta_R(a_6)) \cong \mathbf{F}_2[s, t]/(s^2, t^2) = E(s, t).$$

To emphasize the connection to homotopy theory, we write  $h_1$  for  $s$  and  $h_{21}$  for  $t$ . Now, the map  $\overline{D} \rightarrow \mathbf{F}_2 \otimes_{\overline{D}} \overline{\Sigma} = \mathbf{F}_2[h_1, h_{21}]$  given by sending  $a_2$  to  $h_1^2$  and  $a_6$  to  $h_{21}^2$  is faithfully flat, and defines an isomorphism  $(\overline{D}, \overline{\Sigma}) \rightarrow (\mathbf{F}_2, C)$  of Hopf algebroids. Moreover, the Hopf algebroid  $(\mathbf{F}_2, C)$  presents the (graded) stack  $B\alpha_2 \times B\alpha_2$  over  $\text{Spec}(\mathbf{F}_2)$ . It follows that

$$\text{Ext}_{\overline{\Sigma}}^{p,n}(\overline{D}, \text{Sym}_{\overline{D}}^q(I/I^2)) \cong \text{Ext}_C^{p,n}(\mathbf{F}_2, \text{Sym}_{\mathbf{F}_2}^q(V)) = H^{p,n}(B\alpha_2 \times B\alpha_2; \text{Sym}^q(V)).$$

The comodule structure on  $I/I^2$  appearing in Equation (7) is a representation of  $\alpha_2 \times \alpha_2$  on  $V$ , and so:

$$(8) \quad \text{Ext}_C^{0,*}(\mathbf{F}_2, \text{Sym}_{\mathbf{F}_2}^*(V)) = V^{\alpha_2 \times \alpha_2} = \mathbf{F}_2[\overline{a}_0, \overline{a}_1^2, \overline{a}_0\overline{a}_4 + \overline{a}_1\overline{a}_3, \overline{a}_3^2, \overline{a}_4^2];$$

indeed, these are the invariants under the  $\alpha_2 \times \alpha_2$ -action on  $V$ . Moreover, by looking at the expressions for  $b_2, b_4, b_6$ , and  $b_8$ , we find that they are represented by  $\overline{a}_1^2, \overline{a}_0\overline{a}_4 + \overline{a}_1\overline{a}_3, \overline{a}_3^2$ , and  $\overline{a}_4^2$ , respectively; in particular, all of the generators of  $V^{\alpha_2 \times \alpha_2}$  are permanent cycles in the Bockstein spectral sequence. Moreover,  $H^{*,*}(B\alpha_2 \times B\alpha_2; \text{Sym}^q(V)) \cong \mathbf{F}_2$  for  $q \geq 1$ , and for  $q = 0$  we have  $H^{*,*}(B\alpha_2 \times B\alpha_2; \text{Sym}^0(V)) \cong \mathbf{F}_2[h_1, h_{21}]$ , where  $h_1 = [s]$  and  $h_{21} = [t]$ . Together with Equation (8), we obtain a complete calculation of the  $E_1$ -page of the Bockstein spectral sequence (6).

The right unit in Equation (4) gives Bockstein differentials  $a_1 \mapsto 2h_1$  and  $a_3 \mapsto 2h_{21}$ , which correspond to  $2\eta$  and  $2\sigma_1$  being null, respectively. Moreover,

$$d(a_4^2) = a_3^2s^2 + a_1^2t^2 + 4s^2t^2 + 4a_1st^2 - 2a_1a_3st - 4a_3s^2t$$

in the cobar complex, and

$$s^2b_6 + t^2b_2 = a_3^2s^2 + a_1^2t^2 + 4a_6s^2 + 4a_2t^2.$$

Since  $a_2 = 0$  and  $a_6 = 0$ , we have  $a_1s = -s^2$  and  $a_3t = -t^2$ . This produces a differential with target  $h_1^2b_6 + h_{21}^2b_2$ . Combining together all of these facts gives Equation (5) as the cohomology

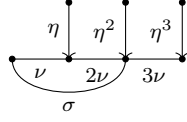


FIGURE 1. 15-skeleton of  $A$  at the prime 2 shown horizontally, with 0-cell on the left. The element  $\sigma_1$  is shown.

of the moduli stack  $M_{\text{cub}}^{\text{dR}}$ . As in Corollary 3.9, this is the  $E_2$ -page of the Adams–Novikov spectral sequence for  $\text{tmf}/\nu$ . We now calculate the Adams–Novikov differentials. The 15-skeleton of  $A$  is as shown in Figure 1.

We know that  $\eta^3 = 4\nu$  vanishes in  $\pi_*A$ , so  $h_1^3$  must die in the Adams–Novikov spectral sequence for  $\text{tmf}/\nu$ . There is only one possibility, namely the  $d_3$ -differential  $d_3(b_2) = h_1^3$ . (Note that this differential already exists in the Adams–Novikov spectral sequence for  $A$ , where  $b_2$  is represented by  $v_1^2$ , i.e., the class  $[y_2]$  in the cobar complex (via Proposition 3.3).) As a consequence, both  $\eta$  and  $\sigma_1$  are permanent cycles in the Adams–Novikov spectral sequence for  $A$ .

Next, we know from Proposition 3.3 that  $\sigma_1$  is detected in the Adams–Novikov spectral sequence for  $A$  by  $h_{21}$ . Since  $\sigma_1 \in \langle \eta, \nu, 1_A \rangle$ , one has that  $\eta^2\sigma_1 = 0$  in  $\pi_*(\text{tmf}/\nu)$ , so  $h_1^2h_{12}$  must die. There is no possibility other than  $d_3(b_4)$  for a differential to kill  $h_1^2h_{21}$  (except for a  $d_3$ -differential on  $b_2^2$ , but this cannot kill  $h_1^2h_{12}$ ). Note that  $h_1h_{12}$  is a permanent cycle and represents  $\eta\sigma_1$ .

For the third differential, note that since there is a  $d_3$ -differential  $d_3(b_2) = h_1^3$ , we have  $d_3(h_{21}^2b_2) = h_1^3h_{21}^2$ . But there is a relation  $h_1^2b_6 = h_{21}^2b_2$ , so  $d_3(h_1^2b_6) = h_1^3h_{21}^2$ , which forces  $d_3(b_6) = h_1h_{21}^2$ . Therefore,  $\eta\sigma_1^2 = 0$ . (Note that  $h_{12}^2$  is a permanent cycle and represents  $\sigma_1^2$ .)

Finally,  $\eta\sigma_1^3 = 0$  in  $\pi_*\text{tmf}/\nu$ , because  $\sigma_1^3$  lives inside the Toda bracket depicted by the arrow in Figure 1 labeled by  $\eta^3$ . It follows that the element  $h_1h_{21}^3$  must be the target of a differential in the Adams–Novikov spectral sequence for  $\text{tmf}/\nu$ . The only possibilities are a  $d_3$ -differential on  $h_1b_4^2$ ,  $h_1b_2b_6$ , or  $h_1b_8$ . Only  $h_1b_8$  can kill  $h_1h_{21}^3$ , and this forces  $d_3(b_8) = h_{21}^3$ . At this point, there are no more possibilities for differentials in the descent/Adams–Novikov spectral sequence for  $\text{tmf}/\nu$ , and the spectral sequence collapses at the  $E_4$ -page.  $\square$

## References

- [AB19] O. Antolín-Camarena and T. Barthel. A simple universal property of Thom ring spectra. *J. Topol.*, 12(1):56–78, 2019. (Cited on page 4.)
- [ABG10] M. Ando, A. Blumberg, and D. Gepner. Twists of  $K$ -theory and TMF. In *Superstrings, geometry, topology, and  $C^*$ -algebras*, volume 81 of *Proc. Sympos. Pure Math.*, pages 27–63. Amer. Math. Soc., Providence, RI, 2010. (Cited on page 5.)
- [AHR10] M. Ando, M. Hopkins, and C. Rezk. Multiplicative orientations of  $KO$ -theory and of the spectrum of topological modular forms. <http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf>, May 2010. (Cited on pages 1 and 8.)
- [Bau08] T. Bauer. Computation of the homotopy of the spectrum  $\text{tmf}$ . In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008. (Cited on pages 5 and 7.)
- [Dev18] S. Devalapurkar. An equivariant version of Wood’s theorem. <http://www.mit.edu/~sanathd/wood.pdf>, 2018. (Cited on page 6.)
- [Dev19] S. Devalapurkar. The Ando-Hopkins-Rezk orientation is surjective. <https://arxiv.org/abs/1911.10534>, 2019. (Cited on pages 4, 5, and 8.)
- [DFHH14] C. Douglas, J. Francis, A. Henriques, and M. Hill. *Topological Modular Forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2014. (Cited on page 5.)

- [HBJ92] F. Hirzebruch, T. Berger, and R. Jung. *Manifolds and modular forms*. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum. (Cited on page 8.)
- [Hop02] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317. Higher Ed. Press, Beijing, 2002. (Cited on pages 1 and 5.)
- [Kat73] N. Katz.  $p$ -adic properties of modular schemes and modular forms. pages 69–190. Lecture Notes in Mathematics, Vol. 350, 1973. (Cited on pages 2, 3, and 8.)
- [Kat81] N. Katz. Crystalline cohomology, Dieudonné modules, and Jacobi sums. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 165–246. Tata Inst. Fundamental Res., Bombay, 1981. (Cited on page 3.)
- [KM85] N. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985. (Cited on page 4.)
- [KZ95] M. Kaneko and D. Zagier. A generalized Jacobi theta function and quasimodular forms. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 165–172. Birkhäuser Boston, Boston, MA, 1995. (Cited on page 7.)
- [Mat16] A. Mathew. The homology of  $\mathrm{tmf}$ . *Homology Homotopy Appl.*, 18(2):1–20, 2016. (Cited on pages 5 and 6.)
- [Mov12] H. Movasati. Quasi-modular forms attached to elliptic curves, I. *Ann. Math. Blaise Pascal*, 19(2):307–377, 2012. (Cited on page 7.)
- [Poo20] B. Poonen. Algebraic de Rham cohomology of an elliptic curve. [http://math.mit.edu/~poonen/papers/deRham\\_for\\_curve.pdf](http://math.mit.edu/~poonen/papers/deRham_for_curve.pdf), 2020. (Cited on page 3.)
- [Rez13] C. Rezk. Power operations in Morava  $E$ -theory: structure and calculations (draft). <https://faculty.math.illinois.edu/~rezk/power-ops-ht-2.pdf>, 2013. (Cited on page 5.)
- [Sil86] J. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986. (Cited on page 9.)

Email address: `sanathd@mit.edu`