Our goal in this lecture is to describe a proof of the following theorem, which we stated last time:

Theorem 1 ([Kal09, Mat20])**.** *Let k be a perfect field, and let $\mathcal{C}$ be a smooth and proper k-linear stable ∞-category. If $\mathbf{F}_p \subseteq k$, assume that $\mathcal{C}$ lifts to $W_2(k)$ and that $\pi_n \mathrm{HH}(\mathcal{C}/k) = 0$ if $|n| > p$. Then the Tate spectral sequence for $\mathrm{HP}(\mathcal{C}/k)$ degenerates at the E_2-page.*

The case when $\mathbf{Q} \subseteq k$ can be proven by a spreading-out argument once the result is known for k of characteristic $p > 0$. We will therefore focus on the case when k is a perfect field of characteristic p. To explain the proof of Theorem 1, let us examine the structure of the Deligne-Illusie proof of Hodge-de Rham degeneration from last time.

Let X be a smooth and proper k-scheme. The key idea in the Deligne-Illusie proof was to consider a *different* filtration on the de Rham complex $\Omega^\bullet_{X/k}$, given by the conjugate filtration. More precisely, we have a square of filtrations and degenerations/associated gradeds:

(1)
$$\begin{tikzcd}
 & \mathrm{F}^\star_{\mathrm{Hdg}}\Omega^\bullet_{X/k} = \Omega^{\bullet \geq \star}_{X/k} \arrow[dl, squiggly, "\mathrm{gr}"'] \arrow[dr, squiggly, "\mathrm{und}"] & \\
\Omega^*_{X/k} & & \Omega^\bullet_{X/k} \\
 & \mathrm{F}^\star_{\mathrm{conj}}\Omega^\bullet_{X/k} = \tau_{\leq \star}\Omega^\bullet_{X/k} \arrow[ul, squiggly, "\mathrm{gr}", "\text{Frob twist}"'] \arrow[ur, squiggly, "\mathrm{und}"'] &
\end{tikzcd}$$

Recall that if $A^\bullet$ is a complex, we write A^* to denote its underlying graded module. Moreover, the bottom-left degeneration is given by the Cartier isomorphism. The Deligne-Illusie proof used the fact that the Frobenius twist of the associated graded of conjugate filtration agreed with the associated graded of the Hodge filtration. It is therefore natural to abstract out the proof by splitting it into two separate results (as we essentially did in Lecture V):

Proposition 2. *Let X be a smooth k-scheme. Then the following statements are true:*

(a) *Let $\mathcal{F}^\bullet$ be a complex of quasicoherent $\mathcal{O}_X$-modules equipped with two filtrations $\mathrm{F}^\star_{\mathrm{Hdg}}\mathcal{F}^\bullet$ and $\mathrm{F}^\star_{\mathrm{conj}}\mathcal{F}^\bullet$ such that $\mathrm{gr}^i(\mathrm{F}^\star_{\mathrm{conj}}\mathcal{F}^\bullet)$ is Frobenius-equivariantly isomorphic*[1] *to $\mathrm{gr}^i(\mathrm{F}^\star_{\mathrm{Hdg}}\mathcal{F}^\bullet)$. If the "conjugate" spectral sequence*
$$E_1^{*,*} = \mathrm{H}^*(X; \mathrm{gr}(\mathrm{F}^\star_{\mathrm{conj}}\mathcal{F}^\bullet)) \Rightarrow \mathrm{H}^*(X; \mathcal{F}^\bullet)$$
degenerates at the E_1-page, then so does the spectral sequence
$$E_1^{*,*} = \mathrm{H}^*(X; \mathrm{gr}(\mathrm{F}^\star_{\mathrm{Hdg}}\mathcal{F}^\bullet)) \Rightarrow \mathrm{H}^*(X; \mathcal{F}^\bullet).$$

(b) *If X is proper, $\dim(X) < p$, and X lifts to $W_2(k)$, then $\mathrm{F}^\star_{\mathrm{Hdg}}\Omega^\bullet_{X/k}$ and $\mathrm{F}^\star_{\mathrm{conj}}\Omega^\bullet_{X/k}$ satisfy condition (a).*

[1] We are abusing notation here by not writing down the symbol Frob_*; this is solely for readability.

We cannot directly apply Proposition 2 to prove Theorem 1, but some further massaging suggests a possible direction of attack. To explain this massaging, we must recall a general result.

Construction 3 (Rees construction)**.** Let k be a commutative ring, and let $\mathrm{F}^\star M$ be a filtered k-module. Let $k[\lambda]$ be the polynomial ring on a generator λ, called the *Rees variable*; equip $k[\lambda]$ with the $\mathbf{G}_m$-action where λ is given weight 1. Then $\mathrm{F}^\star M$ defines a $\mathbf{G}_m$-equivariant $k[\lambda]$-module $\bigoplus_{n \in \mathbf{Z}} (\mathrm{F}^n M)\lambda^n \subseteq M[\lambda]$. This module is denoted $\mathcal{R}(\mathrm{F}^\star M)$, and is called the *Rees construction* on $\mathrm{F}^\star M$. If we wish to make the Rees variable explicit, we will write $\mathcal{R}_\lambda(\mathrm{F}^\star M)$ instead. One can check that $\mathcal{R}(\mathrm{F}^\star M)/\lambda \cong \mathrm{gr}(\mathrm{F}^\star M)$, where the nth graded piece corresponds to the weight n piece of $\mathcal{R}(\mathrm{F}^\star M)/\lambda$. Furthermore, setting $\lambda = 1$ in $\mathcal{R}(\mathrm{F}^\star M)$ evidently produces the underlying k-module, i.e., M.

In fact, if one redefines a "filtered k-module" to be a $\mathbf{Z}$-indexed sequence $\cdots \to M_n \to M_{n-1} \to \cdots$ of k-module maps which are not necessarily injective, then:

Proposition 4. *The ∞-category $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ is equivalent to the ∞-category of filtered k-modules via the above construction; the pointwise tensor product on $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ is sent to the Day convolution tensor product on filtered k-modules. Similarly, the ∞-category $\mathrm{QCoh}(B\mathbf{G}_m)$ is equivalent to the ∞-category of graded k-modules. Furthermore, pullback along the morphism $B\mathbf{G}_m \to \mathbf{A}^1/\mathbf{G}_m$ sends a filtered k-module to its associated graded.*

Remark 5. The statement of Proposition 4 goes through verbatim for any $\mathbf{E}_\infty$-ring k (see [Mou19]). Moreover, the ∞-category $\mathrm{QCoh}(B\mathbf{G}_m)$ acquires a symmetric monoidal structure; this transfers to the *pointwise* tensor product on ∞-category of graded k-modules. If k admits an $\mathbf{E}_\infty$*-complex-orientation*, then the stack $\mathbf{A}^1/\mathbf{G}_m$ admits the structure of a group object in stacks over k; in other words, $\mathbf{A}^1$ admits a group structure in $\mathbf{G}_m$-equivariant derived k-schemes. Therefore, the ∞-category $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ acquires a symmetric monoidal structure. As before, this transfers to the pointwise tensor product on ∞-category of filtered k-modules.

Remark 6. Continuing Remark 5, $\mathbf{A}^1$ does *not* admit a group structure over $\mathbb{S}$! This is due to the existence of nontrivial spherical power operations, and is proved in [Lur18, Proposition 1.6.20]. In general, there is a power operation $\mathrm{Sq}_1 : \pi_{2n}(A) \to \pi_{4n+1}(A)$ defined on any $\mathbf{E}_\infty$-ring A, known as the "cup-1 square". This power operation satisfies the relation

$$\mathrm{Sq}_1(x+y) = \mathrm{Sq}_1(x) + \mathrm{Sq}_1(y) + \left(1 + \frac{|x|}{2}\right)\eta xy.$$

In particular, if $|x| = |y| = 0$, then the final term is ηxy. Now suppose $\mathbf{A}^1 = \operatorname{Spec} \mathbb{S}[\lambda]$ has a group structure restricting to the ordinary group structure on its underlying $\mathbf{Z}$-scheme $\mathbf{G}_a$. This is equivalent to claiming that there is a map $\mathbb{S}[\lambda] \to \mathbb{S}[\lambda_1, \lambda_2]$ of $\mathbf{E}_\infty$-rings such that on π_0, the map sends $\lambda \mapsto \lambda_1 + \lambda_2$. Because η-multiplication induces isomorphisms

$$\mathbf{F}_2[\lambda] \cong \pi_0(\mathbb{S}[\lambda])/2 \xrightarrow{\sim} \pi_1(\mathbb{S}[\lambda]),$$

and similarly for $\mathbb{S}[\lambda_1, \lambda_2]$, we see that there is an element $f(\lambda) \in \mathbf{Z}[\lambda]$ which is unique mod 2 such that $\eta f(\lambda) = \mathrm{Sq}_1(\lambda)$. But then

$$\begin{aligned}\eta f(\lambda_1 + \lambda_2) &= \mathrm{Sq}_1(\lambda_1 + \lambda_2)\\ &= \mathrm{Sq}_1(\lambda_1) + \mathrm{Sq}_1(\lambda_2) + \eta\lambda_1\lambda_2\\ &= \eta\left(f(\lambda_1) + f(\lambda_2) + \lambda_1\lambda_2\right),\end{aligned}$$

where all of these equalities are taken modulo 2. But the coefficient of $\lambda_1\lambda_2$ in $f(\lambda_1+\lambda_2)$ must vanish modulo 2, which gives a contradiction.

Let us now apply Construction 3 to (1), suggestively using $\hbar$ for the Rees variable of the Hodge filtration and σ for the Rees variable of the conjugate filtration:

(2)
$$\begin{tikzcd}
 & \mathcal{R}_\hbar(\mathrm{F}^\star_{\mathrm{Hdg}}\Omega^\bullet_{X/k}) = \bigoplus_{n\geq 0} \Omega^{\bullet \geq -n}_{X/k}\hbar^n \arrow[dl, squiggly, "\hbar\mapsto 0"'] \arrow[dr, squiggly, "\hbar\mapsto 1"] & \\
\Omega^*_{X/k} & & \Omega^\bullet_{X/k} \\
 & \mathcal{R}_\sigma(\mathrm{F}^\star_{\mathrm{conj}}\Omega^\bullet_{X/k}) = \bigoplus_{n\geq 0} \tau_{\leq n}\Omega^\bullet_{X/k}\sigma^n \arrow[ul, squiggly, "\sigma\mapsto 0", "\text{Frob twist}"'] \arrow[ur, squiggly, "\sigma\mapsto 1"'] &
\end{tikzcd}$$

Let us look at the top span in Equation (2): if $\hbar$ is placed in homological degree -2, then $\Omega^{\bullet\geq -n}_{X/k}\hbar^n$ is a copy of $\Omega^{\bullet\geq -n}_{X/k}$ placed in degree $-2n$. As we stated last time, $\mathrm{HC}^-(X/k)$

admits a bifiltration $\mathrm{F}^\star_{\mathbf{C}P}\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(X/k)$ such that $\mathrm{gr}^n(\mathrm{F}^\star_{\mathbf{C}P}\mathrm{HC}^-(X/k)) \simeq \mathrm{HH}(X/k)\cdot\hbar^n$, and such that the B-filtration induces the HKR filtration on $\mathrm{HH}(X/k)$. If $\hbar$ is placed in nontrivial homological degree, then it is no longer sensible to set $\hbar = 1$; however, it is completely valid to *invert* $\hbar$ instead. The resulting object is no longer an ordinary (i.e., unfiltered) k-module, but is a new filtered k-module. This discussion tells us that the appropriate analogue of the top span in Equation (2) is the span

$$\mathrm{HH}(X/k) \xleftarrow{\hbar\mapsto 0} \mathrm{F}^\star_{\mathbf{C}P}\mathrm{HC}^-(X/k) \xrightarrow{\hbar^{-1}} \mathrm{F}^\star_{\mathbf{C}P}\mathrm{HC}^-(X/k)[\hbar^{-1}] \simeq \mathrm{F}^\star_{\mathbf{C}P}\mathrm{HP}(X/k).$$

The final equivalence is due to the fact that $A^{hS^1}[\hbar^{-1}] \simeq A^{tS^1}$ for any k-module A when k is complex-oriented.

Our goal in the remainder of this lecture is to describe the appropriate analogue of the *bottom* span in Equation (2). Using a generalization of Proposition 2(a), this will imply Theorem 1. We begin by describing the category in which the desired generalization of Equation (2) sits: roughly, this category consists of a $k[\sigma]$-module $\mathcal{M}$, a $k[\![\hbar]\!]$-module $\mathcal{N}$, and a diagram of the form

$$\begin{tikzcd}
 & \mathcal{N} \arrow[dl, squiggly, "\hbar\mapsto 0"'] \arrow[dr, squiggly, "\hbar\mapsto 1"] & \\
\mathcal{N}/\hbar \simeq \mathcal{M}/\sigma & & \mathcal{N}[\hbar^{-1}] \simeq_{\mathrm{Frob}} \mathcal{M}[\sigma^{-1}] \\
 & \mathcal{M} \arrow[ul, squiggly, "\sigma\mapsto 0", "\text{Frob twist}"'] \arrow[ur, squiggly, "\sigma\mapsto 1"'] &
\end{tikzcd}$$

Note that we have made a different choice of which equivalence is to be Frobenius-twisted. To describe this category precisely, we need some general results.

Definition 7. Let k be an $\mathbf{E}_\infty$-ring. Let k^{hS^1} denote the homotopy fixed points of the trivial S^1-action on k. Let $\mathrm{F}^\star_{\mathbf{C}P}k^{hS^1}$ denote the filtration on k^{hS^1} given by the homotopy fixed points spectral sequence. More invariantly, one can realize k^{hS^1} as the totalization of the cosimplicial diagram $k[(S^1)^{\times\bullet}]$; then, $\mathrm{F}^\star_{\mathbf{C}P}k^{hS^1}$ can be understood as the filtration given by $\mathrm{Tot}^{\geq n}\, k[(S^1)^{\times\bullet}]$. It follows from Proposition 8 that $\mathrm{F}^\star_{\mathbf{C}P}k^{hS^1}$ acquires the structure of an $\mathbf{E}_2$-algebra in filtered k-modules.

Proposition 8 ([Lur15, Theorem 5.3.1])**.** *The filtration $\{\mathbf{C}P^n\}$ on $\mathbf{C}P^\infty$ defines an $\mathbf{E}_2$-coalgebra in filtered spaces.*

Remark 9. Write $k[\![\hbar]\!]$ to the graded $\mathbf{E}_2$-k-algebra $\mathrm{gr}(\mathrm{F}^\star_{\mathbf{C}P}k^{hS^1})$, and let $\mathrm{F}^\star_\hbar k[\![\hbar]\!]$ denote the filtered $\mathbf{E}_2$-k-algebra associated to $k[\![\hbar]\!]$. Then one can show that $\mathrm{F}^\star_\hbar k[\![\hbar]\!]$ upgrades to a filtered $\mathbf{E}_\infty$-k-algebra. If k is $\mathbf{E}_\infty$-complex-oriented, then there is an equivalence $\mathrm{F}^\star_{\mathbf{C}P}k^{hS^1} \simeq \mathrm{F}^\star_\hbar k[\![\hbar]\!]$ of filtered $\mathbf{E}_2$-k-algebras, and we will often abusively write k^{hS^1} to denote $k[\![\hbar]\!]$.

Remark 10. The filtered space $\{\mathbf{C}P^n\}$ does *not* admit a refinement to an $\mathbf{E}_3$-coalgebra in filtered spaces. Indeed, this would imply that $\mathrm{F}^\star_{\mathbf{C}P}\mathbb{S}^{hS^1}$ admits the structure of an $\mathbf{E}_3$-algebra in filtered spectra, and therefore that $\mathbb{S}[\![\hbar]\!]$ admits the structure of an $\mathbf{E}_3$-algebra in graded spectra. Forgetting the grading, it suffices to just show this claim for the underlying $\mathbf{E}_2$-algebra. Assume for the sake of contradiction that $\mathbb{S}[\![\hbar]\!]$ does admit an $\mathbf{E}_3$-algebra refinement, and let $\hbar : S^{-2} \to \mathbb{S}[\![\hbar]\!]$ denote the map detecting $\hbar$. Then the $\mathbf{E}_3$-multiplication on $\hbar$ defines a map $(\mathrm{Conf}_2(\mathbf{R}^3)_+ \otimes (S^{-2})^{\otimes 2})_{h\mathbf{Z}/2} \to \mathbb{S}[\![\hbar]\!]$. Because $\mathrm{Conf}_2(\mathbf{R}^3)_+ \otimes (S^{-2})^{\otimes 2} \simeq \Sigma^{-2}\mathbf{R}P^0_{-2}$, we obtain a map $f : \Sigma^{-2}\mathbf{R}P^0_{-2} \to \mathbb{S}[\![\hbar]\!]$ which detects $\hbar^2$ on the bottom cell of the source. Since the (-4)-cell of $\mathbb{S}[\![\hbar]\!]$ is unattached, the map f would give a splitting of the bottom cell of $\Sigma^{-2}\mathbf{R}P^0_{-2}$. The cell structure of $\Sigma^{-2}\mathbf{R}P^0_{-2} \simeq \Sigma^{-3}\mathbf{D}(\mathbf{R}P^1_{-1})$, drawn in Figure 1, shows that this is impossible. Note that

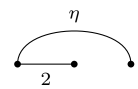

FIGURE 1. $\Sigma^{-2}\mathbf{R}P^0_{-2}$ shown horizontally, with the bottom (-4)-cell on the left and the top (-2)-cell on the right.

the obstruction to $\mathbb{S}[\![\hbar]\!]$ being a graded $\mathbf{E}_3$-algebra stems from the fact that the map η (which attaches the (-2)-cell of $\Sigma^{-2}\mathbf{R}P^0_{-2}$) is *not* null-homotopic in $\mathbb{S}$.

We now define $k[\sigma]$.

Definition 11. Let k be an $\mathbf{E}_\infty$-ring. Let $k/\!\!/\eta$ denote the $\mathbf{E}_2$-k-algebra defined as the Thom spectrum of the map $\Omega S^3 \xrightarrow{\eta} \mathrm{BGL}_1(k)$ which is given by $\eta \in \pi_1(k)$ on the bottom cell of the source. Let $\mathrm{F}^\star_\eta k/\!\!/\eta$ denote the filtration on $k/\!\!/\eta$ given by the James filtration $\{J_n(S^2)\}$ on ΩS^3. It follows from Proposition 12 that $\mathrm{F}^\star_\eta k/\!\!/\eta$ acquires the structure of an $\mathbf{E}_2$-algebra in filtered k-modules.

Proposition 12. *The James filtration $\{J_n(S^2)\}$ on ΩS^3 defines an $\mathbf{E}_2$-algebra in filtered spaces*[2]*.*

[2] See [HY19] for several results along these lines. Thanks to Mike Hopkins for suggesting the paper [CMM78] of Cohen-Mahowald-Milgram, which essentially contains the argument below.

Proof. We need to construct a multiplication

$$\mathrm{Conf}_d(\mathbf{C}) \times_{\Sigma_d} (J_{n_1}(S^2) \times \cdots \times J_{n_d}(S^2)) \to J_{n_1+\cdots+n_d}(S^2) \tag{3}$$

which sends $\mathrm{Conf}_d(\mathbf{C}) \times_{\Sigma_d} *$ to the basepoint of $J_{n_1+\cdots+n_d}(S^2)$. We will just define the map when $n_1 = \cdots = n_d = 1$; this essentially specifies the desired map for all n_i. We will view S^2 as $\mathbf{C}P^1\infty = \mathbf{C}\cup\{\infty\}$. Let $(z_1, \cdots, z_d) \in \mathrm{Conf}_d(\mathbf{C})$ and $(x_1, \cdots, x_d) \in (S^2)^{\times d}$, so that some of the x_i are ∞. Permuting the z_i amounts to applying the same permutation on the x_i. So assume without loss of generality that $1 \leq i \leq d$ is such that $x_j = \infty$ for $j > i$ and $x_j \neq \infty$ for $j \leq i$. Then, we send

$$\mathrm{Conf}_d(\mathbf{C}) \times (\mathbf{C}P^1)^{\times d} \ni (z_1, \cdots, z_d), (x_1, \cdots, x_d) \mapsto \begin{pmatrix} \sum_{j=1}^i x_j \\ \sum_{j=1}^i z_j x_j \\ \vdots \\ \sum_{j=1}^i z_j^{i-1} x_j \\ \infty \\ \vdots \\ \infty \end{pmatrix} \in (\mathbf{C}P^1)^{\times d},$$

which can then be sent to a point of $J_d(S^2)$ via the canonical map $(\mathbf{C}P^1)^{\times d} \to J_d(S^2)$. Note that $\left(\sum_{j=1}^i x_j, \sum_{j=1}^i z_j x_j, \cdots, \sum_{j=1}^i z_j^{i-1} x_j\right)$ is the image of $(x_1, \cdots, x_i)$ under the (invertible!) $i \times i$-Vandermonde matrix associated to $(z_1, \cdots, z_i) \in \mathrm{Conf}_i(\mathbf{C})$. $\square$

Remark 13. The space $\mathrm{Conf}_d(\mathbf{C}) \times_{\Sigma_d} \mathbf{C}^{\times d}$ defines a rank d complex vector bundle over $\mathrm{Conf}_d(\mathbf{C})/\Sigma_d$. If Br_k denotes the braid group on d strands, then $\mathrm{Conf}_d(\mathbf{C})/\Sigma_d$ is the classifying space $B\mathrm{Br}_d$, and the above rank d complex vector bundle is classified by the composite

$$B\mathrm{Br}_d \to B\Sigma_d \to \mathrm{BO}(d) \to \mathrm{BU}(d).$$

However, this composite is nullhomotopic, and therefore defines a trivialization of the aforementioned complex vector bundle; a choice of trivialization is given by the Vandermonde matrix.

Remark 14. The bar construction in filtered spaces of the filtered $\mathbf{E}_2$-space $\{J_n(S^2)\}$ is the filtration $* \to S^3 \to S^3 \to \cdots$ of $\mathrm{B}\Omega S^3 \simeq S^3$. Applying the bar construction again, this deloops in filtered spaces to the cellular filtration $\{\mathbf{H}P^n\}$ of $\mathrm{B}S^3 \simeq \mathbf{H}P^\infty$.

Remark 15. Write $k[\sigma]$ to the graded $\mathbf{E}_2$-k-algebra $\mathrm{gr}(\mathrm{F}^\star_\eta k/\!\!/\eta)$, and let $\mathrm{F}^\star_\sigma k[\sigma]$ denote the filtered $\mathbf{E}_2$-k-algebra associated to $k[\sigma]$. Note that $k[\sigma] = k \otimes \Omega S^3_+$. One can show that $\mathrm{F}^\star_\sigma k[\sigma]$ upgrades to a filtered $\mathbf{E}_\infty$-k-algebra. If k is $\mathbf{E}_\infty$-complex-oriented, then there is an equivalence $\mathrm{F}^\star_\eta k/\!\!/\eta \simeq \mathrm{F}^\star_\sigma k[\sigma]$ of $\mathbf{E}_2$-k-algebras.

Remark 16. Just as with Remark 10, the filtered space $\{J_n(S^2)\}$ does *not* refine to an $\mathbf{E}_3$-algebra in filtered spaces. Indeed, this would imply that the filtered $\mathbf{E}_2$-algebra structure on $\mathrm{F}^\star_\eta \mathbb{S}/\!\!/\eta$ refines to a filtered $\mathbf{E}_3$-algebra structure. We will show that this is not possible; see [Law19] as well. In fact, we prove that $\mathbb{S}/\!\!/\eta$ cannot be refined to a $\mathbf{E}_3$-ring. This can be checked on mod 2 homology: there is a canonical map $\mathbb{S}/\!\!/\eta \to \mathbf{F}_2$, and its image on mod 2 homology is $\mathrm{H}_*(\mathbb{S}/\!\!/\eta; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^2] \subseteq \mathbf{F}_2[\zeta_1, \zeta_2, \cdots] = \mathrm{H}_*(\mathbf{F}_2; \mathbf{F}_2)$. If $\mathbb{S}/\!\!/\eta$ was to be an $\mathbf{E}_3$-ring, then $\mathbf{F}_2[\zeta_1^2]$ would be closed under $\mathbf{E}_3$-Dyer-Lashof operations; however, $Q_2(\zeta_1^2) = \zeta_2^2 \notin \mathbf{F}_2[\zeta_1^2]$, giving the desired contradiction.

Remark 17. Suppose k is a perfect field of characteristic $p > 0$. As we will discuss next time, one can identify the $\mathbf{E}_2$-k-algebra $k[\sigma]$ with $\mathrm{THH}(k)$. In particular, if R is an $\mathbf{E}_\infty$-ring with a map $R \to k$, then there is an induced map $\mathrm{THH}(R) \to k[\sigma]$.

Summary 18. If k is an $\mathbf{E}_\infty$-ring, one can define $k/\!\!/\eta, k[\sigma], k^{hS^1}$, and $k[\![\hbar]\!]$. If k is $\mathbf{E}_\infty$-complex-oriented, then $k/\!\!/\eta \simeq k[\sigma]$ and $k^{hS^1} \simeq k[\![\hbar]\!]$.

The final bit of preparation required is the following result, whose proof we will defer to a future lecture.

Proposition 19. *Let k be an $\mathbf{E}_\infty$-ring. Then there is an equivalence $k[\sigma^{\pm 1}] \xrightarrow{\sim} k(\!(\hbar)\!)$ of graded $\mathbf{E}_2$-k-algebras.*

Definition 20. Let k be a complex-oriented $\mathbf{E}_\infty$-ring equipped with an $\mathbf{E}_\infty$-*automorphism* $F : k \to k$; composing with Proposition 19, we get an equivalence $k[\sigma^{\pm 1}] \xrightarrow{\sim}_F k(\!(\hbar)\!)$. A *weak cyclotomic structure* over k is a tuple $(\mathcal{M}, \mathcal{N}, \varphi)$ of a (graded) $k[\sigma]$-module $\mathcal{M}$, a (graded) k^{hS^1}-module $\mathcal{N}$, and equivalences

$$k[\sigma^{\pm 1}] \circlearrowleft \mathcal{M}[1/\sigma] \xrightarrow{\sim}_F \mathcal{N}[1/\hbar] \circlearrowright k(\!(\hbar)\!),$$
$$\mathcal{M}/\sigma \simeq \mathcal{N}/\hbar,$$

where the first equivalence is F-linear. These equivalences are part of the data, and are what we mean by the symbol φ. Weak cyclotomic structures over k assemble into an ∞-category, which we will denote $\mathrm{Cyc}_k^{\mathrm{wk}}$.

Remark 21. Let k be a perfect field of characteristic $p > 0$, and let $F : k \to k$ denote the Frobenius on k. Then $\mathrm{THH}(k) \simeq k[\sigma]$ as $\mathbf{E}_1$-k-algebras. This equivalence implies that $\mathrm{Cyc}_k^{\mathrm{wk}}$ is almost equivalent to the ∞-category $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ (we will introduce the ∞-category CycSp in the next lecture). More precisely, there is a functor from the ∞-category of *dualizable* objects in $\mathrm{Mod}_{\mathrm{THH}(k)}(\mathrm{CycSp})$ to $\mathrm{Cyc}_k^{\mathrm{wk}}$, which sends a dualizable $\mathrm{THH}(k)$-module X to the weak cyclotomic structure whose underlying $k[\sigma]$-model is X. This functor is not an equivalence, but it is conservative; the only extra data needed to recover a $\mathrm{THH}(k)$-module in cyclotomic spectra from an object $(\mathcal{M}, \mathcal{N}, \varphi) \in \mathrm{Cyc}_k^{\mathrm{wk}}$ is an S^1-action on $\mathcal{M}$ (which makes it an S^1-equivariant $\mathrm{THH}(k) \simeq k[\sigma]$-module) and the S^1-equivariance of φ. Note, however, that the S^1-action on $\mathrm{THH}(k) \simeq k[\sigma]$ is very nontrivial (for example, it depends on the characteristic of k).

The desired analogue of Proposition 2 is the following result (which is due to Mathew, albeit without using the phrase "weak cyclotomic structure"), whose first and fourth parts together imply Theorem 1.

Theorem 22. *Let k be a perfect $\mathbf{F}_p$-algebra, and let $F : k \to k$ denote the Frobenius on k. The following statements are true:*

(a) *Let $(\mathcal{M}, \mathcal{N}, \varphi)$ be a weak cyclotomic structure over k. Suppose that the σ-adic spectral sequence*
$$E_1^{*,*} = \pi_*(\mathcal{M}/\sigma)[\sigma] \Rightarrow \pi_*\mathcal{M}$$
degenerates at the E_1-page. Then the Tate spectral sequence
$$E_2^{*,*} = (\pi_*\mathcal{N}/\hbar)((\hbar)) \Rightarrow \pi_*\mathcal{N}[1/\hbar]$$
degenerates at the E_2-page.

(b) *Suppose $\mathcal{M}$ is a perfect $k[\sigma]$-module with* Tor*-amplitude in $[-p, p]$. If $\mathcal{M}$ lifts to a perfect $\mathrm{THH}(W_2(k))$-module along the canonical map $\mathrm{THH}(W_2(k)) \to k[\sigma]$, then the σ-adic spectral sequence degenerates at the E_1-page.*

(c) *Let $\mathcal{C}$ be a smooth and proper k-linear stable ∞-category. Then the pair $(\mathrm{THH}(\mathcal{C}), \mathrm{HC}^-(\mathcal{C}/k))$ can be upgraded to a weak cyclotomic structure over k.*

(d) *If $\mathcal{C}$ lifts to $W_2(k)$ and $\pi_n \mathrm{HH}(\mathcal{C}/k) = 0$ for $|n| > p$, then this weak cyclotomic structure satisfies condition (a).*

Proof. Part (a) is clear by dimension-counting. Part (b) is proved in [Mat20]; we will recall the proof here. Since $k[\sigma]$ is a PID (owing to k being a field), we can write $\mathcal{M}$ as a direct sum of free $k[\sigma]$-modules and shifts of the form $\mathcal{M}_{i,j} := \Sigma^i k[\sigma]/\sigma^j$. Let us make some general observations about $\mathcal{M}_{i,j}$:

- $\mathcal{M}_{i,j}$ has Tor-amplitude in $[i, i + 2j + 1]$ because $|\sigma| = 2$.
- The multiplication $\sigma : \pi_{n-2}\mathcal{M}_{i,j} \to \pi_n\mathcal{M}_{i,j}$ is an equivalence for $i + 2 \leq n \leq i + 2j - 2$.

We now recall [Mat20, Proposition 3.7] (whose proof uses the structure of $\mathrm{THH}(W_2(k))$ in low degrees), which states if that N is a $\mathrm{THH}(k)$-module which lifts to $\mathrm{THH}(W_2(k))$ such that $\pi_i(N) = 0$ for $i < i_0$, then the map $\sigma : \pi_{n-2}\mathcal{M} \to \pi_n\mathcal{M}$ is injective for $n \leq i_0 + 2p - 2$. Since $\mathcal{M}$ lifts to $\mathrm{THH}(W_2(k))$, the map $\sigma : \pi_{n-2}\mathcal{M} \to \pi_n\mathcal{M}$ is injective for $n \leq -p + 2p - 2 = p - 2$. If $\mathcal{M}_{i,j}$ is a summand of $\mathcal{M}$, then the second bullet implies that $i + 2j - 2 > p - 2$, i.e., $i + 2j + 1 \geq p$. But since $\mathcal{M}$ has Tor-amplitude in $[-p, p]$, the first bullet implies that $i + 2j + 1 \leq p$[3], too, so $i + 2j + 1 = p$.

[3] Note that if we assumed instead that $\mathcal{M}$ had Tor-amplitude in $[-p+1, p-1]$, then we would be requiring $i+2j+1 < p$. Since $i + 2j + 1 \geq p$, this is not possible, so (b) follows.

Let $\mathbf{D}(\mathcal{M})$ denote the $k[\sigma]$-linear dual of $\mathcal{M}$. If $\mathcal{M}_{i,j}$ is a summand of $\mathcal{M}$, then $\mathbf{D}(\mathcal{M}_{i,j}) \simeq \mathcal{M}_{-i-2j-1,j}$ is a summand of $\mathbf{D}(\mathcal{M})$. Therefore, the same argument as above shows that $-i = p$. But there are no integers i, j which satisfy $-i = p$ and $i + 2j + 1 = p$, giving the desired contradiction.

We will prove part (c) next time, since it is a good segue into cyclotomic spectra. Part (d) is immediate from part (c). $\square$

Remark 23. We motivated the introduction of THH in the above discussion by noting that $\mathcal{R}_\hbar(\mathrm{F}^\star_{\mathrm{Hdg}}\Omega^\bullet_{X/k}) = \bigoplus_{n\geq 0} \Omega^{\bullet \geq -n}_{X/k}\hbar^n$ is the associated graded of the B-filtration on $\mathrm{HC}^-(X/k)$, and asking for an analogue of $\mathrm{HC}^-(X/k)$ for the Rees construction $\mathcal{R}_\sigma(\mathrm{F}^\star_{\mathrm{conj}}\Omega^\bullet_{X/k}) = \bigoplus_{n\geq 0} \tau_{\leq n}\Omega^\bullet_{X/k}\sigma^n$ of the conjugate filtration. To complete this line of thought, let us therefore state a result relating the conjugate filtration on the de Rham complex to the σ-adic filtration on THH. In the next lecture, we will show that there is a Frobenius-linear map $\mathrm{THH}(X) \to \mathrm{THH}(X)^{t\mathbf{Z}/p}$, and that there is a map $\mathrm{THH}(X)^{t\mathbf{Z}/p} \to \mathrm{HP}(X/k)$, the latter of which is an equivalence if X is a smooth and proper k-scheme. In particular, there is a Frobenius-linear map $\varphi : \mathrm{THH}(X) \to \mathrm{HP}(X/k)$

for any (smooth) k-scheme X. Then, [BMS19, Corollary 8.18] states that if A is a smooth k-algebra and $\mathrm{F}^\star_\sigma\mathrm{THH}(A)$ is the σ-adic filtration on $\mathrm{THH}(A)$, then:

- There is an equivalence $\mathrm{gr}^n(\mathrm{F}^\star_\sigma\mathrm{THH}(A)) \simeq \tau_{\leq n}\Omega^\bullet_{A/k}\sigma^n$.
- The map $\varphi : \mathrm{THH}(A) \to \mathrm{HP}(A/k)$ lifts to a filtered (Frobenius-linear) map $\mathrm{F}^\star_\sigma\mathrm{THH}(A) \to \mathrm{F}^\star_\mathrm{B}\mathrm{HP}(A/k)$, which on gr^n is given by the inclusion $\tau_{\leq n}\Omega^\bullet_{A/k}\sigma^n \to \Omega^\bullet_{A/k}\hbar^{-n}$ of the conjugate filtration and the mapping $\varphi : \sigma \mapsto \hbar^{-1}$.

Therefore, $\mathrm{THH}(A)$ may be regarded as the homotopical analogue of the Rees construction $\mathcal{R}_\sigma(\mathrm{F}^\star_{\mathrm{conj}}\Omega^\bullet_{X/k})$ of the conjugate filtration.

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