Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$

S. K. Devalapurkar

ABSTRACT. In this note, we study the local relative geometric Langlands conjecture of Ben-Zvi–Sakellaridis–Venkatesh for the spherical subgroup $\mathrm{PGL}_2^{\mathrm{diag}}$ of the triple product $\mathrm{PGL}_2^{\times 3}$ (and also for the spherical subgroup G_2 of SO_8/μ_2), whose corresponding Langlands dual $\mathrm{SL}_2^{\times 3}$-variety can be identified with the symplectic vector space $(\mathbf{A}^2)^{\otimes 3} \cong \mathbf{A}^8$ of $2 \times 2 \times 2$-cubes. Our analysis relies on a construction of Bhargava relating $2 \times 2 \times 2$-cubes to Gauss composition on quadratic forms, arising here as the moment map for the Hamiltonian $\mathrm{SL}_2^{\times 3}$-action on $(\mathbf{A}^2)^{\otimes 3}$, and the Cayley hyperdeterminant as studied by Gelfand-Kapranov-Zelevinsky.

1. Introduction

The goal of this brief note is to study the geometrization of a story from the arithmetic context pioneered by Jacquet, Kudla-Harris, and Ichino among many others (see, e.g., [**HK91, Ich08**]). Fix an eighth root of unity ζ_8, let i be the resulting square root of -1, and write $k := \mathbf{Q}(\zeta_8) \cong \mathbf{Q}(i, \sqrt{2})$.

Notation 1.1. Let std denote the standard representation of SL_2, so that $\mathrm{std}^{\otimes 3}$ consists of cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] \arrow[dd, dash] && d_3 \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] && b_1 \arrow[ur, dash] \arrow[dd, dash] & \\
& d_1 \arrow[rr, dash] && c. \\
b_3 \arrow[ur, dash] \arrow[rr, dash] && d_2 \arrow[ur, dash] &
\end{tikzcd}$$

Fix an integer n. Equip $\mathrm{std}^{\otimes 3}$ with the grading where the entries of a cube have the following weights: a lives in weight $-4n$, each b_i lives in weight $-2n$, c lives in weight $2n$, and each d_i lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the corresponding graded variety.

Similarly, equip SL_2 with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. Write $\mathrm{SL}_2(-2n\rho)$ to denote this graded group. Then there is a natural graded action of $\mathrm{SL}_2(-2n\rho)^{\times 3}$ on $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$.

Recall that the process of *shearing* discussed in [**Rak20, Lur15**], as well as [**Dev23**, Section 2.1], converts gradings into homological shifts (more precisely, it sends a module in weight n to the same module shifted homologically by n). This functor is symmetric monoidal when restricted to the subcategory of modules in *even* weights, and therefore extends to an operation on evenly graded

Part of this work was done when the author was supported by NSF DGE-2140743.

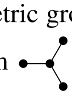
stacks. As in [**Dev23**], we will state all of our results with "arithmetic shearing" in the sense of [**BZSV23**, Section 6.7].

Theorem 1.2 (Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$)**.** *Suppose that the* $\mathrm{PGL}_2^{\times 3}[\![t]\!]$*-action on* $\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!)$ *is optimal in the sense of* [**Dev23**, Hypothesis 3.5.22]. *There is an equivalence*[1]

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!)); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ *denotes the* ∞*-category of perfect complexes on the shearing of the quotient stack* $\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}$. *Moreover, this equivalence is equivariant for the action of the spherical Hecke category for* $\mathrm{PGL}_2^{\times 3}$.

Remark 1.3. Let $\mathrm{PSO}_{2n} := \mathrm{SO}_{2n}/\mu_2$. Then, the embedding $\mathrm{PGL}_2^{\mathrm{diag}} \subseteq \mathrm{PGL}_2^{\times 3}$ can be identified with the diagonal embedding $\mathrm{SO}_3 \subseteq \mathrm{SO}_3 \times \mathrm{PSO}_4$; and similarly, the action of $\mathrm{SL}_2^{\times 3}$ on $\mathrm{std}^{\otimes 3}$ can be identified with the action of $\mathrm{Spin}_4 \times \mathrm{Sp}_2$ on the tensor product of their respective defining representations. From this perspective, Theorem 1.2 could be viewed as a special case of the geometrized analogue of the Gan-Gross-Prasad period (or at least a period isogenous to it).

A similar argument shows a variant for PSO_8. Namely, there is an embedding $G_2 \subseteq \mathrm{PSO}_8$ given by triality, which exhibits G_2 as a spherical subgroup of PSO_8. To see that this situation is analogous to that of Theorem 1.2, note that the Dynkin diagram • of A_1 is obtained from the Dynkin diagram • • • of $A_1^{\times 3}$ by folding with respect to the obvious action of the symmetric group Σ_3. In the same way, the Dynkin diagram of G_2 is obtained from the Dynkin diagram  of D_4 by folding with respect to the action of Σ_3 permuting the three vertices around the branching vertex.

Theorem 1.4 (Derived geometric Satake for PSO_8/G_2)**.** *Suppose that the* $\mathrm{PSO}_8[\![t]\!]$*-action on* $\mathrm{PSO}_8(\!(t)\!)/G_2(\!(t)\!)$ *is optimal in the sense of* [**Dev23**, Hypothesis 3.5.22]. *Then there is an equivalence*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PSO}_8[\![t]\!]}(\mathrm{PSO}_8(\!(t)\!)/G_2(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(12, \vec{6}, -6, \vec{0})/\mathrm{SL}_2(-6\rho)^{\times 3} \times \mathbf{A}^1(4)).$$

In other words, the spherical subgroups $\mathrm{PGL}_2^{\times 2} \subseteq \mathrm{PGL}_2^{\times 4}$ (given by $(g, h) \mapsto (g, g, g, h)$) and $G_2 \subseteq \mathrm{PSO}_8$ have the same dual quotient stacks (namely, $(\mathrm{std})^{\otimes 3}/\mathrm{SL}_2^{\times 3} \times \mathbf{A}^1$) up to grading. Therefore, they fit into the paradigm of [**Dev23**, Remark 4.1.5].

The proofs of Theorem 1.2 and Theorem 1.4 reduce to showing that the conditions of [**Dev23**, Theorem 3.5.24] are met. This ultimately relies on studying Bhargava's construction from [**Bha04**] relating $2 \times 2 \times 2$-matrices to quadratic forms, and the work [**GKZ94**] of Gelfand-Kapranov-Zelevinsky describing the relationship to Cayley's hyperdeterminant.

Remark 1.5. The arguments of this article should continue to hold if one considers sheaves with coefficients in $\mathbf{Z}[i, \frac{1}{\sqrt{2}}]$; we have not checked this explicitly, but it seems likely to be true. In fact, we expect that the results of this article should continue to hold for sheaves with coefficients in $\mathbf{Z}$ itself. This, however, is a rather more subtle question: the prime 2 is an interesting one (see Remark 2.7).

More generally, following the philosophy of [**Dev23**], it should also be possible to use a variant of the methods of this article to prove analogues of Theorem 1.2 and Theorem 1.4 for sheaves with coefficients in connective complex K-theory ku. We have not attempted to do this, but we expect the corresponding 1-parameter deformation of $\mathrm{std}^{\otimes 3}$ over $\pi_*(\mathrm{ku}) \cong \mathbf{Z}[\beta]$ to be a rather interesting ku-Hamiltonian $\mathrm{SL}_2^{\times 3}$-variety.

[1] The ∞-category on the left-hand side is as in [**Dev23**, Definition 3.5.15]; see Definition 3.1 for a quick review.

Remark 1.6. The equivalence of Theorem 1.2 can heuristically be viewed as geometric Langlands for PGL_2 on the "doubled raviolo", obtained by gluing three formal disks along their common punctured disk. We expect Theorem 1.2 to be related to the work of [**MT12**].

Remark 1.7. The quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is also studied (in different language, of course) in quantum information theory; see Remark 2.15 below.

Theorem 1.2 and Theorem 1.4 are predicted by (the Betti version of) the local geometric conjecture of Ben-Zvi–Sakellaridis–Venkatesh; see [**BZSV23**, Conjecture 7.5.1]. My homotopy-theoretic interpretation of their conjecture is as follows. Suppose G is a reductive group over $\mathbf{C}$ and G/H is an affine homogeneous spherical G-variety (meaning that it admits an open B-orbit for its natural left $B \subseteq G$-action). Then, there should be a dual graded $\check{G}$-variety $\check{M}$ equipped with a moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$, and an equivalence of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{C}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ denotes the ∞-category of perfect complexes on the shearing of $\check{M}/\check{G}$ with respect to its given grading. In fact, [**BZSV23**, Section 4] gives an explicit construction of this predicted dual $\check{M}$, and in the examples $(G, H) = (\mathrm{PGL}_2^{\times 3}, \mathrm{PGL}_2^{\mathrm{diag}})$ and (PSO_8, G_2), one can compute that the stacky quotient $\check{M}/\check{G}$ is isomorphic to the right-hand sides of Theorem 1.2 and Theorem 1.4 respectively.[2]

Lest Theorem 1.2 seem like an oddly specific example to focus on, we note that it is essentially the *only* "new" example of a spherical pair (G, H) of the form $(H^{\times j}, H^{\mathrm{diag}})$, as shown by the following lemma.

Lemma 1.8. *Suppose H is a simple linear algebraic group over $\mathbf{C}$. Then the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical if and only if:*

(a) $j = 2$, and H arbitrary;
(b) $j = 3$ and H is of type A_1.

Proof. If the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical, there is an open H^{diag}-orbit on the flag variety of $H^{\times j}$. This implies that the dimension of H must be at least $j|\Phi^+|$, where Φ^+ is the set of positive roots; equivalently, one needs $\mathrm{rank}(H) \geq (j-2)|\Phi^+|$. Of course, this is always satisfied if $j = 2$ (this is the group case corresponding to the symmetric subgroup $H^{\mathrm{diag}} \subseteq H \times H$). Using the classification of simple linear algebraic groups over $\mathbf{C}$, it is easy to see that the only other case when the above inequality can hold is when $j = 3$ and H is of type A_1; one can then check by hand that the diagonal subgroup in this case is indeed spherical. ☐

In the first case of Lemma 1.8, [**BZSV23**, Conjecture 7.5.1] is precisely the derived geometric Satake equivalence of [**BF08**]. Therefore, the only other case of Lemma 1.8 is when H is simple of type A_1, and Theorem 1.2 precisely addresses [**BZSV23**, Conjecture 7.5.1] for the adjoint form PGL_2 of H.

[2] In the first case, this computation is straightforward given the prescription of [**BZSV23**, Section 4]; see [**Sak13**, Example 7.2.4] for a reference. The computation in the second case goes as follows. As in [**BZSV23**, Remark 7.1.1], the quotient stack $\check{M}/\check{G}$ can be identified with the quotient $\check{V}_X/\check{G}_X$, where $\check{G}_X$ is the Gaitsgory-Nadler/Sakellaridis-Venkatesh/Knop-Schalke dual group of X and $\check{V}_X$ is constructed in [**BZSV23**, Section 4.5]. In the case $X = \mathrm{PSO}_8/G_2$, a calculation shows that $\check{G}_X$ is the Levi subgroup of the maximal parabolic subgroup of PSO_8 corresponding to the central vertex of the D_4 Dynkin diagram; so $\check{G}_X \cong \mathrm{SL}_2^{\times 3}$. Using the prescription of [**BZSV23**, Section 4.5], one can check that $\check{V}_X \cong \mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1$, where $\check{G}_X$ acts only on the first factor. See, e.g., [**Sak13**, Line 9 of Table in Appendix A].

1.1. Acknowledgements. I am very grateful to Yiannis Sakellaridis for his support and help in answering my numerous questions about [**BZSV23**], to Akshay Venkatesh for interesting conversations, to Alison Miller for directing me (via MathOverflow) to the work of Bhargava, and to Charles Fu and Jesper Grodal for helpful suggestions.

2. Some properties of $\mathrm{std}^{\otimes 3}$

In this section, we establish some basic properties of $\mathrm{std}^{\otimes 3}$ as a $\mathrm{SL}_2^{\times 3}$-variety; our base field will always be k, and we will write $\check{G} = \mathrm{SL}_2^{\times 3}$. Some of this material appears in [**Bha04**]. In particular, Construction 2.3 is due to Bhargava.

Observation 2.1. An element $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2$ can be identified with a binary quadratic form $q_A(x, y) = cx^2 + 2iaxy + by^2$. Under this identification, the adjoint action of $g \in \mathrm{SL}_2$ on $\mathfrak{sl}_2$ is given by the action on (x, y) of the conjugate of g by the matrix $\mathrm{diag}(\zeta_8, \zeta_8^{-1})$. Explicitly, if $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, the action sends

$$\begin{aligned} x &\mapsto i\delta x + \beta y, \\ y &\mapsto \gamma x - i\alpha y. \end{aligned}$$

Note, moreover, that the discriminant of $q_A(x, y)$ is $4\det(A)$.

Warning 2.2. Note that under Observation 2.1, the element of $\mathfrak{sl}_2$ associated to a binary quadratic form $bx^2 + axy + cy^2$ is *not* the symmetric matrix associated to the quadratic form! Indeed, the associated symmetric matrix is $\left(\begin{smallmatrix} b & a/2 \\ a/2 & c \end{smallmatrix}\right)$, while the associated element of $\mathfrak{sl}_2$ is $\left(\begin{smallmatrix} -ai/2 & c \\ b & ai/2 \end{smallmatrix}\right)$.

Note, also, that we are relying quite heavily on the assumption that 2 is invertible in k. Over $\mathbf{Z}$, one can in fact identify the space of binary quadratic forms with the *coadjoint* representation $\mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$ of SL_2. Working over $\mathbf{Z}$ and keeping track of the difference between $\mathfrak{sl}_2$ and $\mathfrak{sl}_2^*$ has the effect of eliminating extraneous factors of 2 in our discussion below; but working over $\mathbf{Z}$ also introduces new complications (see Remark 2.7) which we do not wish to address in the present article.

Construction 2.3. The affine space $\mathbf{A}^8 = \mathrm{std}^{\otimes 3}$ can be regarded as parametrizing cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] & & d_3 \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & b_1 \arrow[ur, dash] \arrow[dd, dash] & \\
& d_1 \arrow[uu, dash] \arrow[rr, dash] & & c, \\
b_3 \arrow[ur, dash] \arrow[rr, dash] & & d_2 \arrow[ur, dash] &
\end{tikzcd}$$

which we will represent by a tuple $(a, \vec{b}, c, \vec{d})$; we will often use the symbol $\mathcal{C}$ to denote such a cube. If $\{e_1, e_2\}$ are a basis for std, the above cube corresponds to the element of $\mathrm{std}^{\otimes 3}$ given by

$$\begin{gathered} ae_1 \otimes e_1 \otimes e_1 + b_1 e_2 \otimes e_1 \otimes e_1 + b_2 e_1 \otimes e_2 \otimes e_1 + b_3 e_1 \otimes e_1 \otimes e_2 \\ + d_1 e_1 \otimes e_2 \otimes e_2 + d_2 e_2 \otimes e_1 \otimes e_2 + d_3 e_2 \otimes e_2 \otimes e_1 + c e_2 \otimes e_2 \otimes e_2. \end{gathered}$$

Associated to a cube $\mathcal{C}$ are three pairs of matrices, given by slicing along the top, leftmost, or front faces:

$$\begin{aligned} M_1 &= \left(\begin{smallmatrix} a & b_2 \\ b_3 & d_1 \end{smallmatrix}\right), N_1 = \left(\begin{smallmatrix} b_1 & d_3 \\ d_2 & c \end{smallmatrix}\right), \\ M_2 &= \left(\begin{smallmatrix} a & b_1 \\ b_3 & d_2 \end{smallmatrix}\right), N_2 = \left(\begin{smallmatrix} b_2 & d_3 \\ d_1 & c \end{smallmatrix}\right), \\ M_3 &= \left(\begin{smallmatrix} a & b_1 \\ b_2 & d_3 \end{smallmatrix}\right), N_3 = \left(\begin{smallmatrix} b_3 & d_2 \\ d_1 & c \end{smallmatrix}\right); \end{aligned}$$

each of these defines a binary quadratic form

$$q_i(x,y) = -\det(M_i x + N_i y).$$

Explicitly,

$$\begin{aligned} q_1(x,y) &= \det(M_1)x^2 + (ac + b_1d_1 - b_2d_2 - b_3d_3)xy + \det(N_1)y^2, \\ q_2(x,y) &= \det(M_2)x^2 + (ac - b_1d_1 + b_2d_2 - b_3d_3)xy + \det(N_2)y^2, \\ q_3(x,y) &= \det(M_3)x^2 + (ac - b_1d_1 - b_2d_2 + b_3d_3)xy + \det(N_3)y^2. \end{aligned}$$

Viewing $\mathfrak{sl}_2$ as the space of binary quadratic forms as in Observation 2.1, these three quadratic forms define a map

$$\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}.$$

An easy check shows that this map is $\check{G}$-equivariant.

Lemma 2.4 (Cayley). *The composite*

$$\mathrm{std}^{\otimes 3} \xrightarrow{\mu} \mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$$

factors through the diagonal inclusion $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$. In fact, the induced map $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ defines an isomorphism

$$\mathrm{std}^{\otimes 3}/\!\!/\check{G} \xrightarrow{\sim} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1/\!\!/(\mathbf{Z}/2).$$

PROOF. The map $\mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$ sends a triple of matrices to their determinants, or equivalently a triple of quadratic forms to their discriminants. Therefore, we need to check that the three quadratic forms of Construction 2.3 have the same discriminant. This is easy: one finds that their common discriminant is

$$\begin{aligned} \det(q_i) = a^2c^2 + b_1^2d_1^2 + b_2^2d_2^2 + b_3^2d_3^2 - 2(ab_1cd_1 + ab_2cd_2 + ab_3cd_3 \\ + b_1b_2d_1d_2 + b_1b_3d_1d_3 + b_2b_3d_2d_3) + 4(ad_1d_2d_3 + b_1b_2b_3c). \end{aligned} \tag{1}$$

It remains to show that the map $\mathrm{std}^{\otimes 3}/\!\!/\check{G} \to \mathbf{A}^1$ defined by this polynomial is an isomorphism. This is stated/proved in [**GKZ94**, Proposition 1.7 in Chapter 14], and is due to Cayley. □

Notation 2.5. Write $\det$ to denote the map $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ from Lemma 2.4, so that if $\mathcal{C}$ is a cube, $\det(\mathcal{C})$ is the quantity of (1).

Remark 2.6. The standard SL_2-equivariant symplectic structure on std defines an $\mathrm{SL}_2^{\times 3}$-equivariant symplectic structure on $\mathrm{std}^{\otimes 3}$. This action is Hamiltonian, and one can verify that the map $\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3} \cong (\mathfrak{sl}_2^*)^{\times 3}$ from Construction 2.3 is in fact the moment map for this $\mathrm{SL}_2^{\times 3}$-action. This gives a more "invariant" way to think about Bhargava's three quadratic forms. Along these lines, let us remark that [**Bha04**, Theorem 1] implies that the span

$$\begin{tikzcd} & \mathrm{std}^{\otimes 3} \arrow[dl, "\mu_1 \times \mu_2"'] \arrow[dr, "\mu_3"] & \\ \mathfrak{sl}_2 \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \mathfrak{sl}_2 & & \mathfrak{sl}_2 \end{tikzcd}$$

given by the moment maps *encodes* Gauss composition on quadratic forms, in the sense that given two (SL_2-orbits of) quadratic forms q_1 and q_2 with the same discriminant, the (SL_2-orbit of) the Gauss composition $-(q_1 + q_2)$ is given by $\mu_3((\mu_1 \times \mu_2)^{-1}(q_1, q_2))$.

Remark 2.7. Lemma 2.4 is not quite true over $\mathbf{F}_2$ (and hence not over $\mathbf{Z}$). One can already see the subtlety that arises over $\mathbf{F}_2$ from the formula (1): namely, the Cayley hyperdeterminant (appropriately normalized) admits a square root over $\mathbf{F}_2$. Explicitly, if $\mathcal{C} = (a, \vec{b}, c, \vec{d})$ is a cube and $\det(\mathcal{C})$ is defined by the formula (1), one has

$$\frac{\det(\mathcal{C})}{2} \equiv \frac{(ac + b_1 d_1 + b_2 d_2 + b_3 d_3)^2}{2} \pmod 2.$$

This means that the Cayley hyperdeterminant cannot possibly define an isomorphism $\mathrm{std}^{\otimes 3} /\!\!/ \check{G} \xrightarrow{\sim} \mathbf{A}^1$ over $\mathbf{F}_2$. Instead, we expect that there is an isomorphism $\mathrm{std}^{\otimes 3} /\!\!/ \check{G} \xrightarrow{\sim} \mathfrak{sl}_2^* /\!\!/ \mathrm{SL}_2$ (even over $\mathbf{Z}$!), where the double quotient on both sides denotes the *derived* invariant-theoretic quotient (i.e., $V /\!\!/ H = \operatorname{Spec} R\Gamma(BH; \mathrm{Sym}(V^*))$).

We will now define an analogue of the Kostant slice, as it will be needed to apply [**Dev23**, Theorem 3.5.24] (see [**Dev23**, Strategy 1.2.1(b)]). For the purposes of our discussion, one should view this Kostant section as an analogue of the construction of the companion matrix associated to a characteristic polynomial.

Construction 2.8. If n is an integer, let $\vec{n}$ denote the triple (n, n, n). Let

$$\kappa : \mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 \cong \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2) \cong \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$$

denote the map sending $a^2 \mapsto (a^2, \vec{0}, 0, \vec{1})$. This corresponds to the cube

$$\begin{tikzcd}
& 0 \arrow[rr, dash] \arrow[dd, dash] & & 1 \arrow[dd, dash] \\
a^2 \arrow[ur, dash] \arrow[rr, dash, crossing over] \arrow[dd, dash] & & 0 \arrow[ur, dash] & \\
& 1 \arrow[rr, dash] & & 0. \\
0 \arrow[ur, dash] \arrow[rr, dash] & & 1 \arrow[ur, dash] \arrow[uu, dash, crossing over] &
\end{tikzcd}$$

In this case, $\det(\kappa(a^2)) = 4a^2$, so that κ defines a section of $\det$ (at least up to the unit $4 \in k^\times$). The associated quadratic forms are all equal, and are given by

$$q_1(x, y) = q_2(x, y) = q_3(x, y) = a^2 x^2 - y^2,$$

which corresponds to the traceless matrix $\left(\begin{smallmatrix} 0 & -1 \\ a^2 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$. (Note that this is exactly the companion matrix associated to the characteristic polynomial $y^2 - a^2$.)

One of the key properties of the Kostant section/companion matrices is that a matrix $A \in \mathfrak{sl}_2$ is conjugate to $\kappa(\det(A))$ if and only if A is regular (i.e., the minimal polynomial of A agrees with its characteristic polynomial), if and only if A is nonzero. We will now prove an analogous result concerning $\kappa : \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$.

Proposition 2.9. *The $\check{G}$-orbit of the image of κ is a dense open subscheme whose complement has codimension* 3.

Proof. We will use the classification of $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$ as in [**GKZ94**, Example 4.5 in Chapter 14]; see Figure 1 for a graph of the seven orbits of $\check{G}$ on $\mathrm{std}^{\otimes 3}$. Namely, if $\lambda \neq 0$, all elements of $\det^{-1}(\lambda)$ are in a single $\check{G}$-orbit. (In fact, all elements in the fiber $\det^{-1}(1)$ are in the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.) The $\check{G}$-orbit of $\det^{-1}(\mathbf{G}_m)$ is open and dense, and hence is 8-dimensional; moreover, it agrees with the $\check{G}$-orbit of $\kappa(\mathbf{G}_m)$. Next, there is a maximal $\check{G}$-orbit inside the fiber $\det^{-1}(0)$, given by the orbit of $(0, \vec{0}, 0, \vec{1}) = \kappa(0)$. This orbit is 7-dimensional, and the largest $\check{G}$-orbits contained in the complement $\det^{-1}(0) - \check{G} \cdot \kappa(0)$ have dimension 5. In particular, the complement of $\check{G} \cdot \kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ has dimension 5, i.e., codimension $8 - 5 = 3$. $\square$

$$\begin{tikzcd}
& & (1,\vec{0},0,(1,0,0));5 \arrow[dr, no head] & & \\
\kappa(\mathbf{G}_m);8 \arrow[r, no head] & \kappa(0);7 \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & (1,\vec{0},0,(0,1,0));5 \arrow[r, no head] & (1,\vec{0},0,\vec{0});4 \arrow[r, no head] & (0,\vec{0},0,\vec{0});0 \\
& & (1,\vec{0},0,(0,0,1));5 \arrow[ur, no head] & &
\end{tikzcd}$$

FIGURE 1. $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$, representatives, and their dimensions (indicated after the semicolon), connected by closure. Note that $\kappa(0) = (0, \vec{0}, 0, \vec{1})$, and that the $\check{G}$-orbit of $\kappa(1) = (1, \vec{0}, 0, \vec{1})$ is the same as the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.

Remark 2.10. As explained in [**GKZ94**, Example 4.5 in Chapter 14], the closure of the associated orbits inside $\mathbf{P}(\mathrm{std}^{\otimes 3}) = \mathbf{P}^7$ can be described as follows. First, the closure of the generic orbit is $\mathbf{P}^7$. Next, the closure of the orbit of next smallest dimension is the zero locus of $\det$, which cuts out the dual variety of the Segre embedding $(\mathbf{P}^1)^{\times 3} \hookrightarrow \mathbf{P}^7$ (just as the usual determinant for 2×2-matrices cuts out the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$). The projective orbit associated to $(1, \vec{0}, 0, (0, 1, 0))$, say, is cut out inside the locus $\{\det = 0\}$ by the Segre embedding $\mathbf{P}(\mathrm{std}) \times \mathbf{P}(\mathrm{std}^{\otimes 2}) = \mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^7$. Finally, the minimal nonzero orbit is cut out by the Segre embedding $(\mathbf{P}^1)^{\times 3} \to \mathbf{P}^7$.

Proposition 2.11. *Let $\check{J}$ denote the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \operatorname{Spec} k[a^2]$ of regular centralizers for SL_2, so that*

$$\check{J} \cong \operatorname{Spec} k[a, \alpha^{\pm 1}, \tfrac{\alpha - \alpha^{-1}}{a}]^{\mathbf{Z}/2},$$

where the action of $\mathbf{Z}/2$ sends $a \mapsto -a$ and $\alpha \mapsto \alpha^{-1}$, and the group structure is such that α is grouplike. Then there is an isomorphism

$$\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J})$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 = \operatorname{Spec} k[a^2]$.

PROOF. The fiber product on the left identifies with the subgroup of $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times \check{G}$ of those $(a^2, \vec{g})$ such that $\vec{g} = (g_1, g_2, g_3) \in \mathrm{SL}_2^{\times 3}$ stabilizes $\kappa(a^2)$. The trick to determining this stabilizer is to use Bhargava's construction from Construction 2.3: if $\vec{g}$ stabilizes a cube $\mathcal{C}$, it must also stabilize the corresponding triple $\mu(\mathcal{C}) \in \mathfrak{sl}_2^{\times 3}$ of quadratic forms.

First, a simple calculation shows that if a is a unit, the triple of matrices

$$\vec{g} = \left(\sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & a^{-1} \\ a & 1 \end{pmatrix} \right) \in \mathrm{SL}_2^{\times 3}$$

sends

$$\kappa(a^2) \mapsto -\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0}).$$

The triple $\vec{g}$ can be thought of as "diagonalizing" $\kappa(a^2)$. The stabilizer of the cube $-\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0})$ precisely consists of triples of matrices of the form

$$\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3^{-1} \end{pmatrix} \right) \text{ with } \alpha_1\alpha_2\alpha_3 = 1. \tag{2}$$

For $\alpha \in \mathbf{G}_m$, let $h(\alpha)$ denote the matrix

$$h(\alpha) = \frac{1}{2} \begin{pmatrix} \alpha + \alpha^{-1} & \frac{\alpha^{-1} - \alpha}{a} \\ a^2 \cdot \frac{\alpha^{-1} - \alpha}{a} & \alpha + \alpha^{-1} \end{pmatrix} \in \mathrm{SL}_2.$$

Conjugating (2) by the element $\vec{g} \in \check{G}$, we find that the triple $(h(\alpha_1), h(\alpha_2), h(\alpha_3))$ of matrices stabilizes $\kappa(a^2)$ as long as $\alpha_1\alpha_2\alpha_3 = 1$ and $a^2 \in \mathbf{G}_m \subseteq \mathbf{A}^1$. (See [**BFM05**, Section 3.2] for a slight variant of this calculation.) Note that the subgroup of such triples is 2-dimensional, and therefore the associated homogeneous $\check{G}$-space is $9 - 2 = 7$-dimensional. Using that the $\check{G}$-orbit of $\kappa(a^2)$ is also 7-dimensional (e.g., by [**GKZ94**, Example 4.5 in Chapter 14]), it is not hard to see from this calculation (by a limiting argument for $a \to 0$) that the stabilizer of the family $\kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ is precisely the claimed group scheme. □

Remark 2.12. A direct calculation shows that the stabilizer of $\kappa(0)$ is isomorphic to the subgroup of triples of matrices of the form $\left(\begin{smallmatrix} a_i & \gamma_i \\ 0 & a_i^{-1} \end{smallmatrix}\right)$ for $1 \leq i \leq 3$ with $(a_1, a_2, a_3) \in \mu_2^{\times 3}$ such that $a_1a_2a_3 = 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = 0$. This subgroup is, of course, isomorphic to $(\mu_2 \times \mathbf{G}_a)^{\times 2}$; it is also isomorphic to the fiber over $a = 0$ of the group scheme of Proposition 2.11.

As in [**Bha04**], understanding the $\mathrm{SL}_2^{\times 3}$-equivariant geometry of cubes can be specialized to understand variant situations. To illustrate this, we sketch the following result, which gives an analogue of Lemma 2.4, Proposition 2.9, and Proposition 2.11 for the example of binary cubic forms. Note, however, that this example does *not* fit into the formalism of [**BZSV23**], since it is not hyperspherical in the sense of *loc. cit.* (see [**BZSV23**, Example 5.1.10]), so we do not have a Langlands dual description.

Corollary 2.13. *Let $V = \mathrm{Sym}^3(\mathbf{A}^2)$ denote the 4-dimensional affine space of binary cubic forms, so that V admits an action of SL_2. Then:*

(a) Let $\Delta : V \to \mathbf{A}^1$ denote the map sending a binary cubic form $f = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ to its discriminant

$$\Delta(f) = a^2d^2 - 6abcd - 3b^2c^2 + 4(ac^3 + b^3d).$$

Then Δ defines an isomorphism $V /\!\!/ \mathrm{SL}_2 \cong \mathbf{A}^1$.

(b) The closed immersion $\kappa : \mathbf{A}^1 \to V$ sending $a \mapsto -\frac{a}{4}x^3 + 3xy^2$ defines a section of Δ, and the SL_2-orbit of the image of κ has complement of codimension ≥ 2.

(c) Identify $\mathbf{A}^1 = \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$, let $\check{J}$ denote the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ of regular centralizers for SL_2, and let $\check{J}[3]$ denote its 3-torsion subgroup. Then there is an isomorphism

$$\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \check{J}[3]$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. In particular, the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[3]$ is SL_2-equivariantly isomorphic to V.

PROOF. The first statement is in [**PV89**, Section 0.12], and the second statement can be deduced similarly. For the final statement, recall as in [**Bha04**] that there is a closed immersion $V \subseteq \mathrm{std}^{\otimes 3}$ given by $ax^3 + 3bx^2y + 3cxy^2 + dy^3 \mapsto (a, \vec{b}, d, \vec{c})$. This embedding is SL_2-equivariant for the natural action on V and the diagonally embedded $\mathrm{SL}_2^{\mathrm{diag}} \subseteq \mathrm{SL}_2^{\times 3} = \check{G}$ acting on $\mathrm{std}^{\otimes 3}$. Moreover, the composite

$$V \subseteq \mathrm{std}^{\otimes 3} \xrightarrow{\det} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$$

sends $f \mapsto \Delta(f)$. This implies that $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ can be identified with the intersection $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ with the diagonally embedded $\mathrm{SL}_2^{\mathrm{diag}} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \subseteq \mathrm{SL}_2^{\times 3} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. By Proposition 2.11, we find that

$$\begin{aligned} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 &\cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J}) \cap (\mathrm{SL}_2^{\mathrm{diag}} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2) \\ &\cong \check{J}[3]. \end{aligned}$$

The claim about the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[3]$ follows from (b). □

Remark 2.14. There is a variant of Corollary 2.13 for binary quartic forms; since we will not need this result, and the proof is somewhat orthogonal to the methods of this article, we will simply state the relevant facts. Let $V = \mathrm{Sym}^4(\mathbf{A}^2)$ denote the 5-dimensional affine space of binary quartic forms, so that V admits an action of PGL_2.

(a) Let $\pi : V \to \mathbf{A}^2$ denote the map sending a binary quartic form $f = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^3$ to the invariants

$$I = ae - 4bd + 3c^2,$$
$$J = ace + 2bcd - ad^2 - b^2e - c^3.$$

Then π defines an isomorphism $V/\!\!/\mathrm{PGL}_2 \cong \mathbf{A}^2$.

(b) The closed immersion $\kappa : \mathbf{A}^2 \to V$ sending $(a, b) \mapsto 4x^3y + dxy^3 + ey^4$ defines a section of π, and the PGL_2-orbit of the image of κ has complement of codimension ≥ 2. In fact, the PGL_2-orbit consists of those binary quartic forms with at least one root of multiplicity 1.

(c) Let $\mathcal{E}$ denote the elliptic curve over $\mathbf{A}^2 = \mathrm{Spec}\, k[d, e] \cong \mathfrak{sl}_3/\!\!/\mathrm{SL}_3$ given by $y^2 = x^3 + dx + e$, and let $\mathcal{E}[2]$ denote its 2-torsion subgroup. Then there is an isomorphism

$$\mathbf{A}^2 \times_{V/\mathrm{PGL}_2} \mathbf{A}^2 \cong \mathcal{E}[2]$$

of group schemes over $\mathbf{A}^2$. In particular, the affine closure of $(\mathrm{PGL}_2 \times \mathbf{A}^2)/\mathcal{E}[2]$ is PGL_2-equivariantly isomorphic to V.

Parts (a) and (b) are not difficult calculations, and part (c) can be proved as in [**CF09**, Sections 3-5] and [**BS15**, Theorem 3.2]. Finally, we note that if $V = \mathrm{Sym}^j(\mathbf{A}^2)$ denote the $(j+1)$-dimensional affine space of binary j-forms, so that V admits an action of SL_2, the invariant-theoretic quotient $V/\!\!/\mathrm{SL}_2$ is *not* an affine space if $j \geq 5$ (see [**PV89**, Example 1 in Section 8.2]).

Remark 2.15. As mentioned in Remark 1.7, the quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is studied in quantum information theory. For instance, in [**DVC00**], Dür-Vidal-Cirac study the orbit structure of $\mathrm{SL}_2^{\times 3}$ acting on $\mathrm{std}^{\otimes 3}$ (in particular, they recover Figure 1 independently of [**GKZ94**]). For the interested reader, let us describe the translation between our notation/terminology and that of quantum information theory. Our base field will now be $k = \mathbf{C}$. An element of $\mathrm{std}^{\otimes n}$ (really, of the projective space $\mathbf{P}(\mathrm{std}^{\otimes n}) \cong \mathbf{P}^{2^n-1}$) is called an *$n$-qubit*, and the action of $\mathrm{SL}_2^{\times n}$ is via *stochastic local operations and classical communication* (SLOCC) operators (replacing $\mathrm{SL}_2^{\times n}$ by $\mathrm{GL}_2^{\times n}$ simply amounts to dropping the word "stochastic"). The space std is equipped with a basis $\{|0\rangle, |1\rangle\}$, and a cube $\mathcal{C} = (a, \vec{b}, c, \vec{d}) \in \mathrm{std}^{\otimes 3}$ corresponds to the three-qubit[3]

$$a|000\rangle + b_1|100\rangle + b_2|010\rangle + b_3|001\rangle$$
$$+d_1|011\rangle + d_2|101\rangle + d_3|110\rangle + c|111\rangle.$$

Here, the bra-ket notation $|ijk\rangle$ means $|i\rangle \otimes |j\rangle \otimes |k\rangle$. The state

$$\tfrac{1}{\sqrt{2}}(1, \vec{0}, 1, \vec{0}) = \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

is known as the *Greenberger–Horne–Zeilinger* (GHZ) state, and the state

$$\tfrac{1}{\sqrt{3}}\kappa(0) = \tfrac{1}{\sqrt{3}}(0, \vec{1}, 0, \vec{0}) = \tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

is called the *W* state. These two states are known to represent two very different kinds of quantum entanglement; from the perspective of this article, the reason for this is simply that the Cayley

[3]Technically, a qubit is required to have norm 1, so one must rescale $\mathcal{C}$ by $\sqrt{a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2}$; but this could in theory introduce a singularity when $a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2 = 0$. We will ignore this point below.

hyperdeterminant of the GHZ state is nonzero, but the Cayley hyperdeterminant of the W state vanishes. Nevertheless, the proof of Proposition 2.11 shows that there is a natural *degeneration* of (the SLOCC/$\mathrm{SL}_2^{\times 3}$-equivalence class of) the GHZ state into the W state. Indeed, the GHZ state can be transformed into the cube $\frac{1}{2}\kappa(1)$, which admits a natural degeneration to the W state via the one-parameter family

$$\tfrac{1}{\sqrt{a^4+3}}\kappa(a^2) = \tfrac{1}{\sqrt{a^4+3}}\left(a^2|000\rangle + |011\rangle + |101\rangle + |110\rangle\right).$$

In fact, this state already appears as [**DVC00**, Equation 20].

Remark 2.16. Fix an integer n. Then the $\check{G}$-variety $\mathrm{std}^{\otimes 3}$ admits a natural grading, where the entries of a cube $(a, \vec{b}, c, \vec{d})$ have the following weights: a lives in weight $-4n$, b lives in weight $-2n$, c lives in weight $2n$, and d lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the associated graded variety. Equip $\mathfrak{sl}_2$ with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ have the following weights: a lives in weight $-2n$, b lives in weight 0, and c lives in weight $-4n$. Similarly, equip SL_2 with the grading coming from $2n\rho$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. With these gradings, the $\mathrm{SL}_2^{\times 3}$-equivariant map $\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}$ is a *graded* map, and κ defines a graded map $\mathfrak{sl}_2(2n)/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1(4n) \to \mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$. The cases $n = 1$ and $n = 3$ will be relevant below (corresponding to Theorem 1.2 and Theorem 1.4, respectively).

3. The proof

Before proceeding, let us remind the reader of the definition of the left-hand side of the equivalence of Theorem 1.2, following [**Dev23**, Definition 3.5.15].

Definition 3.1. Let G be a compact Lie group, and let $H \subseteq G$ be a closed subgroup such that $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a spherical subgroup. Let $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$ denote the ∞-category of $G[\![t]\!]$-equivariant sheaves of $\mathbf{Q}$-modules on $G(\!(t)\!)/H(\!(t)\!)$ which are constructible for the orbit stratification on $G(\!(t)\!)/H(\!(t)\!)$. Note that since the orbit stratification is countable (by assumption that $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a spherical subgroup and [**GN10**, Theorem 3.2.1]), the ∞-category $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$ is well-behaved. There is a natural left-action of the $\mathbf{E}_3$-monoidal ∞-category $\mathrm{Shv}^c_{(G\times G)[\![t]\!]}(G(\!(t)\!); \mathbf{Q})$ on $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$, and in particular, a left-action of $\mathrm{Rep}(\check{G})$ by the abelian geometric Satake theorem of [**MV07**]. Let

$$\mathrm{IC}_0 \in \mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$$

denote the pushforward $i_!\underline{\mathbf{Q}}$ of the constant sheaf along the inclusion $(G/H)(\mathbf{C}[\![t]\!]) \to (G/H)(\mathbf{C}(\!(t)\!))$. Let

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q}) \subseteq \mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$$

denote the full subcategory generated by IC_0 under the action of $\mathrm{Rep}(\check{G})$. If k is any $\mathbf{Q}$-algebra, base-changing along the unit map defines the ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k)$.

Proof of Theorem 1.2. It suffices to verify conditions (a) and (b) of [**Dev23**, Theorem 3.5.24], which gives a criterion for establishing an equivalence of k-linear ∞-categories of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}).$$

The map κ is given by the map $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \to \mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})$ from Construction 2.8. For condition (a) of [**Dev23**, Theorem 3.5.24], we need to show that if $\check{J}_X = \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})/\check{G}} \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2$, the ring of regular functions on the quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ is isomorphic (as a graded algebra) to $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$. The quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ identifies with the $\check{G}$-orbit of the image of κ, which has complement of codimension 3 in $\mathrm{std}^{\otimes 3}$ by Proposition 2.9; therefore, the algebraic Hartogs theorem implies that there is a graded isomorphism $\mathcal{O}_{(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2\times\check{G})/\check{J}_X} \cong \mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$.

For condition (b) of [**Dev23**, Theorem 3.5.24], we need to check that there is an isomorphism

$$\check{J}_X \cong \operatorname{Spec} \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k)$$

of graded group schemes over $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \cong \operatorname{Spec} \mathrm{H}^*_{\mathrm{PGL}_2}(*; k)$. There is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k) \cong \operatorname{Spec} k[a, \alpha^{\pm 1}, \tfrac{\alpha-\alpha^{-1}}{a}]^{\mathbf{Z}/2} \cong \check{J}, \tag{3}$$

and the action of the $\mathbf{Z}/2$ on the middle term sends $a \mapsto -a$ and $\alpha \mapsto \alpha^{-1}$. This is proved, e.g., in [**BFM05**], and also follows from [**Dev23**, Example 3.6.16]. The Künneth theorem implies that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}); k) \cong \operatorname{Spec} k[a, \alpha_i^{\pm 1}, \tfrac{\alpha_i-\alpha_i^{-1}}{a} | 1 \le i \le 3]^{\mathbf{Z}/2},$$

and the fiber sequence

$$\mathrm{PGL}_2^{\mathrm{diag}} \to \mathrm{PGL}_2^{\times 3} \to \mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$$

implies that

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k) \cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J}).$$

The desired isomorphism now follows from this observation and Proposition 2.11. □

Remark 3.2. The proof of Theorem 1.4 is exactly the same as the proof of Theorem 1.2 above. Indeed, one only needs to observe that PSO_8/G_2 is homotopy equivalent to $\mathbf{R}P^7 \times \mathbf{R}P^7$ (which follows, e.g., from the fact that $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$)[4]. The replacement of (3) is given by [**Dev23**, Proposition 4.8.6], which gives an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{G_2}(\Omega\mathbf{R}P^7; k) \cong \operatorname{Spec} k[a, b, \alpha^{\pm 1}, \tfrac{\alpha-\alpha^{-1}}{a}]^{\mathbf{Z}/2}$$

where a is in weight -6 and b is in weight -4.

Remark 3.3. Remark 2.6 guarantees that the equivalence of Theorem 1.2 is compatible with the action of the spherical Hecke category $\mathrm{Shv}^{c,\mathrm{Sat}}_{(\mathrm{PGL}_2^{\times 3}\times\mathrm{PGL}_2^{\times 3})[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2-2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho))$. Namely, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}^{c,\mathrm{Sat}}_{(\mathrm{PGL}_2^{\times 3}\times\mathrm{PGL}_2^{\times 3})[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!); k) \arrow[r, "\sim", "\text{[BF08]}"'] \arrow[d, "\text{act on } \mathrm{IC}_0"] & \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2-2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho)) \arrow[d, "\mu^*"] \\
\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \arrow[r, "\sim", "\text{Theorem 1.2}"'] & \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})/\mathrm{SL}_2^{\times 3}(-2\rho)),
\end{tikzcd}$$

---

[4] Perhaps the most "conceptual" way to see that $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$ is as follows. Using triality, one can identify Spin_8 with the subgroup of $\mathrm{SO}_8^{\times 3}$ of those triples (A_1, A_2, A_3) such that $A_1(x_1)A_2(x_2) = A_3(x_1 x_2)$ for octonions x_1, x_2. Under this presentation, G_2 corresponds to the subgroup where $A_1 = A_2 = A_3$. The subgroups where $A_1 = A_3$ (resp. $A_2 = A_3$) are both isomorphic to $\mathrm{Spin}(7)$; these are sometimes denoted $\mathrm{Spin}^{\pm}(7)$. The action of Spin_8 on $S^7 \times S^7$ sends $(x, y) \mapsto (A_1 x, A_2 y)$; one can check that this is transitive, and that the stabilizer of the point $(1, 1)$ is precisely $\mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7) \cong G_2$.

That there is an equivalence $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$ at the level of cohomology with $\mathbf{Z}[1/2]$-coefficients, at least, is much simpler: on group cohomology, the map $G_2 \to \mathrm{Spin}_8$ is given by the map $\mathbf{Z}[1/2, p_1, p_2, p_3, c_4] \to \mathbf{Z}[1/2, c_2, c_6]$ sending $p_1 \mapsto -c_2, p_2 \mapsto 0, p_3 \mapsto -c_6$, and $c_4 \mapsto 0$. The Serre spectral sequence for the fibration $\mathrm{Spin}_8/G_2 \to BG_2 \to B\mathrm{Spin}_8$ implies that $\mathrm{H}^*(\mathrm{Spin}_8/G_2; \mathbf{Z}[1/2]) \cong \mathbf{Z}[1/2, \sigma(p_2), \sigma(c_4)]/(\sigma(p_2)^2, \sigma(c_4)^2)$, where $\sigma(p_2)$ and $\sigma(c_4)$ both live in (homological) weight -7. This is precisely the cohomology of $S^7 \times S^7$, as desired.

where μ^* is given by pullback along the moment map for the Hamiltonian $\mathrm{SL}_2^{\times 3}$-action on $\mathrm{std}^{\otimes 3}$.

Remark 3.4. Theorem 1.2 does not need the full strength of optimality in the sense [**Dev23**, Hypothesis 3.5.22]. Indeed, the first and second assumptions in [**Dev23**, Hypothesis 3.5.22] are included to ensure formality of the algebra from [**Dev23**, Equation 16 in the proof of Theorem 3.5.24]. However, as in [**Dev23**, Remark 3.2.22], the formality of this algebra is *guaranteed* in our case: since Theorem 1.2 shows that the homotopy of the algebra in question is $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$, i.e., is polynomial on classes in even weights. This algebra admits an $\mathbf{E}_3$-structure (essentially from factorization; see, e.g., [**BZSV23**, Proposition 16.1.4]), and is therefore automatically formal by [**Dev23**, Lemma 2.1.9]. Note, however, that since $\mathrm{Ind}_{\mathrm{SL}_2^{\times 3}}^{\mathrm{Spin}_8}(\mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1)$ is not an affine space, this argument does not go through in the case of Theorem 1.4 to prove formality of the algebra from [**Dev23**, Equation 16 in the proof of Theorem 3.5.24].

Remark 3.5. Let $k = \mathbf{Q}_2(\zeta_8)$. The theory of 2-compact groups as studied, e.g., in [**AG09**], suggests viewing the Dwyer-Wilkerson space DW_3 from [**DW93**] as an analogue of the groups $\mathrm{SO}_3 \cong \mathrm{PGL}_2$ and G_2; see Table 1. The 2-complete space DW_3 is equipped with an $\mathbf{E}_1$-structure, and it has finite mod 2 cohomology. It is therefore natural to ask whether there is an analogue of Theorem 1.2 and Theorem 1.4, where PGL_2 and G_2 are replaced by DW_3; this is closely related to [**Dev23**, Appendix C(p)].

Group G_n	Rank n	Dimension $(2^{n+1}-1)n$	$\mathbf{F}_2$-cohomology of BG $\widehat{\mathrm{Sym}}^*(\mathbf{F}_2^{n+1}(-1))^{\mathrm{GL}_{n+1}(\mathbf{F}_2)}$	Weyl group $\mathbf{Z}/2 \times \mathrm{GL}_n(\mathbf{F}_2)$
PGL_2	1	3	$\mathbf{F}_2[\![w_2, w_3]\!]$	$\mathbf{Z}/2$
G_2	2	14	$\mathbf{F}_2[\![w_4, w_6, w_7]\!]$	$\mathbf{Z}/2 \times \Sigma_3$
DW_3	3	45	$\mathbf{F}_2[\![w_8, w_{12}, w_{14}, w_{15}]\!]$	$\mathbf{Z}/2 \times \mathrm{PSL}_2(\mathbf{F}_7)$

TABLE 1. Analogies between the (2-compact) groups $\mathrm{PGL}_2 = \mathrm{SO}_3$, G_2, and DW_3; all of these are Poincaré duality complexes of dimension indicated in the third column. Here, w_n denotes the nth Stiefel-Whitney class, and the ring in the fourth column is known as the algebra of rank $n+1$ Dickson invariants. Note, also, that the Weyl group of DW_3 is called G_{24} in the Shephard-Todd classification.

It is difficult to answer this question since the representation theory of DW_3 is not well-understood. For instance, one can ask whether there is a 2-compact group G with an $\mathbf{E}_1$-map $\mathrm{DW}_3 \to G$ such that $G/\mathrm{DW}_3 \simeq \mathbf{R}P^{15} \times \mathbf{R}P^{15}$ (analogous to the equivalences $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2 \cong \mathbf{R}P^3 \times \mathbf{R}P^3$ and $\mathrm{PSO}_8/G_2 \cong \mathbf{R}P^7 \times \mathbf{R}P^7$). If such a G exists, and there is a good theory of $G[\![t]\!]$-equivariant sheaves of k-modules, it seems reasonable to expect that there is an equivalence of the form

$$\mathrm{Shv}_{G[\![t]\!]}^{c,\mathrm{Sat}}(G(\!(t)\!)/\mathrm{DW}_3(\!(t)\!); k) \cong \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(28, \vec{14}, -14, \vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3} \times \mathbf{A}^2(8, 12)).$$

Here, the "Whittaker" factor $\mathbf{A}^2(8, 12)$ on the right-hand side comes from the isomorphism

$$\mathrm{Spf}\,\mathrm{H}^*(B\mathrm{DW}_3; k) := \mathrm{Spf}\,\mathrm{H}^*(B\mathrm{DW}_3; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong \widehat{\mathbf{A}}^3(8, 12, 28),$$

which follows from running the Bockstein spectral sequence on

$$\mathrm{H}^*(B\mathrm{DW}_3; \mathbf{F}_2) \cong \mathbf{F}_2[\![w_8, w_{12}, w_{14}, w_{15}]\!],$$

and the fact that the Bockstein sends $w_{14} \mapsto w_{15}$. One can check that such a G, if it existed, would have rational cohomology given by

$$\mathrm{H}^*(BG; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong k[\![c_4, c_6, c_{14}, x, y]\!],$$

where both x and y live in cohomological degree 16.

In an email, Jesper Grodal told me that such a G cannot exist (it would have to be the 2-completion of a compact Lie group, but no compact Lie group has the desired cohomology). Despite this, one can wonder about the analogue of the "regular centralizer" group scheme calculation from Theorem 1.2:

- Is there a good notion of genuine *equivariant* DW_3-cohomology (with coefficients in $k = \mathbf{Q}_2(\zeta_8)$, say)? One should have $\mathrm{Spec}\,\mathrm{H}^*_{\mathrm{DW}_3}(*; k) \cong \mathbf{A}^3(8, 12, 28)$.
- Is there a faithful (basepoint-preserving) action of DW_3 on S^{15}, which descends to an action of DW_3 on (the 2-completion of) $\mathbf{R}P^{15}$?
- For the above expected action, is there an isomorphism
$$\mathrm{Spec}\,\mathrm{H}_*^{\mathrm{DW}_3}(\Omega(\mathbf{R}P^{15}\times\mathbf{R}P^{15}); k) \cong \mathbf{A}^2(8,12)\times(\mathbf{A}^1(28)\times_{\mathrm{std}^{\otimes 3}(28,\vec{14},-14,\vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3}}\mathbf{A}^1(28))$$
of graded group schemes over k?

References

[AG09] K. Andersen and J. Grodal. The classification of 2-compact groups. *J. Amer. Math. Soc.*, 22(2):387–436, 2009.

[BF08] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005.

[Bha04] M. Bhargava. Higher composition laws. I. A new view on Gauss composition, and quadratic generalizations. *Ann. of Math. (2)*, 159(1):217–250, 2004.

[BS15] M. Bhargava and A. Shankar. Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves. *Ann. of Math. (2)*, 181(1):191–242, 2015.

[BZSV23] D. Ben Zvi, Y. Sakellaridis, and A. Venkatesh. Relative Langlands duality. `https://www.math.ias.edu/~akshay/research/BZSVpaperV1.pdf`, 2023.

[CF09] J. Cremona and T. Fisher. On the equivalence of binary quartics. *J. Symbolic Comput.*, 44(6):673–682, 2009.

[Dev23] S. Devalapurkar. ku-theoretic spectral decompositions for spheres and projective spaces. `https://sanathdevalapurkar.github.io/files/hyperboloid_spectral_decomp.pdf`, 2023.

[DVC00] W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A (3)*, 62(6):062314, 12, 2000.

[DW93] W. G. Dwyer and C. W. Wilkerson. A new finite loop space at the prime two. *J. Amer. Math. Soc.*, 6(1):37–64, 1993.

[GKZ94] I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[GN10] D. Gaitsgory and D. Nadler. Spherical varieties and Langlands duality. *Mosc. Math. J.*, 10(1):65–137, 271, 2010.

[HK91] M. Harris and S. Kudla. The central critical value of a triple product L-function. *Ann. of Math. (2)*, 133(3):605–672, 1991.

[Ich08] A. Ichino. Trilinear forms and the central values of triple product L-functions. *Duke Math. J.*, 145(2):281–307, 2008.

[Lur15] J. Lurie. Rotation invariance in algebraic K-theory. `https://www.math.ias.edu/~lurie/papers/Waldhaus.pdf`, 2015.

[MT12] G. Moore and Y. Tachikawa. On 2d TQFTs whose values are holomorphic symplectic varieties. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 191–207. Amer. Math. Soc., Providence, RI, 2012.

[MV07] I. Mirkovic and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[PV89] V. Popov and E. Vinberg. Invariant theory. In *Algebraic geometry, 4 (Russian)*, Itogi Nauki i Tekhniki, pages 137–314, 315. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.

[Rak20] A. Raksit. Hochschild homology and the derived de Rham complex revisited. `https://arxiv.org/abs/2007.02576`, 2020.

[Sak13] Y. Sakellaridis. Spherical functions on spherical varieties. *Amer. J. Math.*, 135(5):1291–1381, 2013.

1 OXFORD ST, CAMBRIDGE, MA 02139

Email address: `sdevalapurkar@math.harvard.edu, December 9, 2023`