TRIVIAL NOTIONS TALK: MIRROR SYMMETRY FOR TORIC VARIETIES

Homological mirror symmetry is a supply of incredibly rich ideas relating algebraic geometry and symplectic geometry, first suggested by Kontsevich in his famous 1994 ICM address. Unfortunately, giving the usual form of the statement would requires us to introduce the Fukaya category, which I don't understand. However, if M is a smooth manifold, then (some version of) the Fukaya category of its cotangent bundle T^*M is supposed to be related to the category $\mathrm{Cbl}(M; \mathbf{C})$ of constructible sheaves of $\mathbf{C}$-vector spaces on M (this is made precise in [GPS18, Theorem 1.1]). So, for the purposes of this talk, we will adopt a more simplistic perspective on homological mirror symmetry: we will view it as a bridge between the category $\mathrm{QCoh}(X)$ for an algebraic variety X and the category $\mathrm{Cbl}(X^\vee; \mathbf{C})$ on a "dual" complex manifold $X^\vee$.

In this talk, we will explain a simple example of particularly well-behaved case of such a bridge, called the *coherent-constructible correspondence* (proved in [NZ09, Nad09, FLTZ11a, FLTZ11b, FLTZ11c]). I accidentally stumbled upon a baby case of this result when writing up a paper, and asked Ben Gammage about it, who referred me to the above papers. Since I'm no expert in mirror symmetry, I thought I could tell you how I bumped into this fascinating collection of ideas; hopefully my lack of experience translates into the talk being more elementary/accessible. An appropriate subtitle for this talk could be "1-dimensional topology is combinatorics".

To explain my starting point, I want to begin with a simple example. Let's consider the much simpler subcategory of local systems on $X^\vee$. The simplest interesting case is $X^\vee = S^1$. It's a classical fact that local systems of $\mathbf{C}$-vector spaces on S^1 are determined by their monodromy $\pi_1(S^1) \cong \mathbf{Z} \to \mathrm{GL}(V)$ with $V \in \mathrm{Vect}_{\mathbf{C}}$. At the level of categories, this says that $\mathrm{Loc}(S^1; \mathbf{C}) \simeq \mathrm{Mod}(\mathbf{C}[\mathbf{Z}])$. Since $\mathbf{C}[\mathbf{Z}]$ is the ring of functions on the affine group scheme $\mathbf{G}_{m,\mathbf{C}}$ over $\mathbf{C}$, we see that $\mathrm{Loc}(S^1; \mathbf{C}) \simeq \mathrm{QCoh}(\mathbf{G}_{m,\mathbf{C}})$. More generally, if T is a torus, then we have an equivalence $\mathrm{Loc}(T; \mathbf{C}) \simeq \mathrm{QCoh}(\check{T}_{\mathbf{C}})$, where $\check{T}_{\mathbf{C}} = \mathrm{Spec}\, \mathbf{C}[\pi_1(T)]$. (This equivalence is even symmetric monoidal, where the tensor product of quasicoherent sheaves on $\check{T}_{\mathbf{C}}$ corresponds to convolution of local systems on T.) What happens if we take local systems on T with coefficients in a different field, like $\mathbf{F}_p$? It is easy to see that the same argument gives an equivalence $\mathrm{Loc}(T; \mathbf{F}_p) \simeq \mathrm{QCoh}(\check{T}_{\mathbf{F}_p})$, where $\check{T}_{\mathbf{F}_p} = \mathrm{Spec}\, \mathbf{F}_p[\pi_1(T)]$. This can be interpreted as a particularly simple case of the aforementioned bridge between the coherent and constructible worlds. It is useful to view the above equivalence as broken down into two steps:

- There is an equivalence $\mathrm{Loc}(T; \mathbf{C}) \simeq \mathrm{Fun}(B\pi_1(T), \mathrm{Vect}_{\mathbf{C}})$, where $B\pi_1(T)$ is the one-object category whose morphisms are given by $\pi_1(T)$.
- There is an equivalence $\mathrm{Fun}(B\pi_1(T), \mathrm{Vect}_{\mathbf{C}}) \simeq \mathrm{QCoh}(\check{T}_{\mathbf{C}})$.

There are (at least) two takeaways from this baby example:

(a) If one considers local systems on $X^\vee = T$ with coefficients in some ring k, then the corresponding algebraic object X should be a k-scheme. In other words, the coefficients on the constructible side is the base on the coherent side.
(b) All the topological data of the torus T is determined by the combinatorial object $\pi_1(T)$, which can be used to build the algebraic object $\check{T}$ which "controls" the topology of T.

Part (a) can be turned on its head: it can be viewed as giving a large collection of examples of "nice" algebraic varieties over k. This is a particularly useful perspective if k is a more exotic sort of ring (like an $\mathbf{E}_\infty$-ring, an example of which is the sphere spectrum).

Let me now turn to the way in which I stumbled upon this circle of ideas. Let's fix a base ring k throughout, and write $*$ to denote $\mathrm{Spec}(k)$. In the process of writing up a paper, I had the good fortune to encounter the quotient stack $\mathbf{A}^1/\mathbf{G}_m$, where $\mathbf{G}_m$ acts on $\mathbf{A}^1$ by scaling. If you're not familiar with stacks, that's OK: all we will care about today is the category $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$, which can be understood as the category of $\mathbf{G}_m$-equivariant quasicoherent sheaves on $\mathbf{A}^1$. (Quasicoherent sheaves on $\mathbf{A}^1$ do have an alternative simpler description, but let me belabor the point a bit.) Explicitly, an object of $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ is specified by a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a quasicoherent sheaf on $\mathbf{A}^1$, and α is an isomorphism $\sigma^*\mathcal{F} \cong \mathrm{pr}^*\mathcal{F}$, where $\sigma : \mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1$ is the scaling map and $\mathrm{pr} : \mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1$ is the projection.

To understand $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ better, let us first consider the category $\mathrm{QCoh}(*/\mathbf{G}_m)$ of $\mathbf{G}_m$-equivariant quasicoherent sheaves on $* = \mathrm{Spec}(k)$ (with $\mathbf{G}_m$ acting trivially on $*$). This is the same thing as a k-module with a $\mathbf{G}_m$-action, i.e., a graded k-module; in other words, $\mathrm{QCoh}(*/\mathbf{G}_m) \simeq \mathrm{Mod}_k^{\mathrm{gr}}$. If $\mathbf{Z}^{\mathrm{ds}}$ denotes the category whose objects are integers and whose only morphisms are the identity, then $\mathrm{Mod}_k^{\mathrm{gr}} \simeq \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathrm{Mod}_k)$; this exchanges the tensor product of graded k-modules with Day convolution. We conclude that there is a symmetric monoidal equivalence

$$\mathrm{QCoh}(*/\mathbf{G}_m) \simeq \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathrm{Mod}_k). \tag{1}$$

We can now extend (1) to describe $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$. An object of $\mathrm{QCoh}(\mathbf{A}^1)$ is a module over $k[t]$, i.e., a k-module M with an endomorphism $t : M \to M$. If we put in $\mathbf{G}_m$-equivariance, then M acquires a grading $\bigoplus_n M_n$. Since $\mathbf{G}_m$ acts on $\mathbf{A}^1$ with weight 1, the endomorphism $t : M \to M$ must increase the grading by 1. In other words, t is a map $M_n \to M_{n+1}$. So an object of $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ is just a graded k-module $\bigoplus_n M_n$ with a map $M_n \to M_{n+1}$, i.e., is a *filtered* k-module. (Except that the maps between each filtration step need not be injective.) We can state this in a manner similar to (1): let $(\mathbf{Z}, \geq)$ denote the poset of integers with ordering $n \leq n+1$, viewed as a category. Then there is a symmetric monoidal equivalence

$$\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathrm{Fun}((\mathbf{Z}, \geq), \mathrm{Mod}_k), \tag{2}$$

where the symmetric monoidal structure on the right-hand side is via Day convolution.

One should compare (2) to the story with the torus above, where we said that $\mathrm{Fun}(B\pi_1(T), \mathrm{Vect}_{\mathbf{C}}) \simeq \mathrm{QCoh}(\check{T}_{\mathbf{C}})$: the category $(\mathbf{Z}, \geq)$ plays the role of the category $B\pi_1(T)$. Perhaps, then, there is also an analogue of the equivalence $\mathrm{Loc}(T; \mathbf{C}) \simeq \mathrm{Fun}(B\pi_1(T), \mathrm{Vect}_{\mathbf{C}})$!

Again, let us look at (1) as a motivating example. Recall that if Y is a topological space, then $\mathrm{Loc}(Y; k) \simeq \mathrm{Fun}(\Pi_{\leq 1}(Y), \mathrm{Mod}_k)$ where $\Pi_{\leq 1}(Y)$ is the fundamental groupoid of Y. In the case of the torus, the point was that $\Pi_{\leq 1}(T) \simeq B\pi_1(T)$. In the case of (1), one can say something very similar: the fundamental groupoid of the *discrete space* $\mathbf{Z}$ is precisely $\mathbf{Z}^{\mathrm{ds}}$. So (1) can be interpreted as an equivalence

$$\mathrm{QCoh}(*/\mathbf{G}_m) \simeq \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}}, \mathrm{Mod}_k) \simeq \mathrm{Loc}(\mathbf{Z}; k). \tag{3}$$

Now, the case of the torus and (3) give us a clue: perhaps the secret lies in finding a procedure which goes *back* from the categories $\mathbf{Z}^{\mathrm{ds}}$ (resp. $B\pi_1(T)$) to the spaces $\mathbf{Z}$ (resp. T). There is indeed such a procedure, known as geometric realization: it is the composite of the nerve functor $N : \mathrm{Cat} \to \mathrm{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ with the geometric realization functor $|-| : \mathrm{Set}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{S}$. (Intuitively, the geometric realization of a simplicial set $X_\bullet$ replaces n-simplices with topological n-simplices.)

One might therefore expect that $\mathrm{Fun}((\mathbf{Z}, \geq), \mathrm{Mod}_k)$ is equivalent to $\mathrm{Loc}(|(\mathbf{Z}, \geq)|; k)$. However, geometric realization fails us here! Indeed, the geometric realization of the poset $(\mathbf{Z}, \geq)$ is homeomorphic to the real line $\mathbf{R}$. But $\mathbf{R}$ is contractible, so all local systems on $\mathbf{R}$ are constant. In other words, $\mathrm{Loc}(|(\mathbf{Z}, \geq)|; k) \simeq \mathrm{Mod}_k$, which is certainly not equivalent to $\mathrm{Fun}((\mathbf{Z}, \geq), \mathrm{Mod}_k)$. What's happened is that geometric realization has lost too much information: for example, suppose $(\mathbf{Z}, \leq)$ is the opposite of $(\mathbf{Z}, \geq)$, i.e., we equip $\mathbf{Z}$ with the opposite ordering. Then the geometric realizations $|(\mathbf{Z}, \geq)|$ and $|(\mathbf{Z}, \leq)|$ are both homeomorphic to $\mathbf{R}$, since the geometric realization forgets the direction of morphisms. *However*, one may expect that the extra data of the direction of morphisms in $(\mathbf{Z}, \geq)$ translates into some additional structure on $\mathbf{R}$, which we can use to modify the category of local systems.

The appropriate modification is given by the category of "constructible sheaves (on $|(\mathbf{Z}, \geq)| = \mathbf{R}$) with specified singular support". I didn't know what this meant until last week, but it's a simple idea at its core. Let M be a smooth manifold. Let $\mathcal{F}$ be a sheaf of k-modules on M; we will define a subset $\mathrm{SS}(\mathcal{F}) \subseteq T^*M$. Suppose we have a point $(x, \xi) \in T^*M$, and $\phi : U \to \mathbf{R}$ be a differentiable function on a neighborhood $x \in U \subseteq M$ such that $d\phi = \xi$ and $\phi(x) = 0$. The inclusion $U \cap \phi^{-1}(-\infty, 0) \to U$ induces a map $\mathrm{Res} : \Gamma(U; \mathcal{F}) \to \Gamma(U \cap \phi^{-1}(-\infty, 0); \mathcal{F})$. We say that $(x, \xi) \in \mathrm{SS}(\mathcal{F})$ if Res is *not* an equivalence for some sufficiently small neighborhood U. One can think of the failure of Res to be an isomorphism as being the failure of a section of $\mathcal{F}$ over $U \cap \phi^{-1}(-\infty, 0)$ to extend across $\phi^{-1}(0)$. It turns out that $\mathrm{SS}(\mathcal{F}) \subseteq T^*M$ is conical and Lagrangian. If $\mathbb{L} \subseteq T^*M$ is a conical Lagrangian subspace, we will write $\mathrm{Shv}_{\mathbb{L}}(M; k)$ to denote the full subcategory of $\mathrm{Shv}(M; k)$ spanned by those sheaves $\mathcal{F}$ with $\mathrm{SS}(\mathcal{F}) \subseteq \mathbb{L}$.

Let us do some examples:

(a) Suppose $\mathcal{F}$ is a sheaf on M such that $\mathrm{SS}(\mathcal{F})$ is the zero section of T^*M. Then $\mathrm{Res} : \Gamma(U; \mathcal{F}) \xrightarrow{\sim} \Gamma(U \cap \phi^{-1}(-\infty, 0); \mathcal{F})$ for evey neighborhood $x \in U \subseteq M$ and every differentiable

$\phi : U \to \mathbf{R}$ such that $\phi(x) = 0$. Therefore, $\mathcal{F}$ is locally constant. In fact, $\mathcal{F}$ is locally constant if and only if $\mathrm{SS}(\mathcal{F})$ is the zero section. This implies that $\mathrm{Shv}_M(M; k) \simeq \mathrm{Loc}(M; k)$.

(b) Let $M = \mathbf{R}$, and suppose $\mathbb{L}$ is the union $\mathbf{R} \cup T_0^{\geq 0}\mathbf{R}$. Then, we can consider the function $\phi(t) = t$ on $\mathbf{R}$ (which has a zero at 0). Since $(0, d\phi) \in \mathbb{L}$, we see that $\mathcal{F} \in \mathrm{Shv}_{\mathbb{L}}(\mathbf{R}; k)$ if $\Gamma((-\infty, \epsilon); \mathcal{F}) \to \Gamma((-\infty, 0); \mathcal{F})$ is *not* an isomorphism for $\epsilon > 0$, but $\mathcal{F}$ is locally constant on $(-\infty, 0)$ and $(0, \infty)$. In other words, $\mathcal{F}$ is determined entirely by the modules $\mathcal{M}_1 := \Gamma((-\infty, \epsilon); \mathcal{F})$, $\mathcal{M}_2 := \Gamma((-\infty, 0); \mathcal{F})$, and the restriction map $\mathcal{M}_1 \to \mathcal{M}_2$.

An example of such a sheaf $\mathcal{F}$ is the pushforward of the constant sheaf on $Z = [0, \infty)$. Indeed, if V is a small neighborhood of 0, then the map $\Gamma(V; k_Z) \to \Gamma(V \cap \phi^{-1}(-\infty, 0); k_Z)$ is just the map $k \to 0$. Therefore, $(0, dt) \in \mathrm{SS}(k_Z)$, as expected. We also see that $(0, -dt) \notin \mathrm{SS}(k_Z)$, since taking $\psi(t) = -t$, the map $\Gamma(V; k_Z) \to \Gamma(V \cap \psi^{-1}(-\infty, 0); k_Z)$ is the identity map $k \to k$.

In fact, suppose in general that $\phi : M \to \mathbf{R}$ is a function such that $d\phi \neq 0$ on the zero locus of ϕ. Let $Z \subseteq M$ denote the subset $\phi^{-1}[0, \infty)$. Then a little bit of thought reveals that

$$\mathrm{SS}(k_Z) = Z \cup \{(x, \lambda \cdot d\phi_x) | \phi(x) = 0,\ \lambda \geq 0\}.$$

If U is the complement of Z, one can consider the $*$-extension k_U of the constant sheaf on U. Then there is a recollement sequence $k_U \to k \to k_Z$, using which one can argue that

$$\mathrm{SS}(k_U) = U \cup \{(x, \lambda \cdot d\phi_x) | \phi(x) = 0,\ \lambda \leq 0\}.$$

(c) Let $x_1, \cdots, x_n \in M$, and suppose $\mathbb{L}$ is the union $M \cup \bigcup_{i=1}^n T^*_{x_n} M$. Then $\mathrm{Shv}_{\mathbb{L}}(M; k)$ consists of those sheaves on M which are locally constant away from the points $x_1, \cdots, x_n$.

The notion of singular support is very rich, but for our purposes, it suffices to understand what happens when M is a 1-manifold. In this case, as with everything 1-dimensional, one can essentially encode singular support with combinatorial data. Recall from (b) above that if $\mathbb{L}$ is the union $\mathbf{R} \cup T_0^{\geq 0}\mathbf{R}$, then $\mathcal{F} \in \mathrm{Shv}_{\mathbb{L}}(\mathbf{R}; k)$ is determined entirely by the modules $\mathcal{M}_1 := \Gamma((-\infty, \epsilon); \mathcal{F})$, $\mathcal{M}_2 := \Gamma((-\infty, 0); \mathcal{F})$, and the restriction map $\mathcal{M}_1 \to \mathcal{M}_2$. In fact, there is an equivalence $\mathrm{Shv}_{\mathbb{L}}(\mathbf{R}; k) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Mod}_k)$, where Δ^1 is the poset $\{0 < 1\}$.

This works in general. Let us assume for simplicity that $M = \mathbf{R}$, and let $\mathbb{L} \subseteq T^*\mathbf{R}$ be a conical Lagrangian. Since $T^*\mathbf{R} = \mathbf{R} \oplus \mathbf{R}$, the complement $T^*\mathbf{R} - \mathbf{R}$ has two components, and we will choose one of them to be the "positive"/"up" part. Now $\mathbb{L} - (\mathbf{R} \cap \mathbb{L})$ is divided into "up" and "down" rays. If ℓ is such a ray, it intersects the zero section $\mathbf{R}$ at a point which we will call the "source" of ℓ. From this division, we can produce a directed graph $P(\mathbb{L})$ whose vertices are given the following:

(a) Points $x \in \mathbf{R}$ which are the sources of rays ℓ such that $\ell, -\ell \in \mathbb{L} - (\mathbf{R} \cap \mathbb{L})$.
(b) Intervals $[x, y]$ such that both of the following conditions are satisfied:
   - x is the source of a down ray but not an up ray;
   - y is the source of an up ray but not a down ray.
(c) Intervals $(-\infty, x)$ if x is the source of a down ray.
(d) Intervals $(-\infty, x]$ if x is the source of an up ray.
(e) Intervals (x, ∞) if x is the source of an up ray.
(f) Intervals $[x, \infty)$ if x is the source of a down ray.

If $i, j \in P(\mathbb{L})$, we draw a path $i \to j$ if $\overline{j} \cap i \neq \emptyset$. For example, say $\mathbb{L} - (\mathbf{R} \cap \mathbb{L})$ is the union of n rays ℓ. Then $P(\mathbb{L})$ has $n+1$ vertices corresponding to the intervals in $\mathbf{R}$ between the sources of the rays. There is one edge for each ray ℓ: if ℓ is an up ray, then we draw a right arrow, and if ℓ is a down ray, then we draw a left arrow. Figure 1 illustrates an example.

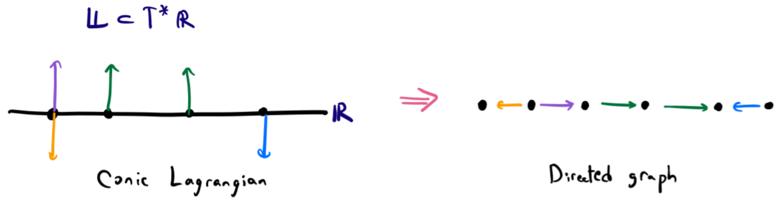

FIGURE 1. Going from a conic Lagrangian $\mathbb{L} \subseteq T^*\mathbf{R}$ to a directed graph $P(\mathbb{L})$. The colors indicate the arrows in $P(\mathbb{L})$ corresponding to the rays in $\mathbb{L}$.

[STZ14, Theorem 3.7] says that there is an equivalence

$$\mathrm{Shv}_{\mathbb{L}}(\mathbf{R};k) \simeq \mathrm{Fun}(P(\mathbb{L}), \mathrm{Mod}_k). \tag{4}$$

Let us do some sanity checks:

(a) If $\mathbb{L}$ is the zero section, then $P(\mathbb{L})$ just has the one vertex, corresponding to $(-\infty,\infty) = \mathbf{R}$. Therefore, we recover the fact that $\mathrm{Shv}_{\mathbf{R}}(\mathbf{R};k) = \mathrm{Loc}(\mathbf{R};k) \simeq \mathrm{Mod}_k$.

(b) If $\mathbb{L}$ is the union $\mathbf{R} \cup T_0^{\geq 0}\mathbf{R}$, then there is one ray (given by the cotangent fiber $\mathbf{R}_{>0}$ at 0). So $P(\mathbb{L})$ has 2 vertices $\{(-\infty,0],(0,\infty)\}$, and a single right arrow $(-\infty,0] \to (0,\infty)$. In other words, $P(\mathbb{L}) = \Delta^1$; so we recover our calculation that $\mathrm{Shv}_{\mathbb{L}}(\mathbf{R};k) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Mod}_k)$. In fact, this calculation is essentially the main input into [STZ14, Theorem 3.7], which is just a combinatorial generalization.

Let's finally return to our question of trying to interpret $\mathrm{Fun}((\mathbf{Z},\geq),\mathrm{Mod}_k)$ in terms of sheaves. According to (4), we just need to find a conical Lagrangian $\mathbb{L} \subseteq T^*\mathbf{R}$ such that $P(\mathbb{L}) = (\mathbf{Z},\geq)$. But this is straightforward given our definition of $P(\mathbb{L})$: namely, define

$$\mathbb{L}_{\mathbf{Z}} := \mathbf{R} \cup \bigcup_{n\in\mathbf{Z}} T_n^{\geq 0}\mathbf{R}.$$

In other words, if we write $\mathbf{R}$ as the geometric realization of $(\mathbf{Z},\geq)$, then $\mathbb{L}_{\mathbf{Z}}$ is obtained by drawing an up ray at each 0-simplex of the resulting "triangulation" of $\mathbf{R}$. See Figure 2.

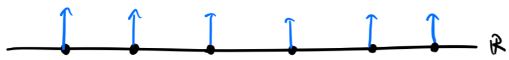

FIGURE 2. The conic Lagrangian $\mathbb{L}_{\mathbf{Z}} \subseteq T^*\mathbf{R}$.

Combining (4) with (2), we finally obtain:

Theorem 1. *View $\mathbf{A}^1$ and $\mathbf{G}_m$ as k-schemes. Then there are (symmetric monoidal) equivalences*

$$\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m) \simeq \mathrm{Fun}((\mathbf{Z},\geq),\mathrm{Mod}_k) \simeq \mathrm{Shv}_{\mathbb{L}_{\mathbf{Z}}}(\mathbf{R};k),$$

where the symmetric monoidal structures are tensor product of quasicoherent sheaves, Day convolution, and convolution product for the addition on $\mathbf{R}$, respectively.

(This was the result that I'd worked out before talking to Ben, but with a much cruder way of defining $\mathrm{Shv}_{\mathbb{L}_{\mathbf{Z}}}(\mathbf{R};k)$.) There is an action of $\mathbf{Z}$ on $\mathbf{R}$, which clearly respects the conic Lagrangian $\mathbb{L}_{\mathbf{Z}}$; so it must translate to an action of $\mathbf{Z}$ on $\mathrm{Fun}((\mathbf{Z},\geq),\mathrm{Mod}_k)$ and $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$. It is easy to say what this action is on $(\mathbf{Z},\geq)$: it is simply given by the automorphism $n \mapsto n+1$. We showed in (1) that $\mathrm{QCoh}(*/\mathbf{G}_m) \simeq \mathrm{Fun}(\mathbf{Z}^{\mathrm{ds}},\mathrm{Mod}_k)$. The action of $\mathrm{QCoh}(*/\mathbf{G}_m)$ acts on $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$ describes the $\mathbf{Z}$-action on $\mathrm{QCoh}(\mathbf{A}^1/\mathbf{G}_m)$.

It follows that the equivalence of Theorem 1 is $\mathbf{Z}$-equivariant, so we may take quotients by this $\mathbf{Z}$-action. The quotient $(\mathbf{Z},\geq)/\mathbf{Z}^{\mathrm{ds}}$ is just $B\mathbf{N}$, while the quotient $\mathbf{R}/\mathbf{Z}$ is S^1. (Note that this observation actually proves that $B\mathbf{N} \simeq S^1$, which is not *a priori* obvious.) Thus:

Corollary 2. *View $\mathbf{A}^1$ as a k-scheme. Let $\mathbb{L}_{S^1} \subseteq T^*S^1$ be the conic Lagrangian $S^1 \cup T_1^{\geq 0}S^1$. Then there are (symmetric monoidal) equivalences*

$$\mathrm{QCoh}(\mathbf{A}^1) \simeq \mathrm{Fun}(B\mathbf{N};\mathrm{Mod}_k) \simeq \mathrm{Shv}_{\mathbb{L}_{S^1}}(S^1;k),$$

where the symmetric monoidal structures are tensor product of quasicoherent sheaves, Day convolution, and convolution product for the addition on $\mathbf{R}$, respectively.

Note that this is very close to our discussion from the very beginning, where we said that

$$\mathrm{QCoh}(\mathbf{G}_m) \simeq \mathrm{Fun}(B\mathbf{Z};\mathrm{Mod}_k) \simeq \mathrm{Loc}(S^1;k).$$

In other words, the difference between $\mathbf{N}$ and $\mathbf{Z}$ is detected algebraically by the difference between $\mathbf{A}^1$ and $\mathbf{G}_m$, and is detected topologically by the difference between $\mathbb{L}_{S^1}$ and the zero section $S^1 \subseteq T^*S^1$.

One can imagine a purely combinatorial generalization of this picture, where one replaces $\mathbf{Z}$ with a lattice Λ. Given a sub-monoid σ of Λ, we may define a poset $(\Lambda,\geq_\sigma)$ by declaring that $a \geq_\sigma b$ if $a - b \in \sigma$. If σ is sufficiently nice, then there is a generalization of Theorem 1, relating

$\mathrm{QCoh}(\operatorname{Spec} k[\sigma]/\operatorname{Spec} k[\Lambda])$ to $\mathrm{Fun}((\Lambda, \geq_\sigma), \mathrm{Mod}_k)$ and to $\mathrm{Shv}_{\mathbb{L}_\sigma}(|(\Lambda, \geq_\sigma)|; k)$, where $\mathbb{L}_\sigma$ is a particular Lagrangian in $T^*|(\Lambda, \geq_\sigma)|$. The quotient stack $\operatorname{Spec} k[\sigma]/\operatorname{Spec} k[\Lambda]$ is an example of a toric stack. The generalization of Theorem 1 to general toric stacks is part of the coherent-constructible correspondence. In [She21], Shende gave a much shorter proof of this combinatorial generalization using only Theorem 1 as input. However, his argument didn't use the above perspective on $\mathbf{R}$ as the geometric realization of $(\mathbf{Z}, \geq)$, which might carry some merit in other situations.

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