Talk VII: Cyclotomic spectra

In this talk, we will describe the ∞-category of cyclotomic spectra, following [NS18]. We will establish some general properties of cyclotomic spectra, but we will then set this subject aside for the next few talks (where we will discuss Koszul duality and deformation quantization). Let us begin by recalling from last time that a weak cyclotomic structure over a complex-oriented $\mathbf{E}_\infty$-ring k equipped with an $\mathbf{E}_\infty$-automorphism $F : k \to k$ is a tuple $(\mathcal{M}, \mathcal{N}, \varphi)$ of a (graded) $k[\sigma]$-module $\mathcal{M}$, a (graded) k^{hS^1}-module $\mathcal{N}$, and equivalences

$$k[\sigma^{\pm 1}] \circlearrowright \mathcal{M}[1/\sigma] \xrightarrow{\sim}_F \mathcal{N}[1/\hbar] \circlearrowleft k((\hbar)),$$
$$\mathcal{M}/\sigma \simeq \mathcal{N}/\hbar,$$

where the first equivalence is F-linear. Our proof of noncommutative Hodge-de Rham degeneration (i.e., degeneration of the Tate spectral sequence) from the previous talk required the following result, whose proof we deferred to this talk:

Theorem 1. *Let k be a perfect field of characteristic $p > 0$, and let $F : k \to k$ be the Frobenius on k. If $\mathcal{C}$ is a smooth and proper k-linear stable ∞-category, then the pair $(\mathrm{THH}(\mathcal{C}), \mathrm{HC}^-(\mathcal{C}/k))$ can be upgraded to a weak cyclotomic structure over k.*

In fact, we will prove a stronger version of Theorem 1, which will require introducing cyclotomic spectra. Recall that if X is a spectrum with S^1-action, then $X^{h\mathbf{Z}/n}$ admits an action of $S^1/\mathbf{Z}/n \simeq S^1$. This induces an S^1-action on $X_{h\mathbf{Z}/n}$, too, such that the norm $X_{h\mathbf{Z}/n} \to X^{h\mathbf{Z}/n}$ is S^1-equivariant. In particular, the Tate construction $X^{t\mathbf{Z}/n}$ also admis an S^1-action.

Definition 2. A *cyclotomic spectrum* (à la [NS18]) is a spectrum $\mathcal{M}$ with an S^1-action and S^1-equivariant maps $\varphi_n : \mathcal{M} \to \mathcal{M}^{t\mathbf{Z}/n}$ for each integer $n \geq 1$, which satisfy compatibility conditions corresponding to the divisibility poset $\mathbf{Z}^\times_{\geq 0}$. Similarly, a *$p$-typical cyclotomic spectrum* is a spectrum $\mathcal{M}$ with an S^1-action and an S^1-equivariant map $\varphi : \mathcal{M} \to \mathcal{M}^{t\mathbf{Z}/p}$. Let CycSp denote the ∞-category of cyclotomic spectra, and let $\mathrm{CycSp}_{(p)}$ denote the ∞-category of p-typical cyclotomic spectra.

Since there are no compatibility conditions imposed on p-typical cyclotomic spectra, we will often just work with these objects. In fact, we will abusively fix a prime p and simply refer to p-typical cyclotomic spectra as "cyclotomic spectra". Many statements generalize to the integral setting, too, though. Our goal in this talk is to prove the following result[1], which implies Theorem 1:

[1] Parts (a) and (b) are due to [NS18]; part (c) is a calculation due to Bökstedt, which is recalled (for instance) in [HN20]; and part (d) is due to [Mat20], albeit in different language.

Proposition 3. *The following statements are true:*

(a) *The symmetric monoidal functor $\mathrm{THH} : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{Sp}^{S^1}$ refines to a symmetric monoidal functor $\mathrm{THH} : \mathrm{Cat}^{\mathrm{ex}}_\infty \to \mathrm{CycSp}$. In particular, if k is an $\mathbf{E}_\infty$-ring, then $\mathrm{THH}(k)$ is an $\mathbf{E}_\infty$-algebra in CycSp; and if $\mathcal{C}$ is a k-linear ∞-category, then $\mathrm{THH}(\mathcal{C})$ is a $\mathrm{THH}(k)$-module in cyclotomic spectra.*

(b) *If $\mathcal{C}$ is a smooth and proper k-linear stable ∞-category, then $\mathrm{THH}(\mathcal{C})$ is a dualizable $\mathrm{THH}(k)$-module in cyclotomic spectra.*

(c) *Let k be a perfect field of characteristic $p > 0$. Then $\mathrm{THH}(k) \cong k[\sigma]$. Furthermore, there is an S^1-equivariant k-linear equivalence $\mathrm{THH}(\mathcal{C})/\sigma \simeq \mathrm{HH}(\mathcal{C}/k)$.*

(d) *If $\mathcal{M}$ is a dualizable $\mathrm{THH}(k)$-module in cyclotomic spectra, then $(\mathcal{M}, (\mathcal{M}/\sigma)^{hS^1})$ refines to a weak cyclotomic structure over k.*

We will prove each part of Proposition 3 in the remainder of this section. For the moment, though, let us just remark that part (b) is an immediate consequence of the symmetric monoidality of part (a).

Proposition 3(a) is mostly formal, once one has the Tate diagonal.

Lemma 4. *Let $T_p : \mathrm{Sp} \to \mathrm{Sp}$ denote the functor $X \mapsto (X^{\otimes p})^{t\mathbf{Z}/p}$. Then:*

(a) *T_p is an exact functor.*
(b) *$T_p(\mathbb{S})$ is equivalent to the p-completion of $\mathbb{S}$.*

Proof. Let us first show that T_p preserves finite direct sums. For this, one observes that

$$(X_0 \oplus X_1)^{\otimes p} \simeq \bigoplus_{I \subseteq \{0,1\}^p} \bigotimes_{i \in I} X_i \simeq \bigoplus_{[I] \in \{0,1\}^p/\mathbf{Z}/p} \bigoplus_{I \in [I]} \bigotimes_{i \in I} X_i, \tag{1}$$

where $[I]$ runs over orbits of the $\mathbf{Z}/p$-action on $\{0,1\}^p$. The only orbits which are not isomorphic to $\mathbf{Z}/p$ itself are the orbits of $(0,\cdots,0)$ and $(1,\cdots,1)$; these orbits are trivial. Therefore, taking the $\mathbf{Z}/p$-Tate construction kills all orbits except those contributing to the terms $(X_0^{\otimes p})^{t\mathbf{Z}/p}$ and $(X_1^{\otimes p})^{t\mathbf{Z}/p}$. This implies that T_p preserves finite direct sums. In general, if $X \to Y \to Z$ is a cofiber sequence, then Y admits a filtration whose graded pieces each look like the terms in (1); the above argument implies that T_p is exact, as desired.

The second claim is much harder, and we will not prove it here. It follows from the definition that $T(\mathbb{S}) \simeq \mathbb{S}^{t\mathbf{Z}/p}$, so the claim is that $\mathbb{S}^{t\mathbf{Z}/p}$ is the p-completion of $\mathbb{S}$. This is known as the Segal conjecture, and this particular case was proved by Adams. ☐

Construction 5. There is a canonical natural transformation $\mathbb{S} \to \mathbb{S}^{t\mathbf{Z}/p}$, given by p-completion. This may be viewed as a natural transformation $\mathbb{S} \to T_p(\mathbb{S})$. A general result states that if $F : \mathrm{Sp} \to \mathrm{Sp}$ is an exact functor, then $\Omega^\infty F(\mathbb{S})$ is equivalent to the space of natural transformations $\mathrm{id}_{\mathrm{Sp}} \to F$. It then follows that there is a natural transformation $\Delta_p : \mathrm{id}_{\mathrm{Sp}} \to T_p$, called the *Tate diagonal*.

Suppose R is an $\mathbf{E}_\infty$-ring. Then the multiplication $R^{\otimes p} \to R$ is $\mathbf{Z}/p$-equivariant for the canonical $\mathbf{Z}/p$-action on $R^{\otimes p}$ and the trivial $\mathbf{Z}/p$-action on R. Therefore, the Tate diagonal gives a map

$$R \xrightarrow{\Delta_p} (R^{\otimes p})^{t\mathbf{Z}/p} \to R^{t\mathbf{Z}/p},$$

which is known as the *Tate-valued Frobenius*. Note that this map does not exist if R is not assumed to be an $\mathbf{E}_\infty$-ring.

Remark 6. Let X be any spectrum, equipped with the trivial S^1-action. Then there is an interesting (nontrivial) $S^1 \simeq S^1/\mathbf{Z}/p$-action on $X^{t\mathbf{Z}/p}$. Moreover, there is a canonical map $\mathrm{can} : X \to X^{h\mathbf{Z}/p} \to X^{t\mathbf{Z}/p}$ which equips X with the structure of a cyclotomic spectrum. Suppose now that R is an $\mathbf{E}_\infty$-ring, so R is equipped with the Tate-valued Frobenius $\mathrm{Frob} : R \to R^{t\mathbf{Z}/p}$. It is then natural to ask: does the trivial S^1-action on R and the resulting nontrivial S^1-action on $R^{t\mathbf{Z}/p}$ equip R with the structure of a cyclotomic spectrum?

The question is equivalent to asking whether Frob is S^1-equivariant. The answer is *no* in general. To understand why, assume that R is connective. Then there is an equivalence $(R^{t\mathbf{Z}/p})^{hS^1} \simeq R^{tS^1}$. If Frob is S^1-equivariant, then there would be a commutative diagram

$$\begin{tikzcd}
R^{hS^1} \arrow[r, "\mathrm{Frob}^{hS^1}"] \arrow[d] & (R^{t\mathbf{Z}/p})^{hS^1} \simeq R^{tS^1} \arrow[d] \\
R \arrow[r, "\mathrm{Frob}"'] & R^{t\mathbf{Z}/p},
\end{tikzcd}$$

where the right-vertical map $R^{tS^1} \to R^{t\mathbf{Z}/p}$ is the canonical map. However, such a diagram cannot exist in general. For instance, suppose that $R = \mathbf{F}_p$; then, the above diagram induces the following one on homotopy:

$$\begin{tikzcd} \mathbf{F}_p[[\hbar]] \arrow[r, "\mathrm{Frob}^{hS^1}"] \arrow[d] & \mathbf{F}_p((\hbar)) \arrow[d] \\ \mathbf{F}_p \arrow[r, "\mathrm{Frob}"'] & \mathbf{F}_p((\hbar))[\epsilon]/\epsilon^2, \end{tikzcd}$$

where ϵ lives in homological degree -1. This diagram cannot commute, since te left-vertical map kills $\hbar$, but the top map sends $\hbar$ to $\hbar$.

On the other hand, Frob is S^1-equivariant if R is a monoid $\mathbf{E}_\infty$-ring, i.e., if $R = \mathbb{S}[X]$ for some $\mathbf{E}_\infty$-monoid X in spaces. This is because the Tate-diagonal on R can be viewed as the composite

$$\mathbb{S}[X] \to \mathbb{S}[X^{\times p}]^{h\mathbf{Z}/p} \to \mathbb{S}[X^{\times p}]^{t\mathbf{Z}/p},$$

where the first map is induced by the space-level diagonal on X. Therefore, the Tate-valued Frobenius on R is the map $\mathbb{S}[X] \to \mathbb{S}[X]^{t\mathbf{Z}/p}$ which sends $x \in X$ to $x^p \in X$ (under the multiplication on X). The Segal conjecture allows us to identify $\mathbb{S}[X]^{t\mathbf{Z}/p}$ with the p-completion $(\mathbb{S}[X])^\wedge_p$; moreover, this equivalence is S^1-equivariant for the nontrivial S^1-action on $\mathbb{S}[X]^{t\mathbf{Z}/p}$ and the trivial S^1-action on $(\mathbb{S}[X])^\wedge_p$. Therefore, the S^1-equivariance of Frob is just the fact that the map $\mathbb{S}[X] \to (\mathbb{S}[X])^\wedge_p$ sending $x \mapsto x^p$ is S^1-equivariant for the trivial S^1-actions on the source and target.

In fact, if $\mathrm{Frob} : R \to R^{t\mathbf{Z}/p}$ is S^1-equivariant, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{THH}(R) \arrow[r, "\varphi"] \arrow[d] & \mathrm{THH}(R)^{t\mathbf{Z}/p} \arrow[d] \\ R \arrow[r, "\mathrm{Frob}"'] & R^{t\mathbf{Z}/p}, \end{tikzcd}$$

and one can consequently refine $\mathrm{THH}(-/R) : \mathrm{LinCat}_R \to \mathrm{Sp}^{S^1}$ to a functor landing in cyclotomic spectra.

Proof of Proposition 3(a). Recall that if Λ is the cyclic category, then a stable ∞-category $\mathcal{C}$ defines a functor $\mathcal{C}^\natural : \Lambda^{\mathrm{op}} \to \mathrm{Sp}$ by the assignment

$$[n]_\Lambda \mapsto \mathrm{colim}_{x_0,\cdots,x_n \in \mathcal{C}} \mathrm{Map}_{\mathcal{C}}(x_0, x_1) \otimes \cdots \otimes \mathrm{Map}_{\mathcal{C}}(x_{n-1}, x_n) \otimes \mathrm{Map}_{\mathcal{C}}(x_n, x_0). \tag{2}$$

Then, $\mathrm{THH}(\mathcal{C})$ is the geometric realization of the functor $\mathcal{C}^\natural|_{\Delta^{\mathrm{op}}}$. Let Λ_∞ denote the paracyclic category, so that Λ_∞ has an action of $B\mathbf{Z}$, and $\Lambda_\infty/B\mathbf{Z} \simeq \Lambda$. For each prime p, let Λ_p denote the quotient $\Lambda_\infty/B(p\mathbf{Z})$, so that there is an action of $B\mathbf{Z}/p$ on Λ_p such that $\Lambda_p/B\mathbf{Z}/p \simeq \Lambda$.

There is another functor $\Lambda_p \to \Lambda$, sending $[n]_{\Lambda_p} \mapsto [np]_\Lambda$. This functor will be denoted sd_p, and is called the edgewise subdivision. Restricting $\mathcal{C}^\natural$ along sd_p defines a functor $\mathrm{sd}_p\mathcal{C}^\natural : \Lambda_p^{\mathrm{op}} \to \mathrm{Sp}$. Using Equation (2), one sees that the Tate diagonal defines a natural transformation $\mathcal{C}^\natural \to (\mathrm{sd}_p\mathcal{C}^\natural)^{t\mathbf{Z}/p}$ of functors $\Lambda^{\mathrm{op}} \to \mathrm{Sp}$. The geometric realization gives an S^1-equivariant map

$$\mathrm{THH}(\mathcal{C}) = |\mathcal{C}^\natural| \to |(\mathrm{sd}_p\mathcal{C}^\natural)^{t\mathbf{Z}/p}| \to |\mathrm{sd}_p\mathcal{C}^\natural|^{t\mathbf{Z}/p} \simeq \mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p};$$

this is the Tate-valued Frobenius on $\mathrm{THH}(\mathcal{C})$. Proving that this construction defines a symmetric monoidal functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{CycSp}$ is somewhat technical, so we omit the argument. $\square$

Remark 7. If $\mathcal{C} = \mathrm{Mod}_R$ for an $\mathbf{E}_\infty$-ring R, then $\mathrm{THH}(R) := \mathrm{THH}(\mathcal{C})$ is the geometric realization of the simplicial diagram $|R^{\otimes \bullet+1}|$; it may be understood as the tensoring $S^1 \otimes R$ in the ∞-category CAlg of $\mathbf{E}_\infty$-rings. The datum of the S^1-equivariant $\mathbf{E}_\infty$-map $\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p}$ is then equivalent to the datum of an $\mathbf{E}_\infty$-algebra map $R \to \mathrm{THH}(R)^{t\mathbf{Z}/p}$. This $\mathbf{E}_\infty$-map is given by the composite $R \xrightarrow{\Delta_p} (\mathbf{Z}/p \otimes R)^{t\mathbf{Z}/p} \to (S^1 \otimes R)^{t\mathbf{Z}/p}$.

We must now prove Proposition 3(c) and (d); these are the key non-formal inputs into the theory of cyclotomic spectra. To prove Proposition 3(c), we will need the following two results:

Theorem 8 (Hopkins-Mahowald)**.** *Let $\mu : S^1 \to \mathrm{BGL}_1(\mathbb{S}_{(p)})$ be the p-local stable spherical fibration associated to the element $1 - p \in \pi_1 \mathrm{BGL}_1(\mathbb{S}_{(p)}) \cong \mathbf{Z}_{(p)}^\times$. Then μ extends to a map $\mu : \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_{(p)})$. The associated Thom spectrum is equivalent as an $\mathbf{E}_2$-algebra to the Eilenberg-Maclane spectrum/discrete commutative ring $\mathbf{F}_p$.*

Theorem 9 ([BCS10])**.** *Let X be an $\mathbf{E}_2$-space, and let $\mu : X \to \mathrm{BGL}_1(\mathbb{S}_{(p)})$ be an $\mathbf{E}_2$-map classifying a stable spherical fibration on X. If X^μ is the Thom spectrum of μ, then $\mathrm{THH}(X^\mu) \simeq X^\mu \otimes (BX)^{\eta B\mu}$, where $\eta B\mu$ is the composite*

$$BX \xrightarrow{B\mu} B^2\mathrm{GL}_1(\mathbb{S}_{(p)}) \xrightarrow{\eta} \mathrm{BGL}_1(\mathbb{S}_{(p)}).$$

Equivalently, $\mathrm{THH}(X^\mu)$ is the Thom spectrum of the composite

$$BX \xrightarrow{B\mu} B^2\mathrm{GL}_1(\mathbb{S}_{(p)}) \xrightarrow{\eta} \mathrm{BGL}_1(\mathbb{S}_{(p)}) \xrightarrow{\text{unit}} \mathrm{BGL}_1(X^\mu).$$

Proof of Proposition 3(c). It follows from Theorem 8 and Theorem 9 that $\mathrm{THH}(\mathbf{F}_p)$ may be identified with the Thom spectrum of the composite

$$\Omega S^3 \simeq B\Omega^2 S^3 \xrightarrow{B\mu} B^2\mathrm{GL}_1(\mathbb{S}_{(p)}) \xrightarrow{\eta} \mathrm{BGL}_1(\mathbb{S}_{(p)}) \xrightarrow{\text{unit}} \mathrm{BGL}_1(\mathbf{F}_p).$$

This composite is an $\mathbf{E}_1$-map, and is therefore determined by a single element in $\pi_2 \mathrm{BGL}_1(\mathbf{F}_p) \cong \pi_1 \mathbf{F}_p = 0$. In other words, this composite is nullhomotopic, which means that its Thom spectrum is just $\mathbf{F}_p \otimes \Omega S^3_+$. This was our definition of $\mathbf{F}_p[\sigma]$, thereby proving the first half of Proposition 3(c).

For the second half of part (c), we must show that there is an S^1-equivariant k-linear equivalence $\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(k)} k \simeq \mathrm{HH}(\mathcal{C}/k)$, where $\mathrm{THH}(k) \to k$ is the augmentation. This is in fact a special case of a more general base-change statement (which follows from the symmetric monoidality of Hochschild homology): if $A \to B$ is a map of $\mathbf{E}_\infty$-rings and $\mathcal{C}$ is a B-linear ∞-category, then there is an S^1-equivariant B-linear equivalence $\mathrm{HH}(\mathcal{C}/A) \otimes_{\mathrm{HH}(B/A)} B \simeq \mathrm{HH}(\mathcal{C}/B)$. □

Remark 10. In [KN19], a version of Proposition 3(c) for complete DVRs is proved; this result is attributed to Bhatt-Lurie-Scholze in *loc. cit.* The statement is the following: let A be a complete DVR with uniformizer u, and assume that its residue field k is perfect of characteristic $p > 0$. Then $\pi_* \mathrm{THH}(A/\mathbb{S}[t])^\wedge_p \cong A[\sigma]$. (Using this result, many statements in this seminar can be generalized to the mixed-characteristic setting; however, this introduces several additional technical issues coming from potential u-torsion, so we have opted not to work in maximal generality in these notes.) One can give a reproof of this result using methods similar to the above proof of Proposition 3(c), as we now sketch[2]. We will assume A is of mixed characteristic, and we will actually compute $\mathrm{THH}(A/\mathbb{S}_{W(k)}[[t]])$ instead, where $\mathbb{S}_{W(k)}$ denotes the spherical Witt vectors. This gives

[2] I recently learned that this argument was also used in [Mao20] to prove (variations of) the below result on "integral" Thom spectra.

the desired result (in mixed characteristic) thanks to the fact that THH satisfies base-change and the calculations that

$$\mathrm{THH}(\mathbb{S}[[t]]/\mathbb{S}[t])^\wedge_p \cong \mathbb{S}[[t]]^\wedge_p,\ \mathrm{THH}(\mathbb{S}_{W(k)}[[t]]/\mathbb{S}[[t]])^\wedge_p \cong \mathbb{S}_{W(k)}[[t]].$$

First, Theorem 9 can be refined to the following statement. Let X be an $\mathbf{E}_2$-space, and let R be an $\mathbf{E}_\infty$-ring. Let $\mu : X \to \mathrm{BGL}_1(R)$ be an $\mathbf{E}_2$-map. If X^μ is the Thom R-module of μ, then $\mathrm{THH}(X^\mu/R) \simeq X^\mu \otimes_R (BX)^{\eta B\mu}$, where $\eta B\mu$ is the composite

$$BX \xrightarrow{B\mu} B^2\mathrm{GL}_1(R) \xrightarrow{\eta} \mathrm{BGL}_1(R).$$

Equivalently, $\mathrm{THH}(X^\mu/R)$ is the Thom spectrum of the composite

$$BX \xrightarrow{B\mu} B^2\mathrm{GL}_1(R) \xrightarrow{\eta} \mathrm{BGL}_1(R) \xrightarrow{\text{unit}} \mathrm{BGL}_1(X^\mu).$$

To calculate $\mathrm{THH}(A/\mathbb{S}[t])$, we set $R = \mathbb{S}_{W(k)}[[t]]$; then, the claimed calculation of $\mathrm{THH}(A/\mathbb{S}[t])$ would follow once we prove the following analogue of Theorem 8: the discrete $\mathbf{E}_\infty$-ring A is equivalent (as an $\mathbf{E}_2$-$\mathbb{S}_{W(k)}[[t]]$-algebra) to the Thom spectrum of an $\mathbf{E}_2$-map $\nu : \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_{W(k)}[[t]])$.

To prove this, note that an $\mathbf{E}_2$-map ν as above can be identified with an element

$$f(t) \in \pi_1 \mathrm{BGL}_1(\mathbb{S}_{W(k)}[[t]]) \cong W(k)[[t]]^\times \cong W(k)^\times + tW(k)[[t]].$$

To identify $f(t)$, choose a uniformizer $u \in A$, and let $g(t) \in W(k)[[t]]$ denote its minimal polynomial. Then $f(t) := 1 + g(t)$. This is invertible in $W(k)[[t]]^\times$: if e is the absolute ramification index of A, then $g(t) = \sum_{i=0}^{e-1} a_i t^i + t^e$, where $p|a_i$ and $p^2 \nmid a_0$; therefore, $f(t)$ is invertible, since its constant term is $1 + a_0 \in W(k)^\times$. Having defined $f(t)$, note that the universal property of Thom spectra gives a canonical map $\Phi : (\Omega^2 S^3)^\nu \to A$ of $\mathbf{E}_2$-$\mathbb{S}_{W(k)}[[t]]$-algebras which sends t to u. To prove that Φ is an equivalence, it suffices to note that $\pi_0(\Omega^2 S^3)^\nu \cong A$ is u-complete, so Φ is a map of t-complete $\mathbf{E}_2$-$\mathbb{S}_{W(k)}[[t]]$-algebras. It is therefore an equivalence if Φ/t is an equivalence. But standard base-change properties of Thom spectra tell us that Φ/t is the map $(\Omega^2 S^3)^\mu \to k$, which we know to be an equivalence by Theorem 8.

Remark 11. One can prove (what seems to be) a sheared analogue of the computation in Remark 10 using a result from [Dev20], but it is not clear to us how/whether the relationship between these results can be made precise. The results of *loc. cit.* can be used to show that $\pi_* \mathrm{THH}(\mathbf{Z}/\mathbb{S}/\!\!/\eta) \simeq \mathbf{Z}[\gamma]$ with $|\gamma| = 4$. Since $\mathbb{S}/\!\!/\eta$ is the Thom spectrum of the map $\Omega S^3 \to \mathrm{BGL}_1(\mathbb{S})$ extending $\eta \in \pi_1(\mathbb{S})$, it can be viewed as an "un-shearing of $\mathbb{S}[t]$". (We do not know how to make this precise.) This suggests that the calculation of $\pi_* \mathrm{THH}(\mathbf{Z}/\mathbb{S}/\!\!/\eta)$ is an un-shearing of $\pi_* \mathrm{THH}(\mathbf{Z}/\mathbb{S}[t])$. It would be interesting to explore this further, since the calculation of $\mathrm{THH}(\mathbf{Z}/\mathbb{S}/\!\!/\eta)$ admits higher chromatic analogues.

Let us now turn to Proposition 3(d). We first need some calculational input:

Lemma 12. *The following statements are true:*

(a) *The Tate-valued Frobenius induces an equivalence* $\mathrm{THH}(\mathbf{F}_p)[1/\sigma] \xrightarrow{\sim}_{\mathrm{Frob}} \mathrm{THH}(\mathbf{F}_p)^{t\mathbf{Z}/p}$.

(b) *If* $\mathrm{TP}(k) = \mathrm{THH}(\mathbf{F}_p)^{tS^1}$, *then* $\pi_* \mathrm{TP}(\mathbf{F}_p) \cong \mathbf{Z}_p((\hbar))$, *and the canonical map* $\mathrm{TP}(\mathbf{F}_p) \to \mathbf{F}_p^{tS^1}$ *is given by reduction mod* p.

Proof. The first part follows from the second, via the following general claim: if N is a $\mathbf{Z}$-module with S^1-action, then $N^{tS^1}/p \simeq N^{t\mathbf{Z}/p}$. We will therefore prove the second claim. For this, recall that the Tate spectral sequence runs

$$E_2^{*,*} = \hat{\mathrm{H}}^*(BS^1; \pi_* \mathrm{THH}(\mathbf{F}_p)) = k[\sigma, u^{\pm 1}] \Rightarrow \pi_* \mathrm{TP}(\mathbf{F}_p),$$

where we are writing u to denote the generator of $\mathrm{H}^2(BS^1;\mathbf{F}_p)$. The entire E_2-page is concentrated in even degrees, so the spectral sequence collapses. Therefore, the desired claim for $\pi_*\mathrm{TP}(\mathbf{F}_p)$ follows from the claim that there is a multiplicative extension $u\sigma = p$ on the E_∞-page of this spectral sequence: the element $\hbar$ is represented by u. To prove this claim, let us begin by making some reductions: first, it suffices to prove the desired multiplicative extension in the spectral sequence for $\pi_*\mathrm{THH}(\mathbf{F}_p)^{hS^1}$. Second, since $\tau_{\le 2}\mathrm{THH}(\mathbf{F}_p) \simeq \tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})$, it suffices to check that this multiplicative extension holds in the spectral sequence for $\pi_*\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})^{hS^1}$. In fact, it suffices to check that this multiplicative extension holds in the spectral sequence for $\pi_*(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}))^{hS^1}$.

It is straightforward to describe the homotopy fixed points spectral sequence for $\pi_*(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}))^{hS^1}$. Indeed, the HKR theorem tells us that there is an extension

$$(\mathbf{F}_p\cdot\sigma)[2] \simeq L_{\mathbf{F}_p/\mathbf{Z}}[1] \to \tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}) \to \mathbf{F}_p,$$

so that $\pi_*\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}) = \mathbf{F}_p[\sigma]/\sigma^2$. Therefore, the $E_2 = E_\infty$-page of the homotopy fixed points spectral sequence for $\pi_*(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}))^{hS^1}$ is $\mathbf{F}_p[u,\sigma]/\sigma^2$. This spectral sequence collapses (again by evenness). Looking at the E_∞-page, one sees that $\pi_0\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})$ is either $\mathbf{F}_p\oplus\mathbf{F}_p$ or $\mathbf{Z}/p^2$. In the latter case, the element $p\in\pi_0(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}))^{hS^1}$ must be represented by $u\sigma$. Therefore, it suffices to show that $\pi_0(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}))^{hS^1} \not\cong \mathbf{F}_p\oplus\mathbf{F}_p$.

Reducing mod p, it suffices to show that the $\mathbf{F}_p$-vector space $\pi_0\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})/p$ is 1-dimensional. Since $\pi_*\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}) = \mathbf{F}_p[\sigma]/\sigma^2$, we see that $\pi_*\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})/p = \mathbf{F}_p[\sigma,\alpha]/(\sigma^2,\alpha^2)$ where $|\alpha| = 1$. Therefore, the homotopy fixed points spectral sequence runs

$$E_2^{*,*} \cong \mathbf{F}_p[u,\sigma,\alpha]/(\sigma^2,\alpha^2) \Rightarrow \pi_*(\tau_{\le 2}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})/p)^{hS^1}.$$

The d_2-differential in this spectral sequence is specified by the S^1-action on $\mathrm{HH}(\mathbf{F}_p/\mathbf{Z})$, and is therefore given by

$$d_2(\alpha) = u\sigma.$$

It follows that

$$E_3^{*,*} = \mathbf{F}_p[u,\sigma,\beta]/(\sigma^2,\beta^2,u\sigma,u\beta),$$

where β is represented by $\alpha\sigma$ and lives in degree $(0,3)$. This implies that the degree 0 part of the E_3-page (and hence the E_∞-page) of this spectral sequence must be $\mathbf{F}_p$, as desired. $\square$

Remark 13. The calculation of Lemma 12 in fact proves that if $\mathrm{TC}^-(\mathbf{F}_p) := \mathrm{THH}(\mathbf{F}_p)^{hS^1}$, then $\pi_*\mathrm{TC}^-(\mathbf{F}_p) \cong \mathbf{Z}_p[\hbar,\sigma]/(\hbar\sigma - p)$.

Proof of Proposition 3(d). Let $\mathcal{M}$ be a dualizable $\mathrm{THH}(k)$-module in cyclotomic spectra. To show that $(\mathcal{M},(\mathcal{M}/\sigma)^{hS^1})$ refines to a weak cyclotomic structure over k, it suffices to prove that there is a Frobenius-linear equivalence $\mathcal{M}[1/\sigma] \xrightarrow{\sim}_{\mathrm{Frob}} (\mathcal{M}/\sigma)^{tS^1}$. Let us first construct a map as indicated: since $\mathcal{M}$ is a cyclotomic spectrum, it admits a Tate-valued Frobenius $\varphi : \mathcal{M} \to \mathcal{M}^{t\mathbf{Z}/p}$. It follows from Lemma 12 that $\mathcal{M}^{t\mathbf{Z}/p}$ is a $k[\sigma^{\pm 1}]$-module, so φ extends to a map $\Phi : \mathcal{M}[1/\sigma] \to \mathcal{M}^{t\mathbf{Z}/p}$. We now claim that there is a natural equivalence $\mathcal{M}^{t\mathbf{Z}/p} \simeq (\mathcal{M}/\sigma)^{tS^1}$. For this, we observe:

$$(\mathcal{M}/\sigma)^{tS^1} \simeq (\mathcal{M}\otimes_{\mathrm{THH}(k)} k)^{tS^1} \simeq \mathcal{M}^{tS^1}\otimes_{\mathrm{TP}(k)} k^{tS^1} \simeq \mathcal{M}^{tS^1}/p \simeq \mathcal{M}^{t\mathbf{Z}/p},$$

where the final equivalence was also used in Lemma 12.

The above discussion produces a Frobenius-linear map $\Phi : \mathcal{M}[1/\sigma] \to \mathcal{M}^{t\mathbf{Z}/p}$. Proposition 3(d) claims that this map is an equivalence if $\mathcal{M}$ is dualizable. For this, we appeal to [AMN18, Proposition 4.6], which states that if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal ∞-categories, $F, G : \mathcal{C} \to \mathcal{D}$ are symmetric monoidal functors, then a symmetric monoidal transformation $F \to G$ is an equivalence if every object of $\mathcal{C}$ is dualizable. In particular, since $\Phi : \mathcal{M}[1/\sigma] \to \mathcal{M}^{t\mathbf{Z}/p}$ is a natural transformation from the symmetric monoidal ∞-category of dualizable $\mathrm{THH}(k)$-modules in CycSp to $\mathrm{THH}(k)^{t\mathbf{Z}/p}$-modules, we conclude that Φ must be an equivalence. $\square$

Remark 14. In particular, we see from the above proof of Proposition 3(d) that if k is a perfect field of characteristic $p > 0$ and $\mathcal{C}$ is a smooth and proper k-linear ∞-category, then $\mathrm{HP}(\mathcal{C}/k) \simeq \mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p}$ (with no Frobenius twist).

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