Equivariant homotopy theory and geometric Langlands

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Overview

1. Motivation
2. Equivariance
3. Proofs and generalizations

Algebra and topology

The very nature of the field of algebraic topology is such that there is a tight relationship between algebra and topology. Here is an example of this.

Theorem (Antiquity)

There is an equivalence of categories

$$\mathrm{Shv}(\mathrm{pt}; k) \simeq \mathrm{Mod}(k) \simeq \mathrm{QCoh}(\mathsf{Spec}\, k).$$

Here, k is any (commutative) ring (spectrum).

This might seem sort of silly, but it encodes the "dimension axiom" in the Eilenberg-Steenrod picture. It also forms the basis of many other dualities, such as Stone duality.

Discrete Fourier transform

The preceding result can be generalized to yield something nontrivial as follows.

Theorem

Let A be a finite abelian group, and let μ_A denote its Cartier dual, so that $\mu_A = \mathrm{Hom}(A, \mathbf{G}_m)$. Then there is an equivalence

$$\mathrm{Shv}(A; k) \simeq \mathrm{QCoh}(B\mu_A).$$

The proof is easy:

$$\mathrm{Shv}(A; k) \simeq \bigoplus_{A} \mathrm{Shv}(\mathrm{pt}; k) \simeq \bigoplus_{\mu_A \to \mathbf{G}_m} \mathrm{QCoh}(\mathrm{Spec}\, k) \simeq \mathrm{QCoh}(B\mu_A).$$

Not-necessarily discrete Fourier transform

It was not really necessary to assume A was finite. If $A = \mathbf{Z}$, the same result holds as long as one interprets "$\mathrm{Shv}(A; k)$" correctly. To generalize this, note that $\mathbf{Z} = \Omega S^1$. So:

Theorem

Let T be a compact torus, and let $\mathbb{X}_*(T) = \mathrm{Hom}(S^1, T)$. Let $\check{T} = \mathrm{Hom}(\mathbb{X}_*(T), \mathbf{G}_m)$. Then there is an equivalence

$$\mathrm{Shv}(\Omega T; k) \simeq \mathrm{QCoh}(B\check{T}).$$

Note that $\check{T}$ is an *algebraic* torus whose *characters* are the *cocharacters* of T.

Therefore, $\check{T}$ is not quite the "algebraization" of T; but it is "dual" to T. One calls $\check{T}$ the **Langlands dual** of T.

Generalization?

As mentioned before, previous work in chromatic homotopy theory has been concerned with replacing A by $K(A, n)$; work of Hopkins-Kuhn-Ravenel, Ravenel-Wilson, Hopkins-Lurie, Barthel-Carmeli-Schlank-Yanovski, ..., studies this question when k is a Morava E-theory.

My goal in this talk is different: following the geometric representation theory literature, I would like to discuss the story when ΩT is replaced by ΩG for *connected* compact Lie groups G. It turns out that studying $\mathrm{Shv}(\Omega G; k)$ is very difficult, for a few reasons:

- The category of all sheaves is too big.
- One is immediately forced to think "derivedly", because ΩG has infinitely many cells, but also infinitely many homotopy groups.

But equivariance fixes the difficulties!

Equivariant sheaves

The space ΩG has a *lot* of structure. It's a double loop space, and it has an action of G by conjugating the loop pointwise. The **derived geometric Satake** theorem describes a Fourier transform for the category $\mathrm{Shv}^c_G(\Omega G; k)$ when k is a localization of $\mathbf{Z}$.

The case of tori

Let $G = T$ be a compact torus. Then T acts trivially on ΩT. We find:

$$\mathrm{Shv}_T(\Omega T; k) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Shv}_T(\mathrm{pt}; k) \simeq \bigoplus_{\check{T} \to \mathbf{G}_m} \mathrm{Shv}_T(\mathrm{pt}; k).$$

So we need to understand $\mathrm{Shv}_T(\mathrm{pt}; k)$, generalizing our theorem from antiquity.

Complex orientations

It is not hard to see that there is an equivalence [1]

$$\mathrm{Shv}_T(\mathrm{pt}; k) \simeq \mathrm{Mod}(C_T^*(\mathrm{pt}; k)).$$

The ring $\mathrm{H}_T^*(\mathrm{pt}; k)$ is determined by the case $T = S^1$, in which case $\mathrm{H}_{S^1}^*(\mathrm{pt}; k)$ is very close to being $\mathrm{H}^*(BS^1; k) = \mathrm{H}^*(\mathbf{C}P^\infty; k)$. The ring structure on $\mathrm{H}^*(\mathbf{C}P^\infty; k)$ is determined by a **complex orientation** on k.

[1] In what follows, I'll ignore the difference between $C_T^*(\mathrm{pt}; k)$ and $\mathrm{H}_T^*(\mathrm{pt}; k)$.

T-equivariance

Quillen showed that Spf $\mathrm{H}^*(\mathbf{C}P^\infty; k)$ is a (1-dimensional) *formal group* over k; and Atiyah and Segal taught us that Spec $\mathrm{H}^*_{S^1}(\mathrm{pt}; k)$ is a *decompletion* of this formal group. I will focus on two main examples in this talk:

- $k = \mathbf{Z}$; then Spf $\mathrm{H}^*(\mathbf{C}P^\infty; k) = \widehat{\mathbf{G}}_a$, and Spec $\mathrm{H}^*_{S^1}(\mathrm{pt}; k) = \mathbf{G}_a$. For a general torus, this implies that

$$\mathrm{Spec}\, \mathrm{H}^*_T(\mathrm{pt}; k) \cong \mathfrak{t} \cong \check{\mathfrak{t}}^*,$$

  the dual of the Lie algebra of $\check{T}$.
- $k = \mathrm{KU}$; then Spf $\mathrm{H}^*(\mathbf{C}P^\infty; \mathrm{KU}) = \widehat{\mathbf{G}}_m$, and Spec $\mathrm{H}^*_{S^1}(\mathrm{pt}; \mathrm{KU}) = \mathbf{G}_m$. For a general torus, this implies that

$$\mathrm{Spec}\, \mathrm{H}^*_T(\mathrm{pt}; \mathrm{KU}) \cong T,$$

  but now viewed as an algebraic variety over $\pi_* \mathrm{KU} \cong \mathbf{Z}[\beta^{\pm 1}]$.

T-equivariant sheaves

The case of tori, continued

This implies that

$$\mathrm{Shv}_T(\mathrm{pt}; k) \simeq \begin{cases} \mathrm{QCoh}(\check{\mathfrak{t}}^*) & k = \mathbf{Z}, \\ \mathrm{QCoh}(T) & k = \mathrm{KU}. \end{cases}$$

It follows that

$$\mathrm{Shv}_T(\Omega T; k) \simeq \bigoplus_{\check{T} \to \mathbf{G}_m} \mathrm{Shv}_T(\mathrm{pt}; k) \simeq \begin{cases} \mathrm{QCoh}(\check{\mathfrak{t}}^* \times B\check{T}) & k = \mathbf{Z}, \\ \mathrm{QCoh}(T \times B\check{T}) & k = \mathrm{KU}. \end{cases}$$

There is an analogue for elliptic cohomology, too.

The first line is the derived geometric Satake theorem/Fourier transform for tori. Note that the right-hand side can be viewed as the coadjoint quotient stack $\check{\mathfrak{t}}^*/\check{T}$ since $\check{T}$ is commutative.

The Langlands dual group

With this setup, we are almost ready to state the main result in the nonabelian case. Recall that a (connected) compact Lie group G is classified by its *root data*, which consists of a weight lattice, a coweight lattice, roots, and coroots.

Swapping weights/roots with coweights/coroots defines a *new* algebraic group, denoted $\check{G}$. This is called the **Langlands dual** of G; if $G = \mathrm{SU}(n)$, $\check{G} = \mathrm{PGL}_n$.

G-equivariant sheaves

Theorem (**derived geometric Satake**; Bezrukavnikov-Finkelberg, building on Lusztig, Mirkovic-Vilonen, Drinfeld, Ginzburg)

Let G be a connected semisimple compact Lie group, acting on ΩG by conjugation. Then there is an equivalence

$$\mathrm{Shv}_G^c(\Omega G; \mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G}),$$

where $\check{G}$ denotes the Langlands dual of G, $\check{\mathfrak{g}}^*$ is the dual of its Lie algebra, and $\check{\mathfrak{g}}^*/\check{G}$ is the quotient stack for the coadjoint action. (Sort of; there's a shift which I am suppressing.)

Question: What about KU-coefficients?

Globalizing to curves

Before discussing KU, let me briefly mention how this relates to the geometric Langlands conjecture. (Feel free to ignore this slide if this is not of interest to you.) It is not hard to show that

$$\mathrm{Bun}_G(\mathbf{C}P^1) \simeq G \backslash \Omega G.$$

Similarly, one can show that

$$\mathrm{Loc}_{\check{G}}(\mathbf{P}^1) \cong \check{\mathfrak{g}}^* / \check{G}.$$

(Again, only sort of; there are shifts everywhere which I am ignoring.) The left-hand side is the moduli stack of $\check{G}$-local systems on $\mathbf{P}^1$. Therefore, derived geometric Satake suggests that

$$\text{“}\mathrm{Shv}(\mathrm{Bun}_G(\mathbf{C}P^1); \mathbf{Q}) \simeq \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}(\mathbf{P}^1))\text{”}.$$

The generalization with $\mathbf{P}^1$ replaced by an algebraic curve is (the naïve form of) the geometric Langlands conjecture. The special case of $\mathbf{P}^1$ was proved by V. Lafforgue using derived geometric Satake.

KU-coefficients

Let us return to homotopy theory. Recall that derived geometric Satake gave an equivalence

$$\mathrm{Shv}^c_G(\Omega G; \mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G}),$$

and when $G = T$, we showed

$$\mathrm{Shv}^c_T(\Omega T; \mathrm{KU}) \simeq \mathrm{QCoh}(T \times B\check{T}).$$

These can be simultaneously generalized:

Theorem (D., earlier this year, building on work of Bezrukavnikov-Finkelberg-Mirkovic)

Let G be a simply-connected, simply-laced, semisimple compact Lie group (e.g., $G = \mathrm{SU}(n)$, so $\check{G} = \mathrm{PGL}_n$). Then there is an equivalence

$$\mathrm{Shv}^c_G(\Omega G; \mathrm{KU}) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\check{G}^{\mathrm{sc}}/\check{G})^{\text{2-periodified}},$$

where $\check{G}^{\mathrm{sc}}$ is the simply-connected cover of $\check{G}$.

Comparison

Let us just compare the two main results:

$$\mathrm{Shv}^c_G(\Omega G; \mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$$

$$\mathrm{Shv}^c_G(\Omega G; \mathrm{KU}) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\check{G}^{\mathrm{sc}}/\check{G})^{\text{2-periodified}}.$$

Some observations:

- The Langlands dual group remains the same, no matter the coefficients.
- Say $\check{G} = \mathrm{SL}_n$ (result still holds). Then

$$\mathfrak{sl}_n = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \end{pmatrix} \right\} \times \{\text{diagonal}\} \times \left\{ \begin{pmatrix} 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\}.$$

  Similarly, have a "big cell" (open)

$$\mathrm{SL}_n \supseteq \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ * & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 1 \end{pmatrix} \right\} \times \{\text{diagonal}\} \times \left\{ \begin{pmatrix} 1 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

Comparison, continued

- The "nilpotent" and "unipotent" parts of $\mathfrak{sl}_n$ and SL_n are isomorphic, but the *diagonal*/semisimple parts are different:

$$\text{“Diagonal” matrices in } \mathfrak{sl}_n = \mathfrak{t}^{n-1} = \operatorname{Spec} \mathrm{H}^*_{T^{n-1}}(\mathrm{pt}; k);$$

$$\text{“Diagonal” matrices } \mathrm{SL}_n = T^{n-1} = \operatorname{Spec} \mathrm{H}^*_{T^{n-1}}(\mathrm{pt}; \mathrm{KU}).$$

Slogan

The behaviour of *Chern classes* on the **topological** side (sometimes called the "automorphic" or A-side) determines the meaning of *semisimplicity* on the **algebraic** side (sometimes called the "spectral" or B-side); and the meaning of "nil/unipotence" is unaffected.

Many results in geometric representation theory should (and do!) have KU*-theoretic analogues; and this should generalize to higher chromatic heights!*

Elliptic cohomology

The preceding slogan also holds for elliptic cohomology, and one finds:

Theorem (D., earlier this year)

Same setup as before; if E is an elliptic curve associated to an elliptic cohomology theory A, there is an equivalence

$$\mathrm{Shv}^{c}_{G/Z(G)}(\Omega G; A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss},0}_{\check{G}}(E)) \otimes_{\pi_0(A)} \pi_*(A_{\mathbf{Q}}),$$

where $Z(G)$ is the center of G, and $\mathrm{Bun}^{\mathrm{ss},0}_{\check{G}}$ denotes the moduli *stack* of semistable, degree zero $\check{G}$-bundles.

The argument

Suppose we focused on the subcategory of *locally constant* sheaves. Recall that if X is a connected space, there is an equivalence

$$\mathrm{Loc}(X; k) \simeq \mathrm{coMod}(C_*(X; k)).$$

This also works equivariantly, and one finds

$$\mathrm{Loc}_G(\Omega G; k) \simeq \mathrm{coMod}(C_*^G(\Omega G; k)),$$

where the right-hand side is comodules in $C_G^*(\mathrm{pt}; k)$-modules. Here, $C_*^G(\Omega G; k)$ is the *equivariant homology* of ΩG; but perhaps not in the usual sense. Its Borel-equivariant analogue would be $C_*(\Omega G; k)^{hG}$.

In any case, the key point will be to compute $\mathrm{H}_*^G(\Omega G; k)$; exactly the kind of thing algebraic topologists love to do! As with all such calculations, one does this by first computing $\mathrm{H}_*^T(\Omega G; k)$ for a maximal torus $T \subseteq G$. I would like to illustrate this when $G = \mathrm{SU}_2$.

The case $G = \mathrm{SU}_2$

Recall $\mathrm{SU}_2 = S^3$, and $T = S^1 \subseteq \mathrm{SU}_2$ is a maximal torus. We will describe $\mathrm{H}_*^{S^1}(\Omega S^3; k)$, or rather $\mathrm{H}_*^{S^1/Z(\mathrm{SU}_2)}(\Omega S^3; k)$. Atiyah-Bott localization + Goresky-Kottwitz-Macpherson, or the Serre spectral sequence, shows that

$$\operatorname{Spec} \mathrm{H}_*^{S^1/\mu_2}(\Omega S^3; k) \cong \begin{cases} \operatorname{Spec} \mathbf{Z}[x, a^{\pm 1}, \frac{a-1}{x}] & k = \mathbf{Z}, \\ \operatorname{Spec} \mathbf{Z}[\beta^{\pm 1}][y^{\pm 1}, a^{\pm 1}, \frac{a-1}{y-1}] & k = \mathrm{KU}. \end{cases}$$

Remark

This calculation, and what's described below, was first done by Bezrukavnikov-Finkelberg-Mirkovic by explicitly studying cycles/vector bundles; but one needs a new strategy to generalize to elliptic cohomology.

On the other hand, this group scheme also shows up when studying $\check{G} = \mathrm{PGL}_2$. Let me focus only on $k = \mathbf{Z}$.

The case $G = \mathrm{SU}_2$

Consider the map

$$\mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}[x] \xrightarrow{\kappa} \mathbf{A}^2 = \operatorname{Spec} \mathbf{Z}[x, y]$$

$$x \mapsto (x, 1).$$

This is called a **Kostant slice** (but I won't have time to explain it's called that). There is an action of the group $B = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)\}$ on $\mathbf{A}^2$ given by

$$(x, y) \mapsto (x, ay - bx).$$

The stabilizer of $\kappa(x) = (x, 1)$ consists of those matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in B$ with

$$a - bx = 1 \Rightarrow b = \frac{a-1}{x}.$$

In algebro-geometric language:

$$\mathbf{A}^1 \times_{\mathbf{A}^2/B} \mathbf{A}^1 \cong \operatorname{Spec} \mathbf{Z}[x, a^{\pm 1}, \tfrac{a-1}{x}].$$

The case $G = \mathrm{SU}_2$

In summary, we find that

$$\operatorname{Spec} \mathrm{H}_*^{S^1/\mu_2}(\Omega S^3; \mathbf{Z}) \cong \mathbf{A}^1 \times_{\mathbf{A}^2/B} \mathbf{A}^1.$$

We were interested in $\mathrm{H}_*^{\mathrm{SU}_2/\mu_2}(\Omega S^3; \mathbf{Z})$. One can write

$$\operatorname{Spec} \mathrm{H}^*_{\mathrm{SU}_2/\mu_2}(\mathrm{pt}; \mathbf{Z}) = \operatorname{Spec} \mathrm{H}^*_{S^1/\mu_2}(\mathrm{pt}; \mathbf{Z})^{\mathbf{Z}/2} = \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2).$$

This, along with "Grothendieck-Springer theory", implies that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{SU}_2/\mu_2}(\Omega S^3; \mathbf{Z}) \cong \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2) \times_{\mathfrak{pgl}_2/\mathrm{PGL}_2} \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2).$$

The map $\kappa : \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2) \to \mathfrak{pgl}_2$ appearing here is (also) called the **Kostant slice**: it sends

$$x^2 \mapsto \begin{pmatrix} 0 & -1 \\ x^2 & 0 \end{pmatrix}.$$

The case $G = \mathrm{SU}_2$

In the last slide, we stated that

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{SU}_2/\mu_2}(\Omega S^3; \mathbf{Z}) \cong \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2) \times_{\mathfrak{pgl}_2/\mathrm{PGL}_2} \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2).$$

To describe $\mathrm{Loc}_{\mathrm{SU}_2/\mu_2}(\Omega S^3; \mathbf{Z})$, we were interested in $\mathrm{H}_*^{\mathrm{SU}_2/\mu_2}(\Omega S^3; \mathbf{Z})$-comodules in $\mathrm{H}^*_{\mathrm{SU}_2/\mu_2}(\mathrm{pt}; \mathbf{Z})$-modules.

Or, algebro-geometrically, we were interested in the descent of the above group scheme along its obvious map to $\operatorname{Spec} \mathrm{H}^*_{\mathrm{SU}_2/\mu_2}(\mathrm{pt}; \mathbf{Z}) = \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2)$.

But this is just the *orbit* of $\kappa : \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2) \to \mathfrak{pgl}_2$ under the PGL_2-action. Simple results on companion matrices imply that this is (**nearly**) all of $\mathfrak{pgl}_2$. (Getting *all* of $\mathfrak{pgl}_2$ corresponds to working with all constructible sheaves, not just locally constant ones.)

Exactly the same argument works for KU, and for elliptic cohomology; the "Kostant slice" in the latter case corresponds to understanding Atiyah's classification of vector bundles on an elliptic curve.

Some complements

Here are some extensions of the above discussion.

- Extension to *connective* ku: same setup as before; then there is an equivalence

$$\mathrm{Shv}^c_G(\Omega G; \mathrm{ku}) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\check{G}^{\mathrm{sc}}_\beta / \check{G}),$$

  where $\check{G}^{\mathrm{sc}}_\beta$ over $\mathrm{Spec}\,\mathbf{Q}[\beta]$ is the canonical degeneration of $\check{G}^{\mathrm{sc}}$ into its Lie algebra. For example, if $\check{G} = \mathrm{SL}_2$, it consists of 2×2-matrices A such that $\frac{\det(I+\beta A)-1}{\beta} = 0$.

- Extensions to other based G-spaces: for instance, if $\mathrm{SO}_{2n} \subseteq \mathrm{SO}_{2n+1}$ acts on $S^{2n} = \mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$, one has

$$\mathrm{Shv}^c_{\mathrm{SO}_{2n}}(\Omega S^{2n}; \mathbf{Q}) \simeq \mathrm{QCoh}(T^*(\mathbf{A}^2)/\mathrm{SL}_2 \times \mathbf{A}^{n-1}).$$

  This is an analogue of *spherical harmonics*; it proves base cases of the relative Langlands conjectures of Ben-Zvi–Sakellaridis–Venkatesh. There are analogues with ku-coefficients. Also, I'm suppressing shifts; but here, that's the most interesting part of the story (has to do with L-functions)!

Philosophizing

- (Equivariant) generalized cohomology is an incredibly powerful tool;
- There are many interesting examples of spaces which encode deep algebro-geometric data, often equipped with actions of (p-)compact Lie groups (such as G acting on ΩG);
- For such spaces X and (equivariant) cohomology theories A^G, describing $\operatorname{Spec} A_*^G(X)$ in the language of algebraic geometry generally leads to interesting theorems.

This is not a new philosophy, of course. Homotopy theorists are uniquely equipped to understand such questions. Things about G, A, and X which seem obvious from the topological perspective tend to be quite nontrivial when thought about from the perspective of $\operatorname{Spec} A_*^G(X)$, and conversely. Hopefully there will be many more interactions between homotopy theory and geometric representation theory, etc.!

Thank you!