DISCRETE ADIC SPACES

1. Recollection

The goal of this talk will be to review some of the theory of adic geometry. Most of what we will talk about is independent of the previous talks, so we will isolate the parts which rely on previous talks (mainly Peter (Haine)'s, from the past two weeks) in this section.

Recall that a preanalytic ring $\mathcal{A}$ is given by the datum of a condensed ring $\underline{\mathcal{A}}$ and a functor $S \mapsto \mathcal{A}[S]$ from extremally disconnected spaces $\mathrm{Top}^{\text{extr. disc.}}$ to condensed $\underline{\mathcal{A}}$-modules $\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$. A preanalytic ring $\mathcal{A}$ is said to be analytic if the $\underline{\mathcal{A}}$-module $\mathcal{A}[S]$ behaves like $\underline{\mathcal{A}}[S]$, at least upon mapping out to certain test $\underline{\mathcal{A}}$-modules. More precisely, $\mathcal{A}$ is said to be analytic if for every complex $C_\bullet$ of $\underline{\mathcal{A}}$-modules such that each C_i is a direct sum of modules of the form $\mathcal{A}[S]$ with S extremally disconnected, the induced map $\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S'], C) \to \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S'], C)$ is an isomorphism for all extremally disconnected S'.

In the previous two talks, Peter showed that if $f : \mathrm{Spec}(A) \to \mathrm{Spec}(\mathbf{Z})$ was a morphism of finite type, then one could obtain a good formalism of functors $f_*, f^*, f_!, f^!$ between $D(A_\blacksquare)$ and $D(\mathbf{Z}_\blacksquare)$; this corresponds to Lecture 8 of Scholze's notes. We will state, but not prove (since the arguments are similar), a relativization of these results; see the appendix to Scholze's Lecture 8.

Let $R \to A$ be a morphism of finitely generated $\mathbf{Z}$-algebras. We can then define a preanalytic ring $(A, R)_\blacksquare$ given by the functor $\mathrm{Top}^{\text{extr. disc.}} \to \mathrm{Mod}^{\mathrm{cond}}_A$ (where A is regarded as a discrete ring) sending $S \mapsto R_\blacksquare[S] \otimes_R A$. The tensor product is implicitly derived, but this does not make a difference in the given expression.

Theorem 1. (a) *The preanalytic ring $(A, R)_\blacksquare$ is analytic.*

(b) *If $R \to S \to A$ are morphisms of finitely generated $\mathbf{Z}$-algebras, then the forgetful functor $j_* : D((A, S)_\blacksquare) \to D((A, R)_\blacksquare)$ has a left adjoint j^*, given by base-change along the morphism $(A, R)_\blacksquare \to (A, S)_\blacksquare$. Moreover, j^* admits a left adjoint $j_!$ such that the following holds:*

$$j_! j^* M \cong M \otimes_{(A,R)_\blacksquare} j_! A.$$

(c) *If $f : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is a morphism of $\mathbf{Z}$-schemes of finite type, define $f_! : D(A_\blacksquare) \to D(R_\blacksquare)$ via the composite*

$$D(A_\blacksquare) \xrightarrow{j_!} D((A, R)_\blacksquare) \to D(R_\blacksquare),$$

where the first functor is a left adjoint to j^* (base-change along $(A, R)_\blacksquare \to A_\blacksquare$), and the second functor is the forgetful functor. Then $f_!$ preserves direct sums and satisfies the projection formula

$$f_!((M \otimes_{R_\blacksquare} A_\blacksquare) \otimes_{A_\blacksquare} N) \cong M \otimes_{R_\blacksquare} f_! N,$$

where all tensor products are derived. Moreover, $f_!$ preserves compact objects if f has finite Tor-amplitude.

(d) *The functor $f_!$ also admits a right adjoint $f^!$. The object $f^! R \in D(A_\blacksquare)$ is discrete and left-bounded complex of finitely generated A-modules. If f is of finite Tor-amplitude, then $f^! R$ is also bounded, commutes with all direct sums, and is given by*

$$f^! M = (M \otimes_{R_\blacksquare} A_\blacksquare) \otimes_{A_\blacksquare} f^! R.$$

Date: November 9, 2020.

If f is a complete intersection, then $f^!R$ is also invertible.

Let's briefly recall one of the key observations that leads to this nice formalism in the condensed setting. We'll specialize to $R = \mathbf{Z}$ and $A = \mathbf{Z}[t]$, so that we're in the context of Peter's lecture. In that case, the functor $f_!$ is given by the composite

$$D(\mathbf{Z}[t]_\blacksquare) \xrightarrow{j_!} D((\mathbf{Z}[t], \mathbf{Z})_\blacksquare) \to D(\mathbf{Z}_\blacksquare).$$

This is reminiscent of the classical use of Nagata's compactification to define the exceptional pushforward functor: if $g : X \to Y$ is a separated finite-type morphism of qcqs schemes, then g can be factored as the composite of an open immersion $j : X \hookrightarrow \overline{X}$ and a proper morphism $\overline{g} : \overline{X} \to S$; one then defines $g_! : D(X) \to D(Y)$ via $\overline{g}_* j_!$.

One should therefore think of the morphism $(A, \mathbf{Z})_\blacksquare \to A_\blacksquare$ as providing a "canonical" compactification of $\mathrm{Spec}(A)$. Evidently, this can't be made sense of in the classical setting of algebraic geometry: we need to work in the condensed world. In any case, the morphism $j_! : D(\mathbf{Z}[t]_\blacksquare) \to D((\mathbf{Z}[t], \mathbf{Z})_\blacksquare)$ sends $\mathbf{Z}[t]$ to $(\mathbf{Z}(\!(t^{-1})\!)/\mathbf{Z}[t])[-1]$. The ring $\mathbf{Z}(\!(t^{-1})\!)$ consists of power series in t^{-1}, i.e., formal expressions $\sum a_n t^{-n}$ with possibly infinitely many powers of t^{-1} but only finitely many powers of t. The quotient $\mathbf{Z}(\!(t^{-1})\!)/\mathbf{Z}[t]$ should therefore be thought of as functions on $(\mathbf{Z}[t], \mathbf{Z})_\blacksquare$ which are "supported at ∞".

This canonical compactification has a natural home in the setting of adic geometry. Explaining this statement will be our goal for the remainder of this talk.

2. Affinoid adics

Let A be a finitely generated $\mathbf{Z}$-algebra; as mentioned in the previous section, we would like $(A, \mathbf{Z})_\blacksquare$ to be a compactification of A. Moving to spectra (in the algebro-geometric sense), we would like some "space", which we will write (suggestively, if you have seen this before) as $\mathrm{Spa}(A, \mathbf{Z})$, such that there is an open immersion $\mathrm{Spec}(A) \hookrightarrow \mathrm{Spa}(A, \mathbf{Z})$, and $\mathrm{Spa}(A, \mathbf{Z}) \to \mathrm{Spec}(\mathbf{Z})$ is "proper".

In what sense is $\mathrm{Spa}(A, \mathbf{Z}) \to \mathrm{Spec}(\mathbf{Z})$ supposed to be proper? To understand this, we can use the valuative criterion for properness. Namely, let V be a valuation ring, and let K be its fraction field. Then a morphism $g : X \to Y$ is proper if and only if every commutative square

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ \mathrm{Spec}(V) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

admits a unique dotted arrow filling in the diagram as indicated. Applying this to $\mathrm{Spa}(A, \mathbf{Z}) \to \mathrm{Spec}(\mathbf{Z})$, we would want the following to be satisfied: for every valuation ring V with fraction field K, every morphism $\mathrm{Spec}(K) \to \mathrm{Spa}(A, \mathbf{Z})$ can be extended a morphism $\mathrm{Spec}(V) \to \mathrm{Spa}(A, \mathbf{Z})$.

One now uses the usual idea that an object satisfying a certain property is just the moduli of all objects satisfying the property. Namely, it is natural to just define the underlying set of the underlying space of $\mathrm{Spa}(A, \mathbf{Z})$ to be the collection of all valuation rings V along with a map $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ from its fraction field to A. If $|\cdot| : K \to \Gamma \cup \{0\}$ is the valuation on K, with Γ the totally ordered value group, then each map $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ gives a valuation $|\cdot| : A \to \Gamma \cup \{0\}$. (Recall that a valuation $|\cdot| : A \to \Gamma \cup \{0\}$ is a map such that $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$, and $|x + y| \leq \max(|x|, |y|)$.)

The above discussion suggests that the following definition is natural.

Definition 2. Let $\mathrm{Spa}(A, \mathbf{Z})$ denote the space of equivalence classes[1] of valuations $|\cdot| : A \to \Gamma \cup \{0\}$ (with Γ a totally ordered abelian group), equipped with the following topology: a basis of quasicompact opens is given by the so-called "rational subsets"

$$U\left(\frac{g_1, \cdots, g_n}{f}\right) = \{\text{valuations } |\cdot| \text{ such that } |g_i| \leq |f| \neq 0\},$$

for $f, g_1, \cdots, g_n \in A$.

It is more common to use the following notation: a point of $\mathrm{Spa}(A, \mathbf{Z})$, corresponding to a valuation $|\cdot|$, is denoted x, and if $f \in A$, then the element $|f| \in \Gamma \cup \{0\}$ is denoted $|f(x)|$.

Definition 2 is incomplete in at least two ways: first, we only described $\mathrm{Spa}(A, \mathbf{Z})$ as a topological space, but it should have an associated sheaf of functions if it is to be considered as an algebro-geometric object; second, we would like to define $\mathrm{Spa}(A, R)$ for an arbitrary map $R \to A$ of finitely generated $\mathbf{Z}$-algebras. We will begin by addressing the second concern.

Let A^+ be the integral closure of R in A. Then, $(A, R)_\blacksquare \cong (A, A^+)_\blacksquare$. Indeed, we need to see that $R_\blacksquare[S] \otimes_R A \cong A^+_\blacksquare[S] \otimes_{A^+} A$ as condensed A-modules. This is because A^+ is finite over R (since R and A are finitely generated $\mathbf{Z}$-algebras), and so $\prod_I R \otimes_R A^+ = \prod_I A^+$; this implies that $\prod_I R \otimes_R A \cong \prod_I A^+ \otimes_{A^+} A$, as desired.

We can therefore just consider objects of the following kind:

Definition 3. A discrete Huber pair is a pair (A, A^+) of a discrete ring A and a integrally closed subalgebra $A^+ \subseteq A$.

Note that neither A nor A^+ are required to be finitely generated in the definition, but there is a functor from discrete Huber pairs to analytic rings sending (A, A^+) to the colimit $\mathrm{colim}(B, B^+)_\blacksquare$ over morphisms $(B, B^+) \to (A, A^+)$ of discrete Huber pairs with B and B^+ finitely generated $\mathbf{Z}$-algebras. We can then modify the definition of $\mathrm{Spa}(A, \mathbf{Z})$ to the following:

Definition 4. Let (A, A^+) be a discrete Huber pair. Let $\mathrm{Spa}(A, A^+)$ denote the space of equivalence classes of valuations $|\cdot| : A \to \Gamma \cup \{0\}$ (with Γ a totally ordered abelian group) such that $|A^+| \leq 1$, equipped with the topology whose basis of quasicompact opens is given by the rational subsets

$$U\left(\frac{g_1, \cdots, g_n}{f}\right) = \{x \in \mathrm{Spa}(A, A^+) \text{ such that } |g_i(x)| \leq |f(x)| \neq 0\},$$

for $f, g_1, \cdots, g_n \in A$.

If (A, A^+) is the discrete Huber pair $(A, \widetilde{\mathbf{Z}})$, where $\widetilde{\mathbf{Z}}$ is the integral closure of $\mathbf{Z}$ in A, then $\mathrm{Spa}(A, \widetilde{\mathbf{Z}})$ is just the space $\mathrm{Spa}(A, \mathbf{Z})$ from above. In general, pairs (A, A^+) sometimes go by the term "affinoids", and $\mathrm{Spa}(A, A^+)$ by "affinoid adic spaces".

Let us give some examples of affinoid adic spaces. These don't exactly fit into our setup above, because we required (A, A^+) to be, in particular, a pair of *finitely generated* $\mathbf{Z}$-algebras. But there is a more general setup of adic geometry, where the rings A and A^+ are allowed to have topologies, and these examples make sense in that setting. The purpose of these examples is just to illustrate how one might think about affinoid adics.

(a) Let $A = \mathbf{Q}_p$ and $A^+ = \mathbf{Z}_p$. Then $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ has only one point, given by the p-adic valuation $|\cdot|_p : \mathbf{Q}_p \to \mathbf{R}_{>0} \cup \{0\}$. Indeed, any valuation on $\mathbf{Q}_p$ has to be trivial on $\mathbf{Z}_p^\times$, and so it is determined entirely by $|p|$. But p is topologically nilpotent, so $|p| < 1$; it is a unit, so $|p| > 0$. It follows that $0 < |p| < 1$, which forces the valuation to be equivalent to the p-adic one.

[1] Two valuations $|\cdot| : A \to \Gamma \cup \{0\}$ and $|\cdot|' : A \to \Gamma' \cup \{0\}$ are equivalent if $|a| \leq |b|$ if and only if $|a|' \leq |b|'$ for all $a, b \in A$.

(b) If $A = \mathbf{Z}_p$ and $A^+ = \mathbf{Z}_p$, then $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$ is homeomorphic to $\mathrm{Spec}(\mathbf{Z}_p)$. Indeed, there are only two points in $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$: one is the trivial valuation $|\cdot|_{\mathrm{triv}} : \mathbf{Z}_p \to \mathbf{F}_p$ given by reducing mod p, and the other is the p-adic valuation. The difference between this and the previous example is that p is not a unit, which means that $|p|$ can be zero; this is why $|\cdot|_{\mathrm{triv}}$ is in fact a valuation. To get the topology (namely, that $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$ is homeomorphic to the Sierpinski space), it suffices to show that $|\cdot|_p$ is open, and that $|\cdot|_{\mathrm{triv}}$ is not open. This follows from the observation that the p-adic valuation is given by the rational open $\{|p| \neq 0\} \subseteq \mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$.

(c) Let $A = \mathbf{Q}_p\langle t\rangle$ (the algebra of convergent power series) and $A^+ = \mathbf{Z}_p\langle t\rangle$. The associated space $\mathrm{Spa}(\mathbf{Q}_p\langle t\rangle, \mathbf{Z}_p\langle t\rangle)$ is the closed adic disc.

3. Localization to globalize

We will now turn to addressing the first concern raised above: how do we define a structure sheaf on $\mathrm{Spa}(A, A^+)$ which encodes the pair (A, A^+)? By analogy with the Zariski spectrum, we need to study localizations of the Huber pair (A, A^+). Namely, we specified a basis of opens of $\mathrm{Spa}(A, A^+)$, given by the rational subsets $U\left(\frac{g_1,\cdots,g_n}{f}\right)$; in order for localization to be a viable procedure, we need this rational subset to depend only on a Huber pair built using the data of $A, A^+, f, g_1, \cdots, g_n$. We first need a lemma (which is an "adic nullstellensatz").

Lemma 5. *There is a bijection*

$$\{\textit{integrally closed subrings } A^+ \subseteq A\} \leftrightarrow \{\textit{subsets } U \subseteq \mathrm{Spa}(A, \mathbf{Z}) \textit{ such that } U = \bigcap U_{1,f}\}.$$

Given an integrally closed subring $A^+ \subseteq A$, define $U = \mathrm{Spa}(A, A^+)$; this is $\bigcap_{f\in A^+} U_{1,f}$. Conversely, given $U \subseteq \mathrm{Spa}(A, \mathbf{Z})$ as above, set

$$A^+ = \{f \in A|\ \textit{for all } x \in U, |f(x)| \leq 1\}.$$

Proof. To see that the correspondence as described above is well-defined, we need to show that if $U \subseteq \mathrm{Spa}(A, \mathbf{Z})$ is a subset such that $U = \bigcap U_{1,f}$, then the associated subring A^+ as defined above is integrally closed in A. Indeed, if $f \in A$ solves $f^n - a_{n-1}f^{n-1} - \cdots - a_0 = 0$ with all $a_i \in A^+$, then the inequality $|x + y| \leq \max(|x|, |y|)$ allows us to conclude that $|f(x)|^n \leq \max(|a_i(x)||f(x)|^i)$. Since each $|a_i(x)| \leq 1$, we see that $|f(x)| \leq 1$, as desired.

We now show that the maps defined above are inverse to each other. First, given a subset $U \subseteq \mathrm{Spa}(A, \mathbf{Z})$ such that $U = \bigcap_{f\in I} U_{1,f}$, we can define A^+ to be the integral closure of the subring of A generated by $f \in I$; this shows that the assignment $A^+ \mapsto \mathrm{Spa}(A, A^+)$ is surjective. To conclude, it therefore suffices to show that for any integrally closed $A^+ \subseteq A$, we have

$$A^+ = \{f \in A|\ \text{for all } x \in \mathrm{Spa}(A, A^+), |f(x)| \leq 1\}.$$

Suppose that $f \notin A^+$; we will show that there is a valuation $x \in \mathrm{Spa}(A, A^+)$ such that $|f(x)| > 1$; since the right hand side clearly contains A^+, this will prove the claim. Since $f \notin A^+$, we see that f cannot lie in $A^+[1/f]$; otherwise, f would solve $x^2 - a_0 = 0$ for some $a_0 \in A^+$, and hence would be integral over A^+ (which would be a contradiction since A^+ is integrally closed).

We can therefore find a prime $\mathfrak{p} \subseteq A^+[1/f]$ which contains $1/f$. Let $\mathfrak{q}$ be a minimal prime of $A^+[1/f]$ which is contained in $\mathfrak{p}$. Then there is a valuation ring V and a map $\mathrm{Spec}(V) \to \mathrm{Spec}\, A^+[1/f]$ which sends the generic point to $\mathfrak{q}$ and the special point to $\mathfrak{p}$. The image of the map $\mathrm{Spec}\, A[1/f] \to \mathrm{Spec}\, A^+[1/f]$ contains $\mathfrak{q}$, and so (since the point $\mathfrak{p}$ is a specialization of the point containing $\mathfrak{q}$) the valuation corresponding to $\mathrm{Spec}(V) \to \mathrm{Spec}\, A^+[1/f]$ can be lifted to a valuation on $A[1/f]$. By construction, this valuation satisfies $|A^+| \leq 1$, but since $1/f \in \mathfrak{p}$, satisfies $|f(x)| \geq 1$, as desired. □

Finally, the statement making localization possible is given by the following proposition:

Proposition 6. *Let $f : (A, A^+) \to (B, B^+)$ be a morphism of discrete Huber pairs such that there is a factorization*

$$\begin{tikzcd} \mathrm{Spa}(B, B^+) \arrow[r] \arrow[dr, dashed] & \mathrm{Spa}(A, A^+) \\ & U\left(\frac{g_1, \cdots, g_n}{f}\right) \arrow[u, hook] \end{tikzcd}$$

Then the map f factors uniquely through the pair $(A[1/f], \widetilde{A^+[g_i/f]})$, where $\widetilde{A^+[g_i/f]}$ is the integral closure of $A^+[g_1/f, \cdots, g_n/f]$ in $A[1/f]$. Moreover, the map $\mathrm{Spa}(A[1/f], \widetilde{A^+[g_i/f]}) \to U\left(\frac{g_1, \cdots, g_n}{f}\right)$ is a homeomorphism.

Proof. We first need to show that the map $A \to B$ factors through $A[1/f]$. Since $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through $U\left(\frac{g_1, \cdots, g_n}{f}\right)$, there is no $y \in \mathrm{Spa}(B, B^+)$ with $|f(y)| = 0$. This implies that f is invertible on B. Indeed, if not, then we can take any point of $\mathrm{Spec}(B/f)$ and compose it with the map $\mathrm{Spec}(B) \subseteq \mathrm{Spa}(B, B^+)$ which sends $\mathfrak{p}$ to $B \to \mathrm{Frac}(B/\mathfrak{p}) \to \{0, 1\}$. Clearly, for any point $y \in \mathrm{Spa}(B, B^+)$ obtained in this way, we have $|f(y)| = 0$. Since f is invertible on B, we get the desired map $A[1/f] \to B$. It remains to show that there is a map $\widetilde{A^+[g_i/f]} \to B^+$. Since $g_i/f \in B$ satisfies $|g_i(y)/f(y)| \leq 1$ for all $y \in \mathrm{Spa}(B, B^+)$, we see from Lemma 5 that $g_i/f \in B^+$. This gives a map $A^+[g_i/f] \to B^+$, which extends over the integral closure. To get the last sentence of the proposition, one just unwinds definitions and uses Lemma 5. □

We can finally define our sought-after sheaf:

Definition 7. Let (A, A^+) be a discrete Huber pair, and let $X = \mathrm{Spa}(A, A^+)$. Define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on an element $U = U\left(\frac{g_1, \cdots, g_n}{f}\right)$ of the basis of rational opens of X by

$$\mathcal{O}_X(U) = A[1/f],\ \mathcal{O}_X^+(U) = \widetilde{A^+[g_i/f]}.$$

Lemma 8. *The presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on $X = \mathrm{Spa}(A, A^+)$ are sheaves.*

Proof. For each $x \in \mathrm{Spa}(A, A^+)$, the valuation $f \mapsto |f(x)|$ extends to $\mathcal{O}_X(U)$ if $x \in U$. By passing to the colimit, we see that $f \mapsto |f(x)|$ extends to a valuation on the stalk $\mathcal{O}_{X,x}$. By Lemma 5, we also see that $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) | \text{ for all } x \in U, |f(x)| \leq 1\}$. This implies that if $\mathcal{O}_X$ is a sheaf, then so is $\mathcal{O}_X^+$. To check that $\mathcal{O}_X$ is a sheaf, note that there is a map $\mathrm{Spec}(A) \subseteq \mathrm{Spa}(A, A^+)$ which sends $\mathfrak{p}$ to $A \to \mathrm{Frac}(A/\mathfrak{p}) \to \{0, 1\}$ (which we already used above). The preimage of $U\left(\frac{g_1, \cdots, g_n}{f}\right)$ under this map is the distinguished open $D(f)$, and therefore $\mathcal{O}_X$ is the pushforward of $\mathcal{O}_{\mathrm{Spec}(A)}$. This implies that $\mathcal{O}_X$ is itself a sheaf. □

In the adic geometry literature, one often considers pairs (A, A^+), where A and A^+ are equipped with topologies. In this case, it is not always true that $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves (this property is called being "sheafy"). However, in our situation, A is assumed to be a finitely generated $\mathbf{Z}$-algebra, and thus has discrete topology; it is known that if (A, A^+) are discrete, then $\mathcal{O}_X$ is indeed sheafy.

In any case, the above discussion allows us to globalize the notion of a discrete Huber pair.

Definition 9. A discrete adic space is a triple $(X, \mathcal{O}_X, \{|\cdot|_x\}_{x\in X})$ where $(X, \mathcal{O}_X)$ is a locally ringed topological space, and for each $x \in X$, $|\cdot|_x$ is a valuation on $\mathcal{O}_{X,x}$. Moreover, $(X, \mathcal{O}_X, \{|\cdot|_x\}_{x\in X})$ must locally be of the form $(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, \{|\cdot|_x\}_{x\in \mathrm{Spa}(A,A^+)})$ for some discrete Huber pair (A, A^+).

If $X = \operatorname{Spec}(A)$ is an affine scheme over $\operatorname{Spec}(R)$, we can construct two adic spaces associated to X:

(a) X^{ad}, given by $\operatorname{Spa}(A, A)$. The points of X^{ad} are maps $\operatorname{Spec}(V) \to \operatorname{Spec}(A)$ from valuation rings, up to equivalence. Here, two maps $\operatorname{Spec}(V) \to \operatorname{Spec}(A)$ and $\operatorname{Spec}(W) \to \operatorname{Spec}(A)$ are equivalent if there is a faithfully flat map $\operatorname{Spec}(W) \to \operatorname{Spec}(V)$ making the obvious diagram commute. Note that a map $\operatorname{Spec}(W) \to \operatorname{Spec}(V)$ is faithfully flat if and only if it is surjective.

(b) $X^{\mathrm{ad}/R}$, given by $\operatorname{Spa}(A, \widetilde{R})$ (where $\widetilde{R}$ is the integral closure of R in A). Points of $X^{\mathrm{ad}/R}$ are given by valuation rings V which are R-algebras, along with a map $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ over $\operatorname{Spec}(R)$ (where K is the fraction field of V), all taken up to the same notion of equivalence. When $R = \mathbf{Z}$, this is the space $\operatorname{Spa}(A, \widetilde{\mathbf{Z}})$ discussed earlier.

Lemma 10. *There is a canonical map $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$. If X is separated and of finite type over* $\operatorname{Spec}(R)$*, then $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is an open immersion. If X is also proper over* $\operatorname{Spec}(R)$*, then $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is an isomorphism.*

Proof. The first sentence is obvious. For the second sentence, it suffices to check the claim when X is affine. In this case, it is the valuative criterion for separatedness. For the final sentence, one just notes that this is the valuative criterion for properness (as we had already used for motivation). $\square$

4. Gluing modules

If $X = \operatorname{Spa}(A, A^+)$, we would like the derived category of "quasicoherent sheaves" on X to be $D((A, A^+)_{\blacksquare})$. The part in Lemma 10 about properness can be interpreted as follows: if X is a scheme over $\operatorname{Spec}(R)$, then $X^{\mathrm{ad}/R}$ is the "canonical compactification" of X^{ad}. The functor $j_! : D(A_{\blacksquare}) \to D((A, R)_{\blacksquare})$ corresponds to exceptional pushforward along the canonical morphism $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$, and is therefore the identity when X is proper.

For a general discrete adic space, we would like to define a derived category of quasicoherent sheaves on X, but this procedure is bound to fail if we work in the setting of 1-categories. Indeed, in order for gluing the derived categories to behave well, we need the localization functor $- \otimes_{(A,A^+)_{\blacksquare}} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare}$ to be exact, where $X = \operatorname{Spa}(A, A^+)$ and $U \subseteq X$ is a rational open. This is not true in general. For instance, let $(A, A^+) = (\mathbf{F}_p[t], \mathbf{F}_p)$, and let $U = \{|t| \neq 0\}$. Since the only valuation on $\mathbf{F}_p$ is the trivial one, we see that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (\mathbf{F}_p[t], \mathbf{F}_p[t])$. Note that $(\mathbf{F}_p[t], \mathbf{F}_p[t])_{\blacksquare} = \mathbf{F}_p[t]_{\blacksquare}$. We claim that

$$\mathbf{F}_p(\!(t^{-1})\!) \otimes_{(\mathbf{F}_p[t], \mathbf{F}_p)_{\blacksquare}} \mathbf{F}_p[t]_{\blacksquare} = 0.$$

It follows that the injection $\mathbf{F}_p[t] \subseteq \mathbf{F}_p(\!(t^{-1})\!)$ in $D((\mathbf{F}_p[t], \mathbf{F}_p)_{\blacksquare})$ base-changes to the map $\mathbf{F}_p[t] \to 0$, which is obviously not injective.

The claimed isomorphism is in fact a consequence of the more general isomorphism

$$\mathbf{Z}(\!(t^{-1})\!) \otimes_{(\mathbf{Z}[t], \mathbf{Z})_{\blacksquare}} \mathbf{Z}[t]_{\blacksquare} = 0,$$

which we used in previous talks. Intuitively, $\mathbf{Z}(\!(t^{-1})\!)$ consists of functions "supported at ∞" on the compactification of $\operatorname{Spec} \mathbf{Z}[t]$, while $\mathbf{Z}[t]$ is, well, functions supported on $\operatorname{Spec} \mathbf{Z}[t]$. The supports of these classes of functions have no common overlap, and so the desired tensor product is zero.

The failure of exactness is remedied by working with derived ∞-categories instead. Dori will discuss the following theorem in the next talk.

Theorem 11. *Let X be a discrete adic space. Then the functor assigning to each open affinoid $U = \operatorname{Spa}(A, A^+) \subseteq X$ the ∞-category $\mathcal{D}((A, A^+)_{\blacksquare})$ defines a sheaf of ∞-categories on X.*

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