Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$

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Abstract. In this note, we study the local relative geometric Langlands conjecture of Ben-Zvi–Sakellaridis–Venkatesh for the spherical subgroup $\mathrm{PGL}_2^{\mathrm{diag}}$ of the triple product $\mathrm{PGL}_2^{\times 3}$ (and also for the spherical subgroup G_2 of SO_8/μ_2), whose corresponding Langlands dual $\mathrm{SL}_2^{\times 3}$-variety can be identified with the symplectic vector space $(\mathbf{A}^2)^{\otimes 3} \cong \mathbf{A}^8$ of $2\times 2\times 2$-cubes. Our analysis relies on a construction of Bhargava relating $2\times 2\times 2$-cubes to Gauss composition on quadratic forms, arising here as the moment map for the Hamiltonian $\mathrm{SL}_2^{\times 3}$-action on $(\mathbf{A}^2)^{\otimes 3}$, and the Cayley hyperdeterminant as studied by Gelfand-Kapranov-Zelevinsky.

1. Introduction

The goal of this brief note is to study the geometrization of a story from the arithmetic context pioneered by Jacquet, Kudla-Harris, Ichino, and Prasad among many others (see, e.g., [**HK, Ich, Pra**]). Fix an eighth root of unity ζ_8, let i be the resulting square root of -1, and write $k := \mathbf{Q}(\zeta_8) \cong \mathbf{Q}(i, \sqrt{2})$.

Notation 1.1. Let std denote the standard representation of SL_2, so that $\mathrm{std}^{\otimes 3}$ consists of cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] \arrow[dd, dash] & & d_3 \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & b_1 \arrow[ur, dash] \arrow[dd, dash] & \\
& d_1 \arrow[rr, dash] & & c. \\
b_3 \arrow[ur, dash] \arrow[rr, dash] & & d_2 \arrow[ur, dash] &
\end{tikzcd}$$

Fix an integer n. Equip $\mathrm{std}^{\otimes 3}$ with the grading where the entries of a cube have the following weights: a lives in weight $-4n$, each b_i lives in weight $-2n$, c lives in weight $2n$, and each d_i lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the corresponding graded variety.

Similarly, equip SL_2 with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. Write $\mathrm{SL}_2(-2n\rho)$ to denote this graded group. Then there is a natural graded action of $\mathrm{SL}_2(-2n\rho)^{\times 3}$ on $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$.

Recall that the process of *shearing* (denoted $\mathrm{sh}^{1/2}$) discussed in [**Rak, Lur**], as well as [**Dev**, Section 2.1], converts gradings into homological shifts (more precisely, it sends

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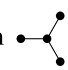
a module in weight n to the same module shifted homologically by n). This functor is symmetric monoidal when restricted to the subcategory of modules in *even* weights, and therefore extends to an operation on evenly graded stacks. If Y is a graded stack, let $\mathrm{Perf}^{\mathrm{sh}}(Y)$ denote $\mathrm{Perf}(\mathrm{sh}^{1/2}Y)$. As in [**Dev**], we will state all of our results with "arithmetic shearing" in the sense of [**BZSV**, Section 6.7].

Theorem 1.2 (Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$)**.** *Suppose that the* $\mathrm{PGL}_2^{\times 3}[\![t]\!]$*-action on* $\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!)$ *is optimal in the sense of* [**Dev**, Hypothesis 3.6.3]. *There is an equivalence*[1]

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}).$$

Moreover, this equivalence is equivariant for the action of the spherical Hecke category for $\mathrm{PGL}_2^{\times 3}$.

Remark 1.3. Let $\mathrm{PSO}_{2n} := \mathrm{SO}_{2n}/\mu_2$. Then, the embedding $\mathrm{PGL}_2^{\mathrm{diag}} \subseteq \mathrm{PGL}_2^{\times 3}$ can be identified with the diagonal embedding $\mathrm{SO}_3 \subseteq \mathrm{SO}_3 \times \mathrm{PSO}_4$; and similarly, the action of $\mathrm{SL}_2^{\times 3}$ on $\mathrm{std}^{\otimes 3}$ can be identified with the action of $\mathrm{Spin}_4 \times \mathrm{Sp}_2$ on the tensor product of their respective defining representations. From this perspective, Theorem 1.2 could be viewed as a special case of the geometrized analogue of the Gan-Gross-Prasad period (or at least a period isogenous to it).

A similar argument shows a variant for PSO_8. Namely, there is an embedding $\mathrm{G}_2 \subseteq \mathrm{PSO}_8$ given by triality, which exhibits G_2 as a spherical subgroup of PSO_8. To see that this situation is analogous to that of Theorem 1.2, note that the Dynkin diagram • of A_1 is obtained from the Dynkin diagram • • • of $A_1^{\times 3}$ by folding with respect to the obvious action of the symmetric group Σ_3. In the same way, the Dynkin diagram of G_2 is obtained from the Dynkin diagram  of D_4 by folding with respect to the action of Σ_3 permuting the three vertices around the branching vertex.

Theorem 1.4 (Derived geometric Satake for $\mathrm{PSO}_8/\mathrm{G}_2$)**.** *Suppose that the* $\mathrm{PSO}_8[\![t]\!]$*-action on* $\mathrm{PSO}_8(\!(t)\!)/\mathrm{G}_2(\!(t)\!)$ *is optimal in the sense of* [**Dev**, Hypothesis 3.6.3]. *Then there is an equivalence*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PSO}_8[\![t]\!]}(\mathrm{PSO}_8(\!(t)\!)/\mathrm{G}_2(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(12, \vec{6}, -6, \vec{0})/\mathrm{SL}_2(-6\rho)^{\times 3} \times \mathbf{A}^1(4)).$$

In other words, the spherical subgroups $\mathrm{PGL}_2^{\times 2} \subseteq \mathrm{PGL}_2^{\times 4}$ (given by $(g,h) \mapsto (g,g,g,h)$) and $\mathrm{G}_2 \subseteq \mathrm{PSO}_8$ have the same dual quotient stacks (namely, $(\mathrm{std})^{\otimes 3}/\mathrm{SL}_2^{\times 3} \times \mathbf{A}^1$) up to grading. Therefore, they fit into the paradigm of [**Dev**, Remark 4.1.5].

The proofs of Theorem 1.2 and Theorem 1.4 reduce to showing that the conditions of [**Dev**, Theorem 3.6.5] are met. This ultimately relies on studying Bhargava's construction from [**Bha**] relating $2 \times 2 \times 2$-matrices to quadratic forms, and the work [**GKZ**] of Gelfand-Kapranov-Zelevinsky describing the relationship to Cayley's hyperdeterminant. The work presented in this article indicates that there is much more to explore regarding the relationship between (relative) geometric Langlands (following [**BZSV**]), prehomogeneous vector spaces (following [**SK**]), and the progress in arithmetic invariant theory (using the terminology of [**BG**]) over the past 20 years spurred by Bhargava's thesis.

Remark 1.5. The arguments of this article should continue to hold if one considers sheaves with coefficients in $\mathbf{Z}[i, \frac{1}{\sqrt{2}}]$; we have not checked this explicitly, but it seems likely to be

[1] The ∞-category on the left-hand side is as in [**Dev**, Definition 3.6.1]; see Definition 3.1 for a quick review.

true. In fact, we expect that the results of this article should continue to hold for sheaves with coefficients in $\mathbf{Z}$ itself. This, however, is a rather more subtle question: the prime 2 is an interesting one (see Remark 2.8).

More generally, following the philosophy of [**Dev**], it should also be possible to use a variant of the methods of this article to prove analogues of Theorem 1.2 and Theorem 1.4 for sheaves with coefficients in connective complex K-theory ku. We have not attempted to do this, but we expect the corresponding 1-parameter deformation of $\mathrm{std}^{\otimes 3}$ over $\pi_*(\mathrm{ku}) \cong \mathbf{Z}[\beta]$ to be a rather interesting ku-Hamiltonian $\mathrm{SL}_2^{\times 3}$-variety.

Remark 1.6. The equivalence of Theorem 1.2 can heuristically be viewed as geometric Langlands for PGL_2 on the "doubled raviolo", obtained by gluing three formal disks along their common punctured disk. I expect Theorem 1.2 to be related to the work of [**MT**], and hope to address this relationship in joint work with Ben-Zvi and Gunningham.

In the final section of this article, we suggest some variants of Theorem 1.2 with $\mathrm{PGL}_2^{\times 3}$ replaced by variants. Namely, we expect:

- Let $G = \mathrm{Res}_{\mathbf{C}[\![t^{1/3}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$, where PGL_2 is viewed as a constant group scheme over $\mathbf{C}[\![t^{1/3}]\!]$. Then there should be an equivalence between $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!);k)$ and the ∞-category of perfect complexes on a shearing of the quotient stack $\mathrm{Sym}^3(\mathrm{std})/\mathrm{SL}_2$. The grading with respect to which the shearing is taken is described in Conjecture 4.7.
- Let $G = \mathrm{Res}_{(\mathbf{C}[\![t^{1/2}]\!]\times\mathbf{C}[\![t]\!])/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$. Then there should be an equivalence between $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!);k)$ and the ∞-category of perfect complexes on a shearing of the quotient stack $(\mathrm{std}\otimes\mathfrak{sl}_2)/\mathrm{SL}_2^{\times 2}$. The grading with respect to which the shearing is taken is described in Conjecture 4.12.

Remark 1.7. The quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is also studied (in different language, of course) in quantum information theory; see Remark 2.15 below.

Theorem 1.2 and Theorem 1.4 are predicted by (the Betti version of) the local geometric conjecture of Ben-Zvi–Sakellaridis–Venkatesh; see [**BZSV**, Conjecture 7.5.1]. My homotopy-theoretic interpretation of their conjecture is as follows. Suppose G is a reductive group over $\mathbf{C}$ and G/H is an affine homogeneous spherical G-variety (meaning that it admits an open B-orbit for its natural left $B \subseteq G$-action). Then, there should be a dual graded $\check{G}$-variety $\check{M}$ equipped with a moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$, and an equivalence of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!);\mathbf{C}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ denotes the ∞-category of perfect complexes on the shearing of $\check{M}/\check{G}$ with respect to its given grading. In fact, [**BZSV**, Section 4] gives an explicit construction of this predicted dual $\check{M}$, and in the examples $(G,H) = (\mathrm{PGL}_2^{\times 3}, \mathrm{PGL}_2^{\mathrm{diag}})$ and $(\mathrm{PSO}_8, \mathrm{G}_2)$, one can compute that the stacky quotient $\check{M}/\check{G}$ is isomorphic to the right-hand sides of Theorem 1.2 and Theorem 1.4 respectively.[2]

[2] In the first case, this computation is straightforward given the prescription of [**BZSV**, Section 4]; see [**Sak**, Example 7.2.4] for a reference. The computation in the second case goes as follows. As in [**BZSV**, Remark 7.1.1], the quotient stack $\check{M}/\check{G}$ can be identified with the quotient $\check{V}_X/\check{G}_X$, where $\check{G}_X$ is the Gaitsgory-Nadler/Sakellaridis-Venkatesh/Knop-Schalke dual group of X and $\check{V}_X$ is constructed in [**BZSV**, Section 4.5]. In the case $X = \mathrm{PSO}_8/\mathrm{G}_2$, a calculation shows that $\check{G}_X$ is the Levi subgroup of the maximal parabolic subgroup of PSO_8 corresponding to the central vertex of the D_4 Dynkin diagram; so $\check{G}_X \cong \mathrm{SL}_2^{\times 3}$. Using the prescription of [**BZSV**, Section 4.5], one can check that $\check{V}_X \cong \mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1$, where $\check{G}_X$ acts only on the first factor. See, e.g., [**Sak**, Line 9 of Table in Appendix A].

Lest Theorem 1.2 seem like an oddly specific example to focus on, we note that it is essentially the *only* "new" example of a spherical pair (G, H) of the form $(H^{\times j}, H^{\mathrm{diag}})$, as shown by the following lemma.

Lemma 1.8. *Suppose H is a simple linear algebraic group over $\mathbf{C}$. Then the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical if and only if:*

(a) $j = 2$, and H arbitrary;
(b) $j = 3$ and H is of type A_1.

PROOF. If the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical, there is an open H^{diag}-orbit on the flag variety of $H^{\times j}$. This implies that the dimension of H must be at least $j|\Phi^+|$, where Φ^+ is the set of positive roots; equivalently, one needs $\mathrm{rank}(H) \geq (j-2)|\Phi^+|$. Of course, this is always satisfied if $j = 2$ (this is the group case corresponding to the symmetric subgroup $H^{\mathrm{diag}} \subseteq H \times H$). Using the classification of simple linear algebraic groups over $\mathbf{C}$, it is easy to see that the only other case when the above inequality can hold is when $j = 3$ and H is of type A_1; one can then check by hand that the diagonal subgroup in this case is indeed spherical. $\square$

In the first case of Lemma 1.8, [**BZSV**, Conjecture 7.5.1] is precisely the derived geometric Satake equivalence of [**BF**]. Therefore, the only other case of Lemma 1.8 is when H is simple of type A_1, and Theorem 1.2 precisely addresses [**BZSV**, Conjecture 7.5.1] for the adjoint form PGL_2 of H.

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2. Some properties of $\mathrm{std}^{\otimes 3}$

In this section, we establish some basic properties of $\mathrm{std}^{\otimes 3}$ as a $\mathrm{SL}_2^{\times 3}$-variety; our base field will always be k, and we will write $\check{G} = \mathrm{SL}_2^{\times 3}$. Some of this material appears in [**Bha**]. In particular, Construction 2.3 is due to Bhargava.

Observation 2.1. An element $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2$ can be identified with a binary quadratic form $q_A(x,y) = cx^2 + 2iaxy + by^2$. Under this identification, the adjoint action of $g \in \mathrm{SL}_2$ on $\mathfrak{sl}_2$ is given by the action on (x, y) of the conjugate of g by the matrix $\mathrm{diag}(\zeta_8, \zeta_8^{-1})$. Explicitly, if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the action sends

$$\begin{aligned} x &\mapsto i\delta x + \beta y, \\ y &\mapsto \gamma x - i\alpha y. \end{aligned}$$

Note, moreover, that the discriminant of $q_A(x, y)$ is $4\det(A)$.

Warning 2.2. Note that under Observation 2.1, the element of $\mathfrak{sl}_2$ associated to a binary quadratic form $bx^2 + axy + cy^2$ is *not* the symmetric matrix associated to the quadratic form! Indeed, the associated symmetric matrix is $\begin{pmatrix} b & a/2 \\ a/2 & c \end{pmatrix}$, while the associated element of $\mathfrak{sl}_2$ is $\begin{pmatrix} -ai/2 & c \\ b & ai/2 \end{pmatrix}$.

Note, also, that we are relying quite heavily on the assumption that 2 is invertible in k. Over $\mathbf{Z}$, one can in fact identify the space of binary quadratic forms with the *coadjoint* representation $\mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$ of SL_2. Working over $\mathbf{Z}$ and keeping track of the difference between $\mathfrak{sl}_2$ and $\mathfrak{sl}_2^*$ has the effect of eliminating extraneous factors of 2 in our discussion below; but working over $\mathbf{Z}$ also introduces new complications (see Remark 2.8) which we do not wish to address in the present article.

Construction 2.3. The affine space $\mathbf{A}^8 = \mathrm{std}^{\otimes 3}$ can be regarded as parametrizing cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & d_3 \arrow[dl, dash] \arrow[dd, dash] \\
a \arrow[rr, dash] \arrow[dd, dash] & & b_1 \arrow[dd, dash] & \\
& d_1 \arrow[rr, dash] \arrow[dl, dash] & & c, \arrow[dl, dash] \\
b_3 \arrow[rr, dash] & & d_2 &
\end{tikzcd}$$

which we will represent by a tuple $(a, \vec{b}, c, \vec{d})$; we will often use the symbol $\mathcal{C}$ to denote such a cube. If $\{e_1, e_2\}$ are a basis for std, the above cube corresponds to the element of $\mathrm{std}^{\otimes 3}$ given by

$$\begin{gathered}
ae_1 \otimes e_1 \otimes e_1 + b_1 e_2 \otimes e_1 \otimes e_1 + b_2 e_1 \otimes e_2 \otimes e_1 + b_3 e_1 \otimes e_1 \otimes e_2 \\
+ d_1 e_1 \otimes e_2 \otimes e_2 + d_2 e_2 \otimes e_1 \otimes e_2 + d_3 e_2 \otimes e_2 \otimes e_1 + c e_2 \otimes e_2 \otimes e_2.
\end{gathered}$$

Associated to a cube $\mathcal{C}$ are three pairs of matrices, given by slicing along the top, leftmost, or front faces:

$$\begin{gathered}
M_1 = \left(\begin{smallmatrix} a & b_2 \\ b_3 & d_1 \end{smallmatrix}\right), N_1 = \left(\begin{smallmatrix} b_1 & d_3 \\ d_2 & c \end{smallmatrix}\right), \\
M_2 = \left(\begin{smallmatrix} a & b_1 \\ b_3 & d_2 \end{smallmatrix}\right), N_2 = \left(\begin{smallmatrix} b_2 & d_3 \\ d_1 & c \end{smallmatrix}\right), \\
M_3 = \left(\begin{smallmatrix} a & b_1 \\ b_2 & d_3 \end{smallmatrix}\right), N_3 = \left(\begin{smallmatrix} b_3 & d_2 \\ d_1 & c \end{smallmatrix}\right);
\end{gathered}$$

each of these defines a binary quadratic form

$$q_i(x, y) = -\det(M_i x + N_i y).$$

Explicitly,

$$\begin{aligned}
q_1(x, y) &= \det(M_1)x^2 + (ac + b_1 d_1 - b_2 d_2 - b_3 d_3)xy + \det(N_1)y^2, \\
q_2(x, y) &= \det(M_2)x^2 + (ac - b_1 d_1 + b_2 d_2 - b_3 d_3)xy + \det(N_2)y^2, \\
q_3(x, y) &= \det(M_3)x^2 + (ac - b_1 d_1 - b_2 d_2 + b_3 d_3)xy + \det(N_3)y^2.
\end{aligned}$$

Viewing $\mathfrak{sl}_2$ as the space of binary quadratic forms as in Observation 2.1, these three quadratic forms define a map

$$\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}.$$

An easy check shows that this map is $\check{G}$-equivariant.

Lemma 2.4 (Cayley)**.** *The composite*

$$\mathrm{std}^{\otimes 3} \xrightarrow{\mu} \mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3} /\!\!/ \check{G}$$

factors through the diagonal inclusion $\mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 \to \mathfrak{sl}_2^{\times 3} /\!\!/ \check{G}$*. In fact, the induced map* $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2$ *defines an isomorphism*

$$\mathrm{std}^{\otimes 3} /\!\!/ \check{G} \xrightarrow{\sim} \mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 \cong \mathbf{A}^1 /\!\!/ (\mathbf{Z}/2).$$

PROOF. The map $\mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3} /\!\!/ \check{G}$ sends a triple of matrices to their determinants, or equivalently a triple of quadratic forms to their discriminants. Therefore, we need to check that the three quadratic forms of Construction 2.3 have the same discriminant. This is easy: one finds that their common discriminant is

$$\begin{aligned}\det(q_i) = a^2c^2 + b_1^2d_1^2 + b_2^2d_2^2 + b_3^2d_3^2 - 2(ab_1cd_1 + ab_2cd_2 + ab_3cd_3 \\ + b_1b_2d_1d_2 + b_1b_3d_1d_3 + b_2b_3d_2d_3) + 4(ad_1d_2d_3 + b_1b_2b_3c).\end{aligned} \tag{1}$$

It remains to show that the map $\mathrm{std}^{\otimes 3}/\!\!/\check{G} \to \mathbf{A}^1$ defined by this polynomial is an isomorphism. This is stated/proved in [**GKZ**, Proposition 1.7 in Chapter 14], and is due to Cayley. □

Notation 2.5. Write $\det$ to denote the map $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ from Lemma 2.4, so that if $\mathcal{C}$ is a cube, $\det(\mathcal{C})$ is the quantity of (1).

Remark 2.6. The standard SL_2-equivariant symplectic structure on std defines an $\mathrm{SL}_2^{\times 3}$-equivariant symplectic structure on $\mathrm{std}^{\otimes 3}$. This action is Hamiltonian, and one can verify that the map $\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3} \cong (\mathfrak{sl}_2^*)^{\times 3}$ from Construction 2.3 is in fact the moment map for this $\mathrm{SL}_2^{\times 3}$-action. This gives a more "invariant" way to think about Bhargava's three quadratic forms. Along these lines, let us remark that [**Bha**, Theorem 1] implies that the span

$$\begin{tikzcd} & \mathrm{std}^{\otimes 3} \arrow[dl, "\mu_1 \times \mu_2"'] \arrow[dr, "\mu_3"] & \\ \mathfrak{sl}_2^* \times_{\mathfrak{sl}_2^*/\!\!/\mathrm{SL}_2} \mathfrak{sl}_2^* & & \mathfrak{sl}_2^* \end{tikzcd}$$

given by the moment maps *encodes* Gauss composition on quadratic forms, in the sense that given two (SL_2-orbits of) quadratic forms q_1 and q_2 with the same discriminant, the (SL_2-orbit of) the Gauss composition $-(q_1 + q_2)$ is given by $\mu_3((\mu_1 \times \mu_2)^{-1}(q_1, q_2))$.

Remark 2.7. An alternative way of constructing $\det(\mathcal{C})$ is as follows. Write $\mathcal{C} = e_1 \otimes v_1 + e_2 \otimes v_2$ with $v_1, v_2 \in \mathrm{std}^{\otimes 2} \cong \mathbf{A}^4$, and consider the symmetric bilinear form on $\mathrm{std}^{\otimes 2}$ given by

$$\langle e_1 \otimes e_1, e_2 \otimes e_2 \rangle = -\langle e_1 \otimes e_2, e_2 \otimes e_1 \rangle = 1,$$

and all other pairings zero. This is the symmetric form on $\mathrm{std}^{\otimes 2}$ induced from the standard symplectic form on std. Then, one can identify

$$\det(\mathcal{C}) = \det\begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix}.$$

Remark 2.8. Lemma 2.4 is not quite true over $\mathbf{F}_2$ (and hence not over $\mathbf{Z}$). One can already see the subtlety that arises over $\mathbf{F}_2$ from the formula (1): namely, the Cayley hyperdeterminant (appropriately normalized) admits a square root over $\mathbf{F}_2$. Explicitly, if $\mathcal{C} = (a, \vec{b}, c, \vec{d})$ is a cube and $\det(\mathcal{C})$ is defined by the formula (1), one has

$$\frac{\det(\mathcal{C})}{2} \equiv \frac{(ac + b_1d_1 + b_2d_2 + b_3d_3)^2}{2} \pmod 2.$$

In fact, over an $\mathbf{F}_2$-algebra, there is an analogue of Lemma 2.4 which states that the composite

$$\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{*,\times 3} \cong \mathfrak{pgl}_2^{\times 3} \xrightarrow{\mathrm{Tr}} (\mathbf{A}^1)^{\times 3}$$

factors through the diagonal $\mathbf{A}^1 \subseteq (\mathbf{A}^1)^{\times 3}$; the resulting map $\mathrm{std}^{\otimes 3} \to \mathbf{A}^1$ is given by the $\check{G}$-invariant function

$$\mathcal{C} \mapsto \mathrm{Tr}(\mathcal{C}) := ac + b_1d_1 + b_2d_2 + b_3d_3.$$

I expect that Tr defines an isomorphism $\mathrm{std}^{\otimes 3}/\!\!/\check{G} \to \mathbf{A}^1$ over $\mathbf{F}_2$. In particular, this means that the Cayley hyperdeterminant does *not* define an isomorphism $\det : \mathrm{std}^{\otimes 3}/\!\!/\check{G} \to \mathbf{A}^1$ over $\mathbf{F}_2$.

However, we expect more to be true. Namely, there should be an isomorphism $\mathrm{std}^{\otimes 3}/\!\!/_{\mathrm{der}}\check{G} \xrightarrow{\sim} \mathfrak{sl}_2^*/\!\!/_{\mathrm{der}}\mathrm{SL}_2$ (even over $\mathbf{Z}$!), where the symbol $/\!\!/_{\mathrm{der}}$ denotes the *derived* invariant-theoretic quotient (i.e., $V/\!\!/_{\mathrm{der}}H = \operatorname{Spec} R\Gamma(BH; \mathrm{Sym}(V^*))$). We also expect that $\mathrm{H}^*(\mathrm{SL}_{2,\mathbf{Z}}; \mathfrak{sl}_{2,\mathbf{Z}}^*(2)) \cong \mathrm{H}^*_{\mathrm{SO}_3}(*; \mathbf{Z})$. There is an isomorphism $\mathrm{H}^*_{\mathrm{SO}_3}(*; \mathbf{Z}) \cong \mathbf{Z}[p_1, e]/2e$, where e lives in cohomological degree 3; the class p_1 should correspond to the determinant $\mathfrak{sl}_2^* \to \mathbf{A}^1$, and the class e should correspond to the nontrivial extension of $\mathfrak{sl}_2^*$ given by $\mathfrak{gl}_2$.

We will now define an analogue of the Kostant slice, as it will be needed to apply [**Dev**, Theorem 3.6.5] (see [**Dev**, Strategy 1.2.1(b)]). For the purposes of our discussion, one should view this Kostant section as an analogue of the construction of the companion matrix associated to a characteristic polynomial.

Construction 2.9. If n is an integer, let $\vec{n}$ denote the triple (n, n, n). Let

$$\kappa : \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1/\!\!/(\mathbf{Z}/2) \cong \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$$

denote the map sending $a^2 \mapsto (a^2, \vec{0}, 0, \vec{1})$. This corresponds to the cube

$$\begin{tikzcd}
& 0 \arrow[rr, dash] \arrow[dd, dash] & & 1 \arrow[dd, dash] \\
a^2 \arrow[ur, dash] \arrow[rr, dash, crossing over] \arrow[dd, dash] & & 0 \arrow[ur, dash] & \\
& 1 \arrow[rr, dash] & & 0. \\
0 \arrow[ur, dash] \arrow[rr, dash] & & 1 \arrow[ur, dash] \arrow[uu, dash, crossing over] &
\end{tikzcd}$$

In this case, $\det(\kappa(a^2)) = 4a^2$, so that κ defines a section of $\det$ (at least up to the unit $4 \in k^\times$). The associated quadratic forms are all equal, and are given by

$$q_1(x, y) = q_2(x, y) = q_3(x, y) = a^2x^2 - y^2,$$

which corresponds to the traceless matrix $\left(\begin{smallmatrix} 0 & -1 \\ a^2 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$. (Note that this is exactly the companion matrix associated to the characteristic polynomial $y^2 - a^2$.)

One of the key properties of the Kostant section/companion matrices is that a matrix $A \in \mathfrak{sl}_2$ is conjugate to $\kappa(\det(A))$ if and only if A is regular (i.e., the minimal polynomial of A agrees with its characteristic polynomial), if and only if A is nonzero. We will now prove an analogous result concerning $\kappa : \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$.

Proposition 2.10. *The $\check{G}$-orbit of the image of κ is a dense open subscheme whose complement has codimension* 3.

PROOF. We will use the classification of $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$ as in [**GKZ**, Example 4.5 in Chapter 14]; see Figure 1 for a graph of the seven orbits of $\check{G}$ on $\mathrm{std}^{\otimes 3}$. Namely, if $\lambda \neq 0$, all elements of $\det^{-1}(\lambda)$ are in a single $\check{G}$-orbit. (In fact, all elements in the fiber $\det^{-1}(1)$ are in the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.) The $\check{G}$-orbit of $\det^{-1}(\mathbf{G}_m)$ is open and dense, and hence is 8-dimensional; moreover, it agrees with the $\check{G}$-orbit of $\kappa(\mathbf{G}_m)$. Next, there is a maximal $\check{G}$-orbit inside the fiber $\det^{-1}(0)$, given by the orbit of $(0, \vec{0}, 0, \vec{1}) = \kappa(0)$. This orbit is 7-dimensional, and the largest $\check{G}$-orbits contained in the complement $\det^{-1}(0) - \check{G}\cdot\kappa(0)$ have dimension 5. In particular, the complement of $\check{G}\cdot\kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ has dimension 5, i.e., codimension $8 - 5 = 3$. $\square$

$$\begin{tikzcd}
 & & (1,\vec{0},0,(1,0,0));5 \arrow[dl, dash] \arrow[dr, dash] & & \\
\kappa(\mathbf{G}_m);8 \arrow[r, dash] & \kappa(0);7 \arrow[r, dash] & (1,\vec{0},0,(0,1,0));5 \arrow[r, dash] & (1,\vec{0},0,\vec{0});4 \arrow[r, dash] & (0,\vec{0},0,\vec{0});0 \\
 & & (1,\vec{0},0,(0,0,1));5 \arrow[ul, dash] \arrow[ur, dash] & &
\end{tikzcd}$$

FIGURE 1. $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$, representatives, and their dimensions (indicated after the semicolon), connected by closure. Note that $\kappa(0) = (0, \vec{0}, 0, \vec{1})$, and that the $\check{G}$-orbit of $\kappa(1) = (1, \vec{0}, 0, \vec{1})$ is the same as the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.

Remark 2.11. As explained in [**GKZ**, Example 4.5 in Chapter 14], the closure of the associated orbits inside $\mathbf{P}(\mathrm{std}^{\otimes 3}) = \mathbf{P}^7$ can be described as follows. First, the closure of the generic orbit is $\mathbf{P}^7$. Next, the closure of the orbit of next smallest dimension is the zero locus of $\det$, which cuts out the dual variety of the Segre embedding $(\mathbf{P}^1)^{\times 3} \hookrightarrow \mathbf{P}^7$ (just as the usual determinant for 2×2-matrices cuts out the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$). The projective orbit associated to $(1, \vec{0}, 0, (0, 1, 0))$, say, is cut out inside the locus $\{\det = 0\}$ by the Segre embedding $\mathbf{P}(\mathrm{std}) \times \mathbf{P}(\mathrm{std}^{\otimes 2}) = \mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^7$. Finally, the minimal nonzero orbit is cut out by the Segre embedding $(\mathbf{P}^1)^{\times 3} \to \mathbf{P}^7$.

Remark 2.12. More generally, let std_n denote the standard n-dimensional representation of SO_n, so that the symplectic vector space $\mathrm{std} \otimes \mathrm{std}_n$ is equipped with an action of $\mathrm{SL}_2 \times \mathrm{SO}_n$. Using [**SK**, Section 7], one finds that the obvious analogue of the formula for $\det(\mathcal{C})$ in Remark 2.7 defines a map $\mathrm{std} \otimes \mathrm{std}_n \to \mathbf{A}^1/\!\!/(\mathbf{Z}/2)$ which induces an isomorphism

$$(\mathrm{std} \otimes \mathrm{std}_n)/\!\!/(\mathrm{SL}_2 \times \mathrm{SO}_n) \cong \mathbf{A}^1/\!\!/(\mathbf{Z}/2).$$

Proposition 2.10 admits an analogue in this more general setting (at least if one works over $\mathbf{C}$): there is a Kostant slice $\kappa : \mathbf{A}^1/\!\!/(\mathbf{Z}/2) \to \mathrm{std} \otimes \mathrm{std}_n$ whose $\mathrm{SL}_2 \times \mathrm{SO}_n$-orbit is open and has complement of codimension 3. Namely, assume $n = 2j$ is even for simplicity (a slight variant of this construction will work for odd n), so that without loss of generality, the symmetric bilinear form on std_n is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{\oplus j}$. If $e_1, \cdots, e_{2j}$ is a basis for std_n, let $v_1 = a^2 e_1 + e_2$, and let $v_2 = \sum_{i=2}^{j} (e_{2i-1} + e_{2i})$. Then $\langle v_1, v_1 \rangle = 2a^2$, $\langle v_2, v_2 \rangle = 2(j-1)$, and $\langle v_1, v_2 \rangle = 0$. If e_1, e_2 is a basis for std, the Kostant slice sends

$$\kappa : \mathbf{A}^1/\!\!/(\mathbf{Z}/2) \to \mathrm{std} \otimes \mathrm{std}_n, \ a^2 \mapsto \tfrac{1}{\sqrt{2}} e_1 \otimes v_1 + \tfrac{1}{\sqrt{2(j-1)}} e_2 \otimes v_2.$$

It is easy to check that this map does indeed give a section of $\det$. To check that the $\mathrm{SL}_2 \times \mathrm{SO}_n$-orbit of κ has complement of codimension 3, we need an analogue of Remark 2.11. This succumbs to an analysis similar to that of [**GKZ**, Chapter 14]. One finds that if $n \geq 5$, the poset of closures of $\mathrm{SL}_2 \times \mathrm{SO}_n$-orbits in $\mathbf{P}(\mathrm{std} \otimes \mathrm{std}_n) \cong \mathbf{P}^{2n-1}$ is as shown in Figure 2. The case $n = 4$ is "degenerate" and one instead gets Figure 1.

$$\begin{tikzcd}
& & \mathbf{P}^1 \times \mathbf{P}^{n-1} \arrow[dr, dash] & \\
\mathbf{P}(\mathrm{std} \otimes \mathrm{std}_n) \arrow[r, dash] & \{\det = 0\} \arrow[ur, dash] \arrow[dr, dash] & & X \cap (\mathbf{P}^1 \times \mathbf{P}^{n-1}) \\
& & X \arrow[ur, dash] &
\end{tikzcd}$$

FIGURE 2. $\mathrm{SL}_2 \times \mathrm{SO}_n$-orbit closures on $\mathrm{std} \otimes \mathrm{std}_n$, connected by closure. The generic orbit is given by the nonvanishing of $\det$. If an element of $\mathrm{std} \otimes \mathrm{std}_n$ is given by $e_1 \otimes v_1 + e_2 \otimes v_2$ with $v_1, v_2 \in \mathrm{std}_n$, the subvariety X has codimension 3, and is cut out by $\begin{pmatrix} \langle v_1,v_1\rangle & \langle v_1,v_2\rangle \\ \langle v_2,v_1\rangle & \langle v_2,v_2\rangle \end{pmatrix} = 0$. Moreover, the inclusion $\mathbf{P}^1 \times \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^{2n-1}$ is the Segre embedding, i.e., is cut out by $v_1 \wedge v_2 = 0 \in \wedge^2 \mathrm{std}_n$.

The motivation for this example comes from attempting to generalize the discussion in [**Bha**, Section 4]. Indeed, removing the vertex in the Dynkin diagram of type D_{j+2} which is connected to the affine root in the extended Dynkin diagram defines a maximal parabolic subgroup P of SO_{2j+2}, and the simply-connected form of its Levi quotient L is $\mathrm{SL}_2 \times \mathrm{SO}_{2j}$. If U denotes the unipotent radical of P, then L acts on the vector space $U/[U,U]$ by conjugation, and the Lie bracket on U defines a symplectic form on $U/[U,U]$. One can check that $U/[U,U] \cong \mathrm{std} \otimes \mathrm{std}_{2j}$ as a symplectic $L \cong \mathrm{SL}_2 \times \mathrm{SO}_{2j}$-representation. A similar construction with the Dynkin diagram of type B_{j+2} produces $\mathrm{SL}_2 \times \mathrm{SO}_{2j+1}$ acting on $\mathrm{std} \otimes \mathrm{std}_{2j+1}$. Doing this procedure for the other Dynkin diagrams produces some of the "vectorial" examples in one of the columns in [**BZSV**, Table 1.5.1]; for example, the type A_{n+1} Dynkin diagram produces GL_n acting on $T^*(\mathrm{std}_n)$, and I believe the type E_6 Dynkin diagram will produce SL_6 acting on $\wedge^3 \mathrm{std}_6$. I do not yet understand the significance of this observation in the context of relative geometric Langlands.

Proposition 2.13. *Let $\check{J}$ denote the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathrm{Spec}\, k[a^2]$ of regular centralizers for SL_2, so that*

$$\check{J} \cong \mathrm{Spec}\, k[a, \alpha^{\pm 1}, \tfrac{\alpha - \alpha^{-1}}{a}]^{\mathbf{Z}/2},$$

where the action of $\mathbf{Z}/2$ sends $a \mapsto -a$ and $\alpha \mapsto \alpha^{-1}$, and the group structure is such that α is grouplike. Then there is an isomorphism

$$\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J})$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 = \mathrm{Spec}\, k[a^2]$; of course, this group scheme is in turn isomorphic to $\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J}$.

PROOF. The fiber product on the left identifies with the subgroup of $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times \check{G}$ of those $(a^2, \vec{g})$ such that $\vec{g} = (g_1, g_2, g_3) \in \mathrm{SL}_2^{\times 3}$ stabilizes $\kappa(a^2)$. The trick to determining this stabilizer is to use Bhargava's construction from Construction 2.3: if $\vec{g}$ stabilizes a cube $\mathcal{C}$, it must also stabilize the corresponding triple $\mu(\mathcal{C}) \in \mathfrak{sl}_2^{\times 3}$ of quadratic forms.

First, a simple calculation shows that if a is a unit, the triple of matrices

$$\vec{g} = \left(\sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & a^{-1} \\ a & 1 \end{pmatrix} \right) \in \mathrm{SL}_2^{\times 3}$$

sends

$$\kappa(a^2) \mapsto -\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0}).$$

The triple $\vec{g}$ can be thought of as "diagonalizing" $\kappa(a^2)$. The stabilizer of the cube $-\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0})$ precisely consists of triples of matrices of the form

$$\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3^{-1} \end{pmatrix}\right) \text{ with } \alpha_1\alpha_2\alpha_3 = 1. \tag{2}$$

For $\alpha \in \mathbf{G}_m$, let $h(\alpha)$ denote the matrix

$$h(\alpha) = \frac{1}{2}\begin{pmatrix} \alpha + \alpha^{-1} & \frac{\alpha^{-1}-\alpha}{a} \\ a^2 \cdot \frac{\alpha^{-1}-\alpha}{a} & \alpha + \alpha^{-1} \end{pmatrix} \in \mathrm{SL}_2.$$

Conjugating (2) by the element $\vec{g} \in \check{G}$, we find that the triple $(h(\alpha_1), h(\alpha_2), h(\alpha_3))$ of matrices stabilizes $\kappa(a^2)$ as long as $\alpha_1\alpha_2\alpha_3 = 1$ and $a^2 \in \mathbf{G}_m \subseteq \mathbf{A}^1$. (See [**BFM**, Section 3.2] for a slight variant of this calculation.) Note that the subgroup of such triples is 2-dimensional, and therefore the associated homogeneous $\check{G}$-space is $9-2 = 7$-dimensional. Using that the $\check{G}$-orbit of $\kappa(a^2)$ is also 7-dimensional (e.g., by [**GKZ**, Example 4.5 in Chapter 14]), it is not hard to see from this calculation (by a limiting argument for $a \to 0$) that the stabilizer of the family $\kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ is precisely the claimed group scheme. □

Remark 2.14. A direct calculation shows that the stabilizer of $\kappa(0)$ is isomorphic to the subgroup of triples of matrices of the form $\left(\begin{smallmatrix} a_i & \gamma_i \\ 0 & a_i^{-1} \end{smallmatrix}\right)$ for $1 \leq i \leq 3$ with $(a_1, a_2, a_3) \in \mu_2^{\times 3}$ such that $a_1a_2a_3 = 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = 0$. This subgroup is, of course, isomorphic to $(\mu_2 \times \mathbf{G}_a)^{\times 2}$; it is also isomorphic to the fiber over $a = 0$ of the group scheme of Proposition 2.13.

In the more general setting of Remark 2.12, the stabilizer $\mathbf{A}^1/\!\!/(\mathbf{Z}/2) \times_{(\mathrm{std}\otimes\mathrm{std}_n)/(\mathrm{SL}_2\times\mathrm{SO}_n)} \mathbf{A}^1/\!\!/(\mathbf{Z}/2)$ of the Kostant slice is a group scheme over $\mathbf{A}^1/\!\!/(\mathbf{Z}/2)$ of relative dimension $\binom{n-2}{2} + 1$. The case $n = 3$ is of course the simplest, and we discuss it below in Proposition 4.13 (if one replaces SO_3 by $\mathrm{Spin}_3 \cong \mathrm{SL}_2$). Proposition 2.13 corresponds to the case $n = 4$ (if one replaces SO_4 by Spin_4). I believe $n = 3, 4$ are the only cases where this stabilizer group scheme is commutative. It would be interesting to understand this stabilizer group scheme as a subgroup of $\mathrm{SL}_2 \times \mathrm{SO}_n \times \mathbf{A}^1/\!\!/(\mathbf{Z}/2)$ for more general n.

Remark 2.15. As mentioned in Remark 1.7, the quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is studied in quantum information theory. For instance, in [**DVC**], Dür-Vidal-Cirac study the orbit structure of $\mathrm{SL}_2^{\times 3}$ acting on $\mathrm{std}^{\otimes 3}$ (in particular, they recover Figure 1 independently of [**GKZ**]). See also [**CKW**], where the Cayley hyperdeterminant is rediscovered as [**CKW**, Equations 20 and 21].

For the interested reader, let us describe the translation between our notation/terminology and that of quantum information theory. Our base field will now be $k = \mathbf{C}$. An element of $\mathrm{std}^{\otimes n}$ (really, of the projective space $\mathbf{P}(\mathrm{std}^{\otimes n}) \cong \mathbf{P}^{2^n-1}$) is called an *$n$-qubit*, and the action of $\mathrm{SL}_2^{\times n}$ is via *stochastic local operations and classical communication* (SLOCC) operators (replacing $\mathrm{SL}_2^{\times n}$ by $\mathrm{GL}_2^{\times n}$ simply amounts to dropping the word "stochastic"). The space std is equipped with a basis $\{|0\rangle, |1\rangle\}$, and a cube $\mathcal{C} = (a, \vec{b}, c, \vec{d}) \in \mathrm{std}^{\otimes 3}$

corresponds to the three-qubit[3]

$$a|000\rangle + b_1|100\rangle + b_2|010\rangle + b_3|001\rangle$$
$$+d_1|011\rangle + d_2|101\rangle + d_3|110\rangle + c|111\rangle.$$

Here, the bra-ket notation $|ijk\rangle$ means $|i\rangle \otimes |j\rangle \otimes |k\rangle$. The state

$$\tfrac{1}{\sqrt{2}}(1, \vec{0}, 1, \vec{0}) = \tfrac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

is known as the *Greenberger–Horne–Zeilinger* (GHZ) state, and the state

$$\tfrac{1}{\sqrt{3}}\kappa(0) = \tfrac{1}{\sqrt{3}}(0, \vec{1}, 0, \vec{0}) = \tfrac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

is called the *W* state. These two states are known to represent two very different kinds of quantum entanglement; from the perspective of this article, the reason for this is simply that the Cayley hyperdeterminant of the GHZ state is nonzero, but the Cayley hyperdeterminant of the W state vanishes. Nevertheless, the proof of Proposition 2.13 shows that there is a natural *degeneration* of (the SLOCC/$\mathrm{SL}_2^{\times 3}$-equivalence class of) the GHZ state into the W state. Indeed, the GHZ state can be transformed into the cube $\frac{1}{2}\kappa(1)$, which admits a natural degeneration to the W state via the one-parameter family

$$\tfrac{1}{\sqrt{a^4+3}}\kappa(a^2) = \tfrac{1}{\sqrt{a^4+3}}\left(a^2|000\rangle + |011\rangle + |101\rangle + |110\rangle\right).$$

In fact, this state already appears as [**DVC**, Equation 20].

We need one final algebraic construction.

Construction 2.16. Fix an integer n. Then the $\check{G}$-variety $\mathrm{std}^{\otimes 3}$ admits a natural grading, where the entries of a cube $(a, \vec{b}, c, \vec{d})$ have the following weights: a lives in weight $-4n$, b lives in weight $-2n$, c lives in weight $2n$, and d lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the associated graded variety. Equip $\mathfrak{sl}_2$ with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ have the following weights: a lives in weight $-2n$, b lives in weight 0, and c lives in weight $-4n$. Similarly, equip SL_2 with the grading coming from $2n\rho$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. With these gradings, the $\mathrm{SL}_2^{\times 3}$-equivariant map $\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}$ is a *graded* map, and κ defines a graded map $\mathfrak{sl}_2(2n)/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1(4n) \to \mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$. The cases $n = 1$ and $n = 3$ will be relevant below (corresponding to Theorem 1.2 and Theorem 1.4, respectively).

3. The proof

Before proceeding, let us remind the reader of the definition of the left-hand side of the equivalence of Theorem 1.2, following [**Dev**, Definition 3.6.1].

Definition 3.1. Let G be a complex reductive group, and let $H \subseteq G$ be a closed subgroup such that $H \subseteq G$ is a spherical subgroup. Let $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$ denote the ∞-category of $G[\![t]\!]$-equivariant sheaves of $\mathbf{Q}$-modules on $G(\!(t)\!)/H(\!(t)\!)$ which are constructible for the orbit stratification on $G(\!(t)\!)/H(\!(t)\!)$. Note that since the orbit stratification is countable (by assumption that $H \subseteq G$ is a spherical subgroup and [**GN**, Theorem 3.2.1]), the ∞-category $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$ is well-behaved. There is a natural left-action of the $\mathbf{E}_3$-monoidal ∞-category $\mathrm{Shv}^c_{(G\times G)[\![t]\!]}(G(\!(t)\!); \mathbf{Q})$ on $\mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$,

[3]Technically, a qubit is required to have norm 1, so one must rescale $\mathcal{C}$ by $\sqrt{a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2}$; but this could in theory introduce a singularity when $a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2 = 0$. We will ignore this (important!) point below.

and in particular, a left-action of $\mathrm{Rep}(\check{G})$ by the abelian geometric Satake theorem of [**MV**]. Let

$$\mathrm{IC}_0 \in \mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$$

denote the pushforward $i_! \underline{\mathbf{Q}}$ of the constant sheaf along the inclusion $(G/H)(\mathbf{C}[\![t]\!]) \to (G/H)(\mathbf{C}(\!(t)\!))$. Let

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q}) \subseteq \mathrm{Shv}^c_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); \mathbf{Q})$$

denote the full subcategory generated by IC_0 under the action of $\mathrm{Rep}(\check{G})$. If k is any $\mathbf{Q}$-algebra, base-changing along the unit map defines the ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k)$.

PROOF OF THEOREM 1.2. It suffices to verify conditions (a) and (b) of [**Dev**, Theorem 3.6.5], which gives a criterion for establishing an equivalence of k-linear ∞-categories of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}).$$

The map κ is given by the map $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \to \mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})$ from Construction 2.9. For condition (a) of [**Dev**, Theorem 3.6.5], we need to show that if $\check{J}_X = \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})/\check{G}} \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2$, the ring of regular functions on the quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ is isomorphic (as a graded algebra) to $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$. The quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ identifies with the $\check{G}$-orbit of the image of κ, which has complement of codimension 3 in $\mathrm{std}^{\otimes 3}$ by Proposition 2.10; therefore, the algebraic Hartogs theorem implies that there is a graded isomorphism $\mathcal{O}_{(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X} \cong \mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$.

For condition (b) of [**Dev**, Theorem 3.6.5], we need to check that there is an isomorphism

$$\check{J}_X \cong \mathrm{Spec}\, \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k)$$

of graded group schemes over $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \cong \mathrm{Spec}\, \mathrm{H}^*_{\mathrm{PGL}_2}(*; k)$. There is an isomorphism

$$\mathrm{Spec}\, \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k) \cong \mathrm{Spec}\, k[a, \alpha^{\pm 1}, \tfrac{\alpha - \alpha^{-1}}{a}]^{\mathbf{Z}/2} \cong \check{J}, \tag{3}$$

and the action of the $\mathbf{Z}/2$ on the middle term sends $a \mapsto -a$ and $\alpha \mapsto \alpha^{-1}$. This is proved, e.g., in [**BFM**], and also follows from [**Dev**, Example 3.6.16]. (As in Proposition 2.13, $\check{J}$ denotes the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ of regular centralizers for SL_2.) The Künneth theorem implies that there is an isomorphism

$$\mathrm{Spec}\, \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}); k) \cong \mathrm{Spec}\, k[a, \alpha_i^{\pm 1}, \tfrac{\alpha_i - \alpha_i^{-1}}{a} | 1 \leq i \leq 3]^{\mathbf{Z}/2},$$

and the fiber sequence

$$\mathrm{PGL}_2^{\mathrm{diag}} \to \mathrm{PGL}_2^{\times 3} \to \mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$$

implies that

$$\mathrm{Spec}\, \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k) \cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J}).$$

The desired isomorphism now follows from this observation and Proposition 2.13. □

Remark 3.2. Specializing [**Dev**, Remark 3.6.10] to the present case, I expect that an object of $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k)$ is compact if its image under the equivalence of Theorem 1.2 is set-theoretically supported on the vanishing locus of the Cayley hyperdeterminant.

Remark 3.3. The proof of Theorem 1.4 is exactly the same as the proof of Theorem 1.2 above. Indeed, one only needs to observe that $\mathrm{PSO}_8/\mathrm{G}_2$ is homotopy equivalent to $\mathbf{R}P^7 \times \mathbf{R}P^7$ (which follows, e.g., from the fact that $\mathrm{Spin}_8/\mathrm{G}_2 \simeq S^7 \times S^7$)[4]. The replacement of (3) is given by [**Dev**, Proposition 4.8.6], which gives an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{G}_2}(\Omega \mathbf{R}P^7; k) \cong \operatorname{Spec} k[a, b, \alpha^{\pm 1}, \tfrac{\alpha-\alpha^{-1}}{a}]^{\mathbf{Z}/2}$$

where a is in weight -6 and b is in weight -4.

Remark 3.4. Remark 2.6 guarantees that the equivalence of Theorem 1.2 is compatible with the action of the spherical Hecke category $\mathrm{Shv}^{c,\mathrm{Sat}}_{(\mathrm{PGL}_2^{\times 3}\times \mathrm{PGL}_2^{\times 3})[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2-2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho))$. Namely, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}^{c,\mathrm{Sat}}_{(\mathrm{PGL}_2^{\times 3}\times \mathrm{PGL}_2^{\times 3})[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!); k) \arrow[r, "\sim", "\text{[BF]}"'] \arrow[d, "\text{act on } \mathrm{IC}_0"] & \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2-2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho)) \arrow[d, "\mu^*"] \\
\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \arrow[r, "\sim", "\text{Theorem 1.2}"'] & \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2^{\times 3}(-2\rho)),
\end{tikzcd}$$

where μ^* is given by pullback along the moment map for the Hamiltonian $\mathrm{SL}_2^{\times 3}$-action on $\mathrm{std}^{\otimes 3}$.

Let us also note that taking cohomology (i.e., pushforward to a point) defines a functor

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \to \mathrm{Shv}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(*; k),$$

which, as discussed in [**Dev**, Remark 3.5.9], factors through the functor $\mathrm{Shv}_{\mathrm{PGL}_2^{\mathrm{diag}}[\![t]\!]}(*; k) \to \mathrm{Shv}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(*; k)$. Under Theorem 1.2 and the equivalence $\mathrm{Shv}_{\mathrm{PGL}_2^{\mathrm{diag}}[\![t]\!]}(*; k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2)$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \arrow[r, "\sim", "\text{Theorem 1.2}"'] \arrow[d, "\text{cohomology}"'] & \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2^{\times 3}(-2\rho)) \arrow[d, "\kappa^*"] \\
\mathrm{Shv}_{\mathrm{PGL}_2^{\mathrm{diag}}[\![t]\!]}(*; k) \arrow[r, "\sim"'] & \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2),
\end{tikzcd}$$

where κ is the Kostant slice of Construction 2.9.

Remark 3.5. Theorem 1.2 does not need the full strength of optimality in the sense [**Dev**, Hypothesis 3.6.3]. Indeed, the first and second assumptions in [**Dev**, Hypothesis 3.6.3] are included to ensure formality of the algebra from [**Dev**, Equation 16 in the proof of Theorem 3.6.5]. However, as in [**Dev**, Remark 3.2.22], the formality of this algebra is

[4]Perhaps the most "conceptual" way to see that $\mathrm{Spin}_8/\mathrm{G}_2 \simeq S^7 \times S^7$ is as follows. Using triality, one can identify Spin_8 with the subgroup of $\mathrm{SO}_8^{\times 3}$ of those triples (A_1, A_2, A_3) such that $A_1(x_1)A_2(x_2) = A_3(x_1x_2)$ for octonions x_1, x_2. Under this presentation, G_2 corresponds to the subgroup where $A_1 = A_2 = A_3$. The subgroups where $A_1 = A_3$ (resp. $A_2 = A_3$) are both isomorphic to $\mathrm{Spin}(7)$; these are sometimes denoted $\mathrm{Spin}^{\pm}(7)$. The action of Spin_8 on $S^7 \times S^7$ sends $(x, y) \mapsto (A_1x, A_2y)$; one can check that this is transitive, and that the stabilizer of the point $(1, 1)$ is precisely $\mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7) \cong \mathrm{G}_2$.

That there is an equivalence $\mathrm{Spin}_8/\mathrm{G}_2 \simeq S^7 \times S^7$ at the level of cohomology with $\mathbf{Z}[1/2]$-coefficients, at least, is much simpler: on group cohomology, the map $\mathrm{G}_2 \to \mathrm{Spin}_8$ is given by the map $\mathbf{Z}[1/2, p_1, p_2, p_3, c_4] \to \mathbf{Z}[1/2, c_2, c_6]$ sending $p_1 \mapsto -c_2$, $p_2 \mapsto 0$, $p_3 \mapsto -c_6$, and $c_4 \mapsto 0$. The Serre spectral sequence for the fibration $\mathrm{Spin}_8/\mathrm{G}_2 \to B\mathrm{G}_2 \to B\mathrm{Spin}_8$ implies that $\mathrm{H}^*(\mathrm{Spin}_8/\mathrm{G}_2; \mathbf{Z}[1/2]) \cong \mathbf{Z}[1/2, \sigma(p_2), \sigma(c_4)]/(\sigma(p_2)^2, \sigma(c_4)^2)$, where $\sigma(p_2)$ and $\sigma(c_4)$ both live in (homological) weight -7. This is precisely the cohomology of $S^7 \times S^7$, as desired.

guaranteed in our case: since Theorem 1.2 shows that the homotopy of the algebra in question is $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$, i.e., is polynomial on classes in even weights. This algebra admits an $\mathbf{E}_3$-structure (essentially from factorization; see, e.g., [**BZSV**, Proposition 16.1.4]), and is therefore automatically formal by [**Dev**, Lemma 2.1.9]. Note, however, that since $\mathrm{Ind}_{\mathrm{SL}_2^{\times 3}}^{\mathrm{Spin}_8}(\mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1)$ is not an affine space, this argument does not go through in the case of Theorem 1.4 to prove formality of the algebra from [**Dev**, Equation 16 in the proof of Theorem 3.6.5].

Remark 3.6. Let $k = \mathbf{Q}_2(\zeta_8)$. The theory of 2-compact groups as studied, e.g., in [**AG**], suggests viewing the Dwyer-Wilkerson space DW_3 from [**DW**] as an analogue of the groups $\mathrm{SO}_3 \cong \mathrm{PGL}_2$ and G_2; see Table 1. The 2-complete space DW_3 is equipped with an $\mathbf{E}_1$-structure, and it has finite mod 2 cohomology. It is therefore natural to ask whether there is an analogue of Theorem 1.2 and Theorem 1.4, where PGL_2 and G_2 are replaced by DW_3; this is closely related to [**Dev**, Appendix C(p)].

Group G_n	Rank n	Dimension $(2^{n+1}-1)n$	$\mathbf{F}_2$-cohomology of BG $\widehat{\mathrm{Sym}}^*(\mathbf{F}_2^{n+1}(-1))^{\mathrm{GL}_{n+1}(\mathbf{F}_2)}$	Weyl group $\mathbf{Z}/2 \times \mathrm{GL}_n(\mathbf{F}_2)$
PGL_2	1	3	$\mathbf{F}_2[\![w_2, w_3]\!]$	$\mathbf{Z}/2$
G_2	2	14	$\mathbf{F}_2[\![w_4, w_6, w_7]\!]$	$\mathbf{Z}/2 \times \Sigma_3$
DW_3	3	45	$\mathbf{F}_2[\![w_8, w_{12}, w_{14}, w_{15}]\!]$	$\mathbf{Z}/2 \times \mathrm{PSL}_2(\mathbf{F}_7)$

TABLE 1. Analogies between the (2-compact) groups $\mathrm{PGL}_2 = \mathrm{SO}_3$, G_2, and DW_3; all of these are Poincaré duality complexes of dimension indicated in the third column. Here, w_n denotes the nth Stiefel-Whitney class, and the ring in the fourth column is known as the algebra of rank $n+1$ Dickson invariants. Note, also, that the Weyl group of DW_3 is called G_{24} in the Shephard-Todd classification.

It is difficult to answer this question since the representation theory of DW_3 is not well-understood. For instance, one can ask whether there is a 2-compact group G with an $\mathbf{E}_1$-map $\mathrm{DW}_3 \to G$ such that G/DW_3 is the 2-completion of a framed 30-manifold with Kervaire invariant one; see [**Jon**] for a construction of such a 30-manifold. This desideratum is analogous to the equivalences $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2 \cong \mathbf{R}P^3 \times \mathbf{R}P^3$ and $\mathrm{PSO}_8/\mathrm{G}_2 \cong \mathbf{R}P^7 \times \mathbf{R}P^7$. If such a G exists, and there is a good theory of $G[\![t]\!]$-equivariant sheaves of k-modules, it seems reasonable to expect that there is an equivalence of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{DW}_3(\!(t)\!);k) \cong \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(28,\vec{14},-14,\vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3} \times \mathbf{A}^2(8,12)).$$

Here, the "Whittaker" factor $\mathbf{A}^2(8,12)$ on the right-hand side comes from the isomorphism

$$\mathrm{Spf}\,\mathrm{H}^*(B\mathrm{DW}_3;k) := \mathrm{Spf}\,\mathrm{H}^*(B\mathrm{DW}_3;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong \widehat{\mathbf{A}}^3(8,12,28),$$

which follows from running the Bockstein spectral sequence on

$$\mathrm{H}^*(B\mathrm{DW}_3;\mathbf{F}_2) \cong \mathbf{F}_2[\![w_8, w_{12}, w_{14}, w_{15}]\!],$$

and the fact that the Bockstein sends $w_{14} \mapsto w_{15}$.[5] Despite this, one can wonder about the analogue of the "regular centralizer" group scheme calculation from Theorem 1.2:

[5] A previous version of this remark asked for G/DW_3 to be $\mathbf{R}P^{15} \times \mathbf{R}P^{15}$. One can check that such a G, if it existed, would have rational cohomology given by

$$\mathrm{H}^*(BG;\mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong k[\![c_4, c_6, c_{14}, x, y]\!],$$

- Is there a good notion of *genuine* equivariant DW_3-cohomology (with coefficients in $k = \mathbf{Q}_2(\zeta_8)$, say)? One should have $\mathrm{Spec}\,\mathrm{H}^*_{\mathrm{DW}_3}(*; k) \cong \mathbf{A}^3(8, 12, 28)$.
- Is there a faithful (basepoint-preserving) action of DW_3 on S^{15}? Similarly, is there a faithful (basepoint-preserving) action of DW_3 on the 2-completion of a framed 30-manifold M^{30} with Kervaire invariant one?
- For the above expected action, is there an isomorphism

$$\mathrm{Spec}\,\mathrm{H}_*^{\mathrm{DW}_3}(\Omega M^{30}; k) \cong \mathbf{A}^2(8, 12) \times (\mathbf{A}^1(28) \times_{\mathrm{std}^{\otimes 3}(28, \vec{14}, -14, \vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3}} \mathbf{A}^1(28))$$

of graded group schemes over k?

4. Variants

In this section, we will take the base field k to be $\mathbf{C}$.

Remark 4.1. As in [**Bha**], understanding the $\mathrm{SL}_2^{\times 3}$-equivariant geometry of cubes can be specialized to understand variant situations. We will sketch some such variants below. Somewhat more precisely, let $D \in \mathbf{Z}$ be a fundamental discriminant, and let $\delta \in \mathbf{A}^1/\!\!/(\mathbf{Z}/2)$. The observations motivating the discussion in this section are the main results of [**Bha**], and the analogy, likely already observed by the reader familiar with Bhargava's work, between the fiber $\{\delta\} \times_{\mathbf{A}^1/\!\!/(\mathbf{Z}/2)} B_{\mathbf{A}^1/\!\!/(\mathbf{Z}/2)}\check{J}$ and the narrow class group of the quadratic extension $\mathbf{Q}(\sqrt{D})$ of $\mathbf{Q}$.

Let us begin with the case of squares/degree 2 extensions. More precisely, recall from [**BF**] that the (arithmetically sheared; see [**BZSV**]) derived geometric Satake equivalence for PGL_2 states:

Theorem 4.2 (Bezrukavnikov-Finkelberg)**.** *There is an equivalence of ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 2}[\![t]\!]}(\mathrm{PGL}_2^{\times 2}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^*(2-2\rho)/\mathrm{SL}_2(-2\rho)).$$

The latter can alternatively be understood as the ∞-category of perfect complexes on a shearing of $T^*(\mathrm{SL}_2)/\mathrm{SL}_2^{\times 2}$.

As described in [**Dev**, Section 3.2], one key input into Theorem 4.2 is that if $\check{J}$ denotes the group scheme of regular centralizers for SL_2, with the embedding $\check{J} \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ via $g \mapsto (g, g^{-1})$, the affine closure of the quotient $(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}$ is isomorphic to $T^*\mathrm{SL}_2$.

One can ask for a *variant* of Theorem 4.2, where the embedding $\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!) \hookrightarrow \mathrm{PGL}_2^{\times 2}(\!(t)\!)$ is replaced by the embedding of $\mathrm{PGL}_2(\!(t)\!)$ into $(\mathrm{Res}_{\mathbf{C}[\![t^{1/2}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2)(\!(t)\!)$. The latter is the base-change to $\mathbf{C}(\!(t)\!)$ of the Weil restriction of the constant group scheme PGL_2 along $\mathbf{C}[\![t]\!] \subseteq \mathbf{C}[\![t^{1/2}]\!]$. In this case, we have the following expectation.

Conjecture 4.3. *Let $G = \mathrm{Res}_{\mathbf{C}[\![t^{1/2}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$. Then there is an equivalence of ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(B\mu_2 \times \mathfrak{sl}_2^*(2)/\!\!/\mathrm{SL}_2),$$

where the left-hand side is the full subcategory of $\mathrm{Shv}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k)$ generated by $\mathrm{IC}_0 = i_!\underline{\mathbf{Q}}$ under the action of $\mathrm{Rep}(\check{G})$, where $i : G[\![t]\!]/\mathrm{PGL}_2[\![t]\!] \hookrightarrow G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!)$.

---

where both x and y live in cohomological degree 16. In an email, Jesper Grodal told me that such a G cannot exist (it would have to be the 2-completion of a compact Lie group, but no compact Lie group has the desired cohomology).

Although [**Dev**, Theorem 3.6.5] does not apply in this situation, because G is not the base-change of a constant group scheme over $\mathbf{C}$, we can nevertheless attempt to compute the analogue of the regular centralizer group scheme for the pair $\mathrm{PGL}_2 \hookrightarrow G$. The following is a helpful tool in understanding these variant cases.

Lemma 4.4. *Let n be an integer, so that the degree n map $S^2 \to S^2$ induces an $\mathbf{E}_1$-endomorphism*[6] *$[n]$ of $\Omega^2 B\mathrm{PGL}_2 \simeq \Omega\mathrm{PGL}_2$. Under the homotopy equivalence $\Omega\mathrm{PGL}_2 \simeq \mathrm{PGL}_2(\!(t)\!)/\mathrm{PGL}_2[\![t]\!]$, the map $[n]$ is induced by the map $\mathbf{C}(\!(t)\!) \to \mathbf{C}(\!(t^{1/n})\!)$. Moreover, under the isomorphism*

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k) \cong \check{J}$$

of (3)*, the map $[n]^* : \mathcal{O}_{\check{J}} \to \mathcal{O}_{\check{J}}$ induced by $[n] : \Omega\mathrm{PGL}_2 \to \Omega\mathrm{PGL}_2$ is given by multiplication by n on the ring of functions $\mathcal{O}_{\check{J}}$.*

Remark 4.5. Although the map $[n] : \Omega\mathrm{PGL}_2 \to \Omega\mathrm{PGL}_2$ is only an $\mathbf{E}_1$-endomorphism, it can be shown that the induced endomorphism of $C_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k)$ is one of $\mathbf{E}_2$-k-algebras. Upon Borel-completion (so $C_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k)$ is replaced by $k[\Omega\mathrm{PGL}_2]^{h\mathrm{PGL}_2}$), this is a consequence of the observation that $k[\Omega\mathrm{PGL}_2]^{h\mathrm{PGL}_2}$ is an $\mathbf{E}_3$-k-algebra: it is the $\mathbf{E}_2$-Hochschild cohomology of $k[\Omega\mathrm{PGL}_2]$, and the $\mathbf{E}_3$-structure comes from the Deligne conjecture. Alternatively, the completion of $C_*^{\mathrm{PGL}_2}(\Omega\mathrm{PGL}_2; k)$ at the cellular filtration of $\Omega\mathrm{PGL}_2$ can be identified with the $\mathbf{E}_2$-Hochschild cohomology of $C^*_{\mathrm{PGL}_2}(*; k)$, and the $\mathbf{E}_3$-structure again comes from the Deligne conjecture. See [**Dev**, Corollary 3.5.11].

Let us now explain our reasoning behind Conjecture 4.3.

Remark 4.6. Lemma 4.4 suggests that the analogue of the regular centralizer group scheme for Conjecture 4.3 is the the 2-torsion subgroup $\check{J}[2]$ of $\check{J}$; following [**Dev**, Theorem 3.6.5], one expects $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k)$ to be the ∞-category of perfect complexes on (a shearing of) the SL_2-quotient of the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[2]$. Now, as described in [**BFM**], $\check{J}$ can be viewed as the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \operatorname{Spec} k[y]$ of matrices of the form $g = \left(\begin{smallmatrix} a & b \\ by & a \end{smallmatrix}\right)$ with $\det(g) = a^2 - b^2 y = 1$.

To understand $\check{J}[2]$, note that $g^2 = \left(\begin{smallmatrix} a^2+b^2y & 2ab \\ 2aby & a^2+b^2y \end{smallmatrix}\right)$, and since 2 is a unit in k, we find that g is 2-torsion if and only if $ab = 0$ and $a^2 + b^2 y = 1$. But since $\det(g) = 1$, this forces $a^2 = 1$ and $b = 0$. In other words, $\check{J}[2]$ is isomorphic to the constant group scheme $\mu_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. It follows that $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[2] \cong \mathrm{PGL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. No affine closure is necessary, since this is already affine. This suggests that $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k)$ might be equivalent to the ∞-category of perfect complexes on a shearing of the SL_2-quotient of $\mathrm{PGL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$, i.e., the ∞-category $\mathrm{Perf}^{\mathrm{sh}}((\mathrm{PGL}_2 \times \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2)/\mathrm{SL}_2) \simeq \mathrm{Perf}^{\mathrm{sh}}(B\mu_2 \times \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2)$.

Let us now turn to the cubic case. Here, Theorem 1.2 gives an equivalence (under the assumption of optimality)

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2^{\mathrm{diag}}(\!(t)\!)); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}).$$

One can ask for variants of Theorem 1.2. Here, there are two choices: the group scheme $\mathrm{PGL}_2^{\times 3}$ over $\mathbf{C}[\![t]\!]$ can be replaced either by the Weil restriction $\mathrm{Res}_{\mathbf{C}[\![t^{1/3}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$

[6] This is an analogue for $\mathbf{E}_2$-spaces of the observation that if X is a space, the degree n map $S^1 \to S^1$ induces a map $\Omega X \to \Omega X$ which sends $\gamma \to \gamma^n$. This is only a pointed map, and not necessarily an $\mathbf{E}_1$-map, since taking powers is generally not a map of monoids. However, if X is itself an $\mathbf{E}_1$-space, so that ΩX is an $\mathbf{E}_2$-space, the map $\gamma \to \gamma^n$ is one of $\mathbf{E}_1$-spaces.

along the cubic extension $\mathbf{C}[\![t]\!] \subseteq \mathbf{C}[\![t^{1/3}]\!]$, or by the Weil restriction $\mathrm{Res}_{\mathbf{C}[\![t^{1/2}]\!]\times\mathbf{C}[\![t]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$ along the extension $\mathbf{C}[\![t]\!] \subseteq \mathbf{C}[\![t^{1/2}]\!] \times \mathbf{C}[\![t]\!]$. For each of these cases, we have an expected answer (in parallel to parts of [**Bha**, Section 2]). In the first case, here is a variant of Theorem 1.2.

Conjecture 4.7. *Let $\mathrm{Sym}^3(\mathrm{std})(4,2,0,-2)$ denote the graded vector space of binary cubic forms, where such a form is viewed as a function $\mathbf{A}^2(-2,0) \to \mathbf{A}^1(-2)$. In other words, the coefficients of $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ have the following weights: a lives in weight -4, b lives in weight -2, c lives in weight 0, and d lives in weight 2. Let $G = \mathrm{Res}_{\mathbf{C}[\![t^{1/3}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$. Then there is an equivalence of ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!);k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{Sym}^3(\mathrm{std})(4,2,0,-2)/\mathrm{SL}_2(-2\rho)).$$

Unfortunately, [**Dev**, Theorem 3.6.5] does not apply in this situation, because G is not the base-change of a constant group scheme over $\mathbf{C}$. Nevertheless, we expect that Conjecture 4.7 is a consequence of Lemma 4.4 and Proposition 4.8 below; together, these results should show an analogue of the criteria of [**Dev**, Theorem 3.6.5] in the present situation. Note, also, that the example of SL_2 acting on $\mathrm{Sym}^3(\mathrm{std})$ does *not* fit into the formalism of [**BZSV**], since it is not hyperspherical in the sense of *loc. cit.* (see [**BZSV**, Example 5.1.10]).

Proposition 4.8. *Let $V = \mathrm{Sym}^3(\mathrm{std})$ denote the 4-dimensional affine space of binary cubic forms, so that V admits an action of SL_2. Then:*

(a) Let $\Delta : V \to \mathbf{A}^1$ denote the map sending a binary cubic form $f = ax^3+3bx^2y+3cxy^2 + dy^3$ to its discriminant

$$\Delta(f) = a^2d^2 - 6abcd - 3b^2c^2 + 4(ac^3 + b^3d).$$

Then Δ defines an isomorphism $V/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$.

(b) The closed immersion $\kappa : \mathbf{A}^1 \to V$ sending $a \mapsto -\frac{a}{4}x^3 + 3xy^2$ defines a section of Δ, and the SL_2-orbit of the image of κ has complement of codimension ≥ 2.

(c) Identify $\mathbf{A}^1 = \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$, let $\check{J}$ denote the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ of regular centralizers for SL_2, and let $\check{J}[3]$ denote its 3-torsion subgroup. Then there is an isomorphism

$$\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \check{J}[3]$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. In particular, the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[3]$ is SL_2-equivariantly isomorphic to V.

PROOF. The first statement is in [**PV**, Section 0.12], and the second statement can be deduced similarly. For the final statement, recall as in [**Bha**] that there is a closed immersion $V \subseteq \mathrm{std}^{\otimes 3}$ given by $ax^3 + 3bx^2y + 3cxy^2 + dy^3 \mapsto (a, \vec{b}, d, \vec{c})$. This corresponds to the triply-symmetric cube

$$\begin{tikzcd}
& b \arrow[rr, dash] & & c \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[ur, dash] \arrow[dd, dash] & \\
& c \arrow[uu, dash] \arrow[rr, dash] & & d. \\
b \arrow[ur, dash] \arrow[rr, dash] & & c \arrow[ur, dash] &
\end{tikzcd}$$

The above embedding is SL_2-equivariant for the natural action on V and the diagonally embedded $\mathrm{SL}_2^{\mathrm{diag}} \subseteq \mathrm{SL}_2^{\times 3} = \check{G}$ acting on $\mathrm{std}^{\otimes 3}$. Moreover, the composite

$$V \subseteq \mathrm{std}^{\otimes 3} \xrightarrow{\det} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$$

sends $f \mapsto \Delta(f)$. This implies that $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ can be identified with the intersection $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ with the diagonally embedded $\mathrm{SL}_2^{\mathrm{diag}} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \subseteq \mathrm{SL}_2^{\times 3} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. By Proposition 2.13, we find that

$$\begin{aligned}\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/\mathrm{SL}_2} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 &\cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J}) \cap (\mathrm{SL}_2^{\mathrm{diag}} \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2) \\ &\cong \check{J}[3].\end{aligned}$$

The claim about the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[3]$ follows from (b). □

Remark 4.9. The poset of SL_2-orbit closures in $\mathbf{P}(\mathrm{Sym}^3(\mathrm{std}))$ is shown in Figure 3.

$$\mathbf{P}(\mathrm{Sym}^3(\mathrm{std})) \;\text{——}\; \{\Delta = 0\} \;\text{——}\; \mathbf{P}^1$$

FIGURE 3. SL_2-orbit closures on $\mathrm{Sym}^3(\mathrm{std})$, connected by closure. The inclusion $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$ is the embedding of the twisted cubic.

Remark 4.10. More generally, one might expect that if n is a positive integer and $G = \mathrm{Res}_{\mathbf{C}[\![t^{1/n}]\!]/\mathbf{C}[\![t]\!]}\mathrm{PGL}_2$, the ∞-category $\mathrm{Shv}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k)$ is equivalent to the ∞-category of perfect complexes on a shearing of the affine closure of $(\mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}[n]$. This affine closure should be a 4-dimensional SL_2-scheme, and its invariant-theoretic quotient by this SL_2-action is $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. However, I do not have a more explicit description of this affine closure for general n (for instance, I do not even know whether it is a scheme of finite type!).

Remark 4.11. For completeness, let us describe the story for more general binary forms. Let us begin with a variant of Proposition 4.8 for binary *quartic* forms; since we will not need this result, and the proof is somewhat orthogonal to the methods of this article, we will simply state the relevant facts. Let $V = \mathrm{Sym}^4(\mathbf{A}^2)$ denote the 5-dimensional affine space of binary quartic forms, so that V admits an action of PGL_2.

(a) Let $\pi : V \to \mathbf{A}^2$ denote the map sending a binary quartic form $f = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^3$ to the invariants
$$\begin{aligned}I &= ae - 4bd + 3c^2, \\ J &= ace + 2bcd - ad^2 - b^2e - c^3.\end{aligned}$$
Then π defines an isomorphism $V/\!\!/\mathrm{PGL}_2 \cong \mathbf{A}^2$.

(b) The closed immersion $\kappa : \mathbf{A}^2 \to V$ sending $(a, b) \mapsto 4x^3y + dxy^3 + ey^4$ defines a section of π, and the PGL_2-orbit of the image of κ has complement of codimension ≥ 2. In fact, the PGL_2-orbit consists of those binary quartic forms with at least one root of multiplicity 1.

(c) Let $\mathcal{E}$ denote the elliptic curve over $\mathbf{A}^2 = \mathrm{Spec}\, k[d, e] \cong \mathfrak{sl}_3/\!\!/\mathrm{SL}_3$ given by $y^2 = x^3 + dx + e$, and let $\mathcal{E}[2]$ denote its 2-torsion subgroup. Then there is an isomorphism
$$\mathbf{A}^2 \times_{V/\mathrm{PGL}_2} \mathbf{A}^2 \cong \mathcal{E}[2]$$
of group schemes over $\mathbf{A}^2$. In particular, the affine closure of $(\mathrm{PGL}_2 \times \mathbf{A}^2)/\mathcal{E}[2]$ is PGL_2-equivariantly isomorphic to V.

Parts (a) and (b) are not difficult calculations, and part (c) can be proved as in [**CF**, Sections 3-5] and [**BS**, Theorem 3.2]. Finally, we note that if $V = \mathrm{Sym}^j(\mathbf{A}^2)$ denote the $(j+1)$-dimensional affine space of binary j-forms, so that V admits an action of SL_2, the invariant-theoretic quotient $V/\!\!/\mathrm{SL}_2$ is *not* an affine space if $j \geq 5$ (see [**PV**, Example 1 in Section 8.2]). In other words, the action of SL_2 on V is not coregular, and hence the quotient stack V/SL_2 *cannot* appear on the spectral side of [**BZSV**, Conjecture 7.5.1] (the coregularity restriction is explained in the discussion following [**Dev**, Conjecture 3.5.9]).

Along similar lines, here is a variant of Theorem 1.2 for the other variant of $\mathrm{PGL}_2^{\times 3}(\!(t)\!)$.

Conjecture 4.12. *Let $(\mathrm{std} \otimes \mathfrak{sl}_2)(4, 2, 0, -2)$ denote the graded vector space of pairs of binary quadratic forms, where the coefficients of a pair $(q_1, q_2) = (ax^2+2bxy+cy^2, dx^2+2exy+fy^2)$ have the following weights: a lives in weight -4, b lives in weight -2, c lives in weight 0, d lives in weight -2, e lives in weight 0, and f lives in weight 2. In other words, $(\mathrm{std} \otimes \mathfrak{sl}_2)(4, 2, 0, -2) \cong \mathbf{A}^2(2, 0) \otimes \mathfrak{sl}_2(-2\rho)$. Let $G = \mathrm{Res}_{\mathbf{C}[\![t^{1/2}]\!] \times \mathbf{C}[\![t]\!]/\mathbf{C}[\![t]\!]} \mathrm{PGL}_2$. Then there is an equivalence of ∞-categories*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k) \simeq \mathrm{Perf}^{\mathrm{sh}}((\mathrm{std} \otimes \mathfrak{sl}_2)(4, 2, 0, -2)/\mathrm{SL}_2(-2\rho)^{\times 2}).$$

Again, [**Dev**, Theorem 3.6.5] does not apply in this situation, because G is not the base-change to $\mathbf{C}(\!(t)\!)$ of a constant group scheme over $\mathbf{C}$. Nevertheless, we expect that Conjecture 4.7 is a consequence of Lemma 4.4 and Proposition 4.13 below; together, these results should give an analogue of the criteria of [**Dev**, Theorem 3.6.5].

Proposition 4.13. *Let $V = \mathrm{std} \otimes \mathfrak{sl}_2$, equipped with an action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ via the SL_2-actions on std and $\mathfrak{sl}_2$. Then:*

(a) Let $\Delta : V \to \mathbf{A}^1$ denote the map sending a pair of binary quadratic forms $(q_1, q_2) = (ax^2 + 2bxy + cy^2, dx^2 + 2exy + fy^2)$ to the function

$$\Delta(q_1, q_2) = a^2f^2 + c^2d^2 - 2acdf + 4(ae - bd)(ce - bf).$$

Then Δ defines an isomorphism $V/\!\!/(\mathrm{SL}_2 \times \mathrm{SL}_2) \cong \mathbf{A}^1$.

(b) The closed immersion $\kappa : \mathbf{A}^1 \to V$ sending $a \mapsto (\frac{a}{4}x^2 + y^2, 2xy)$ defines a section of Δ, and the $\mathrm{SL}_2 \times \mathrm{SL}_2$-orbit of the image of κ has complement of codimension ≥ 2.

(c) Identify $\mathbf{A}^1 = \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$, let $\check{J}$ denote the group scheme over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ of regular centralizers for SL_2, and define the embedding

$$\check{J} \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2,\ g \mapsto (g^{-2}, g).$$

Note that this is indeed a homomorphism since $\check{J}$ is commutative. Then there is an isomorphism

$$\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/(\mathrm{SL}_2 \times \mathrm{SL}_2)} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \check{J} \subseteq \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2$. In particular, the affine closure of $(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)/\check{J}$ is $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariantly isomorphic to V.

PROOF SKETCH. These statements follow exactly as in Proposition 4.8. Indeed, as described in [**Bha**], there is a closed immersion $V \subseteq \mathrm{std}^{\otimes 3}$ given by

$$(ax^2 + 2bxy + cy^2, dx^2 + 2exy + fy^2) \mapsto (a, (b, d, b), f, (e, c, e)).$$

This corresponds to the doubly-symmetric cube

$$\begin{tikzcd}
& d \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & e \arrow[dl, dash] \arrow[dd, dash] \\
a \arrow[rr, dash, crossing over] \arrow[dd, dash] & & b \arrow[dd, dash] & \\
& e \arrow[rr, dash] \arrow[dl, dash] & & f. \arrow[dl, dash] \\
b \arrow[rr, dash] & & c &
\end{tikzcd}$$

The above morphism is $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariant for the natural action on V and the embedding

$$\iota : \mathrm{SL}_2 \times \mathrm{SL}_2 \subseteq \mathrm{SL}_2^{\times 3} = \check{G},\ (g,h) \mapsto (g,h,h).$$

One can check that the composite

$$V \subseteq \mathrm{std}^{\otimes 3} \xrightarrow{\det} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$$

sends $(q_1, q_2) \mapsto \Delta(q_1, q_2)$. Finally, as in Proposition 4.8, we find that

$$\begin{aligned}
\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times_{V/(\mathrm{SL}_2\times\mathrm{SL}_2)} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 &\cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{prod}} \check{J}) \cap (\iota(\mathrm{SL}_2 \times \mathrm{SL}_2) \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2) \\
&\cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{(g,h)\mapsto gh^2} \check{J}) \\
&\cong \check{J},
\end{aligned}$$

where $\check{J}$ is a subgroup of $(\iota(\mathrm{SL}_2 \times \mathrm{SL}_2) \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2)$ via $g \mapsto (g^{-2}, g)$, as desired. $\square$

Remark 4.14. The inclusion $\check{J} \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ from Proposition 4.13(c) can be alternatively described as the composite inclusion

$$\check{J} \cong \ker(\check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \xrightarrow{\mathrm{id}\times[2]^*} \check{J}) \hookrightarrow \check{J} \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \check{J} \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathfrak{sl}_2/\!\!/\mathrm{SL}_2.$$

Remark 4.15. The poset of $\mathrm{SL}_2 \times \mathrm{SL}_2$-orbit closures in $\mathbf{P}(\mathrm{std} \otimes \mathfrak{sl}_2)$ is shown in Figure 4.

$$\mathbf{P}(\mathrm{std} \otimes \mathfrak{sl}_2) \text{ ——— } \{\Delta = 0\} \text{ ——— } \mathbf{P}^1 \times \mathbf{P}^2 \text{ ——— } \mathbf{P}^1 \times \mathbf{P}^1$$

FIGURE 4. $\mathrm{SL}_2 \times \mathrm{SL}_2$-orbit closures on $\mathbf{P}(\mathrm{std} \otimes \mathfrak{sl}_2)$, connected by closure. The inclusion $\mathbf{P}^1 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ is the Segre embedding, and the embedding $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2$ is induced by the inclusion of a quadric $\mathbf{P}^1 \subseteq \mathbf{P}^2$.

Remark 4.16. Let us conclude by mentioning that there is a variant of Proposition 4.13 for the vector space $V = \mathrm{std} \otimes \mathrm{Sym}^3(\mathrm{std})$ of pairs of binary *cubic* forms, equipped with its natural $\mathrm{SL}_2 \times \mathrm{SL}_2$-action. Again, we will not use this, and they follow from the work of Bhargava-Ho in [**BH**], so we will just state the relevant results for the sake of completeness. It turns out that there is an isomorphism $V/\!\!/(\mathrm{SL}_2 \times \mathrm{SL}_2) \cong \mathbf{A}^2$, where the invariants, denoted a_1 and a_3, are of degrees 2 and 6, respectively. In particular, there is an isomorphism

$$V(2)/\!\!/(\mathrm{SL}_2 \times \mathrm{SL}_2) \cong \mathbf{A}^2(4, 12) \cong \mathrm{Spec}\, \mathrm{H}^*_{\mathrm{G}_2}(*; \mathbf{Q}),$$

where the symbol $V(2)$ means that the coordinates of V are placed in weight -2. In fact, the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on V descends to an action of SO_4, and it turns out that $V \cong \mathfrak{so}_4^\perp \subseteq \mathfrak{g}_2^*$ under the embedding $\mathrm{SO}_4 \subseteq \mathrm{G}_2$.

As in Remark 4.11, there is a Kostant slice $\kappa : \mathbf{A}^2 \cong V/\!\!/(\mathrm{SL}_2 \times \mathrm{SL}_2) \to V$, and it turns out that there is an isomorphism

$$\mathbf{A}^2 \times_{V/(\mathrm{SL}_2\times\mathrm{SL}_2)} \mathbf{A}^2 \cong \mathcal{E}_{\Gamma_1(3)}[2],$$

where $\mathcal{E}_{\Gamma_1(3)}$ is the universal elliptic curve $y^2 + a_1 xy + a_3 y = x^3$ with a $\Gamma_1(3)$-level structure over $\mathbf{A}^2$. The reader is referred to [**BH**] (see in particular [**BH**, Line 6 of Table 1]) for a detailed study of this example, from which the above claims can be deduced.

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