Integrable systems

S. K. Devalapurkar

Lecture 2: The Legendre transform and Hamiltonian mechanics

Simple dualities often have profound mathematical consequences. In this lecture, we will study in a bit more detail the Legendre transform introduced last time, and discuss its implications for modeling mechanics. Previously, we introduced the Legendre transform as an operation on a vector bundle $\mathcal{E}$ on a manifold M associated to a map $L : \mathcal{E} \to \mathbf{R}$. As always with the theory of vector bundles, this is the globalization of some simple procedure at the level of vector spaces, so let us study that first.

Definition 1. Let V be a vector space (think of as a vector bundle over a point, if you like), and let $L : V \to \mathbf{R}$ be a map. The *Legendre transform* of L is the map $\Phi_L : V \to V^*$ which sends v to the linear map $w \mapsto \frac{d}{dt} L(v + tw)\big|_{t=0}$, i.e., the directional derivative. If you wish, $\Phi_L(v)$ is the Jacobian of L, evaluated at v.

Can the map Φ_L be inverted? Generally not; for example, say $L : \mathbf{R} \to \mathbf{R}$ is the map $\exp(x)$. Then $\Phi_L : \mathbf{R} \to \mathbf{R}$ is again the map $x \mapsto \exp(x)$, so its image is $(0, \infty)$. Let us try to understand what it means to invert Φ_L in the case when $V = \mathbf{R}$. The map $\Phi_L : V \to V^*$ is just the map $x \mapsto L'(x)$, so we need to find a composition inverse to L'. Ideally, we could do this by constructing a function $L^* : V^* \to \mathbf{R}$ such that $\Phi_{L^*} = \Phi_L^{-1}$, i.e., such that

$$(L^*)'(x) = (L')^{-1}(x).$$

Let us write $f(x) = (L')^{-1}(x)$. Then $f(x)$ is a critical point of the function

$$y \mapsto xy - L(y) \tag{1}$$

because the derivative of this function is $x - L'(y)$, which vanishes when $y = f(x)$. Let us therefore make an ansatz:

$$L^*(x) := xf(x) - L(f(x)).$$

Then

$$(L^*)'(x) = f(x) + xf'(x) - L'(f(x))f'(x) = f(x) = (L')^{-1}(x),$$

as desired. But how do we make L^* more implicitly defined in terms of x and L? Because $f(x)$ is a critical point of (1), we could simply try to define a function

---

Part of this work was done when the author was supported by NSF DGE-2140743.

$L^* : \mathbf{R} \to \mathbf{R}$ by

$$L^*(x) := \sup_{y \in \mathbf{R}} (xy - L(y)). \tag{2}$$

Of course, one needs some assumptions to know that L^* is well-defined. If $L(y)$ is *convex*, then L^* is always well-defined. Let us make the definition for a general vector space:

Definition 2. Given a function $L : V \to \mathbf{R}$, let $L^* : V^* \to \mathbf{R}$ denote the function

$$L^*(p) = \sup_{q \in V} (\langle q, p \rangle - L(q)).$$

We will not prove the next result, but the main idea is already visible in the 1-dimensional case, where it appears as Theorem 14.C in Arnold's book.

Theorem 3. *Suppose $L : V \to \mathbf{R}$ is strongly convex, meaning that the Hessian of L is a positive-definite matrix (at each point of V). Then L^* is well-defined on the image of Φ_L and strongly convex, $\Phi_{L^*} = \Phi_L^{-1}$ (defined on the image of Φ_L), and $(L^*)^* = L$. In particular, Φ_L is a diffeomorphism onto its image.*

Furthermore, if L has quadratic growth at ∞ (i.e., there is a positive-definite quadratic form Q on V and a constant C such that $L(v) \geq Q(v) - C$ for all $v \in V$), then Φ_L in fact defines an isomorphism $V \xrightarrow{\sim} V^*$. This is [**Can01**, Exercise 5 on page 126].

Exactly the same result holds for vector bundles. Namely:

Theorem 4. *Let $\mathcal{E}$ be a vector bundle on a smooth manifold, and suppose $L : \mathcal{E} \to \mathbf{R}$ is strongly convex and has quadratic growth at ∞. Then Φ_L defines a diffeomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^\vee$, and its inverse is given by the map $L^* : \mathcal{E}^* \to \mathbf{R}$ defined as*

$$L^*(v) = \langle v, \Phi_L^{-1}(v) \rangle - L(\Phi_L^{-1}(v)).$$

Naturally, we are interested in what this says when $\mathcal{E} = TM$ and L is a Lagrangian. Let us therefore assume throughout that L is strongly convex and that it has quadratic growth at ∞, and let us write

$$H : T^*M \to \mathbf{R}$$

to denote the Legendre transform L^*; this will be called the *Hamiltonian.* If (q, v) are coordinates on TM, the coordinates on T^*M will be denoted (q, p). Therefore,

$$H(q, p) = \langle p, v \rangle - L(x, v),$$

where $p = \Phi_L(v) = \frac{\partial L}{\partial v}$ is the *conjugate momentum.*

Remark 5. The cotangent bundle T^*M is called the *phase space*, and M is called the *configuration space* of the system.

Example 6. Suppose that $L : TM \to \mathbf{R}$ is given by $\langle \dot{q}, \dot{q} \rangle / 2$ for some metric on M. Then the formula for the Legendre transform tells us that $H : T^*M \to \mathbf{R}$ is given by $\langle p, p \rangle / 2$.

How does Lagrangian mechanics as we studied it last lecture translate under this Legendre transform? Let us begin by rephrasing the Euler-Lagrange equations. Recall that these equations stated that $q : \mathbf{R} \to M$ minimizes the action if

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q};$$

note that the left-hand side is $\dot{p}$. Therefore, we find that

$$dL(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} = \langle \dot{p}, dq \rangle + \langle p, d\dot{q} \rangle,$$

which means that

$$dH(q(t), p(t)) = d(\langle p, \dot{q} \rangle - L) = \langle dp, \dot{q} \rangle + \langle p, d\dot{q} \rangle - dL = \langle dp, \dot{q} \rangle - \langle \dot{p}, dq \rangle.$$

But we can also expand $dH(q(t), p(t))$ directly as

$$dH(q(t), p(t)) = \left\langle \frac{\partial H}{\partial p}, dp \right\rangle + \left\langle \frac{\partial H}{\partial q}, dq \right\rangle.$$

But this implies that

$$\dot{q} = \frac{\partial H}{\partial p}, \ \dot{p} = -\frac{\partial H}{\partial q}. \tag{3}$$

These equations together are called *Hamilton's equations.*

Remark 7. Suppose M is a vector space. Hamilton's equations display a remarkable symmetry in p and q: namely, these equations remain invariant under interchanging $(p, q) \mapsto (-q, p)$. In particular, these equations treat q and p on equal footing, and suggests that one should really think of classical mechanics on a manifold M as describing solutions to the above equations for a path $(q(t), p(t)) : \mathbf{R} \to T^*M$, where H is some smooth function $T^*M \to \mathbf{R}$. Said more succinctly: the Euler-Lagrange equations are second-order differential equations describing paths in M, while Hamilton's equations are first-order differential equations describing paths in T^*M.

Here is a slightly different way of thinking about these things. One can write down the Euler-Lagrange equation for paths $q : \mathbf{R} \to TM$ sending $t \mapsto (q(t), \dot{q}(t))$, and when we think about such a path as coming from a path on M, we are imposing the condition that $\dot{q} = \frac{dq}{dt}$. Therefore, one could really view the Euler-Lagrange equation for paths on M as describing the *constrained* action

$$S = \int_{t_0}^{t_1} \left(p\frac{dq}{dt} - H(p, q) \right) dt = \int_{t_0}^{t_1} \left(L(q, \dot{q}) - p\left(\dot{q} - \frac{dq}{dt} \right) \right) dt \tag{4}$$

Varying this action in the usual manner, one finds that

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{dp}{dt} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} - p \right) \delta \dot{q} + \left(\frac{dq}{dt} - \dot{q} \right) \delta p dt.$$

In particular, $\delta S = 0$ if and only if

$$\frac{\partial L}{\partial q} = \dot{p}, \ \frac{\partial L}{\partial \dot{q}} = p, \ \frac{dq}{dt} - \dot{q}.$$

The first two equations are the Euler-Lagrange equations and the final is the constraint discussed above. Since the Euler-Lagrange equations are *equivalent* to Hamilton's equations, one finds:

Lemma 8. *Stationary variations of the constrained action I from* (4) *describe Hamilton's equations* (3).

We actually saw the constrained action I in the very first lecture (on Zoom). Namely, observe that one can rewrite

$$S = \int_{q(t_0)}^{q(t_1)} pdq - \int_{t_0}^{t^1} Hdt = \oint dp \wedge dq - \int_{t_0}^{t^1} Hdt,$$

and we saw that in the case of the harmonic oscillator, $\oint dp \wedge dq$ was the as the area of the ellipse traced out by motion in phase space. I will return to this point later.

It will be convenient to make the following observation. Define a vector field X_H on T^*M by

$$X_H = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}.$$

Then:

Proposition 9. *A curve $f : \mathbf{R} \to T^*M$ satisfies the flow equation*

$$\dot{f} = X_H(f)$$

if and only if $\Phi_L^{-1}(f) : \mathbf{R} \to TM$ satisfies the Euler-Lagrange equation.

Proof. Suppose that f satisfies the flow equation. Since

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}$$

and

$$X_H(f) = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p},$$

we see that f satisfies the Hamilton's equation (3). The converse is similar. $\square$

We have described how to translate the Euler-Lagrange equations under the Legendre transform; now we will see how to translate Noether's theorem. Proposition 9 suggests that we should think about symmetries of vector fields. Let's recall that Noether's theorem says the following. Fix a 1-parameter family of symmetries $\{f_s\}$ of M; these symmetries were only required to be infinitesimal, so we will actually think of this as described by a vector field ξ on M (so $\delta q = \xi(q)$). Then, there is a conserved quantity $\mathcal{N}_\xi : TM \to \mathbf{R}$ given by $\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle$. Remember what this means: for any curve $\gamma : \mathbf{R} \to TM$, the quantity $\mathcal{N}_\xi(\gamma) : \mathbf{R} \to \mathbf{R}$ has vanishing derivative. Note that the quantity $\frac{\partial L}{\partial \dot{q}}$ is the Legendre transform of $\dot{q}$, so we can think of $\mathcal{N}_\xi$ as the function

$$\Phi_L(\mathcal{N}_\xi) : T^*M \to \mathbf{R},\ (p,q) \mapsto \langle p, \delta q \rangle.$$

Let us call this function J_ξ. Here are two observations about J_ξ.

Observation 10. One could think of the function $J_\xi(p,q)$ as the pairing of the 1-form pdq on T^*M with the vector field ξ (pulled back to T^*M from M via $T^*M \to M$). This means that the 1-form dJ_ξ on T^*M is can be thought of as the pairing of the 2-form $dp \wedge dq$ on T^*M with ξ. Note that we have already seen this 2-form $dp \wedge dq$ before (in studying the constrained action).

Observation 11. If ξ is a *family* of symmetries, given say by the action of a Lie algebra $\mathfrak{g}$ on M by vector fields (i.e., by a map $\mathfrak{g} \to T_M$), then we could think of the assignment $\xi \mapsto J_\xi$ as a map

$$T^*M \xrightarrow{\mu} \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbf{R}),\ (q,p) \mapsto [\xi \mapsto J_\xi(p,q)].$$

This is an example of a *moment map*. Let us note the following basic property of the moment map, coming from the preceding observation. If $\xi \in \mathfrak{g}$, then one has the following equality of 1-forms on T^*M:

$$d\langle \mu, \xi \rangle = \langle dp \wedge dq, \xi \rangle.$$

Here, $\langle \mu, \xi \rangle : T^*M \to \mathbf{R}$ is the conserved quantity J_ξ from before. The above equation is extremely important (perhaps not evidently so now), and we will study it in greater detail later when talking about more general moment maps.

Example 12. Recall that we considered rotations of $\mathbf{R}^3$ via the infinitesimal action of $\mathfrak{so}_3$ on $\mathbf{R}^3$. In this case, we computed that when $L = \langle \dot{q}, \dot{q} \rangle / 2$, so that $H = \langle p, p \rangle / 2$, the moment map $T^*\mathbf{R}^3 \to \mathfrak{so}_3^* \cong \mathbf{R}^3$ was given by the cross product $(p, q) \mapsto p \times q$.

In the next lecture, we will introduce symplectic manifolds, give an interpretation of the assignment $H \mapsto X_H$ of functions to vector fields, and talk about Poisson brackets. This will give us a nice way of thinking about *Liouville's theorem*, which we will also discuss next time.

References

[Can01] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

1 Oxford St, Cambridge, MA 02139
Email address: `sdevalapurkar@math.harvard.edu, February 2, 2024`