NILPOTENT SINGULAR SUPPORT AND THE HECKE ACTION

At the moment, I've only really finished writing up the proof of part (a) of the main theorem. You can look at the proof of part (b) if you wish, but it'll be cleaned up over the coming week (it's "real math").

1. Main results and notation

Let us begin with some notation. Let k be a fixed algebraically closed ground field, and let X be a (nice enough) stack over X. Then $\mathrm{Shv}(X)$ can refer to Ind of any of the following: constructible sheaves on the underlying space of X if $k = \mathbf{C}$ (Betti context); holonomic D-modules on X if $\mathbf{Q} \subseteq k$ (de Rham context); constructible $\overline{\mathbf{Q}}_\ell$-adic sheaves on X if $\ell \in k^\times$ (ℓ-adic context). In each of these cases, $\mathrm{QLisse}(X)$ denotes the full subcategory of $\mathrm{Shv}(X)$ spanned by those $\mathcal{F}$ whose cohomology is a local system of finite rank in the Betti context, a $\mathcal{O}$-coherent D-module (i.e., vector bundle with flat connection) in the de Rham context, or an ℓ-adic local system in the ℓ-adic context.

If $\mathcal{Y}$ is an algebraic stack and a conical Zariski-closed subset $\mathcal{N} \subseteq T^*(\mathcal{Y})$ (I believe this means cotangent space, and not the derived bundle associated to the cotangent complex), one can associate the full subcategory $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subseteq \mathrm{Shv}(\mathcal{Y})$ of sheaves with singular support contained in $\mathcal{N}$ (defined as sheaves whose pullback along any smooth map from a scheme has singular support contained in the preimage of $\mathcal{N}$). One can view $\mathrm{QLisse}(X)$ as $\mathrm{Shv}_{\{0\}}(X)$, where $\{0\} \subseteq T^*(X)$ denotes the zero section. We will need the following lemma:

Lemma 1.1. *Let X be a smooth proper scheme, and let $\mathcal{N} \subseteq T^*(\mathcal{Y})$ be half-dimensional. Denote by $\mathcal{N}'$ the product $\mathcal{N} \times \{0\} \subseteq T^*(\mathcal{Y} \times X)$. Then $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}(X) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{N}'}(\mathcal{Y} \times X)$.*

Let X be a smooth projective curve, and let G be a reductive group. Here is the main result. Let $\mathcal{N} \subseteq T^*(\mathrm{Bun}_G)$ denote the nilpotent cone.

Theorem 1.2. *The following statements hold:*

(a) *The Hecke functor $\mathrm{H}(-,-) : \mathrm{Rep}(G^\vee) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X)$ sends $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ to $\mathrm{Shv}_{\mathcal{N}'}(\mathrm{Bun}_G \times X)$.*

(b) *Conversely, if $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is such that $\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$ has singular support contained in $T^*(\mathrm{Bun}_G) \times \{0\}$, then $\mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$.*

The goal of this talk is to prove this theorem, the first part of which is due to Nadler and Yun (granting the lemma, which we will not prove). Before doing so, let us just mention its importance: it is firstly an important conceptual result, telling us that the condition of nilpotent singular support is essentially characterized by its relationship with the Hecke functors; and second, using "Beilinson's spectral projector", whatever that is, you can get a bunch of structural results about Hecke categories.

2. Proof of part (a)

Let us begin by proving part (a) of the theorem, i.e., that the condition of nilpotent singular support is preserved by the Hecke action. Recall how the Hecke functor is defined: let Hecke_X denote the global Hecke stack (of pairs of G-bundles $(\mathcal{P}, \mathcal{P}')$ on X along with a point $x \in X$ and an isomorphism $\alpha : \mathcal{P}|_{X-x} \xrightarrow{\sim} \mathcal{P}'|_{X-x}$ of G-bundles away from x). Then, we have morphisms

$$\begin{tikzcd} \mathrm{Bun}_G & \mathrm{Hecke}_X \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[d, "s"] & \mathrm{Bun}_G \\ & X & \end{tikzcd}$$

which project onto each of the components of the tuple $(\mathcal{P}, \mathcal{P}', x)$. The morphism s factors through the local Hecke stack $\mathrm{Hecke}_X^{\mathrm{loc}}$, where $\mathcal{P}$, $\mathcal{P}'$, and α are only defined on the formal disk D_x around the point

$x \in X$. Noting that sheaves on $\mathrm{Bun}_G(D_x)$ is (essentially) $G(\mathcal{O})$-equivariant sheaves on Gr_G, we can view $V \in \mathrm{Rep}(G^\vee)$ as giving rise to a sheaf $\mathrm{Sat}_V \in \mathrm{Shv}(\mathrm{Hecke}_X^{\mathrm{loc}})$ via geometric Satake. The map s then factors as $\mathrm{Hecke}_X \xrightarrow{r} \mathrm{Hecke}_X^{\mathrm{loc}} \xrightarrow{\pi} X$, and the Hecke functor is given by $\mathrm{H}(V, \mathcal{F}) = (\overrightarrow{h} \times s)_*(\overrightarrow{h} \times r)^!(\mathcal{F} \boxtimes \mathrm{Sat}_V)$. Our diagram above can be extended as follows:

$$\begin{tikzcd} \mathrm{Bun}_G & \mathrm{Hecke}_X \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[d, "r"] & \mathrm{Bun}_G \\ & \mathrm{Hecke}_X^{\mathrm{loc}} \arrow[d, "\pi"] & \\ & X & \end{tikzcd}$$

Importantly, $\overrightarrow{h} \times r$ is pro-smooth, while $\overleftarrow{h} \times s$ is ind-proper. Using the appropriate pro/ind modification of the following lemma, we can rephrase the condition in the theorem.

Lemma 2.1. *The following statements are true:*

- *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map, and let $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$. Then $\mathrm{SingSupp}(f^! \mathcal{F})$ is the image of $\mathrm{SingSupp}(\mathcal{F}) \times_{\mathcal{Y}} \mathcal{X}$ along $Df : T^*(\mathcal{Y}) \times_{\mathcal{Y}} \mathcal{X} \to T^*(\mathcal{X})$.*
- *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper map, and let $\mathcal{F} \in \mathrm{Shv}(\mathcal{X})$. Then $\mathrm{SingSupp}(f_* \mathcal{F})$ lives inside the image of $(Df)^{-1}\mathrm{SingSupp}(\mathcal{F})$ along the projection $T^*(\mathcal{Y}) \times_{\mathcal{Y}} \mathcal{X} \to T^*(\mathcal{Y})$.*

We can now rephrase the condition in the theorem as follows. We want to prove something about the Hecke action on $\mathcal{P} \in \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ indexed by $V \in \mathrm{Rep}(G^\vee)$, so assume that $\xi \in T^*_{\mathcal{P}}(\mathrm{Bun}_G)$ is nilpotent, and suppose that $\xi_H \in T^*_{(x, \mathcal{P}, \mathcal{P}', \alpha)} \mathrm{Hecke}_X^{\mathrm{loc}}$ is in $\mathrm{SingSupp}(\mathrm{Sat}_V)$. Let $\xi' \in T^*_{\mathcal{P}'}(\mathrm{Bun}_G)$ and $\xi_X \in T^*_x X$ be such that

$$(d\overleftarrow{h})^*(\xi) + (dr)^*(\xi_H) = (d\overrightarrow{h})^*(\xi') + (ds)^*(\xi_X).$$

(This just says that $(\xi', \xi_X) \in T^*_{(\mathcal{P}', x)}(\mathrm{Bun}_G \times X)$ is in the singular support of $\mathrm{H}(V, \mathcal{P})$.) We need to show that ξ' is nilpotent, and that $\xi_X = 0$.

To show that ξ' is nilpotent, observe that we can work locally (i.e., just fix $x \in X$ for now), and that our condition above becomes

$$(d\overleftarrow{h}_x)^*(\xi) + (dr_x)^*(\xi_H) = (d\overrightarrow{h}_x)^*(\xi'). \tag{1}$$

We need to understand the cotangent space of $\mathrm{Hecke}_x = \mathrm{Hecke}_X^{\mathrm{loc}}|_{D_x}$. Since Hecke_x consists of tuples $(\mathcal{P}, \mathcal{P}', \alpha)$ with $\mathcal{P}$ and $\mathcal{P}'$ being G-bundles on D_x and $\alpha : \mathcal{P}|_{\mathring{D}_x} \xrightarrow{\sim} \mathcal{P}'|_{\mathring{D}_x}$, and we know that infinitesimally varying $\mathcal{P}$ just gives rise to a copy of $T^*_{\mathcal{P}}(\mathrm{Bun}_G) \cong \Gamma(D_x; \mathfrak{g}^*_{\mathcal{P}} \otimes \omega_X)$ (and similarly for $\mathcal{P}'$), we only need to understand what varying α gets us. But this is easy: if $j : \mathring{D}_x \to D_x$ is the inclusion, then α induces a map

$$\mathfrak{g}_{\mathcal{P}} \oplus \mathfrak{g}_{\mathcal{P}'} \to j_*(j^* \mathfrak{g}_{\mathcal{P}} \xrightarrow{\sim} j^* \mathfrak{g}_{\mathcal{P}'})$$

which sends $(v, w) \mapsto v - w$. The above diagram can be viewed as a complex K, and

$$T^*_{(\mathcal{P}, \mathcal{P}', \alpha)} = R^1\Gamma(X; K)^* = \Gamma(X; \mathrm{H}^1(K))^*.$$

Explicitly, this consists of pairs $(\xi_1, \xi_2) \in \mathfrak{g}^*_{\mathcal{P}} \oplus \mathfrak{g}^*_{\mathcal{P}'}$ such that $\alpha(\xi_1) = \xi_2$ in $\Gamma(\mathring{D}_x; \mathfrak{g}^*_{\mathcal{P}'} \otimes \omega_X)$.

Returning to our situation above, we see that (1) tells us that $\alpha(\xi|_{\mathring{D}_x}) = \xi'|_{\mathring{D}_x}$. Since ξ is nilpotent (on D_x itself), we see that $\xi'|_{\mathring{D}_x}$ is nilpotent, and therefore that ξ' is itself nilpotent, as desired.

Let us now show that $\xi_X = 0$. For this, we first claim that the short exact sequence

$$0 \to T^*_x X \to T^*_{(x, \mathcal{P}, \mathcal{P}', \alpha)} \mathrm{Hecke}_X \to T^*_{(\mathcal{P}, \mathcal{P}', \alpha)} \mathrm{Hecke}_x \to 0$$

admits a canonical splitting (similarly for $\mathrm{Hecke}_X^{\mathrm{loc}}$); let's call this splitting $d\pi$. To see where this splitting comes from, recall the definition of a crystal over X: this is a prestack $Y \to X$ such that there exists some $Y' \to X_{\mathrm{dR}}$ for which there is a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "\pi"] & Y' \arrow[d] \\ X \arrow[r, "q"'] & X_{\mathrm{dR}}; \end{tikzcd}$$

in other words, Y descends along the quotient $X \to X_{\mathrm{dR}}$. The condition of being a crystal can be understood as follows: if x, x' are two R-points of X for some ring R whose R^{red}-points agree, then lifting x to $y \in Y(R)$ is equivalent to lifting x' to (necessarily the "same") $y \in Y(R)$.

By standard base-change results, we know that $T_{Y/X} = q^* T_{Y'/X_{\mathrm{dR}}}$. Since X_{dR} has no tangent space, we see that $T_{Y'/X_{\mathrm{dR}}} = TY'$. Therefore, one gets a map $q^*TY' \cong T_{Y/X} \to TY$. This map splits the exact sequence

$$0 \to T_{q(y)}Y' \to T_y Y \to T_{\pi(x)} X \to 0.$$

Taking duals, we see that the corresponding short exact sequence of cotangent spaces splits. Therefore, to get the splitting $d\pi$ for the Hecke stack, it suffices to show that Hecke_X admits the structure of a crystal over X. Indeed, if $x, x' \in X$ are two R-points of X (for some ring R) whose R^{red}-points agree, then the (punctured) formal disks around x and x' are the same, as do the complements $X \times \mathrm{Spec}(R) - \Gamma_x$ and $X \times \mathrm{Spec}(R) - \Gamma_{x'}$. Therefore, the data of a pair of G-bundles on X and an isomorphism on X away from x is the same as the data of a pair of G-bundles on X and an isomorphism on X away from x' (and similarly for the local Hecke stack). This is exactly what it means to be a crystal.

Returning back to our story: the existence of the splitting $d\pi : T^*_{(x,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}_X \to T^*_x X$ implies that the composite of the maps

$$(d\overleftarrow{h})^* : T^*_{\mathcal{P}}\mathrm{Bun}_G \to T^*_{(x,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}_X, \; (d\overrightarrow{h})^* : T^*_{\mathcal{P}'}\mathrm{Bun}_G \to T^*_{(x,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}_X$$

with $d\pi$ are zero.

Now, ξ_X is the image of $\xi_H \in T^*_{(x,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}_X$ under the aforementioned splitting. This means that $\xi_X = 0$ if we show the following: if $\xi_H \in T^*_{(x,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}^{\mathrm{loc}}_X$ lies in $\mathrm{SingSupp}(\mathrm{Sat}_V)$ and the corresponding pair $(\xi, \xi') \in \Gamma(D_x, (\mathfrak{g}^*_{\mathcal{P}} \oplus \mathfrak{g}^*_{\mathcal{P}'}) \otimes \omega_X)$ is nilpotent, then the image of ξ_H along the splitting $d\pi$ is zero. This claim is local around $x \in X$, so we might as well assume $X = \mathbf{A}^1$ and x is the origin.

Recall that we had a splitting map $T^*_{(0,\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}_{\mathbf{A}^1} \to T^*_0\mathbf{A}^1 = k$, and we want to show that the image of a nilpotent ξ_H under this map is zero. There is *another* splitting map, given by the isomorphism $\mathrm{Hecke}^{\mathrm{loc}}_{\mathbf{A}^1} \simeq \mathrm{Hecke}^{\mathrm{loc}}_0 \times \mathbf{A}^1$, where $\mathrm{Hecke}^{\mathrm{loc}}_0 = L^+G\backslash LG/L^+G$. The difference between these two splittings can be viewed as a map $f : T^*_{(\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}^{\mathrm{loc}}_0 \to T^*_0\mathbf{A}^1 = k$, and it suffices to show that the image of a nilpotent ξ_H under *this* map f is zero. (Indeed, if a, b are elements in a Lie algebra such that a and $b - a$ are nilpotent, then b is nilpotent.)

We therefore need to determine the map f. For this, let X be a curve, and let LG_X denote the group scheme over X of formal loops on G. Let LG_x be the fiber at $x \in X$. Then for any $g \in LG_x$, the exact sequence

$$0 \to T_g(LG_x) \to T_g(LG_X) \to T_x X \to 0$$

splits (for the same reason as above). If $X = \mathbf{A}^1$, there is a canonical equivalence $LG_{\mathbf{A}^1} = LG_0 \times \mathbf{A}^1$, which gives another splitting of the above exact sequence. The difference between these two splittings is a k-linear map $k = T_0\mathbf{A}^1 \to T_g(LG_0)$, i.e., an element of $T_g(LG_0)$. If u is the coordinate on $\mathbf{A}^1$, then $T_g(LG_0)$ is equivalent (by left-translation via g) to $\mathfrak{g}(\!(u)\!)$, and the corresponding element of $T_g(LG_0)$ described above is just $g^{-1}\frac{dg}{du}$. (The g^{-1} is the left-translation.)

By taking duals, this determines the map $f : T^*_{(\mathcal{P},\mathcal{P}',\alpha)}\mathrm{Hecke}^{\mathrm{loc}}_0 \to T^*_0\mathbf{A}^1 = k$. More precisely, suppose $g \in LG$ is a representative of $(\mathcal{P}, \mathcal{P}', \alpha) \in \mathrm{Hecke}^{\mathrm{loc}}_0$. Then

$$T^*_g\mathrm{Hecke}^{\mathrm{loc}}_0 = \mathfrak{g}^*[\![u]\!]du \cap \mathrm{Ad}_g(\mathfrak{g}^*[\![u]\!]du) \subseteq \mathfrak{g}^*(\!(u)\!)du.$$

Then, the image of an element ξ_H under $f : T^*_g\mathrm{Hecke}^{\mathrm{loc}}_0 \to k$ is given by pairing with $g^{-1}\frac{dg}{du} \in T_g(LG_0) = \mathfrak{g}(\!(u)\!)$ under the residue map $\mathfrak{g}^*(\!(u)\!)du \otimes \mathfrak{g}(\!(u)\!) \to k$. We need to show that if ξ_H is nilpotent, then $\langle \xi_H, g^{-1}\frac{dg}{du}\rangle_{\mathrm{Res}} = 0$.

Since $\mathrm{Hecke}_0^{\mathrm{loc}} = L^+G\backslash LG/L^+G$, we can assume (by the usual Cartan decomposition) that $g = u^\lambda$ for some dominant coweight $\lambda \in \mathbf{X}_+^*$. In this case, $g^{-1}\frac{dg}{du} = \lambda \cdot u^{-1} \in \mathfrak{t}(\!(u)\!)$, so we need to show that $\langle \xi_H, \lambda u^{-1}\rangle_{\mathrm{Res}} = 0$. By definition of the residue pairing, we only need to concern ourself with $\xi_0 := \xi_H \pmod{u} \in \mathfrak{g}^* du$. We will identify $\mathfrak{g}^* du$ with $\mathfrak{g}^*$, so $\langle \xi_H, \lambda u^{-1}\rangle_{\mathrm{Res}} = \langle \xi_0, \lambda\rangle$.

Because $\xi_H \in \mathrm{Ad}_{u^\lambda}\mathfrak{g}^*[\![u]\!]du$, we see that $\xi_0 \in \mathfrak{p}_\lambda^+$, where $\mathfrak{p}_\lambda^+ \subseteq \mathfrak{g}$ is the parabolic in $\mathfrak{g}$ defined by the roots α with $\langle \alpha, \lambda\rangle \geq 0$. Let $\mathfrak{l}_\lambda$ be the Levi factor of $\mathfrak{p}_\lambda^+$ (so it is defined by the roots α with $\langle \alpha, \lambda\rangle = 0$), and let ξ_0^0 be the projection of ξ_0 into $\mathfrak{l}_\lambda$. Then $\langle \xi_0, \lambda\rangle = \langle \xi_0^0, \lambda\rangle_{\mathfrak{l}_\lambda}$. But λ is central in $\mathfrak{l}_\lambda$, while ξ_0^0 is nilpotent (since ξ_0 is nilpotent, owing to ξ_H being nilpotent). This implies that $\langle \xi_0^0, \lambda\rangle_{\mathfrak{l}_\lambda} = 0$, as desired.

3. Proof of part (b)

Let us now prove part (b) of the main theorem. I'll only discuss the proof for $G = \mathrm{GL}_2$, mostly because I haven't had time to fully understand the general case. Let us begin with the most elementary case: $G = \mathbf{G}_m$. In this case, $\mathrm{Bun}_G = \mathrm{Pic}$, and the Hecke functor (with respect to the character of weight 1 of $\mathbf{G}_m$) is just pullback along $\mathrm{add} : \mathrm{Pic} \times X \to \mathrm{Pic}$. Therefore, the conditions in the theorem say that $\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic})$ is such that $\mathrm{add}^!\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X)$. To show that $\mathcal{F} \in \mathrm{QLisse}(\mathrm{Pic})$, it suffices to show that $\mathcal{F}|_{\mathrm{Pic}_d} \in \mathrm{QLisse}(\mathrm{Pic}_d)$ for large enough d. We now use the standard trick involving the Abel-Jacobi map: one can show that $\mathrm{add}_d^!\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X^d)$, where $\mathrm{add}_d : \mathrm{Pic} \times X^d \to \mathrm{Pic}$ is the d-fold addition map. Let f denote the composite

$$X^d \to \mathrm{Pic} \times X^d \xrightarrow{\mathrm{add}_d} \mathrm{Pic};$$

then $f^!\mathcal{F} \in \mathrm{QLisse}(X^d)$. For $d > 2g-2$, the map $X^d \to \mathrm{Sym}^d(X) \to \mathrm{Pic}_d$ is flat and surjective (because the Abel-Jacobi map $\mathrm{Sym}^d(X) \to \mathrm{Pic}_d$ is smooth and surjective), so we would see that $\mathcal{F}|_{\mathrm{Pic}_d} \in \mathrm{QLisse}(\mathrm{Pic}_d)$ once we have the following lemma:

Lemma 3.1. *If the !-pullback of a sheaf $\mathcal{F} \in \mathrm{Shv}(Y)$ along a flat and surjective map $f : X \to Y$ is in $\mathrm{QLisse}(X)$, then $\mathcal{F} \in \mathrm{QLisse}(Y)$.*

The proof of the lemma is easy: some shift of $f^!$ is t-exact because f is flat, and so we may assume that $\mathcal{F} \in \mathrm{Shv}(Y)^\heartsuit$, i.e., that $\mathcal{F}$ is an irreducible perverse sheaf. But then the condition of being lisse is equivalent to its !-fibers over closed points having the same dimension; this follows from the surjectivity of f and the claim that $f^!\mathcal{F} \in \mathrm{QLisse}(X)$.

We will now turn to the case $G = \mathrm{GL}_2$. Let us just write Bun instead of Bun_G. Let $\mathcal{H}$ be the Hecke stack of tuples $(\mathcal{M}, \mathcal{M}', \alpha : \mathcal{M} \hookrightarrow \mathcal{M}')$ where $\mathrm{coker}(\alpha)$ is a torsion sheaf on X of length 1. Then there are morphisms $\overleftarrow{h} : \mathcal{H} \to \mathrm{Bun}$ and $\overrightarrow{h} \times s : \mathcal{H} \to \mathrm{Bun} \times X$ (where s sends the tuple to the support of $\mathrm{coker}(\alpha)$). Our assumption on $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun})$ is that $(\overrightarrow{h} \times s)_* \overleftarrow{h}^!\mathcal{F}$ has singular support in $T^*\mathrm{Bun} \times \{0\} \subseteq T^*(\mathrm{Bun} \times X)$, and we wish to show that the singular support of $\mathcal{F}$ is contained in the nilpotent cone.

An element of $T^*_{\mathcal{M}}\mathrm{Bun}$ can be understood as a $\mathcal{O}_X$-linear map $\mathcal{M} \to \mathcal{M} \otimes \omega_X$, which we will think of as a matrix $A \in \mathrm{End}_X(\mathcal{M}) \otimes \omega_X$. The trace of this matrix lives in $\Gamma(X; \omega_X)$, and we first claim:

Lemma 3.2. *Suppose $(\mathcal{M}, A)$ is in $\mathrm{SingSupp}(\mathcal{F})$. Then we have $\mathrm{Tr}(A) = 0$.*

Proof. Consider the action of Pic on Bun by tensoring; the !-pullback of $\mathcal{F}$ along $\mathrm{Pic} \times \mathrm{Bun} \to \mathrm{Bun}$ lives in $\mathrm{Shv}_{\{0\} \times T^*\mathrm{Bun}}(\mathrm{Pic} \times \mathrm{Bun})$ (by the same argument as in the case $G = \mathbf{G}_m$). This implies that A is in the subspace of $T^*_{\mathcal{M}}\mathrm{Bun}$ which is orthogonal to $\mathrm{im}(T_e \mathrm{Pic} \to T_{\mathcal{M}}\mathrm{Bun})$, i.e., in the subspace of $T^*_{\mathcal{M}}\mathrm{Bun}$ which is dual to $\mathrm{coker}(T_e \mathrm{Pic} \to T_{\mathcal{M}}\mathrm{Bun})$. But $T_e \mathrm{Pic} \cong \mathrm{H}^1(X; \mathfrak{t})$, while $T_{\mathcal{M}}\mathrm{Bun} \cong \mathrm{H}^1(X; \mathfrak{gl}_{\mathcal{P}})$; the map $T_e \mathrm{Pic} \to T_{\mathcal{M}}\mathrm{Bun}$ is induced by the inclusion $\mathfrak{t} \subseteq \mathfrak{gl}_2$. Therefore, the subspace of $T^*_{\mathcal{M}}\mathrm{Bun}$ is precisely the subspace of tracefree matrices. □

Recall we want to show that A is nilpotent. Since A is a two-by-two matrix, this is equivalent to claiming that $\det(A) = 0$. Assume for the sake of contradiction that $\det(A) \neq 0$; then, A is regular semisimple since $\mathrm{Tr}(A) = 0$. (Recall that a regular semisimple endomorphism of a vector space is one which has distinct eigenvalues upon passing to an algebraically closed field extension. In other words, the characteristic polynomial has distinct roots; this is certainly true for any polynomial of the form $x^2 - b$ with $b \neq 0$, at least in characteristic > 2.)

Let $\widetilde{X} \subseteq T^*X$ denote the spectral curve corresponding to A; this means the subspace cut out by the characteristic polynomial of A. Note that since A is two-by-two, the projection $\widetilde{X} \subseteq T^*X \to X$ is generically a finite map of degree 2. Since A is assumed to be regular semisimple, the map $\widetilde{X} \to X$ is étale. In summary, it is finite étale of degree 2. Let $x \in X$, and let $\widetilde{x} \in \widetilde{X}$ be one of the two preimages of x. The point $\widetilde{x}$ is just a pair $(x, \xi_x) \in T^*X$. To get a contradiction, it suffices to construct $\mathcal{M}' \in \mathrm{Bun}$ and $A' \in T^*_{\mathcal{M}'}\mathrm{Bun}$ such that $(A', \xi_x) \in T^*_{(\mathcal{M}',x)}(\mathrm{Bun} \times X)$ lives in $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F}))$. Indeed, since $\xi_x \neq 0$, this contradicts the assumption that $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F})) \subseteq T^*\mathrm{Bun} \times \{0\}$.

We will begin by describing $\mathcal{M}'$ and A', and then show that it is indeed in $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F}))$. The pair $(\mathcal{M}', A')$ can be viewed as a point of $T^*\mathrm{Bun}$. If $(\mathcal{M}, \mathcal{M}', \alpha) \in \mathcal{H}$ is such that $\mathcal{M}'/\mathcal{M}$ is supported at x, then the intersection of $(d\overleftarrow{h})^*(T^*_{\mathcal{M}}\mathrm{Bun})$ with $(d\overrightarrow{h} \times s)^*(T^*_{(\mathcal{M}',x)}(\mathrm{Bun} \times X))$ inside $T^*_{(\mathcal{M},\mathcal{M}',\alpha)}(\mathcal{H})$ is the space of diagrams

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "A"] \arrow[d, "\alpha"] & \mathcal{M} \otimes \omega_X \arrow[d, "\alpha"] \\
\mathcal{M}' \arrow[r, "A'"'] & \mathcal{M}' \otimes \omega_X .
\end{tikzcd}
\tag{2}
$$

The corresponding point of $T^*_x X$ is given by the map $\mathcal{M}'/\mathcal{M} \to \mathcal{M}'/\mathcal{M} \otimes \omega_X$ (recall $\mathcal{M}'/\mathcal{M}$ is supported at x). We would like to construct $\mathcal{M}'$ and A' such that this element is given by $\xi_x \in T^*_x X$.

To do this, observe that $\mathcal{M}$ is the pushforward of a torsion-free sheaf $\mathcal{L}$ on $\widetilde{X}$ which is generically a line bundle (TODO). Then, the datum of a diagram as above is equivalent to a modification $\mathcal{L} \subseteq \mathcal{L}'$ whose support is contained in $\widetilde{X} \times_X \{x\}$. The choice of $\widetilde{x} = (x, \xi_x)$ specifies such a modification, and hence the desired pair $(\mathcal{M}', A')$.

We now need to check that $(A', \xi_x) \in T^*_{(\mathcal{M}',x)}(\mathrm{Bun} \times X)$ indeed lives in $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F}))$. This requires a result, whose proof we omit:

Proposition 3.3. *Let $f : \mathcal{Z} \to \mathcal{Y}$ be a (nice) map of stacks with $\mathcal{Y}$ smooth. Let $y \in \mathcal{Y}$, and let $\mathcal{G} \in \mathrm{Shv}(\mathcal{Z})$. Fix $\psi \neq 0 \in T^*_y\mathcal{Y}$. Then $\psi \in \mathrm{SingSupp}(f_*\mathcal{G})$ if there is a point $z \in f^{-1}(y)$ such that:*

- (a) *The point $(\xi, z) \in T^*\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Z}$ lives in and is isolated in the intersection of $(df^*)^{-1}\mathrm{SingSupp}(\mathcal{G})$ with $\{\psi\} \times f^{-1}(y)$ inside $T^*\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Z}$.*
- (b) *For each cohomological degree m and constructible subobject $\mathcal{G}' \subseteq \mathrm{H}^m(\mathcal{G})$, the subvariety $(df^*)^{-1}\mathrm{SingSupp}(\mathcal{G}') \subseteq T^*\mathcal{Y} \times_{\mathcal{Y}} \mathcal{Z}$ has dimension $\dim(\mathcal{Y})$ at (ψ, z).*

This will be applied to:

$$
\mathcal{Z} = \mathcal{H},\ \mathcal{Y} = \mathrm{Bun} \times X,\ f = \overrightarrow{h} \times s,\ \mathcal{G} = \overleftarrow{h}^*(\mathcal{F}),
$$
$$
z = (x, \alpha : \mathcal{M} \to \mathcal{M}'),\ y = (\mathcal{M}', x),\ \psi = (A', \xi_x).
$$

We will check each of the conditions in the proposition. First, we observe a simplification. The map $\overleftarrow{h}$ is (pro)smooth, so $\mathrm{SingSupp}(\overleftarrow{h}^*\mathcal{F})$ is the image of $\mathrm{SingSupp}(\mathcal{F}) \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H} \subseteq T^*(\mathrm{Bun}) \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H}$ under the map

$$
T^*(\mathrm{Bun}) \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H} \to T^*\mathcal{H}.
$$

We now proceed:

- (a) By unwinding definitions, one sees that the first condition is equivalent to showing that
$$
[\{(A', \xi_x)\} \times (\overrightarrow{h} \times s)^{-1}(\mathcal{M}', x)] \cap [T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H}] \subseteq T^*(\mathcal{H})
$$
is finite. Unwinding further, one sees that the above space is the collection of diagrams of the form (2) for a fixed $(\mathcal{M}', A', x)$; we have already seen that there are only two possibilities.
- (b) For the second part, fix a cohomological degree m and consider a constructible subobject $\mathcal{F}'$ of $\mathrm{H}^m(\mathcal{F})$. Then a theorem of Beilinson says that the irreducible components of $\mathrm{SingSupp}(\mathcal{F}')$ have dimension $\dim(\mathrm{Bun})$. We wish to show that $(d\overrightarrow{h} \times s^*)^{-1}\mathrm{SingSupp}(\mathcal{F}')$ has dimension $\dim(\mathrm{Bun} \times X) = \dim \mathrm{Bun} + 1$, so it suffices to show that the composite map
$$
\mathrm{SingSupp}(\overleftarrow{h}^*\mathcal{F}') \times_{T^*\mathcal{H}} (T^*(\mathrm{Bun} \times X) \times_{\mathrm{Bun} \times X}^{\overrightarrow{h} \times s} \mathcal{H}) \to \mathrm{SingSupp}(\overleftarrow{h}^*\mathcal{F}') \xrightarrow{\overleftarrow{h}} \mathrm{SingSupp}(\mathcal{F}')
$$

has dimension ≤ 1 near $((A', \xi_x), (x, \alpha : \mathcal{M} \to \mathcal{M}'))$. Since the above composite is obtained as the pullback of a diagram

$$\begin{tikzcd}
\mathrm{SingSupp}(\overleftarrow{h}^*\mathcal{F}') \times_{T^*\mathcal{H}} (T^*(\mathrm{Bun} \times X) \times_{\mathrm{Bun}\times X}^{\overrightarrow{h}\times s} \mathcal{H}) \arrow[r] \arrow[d] & (T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H}) \times_{T^*\mathcal{H}} (T^*(\mathrm{Bun} \times X) \times_{\mathrm{Bun}\times X}^{\overrightarrow{h}\times s} \mathcal{H}) \arrow[d] \\
\mathrm{SingSupp}(\overleftarrow{h}^*\mathcal{F}') \arrow[r] \arrow[d, "\overleftarrow{h}"] & T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overleftarrow{h}} \mathcal{H} \arrow[d, "\overleftarrow{h}"] \\
\mathrm{SingSupp}(\mathcal{F}') \arrow[r] & T^*\mathrm{Bun},
\end{tikzcd}$$

it suffices to show that the left vertical composite has dimension ≤ 1 near $((A', \xi_x), (x, \alpha : \mathcal{M} \to \mathcal{M}'))$. We can fix a point $x \in X$; because X is proper, it suffices to show that the composite

$$(T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overleftarrow{h}_x} \mathcal{H}_x) \times_{T^*\mathcal{H}_x} (T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overrightarrow{h}_x} \mathcal{H}_x) \to T^*\mathrm{Bun} \times_{\mathrm{Bun}}^{\overleftarrow{h}_x} \mathcal{H}_x \xrightarrow{\overleftarrow{h}} T^*\mathrm{Bun}$$

is finite near $(A', \alpha : \mathcal{M} \to \mathcal{M}')$. We've pointed out that $\overleftarrow{h}$ is ind-proper, and so it suffices to show that $(A', \alpha : \mathcal{M} \to \mathcal{M}')$ is isolated in its fiber. This is another finiteness assertion, which amounts to the fact that there are only finitely many (two) diagrams of the form (2) for a fixed $(\mathcal{M}', A')$. I haven't had time to work this out in detail.

Email address: `sdevalapurkar@math.harvard.edu`