PRISMATIZATION

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In this talk, we will review the *filtered* prismatization $\mathbf{Z}_p^{\mathcal{N}}$ of $\mathbf{Z}_p$. It turns out to be conceptually easier to understand the filtered prismatization $\mathbf{G}_a^{\mathcal{N}}$ of $\mathbf{G}_a$, which (as a by-product) tells us what $\mathbf{Z}_p^{\mathcal{N}}$ is supposed to be. To illustrate this, let us briefly review Arpon's talk, which described the prismatization $\mathbf{G}_a^{\Delta}$. Symbols like $\mathrm{CAlg}_{\mathbf{Z}_p}$ will always mean ∞-categories of (animated) *p-nilpotent* $\mathbf{Z}_p$-algebras. Throughout, we will make liberal use of the identifications $W/V = \mathbf{G}_a$ and $W[F] = \mathbf{G}_a^\sharp$.

1. Prismatization

Recollection 1.1. If A and B are commutative rings, and we are given a *ring stack* $\mathcal{R} : \mathrm{CAlg}_A \to \mathrm{CAlg}_B$, then any B-scheme X defines an A-stack $X^{\mathcal{R}}$ via the composite

$$\mathrm{CAlg}_A \xrightarrow{\mathcal{R}} \mathrm{CAlg}_B \xrightarrow{X} \mathcal{S}.$$

The global sections $\Gamma(X^{\mathcal{R}}; \mathcal{O}_{X^{\mathcal{R}}}) \in \mathrm{CAlg}_A$ can be regarded as some "cohomology of X" valued in A-algebras. This is known as *transmutation*. The driving principle behind this whole story is that one can fully recover "A-valued cohomology theories" on B-schemes via ring stacks as above.

Recall that if $\overline{A}$ is a p-adic ring, then the de Rham stack associated to $\mathbf{G}_a$ is given by the quotient $\mathbf{G}_a/\mathbf{G}_a^\sharp$. There is a commutative diagram

$$\begin{tikzcd} F_*W \arrow[r, equal] \arrow[d, "V"] & F_*W \arrow[d, "p=F_*V"] \\ W \arrow[r, "F"'] & F_*W; \end{tikzcd}$$

taking cones in every direction (and using the fact that $F : W \to F_*W$ is faithfully flat), we see that there is an isomorphism

$$\mathbf{G}_a/\mathbf{G}_a^\sharp \cong (W/V)/W[F] \cong F_*W/p.$$

When $\overline{A} = k$ is a perfect field of characteristic $p > 0$, the theory of crystalline cohomology produces a cohomology theory taking values in $W(k)$-algebras such that if X is an $\mathbf{F}_p$-scheme, then

$$\Gamma_{\mathrm{crys}}(X/W(k)) \otimes_{W(k),\varphi} k \cong \Gamma_{\mathrm{dR}}(X/k). \tag{1}$$

The existence of crystalline cohomology can be explained by the observation that there is a factorization

$$\begin{tikzcd} \mathrm{CAlg}_{W(k)} \arrow[r, "\mathbf{G}_a^{\mathrm{dR}}"] \arrow[dr, dashed] & \mathrm{CAlg}_{W(k)} \\ & \mathrm{CAlg}_k, \arrow[u, "\epsilon"'] \end{tikzcd}$$

where $\epsilon : \mathrm{CAlg}_k \to \mathrm{CAlg}_{W(k)}$ is the functor induced by the augmentation $W(k) \to k$. This factorization comes from the fact that if $R \in \mathrm{CAlg}_{W(k)}$, then $p = 0 \in \mathbf{G}_a^{\mathrm{dR}}(R) = W(R)/p$. If X is a k-scheme,

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Part of this work was done when the author was supported by the PD Soros Fellowship and NSF DGE-2140743.

then the composite

$$\mathrm{CAlg}_{W(k)} \xrightarrow{\mathbf{G}_a^{\mathrm{dR}}} \mathrm{CAlg}_k \xrightarrow{X} \mathcal{S}$$

is the crystalline stack X^{crys}, whose coherent cohomology is $\Gamma_{\mathrm{crys}}(X/W(k))$. The isomorphism (1) can be encoded in the following observation:

Observation 1.2. The composite

$$\mathrm{CAlg}_k \xrightarrow{\epsilon} \mathrm{CAlg}_{W(k)} \xrightarrow{\varphi} \mathrm{CAlg}_{W(k)} \xrightarrow{W/p} \mathrm{CAlg}_k$$

can be identified with the functor defining the ring stack $\mathbf{G}_a^{\mathrm{dR}}$ over k.

One can generalize the pair $(W(k), p)$ to a more general pair (A, d) such that $A/d = \overline{A}$, and ask for a deformation of de Rham cohomology over A/d to A itself; this would be some version of crystalline cohomology. For instance, we could ask for a functor $\mathcal{R} : \mathrm{CAlg}_A \to \mathrm{CAlg}_{A/d}$ such that if X is an A/d-scheme, the composite

$$\mathrm{CAlg}_A \xrightarrow{\mathcal{R}} \mathrm{CAlg}_{A/d} \xrightarrow{X} \mathcal{S}$$

is somehow related to the de Rham stack of X.

A naive guess for the functor $\mathcal{R}$ might be to consider a stack "W/d", viewed as a functor $\mathrm{CAlg}_A \to \mathrm{CAlg}_{A/d}$ sending $R \mapsto W(R)/d$. To make sense of this, we need to be able to view the element $d \in A$ as an element of $W(A)$; if there were a map $A \to W(A)$, we could simply take the image of d to get the desired element. Having a map $A \to W(A)$ is the same as asking that A be a δ-ring, so let us now assume this. Then, A admits a lift of Frobenius φ, and we can ask that the composite

$$\mathrm{CAlg}_{A/d} \xrightarrow{\epsilon} \mathrm{CAlg}_A \xrightarrow{\varphi} \mathrm{CAlg}_A \xrightarrow{W/d} \mathrm{CAlg}_{A/d}$$

be identified with $\mathbf{G}_a^{\mathrm{dR}}$. This is the same as asking that the composite

$$A \to W(A) \to W(A/d) \xrightarrow{\varphi} W(A/d)$$

send d to a unit multiple of p. This composite sends

$$d \mapsto (d, \delta(d), \cdots) \mapsto (0, \delta(d), \cdots) \mapsto p(\delta(d), \cdots),$$

so we are simply asking that $\delta(d) \in A/d$ be a unit. If we further ask that A be d-complete, then this is the same as asking that $\delta(d)$ be a unit in A.

Combining the discussion above, we end up with the definition of an oriented prism:

Definition 1.3. An *oriented prism* is a pair (A, d) such that A is equipped with a δ-ring structure, A is (p, d)-adically complete, and $\delta(d) \in A$ is a unit.

If (A, d) is an oriented prism, the functor $W/d : \mathrm{CAlg}_A \to \mathrm{CAlg}_{A/d}$ is well-defined, and therefore can be regarded as an analogue of the crystalline stack of $\mathbf{G}_a$; we will denote it by $\mathbf{G}_a^{\mathbb{\Delta}}$, and refer to it as the *prismatization of* $\mathbf{G}_a$. Let us make a few points:

- The "de Rham comparison theorem" is now baked into the construction: namely, there is an isomorphism $F_*\mathbf{G}_a^{\mathbb{\Delta}} \cong \mathbf{G}_a^{\mathrm{dR}}$ as stacks over A/d.
- Similarly, if $d = p$, the "crystalline comparison theorem" is simply the observation that as stacks over A, there is an isomorphism $F_*\mathbf{G}_a^{\mathbb{\Delta}} \cong \mathbf{G}_a^{\mathrm{crys}}$.

This whole picture can be "globalized" over all prisms as follows (see [BL22a, BL22b, Dri22]). Namely, if R is a p-nilpotent ring, let us say that a pair $(I, \alpha : I \to W(R))$ of an invertible $W(R)$-module I and a map α is a *Cartier-Witt divisor* if the composite

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\mathrm{Res}} R$$

is nilpotent, and the composite

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\delta} R$$

generates the unit ideal of R. The functor $R \mapsto \{\text{Cartier-Witt divisors on } R\}$ defines a functor $\mathbf{Z}_p^{\mathbb{A}} : \mathrm{CAlg}_{\mathbf{Z}_p} \to \mathcal{S}$. If (A, d) is a oriented prism, and $A \to R$ is a map, there is a unique δ-ring map $A \to W(R)$; the tensor product $(d) \otimes_A W(R) \to W(R)$ is a Cartier-Witt divisor if (p, d) is nilpotent in R. Therefore, we obtain a map $\mathrm{Spf}(A) \to \mathbf{Z}_p^{\mathbb{A}}$.

Definition 1.4. Let X be a bounded p-adic formal scheme. Let $X^{\mathbb{A}} : \mathrm{CAlg}_{\mathbf{Z}_p} \to \mathcal{S}$ be the functor sending R to the groupoid of Cartier-Witt divisors $I \xrightarrow{\alpha} W(R)$ and a map $\mathrm{Spec}\, W(R)/I \to X$ of $\mathrm{Spf}(\mathbf{Z}_p)$-schemes. By construction, there is a map $X^{\mathbb{A}} \to \mathbf{Z}_p^{\mathbb{A}}$.

Note that by construction, if (A, d) is an oriented prism, the pullback of $\mathbf{G}_a^{\mathbb{A}}$ along the map $\mathrm{Spf}(A) \to \mathbf{Z}_p^{\mathbb{A}}$ is isomorphic to the stack we denoted $\mathbf{G}_a^{\mathbb{A}}$ above.

2. FILTERED PRISMATIZATION AND THE HODGE+CONJUGATE FILTRATIONS

Our goal in this talk is to understand the *filtered* prismatization. Again, the whole story will be modeled after the structures present in crystalline cohomology. As a precursor to this, let us try to understand the structures present in de Rham cohomology over a perfect field k of characteristic $p > 0$: namely, the Hodge and conjugate filtrations. Let X be a smooth k-scheme.

(a) The Hodge filtration on de Rham cohomology is a *decreasing* filtration; the associated filtered k-module has underlying object $\Gamma_{\mathrm{dR}}(X/k)$, and has associated graded given by $\Gamma_{\mathrm{Hdg}}(X/k)$. The ring stack defining de Rham cohomology is

$$\mathbf{G}_a^{\mathrm{dR}} = (W/V)/W[F] = \mathrm{cofib}(\mathbf{G}_a^\sharp \oplus F_* W \xrightarrow{(x,a)\mapsto x+Va} W),$$

while the ring stack defining Hodge cohomology is

$$\mathbf{G}_a^{\mathrm{Hdg}} = \mathbf{G}_a \oplus \mathbf{G}_a^\sharp(-1)[1] \cong W/V \oplus \mathbf{G}_a^\sharp(-1)[1].$$

One natural way to interpolate between these two stacks is by working over $\mathbf{A}^1_\hbar/\mathbf{G}_m$ with coordinate[1] $\hbar$. The universal line bundle $\mathcal{O}(1)$ over $\mathbf{A}^1_\hbar/\mathbf{G}_m$ has a tautological section $\hbar : \mathcal{O} \to \mathcal{O}(1)$. We can then consider the cofiber of the composite

$$\mathbf{G}_a^{\mathrm{dR},+} := \mathrm{cofib}(\mathcal{V}(\mathcal{O}(-1))^\sharp \oplus F_* W \xrightarrow{\hbar^\sharp, \mathrm{id}} \mathbf{G}_a^\sharp \oplus F_* W \xrightarrow{(x,a)\mapsto x+Va} W).$$

It turns out that this quotient is indeed a ring stack over $\mathbf{A}^1_\hbar/\mathbf{G}_m$, and the resulting cohomology theory is Hodge-filtered de Rham cohomology.

(b) The conjugate filtration on de Rham cohomology is an *increasing* filtration; the associated filtered k-module has underlying object $\Gamma_{\mathrm{dR}}(X/k)$, and has associated graded given by $\Gamma_{\mathrm{Hdg}}(X^{(1)}/k)$. Therefore, we are looking for a stack $\mathbf{G}_a^{\mathrm{conj}}$ which interpolates between $\mathbf{G}_a^{\mathrm{dR}}$ and $F_*\mathbf{G}_a^{\mathrm{Hdg}} = F_*\mathbf{G}_a \oplus F_*\mathbf{G}_a^\sharp(1)[1]$. (Note that the weight is $+1$ and not -1, because the filtration is increasing!) To motivate this construction, recall how the Cartier isomorphism comes about in the stacky picture: the map $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ defining $\mathbf{G}_a^{\mathrm{dR}}$ factors as the composite $\mathbf{G}_a^\sharp \twoheadrightarrow \alpha_p \hookrightarrow \mathbf{G}_a$, so that

$$\mathbf{G}_a^{\mathrm{dR}} \cong \mathbf{G}_a/\alpha_p \times B\ker(\mathbf{G}_a^\sharp \twoheadrightarrow \alpha_p) \cong F_*\mathbf{G}_a \oplus F_*\mathbf{G}_a^\sharp[1].$$

This isomorphism is not one of ring stacks, but it does indicate to us that the conjugate filtration on $\mathbf{G}_a^{\mathrm{dR}}$ should be obtained by "degenerating $F_*\mathbf{G}_a^\sharp \xrightarrow{V} \mathbf{G}_a^\sharp$ to zero". More precisely, let us work over the stack $\mathbf{A}^1_\sigma/\mathbf{G}_m$ with coordinate[2] σ in weight -1, and let G_σ be the group scheme

[1] Everywhere a subscript $\hbar$ shows up below, one can replace it by t to obtain the notation used in [Bha22].

[2] Everywhere a subscript σ shows up below, one can replace it by u to obtain the notation used in [Bha22].

over $\mathbf{A}^1_\sigma/\mathbf{G}_m$ defined by the pushout

$$\begin{tikzcd} F_*\mathbf{G}_a^\sharp \arrow[r, "V"] \arrow[d, "\sigma^\sharp"'] & \mathbf{G}_a^\sharp \arrow[d] \\ F_*\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r] & G_\sigma. \end{tikzcd}$$

Note that $G_\sigma/F_*\mathcal{V}(\mathcal{O}(1))^\sharp \cong \alpha_p$. Then, there is a map $G_\sigma \to \mathbf{G}_a$ of group schemes over $\mathbf{A}^1_\sigma/\mathbf{G}_m$, given by the square

$$\begin{tikzcd} F_*\mathbf{G}_a^\sharp \arrow[r, "V"] \arrow[d, "\sigma^\sharp"'] & \mathbf{G}_a^\sharp \arrow[d] \\ F_*\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r, "0"'] & \mathbf{G}_a. \end{tikzcd}$$

The map $G_\sigma \to \mathbf{G}_a$ is a quasi-ideal, and we will write $\mathbf{G}_a^{\mathrm{conj}}$ to denote its cofiber. This is a ring stack, and it encodes the conjugate filtration on de Rham cohomology.

One can translate the preceding discussion to Witt vector models, too. Namely, define a group scheme M_σ over $\mathbf{A}^1_\sigma/\mathbf{G}_m$ defined by the pushout

(2)
$$\begin{tikzcd} \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] \arrow[dr, phantom, "\text{pushout}"] & W \arrow[d] \\ \mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r] & M_\sigma. \end{tikzcd}$$

Note that $M_\sigma/\mathcal{V}(\mathcal{O}(1))^\sharp \cong F_*W$. Then, there is a map $d_\sigma : M_\sigma \to W$ of group schemes over $\mathbf{A}^1_\sigma/\mathbf{G}_m$, given by the square

(3)
$$\begin{tikzcd} \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] & W \arrow[d, "p"] \\ \mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r, "0"'] & W. \end{tikzcd}$$

The map $M_\sigma \to W$ is a quasi-ideal, and F_*W/M_σ can be shown to be isomorphic to $\mathbf{G}_a^{\mathrm{conj}}$. (This is actually not very difficult: it boils down to relating the above squares to the argument we used at the beginning to prove the isomorphism $\mathbf{G}_a^{\mathrm{dR}} \cong F_*W/p$.)

Remark 2.1. The diagram (3) can be extended slightly as follows: there is in fact a commutative diagram whose rows are cofiber sequences

(4)
$$\begin{tikzcd} \mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] \arrow[dr, phantom, "\text{pushout}"] & W \arrow[r, "F"] \arrow[d, "p"] & F_*W \arrow[d, equal] \\ \mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r, "0"'] \arrow[d, "0"'] & M_\sigma \arrow[r, "F"] \arrow[d, "d_\sigma"] & F_*W \arrow[d, "p"] \\ \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W. \end{tikzcd}$$

Our final stop in characteristic p is to understand how to glue the conjugate and Hodge filtrations together. For this, we need to work over a base which encodes *two* filtrations on the same k-module: the most natural candidate is

$$C := (\operatorname{Spec} k[\sigma, \hbar]/\sigma\hbar)/\mathbf{G}_m,$$

where σ has weight -1 and $\hbar$ has weight 1. This looks like the $\mathbf{G}_m$-quotient of two coordinate axes. The universal line bundle $\mathcal{L}$ over C has two maps $\sigma : \mathcal{O} \to \mathcal{L}$ and $\hbar : \mathcal{L} \to \mathcal{O}$; its restriction to $\mathbf{A}^1_\sigma/\mathbf{G}_m$ is $\mathcal{O}(1)$, while its restriction to $\mathbf{A}^1_\hbar/\mathbf{G}_m$ is $\mathcal{O}(-1)$.

We can now define a ring stack $\mathbf{G}_a^C$ which glues the conjugate and Hodge filtrations: this will have the property that

$$F_*\mathbf{G}_a^C|_{\hbar=0} = \mathbf{G}_a^{\mathrm{conj}},\ \mathbf{G}_a^C|_{\sigma=0} = \mathbf{G}_a^{\mathrm{dR},+}.$$

First, note that we can still define M_σ over C via the same pushout square (2). To obtain the Hodge filtration in a manner compatible with the conjugate filtration, we therefore want a deformation $d_{\sigma,\hbar} : M_\sigma \to W$ of the map d_σ (from (b) above) such that:

- When $\sigma = 0$, the map $d_{\sigma,\hbar} : M_\sigma \to W$ can be identified with the composite

$$\mathcal{V}(\mathcal{L})^\sharp \oplus F_*W \xrightarrow{\hbar^\sharp + V} W.$$

- When $\hbar = 0$, the map $d_{\sigma,\hbar} : M_\sigma \to W$ can be identified with d_σ.

Note that when $\sigma = 0$, we can identify M_σ with $\mathcal{V}(\mathcal{O}(-1))$; so we only need to modify the square (3) as follows:

(5)
$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[rr] \arrow[d, "\sigma^\sharp"'] & & W \arrow[d, "p"] \\
\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r, "\hbar^\sharp"'] & \mathbf{G}_a^\sharp \arrow[r] & W.
\end{tikzcd}$$

This pushout defines the desired map $d_{\sigma,\hbar} : M_\sigma \to W$. Note that the composite

$$\mathbf{G}_a^\sharp \xrightarrow{\sigma^\sharp} \mathcal{V}(\mathcal{O}(1))^\sharp \xrightarrow{\hbar^\sharp} \mathbf{G}_a^\sharp$$

is zero, since $\hbar\sigma = 0$.

Remark 2.2. As with the story from $\mathbf{G}_a^{\mathrm{conj}}$, the diagram (5) can be extended slightly as follows: there is in fact a commutative diagram whose rows are cofiber sequences

(6)
$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] \arrow[dr, phantom, "\text{pushout}"] & W \arrow[r, "F"] \arrow[d, "p"] & F_*W \arrow[d, equal] \\
\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r] \arrow[d, "\hbar^\sharp"'] & M_\sigma \arrow[r, "F"] \arrow[d, "d_{\sigma,\hbar}"] & F_*W \arrow[d, "p"] \\
\mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W.
\end{tikzcd}$$

One can check that the map $d_{\sigma,\hbar} : M_\sigma \to W$ defines a quasi-ideal, so that:

Definition 2.3. Let $\mathbf{G}_a^C$ denote the ring stack over C defined by $\operatorname{cofib}(M_\sigma \xrightarrow{d_{\sigma,\hbar}} W)$. Note that

$$\mathbf{G}_a^C|_{\sigma\neq 0} = W/p,\ \mathbf{G}_a^C|_{\hbar\neq 0} = F_*W/p.$$

We will call the inclusions $\operatorname{Spec} k = C_{\sigma\neq 0} \subseteq C$ and $\operatorname{Spec} k = C_{\hbar\neq 0} \subseteq C$ the *Hodge-Tate* and *de Rham* points, respectively.

We can now finally start to study structures on crystalline cohomology, so that all stacks below will live over $W(k)$. The key structure showing up here is the *Nygaard filtration*. If X is a smooth affine k-scheme, it is characterized by the following property: $\mathcal{N}^{\geq j}\Gamma_{\mathrm{crys}}(X/W(k))$ is the subcomplex of $\Gamma_{\mathrm{crys}}(X/W(k))$ on which the crystalline Frobenius φ is divisible by p^j. Using this, one can show that the graded pieces $\mathcal{N}^j\Gamma_{\mathrm{crys}}(X/W(k))$ can be identified with $\mathrm{F}_i^{\mathrm{conj}}\Gamma_{\mathrm{dR}}(X/k)\{i\}$. Here, $\{i\}$ simply denotes tensoring by the ideal $(p^i)/(p^{i+1})$. Another important property of the Nygaard filtration is that if X is F-liftable to a $W(k)$-scheme $\widetilde{X}$, then $\mathcal{N}^{\geq j}\Gamma_{\mathrm{crys}}(X/W(k)) = p^{\max(j-*,0)}\mathrm{F}^*_{\mathrm{Hdg}}\Gamma_{\mathrm{dR}}(\widetilde{X}/W(k))$; in other words, it mixes the Hodge and p-adic filtrations.

We would therefore like to construct a mixed characteristic ring stack $\mathbf{G}_a^{\mathcal{N}}$ which encodes the Nygaard filtration on crystalline cohomology. In particular, the underlying stack of $\mathbf{G}_a^{\mathcal{N}}$ should be $\mathbf{G}_a^{\mathrm{dR}}$ (now over $\mathrm{Spf}\, W(k)$!). Recall that

$$\pi_*\mathrm{TC}^-(k) \cong W(k)[\sigma, \hbar]/(\sigma\hbar - p),$$

and that the resulting $\hbar$-adic filtration on $\mathrm{TC}^-(X)$ encodes the Nygaard filtration on prismatic cohomology. Motivated by this, let us define

$$k^{\mathcal{N}} := \mathrm{Spf}(W(k)[\sigma, \hbar]/(\sigma\hbar - p))/\mathbf{G}_m, \tag{7}$$

where σ has weight -1 and $\hbar$ has weight 1. By construction, $k^{\mathcal{N}} \otimes_{W(k)} k \cong C$, and $\mathrm{QCoh}(k^{\mathcal{N}})$ is precisely the ∞-category of filtered $W(k)$-modules over $(p)^\bullet$. Over $k^{\mathcal{N}}$, the definition of M_σ, etc., still go through, and we can define a map $d_{\sigma,\hbar} : M_\sigma \to W$ via the pushout

$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[rr] \arrow[d, "\sigma^\sharp"'] & & W \arrow[d, "p"] \\
\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r, "\hbar^\sharp"'] & \mathbf{G}_a^\sharp \arrow[r] & W.
\end{tikzcd} \tag{8}$$

Note that the composite

$$\mathbf{G}_a^\sharp \xrightarrow{\sigma^\sharp} \mathcal{V}(\mathcal{O}(1))^\sharp \xrightarrow{\hbar^\sharp} \mathbf{G}_a^\sharp$$

is no longer zero, but is rather p (since $\hbar\sigma = p$).

Remark 2.4. As with the story from $\mathbf{G}_a^{\mathrm{conj}}$ and $\mathbf{G}_a^C$, the diagram (8) can be extended slightly as follows: there is in fact a commutative diagram whose rows are cofiber sequences

$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] \arrow[dd, bend right, "p^\sharp = p"'] \arrow[dr, phantom, "\text{pushout}"] & W \arrow[r, "F"] \arrow[d, "p"] & F_*W \arrow[d, equal] \\
\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r] \arrow[d, "\hbar^\sharp"'] & M_\sigma \arrow[r, "F"] \arrow[d, "d_{\sigma,\hbar}"] & F_*W \arrow[d, "p"] \\
\mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W.
\end{tikzcd} \tag{9}$$

Again, one can check that the map $d_{\sigma,\hbar} : M_\sigma \to W$ defines a quasi-ideal, so that:

Definition 2.5. Let $\mathbf{G}_a^{\mathcal{N}}$ denote the *filtered prismatization* of $\mathbf{G}_a$, defined as the ring stack over $k^{\mathcal{N}}$ given by $\mathrm{cofib}(M_\sigma \xrightarrow{d_{\sigma,\hbar}} W)$. Note that

$$\mathbf{G}_a^{\mathcal{N}}|_{\sigma\neq 0} = W/p = \mathbf{G}_a^{\mathbb{\Delta}},\ \mathbf{G}_a^{\mathcal{N}}|_{\hbar\neq 0} = F_*W/p = \mathbf{G}_a^{\mathrm{crys}},\ \mathbf{G}_a^{\mathcal{N}}|_{p=0} = \mathbf{G}_a^C. \tag{10}$$

We will call the inclusions $\mathrm{Spf}\, W(k) = k^{\mathcal{N}}_{\sigma\neq 0} \subseteq k^{\mathcal{N}}$ and $\mathrm{Spf}\, W(k) = k^{\mathcal{N}}_{\hbar\neq 0} \subseteq k^{\mathcal{N}}$ the *Hodge-Tate* and *de Rham* points, respectively. If X is a k-scheme, we obtain a stack $X^{\mathcal{N}}$ over $k^{\mathcal{N}}$ defined by the functor

$$\mathrm{CAlg}_{k^{\mathcal{N}}} \xrightarrow{\mathbf{G}_a^{\mathcal{N}}} \mathrm{CAlg}_k \xrightarrow{X} \mathcal{S}.$$

Let $\mathcal{H}_{\mathcal{N}}(X) \in \mathrm{QCoh}(k^{\mathcal{N}})$ denote the pushforward of the structure sheaf along the morphism $X^{\mathcal{N}} \to k^{\mathcal{N}}$, and let $\mathcal{N}^{\geq \star}\Gamma_{\Delta}(X/A)$ denote its underlying $W(k)$-module.

Remark 2.6. Let us briefly mention why $\mathbf{G}_a^{\mathcal{N}}$ encodes the Nygaard filtration. Firstly, we need to show that the Frobenius on $\Gamma_{\Delta}(X/W(k))$ factors through $\mathcal{N}^{\geq \star}\Gamma_{\Delta}(X/A)$. This is essentially a consequence of the fact that the map $W \xrightarrow{p} W$ fits into a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "p"] & M_\sigma \arrow[r] \arrow[d, "d_{\sigma,\hbar}"] & F_* W \arrow[d] \\ W \arrow[r, equal] & W \arrow[r, "F"'] & F_* W. \end{tikzcd}$$

Taking vertical cofibers, we obtain a factorization

$$W/p \to \mathbf{G}_a^{\mathcal{N}} \to F_* W/p$$

of the Frobenius on the ring stack W/p. Secondly, we need to show that $\mathcal{N}^j\Gamma_{\mathrm{crys}}(X/W(k))$ can be identified with $\mathrm{F}_i^{\mathrm{conj}}\Gamma_{\mathrm{dR}}(X/k)\{i\}$. This has a rather fun argument; see [Bha22, Theorem 3.3.5(1)]. It is a topological analogue of the observation that $\mathrm{TC}^-(X)/\hbar \simeq \mathrm{THH}(X)$, which encodes the conjugate filtration (this uses that the cyclotomic Frobenius gives an equivalence $\mathrm{THH}(X)[1/\sigma] \xrightarrow{\varphi} \mathrm{THH}(X)^{t\mathbf{Z}/p} \simeq \mathrm{HP}(X/k)$, and that $\mathrm{THH}(X)/\sigma \cong \mathrm{HH}(X/k)$).

Remark 2.7. The Hodge-Tate and de Rham points of $k^{\mathcal{N}}$ can be understood homotopy-theoretically as follows: the Hodge-Tate point is related to the map $\varphi : \mathrm{TC}^-(k)[1/\sigma] \to \mathrm{TP}(k) \simeq W(k)^{tS^1}$ induced by the cyclotomic Frobenius, while the de Rham point is related to the canonical map $\mathrm{can} : \mathrm{TC}^-(k) \to \mathrm{TP}(k)$. The isomorphisms of (10) correspond to the observation that if X is quasisyntomic over k, then $\mathrm{TC}^-(X)[1/\sigma]$ gives a Frobenius untwist of $\mathrm{TP}(X)$; since $\mathrm{TP}(X)$ encodes the crystalline cohomology of X, $\mathrm{TC}^-(X)[1/\sigma]$ encodes a Frobenius untwist of crystalline cohomology. The resulting σ-adic filtration (with respect to the lattice $\mathrm{TC}^-(X) \to \mathrm{TC}^-(X)[1/\sigma]$) encodes the conjugate filtration.

3. Filtered prismatization over $\mathbf{Z}_p$

Let us now turn to mixed characteristic (i.e., deforming from A/d to A, where (A, d) is an oriented prism). Recall from the beginning of the talk that the key idea was deforming the quasi-ideal $W \xrightarrow{p} W$ over A/d to $W \xrightarrow{d} W$ over A. Now, we essentially want to deform the quasi-ideal $M_\sigma \xrightarrow{d_{\sigma,\hbar}} W$. Recall that M_σ sits in an extension

$$0 \to \mathcal{V}(\mathcal{L})^\sharp \to M_\sigma \to F_* W \to 0.$$

This motivates:

Definition 3.1. Let R be a p-nilpotent ring. An *admissible W-module* M is a W-module scheme M which sits in an extension of the form

$$0 \to \mathcal{V}(\mathcal{L})^\sharp \to M \to F_* M' \to 0$$

for some $\mathcal{L} \in \mathrm{Pic}(R)$ and an invertible W-module M'.

Remark 3.2. Every invertible W-module is admissible. Moreover, there is a unique extension witnessing the admissibility of a W-module: indeed, extensions form a torsor for $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, F_* W)$, but this vanishes[3].

[3] Since $F_* W$ has a filtration whose graded pieces are $F_*^n \mathbf{G}_a$, it suffices to show that $\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, F_*^n \mathbf{G}_a) = 0$ for $n > 0$. Such a map is $\mathbf{G}_m$-equivariant (because of the Teichmuller map $\mathbf{G}_m \to W^\times$), so such a map is the same as a *primitive* element of $\mathcal{O}_{\mathbf{G}_a^\sharp} \cong \mathbf{Z}_p\langle t \rangle$ of weight p^n. All such elements are zero.

Construction 3.3. One can prove that there is an equivalence of groupoids $\underline{\mathrm{Pic}}(W(R)) \simeq \mathrm{Map}(\mathrm{Spec}(R), BW^\times)$. Given $I \in \mathrm{Pic}(W(R))$, we obtain an exact sequence

$$0 \to I \otimes_{W(R)} \mathbf{G}_a^\sharp \to I \otimes_{W(R)} W \xrightarrow{F} I \otimes_{W(R)} F_* W \to 0.$$

If $\mathcal{L} \in \mathrm{Pic}(R)$ and $\sigma : I \otimes_{W(R)} R \to \mathcal{L}$ is a map of line bundles over R, then define M_σ via the pushout

$$\begin{tikzcd} I \otimes_{W(R)} \mathbf{G}_a^\sharp \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & I \otimes_{W(R)} W \arrow[d] \\ \mathcal{V}(\mathcal{L})^\sharp \arrow[r] & M_\sigma. \end{tikzcd}$$

There is then a cofiber sequence

$$0 \to \mathcal{V}(\mathcal{L})^\sharp \to M_\sigma \xrightarrow{F} I \otimes_{W(R)} F_* W \to 0,$$

and M_σ is an admissible W-module over R. In fact, fpqc-locally on R, every admissible W-module arises in this way.

Motivated by this construction, we are led to consider:

Definition 3.4. Let R be a p-nilpotent ring. A *filtered Cartier-Witt divisor* on R is an admissible W-module M and a map $d : M \to W$ of admissible W-modules, such that the induced map $F_* M' \to F_* W$ is obtained as F_* of a Cartier-Witt divisor over R. Let $\mathbf{Z}_p^{\mathcal{N}}$ denote the functor $\mathrm{CAlg} \to \mathcal{S}$ sending $R \mapsto \{\text{filtered Cartier-Witt divisors on } R\}$.

Example 3.5. Let $I \xrightarrow{\alpha} W(R)$ be a Cartier-Witt divisor. Then, we obtain a map $d_\alpha : I \otimes_{W(R)} W \to W$, which is a filtered Cartier-Witt divisor. Indeed, $M := I \otimes_{W(R)} W$ is admissible (in fact, invertible!) by Construction 3.3, and the map $M' \to W$ is simply given by the map

$$M' = F^* I \otimes_{W(R)} W \xrightarrow{\alpha \otimes \mathrm{id}} W(R) \otimes_{W(R)} W = W.$$

This is indeed a Cartier-Witt divisor. This construction produces a map $j_{\mathrm{HT}} : \mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\mathcal{N}}$, which exhibits it as an open substack of $\mathbf{Z}_p^{\mathcal{N}}$.

Example 3.6. Let $d : M \to W$ be a filtered Cartier-Witt divisor over R, so that there is a map of admissible sequences

$$\begin{tikzcd} \mathcal{V}(\mathcal{L})^\sharp \arrow[r] \arrow[d, "d^\sharp"] & M \arrow[r] \arrow[d, "d"] & F_* M' \arrow[d, "F_* d'"] \\ \mathbf{G}_a^\sharp \arrow[r] & W \arrow[r] & F_* W. \end{tikzcd} \tag{11}$$

It turns out that that the map $d^\sharp$ arises via an actual morphism $\hbar(d) : \mathcal{L} \to \mathbf{G}_a$ of line bundles[4], so that we obtain a map $\hbar : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{A}^1_\hbar/\mathbf{G}_m$. The fiber $(\mathbf{Z}_p^{\mathcal{N}})_{\hbar \neq 0}$ over $\mathbf{G}_m/\mathbf{G}_m$ consists of those Cartier-Witt divisors for which d is nonzero, i.e., $d^\sharp$ is an isomorphism. In this case, the Cartier-Witt divisor $d : M \to W$ is encoded entirely by the Cartier-Witt divisor $d' : M' \to W$, so that we obtain an isomorphism

$$j_{\mathrm{dR}} : \mathbf{Z}_p^{\Delta} \cong (\mathbf{Z}_p^{\mathcal{N}})_{\hbar \neq 0} \subseteq \mathbf{Z}_p^{\mathcal{N}},$$

[4] It suffices to observe that

$$\underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \cong \underline{\mathrm{Hom}}_{\mathbf{G}_a}(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \cong \underline{\mathrm{Hom}}_{\mathbf{G}_a}(\mathbf{G}_a, \mathbf{G}_a) \cong \mathbf{G}_a.$$

The first isomorphism comes from the fact that the W-action on $\mathbf{G}_a^\sharp$ factors through $W \to \mathbf{G}_a$; the second isomorphism comes from Cartier duality over $B\mathbf{G}_m$; the third isomorphism is obvious.

exhibiting $\mathbf{Z}_p^{\Delta}$ as an open substack of $\mathbf{Z}_p^{\mathcal{N}}$. Note that j_{dR} and j_{HT} are disjoint — for any filtered Cartier-Witt divisor in the image of j_{HT}, the map $d^\sharp$ is nilpotent!

Remark 3.7. In homotopy theory, the map $\hbar : \mathbf{Z}_p^{\mathcal{N}} \to \mathbf{A}^1_\hbar/\mathbf{G}_m$ encodes the filtration on $\mathrm{TC}^-(\mathbf{Z}_p)$ arising via the homotopy fixed points spectral sequence. The points j_{HT} and j_{dR} are supposed to correspond to the maps $\mathrm{TC}^- \rightrightarrows \mathrm{TP}$ given by the cyclotomic Frobenius and the canonical map, respectively. Note that σ does not actually exist in $\pi_2\mathrm{TC}^-(\mathbf{Z}_p)$ – rather, the advantage of the stacky perspective is that we can do everything locally. For instance, there *is* a cover $\mathrm{TC}^-(\mathbf{Z}_p) \to \mathrm{TC}^-(\mathbf{Z}_p/S[\![\widetilde{p}]\!])$, where the map $S[\![\widetilde{p}]\!] \to \mathbf{Z}_p$ sends $\widetilde{p} \mapsto p$, and the $\mathbf{E}_\infty$-ring $\mathrm{TC}^-(\mathbf{Z}_p/S[\![\widetilde{p}]\!])$ is even[5]: its homotopy groups are given by $\mathbf{Z}_p[\![\widetilde{p}]\!][\sigma,\hbar]/(\sigma\hbar - (\widetilde{p} - p))$. We can therefore construct the localization $\mathrm{TC}^-(\mathbf{Z}_p/S[\![\widetilde{p}]\!])[1/\sigma]$; as long as we can extend this localization to the entire cosimplicial diagram induced by the cover $\mathrm{TC}^-(\mathbf{Z}_p) \to \mathrm{TC}^-(\mathbf{Z}_p/S[\![\widetilde{p}]\!])$, we can localize the stack associated to the even filtration[6] on $\mathrm{TC}^-(\mathbf{Z}_p)$, as well.

It turns out that if $d : M \to W$ is a filtered Cartier-Witt divisor, then d defines a quasi-ideal; we will not prove this here. This implies that the quotient W/M is in fact a *ring* stack. In particular:

Definition 3.8. Let $\mathbf{G}_a^{\mathcal{N}}$ denote the ring stack over $\mathbf{Z}_p^{\mathcal{N}}$ given locally by the assignment

$$(d : M \to W) \in \mathbf{Z}_p^{\mathcal{N}}(R) \mapsto (W/M)(R) \in \mathrm{CAlg}.$$

This will be called the *filtered prismatization* of the affine line. Using Recollection 1.1, we can now define the filtered prismatization of any bounded p-adic formal scheme X. Let us assume that $X = \mathrm{Spf}(A)$ is affine, for simplicity. Then, $X^{\mathcal{N}} \to \mathbf{Z}_p^{\mathcal{N}}$ is the stack whose functor of points is given by

$$\mathrm{CAlg} \ni R \mapsto \{\text{filtered CW-divisors } d : M \to W, \text{ and } A \to (W/M)(R)\} \in \mathcal{S}.$$

We will close with two results.

Proposition 3.9. *The filtered prismatization $k^{\mathcal{N}}$ of Definition 3.8 agrees with the stack $\mathrm{Spf}(\pi_*\mathrm{TC}^-(k))/\mathbf{G}_m$ of* (7).

Proof. Let us write $k^{\mathcal{N}'} := \mathrm{Spf}(\pi_*\mathrm{TC}^-(k))/\mathbf{G}_m$, so that if R is a p-nilpotent ring, then $k^{\mathcal{N}'}(R)$ is the groupoid of tuples $(\mathcal{L}, \sigma, \hbar)$ of $\mathcal{L} \in \mathrm{Pic}(R)$, $\sigma : \mathcal{O} \to \mathcal{L}$, and $\hbar : \mathcal{L} \to \mathcal{O}$ such that $\sigma\hbar = p$. We will build maps $k^{\mathcal{N}'} \to k^{\mathcal{N}}$ and $k^{\mathcal{N}} \to k^{\mathcal{N}'}$ (which will clearly be inverse to each other) as follows:

- To define a map $k^{\mathcal{N}} \to k^{\mathcal{N}'}$, we need to define a map $k^{\mathcal{N}}(R) \to k^{\mathcal{N}'}(R)$ for every p-nilpotent ring R. Suppose we are given a point of $k^{\mathcal{N}}(R)$, i.e., a filtered Cartier-Witt divisor $d : M \to W$ and $k \to (W/M)(R)$. Then, this lifts uniquely to the dotted arrows in the following diagram, whose columns are cofiber sequences:

(12)
$$\begin{tikzcd} W(k) \arrow[r, dashed, "\alpha"] \arrow[d, "p"] & M(R) \arrow[d, "d"] \\ W(k) \arrow[r, dashed] \arrow[d] & W(R) \arrow[d] \\ k \arrow[r] & (W/M)(R). \end{tikzcd}$$

  This can be understood as a map

$$(W \xrightarrow{p} W) \to (M \xrightarrow{d} W)$$

[5]In fact, it is equivalent (at least) as an $\mathbf{E}_1$-ring to $(\tau_{\geq 0}\ell^{t\mathbf{Z}/p})^{hS^1}$. Using this cover of $\mathrm{TC}^-(\mathbf{Z}_p)$, one can even show that $\mathrm{TC}^-(\mathbf{Z}_p)$ is closely related to the complex image of J spectrum $j_{\mathbf{C}} = \mathrm{fib}(\ell \xrightarrow{\psi-1} \Sigma^{2p-2}\ell)$.

[6]See [HRW22].

of filtered CW-divisors over R, and hence a map of admissible sequences

$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[r] \arrow[d, "\sigma^\sharp"'] \arrow[dd, bend right=60, "p^\sharp=p"'] & W \arrow[r, "F"] \arrow[d, "\alpha"] \arrow[dd, bend right=40, "p"'] & F_*W \arrow[d, "\alpha'"] \\
\mathcal{V}(\mathcal{O}(1))^\sharp \arrow[r] \arrow[d, "\hbar^\sharp"'] & M \arrow[r, "F"] \arrow[d, "d"] & F_*M' \arrow[d, "d'"] \\
\mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W.
\end{tikzcd}$$

Note that by Footnote 4, the top left vertical map can be identified as $\sigma^\sharp : \mathbf{G}_a^\sharp \to \mathcal{V}(\mathcal{L})^\sharp$ for a unique map $\sigma : \mathcal{O} \to \mathcal{L}$; similarly, the bottom left vertical map can be identified as $\hbar^\sharp : \mathcal{V}(\mathcal{L})^\sharp \to \mathbf{G}_a^\sharp$ for a unique map $\hbar : \mathcal{L} \to \mathcal{O}$. The right vertical column can be viewed as a map $(W \xrightarrow{p} W) \to (M' \xrightarrow{d'} W)$ of Cartier-Witt divisors, which by rigidity means that the map $\alpha' : W \to M'$ is an isomorphism.

In particular, the line bundle $\mathcal{L} \in \mathrm{Pic}(R)$ associated to M is equipped with maps $\sigma : \mathcal{O} \to \mathcal{L}$ and $\hbar : \mathcal{L} \to \mathcal{O}$ such that $\sigma\hbar = p$; this defines an R-point of $k^{\mathcal{N}'}$, as desired.

- Suppose we are given an R-point $(\mathcal{L}, \sigma, \hbar)$ of $k^{\mathcal{N}'}$. Define M_σ and the map $M_\sigma \xrightarrow{d_{\sigma,\hbar}} W$ via the square (6). Then, we obtain a map

$$(W \xrightarrow{p} W) \xrightarrow{\alpha} (M_\sigma \xrightarrow{d_{\sigma,\hbar}} W).$$

of filtered Cartier-Witt divisors over R. In particular, this is a map of quasi-ideals over R, so that we obtain a map

$$k = W(k)/p \to W(R)/p \xrightarrow{\alpha} (W/M_\sigma)(R).$$

The data of $d_{\sigma,\hbar}$ along with this map $k \to (W/M_\sigma)(R)$ is precisely an R-point of $k^{\mathcal{N}}$, so that we obtain the desired map $k^{\mathcal{N}'} \to k^{\mathcal{N}}$.

□

The same argument shows that if R is a perfectoid ring, the filtered prismatization $R^{\mathcal{N}}$ of Definition 3.8 agrees with the stack $\mathrm{Spf}(\pi_*\mathrm{TC}^-(R))/\mathbf{G}_m$.

Proposition 3.10. *There is an isomorphism* $(\mathbf{Z}_p^{\mathcal{N}})_{\hbar=0} \cong \mathbf{G}_a^{\mathrm{dR}}/\mathbf{G}_m$.

Proof. Suppose that $d : M \to W$ is a filtered Cartier-Witt divisor over a p-nilpotent ring R such that $\hbar(d) = 0$ (so $d^\sharp = 0$). Recall the map of exact sequences (11):

$$\begin{tikzcd}
\mathcal{V}(\mathcal{L})^\sharp \arrow[r] \arrow[d, "d^\sharp=0"] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[d, "F_*d'"] \arrow[dl, dashed] \\
\mathbf{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W.
\end{tikzcd}$$

Since the left vertical map is zero, there is a dotted map $\widetilde{d} : F_*M' \to W$ as indicated. We claim:

$(*)$ $\widetilde{d}$ has to factor as

$$\widetilde{d} : F_*M' \xrightarrow{F_*\xi} F_*W \xrightarrow{V} W$$

for some $\xi : M' \to W$.

We will prove $(*)$ below. First, note that it implies that ξ can be viewed as a map

$$\xi : (M' \to W) \to (W \xrightarrow{FV=p} W)$$

of Cartier-Witt divisors; in particular, $\xi : M' \to W$ must be an isomorphism by rigidity. Therefore, M is necessarily an extension of F_*W by $\mathcal{V}(\mathcal{L})^\sharp$. We claim that

$(**)$ There is an isomorphism $\underline{\mathrm{Ext}}^1_W(F_*W, \mathbf{G}_a^\sharp) \cong \mathbf{G}_a/\mathbf{G}_a^\sharp \cong \mathbf{G}_a^{\mathrm{dR}}$, which is $\mathbf{G}_m$-equivariant for the standard action on the target $\mathbf{G}_a^{\mathrm{dR}}$, and the action on $\mathbf{G}_a^\sharp$ on the source.

This immediately implies the desired claim, so let us now prove $(*)$ and $(**)$.

Proof of $(*)$. It suffices to show that the map $V : F_*W \to W$ gives an isomorphism

$$\underline{\mathrm{Hom}}_W(F_*W, F_*W) \to \underline{\mathrm{Hom}}_W(F_*W, W).$$

To prove this, first note that the source is

$$\underline{\mathrm{Hom}}_W(F_*W, F_*W) \cong \underline{\mathrm{Hom}}_{F_*W}(F_*W, F_*W) \cong F_*W,$$

where the first isomorphism is because F_*W is a quotient of W. From right to left, this isomorphism sends $x \in F_*W$ to $F_*W \xrightarrow{x} F_*W$. Therefore, we need to show that the map

$$F_*W \to \underline{\mathrm{Hom}}_W(F_*W, W)$$

sending $x \in F_*W$ to $F_*W \xrightarrow{x} F_*W \xrightarrow{V} W$ is an isomorphism. Applying $\underline{\mathrm{Hom}}_W(-, W)$ to the exact sequence

$$0 \to \mathbf{G}_a^\sharp \to W \xrightarrow{F} F_*W \to 0,$$

we obtain

$$0 \to \underline{\mathrm{Hom}}_W(F_*W, W) \to \underline{\mathrm{Hom}}_W(W, W) \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W).$$

The middle term is evidently W, so it suffices to show that the kernel of the map $W \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W)$ is isomorphic to F_*W.

Observe that the map $W \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W)$ factors as

$$W \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W). \tag{13}$$

Indeed, if $x \in W$, the map $\mathbf{G}_a^\sharp \to W$ sending $y \mapsto xy$ lands in $W[F]$ (since $F(xy) = F(x)F(y) = 0$). Therefore, (13) gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_*W \arrow[r, "V"] \arrow[d] & W \arrow[r] \arrow[d, "\sim"] & \mathbf{G}_a \cong \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathrm{Hom}}_W(F_*W, W) \arrow[r] & \underline{\mathrm{Hom}}_W(W, W) \arrow[r] & \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W) &
\end{tikzcd}$$

The map $\mathbf{G}_a \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W)$ is injective, and the map $W \to \mathbf{G}_a$ is surjective. In particular, the kernel of the map $W \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, W)$ can be identified with the kernel of $W \to \mathbf{G}_a$, which is precisely F_*W, as desired. $\square$

Proof of $(**)$. The cofiber sequence

$$\mathbf{G}_a^\sharp \to W \xrightarrow{F} F_*W$$

induces a cofiber sequence

$$\underline{\mathrm{Hom}}_W(W, \mathbf{G}_a^\sharp) \to \underline{\mathrm{Hom}}_W(\mathbf{G}_a^\sharp, \mathbf{G}_a^\sharp) \to \underline{\mathrm{Ext}}^1_W(F_*W, \mathbf{G}_a^\sharp).$$

The first term is simply $\mathbf{G}_a^\sharp$, and the second term can be identified with $\mathbf{G}_a$ by Footnote 4. It follows that there is a cofiber sequence

$$\mathbf{G}_a^\sharp \to \mathbf{G}_a \to \underline{\mathrm{Ext}}^1_W(F_*W, \mathbf{G}_a^\sharp),$$

giving the desired identification. $\square$

$\square$

The isomorphism of Proposition 3.10 is very useful: suppose one has a map $X \to \mathbf{Z}_p^{\mathcal{N}}$ of stacks over $\mathbf{A}^1_\hbar/\mathbf{G}_m$ which one wants to prove is an isomorphism. Let $\mathcal{I} \to \mathcal{O}_X$ denote the ideal given by the zero locus of $\hbar$, and suppose that $\mathcal{O}_X$ is $\mathcal{I}$-complete. If the induced map $X_{\hbar=0} \to (\mathbf{Z}_p^{\mathcal{N}})_{\hbar=0}$ is an isomorphism, then completeness implies that the original map $X \to \mathbf{Z}_p^{\mathcal{N}}$ is itself an isomorphism. It often turns out to be much easier to study $X_{\hbar=0}$. For instance, one can argue in this manner to show that the stack associated to the even filtration ([HRW22]) on $\mathrm{TC}^-(\mathbf{Z}_p)$ is isomorphic to $\mathbf{Z}_p^{\mathcal{N}}$, and even relate $\mathbf{Z}_p^{\mathcal{N}}$ to the complex connective image-of-J spectrum.

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