Lecture IV: Bogomolov-Tian-Todorov

In the previous two lectures, we saw a proof of Kontsevich's formality theorem for smooth manifolds: namely, that any Poisson structure on a smooth manifold M admits a deformation quantization. The proof proceeded by showing that the formal moduli problem associated to the Hochschild cohomology $\mathrm{HC}(\mathcal{O}_M)$ can always be solved. This was done in two parts: first, one showed that the formal moduli problem associated to deformations of the Poisson bracket is controlled by $\pi_{*-1}\mathrm{HC}(\mathcal{O}_M)$, and that it is always solvable; second, one showed that the $\mathbf{E}_2$-algebra $\mathrm{HC}(\mathcal{O}_M)[1]$ is formal, i.e., is equivalent as an $\mathbf{E}_2$-algebra to $\pi_{*-1}\mathrm{HC}(\mathcal{O}_M)$.

Our goal in this talk is to prove a *holomorphic* version of this statement, known as the *Bogomolov-Tian-Todorov theorem* (often shortened to "BTT theorem")[1]. In order to state this result, we need to recall the notion of a Calabi-Yau variety. As the definition makes clear, one should think of Calabi-Yau varieties as algebraic analogues of orientable manifolds.

[1] Warning to the reader: there are some issues with the grading that I garbled in the first draft of this document. I'll fix this soon.

Definition 1. Let k be a commutative ring, and let X be a smooth and proper scheme over k. Say that X is a *Calabi-Yau variety*[2] if there is an isomorphism $K_X \cong \mathcal{O}_X$, where $K_X = \wedge^{\dim X}\Omega^1_{X/k}$ is the canonical bundle on X. The *datum* of an isomorphism $K_X \cong \mathcal{O}_X$ is called a volume form on X.

[2] The definition of a Calabi-Yau variety sometimes includes the additional condition that $\mathrm{H}^i(X;\mathcal{O}_X) = 0$ for $0 < i < \dim(X)$, but this will not be necessary for the moment.

Example 2. Suppose X is a smooth and proper curve of genus g, and assume that k is an algebraically closed field. Then the Riemann-Roch formula tells us that $\deg K_X = 2g-2$, so X is Calabi-Yau if and only if $g = 1$, i.e., X is an elliptic curve. More generally, the adjunction formula and the fact that $K_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-n-1)$ implies that any smooth hypersurface of degree $n+1$ in $\mathbf{P}^n$ is Calabi-Yau.

Theorem 3 (BTT)**.** *Let X be a Calabi-Yau variety over $\mathbf{C}$ of dimension n such that $\mathrm{H}^0(X;T_{X/\mathbf{C}}) = 0$ and $\mathrm{H}^i(X;\mathcal{O}_X) = 0$ for $0 < i < n$. Then the deformation theory of X is unobstructed. More precisely, the functor $\mathrm{Def}_{X/\mathbf{C}} : \mathrm{Art}_{\mathbf{C}} \to \mathrm{Set}$ is pro-represented by the formal completion of the ring $k[\mathrm{H}^1(X;T_{X/\mathbf{C}})] = k[t_1, \cdots, t_{\dim \mathrm{H}^1(X;T_{X/\mathbf{C}})}]$.*

The Calabi-Yau condition on X implies that $T_{X/\mathbf{C}} \cong \Omega^{n-1}_{X/\mathbf{C}}$, so $\mathrm{H}^0(X;T_{X/\mathbf{C}}) \cong \mathrm{H}^0(X;\Omega^{n-1}_{X/\mathbf{C}})$. Hodge theory tells us that $\dim \mathrm{H}^0(X;\Omega^{n-1}_{X/\mathbf{C}}) = \dim \mathrm{H}^1(X;\mathcal{O}_X)$, so the condition that $\mathrm{H}^0(X;T_{X/\mathbf{C}}) = 0$ is satisfied if and only if $\mathrm{H}^1(X;\mathcal{O}_X) = 0$. Note further that $\mathrm{H}^0(X;T_{X/\mathbf{C}})$ vanishing amounts to asking that the deformations of X have no nontrivial automorphisms: this is what allows us to say that $\mathrm{Def}_{X/\mathbf{C}}$ lands in sets, as opposed to groupoids.

The deformation problem $\mathrm{Def}_{X/\mathbf{C}}$ associated to X is controlled by the Kodaira-Spencer differential graded Lie algebra $\mathfrak{g} = \Gamma(X;\wedge^{*+1}T_{X/\mathbf{C}})$[3]. Here (and always), Γ denotes *derived* global sections. Using the general yoga of deformation theory as presented in Lecture II, one can then prove Theorem 3 by showing the following:

[3] This is a slight lie: the deformation problem $\mathrm{Def}_{X/\mathbf{C}}$ is really controlled by the sub-Lie algebra of $\mathfrak{g}$ given by $\Gamma(X;T_{X/\mathbf{C}})$. However, for the sake of Theorem 4 below, it suffices to work with $\mathfrak{g}$ instead.

Theorem 4. *The dg-Lie algebra $\mathfrak{g}$ is homotopy abelian, i.e., $\mathfrak{g}$ is (quasi-)isomorphic to a dg-Lie algebra whose Lie bracket is zero.*

For future discussion, it will be useful to place the argument for this claim in a broader context.

Construction 5. Let $\mathcal{A}$ denote the *graded* dg-algebra $\Gamma(X;\wedge^*T_{X/\mathbf{C}})$, where a polyvector field of degree i is placed in graded weight $-i$ and homological degree i. If $\Sigma^{m,n}$ denotes a shift by homological degree m and graded weight n, then forgetting the grading on $\Sigma^{1,0}\mathcal{A}$ produces $\mathfrak{g}$. One can construct $\mathcal{A}$ more invariantly using the "Koszul sign rule equivalence" $\mathrm{Sym}^n_R(M[1]) \cong (\wedge^n M)[n]$ for any (graded) commutative ring R and any (graded) R-module M. Then, $\mathcal{A} = \Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Sigma^{1,-1}T_{X/\mathbf{C}}))$, where the parenthesis denotes the graded weight. In particular, $\mathcal{A}$ has the structure of a graded commutative $\mathbf{C}$-algebra.

The Lie bracket on $\mathfrak{g}$ further defines a 1-*shifted Lie bracket* on $\mathcal{A}$: in other words, if a and b are homotopy classes in $\mathcal{A}$ of degrees (m, i) and (n, j) regarded as polyvector fields on X, then the Schouten-Nijenhuis bracket $[a, b]$ is a homotopy class in $\mathcal{A}$ of degree $(m + n + 1, i + j + 1)$.

Placing a grading on $\mathcal{A}$ is not artificial: indeed, the HKR theorem tells us that the Whitehead filtration $\tau_{\geq -\star}\mathrm{HC}(X/\mathbf{C})$ on the Hochschild cohomology of X has associated graded given by

$$\Sigma^{0,i}\mathrm{gr}^{-i}(\tau_{\geq -\star}\mathrm{HC}(X/\mathbf{C})) = \pi_{-i}\mathrm{HC}(X/\mathbf{C}) \cong \Gamma(X; (\wedge^i T_{X/\mathbf{C}})[-i]). \tag{1}$$

This equivalence can be refined in two ways:

- The Deligne conjecture tells us that $\mathrm{HC}(X/\mathbf{C})$ is an $\mathbf{E}_2$-algebra over $\mathbf{C}$. In particular, $\tau_{\geq -\star}\mathrm{HC}(X/\mathbf{C})$ is a *filtered* $\mathbf{E}_2$-algebra. Then, (1) upgrades to a multiplicative equivalence of graded commutative $\mathbf{C}$-algebras between the associated graded of $\tau_{\geq -\star}\mathrm{HC}(X/\mathbf{C})$ and $\mathcal{A}$.
- Let B be any $\mathbf{E}_2$-algebra over $\mathbf{C}$. Then $\pi_* B$ has the structure of a 1-shifted Poisson algebra, where the Poisson bracket is given by the Browder bracket. The Poisson bracket on $\Sigma^{0,\bullet}\mathrm{gr}^{-\bullet}(\tau_{\geq -\star}\mathrm{HC}(X/\mathbf{C}))$ then agrees with the 1-shifted Lie bracket on $\mathcal{A}$.

Observe that $\mathrm{gr}(\tau_{\geq -\star}\Sigma B) \cong \Sigma^{-1,-1}\mathrm{gr}(\tau_{\geq -\star}B)$. Therefore, we can construct $\mathfrak{g}$ as

$$\mathfrak{g} \simeq \mathrm{gr}(\tau_{\geq -\star}\Sigma\mathrm{HC}(X/\mathbf{C})). \tag{2}$$

(This is not quite correct, since the homological degrees do not match up. However, this can be fixed using the *shearing* operation on graded $\mathbf{C}$-modules (which we will discuss in greater detail later), which sends a graded $\mathbf{C}$-module $M_\bullet$ to $M_\bullet[2\bullet]$.) Moreover, $\Sigma\mathrm{HC}(X/\mathbf{C})$ is a dg-Lie algebra over $\mathbf{C}$, and (2) is an equivalence of graded Lie algebras over $\mathbf{C}$. Therefore, Theorem 4 will follow if we can prove that $\Sigma\mathrm{HC}(X/\mathbf{C})$ is a homotopy abelian dg-Lie algebra over $\mathbf{C}$. This is essentially the form of the BTT theorem that we will discuss.

If we wish to prove a version of the BTT theorem over fields of positive characteristic (for instance), we must work with the analogue dg-Lie algebra $\Sigma\mathrm{HC}(X/\mathbf{C})$, or rather the $\mathbf{E}_2$-$\mathbf{C}$-algebra $\mathrm{HC}(X/\mathbf{C})$, itself. However, over characteristic zero, we can use the formality of the $\mathbf{E}_2$-operad from Lecture III to view the $\mathbf{E}_2$-$\mathbf{C}$-algebra $\mathrm{HC}(X/\mathbf{C})$ as the 1-shifted Poisson algebra $\mathcal{A}$. (We will mainly ignore the grading on $\mathcal{A}$, because it can be read off from the homological degree.) We will elect to adopt this point of view for the first part of this lecture, because $\mathcal{A}$ is a more concrete object than $\mathrm{HC}(X/\mathbf{C})$.

In summary, our goal is to show that if X is a Calabi-Yau variety, then $\mathcal{A}$ is a homotopy abelian dg-Lie algebra over $\mathbf{C}$. We will do this by upgrading $\mathcal{A}$ into a Batalin-Vilkovisky algebra.

Construction 6. Choose a volume form ω on X. Then ω defines an isomorphism $\Omega^{n-k}_{X/\mathbf{C}} \cong \wedge^k T_{X/\mathbf{C}}$. In particular, the de Rham differential $d : \Omega^i_{X/\mathbf{C}} \to \Omega^{i+1}_{X/\mathbf{C}}$ induces a map $\Delta : \wedge^{j+1}T_{X/\mathbf{C}} \to \wedge^j T_{X/\mathbf{C}}$, and hence a map $\Delta : \mathcal{A} \to \mathcal{A}$ which increases homological degree (and the graded weight) by 1. Since $d^2 = 0$, the operator Δ also squares to zero. Moreover, $d\Delta + \Delta d = 0$.

The crucial analytic input into Theorem 3 (which we will not prove here) is that the Schouten-Nijenhuis bracket can be expressed in terms of Δ:

$$\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b) = [a, b]. \tag{3}$$

This is known as the *Tian-Todorov lemma*: one can interpret (3) as the claim that the bracket on polyvector fields measures the failure of Δ to be a derivation on $\mathcal{A}$. We can package the algebraic data above into a definition:

Definition 7. A *Batalin-Vilkovisky algebra* (often shortened to $\mathcal{BV}$-algebra) over a field k is a pair (A, Δ) of a commutative and associative differential graded algebra A (with internal differential d) over k and a square-zero map $\Delta : A \to A$ of homological degree 1 such that:

- We have $d\Delta + \Delta d = 0$.
- For each $a \in A$, the operator $[a, -] : A \to A$ defined by Equation (3) is a derivation.

One can compute using Equation (3) that $\Delta[a, b] = [\Delta(a), b] + (-1)^{|a|+1}[a, \Delta(b)]$. Furthermore, one can extend the above notion to the *graded* setting.

Note that (3) tells us that any $\mathcal{BV}$-algebra over k has an underlying 1-shifted Poisson algebra. Recall that 1-shifted Poisson algebras are algebras for the Poisson operad $\mathbf{P}_2$ over k. It turns out that $\mathcal{BV}$-algebras are also algebras for a particular operad over k, which we will denote $\mathcal{BV}_2$; this is true even in the graded setting, where graded $\mathcal{BV}$-algebras are algebras for a graded k-operad $\mathcal{BV}_2^{\mathrm{gr}}$. Rather than giving a generators-and-relations construction of $\mathcal{BV}_2$, we will just cite the following generalization of a theorem of Fred Cohen's:

Theorem 8. *Let $\mathbf{E}_2^{\mathrm{fr}}$ denote the framed $\mathbf{E}_2$-operad whose nth space is the space of embeddings $\amalg_n D^2 \hookrightarrow D^2$ which on each component is a composite of translations, dilations, and rotations. In general, there is a morphism $\mathcal{BV}_2^{\mathrm{gr}} \to \mathrm{gr}(\tau_{\geq\star} C_*(\mathbf{E}_2^{\mathrm{fr}}; k))$ of graded operads over k; this map is an equivalence if k is a field of characteristic zero. One also has a formality statement: over a field k of characteristic zero, the operad $C_*(\mathbf{E}_2^{\mathrm{fr}}; k)$ is isomorphic to its homology.*

In particular, the k-homology of a framed $\mathbf{E}_2$-algebra is a (graded) $\mathcal{BV}$-algebra. (Later, we will describe the additional structure that one acquires on the $\mathcal{BV}$-algebra if $\mathbf{F}_p \subseteq k$.) The BV-structure on $\mathcal{A}$ that we described above in fact comes from a framed $\mathbf{E}_2$-structure on $\mathrm{HC}(X/\mathbf{C})$ itself. We will discuss this further later when discussing the positive characteristic story.

We can finally prove Theorem 4 via the following, which appears in [KKP08]:

Proposition 9 (The "KKP proposition")**.** *Let A be a (graded) $\mathcal{BV}$-algebra over a field k, and let $\hbar$ be a formal variable of homological degree -2 (and graded weight 1). Let $(A[\![\hbar]\!], d+\hbar\Delta)$ denote the differential graded k-module whose underlying k-module is $A[\![\hbar]\!]$ and whose differential is $d + \hbar\Delta$. Suppose that the homology of $(A[\![\hbar]\!], d + \hbar\Delta)$ is a free $k[\![\hbar]\!]$-module[4]. If $\mathbf{Q} \subseteq k$, then the dg-Lie algebra $\mathfrak{g} = \Sigma A$ is homotopy abelian.*

[4] Note that this condition would be vacuous if $\Delta = 0$ on the nose. This would then imply the conclusion of Proposition 9, too, via (3).

Proof of Theorem 4 given Proposition 9. According to Proposition 9, we only need to prove that the homology of $(\mathcal{A}[\![\hbar]\!], d+\hbar\Delta)$ is a free $\mathbf{C}[\![\hbar]\!]$-module. Equivalently, it suffices to show that the spectral sequence going from $E_2^{*,*} = (\mathrm{H}_*\mathcal{A})[\![\hbar]\!]$ to $\mathrm{H}_*(\mathcal{A}[\![\hbar]\!], d + \hbar\Delta)$ degenerates at the E_2-page. This happens if and only if the spectral sequence degenerates after inverting $\hbar$. We claim that the spectral sequence

$$E_2^{*,*} = (\mathrm{H}_*\mathcal{A})(\!(\hbar)\!) \Rightarrow \mathrm{H}_*(\mathcal{A}(\!(\hbar)\!), d + \hbar\Delta) \tag{4}$$

can be identified with a 2-periodified version of the Hodge-de Rham spectral sequence. More precisely, recall that the Calabi-Yau structure on X allows us to identify $\wedge^k T_{X/\mathbf{C}}$ with $\wedge^{n-k}\Omega^1_{X/\mathbf{C}}$. This gives an identification between $\mathcal{A} = \Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/\mathbf{C}}[1]))$ and a shift of $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/\mathbf{C}}[1]))$, which is just (up to the shearing operation described above) the underlying graded vector space of the de Rham complex. Under this identification, the differential $\hbar^{-1}(d+\hbar\Delta) = \hbar^{-1}d + \Delta$ on $\mathcal{A}(\!(\hbar)\!)$ translates to the de Rham differential d_{dR} on $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/\mathbf{C}}[1]))(\!(\hbar)\!)$. Therefore, the spectral sequence (4) can be rewritten (up to some shift) as a 2-periodified version of the Hodge-de Rham spectral sequence[5]

$$E_2^{*,*} = \mathrm{H}^*(X; \Omega^*_{X/\mathbf{C}})(\!(\hbar)\!) \Rightarrow \mathrm{H}^*(X; \Omega^\bullet_{X/\mathbf{C}})(\!(\hbar)\!) = \mathrm{H}^*_{\mathrm{dR}}(X/\mathbf{C})(\!(\hbar)\!).$$

However, it is a well-known fact (e.g., by Hodge theory) that the Hodge-de Rham spectral sequence degenerates at the first page, proving the desired result. □

[5] The classical Hodge-de Rham spectral sequence runs from $E_1^{*,*} = \mathrm{H}^*(X; \Omega^*_{X/\mathbf{C}})$ to $\mathrm{H}^*_{\mathrm{dR}}(X/\mathbf{C})$. Note that it starts at the E_1-page, whereas below we see the E_2-page of some other spectral sequence. These spectral sequences are related by a *shearing* process, which we will discuss later. For now, let us just mention the renumbering $E_r^{i,j} \mapsto E_{r+1}^{2i+j,-i}$ for the spectral sequence of a bicomplex.

Remark 10. An important observation is that in the above proof, the only place where it was crucial that X lived over $\mathbf{C}$ (or a characteristic zero field) was when we invoked the degeneration of the Hodge-de Rham spectral sequence. More generally, the above argument shows that when X is a Calabi-Yau variety over a field k (of arbitary characteristic), the $\mathcal{BV}$-algebra $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$ satisfies the freeness hypothesis of Proposition 9 if and only if the Hodge-de Rham spectral sequence for X degenerates at the E_1-page.

Proof of Proposition 9. We will completely ignore the (external) grading on A in this discussion. First, a specialization argument (i.e., degenerating from the complement of the origin in the formal disk $\mathrm{Spf}\, k[\![\hbar]\!]$ to $\mathrm{Spec}\, k$) reduces one to showing that the the dg-Lie algebra $\Sigma(A(\!(\hbar)\!), d+\hbar\Delta)$ over $k(\!(\hbar)\!)$ is homotopy abelian. To prove this, let us collapse the $\mathbf{Z}\times\mathbf{Z}/2$-grading on $A(\!(\hbar)\!)$ (where the $\mathbf{Z}$-grading is the homological grading on A, and the $\mathbf{Z}/2$-grading comes from $\hbar$) to a $\mathbf{Z}/2$-grading. Let $\delta : A(\!(\hbar)\!) \to A(\!(\hbar)\!)$ denote the $\mathbf{Z}/2$-graded operator on $A(\!(\hbar)\!)$ given by

$$\delta(a) = da + \hbar\Delta(a) + \frac{[a,a]}{2}.$$

In fact, δ is a derivation on $A(\!(\hbar)\!)$. To show that $A(\!(\hbar)\!)$ is homotopy abelian, it suffices to show that there is some $\mathbf{Z}/2$-graded automorphism Φ of $A(\!(\hbar)\!)$ such that

$$\delta(\Phi(a)) = d\Phi(a) + \hbar\Delta\Phi(a). \tag{5}$$

Define $\Phi : A(\!(\hbar)\!) \to A(\!(\hbar)\!)$ to be the $\mathbf{Z}/2$-graded automorphism given by

$$\Phi(\hbar) = \hbar,\ \Phi(a) = \hbar(e^{a/\hbar} - 1) = \sum_{n\geq 1} \frac{a^n}{n!\hbar^{n-1}}.$$

Here, e^x denotes the formal series $\sum_{m\geq 0}\frac{x^m}{m!}$; in particular, if $|x| \equiv 1 \pmod 2$, then $x^2 = 0$, so $e^x - 1$ just means x. Observe that $|a^n/\hbar^{n-1}| = n|a| - 2(n-1) \equiv n|a| \pmod 2$, which agrees with $|a|$ if a is in $\mathbf{Z}/2$-graded weight 0. To show that Φ satisfies Equation (5), we need to compute $\Delta\Phi(a)$. For this, we need a preliminary calculation: an easy induction using $\Delta(x^2) = 2x\Delta(x) + [x,x]$ shows that

$$\Delta(x^n) = nx^{n-1}\Delta(x) + \binom{n}{2}x^{n-2}[x,x].$$

Therefore,

$$\begin{aligned}\Delta\Phi(a) = \hbar\Delta e^{a/\hbar} &= \sum_{n\geq 0}\frac{1}{n!\hbar^{n-1}}\Delta(a^n)\\ &= \left(\Delta(a) + \frac{1}{2}\left[\frac{a}{\hbar}, \frac{a}{\hbar}\right]\right)e^{a/\hbar}.\end{aligned}$$

This implies the desired identity (5):

$$\begin{aligned}(d+\hbar\Delta)\Phi(a) &= \hbar\left(\frac{da}{\hbar}+\hbar\Delta(a)+\frac{\hbar}{2}\left[\frac{a}{\hbar},\frac{a}{\hbar}\right]\right)e^{a/\hbar}\\ &= \left(da+\hbar\Delta(a)+\frac{[a,a]}{2}\right)e^{a/\hbar}\\ &= \delta(a)e^{a/\hbar}=\delta(e^{a/\hbar})=\delta\Phi(a).\end{aligned}$$

□

We now turn to the question of generalizing Theorem 4 to Calabi-Yau varieties over fields of characteristic $p > 0$. We will not be able to prove any such statement in this lecture, but we will have more things to say in a few weeks. Recall from Remark 10 that if X is a Calabi-Yau variety over a field k (of arbitary characteristic), then the $\mathcal{BV}$-algebra $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$ satisfies the freeness hypothesis of Proposition 9 if and only if the Hodge-de Rham spectral sequence for X degenerates at the E_1-page. Therefore, we have two tasks set out for us: first, determine when the Hodge-de Rham spectral sequence for X degenerates at the E_1-page; and second, prove an analogue of Proposition 9.

Unfortunately, it turns out that the correspondence between formal moduli problems and differential graded Lie algebras fails in characteristic $p > 0$: in particular, the Kodaira-Spencer differential graded Lie algebra *no longer* describes the deformation theory of X. However, one remedy for this failure was proposed in [BM19]. Namely, if k is a field of characteristic $p > 0$, there is a correspondence between formal moduli problems (viewed as certain functors from Artinian $\mathbf{E}_\infty$-k-algebras to ∞-groupoids) and "partition Lie algebras". We will not need the precise definition of partition Lie algebras in the sequel, but we will summarize some of its relationships with other, more well-known, notions.

(a) If k is a field of characteristic zero, then partition Lie algebras are just 1-shifted differential graded Lie algebras over k.

(b) Partition Lie algebras are algebras in Mod_k for a monad $\mathrm{Lie}^\pi_{\mathbf{E}_\infty}$. This monad preserves limits and sifted colimits. Partition Lie algebras can be viewed roughly as 1-shifted analogues of restricted differential graded Lie algebras. Let $\mathfrak{g}$ be a differential graded Lie algebra over k with internal differential d, and assume that $\mathfrak{g}$ admits a restricted Lie structure $\varphi : \mathfrak{g} \to \mathfrak{g}$ (in the sense of Lecture I), so that φ sends a class in degree n to a class in degree np, and $d\varphi(x) = \mathrm{ad}_x^{p-1} dx$. Then $\Sigma\mathfrak{g} = \mathfrak{g}[1]$ is a partition Lie algebra: the bracket on $\pi_*\mathfrak{g}[1]$ sends a class in degree n and a class in degree m to a class in degree $n + m - 1$, while the operator φ sends a class in degree n to a class in degree $p(n - 1) + 1 = np - (p - 1)$.

(c) If X is a scheme over k, then $\Gamma(X; T_{X/k}[1])$ is a partition Lie algebra over k. If we abusively attempt to describe the partition Lie algebra as a 1-shifted restricted differential graded Lie algebra, then its Lie bracket is given by the usual bracket of vector fields, while the restricted structure is given by sending a vector field to its pth power (which is still a derivation by the Leibniz formula).

(d) Let A be an $\mathbf{E}_2$-k-algebra. Then $\Sigma^2 A$ admits the structure of a partition Lie algebra. The Lie bracket on $\Sigma^2 A$ is given by the Browder bracket on A, while the restricted Lie structure is given by the first Dyer-Lashof operation $\mathcal{Q}_1$. This description is abusive in several ways: first, we are attempting describe the partition Lie algebra as a 1-shifted restricted differential graded Lie algebra; second, the Browder bracket and the Dyer-Lashof operation $\mathcal{Q}_1$ are both defined on the *homotopy* of A, and are not actual spectrum-level operations on A itself. However, one can make the above description precise: there is a map from the $\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}$-monad to the "double suspension" of the free $\mathbf{E}_2$-algebra monad.

(e) There is a colimit-preserving functor from $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}}(\mathrm{Mod}_k)$ to $\mathrm{Alg}^{\mathrm{aug}}_{\mathbf{E}_2}(\mathrm{Mod}_k)$, which sends a partition Lie algebra $\mathfrak{g}$ over k to its *universal enveloping* $\mathbf{E}_2$-*algebra* $\mathfrak{U}(\mathfrak{g})$; the functor $\mathfrak{U}$ is left adjoint to the forgetful functor $\mathrm{Alg}^{\mathrm{aug}}_{\mathbf{E}_2}(\mathrm{Mod}_k) \to \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}}(\mathrm{Mod}_k)$ from the previous bullet. This construction can be refined to a functor $\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}}(\mathrm{Mod}_k) \to \mathrm{coCAlg}(\mathrm{Alg}^{\mathrm{aug}}_{\mathbf{E}_2}(\mathrm{Mod}_k))$. Moreover, if V is a k-module and $\mathfrak{g}$ is the free partition Lie algebra on V, then $\mathfrak{U}(\mathfrak{g})$ is the free augmented $\mathbf{E}_2$-k-algebra on V.

(f) There is a Koszul duality functor $\mathfrak{D} : \mathrm{CAlg}^{\mathrm{aug}}_k \to \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}}(\mathrm{Mod}_k)$ which sends an augmented k-algebra R to the partition Lie algebra $L^{\vee}_{k/R}[1] \simeq (L_{R/k} \otimes_R k)^{\vee}$. If $\mathfrak{g}$ is a partition Lie algebra, then the associated formal moduli problem $\Psi_{\mathfrak{g}} : \mathrm{CAlg}^{\mathrm{Art}}_k \to \mathcal{S}$ is given by

$$\Psi_{\mathfrak{g}}(R) = \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{\mathbf{E}_\infty}}(\mathrm{Mod}_k)}(\mathfrak{D}(R), \mathfrak{g}).$$

Our goal, therefore, is to prove an analogue of Theorem 4 in the case when $\mathfrak{g}$ is replaced by the partition Lie algebra $\Gamma(X; \mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1]))$. Since any deformation of X defines a deformation of $\mathrm{QCoh}(X)$ as a symmetric monoidal ∞-category, a simpler question is to consider symmetric monoidal deformations of the k-linear (∞-)category $\mathrm{QCoh}(X)$ instead. It is even easier to consider deformations of $\mathrm{QCoh}(X)$ as an *unstructured* k-linear category (we will return to partition Lie algebras and deformations of X itself at the end of this seminar). One of our goals in future talks will be to run the argument for Theorem 4 in this setting. Namely, we will:

- Recall a proof that the noncommutative Hodge-de Rham spectral sequence for the Hochschild homology of $\mathrm{QCoh}(X)$ degenerates at the E_2-page, with some assumptions on $\mathrm{QCoh}(X)$ if $\mathbf{F}_p \subseteq k$.
- Show that an analogue of Proposition 9 holds even in characteristic $p > 0$ (using partition Lie algebras instead), at least with some assumptions on A which allow us to make sense of expressions like the exponential. For this, we will give a reproof of Proposition 9, using *operadic* methods instead.

Together, these results will allow us to prove that if X is a Calabi-Yau variety over a field k of characteristic $p > 0$ such that the Hodge-de Rham spectral sequence of X degenerates and $\dim(X) < p$, then the deformation theory of $\mathrm{QCoh}(X)$ is unobstructed. However, it will take us a few more lectures before arriving at the point where we can prove such a result.

References

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