Geometric Langlands duality for PGL_2 on the nodal curve

S. K. Devalapurkar

Abstract. In this note, we prove the local relative geometric Langlands conjecture of Ben-Zvi–Sakellaridis–Venkatesh for the spherical subgroup $\mathrm{PGL}_2^{\mathrm{diag}}$ of the triple product $\mathrm{PGL}_2^{\times 3}$ (and also for the spherical subgroup G_2 of SO_8/μ_2), whose corresponding dual $\mathrm{SL}_2^{\times 3}$-variety can be identified with the space $(\mathbf{A}^2)^{\otimes 3} \cong \mathbf{A}^8$ of $2 \times 2 \times 2$-cubes. The argument uses a construction of Bhargava relating $2 \times 2 \times 2$-cubes to quadratic forms and the Cayley hyperdeterminant as studied by Gelfand-Kapranov-Zelevinsky.

1. Introduction

The goal of this brief note is to study the geometrization of a story from the arithmetic context pioneered by Jacquet, Kudla-Harris, and Ichino among many others (see, e.g., [**HK91, Ich08**]). Fix an eighth root of unity ζ_8, let i be the resulting square root of -1, and write $k := \mathbf{Q}(\zeta_8) \cong \mathbf{Q}(i, \sqrt{2})$.

Notation 1.1. Let std denote the standard representation of SL_2, so that $\mathrm{std}^{\otimes 3}$ consists of cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] \arrow[dd, dash] & & d_3 \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & b_1 \arrow[ur, dash] \arrow[dd, dash] & \\
& d_1 \arrow[rr, dash] & & c. \\
b_3 \arrow[ur, dash] \arrow[rr, dash] & & d_2 \arrow[ur, dash] &
\end{tikzcd}$$

Fix an integer n. Equip $\mathrm{std}^{\otimes 3}$ with the grading where the entries of a cube have the following weights: a lives in weight $-4n$, each b_i lives in weight $-2n$, c lives in weight $2n$, and each d_i lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the corresponding graded variety.

Similarly, equip SL_2 with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. Write $\mathrm{SL}_2(-2n\rho)$ to denote this graded group. Then there is a natural graded action of $\mathrm{SL}_2(-2n\rho)^{\times 3}$ on $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$.

Recall that the process of *shearing* discussed in [**Rak20, Lur15**], as well as [**Dev23**, Section 2.1], converts gradings into homological shifts (more precisely, it sends a module in weight n to the same module shifted homologically by n). This

Part of this work was done when the author was supported by NSF DGE-2140743.

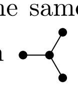
functor is symmetric monoidal when restricted to the subcategory of modules in *even* weights, and therefore extends to an operation on evenly graded stacks. As in [**Dev23**], we will state all of our results with "arithmetic shearing" in the sense of [**BZSV23**, Section 6.7].

Theorem 1.2 (Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$)**.** *There is an equivalence*[1]

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PGL}_2^{\times 3}}(\mathcal{L}(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ *denotes the* ∞*-category of perfect complexes on the shearing of the quotient stack* $\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}$.

Let $\mathrm{PSO}_{2n} := \mathrm{SO}_{2n}/\mu_2$. Then, the embedding $\mathrm{PGL}_2^{\mathrm{diag}} \subseteq \mathrm{PGL}_2^{\times 3}$ can be identified with the diagonal embedding $\mathrm{SO}_3 \subseteq \mathrm{SO}_3 \times \mathrm{PSO}_4$, so Theorem 1.2 could be viewed as a special case of the geometrized analogue of the Gan-Gross-Prasad period.

A similar argument shows a variant for PSO_8. Namely, there is an embedding $G_2 \subseteq \mathrm{PSO}_8$ given by triality, which exhibits G_2 as a spherical subgroup of PSO_8. To see that this situation is analogous to that of Theorem 1.2, note that the Dynkin diagram $\bullet$ of A_1 is obtained from the Dynkin diagram $\bullet\bullet\bullet$ of $A_1^{\times 3}$ by folding with respect to the obvious action of the symmetric group Σ_3. In the same way, the Dynkin diagram of G_2 is obtained from the Dynkin diagram  of D_4 by folding with respect to the action of Σ_3 permuting the three vertices around the branching vertex.

Theorem 1.3 (Derived geometric Satake for PSO_8/G_2)**.** *Suppose that the* $\mathrm{PSO}_8[\![t]\!]$*-action on* $\mathrm{PSO}_8(\!(t)\!)/G_2(\!(t)\!)$ *is optimal in the sense of* [**Dev23**, Hypothesis 3.5.21]. *Then there is an equivalence*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{PSO}_8}(\mathcal{L}(\mathrm{PSO}_8/G_2); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(12, \vec{6}, -6, \vec{0})/\mathrm{SL}_2(-6\rho)^{\times 3} \times \mathbf{A}^1(4)).$$

Remark 1.4. Following the philosophy of [**Dev23**], it should also be possible to use a variant of the methods of this article to prove analogues of Theorem 1.2 and Theorem 1.3 for sheaves with coefficients in connective complex K-theory ku. We have not attempted to do this, but we expect the corresponding 1-parameter deformation of $\mathrm{std}^{\otimes 3}$ over $\pi_*(\mathrm{ku}) \cong \mathbf{Z}[\beta]$ to be rather interesting.

Remark 1.5. The equivalence of Theorem 1.2 can heuristically be viewed as geometric Langlands for PGL_2 on the nodal curve $\mathbf{C}P^1 \vee \mathbf{C}P^1$ (which can be thought of as cut out in $\mathbf{P}^1_{[x_0:x_1]} \times \mathbf{P}^1_{[y_0:y_1]}$ by $x_0 y_0 = 0$). Indeed, if H is an complex algebraic group, and $(\mathbf{C}P^1)^{\vee(n-1)}$ is a wedge sum of $(n-1)$ complex projective lines, there is an isomorphism $\mathrm{Bun}_H((\mathbf{C}P^1)^{\vee(n-1)}) \simeq H^{\times n}\backslash\mathcal{L}(H^{\times n}/H^{\mathrm{diag}})$. Note that if $[n] = \{0, \cdots, n\}$, one may view $(\mathbf{C}P^1)^{\vee(n-1)}$ as the homotopy pushout $* \amalg_{* \amalg_{[n-1]} *} *$.

Remark 1.6. The quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is also studied (in different language, of course) in quantum information theory; see Remark 2.11 below.

[1]The ∞-category on the left-hand side is as in [**Dev23**, Definition 3.5.15]; see Definition 3.1 for a quick review. We expect the results of this article continue to hold if one considers sheaves with coefficients in $\mathbf{Z}[i, \frac{1}{\sqrt{2}}]$.

Theorem 1.2 and Theorem 1.3 are predicted by (the Betti version of) the local geometric conjecture of Ben-Zvi–Sakellaridis–Venkatesh; see [**BZSV23**, Conjecture 7.5.1]. My homotopy-theoretic interpretation of their conjecture is as follows. Suppose G is a reductive group over $\mathbf{C}$ and G/H is an affine homogeneous spherical G-variety (meaning that it admits an open B-orbit for its natural left $B \subseteq G$-action). Then, there should be a dual graded $\check{G}$-variety $\check{M}$ equipped with a moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$, and an equivalence of the form

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{C}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G}),$$

where $\mathrm{Perf}^{\mathrm{sh}}$ denotes the ∞-category of perfect complexes on the shearing of $\check{M}/\check{G}$ with respect to its given grading. In fact, [**BZSV23**, Section 4] gives an explicit construction of this predicted dual $\check{M}$, and in the examples $(G, H) = (\mathrm{PGL}_2^{\times 3}, \mathrm{PGL}_2^{\mathrm{diag}})$ and (PSO_8, G_2), one can compute that the stacky quotient $\check{M}/\check{G}$ is isomorphic to the right-hand sides of Theorem 1.2 and Theorem 1.3 respectively.[2]

Remark 1.7. Lest Theorem 1.2 seem like an oddly specific example to focus on, we note that it is essentially the *only* "new" example of a spherical pair (G, H) of the form $(H^{\times j}, H^{\mathrm{diag}})$, as shown by the following lemma.

Lemma 1.8. *Suppose H is a simple linear algebraic group over $\mathbf{C}$. Then the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical if and only if:*

(a) $j = 2$, and H arbitrary;
(b) $j = 3$ and H is of type A_1.

PROOF. If the subgroup $H^{\mathrm{diag}} \subseteq H^{\times j}$ is spherical, there is an open H^{diag}-orbit on the flag variety of $H^{\times j}$. This implies that the dimension of H must be at least $j|\Phi^+|$, where Φ^+ is the set of positive roots; equivalently, one needs $\mathrm{rank}(H) \geq (j-2)|\Phi^+|$. Of course, this is always satisfied if $j = 2$ (this is the group case corresponding to the symmetric subgroup $H^{\mathrm{diag}} \subseteq H \times H$). Using the classification of simple linear algebraic groups over $\mathbf{C}$, it is easy to see that the only other case when the above inequality can hold is when $j = 3$ and H is of type A_1; one can then check by hand that the diagonal subgroup in this case is indeed spherical. □

In the first two cases of Lemma 1.8, [**BZSV23**, Conjecture 7.5.1] follows from existing results: the first case is precisely the derived geometric Satake equivalence of [**BF08**]. Therefore, the only other case of Lemma 1.8 is when H is simple of type A_1, and Theorem 1.2 precisely addresses [**BZSV23**, Conjecture 7.5.1] for the adjoint form PGL_2 of H.

The proof of Theorem 1.2 reduces to showing that the conditions of [**Dev23**, Theorem 3.5.20] are met. This ultimately relies on studying Bhargava's construction from [**Bha04**] relating $2 \times 2 \times 2$-matrices to quadratic forms, and the work

[2] In the first case, this computation is straightforward given the prescription of [**BZSV23**, Section 4]; see [**Sak13**, Example 7.2.4] for a reference. The computation in the second case goes as follows. As in [**BZSV23**, Remark 7.1.1], the quotient stack $\check{M}/\check{G}$ can be identified with the quotient $\check{V}_X/\check{G}_X$, where $\check{G}_X$ is the Gaitsgory-Nadler/Sakellaridis-Venkatesh/Knop-Schalke dual group of X and $\check{V}_X$ is constructed in [**BZSV23**, Section 4.5]. In the case $X = \mathrm{PSO}_8/G_2$, a calculation shows that $\check{G}_X$ is the Levi subgroup of the maximal parabolic subgroup of PSO_8 corresponding to the central vertex of the D_4 Dynkin diagram; so $\check{G}_X \cong \mathrm{SL}_2^{\times 3}$. Using the prescription of [**BZSV23**, Section 4.5], one can check that $\check{V}_X \cong \mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1$, where $\check{G}_X$ acts only on the first factor. See, e.g., [**Sak13**, Line 9 of Table in Appendix A].

[**GKZ94**] of Gelfand-Kapranov-Zelevinsky describing the relationship to Cayley's hyperdeterminant.

1.1. Acknowledgements. I am very grateful to Yiannis Sakellaridis for his support and help in answering my numerous questions about [**BZSV23**], to Alison Miller for directing me (via MathOverflow) to the work of Bhargava, and to Charles Fu for helpful suggestions.

2. Some properties of $\mathrm{std}^{\otimes 3}$

In this section, we establish some basic properties of $\mathrm{std}^{\otimes 3}$ as a $\mathrm{SL}_2^{\times 3}$-variety; our base field will always be k, and we will write $\check{G} = \mathrm{SL}_2^{\times 3}$. Some of this material appears in [**Bha04**]. In particular, Construction 2.3 is due to Bhargava.

Observation 2.1. An element $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2$ can be identified with a binary quadratic form $q_A(x,y) = cx^2 + 2iaxy + by^2$. Under this identification, the adjoint action of $g \in \mathrm{SL}_2$ on $\mathfrak{sl}_2$ is given by the action on (x,y) of the conjugate of g by the matrix $\mathrm{diag}(\zeta_8, \zeta_8^{-1})$. Explicitly, if $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, the action sends

$$\begin{aligned} x &\mapsto i\delta x + \beta y, \\ y &\mapsto \gamma x - i\alpha y. \end{aligned}$$

Note, moreover, that the discriminant of $q_A(x,y)$ is $4\det(A)$.

Warning 2.2. Note that under Observation 2.1, the element of $\mathfrak{sl}_2$ associated to a binary quadratic form $bx^2 + axy + cy^2$ is *not* the symmetric matrix associated to the quadratic form! Indeed, the associated symmetric matrix is $\left(\begin{smallmatrix} b & a/2 \\ a/2 & c \end{smallmatrix}\right)$, while the associated element of $\mathfrak{sl}_2$ is $\left(\begin{smallmatrix} -ai/2 & c \\ b & ai/2 \end{smallmatrix}\right)$.

Construction 2.3. The affine space $\mathbf{A}^8 = \mathrm{std}^{\otimes 3}$ can be regarded as parametrizing cubes

$$\begin{tikzcd}
& b_2 \arrow[rr, dash] \arrow[dd, dash] & & d_3 \arrow[dd, dash] \\
a \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & b_1 \arrow[ur, dash] \arrow[dd, dash] & \\
& d_1 \arrow[rr, dash] & & c, \\
b_3 \arrow[ur, dash] \arrow[rr, dash] & & d_2 \arrow[ur, dash] &
\end{tikzcd}$$

which we will represent by a tuple $(a, \vec{b}, c, \vec{d})$; we will often use the symbol $\mathcal{C}$ to denote such a cube. If $\{e_1, e_2\}$ are a basis for std, the above cube corresponds to the element of $\mathrm{std}^{\otimes 3}$ given by

$$\begin{aligned} &ae_1 \otimes e_1 \otimes e_1 + b_1 e_2 \otimes e_1 \otimes e_1 + b_2 e_1 \otimes e_2 \otimes e_1 + b_3 e_1 \otimes e_1 \otimes e_2 \\ &+ d_1 e_1 \otimes e_2 \otimes e_2 + d_2 e_2 \otimes e_1 \otimes e_2 + d_3 e_2 \otimes e_2 \otimes e_1 + c e_2 \otimes e_2 \otimes e_2. \end{aligned}$$

Associated to a cube $\mathcal{C}$ are three pairs of matrices, given by slicing along the top, leftmost, or front faces:

$$\begin{aligned} M_1 &= \left(\begin{smallmatrix} a & b_2 \\ b_3 & d_1 \end{smallmatrix}\right),\ N_1 = \left(\begin{smallmatrix} b_1 & d_3 \\ d_2 & c \end{smallmatrix}\right), \\ M_2 &= \left(\begin{smallmatrix} a & b_1 \\ b_3 & d_2 \end{smallmatrix}\right),\ N_2 = \left(\begin{smallmatrix} b_2 & d_3 \\ d_1 & c \end{smallmatrix}\right), \\ M_3 &= \left(\begin{smallmatrix} a & b_1 \\ b_2 & d_3 \end{smallmatrix}\right),\ N_3 = \left(\begin{smallmatrix} b_3 & d_2 \\ d_1 & c \end{smallmatrix}\right); \end{aligned}$$

each of these defines a binary quadratic form

$$q_i(x,y) = -\det(M_i x + N_i y).$$

Explicitly,

$$\begin{aligned}
q_1(x,y) &= \det(M_1)x^2 + (ac + b_1d_1 - b_2d_2 - b_3d_3)xy + \det(N_1)y^2,\\
q_2(x,y) &= \det(M_2)x^2 + (ac - b_1d_1 + b_2d_2 - b_3d_3)xy + \det(N_2)y^2,\\
q_3(x,y) &= \det(M_3)x^2 + (ac - b_1d_1 - b_2d_2 + b_3d_3)xy + \det(N_3)y^2.
\end{aligned}$$

Viewing $\mathfrak{sl}_2$ as the space of binary quadratic forms as in Observation 2.1, these three quadratic forms define a map

$$\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}.$$

An easy check shows that this map is $\check{G}$-equivariant.

Lemma 2.4 (Cayley)**.** *The composite*

$$\mathrm{std}^{\otimes 3} \xrightarrow{\mu} \mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$$

factors through the diagonal inclusion $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$*. In fact, the induced map* $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ *defines an isomorphism*

$$\mathrm{std}^{\otimes 3}/\!\!/\check{G} \xrightarrow{\sim} \mathfrak{sl}_2/\!\!/\mathrm{SL}_2.$$

Proof. The map $\mathfrak{sl}_2^{\times 3} \to \mathfrak{sl}_2^{\times 3}/\!\!/\check{G}$ sends a triple of matrices to their determinants, or equivalently a triple of quadratic forms to their discriminants. Therefore, we need to check that the three quadratic forms of Construction 2.3 have the same discriminant. This is an easy check: one finds that their common discriminant is

$$\begin{aligned}
\det(q_i) = a^2c^2 &+ b_1^2d_1^2 + b_2^2d_2^2 + b_3^2d_3^2 - 2(ab_1cd_1 + ab_2cd_2 + ab_3cd_3\\
&+ b_1b_2d_1d_2 + b_1b_3d_1d_3 + b_2b_3d_2d_3) + 4(ad_1d_2d_3 + b_1b_2b_3c).
\end{aligned} \tag{1}$$

It remains to check that the map $\mathrm{std}^{\otimes 3}/\!\!/\check{G} \to \mathbf{A}^1$ defined by this polynomial is an isomorphism. This is stated/proved in [**GKZ94**, Proposition 1.7 in Chapter 14], and is due to Cayley. □

Notation 2.5. Write det to denote the map $\mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2/\!\!/\mathrm{SL}_2$ from Lemma 2.4, so that if $\mathcal{C}$ is a cube, $\det(\mathcal{C})$ is the quantity of (1).

We will now define an analogue of the Kostant slice, as it will be needed to apply [**Dev23**, Theorem 3.5.20] (see [**Dev23**, Strategy 1.2.1(b)]). For the purposes of our discussion, one should view this Kostant section as an analogue of the construction of the companion matrix associated to a characteristic polynomial.

Construction 2.6. If n is an integer, let $\vec{n}$ denote the triple (n,n,n). Let

$$\kappa : \mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1/\!\!/(\mathbf{Z}/2) \cong \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$$

denote the map sending $a^2 \mapsto (a^2, \vec{0}, 0, \vec{1})$. This corresponds to the cube

$$\begin{tikzcd}
& 0 \arrow[rr, dash] \arrow[dd, dash] & & 1 \arrow[dd, dash] \\
a^2 \arrow[ur, dash] \arrow[rr, dash] \arrow[dd, dash] & & 0 \arrow[ur, dash] \arrow[dd, dash] & \\
& 1 \arrow[rr, dash] & & 0. \\
0 \arrow[ur, dash] \arrow[rr, dash] & & 1 \arrow[ur, dash] &
\end{tikzcd}$$

In this case, $\det(\kappa(a^2)) = 4a^2$, so that κ defines a section of det (at least up to the unit $4 \in k^\times$). The associated quadratic forms are all equal, and are given by

$$q_1(x, y) = q_2(x, y) = q_3(x, y) = a^2x^2 - y^2,$$

which corresponds to the traceless matrix $\left(\begin{smallmatrix} 0 & -1 \\ a^2 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$. (Note that this is exactly the companion matrix associated to the characteristic polynomial $y^2 - a^2$.)

One of the key properties of the Kostant section/companion matrices is that a matrix $A \in \mathfrak{sl}_2$ is conjugate to $\kappa(\det(A))$ if and only if A is regular (i.e., the minimal polynomial of A agrees with its characteristic polynomial), if and only if A is nonzero. We will now prove an analogous result concerning $\kappa : \mathbf{A}^1 \to \mathrm{std}^{\otimes 3}$.

Proposition 2.7. *The $\check{G}$-orbit of the image of κ is a dense open subscheme whose complement has codimension* 3.

Proof. We will use the classification of $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$ as in [**GKZ94**, Example 4.5 in Chapter 14]; see Figure 1 for a graph of the seven orbits of $\check{G}$ on $\mathrm{std}^{\otimes 3}$. Namely, if $\lambda \neq 0$, all elements of $\det^{-1}(\lambda)$ are in a single $\check{G}$-orbit. (In fact, all elements in the fiber $\det^{-1}(1)$ are in the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.) The $\check{G}$-orbit of $\det^{-1}(\mathbf{G}_m)$ is open and dense, and hence is 8-dimensional; moreover, it agrees with the $\check{G}$-orbit of $\kappa(\mathbf{G}_m)$. Next, there is a maximal $\check{G}$-orbit inside the fiber $\det^{-1}(0)$, given by the orbit of $(0, \vec{0}, 0, \vec{1}) = \kappa(0)$. This orbit is 7-dimensional, and the largest $\check{G}$-orbits contained in the complement $\det^{-1}(0) - \check{G} \cdot \kappa(0)$ have dimension 5. In particular, the complement of $\check{G} \cdot \kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ has dimension 5, i.e., codimension $8 - 5 = 3$. $\square$

$$\begin{tikzcd}
 & & (1,\vec{0},0,(1,0,0));5 \arrow[dr, dash] & & \\
\kappa(\mathbf{G}_m);8 \arrow[r, dash] & \kappa(0);7 \arrow[r, dash] \arrow[ur, dash] \arrow[dr, dash] & (1,\vec{0},0,(0,1,0));5 \arrow[r, dash] & (1,\vec{0},0,\vec{0});4 \arrow[r, dash] & (0,\vec{0},0,\vec{0});0 \\
 & & (1,\vec{0},0,(0,0,1));5 \arrow[ur, dash] & &
\end{tikzcd}$$

Figure 1. $\check{G}$-orbits on $\mathrm{std}^{\otimes 3}$, representatives, and their dimensions (indicated after the semicolon), connected by closure. Note that $\kappa(0) = (0, \vec{0}, 0, \vec{1})$, and that the $\check{G}$-orbit of $\kappa(1) = (1, \vec{0}, 0, \vec{1})$ is the same as the $\check{G}$-orbit of $(1, \vec{0}, 1, \vec{0})$.

Remark 2.8. As explained in [**GKZ94**, Example 4.5 in Chapter 14], the closure of the associated orbits inside $\mathbf{P}(\mathrm{std}^{\otimes 3}) = \mathbf{P}^7$ can be described as follows. First, the closure of the generic orbit is $\mathbf{P}^7$. Next, the closure of the orbit of next smallest dimension is the zero locus of det, which cuts out the dual variety of the Segre embedding $(\mathbf{P}^1)^{\times 3} \hookrightarrow \mathbf{P}^7$ (just as the usual determinant for 2×2-matrices cuts out the quadric $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$). The projective orbit associated to $(1, \vec{0}, 0, (0, 1, 0))$, say, is cut out inside the locus $\{\det = 0\}$ by the Segre embedding $\mathbf{P}(\mathrm{std}) \times \mathbf{P}(\mathrm{std}^{\otimes 2}) = \mathbf{P}^1 \times \mathbf{P}^3 \to \mathbf{P}^7$. Finally, the minimal nonzero orbit is cut out by the Segre embedding $(\mathbf{P}^1)^{\times 3} \to \mathbf{P}^7$.

Proposition 2.9. *There is an isomorphism*

$$\mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2 /\!\!/ \mathrm{SL}_2 \cong \operatorname{Spec} k[a, \alpha_i^{\pm 1}, \tfrac{\alpha_i - \alpha_i^{-1}}{a} | 1 \leq i \leq 3]^{\mathbf{Z}/2} / (\alpha_1\alpha_2\alpha_3 = 1)$$

of group schemes over $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 = \operatorname{Spec} k[a^2]$*, where the action of* $\mathbf{Z}/2$ *sends* $a \mapsto -a$ *and* $\alpha_i \mapsto \alpha_i^{-1}$*, and the group structure is such that each* α_i *is grouplike.*

Proof. The fiber product on the left identifies with the subgroup of $\mathfrak{sl}_2/\!\!/\mathrm{SL}_2 \times \check{G}$ of those $(a^2, \vec{g})$ such that $\vec{g} = (g_1, g_2, g_3) \in \mathrm{SL}_2^{\times 3}$ stabilizes $\kappa(a^2)$. The trick to determining this stabilizer is to use Bhargava's construction from Construction 2.3: if $\vec{g}$ stabilizes a cube $\mathcal{C}$, it must also stabilize the corresponding triple $\mu(\mathcal{C}) \in \mathfrak{sl}_2^{\times 3}$ of quadratic forms.

First, a simple calculation shows that if a is a unit, the triple of matrices

$$\vec{g} = \left(\sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & a^{-1} \\ a & 1 \end{pmatrix} \right) \in \mathrm{SL}_2^{\times 3}$$

sends

$$\kappa(a^2) \mapsto -\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0}).$$

The triple $\vec{g}$ can be thought of as "diagonalizing" $\kappa(a^2)$. The stabilizer of the cube $-\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0})$ precisely consists of triples of matrices of the form

$$\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3^{-1} \end{pmatrix} \right) \text{ with } \alpha_1\alpha_2\alpha_3 = 1. \tag{2}$$

For $\alpha \in \mathbf{G}_m$, let $h(\alpha)$ denote the matrix

$$h(\alpha) = \frac{1}{2} \begin{pmatrix} \alpha + \alpha^{-1} & \frac{\alpha^{-1} - \alpha}{a} \\ a^2 \cdot \frac{\alpha^{-1} - \alpha}{a} & \alpha + \alpha^{-1} \end{pmatrix} \in \mathrm{SL}_2.$$

Conjugating (2) by the element $\vec{g} \in \check{G}$, we find that the triple $(h(\alpha_1), h(\alpha_2), h(\alpha_3))$ of matrices stabilizes $\kappa(a^2)$ as long as $\alpha_1\alpha_2\alpha_3 = 1$ and $a^2 \in \mathbf{G}_m \subseteq \mathbf{A}^1$. (See [**BFM05**, Section 3.2] for a slight variant of this calculation.) Note that the subgroup of such triples is 2-dimensional, and therefore the associated homogeneous $\check{G}$-space is $9 - 2 = 7$-dimensional. Using that the $\check{G}$-orbit of $\kappa(a^2)$ is also 7-dimensional (e.g., by [**GKZ94**, Example 4.5 in Chapter 14]), it is not hard to see from this calculation (by a limiting argument for $a \to 0$) that the stabilizer of the family $\kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$ is precisely the claimed group scheme. □

Remark 2.10. A direct calculation shows that the stabilizer of $\kappa(0)$ is isomorphic to the subgroup of triples of matrices of the form $\left(\begin{smallmatrix} \pm 1 & \mp\gamma_i \\ 0 & \pm 1 \end{smallmatrix}\right)$ for $1 \leq i \leq 3$ with $\gamma_1 + \gamma_2 + \gamma_3 = 0$. This subgroup is, of course, isomorphic to $\mu_2^{\times 3} \times \mathbf{G}_a^{\times 2}$; it is also isomorphic to the fiber over $a = 0$ of the group scheme of Proposition 2.9.

Remark 2.11. As mentioned in Remark 1.6, the quotient stack $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ is studied in quantum information theory. For instance, in [**DVC00**], Dür-Vidal-Cirac study the orbit structure of $\mathrm{SL}_2^{\times 3}$ acting on $\mathrm{std}^{\otimes 3}$ (in particular, they recover Figure 1 independently of [**GKZ94**]). For the interested reader, let us describe the translation between our notation/terminology and that of quantum information theory. Our base field will now be $k = \mathbf{C}$. An element of $\mathrm{std}^{\otimes n}$ (really, of the projective space $\mathbf{P}(\mathrm{std}^{\otimes n}) \cong \mathbf{P}^{2^n - 1}$) is called an *n-qubit*, and the action of $\mathrm{SL}_2^{\times n}$ is via *stochastic local operations and classical communication* (SLOCC) operators (replacing $\mathrm{SL}_2^{\times n}$ by $\mathrm{GL}_2^{\times n}$ simply amounts to dropping the word "stochastic"). The space std is equipped with a basis $\{|0\rangle, |1\rangle\}$, and a cube $\mathcal{C} = (a, \vec{b}, c, \vec{d}) \in \mathrm{std}^{\otimes 3}$

corresponds to the three-qubit[3]

$$a|000\rangle + b_1|100\rangle + b_2|010\rangle + b_3|001\rangle$$
$$+d_1|011\rangle + d_2|101\rangle + d_3|110\rangle + c|111\rangle.$$

The state

$$\tfrac{1}{\sqrt{2}}(1, \vec{0}, 1, \vec{0}) = \tfrac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

is known as the *Greenberger–Horne–Zeilinger* (GHZ) state, and the state

$$\tfrac{1}{\sqrt{3}}\kappa(0) = \tfrac{1}{\sqrt{3}}(0, \vec{1}, 0, \vec{0}) = \tfrac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

is called the W state. These two states are known to represent two very different kinds of quantum entanglement; from the perspective of this article, the reason for this is simply that the Cayley hyperdeterminant of the GHZ state is nonzero, but the Cayley hyperdeterminant of the W state vanishes. Nevertheless, the proof of Proposition 2.9 shows that there is a natural *degeneration* of (the SLOCC/$\mathrm{SL}_2^{\times 3}$-equivalence class of) the GHZ state into the W state. Indeed, the GHZ state can be transformed into the cube $\frac{1}{2}\kappa(1)$, which admits a natural degeneration to the W state via the one-parameter family

$$\tfrac{1}{\sqrt{a^4+3}}\kappa(a^2) = \tfrac{1}{\sqrt{a^4+3}}\left(a^2|000\rangle + |011\rangle + |101\rangle + |110\rangle\right).$$

In fact, this state already appears as [**DVC00**, Equation 20].

Remark 2.12. Fix an integer n. Then the $\check{G}$-variety $\mathrm{std}^{\otimes 3}$ admits a natural grading, where the entries of a cube $(a, \vec{b}, c, \vec{d})$ have the following weights: a lives in weight $-4n$, b lives in weight $-2n$, c lives in weight $2n$, and d lives in weight 0. Write $\mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$ to denote the associated graded variety. Equip $\mathfrak{sl}_2$ with the grading where the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ have the following weights: a lives in weight $-2n$, b lives in weight 0, and c lives in weight $-4n$. Similarly, equip SL_2 with the grading coming from $2n\rho$, so that the entries of a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the following weights: a and d live in weight 0, b lives in weight $2n$, and c lives in weight $-2n$. With these gradings, the $\mathrm{SL}_2^{\times 3}$-equivariant map $\mu : \mathrm{std}^{\otimes 3} \to \mathfrak{sl}_2^{\times 3}$ is a *graded* map, and κ defines a graded map $\mathfrak{sl}_2(2n)/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1(4n) \to \mathrm{std}^{\otimes 3}(4n, \vec{2n}, -2n, \vec{0})$. The cases $n = 1$ and $n = 3$ will be relevant below (corresponding to Theorem 1.2 and Theorem 1.3, respectively).

3. The proof

Before proceeding, let us remind the reader of the definition of the left-hand side of the equivalence of Theorem 1.2, following [**Dev23**, Definition 3.5.15].

Definition 3.1. Let G be a compact Lie group, and let $H \subseteq G$ be a closed subgroup such that $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a spherical subgroup. Let $\mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$ denote the ∞-category of G-equivariant sheaves of $\mathbf{Q}$-modules on $\mathcal{L}(G/H)$ which are constructible for the orbit stratification on $\mathcal{L}(G/H)$. Note that since the orbit stratification is countable (by assumption that $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a spherical subgroup and [**GN10**, Theorem 3.2.1]), the ∞-category $\mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$ is well-defined. There is a natural left-action of the $\mathbf{E}_3$-monoidal ∞-category $\mathrm{Shv}^c_{G\times G}(\mathcal{L}G; \mathbf{Q})$ on $\mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$,

[3]Technically, a qubit is required to have norm 1, so one must rescale $\mathcal{C}$ by $\sqrt{a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2}$; but this could in theory introduce a singularity when $a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2 = 0$. We will ignore this point below.

and in particular, a left-action of $\mathrm{Rep}(\check{G})$ by the abelian geometric Satake theorem of [**MV07**]. Let $\mathrm{IC}_0 \in \mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$ denote the pushforward $i_! \underline{\mathbf{Q}}$ of the constant sheaf along the inclusion $i : G/H \hookrightarrow \mathcal{L}(G/H)$ of the constant loops. Note that i is the analytic realization of the natural map $(G/H)(\mathbf{C}[\![t]\!]) \to (G/H)(\mathbf{C}(\!(t)\!))$. Let $\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); \mathbf{Q})$ denote the full subcategory of $\mathrm{Shv}^c_G(\mathcal{L}(G/H); \mathbf{Q})$ generated by IC_0 under the action of $\mathrm{Rep}(\check{G})$. If k is any $\mathbf{Q}$-algebra, base-changing along the unit map defines the ∞-category $\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); k)$.

Proof of Theorem 1.2. It suffices to verify conditions (a) and (b) of [**Dev23**, Theorem 3.5.20], which gives a criterion for establishing an equivalence of k-linear ∞-categories of the form

$$\mathrm{Shv}^{c,\mathrm{Sat}}_G(\mathcal{L}(G/H); k) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2}\check{M}/\check{G}).$$

The map κ is given by the map $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \to \mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})$ from Construction 2.6. First, the condition of the placidity of the $G(\mathbf{C}[\![t]\!])$-action on $G(\mathbf{C}(\!(t)\!))/H(\mathbf{C}(\!(t)\!))$ follows from the placidity of the $(G\times G)(\mathbf{C}[\![t]\!])$-action on $G(\mathbf{C}(\!(t)\!))$ via the Schubert stratification of the affine Grassmannian. For condition (a) of [**Dev23**, Theorem 3.5.20], we need to show that if $\check{J}_X = \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})/\check{G}} \mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2$, the ring of regular functions on the quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ is isomorphic (as a graded algebra) to $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$. The quotient $(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \times \check{G})/\check{J}_X$ identifies with the $\check{G}$-orbit of the image of κ, which has complement of codimension 3 in $\mathrm{std}^{\otimes 3}$ by Proposition 2.7; therefore, the algebraic Hartogs theorem implies that there is a graded isomorphism $\mathcal{O}_{(\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2\times\check{G})/\check{J}_X} \cong \mathcal{O}_{\mathrm{std}^{\otimes 3}(4,\vec{2},-2,\vec{0})}$.

For condition (b) of [**Dev23**, Theorem 3.5.20], we need to check that there is an isomorphism

$$\check{J}_X \cong \mathrm{Spec}\, \mathrm{H}^{\mathrm{PGL}_2}_*(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k)$$

of graded group schemes over $\mathfrak{sl}_2(2)/\!\!/\mathrm{SL}_2 \cong \mathrm{Spec}\, \mathrm{H}^*_{\mathrm{PGL}_2}(*; k)$. There is an isomorphism

$$\mathrm{Spec}\, \mathrm{H}^{\mathrm{PGL}_2}_*(\Omega\mathrm{PGL}_2; k) \cong \mathrm{Spec}\, k[a, \alpha^{\pm 1}, \tfrac{\alpha-\alpha^{-1}}{a}]^{\mathbf{Z}/2}, \tag{3}$$

and the action of the $\mathbf{Z}/2$ on the left-hand side sends $a \mapsto -a$ and $\alpha \mapsto \alpha^{-1}$. This is proved, e.g., in [**BFM05**], and also follows from [**Dev23**, Example 3.6.16]. The Künneth theorem implies that there is an isomorphism

$$\mathrm{Spec}\, \mathrm{H}^{\mathrm{PGL}_2}_*(\Omega(\mathrm{PGL}_2^{\times 3}); k) \cong \mathrm{Spec}\, k[a, \alpha_i^{\pm 1}, \tfrac{\alpha_i-\alpha_i^{-1}}{a} | 1 \leq i \leq 3]^{\mathbf{Z}/2},$$

and the fiber sequence

$$\mathrm{PGL}_2^{\mathrm{diag}} \to \mathrm{PGL}_2^{\times 3} \to \mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$$

implies that

$$\mathrm{Spec}\, \mathrm{H}^{\mathrm{PGL}_2}_*(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k) \cong \mathrm{Spec}\left(k[a, \alpha_i^{\pm 1}, \tfrac{\alpha_i-\alpha_i^{-1}}{a} | 1 \leq i \leq 3]/(\alpha_1\alpha_2\alpha_3 - 1)\right)^{\mathbf{Z}/2},$$

The desired isomorphism now follows from this observation and Proposition 2.9. □

Remark 3.2. The proof of Theorem 1.3 is exactly the same as the proof of Theorem 1.2 above. Indeed, one only needs to observe that PSO_8/G_2 is homotopy equivalent to $\mathbf{R}P^7 \times \mathbf{R}P^7$ (which follows, e.g., from the fact that $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$)[4].

---

[4]Perhaps the most "conceptual" way to see that $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$ is as follows. Using triality, one can identify Spin_8 with the subgroup of $\mathrm{SO}_8^{\times 3}$ of those triples (A_1, A_2, A_3) such that

The replacement of (3) is given by [**Dev23**, Proposition 4.8.6], which gives an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{G_2}(\Omega \mathbf{R}P^7; k) \cong \operatorname{Spec} k[a, b, \alpha^{\pm 1}, \tfrac{\alpha-\alpha^{-1}}{a}]^{\mathbf{Z}/2}$$

where a is in weight -6 and b is in weight -4.

Remark 3.3. The Künneth isomorphism fails to hold for $\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q})$, as can be seen (for instance) from Theorem 1.2. (We will ignore gradings for the purpose of this illustration.) Indeed, let $G = \mathrm{PGL}_2^{\times 3}$, and let $H = \mathrm{PGL}_2^{\mathrm{diag}}$. Then $G/H \cong \mathrm{PGL}_2^{\times 2}$, so if the Künneth isomorphism held, there would be a series of equivalences

$$\begin{aligned} \mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/H); \mathbf{Q}) &\simeq \mathrm{Shv}_H^{c,\mathrm{Sat}}(\Omega(G/H); \mathbf{Q}) \\ &\simeq \mathrm{Shv}_{\mathrm{PGL}_2}^{c,\mathrm{Sat}}((\Omega\mathrm{PGL}_2)^{\times 2}; \mathbf{Q}) \\ &\simeq \mathrm{Shv}_{\mathrm{PGL}_2}^{c,\mathrm{Sat}}(\Omega\mathrm{PGL}_2; \mathbf{Q})^{\otimes_{\mathrm{Shv}_{\mathrm{PGL}_2}^{c,\mathrm{Sat}}(*;\mathbf{Q})} 2} \\ &\simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2/\mathrm{SL}_2 \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \mathfrak{sl}_2/\mathrm{SL}_2). \end{aligned}$$

However, it is obviously *not* true that $(\mathfrak{sl}_2 \times_{\mathfrak{sl}_2/\!\!/\mathrm{SL}_2} \mathfrak{sl}_2)/\mathrm{SL}_2^{\times 2}$ is isomorphic to the true dual $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$ from Theorem 1.2. Nevertheless, these stacks are closely related, e.g., via the construction of Construction 2.3.

Remark 3.4. The theory of 2-compact groups as studied, e.g., in [**AG09**], suggests viewing the Dwyer-Wilkerson space DW_3 from [**DW93**] as an analogue of the groups $\mathrm{SO}_3 \cong \mathrm{PGL}_2$ and G_2; see Table 1. The space DW_3 is a finite CW-complex equipped with an $\mathbf{E}_1$-structure. It is therefore natural to ask whether there is an analogue of Theorem 1.2 and Theorem 1.3, where PGL_2 and G_2 are replaced by DW_3; this is closely related to [**Dev23**, Appendix C(p)].

It is difficult to answer this question since the representation theory of DW_3 is not well-understood. For instance, it does not seem to be known whether there is a 2-compact group G with an $\mathbf{E}_1$-map $\mathrm{DW}_3 \to G$ such that $G/\mathrm{DW}_3 \simeq \mathbf{R}P^{15} \times \mathbf{R}P^{15}$ (analogous to the equivalences $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2 \cong \mathbf{R}P^3 \times \mathbf{R}P^3$ and $\mathrm{PSO}_8/G_2 \cong \mathbf{R}P^7 \times \mathbf{R}P^7$). If such a G exists, and there is a good theory of G-equivariant sheaves of k-modules, it seems reasonable to expect that there is an equivalence of the form

$$\mathrm{Shv}_G^{c,\mathrm{Sat}}(\mathcal{L}(G/\mathrm{DW}_3); k) \cong \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(28, \vec{14}, -14, \vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3} \times \mathbf{A}^2(8, 12)).$$

Here, the "Whittaker" factor $\mathbf{A}^2(8, 12)$ on the right-hand side comes from the isomorphism

$$\operatorname{Spf} \mathrm{H}^*(B\mathrm{DW}_3; k) \cong \widehat{\mathbf{A}}^3(8, 12, 28),$$

$A_1(x_1)A_2(x_2) = A_3(x_1x_2)$ for octonions x_1, x_2. Under this presentation, G_2 corresponds to the subgroup where $A_1 = A_2 = A_3$. The subgroups where $A_1 = A_3$ (resp. $A_2 = A_3$) are both isomorphic to $\mathrm{Spin}(7)$; these are sometimes denoted $\mathrm{Spin}^{\pm}(7)$. The action of Spin_8 on $S^7 \times S^7$ sends $(x, y) \mapsto (A_1x, A_2y)$; one can check that this is transitive, and that the stabilizer of the point $(1, 1)$ is precisely $\mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7) \cong G_2$.

That there is an equivalence $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$ at the level of cohomology with $\mathbf{Z}[1/2]$-coefficients, at least, is much simpler: on group cohomology, the map $G_2 \to \mathrm{Spin}_8$ is given by the map $\mathbf{Z}[1/2, p_1, p_2, p_3, c_4] \to \mathbf{Z}[1/2, c_2, c_6]$ sending $p_1 \mapsto -c_2$, $p_2 \mapsto 0$, $p_3 \mapsto -c_6$, and $c_4 \mapsto 0$. The Serre spectral sequence for the fibration $\mathrm{Spin}_8/G_2 \to BG_2 \to B\mathrm{Spin}_8$ implies that $\mathrm{H}^*(\mathrm{Spin}_8/G_2; \mathbf{Z}[1/2]) \cong \mathbf{Z}[1/2, \sigma(p_2), \sigma(c_4)]/(\sigma(p_2)^2, \sigma(c_4)^2)$, where $\sigma(p_2)$ and $\sigma(c_4)$ both live in (homological) weight -7. This is precisely the cohomology of $S^7 \times S^7$, as desired.

Group G_n	Rank n	Dimension $(2^{n+1}-1)n$	$\mathbf{F}_2$-cohomology of BG $\widehat{\mathrm{Sym}}^*(\mathbf{F}_2^{n+1}(-1))^{\mathrm{GL}_{n+1}(\mathbf{F}_2)}$	Weyl group $\mathbf{Z}/2\times\mathrm{GL}_n(\mathbf{F}_2)$
PGL_2	1	3	$\mathbf{F}_2[\![w_2,w_3]\!]$	$\mathbf{Z}/2$
G_2	2	14	$\mathbf{F}_2[\![w_4,w_6,w_7]\!]$	$\mathbf{Z}/2\times\Sigma_3$
DW_3	3	45	$\mathbf{F}_2[\![w_8,w_{12},w_{14},w_{15}]\!]$	$\mathbf{Z}/2\times\mathrm{PSL}_2(\mathbf{F}_7)$

TABLE 1. Analogies between the (2-compact) groups $\mathrm{PGL}_2 = \mathrm{SO}_3$, G_2, and DW_3; all of these are Poincaré duality complexes of dimension indicated in the third column. Here, w_n denotes the nth Stiefel-Whitney class, and the ring in the fourth column is known as the algebra of rank $n+1$ Dickson invariants. Note, also, that the Weyl group of DW_3 is called G_{24} in the Shephard-Todd classification.

which follows from running the Bockstein spectral sequence on

$$\mathrm{H}^*(B\mathrm{DW}_3;\mathbf{F}_2)\cong\mathbf{F}_2[\![w_8,w_{12},w_{14},w_{15}]\!],$$

and the fact that the Bockstein sends $w_{14}\mapsto w_{15}$. One can check that such a G, if it existed, would have rational cohomology given by

$$\mathrm{H}^*(BG;k)\cong k[\![c_4,c_6,c_{14},x,y]\!],$$

where both x and y live in cohomological degree 16.

References

[AG09] K. Andersen and J. Grodal. The classification of 2-compact groups. *J. Amer. Math. Soc.*, 22(2):387–436, 2009.

[BF08] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005.

[Bha04] M. Bhargava. Higher composition laws. I. A new view on Gauss composition, and quadratic generalizations. *Ann. of Math. (2)*, 159(1):217–250, 2004.

[BZSV23] D. Ben Zvi, Y. Sakellaridis, and A. Venkatesh. Relative Langlands duality. `https://www.math.ias.edu/~akshay/research/BZSVpaperV1.pdf`, 2023.

[Dev23] S. Devalapurkar. ku-theoretic spectral decompositions for spheres and projective spaces. `https://sanathdevalapurkar.github.io/files/hyperboloid_spectral_decomp.pdf`, 2023.

[DVC00] W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A (3)*, 62(6):062314, 12, 2000.

[DW93] W. G. Dwyer and C. W. Wilkerson. A new finite loop space at the prime two. *J. Amer. Math. Soc.*, 6(1):37–64, 1993.

[GKZ94] I. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[GN10] D. Gaitsgory and D. Nadler. Spherical varieties and Langlands duality. *Mosc. Math. J.*, 10(1):65–137, 271, 2010.

[HK91] M. Harris and S. Kudla. The central critical value of a triple product L-function. *Ann. of Math. (2)*, 133(3):605–672, 1991.

[Ich08] A. Ichino. Trilinear forms and the central values of triple product L-functions. *Duke Math. J.*, 145(2):281–307, 2008.

[Lur15] J. Lurie. Rotation invariance in algebraic K-theory. `https://www.math.ias.edu/~lurie/papers/Waldhaus.pdf`, 2015.

[MV07] I. Mirkovic and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[Rak20] A. Raksit. Hochschild homology and the derived de Rham complex revisited. `https://arxiv.org/abs/2007.02576`, 2020.

[Sak13] Y. Sakellaridis. Spherical functions on spherical varieties. *Amer. J. Math.*, 135(5):1291–1381, 2013.

1 Oxford St, Cambridge, MA 02139
Email address: `sdevalapurkar@math.harvard.edu, September 30, 2023`