QUANTIZATION AND CYCLOTOMIC SPECTRA

Lecture I: Overview

Our goal in this seminar is to explore various aspects of quantization. The primary focus will be on *deformation* quantization, because one can port many tools from homotopy theory to prove results in this setup. Let us begin with a brief impressionistic overview of deformation quantization in quantum mechanics.

Let $(\mathcal{M}, \omega)$ be a symplectic manifold, which one should regard as the phase space of a physical system. For instance, if X is a smooth manifold, there is a canonical 1-form on T^*X such that the pair $(T^*X, d\theta)$ is a symplectic manifold which describes the phase space of the *free particle* on X. In physics, the most interesting question about a mechanical system revolves around the *dynamics* of the system; in other words, the evolution of functions on $\mathcal{M}$ (which are known as (classical) observables) with respect to "time". Concretely, this means that one is given a vector field on $\mathcal{M}$, and the object of interest is the gradient flow equation with respect to this vector field.

To be more precise, let $H : \mathcal{M} \to \mathbf{R}$ be a function on $\mathcal{M}$, known as the *Hamiltonian* of the classical system. This function measures the total energy of a given configuration of the system. Then, since ω defines an isomorphism $T^*\mathcal{M} \xrightarrow{\cong} T\mathcal{M}$, the 1-form dH defines a vector field X_H on $\mathcal{M}$. The flow equation associated to this vector field is then

$$\frac{df}{dt} = X_H(f) = \omega(X_H, X_f) = \{H, f\}. \tag{1}$$

This is known as *Hamilton's equation*. Note that $\omega(X_H, X_H) = \{H, H\} = 0$ by skew-symmetry of ω, so $\frac{dH}{dt} = 0$; in other words, the total energy is constant with time (as one expects). For instance, suppose that $X = \mathbf{R}$ with coordinate q, and $\mathcal{M} = (T^*\mathbf{R}, dp \wedge dq)$; then,

$$X_H = \frac{dH}{dp}\partial_q - \frac{dH}{dq}\partial_p,$$

which means (by Equation (1)) that

$$\frac{dq}{dt} = \frac{dH}{dp}, \ \frac{dp}{dt} = -\frac{dH}{dq}.$$

This is the more familiar version of Hamilton's equations.

In deformation quantization, passing from the classical to quantum world involves deforming the *commutative* ring $\mathcal{O}_\mathcal{M}$ of functions on $\mathcal{M}$ to a *flat and associative* ring $\mathcal{A}$ over $\mathbf{R}[\![\hbar]\!]$ such that $\mathcal{A}/\hbar \cong \mathcal{O}_\mathcal{M}$. If $f, g \in \mathcal{O}_\mathcal{M}$ are functions on $\mathcal{M}$, and f, g abusively denote lifts of these functions to elements of $\mathcal{A}$, then one further requires that

$$[f, g] = fg - gf = \hbar\{f, g\}. \tag{2}$$

In other words, the commutator in the associative deformation $\mathcal{A}$ is a deformation of the Poisson bracket on $\mathcal{O}_\mathcal{M}$. The flow equation Equation (1) is then promoted to an equation which describes time evolution in $\mathcal{A}$: namely, one requires that if $B \in \mathcal{A}$, then

$$\frac{dB}{dt} = \frac{i}{\hbar}[H, B]. \tag{3}$$

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This is the *Heisenberg equation*. Note that this is a direct analogue of Equation (1), where one replaces $\{H, f\}$ with $\frac{i}{\hbar}[H, B]$. Up to the factor of i (which can just be absorbed), this is precisely encoding the relation Equation (2). Observe that it was not crucial in this discussion that $\mathcal{M}$ be a *symplectic* manifold, only that $\mathcal{O}_\mathcal{M}$ have a Poisson bracket.

Motivated by the preceding discussion, we make the following definition.

Definition 1. Let k be a field, and let A be a commutative k-algebra equipped with a Poisson bracket $\{-, -\}$. A *quantization* of A is a flat associative $k[\![\hbar]\!]$-algebra $A_\hbar$ such that $A_\hbar/\hbar \cong A$ as k-algebras, and Equation (2) is satisfied.

Note that $A_\hbar$ is, as a $k[\![\hbar]\!]$-module, just $A[\![\hbar]\!]$ (but the product on $A_\hbar$ is usually different). Let $\star$ and $\star'$ be two associative products on $A[\![\hbar]\!]$ which define quantizations of $(A, \{-, -\})$. Say that $\star$ and $\star'$ are *gauge-equivalent* if there is a $k[\![\hbar]\!]$-module automorphism g of $A[\![\hbar]\!]$ such that $g(x) \equiv x \pmod{\hbar}$ such that $g(x \star y) = g(x) \star' g(y)$.

It is natural to ask why we considered $k[\![\hbar]\!]$, as opposed to $k[\hbar]$. There are concrete physical reasons for this, but we will defer discussion of this point to a future talk. Let us look at two concrete examples of Definition 1.

Example 2. Let X be a smooth manifold, and let A be the ring of smooth functions on T^*X equipped with its canonical Poisson bracket. Let $A_\hbar$ denote the ring of "asymptotic" differential operators on X, so the commutator is given in local coordinates by $[\partial_i, x_j] = \hbar\delta_{ij}$. (Specializing to $\hbar = 1$ gives the usual ring of differential operators.) Then $A_\hbar/\hbar$ is isomorphic to A, while $\frac{1}{\hbar}[\partial_i, x_j] = \delta_{ij}$ is equal to $\{p_i, q_j\}$.

The preceding example works in the same way if X is a smooth algebraic variety (over $\mathbf{C}$), and $A = \mathcal{O}_{T^*X} = \mathrm{Sym}_{\mathcal{O}_X}(T_X)$. Then $A_\hbar$ is the ring of asymptotic algebraic differential operators $\mathcal{D}_X$.

Example 3. Let k be a field, and let $\mathfrak{g}$ be a Lie algebra over k with Lie bracket $\{-, -\}$. Then there is a Poisson bracket on $A = \mathrm{Sym}(\mathfrak{g})$, and a quantization of A is given by the asymptotic universal enveloping algebra $U_\hbar(\mathfrak{g})$[1], which satisfies $[x, y] = \hbar\{x, y\}$.

[1] There is an unfortunate clash in both notation and terminology here: this is *not* the quantum group, which is usually denoted $U_q(\mathfrak{g})$.

Our first goal in this seminar is to study the proof of the following important result due to Kontsevich:

Theorem 4 (Kontsevich, [Kon03])**.** *The ring $\mathcal{O}_\mathcal{M}$ of smooth functions on any smooth Poisson manifold $\mathcal{M}$ admits a quantization. In other words, the map from the space of gauge-equivalence classes of quantizations of $\mathcal{O}_\mathcal{M}$ to Poisson brackets on $\mathcal{O}_\mathcal{M}$ is surjective; furthermore, there is an explicit section of this map.*

The proof of Kontsevich's theorem can roughly be described as follows. Let A be a Poisson algebra over a field k; then, one can view the problem of constructing an associative product on $A[\![\hbar]\!]$ as a deformation problem, by reducing to constructing compatible associative products on $A[\![\hbar]\!]/\hbar^n$ for all $n \geq 2$. In fact, one can consider two deformation problems: the first is the one just discussed above, while the second is the problem of constructing a Poisson bracket on $A[\![\hbar]\!]$. In other words, one can consider the problem of deforming the multiplication on A, and the problem of deforming the Poisson bracket on A. Kontsevich's theorem can be understood as saying that these two deformation problems are equivalent, and furthermore that the problem of deforming the Poisson bracket on A can always be solved.

Let us focus on the first part, i.e., that these two deformation problems are equivalent. Assume from now that k is of *characteristic zero*. A well-known philosophy says that deformation/formal moduli problems (which take Artinian k-algebras like $k[\![\hbar]\!]/\hbar^n$ as input) are controlled by differential graded Lie algebras over k. In the case when $A = \mathcal{O}_\mathcal{M}$, one can prove that the formal moduli problem of deforming the multiplication on

A is controlled by the differential graded Lie algebra $\mathrm{HC}(A/k)[1]$ (where HC denotes Hochschild cohomology). Recall that in degree n, this consists of maps $A^{\otimes n+1} \to A$. This is quasi-isomorphic to the subcomplex $\mathcal{D}^\bullet_{\mathcal{M}}$ of $\mathrm{HC}(A/k)[1]$ where we ask that the maps $A^{\otimes n+1} \to A$ are given by polydifferential operators. Similarly, the formal moduli problem of deforming the Poisson bracket on A is controlled by the differential graded Lie algebra $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})[1]$ with the usual Poisson bracket of polyvector fields. Moreover, there is a map $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})[1] \to \mathcal{D}^\bullet_{\mathcal{M}}$ which sends the polyvector field $X_0 \wedge \cdots \wedge X_n$ to the map $A^{\otimes n+1} \to A$ given by

$$f_0 \otimes \cdots f_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^\sigma \prod_{i=1}^n X_{\sigma(i)}(f_i).$$

The Hochschild-Kostant-Rosenberg theorem tells us that the map $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})[1] \to \mathcal{D}^\bullet_{\mathcal{M}}$ is an isomorphism of *complexes* — but one can check that the above map is not one of differential graded Lie algebras. Kontsevich's result may be understood as a refinement of the above map to one of differential graded Lie algebras, which shows that the problem of deforming the multiplication on A and the problem of deforming the Poisson bracket on A are equivalent.

In the course of his argument, Kontsevich actually writes down a map of differential graded Lie algebras $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})[1] \to \mathcal{D}^\bullet_{\mathcal{M}}$, the explicit form of which is motivated by some considerations from quantum field theory. We will probably not discuss this topic here, but we will discuss an alternative proof due to Tamarkin (see [Tam03]). First, using the Deligne conjecture, one shows that the Lie bracket on $\mathrm{HC}(A/k)[1]$ arises via a canonical $\mathbf{E}_2$-algebra structure on $\mathrm{HC}(A/k)$. It is much easier to establish the analogous algebraic claim, that the Lie bracket on $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})[1]$ arises via a canonical $\mathbf{P}_2$-structure on $\Gamma(\mathcal{M}; \wedge^\bullet T_{\mathcal{M}})$, where $\mathbf{P}_2$ is the 1-shifted Poisson operad. It is a well-known result due to Cohen that the homology $\mathrm{H}_*(\mathbf{E}_2; k)$ is isomorphic as an operad in k-modules to $\mathbf{P}_2$. Then, the key step in Tamarkin's argument amounts to showing that there is a quasi-isomorphism $C_*(\mathbf{E}_2; k) \xrightarrow{\sim} \mathbf{P}_2$ of operads in k-modules. We will discuss the construction of this quasi-isomorphism, and discuss how it implies Kontsevich's theorem. It will be clear from this discussion that the claim fails horribly when $\mathbf{F}_p \subseteq k$: the failure stems from an interesting class in $\mathrm{H}_{p-1}(\mathbf{E}_2; \mathbf{F}_p)$.

The upshot of the above discussion is that Kontsevich's theorem is essentially equivalent to the claim that if $k = \mathbf{R}$ and A is the k-algebra of smooth functions on a smooth manifold M, then $\mathrm{HC}(A/k)[1]$ is a *formal* differential graded Lie algebra. There is a complex-analytic version of this result, known as the Bogomolov-Tian-Todorov theorem. To state it, recall that a Calabi-Yau variety (over any field k) is a smooth and proper k-variety X such that the canonical line bundle $K_X = \wedge^{\dim(X)} \Omega^1_{X/k}$ is trivial. Then:

Theorem 5 (Bogomolov-Tian-Todorov, [Bog78, Tia87, Tod89, KKP08])**.** *Let k be a field of characteristic zero, and let X be a Calabi-Yau variety over k. Then the differential graded Lie algebra structure on $\mathrm{HC}(X/k)[1]$ is homotopy abelian (i.e., is quasi-isomorphic to an abelian differential graded Lie algebra). As a consequence, the deformation theory $\mathrm{Def}_X : \mathrm{Art}_k \to \mathrm{Set}$ of X is formally smooth, i.e., is represented by $\mathrm{Spf}\, k[\![\mathrm{H}^1(X; T_{X/k})]\!]$.*

Grothendieck showed that if X is a smooth and proper k-variety such that $\mathrm{H}^0(X; T_{X/k}) = 0$, then the deformation theory of X is pro-representable, so the interesting part is the *formal smoothness.* There are many proofs of the Bogomolov-Tian-Todorov theorem, but we will discuss a particular argument due to Katzarkov-Kontsevich-Pantev ([KKP08]), which serves as a useful blueprint for later discussion. An outline of this argument runs as follows. Since X is Calabi-Yau, one can apply a holomorphic variant of Poincaré duality to conclude that $\mathrm{HC}(X/k) \xrightarrow{\sim} \Sigma^{\dim X}\mathrm{HH}(X/k)$, where HH denotes Hochschild homology. There is an S^1-action on $\mathrm{HH}(X/k)$, which transports to an S^1-action on $\mathrm{HC}(X/k)$. Therefore, $\mathrm{HC}(X/k)$ is an $\mathbf{E}_2$-k-algebra with an S^1-action; these two structures are compatible with each other, in the sense that $\mathrm{HC}(X/k)$ is a *framed* $\mathbf{E}_2$-algebra when X is Calabi-Yau. Since k is of characteristic zero, this compatibility can be expressed as follows: the S^1-action on $\mathrm{HC}(X/k)$ gives rise to a differential Δ on $\pi_*\mathrm{HC}(X/k)$, and we have

$$\Delta(xy) - \Delta(x)y - (-1)^? x\Delta(y) = [x, y] \tag{4}$$

at the level of $\pi_*\mathrm{HC}(X/k)$. In other words, the Lie bracket on $\pi_{*-1}\mathrm{HC}(X/k)$ is determined by the S^1-action on $\mathrm{HC}(X/k)$.

The idea behind the proof of the Bogomolov-Tian-Todorov theorem is to show that the S^1-action on $\mathrm{HC}(X/k)$ is homotopically trivializable (and that the choice of a holomorphic volume form on X gives such a trivialization), and then show that a *chain-level* version of Equation (4) implies that the Lie bracket on $\mathrm{HC}(X/k)[1]$ is trivializable. Since the S^1-action on $\mathrm{HC}(X/k)$ is inherited from the S^1-action on $\mathrm{HH}(X/k)$, the first step amounts to showing that the S^1-action on $\mathrm{HH}(X/k)$ is trivializable, i.e., that the Tate spectral sequence for $\mathrm{HH}(X/k)$ degenerates at the E_2-page. This is precisely the degeneration of the noncommutative Hodge-de Rham spectral sequence (in characteristic zero, this is the *same* as Hodge-de Rham degeneration for X). The second claim requires further work, but it is a general operadic fact that is proven using the "quantum master equation".

This proof is quite satisfying: it relates Hodge-de Rham degeneration/triviality of the S^1-action on $\mathrm{HH}(X/k)$ (which holds for *any* smooth and proper k-variety) to the abelian-ness of the differential graded Lie algebra underlying the $\mathbf{E}_2$-algebra $\mathrm{HC}(X/k)$. One of the main theses of the second part of this seminar is that this phenomenon is rather general: S^1-actions are "dual" to $\mathbf{E}_2$-algebras, and framed $\mathbf{E}_2$-algebras straddle between these two.

To explain this further, let us reconsider the characteristic zero story. As we have mentioned, we can then canonically identify an S^1-action on a differential graded k-algebra (A, d) with a *mixed differential* Δ on A, and we can noncanonically identify an $\mathbf{E}_2$-algebra structure on a differential graded k-algebra (A, d) with a 1-shifted Poisson structure on A (by Kontsevich-Tamarkin formality). The key motivating example is then the following:

Example 6. Let $\mathfrak{g}$ be a differential graded Lie algebra over k, and let d denote the internal differential on $\mathfrak{g}$. The linear dual of the 1-shifted Poisson k-algebra $\mathrm{Sym}(\mathfrak{g}[1])$ is the cocommutative k-algebra $\mathrm{Sym}(\mathfrak{g}^*[-1])$[2] equipped with the mixed differential Δ that is characterized by the following property: $\Delta : \mathfrak{g}^*[1] \to \mathrm{Sym}^2(\mathfrak{g}^*[1]) \cong (\wedge^2\mathfrak{g}^*)[2]$ is the linear dual of the Lie bracket $\wedge^2\mathfrak{g} \to \mathfrak{g}$. The fact that $d([x, y]) = [dx, y] + (-1)^?[x, dy]$ amounts to the claim that d anti-commutes with Δ.

[2] This is *not* true in arbitrary characteristic: the dual of $\mathrm{Sym}(V)$ is generally the divided power algebra $\Gamma(V^*)$, which is isomorphic to $\mathrm{Sym}(V^*)$ *only* in characteristic zero.

Both $\mathrm{Sym}(\mathfrak{g}^*[-1])$ with the mixed differential Δ and the 1-shifted Poisson k-algebra $\mathrm{Sym}(\mathfrak{g}[1])$ have well-known interpretations:

- The 1-shifted Poisson k-algebra $\mathrm{Sym}(\mathfrak{g}[1])$ is a version of the universal enveloping algebra $U(\mathfrak{g})$. More precisely, recall that $U(\mathfrak{g})$ admits a filtration (the PBW filtration) whose associated graded is the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$. Passing from the commutative algebra $\mathrm{Sym}(\mathfrak{g})$ to $U(\mathfrak{g})$ is specified by the Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$, i.e., the Lie bracket on $\mathfrak{g}$. One should therefore regard $\mathrm{Sym}(\mathfrak{g}[1])$ as a 1-shifted analogue of the universal enveloping algebra of $\mathfrak{g}$. In other words, it is reasonable to view $\mathrm{Sym}(\mathfrak{g}[1])$ with its 1-shifted Poisson bracket as a quantization of the commutative k-algebra $\mathrm{Sym}(\mathfrak{g}[1])$ (not viewed as being equipped with a Poisson bracket).

- The Chevalley-Eilenberg complex $C^*(\mathfrak{g})$ is precisely the differential graded k-algebra whose underlying vector space is $\mathrm{Sym}(\mathfrak{g}^*[-1])$, and whose differential is $d+\Delta$. Therefore, it is reasonable to view the cocommutative differential graded k-algebra $(\mathrm{Sym}(\mathfrak{g}^*[-1]), d)$ equipped with the mixed differential Δ as a refinement of $C^*(\mathfrak{g})$.

Example 7. Let k be a field, and let X be a derived k-scheme. Let $T_{X/k}$ denote the *tangent complex* of X, i.e., the dual of the cotangent complex $L_{X/k}$. The linear dual of the 1-shifted Poisson $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ is the cocommutative $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1])$[3] equipped with the mixed differential Δ that is characterized by the following property: $\Delta : L_{X/k}[1] \to \mathrm{Sym}^2_{\mathcal{O}_X}(L_{X/k}[1]) \cong (\wedge^2 L_{X/k})[2]$ is the linear dual of the bracket $\wedge^2 T_{X/k} \to T_{X/k}$ of vector fields. Note that the Cartan formula tells us that Δ is just the usual de Rham differential d_{dR}. Just as in Example 6, both $\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1])$ with the mixed differential d_{dR} and the 1-shifted Poisson $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ have well-known interpretations:

- The 1-shifted Poisson $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ is a version of the sheaf $\mathcal{D}_X$ of differential operators on X. Observe that if X is a *smooth* k-scheme, then $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ is precisely the associated graded of the HKR filtration on Hochschild cohomology $\mathrm{HC}(X/k)$. Moreover, $\mathrm{HC}(X/k)$ is roughly the bar construction on $\mathcal{D}_X$. Therefore, it is reasonable to view $\mathrm{HC}(X/k)$ as a quantization of the commutative $\mathcal{O}_X$-algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ (not viewed as being equipped with a Poisson bracket).
- The Chevalley-Eilenberg complex $C^*(L_{X/k})$ is precisely the derived de Rham complex $\mathrm{dR}_{X/k}$. Therefore, it is reasonable to view the cocommutative differential graded k-algebra $(\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1]), d)$ equipped with the mixed differential d_{dR} as a refinement of $\mathrm{dR}_{X/k}$.

[3] At least, if $L_{X/k}$ is a perfect $\mathcal{O}_X$-module.

In both of the above examples, one observes that Δ is a linear dual of the Poisson bracket. Motivated by this observation, one of our main goals in this seminar is to make the following slogan precise:

Slogan 8. Let k be a ring. Then 1-shifted quantizations of commutative k-algebras are classified by S^1-actions.

The proof of Slogan 8 essentially boils down to *Koszul duality*: more precisely, the fact that the $\mathbf{E}_2$-Koszul dual of the power series ring $k[\![t]\!]$ is the cochains $C^*(\mathbf{C}P^\infty; k)$, and the fact that the $\mathbf{E}_1$-Koszul dual of the chains $C_*(S^1; k)$ is the cochains $C^*(\mathbf{C}P^\infty; k)$.

Remark 9. Observe that the homotopy of $C^*(\mathbf{C}P^\infty; k)$ is $\mathrm{H}^*(\mathbf{C}P^\infty; k) \cong k[u]$ with u in homological degree -2. This is sometimes known as the *Bott element*. Experience (such as Nekrasov's Ω-deformation [NS09] from supersymmetric QFT, as well as the derived geometric Satake equivalence [BF08]) teaches us that when studying quantizations, the parameter $\hbar$ should be viewed as living in *cohomological degree* 2 — in other words, that $u = \hbar$. Therefore, deformation quantization should be understood as not just deformations over an arbitrary power series ring, but rather as deformations over the *cohomology ring* $\mathrm{H}^*(\mathbf{C}P^\infty; k)$. This philosophy is rather well-known to (a certain class of) representation theorists, but has not yet permeated the general literature on deformation quantization. In any case, we will see this explicitly during the seminar in several examples.

The second part of this seminar will be devoted to understanding deformation quantization in characteristic p, as well as proving an analogue of Slogan 8. To explain what this would entail, let us examine the analogues of Example 2 and Example 3:

Example 10. Let k be a field of characteristic $p > 0$, and let $\mathfrak{g}$ be a *restricted* Lie algebra over k, so that $\mathfrak{g}$ is equipped with a pth power map $x \mapsto \varphi(x)$ that satisfies the following conditions:

$$\begin{aligned} \mathrm{ad}_{\varphi(x)}(y) &= \mathrm{ad}_x^p(y), \\ \varphi(x+y) &= \varphi(x) + \varphi(y) + \mathrm{ad}_x^{p-1}(y). \end{aligned} \tag{5}$$

Then $\mathrm{Sym}(\mathfrak{g})$ becomes a *restricted* Poisson algebra, where the map φ is extended to products of generators via a formula

$$\varphi(xy) = \varphi(x)y^p + x^p\varphi(y) + \sum_{0 \le i,j \le p, i+j \le p} x^i y^j \Gamma_{i,j}(x,y), \tag{6}$$

for some rather complicated expression $\Gamma_{i,j}(x,y)$ that does not depend on φ. (For instance, $\Gamma_{1,1}(x,y) = \mathrm{ad}_x(y)^{p-1}$, while for $i \neq 0, p$, $\Gamma_{i,p-i}(x,y)$ is the coefficient of t^{i-1} in the expression $\mathrm{ad}_{tx+y}^{p-1}(x)$.) If $\mathfrak{g}$ is further equipped with an internal differential d, then we require the Leibniz rule

$$d(\varphi(x)) = \mathrm{ad}_x^{p-1}(dx).$$

Following Example 6, let us consider the 1-shifted analogue of this story. Then the 1-shifted enveloping algebra $\mathrm{Sym}(\mathfrak{g}[1])$ acquires the structure of a 1-shifted restricted Poisson algebra. It is natural to ask how this additional structure is reflected on the dual $(\mathrm{Sym}(\mathfrak{g}^*[-1]), d)$ equipped with its mixed differential Δ. This turns out to be a somewhat difficult question to answer, if only because the formulae become quite complicated.

Example 11. Let k be a field of characteristic $p > 0$, and let X be a smooth k-variety. Then $T_{X/k}$ becomes a restricted Lie algebra, where the restricted structure φ is given by sending a derivation to its pth power. This makes $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k})$ into a restricted Poisson $\mathcal{O}_X$-algebra. If X is not smooth, so T_X is further equipped with an internal differential d, then one can check the Leibniz rule $d(\varphi(x)) = \mathrm{ad}_x^{p-1}(dx)$ holds. Again, following Example 7, we can consider the 1-shifted analogue of this story. Then the 1-shifted enveloping algebra $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ acquires the structure of a 1-shifted restricted Poisson $\mathcal{O}_X$-algebra. On the dual $(\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1]), d)$ equipped with its mixed differential d_{dR}, this restricted structure is reflected by the *Cartier operator* (which, in the smooth case, is a map $\mathrm{Fr}_*\Omega^i_{X/k}/d_{\mathrm{dR}}\Omega^{i-1}_{X/k} \to \Omega^1_{X^{(p)}/k}$).

Remark 12. The operad $\mathrm{H}_*(\mathbf{E}_2; k)$ in k-modules nearly mirrors the definition of restricted 1-shifted Poisson k-algebras: the Dyer-Lashof operation Q_1 plays the role of the restricted structure φ, and the relations Equation (5) are satisfied. Therefore, the restricted 1-shifted Poisson structures in both Example 10 and Example 11 can both be viewed as vestiges of an $\mathbf{E}_2$-algebra structure on a more "primeval" object. In the case of Example 11, the HKR theorem tells us that this primeval object is the Hochschild cohomology $\mathrm{HC}(X/k)$; in the case of Example 10, one finds that the primeval object is given by the notion of the $\mathbf{E}_2$*-universal enveloping algebra*[4].

[4] In fact, there is a refinement of the notion of a 1-shifted restricted Lie algebra, which behaves well homotopically: these are known as "partition Lie algebras", and were introduced by Brantner and Mathew in [BM19]. One can then define the $\mathbf{E}_2$-enveloping algebra of any partition Lie algebra.

Motivated by Example 10 and Example 11 (and the preceding remark), one can abstract out the notion of *Frobenius-constant quantizations*, following Bezrukavnikov and Kaledin ([BK08]).

Definition 13. Let k be a field of characteristic $p > 0$, and let A be a commutative k-algebra equipped with a Poisson bracket $\{-,-\}$ and a restricted structure φ. A

Frobenius-constant quantization of A is a quantization $A_\hbar$ of A equipped with a map Φ which satisfies Equation (5) and the following deformation of Equation (6):

$$\Phi(xy) = \varphi(x)y^p + x^p\varphi(y) - \hbar^{p-1}x^{[p]}y^{[p]} + \sum_{0\leq i,j\leq p, i+j\leq p} x^i y^j \Gamma_{i,j}(x,y).$$

The second part of this seminar will be focused on making the following slogan precise:

Slogan 14. Let k be an $\mathbf{F}_p$-algebra. Then 1-shifted Frobenius-constant quantizations of commutative k-algebras are classified by (THH(k)-modules in) *cyclotomic spectra*[5] (à la Nikolaus-Scholze [NS18]).

[5] Somewhat more precisely, one can use cyclotomic spectra to define an analogue of the "F-zips" of Moonen-Wedhorn, and $\mathbf{E}_\infty$-coalgebras in this category correspond to 1-shifted Frobenius-constant quantizations.

Example 15. Let us see how Slogan 14 manifests in the setting of Example 11. As we mentioned, the restricted structure on $\mathrm{Sym}_{\mathcal{O}_X}(T_{X/k}[1])$ leads to the Cartier operator on $(\mathrm{Sym}_{\mathcal{O}_X}(L_{X/k}[-1]), d)$ equipped with its mixed structure d_{dR}. Moreover, it is a well-known philosophy in characteristic p geometry that the Cartier operator is essentially determined by the *conjugate filtration* on the derived de Rham complex. At the level of "primeval" objects, this can be rephrased as saying that the $\mathbf{E}_2$-structure (more specifically, the Dyer-Lashof operation Q_1) on $\mathrm{HC}(X/k)$ is reflected in terms of some filtration on the periodic cyclic homology $\mathrm{HP}(X/k) = \mathrm{HH}(X/k)^{tS^1}$. One can construct such a filtration (see [Mat20]) via the THH(k)-module structure on *topological* Hochschild homology THH(X), which is the typical example of a cyclotomic spectrum. In other words, the refinement of $\mathrm{HH}(X/k)$ to $\mathrm{THH}(X)$ reflects the restricted structure on $\mathrm{HC}(X/k)$.

There is a parallel story involving Example 10. The perspective afforded by Slogan 14 is quite powerful for several reasons:

- Unlike with (Frobenius-constant) quantizations, the theory of cyclotomic spectra works perfectly well if k is an arbitrary $\mathbf{E}_\infty$-ring which is not necessarily discrete. In particular, one can study "deformation quantization over the sphere spectrum".
- Categorifying Slogan 14 is useful in studying classical problems. For instance, categorifying the relationship between $\mathcal{D}_X$ (rather, $\mathrm{HC}(X/k)$) and $\mathrm{dR}_{X/k}$ (rather, THH(X) as a cyclotomic spectrum) allows one to prove a general result pertaining to the p-curvature conjecture (see [Kat72]) when the variety X lifts to the sphere spectrum.
- Many results about deformation quantization (in arbitrary characteristic) can be viewed as equipping the objects involved with S^1-actions/cyclotomic structures. This has some applications in representation theory (such as Lonergan's proof [Lon18] that Coulomb branch quantizations admit Frobenius-constant structures, and Ben-Zvi–Gunningham's recent interpretation [BG17] of work of Ngo). Moreover, analogues of many formality results (such as the Bogomolov-Tian-Todorov theorem) can be proved in characteristic $p > 0$.
- There are also several applications to integrable systems: for instance, one can quantize the Calogero-Moser, Ruijsenaars-Schneider, and "double elliptic" integrable systems by describing them in terms of (variants of) free loop spaces and using the resulting S^1-action. This is essentially a reinterpretation of Nekrasov's Ω-deformation from a more homotopical lens.

Time permitting, we will discuss some of these applications in this seminar.

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