

# Derived geometric Satake for $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$

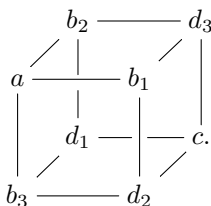
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**ABSTRACT.** In this note, we study the local relative geometric Langlands conjecture of Ben-Zvi–Sakellaridis–Venkatesh for the spherical subgroup  $\mathrm{PGL}_2^{\mathrm{diag}}$  of the triple product  $\mathrm{PGL}_2^{\times 3}$  (and also for the spherical subgroup  $G_2$  of  $\mathrm{SO}_8/\mu_2$ ), whose corresponding Langlands dual  $\mathrm{SL}_2^{\times 3}$ -variety can be identified with the space  $(\mathbf{A}^2)^{\otimes 3} \cong \mathbf{A}^8$  of  $2 \times 2 \times 2$ -cubes. Our analysis relies on a construction of Bhargava relating  $2 \times 2 \times 2$ -cubes to quadratic forms and the Cayley hyperdeterminant as studied by Gelfand–Kapranov–Zelevinsky.

## 1. Introduction

The goal of this brief note is to study the geometrization of a story from the arithmetic context pioneered by Jacquet, Kudla–Harris, and Ichino among many others (see, e.g., [HK91, Ich08]). Fix an eighth root of unity  $\zeta_8$ , let  $i$  be the resulting square root of  $-1$ , and write  $k := \mathbf{Q}(\zeta_8) \cong \mathbf{Q}(i, \sqrt{2})$ .

**Notation 1.1.** Let  $\mathrm{std}$  denote the standard representation of  $\mathrm{SL}_2$ , so that  $\mathrm{std}^{\otimes 3}$  consists of cubes



Fix an integer  $n$ . Equip  $\mathrm{std}^{\otimes 3}$  with the grading where the entries of a cube have the following weights:  $a$  lives in weight  $-4n$ , each  $b_i$  lives in weight  $-2n$ ,  $c$  lives in weight  $2n$ , and each  $d_i$  lives in weight  $0$ . Write  $\mathrm{std}^{\otimes 3}(4n, 2\vec{n}, -2n, \vec{0})$  to denote the corresponding graded variety.

Similarly, equip  $\mathrm{SL}_2$  with the grading where the entries of a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  have the following weights:  $a$  and  $d$  live in weight  $0$ ,  $b$  lives in weight  $2n$ , and  $c$  lives in weight  $-2n$ . Write  $\mathrm{SL}_2(-2n\rho)$  to denote this graded group. Then there is a natural graded action of  $\mathrm{SL}_2(-2n\rho)^{\times 3}$  on  $\mathrm{std}^{\otimes 3}(4n, 2\vec{n}, -2n, \vec{0})$ .

Recall that the process of *shearing* discussed in [Rak20, Lur15], as well as [Dev23, Section 2.1], converts gradings into homological shifts (more precisely, it sends a module in weight  $n$  to the same module shifted homologically by  $n$ ). This

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functor is symmetric monoidal when restricted to the subcategory of modules in *even* weights, and therefore extends to an operation on evenly graded stacks. As in [Dev23], we will state all of our results with “arithmetic shearing” in the sense of [BZSV23, Section 6.7].

**Theorem 1.2** (Derived geometric Satake for  $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$ ). *Suppose that the  $\mathrm{PGL}_2^{\times 3}[[t]]$ -action on  $\mathrm{PGL}_2^{\times 3}((t))/\mathrm{PGL}_2^{\mathrm{diag}}((t))$  is optimal in the sense of [Dev23, Hypothesis 3.5.22]. There is an equivalence<sup>1</sup>*

$$\mathrm{Shv}_{\mathrm{PGL}_2^{\times 3}[[t]]}^{\mathrm{c}, \mathrm{Sat}}(\mathrm{PGL}_2^{\times 3}((t))/\mathrm{PGL}_2^{\mathrm{diag}}((t)); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}),$$

where  $\mathrm{Perf}^{\mathrm{sh}}$  denotes the  $\infty$ -category of perfect complexes on the shearing of the quotient stack  $\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2(-2\rho)^{\times 3}$ . Moreover, this equivalence is equivariant for the action of the spherical Hecke category for  $\mathrm{PGL}_2^{\times 3}$ .

Let  $\mathrm{PSO}_{2n} := \mathrm{SO}_{2n}/\mu_2$ . Then, the embedding  $\mathrm{PGL}_2^{\mathrm{diag}} \subseteq \mathrm{PGL}_2^{\times 3}$  can be identified with the diagonal embedding  $\mathrm{SO}_3 \subseteq \mathrm{SO}_3 \times \mathrm{PSO}_4$ , so Theorem 1.2 could be viewed as a special case of the geometrized analogue of the Gan-Gross-Prasad period.

A similar argument shows a variant for  $\mathrm{PSO}_8$ . Namely, there is an embedding  $G_2 \subseteq \mathrm{PSO}_8$  given by triality, which exhibits  $G_2$  as a spherical subgroup of  $\mathrm{PSO}_8$ . To see that this situation is analogous to that of Theorem 1.2, note that the Dynkin diagram  $\bullet$  of  $A_1$  is obtained from the Dynkin diagram  $\bullet \bullet \bullet$  of  $A_1^{\times 3}$  by folding with respect to the obvious action of the symmetric group  $\Sigma_3$ . In the same way, the Dynkin diagram  $\bullet \rightleftharpoons \bullet$  of  $G_2$  is obtained from the Dynkin diagram  $\bullet \begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$  of  $D_4$  by folding with respect to the action of  $\Sigma_3$  permuting the three vertices around the branching vertex.

**Theorem 1.3** (Derived geometric Satake for  $\mathrm{PSO}_8/G_2$ ). *Suppose that the  $\mathrm{PSO}_8[[t]]$ -action on  $\mathrm{PSO}_8((t))/G_2((t))$  is optimal in the sense of [Dev23, Hypothesis 3.5.22]. Then there is an equivalence*

$$\mathrm{Shv}_{\mathrm{PSO}_8[[t]]}^{\mathrm{c}, \mathrm{Sat}}(\mathrm{PSO}_8/G_2((t)); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(12, \vec{6}, -6, \vec{0})/\mathrm{SL}_2(-6\rho)^{\times 3} \times \mathbf{A}^1(4)).$$

In other words, the spherical subgroups  $\mathrm{PGL}_2^{\mathrm{diag}} \times \mathbf{G}_m \subseteq \mathrm{PGL}_2^{\times 3} \times \mathbf{G}_m$  and  $G_2 \subseteq \mathrm{PSO}_8$  have the same dual quotient stacks (namely,  $(\mathrm{std})^{\otimes 3}/\mathrm{SL}_2^{\times 3} \times \mathbf{A}^1$ ) up to grading. Therefore, they fit into the paradigm of [Dev23, Remark 4.1.5].

The proofs of Theorem 1.2 and Theorem 1.3 reduce to showing that the conditions of [Dev23, Theorem 3.5.24] are met. This ultimately relies on studying Bhargava’s construction from [Bha04] relating  $2 \times 2 \times 2$ -matrices to quadratic forms, and the work [GKZ94] of Gelfand-Kapranov-Zelevinsky describing the relationship to Cayley’s hyperdeterminant.

**Remark 1.4.** Following the philosophy of [Dev23], it should also be possible to use a variant of the methods of this article to prove analogues of Theorem 1.2 and Theorem 1.3 for sheaves with coefficients in connective complex K-theory  $\mathrm{ku}$ . We have not attempted to do this, but we expect the corresponding 1-parameter

<sup>1</sup>The  $\infty$ -category on the left-hand side is as in [Dev23, Definition 3.5.15]; see Definition 3.1 for a quick review. We expect the results of this article continue to hold if one considers sheaves with coefficients in  $\mathbf{Z}[i, \frac{1}{\sqrt{2}}]$ .

deformation of  $\mathrm{std}^{\otimes 3}$  over  $\pi_*(\mathrm{ku}) \cong \mathbf{Z}[\beta]$  to be a rather interesting  $\mathrm{ku}$ -Hamiltonian  $\mathrm{SL}_2^{\times 3}$ -variety.

**Remark 1.5.** The equivalence of Theorem 1.2 can heuristically be viewed as geometric Langlands for  $\mathrm{PGL}_2$  on the “doubled raviolo”, obtained by gluing three formal disks along their common punctured disk. We expect Theorem 1.2 to be related to the work of [MT12].

**Remark 1.6.** The quotient stack  $\mathrm{std}^{\otimes 3}/\mathrm{SL}_2^{\times 3}$  is also studied (in different language, of course) in quantum information theory; see Remark 2.12 below.

Theorem 1.2 and Theorem 1.3 are predicted by (the Betti version of) the local geometric conjecture of Ben-Zvi–Sakellaridis–Venkatesh; see [BZSV23, Conjecture 7.5.1]. My homotopy-theoretic interpretation of their conjecture is as follows. Suppose  $G$  is a reductive group over  $\mathbf{C}$  and  $G/H$  is an affine homogeneous spherical  $G$ -variety (meaning that it admits an open  $B$ -orbit for its natural left  $B \subseteq G$ -action). Then, there should be a dual graded  $\check{G}$ -variety  $\check{M}$  equipped with a moment map  $\mu : \check{M} \rightarrow \check{\mathfrak{g}}^*$ , and an equivalence of the form

$$\mathrm{Shv}_{G[[t]]}^{c,\mathrm{Sat}}(G((t))/H((t)); \mathbf{C}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G}),$$

where  $\mathrm{Perf}^{\mathrm{sh}}$  denotes the  $\infty$ -category of perfect complexes on the shearing of  $\check{M}/\check{G}$  with respect to its given grading. In fact, [BZSV23, Section 4] gives an explicit construction of this predicted dual  $\check{M}$ , and in the examples  $(G, H) = (\mathrm{PGL}_2^{\times 3}, \mathrm{PGL}_2^{\mathrm{diag}})$  and  $(\mathrm{PSO}_8, G_2)$ , one can compute that the stacky quotient  $\check{M}/\check{G}$  is isomorphic to the right-hand sides of Theorem 1.2 and Theorem 1.3 respectively.<sup>2</sup>

**Remark 1.7.** Lest Theorem 1.2 seem like an oddly specific example to focus on, we note that it is essentially the *only* “new” example of a spherical pair  $(G, H)$  of the form  $(H^{\times j}, H^{\mathrm{diag}})$ , as shown by the following lemma.

**Lemma 1.8.** *Suppose  $H$  is a simple linear algebraic group over  $\mathbf{C}$ . Then the subgroup  $H^{\mathrm{diag}} \subseteq H^{\times j}$  is spherical if and only if:*

- (a)  $j = 2$ , and  $H$  arbitrary;
- (b)  $j = 3$  and  $H$  is of type  $A_1$ .

**PROOF.** If the subgroup  $H^{\mathrm{diag}} \subseteq H^{\times j}$  is spherical, there is an open  $H^{\mathrm{diag}}$ -orbit on the flag variety of  $H^{\times j}$ . This implies that the dimension of  $H$  must be at least  $j|\Phi^+|$ , where  $\Phi^+$  is the set of positive roots; equivalently, one needs  $\mathrm{rank}(H) \geq (j-2)|\Phi^+|$ . Of course, this is always satisfied if  $j = 2$  (this is the group case corresponding to the symmetric subgroup  $H^{\mathrm{diag}} \subseteq H \times H$ ). Using the classification of simple linear algebraic groups over  $\mathbf{C}$ , it is easy to see that the only other case when the above inequality can hold is when  $j = 3$  and  $H$  is of type  $A_1$ ; one can then check by hand that the diagonal subgroup in this case is indeed spherical.  $\square$

<sup>2</sup>In the first case, this computation is straightforward given the prescription of [BZSV23, Section 4]; see [Sak13, Example 7.2.4] for a reference. The computation in the second case goes as follows. As in [BZSV23, Remark 7.1.1], the quotient stack  $\check{M}/\check{G}$  can be identified with the quotient  $\check{V}_X/\check{G}_X$ , where  $\check{G}_X$  is the Gaitsgory–Nadler/Sakellaridis–Venkatesh/Knop–Schalke dual group of  $X$  and  $\check{V}_X$  is constructed in [BZSV23, Section 4.5]. In the case  $X = \mathrm{PSO}_8/G_2$ , a calculation shows that  $\check{G}_X$  is the Levi subgroup of the maximal parabolic subgroup of  $\mathrm{PSO}_8$  corresponding to the central vertex of the  $D_4$  Dynkin diagram; so  $\check{G}_X \cong \mathrm{SL}_2^{\times 3}$ . Using the prescription of [BZSV23, Section 4.5], one can check that  $\check{V}_X \cong \mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1$ , where  $\check{G}_X$  acts only on the first factor. See, e.g., [Sak13, Line 9 of Table in Appendix A].

In the first case of Lemma 1.8, [BZSV23, Conjecture 7.5.1] is precisely the derived geometric Satake equivalence of [BF08]. Therefore, the only other case of Lemma 1.8 is when  $H$  is simple of type  $A_1$ , and Theorem 1.2 precisely addresses [BZSV23, Conjecture 7.5.1] for the adjoint form  $\mathrm{PGL}_2$  of  $H$ .

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## 2. Some properties of $\mathrm{std}^{\otimes 3}$

In this section, we establish some basic properties of  $\mathrm{std}^{\otimes 3}$  as a  $\mathrm{SL}_2^{\times 3}$ -variety; our base field will always be  $k$ , and we will write  $\check{G} = \mathrm{SL}_2^{\times 3}$ . Some of this material appears in [Bha04]. In particular, Construction 2.3 is due to Bhargava.

**Observation 2.1.** An element  $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2$  can be identified with a binary quadratic form  $q_A(x, y) = cx^2 + 2iaxy + by^2$ . Under this identification, the adjoint action of  $g \in \mathrm{SL}_2$  on  $\mathfrak{sl}_2$  is given by the action on  $(x, y)$  of the conjugate of  $g$  by the matrix  $\mathrm{diag}(\zeta_8, \zeta_8^{-1})$ . Explicitly, if  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ , the action sends

$$\begin{aligned} x &\mapsto i\delta x + \beta y, \\ y &\mapsto \gamma x - i\alpha y. \end{aligned}$$

Note, moreover, that the discriminant of  $q_A(x, y)$  is  $4\det(A)$ .

**Warning 2.2.** Note that under Observation 2.1, the element of  $\mathfrak{sl}_2$  associated to a binary quadratic form  $bx^2 + axy + cy^2$  is *not* the symmetric matrix associated to the quadratic form! Indeed, the associated symmetric matrix is  $\begin{pmatrix} b & a/2 \\ a/2 & c \end{pmatrix}$ , while the associated element of  $\mathfrak{sl}_2$  is  $\begin{pmatrix} -ai/2 & c \\ b & ai/2 \end{pmatrix}$ .

**Construction 2.3.** The affine space  $\mathbf{A}^8 = \mathrm{std}^{\otimes 3}$  can be regarded as parametrizing cubes

$$\begin{array}{ccccc} & & b_2 & \text{---} & d_3 \\ & \swarrow & | & \searrow & \\ a & \text{---} & b_1 & & \\ & \swarrow & | & \searrow & \\ & & d_1 & \text{---} & c, \\ & \swarrow & | & \searrow & \\ b_3 & \text{---} & d_2 & & \end{array}$$

which we will represent by a tuple  $(a, \vec{b}, c, \vec{d})$ ; we will often use the symbol  $\mathcal{C}$  to denote such a cube. If  $\{e_1, e_2\}$  are a basis for  $\mathrm{std}$ , the above cube corresponds to the element of  $\mathrm{std}^{\otimes 3}$  given by

$$\begin{aligned} &ae_1 \otimes e_1 \otimes e_1 + b_1e_2 \otimes e_1 \otimes e_1 + b_2e_1 \otimes e_2 \otimes e_1 + b_3e_1 \otimes e_1 \otimes e_2 \\ &+ d_1e_1 \otimes e_2 \otimes e_2 + d_2e_2 \otimes e_1 \otimes e_2 + d_3e_2 \otimes e_2 \otimes e_1 + ce_2 \otimes e_2 \otimes e_2. \end{aligned}$$

Associated to a cube  $\mathcal{C}$  are three pairs of matrices, given by slicing along the top, leftmost, or front faces:

$$\begin{aligned} M_1 &= \begin{pmatrix} a & b_2 \\ b_3 & d_1 \end{pmatrix}, N_1 = \begin{pmatrix} b_1 & d_3 \\ d_2 & c \end{pmatrix}, \\ M_2 &= \begin{pmatrix} a & b_1 \\ b_3 & d_2 \end{pmatrix}, N_2 = \begin{pmatrix} b_2 & d_3 \\ d_1 & c \end{pmatrix}, \\ M_3 &= \begin{pmatrix} a & b_1 \\ b_2 & d_3 \end{pmatrix}, N_3 = \begin{pmatrix} b_3 & d_2 \\ d_1 & c \end{pmatrix}; \end{aligned}$$

each of these defines a binary quadratic form

$$q_i(x, y) = -\det(M_i x + N_i y).$$

Explicitly,

$$\begin{aligned} q_1(x, y) &= \det(M_1)x^2 + (ac + b_1d_1 - b_2d_2 - b_3d_3)xy + \det(N_1)y^2, \\ q_2(x, y) &= \det(M_2)x^2 + (ac - b_1d_1 + b_2d_2 - b_3d_3)xy + \det(N_2)y^2, \\ q_3(x, y) &= \det(M_3)x^2 + (ac - b_1d_1 - b_2d_2 + b_3d_3)xy + \det(N_3)y^2. \end{aligned}$$

Viewing  $\mathfrak{sl}_2$  as the space of binary quadratic forms as in Observation 2.1, these three quadratic forms define a map

$$\mu : \mathrm{std}^{\otimes 3} \rightarrow \mathfrak{sl}_2^{\times 3}.$$

An easy check shows that this map is  $\check{G}$ -equivariant.

**Remark 2.4.** The standard  $\mathrm{SL}_2$ -equivariant symplectic structure on  $\mathrm{std}$  defines an  $\mathrm{SL}_2^{\times 3}$ -equivariant symplectic structure on  $\mathrm{std}^{\otimes 3}$ . This action is Hamiltonian, and one can verify that the map  $\mu : \mathrm{std}^{\otimes 3} \rightarrow \mathfrak{sl}_2^{\times 3} \cong (\mathfrak{sl}_2^*)^{\times 3}$  from Construction 2.3 is in fact the moment map for this  $\mathrm{SL}_2^{\times 3}$ -action.

**Lemma 2.5** (Cayley). *The composite*

$$\mathrm{std}^{\otimes 3} \xrightarrow{\mu} \mathfrak{sl}_2^{\times 3} \rightarrow \mathfrak{sl}_2^{\times 3} // \check{G}$$

*factors through the diagonal inclusion  $\mathfrak{sl}_2 // \mathrm{SL}_2 \rightarrow \mathfrak{sl}_2^{\times 3} // \check{G}$ . In fact, the induced map  $\mathrm{std}^{\otimes 3} \rightarrow \mathfrak{sl}_2 // \mathrm{SL}_2$  defines an isomorphism*

$$\mathrm{std}^{\otimes 3} // \check{G} \xrightarrow{\sim} \mathfrak{sl}_2 // \mathrm{SL}_2.$$

**PROOF.** The map  $\mathfrak{sl}_2^{\times 3} \rightarrow \mathfrak{sl}_2^{\times 3} // \check{G}$  sends a triple of matrices to their determinants, or equivalently a triple of quadratic forms to their discriminants. Therefore, we need to check that the three quadratic forms of Construction 2.3 have the same discriminant. This is an easy check: one finds that their common discriminant is

$$\begin{aligned} \det(q_i) &= a^2c^2 + b_1^2d_1^2 + b_2^2d_2^2 + b_3^2d_3^2 - 2(ab_1cd_1 + ab_2cd_2 + ab_3cd_3 \\ (1) \quad &+ b_1b_2d_1d_2 + b_1b_3d_1d_3 + b_2b_3d_2d_3) + 4(ad_1d_2d_3 + b_1b_2b_3c). \end{aligned}$$

It remains to check that the map  $\mathrm{std}^{\otimes 3} // \check{G} \rightarrow \mathbf{A}^1$  defined by this polynomial is an isomorphism. This is stated/proved in [GKZ94, Proposition 1.7 in Chapter 14], and is due to Cayley.  $\square$

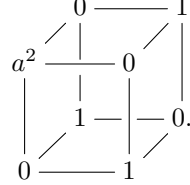
**Notation 2.6.** Write  $\det$  to denote the map  $\mathrm{std}^{\otimes 3} \rightarrow \mathfrak{sl}_2 // \mathrm{SL}_2$  from Lemma 2.5, so that if  $\mathcal{C}$  is a cube,  $\det(\mathcal{C})$  is the quantity of (1).

We will now define an analogue of the Kostant slice, as it will be needed to apply [Dev23, Theorem 3.5.24] (see [Dev23, Strategy 1.2.1(b)]). For the purposes of our discussion, one should view this Kostant section as an analogue of the construction of the companion matrix associated to a characteristic polynomial.

**Construction 2.7.** If  $n$  is an integer, let  $\vec{n}$  denote the triple  $(n, n, n)$ . Let

$$\kappa : \mathfrak{sl}_2 // \mathrm{SL}_2 \cong \mathbf{A}^1 // (\mathbf{Z}/2) \cong \mathbf{A}^1 \rightarrow \mathrm{std}^{\otimes 3}$$

denote the map sending  $a^2 \mapsto (a^2, \vec{0}, 0, \vec{1})$ . This corresponds to the cube



In this case,  $\det(\kappa(a^2)) = 4a^2$ , so that  $\kappa$  defines a section of  $\det$  (at least up to the unit  $4 \in k^\times$ ). The associated quadratic forms are all equal, and are given by

$$q_1(x, y) = q_2(x, y) = q_3(x, y) = a^2 x^2 - y^2,$$

which corresponds to the traceless matrix  $\begin{pmatrix} 0 & -1 \\ a^2 & 0 \end{pmatrix} \in \mathfrak{sl}_2$ . (Note that this is exactly the companion matrix associated to the characteristic polynomial  $y^2 - a^2$ .)

One of the key properties of the Kostant section/companion matrices is that a matrix  $A \in \mathfrak{sl}_2$  is conjugate to  $\kappa(\det(A))$  if and only if  $A$  is regular (i.e., the minimal polynomial of  $A$  agrees with its characteristic polynomial), if and only if  $A$  is nonzero. We will now prove an analogous result concerning  $\kappa : \mathbf{A}^1 \rightarrow \mathrm{std}^{\otimes 3}$ .

**Proposition 2.8.** *The  $\check{G}$ -orbit of the image of  $\kappa$  is a dense open subscheme whose complement has codimension 3.*

**PROOF.** We will use the classification of  $\check{G}$ -orbits on  $\mathrm{std}^{\otimes 3}$  as in [GKZ94, Example 4.5 in Chapter 14]; see Figure 1 for a graph of the seven orbits of  $\check{G}$  on  $\mathrm{std}^{\otimes 3}$ . Namely, if  $\lambda \neq 0$ , all elements of  $\det^{-1}(\lambda)$  are in a single  $\check{G}$ -orbit. (In fact, all elements in the fiber  $\det^{-1}(1)$  are in the  $\check{G}$ -orbit of  $(1, \vec{0}, 1, \vec{0})$ .) The  $\check{G}$ -orbit of  $\det^{-1}(\mathbf{G}_m)$  is open and dense, and hence is 8-dimensional; moreover, it agrees with the  $\check{G}$ -orbit of  $\kappa(\mathbf{G}_m)$ . Next, there is a maximal  $\check{G}$ -orbit inside the fiber  $\det^{-1}(0)$ , given by the orbit of  $(0, \vec{0}, 0, \vec{1}) = \kappa(0)$ . This orbit is 7-dimensional, and the largest  $\check{G}$ -orbits contained in the complement  $\det^{-1}(0) - \check{G} \cdot \kappa(0)$  have dimension 5. In particular, the complement of  $\check{G} \cdot \kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$  has dimension 5, i.e., codimension  $8 - 5 = 3$ .  $\square$

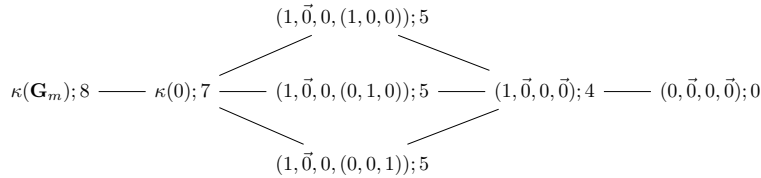


FIGURE 1.  $\check{G}$ -orbits on  $\mathrm{std}^{\otimes 3}$ , representatives, and their dimensions (indicated after the semicolon), connected by closure. Note that  $\kappa(0) = (0, \vec{0}, 0, \vec{1})$ , and that the  $\check{G}$ -orbit of  $\kappa(1) = (1, \vec{0}, 0, \vec{1})$  is the same as the  $\check{G}$ -orbit of  $(1, \vec{0}, 1, \vec{0})$ .

**Remark 2.9.** As explained in [GKZ94, Example 4.5 in Chapter 14], the closure of the associated orbits inside  $\mathbf{P}(\mathrm{std}^{\otimes 3}) = \mathbf{P}^7$  can be described as follows. First, the closure of the generic orbit is  $\mathbf{P}^7$ . Next, the closure of the orbit of next smallest dimension is the zero locus of  $\det$ , which cuts out the dual variety of the Segre embedding  $(\mathbf{P}^1)^{\times 3} \hookrightarrow \mathbf{P}^7$  (just as the usual determinant for  $2 \times 2$ -matrices cuts out the quadric  $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$ ). The projective orbit associated to  $(1, \vec{0}, 0, (0, 1, 0))$ , say, is cut out inside the locus  $\{\det = 0\}$  by the Segre embedding  $\mathbf{P}(\mathrm{std}) \times \mathbf{P}(\mathrm{std}^{\otimes 2}) = \mathbf{P}^1 \times \mathbf{P}^3 \rightarrow \mathbf{P}^7$ . Finally, the minimal nonzero orbit is cut out by the Segre embedding  $(\mathbf{P}^1)^{\times 3} \rightarrow \mathbf{P}^7$ .

**Proposition 2.10.** *There is an isomorphism*

$$\mathfrak{sl}_2//\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}/\check{G}} \mathfrak{sl}_2//\mathrm{SL}_2 \cong \mathrm{Spec} k[a, \alpha_i^{\pm 1}, \frac{\alpha_i - \alpha_i^{-1}}{a} | 1 \leq i \leq 3]^{\mathbf{Z}/2} / (\alpha_1 \alpha_2 \alpha_3 = 1)$$

of group schemes over  $\mathfrak{sl}_2//\mathrm{SL}_2 = \mathrm{Spec} k[a^2]$ , where the action of  $\mathbf{Z}/2$  sends  $a \mapsto -a$  and  $\alpha_i \mapsto \alpha_i^{-1}$ , and the group structure is such that each  $\alpha_i$  is grouplike.

PROOF. The fiber product on the left identifies with the subgroup of  $\mathfrak{sl}_2//\mathrm{SL}_2 \times \check{G}$  of those  $(a^2, \vec{g})$  such that  $\vec{g} = (g_1, g_2, g_3) \in \mathrm{SL}_2^{\times 3}$  stabilizes  $\kappa(a^2)$ . The trick to determining this stabilizer is to use Bhargava's construction from Construction 2.3: if  $\vec{g}$  stabilizes a cube  $\mathcal{C}$ , it must also stabilize the corresponding triple  $\mu(\mathcal{C}) \in \mathfrak{sl}_2^{\times 3}$  of quadratic forms.

First, a simple calculation shows that if  $a$  is a unit, the triple of matrices

$$\vec{g} = \left( \sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \sqrt{\frac{i}{2}} \begin{pmatrix} -i & a^{-1} \\ ia & 1 \end{pmatrix}, \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & a^{-1} \\ a & 1 \end{pmatrix} \right) \in \mathrm{SL}_2^{\times 3}$$

sends

$$\kappa(a^2) \mapsto -\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0}).$$

The triple  $\vec{g}$  can be thought of as “diagonalizing”  $\kappa(a^2)$ . The stabilizer of the cube  $-\sqrt{2}(a^2, \vec{0}, a^{-1}, \vec{0})$  precisely consists of triples of matrices of the form

$$(2) \quad \left( \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3^{-1} \end{pmatrix} \right) \text{ with } \alpha_1 \alpha_2 \alpha_3 = 1.$$

For  $\alpha \in \mathbf{G}_m$ , let  $h(\alpha)$  denote the matrix

$$h(\alpha) = \frac{1}{2} \begin{pmatrix} \alpha + \alpha^{-1} & \frac{\alpha^{-1} - \alpha}{a} \\ a^2 \cdot \frac{\alpha^{-1} - \alpha}{a} & \alpha + \alpha^{-1} \end{pmatrix} \in \mathrm{SL}_2.$$

Conjugating (2) by the element  $\vec{g} \in \check{G}$ , we find that the triple  $(h(\alpha_1), h(\alpha_2), h(\alpha_3))$  of matrices stabilizes  $\kappa(a^2)$  as long as  $\alpha_1 \alpha_2 \alpha_3 = 1$  and  $a^2 \in \mathbf{G}_m \subseteq \mathbf{A}^1$ . (See [BFM05, Section 3.2] for a slight variant of this calculation.) Note that the subgroup of such triples is 2-dimensional, and therefore the associated homogeneous  $\check{G}$ -space is  $9 - 2 = 7$ -dimensional. Using that the  $\check{G}$ -orbit of  $\kappa(a^2)$  is also 7-dimensional (e.g., by [GKZ94, Example 4.5 in Chapter 14]), it is not hard to see from this calculation (by a limiting argument for  $a \rightarrow 0$ ) that the stabilizer of the family  $\kappa(\mathbf{A}^1) \subseteq \mathrm{std}^{\otimes 3}$  is precisely the claimed group scheme.  $\square$

**Remark 2.11.** A direct calculation shows that the stabilizer of  $\kappa(0)$  is isomorphic to the subgroup of triples of matrices of the form  $\begin{pmatrix} \pm 1 & \mp \gamma_i \\ 0 & \pm 1 \end{pmatrix}$  for  $1 \leq i \leq 3$  with  $\gamma_1 + \gamma_2 + \gamma_3 = 0$ . This subgroup is, of course, isomorphic to  $\mu_2^{\times 3} \times \mathbf{G}_a^{\times 2}$ ; it is also isomorphic to the fiber over  $a = 0$  of the group scheme of Proposition 2.10.

**Remark 2.12.** As mentioned in Remark 1.6, the quotient stack  $\text{std}^{\otimes 3}/\text{SL}_2^{\times 3}$  is studied in quantum information theory. For instance, in [DVC00], Dür-Vidal-Cirac study the orbit structure of  $\text{SL}_2^{\times 3}$  acting on  $\text{std}^{\otimes 3}$  (in particular, they recover Figure 1 independently of [GKZ94]). For the interested reader, let us describe the translation between our notation/terminology and that of quantum information theory. Our base field will now be  $k = \mathbf{C}$ . An element of  $\text{std}^{\otimes n}$  (really, of the projective space  $\mathbf{P}(\text{std}^{\otimes n}) \cong \mathbf{P}^{2^n-1}$ ) is called an  $n$ -qubit, and the action of  $\text{SL}_2^{\times n}$  is via *stochastic local operations and classical communication* (SLOCC) operators (replacing  $\text{SL}_2^{\times n}$  by  $\text{GL}_2^{\times n}$  simply amounts to dropping the word “stochastic”). The space  $\text{std}$  is equipped with a basis  $\{|0\rangle, |1\rangle\}$ , and a cube  $\mathcal{C} = (a, \vec{b}, c, \vec{d}) \in \text{std}^{\otimes 3}$  corresponds to the three-qubit<sup>3</sup>

$$a|000\rangle + b_1|100\rangle + b_2|010\rangle + b_3|001\rangle \\ + d_1|011\rangle + d_2|101\rangle + d_3|110\rangle + c|111\rangle.$$

The state

$$\frac{1}{\sqrt{2}}(1, \vec{0}, 1, \vec{0}) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

is known as the *Greenberger–Horne–Zeilinger* (GHZ) state, and the state

$$\frac{1}{\sqrt{3}}\kappa(0) = \frac{1}{\sqrt{3}}(0, \vec{1}, 0, \vec{0}) = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

is called the *W* state. These two states are known to represent two very different kinds of quantum entanglement; from the perspective of this article, the reason for this is simply that the Cayley hyperdeterminant of the GHZ state is nonzero, but the Cayley hyperdeterminant of the W state vanishes. Nevertheless, the proof of Proposition 2.10 shows that there is a natural *degeneration* of (the SLOCC/ $\text{SL}_2^{\times 3}$ -equivalence class of) the GHZ state into the W state. Indeed, the GHZ state can be transformed into the cube  $\frac{1}{2}\kappa(1)$ , which admits a natural degeneration to the W state via the one-parameter family

$$\frac{1}{\sqrt{a^4+3}}\kappa(a^2) = \frac{1}{\sqrt{a^4+3}}(a^2|000\rangle + |011\rangle + |101\rangle + |110\rangle).$$

In fact, this state already appears as [DVC00, Equation 20].

**Remark 2.13.** Fix an integer  $n$ . Then the  $\check{G}$ -variety  $\text{std}^{\otimes 3}$  admits a natural grading, where the entries of a cube  $(a, \vec{b}, c, \vec{d})$  have the following weights:  $a$  lives in weight  $-4n$ ,  $b$  lives in weight  $-2n$ ,  $c$  lives in weight  $2n$ , and  $d$  lives in weight  $0$ . Write  $\text{std}^{\otimes 3}(4n, 2\vec{n}, -2n, \vec{0})$  to denote the associated graded variety. Equip  $\mathfrak{sl}_2$  with the grading where the entries of a matrix  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$  have the following weights:  $a$  lives in weight  $-2n$ ,  $b$  lives in weight  $0$ , and  $c$  lives in weight  $-4n$ . Similarly, equip  $\text{SL}_2$  with the grading coming from  $2n\rho$ , so that the entries of a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  have the following weights:  $a$  and  $d$  live in weight  $0$ ,  $b$  lives in weight  $2n$ , and  $c$  lives in weight  $-2n$ . With these gradings, the  $\text{SL}_2^{\times 3}$ -equivariant map  $\mu : \text{std}^{\otimes 3} \rightarrow \mathfrak{sl}_2^{\times 3}$  is a *graded* map, and  $\kappa$  defines a graded map  $\mathfrak{sl}_2(2n)/\text{SL}_2 \cong \mathbf{A}^1(4n) \rightarrow \text{std}^{\otimes 3}(4n, 2\vec{n}, -2n, \vec{0})$ . The cases  $n = 1$  and  $n = 3$  will be relevant below (corresponding to Theorem 1.2 and Theorem 1.3, respectively).

<sup>3</sup>Technically, a qubit is required to have norm 1, so one must rescale  $\mathcal{C}$  by  $\sqrt{a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2}$ ; but this could in theory introduce a singularity when  $a^2 + \|\vec{b}\|^2 + c^2 + \|\vec{d}\|^2 = 0$ . We will ignore this point below.



### 3. The proof

Before proceeding, let us remind the reader of the definition of the left-hand side of the equivalence of Theorem 1.2, following [Dev23, Definition 3.5.15].

**Definition 3.1.** Let  $G$  be a compact Lie group, and let  $H \subseteq G$  be a closed subgroup such that  $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$  is a spherical subgroup. Let  $\mathrm{Shv}_{G[[t]]}^c(G((t))/H((t)); \mathbf{Q})$  denote the  $\infty$ -category of  $G[[t]]$ -equivariant sheaves of  $\mathbf{Q}$ -modules on  $G((t))/H((t))$  which are constructible for the orbit stratification on  $G((t))/H((t))$ . Note that since the orbit stratification is countable (by assumption that  $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$  is a spherical subgroup and [GN10, Theorem 3.2.1]), the  $\infty$ -category  $\mathrm{Shv}_{G[[t]]}^c(G((t))/H((t)); \mathbf{Q})$  is well-defined. There is a natural left-action of the  $\mathbf{E}_3$ -monoidal  $\infty$ -category  $\mathrm{Shv}_{(G \times G)[[t]]}^c(G((t)); \mathbf{Q})$  on  $\mathrm{Shv}_{G[[t]]}^c(G((t))/H((t)); \mathbf{Q})$ , and in particular, a left-action of  $\mathrm{Rep}(\check{G})$  by the abelian geometric Satake theorem of [MV07]. Let  $\mathrm{IC}_0 \in \mathrm{Shv}_{G[[t]]}^c(G((t))/H((t)); \mathbf{Q})$  denote the pushforward  $i_* \mathbf{Q}$  of the constant sheaf along the inclusion  $(G/H)(\mathbf{C}[[t]]) \rightarrow (G/H)(\mathbf{C}((t)))$ . Let  $\mathrm{Shv}_{G[[t]]}^{c, \mathrm{Sat}}(G((t))/H((t)); \mathbf{Q})$  denote the full subcategory of  $\mathrm{Shv}_{G[[t]]}^c(G((t))/H((t)); \mathbf{Q})$  generated by  $\mathrm{IC}_0$  under the action of  $\mathrm{Rep}(\check{G})$ . If  $k$  is any  $\mathbf{Q}$ -algebra, base-changing along the unit map defines the  $\infty$ -category  $\mathrm{Shv}_{G[[t]]}^{c, \mathrm{Sat}}(G((t))/H((t)); k)$ .

**PROOF OF THEOREM 1.2.** It suffices to verify conditions (a) and (b) of [Dev23, Theorem 3.5.24], which gives a criterion for establishing an equivalence of  $k$ -linear  $\infty$ -categories of the form

$$\mathrm{Shv}_{G[[t]]}^{c, \mathrm{Sat}}(G((t))/H((t)); k) \simeq \mathrm{Perf}(\mathrm{sh}^{1/2} \check{M}/\check{G}).$$

The map  $\kappa$  is given by the map  $\mathfrak{sl}_2(2)//\mathrm{SL}_2 \rightarrow \mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})$  from Construction 2.7. For condition (a) of [Dev23, Theorem 3.5.24], we need to show that if  $\check{J}_X = \mathfrak{sl}_2(2)//\mathrm{SL}_2 \times_{\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\check{G}} \mathfrak{sl}_2(2)//\mathrm{SL}_2$ , the ring of regular functions on the quotient  $(\mathfrak{sl}_2(2)//\mathrm{SL}_2 \times \check{G})/\check{J}_X$  is isomorphic (as a graded algebra) to  $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})}$ . The quotient  $(\mathfrak{sl}_2(2)//\mathrm{SL}_2 \times \check{G})/\check{J}_X$  identifies with the  $\check{G}$ -orbit of the image of  $\kappa$ , which has complement of codimension 3 in  $\mathrm{std}^{\otimes 3}$  by Proposition 2.8; therefore, the algebraic Hartogs theorem implies that there is a graded isomorphism  $\mathcal{O}_{(\mathfrak{sl}_2(2)//\mathrm{SL}_2 \times \check{G})/\check{J}_X} \cong \mathcal{O}_{\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})}$ .

For condition (b) of [Dev23, Theorem 3.5.24], we need to check that there is an isomorphism

$$\check{J}_X \cong \mathrm{Spec} H_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k)$$

of graded group schemes over  $\mathfrak{sl}_2(2)//\mathrm{SL}_2 \cong \mathrm{Spec} H_{\mathrm{PGL}_2}^*(\ast; k)$ . There is an isomorphism

$$(3) \quad \mathrm{Spec} H_*^{\mathrm{PGL}_2}(\Omega \mathrm{PGL}_2; k) \cong \mathrm{Spec} k[a, \alpha^{\pm 1}, \frac{\alpha - \alpha^{-1}}{a}]^{\mathbf{Z}/2},$$

and the action of the  $\mathbf{Z}/2$  on the left-hand side sends  $a \mapsto -a$  and  $\alpha \mapsto \alpha^{-1}$ . This is proved, e.g., in [BFM05], and also follows from [Dev23, Example 3.6.16]. The Künneth theorem implies that there is an isomorphism

$$\mathrm{Spec} H_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}); k) \cong \mathrm{Spec} k[a, \alpha_i^{\pm 1}, \frac{\alpha_i - \alpha_i^{-1}}{a} | 1 \leq i \leq 3]^{\mathbf{Z}/2},$$

and the fiber sequence

$$\mathrm{PGL}_2^{\mathrm{diag}} \rightarrow \mathrm{PGL}_2^{\times 3} \rightarrow \mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}$$

implies that

$$\mathrm{Spec} H_*^{\mathrm{PGL}_2}(\Omega(\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2^{\mathrm{diag}}); k) \cong \mathrm{Spec} \left( k[a, \alpha_i^{\pm 1}, \frac{\alpha_i - \alpha_i^{-1}}{a} \mid 1 \leq i \leq 3] / (\alpha_1 \alpha_2 \alpha_3 - 1) \right)^{\mathbf{Z}/2},$$

The desired isomorphism now follows from this observation and Proposition 2.10.  $\square$

**Remark 3.2.** The proof of Theorem 1.3 is exactly the same as the proof of Theorem 1.2 above. Indeed, one only needs to observe that  $\mathrm{PSO}_8/G_2$  is homotopy equivalent to  $\mathbf{R}P^7 \times \mathbf{R}P^7$  (which follows, e.g., from the fact that  $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$ )<sup>4</sup>. The replacement of (3) is given by [Dev23, Proposition 4.8.6], which gives an isomorphism

$$\mathrm{Spec} H_*^{G_2}(\Omega \mathbf{R}P^7; k) \cong \mathrm{Spec} k[a, b, \alpha^{\pm 1}, \frac{\alpha - \alpha^{-1}}{a}]^{\mathbf{Z}/2}$$

where  $a$  is in weight  $-6$  and  $b$  is in weight  $-4$ .

**Remark 3.3.** Remark 2.4 guarantees that the equivalence of Theorem 1.2 is compatible with the action of the spherical Hecke category  $\mathrm{Shv}_{(\mathrm{PGL}_2^{\times 3} \times \mathrm{PGL}_2^{\times 3})[[t]]}^{c, \mathrm{Sat}}(\mathrm{PGL}_2^{\times 3}((t)); k) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2 - 2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho))$ . Namely, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}_{(\mathrm{PGL}_2^{\times 3} \times \mathrm{PGL}_2^{\times 3})[[t]]}^{c, \mathrm{Sat}}(\mathrm{PGL}_2^{\times 3}((t)); k) & \xrightarrow[\mathrm{[BF08]}]{\sim} & \mathrm{Perf}^{\mathrm{sh}}(\mathfrak{sl}_2^{\times 3}(2 - 2\rho)/\mathrm{SL}_2^{\times 3}(-2\rho)) \\ \downarrow \text{act on IC}_0 & & \downarrow \mu^* \\ \mathrm{Shv}_{\mathrm{PGL}_2^{\times 3}[[t]]}^{c, \mathrm{Sat}}(\mathrm{PGL}_2^{\times 3}((t))/\mathrm{PGL}_2^{\mathrm{diag}}((t)); k) & \xrightarrow[\text{Theorem 1.2}]{\sim} & \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})/\mathrm{SL}_2^{\times 3}(-2\rho)), \end{array}$$

where  $\mu^*$  is given by pullback along the moment map for the Hamiltonian  $\mathrm{SL}_2^{\times 3}$ -action on  $\mathrm{std}^{\otimes 3}$ .

**Remark 3.4.** Theorem 1.2 does not need the full strength of optimality in the sense [Dev23, Hypothesis 3.5.22]. Indeed, the first and second assumptions in [Dev23, Hypothesis 3.5.22] are included to ensure formality of the algebra from [Dev23, Equation 16 in the proof of Theorem 3.5.24]. However, as in [Dev23, Remark 3.2.22], the formality of this algebra is *guaranteed* in our case: since Theorem 1.2 shows that the homotopy of the algebra in question is  $\mathcal{O}_{\mathrm{std}^{\otimes 3}(4, \vec{2}, -2, \vec{0})}$ , i.e., is polynomial on classes in even weights. It is therefore automatically formal by [Dev23, Lemma 2.1.9]. Note that since  $\mathrm{Ind}_{\mathrm{SL}_2^{\times 3}}^{\mathrm{Spin}_8}(\mathrm{std}^{\otimes 3} \oplus \mathbf{A}^1)$  is not an affine space, this argument does not go through in the case of Theorem 1.3 to prove formality of the algebra from [Dev23, Equation 16 in the proof of Theorem 3.5.24].

<sup>4</sup>Perhaps the most “conceptual” way to see that  $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$  is as follows. Using triality, one can identify  $\mathrm{Spin}_8$  with the subgroup of  $\mathrm{SO}_8^{\times 3}$  of those triples  $(A_1, A_2, A_3)$  such that  $A_1(x_1)A_2(x_2) = A_3(x_1x_2)$  for octonions  $x_1, x_2$ . Under this presentation,  $G_2$  corresponds to the subgroup where  $A_1 = A_2 = A_3$ . The subgroups where  $A_1 = A_3$  (resp.  $A_2 = A_3$ ) are both isomorphic to  $\mathrm{Spin}(7)$ ; these are sometimes denoted  $\mathrm{Spin}^{\pm}(7)$ . The action of  $\mathrm{Spin}_8$  on  $S^7 \times S^7$  sends  $(x, y) \mapsto (A_1x, A_2y)$ ; one can check that this is transitive, and that the stabilizer of the point  $(1, 1)$  is precisely  $\mathrm{Spin}^+(7) \cap \mathrm{Spin}^-(7) \cong G_2$ .

That there is an equivalence  $\mathrm{Spin}_8/G_2 \simeq S^7 \times S^7$  at the level of cohomology with  $\mathbf{Z}[1/2]$ -coefficients, at least, is much simpler: on group cohomology, the map  $G_2 \rightarrow \mathrm{Spin}_8$  is given by the map  $\mathbf{Z}[1/2, p_1, p_2, p_3, c_4] \rightarrow \mathbf{Z}[1/2, c_2, c_6]$  sending  $p_1 \mapsto -c_2$ ,  $p_2 \mapsto 0$ ,  $p_3 \mapsto -c_6$ , and  $c_4 \mapsto 0$ . The Serre spectral sequence for the fibration  $\mathrm{Spin}_8/G_2 \rightarrow BG_2 \rightarrow B\mathrm{Spin}_8$  implies that  $H^*(\mathrm{Spin}_8/G_2; \mathbf{Z}[1/2]) \cong \mathbf{Z}[1/2, \sigma(p_2), \sigma(c_4)]/(\sigma(p_2)^2, \sigma(c_4)^2)$ , where  $\sigma(p_2)$  and  $\sigma(c_4)$  both live in (homological) weight  $-7$ . This is precisely the cohomology of  $S^7 \times S^7$ , as desired.

**Remark 3.5.** Let  $k = \mathbf{Q}_2(\zeta_8)$ . The theory of 2-compact groups as studied, e.g., in [AG09], suggests viewing the Dwyer-Wilkerson space  $\mathrm{DW}_3$  from [DW93] as an analogue of the groups  $\mathrm{SO}_3 \cong \mathrm{PGL}_2$  and  $G_2$ ; see Table 1. The 2-complete space  $\mathrm{DW}_3$  is equipped with an  $\mathbf{E}_1$ -structure, and it has finite mod 2 cohomology. It is therefore natural to ask whether there is an analogue of Theorem 1.2 and Theorem 1.3, where  $\mathrm{PGL}_2$  and  $G_2$  are replaced by  $\mathrm{DW}_3$ ; this is closely related to [Dev23, Appendix C(p)].

Group	Rank	Dimension	$\mathbf{F}_2$ -cohomology of $BG$	Weyl group
$G_n$	$n$	$(2^{n+1} - 1)n$	$\widehat{\mathrm{Sym}}^*(\mathbf{F}_2^{n+1}(-1))^{\mathrm{GL}_{n+1}(\mathbf{F}_2)}$	$\mathbf{Z}/2 \times \mathrm{GL}_n(\mathbf{F}_2)$
$\mathrm{PGL}_2$	1	3	$\mathbf{F}_2[w_2, w_3]$	$\mathbf{Z}/2$
$G_2$	2	14	$\mathbf{F}_2[w_4, w_6, w_7]$	$\mathbf{Z}/2 \times \Sigma_3$
$\mathrm{DW}_3$	3	45	$\mathbf{F}_2[w_8, w_{12}, w_{14}, w_{15}]$	$\mathbf{Z}/2 \times \mathrm{PSL}_2(\mathbf{F}_7)$

TABLE 1. Analogies between the (2-compact) groups  $\mathrm{PGL}_2 = \mathrm{SO}_3$ ,  $G_2$ , and  $\mathrm{DW}_3$ ; all of these are Poincaré duality complexes of dimension indicated in the third column. Here,  $w_n$  denotes the  $n$ th Stiefel-Whitney class, and the ring in the fourth column is known as the algebra of rank  $n + 1$  Dickson invariants. Note, also, that the Weyl group of  $\mathrm{DW}_3$  is called  $G_{24}$  in the Shephard-Todd classification.

It is difficult to answer this question since the representation theory of  $\mathrm{DW}_3$  is not well-understood. For instance, it does not seem to be known whether there is a 2-compact group  $G$  with an  $\mathbf{E}_1$ -map  $\mathrm{DW}_3 \rightarrow G$  such that  $G/\mathrm{DW}_3 \simeq \mathbf{R}P^{15} \times \mathbf{R}P^{15}$  (analogous to the equivalences  $\mathrm{PGL}_2^{\times 3}/\mathrm{PGL}_2 \cong \mathbf{R}P^3 \times \mathbf{R}P^3$  and  $\mathrm{PSO}_8/G_2 \cong \mathbf{R}P^7 \times \mathbf{R}P^7$ ). If such a  $G$  exists, and there is a good theory of  $G[t]$ -equivariant sheaves of  $k$ -modules, it seems reasonable to expect that there is an equivalence of the form

$$\mathrm{Shv}_{G[t]}^{c, \mathrm{Sat}}(G((t))/\mathrm{DW}_3((t)); k) \cong \mathrm{Perf}^{\mathrm{sh}}(\mathrm{std}^{\otimes 3}(28, \vec{14}, -14, \vec{0})/\mathrm{SL}_2(-14\rho)^{\times 3} \times \mathbf{A}^2(8, 12)).$$

Here, the “Whittaker” factor  $\mathbf{A}^2(8, 12)$  on the right-hand side comes from the isomorphism

$$\mathrm{Spf} H^*(BDW_3; k) := \mathrm{Spf} H^*(BDW_3; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong \widehat{\mathbf{A}}^3(8, 12, 28),$$

which follows from running the Bockstein spectral sequence on

$$H^*(BDW_3; \mathbf{F}_2) \cong \mathbf{F}_2[w_8, w_{12}, w_{14}, w_{15}],$$

and the fact that the Bockstein sends  $w_{14} \mapsto w_{15}$ . One can check that such a  $G$ , if it existed, would have rational cohomology given by

$$H^*(BG; \mathbf{Z}_2) \otimes_{\mathbf{Z}_2} k \cong k[c_4, c_6, c_{14}, x, y],$$

where both  $x$  and  $y$  live in cohomological degree 16.

In an email, Jesper Grodal told me that such a  $G$  cannot exist (it would have to be the 2-completion of a compact Lie group, but no compact Lie group has the desired cohomology). Despite this, one can wonder about the analogue of the “regular centralizer” group scheme calculation from Theorem 1.2:

- Is there a good notion of genuine *equivariant*  $DW_3$ -cohomology (with coefficients in  $k = \mathbf{Q}_2(\zeta_8)$ , say)? One should have  $\mathrm{Spec} H_{DW_3}^*(*; k) \cong \mathbf{A}^3(8, 12, 28)$ .
- Is there a faithful (basepoint-preserving) action of  $DW_3$  on  $S^{15}$ , which descends to an action of  $DW_3$  on (the 2-completion of)  $\mathbf{R}P^{15}$ ?
- For the above expected action, is there an isomorphism

$\mathrm{Spec} H_*^{DW_3}(\Omega(\mathbf{R}P^{15} \times \mathbf{R}P^{15}); k) \cong \mathbf{A}^2(8, 12) \times (\mathbf{A}^1(28) \times_{\mathrm{std} \otimes^3(28, \bar{1}\bar{4}, -14, \bar{0})/\mathrm{SL}_2(-14\rho) \times^3} \mathbf{A}^1(28))$   
of graded group schemes over  $k$ ?

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