NONABELIAN FOURIER TRANSFORM/BI-WHITTAKER REDUCTION

1. Introduction

Our goal in this talk is to describe a Fourier transform for the universal centralizer group scheme, following [Lon18, Gin18]. Let us begin by recalling a classical construction of the Fourier transform:

Recollection 1.1. Fix a field k (of any characteristic). Let V be a vector space over k, and let V^* denote its dual vector space. We will (unfortunately) abusively use the same symbol V to denote both the affine space over k and the k-module. The translation is provided by the isomorphism $\mathcal{O}_V = \mathrm{Sym}_k(V^*)$. The classical limit of the Fourier transform is given by the evident isomorphism

$$T^*V = V \oplus V^* \cong T^*(V^*).$$

Recall that T^*V is quantized by the sheaf $\mathcal{D}_V$ of (crystalline) differential operators on V. It will be useful to include a quantum parameter, denoted $\hbar$, in the differential operators (defining the so-called "asymptotic" differential operators). More precisely, recall that $\mathcal{O}_{T^*V} \cong \mathrm{Sym}_k(V^* \oplus V)$; equip this ring with a grading by declaring that the generators from V live in weight 1. The sheaf of asymptotic differential operators $\mathcal{D}_V^\hbar$ is defined as

$$\mathcal{D}_V^\hbar = k[\hbar]\langle V^* \oplus V\rangle/([v, f] = \hbar f(v) \text{ for all } v \in V, f \in V^*),$$

where both $\hbar$ and V live in weight 1. It is then clear that $\mathcal{D}_V^\hbar/\hbar$ is isomorphic (as a graded ring) to $\mathcal{O}_{T^*V}$. The (quantized) Fourier transform is given by the isomorphism $\mathcal{D}_V^\hbar \cong \mathcal{D}_{V^*}^\hbar$ which flips the role of V and V^*. (As written, this isomorphism does not respect the grading. Since the gradings do not play a major role in what follows, we will ignore this issue. In particular, the reader should assume that $\hbar$ is just some parameter in $\mathbf{A}^1_k$.) Note, in particular, this implies that $\mathrm{DMod}_\hbar(V) \simeq \mathrm{DMod}_\hbar(V^*)$ where $\mathrm{DMod}_\hbar(V) = \mathrm{LMod}_{\mathcal{D}_V^\hbar}$.

We will study a modification of the above to tori. Since it will be useful in a moment, let us just set up some notation.

Notation 1.2. We will let G denote a semisimple connected and simply-connected algebraic group over $k = \mathbf{C}$. (For much of this story, one can assume that k is of characteristic $p > 0$, as long as p is large enough.) Presumably one does not need all these assumptions. We will also let $B \subseteq G$ be a Borel, $N \subseteq B$ be its unipotent radical, and $T \subseteq B$ a maximal torus. Moreover, Λ will denote the weight lattice (of any given torus, not necessarily one that manifests as a maximal torus), Λ^* the coweight lattice, Λ^{pos} the dominant weights, $\Phi \subseteq \Lambda$ the subset of roots, $\Phi^{\mathrm{pos}} \subseteq \Lambda^{\mathrm{pos}}$ the subset of positive roots determined by B, $\Delta \subseteq \Phi$ a subset of simple roots, W the Weyl group, $\mathfrak{t}$ the Lie algebra of T, $\mathfrak{b}$ the Lie algebra of B, $\mathfrak{n}$ the Lie algebra of N, and $\mathfrak{g}$ the Lie algebra of G.

Construction 1.3. Let k be a field, and let T be a torus with weight lattice Λ. Then $T = \mathrm{Spec}\, k[\Lambda]$, and $\mathfrak{t}^* = \Lambda \otimes_{\mathbf{Z}} k$. Then $T^*T = T \times \mathfrak{t}^*$; this is quantized by the sheaf of asymptotic (crystalline) differential operators

$$\mathcal{D}_T^\hbar = k[\hbar]\langle x_\lambda, \delta_\lambda | \lambda \in \Lambda\rangle/([x_\lambda, \delta_\lambda] = \hbar x_\lambda),$$

where it is implicit that all other commutators are set to zero. Here, δ_λ is to be understood as the scaling-invariant differential operator $x_\lambda \partial_{x_\lambda}$. To describe the Fourier transform, let us just flip the roles of x and δ, and rewrite the above relation as

$$x_\lambda \delta_\lambda = (\delta_\lambda + \hbar) x_\lambda.$$

Thinking of δ_λ as a coordinate on the affine space $\mathfrak{t}^*_{k[\hbar]} := \mathfrak{t}^* \otimes_k k[\hbar] \cong \Lambda \otimes_{\mathbf{Z}} \mathbf{A}^1_{k[\hbar]}$, we may understand $\mathcal{D}_T^\hbar$ as the semidirect product $\mathcal{O}_{\mathfrak{t}^*_{k[\hbar]}} \rtimes \Lambda$, with Λ acting on X by translation. This implies that there is an equivalence

$$\mathrm{LMod}_{\mathcal{D}_T^\hbar} \simeq \mathrm{QCoh}^\Lambda(\mathfrak{t}^*_{k[\hbar]}) = \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda). \tag{1}$$

Here, the right-hand side is to be understood as Λ-equivariant quasicoherent sheaves on $\mathfrak{t}^*_{k[\hbar]}$. We will view (1) as the Fourier transform for the torus. Note that when you force $\hbar = 0$, the action of Λ on $\mathfrak{t}^*_{k[\hbar]} \otimes_{k[\hbar]} k$ becomes trivial, and so the stacky quotient $\mathfrak{t}^*_k/\Lambda$ is just equivalent to $\mathfrak{t}^*_k \times B\Lambda$. However, we may identify $B\Lambda$ with T, so we recover the equivalence $\mathrm{QCoh}(T^*T) \simeq \mathrm{QCoh}(T \times \mathfrak{t}^*)$.

The question we will attempt to answer in this talk is whether there is a noncommutative analogue of this result. So assume that G, B, etc. is as in Notation 1.2. Then W acts on T (and hence on $\mathcal{D}_T^\hbar$), and it is not difficult to see that (1) upgrades to an equivalence

$$\mathrm{LMod}_{(\mathcal{D}_T^\hbar)^W} \simeq \mathrm{QCoh}^{\Lambda \rtimes W}(\mathfrak{t}^*_{k[\hbar]}) = \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W). \tag{2}$$

Thanks to the fact that $(\mathcal{D}_T^\hbar)^W$ is Morita equivalent to $\mathcal{D}_T^\hbar \rtimes W$, we can further rewrite this as an equivalence

$$\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W) \simeq \mathrm{LMod}_{\mathcal{D}_R^\hbar \rtimes W}. \tag{3}$$

This is not terribly satisfactory, since $(\mathcal{D}_T^\hbar)^W$ does not have a good geometric interpretation. To understand an appropriate modification, let us force $\hbar = 0$, which degenerates our algebra to functions on the GIT quotient $(T^*T)/\!\!/W$. This does not contain much information about G. A much more interesting object is the universal regular centralizer, introduced in Ben's talk; this will be the replacement of T^*T.

2. The universal regular centralizer

Let us now introduce/review some properties of the universal regular centralizer. We will assume from now that the base field k is $\mathbf{C}$.

Definition 2.1. Let J denote the commutative group scheme of regular centralizers associated to G. To define this precisely, consider an auxiliary group scheme I over $\mathfrak{g}$, defined as follows. The action of G on $\mathfrak{g}$ defines a map $G \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ which sends $(g, x) \mapsto (\mathrm{Ad}_g(x), x)$. This map is G-equivariant for the diagonal action of G on G (resp. $\mathfrak{g}$) by conjugation (resp. the adjoint action). Define I via the Cartesian square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & G \times \mathfrak{g} \arrow[d] \\ \mathfrak{g} \arrow[r, "\Delta"'] & \mathfrak{g} \times \mathfrak{g}. \end{tikzcd}$$

It is clear that if $x \in \mathfrak{g}$, the fiber of I over x is the quotient $Z_G(x)$. One can prove (we will sketch this below) that I descends to a group scheme over the GIT quotient $\mathfrak{g}/\!\!/G$; this group scheme will be denoted J. It is much easier to see that I descends to the stacky quotient $\mathfrak{g}/G$, because all the maps in the above diagram are G-equivariant.

To descend to $\mathfrak{g}/\!\!/G$, let us recall the Kostant section of the map $\mathfrak{g} \to \mathfrak{g}/\!\!/G$.

Construction 2.2. Let e be a principal nilpotent in $\mathfrak{n} \subseteq \mathfrak{g}$. (All of these are equivalent up to G-conjugacy; one particular choice is given by $\sum_{\alpha \in \Delta} e_\alpha$, where e_α is a nonzero vector in the root space $\mathfrak{g}_\alpha$. For $G = \mathrm{SL}_n$, this is just the $n \times n$-matrix with ones on the superdiagonal.) Then the Jacobson-Morozov theorem tells us that e determines an $\mathfrak{sl}_2$-triple $\mathfrak{sl}_2 \to \mathfrak{g}$ which sends $e \in \mathfrak{sl}_2$ to $e \in \mathfrak{g}$. Let $f \in \mathfrak{n}_-$ denote the image of $f \in \mathfrak{sl}_2$; then, the Kostant slice $\mathcal{S}$ is defined as $f + \mathfrak{g}^e \subseteq \mathfrak{g}$, where $\mathfrak{g}^e$ is the centralizer of e in $\mathfrak{g}$. The reason this is known as a slice is because the composite

$$\mathcal{S} = f + \mathfrak{g}^e \subseteq \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\!\!/G$$

is an isomorphism; therefore, $\mathcal{S}$ defines a section of the map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\!\!/G$. In fact, $\mathcal{S}$ is contained in the *regular* locus of $\mathfrak{g}$ (i.e., those $x \in \mathfrak{g}$ such that $\dim Z_G(x) = \dim T$).

A little more is true. Namely, the unipotent subgroup N of B acts on $f + \mathfrak{g}^e$, and one can prove that the action map

$$N \times (f + \mathfrak{g}^e) \to f + \mathfrak{b}$$

is an isomorphism. In particular, $f + \mathfrak{g}^e$ is isomorphic to the *stacky* quotient $(f + \mathfrak{b})/N$. To summarize, there are isomorphisms

$$\mathcal{S} = f + \mathfrak{g}^e \xrightarrow{\sim} (f + \mathfrak{b})/N \xrightarrow{\sim} \mathfrak{g}/\!\!/G.$$

Let us denote the Kostant section $\mathfrak{g}/\!\!/G \to \mathfrak{g}$ by κ.

The following is a restatement of the above discussion.

Lemma 2.3. *The Kostant slice $\mathcal{S} \subseteq \mathfrak{g}$ intersects each regular G-orbit on $\mathfrak{g}$ exactly once, and does so transversally.*

Remark 2.4. If $\mathcal{F}$ is the space of fields in a gauge theory and G is the gauge group, then the space of physical fields is $\mathcal{F}/\!\!/G$. To do any computation in quantum gauge theory (e.g., in the BRST formalism), one often chooses a section of the quotient $\mathcal{F} \to \mathcal{F}/\!\!/G$. (Physicists often only do so locally, which is OK for *perturbative* calculations. However, it is generally impossible to choose such a section locally (as a mathematician would expect); in physics, this is known as a *Gribov ambiguity.*) One might therefore think of the Kostant section κ as analogous to gauge fixing (the choice of the nilpotent element f is a *particular* choice of gauge). In fact, this statement is literally true for some particular (quantum) gauge theories.

Remark 2.5. Another way of saying that the action map $N \times (f + \mathfrak{g}^e) \to f + \mathfrak{b}$ is an isomorphism is that the stacky quotient $(f + \mathfrak{b})/N$ is a scheme. (This is the same statement once you observe that this implies $(f + \mathfrak{b})/N$ must be affine by general principles, and then note that the GIT quotient is $f + \mathfrak{g}^e$.) How can this be proved? An alternate way of stating this fact is that the group cohomology of N in the representation given by $f + \mathfrak{b}$ is concentrated in degree 0. In other words: choose an invariant symmetric bilinear form on $\mathfrak{g}$, identify $\mathfrak{n}$ with $\mathfrak{n}^*$ under the resulting pairing, and thereby view f as an additive character $\psi : N \to \mathbf{G}_a$. The claim is then equivalent to the statement that $C^*(\mathfrak{n}; \psi \otimes U(\mathfrak{g}))$ is concentrated in degree 0.

Example 2.6. In general, $\mathfrak{g}/\!\!/G$ is isomorphic to an affine space of dimension $\dim(T)$. Let $G = \mathrm{SL}_2$, so that $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\mathfrak{g}/\!\!/G \cong \mathbf{C}$, and the map $\mathfrak{g} \to \mathfrak{g}/\!\!/G$ sends a traceless 2×2-matrix to its determinant. (If $G = \mathrm{SL}_n$, the map $\mathfrak{g} \to \mathfrak{g}/\!\!/G \cong \mathbf{C}^{n-1}$ sends a traceless $n \times n$-matrix to the nonzero coefficients of its characteristic polynomial.) The Kostant section $\mathbf{C} \to \mathfrak{g}$ sends $\lambda \in \mathbf{C}$ to the matrix $\left(\begin{smallmatrix} 0 & -\lambda \\ 1 & 0 \end{smallmatrix}\right)$, which evidently has determinant λ. More generally, for SL_n, one gets companion matrices.

Descending $I \to \mathfrak{g}$ to $\mathfrak{g}/\!\!/G$ is now easy: one can just restrict to the Kostant slice $\mathcal{S} \subseteq \mathfrak{g}$. Since this might be a bit opaque unless the reader is comfortable with the Kostant slice, let us unwind what this means.

Remark 2.7. Let $\chi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ be the quotient map, and let Z_G be the sheaf of groups on $\mathfrak{g}$ whose fiber over any $x \in \mathfrak{g}$ is $Z_G(x)$. By construction, J is characterized by the following two properties: it has a canonical G-equivariant map $\chi^* J \to Z_G$ of group schemes over $\mathfrak{g}$ which is an isomorphism over $\mathfrak{g}^{\mathrm{reg}}$.

Example 2.8. The above story goes through even if we only assume that G is reductive. Let $G = \mathrm{GL}_n$, so that the map $\mathfrak{gl}_n \to \mathfrak{gl}_n/\!\!/\mathrm{GL}_n \cong \mathbf{C}^n$ is given by taking coefficients of the characteristic polynomial (i.e., $x \mapsto \mathrm{coeff}(\chi_x(t))$). Then the fiber of $J \times_{\mathfrak{g}/\!\!/G} \mathfrak{g}$ is over $x \in \mathfrak{g}$ is the group of invertible elements in $\mathbf{C}[t]/\chi_x(t)$. There is a canonical (G-equivariant) map from this group to $Z_G(x)$ by the Cayley-Hamilton theorem (informally, $\chi_x(x) = 0$), which is an isomorphism when x is regular.

The description of the Kostant slice gives an alternative interpretation of $\mathfrak{g}/\!\!/G$. Namely, let us choose an invariant symmetric bilinear form on $\mathfrak{g}$, giving an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. Then $\mathfrak{g}^*$ admits a symplectic form, and the action of N_- on $\mathfrak{g}^*$ defines a moment map $\mu : \mathfrak{g}^* \to \mathfrak{n}_-^*$. (This is just the projection map dual to the inclusion $\mathfrak{n}_- \subseteq \mathfrak{g}$.) The nilpotent $f \in \mathfrak{n}_-$ dualizes to a character $\psi \in \mathfrak{n}_-^*$, and the resulting Hamiltonian reduction $\mathfrak{g}^* /\!\!/_\psi N_- := \mu^{-1}(\psi)/N_-$ is isomorphic to $\mathfrak{g}/\!\!/G$. This is just a restatement of the isomorphism $(f + \mathfrak{b})/N \xrightarrow{\sim} \mathfrak{g}/\!\!/G$.

Remark 2.9. If X is a symplectic N_--variety with moment map $\mu : X \to \mathfrak{n}_-^*$, the quotient/symplectic reduction $\mu^{-1}(\psi)/N_-$ is also known as the *Whittaker reduction* of X.

Since $J = I \times_{\mathfrak{g}} \mathcal{S}$, we see that each square in the following diagram is Cartesian:

$$\begin{tikzcd}
J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & G \times \mathfrak{g}^* \cong T^*G \arrow[d] \\
\mathcal{S} \arrow[r, hook] & \mathfrak{g}^* \arrow[r, "\Delta"'] & \mathfrak{g}^* \times \mathfrak{g}^*.
\end{tikzcd}$$

Using the fact that $\mathcal{S} \cong \mu^{-1}(\psi)/\mathfrak{n}_-^*$, one can conclude that J is the Hamiltonian reduction of T^*G by the $N_- \times N_-$-action at the point (ψ, ψ). In other words:

Proposition 2.10. *The group scheme J is the bi-Whittaker reduction of T^*G by the adjoint $N_- \times N_-$-action.*

Being a Hamiltonian reduction, J itself admits a symplectic structure. Grant's talk next week will prove the following.

Proposition 2.11 (Bezrukavnikov-Finkelberg-Mirkovic [BFM05]). *Let $G^\vee$ denote the Langlands dual of G. Then there is an isomorphism $\mathcal{O}_J \cong \mathrm{H}_*^{G^\vee}(\mathrm{Gr}_{G^\vee}; \mathbf{C})$ of cocommutative coalgebras. Furthermore, $C_*^{G^\vee}(\mathrm{Gr}_{G^\vee}; \mathbf{C})$ admits the structure of an $\mathbf{E}_3$-algebra, so that $\mathrm{H}_*^{G^\vee}(\mathrm{Gr}_{G^\vee}; \mathbf{C})$ admits the structure of a 2-shifted Poisson algebra. The isomorphism with $\mathcal{O}_J$ respects the shifted Poisson structure (ignoring the gradings). Finally, there is an isomorphism*

$$\mathrm{H}_*^{G^\vee}(\mathrm{Gr}_{G^\vee}; \mathbf{C}) \cong \mathcal{O}_{T\times\mathfrak{t}^*}[\tfrac{e^\alpha-1}{\alpha^\vee}|\alpha \in \Phi]^W.$$

In other words, J is an affine blowup of T^*T at the locus cut out by $e^\alpha - 1$ and $\alpha^\vee$.

Remark 2.12. One can also prove that $\mathrm{Lie}(J) = T^*(\mathfrak{g}/\!\!/G)$ as commutative Lie algebras over $\mathfrak{g}/\!\!/G$.

Example 2.13. Let us describe an example. Suppose $G = \mathrm{PGL}_2$, so that $G^\vee = \mathrm{SL}_2$. Then the above theorem tells us that

$$J = \mathrm{Spec}(\mathbf{C}[t^{\pm 1}, \delta, \tfrac{t+t^{-1}}{\delta}]^{\mathbf{Z}/2}),$$

where $\mathbf{Z}/2$ acts by $t \mapsto t^{-1}$ and $\delta \mapsto -\delta$. (Note that $\frac{t+t^{-1}}{\delta} = t^{-1}\cdot\frac{t^2+1}{\delta}$.) The ring on the inside (forgetting the $\mathbf{Z}/2$-fixed points) is the ring of functions on the blowup of $\mathbf{A}^1 \times \mathbf{G}_m$ blown up at $(0, \pm 1)$, with the proper transform of $\delta = 0$ removed.

Proposition 2.10 suggests a quantization of J.

Definition 2.14. The *quantized universal regular centralizer* is defined as the quantum Hamiltonian reduction of $\mathcal{D}_G^\hbar$ by the adjoint $N_-\times N_-$-action, taken at the character $U_\hbar(\mathfrak{n}_-)\otimes U_\hbar(\mathfrak{n}_-) \to \mathbf{C}$ defined by ψ. Following [Gin18], we will denote this object by $\mathbf{W}_\hbar$. Note that the $\mathbf{C}[\![\hbar]\!]$-linear structure can be viewed as defining a filtration on $\mathbf{W} := \mathbf{W}_\hbar|_{\hbar=1}$.

Proposition 2.15 (Bezrukavnikov-Finkelberg [BF08]). *Let $G^\vee$ denote the Langlands dual of G. Then there is an isomorphism $\mathbf{W}_\hbar \cong \mathrm{H}_*^{G^\vee\rtimes\mathbf{C}^\times}(\mathrm{Gr}_{G^\vee}; \mathbf{C})$ of associative algebras in cocommutative coalgebras. Here, the parameter $\hbar$ in $\mathbf{W}_\hbar$ corresponds to the generator of $\mathrm{H}^*_{\mathbf{C}^\times}(*; \mathbf{C}) \cong \mathbf{C}[\![\hbar]\!]$.*

3. The Fourier transform

The main result is the following.

Theorem 3.1 (Ginzburg, Lonergan). *Let $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}}$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)$ spanned by those $\Lambda \rtimes W$-equivariant quasicoherent sheaves over $\mathfrak{t}^*_{k[\hbar]}$ whose pullback to $\mathfrak{t}^*_{k[\hbar]}$ descends to the GIT quotient $\mathfrak{t}^*_{k[\hbar]}/\!\!/W$. Then there is an equivalence $\mathrm{LMod}_{\mathbf{W}} \simeq \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}}$.*

There is an evident inclusion $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}} \hookrightarrow \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)$. By (3), the target is equivalent to $\mathrm{LMod}_{\mathcal{D}_T^\hbar\rtimes W} \simeq \mathrm{LMod}_{(\mathcal{D}_T^\hbar)^W}$. The equivalence of Theorem 3.1 should fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{LMod}_? \arrow[r, "\text{Morita}", "\sim"'] \arrow[d, "F'"] & \mathrm{LMod}_{\mathbf{W}_\hbar} \arrow[r] \arrow[d, "F"] & \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}} \arrow[d, hook] \\
\mathrm{LMod}_{\mathcal{D}_T^\hbar\rtimes W} \arrow[r, "\text{Morita}", "\sim"'] & \mathrm{LMod}_{(\mathcal{D}_T)^W} \arrow[r, "(3)", "\sim"'] & \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W).
\end{tikzcd}$$

We have not yet specified the functor F; in fact, its construction is rather indirect[1]. As indicated in the above diagram, the idea is to describe some object in the place denoted "?", which is Morita equivalent to $\mathbf{W}$, and characterize the image of the functor F'.

Before we describe "?", let us just unwind the essential image of $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}}$ in $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)$ under the Fourier equivalences on the bottom row of the above diagram. Namely, let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)$. Then the following are equivalent:

(a) $\mathcal{F}$ lives in $\mathrm{QCoh}(\mathfrak{t}^*_{k[\hbar]}/\Lambda \rtimes W)^{\mathrm{Weyl-desc}}$.
(b) Use the same symbol to denote the image of $\mathcal{F}$ in $\mathrm{LMod}_{(\mathcal{D}_T^\hbar)^W}$. Then the following map (induced by the W-equivariant inclusion $\mathrm{Sym}(\mathfrak{t}) \subseteq \mathcal{D}_T^\hbar$) is an isomorphism:

$$\mathrm{Sym}(\mathfrak{t}) \otimes_{(\mathrm{Sym}\,\mathfrak{t})^W} \mathcal{F} \xrightarrow{\sim} \mathcal{D}_T^\hbar \otimes_{(\mathcal{D}_T^\hbar)^W} \mathcal{F}.$$

[1]In his paper, Ginzburg says he is not aware of a direct construction of a map $(\mathcal{D}_T^\hbar)^W \to \mathbf{W}_\hbar$, if one takes the definition of $\mathbf{W}_\hbar$ to be the quantum bi-Whittaker reduction from Definition 2.14.

(c) Use the same symbol to denote the image of $\mathcal{F}$ in $\mathrm{LMod}_{\mathcal{D}_T^\hbar \rtimes W}$. Then the following map (induced by the W-equivariant inclusion $\mathrm{Sym}(\mathfrak{t}) \subseteq \mathcal{D}_T^\hbar$) is an isomorphism:

$$\mathrm{Sym}(\mathfrak{t}) \otimes_{(\mathrm{Sym}\,\mathfrak{t})^W} M^W \xrightarrow{\sim} M. \tag{4}$$

To summarize:

Desiderata 3.2. We wish to define an algebra "?" such that "?" is Morita equivalent to $\mathbf{W}_\hbar$, there is a map $\mathcal{D}_T^\hbar \rtimes W \to$ "?″ which induces a forgetful functor $\mathrm{LMod}_? \to \mathrm{LMod}_{\mathcal{D}_T^\hbar \rtimes W}$ whose image is characterized by part (c) above.

It turns out that Kostant and Kumar's *affine nil-Hecke algebra* $\mathbf{H}_\hbar$ satisfies these properties.

Definition 3.3. Let $I^\vee \subseteq G^\vee(\mathcal{O})$ be the Iwahori subgroup associated to the Borel $B^\vee \subseteq G^\vee$; then the affine flag variety is defined to be $\mathcal{F}\ell^\vee = G^\vee(\mathcal{O})/I^\vee$.

Kostant and Kumar computed $\mathbf{H}_\hbar := \mathrm{H}_*^{I^\vee \rtimes \mathbf{C}^\times}(\mathcal{F}\ell^\vee; \mathbf{C})$. We will delay describing it explicitly for the moment.

Remark 3.4. Note that $\mathbf{H}_\hbar$ has a left and right action of W, which geometrically comes from the fact that there is a canonical map $\mathcal{F}\ell^\vee \to \mathrm{Gr}_{G^\vee}$ which exhibits $\mathcal{F}\ell^\vee$ as a $G^\vee/B^\vee$-bundle over the affine Grassmannian. This implies that if $e = \frac{1}{|W|}\sum_{w\in W} w \in \mathbf{C}[W]$, then there is an isomorphism

$$e\mathbf{H}_\hbar e \cong \mathbf{W}_\hbar = \mathrm{H}_*^{G^\vee \rtimes \mathbf{C}^\times}(\mathrm{Gr}_{G^\vee}; \mathbf{C}).$$

The subalgebra of $\mathbf{H}_\hbar$ defined by $e\mathbf{H}_\hbar e$ is called the *spherical subalgebra*. Moreover,

$$\mathrm{H}_*^{T^\vee}(\mathcal{F}\ell^\vee; \mathbf{C}) \cong \mathrm{H}_*^{I^\vee}(\mathcal{F}\ell^\vee; \mathbf{C}) \cong \mathbf{H}_\hbar|_{\hbar=0} = \mathcal{O}_{T\times\mathfrak{t}^*}[\tfrac{e^\alpha - 1}{\alpha^\vee} | \alpha \in \Phi] \rtimes W.$$

This can be proved in several ways; in fact, one approach uses an ind-version of the Goresky-Kottwitz-MacPherson recipe for computing torus-equivariant homology of certain varieties, and it implies the Bezrukavnikov-Finkelberg-Mirkovic calculation from above. This requires knowing the fixed point set $(\mathcal{F}\ell^\vee)^{T^\vee}$, which is $\Lambda \rtimes W$, as well as the 1-dimensional $T^\vee$-orbits. Since Grant may take this approach to proving Proposition 2.11, we will not go into further details.

Observation 3.5. There is a canonical inclusion $\mathcal{O}_{T^*T} \rtimes W = \mathcal{O}_{T\times\mathfrak{t}^*} \rtimes W \hookrightarrow \mathbf{H}_\hbar|_{\hbar=0}$. This quantizes to an inclusion $\mathcal{D}_T^\hbar \rtimes W \hookrightarrow \mathbf{H}_\hbar$; this is the second piece of Desiderata 3.2).

Remark 3.6. The algebras $\mathbf{H}_\hbar$ and $\mathbf{W}_\hbar$ are Morita equivalent (so $\mathbf{H}_\hbar$ satisfies the first piece of Desiderata 3.2). In fact, there is an explicit $(\mathbf{H}_\hbar, \mathbf{W}_\hbar)$-bimodule which witnesses this equivalence, called the "Miura bimodule". As discussed in [Gin18, Section 6.2], one explicit description of this is $\mathrm{Sym}(\mathfrak{t}) \otimes_{Z(U(\mathfrak{g}))} \mathbf{W}_\hbar$, which is *a priori* only a $(\mathcal{D}_T^\hbar \rtimes W, \mathbf{W}_\hbar)$-bimodule. However, using the general criterion of Proposition 3.7 below, one can extend this to a $(\mathbf{H}_\hbar, \mathbf{W}_\hbar)$-bimodule.

The only thing that remains is the third part of Desiderata 3.2:

Proposition 3.7. *Let M be a $\mathcal{D}_T^\hbar \rtimes W$-module. Then the map* (4) *is an isomorphism if and only if the $\mathcal{D}_T^\hbar \rtimes W$-action on M extends along the map $\mathcal{D}_T^\hbar \rtimes W \hookrightarrow \mathbf{H}_\hbar$.*

The basic idea is to use an explicit presentation for $\mathbf{H}_\hbar$, i.e., unwinding the phrase "affine nil-Hecke algebra". Let us begin by exploring consequences of the map (4) being an isomorphism.

Construction 3.8. Let $\mathcal{H}(W)$ denote the *nil-Hecke algebra*, defined to be the $\mathbf{C}$-algebra with generators t_α for $\alpha \in \Delta$, such that

$$t_\alpha^2 = 0,\ (t_\alpha t_\beta)^{m_{\alpha,\beta}} = (t_\beta t_\alpha)^{m_{\alpha,\beta}} \text{ for all } \alpha, \beta \in \Delta.$$

Here, $m_{\alpha,\beta}$ is the order of $s_\alpha s_\beta \in W$.

Let $\alpha \in \Phi^\vee$ be a coroot. Define $\theta_\alpha = \frac{s_\alpha - 1}{\alpha^\vee} \in \mathrm{Frac}(\mathrm{Sym}(\mathfrak{t})) \rtimes W$. Then there is a map[2] $\mathcal{H}(W) \to \mathrm{Frac}(\mathrm{Sym}(\mathfrak{t})) \rtimes W$ sending $t_\alpha \mapsto \theta_\alpha$, and one defines $\mathcal{H}(\mathfrak{t}, W)$ to be the free left $\mathrm{Sym}(\mathfrak{t})$-submodule of $\mathrm{Frac}(\mathrm{Sym}(\mathfrak{t})) \rtimes W$ with basis θ_w for $w \in W$. Kumar showed that $\mathcal{H}(\mathfrak{t}, W)$ is generated by $\mathcal{H}(W)$ and $\mathrm{Sym}(\mathfrak{t})$ subject to

$$\theta_\alpha \cdot s_\alpha(x) - x \cdot \theta_\alpha = \langle \alpha, x \rangle \text{ for all } x \in \mathfrak{t}, \alpha \in \Delta.$$

[2]To make sure this map is well-defined, we need to check that the θ_α satisfy the relations in the nil-Hecke algebra. For instance,

$$\theta_\alpha^2 = \left(\frac{s_\alpha - 1}{\alpha^\vee}\right) = \frac{1}{\alpha^\vee} s_\alpha \left(\frac{s_\alpha}{\alpha^\vee}\right) - \frac{1}{\alpha^\vee} s_\alpha \left(\frac{1}{\alpha^\vee}\right) - \frac{1}{\alpha^\vee}\frac{s_\alpha}{\alpha^\vee} + \frac{1}{(\alpha^\vee)^2}.$$

But the first and last terms cancel, since $s_\alpha\left(\frac{s_\alpha}{\alpha^\vee}\right) = -\frac{1}{\alpha^\vee}$ owing to $s_\alpha^2 = 1$ and $s_\alpha(\alpha^\vee) = -\alpha^\vee$. Similarly, the second and third terms cancel, so $\theta_\alpha^2 = 0$. The other relation is checked similarly.

Remark 3.9. One can then prove using the finiteness of W that $\mathcal{H}(\mathfrak{t}, W)$ is isomorphic as an algebra to $\mathrm{End}_{(\mathrm{Sym}\,\mathfrak{t})^W}(\mathrm{Sym}(\mathfrak{t}))$. By Chevalley-Shepard-Todd, $\mathrm{Sym}(\mathfrak{t})$ is a free $(\mathrm{Sym}(\mathfrak{t}))^W$-module (of finite rank); therefore, $\mathcal{H}(\mathfrak{t}, W)$ is a finite-dimensional matrix algebra over $\mathrm{Sym}(\mathfrak{t})^W$, and hence is Morita equivalent to $\mathrm{Sym}(\mathfrak{t})^W$. General principles of Morita theory now tell us that for a $\mathrm{Sym}(\mathfrak{t}) \rtimes W$-module M, the following are equivalent:

(a) the map (4) is an isomorphism;
(b) the $\mathrm{Sym}(\mathfrak{t}) \rtimes W$-action on M extends to an action of $\mathcal{H}(\mathfrak{t}, W)$.

Let us return to Proposition 3.7. Suppose that M is a $\mathcal{D}_T^\hbar \rtimes W$-module such the map (4) is an isomorphism. The above remark tells us that the action of $\mathrm{Sym}(\mathfrak{t}) \rtimes W$ on M extends to an action of $\mathcal{H}(\mathfrak{t}, W)$. This essentially finishes our task, as we now explain. For simplicity, let us set $\hbar = 0$ (it is a bit more difficult to argue when $\hbar \neq 0$). Then $\mathcal{D}_T^\hbar|_{\hbar=0} = \mathcal{O}_{T\times\mathfrak{t}^*}$, and $\mathbf{H}_\hbar|_{\hbar=0}$ is $\mathcal{O}_{T\times\mathfrak{t}^*}[\frac{e^\alpha-1}{\alpha^\vee}|\alpha\in\Phi] \rtimes W$. Given a $\mathcal{O}_{T\times\mathfrak{t}^*} \rtimes W$-module M such that (4) is an isomorphism, we need to describe how $\frac{e^\alpha-1}{\alpha^\vee}$ acts on M.

We already know that $\theta_\alpha = \frac{s_\alpha-1}{\alpha^\vee}$ acts on M by the preceding discussion. If $\lambda \in \Lambda$, let e^λ denote the function on T associated to λ. Then we have

$$e^\lambda s_\alpha e^{-\lambda} s_\alpha = e^{\langle\lambda,\alpha^\vee\rangle\alpha}$$

for $\alpha \in \Phi$. This implies that

$$\begin{aligned}\frac{e^{\langle\mu,\alpha^\vee\rangle\alpha}-1}{\alpha^\vee} &= \frac{e^{\langle\mu,\alpha^\vee\rangle\alpha}-1}{\alpha^\vee} + \frac{s_\alpha-1}{\alpha^\vee} \\ &= e^\lambda \frac{s_\alpha-1}{\alpha^\vee} e^{-\lambda} s_\alpha + \frac{s_\alpha-1}{\alpha^\vee} \\ &= e^\lambda \theta_\alpha e^{-\lambda} s_\alpha + \theta_\alpha.\end{aligned}$$

It follows that once we know that the action of $\mathrm{Sym}(\mathfrak{t}) \rtimes W$ on M extends to an action of $\mathcal{H}(\mathfrak{t}, W)$, we can use the action of $e^\lambda \in \mathcal{O}_T$ and the resulting action of the $\theta_\alpha \in \mathcal{H}(\mathfrak{t}, W)$ on M to define how $\frac{e^\alpha-1}{\alpha^\vee}$ acts on M.

References

[BF08] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005.

[Gin18] V. Ginzburg. Nil-Hecke algebras and Whittaker $\mathcal{D}$-modules. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 137–184. Birkhäuser/Springer, Cham, 2018.

[Lon18] G. Lonergan. A Fourier transform for the quantum Toda lattice. *Selecta Math. (N.S.)*, 24(5):4577–4615, 2018.

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