Topological Hochschild homology, truncated Brown-Peterson spectra, and a topological Sen operator

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Abstract. In this article, we study the topological Hochschild homology of $\mathbf{E}_3$-forms of truncated Brown-Peterson spectra, taken relative to certain Thom spectra $X(p^n)$ (introduced by Ravenel and used by Devinatz-Hopkins-Smith in the proof of the nilpotence theorem). We prove analogues of Bökstedt's calculations $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{F}_p[\Omega S^3]$ and $\mathrm{THH}(\mathbf{Z}_p) \simeq \mathbf{Z}_p[\Omega S^3\langle 3\rangle]$. We also construct a topological analogue of the Sen operator of Bhatt-Lurie-Drinfeld, and study a higher chromatic extension. The behavior of these "topological Sen operators" is dictated by differentials in the Serre spectral sequence for Cohen-Moore-Neisendorfer fibrations.

Contents

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1. Introduction

1.1. Summary. Fix a prime p. A fundamental calculation of Bökstedt's [**Bö85**] says that $\pi_*\mathrm{THH}(\mathbf{F}_p)$ is isomorphic to a polynomial ring $\mathbf{F}_p[\sigma]$ with $|\sigma| = 2$. Recent work of Hahn-Wilson shows that this polynomiality phenomenon persists at higher heights, provided one works relative to MU instead of the sphere. Namely, [**HW20**, Theorem E] states that if $\mathrm{BP}\langle n\rangle$ is an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum, then $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is a polynomial algebra over $\pi_*\mathrm{BP}\langle n\rangle$ on generators in even degree. Moreover, the first such generator is the double suspension $\sigma^2(v_{n+1})$.

In this article, we will show that the "polynomial THH" phenomenon persists if one instead considers THH relative to the Ravenel spectra $X(p^n)$, introduced in [**Rav84**] and used by [**DHS88**] in the proof of the nilpotence theorem. Motivated by [**Dev23a**], the thesis of this article is that many statements involving the study of $\mathbf{F}_p$- or $\mathbf{Z}_p$-algebras relative to the sphere spectrum admit natural generalizations when studying $\mathrm{BP}\langle n-1\rangle$- or $\mathrm{BP}\langle n\rangle$-algebras relative to $X(p^n)$. Many of the results presented here were motivated by the perspective that there should be a chromatic analogue of integral p-adic Hodge theory (where p is replaced by the chromatic element v_n; see Figure 1)[1].

The $\mathbf{E}_2^{\mathrm{fr}}$-ring $X(p^n)$ is the Thom spectrum of the $\mathbf{E}_2^{\mathrm{fr}}$-map $\Omega\mathrm{SU}(p^n) \to \Omega\mathrm{SU} \simeq \mathrm{BU}$, so that $X(1) = S^0$ and $X(\infty) = \mathrm{MU}$. Just as $\mathrm{MU}_{(p)}$ splits as a direct sum of shifts of BP, the spectrum $X(p^n)_{(p)}$ splits into a direct sum of shifts of an $\mathbf{E}_1$-ring denoted[2] $T(n)$. If $\mathcal{C}$ is a left $X(p^n)$-linear ∞-category, then [**DHL$^+$23**, Corollary 2.9 and Corollary 3.7] ensures that it makes sense to define the relative topological Hochschild homology $\mathrm{THH}(\mathcal{C}/X(p^n))$, and furthermore that $\mathrm{THH}(\mathcal{C}/X(p^n))$ admits an S^1-action.[3]

Our main result is an analogue of Bökstedt's calculation. If R is a ring spectrum, let $R[B\Delta_n]$ denote the free R-module whose homotopy groups are isomorphic to a divided power algebra $\pi_*(R)\langle y_i | 1 \leq i \leq p^n - 1, i \neq p^k\rangle$ where $|y_j| = 2j$.[4] Morally, $R[B\Delta_n]$ is the R-chains on the "classifying space of $\prod_{i=1}^n \mathrm{SU}(p^i - 1)/\mathrm{SU}(p^{i-1})$"; so, if X is another space, we will write $R[B\Delta_n \times X]$ to denote $R[B\Delta_n] \otimes_R R[X]$. Fix an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum $\mathrm{BP}\langle n-1\rangle$ (which exists by [**HW20**, Theorem A]). Motivated by the results of [**Dev23a**], and using the calculations of [**AR05, BR05**], we show:

Theorem (Theorem 2.2.4(a))**.** *There is a p-complete equivalence*

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n-1\rangle[B\Delta_n \times \Omega S^{2p^n+1}]$$

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[1] I'd also like to direct the reader to `https://www.royalacademy.org.uk/art-artists/work-of-art/prismatic-colour-wheel`; but I hope our Figure 1 is more mathematically informative!

[2] This is *not* the telescope of a v_n-self map! See Warning 2.1.6.

[3] We warn the reader that even if $\mathcal{C}$ admits the structure of a monoidal ∞-category, $\mathrm{THH}(\mathcal{C}/X(p^n))$ rarely inherits any multiplicative structure from $\mathcal{C}$, since $X(p^n)$ does not admit the structure of an $\mathbf{E}_3$-ring (see Remark 2.1.3).

[4] The contribution $B\Delta_n$ plays essentially no practical/meaningful role in this article. Its appearance in the equivalences below can be removed if $T(n) \subseteq X(p^n)_{(p)}$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-algebra. We strongly believe this to be possible (enough to state it as Conjecture 2.1.9!), so we suggest the reader ignore $B\Delta_n$ — and simultaneously replace $X(p^n)$ by $T(n)$ — on a first pass.

of $\mathrm{BP}\langle n-1\rangle$*-modules; in particular, there is a p-complete isomorphism*

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \pi_*\mathrm{BP}\langle n-1\rangle[B\Delta_n][\theta_n],$$

where $\theta_n \in \pi_{2p^n}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ *is* $\sigma^2(v_n)$.

Moreover, there are p-complete isomorphisms

$$\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)) \cong \pi_*(\mathrm{BP}\langle n\rangle[B\Delta_n])[\![\hbar]\!][\theta_n]/(\theta_n\hbar - v_n),$$
$$\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \cong \pi_*(\mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n]),$$

where $\hbar \in \pi_{-2}\mathrm{BP}\langle n\rangle^{hS^1}$*. Under the map* $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$*, the image of* $v_n \in \pi_{2p^n-2}\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ *can be identified with the image of* $v_n \in \pi_{2p^n-2}\mathrm{MU}^{tS^1}$ *under the map* $\mathrm{MU}^{tS^1} \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$.

Remark 1.1.1. If $T(n) \subseteq X(p^n)_{(p)}$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-algebra (Conjecture 2.1.9), then Theorem 2.2.4 would give the cleaner statements that $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \simeq \mathrm{BP}\langle n-1\rangle[\Omega S^{2p^n+1}]$, and that $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/T(n)) \cong \pi_*\mathrm{BP}\langle n\rangle^{tS^1}$. The map $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \to \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$ is injective, and exhibits the source as the submodule $\pi_*\mathrm{BP}\langle n-1\rangle[\sigma^2(v_n)]$ of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$.

Theorem 2.2.4 implies the following result, which, for $n=0$, is a very special case of the main result of [**PV19**]:

Corollary (Proposition 3.3.8)**.** *Let* $R = \mathrm{BP}\langle n\rangle[\mathbf{Z}^j_{\geq 0}]$ *be a flat polynomial ring over* $\mathrm{BP}\langle n\rangle$*, viewed as a* $\mathbf{Z}^j_{\geq 0}$*-graded* $\mathbf{E}_2^{\mathrm{fr}}$-$\mathrm{BP}\langle n\rangle$*-algebra. Then there is a p-complete isomorphism of* $\mathbf{Z}^j_{\geq 0}$*-graded modules equipped with a map from* $\pi_*\mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n] \cong \pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$*:*

$$\pi_*\mathrm{TP}^{\mathrm{gr}}((R/v_n)/X(p^n)) \cong \pi_*\mathrm{HP}^{\mathrm{gr}}(R/\mathrm{BP}\langle n\rangle)[B\Delta_n].$$

Here, the superscript gr *denotes the Tate construction taken in* $\mathbf{Z}^j_{\geq 0}$-graded *spectra.*

Remark 1.1.2. Theorem 2.2.4 quickly implies redshift for $K(\mathrm{BP}\langle n-1\rangle)$ (see Corollary 2.2.9). When $n=0$, the first part of Theorem 2.2.4(a) recovers Bökstedt's calculation of $\mathrm{THH}(\mathbf{F}_p)$, since $\mathrm{BP}\langle -1\rangle = \mathbf{F}_p$ and $X(0) = S^0$. When $p=2$, the statement of Theorem 2.2.4 can be simplified using [**Dev23a**, Remark 3.1.9]; for instance, we obtain the following *additive* equivalences and isomorphisms: for $n=1$, we have

$$\mathrm{THH}(\mathbf{Z}_2/T(1)) \simeq \mathbf{Z}_2[\sigma^2(v_1)],\ \pi_*\mathrm{TP}(\mathbf{Z}_2/T(1))^\wedge_2 \simeq \pi_*(\mathrm{ku}^{tS^1})^\wedge_2.$$

Since $\mathrm{tmf}_1(3)$ is a form of $\mathrm{BP}\langle 2\rangle$ by [**LN14**], for $n=2$, we have

$$\mathrm{THH}(\mathrm{ku}_2^\wedge/T(2)) \simeq \mathrm{ku}_2^\wedge[\sigma^2(v_2)],\ \pi_*\mathrm{TP}(\mathrm{ku}_2^\wedge/T(2))^\wedge_2 \simeq \pi_*(\mathrm{tmf}_1(3)^{tS^1})^\wedge_2.$$

We also prove an analogue of Bökstedt's calculation [**Bö85**] of $\mathrm{THH}(\mathbf{Z}_p)$:

Theorem (Theorem 2.2.4(b))**.** *There is an equivalence of* $\mathrm{BP}\langle n\rangle$*-modules*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))^\wedge_p \cong \mathrm{BP}\langle n\rangle[B\Delta_n]^\wedge_p \oplus \left(\bigoplus_{j\geq 1} \Sigma^{2jp^{n+1}-1}\mathrm{BP}\langle n\rangle[B\Delta_n]/pj\right)^\wedge_p.$$

Moreover, $\pi_{2p^{n+1}-3}\mathrm{TC}^-(\mathrm{BP}\langle n\rangle/X(p^n))^\wedge_p$ *detects the class* $\sigma_n \in \pi_{2p^{n+1}-3}X(p^n)$ *from* [**Dev23a**, Lemma 3.1.12].

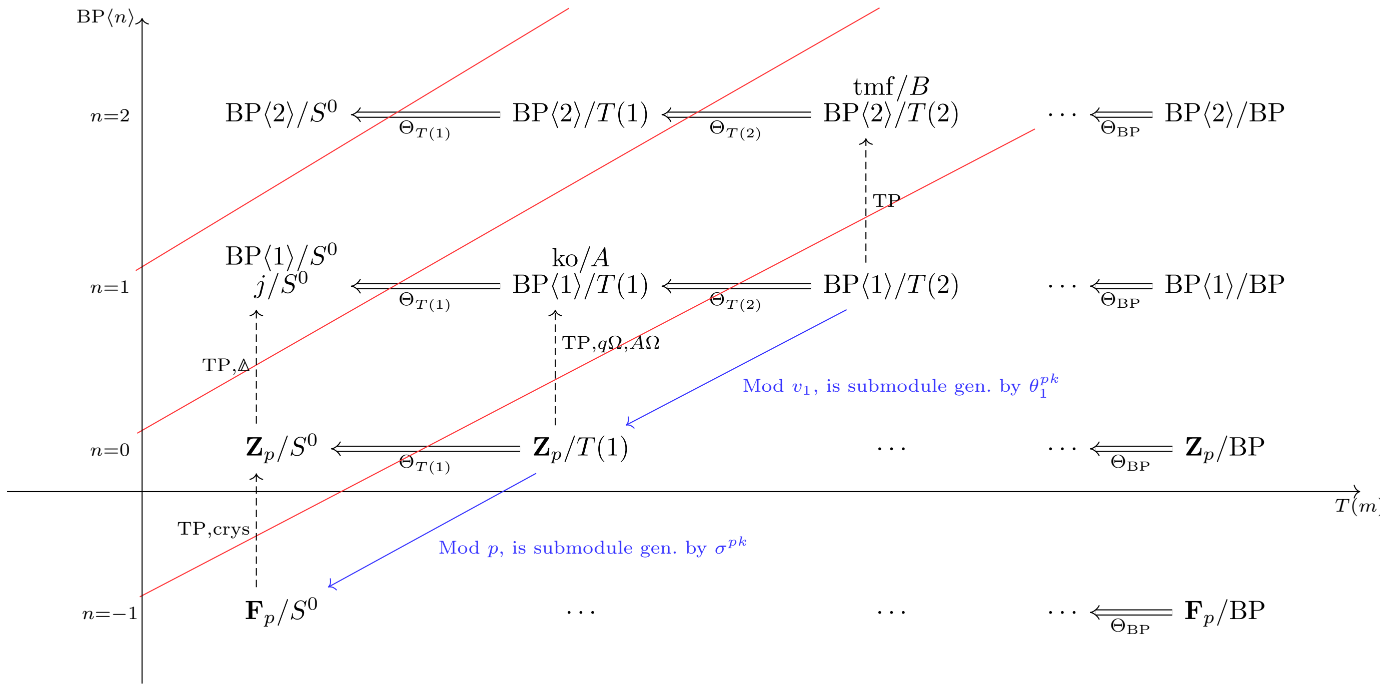

FIGURE 1. Heuristic picture suggested by this article, where we have assumed for simplicity that $T(m)$ admits the structure of a framed $\mathbf{E}_2$-ring.

• The spectra sandwiched between diagonal lines of slope 1 (partitioned by a red line) display similar structural behaviour. Here, A and B are studied in [**Mah79**] (where A is denoted X_5), [**Dev19**, Construction 3.1], and [**HM02**].

• The horizontal double arrows indicate the topological Sen operators of Theorem 3.1.4, i.e., the descent spectral sequence for the map $\mathrm{THH}(-/T(n-1)) \to \mathrm{THH}(-/T(n))$. This is closely related to the Cohen-Moore-Neisendorfer map $\Omega^2 S^{2p^n+1} \to S^{2p^n-1}$.

• The (slightly offset) vertical dashed lines going from $(n, n-1)$ to (n, n) indicate the p-completed isomorphism $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/T(n)) \cong \pi_*\mathrm{BP}\langle n\rangle^{tS^1}$ of Theorem 2.2.4. The other vertical arrow from $(0, 0)$ to $(0, 1)$ is the identification of $\mathrm{THH}(\mathbf{Z}_p)$ with $\tau_{\geq 0}(j^{t\mathbf{Z}/p})$, which will appear in future work with Arpon Raksit. (Here, j is the connective complex image-of-J spectrum.) This equivalence is already predicted by the pioneering work of Bökstedt-Madsen in [**BM94**].

• The downwards-sloping blue arrows indicate that $\mathrm{THH}(\mathrm{BP}\langle n\rangle/T(n+1))/v_n$ is a submodule of $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))$ generated by θ_n^{pk} for $k \geq 0$. See Example 4.2.2, Remark 4.2.4, and Example 4.2.6 for an explanation of this phenomenon using the EHP sequence.

• The columns continue infinitely far out (i.e., $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(m))$ for $m > n$). However, the drawing is truncated because these terms do not detect any more information than $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))$ itself. The "exception" is the final column, where the descent from $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{BP})$ to $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$ can be described algebro-geometrically via the p-typical Witt ring scheme.

Remark 1.1.3. If one replaces $X(p^n)$ in the left-hand side of Theorem 2.2.4(b) with $X(p^{n+1}-1)$, the only change to the right-hand side is that $B\Delta_n$ is replaced by $B\Delta_{n+1}$. Let us mention the following mild variant of Theorem 2.2.4 (see (4)): the $\mathbf{F}_p[v_{n-j}, \cdots, v_{n-1}]$-module $\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))/(p, \cdots, v_{n-1-j})$ is isomorphic to the tensor product of $\mathrm{BP}\langle n-1\rangle[\Omega S^{2p^n+1} \times B\Delta_j]/(p, \cdots, v_{n-1-j})_*$ with an exterior algebra on classes $\lambda_{j+1}, \cdots, \lambda_n$, where $|\lambda_m| = 2p^m - 1$. We also prove an analogue of Theorem 2.2.4 for ko and tmf in Appendix A. For example, if the spectra A and B [**Dev23a**, Section 3] lift to $\mathbf{E}_2^{\mathrm{fr}}$-rings, there are 2-complete equivalences

$$\mathrm{THH}(\mathrm{ko}/A) \simeq \mathrm{ko} \oplus \left(\bigoplus_{j \geq 1} \Sigma^{8j-1} \mathrm{ko}/2j \right),$$

$$\mathrm{THH}(\mathrm{tmf}/B) \simeq \mathrm{tmf} \oplus \left(\bigoplus_{j \geq 1} \Sigma^{16j-1} \mathrm{tmf}/2j \right).$$

Remark 1.1.4. If Conjecture 2.1.9 (or rather, a weaker version which only asks that $T(n)$ admit the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring) were true, then the contribution of $B\Delta_n$ could be eliminated from Theorem 2.2.4(b): namely, there would be a p-complete equivalence

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/T(n)) \simeq \mathrm{BP}\langle n\rangle \oplus \bigoplus_{j \geq 1} \Sigma^{2jp^{n+1}-1} \mathrm{BP}\langle n\rangle/pj.$$

We warn the reader that all the equivalences proved above are only additive, so one cannot directly use them to study the stacks associated to THH (defined via the even filtration of [**HRW22**]). As a perhaps more digestible example of this phenomenon (see Remark 2.3.16), note that since $\mathbf{F}_2$ is the Thom spectrum of an $\mathbf{E}_1$-map $\mathrm{U}(2) \to \mathrm{BGL}_1(\mathrm{ku})$, there is an equivalence $\mathrm{HH}(\mathbf{F}_2/\mathrm{ku}) \simeq \mathbf{F}_2[\mathrm{BU}(2)]$; however, this cannot be upgraded to an equivalence of $\mathbf{F}_2$-algebras, since the right-hand side is not even obviously a ring!

In Example 4.2.2, Remark 4.2.4, and Example 4.2.6, we use the EHP sequence to explain the similarity in the calculation of $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^{n+1}))$ and $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ given by Theorem 2.2.4. This discussion in fact yields the following more general structural uniformity in the truncated Brown-Peterson spectra (see Figure 1 for a visual illustration):

Slogan 1.1.5 (Remark 4.2.3 and Remark 4.2.5 for precise statements)**.** If $n \geq j-1$, the structure of $\mathrm{BP}\langle n\rangle$ as an $\mathbf{E}_1$-$X(p^j)$-algebra (i.e., $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^j))$) mirrors the structure of $\mathrm{BP}\langle n-1\rangle$ as an $\mathbf{E}_1$-$X(p^{j-1})$-algebra (i.e., $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^{j-1}))$), which in turn mirrors the structure of $\mathrm{BP}\langle n-j\rangle$ as an $\mathbf{E}_1$-algebra over the sphere (i.e., $\mathrm{THH}(\mathrm{BP}\langle n-j\rangle)$).

Let $\mathcal{C}$ be a left $X(p^n)$-linear ∞-category. Then, the descent spectral sequence from THH relative to $X(p^n)$ to THH relative to $X(p^n-1)$ runs:

Theorem (Theorem 3.1.4)**.** *There is a map* $\Theta_{\mathcal{C}} : \Sigma^{-2p^n} \mathrm{THH}(\mathcal{C}/X(p^n)) \to \mathrm{THH}(\mathcal{C}/X(p^n))$ *such that there is a cofiber sequence*

$$\mathrm{THH}(\mathcal{C}/X(p^n-1)) \xrightarrow{\iota} \mathrm{THH}(\mathcal{C}/X(p^n)) \xrightarrow{\Theta_{\mathcal{C}}} \Sigma^{2p^n} \mathrm{THH}(\mathcal{C}/X(p^n)), \tag{1}$$

where the map ι *is* S^1*-equivariant, and the cofiber of* ι *is (at least nonequivariantly) identified with* $\Sigma^{2p^n} \mathrm{THH}(\mathcal{C}/X(p^n))$.

Remark 1.1.6. Motivated by [**BL22a, Dri22**], we dub the map $\Theta_{\mathcal{C}}$ the *topological Sen operator*; its construction is motivated by the work of [**Dev23a**] relating $\mathrm{BP}\langle n\rangle$ to Cohen-Moore-Neisendorfer type fiber sequences (11). When $\mathcal{C} = \mathrm{LMod}_{\mathrm{BP}\langle n-1\rangle}$, Theorem 2.2.4 implies that the map Θ sends

$$\Theta : \theta_n^j \mapsto jp\theta_n^{j-1}.$$

When $n = 1$, it therefore behaves like the Sen operator on the diffracted Hodge complex of $\mathbf{Z}_p$ which computes $\overline{\mathbb{\Delta}}_{\mathbf{Z}_p}\{*\}$.

Remark 1.1.7. In Appendix A (see Remark A.24), we describe a quaternionic analogue of (1), obtained by replacing $X(n)$ by the Thom spectrum $X_{\mathbf{H}}(n)$ of the tautological symplectic bundle over $\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n))$ obtained via the map $\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)) \to \Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp}$ given by Bott periodicity.

In Construction 2.3.1, we define an $\mathbf{E}_2^{\mathrm{fr}}$-ring $J(p)$ which admits an $\mathbf{E}_2^{\mathrm{fr}}$-map $J(p) \to X(p)$ such that $\mathrm{THH}(T(1)/J(p)) \simeq T(1)[J_{p-1}(S^2)]$. The underlying $\mathbf{E}_1$-ring of $J(p)$ is $S[\mathbf{Z}] = S[t^{\pm 1}]$ with $|t| = 0$, but they differ as $\mathbf{E}_2^{\mathrm{fr}}$-rings. The *raison d'être* for $J(p)$ is that $\mathrm{THH}(\mathbf{Z}_p/J(p))$ is polynomial on a class x in degree 2 which is a pth root of $\theta \in \pi_{2p}\mathrm{THH}(\mathbf{Z}_p/X(p))$. More precisely, there is an equivalence $\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[\Omega S^3]$ such that the map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/X(p))$ is induced by $\mathbf{Z}_p$-chains of the Hopf map $\Omega S^3 \to \Omega S^{2p+1}$. In Construction 2.3.9, we also construct two $\mathbf{E}_2^{\mathrm{fr}}$-rings (as Thom spectra over $\Omega\mathrm{U}(2)$ and $\Omega\mathrm{Spin}(4)$) which play the role of $J(p)$ for ku when $p = 2$.

We construct the following cofiber sequence analogous to (1) for any $J(p)$-linear ∞-category $\mathcal{C}$:

$$\mathrm{THH}(\mathcal{C}) \xrightarrow{\iota} \mathrm{THH}(\mathcal{C}/J(p)) \xrightarrow{\Theta'_{\mathcal{C}}} \Sigma^2\mathrm{THH}(\mathcal{C}/J(p)).$$

It turns out that upon reducing the above cofiber sequence mod p, one obtains the following important example:

Example 1.1.8. If $\mathcal{C}$ is a $\mathbf{Z}_p$-linear ∞-category, there is a cofiber sequence (see Variant 3.1.10)

$$\mathrm{THH}(\mathcal{C}) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \xrightarrow{\iota} \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \xrightarrow{\Theta'} \Sigma^2\mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p). \tag{2}$$

When $\mathcal{C} = \mathrm{Mod}_{\mathbf{Z}_p}$, the effect of the map Θ' on homotopy is given by the map $\mathbf{F}_p[\sigma] \to \Sigma^2\mathbf{F}_p[\sigma]$ which sends $\sigma^j \mapsto j\sigma^{j-1}$. There is also a cofiber sequence

$$\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p \xrightarrow{\iota} \mathrm{HP}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p/\mathbf{F}_p) \xrightarrow{\Theta'} \mathrm{HP}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p/\mathbf{F}_p). \tag{3}$$

If $\mathcal{C} = \mathrm{Mod}_R$ for an animated $\mathbf{Z}_p$-algebra R, we expect the maps in (2) and (3) to respect the motivic filtrations. Taking $\mathrm{gr}^i_{\mathrm{mot}}[-2i]$ would then produce the following cofiber sequences involving the associated graded pieces of the Nygaard filtration on the prismatic cohomologies of R and R/p:

$$(\mathcal{N}^i\hat{\mathbb{\Delta}}_R)/p \to \mathrm{F}_i^{\mathrm{conj}}\mathrm{dR}_{(R/p)/\mathbf{F}_p} \to \mathrm{F}_{i-1}^{\mathrm{conj}}\mathrm{dR}_{(R/p)/\mathbf{F}_p},$$
$$\overline{\mathbb{\Delta}}_R/p \to \mathrm{dR}_{(R/p)/\mathbf{F}_p} \to \mathrm{dR}_{(R/p)/\mathbf{F}_p}.$$

Such cofiber sequences on Hodge-Tate cohomology do indeed exist, and can be constructed purely algebraically using the methods of [**BM22**] and [**BL22a**, Proposition 6.4.8]; see (19) and (21).

We also show by explicit calculation:

Proposition (Example 3.3.3 and Proposition 3.3.11 for precise statements)**.** *There is an isomorphism* $\pi_* \mathrm{TP}(\mathbf{Z}_p[t]/X(p)) \cong \pi_* \mathrm{HP}(\mathrm{BP}\langle 1\rangle[t]/\mathrm{BP}\langle 1\rangle)$.

Furthermore, the map $\mathrm{TP}^{\mathrm{gr}}(\mathrm{BP}\langle n-1\rangle[t]/X(p^n)) \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ *is an equivalence after* $K(n)$*-localization, and Conjecture 2.2.18 implies that (up to a Nygaard-type completion)* $L_{K(n)}\mathrm{TP}(-/X(p^n))$ *is* $\mathbf{A}^1$*-invariant.*

We also have:

Conjecture (Conjecture 3.1.14 and Conjecture 3.3.5)**.** *Let* R *be an animated* $\mathbf{Z}_p$*-algebra, and let* $\mathrm{F}_\star^{\mathrm{conj}}\widehat{\Omega}_R^{\not D}$ *denote the conjugate-filtered (p-completed) diffracted Hodge complex of* [**BL22a**, Construction 4.7.1]. *Then* $\mathrm{THH}(R/J(p))$ *admits a motivic filtration such that* $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/J(p)) \simeq (\mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}_R^{\not D})[2i]$*, and such that the map* $\Theta'_R : \mathrm{THH}(R/J(p)) \to \Sigma^2\mathrm{THH}(R/J(p))$ *respects the motivic filtration and induces the map* $\Theta + i : \mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}_R^{\not D} \to \mathrm{F}_{i-1}^{\mathrm{conj}}\widehat{\Omega}_R^{\not D}$ *on* $\mathrm{gr}^i_{\mathrm{mot}}$.

Similarly, $\mathrm{THH}(R/X(p))$ *admits a motivic filtration such that* $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/X(p)) \simeq (\mathrm{F}_{pi}^{\mathrm{conj}}\widehat{\Omega}_R^{\not D})[2pi] \otimes_R R[\mathrm{BSU}(p-1)]$*. Moreover,* $\mathrm{TP}(R/X(p))$ *admits a motivic filtration* $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TP}(R/X(p))$ *such that* $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TP}(R/X(p)) \simeq \hat{\mathbb{\Delta}}_{R/\mathbf{Z}_p[\![\widetilde{p}]\!]}[2i] \otimes_R \epsilon^R$*, where* $\hat{\mathbb{\Delta}}_{R/\mathbf{Z}_p[\![\widetilde{p}]\!]}$ *is the Nygaard completion of* $\widetilde{p}\Omega_R$.

In Section 3, we supplement Conjecture 3.1.14 with some examples (such as R being a p-complete perfectoid ring, $R = \mathbf{Z}/p^n$ for odd p, R being a complete DVR of mixed characteristic $(0,p)$, and $R = \mathbf{Z}_p[t]$).

Remark 1.1.9. For the case $R = \mathbf{Z}/p^n$, we give "two" calculations of the diffracted Hodge complex $\widehat{\Omega}^{\not D}_{\mathbf{Z}/p^n}$; one uses abstract properties of the diffracted Hodge complex (and was explained to us by Bhatt), and the other (provided in Appendix B) is via concrete calculations in the ring $W(\mathbf{Z}_p)$. In particular, in Corollary 3.2.15, we refine the calculation of [**BL22b**, Example 5.15] to show that there is an equivalence $\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}/p^n} \cong \mathbf{G}_a^\sharp/\mathbf{G}_m^\sharp$ of stacks over $\mathbf{Z}/p^n$.

In Section 3.4, we also study an analogue of the Segal conjecture for THH relative to $J(p)$ and $T(n)$. One interesting consequence (Proposition 3.4.7) is that if R is a p-torsionfree discrete commutative ring such that R/p is regular Noetherian and $L\Omega^n_R = 0$ for $n \gg 0$, then [**BL22a**, Remark 4.7.4] and Conjecture 3.1.14 imply that R satisfies a version of the Segal conjecture for THH relative to $J(p)$.

In Proposition 3.5.3, we prove an analogue of the Cartier isomorphism in Hochschild homology for a flat polynomial algebra over any $\mathbf{E}_2$-ring, and show that it specializes to homotopical analogues of several known examples of the Cartier isomorphism. (This is quite likely well-known to some experts, but we could not find a source.)

Proposition (Proposition 3.5.3)**.** *Let* R *be an* $\mathbf{E}_2$*-ring. Then there is a* S^1*-equivariant map* $\mathfrak{C} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) \to \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ *sending* $t \mapsto t^p$*, where* S^1 *acts on* $\mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ *via the residual* S^1/μ_p*-action, and on* $\mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p})$ *via the diagonal action on Hochschild homology and residual* S^1/μ_p*-action on* $R^{t\mathbf{Z}/p}$*. If* t *is given weight* 1*, then* $\mathfrak{C}$ *induces an* S^1*-equivariant equivalence* $\mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p})_{\mathrm{wt}\leq m} \to (\mathrm{HH}(R[t]/R)_{\mathrm{wt}\leq mp})^{t\mathbf{Z}/p}$ *of graded* $R^{t\mathbf{Z}/p}$*-modules.*

In Section 4, we describe the topological Sen operator from the perspective of the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups. We begin by describing an algebraic analogue of THH. This is given by an Adams-Novikov analogue of the Bökstedt spectral sequence: if R is a p-local homotopy commutative ring such that $\mathrm{MU}_*(R)$ is concentrated in even degrees, one can define a stack $\mathcal{M}_R$ whose coherent cohomology is the E_2-page of the Adams-Novikov spectral sequence for R (see [**DFHH14**, Chapter 9]).

Proposition (Remark 4.1.5)**.** *If* $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(R)$ *denotes the associated graded of the even filtration of* [**HRW22**] *on* $\mathrm{THH}(R)$ *(which recovers the motivic filtration on* $\mathrm{THH}(R)$ *of* [**BMS19**] *when R is quasisyntomic), then there is a spectral sequence:*

$$\pi_*\mathrm{HH}(\mathcal{M}_R/\mathcal{M}_{\mathrm{FG}}) \Rightarrow \pi_*\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(R).$$

There is also an analogue for relative THH*.*

This spectral sequence behaves essentially like the Bökstedt spectral sequence in most examples. In particular, if $R \to R'$ is a map of p-local homotopy commutative rings whose MU-homologies are concentrated in even degrees, then $\mathrm{HH}(\mathcal{M}_R/\mathcal{M}_{R'})$ can be viewed as the "Adams-Novikov-Bökstedt associated graded" of $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(R/R')$. Motivated by this perspective, we describe an analogue of the topological Sen operator of Theorem 3.1.4 as a Gauss-Manin connection on stacks related to $\mathcal{M}_{\mathrm{FG}}$ (see Example 4.1.11):

Theorem (Example 4.1.11 and Variant 4.1.13)**.** *The stack $\mathcal{M}_{T(n)}$ is isomorphic to the moduli stack of graded p-typical formal groups equipped with a p-typical coordinate of order $\leq p^n$. Moreover, the Adams-Novikov analogue of Theorem 3.1.4 is a fiber sequence*

$$\mathrm{HH}(X/\mathcal{M}_{T(n-1)}) \to \mathrm{HH}(X/\mathcal{M}_{T(n)}) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{2p^n,p^n}\mathrm{HH}(X/\mathcal{M}_{T(n)})$$

associated to any stack $X \to \mathcal{M}_{T(n)}$, where $\Sigma^{n,w}$ denotes a shift by homological degree n and weight w.

Similarly, there is a fiber sequence

$$\mathrm{HH}(X/\mathcal{M}_{\mathrm{FG}}) \to \mathrm{HH}(X/\mathcal{M}_{J(p)}) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{2,1}\mathrm{HH}(X/\mathcal{M}_{J(p)})$$

associated to any stack $X \to \mathcal{M}_{J(p)}$.

Remark 1.1.10. In Appendix A (Proposition A.23 and Remark A.24), we also study a quaternionic analogue of the above fiber sequence. This description crucially relies on the twistor fibration $\mathbf{C}P^{2n-1} \to \mathbf{H}P^{n-1}$, which is given in coordinates by the map $[z_1 : \cdots : z_{2n}] \mapsto [z_1 + z_2\mathbf{j} : \cdots : z_{2n-1} + z_{2n}\mathbf{j}]$.

1.2. Some complements. In Conjecture 2.2.18, we suggest that the identification of $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ can be extended to an equivalence $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n]$ of spectra:

Conjecture (Conjecture 2.2.18)**.** *The spectrum* $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ *admits the structure of an S^1-equivariant* $\mathrm{BP}\langle n\rangle$*-module, and the isomorphism* $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/T(n)) \cong \pi_*\mathrm{BP}\langle n\rangle^{tS^1}$ *lifts to an equivalence of spectra* $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/T(n)) \simeq \mathrm{BP}\langle n\rangle^{tS^1}$*.*

This discussion suggests viewing the pair $(\pi_0\mathrm{BP}\langle n\rangle^{tS^1}, (\frac{[p](\hbar)}{\hbar}))$ as a higher chromatic analogue of the crystalline prism $(\mathbf{Z}_p, (p))$, where $\hbar$ is the complex orientation of $\mathrm{BP}\langle n\rangle$ and $[p](\hbar)$ is its p-series. Note that the pair $(\pi_0\mathrm{BP}\langle n\rangle^{tS^1}, (\frac{[p](\hbar)}{\hbar}))$ has no reason to naturally admit the structure of a prism.

Finally, it would be interesting to know whether Slogan 1.1.5 can be used to prove [**Lee22**, Conjecture 6.1]. A first step in this direction would be to show that the topological Sen operators on $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^j))$, $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^{j-1}))$, ..., and $\mathrm{THH}(\mathrm{BP}\langle n-j\rangle)$ can also be matched up under the structural uniformity of Slogan 1.1.5. (Also see Remark 2.2.5.)

This article suggests several directions in which the work presented here can be extended; we have recorded these as Conjecture 2.1.9, Conjecture 2.2.18, Conjecture 2.3.22, Conjecture 3.1.14, the closely related Conjecture 3.3.5, Conjecture 3.3.16, and Conjecture A.2. We wish to emphasize that, unlike [**Dev23a**, Theorem A and Corollary B], the main results of this article are entirely unconditional, and can be viewed as (in our opinion, substantial) evidence for the conjectures presented here and in [**Dev23a**].

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2. Calculation of THH

2.1. Review of $X(p^n)$.

Definition 2.1.1 (Ravenel, [**Rav84**, Section 3])**.** Let $X(n)$ denote the Thom spectrum of the $\mathbf{E}_2$-map $\Omega\mathrm{SU}(n) \subseteq \mathrm{BU} \xrightarrow{J} B\mathrm{GL}_1(S)$, where the first map arises from Bott periodicity.

Example 2.1.2. The $\mathbf{E}_2$-ring $X(1)$ is the sphere spectrum, while $X(\infty)$ is MU. Since the map $\Omega\mathrm{SU}(n) \to \mathrm{BU}$ is an equivalence in dimensions $\leq 2n-2$, the same is true for the map $X(n) \to \mathrm{MU}$; the first dimension in which $X(n)$ has an element in its homotopy which is not detected by MU is $2n-1$. In other words, writing $\pi_*\mathrm{MU} = \mathbf{Z}[b_1, b_2, \cdots]$ with $|b_i| = 2i$, the classes $b_1, \cdots, b_{n-1}$ lift to $X(n)$; there is an inclusion $\mathbf{Z}[b_1, \cdots, b_{n-1}] \subseteq \pi_*X(n)$.

Remark 2.1.3. The $\mathbf{E}_2$-structure on $X(n)$ does *not* extend to an $\mathbf{E}_3$-structure (see [**Law19**, Example 1.5.31]).

After localizing at a prime p, the spectrum MU splits as a wedge of suspensions of BP; this splitting comes from the Quillen idempotent on MU. The same is true of the $X(n)$ spectra, as explained in [**Rav86**, Section 6.5]: a multiplicative map $X(n)_{(p)} \to X(n)_{(p)}$ is determined by a polynomial $f(x) = \sum_{0\leq i\leq n-1} a_i x^{i+1}$, with $a_0 = 1$ and $a_i \in \pi_{2i}(X(n)_{(p)})$. One can use this to define a truncated form of the Quillen idempotent ϵ_n on $X(n)_{(p)}$ (see [**Hop84**, Proposition 1.3.7]), and thereby obtain a summand of $X(n)_{(p)}$. We summarize the necessary results in the following theorem.

Theorem 2.1.4. *Let n be such that $p^n \leq k \leq p^{n+1}-1$. Then $X(k)_{(p)}$ splits as a wedge of suspensions of the spectrum $T(n) = \epsilon_{p^n} \cdot X(p^n)_{(p)}$.*

- *$T(n)$ admits the structure of an $\mathbf{E}_1$-ring such that the map $T(n) \to X(p^n)$ is a map of $\mathbf{E}_1$-rings (see* [**BL21**, Section 7.5]*).*
- *The map $T(n) \to \mathrm{BP}$ is an equivalence in dimensions $\leq |v_{n+1}| - 2$, so there is an indecomposable element $v_i \in \pi_*T(n)$ which maps to an indecomposable element in $\pi_*\mathrm{BP}$ for $0 \leq i \leq n$. In particular (by (a)), there is an inclusion $\mathbf{Z}_{(p)}[v_1, \cdots, v_n] \subseteq \pi_*T(n)$.*
- *The map $T(n) \to \mathrm{BP}$ induces the inclusion $\mathrm{BP}_*T(n) = \mathrm{BP}_*[t_1, \cdots, t_n] \subseteq \mathrm{BP}_*(\mathrm{BP})$ on BP-homology, and the inclusions $\mathbf{F}_2[\zeta_1^2, \cdots, \zeta_n^2] \subseteq \mathbf{F}_2[\zeta_1^2, \zeta_2^2, \cdots]$ and $\mathbf{F}_p[\zeta_1, \cdots, \zeta_n] \subseteq \mathbf{F}_p[\zeta_1, \zeta_2, \cdots]$ on mod 2 and mod p homology, respectively.*

Example 2.1.5. The $\mathbf{E}_1$-ring $T(1)$ is the Thom spectrum of the $\mathbf{E}_1$-map $\Omega S^{2p-1} \to \mathrm{BGL}_1(S)$ which detects $\alpha_1 \in \pi_{2p-2}\mathrm{BGL}_1(S) \cong \pi_{2p-3}S$ on the bottom cell of ΩS^{2p-1}. Since $p\alpha_1 = 0$, a nullhomotopy of $p\alpha_1$ defines a class $v_1 \in \pi_{2p-2}T(1)$. Under the unit map $T(1) \to \mathrm{BP}$, this class is sent to the eponymous class $v_1 \in \pi_{2p-2}\mathrm{BP}$.

Warning 2.1.6. Unfortunately, Theorem 2.1.4 leads to an egregious clash of notation, since $T(n)$ is also often used to denote the telescope of a v_n-self map of a finite type n spectrum. In this article, we will *only* use $T(n)$ to mean the $\mathbf{E}_1$-ring from Theorem 2.1.4. We propose using the notation $\mathrm{Tel}(n)$ to denote the telescope of a v_n-self map.

Notation 2.1.7. If R is a commutative ring, we write $\Lambda_R(x)$ to denote an exterior R-algebra on a class x, and $R\langle x\rangle$ to denote a divided power R-algebra on a class x. The notation $\gamma_j(x)$ denotes the jth divided power of x, so that $j!\gamma_j(x) = x^j$. We will also often write $R\langle x\rangle$ to denote the underlying R-module of $R\langle x\rangle$.

Construction 2.1.8. Define a space Δ_n by

$$\Delta_n = \prod_{i=1}^{n} \mathrm{SU}(p^i - 1)/\mathrm{SU}(p^{i-1}),$$

and let $\overline{\Delta}_i$ denote the ith term in this product. If R is a ring spectrum, write $R[\Omega\Delta_n]$ to denote the $\mathbf{E}_2$-polynomial R-algebra $R[x_i|1 \leq i \leq p^n - 1, i \neq p^k - 1]$, where $|x_i| = 2i$. Let $R[B\Delta_n]$ denote the 2-fold bar construction of the augmentation $R[\Omega\Delta_n] \to R$, so that it is an $\mathbf{E}_2$-R-coalgebra whose homotopy groups are isomorphic to $\pi_*(R)\langle y_i|1 \leq i \leq p^n - 1, i \neq p^k\rangle$ where $|y_j| = 2j$. As mentioned in the introduction, $R[B\Delta_n]$ morally should be viewed as the R-chains on the "classifying space of $\prod_{i=1}^n \mathrm{SU}(p^i-1)/\mathrm{SU}(p^{i-1})$"; to this end, if X is another space, we will write $R[B\Delta_n \times X]$ to denote $R[B\Delta_n] \otimes_R R[X]$; and if R is a discrete ring, we will often write $\mathrm{H}_*(B\Delta_n; R)$ to denote $\pi_* R[B\Delta_n]$. The factor $R[B\Delta_n]$ will primarily be an unfortunate annoyance in this article. Note that $\Delta_1 = \mathrm{SU}(p-1)$. Then, we have $X(p^n) = T(n)[\Omega\Delta_n]$ and $X(p^n - 1) = T(n-1)[\Omega\Delta_n]$, so that

$$\mathrm{H}_*(X(2^n); \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^2, \cdots, \zeta_n^2] \otimes_{\mathbf{F}_p} \mathrm{H}_*(\Omega\Delta_n; \mathbf{F}_p),$$
$$\mathrm{H}_*(X(p^n); \mathbf{F}_p) \cong \mathbf{F}_p[\zeta_1, \cdots, \zeta_n] \otimes_{\mathbf{F}_p} \mathrm{H}_*(\Omega\Delta_n; \mathbf{F}_p),$$

and similarly for $X(p^n - 1)$.

It is believed that $T(n)$ admits more structure (see also **[AQ17]** for some discussion):

Conjecture 2.1.9. *The $\mathcal{Q}_1$-ring structure on $T(n)$ extends to a framed $\mathbf{E}_2$-ring structure.*

Remark 2.1.10. When $p = 2$, both $X(2) = T(1)$ and $T(2)$ admit the structure of $\mathbf{E}_2^{\mathrm{fr}}$-algebras by **[DHL+23**, Remark 3.8]: they are Thom spectra of U-bundles over $\Omega\mathrm{Sp}(1) \simeq \Omega S^3$ and $\Omega\mathrm{Sp}(2)$, respectively. These U-bundles are defined via double loops of the the composite

$$\mathrm{BSp}(n) \to \mathrm{BSU}(2n) \to \mathrm{BSU} \simeq B^3\mathrm{U}.$$

Proposition 2.1.11 ([**DHL+23**, Corollary 2.9 and Corollary 3.7])**.** *The $\mathbf{E}_2$-structure on $X(n)$ refines to an $\mathbf{E}_2^{\mathrm{fr}}$-structure.*

Corollary 2.1.12. *Let $\mathcal{C}$ be an $X(n)$-linear ∞-category. Then $\mathrm{THH}(\mathcal{C}/X(n))$ acquires the structure of an S^1-equivariant spectrum with an S^1-equivariant unit map $X(n) \to \mathrm{THH}(\mathcal{C}/X(n))$.*

2.2. Computation of THH relative to $X(p^n)$.

Unless explicitly stated otherwise, all fiber sequences in this section (as well as the following sections) will be localized at p.

Recollection 2.2.1. There are isomorphisms

$$\begin{aligned}
\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_2) &\cong \mathbf{F}_2[\zeta_1^2, \cdots, \zeta_n^2, \zeta_{n+1}, \cdots] \\
&\cong \mathrm{H}_*(T(n); \mathbf{F}_2) \otimes_{\mathbf{F}_2} \mathbf{F}_2[\zeta_j | j \geq n+1], \\
\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_p) &\cong \Lambda_{\mathbf{F}_p}[\tau_j | j \geq n] \otimes_{\mathbf{F}_p} \mathbf{F}_p[\zeta_1, \zeta_2, \cdots] \\
&\cong \mathrm{H}_*(T(n); \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p[\zeta_j | j \geq n+1] \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}[\tau_j | j \geq n], p > 2.
\end{aligned}$$

We note that the "Q_0-Margolis homology" of $\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_2)$ (i.e., the homology of Sq^1 viewed as a differential acting on $\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_2)$) is precisely $\mathrm{H}_*(T(n); \mathbf{F}_2)$, because Sq^1 is a derivation and $\mathrm{Sq}^1(\zeta_j) = \zeta_{j-1}^2$.

Recollection 2.2.2. We need to recall some results from [**HW20**]. First, [**HW20**, Theorem A] tells us that there exists an $\mathbf{E}_3$-form of $\mathrm{BP}\langle n\rangle$. First, [**HW20**, Theorem 2.5.4] states that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$ is isomorphic to a polynomial algebra over $\pi_*\mathrm{BP}\langle n-1\rangle$ on infinitely many generators, the first of which is denoted $\sigma^2(v_n)$. The class $\sigma^2(v_n)$ lives in degree $2p^n$. Finally, [**HW20**, Theorem 5.0.1] states that there is an isomorphism $\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/\mathrm{MU}) \simeq (\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU}))[\![\hbar]\!]$ of $\mathbf{Z}_p[v_1, \cdots, v_{n-1}]$-algebras. Moreover, under the map $\mathrm{MU}^{hS^1} \to \mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$, the class $v_n \in \pi_*\mathrm{MU}^{hS^1} \cong (\pi_*\mathrm{MU})[\![\hbar]\!]$ is sent to $\sigma^2(v_n)\hbar$. In particular, $\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$ detects the classes $p, v_1, \cdots, v_{n-1}, v_n := \sigma^2(v_n)\hbar$. Similarly, $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$ detects the classes $p, \cdots, v_n$ under the map $\mathrm{MU}^{tS^1} \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$, and $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})^{t\mathbf{Z}/p}$ detects the classes $p, \cdots, v_{n-1}$ under the map $\mathrm{MU}^{t\mathbf{Z}/p} \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})^{t\mathbf{Z}/p}$.

Notation 2.2.3. If R is a complex-oriented ring spectrum, we will write $\hbar$ to denote the complex orientation of R, viewed as a class in $\pi_{-2}R^{hS^1}$. The motivation for this notation comes from geometric representation theory (in the case where R is a $\mathbf{Z}_p$-algebra), where the complex orientation $\hbar \in \mathrm{H}^2(\mathbf{C}P^\infty; R)$ plays the role of a quantization parameter.

The main result of this section is the following analogue of Bökstedt's theorem on $\mathrm{THH}(\mathbf{F}_p)$ and $\mathrm{THH}(\mathbf{Z}_p)$.

Theorem 2.2.4. *Fix $\mathbf{E}_3$-forms of the truncated Brown-Peterson spectra $\mathrm{BP}\langle n-1\rangle$ and $\mathrm{BP}\langle n\rangle$. We have:*

(a) There is a p-complete equivalence of $\mathrm{BP}\langle n-1\rangle$-modules:

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n-1\rangle[B\Delta_n \times \Omega S^{2p^n+1}].$$

Write $\theta_n \in \pi_{2p^n}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ to denote the class corresponding to the map $E : S^{2p^n} \to \Omega S^{2p^n+1}$. Under the S^1-equivariant map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU})$, the class θ_n is sent to the class $\sigma^2(v_n)$ from Recollection 2.2.2. There are also p-complete isomorphisms

$$\begin{aligned}
\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m} &\cong \mathrm{BP}\langle n\rangle^{t\mathbf{Z}/m}[B\Delta_n]_*, \\
\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)) &\cong \mathrm{BP}\langle n\rangle[B\Delta_n]_*[\![\hbar]\!][\tfrac{v_n}{\hbar}] \\
&\cong \mathrm{BP}\langle n\rangle[B\Delta_n]_*[\![\hbar]\!][\theta_n]/(\theta_n\hbar - v_n), \\
\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) &\cong \mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n]_* \\
&\cong \mathrm{BP}\langle n\rangle[B\Delta_n]_*(\!(\hbar)\!).
\end{aligned}$$

Here, the equation $\theta_n \hbar = v_n$ *is to be understood modulo decomposables. These isomorphisms satisfy the following property: under the maps*

$$\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)) \to \mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/\mathrm{MU}),$$
$$\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/\mathrm{MU}),$$

the classes $\{v_i\}_{0\leq i\leq n}$ *on the left-hand side are sent to the eponymous classes in the right-hand side (via Recollection 2.2.2).*

(b) *There is an equivalence of* $\mathrm{BP}\langle n\rangle$*-modules:*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))^\wedge_p \cong \mathrm{BP}\langle n\rangle[B\Delta_n]^\wedge_p \oplus \left(\bigoplus_{j\geq 1} \Sigma^{2jp^{n+1}-1}\mathrm{BP}\langle n\rangle[B\Delta_n]/p^{v_p(j)+1}\right)^\wedge_p.$$

In particular, there is an additive equivalence

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))/p \cong \mathrm{BP}\langle n\rangle[S^{2p^{n+1}-1} \times \Omega S^{2p^{n+1}+1} \times B\Delta_n]/p.$$

Moreover, $\pi_{2p^{n+1}-3}\mathrm{TC}^-(\mathrm{BP}\langle n\rangle/X(p^n))^\wedge_p$ *detects the class* $\sigma_n \in \pi_{2p^{n+1}-3}X(p^n)$ *from* [**Dev23a**, Lemma 3.1.12].

Remark 2.2.5. Let $v_{[j,m)}$ denote the regular sequence $v_j, \cdots, v_{m-1}$ in $\pi_*\mathrm{BP}$. Then the argument used to prove Theorem 2.2.4 in fact shows the following (somewhat more general) result: for $j \leq n$, there is an isomorphism of $\mathrm{BP}\langle n-1\rangle_*$-modules

(4)
$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))/v_{[0,n-j)} \cong \mathrm{BP}\langle n-1\rangle[B\Delta_j]_*[\theta_n]/v_{[0,n-j)} \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}(\lambda_{j+1}, \cdots, \lambda_n),$$

where $|\lambda_i| = 2p^i - 1$. When $j = 0$, (4) recovers [**ACH21**, Proposition 2.9]. For brevity, the discussion below only includes the cases $j = n$ and $j = n-1$. Similarly, using that $T(1)$ (resp. $T(2)_{(2)}$) is a Thom spectrum over ΩS^{2p+1} (resp. $\Omega\mathrm{Sp}(2)$), there are equivalences

$$\mathrm{THH}(\mathbf{Z}_p)/p \simeq \mathbf{F}_p[S^{2p-1} \times \Omega S^{2p+1}],$$
$$\mathrm{THH}(\mathrm{ku})/(2,\beta) \simeq \mathbf{F}_2[\mathrm{Sp}(2) \times \Omega S^9].$$

Remark 2.2.6. If we write $\pi_*\mathrm{MU} = \mathbf{Z}[x_1, x_2, \cdots]$ where $|x_i| = 2i$, and define $\mathrm{MU}\langle n-1\rangle = \mathrm{MU}/(x_n, x_{n+1}, \cdots)$, then one can similarly prove an analogue of Theorem 2.2.4 with $\mathrm{BP}\langle n-1\rangle$ replaced by $\mathrm{MU}\langle n-1\rangle$. Namely, if n is a power of p, there is an equivalence

$$\mathrm{THH}(\mathrm{MU}\langle n-1\rangle/X(n))^\wedge_p \simeq \mathrm{MU}\langle n-1\rangle[\Omega S^{2n+1}]^\wedge_p$$

of $\mathrm{MU}\langle n-1\rangle$-modules. There is also a p-complete isomorphism

$$\pi_*\mathrm{TP}(\mathrm{MU}\langle n-1\rangle/X(n))^\wedge_p \cong \pi_*(\mathrm{MU}\langle n\rangle^{tS^1})^\wedge_p.$$

We expect (see Conjecture 2.2.18 below) that this refines to a p-complete equivalence $\mathrm{TP}(\mathrm{MU}\langle n-1\rangle/X(n))^\wedge_p \simeq (\mathrm{MU}\langle n\rangle^{tS^1})^\wedge_p$.

Example 2.2.7. One can make Theorem 2.2.4(a) very explicit for $\mathbf{Z}_p$ (note that Theorem 2.2.4(b) for $\mathbf{Z}_p$ is Bökstedt's result). For instance,

$$\pi_*\mathrm{TC}^-(\mathbf{Z}_p/T(1)) \cong \mathbf{Z}_p[v_1][\![\hbar]\!][\theta]/(\hbar\theta = v_1).$$

Let us view $\mathrm{BP}\langle 1\rangle$ as $(\mathrm{ku}^\wedge_p)^{h\mathbf{F}_p^\times}$, and let $\beta \in \pi_2\mathrm{ku}$ be the Bott class. Then, $\pi_*\mathrm{ku}^{tS^1} \cong \mathbf{Z}[\beta](\!(\hbar)\!)$ is isomorphic to $\mathbf{Z}[\![q-1]\!](\!(\hbar)\!)$, where $q = 1 + \beta\hbar$ lives in degree 0. If $\mathbf{Z}_p[\![\widetilde{p}]\!]$ is as in [**BL22a**, Corollary 3.8.8], then $\pi_*\mathrm{BP}\langle 1\rangle^{tS^1} \cong \mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!)$. If we assume (for simplicity) that $T(1)$ is an $\mathbf{E}_2^{\mathrm{fr}}$-algebra, then replacing $X(p)$ by $T(1)$, we obtain:

$$\pi_* \mathrm{TP}(\mathbf{Z}_p/T(1)) \cong \mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!).$$

Here, $\mathbf{F}_p^\times$ acts on $\mathbf{Z}_p[\![q-1]\!]$ as specified before [**BL22a**, Proposition 3.8.6]; indeed, the $\mathbf{Z}_p^\times$-action on $\mathbf{Z}_p[\![q-1]\!] = \pi_0(\mathrm{ku}_p^\wedge)^{tS^1}$ agrees with the action of the Adams operations on $\pi_*(\mathrm{ku}_p^\wedge)^{tS^1}$, as one can check by calculating the Adams operations on the p-completed complex K-theory of $\mathbf{C}P^\infty$. Indeed, if $g \in \mathbf{Z}_p^\times$, then

$$\psi^g(\hbar) = \frac{1}{g} \sum_{j \geq 1} \binom{g}{j} \beta^{j-1} \hbar^j = \frac{1}{g} \frac{(1+\beta\hbar)^g - 1}{\beta},$$

so that

$$\psi^g(q) = \psi^g(1+\beta\hbar) = 1 + g\beta\psi^g(\hbar) = (1+\beta\hbar)^g = q^g.$$

Remark 2.2.8. Recall from [**Lur15**, Section 3.4] that there is an $\mathbf{E}_2$-monoidal functor $\mathrm{sh} : \mathrm{Sp}^{\mathrm{gr}} \to \mathrm{Sp}^{\mathrm{gr}}$ given by shearing: this functor sends $M_\bullet \mapsto M_\bullet[2\bullet]$. Assume for simplicity that $T(n)$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-algebra. From this perspective, part of Theorem 2.2.4(a) simply states that there is an equivalence of ungraded $\mathrm{BP}\langle n-1\rangle$-modules

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \simeq \mathrm{sh}(\mathrm{gr}_{v_n} \mathrm{BP}\langle n\rangle),$$

where $\mathrm{sh}(\mathrm{gr}_{v_n} \mathrm{BP}\langle n\rangle)$ denotes the shearing of the associated graded of the v_n-adic filtration $\mathrm{F}^\star_{v_n} \mathrm{BP}\langle n\rangle$ on $\mathrm{BP}\langle n\rangle$.

An immediate implication of Theorem 2.2.4 is the following.

Corollary 2.2.9 ([**HW20**, Corollary 5.0.2])**.** *Fix an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum $\mathrm{BP}\langle n-1\rangle$. We have $L_{K(n)}K(\mathrm{BP}\langle n-1\rangle) \neq 0$.*

Proof. There is a trace map $K(\mathrm{BP}\langle n-1\rangle) \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle)$, which is a map of $\mathbf{E}_2$-rings. It therefore suffices to exhibit a nonzero module over $L_{K(n)}\mathrm{TP}(\mathrm{BP}\langle n-1\rangle)$ — but we may take the module $L_{K(n)}\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$, which is nonzero by Theorem 2.2.4(a). (In fact, Theorem 2.2.4(a) implies $\pi_* L_{K(n)}\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ is isomorphic to $\mathbf{Z}_p[v_1, \cdots, v_{n-1}, v_n^{\pm 1}]^\wedge_{(p,\cdots,v_{n-1})}(\!(\hbar)\!)$ tensored with the $\mathbf{Z}_p$-homology of $B\Delta_n$.) ∎

Remark 2.2.10. It is easy to see that $T(n) \to \mathrm{BP}\langle n\rangle$ is a nilpotent extension. This implies in particular that the following square is Cartesian by the Dundas-Goodwillie-McCarthy theorem [**DGM13**, Theorem 7.2.2.1]:

$$\begin{tikzcd}
K(T(n)) \arrow[r] \arrow[d] & K(\mathrm{BP}\langle n\rangle) \arrow[d] \\
\mathrm{TC}(T(n)) \arrow[r] & \mathrm{TC}(\mathrm{BP}\langle n\rangle).
\end{tikzcd}$$

Note that there is also a commutative square

$$\begin{tikzcd}
\mathrm{TC}(T(n)) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{BP}\langle n\rangle) \arrow[d] \\
\mathrm{TC}^-(T(n)) \arrow[r] & \mathrm{TC}^-(\mathrm{BP}\langle n\rangle),
\end{tikzcd}$$

and Theorem 2.2.4 and Theorem 3.1.4 give an inductive approach to calculating the bottom row. One might therefore view the results of this article as a first step to fully computing $K(\mathrm{BP}\langle n\rangle)$. It would be very interesting to describe $\mathrm{TC}(T(n))$. For example, we expect that for a general odd prime, the spectrum $\mathrm{TP}(T(1))$ is closely related to the $\mathbf{E}_1$-quotient $S/\!\!/\alpha_{p/p}$. (Here, $\alpha_{p/p} \in \pi_{2p(p-1)-1}(S)$ is an element in the α-family.)

However, more is true about the map $T(n) \to \mathrm{BP}\langle n\rangle$: in fact, every element in $\ker(\pi_* T(n) \to \pi_* \mathrm{BP}\langle n\rangle)$ is nilpotent. To see this, first observe that this map is a rational equivalence (indeed, it is an equivalence on Q_0-Margolis homology), so $\mathrm{fib}(T(n) \to \mathrm{BP}\langle n\rangle)$ is torsion. Moreover, the map $T(n) \to \mathrm{BP}\langle n\rangle$ is surjective on homotopy (since it is a ring map, and the generators $p, v_1, \cdots, v_n \in \pi_* \mathrm{BP}\langle n\rangle$ lift to $T(n)$), so that the map $\mathrm{fib}(T(n) \to \mathrm{BP}\langle n\rangle) \to T(n)$ induces an injection on homotopy. If $x \in \pi_* T(n)$ is in the image of the map $\mathrm{fib}(T(n) \to \mathrm{BP}\langle n\rangle) \to T(n)$, then the image of x under the Hurewicz map $\pi_* T(n) \to \mathrm{MU}_* T(n)$ is also torsion; but $\mathrm{MU}_* T(n) \cong \mathrm{MU}_*[t_1, \cdots, t_n]$ is torsion-free, so x must be nilpotent by the main theorem of [**DHS88**][5]. This is the desired claim.

More generally, recall [**Dev23a**, Table 1], reproduced here as Table 1 (for the definitions of these spectra, see [**Mah79**] for A, where it is denoted X_5; [**Dev19**, Construction 3.1] and [**HM02**] for B; [**MRS01**] for $y(n)$; and [**AQ19**] for $y_{\mathbf{Z}}(n)$).

Height	0	1	2	n	n	n
Base $\mathbf{E}_1$-ring R	$(S^0)^\wedge_p$	A	B	$T(n)$	$y(n)$	$y_{\mathbf{Z}}(n)$
Designer chromatic spectrum $\Theta(R)$	$\mathbf{Z}_p$	bo	tmf	$\mathrm{BP}\langle n\rangle$	$k(n)$	$k_{\mathbf{Z}}(n)$

TABLE 1. The relation between R and $\Theta(R)$ is analogous to the relationship between $T(n)$ and $\mathrm{BP}\langle n\rangle$.

In a manner similar to above, if R is an $\mathbf{E}_1$-ring as in the second line of Table 1, and $\Theta(R)$ is the associated designer spectrum, one can show that every element in $\ker(\pi_* R \to \pi_* \Theta(R))$ is nilpotent. It follows, for example, that there is a Cartesian square

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & K(\Theta(R)) \arrow[d] \\ \mathrm{TC}(R) \arrow[r] & \mathrm{TC}(\Theta(R)). \end{tikzcd}$$

Moreover, the proof of Theorem 2.2.4 shows that were R to admit the structure of an $\mathbf{E}_2$-ring (which is generally *not true*[6], $\mathrm{THH}(\Theta(R)/R)$ would be p-completely equivalent to $R \oplus \bigoplus_{j\geq 1} \Sigma^{2jp^{n+1}-1} R/pj$ (where n is the "height" of R). If $R = y(n)$ or $y_{\mathbf{Z}}(n)$, this result is literally true by Theorem 2.2.4, as long as one assumes Conjecture 2.1.9 and interprets $\mathrm{THH}(\Theta(R)/R)$ to mean $\mathrm{THH}(\mathrm{BP}\langle n\rangle/T(n)) \otimes_{T(n)} R$. This does not cover the cases $R = A, B$, though; see Appendix A for further discussion of these cases.

[5]In some sense, this is a rather perverse argument, because the heart of the proof of the nilpotence theorem relies crucially on showing that every element in $\ker(\pi_* T(n) \to \pi_* \mathrm{BP}\langle n\rangle)$ is nilpotent.

[6]For instance, $y(n)$ cannot admit the structure of an $\mathbf{E}_2$-ring, thanks to the Steinberger identity on the action of the Dyer-Lashof operation Q_1 on the dual Steenrod algebra (see [**BMMS86**, Theorems III.2.2 and III.2.3]).

Remark 2.2.11. It is natural to ask whether Theorem 2.2.4 can be generalized to describe $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^m))$ if $m \neq n$. For $m < n$, we do not know a full description (after killing $p, \cdots, v_{n-m-1}$, see Remark 2.2.5); but the techniques of Theorem 3.1.4 below provide a conceptual approach to addressing this question. For $m > n$, the proof of Theorem 2.2.4 easily implies that there is an additive isomorphism

$$\begin{aligned}\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^m)) &\cong \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \otimes_{\mathrm{BP}\langle n-1\rangle_*} \mathrm{BP}\langle n-1\rangle_*\langle y_i | p^n < i \leq p^m\rangle \\ &\cong \mathrm{BP}\langle n-1\rangle[\Omega S^{2p^n+1}]_*\langle y_i | 1 \leq i \leq p^m \text{ such that } i \neq p^k \text{ for } 0 \leq k \leq n\rangle.\end{aligned}$$

Here, y_i lives in degree $2i$. For example, if $n = 0$, the divided power factor is just $\mathrm{BP}\langle n-1\rangle_*[\mathrm{BSU}(p^m)]$. For instance, in the limit as $m \to \infty$, we recover the statement that $\pi_*\mathrm{THH}(\mathbf{F}_p/\mathrm{MU}) \simeq \mathbf{F}_p[\mathrm{BSU} \times \Omega S^3]_*$.

Remark 2.2.12. Theorem 2.2.4(b) implies that

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n-1)) \cong \mathrm{BP}\langle n-1\rangle[B\Delta_n]_* \oplus \bigoplus_{j\geq 1} \mathrm{BP}\langle n-1\rangle[B\Delta_n]_{*-2jp^n+1}/p^{v_p(j)+1}.$$

This can be compared to Theorem 2.2.4(a) (we will study in this in further detail in Section 3): the complexity of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n-1))$ compared to $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ can be understood as arising via the descent spectral sequence for the map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n-1)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$. Note that $X(p^n) \otimes_{X(p^n-1)} X(p^n) \simeq X(p^n)[\Omega S^{2p^n-1}]$; using this, one can calculate using methods similar to the proof of Theorem 2.2.4 that the E_2-page of the descent spectral sequence is

$$E_2^{*,*} \cong \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))[\epsilon]/\epsilon^2,$$

where $|\epsilon| = 2p^n - 1$. Calculating the differentials gives an "alternative" proof of Theorem 2.2.4(b) given Theorem 2.2.4(a); we will expand on this below in Remark 2.2.17. In fact, inductively studying THH of $\mathrm{BP}\langle n-1\rangle$ relative to $X(p^j)$ for $j \leq n$ gives a conceptual explanation for the families of differentials visible in the calculations of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$ in [**AR05**, Section 8], [**MS93**], and [**AHL10**]; see Theorem 3.1.4 and Example 4.1.11.

The proof of Theorem 2.2.4 will be broken into several components. Let us begin by illustrating Theorem 2.2.4(a) in the case $n = 0, 1$.

Proof of Theorem 2.2.4(a) for $n = 0, 1$. We need to show that there are equivalences of spectra $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{F}_p[\Omega S^3]$ and $\mathrm{THH}(\mathbf{Z}_p/X(p)) \simeq \mathbf{Z}_p[\mathrm{BSU}(p-1) \times \Omega S^{2p+1}]$. The first equivalence is classical (see [**Bö85**]), so we argue the second equivalence. There is a p-local map $f : \mathrm{SU}(p) \to \Omega S^3\langle 3\rangle$ of spaces given by the composite

$$\mathrm{SU}(p) \to \mathrm{SU}(p)/\mathrm{SU}(p-1) \simeq S^{2p-1} \xrightarrow{\alpha_1} \Omega S^3\langle 3\rangle.$$

In [**Dev23a**, Remark 4.1.4], we described a fiber sequence (which was also known to Toda in [**Tod62**])

$$S^{2p-1} \xrightarrow{\alpha_1} \Omega S^3\langle 3\rangle \to \Omega S^{2p+1}. \tag{5}$$

This induces a fiber sequence of $\mathbf{E}_1$-spaces

$$\Omega\mathrm{SU}(p) \xrightarrow{f} \Omega^2 S^3\langle 3\rangle \to \mathrm{SU}(p-1) \times \Omega^2 S^{2p+1}.$$

We now compute:

$$\begin{aligned}\mathrm{THH}(\mathbf{Z}_p/X(p)) &\simeq \mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathrm{THH}(X(p))} X(p)\\ &\simeq \mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p\otimes\mathrm{THH}(X(p))}\mathbf{Z}_p.\end{aligned}$$

The map $X(p)\to\mathbf{Z}_p$ is precisely the map induced by $f:\Omega\mathrm{SU}(p)\to\Omega^2S^3\langle 3\rangle$, so the above tensor product is given by $\mathbf{Z}_p[\Omega S^{2p+1}\times\mathrm{BSU}(p-1)]$, as desired. □

Remark 2.2.13. Recall that the calculation $\mathrm{THH}(\mathbf{F}_p)\simeq\mathbf{F}_p[\Omega S^3]$ follows from [**BCS10**] and the Hopkins-Mahowald theorem that $\mathbf{F}_p$ is the Thom spectrum of the $\mathbf{E}_2$-map $\Omega^2S^3\to\mathrm{BGL}_1(S^\wedge_p)$ which detects $1-p\in\pi_1\mathrm{BGL}_1(S^\wedge_p)\cong\mathbf{Z}_p^\times$ on the bottom cell of Ω^2S^3. In [**Dev23a**, Corollary B], we prove (unconditionally!) that $\mathbf{Z}_p$ is the Thom spectrum of a map $\mu:\Omega^2S^{2p+1}\to\mathrm{BGL}_1(T(1))$ which detects $v_1\in\pi_{2p-1}\mathrm{BGL}_1(T(1))\cong\pi_{2p-2}T(1)$ on the bottom cell of Ω^2S^{2p+1}. (Unlike in the classical Hopkins-Mahowald theorem, the map μ is not an $\mathbf{E}_2$-map.) This result implies that $\mathbf{Z}_p$ is the Thom spectrum of a map $\mathrm{SU}(p-1)\times\Omega^2S^{2p+1}\to\mathrm{BGL}_1(X(p))$, which can also be used to prove Theorem 2.2.4(a) for $n=1$.

We now turn to Theorem 2.2.4(a) in the general case; the strategy is to compute the homology of each of the spectra under consideration, and run the Adams spectral sequence. In the case of $\mathrm{THH}^{t\mathbf{Z}/m}$, TC^-, and TP, we will need the "continuous homology" of [**BR05**, Equation 2.3].

Proposition 2.2.14. *(a) There are isomorphisms*

$$\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n));\mathbf{F}_p)\cong\begin{cases}\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle[B\Delta_n];\mathbf{F}_2)[\sigma(\zeta_{n+1})] & p=2,\\ \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle[B\Delta_n];\mathbf{F}_p)[\sigma(\tau_n)] & p>2.\end{cases}$$

(b) There are isomorphisms

$$\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n));\mathbf{F}_p)\cong\begin{cases}\mathrm{H}_*(\mathrm{BP}\langle n\rangle[B\Delta_n];\mathbf{F}_2)[\sigma(\zeta_{n+2})]\otimes_{\mathbf{F}_2}\Lambda_{\mathbf{F}_2}(\sigma(\zeta_{n+1}^2)) & p=2,\\ \mathrm{H}_*(\mathrm{BP}\langle n\rangle[B\Delta_n];\mathbf{F}_p)[\sigma(\tau_{n+1})]\otimes_{\mathbf{F}_p}\Lambda_{\mathbf{F}_p}(\sigma(\zeta_{n+1})) & p>2.\end{cases}$$

Moreover, there is a Bockstein $\beta:\sigma(\zeta_{n+2})\mapsto\sigma(\zeta_{n+1}^2)$ for $p=2$, and a Bockstein $\beta:\sigma(\tau_{n+1})\mapsto\sigma(\zeta_{n+1})$ for $p>2$.

Proof. We begin by proving (a). We will use the Bökstedt spectral sequence, which runs

$$E^2_{*,*}=\mathrm{HH}_*(\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle;\mathbf{F}_p)/\mathrm{H}_*(X(p^n);\mathbf{F}_p))\Rightarrow\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n));\mathbf{F}_p).$$

Since $\mathrm{H}_*(X(p^n);\mathbf{F}_p)\cong\mathrm{H}_*(T(n);\mathbf{F}_p)\otimes_{\mathbf{F}_p}\mathrm{H}_*(\Omega\Delta_n;\mathbf{F}_p)$ and the action of $\mathrm{H}_*(X(p^n);\mathbf{F}_p)$ on $\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle;\mathbf{F}_p)$ factors through the map $\mathrm{H}_*(X(p^n);\mathbf{F}_p)\to\mathrm{H}_*(T(n);\mathbf{F}_p)$ induced by the map crushing $\Omega\Delta_n$ to a point, we will ignore the contribution from Δ_n in this discussion. The final contribution from these terms will only be $\mathrm{H}_*(B\Delta_n;\mathbf{F}_p)$. (The following may therefore be interpreted as a computation of $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n));\mathbf{F}_p)$; however, since Conjecture 2.1.9 is not known to be true, the spectrum $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))$ cannot yet be defined.) We will continue to write $E^2_{*,*}$ to denote the Hochschild homology groups of $\mathrm{H}_*(\mathrm{BP}\langle n-1\rangle;\mathbf{F}_p)$ over $\mathrm{H}_*(T(n);\mathbf{F}_p)$.

Recall that if R is any discrete commutative ring, there are isomorphisms $\pi_*\mathrm{HH}(R[x]/R)\simeq R[x]\otimes\Lambda_R(\sigma x)$ and $\pi_*\mathrm{HH}(\Lambda_R(x))\simeq\Lambda_R(x)\otimes R\langle\sigma x\rangle$. It therefore follows from Recollection 2.2.1 that we have

$$E^2_{*,*} = \begin{cases} \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma\zeta_j | j \geq n+1) & p=2, \\ \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}(\sigma\zeta_j | j \geq n+1) \otimes_{\mathbf{F}_p} \mathbf{F}_p\langle \sigma\tau_j | j \geq n\rangle & p>2. \end{cases}$$

The map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ induces a map from the Bökstedt spectral sequence computing $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle))$ to our spectral sequence. The differentials in the Bökstedt spectral sequence computing $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle))$ are calculated in [**AR05**, Proposition 5.6], where it is shown that for p odd, $j \geq p$, and $m \geq n$, there are differentials

$$d^{p-1}(\gamma_j(\sigma\tau_m)) = \sigma(\zeta_{m+1})\gamma_{j-p}(\sigma\tau_m). \tag{6}$$

The argument of [**AR05**, Proposition 5.7] implies that

$$E^\infty_{*,*} = \begin{cases} \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma\zeta_j | j \geq n+1) & p=2, \\ \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p[\sigma\tau_j | j \geq n]/(\sigma\tau_j)^p & p>2. \end{cases}$$

The extensions on the E^∞-page of the Bökstedt spectral sequence computing $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle))$ are determined by [**AR05**, Theorem 5.12]: there, it is shown that for $j \geq n+1$, we have $(\sigma\zeta_j)^2 = \sigma\zeta_{j+1}$ when $p=2$, and $(\sigma\tau_j)^p = \sigma\tau_{j+1}$. These imply extensions on the E^∞-page of the Bökstedt spectral sequence for $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$, and the resulting answer is that of the proposition.

We now turn to (b). The calculation is similar to (a), the only difference being that the E^2-page of the Bökstedt spectral sequence is now

$$E^2_{*,*} = \begin{cases} \mathrm{H}_*(\mathrm{BP}\langle n\rangle; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma(\zeta_{n+1}^2), \sigma\zeta_j | j \geq n+2) & p=2, \\ \mathrm{H}_*(\mathrm{BP}\langle n\rangle; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}(\sigma\zeta_j | j \geq n+1) \otimes_{\mathbf{F}_p} \mathbf{F}_p\langle \sigma\tau_j | j \geq n+1\rangle & p>2. \end{cases}$$

Again, the differentials in the Bökstedt spectral sequence computing $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle))$ give rise to differentials in the above Bökstedt spectral sequence, and we have

$$E^\infty_{*,*} = \begin{cases} \mathrm{H}_*(\mathrm{BP}\langle n\rangle; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma(\zeta_{n+1}^2), \sigma\zeta_j | j \geq n+2) & p=2, \\ \mathrm{H}_*(\mathrm{BP}\langle n-1\rangle; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p[\sigma\tau_j | j \geq n+1]/(\sigma\tau_j)^p \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}(\sigma\zeta_{n+1}) & p>2. \end{cases}$$

Again, the extensions on the E^∞-page of the Bökstedt spectral sequence computing $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle))$ imply extensions on the above E^∞-page, and the resulting answer is that of the proposition. The Bockstein follows from the fact that $\beta(\tau_i) = \zeta_i$ for p odd and $\beta(\zeta_i) = \zeta_{i-1}^2$ for $p=2$. □

Proposition 2.2.15. *There are isomorphisms*

$$\mathrm{H}^c_*(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p) \cong \mathrm{H}_*(\mathrm{BP}\langle n\rangle[B\Delta_n]; \mathbf{F}_p)[\![\hbar]\!] \oplus \hbar\text{-torsion},$$
$$\mathrm{H}^c_*(\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p) \cong \mathrm{H}_*(\mathrm{BP}\langle n\rangle[B\Delta_n]; \mathbf{F}_p)(\!(\hbar)\!),$$
$$\mathrm{H}^c_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m}; \mathbf{F}_p) \cong \mathrm{H}_*(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/m}[B\Delta_n]; \mathbf{F}_p)(\!(\hbar)\!).$$

Here, $|\hbar| = -2$, and the $\hbar$-torsion terms will be specified in the proof.

Proof. As in Proposition 2.2.14, the contribution from Δ_n is just the $\mathbf{F}_p$-homology of $B\Delta_n$, and we will ignore this term in the calculations. Moreover, the calculation for $\mathrm{H}^c_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p^k}; \mathbf{F}_p)$ is similar to the calculation of $\mathrm{H}^c_*(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$ (and $\mathrm{H}^c_*(\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$), so we will only do the latter. (The only difference is that $\mathbf{F}_p(\!(\hbar)\!)$ below is replaced by $\mathbf{F}_p(\!(\hbar)\!)[\epsilon_k]/\epsilon_k^2$.) The E^2-page of the homological homotopy fixed points spectral sequence computing $\mathrm{H}_*^c(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$ is given by

$$\begin{aligned} E^2_{*,*} &\cong \mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p[\![\hbar]\!] \\ &\cong \begin{cases} \mathbf{F}_2[\sigma(\zeta_{n+1}), \hbar, \zeta_1^2, \cdots, \zeta_n^2, \zeta_j | j \geq n+1] & p = 2 \\ \mathbf{F}_p[\sigma(\tau_n), \hbar, \zeta_i | i \geq 1] \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}[\tau_j | j \geq n] & p > 2. \end{cases} \end{aligned}$$

There is a map to the above spectral sequence from the homological homotopy fixed points spectral sequence computing $\mathrm{H}_*^c(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle); \mathbf{F}_p)$, and [**BR05**, Proposition 6.1] calculates that there are differentials $d^2(x) = \hbar\sigma(x)$ for every $x \in \mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$. For $j \geq n$, the following classes survive to the E^3-page:

$$\begin{aligned} \zeta'_{j+1} &= \zeta_{j+1} + \zeta_j\sigma(\zeta_j) = \zeta_{j+1} + \zeta_j\sigma(\zeta_{n+1})^{2^{n+1-j}}, \ p = 2 \\ \tau'_{j+1} &= \tau_{j+1} + \tau_j\sigma(\tau_j)^{p-1}, \ p > 2. \end{aligned}$$

Moreover, (powers of) the classes $\sigma(\zeta_{n+1})$ at $p = 2$ and $\sigma(\tau_n)$ at $p > 2$ are simple $\hbar$-torsion: for example, $\hbar\sigma(\zeta_{n+1})^{2^{n+1-j}}$ is killed by a d^2-differential on ζ_j, and the case for a general power of $\sigma(\zeta_{n+1})$ follows from taking a binary expansion of the exponent. This leaves

$$\begin{aligned} E^3_{*,*} &\cong \mathbf{F}_2[\![\hbar]\!][\zeta_1^2, \cdots, \zeta_n^2, \zeta_{n+1}^2, \zeta'_{j+1} | j \geq n+1], \ p = 2, \\ E^3_{*,*} &\cong \mathbf{F}_p[\![\hbar]\!][\zeta_i | i \geq 1] \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}[\tau'_{j+1} | j \geq n], \ p > 2, \end{aligned}$$

and the image of σ in filtration zero (these classes being simple $\hbar$-torsion). We claim that the spectral sequence degenerates at the E^3-page, which then implies the desired result. (In the case of $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p}$, for instance, the class $\epsilon_1\hbar^{1-p^n}$ plays the role of τ_n in $\mathrm{H}_*^c(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p}; \mathbf{F}_p)$ for p odd.) As with the proof of Proposition 2.2.14, this follows from [**BR05**, Proposition 6.1]: were there any differentials in the homological homotopy fixed points spectral sequence for $\mathrm{H}_*(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$, there would also exist corresponding differentials in the homological homotopy fixed points spectral sequence for $\mathrm{H}_*(\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle); \mathbf{F}_p)$.

However, the statement of [**BR05**, Proposition 6.1] assumes that $\mathrm{BP}\langle n-1\rangle$ admits the structure of an $\mathbf{E}_\infty$-algebra; this is not necessary, since their appeal to [**BR05**, Proposition 5.1] only uses the existence of the Dyer-Lashof operations Q_0 and Q_1 on $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle); \mathbf{F}_p)$, which already exist in the homology of any $\mathbf{E}_2$-algebra. It therefore suffices to know that $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$ admits the structure of an $\mathbf{E}_2$-algebra, which is a consequence of our assumption that $\mathrm{BP}\langle n-1\rangle$ is an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum. □

Proof of Theorem 2.2.4(a). We will ignore the contribution from $B\Delta_n$ below: the contribution from this term is simply its homology. We will first calculate $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ via the Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbf{F}_p, \mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)) \Rightarrow \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^\wedge_p.$$

Using Proposition 2.2.14(a), there is a change-of-rings isomorphism

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{\mathcal{E}(n-1)_*}(\mathbf{F}_p, \mathbf{F}_p[\sigma(\zeta_{n+1})]) \cong \mathbf{F}_p[\sigma(\zeta_{n+1}), v_j | 0 \leq j \leq n-1],$$

where v_j lives in bidegree $(s, t-s) = (1, 2p^j - 2)$. The Adams spectral sequence is concentrated in even total degree (and therefore degenerates at the E_2-page). The class $\sigma(\zeta_{n+1})$ in degree $|\zeta_{n+1}|+1 = 2p^n$ is denoted θ_n, so that the above calculation says that there is an isomorphism

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n-1\rangle[B\Delta_n]_*[\theta_n].$$

Since $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \otimes_{\mathrm{THH}(X(p^n))} X(p^n)$, we see that $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ admits the structure of a $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$-module. There is an $\mathbf{E}_2$-map $\mathrm{BP}\langle n-1\rangle \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$, so that $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ acquires the structure of a $\mathrm{BP}\langle n-1\rangle$-module by restriction of scalars. Therefore, each of the $\mathrm{BP}\langle n-1\rangle_*$-module generators of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ lift to maps of spectra from shifts of $\mathrm{BP}\langle n-1\rangle$ to $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$. Moreover, the resulting map $\mathrm{BP}\langle n-1\rangle[B\Delta_n \times \Omega S^{2p^n+1}] \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ induces an isomorphism on homotopy by construction, so we obtain the first part of Theorem 2.2.4(a).

The calculation for $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m}$ is similar to the calculation of $\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n))$ (and $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$); moreover, it will be illustrative to calculate $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$, since the case of $\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n))$ will just involve bookkeeping of the $\hbar$-torsion terms in Proposition 2.2.15. There is an Adams spectral sequence

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbf{F}_p, \mathrm{H}^c_*(\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)) \Rightarrow \pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))^\wedge_p,$$

which is in general only conditionally convergent, but is strongly convergent in this case. (This is because $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)); \mathbf{F}_p)$ is bounded-below and of finite type.) By Proposition 2.2.15, there is a change-of-rings isomorphism

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{\mathcal{E}(n)_*}(\mathbf{F}_p, \mathbf{F}_p(\!(\hbar)\!)) \cong \mathbf{F}_p[v_j | 0 \leq j \leq n](\!(\hbar)\!),$$

so that the Adams spectral sequence is concentrated in even total degree (and therefore degenerates at the E_2-page); this gives the desired calculation. □

Remark 2.2.16. The homotopy fixed points spectral sequence for $\pi_*\mathrm{TC}^-(\mathrm{BP}\langle n-1\rangle/X(p^n))$ has E_2-page given by

$$E_2^{*,*} = \mathrm{BP}\langle n-1\rangle[B\Delta_n]_*[\theta_n][\![\hbar]\!].$$

By evenness, this spectral sequence degenerates at the E_2-page. The calculation of Theorem 2.2.4(a) tells us that the class $\hbar\theta_n$ on the E_∞-page represents the class $v_n \in \pi_*\mathrm{BP}\langle n\rangle$ (modulo decomposables).

Note that Theorem 2.2.4(a) says in particular that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p} \cong \pi_*\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p}[B\Delta_n]$. There is an isomorphism $\pi_*\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p} \cong \pi_*\mathrm{BP}\langle n-1\rangle^{tS^1}$ (which was proved in [**AMS98**, Proposition 2.3], and conjectured to lift to an equivalence of spectra in [**DJK$^+$86**, Conjecture 1.2]), so that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p} \cong \mathrm{BP}\langle n-1\rangle[B\Delta_n]_*(\!(\hbar)\!)$. Note that unless $n = 0$, this is *not* isomorphic to $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))[\theta_n^{-1}]$, since $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))[\theta_n^{-1}]$ is $2p^n$-periodic, while $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/p}$ is 2-periodic.

Proof of Theorem 2.2.4(b). We now calculate $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$, this time with the use of Bockstein spectral sequences. (Similar arguments can be found in [**ACH21**].) Again, we will ignore the contribution from $B\Delta_n$ below: the contribution from this term is simply its homology. For simplicity, let us write

$$x = \begin{cases} \sigma(\zeta_{n+1}^2) & p = 2, \\ \sigma(\zeta_{n+1}) & p > 2 \end{cases}, \quad y = \begin{cases} \sigma(\zeta_{n+2}) & p = 2, \\ \sigma(\tau_{n+1}) & p > 2, \end{cases}$$

so that $|x| = 2p^{n+1} - 1$ and $|y| = 2p^{n+1}$. If M is a (left) $\mathrm{BP}\langle n\rangle$-module, let $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); M)$ denote $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n)) \otimes_{\mathrm{BP}\langle n\rangle} M$, so that we may informally view $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); \mathbf{F}_p)$ as $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))/(p, \cdots, v_n)$. Using Proposition 2.2.14(b), one can show that

$$\pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); \mathbf{F}_p) \cong \mathbf{F}_p[x, y]/x^2;$$

we will compute $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); \mathrm{BP}\langle n\rangle)$ using this calculation and $n+1$ Bockstein spectral sequences. The v_0-Bockstein spectral sequence is given by

$$E_1^{*,*} = \pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); \mathbf{F}_p)[v_0] \cong \mathbf{F}_p[v_0, x, y]/x^2 \Rightarrow \pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n); \mathbf{Z}_p). \tag{7}$$

It follows from the Bockstein calculation in Proposition 2.2.14(b) that there is a d_1-differential

$$d_1(y) = v_0 x, \tag{8}$$

which implies $d_1(y v_0^n) = v_0^{n+1} x$ (by $\mathbf{F}_p[v_0]$-linearity). However, (8) does not immediately imply differentials on powers of y, since $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ does not admit the structure of a ring (so the spectral sequence is not multiplicative). However, this is easily resolved: there is a map to the above Bockstein spectral sequence from the Bockstein spectral sequence computing $\pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{Z}_p)$, whose E_1-page is

$$'E_1^{*,*} \cong \pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{F}_p)[v_0].$$

The calculation of $\mathrm{H}_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle); \mathbf{F}_p)$ is described in [**AR05**, Theorem 5.12]; from this, one can compute $\pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{F}_p)$. Here, we will only need to observe that the classes $x, y \in E_1^{*,*}$ lift along the map $'E_1^{*,*} \to E_1^{*,*}$. We will continue to denote these lifts by x and y; there is still a d_1-differential $d_1(y) = v_0 x$ in $'E_1^{*,*}$. Since $\mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{Z}_p)$ admits the structure of an $\mathbf{E}_2$-ring, the above spectral sequence is multiplicative. Therefore, we may appeal to [**May70**, Proposition 6.8], which gives higher differentials on powers of y. In particular, we claim:

$$d_{v_p(j)+1}(y^j) = v_0^{v_p(j)+1} x y^{j-1}, \tag{9}$$

up to a unit in $\mathbf{F}_p^\times$. By taking base-p expansions, it suffices to prove this differential when j is a power of p, say $j = p^k$: then, (9) says that $d_{k+1}(y^{p^k}) = v_0^{k+1} x y^{p^k - 1}$. Using [**May70**, Proposition 6.8] for $k > 1$, we have

$$d_{k+1}((y^{p^{k-1}})^p) = v_0 (y^{p^{k-1}})^{p-1} d_k(y^{p^{k-1}}) = v_0 y^{p^k - p^{k-1}} d_k(y^{p^{k-1}});$$

this inductively implies (9) once we establish the case $k = 1$.

For $p = 2$, [**May70**, Proposition 6.8] says that

$$d_2(y^2) = v_0 y d_1(y) + Q_1(d_1(y)) = v_0^2 x y^2 + Q_1(v_0 x).$$

But

$$Q_1(x) = Q_1(\sigma(\zeta_{n+1}^2)) = \sigma(Q_2(\zeta_{n+1}^2)) = \sigma(\zeta_{n+2}^2),$$

which is zero. Therefore, we see that $d_2(y^2) = v_0 x y^2$, as desired. For $p > 2$, [**May70**, Proposition 6.8] says that

$$d_2(y^p) = v_0 y^{p-1} d_1(y) + \sum_{1 \le j \le r} j[d_1(y) y^{j-1}, d_1(y) y^{p-j-1}],$$

for some integer r. The "correction" term is a v_0-multiple of sum of terms of the form $[xy^{j-1}, xy^{p-j-1}]$. Note that this class lives in $\pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{F}_p)$, but for the calculation of (7), we are only concerned with the image of this class in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$. We claim that the image of $[xy^{j-1}, xy^{p-j-1}]$ in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$ vanishes, so the correction terms above vanish. To prove this, observe that the Leibniz rule implies that, in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbf{F}_p)$, we have

$$\begin{aligned}[xy^{j-1}, xy^{p-j-1}] &= x[y^{j-1}, xy^{p-j-1}] + y^{j-1}[x, xy^{p-j-1}]\\ &= x^2[y^{j-1}, y^{p-j-1}] + xy^{p-j-1}[y^{j-1}, x] + y^{p-2}[x,x] + xy^{j-1}[x, y^{p-j-1}].\end{aligned}$$

Here, all terms are written up to sign; this will not matter, since we will show that each of the terms in the sum above vanish. The first term vanishes since $x^2 = 0$, and the third term vanishes since $[x,x] = 0$. For the second and fourth term, we will argue more generally that the image of $[x, y^k]$ in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$ vanishes for any $k \geq 0$. The Leibniz rule implies that $[x, y^k] = ky^{k-1}[x,y]$, so it suffices to show that the image of $[x,y]$ in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$ vanishes.

Since $[x,y]$ lives in degree $|x| + |y| + 1 = (2p^{n+1} - 1) + 2p^{n+1} + 1 = 4p^{n+1}$ and $\pi_{4p^{n+1}}\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p) \cong \mathbf{F}_p\{y^2\}$, we must have $[x,y] \doteq y^2$ in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$ if $[x,y]$ is nonzero. To show that $[x,y] \not\doteq y^2$, we observe that the $\mathbf{E}_2$-map ι : $\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbf{F}_p) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU};\mathbf{F}_p)$ factors through $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$. The classes x and y are in the image of the map $\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbf{F}_p) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$, and x is killed by the map ι. Since ι is an $\mathbf{E}_2$-map, we must have $\iota([x,y]) = [\iota(x), \iota(y)] = 0$; however, $\iota(y^2) = \iota(y)^2$ is nonzero. Therefore, $[x,y] \not\doteq y^2$; but since $\pi_{4p^{n+1}}\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{F}_p)$ is a 1-dimensional $\mathbf{F}_p$-vector space spanned by y^2, we must have $[x,y] = 0$.

The upshot of this discussion is that the E_r-page of (7) is given by

$$E_r^{*,*} = \mathbf{F}_p[v_0, y^{p^{r-1}}]\{1, x, xy, xy^2, \cdots\}/(v_0^i xy^{p^{i-1}j-1}, 1 \leq i \leq r-1, 1 \leq j \leq p-1).$$

In particular, no power of y survives to the E_∞-page, and since v_0 represents p, we can resolve the v_0-extensions to conclude that

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n);\mathbf{Z}_p) \cong \mathbf{Z}_p \oplus \bigoplus_{j\geq 1} \mathbf{Z}_p/p^{v_p(j)+1}\{xy^{j-1}\}. \tag{10}$$

Note that $|xy^{j-1}| = 2jp^{n+1} - 1$.

The higher Bockstein spectral sequences (for $v_1, \cdots, v_n$) all collapse at the E_1-page for degree reasons, as we now explain. For the v_m-Bockstein spectral sequence with $1 \leq m \leq n$, one can argue by induction on m (the base case is the same argument as the inductive step). First, observe that $v_1, \cdots, v_n$ survive the Bockstein spectral sequence, since $\mathrm{BP}\langle n\rangle$ splits off $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$. In particular, there cannot be any differential with target given by a product of monomials in the v_is. By $\mathbf{Z}_p[v_1, \cdots, v_m]$-linearity, any differential must therefore be of the form

$$d_r(xy^{j-1}) = v_{i_1}^{r_1} \cdots v_{i_a}^{r_a} v_m^r xy^{k-1}$$

for some j, k, exponents $r_1, \cdots, r_a$, and $1 \leq i_1, \cdots, i_a < m$. (More precisely, it will be a sum of monomials of the above form, but this point will not matter.) But $d_r(xy^{j-1})$ has bidegree $(t-s, s) = (2jp^{n+1} - 2, r)$, while $v_{i_1}^{r_1} \cdots v_{i_a}^{r_a} v_m^r xy^{k-1}$ has bidegree $(t-s, s) = (2r_1(p^{i_1} - 1) + \cdots + 2r_a(p^{i_a} - 1) + 2r(p^m - 1) + 2kp^{n+1} - 1, r)$. Such a differential is therefore not possible, since $2jp^{n+1} - 2$ is even, while $2r_1(p^{i_1} - 1) + \cdots + 2r_a(p^{i_a} - 1) + 2r(p^m - 1) + 2kp^{n+1} - 1$ is odd. The calculation of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ now follows from (10).

Since $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n)) \simeq \mathrm{THH}(\mathrm{BP}\langle n\rangle) \otimes_{\mathrm{THH}(X(p^n))} X(p^n)$, we see that $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ admits the structure of a $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$-module. There is an

$\mathbf{E}_2$-map $\mathrm{BP}\langle n\rangle \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)$, so that $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ acquires the structure of a $\mathrm{BP}\langle n\rangle$-module by restriction of scalars. Therefore, each of the $\mathrm{BP}\langle n\rangle_*$-module generators of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ lift to maps of spectra from shifts of $\mathrm{BP}\langle n\rangle$ to $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$. Moreover, the resulting map $\mathrm{BP}\langle n\rangle[B\Delta_n] \oplus \bigoplus_{j\geq 1} \Sigma^{2jp^{n+1}-1}\mathrm{BP}\langle n\rangle[B\Delta_n]/p^{v_p(j)+1} \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n))$ induces an isomorphism on homotopy by construction, so we obtain Theorem 2.2.4(b). $\square$

Remark 2.2.17. When $n = 0$, one may view the Bockstein calculation of Theorem 2.2.4(b) as a translation of the Serre spectral sequence for the fibration (5). Assume that $p > 2$. Indeed, the Serre spectral sequence is given by

$$E^2_{*,*} = \mathrm{H}_*(S^{2p-1}; \mathbf{Z}_p) \otimes \mathrm{H}_*(\Omega S^{2p+1}; \mathbf{Z}_p) \cong \mathbf{Z}_p[x, y]/x^2 \Rightarrow \mathrm{H}_*(\Omega S^3\langle 3\rangle; \mathbf{Z}_p).$$

There is a single family of differentials, determined multiplicatively from

$$d^{2p}(y) = px;$$

this implies that $d^{2p}(y^m) = mpy^{m-1}x$. The Serre spectral sequence collapses at the E^{2p+1}-page, and the resulting answer is precisely (10). In fact, if $\phi_n : \Omega^2 S^{2p^n+1} \to S^{2p^n-1}$ is a charming map in the sense of [**Dev23a**, Definition 4.1.1] (such as the Cohen-Moore-Neisendorfer map of [**CMN79a, CMN79b, Nei81**]), the proof of Theorem 2.2.4(b) can be understood as a calculation of $\pi_*\mathrm{BP}\langle n-1\rangle[B\,\mathrm{fib}(\phi_n)]$ using the Serre spectral sequence for the Cohen-Moore-Neisendorfer type fibration

$$S^{2p^n-1} \to B\,\mathrm{fib}(\phi_n) \to \Omega S^{2p^n+1}. \tag{11}$$

The Serre spectral sequence for (11) is exactly the same as that of (5): the E^2-page is given by

$$E^2_{*,*} = \mathrm{H}_*(S^{2p^n-1}; \mathbf{Z}_p) \otimes \mathrm{H}_*(\Omega S^{2p^n+1}; \mathbf{Z}_p) \cong \mathbf{Z}_p[x, y]/x^2 \Rightarrow \mathrm{H}_*(B\,\mathrm{fib}(\phi_n); \mathbf{Z}_p).$$

There is a single family of differentials, determined multiplicatively from

$$d^{2p^n}(y) = px;$$

this implies that $d^{2p^n}(y^m) = mpy^{m-1}x$, and the Serre spectral sequence collapses at the E^{2p^n+1}-page. The upshot is that

$$\mathrm{H}_i(B\,\mathrm{fib}(\phi_n); \mathbf{Z}_p) \cong \begin{cases} \mathbf{Z}_p & i = 0, \\ \mathbf{Z}_p/pk & 2kp^n - 1, \\ 0 & \text{else.} \end{cases}$$

In fact, Theorem 2.2.4(b) implies that there is an equivalence of $\mathrm{BP}\langle n-1\rangle$-modules

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^{n-1})) \simeq \mathrm{BP}\langle n-1\rangle[B\Delta_{n-1} \times B\,\mathrm{fib}(\phi_n)].$$

The calculations of Theorem 2.2.4 can be predicted from the results of [**Dev23a**]. Let us suppose that p is odd for simplicity. Assuming [**Dev23a**, Conjectures D and E], [**Dev23a**, Corollary B] implies that there is a map $\Omega^2 S^{2p^n+1} \to \mathrm{BGL}_1(X(p^n))$ whose Thom spectrum is $\mathrm{BP}\langle n-1\rangle[\Omega\Delta_n]$. This implies that there is an equivalence of spectra $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n-1\rangle[B\Delta_n \times \Omega S^{2p^n+1}]$; this is precisely the first part of Theorem 2.2.4(a). Moreover, [**Dev23a**, Theorem A] says (still assuming the aforementioned conjectures) that the Thom spectrum of the composite $\mathrm{fib}(\phi_n) \to \Omega^2 S^{2p^n+1} \to \mathrm{BGL}_1(X(p^n))$ is $\mathrm{BP}\langle n\rangle[\Omega\Delta_n]$. This can be shown to imply that $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \pi_*\mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n]$, which is indeed confirmed by Theorem 2.2.4(a). This result also implies that there is an equivalence of spectra $\mathrm{THH}(\mathrm{BP}\langle n\rangle/X(p^n)) \simeq \mathrm{BP}\langle n\rangle[B\Delta_n \times B\,\mathrm{fib}(\phi_{n+1})]$, which is indeed true by Theorem 2.2.4(b). We will state the results predicted by this discussion as a conjecture.

Conjecture 2.2.18. *Fix an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum $\mathrm{BP}\langle n-1\rangle$. Then $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ admits the structure of an S^1-equivariant $\mathrm{BP}\langle n\rangle$-module (where S^1 acts trivially on $\mathrm{BP}\langle n\rangle$), and the equivalences of Theorem 2.2.4(a) refine to p-complete equivalences of spectra*

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m} \simeq \mathrm{BP}\langle n\rangle^{t\mathbf{Z}/m}[B\Delta_n],$$
$$\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \simeq \mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n].$$

The first equivalence is S^1-equivariant for the residual S^1/μ_m-action on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m}$ and $\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/m}$.

Remark 2.2.19. The primary difficulty with proving Conjecture 2.2.18 is that it is not clear how to endow $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ or $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m}$ with the structure of $\mathrm{BP}\langle n\rangle$-modules. Nevertheless, a small part of the final equivalence in Conjecture 2.2.18 can be proved unconditionally when $n=1$. Namely, there is a map $\mathrm{TP}(\mathbf{Z}_p/X(p)) \to \bigoplus_{j>-(p-1)} \Sigma^{2j}\mathrm{BP}\langle 1\rangle$ which induces the inclusion of summands on mod p cohomology. (This is the "easy" range, since the first predicted summand of $\mathrm{TP}(\mathbf{Z}_p/X(p))$ which is not covered by this claim is $\Sigma^{-2(p-1)}\mathrm{BP}\langle 1\rangle$; but π_0 of this spectrum this is exactly where the class v_1 lives.) We computed the mod p homology of $\mathrm{TP}(\mathbf{Z}_p/X(p))$ in Proposition 2.2.15. This implies that $\mathrm{H}^{*,c}(\mathrm{TP}(\mathbf{Z}_p/X(p)); \mathbf{F}_p) \cong \mathrm{H}^*(\mathrm{BP}\langle 1\rangle; \mathbf{F}_p)(\!(\hbar)\!) \otimes_{\mathbf{F}_p} \mathrm{H}^*(\mathrm{BSU}(p-1); \mathbf{F}_p)$. There is an Adams spectral sequence

$$\mathrm{Ext}^{s,t+2j}_{\mathcal{A}_*}(\mathcal{A}/\!\!/\mathcal{E}(1), \mathcal{A}/\!\!/\mathcal{E}(1))(\!(\hbar)\!)\otimes_{\mathbf{F}_p}\mathrm{H}^*(\mathrm{BSU}(p-1); \mathbf{F}_p) \Rightarrow \pi_0\mathrm{Map}(\mathrm{TP}(\mathbf{Z}_p/X(p)), \Sigma^{2j}\mathrm{BP}\langle 1\rangle)^\wedge_p.$$

We wish to show that for $j > -(p-1)$, any class in bidegree $(s, t-s) = (0, 2j)$ survives to the E_∞-page. For this, it suffices to show that there can be no nonzero d_r-differential off this class for $r \geq 2$. This differential would necessarily land in $(r, 2j-1)$. By [**AP76**, Proposition 4.1], $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathcal{A}/\!\!/\mathcal{E}(1), \mathcal{A}/\!\!/\mathcal{E}(1))$ vanishes for $s \geq 1$, $t-s$ odd, and $t-s \geq -2(p-1)$. In particular, we see that taking $(s, t-s) = (r, 2j-1)$, we have $2j-1 \geq -2(p-1)$ precisely when $j > -(p-1)$. Therefore, we get a map $\mathrm{TP}(\mathbf{Z}_p/X(p)) \to \Sigma^{2j}\mathrm{BP}\langle 1\rangle$ for every $j > -(p-1)$, which gives the desired claim.

2.3. Variant: THH over a deeper base.

In Theorem 2.2.4, we saw a "polynomial" generator in degree $2p^n$, where n is the height. When $n=0$, this reduces the Bökstedt generator in degree 2; we will now discuss a variant of Theorem 2.2.4 when $n=1$, where one obtains a generator in degree 2.

Construction 2.3.1. Let $\mathrm{U}(1) \to \mathrm{SU}(p)$ denote the inclusion given by the homomorphism

$$\lambda \mapsto \mathrm{diag}(\lambda, \cdots, \lambda, \lambda^{1-p}).$$

There is an induced map $\mathrm{BU}(1) \to \mathrm{BSU}(p)$, which defines an $\mathbf{E}_2$-map $\Omega\mathrm{U}(1) \simeq \mathbf{Z} \to \Omega\mathrm{SU}(p)$. Let $J(p)$ denote the Thom spectrum of the composite $\mathbf{E}_2$-map $\mu : \Omega\mathrm{U}(1) \to \Omega\mathrm{SU}(p) \to \Omega\mathrm{SU} \simeq \mathrm{BU}$. Then $J(p)$ admits an $\mathbf{E}_2^{\mathrm{fr}}$-structure by Proposition 2.1.11 such that there is an $\mathbf{E}_2^{\mathrm{fr}}$-algebra map $J(p) \to X(p)$. Note that the underlying $\mathbf{E}_1$-map of μ is null, since $B\mu : S^1 \to \mathrm{B}^2\mathrm{U} \simeq \mathrm{SU}$ is a class in $\pi_1(\mathrm{SU}) = 0$. Therefore, the underlying $\mathbf{E}_1$-ring of $J(p)$ is $S[\mathbf{Z}] = S[t^{\pm 1}]$. Moreover, the underlying $\mathbf{E}_1$-map of $J(p) \to X(p) \to \mathbf{Z}_p$ is the map $S[t^{\pm 1}] \to \mathbf{Z}_p$ sending $t \mapsto 1$.

Proposition 2.3.2. *There is an equivalence* $\mathrm{THH}(T(1)/J(p)) \simeq T(1)[J_{p-1}(S^2)]$. *Similarly,* $\mathrm{THH}(X(p)/J(p)) \simeq X(p)[J_{p-1}(S^2) \times \mathrm{SU}(p-1)]$.

Proof. Indeed, $\mathrm{THH}(T(1)/J(p)) \simeq \mathrm{THH}(T(1)) \otimes_{\mathrm{THH}(J(p))} J(p)$ is equivalent to $T(1)[S^{2p-1}] \otimes_{T(1)[S^1]} T(1)$; but there is a fiber sequence

$$S^1 \to S^{2p-1} \to S^{2p-1}/S^1 = \mathbf{C}P^{p-1} \simeq J_{p-1}(S^2),$$

from which the desired claim follows. $\square$

Proposition 2.3.3. *The following statements are true:*

(a) There is an equivalence $\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[\Omega S^3]$. *In particular,* $\pi_*\mathrm{THH}(\mathbf{Z}_p/J(p)) \cong \mathbf{Z}_p[x]$ *with* $|x| = 2$. *On homotopy, the map* $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/X(p))$ *is given by*

$$x^j \mapsto \begin{cases} \theta^{j/p} & j \in p\mathbf{Z}, \\ 0 & \textit{else.} \end{cases}$$

(b) The canonical map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{F}_p/J(p))$ *factors through the unit* $\mathrm{THH}(\mathbf{F}_p) \to \mathrm{THH}(\mathbf{F}_p/J(p))$, *and defines an equivalence* $\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{Z}_p/J(p)) \xrightarrow{\sim} \mathrm{THH}(\mathbf{F}_p)$ *of* $\mathrm{THH}(\mathbf{Z}_p)$*-modules.*

Proof. For part (a), we begin by observing that there is an equivalence

$$\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathrm{THH}(\mathbf{Z}_p) \otimes_{\mathrm{THH}(J(p))} J(p) \simeq \mathbf{Z}_p[\Omega S^3\langle 3\rangle] \otimes_{\mathbf{Z}_p[\mathrm{U}(1)]} \mathbf{Z}_p.$$

The map $\mathbf{Z}_p \otimes_{J(p)} \mathrm{THH}(J(p)) \to \mathrm{THH}(\mathbf{Z}_p)$ factors through $\mathbf{Z}_p \otimes_{X(p)} \mathrm{THH}(X(p)) \to \mathrm{THH}(\mathbf{Z}_p)$, and can be identified with $\mathbf{Z}_p$-chains of the composite

$$\mathrm{U}(1) \to \mathrm{SU}(p) \to S^{2p-1} \xrightarrow{\alpha_1} \Omega S^3\langle 3\rangle.$$

Note that the map $\mathrm{U}(1) \to S^{2p-1}$ is the fiber of the map $S^{2p-1} \to \mathbf{C}P^{p-1}$. This composite can be identified with action of S^1 on $\Omega S^3\langle 3\rangle$. Since there is a fiber sequence

$$S^1 \to \Omega S^3\langle 3\rangle \to \Omega S^3,$$

we see that $\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[\Omega S^3]$. To identify the map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/X(p))$, observe that $\mathbf{C}P^{p-1} \simeq J_{p-1}(S^2)$ and that there is a square where each row and column is a fiber sequence:

$$\begin{tikzcd}
\Omega(\mathrm{SU}(p-1) \times \mathbf{C}P^{p-1}) \simeq \Omega\mathrm{SU}(p)/S^1 \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \mathbf{C}P^{p-1} \times \mathrm{SU}(p-1) \arrow[d] \\
S^1 \arrow[r] \arrow[d] & \Omega S^3\langle 3\rangle \arrow[r] \arrow[d] & \Omega S^3 \arrow[d, "H_p"] \\
\mathrm{SU}(p) \arrow[r] & \Omega S^3\langle 3\rangle \arrow[r] & \Omega S^{2p+1} \times \mathrm{BSU}(p-1).
\end{tikzcd}$$

The effect of the map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/X(p))$ is dictated by the bottom-right vertical map, which is induced by the James-Hopf map $H_p : \Omega S^3 \to \Omega S^{2p+1}$. On $\mathbf{Z}_p$-homology, the effect of the James-Hopf map is as stated in Proposition 2.3.3(a).

For part (b), there is an equivalence

$$\mathrm{THH}(\mathbf{F}_p/J(p)) \simeq \mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(J(p))} J(p) \simeq \mathbf{F}_p[\Omega S^3] \otimes_{\mathbf{F}_p[\mathrm{U}(1)]} \mathbf{F}_p.$$

However, the map $\mathbf{F}_p \otimes_{J(p)} \mathrm{THH}(J(p)) \to \mathrm{THH}(\mathbf{F}_p)$ factors through $\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{Z}_p) \to \mathrm{THH}(\mathbf{F}_p)$, and can be identified with $\mathbf{F}_p$-chains of the composite of $\mathrm{U}(1) \to \Omega S^3\langle 3\rangle$

with the canonical map $\Omega S^3\langle 3\rangle \to \Omega S^3$. This composite is null as an $\mathbf{E}_1$-map (in fact, as an $\mathbf{E}_2$-map), since there is a fiber sequence of $\mathbf{E}_1$-spaces

$$\mathrm{BU}(1) \simeq \mathbf{C}P^\infty \to S^3\langle 3\rangle \to S^3.$$

Therefore, we see that

$$\mathrm{THH}(\mathbf{F}_p/J(p)) \simeq \mathbf{F}_p[\Omega S^3] \otimes_{\mathbf{F}_p} (\mathbf{F}_p \otimes_{\mathbf{F}_p[\mathrm{U}(1)]} \mathbf{F}_p) \simeq \mathbf{F}_p[\Omega S^3 \times \mathbf{C}P^\infty].$$

This implies that the map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{F}_p/J(p))$ factors through $\mathrm{THH}(\mathbf{F}_p) \to \mathrm{THH}(\mathbf{F}_p/J(p))$. In turn, we obtain a map $\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{F}_p)$ which sends the generators in $\pi_*(\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{Z}_p/J(p))) \cong \mathbf{F}_p[x]$ to the generators in $\pi_*\mathrm{THH}(\mathbf{F}_p) \cong \mathbf{F}_p[\sigma]$. Therefore, the map $\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{F}_p)$ is an equivalence, as desired. □

Remark 2.3.4. The map $J(p) \to X(p)$ induces a map $u : \mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/X(p))$. Under Theorem 2.2.4 and Proposition 2.3.3, the map u can be identified with the $\mathbf{Z}_p$-chains of the composite

$$\Omega S^3 \to \Omega S^{2p+1} \to \Omega S^{2p+1} \times \mathrm{BSU}(p-1);$$

here, the map $\Omega S^3 \to \Omega S^{2p+1}$ is the Hopf map. This claim follows from the proof of Proposition 2.3.3, Proposition 2.3.2, and the EHP fibration

$$J_{p-1}(S^2) \to \Omega S^3 \to \Omega S^{2p+1}.$$

In particular, the map u induces the map $\mathbf{Z}_p[x] \to \mathbf{Z}_p[\theta] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\mathrm{BSU}(p-1)]$ which sends $x^m \mapsto \theta^{m/p}$ if $p \mid m$ and $x^m \mapsto 0$ otherwise.

Note that if $T(1)$ were an $\mathbf{E}_2^{\mathrm{fr}}$-algebra, the map u would factor through $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{Z}_p/T(1))$; and under the equivalences of Theorem 2.2.4 and Proposition 2.3.3, this would identify with the $\mathbf{Z}_p$-chains of the Hopf map.

Remark 2.3.5. Proposition 2.3.3 demonstrates the dependence of $\mathrm{THH}(R'/R)$ on the $\mathbf{E}_1$-R-algebra structure on R'. Indeed, recall that the underlying $\mathbf{E}_1$-map of the $\mathbf{E}_2$-map $J(p) \to X(p) \to \mathbf{Z}_p$ is the map $S[t^{\pm 1}] \to \mathbf{Z}_p$ sending $t \mapsto 1$. Proposition 2.3.3 states that $\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[\Omega S^3]$. However, suppose that $S[t^{\pm 1}] = S[\mathbf{Z}]$ is equipped with its standard $\mathbf{E}_2$-structure, and $\mathbf{Z}_p$ is viewed as an $\mathbf{E}_1$-$S[\mathbf{Z}]$-algebra via the composite $S[\mathbf{Z}] \to S \to \mathbf{Z}_p$. Then $\mathrm{THH}(\mathbf{Z}_p/S[\mathbf{Z}]) \simeq \mathrm{THH}(\mathbf{Z}_p) \otimes S[\mathbf{C}P^\infty] \simeq \mathbf{Z}_p[\Omega S^3\langle 3\rangle \times \mathbf{C}P^\infty]$. Since $\mathbf{Z}_p[\Omega S^3\langle 3\rangle \times \mathbf{C}P^\infty] \not\simeq \mathbf{Z}_p[\Omega S^3]$, we conclude that $\mathrm{THH}(\mathbf{Z}_p/S[\mathbf{Z}]) \not\simeq \mathrm{THH}(\mathbf{Z}_p/J(p))$.

Corollary 2.3.6. *There is an isomorphism* $\pi_*\mathrm{TP}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[t^{\pm 1}]^\wedge_{(t-1)}(\!(\hbar)\!)$ *with* $|\hbar| = -2$.

Corollary 2.3.7. *If $\mathcal{C}$ is a $\mathbf{Z}_p$-linear ∞-category, there is a (non-S^1-equivariant) equivalence* $\mathrm{THH}(\mathcal{C}/J(p)) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \simeq \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)$.

Proof. By Proposition 2.3.3(b), there is an equivalence $\mathrm{THH}(\mathbf{Z}_p/J(p)) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \simeq \mathrm{THH}(\mathbf{F}_p)$ of $\mathrm{THH}(\mathbf{Z}_p)$-modules. It follows that

$$\begin{aligned}\mathrm{THH}(\mathcal{C}/J(p)) \otimes_{\mathbf{Z}_p} \mathbf{F}_p &\simeq \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathrm{THH}(\mathbf{Z}_p/J(p)) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \\ &\xrightarrow{\sim} \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathrm{THH}(\mathbf{F}_p) \simeq \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p),\end{aligned}$$

as desired. □

Remark 2.3.8. Recall from [**AMN18**, Theorem 3.5] that if $S[z] = S[\mathbf{Z}_{\geq 0}]$ denotes the flat polynomial ring on a class in degree 0, then there is an isomorphism $\pi_* \mathrm{THH}(\mathbf{Z}_p/S[z]) \cong \mathbf{Z}_p[\sigma^2(z-p)]$, where the $\mathbf{E}_\infty$-map $S[z] \to \mathbf{Z}_p$ sends $z \mapsto p$. This implies that $\pi_* \mathrm{TP}(\mathbf{Z}_p/S[z]) \cong \mathbf{Z}_p[z]^\wedge_{(z-p)}(\!(\hbar)\!)$. Similarly, there is an isomorphism $\pi_* \mathrm{TP}(\mathbf{Z}_p/S[\![\widetilde{p}]\!]) \cong \mathbf{Z}_p[\![\widetilde{p}]\!]^\wedge_{(\widetilde{p}-p)}(\!(\hbar)\!)$, where $\widetilde{p} \mapsto p$ and $S[\![\widetilde{p}]\!] = \left(S[q^{\pm 1}]^\wedge_{(p,q-1)}\right)^{h\mathbf{F}_p^\times}$.

In the same way, there is an isomorphism $\pi_* \mathrm{THH}(\mathbf{Z}_p/S[t^{\pm 1}]) \cong \mathbf{Z}_p[\sigma^2(t + p - 1)]$, where the $\mathbf{E}_\infty$-map $S[t^{\pm 1}] \to \mathbf{Z}_p$ sends $t \mapsto 1-p$. This implies that $\pi_* \mathrm{TP}(\mathbf{Z}_p/S[t^{\pm 1}]) \cong \mathbf{Z}_p[t^{\pm 1}]^\wedge_{(t+p-1)}(\!(\hbar)\!)$. In light of the obvious analogy to Proposition 2.3.3 and Corollary 2.3.6, it is natural to ask: what is the role of $J(p)$?

To answer this, let us assume for simplicity that $T(1)$ admits the structure of an $\mathbf{E}_2$-ring. The main utility of $J(p)$ is that it admits, by construction, a direct comparison to $T(1)$; one can view $J(p)$ as containing roughly the same "height 1" information as $T(1)$. On the other hand, we do not know how to directly compare $S[t^{\pm 1}]$ (with the standard $\mathbf{E}_2$-structure) to $T(1)$. (Both admit $\mathbf{E}_1$-algebra maps to $T(1)[t^{\pm 1}]$, but this is somewhat unsatisfactory.) One can therefore view Construction 2.3.1 as an explicit modification of the $\mathbf{E}_2$-structure on $S[t^{\pm 1}]$ such that the resulting $\mathbf{E}_2$-algebra admits an interesting map to $T(1)$.

It is natural to ask if Proposition 2.3.3 admits a generalization to $\mathrm{BP}\langle n-1\rangle$. At height 1 and $p=2$, we can explicitly construct some $\mathbf{E}_2^{\mathrm{fr}}$-rings which give higher analogues of $J(p)$, but a general construction at higher heights and other primes eludes us.

Construction 2.3.9. Recall from Remark 2.1.10 that there is an $\mathbf{E}_2$-map $\Omega\mathrm{Sp}(2) \to \mathrm{BU}$ whose Thom spectrum is equivalent to $T(2)$ at $p=2$. Let $T_2(2)$ denote the $\mathbf{E}_2^{\mathrm{fr}}$-ring defined as the Thom spectrum of the composite $\mathbf{E}_2$-map

$$\Omega\mathrm{Spin}(4) \to \Omega\mathrm{Sp}(2) \to \mathrm{BU},$$

where the first map is induced by the inclusion $\mathrm{Spin}(4) \subseteq \mathrm{Spin}(5) \cong \mathrm{Sp}(2)$. Similarly, let $T_4(2)$ denote the $\mathbf{E}_2^{\mathrm{fr}}$-ring defined as the Thom spectrum of the composite $\mathbf{E}_2$-map

$$\Omega\mathrm{U}(2) \to \Omega\mathrm{Sp}(2) \to \mathrm{BU},$$

where the first map is induced by the inclusion $\mathrm{U}(2) \subseteq \mathrm{Sp}(2)$. Note that this inclusion factors as $\mathrm{U}(2) \to \mathrm{Spin}(4) \to \mathrm{Sp}(2)$, so that there is a composite map of $\mathbf{E}_2^{\mathrm{fr}}$-rings

$$T_4(2) \to T_2(2) \to T(2).$$

Remark 2.3.10. There is a fiber sequence

$$\Omega S^3 \to \Omega\mathrm{Spin}(4) \to \Omega S^3,$$

which implies that $\mathrm{MU}_*(T_2(2)) \simeq \mathrm{MU}_*[t_1, x_2]$ where $|x_2| = 2$. Similarly, there is a fiber sequence

$$\Omega S^3 \to \Omega\mathrm{U}(2) \to \Omega S^1 \simeq \mathbf{Z},$$

which implies that $\mathrm{MU}_*(T_4(2)) \simeq \mathrm{MU}_*[t_1, x_0^{\pm 1}]$ where $|x_0| = 0$.

Lemma 2.3.11. *There is a diffeomorphism* $\mathrm{Sp}(2)/\mathrm{Spin}(4) \cong S^4$*, as well as a homotopy equivalence* $\mathrm{Sp}(2)/\mathrm{U}(2) \simeq J_3(S^2)$.

Proof. The first diffeomorphism follows immediately from the isomorphism $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ and the resulting chain

$$\mathrm{Sp}(2)/\mathrm{Spin}(4) \cong \mathrm{Spin}(5)/\mathrm{Spin}(4) \cong \mathrm{SO}(5)/\mathrm{SO}(4) \cong S^4.$$

To prove the second equivalence, the key input is [**Ame18**, Proposition 4.3], which says that there is a fiber sequence

$$V_2(\mathbf{R}^5) \to J_3(S^2) \to \mathbf{C}P^\infty;$$

in other words, there is an S^1-action on the Stiefel manifold $V_2(\mathbf{R}^5)$ such that $V_2(\mathbf{R}^5)/S^1 \cong J_3(S^2)$. Recall that $V_2(\mathbf{R}^5)$ is diffeomorphic to $\mathrm{SO}(5)/\mathrm{SO}(3) \cong \mathrm{Spin}(5)/\mathrm{SU}(2)$. It is not difficult to see that the claimed S^1-action on $V_2(\mathbf{R}^5)$ via the above fiber sequence is precisely the residual action of $\mathrm{U}(2)/\mathrm{SU}(2) \cong S^1$ on $\mathrm{Spin}(5)/\mathrm{SU}(2)$; in particular, we may identify $J_3(S^2) \simeq \mathrm{Spin}(5)/\mathrm{U}(2)$, as desired. □

Remark 2.3.12. The quotient $\mathrm{Sp}(2)/\mathrm{U}(2)$ is also known as the complex Lagrangian Grassmannian $\mathrm{Gr}_2^{\mathrm{Lag}}(T^*\mathbf{C}^2)$ of Lagrangian subspaces of $T^*\mathbf{C}^2$.

Warning 2.3.13. One should not confuse $\mathrm{Sp}(2)/\mathrm{U}(2)$ with the quotient $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times \mathrm{U}(1))$: indeed, Lemma 2.3.11 says that the former is homotopy equivalent to $J_3(S^2)$, while the latter is diffeomorphic to $S^7/\mathrm{U}(1) = \mathbf{C}P^3$. These spaces are not homotopy equivalent (although they do become equivalent after inverting 6).

Lemma 2.3.11 has the following amusing (inconsequential?) consequence:

Corollary 2.3.14. *Let $Q \subseteq \mathbf{C}P^4$ be a complex quadric, and let $\mathrm{Gr}_2^+(\mathbf{R}^5)$ denote the Grassmannian of oriented 2-planes in $\mathbf{R}^5$. Then, there are diffeomorphisms $Q \cong \mathrm{Gr}_2^{\mathrm{Lag}}(T^*\mathbf{C}^2) \cong \mathrm{Gr}_2^+(\mathbf{R}^5)$, and these are homotopy equivalent to $J_3(S^2)$.*

Proof. Since $\mathrm{Sp}(2)/\mathrm{U}(2) \cong \mathrm{SO}(5)/(\mathrm{SO}(3)\cdot\mathrm{SO}(2))$, we can identify $\mathrm{Sp}(2)/\mathrm{U}(2) = \mathrm{Gr}_2^{\mathrm{Lag}}(T^*\mathbf{C}^2)$ with $\mathrm{Gr}_2^+(\mathbf{R}^5)$. Therefore, Lemma 2.3.11 gives a homotopy equivalence $\mathrm{Gr}_2^+(\mathbf{R}^5) \simeq J_3(S^2)$. The desired claim now follows from the observation that $\mathrm{Gr}_2^+(\mathbf{R}^5)$ is diffeomorphic to a quadric $Q \subseteq \mathbf{C}P^4$ via the map $\mathrm{Gr}_2^+(\mathbf{R}^5) \to \mathrm{Gr}_1(\mathbf{C}^5) \cong \mathbf{C}P^4$ induced by the isomorphism $\mathbf{R}^{10} \xrightarrow{\sim} \mathbf{C}^5$; see [**KN96**, Example 10.6, Page 280]. □

Remark 2.3.15. There is a fibration[7] (see (57) for a more general statement)

$$S^2 \to J_3(S^2) \to S^4, \tag{12}$$

which, under the diffeomorphism

$$\mathrm{Spin}(4)/\mathrm{U}(2) \cong (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{U}(2) \cong \mathrm{SU}(2)/\mathrm{U}(1) \cong S^2,$$

can be identified via Lemma 2.3.11 with the fibration

$$\mathrm{Spin}(4)/\mathrm{U}(2) \to \mathrm{Sp}(2)/\mathrm{U}(2) \to \mathrm{Sp}(2)/\mathrm{Spin}(4).$$

[7] The fibration (12) is analogous to the "twistor" fibration (see (62)) $S^2 \to \mathbf{C}P^3 \to S^4$.

There is also a commutative diagram where each row and column is a fibration:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & \mathrm{U}(2) \arrow[r] \arrow[d] & S^3 \arrow[d] \\
\mathrm{Sp}(1) \arrow[r] \arrow[d] & \mathrm{Sp}(2) \arrow[r] \arrow[d] & S^7 \arrow[d] \\
S^2 \arrow[r] & J_3(S^2) \arrow[r] & S^4;
\end{tikzcd}$$

the rightmost vertical fiber sequence is the Hopf fibration. This diagram captures the relationships between $J(2)$, $T_4(2)$, $T(1)$, and $T(2)$.

Remark 2.3.16. The equivalence $\mathrm{Sp}(2)/\mathrm{U}(2) = \mathrm{Gr}_2^{\mathrm{Lag}}(T^*\mathbf{C}^2) \simeq J_3(S^2)$ of Lemma 2.3.11 can be used to understand the relationship between $T(2)$ and the Mahowald-Ravenel-Shick spectrum $y(2)$ from [**MRS01**] (at the prime 2).[8] Recall from Remark 2.1.10 that there is an $\mathbf{E}_2$-map $\Omega\mathrm{Sp}(2) \to \mathrm{BU}$ whose Thom spectrum is equivalent to $T(2)$ at $p = 2$. Similarly, recall that $y(2)$ is the Thom spectrum of the bundle determined by the map $\mu : \Omega J_3(S^2) \to \Omega^2 S^3 \to \mathrm{BO}$, where the second map is the extension of the Möbius bundle $S^1 \to \mathrm{BO}$. Under the equivalence $\mathrm{Sp}(2)/\mathrm{U}(2) \simeq J_3(S^2)$, the map $\mu : \Omega J_3(S^2) \to \mathrm{BO}$ can be identified with the composite

$$\Omega(\mathrm{Sp}(2)/\mathrm{U}(2)) \to \Omega(\mathrm{Sp}/\mathrm{U}) \to \mathrm{B}^2\mathrm{O} \xrightarrow{\eta} \mathrm{BO};$$

the middle map is obtained via Bott periodicity. Applying [**Dev23a**, Proposition 2.1.6] to loops on the fibration

$$\mathrm{Sp}(2) \to J_3(S^2) \to \mathrm{BU}(2),$$

we conclude that $y(2) = \Omega J_3(S^2)^\mu$ is equivalent as an $\mathbf{E}_1$-ring to the Thom spectrum of an $\mathbf{E}_1$-map $\mathrm{U}(2) \to \mathrm{BGL}_1(T(2))$. This implies, for instance, that $\mathrm{THH}(y(2)/T(2)) \simeq y(2)[\mathrm{BU}(2)]$. Since $k(2) \simeq y(2) \otimes_{T(2)} \mathrm{BP}\langle 2\rangle$, this implies that $\mathrm{THH}(k(2)/\mathrm{BP}\langle 2\rangle) \simeq k(2)[\mathrm{BU}(2)]$. Similarly, since $y(2) \otimes_{T(2)} \mathrm{ku} \simeq \mathbf{F}_2$, we also recover the observation that $\mathbf{F}_2$ is equivalent as an $\mathbf{E}_1$-ring to the Thom spectrum of an $\mathbf{E}_1$-map $\mathrm{U}(2) \to \mathrm{BGL}_1(\mathrm{ku})$, and hence that $\mathrm{HH}(\mathbf{F}_2/\mathrm{ku}) \simeq \mathbf{F}_2[\mathrm{BU}(2)]$ as $\mathbf{F}_2$-modules.

Proposition 2.3.17. *There is an equivalence* $\mathrm{THH}(T(2)/T_2(2)) \simeq T(2)[S^4]$*, as well as an equivalence* $\mathrm{THH}(T(2)/T_4(2)) \simeq T(2)[J_3(S^2)]$.

Proof. Note that η is nullhomotopic in $T_4(2)$ (and hence in $T_2(2)$), since the inclusion $\mathrm{SU}(2) \to \mathrm{U}(2)$ defines a map $S^2 \to \Omega\mathrm{U}(2)$, which in turn Thomifies to

[8] A simpler version of this discussion simply states that if $\Omega S^2 \to \mathrm{BO}$ is the map extending the Möbius bundle $S^1 \to \mathrm{BO}$, then [**Dev23a**, Proposition 2.1.6] along with loops on the fibration

$$S^3 \xrightarrow{\eta} S^2 \to \mathbf{C}P^\infty$$

implies that there is a map $S^1 \to \mathrm{BGL}_1(T(1))$ whose Thom spectrum is the $\mathbf{E}_1$-quotient $S/\!\!/2 = y(1)$. The map $S^1 \to \mathrm{BGL}_1(T(1))$ detects $1-2 \in \pi_0(T(1))^\times$ on the bottom cell of the source, so we recover the fact that $T(1)/2 \simeq y(1)$. In particular, $\mathrm{HH}(y(1)/T(1)) \simeq y(1)[\mathbf{C}P^\infty]$. Since $y(1) \otimes_{T(1)} \mathbf{Z}_2 \simeq \mathbf{F}_2$, this recovers the well-known observation that $\mathrm{HH}(\mathbf{F}_2/\mathbf{Z}_2) \simeq \mathbf{F}_2[\mathbf{C}P^\infty]$, at least as *modules* over $\mathbf{F}_2$. This argument does not give the $\mathbf{F}_2$-*algebra* structure, since $\mathrm{HH}(y(1)/T(1))$ is not a ring.

a map $C\eta \to T_4(2)$ which factors the unit. By Lemma 2.3.11, there are fiber sequences of $\mathbf{E}_1$-spaces

$$\Omega\mathrm{Spin}(4) \to \Omega\mathrm{Sp}(2) \to \Omega S^4,$$
$$\Omega\mathrm{U}(2) \to \Omega\mathrm{Sp}(2) \to \Omega J_3(S^2),$$

which by [**Dev23a**, Proposition 2.1.6] (see also [**Bea17**]) imply that $T(2)$ is a Thom spectrum of an $\mathbf{E}_1$-map $\Omega S^4 \to \mathrm{BGL}_1(T_2(2))$ (resp. $\Omega J_3(S^2) \to \mathrm{BGL}_1(T_4(2))$). Together with [**BCS10**], this implies the desired claim. ☐

Remark 2.3.18. Recall that $\mathrm{SU}(4)/\mathrm{Sp}(2) \cong S^5$. It follows that $\mathrm{THH}(X(4)/T(2)) \simeq X(4)[S^5]$. Similarly, recall that $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$; therefore, there is an diffeomorphism

$$\mathrm{SU}(4)/\mathrm{Spin}(4) \cong \mathrm{Spin}(6)/\mathrm{Spin}(4) \cong \mathrm{SO}(6)/\mathrm{SO}(4) \cong V_2(\mathbf{R}^6).$$

It follows that $\mathrm{THH}(X(4)/T_2(2)) \simeq X(4)[V_2(\mathbf{R}^6)]$. (Note also that $\mathrm{SU}(4)/\mathrm{Spin}(4) \cong \mathrm{SU}(4)/(\mathrm{SU}(2) \times \mathrm{SU}(2))$ can be viewed as an "oriented complex Grassmannian" $\widetilde{\mathrm{Gr}}_2(\mathbf{C}^4)$.) Finally, $\mathrm{THH}(X(4)/T_2(2)) \simeq X(4)[\mathrm{SU}(4)/\mathrm{U}(2)]$.

Corollary 2.3.19. *There are* 2*-complete equivalences of* ku*-modules*

$$\mathrm{THH}(\mathrm{ku}/T_2(2)) \simeq \mathrm{ku}[\Omega S^5],$$
$$\mathrm{THH}(\mathrm{ku}/T_4(2)) \simeq \mathrm{ku}[\Omega S^3].$$

Under these equivalences, the maps

$$\mathrm{THH}(\mathrm{ku}/T_4(2)) \to \mathrm{THH}(\mathrm{ku}/T_2(2)) \to \mathrm{THH}(\mathrm{ku}/T(2))$$

are induced by taking ku*-chains of the Hopf maps*

$$\Omega S^3 \xrightarrow{H} \Omega S^5 \xrightarrow{H} \Omega S^9.$$

Proof. Using Proposition 2.3.17, this follows from Theorem 2.2.4(a) (more precisely, the version with $p = 2$ and $n = 2$ for $\mathrm{THH}(\mathrm{BP}\langle 1\rangle/T(2)) \simeq \mathrm{ku}[\Omega S^9]$), and the fiber sequences of $\mathbf{E}_1$-spaces

$$\Omega S^4 \simeq \Omega(\mathrm{Sp}(2)/\mathrm{Spin}(4)) \to \Omega^2 S^5 \to \Omega^2 S^9,$$
$$\Omega J_3(S^2) \simeq \Omega(\mathrm{Sp}(2)/\mathrm{U}(2)) \to \Omega^2 S^3 \to \Omega^2 S^9$$

obtained by looping the 2-local EHP fiber sequences for S^4 and S^2. The identification of the maps $\mathrm{THH}(\mathrm{ku}/T_4(2)) \to \mathrm{THH}(\mathrm{ku}/T_2(2))$ and $\mathrm{THH}(\mathrm{ku}/T_2(2)) \to \mathrm{THH}(\mathrm{ku}/T(2))$ is an immediate consequence. ☐

Remark 2.3.20. Recall from Theorem 2.2.4(a) that the generator $\theta_2 \in \pi_8\mathrm{THH}(\mathrm{ku}/T(2))$ can be understood as $\sigma^2(v_2)$ (up to decomposables). Taking THH relative to the Thom spectrum $T_2(2)$ over $\Omega\mathrm{Spin}(4)$ can be regarded as extracting a square root of $\theta_2 \in \pi_8\mathrm{THH}(\mathrm{ku}/T(2))$. Similarly, taking THH relative to the Thom spectrum $T_4(2)$ over $\Omega\mathrm{U}(2)$ can be regarded as extracting a fourth root of $\theta_2 \in \pi_8\mathrm{THH}(\mathrm{ku}/T(2))$; hence the subscript 4. (Roughly, the generator of $\pi_4\mathrm{THH}(\mathrm{ku}/T_2(2))$ can be thought of as $\sigma^2(v_1)$; and the generator of $\pi_2\mathrm{THH}(\mathrm{ku}/T_4(2))$ can be thought of as $\sigma^2(2)$.) In particular, one should regard $T_4(2) = (\Omega\mathrm{U}(2))^\mu$ as the appropriate analogue of $J(p)$ at height 1 and $p = 2$.

Remark 2.3.21. Corollary 2.3.19 suggests that $\mathrm{ku}_2^\wedge$ is equivalent to the Thom spectrum of an $\mathbf{E}_1$-map $\Omega^2 S^3 \to \mathrm{BGL}_1(T_4(2))$. This could also be rephrased in a manner similar to the results of [**Dev23a**]: assuming [**Dev23a**, Conjectures D and E], [**Dev23a**, Corollary B] says that $\mathrm{ku}_2^\wedge$ is the Thom spectrum of a map $\Omega^2 S^9 \to \mathrm{BGL}_1(T(2))$. It follows from Proposition 2.3.17 that $T(2) \simeq \mathrm{colim}_{\Omega J_3(S^2)} T_4(2)$, so that [**Dev23a**, Corollary B] implies

$$\mathrm{ku}_2^\wedge \simeq \mathrm{colim}_{\Omega^2 S^9} T(2) \simeq \mathrm{colim}_{\Omega^2 S^9} \mathrm{colim}_{\Omega J_3(S^2)} T_4(2) \simeq \mathrm{colim}_{\Omega^2 S^3} T_4(2),$$

where the final equivalence comes from the $\mathbf{E}_1$-equivalence $\mathrm{colim}_{\Omega^2 S^9} \Omega J_3(S^2) \simeq \Omega^2 S^3$ arising from the EHP sequence.

This leads to the following, which we only state for $T(n)$; there is an analogue for $X(p^n)$, too.

Conjecture 2.3.22. *Fix a prime p and $n \geq 0$. For each $0 \leq j \leq n$, there are $\mathbf{E}_2^{\mathrm{fr}}$-rings $T_{p^j}(n)$ equipped with $\mathbf{E}_2^{\mathrm{fr}}$-maps*

$$T_{p^n}(n) \to \cdots \to T_{p^j}(n) \to T_{p^{j-1}}(n) \to \cdots \to T_0(n) = T(n)$$

such that there are p-complete equivalences

$$\mathrm{THH}(T(n)/T_{p^j}(n)) \simeq \mathrm{BP}\langle n-1\rangle[J_{p^j-1}(S^{2p^{n-j}})],$$
$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^j}(n)) \simeq \mathrm{BP}\langle n-1\rangle[\Omega S^{2p^{n-j}+1}].$$

The map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^j}(n)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^{j-1}}(n))$ induced by the $\mathbf{E}_2^{\mathrm{fr}}$-map $T_{p^j}(n) \to T_{p^{j-1}}(n)$ is given by $\mathrm{BP}\langle n-1\rangle$-chains on the Hopf map $\Omega S^{2p^{n-j}+1} \to \Omega S^{2p^{n-j+1}+1}$. In other words, if $\theta_n^{1/p^j} \in \pi_{2p^{n-j}}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^j}(n))$ denotes the generator (roughly, thought of as $\sigma^2(v_{n-j})$), then

$$\pi_{2p^{n-j}}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^j}(n)) \ni \theta_n^{1/p^j} \mapsto (\theta_n^{1/p^{j-1}})^p \in \pi_{2p^{n-j}}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^{j-1}}(n)).$$

In particular, Conjecture 2.3.22 says that for the putative $\mathbf{E}_2^{\mathrm{fr}}$-ring $T_{p^n}(n)$, there is an equivalence $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T_{p^n}(n)) \simeq \mathrm{BP}\langle n-1\rangle[\sigma]$ with $|\sigma| = 2$.

Example 2.3.23. There is an inclusion $\mathrm{Spin}^c(5) \cong \mathrm{Sp}(2) \cdot \mathrm{U}(1) \subseteq \mathrm{Sp}(3)$ (whose quotient is $\mathbf{C}P^5$), so that composition with the inclusion $\mathrm{Sp}(3) \subseteq \mathrm{SU}(6)$ defines an inclusion $\mathrm{Sp}(2) \cdot \mathrm{U}(1) \subseteq \mathrm{SU}(6)$. In particular, we obtain an $\mathbf{E}_2$-map $\Omega(\mathrm{Sp}(2) \cdot \mathrm{U}(1)) \to \Omega\mathrm{SU}(6)$. The Thom spectrum of the resulting composite $\mathbf{E}_2$-map

$$\Omega(\mathrm{Sp}(2) \cdot \mathrm{U}(1)) \to \Omega\mathrm{SU}(6) \to \Omega\mathrm{SU} \simeq \mathrm{BU}$$

defines an $\mathbf{E}_2^{\mathrm{fr}}$-ring, which we expect can be identified with $T_8(3)$ for $p = 2$.

3. The topological Sen operator

3.1. Constructing the topological Sen operator. There is a much simpler description of the descent spectral sequence of Remark 2.2.12, following the perspective of Remark 2.2.17 that Theorem 2.2.4(b) is essentially a calculation of a Serre spectral sequence. We will continue to fix $\mathbf{E}_3$-forms of the truncated Brown-Peterson spectra $\mathrm{BP}\langle n-1\rangle$ and $\mathrm{BP}\langle n\rangle$.

Notation 3.1.1. Let R be an $\mathbf{E}_\infty$-$\mathbf{Z}_p$-algebra. We will write ϵ^R to denote $R[\mathrm{BSU}(p-1)]$ and ϵ^R_* to denote $\pi_*\epsilon^R$. (The notation is meant to indicate that ϵ only plays a "small" role in the below discussion.)

Definition 3.1.2 (Spectral Gysin sequence)**.** Suppose $S^{n-1} \to E \to B$ is a fibration. Since $E \simeq \mathrm{hocolim}_B\, S^{n-1}$ in pointed spaces, we have $E_+ \simeq \mathrm{hocolim}_B\, S^{n-1}_+$. There is a cofiber sequence $S^{n-1}_+ \to S^0 \to S^n$, so we obtain a cofiber sequence

$$E_+ \to \mathrm{hocolim}_B(S^0) \simeq B_+ \to \mathrm{hocolim}_B(S^n) \simeq \Sigma^n(B_+).$$

If R is an $\mathbf{E}_1$-ring, we get a cofiber sequence of left R-modules:

$$R[E] \to R[B] \to \Sigma^n R[B].$$

Construction 3.1.3 (Topological Sen operator)**.** Let $\mathcal{C}$ be an $X(n)$-linear ∞-category. There is an S^1-equivariant equivalence

$$\begin{aligned}\mathrm{THH}(\mathcal{C}/X(n-1)) &\simeq \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(X(n-1))} X(n-1)\\ &\simeq \mathrm{THH}(\mathcal{C}) \otimes_{X(n)\otimes_{X(n-1)}\mathrm{THH}(X(n-1))} X(n),\end{aligned}$$

a tautological S^1-equivariant equivalence

$$\mathrm{THH}(\mathcal{C}/X(n)) \simeq \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(X(n))} X(n).$$

Since $\mathrm{THH}(X(n-1)) \simeq X(n-1)[\mathrm{SU}(n-1)]$, there is an equivalence $X(n) \otimes_{X(n-1)} \mathrm{THH}(X(n-1)) \simeq X(n)[\mathrm{SU}(n-1)]$. Note that $X(n) \otimes_{X(n-1)} \mathrm{THH}(X(n-1))$ admits the structure of an $\mathbf{E}_1$-ring, and that the $\mathbf{E}_1$-algebra map $\mathrm{THH}(X(n-1)) \to \mathrm{THH}(X(n))$ induces an $\mathbf{E}_1$-algebra map $X(n) \otimes_{X(n-1)} \mathrm{THH}(X(n-1)) \to \mathrm{THH}(X(n)) \simeq X(n)[\mathrm{SU}(n)]$. The fiber sequence

$$S^{2n-1} \to \mathrm{BSU}(n-1) \to \mathrm{BSU}(n)$$

implies:

Theorem 3.1.4. *Let $\mathcal{C}$ be a left $X(n)$-linear ∞-category. Then there is a cofiber sequence*

$$\mathrm{THH}(\mathcal{C}/X(n-1)) \xrightarrow{\iota} \mathrm{THH}(\mathcal{C}/X(n)) \xrightarrow{\Theta_{\mathcal{C}}} \Sigma^{2n}\mathrm{THH}(\mathcal{C}/X(n)), \tag{13}$$

where the map ι is S^1-equivariant, and the cofiber of ι is (at least nonequivariantly) identified with $\Sigma^{2n}\mathrm{THH}(\mathcal{C}/X(n))$. We will call the map $\Theta_{\mathcal{C}} : \Sigma^{-2n}\mathrm{THH}(\mathcal{C}/X(n)) \to \mathrm{THH}(\mathcal{C}/X(n))$ the topological Sen operator.

Remark 3.1.5. A simpler analogue of Theorem 3.1.4 can be described as follows. Let A be an $\mathbf{E}_2^{\mathrm{fr}}$-ring, and let $A[t]$ be the flat polynomial ring over A on a generator in degree 0. Suppose $\mathcal{C}$ is an $A[t]$-linear ∞-category. The nonequivariant equivalence $\mathrm{HH}(A[t]/A) \simeq A[t][S^1]$ defines a cofiber sequence

$$\mathrm{HH}(\mathcal{C}/A) \to \mathrm{HH}(\mathcal{C}/A[t]) \xrightarrow{\nabla} \Sigma^2\mathrm{HH}(\mathcal{C}/A[t]) \tag{14}$$

analogous to Theorem 3.1.4, which exhibits $\nabla : \mathrm{HH}(\mathcal{C}/A[t]) \to \Sigma^2\mathrm{HH}(\mathcal{C}/A[t])$ as a "Gauss-Manin connection". This cofiber sequence is often quite useful; for example, if we regard $\mathbf{Z}_p$ as a $S[\![t]\!]$-algebra by the $\mathbf{E}_\infty$-map $S[\![t]\!] \to \mathbf{Z}_p$ sending $t \mapsto p$, we have $\pi_*\mathrm{THH}(\mathbf{Z}_p/S[\![t]\!]) \simeq \mathbf{Z}_p[y]$ with $|y| = 2$ (more precisely, $y = \sigma^2(t-p)$); see **[KN19]**. It is not difficult to show that the map $\nabla : \mathrm{THH}(\mathbf{Z}_p/S[\![t]\!]) \to \Sigma^2\mathrm{THH}(\mathbf{Z}_p/S[\![t]\!])$ sends $y^n \mapsto ny^{n-1}$, which implies Bökstedt's calculation of $\pi_*\mathrm{THH}(\mathbf{Z}_p)$.

Just as in Theorem 3.1.4, the map $\mathrm{HH}(\mathcal{C}/A) \to \mathrm{HH}(\mathcal{C}/A[t])$ in (14) is S^1-equivariant, but we can only nonequivariantly identify its cofiber with $\Sigma^2\mathrm{HH}(\mathcal{C}/A[t])$. To identify the cofiber equivariantly, observe that if λ denotes the rotation representation of S^1, then $\mathrm{HH}(A/A[t]) \simeq A[B^\lambda\mathbf{Z}_{\geq 0}]$. Here, $B^\lambda\mathbf{Z}_{\geq 0}$ is the λ-delooping of $\mathbf{Z}_{\geq 0}$. This implies that there is an *equivariant* cofiber sequence

$$\mathrm{HH}(\mathcal{C}/A) \to \mathrm{HH}(\mathcal{C}/A[t]) \xrightarrow{\nabla} \Sigma^\lambda\mathrm{HH}(\mathcal{C}/A[t]). \tag{15}$$

See Corollary 3.1.19 for some further discussion.

Remark 3.1.6. At the level of homotopy, the map Θ in (13) for $\mathcal{C} = \mathrm{LMod}_{\mathrm{BP}\langle n-1\rangle}$ can be identified using Theorem 2.2.4. Namely, recall that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n)) \cong \mathrm{BP}\langle n-1\rangle[B\Delta_n]_*[\theta_n]$ by Theorem 2.2.4(a); it then follows from Theorem 2.2.4(b) that Θ must send

$$\Theta : \theta_n^j \mapsto jp\theta_n^{j-1}.$$

Therefore, we may informally write $\Theta = p\partial_{\theta_n}$.[9] From the point of view of Remark 2.2.17, the map Θ can be interpreted as the d^{2p^n}-differential in the Serre spectral sequence computing the $\mathrm{BP}\langle n-1\rangle$-homology of the total space of the fibration (11). Determining the action of Θ on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))$ for $j \leq n-1$ can therefore be viewed as an analogue of determining the differentials in the Serre spectral sequence/Gysin sequence of a putative analogue of the Cohen-Moore-Neisendorfer fibration (11) (where p is replaced by v_{n-j}).

One can make some qualitative observations about the action of Θ on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))$ for $j \leq n-1$. Indeed, recall from (4) that there is an isomorphism

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))/v_{[0,n-j)} \cong \mathrm{BP}\langle n-1\rangle[B\Delta_j]_*[\theta_n]/v_{[0,n-j)} \otimes_{\mathbf{F}_p} \Lambda_{\mathbf{F}_p}(\lambda_{j+1}, \cdots, \lambda_n).$$

An easy calculation shows that there is an isomorphism

$$\pi_*\mathrm{THH}(X(p^n)/X(p^j)) \cong X(p^n)\left[\prod_{i=j+1}^{n} \overline{\Delta}_i\right]_* \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}(\lambda_{j+1}, \cdots, \lambda_n).$$

Therefore, the calculation of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))/v_{[0,n-j)}$ implies that the image of a class $y \in \pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))$ under $\Theta : \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j)) \to \Sigma^{2p^j}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^j))$ lives in the ideal generated by $v_{[0,n-j+1)} = (p, \cdots, v_{n-j})$.

Remark 3.1.7. The fact that the cofiber of the S^1-equivariant map $\iota : \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n - 1)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ is (at least nonequivariantly) identified with $\Sigma^{2p^n}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ makes it more difficult to determine $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ (even modulo v_{n-1}) from our calculation of $\pi_*\mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ in Theorem 2.2.4 and the preceding description of Θ as an endomorphism of $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))$. One

[9] This action of Θ on $\theta_n = \sigma^2(v_n)$ is related to the observation from **[Lee22**, Lemma 3.2.8(d)**]** that there is a choice of v_n such that the right unit $\eta_R : \mathrm{BP}_* \to \mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \cdots]$ satisfies $d(v_n) = \eta_R(v_n) - v_n \equiv pt_n \pmod{t_1, \cdots, t_{n-1}}$.

fundamental question is therefore to describe the S^1-action on $\mathrm{cofib}(\iota)$. This is already complicated modulo p when $n=1$, and a description of $\mathrm{TP}(\mathbf{Z}_p/X(p-1)) \simeq \mathrm{TP}(\mathbf{Z}_p)[\mathrm{BSU}(p-1)]$ from $\mathrm{TP}(\mathbf{Z}_p/X(p))$ was essentially done in [**BM94**, Conjecture 4.3] and [**Tsa97**, Theorem 7.4]. Recall from Theorem 2.2.4(a) that there is an isomorphism

$$\pi_*\mathrm{TP}(\mathbf{Z}_p/X(p))/p \cong \mathbf{F}_p[v_1, \hbar^{\pm 1}] \otimes_{\mathbf{F}_p} \epsilon_*^{\mathbf{F}_p} \cong \pi_* k(1)^{tS^1}[\mathrm{BSU}(p-1)].$$

Then, the map $\pi_*\mathrm{TP}(\mathbf{Z}_p/X(p))/p \to \pi_{*-2p}\mathrm{TP}(\mathbf{Z}_p/X(p))/p$ is given by

$$\hbar^{p^k} \mapsto \hbar^{p^k(p+1)} v_1^{\frac{p^{k+1}-p}{p-1}},\ v_1^k \mapsto 0.$$

This is a direct consequence of [**Tsa97**, Theorem 7.4], once one notes that the the formula $t^{p^k+\phi(k+1)} f^{\phi(k)}$ from *loc. cit.* becomes precisely $\hbar^{p^k(p+1)} v_1^{\frac{p^{k+1}-p}{p-1}}$, via the translation in notation given by

$$t \rightsquigarrow \hbar,\ f \rightsquigarrow \sigma^2(v_1),\ tf \rightsquigarrow v_1,\ \phi(k) = \frac{p^{k+1}-p}{p-1} = v_p((p^k)!^p).$$

One could also prove this using an argument similar to [**HRW22**, Theorem 6.5.1].

Moreover, the image of $\hbar$ under the boundary map $\pi_{-2}\mathrm{TP}(\mathbf{Z}_p/X(p))/p \to \pi_{2p-3}\mathrm{TP}(\mathbf{Z}_p/X(p-1))/p$ is the class $\alpha_1 \in \pi_{2p-3}\mathrm{TP}(\mathbf{Z}_p)/p$; note that since $\hbar$ lives in $\pi_{-2}\mathrm{TP}(\mathbf{Z}_p/X(p))$, the class α_1 in fact extends to an element of $\pi_{2p-3}\mathrm{TP}(\mathbf{Z}_p/X(p-1))$. The problem of calculating $\pi_*\mathrm{TP}(\mathbf{Z}_p)$ from $\mathrm{TP}(\mathbf{Z}_p/X(p))$ is very similar to the problem of $\pi_*\mathrm{TP}(\mathbf{Z}_p)$ from $\mathrm{TP}(\mathbf{Z}_p/S[\![t]\!])$, discussed in [**LW20**] (see Remark 3.1.5).

If we assume Conjecture 2.1.9, then Theorem 3.1.4 can be refined: namely, if $\mathcal{C}$ is a left $T(n)$-linear ∞-category, then there is a cofiber sequence

$$\mathrm{THH}(\mathcal{C}/T(n-1)) \xrightarrow{\iota} \mathrm{THH}(\mathcal{C}/T(n)) \xrightarrow{\Theta_{\mathcal{C}}} \Sigma^{2p^n}\mathrm{THH}(\mathcal{C}/T(n)). \tag{16}$$

Remark 3.1.8. Suppose $n=1$ and $\mathcal{C} = \mathrm{Mod}_{\mathbf{Z}_p}$ for p odd. Then there is a map $\mathrm{TP}(\mathbf{Z}_p) \to \mathrm{TP}(\mathbf{Z}_p/T(1))$, and a trace map $K(\mathbf{Z}_p) \to \mathrm{TP}(\mathbf{Z}_p)$. Let $j = \tau_{\geq 0} L_{K(1)} S$; upon p-adic completion, there is an equivalence (see [**BM94**, Theorem 9.17])

$$K(\mathbf{Z}_p)^\wedge_p \simeq j \vee \Sigma j \vee \Sigma^3 \mathrm{ku}.$$

The summand j is the unit component, i.e., there is an $\mathbf{E}_\infty$-ring map $j \to K(\mathbf{Z}_p)^\wedge_p$. It follows that after p-completion, there is a ring map $j \to \mathrm{TP}(\mathbf{Z}_p)$. Assuming the equivalence $\mathrm{TP}(\mathbf{Z}_p/T(1)) \simeq \mathrm{BP}\langle 1\rangle^{tS^1}$ of Conjecture 2.2.18, the following diagram commutes:

$$\begin{tikzcd} j \arrow[r] \arrow[d, "\mathrm{unit}"'] & \mathrm{BP}\langle 1\rangle \arrow[d, "\mathrm{unit}"] \\ \mathrm{TP}(\mathbf{Z}_p) \arrow[r] & \mathrm{TP}(\mathbf{Z}_p/T(1)). \end{tikzcd}$$

Let ℓ be a topological generator of $\mathbf{Z}_p^\times$, and let $\psi^\ell : \mathrm{BP}\langle 1\rangle \to \Sigma^{2p-2}\mathrm{BP}\langle 1\rangle$ be the associated Adams operation. Then, the fiber of $\psi^\ell - 1$ is j. Based on the above commutative diagram, one expects that under the equivalence $\mathrm{TP}(\mathbf{Z}_p/T(1)) \simeq \mathrm{BP}\langle 1\rangle^{tS^1}$ of Conjecture 2.2.18, the map $\psi^\ell - 1$ is closely related to $\Theta^{tS^1}_{\mathbf{Z}_p}$. Note, for example, that if we take $\ell = p+1$, the map $\psi^\ell - 1$ sends $v_1^j \mapsto p^{v_p(j)+1} v_1^j$ up to p-adic units; this should be compared to the fact that $\Theta_{\mathbf{Z}_p}$ sends $\theta_1^j \mapsto jp\theta_1^{j-1}$ by Remark 3.1.6. This discussion, as well as the classical discussion in

[**BM94**], suggests that $\mathrm{TP}(\mathbf{Z}_p)^\wedge_p \simeq (j^{tS^1})^\wedge_p$. In fact, something stronger is true: in forthcoming work [**DR23**] with Arpon Raksit, we will show that $\mathrm{THH}(\mathbf{Z}_p) = \tau_{\geq 0}(j^{t\mathbf{Z}/p})$ as cyclotomic $\mathbf{E}_\infty$-rings.

Example 3.1.9. Let $n = 1$, and let $\mathcal{C} = \mathrm{Mod}_{\mathrm{BP}\langle 1\rangle}$. Then Theorem 3.1.4 gives a cofiber sequence

$$\mathrm{THH}(\mathrm{BP}\langle 1\rangle/X(p-1)) \to \mathrm{THH}(\mathrm{BP}\langle 1\rangle/X(p)) \xrightarrow{\Theta_{\mathrm{BP}\langle 1\rangle}} \Sigma^{2p}\mathrm{THH}(\mathrm{BP}\langle 1\rangle/X(p)).$$

Moreover, recall from Theorem 2.2.4(b) that there is a p-complete equivalence

$$\mathrm{THH}(\mathrm{BP}\langle 1\rangle/X(p)) \simeq \mathrm{BP}\langle 1\rangle[\mathrm{BSU}(p-1)] \oplus \bigoplus_{j\geq 1} \Sigma^{2jp^2-1}\mathrm{BP}\langle 1\rangle[\mathrm{BSU}(p-1)]/pj.$$

Let a_j denote the $\mathrm{BP}\langle 1\rangle$-module generator of the summand $\Sigma^{2jp^2-1}\mathrm{BP}\langle 1\rangle/pj$. Since $\mathrm{THH}(\mathrm{BP}\langle 1\rangle/X(p-1)) \simeq \mathrm{THH}(\mathrm{BP}\langle 1\rangle)[\mathrm{BSU}(p-1)]$, the calculations of [**AHL10**, Section 6] can be rephrased as follows. For $0 \leq k \leq v_p(j)$, $\Theta_{\mathrm{BP}\langle 1\rangle}$ is given on homotopy by

$$\Theta_{\mathrm{BP}\langle 1\rangle} : p^k a_j \mapsto \left(\frac{j}{p^k} - 1\right) a_{j-p^k} v_1^{p^{\frac{p^{k+1}-1}{p-1}}},$$

up to p-adic units. A different perspective on this computation is given in [**Lee22**].

Variant 3.1.10. One can prove a variant of Theorem 3.1.4 by replacing $X(p)$ with $J(p)$. If $\mathcal{C}$ is a left $J(p)$-linear ∞-category, then Proposition 2.3.2 produces a cofiber sequence:

$$\mathrm{THH}(\mathcal{C}) \xrightarrow{\iota} \mathrm{THH}(\mathcal{C}/J(p)) \xrightarrow{\Theta'} \Sigma^2\mathrm{THH}(\mathcal{C}/J(p)). \tag{17}$$

Here, the map ι is S^1-equivariant, and $\mathrm{cofib}(\iota)$ is (at least nonequivariantly) identified with $\Sigma^2\mathrm{THH}(\mathcal{C}/J(p))$. Proposition 2.3.3 shows that $\mathrm{THH}(\mathbf{Z}_p/J(p)) \simeq \mathbf{Z}_p[\Omega S^3]$. On homotopy, the map $\mathrm{THH}(\mathbf{Z}_p/J(p)) \to \Sigma^2\mathrm{THH}(\mathbf{Z}_p/J(p))$ is given by the d^2-differential in the Serre spectral sequence for the fibration

$$S^1 \to \Omega S^3\langle 3\rangle \to \Omega S^3.$$

For example, under the isomorphism $\pi_*\mathrm{THH}(\mathbf{Z}_p/J(p)) \cong \mathbf{Z}_p[x]$ with $|x| = 2$, the map Θ' in the cofiber sequence (17) for $n = 1$ sends $x^j \mapsto jx^{j-1}$.

Suppose $\mathcal{C}$ is in fact a $\mathbf{Z}_p$-linear ∞-category. Base-changing (17) along the map $\mathbf{Z}_p \to \mathbf{F}_p$ and using Corollary 2.3.7, we obtain a cofiber sequence

$$\mathrm{THH}(\mathcal{C}) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \xrightarrow{\iota} \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \xrightarrow{\Theta'} \Sigma^2\mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p). \tag{18}$$

Note that the map $\Theta' : \mathrm{THH}(\mathbf{F}_p) \to \Sigma^2\mathrm{THH}(\mathbf{F}_p)$ sends $\sigma^j \mapsto j\sigma^{j-1}$ on homotopy. It follows that upon composition with $\sigma : \Sigma^2\mathrm{THH}(\mathcal{C}\otimes_{\mathbf{Z}_p}\mathbf{F}_p) \to \mathrm{THH}(\mathcal{C}\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$, Θ' acts by multiplication by j on the homotopy of the jth graded piece $\mathrm{gr}^j_\sigma\mathrm{THH}(\mathcal{C}\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$ of the σ-adic filtration on $\mathrm{THH}(\mathcal{C}\otimes_{\mathbf{Z}_p}\mathbf{F}_p)$.

Remark 3.1.11. Let $p = 2$. Using the fiber sequence

$$S^3 \to \mathrm{BU}(1) \to \mathrm{BU}(2),$$

one can similarly show that if $T_4(2)$ denotes the $\mathbf{E}_2^{\mathrm{fr}}$-ring from Construction 2.3.9 and $\mathcal{C}$ is a left $T_4(2)$-linear ∞-category, there is a cofiber sequence

$$\mathrm{THH}(\mathcal{C}/J(2)) \to \mathrm{THH}(\mathcal{C}/T_4(2)) \to \Sigma^4\mathrm{THH}(\mathcal{C}/T_4(2)).$$

Remark 3.1.12. Let R be an animated $\mathbf{Z}_p$-algebra. Let $\hat{\Delta}_R$ denote the Nygaard-completed prismatic cohomology of R, and $\mathcal{N}^i\hat{\Delta}_R$ denote the ith graded piece of the Nygaard filtration $\mathcal{N}^{\geq\star}(\hat{\Delta}_R)$. Note that [**BL22a**, Remark 5.5.15] gives an isomorphism $\mathcal{N}^i(\hat{\Delta}_R\{i\}) \cong \mathcal{N}^i\hat{\Delta}_R$, where $\hat{\Delta}_R\{i\}$ denotes the Breuil-Kisin twisted prismatic cohomology of R. Using the methods of [**BM22**], one can construct a cofiber sequence

$$(\mathcal{N}^i\hat{\Delta}_R)/p \to \mathrm{F}_i^{\mathrm{conj}}\mathrm{dR}_{(R/p)/\mathbf{F}_p} \cong \mathcal{N}^i\hat{\Delta}_{R/p} \to \mathrm{F}_{i-1}^{\mathrm{conj}}\mathrm{dR}_{(R/p)/\mathbf{F}_p}. \tag{19}$$

As explained in *loc. cit.*, the second map is closely related to the Sen operator. Recall (see [**BL22a**, Example 6.4.17] and [**BMS19**]) that $\mathrm{THH}(R/p)$ admits a motivic filtration such that $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/p) = \mathcal{N}^i(\hat{\Delta}_{R/p})[2i]$. Taking $\mathcal{C} = \mathrm{Mod}_R$, (18) says that there is a self-map $\Theta' : \mathrm{THH}(R/p) \to \Sigma^2\mathrm{THH}(R/p)$ whose fiber is $\mathrm{THH}(R)/p$. Presumably, the cofiber sequence (18) can be shown to respect the motivic filtration, so taking graded pieces would recover the cofiber sequence (19). Given this discussion, it is natural to ask if $\mathrm{THH}(R/J(p))$ admits a motivic filtration such that (17) is a cofiber sequence of motivically-filtered spectra.

Recollection 3.1.13. Let $(\mathbf{Z}_p[\![\widetilde{p}]\!], \widetilde{p})$ denote the prism of [**BL22a**, Notation 3.8.9], and if R is a p-complete animated $\mathbf{Z}_p$-algebra, let $\widetilde{p}\Omega_R$ denote $\Delta_{R/\mathbf{Z}_p[\![\widetilde{p}]\!]}$. In particular, $\widetilde{p}\Omega_R \simeq (q\Omega_R)^{h\mathbf{F}_p^\times}$, via the $\mathbf{F}_p^\times$-action on the prism $(\mathbf{Z}_p[\![q-1]\!], [p]_q)$. Let $\widehat{\Omega}^{\not{D}}_R$ denote the diffracted Hodge complex of [**BL22a**, Construction 4.7.1], so that $\widehat{\Omega}^{\not{D}}_R$ is isomorphic to $\widetilde{p}\Omega_R/\widetilde{p}$ by [**BL22a**, Remark 4.8.6]. Recall the cofiber sequence of [**BL22a**, Remark 5.5.8]:

$$\mathcal{N}^i\hat{\Delta}_R \to \mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R \xrightarrow{\Theta+i} \mathrm{F}_{i-1}^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R.$$

The mod p reduction of this fiber sequence produces (19).

Since $\mathrm{THH}(R)$ admits a motivic filtration whose graded pieces are $\mathcal{N}^i(\hat{\Delta}_R)[2i]$, the cofiber sequence (17) motivates the following conjecture (which essentially states that $\mathrm{THH}(R/J(p))$ is a sheared Rees construction on the conjugate filtration of $\widehat{\Omega}^{\not{D}}_R$):

Conjecture 3.1.14. *Let R be an animated $\mathbf{Z}_p$-algebra. Then there is a filtration $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{THH}(R/J(p))$ on $\mathrm{THH}(R/J(p))$ such that:*

- $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/J(p)) \simeq (\mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R)[2i]$*; and*
- *the map $\Theta'_R : \mathrm{THH}(R/J(p)) \to \Sigma^2\mathrm{THH}(R/J(p))$ respects the motivic filtration and induces the map $\Theta + i : \mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R \to \mathrm{F}_{i-1}^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R$ on $\mathrm{gr}^i_{\mathrm{mot}}$; and*
- $\mathrm{gr}^i_{\mathrm{mot}}(\mathrm{THH}(R/J(p))[x^{-1}]) \simeq \widehat{\Omega}^{\not{D}}_R[2i]$*, such that the localization map $\mathrm{THH}(R/J(p)) \to \mathrm{THH}(R/J(p))[x^{-1}]$ induces the inclusion $(\mathrm{F}_i^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R)[2i] \to \widehat{\Omega}^{\not{D}}_R[2i]$ on $\mathrm{gr}^i_{\mathrm{mot}}$.*

Remark 3.1.15. Recall from Proposition 2.3.2 that there is an equivalence $\mathrm{THH}(X(p)/J(p)) \simeq X(p)[\mathrm{SU}(p-1) \times J_{p-1}(S^2)]$. Using this, it is not difficult to show that Conjecture 3.1.14 implies that if R is an animated $\mathbf{Z}_p$-algebra, then $\mathrm{THH}(R/X(p))$ admits a motivic filtration such that $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/X(p)) \simeq (\mathrm{F}_{pi}^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R)[2pi] \otimes_R \epsilon^R$. If Conjecture 2.1.9 were true[10] for $n = 1$, then $\mathrm{THH}(R/T(1))$ would admit a motivic filtration such that $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/T(1)) \simeq (\mathrm{F}_{pi}^{\mathrm{conj}}\widehat{\Omega}^{\not{D}}_R)[2pi]$. Therefore, $\mathrm{THH}(R/X(p))$ precisely extracts the pieces of the conjugate filtration on $\widehat{\Omega}^{\not{D}}_R$ which are not automatically split by the Sen operator. From the point of view of Conjecture 3.1.14, the

[10] Or at least the weaker statement that $T(1)$ admits the structure of an $\mathbf{E}_2$-ring.

utility of the discussion in Construction 2.3.9 is that although describing a higher chromatic analogue of $J(p)$ is tricky (see Conjecture 2.3.22), $\mathrm{THH}(\mathcal{C}/X(p^n))$ furnishes a natural higher chromatic and noncommutative analogue of the diffracted Hodge complex when $\mathcal{C}$ is a left $\mathrm{BP}\langle n\rangle$-linear ∞-category.

Remark 3.1.16. We collect some further evidence for Conjecture 3.1.14:

(a) Recall that if $\mathcal{D}$ is an $\mathbf{F}_p$-linear ∞-category, then the canonical map $\mathrm{THH}(\mathcal{D}) \to \mathrm{HH}(\mathcal{D}/\mathbf{F}_p)$ is given by quotienting by $\sigma \in \pi_2 \mathrm{THH}(\mathbf{F}_p)$. Moreover, if R is an animated $\mathbf{F}_p$-algebra, then $\mathrm{gr}^i_{\mathrm{mot}} \mathrm{THH}(R) \simeq (\mathrm{F}^{\mathrm{conj}}_i \mathrm{dR}_{R/\mathbf{F}_p})[2i]$, and $\mathrm{F}^\sigma_\star \mathrm{THH}(R)$ is a noncommutative analogue of the conjugate filtration $\mathrm{F}^{\mathrm{conj}}_\star \mathrm{dR}_{R/\mathbf{F}_p}$. In particular, the induced motivic filtration on $\mathrm{THH}(R)/\sigma$ has $\mathrm{gr}^i_{\mathrm{mot}}(\mathrm{THH}(R)/\sigma) \simeq L\Omega^i_{R/\mathbf{F}_p}[-i]$.

This picture admits an analogue over $J(p)$. Recall from Proposition 2.3.3(a) that $\pi_* \mathrm{THH}(\mathbf{Z}_p/J(p)) \cong \mathbf{Z}_p[x]$ with $|x| = 2$. Let $\mathcal{C}$ be a $\mathbf{Z}_p$-linear ∞-category. One could attempt to define the quotient $\mathrm{THH}(\mathcal{C}/J(p))/x$ as a relative tensor product of $\mathrm{THH}(\mathcal{C}/J(p))$ with $\mathbf{Z}_p$ over $\mathrm{THH}(\mathbf{Z}_p/J(p))$. Unfortunately, this tensor product does not make sense, since $\mathrm{THH}(\mathbf{Z}_p/J(p))$ does not naturally acquire the structure of an $\mathbf{E}_1$-algebra. However, were $J(p)$ to admit the structure of an $\mathbf{E}_3$-algebra, the above relative tensor product would precisely be computing $\mathrm{HH}(\mathcal{C}/\mathbf{Z}_p) = \mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathbf{Z}_p$. It is therefore reasonable to view the canonical map $\mathrm{THH}(\mathcal{C}/J(p)) \to \mathrm{HH}(\mathcal{C}/\mathbf{Z}_p)$ as a quotient by x. If R is an animated $\mathbf{Z}_p$-algebra, then $\mathrm{HH}(R/\mathbf{Z}_p)$ is a noncommutative analogue of the Hodge complex $\bigoplus_{n\geq 0} L\widehat{\Omega}^n_{R/\mathbf{Z}_p}[-n]$. Under Conjecture 3.1.14, the perspective that the map $\mathrm{THH}(\mathcal{C}/J(p)) \to \mathrm{HH}(\mathcal{C}/\mathbf{Z}_p)$ is given by "killing x" can be regarded as an analogue of [**BL22a**, Remark 4.7.14], which identifies $\mathrm{gr}^{\mathrm{conj}}_i \widehat{\Omega}^{\not{D}}_R \simeq L\widehat{\Omega}^i_{R/\mathbf{Z}_p}[-i]$.

(b) Let R be a smooth $\mathbf{Z}_p$-algebra. Then the prismatic-crystalline comparison theorem (see [**BL22a**, Remark 4.7.18]) implies that the base-change $\mathbf{F}_p \otimes_{\mathbf{Z}_p} \mathrm{F}^{\mathrm{conj}}_\star \widehat{\Omega}^{\not{D}}_R$ can be identified with $\mathrm{Frob}_* \mathrm{F}^{\mathrm{conj}}_\star \Omega^\bullet_{R/p/\mathbf{F}_p}$, where $\mathrm{Frob} : R \to R$ is the absolute Frobenius. Under Conjecture 3.1.14, Corollary 2.3.7 can be viewed as a noncommutative analogue of this result.

(c) By Proposition 2.3.3, the class x is sent to $\sigma \in \pi_2 \mathrm{THH}(\mathbf{F}_p)$ under the map $\iota : \mathrm{THH}(\mathbf{Z}_p/J(p)) \to \mathrm{THH}(\mathbf{F}_p)$. Since the cyclotomic Frobenius induces an equivalence $\varphi : \mathrm{THH}(\mathbf{F}_p)[1/\sigma] \xrightarrow{\sim} \mathrm{THH}(\mathbf{F}_p)^{t\mathbf{Z}/p}$, the cofiber sequence of (18) predicts a cofiber sequence

$$\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p \xrightarrow{\iota} \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^{t\mathbf{Z}/p} \xrightarrow{\Theta'} \mathrm{THH}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)^{t\mathbf{Z}/p}. \tag{20}$$

Such a cofiber sequence does indeed exist, and we will construct it below in Corollary 3.1.19 (albeit using slightly different methods).

Suppose that the cofiber sequence (20) respects the motivic filtration when $\mathcal{C} = \mathrm{Mod}_R$. Since $\mathrm{THH}(R)^{t\mathbf{Z}/p} \simeq \mathrm{HP}((R/p)/\mathbf{F}_p)$ (see [**Mat20**, Proposition 2.12]) and $\mathrm{HP}((R/p)/\mathbf{F}_p)$ has a motivic filtration such that $\mathrm{gr}^i_{\mathrm{mot}} \mathrm{HP}((R/p)/\mathbf{F}_p) \simeq \mathrm{dR}_{(R/p)/\mathbf{F}_p}[2i]$, the cofiber sequence (20) would presumably be related under Conjecture 3.1.14 to the following cofiber sequence related to (19) (whose existence was told to me by Akhil Mathew):

$$\overline{\mathbb{\Delta}}_R/p \to \mathrm{dR}_{(R/p)/\mathbf{F}_p} \to \mathrm{dR}_{(R/p)/\mathbf{F}_p}. \tag{21}$$

For completeness, we give an argument for (21).

Proof of the cofiber sequence (21). Recall from [**BM22**, Corollary 3.16] that if A is an animated $\mathbf{Z}_p[x]$-algebra, there is a cofiber sequence

$$\overline{\Delta}_A\{i\}/x \to \overline{\Delta}_{A/x}\{i\} \to \overline{\Delta}_{A/x}\{i-1\}. \tag{22}$$

This implies (by setting $i=0$ and viewing R/p as the base-change $R\otimes_{\mathbf{Z}_p[x]}\mathbf{Z}_p$, where the map $\mathbf{Z}_p[x] \to R$ sends $x \mapsto p$, and the map $\mathbf{Z}_p[x] \to \mathbf{Z}_p$ is the augmentation) that there is a cofiber sequence

$$\overline{\Delta}_R/p \to \overline{\Delta}_{R/p} \to \overline{\Delta}_{R/p}.$$

The de Rham/crystalline comparison theorems tell us that $\Delta_{R/p} \simeq \Delta_{(R/p)/\mathbf{Z}_p} \simeq (\mathrm{dR}_R)^\wedge_p$, where $\Delta_{(R/p)/\mathbf{Z}_p}$ denotes prismatic cohomology with respect to the crystalline prism $(\mathbf{Z}_p,(p))$ (i.e., the derived crystalline cohomology of R/p). But then $\overline{\Delta}_{R/p} \simeq \mathrm{dR}_{(R/p)/\mathbf{F}_p}$, as desired.

Let us remark that (22) can be constructed using $\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$. Indeed, we can reduce to the case when A is the p-completion of $\mathbf{Z}_p[x] = \mathcal{O}_{\mathbf{G}_a}$. Then, [**BL22b**, Example 9.1] implies that $\mathrm{Spec}(\mathbf{Z}_p)\times_{\mathbf{G}_a}\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a} \cong B(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m^\sharp)$. Let $\alpha : \mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}_p} \to \mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$ be the tautological map, so that it factors through a map $f : \mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}_p} \to \mathrm{Spec}(\mathbf{Z}_p)\times_{\mathbf{G}_a}\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$, which can in turn be identified with the map $B\mathbf{G}_m^\sharp \to B(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m^\sharp)$. It follows that there is a Cartesian square

$$\begin{tikzcd}
\mathbf{G}_a^\sharp \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbf{Z}_p) \arrow[d] \\
\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}_p} \arrow[r, "f"'] & \mathrm{Spec}(\mathbf{Z}_p)\times_{\mathbf{G}_a}\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}.
\end{tikzcd}$$

Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$, and let $\mathcal{F}/x$ be the associated quasicoherent sheaf on $\mathrm{Spec}(\mathbf{Z}_p)\times_{\mathbf{G}_a}\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$. Our goal is to identify the cofiber of the map $\mathcal{F}/x \to f_*\alpha^*\mathcal{F} \simeq f_*f^*(\mathcal{F}/x)$ in the case when $\mathcal{F}$ is the Breuil-Kisin twisting line bundle $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}}\{i\}$ on $\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}$. The preceding Cartesian square along with the cofiber sequence[11]

$$\mathbf{Z}_p \to \mathbf{Z}_p\langle t\rangle = \mathcal{O}_{\mathbf{G}_a^\sharp} \xrightarrow{\partial_t} \mathbf{Z}_p\langle t\rangle,\ \gamma_n(t)\mapsto\gamma_{n-1}(t)$$

implies that $\mathrm{cofib}(\mathcal{F}/x \to f_*\alpha^*\mathcal{F})$ can be identified with $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}_p}}\{-1\}\otimes f_*\alpha^*\mathcal{F}$. Setting $\mathcal{F} = \mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{G}_a}}\{i\}$ and taking global sections produces (22). $\square$

We now construct a more general version of the cofiber sequence (20). We first need the following lemma:

Lemma 3.1.17. *Let $G \subseteq S^1$ be a nontrivial finite subgroup of S^1, and let λ denote the rotation representation of S^1 on $\mathbf{C}$.*

(a) Define $(S^\lambda)^{(1)}$ via the cofiber sequence

$$G_+ \to S^0 \to (S^\lambda)^{(1)}.$$

Then there is a cofiber sequence

$$\Sigma(G_+) \to (S^\lambda)^{(1)} \to S^\lambda.$$

(b) Let X be a spectrum with G-action. Then $X^{tG} \xrightarrow{\sim} (\Sigma^\lambda X)^{tG}$.

[11]Here, we declare $\gamma_{-1}(x) = 0$.

Proof. Part (a) describes an equivariant CW-structure on S^λ; we leave this as an exercise to the reader. Part (b) follows by observing that the cofiber sequence

$$G_+ \otimes X \to X \to (S^\lambda)^{(1)} \otimes X$$

implies that $X^{tG} \xrightarrow{\sim} ((S^\lambda)^{(1)} \otimes X)^{tG}$; and the cofiber sequence

$$X \otimes \Sigma(G_+) \to X \otimes (S^\lambda)^{(1)} \to \Sigma^\lambda X$$

implies that $(X \otimes (S^\lambda)^{(1)})^{tG} \xrightarrow{\sim} (\Sigma^\lambda X)^{tG}$. $\square$

Proposition 3.1.18. *Let $S[\pi] = S[\mathbf{Z}_{\geq 0}]$. For any $S[\pi]$-linear ∞-category $\mathcal{C}$, there are cofiber sequences*

$$\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_{S[\pi]} S \to \mathrm{THH}(\mathcal{C} \otimes_{S[\pi]} S)^{t\mathbf{Z}/p} \xrightarrow{\nabla^{t\mathbf{Z}/p}} \mathrm{THH}(\mathcal{C} \otimes_{S[\pi]} S)^{t\mathbf{Z}/p}, \tag{23}$$

$$\mathrm{TP}(\mathcal{C}) \to \mathrm{TP}(\mathcal{C}/S[\pi]) \xrightarrow{\nabla^{tS^1}} \mathrm{TP}(\mathcal{C}/S[\pi]). \tag{24}$$

Proof. We will use (15) with $A = S$ (here, the variable t is relabeled as π). This gives us an S^1-equivariant cofiber sequence

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C}/S[\pi]) \to \Sigma^\lambda \mathrm{THH}(\mathcal{C}/S[\pi]). \tag{25}$$

To prove the cofiber sequence (23), we first apply $t\mathbf{Z}/p$ to the preceding cofiber sequence:

$$\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \to \mathrm{THH}(\mathcal{C}/S[\pi])^{t\mathbf{Z}/p} \to (\Sigma^\lambda \mathrm{THH}(\mathcal{C}/S[\pi]))^{t\mathbf{Z}/p}.$$

Observe that the tensor product $\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_{S[\pi]} S$ along the augmentation $S[\pi] \to S$ sending $\pi \mapsto 0$ is precisely $\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_{S[\pi]} S$. Similarly, $\mathrm{THH}(\mathcal{C}/S[\pi])^{t\mathbf{Z}/p} \otimes_{S[\pi]} S \simeq \mathrm{THH}(\mathcal{C} \otimes_{S[\pi]} S)^{t\mathbf{Z}/p}$. It therefore suffices to show that $(\Sigma^\lambda \mathrm{THH}(\mathcal{C}/S[\pi]))^{t\mathbf{Z}/p} \simeq \mathrm{THH}(\mathcal{C}/S[\pi])^{t\mathbf{Z}/p}$; but this is exactly Lemma 3.1.17.

The cofiber sequence (24) is even easier to construct: applying tS^1 to (25), we obtain a cofiber sequence

$$\mathrm{TP}(\mathcal{C}) \to \mathrm{TP}(\mathcal{C}/S[\pi]) \to (\Sigma^\lambda \mathrm{THH}(\mathcal{C}/S[\pi]))^{tS^1}.$$

Since there is a cofiber sequence

$$S^1_+ \to S^0 \to S^\lambda,$$

we see that there is an equivalence $X^{tS^1} \xrightarrow{\sim} (\Sigma^\lambda X)^{tS^1}$ for any S^1-spectrum X. In particular, $(\Sigma^\lambda \mathrm{THH}(\mathcal{C}/S[\pi]))^{tS^1} \simeq \mathrm{TP}(\mathcal{C}/S[\pi])$, as desired. $\square$

Corollary 3.1.19. *Let K be a number field, let $\mathfrak{p} \subseteq \mathcal{O}_K$ be a prime ideal over p, and let R denote the localization of $\mathcal{O}_K$ at $\mathfrak{p}$. Denote by $\pi \in R$ a uniformizer, and let $k = R/\pi$ be the residue field, so that there is an $\mathbf{E}_\infty$-map $S[\pi] \to R$ sending $\pi \mapsto \pi$. For any R-linear ∞-category $\mathcal{C}$, there are cofiber sequences*

$$\mathrm{THH}(\mathcal{C})^{t\mathbf{Z}/p} \otimes_R k \to \mathrm{THH}(\mathcal{C} \otimes_R k)^{t\mathbf{Z}/p} \xrightarrow{\nabla^{t\mathbf{Z}/p}} \mathrm{THH}(\mathcal{C} \otimes_R k)^{t\mathbf{Z}/p}, \tag{26}$$

$$\mathrm{TP}(\mathcal{C}) \to \mathrm{TP}(\mathcal{C}/S[\pi]) \xrightarrow{\nabla^{tS^1}} \mathrm{TP}(\mathcal{C}/S[\pi]). \tag{27}$$

Remark 3.1.20. The cofiber sequence (27) was used in **[LW20]** to calculate $\mathrm{TP}(\mathcal{O}_K)$ by computing the resulting endomorphism of $\mathrm{TP}(\mathcal{O}_K/S[\pi])$.

3.2. Some calculations of THH relative to $X(p)$ and Θ. We now calculate the topological Sen operator for perfectoid rings; these calculations lend further evidence for Conjecture 3.1.14.

Recollection 3.2.1. Let R be a perfectoid ring. Recall that $A_{\mathrm{inf}}(R) = W(R^\flat)$, so that $L_{A_{\mathrm{inf}}(R)/\mathbf{Z}_p}$ is p-completely zero. Let $A^+_{\mathrm{inf}}(R)$ denote the spherical Witt vectors $W^+(R^\flat)$ of [**Lur18**, Example 5.2.7].

Lemma 3.2.2. *Let ξ be a generator of the kernel of Fontaine's map $\theta : A_{\mathrm{inf}}(R) \to R$. Let $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+_{\mathrm{inf}}(R))$ denote the $\mathbf{E}_2$-map which detects $1-\xi \in A_{\mathrm{inf}}(R)^\times$ on the bottom cell of the source. Then there is an equivalence of $\mathbf{E}_2$-$A^+_{\mathrm{inf}}(R)$-algebras between the ξ-adic completion of $A_{\mathrm{inf}}(R)$ and the ξ-adic completion of the Thom spectrum of the following composite:*

$$g_\xi : \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to \mathrm{BGL}_1(A^+_{\mathrm{inf}}(R)).$$

In particular, there is an equivalence $\mathrm{THH}(A_{\mathrm{inf}}(R)^\wedge_\xi / A^+_{\mathrm{inf}}(R)^\wedge_\xi) \simeq A_{\mathrm{inf}}(R)^\wedge_\xi[\Omega S^3\langle 3\rangle]$ of $\mathbf{E}_2$-$A_{\mathrm{inf}}(R)^\wedge_\xi$-algebras.

Proof. Recall from [**Mao20**, Theorem 1.13] that the Thom spectrum of the map $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+_{\mathrm{inf}}(R))$ is equivalent to R as an $\mathbf{E}_2$-$A^+_{\mathrm{inf}}(R)$-algebra. The fiber sequence

$$\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to S^1$$

implies that there is a class $\xi \in \pi_0(\Omega^2 S^3\langle 3\rangle)^{g_\xi}$ and a map $S^1 \to \mathrm{BGL}_1(\Omega^2 S^3\langle 3\rangle)^{g_\xi}$ detecting $1-\xi$, such that its Thom spectrum is R. This implies that there is a cofiber sequence

$$(\Omega^2 S^3\langle 3\rangle)^{g_\xi} \xrightarrow{\xi} (\Omega^2 S^3\langle 3\rangle)^{g_\xi} \to R.$$

It follows that the ξ-adic completion $(\Omega^2 S^3\langle 3\rangle)^{g_\xi}$ is equivalent to $A_{\mathrm{inf}}(R)^\wedge_\xi$. The claim about THH follows in the standard manner using [**BCS10**]. $\square$

Remark 3.2.3. In fact, the calculation from [**BMS19**, Theorem 6.1] that $\pi_*\mathrm{THH}(R) \cong R[\sigma]$ is equivalent to [**Mao20**, Theorem 1.13] (which constructs R as the Thom spectrum of the map $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+_{\mathrm{inf}}(R))$). The equivalence between these two statements can be proved similarly to [**KN19**, Remark 1.5].

Proposition 3.2.4. *Let R be a p-complete perfectoid ring. Then there is a p-complete equivalence*

$$\mathrm{THH}(R/X(p)) \simeq R[\mathbf{C}P^\infty \times \Omega S^{2p+1}] \otimes_R \epsilon^R.$$

In particular, if θ denotes the "polynomial[12]" generator in degree $2p$ arising via the James filtration on ΩS^{2p+1} and $R\langle u\rangle = \pi_* R[\mathbf{C}P^\infty]$ is (the underlying R-module of) a divided power algebra on a class u in degree 2, then there is a p-complete isomorphism

$$\pi_*\mathrm{THH}(R/X(p)) \simeq R[\theta]\langle u\rangle \otimes_R \epsilon^R_*.$$

Proof. Let $X(p)_\xi$ denote the ξ-adic completion of the Thom spectrum of the composite

$$\Omega\mathrm{SU}(p) \to \Omega S^{2p-1} \xrightarrow{\alpha_1} \Omega^2 S^3\langle 3\rangle \to \mathrm{BGL}_1(A^+_{\mathrm{inf}}(R)).$$

[12]Recall that $\mathrm{THH}(R/X(p))$ is not a ring; the word polynomial simply means the subspace generated by $R[\Omega S^{2p+1}]_*$.

Then, the map $\mathrm{THH}(X(p)_\xi) \to \mathrm{THH}(X(p)) \otimes A_{\mathrm{inf}}^+(R)^\wedge_\xi$ is a (p,ξ)-complete equivalence: indeed, the above composite is determined as an $\mathbf{E}_1$-map by the composite

$$\mathrm{SU}(p) \to S^{2p-1} \xrightarrow{(1-\xi)\alpha_1} B^2\mathrm{GL}_1(A_{\mathrm{inf}}^+(R)).$$

Since $1-\xi$ is a unit in $\pi_0 A_{\mathrm{inf}}^+(R) \cong A_{\mathrm{inf}}(R)$, it suffices to prove that the map $\mathrm{THH}(A_{\mathrm{inf}}^+(R)^\wedge_\xi) \to A_{\mathrm{inf}}^+(R)^\wedge_\xi$ is a (p,ξ)-complete equivalence. But this is clear: after killing ξ and tensoring with $\mathbf{F}_p$, we obtain the map $\mathrm{HH}(R^\flat/\mathbf{F}_p) \to R^\flat$, which is an equivalence since $R^\flat$ is perfect.

It then follows from Lemma 3.2.2 and the same argument used to prove Theorem 2.2.4(a) that there are (p,ξ)-complete equivalences

$$\mathrm{THH}(A_{\mathrm{inf}}(R)^\wedge_\xi/X(p)) \simeq \mathrm{THH}(A_{\mathrm{inf}}(R)^\wedge_\xi/X(p)_\xi) \simeq A_{\mathrm{inf}}(R)[\Omega S^{2p+1} \times \mathrm{BSU}(p-1)].$$

Therefore, there are p-complete equivalences

$$\begin{aligned}\mathrm{THH}(R/X(p)) &\simeq \mathrm{THH}(R/X(p)_\xi)\\ &\simeq \mathrm{THH}(R/A_{\mathrm{inf}}^+(R)^\wedge_\xi) \otimes_{\mathrm{THH}(A_{\mathrm{inf}}(R)^\wedge_\xi/A_{\mathrm{inf}}^+(R)^\wedge_\xi)} \mathrm{THH}(A_{\mathrm{inf}}(R)^\wedge_\xi/X(p)_\xi)\\ &\simeq \mathrm{THH}(R/A_{\mathrm{inf}}^+(R)^\wedge_\xi) \otimes_{A_{\mathrm{inf}}(R)^\wedge_\xi[\Omega S^3\langle 3\rangle]} A_{\mathrm{inf}}(R)^\wedge_\xi[\Omega S^{2p+1} \times \mathrm{BSU}(p-1)].\end{aligned}$$

Since R is perfectoid, [**BMS19**, Theorem 6.1] implies that $\mathrm{THH}(R/A_{\mathrm{inf}}^+(R)) \simeq R[\Omega S^3]$. The map $\mathrm{THH}(W(R^\flat)) \to \mathrm{THH}(R)$ induced by the unit can be identified with the composite $W(R^\flat)[\Omega S^3\langle 3\rangle] \to R[\Omega S^3]$, induced by Fontaine's map $\theta : A_{\mathrm{inf}}(R) \to R$. There is a p-local Cartesian square

$$\begin{tikzcd} \Omega S^3\langle 3\rangle \arrow[rr] \arrow[d] & & \Omega S^3 \arrow[d, "H_p \times \iota"] \\ \Omega S^{2p+1} \times \mathrm{BSU}(p-1) \arrow[r] & \Omega S^{2p+1} \times \mathbf{C}P^\infty \times \mathrm{BSU}(p-1), & \end{tikzcd} \tag{28}$$

which implies that

$$\mathrm{THH}(R/X(p)) \simeq R[\Omega S^{2p+1} \times \mathbf{C}P^\infty \times \mathrm{BSU}(p-1)],$$

as desired. Alternatively, there are equivalences

$$\begin{aligned}\mathrm{THH}(R/X(p)_\epsilon) &\simeq \mathrm{THH}(R/A_{\mathrm{inf}}^+(R)^\wedge_\xi) \otimes_{\mathrm{THH}(X(p)_\epsilon/A_{\mathrm{inf}}^+(R)^\wedge_\xi)} X(p)_\epsilon\\ &\simeq R[\Omega S^3] \otimes_{R[\mathrm{SU}(p)]} R.\end{aligned}$$

The desired calculation follows from the observation that there is a p-local fibration

$$\mathrm{SU}(p) \simeq \mathrm{SU}(p-1) \times S^{2p-1} \xrightarrow{*\times\alpha_1} \Omega S^3 \xrightarrow{H_p\times\iota} \Omega S^{2p+1} \times \mathbf{C}P^\infty \times \mathrm{BSU}(p-1)$$

which is induced by the Cartesian square (28). □

Remark 3.2.5. Proposition 3.2.4 has the following slight variant: if R is a p-complete perfectoid ring, then there is a p-complete equivalence $\mathrm{THH}(R/J(p)) \simeq R[\Omega S^3 \times \mathbf{C}P^\infty]$. The only modification is that one instead has to use the p-local Cartesian square

$$\begin{tikzcd} \Omega S^3\langle 3\rangle \arrow[r] \arrow[d] & \Omega S^3 \arrow[d] \\ \Omega S^3 \arrow[r] & \Omega S^3 \times \mathbf{C}P^\infty, \end{tikzcd}$$

which supplies a fibration

$$S^1 \to \Omega S^3 \to \Omega S^3 \times \mathbf{C}P^\infty.$$

In particular, the above discussion shows that $\pi_*\mathrm{THH}(R/J(p)) \cong R[x]\langle u\rangle$. This is compatible with Conjecture 3.1.14:

(a) First, $\pi_*\mathrm{THH}(R/J(p))[x^{-1}] \cong R[x^{\pm 1}]\langle \frac{u}{x}\rangle$. Since $\frac{u}{x}$ lives in degree 0, Conjecture 3.1.14 predicts that $\widehat{\Omega}^{\not D}_R \cong R\langle \frac{u}{x}\rangle$. This is indeed true: [**BL22a**, Example 4.7.6] implies that the diffracted Hodge complex of a p-complete perfectoid ring R is a divided power R-algebra on a single class in degree zero.
(b) Second, $\tau_{(2n-2,2n]}\mathrm{THH}(R/J(p))$ is equivalent to $\bigoplus_{0\leq j\leq n} R\cdot\gamma_j(u)x^{n-j}$, so that Conjecture 3.1.14 predicts that $\mathrm{F}^{\mathrm{conj}}_i\widehat{\Omega}^{\not D}_R$ is isomorphic to the R-submodule of $\widehat{\Omega}^{\not D}_R$ generated by $\{\gamma_j(\frac{u}{x})\}_{0\leq j\leq n}$. This is indeed true: see $(*_n)$ in the proof of [**BL22a**, Lemma 5.6.14]. In the same way, $\tau_{(2(n-1)p,2np]}\mathrm{THH}(R/T(1))$ is a free R-module spanned by $\theta^i\gamma_j(u)$ for $(n-1-i)p < j \leq (n-i)p$. This includes $\gamma_j(u)$ for $(n-1)p < j \leq np$, but also terms such as θ^n and $\theta^{n-1}\gamma_p(u)$.

Remark 3.2.6. We can understand the calculation of Proposition 3.2.4 more algebraically as follows. There is a p-local fiber sequence

$$S^{2p-1} \to \Omega S^3 \to \mathbf{C}P^\infty \times \Omega S^{2p+1}, \tag{29}$$

where the second map is given by the product of the canonical map $\Omega S^3 \to \mathbf{C}P^\infty$ with the James-Hopf map $\Omega S^3 \to \Omega S^{2p+1}$. The Serre spectral sequence in $\mathbf{Z}_p$-homology for (29) is given by

$$E^2_{*,*} = \mathbf{Z}_p\langle u\rangle \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\theta,\epsilon]/\epsilon^2 \Rightarrow \pi_*\mathbf{Z}_p[\Omega S^3] \cong \mathbf{Z}_p[\sigma],$$

where ϵ lives in degree $2p-1$. It is not difficult to show that there is a single family of differentials given by

$$d^{2p}(\gamma_{p^n}(u)) = \epsilon\prod_{j=1}^{n-1}\gamma_{p^j}(u)^{p-1},\ d^{2p}(\theta^j) = jp\theta^{j-1}\epsilon.$$

where the equality is to be understood up to p-adic units. The above description implies that the map $d^{2p} : E^2_{2np,0} \to E^2_{2np-2p,2p-1}$ is surjective, and its kernel is a free $\mathbf{Z}_p$-module of rank 1 (for example, one can calculate an explicit $(n+1)\times n$-matrix with coefficients in $\mathbf{Z}_p$ which describes d^{2p}). If R is a perfectoid ring, this discussion determines the Serre spectral sequence in R-homology for (29). Since the d^{2p}-differential in this spectral sequence is just the effect of the topological Sen operator $\Theta_R : \mathrm{THH}(R/X(p)) \to \Sigma^{2p}\mathrm{THH}(R/X(p))$ on homotopy, we see that Θ_R is given (up to p-adic units) by the map

$$\gamma_{p^n}(u) \mapsto \prod_{j=1}^{n-1}\gamma_{p^j}(u)^{p-1}.$$

Treat u as a variable, and write $\frac{u^j}{j!}$ to denote $\gamma_j(u)$; then[13]

$$(u^{1-p}\partial_u)(\gamma_{p^n}(u)) = \frac{u^{p^n-p}}{(p^n-1)!} = \frac{(u^{p^{n-1}})^{p-1}}{(p^n-1)!} = \frac{u^{\sum_{j=1}^{n-1} p^j(p-1)}}{(p^n-1)!} \doteq \prod_{j=1}^{n-1}\left(\frac{u^{p^j}}{p^j!}\right)^{p-1}.$$

Therefore, we may informally write $\Theta_R = u^{1-p}\partial_u$. The division by u^p can be viewed as accounting for the shift by $2p$ in Θ_R. Note that if R is p-torsionfree, this operator can in turn be interpreted as $p\partial_{u^p}$. Similarly, under the isomorphism $\pi_*\mathrm{THH}(R/J(p)) \cong R[x]\langle u\rangle$, the operator $\Theta'_R : \mathrm{THH}(R/J(p)) \to \Sigma^2\mathrm{THH}(R/J(p))$ can be interpreted as ∂_u.

A slight variant of the above discussion proves an analogous statement for $\mathbf{Z}/p^n$.

Definition 3.2.7. Let Y_n denote the fiber of the composite

$$\mathbf{H}P^\infty \to K(\mathbf{Z},4) \to K(\mathbf{Z}/p^{n-1},4).$$

Proposition 3.2.8. *Fix an odd prime p. There are equivalences*

$$\begin{aligned}\mathrm{THH}(\mathbf{Z}/p^n/X(p)) &\simeq \mathbf{Z}/p^n[\Omega S^{2p+1}\times B^2(p^{n-1}\mathbf{Z})]\otimes_{\mathbf{Z}/p^n}\epsilon^{\mathbf{Z}/p^n},\\ \mathrm{THH}(\mathbf{Z}/p^n/J(p)) &\simeq \mathbf{Z}/p^n[\Omega S^3\times B^2(p^{n-1}\mathbf{Z})],\end{aligned}$$

where the map $\mathbf{F}_p\otimes_{\mathbf{Z}/p^n}\mathrm{THH}(\mathbf{Z}/p^n/J(p)) \to \mathrm{THH}(\mathbf{F}_p/J(p))$ is given by the map $\mathbf{F}_p[\Omega S^3\times B^2(p^{n-1}\mathbf{Z})] \to \mathbf{F}_p[\Omega S^3\times B^2(\mathbf{Z})]$ induced by $p^{n-1}\mathbf{Z}\subseteq\mathbf{Z}$.

Proof. In [**Kit20**], it was shown that $\mathbf{Z}/p^n$ is the Thom spectrum of the $\mathbf{E}_2$-map

$$\Omega^3 Y_n \to \Omega^3\mathbf{H}P^\infty \simeq \Omega^2 S^3 \to \mathrm{BGL}_1(S^0),$$

which implies that $\mathrm{THH}(\mathbf{Z}/p^n)\simeq \mathbf{Z}/p^n[\Omega^2 Y_n]$. Note that there is a canonical map $\Omega^2 Y_n\to\Omega S^3$, and hence a map $\Omega^2 Y_n \to \mathbf{C}P^\infty$. Just as with Proposition 3.2.4, we have

$$\begin{aligned}\mathrm{THH}(\mathbf{Z}/p^n/X(p)) &\simeq \mathrm{THH}(\mathbf{Z}/p^n)\otimes_{\mathrm{THH}(X(p))} X(p)\\ &\simeq \mathrm{THH}(\mathbf{Z}/p^n)\otimes_{\mathrm{THH}(\mathbf{Z}_p)}\mathrm{THH}(\mathbf{Z}_p/X(p))\\ &\simeq \mathbf{Z}/p^n[\Omega^2 Y_n]\otimes_{\mathbf{Z}/p^n[\Omega S^3\langle 3\rangle]}\mathbf{Z}/p^n[\Omega S^{2p+1}\times\mathrm{BSU}(p-1)].\end{aligned}$$

There is still a p-local Cartesian square

$$\begin{tikzcd}
\Omega S^3\langle 3\rangle \arrow[r] \arrow[d] & \Omega^2 Y_n \arrow[d] \\
\Omega S^{2p+1}\times\mathrm{BSU}(p-1) \arrow[r] & \Omega S^{2p+1}\times B^2(p^{n-1}\mathbf{Z})\times\mathrm{BSU}(p-1),
\end{tikzcd}$$

which implies the calculation of $\mathrm{THH}(\mathbf{Z}/p^n/X(p))$. The calculation of $\mathrm{THH}(\mathbf{Z}/p^n/J(p))$ is similar. □

[13]For the last equality, note that if $n\geq 1$, then $(p^n-1)!$ is a p-adic unit multiple of $\prod_{j=1}^{n-1}(p^j!)^{p-1}$. Indeed, observe that $p^n-1=\sum_{j=0}^{n-1}p^j(p-1)$. By Legendre's formula for the p-adic valuation of factorials, we have $v_p(p^j!)=\frac{p^j-1}{p-1}$, so that

$$v_p((p^n-1)!) = \frac{p^n-1-n(p-1)}{p-1} = -n+\sum_{j=0}^{n-1}p^j = \sum_{j=1}^{n-1}(p^j-1) = v_p\left(\prod_{j=1}^{n-1}(p^j!)^{p-1}\right),$$

as desired.

Remark 3.2.9. One could also deduce Proposition 3.2.8 for $n \geq 2$ from Proposition 3.2.4 for $\mathbf{F}_p$, using descent and the fact that $\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}/p^n) = \mathbf{F}_p[K(\mathbf{Z}/p^{n-1}, 2)]$. Indeed, the composite $S^1 \xrightarrow{p^{n-1}} S^1 \xrightarrow{1-p} \mathrm{BGL}_1(S)$ detects the class $(1-p)^{p^{n-1}} = 1 - p^n u \in \mathbf{Z}_p^\times$ for some p-adic unit u. Therefore, its Thom spectrum is equivalent to $\mathbf{Z}/p^n$. In turn, [**Dev23a**, Proposition 2.1.6] (or [**Bea17**]) and the fiber sequence

$$S^1 \xrightarrow{p^{n-1}} S^1 \to B\mathbf{Z}/p^{n-1}$$

imply that $\mathbf{F}_p$ is the Thom spectrum of a map $B\mathbf{Z}/p^{n-1} \to \mathrm{BGL}_1(\mathbf{Z}/p^n)$ which detects $1 - p \in (\mathbf{Z}/p^n)^\times$ on the bottom cell of the source. Applying [**BCS10**] implies the desired calculation of $\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}/p^n)$.

Remark 3.2.10. There is a higher chromatic analogue of Proposition 3.2.8. To explain this, recall from [**Lur15**, Construction 3.5.1] that there is an $\mathbf{E}_2$-algebra $S(\!(\hbar)\!)$ over the sphere spectrum with $|\hbar| = -2$. It follows from [**DHL$^+$23**, Corollary 3.12] that $S(\!(\hbar)\!)$ can be upgraded to an $\mathbf{E}_2^{\mathrm{fr}}$-algebra. Tensoring with $X(p^n)$ therefore defines an $\mathbf{E}_2^{\mathrm{fr}}$-ring $X(p^n)(\!(\hbar)\!)$; in particular, one can define THH relative to $X(p^n)(\!(\hbar)\!)$. The $\mathbf{E}_2$-map $X(p^n) \to \mathrm{BP}\langle n-1\rangle \to \mathrm{BP}\langle n-1\rangle^{tS^1}$ factors through an $\mathbf{E}_2$-map $X(p^n)(\!(\hbar)\!) \to \mathrm{BP}\langle n-1\rangle^{tS^1}$, where $\hbar$ is sent to a complex orientation of $\mathrm{BP}\langle n-1\rangle$ (viewed as a class in $\pi_{-2}\mathrm{BP}\langle n-1\rangle^{tS^1}$). The calculation of Theorem 2.2.4 implies that

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle^{tS^1}/X(p^n)(\!(\hbar)\!)) \simeq \mathrm{BP}\langle n-1\rangle^{tS^1}[\Omega S^{2p^n+1} \times B\Delta_n].$$

The spectrum $\mathrm{BP}\langle n-1\rangle^{t\mathbf{Z}/m}$ is the quotient $\mathrm{BP}\langle n-1\rangle^{tS^1}/\frac{[m](\hbar)}{\hbar}$, where $[m](\hbar)$ denotes the m-series of the formal group law over $\mathrm{BP}\langle n-1\rangle_*$. This can be viewed as the Thom spectrum of a map $S^1 \to \mathrm{BGL}_1(\mathrm{BP}\langle n-1\rangle^{tS^1})$ detecting $1 + \frac{[m](\hbar)}{\hbar} \in \pi_0(\mathrm{BP}\langle n-1\rangle^{tS^1})^\times$. It follows that

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle^{t\mathbf{Z}/m}/X(p^n)(\!(\hbar)\!)) \simeq \mathrm{BP}\langle n-1\rangle^{t\mathbf{Z}/m}[BS^1 \times \Omega S^{2p^n+1} \times B\Delta_n]. \tag{30}$$

When $n = 1$, there is an equivalence $\mathrm{BP}\langle 0\rangle^{t\mathbf{Z}/m} \simeq (\mathbf{Z}/m)^{tS^1}$, and (30) can be viewed as the equivalence of Proposition 3.2.8, base-changed along $\mathbf{Z}/m \to (\mathbf{Z}/m)^{tS^1}$.

Since $B^2(p^{n-1}\mathbf{Z}) \cong \mathbf{C}P^\infty$ (more canonically, it is the total space of the line bundle $\mathcal{O}(p^{n-1})$ over the standard $\mathbf{C}P^\infty$), Proposition 3.2.8 implies that $\pi_*\mathrm{THH}(\mathbf{Z}/p^n/J(p)) \cong \mathbf{Z}/p^n[x]\langle u_n\rangle$ with $|u_n| = |x| = 2$. Were Conjecture 3.1.14 to hold, Proposition 3.2.8 would imply that $\widehat{\Omega}^{\not{D}}_{\mathbf{Z}/p^n}$ is a (discrete) divided power algebra over $\mathbf{Z}/p^n$. In [**BL22b**, Example 5.15], it is shown that if $\mathbf{G}_a^\sharp$ denotes the PD-completion of $\mathbf{G}_a$ at the origin, then $\mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}} \otimes \mathbf{F}_p \cong \mathbf{G}_a^\sharp \otimes \mathbf{F}_p$ in the notation of [**BL22b**]. This implies that $\widehat{\Omega}^{\not{D}}_{\mathbf{Z}/p^n} \otimes_{\mathbf{Z}/p^n} \mathbf{F}_p$ is isomorphic to the divided power algebra $\mathbf{F}_p\langle t_n\rangle$ for $|t_n| = 0$. However, as predicted by Conjecture 3.1.14, there is in fact no need to reduce modulo p: Corollary 3.2.15 below says that $\widehat{\Omega}^{\not{D}}_{\mathbf{Z}/p^n}$ is indeed isomorphic to the divided power algebra $\mathbf{Z}/p^n\langle t_n\rangle$ for $|t_n| = 0$.

I am grateful to Bhargav Bhatt for the statement of the following lemma, which is analogous to the calculation that if R is a commutative ring and $x \in R$ is a regular element, then there is a p-complete equivalence $\mathrm{dR}_{R/x/R} \simeq R\langle x\rangle/x$ (see [**Bha12**, Theorem 8.4]). The argument for Lemma 3.2.11 below is my interpretation of Bhatt's explanation. The topological discussion above can be regarded as an analogue of the calculation that $\mathrm{HH}(R/x/R) \simeq R[\mathbf{C}P^\infty]/x$. We will freely use notation from [**BL22a, BL22b**] below.

Lemma 3.2.11. *Let (A, I) be a transversal prism (i.e., A/I is p-torsionfree). Let $x \in A$ be an element such that $x \pmod I$ is regular in $\overline{A} := A/I$, and such that $(x) \subseteq A$ is ϕ-stable. Then $\mathrm{WCart}^{\mathrm{HT}}_{A/(I,x)/A}$ is p-completely isomorphic to $\mathbf{G}_a^\sharp \times \mathrm{Spf}(A/(I,x))$, so that $\overline{\mathbb{\Delta}}_{A/(I,x)/A} \cong A/(I,x)\langle t\rangle$ with $|t| = 0$.*

Proof. By [**BL22b**, Proposition 5.12], the map $\mathrm{WCart}^{\mathrm{HT}}_{A/(I,x)/A} \to \mathrm{Spf}(A/(I,x))$ is a split gerbe, banded by $T_{A/(I,x)/\overline{A}}\{1\}^\sharp$. In this case, since $x \pmod I$ is a regular element of $\overline{A}$, we see that $L_{A/(I,x)/\overline{A}} = (x)/(x^2)[1]$, so that $T_{A/(I,x)/\overline{A}} = \mathrm{Spf}\,\mathrm{Sym}_{A/(I,x)}(L_{A/(I,x)/\overline{A}})^\wedge_p$ is isomorphic to $\Omega\mathbf{G}_a$ over $A/(I,x)$. It follows that $\mathrm{WCart}^{\mathrm{HT}}_{A/(I,x)/A}$ is isomorphic to a trivial $\mathbf{G}_a^\sharp$-torsor over $\mathrm{Spf}(A/(I,x))$. Since $\overline{\mathbb{\Delta}}_{A/(I,x)/A}$ is the global sections of the structure sheaf of $\mathrm{WCart}^{\mathrm{HT}}_{A/(I,x)/A}$, the lemma follows. □

Remark 3.2.12. In fact, the conjugate filtration $\mathrm{F}_i^{\mathrm{conj}}\overline{\mathbb{\Delta}}_{A/(I,x)/A}$ is isomorphic to the divided power filtration on $A/(I,x)\langle t\rangle$ under Lemma 3.2.11.

Remark 3.2.13. Sticking with the assumptions of Lemma 3.2.11, let us mention without proof that Lemma 3.2.11 is also a consequence of [**BS19**, Example 7.9], which states that $\mathbb{\Delta}_{A/(I,x)/A} \cong A\{\frac{x}{I}\}^\wedge_{(p,I)}$. If $I = (d)$ is principal, the p-complete isomorphism

$$\beta : A/(I,x)\langle t\rangle^\wedge_p \xrightarrow{\sim} \overline{\mathbb{\Delta}}_{A/(I,x)/A} \cong A\left\{\frac{x}{I}\right\}^\wedge_p/I$$

leads to an I-adic Bockstein spectral sequence

$$E_1^{*,*} = A/(I,x)\langle t\rangle^\wedge_p[\overline{d}] \cong A\left\langle\frac{x}{d}\right\rangle^\wedge_p[\overline{d}]/d \Rightarrow A\left\{\frac{x}{d}\right\}^\wedge_{(p,d)},$$

where $\overline{d}$ represents d on the E_1-page.

The map β sends $\gamma_{p^n}(t) \mapsto \delta^n(\frac{x}{d})$ (up to p-adic units). This can be proved by showing that in the setting of Lemma 3.2.11, $\phi(\delta^n(\frac{x}{d})) \in (d) \subseteq A\{\frac{x}{d}\}$ if $n \geq 0$ (see Lemma 3.2.14 below). The fact that

$$\phi\left(\delta^n\left(\frac{x}{d}\right)\right) = \delta^n\left(\frac{x}{d}\right)^p + p\delta^{n+1}\left(\frac{x}{d}\right)$$

then implies that $\delta^n(\frac{x}{d})^p \equiv -p\delta^{n+1}(\frac{x}{d}) \pmod d$. Therefore, the elements $\delta^n(\frac{x}{d})$ can be used to define divided powers of $\frac{x}{d} \pmod d$. In particular, we obtain the desired map $\beta : A/(I,x)\langle t\rangle \to A\{\frac{x}{d}\}^\wedge_{(p,d)}/d$, but further work is required to show that it is a p-complete isomorphism.

Lemma 3.2.14. *Fix notation as in Lemma 3.2.11. Then $\phi(\delta^n(\frac{x}{d})) \in (d) \subseteq A\{\frac{x}{d}\}$.*

Proof. Let $t = \frac{x}{d}$. The desired claim can be proved by induction on n. For the base case, we need to show that $\phi(t) \in I$. By reduction to the universal case, we may assume that (p, d) is regular in A. Then [**AL20**, Lemma 3.6] implies that the sequence $(d, \phi(d))$ is regular in A. Since (x) is ϕ-stable, we see that d divides $\phi(x)$; it then follows from the formula $\phi(d)\phi(t) = \phi(x)$ that d divides $\phi(t)$, as desired. For the inductive step, observe that

$$p\phi(\delta^{n+1}(t)) = p\delta(\phi(\delta^n(t))) = \phi^2(\delta^n(t)) - \phi(\delta^n(t))^p.$$

The inductive hypothesis says that $\phi(\delta^n(t)) \in (d)$ for every $k \geq 1$, so that d divides $p\phi(\delta^{n+1}(t))$. Since (p,d) is a regular sequence, this implies that d divides $\phi(\delta^{n+1}(t))$, as desired. $\square$

This implies the following result, which is also proved in [**Pet23**, Lemma 6.13].

Corollary 3.2.15. *There is an isomorphism* $\mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}} \cong \mathbf{G}_a^\sharp \times \mathrm{Spec}(\mathbf{Z}/p^n)$ *of* $\mathbf{Z}/p^n$*-schemes. In particular, the scaling action of* $\mathbf{G}_m^\sharp$ *on* $\mathbf{G}_a^\sharp$ *over* $\mathbf{Z}/p^n$ *gives an isomorphism* $\mathrm{WCart}^{\mathrm{HT}}_{\mathbf{Z}/p^n} \cong \mathbf{G}_a^\sharp/\mathbf{G}_m^\sharp$ *of* $\mathbf{Z}/p^n$*-stacks.*

Proof. Recall that $\overline{\Delta}_{\mathbf{Z}/p^n/\mathbf{Z}_p[\![\widetilde{p}]\!]} = \widehat{\Omega}^{\not{D}}_{\mathbf{Z}/p^n}$. Lemma 3.2.11 implies that $\overline{\Delta}_{\mathbf{Z}/p^n/\mathbf{Z}_p[\![\widetilde{p}]\!]} \cong \mathbf{Z}/p^n\langle t\rangle$ with $|t| = 0$; this gives the desired claim. (It is useful to view $\gamma_{p^m}(t)$ as a p-adic unit multiple of $\delta^m(\frac{p^n}{\widetilde{p}})$, as described in Remark 3.2.13.)

Alternatively, consider the transversal prism $(A, I) = (\mathbf{Z}_p[\![q-1]\!], [p]_q)$, and let $x = (q-1)^{n(p-1)}$. Note that $\phi(x) \in (x)$, so (x) is ϕ-stable. Then $A/I \cong \mathbf{Z}_p[\zeta_p]$, and $A/(I,x)$ is isomorphic to $\mathbf{Z}_p[\zeta_p]/(\zeta_p-1)^{n(p-1)} \cong \mathbf{Z}/p^n[\zeta_p]$ since the p-adic valuation of $(\zeta_p-1)^{n(p-1)}$ is n. It follows from Lemma 3.2.11 that $\overline{\Delta}_{\mathbf{Z}/p^n[\zeta_p]/\mathbf{Z}_p[\![q-1]\!]} \cong \mathbf{Z}/p^n[\zeta_p]\langle t'\rangle$ with $|t'| = 0$. There is an action of $\mathbf{Z}_p^\times$ (and hence $\mathbf{F}_p^\times \subseteq \mathbf{Z}_p^\times$) on (A, I); taking $\mathbf{F}_p^\times$-fixed points produces an isomorphism

$$\overline{\Delta}_{\mathbf{Z}/p^n/\mathbf{Z}_p[\![\widetilde{p}]\!]} \cong (\overline{\Delta}_{\mathbf{Z}/p^n[\zeta_p]/\mathbf{Z}_p[\![q-1]\!]})^{h\mathbf{F}_p^\times} \cong \mathbf{Z}/p^n\langle t\rangle$$

with $|t| = 0$, as desired. Note that as described in Remark 3.2.13, the divided power $\gamma_{p^m}(t')$ can be viewed as a p-adic multiple of $\delta^m(\frac{(q-1)^{n(p-1)}}{[p]_q}) = \delta^m(\frac{(q-1)^{np-n+1}}{q^p-1})$. $\square$

An alternative (and more hands-on) proof of Corollary 3.2.15 is given in Appendix B; this alternative argument is also presented as [**Pet23**, Lemma 6.13].

Example 3.2.16. Let us describe the topological Sen operator on $\mathrm{THH}(\mathbf{Z}/p^n/X(p))$ for $n \geq 2$ (recall that $p > 2$). This is equivalent to describing the Serre spectral sequence in $\mathbf{Z}/p^n$-homology for the fibration

$$S^{2p-1} \to \Omega^2 Y_n \to \Omega S^{2p+1} \times B(p^{n-1}\mathbf{Z}).$$

Note that this fibration is an analogue of the fibration (5).

It will be simpler to analyze the Serre spectral sequence in $\mathbf{Z}_p$-homology, since all the differentials in the Serre spectral sequence in $\mathbf{Z}/p^n$-homology arise from the Serre spectral sequence in $\mathbf{Z}_p$-homology. The analysis is similar to Remark 3.2.6; the Serre spectral sequence runs

$$E^2_{*,*} = \mathbf{Z}_p\langle u_n\rangle \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\theta,\epsilon]/\epsilon^2 \Rightarrow \pi_*\mathbf{Z}_p[\Omega^2 Y_n], \tag{31}$$

where ϵ lives in degree $2p-1$ and u_n lives in degree 2. There are several ways to determine the differentials in this spectral sequence. Our approach will be to describe the pattern of differentials by first calculating $\pi_*\mathbf{Z}_p[\Omega^2 Y_n]$; in turn, we will do this by computing $\pi_* C^*(\Omega^2 Y_n; \mathbf{Z}_p)$. For this, we use the Serre spectral sequence for the fibration

$$B\mathbf{Z}/p^{n-1} \to \Omega^2 Y_n \to \Omega S^3.$$

Since $\mathrm{H}^*(B\mathbf{Z}/p^{n-1}; \mathbf{Z}) \cong \mathbf{Z}[c]/p^{n-1}c$ with $|c| = 2$, the Serre spectral sequence collapses on the E_2-page, and we find that $\pi_* C^*(\Omega^2 Y_n; \mathbf{Z}_p) \cong \mathbf{Z}_p\langle x\rangle[c]/(x - p^{n-1}c)$ with $|x| = 2$. (If $n = 1$, then $\Omega^2 Y_n \simeq \Omega S^3$, and the cohomology ring is $\mathbf{Z}_p\langle x\rangle$.) For

$n \geq 2$, this is isomorphic to $\mathbf{Z}_p\langle y\rangle[c]/y$, where $y = x - p^{n-1}c$. Indeed, observe that if $n \geq 2$, then

$$\gamma_j(y) := \sum_{i=0}^{j} \frac{p^{i(n-1)}}{i!} c^i \gamma_{j-i}(x)$$

is a well-defined class in $\mathbf{Z}_p\langle x\rangle[c]/(x - p^{n-1}c)$ since p has divided powers in $\mathbf{Z}_p$, and that these classes form a basis for $\mathbf{Z}_p\langle x\rangle[c]/(x - p^{n-1}c)$ as a $\mathbf{Z}_p[c]$-module. Recall that in homological grading, there is an equivalence:

$$\mathbf{Z}_p\langle y\rangle/y \simeq \mathbf{Z}_p \oplus \bigoplus_{n\geq 1} \mathbf{Z}_p/n\{\gamma_n(y)\}[-2n],$$

which implies that if $n \geq 2$, then

$$\mathrm{H}^i(\Omega^2 Y_n; \mathbf{Z}_p) \cong \begin{cases} \mathbf{Z}_p \oplus \bigoplus_{j=1}^k \mathbf{Z}_p/j\{\gamma_j(y)c^{k-j}\} & i = 2k \geq 0 \text{ even} \\ 0 & \text{else.} \end{cases}$$

Using the universal coefficients theorem, we find that if $n \geq 2$, then

$$\pi_i \mathbf{Z}_p[\Omega^2 Y_n] \cong \begin{cases} \mathbf{Z}_p & i \in 2\mathbf{Z}_{\geq 0} \\ \bigoplus_{j=1}^k \mathbf{Z}_p/j & i = 2k-1. \end{cases}$$

The generator of $\pi_{2j}\mathbf{Z}_p[\Omega^2 Y_n]$ is the linear dual to $c^j \in \mathbf{Z}_p\langle y\rangle[c]/y$, while the generator of $\pi_{2k-1}\mathbf{Z}_p[\Omega^2 Y_n]$ which is killed by j is dual to $\gamma_j(y)c^{k-j}$. Note that the homotopy groups $\pi_*\mathbf{Z}_p[\Omega^2 Y_n]$ are *independent* of n if $n \geq 2$ (but the generators of these groups do depend on n).

Let us now return to the Serre spectral sequence (31). Comparison with the Serre spectral sequence for the fibration (5) (i.e., with the topological Sen operator on $\mathrm{THH}(\mathbf{Z}_p/X(p))$; see Remark 3.1.6) forces the differentials in (31) to be given by (up to p-adic units):

$$d^{2p}(\gamma_{p^k}(u_n)) = p^{n-1}\epsilon \prod_{j=1}^{k-1} \gamma_{p^j}(u_n)^{p-1} = p^n \epsilon \partial_{u_n^p}(\gamma_{p^k}(u_n)),\ d^{2p}(\theta^j) = jp\theta^{j-1}\epsilon.$$

Reducing modulo p^n, we get the topological Sen operator on $\mathrm{THH}(\mathbf{Z}/p^n/X(p))$ for $n \geq 2$:

$$\Theta : \gamma_{p^k}(u_n) \mapsto p^{n-1} \prod_{j=1}^{k-1} \gamma_{p^j}(u_n)^{p-1},\ \Theta : \theta^j \mapsto jp\theta^{j-1}.$$

Observe that this acts as "$p^n\partial_{u_n^p}$". Of course, one can similarly deduce the action of the topological Sen operator on $\mathrm{THH}(\mathbf{Z}/p^n/J(p))$. This recovers the calculation

$$\pi_j \mathrm{THH}(\mathbf{Z}/p^n) = \begin{cases} \bigoplus_{i=0}^j \mathbf{Z}/\gcd(j, p^n) & \text{even } j \geq 0, \\ \bigoplus_{i=0}^j \mathbf{Z}/\gcd(j, p^n) & \text{odd } j \geq 0, \\ 0 & j < 0. \end{cases}$$

Another example of the topological Sen operator comes from studying complete DVRs, where the relationship between THH relative to $J(p)$ and the diffracted Hodge complex predicted by Conjecture 3.1.14 can be seen directly.

Example 3.2.17. Let R be a p-torsionfree complete DVR of mixed characteristic $(0, p > 0)$ whose residue field k is perfect. Then we have

$$\pi_* \mathrm{THH}(R/X(p)) \cong \mathrm{HH}_*(R/\mathbf{Z}_p)[\theta] \otimes_{\mathbf{Z}_p} \epsilon_*^{\mathbf{Z}_p},$$

and the map $\Theta : \pi_*\mathrm{THH}(R/X(p)) \to \pi_{*-2p}\mathrm{THH}(R/X(p))$ sends $\theta^j \mapsto jp\theta^{j-1}$. To compute the action of the topological Sen operator on the remainder of $\mathrm{THH}(R/X(p))$, it will be simpler to assume that $T(1)$ is an $\mathbf{E}_2$-ring and work instead with $\mathrm{THH}(R/T(1))$; this is merely cosmetic, and it is not difficult to modify the below argument to use $\mathrm{THH}(R/X(p))$ instead. Then, we have $\pi_*\mathrm{THH}(R/T(1)) \cong \mathrm{HH}_*(R/\mathbf{Z}_p)[\theta]$. We will compute $\mathrm{THH}(R)$ using the topological Sen operator on $\mathrm{THH}(R/T(1))$ and (16). Let $\pi \in R$ be a uniformizer, let $E(u) \in W(k)[\![u]\!]$ be its minimal polynomial, and let $E'(u) \in W(k)[\![u]\!]$ denote its derivative with respect to u. Recall that $R = W(k)[\![u]\!]/E(u)$, that $W(k)$ is étale over $\mathbf{Z}_p$, $\pi_*\mathrm{HH}(W(k)[\![u]\!]/W(k)) \cong \Lambda_{W(k)[\![u]\!]}(du)$ with $|du| = 1$, and $\pi_*\mathrm{HH}(R/W(k)[\![u]\!]) \cong R\langle\sigma_E\rangle$, where $\sigma_E := \sigma^2(E(u))$. The transitivity sequence for the composite $W(k) \to W(k)[\![u]\!] \to R$ implies that $\mathrm{HH}(R/W(k)) \simeq \mathrm{HH}(R/\mathbf{Z}_p)$ is the fiber of a map $R\langle\sigma_E\rangle \to \Sigma^2 R\langle\sigma_E\rangle$ sending $\gamma_n(\sigma_E) \mapsto E'(\pi)\gamma_{n-1}(\sigma_E)$. In particular,

$$\pi_n\mathrm{HH}(R/\mathbf{Z}_p) \cong \begin{cases} R & n = 0,\\ R/E'(\pi) & n = 2j+1,\ j \geq 0,\\ 0 & \text{else.}\end{cases}$$

Let us denote the generator of $\pi_{2j-1}\mathrm{HH}(R/\mathbf{Z}_p)$ by z_j, so that $\gamma_{j-1}(\sigma_E) \in \pi_{2j}\Sigma^2 R\langle\sigma_E\rangle$ is sent to z_j under the boundary map $\Sigma^2 R\langle\sigma_E\rangle \to \Sigma\mathrm{HH}(R/\mathbf{Z}_p)$. We then have

$$\pi_n\mathrm{THH}(R/T(1)) \cong \begin{cases} R\cdot\theta^j & n = 2pj, j \geq 0\\ \bigoplus_{0\leq i<j/p} R/E'(\pi)\cdot z_{j-pi}\theta^i & n = 2j-1, j \geq 0,\\ 0 & \text{else.}\end{cases} \tag{32}$$

From this, we can describe the topological Sen operator on $\mathrm{THH}(R/T(1))$. For this, it will be useful to rephrase the above calculations somewhat, and use $J(p)$ instead of $T(1)$. It is easy to compute that $\pi_*\mathrm{THH}(R/J(p)) \cong \mathrm{HH}_*(R/\mathbf{Z}_p)[x]$, where x is the class in degree 2 from Proposition 2.3.3. In other words,

$$\pi_n\mathrm{THH}(R/J(p)) \cong \begin{cases} R\cdot x^j & n = 2j \text{ for } j \geq 0,\\ \bigoplus_{0\leq i<j} R/E'(\pi)\cdot z_{j-i}x^i & n = 2j-1, j \geq 1.\end{cases}$$

Since $\mathrm{HH}(R/\mathbf{Z}_p)$ is the fiber of a map $R\langle\sigma_E\rangle \to \Sigma^2 R\langle\sigma_E\rangle$, it follows that there is a cofiber sequence

$$\mathrm{THH}(R/J(p)) \to R\langle\sigma_E\rangle[x] \xrightarrow{\nabla} \Sigma^2 R\langle\sigma_E\rangle[x], \tag{33}$$

where we have denoted the second map by ∇. The map ∇ is given on homotopy by a derivation, sending $\sigma_E \mapsto E'(\pi)$. Informally, $\mathrm{THH}(R/J(p))$ can be written as $R\langle\sigma_E\rangle[x]^{\nabla=0}$.

The topological Sen operator $\Theta : \mathrm{THH}(R/J(p)) \to \Sigma^2\mathrm{THH}(R/J(p))$ is described on homotopy by the operator on $R\langle\sigma_E\rangle[x]$ sending $x \mapsto nx^{n-1}$. Note that this operator commutes with ∇ (so that it does indeed define an operator on $\pi_*\mathrm{THH}(R/J(p))$). Observe that since $\mathrm{THH}(R)$ is the fiber of $\Theta : \mathrm{THH}(R/J(p)) \to \Sigma^2\mathrm{THH}(R/J(p))$, and $\mathrm{THH}(R/J(p))$ is the fiber of $\nabla : R\langle\sigma_E\rangle[x] \to \Sigma^2 R\langle\sigma_E\rangle[x]$, we can write $\mathrm{THH}(R)$ as the total fiber of the square

$$\begin{tikzcd} R\langle\sigma_E\rangle[x] \arrow[r, "\nabla"] \arrow[d, "\Theta"] & \Sigma^2 R\langle\sigma_E\rangle[x] \arrow[d, "\Theta"] \\ \Sigma^2 R\langle\sigma_E\rangle[x] \arrow[r, "\nabla"] & \Sigma^4 R\langle\sigma_E\rangle[x], \end{tikzcd} \tag{34}$$

where the map denoted Θ sends $x^n \mapsto nx^{n-1}$. In turn, it follows that $\mathrm{THH}(R)$ is also the total fiber of the square

$$\begin{tikzcd} R\langle\sigma_E\rangle[x] \arrow[r, "\nabla"] \arrow[d, "\Theta+\nabla"] & \Sigma^2 R\langle\sigma_E\rangle[x] \arrow[d, "\Theta+\nabla"] \\ \Sigma^2 R\langle\sigma_E\rangle[x] \arrow[r, "\nabla"] & \Sigma^4 R\langle\sigma_E\rangle[x]. \end{tikzcd} \tag{35}$$

The operator $\nabla + \Theta$ acts on $R\langle\sigma_E\rangle[x]$ by

$$\nabla + \Theta : x^n\gamma_m(\sigma_E) \mapsto nx^{n-1}\gamma_m(\sigma_E) + E'(\pi)x^n\gamma_{m-1}(\sigma_E). \tag{36}$$

Let us now invert x, and write $y = \sigma_E x^{-1}$ in $R\langle\sigma_E\rangle[x^{\pm 1}]$. Then y has divided powers and lives in degree 0, and there is an isomorphism $R\langle\sigma_E\rangle[x^{\pm1}] \cong R\langle y\rangle^{\wedge}_{(y)}[x^{\pm1}]$ on homotopy. We can formally define Θ on x^n for $n \leq 0$ by the same formula: $\Theta(x^n) = nx^{n-1}$. It follows from (36) that $\Psi := x(\nabla + \Theta)$ sends

$$\Psi : \gamma_n(y) \mapsto -nx^{-n}\gamma_n(\sigma_E) + E'(\pi)x^{-n+1}\gamma_{n-1}(\sigma_E) = E'(\pi)\gamma_{n-1}(y) - n\gamma_n(y).$$

We claim that the action of $-t\partial_t$ on $R\langle(1-t)E'(\pi)\rangle$ agrees the action of Ψ on $R\langle y\rangle^{\wedge}_{(y)}$, if we identify $y = (1-t)E'(\pi)$. Indeed:

$$\begin{aligned} \gamma_n(y) &= \gamma_n((1-t)E'(\pi)) = -E'(\pi)^n\frac{(1-t)^n}{n!} \\ &\xmapsto{-t\partial_t} E'(\pi)^n\frac{t(1-t)^{n-1}}{(n-1)!} = E'(\pi)^n\left(\frac{(1-t)^{n-1}}{(n-1)!} - n\frac{(1-t)^n}{n!}\right) \\ &= E'(\pi)\gamma_{n-1}((1-t)E'(\pi)) - n\gamma_n((1-t)E'(\pi)) \\ &= E'(\pi)\gamma_{n-1}(y) - n\gamma_n(y). \end{aligned}$$

In particular, we can rewrite the square (35) after inverting x as

$$\begin{tikzcd} R\langle(1-t)E'(\pi)\rangle[x^{\pm1}] \arrow[r] \arrow[d, "t\partial_t"] & R\langle(1-t)E'(\pi)\rangle[x^{\pm1}] \arrow[d, "t\partial_t"] \\ R\langle(1-t)E'(\pi)\rangle[x^{\pm1}] \arrow[r] & R\langle(1-t)E'(\pi)\rangle[x^{\pm1}], \end{tikzcd}$$

where the horizontal maps act by sending $\gamma_n((1-t)E'(\pi)) \mapsto E'(\pi)\gamma_{n-1}((1-t)E'(\pi))$. The fiber of either of the horizontal maps in the above square can be identified with $\mathrm{THH}(R/J(p))[x^{-1}]$.

Using the above description of Θ, one can calculate (with some tedium) that

$$\pi_n\mathrm{THH}(R) = \begin{cases} R & n = 0, \\ R/jE'(\pi) & n = 2j-1 \geq 0, \\ 0 & \text{else.} \end{cases}$$

This is exactly the calculation of $\pi_*\mathrm{THH}(R)$ from [**LM00**, Theorem 5.1] (reproved in [**KN19**, Theorem 4.4]).

The above discussion can be compared to [**BL22b**, Remark 9.7], which says that if $X = \mathrm{Spec}\, R$, then $\mathrm{WCart}^{\mathrm{HT}}_X \cong X \times BG$, where $G = \{(a,t) \in \mathbf{G}_a^\sharp \rtimes \mathbf{G}_m^\sharp | t-1 = E'(\pi)a\}$. The canonical map $\mathrm{WCart}^{\mathrm{HT}}_X \to X \times \mathrm{WCart}^{\mathrm{HT}} \cong (B\mathbf{G}_m^\sharp)_X$ can be identified with the map induced on classifying stacks by the quotient map $G \to \mathbf{G}_m^\sharp$ of group schemes over X. Recall that the diffracted Hodge stack $X^{\not{D}}$ can be identified with $\mathrm{WCart}^{\mathrm{HT}}_X \times_{\mathrm{WCart}^{\mathrm{HT}}} \mathrm{Spec}(\mathbf{Z}_p) \cong \mathrm{WCart}^{\mathrm{HT}}_X \times_{\mathrm{WCart}^{\mathrm{HT}} \times X} X$. In particular, $X^{\not{D}} \cong (\mathbf{G}_m^\sharp)_X/G$, i.e., the classifying stack of the group scheme $\mathbf{G}_a^\sharp[E'(\pi)] = \{a \in \mathbf{G}_a^\sharp | E'(\pi)a = 0\}$. One can show from this description that the cohomology of the diffracted Hodge complex $\widehat{\Omega}^{\not{D}}_R \cong \Gamma(B\mathbf{G}_a^\sharp[E'(\pi)]; \mathcal{O})$ is given by

$$\pi_*\widehat{\Omega}^{\not{D}}_R \cong \begin{cases} R & * = 0, \\ \bigoplus_{n\geq 1} R/E'(\pi) & * = -1. \end{cases}$$

Upon 2-periodification, this can be identified with $\pi_*\mathrm{THH}(R/J(p))[x^{-1}]$ (which is additively the 2-periodification of $\pi_*\mathrm{HH}(R/\mathbf{Z}_p)$), as predicted by Conjecture 3.1.14.

Note that the extensions in the following long exact sequence in homotopy for (16) are *always* nontrivial:

$$\begin{aligned} \cdots \to \pi_{2j}\mathrm{THH}(R/T(1)) \to \pi_{2(j-p)}\mathrm{THH}(R/T(1)) \to \pi_{2j-1}\mathrm{THH}(R) \to \\ \pi_{2j-1}\mathrm{THH}(R/T(1)) \to \pi_{2(j-p)-1}\mathrm{THH}(R/T(1)) \to \pi_{2j-2}\mathrm{THH}(R) \to \cdots \end{aligned} \tag{37}$$

For example, when $j = p$, there is a long exact sequence

$$\pi_{2p}\mathrm{THH}(R/T(1)) \cong R\cdot\theta \to \pi_0\mathrm{THH}(R/T(1)) \cong R \to \pi_{2p-1}\mathrm{THH}(R) \to \pi_{2p-1}\mathrm{THH}(R/T(1)) \to 0,$$

which in particular gives a short exact sequence

$$0 \to R/p \to \pi_{2p-1}\mathrm{THH}(R) \to R/E'(\pi) \cong \pi_{2p-1}\mathrm{THH}(R/T(1)) \to 0.$$

Since $\pi_{2p-1}\mathrm{THH}(R) \cong R/pE'(\pi)$, this extension must be nontrivial.

3.3. Relation to the $\widetilde{p}$-de Rham complex. We now describe some additional calculations which give further evidence for Conjecture 3.1.14.

Remark 3.3.1. Assume that R is the p-completion of $\mathbf{Z}_p[t]$. Forthcoming work of Arpon Raksit ([**Rak20**]) shows that (a completion of) $q\Omega_R$ arises as the associated graded of a motivic filtration on $\mathrm{HP}(\mathrm{ku}[t]/\mathrm{ku})$. In fact, Raksit studies $\mathrm{HP}(A[t]/A)$ for a general $\mathbf{E}_\infty$-ring A with even homotopy groups.

Using Remark 3.3.1, one can show that (a completion of) $\widetilde{p}\Omega_R$ arises as the associated graded of a motivic filtration on $\mathrm{HP}(\mathrm{BP}\langle 1\rangle[t]/\mathrm{BP}\langle 1\rangle)$. Moreover, the class $\widetilde{p}$ is identified as the image of $v_1\hbar^{p-1}$ in the associated graded. For the sake of completeness, let us explicitly compute $\pi_*\mathrm{TP}(\mathbf{Z}_p[t]/X(p))$. As in Example 3.2.17, it will be convenient to assume that $T(1)$ is an $\mathbf{E}_2^{\mathrm{fr}}$-ring and work instead with $\mathrm{THH}(R/T(1))$; again, this is merely cosmetic. We first need the following result, which is a special case of [**Rig21**, Proposition 3.1.1] and S^1-equivariant Poincaré duality for S^1/μ_n.

Lemma 3.3.2. *Let X be a bounded-below spectrum equipped with an action of S^1. Then there is an equivalence*

$$\left(\bigoplus_{n\geq 1} X\otimes (S^1/\mu_n)_+\right)^{tS^1} \simeq \lim_{k\to\infty}\bigoplus_{n\geq 1}(\Sigma\tau_{\leq k}X)^{t\mathbf{Z}/n}.$$

Example 3.3.3. Let S be the sphere spectrum. Recall that $\mathbf{Z}_p[t]\simeq \mathbf{Z}_p\otimes S[t]$, so that $\mathrm{THH}(\mathbf{Z}_p[t]/T(1))\simeq \mathrm{THH}(\mathbf{Z}_p/T(1))\otimes\mathrm{THH}(S[t])$. Let $\mathrm{THH}(S[t],(t))$ denote the fiber of the map $\mathrm{THH}(S[t])\to\mathrm{THH}(S)\simeq S$ induced by the augmentation $S[t]\to S$ sending $t\mapsto 0$; note that the map $\mathrm{THH}(S[t])\to S$ admits an S^1-equivariant splitting. Similarly, we write $\mathrm{THH}(\mathbf{Z}_p[t]/T(1),(t))$ to denote the fiber of the map $\mathrm{THH}(\mathbf{Z}_p[t]/T(1))\to\mathrm{THH}(\mathbf{Z}_p/T(1))$ induced by the augmentation $\mathbf{Z}_p[t]\to\mathbf{Z}_p$. Then $\mathrm{THH}(S[t],(t))\simeq\bigoplus_{n\geq 1}(S^1/\mu_n)_+$, so that

(38)
$$\mathrm{THH}(\mathbf{Z}_p[t]/T(1),(t))\simeq\mathrm{THH}(\mathbf{Z}_p/T(1))\otimes\mathrm{THH}(S[t],(t))\simeq\bigoplus_{n\geq 1}(S^1/\mu_n)_+\otimes\mathrm{THH}(\mathbf{Z}_p/T(1)).$$

It follows from Lemma 3.3.2 that there is an equivalence

$$\mathrm{THH}(\mathbf{Z}_p[t]/T(1),(t))^{tS^1}\simeq\lim_{k\to\infty}\bigoplus_{n\geq 1}\Sigma(\tau_{\leq k}\mathrm{THH}(\mathbf{Z}_p/T(1)))^{t\mathbf{Z}/n}.$$

Using Theorem 2.2.4(a), we have $\tau_{\leq 2kp}\mathrm{THH}(\mathbf{Z}_p/T(1))\simeq\mathbf{Z}_p[J_k(S^{2p})]$. A simple calculation using Theorem 2.2.4(a) shows that there is an isomorphism

$$\pi_*(\tau_{\leq 2kp}\mathrm{THH}(\mathbf{Z}_p/T(1)))^{t\mathbf{Z}/n}\cong\pi_*(\mathrm{BP}\langle 1\rangle/v_1^{k+1})^{t\mathbf{Z}/n}\cong\pi_*(\tau_{\leq 2k(p-1)}\mathrm{BP}\langle 1\rangle)^{t\mathbf{Z}/n}.$$

Let $\langle n\rangle(\hbar):=\frac{[n](\hbar)}{\hbar}$, so that $\pi_*\mathrm{BP}\langle 1\rangle^{t\mathbf{Z}/n}\cong\mathbf{Z}_p[v_1](\!(\hbar)\!)/\langle n\rangle(\hbar)$. In analogy to $q=\beta\hbar+1$, if we define $\widetilde{p}=v_1\hbar^{p-1}$, then $\langle n\rangle(\hbar)$ defines an element of $\mathbf{Z}_p[\![\widetilde{p}]\!]$ which we will denote $\langle n\rangle_{\widetilde{p}}$. We conclude that

$$\pi_*(\tau_{\leq 2kp}\mathrm{THH}(\mathbf{Z}_p/T(1)))^{t\mathbf{Z}/n}\cong\mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!)/(\widetilde{p}^{k+1},\langle n\rangle(\hbar)).$$

It follows that

$$\pi_*\mathrm{THH}(\mathbf{Z}_p[t]/T(1),(t))^{tS^1}\cong\lim_{k\to\infty}\bigoplus_{n\geq 1}\Sigma\mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!)/(\widetilde{p}^{k+1},\langle n\rangle_{\widetilde{p}}),$$

i.e., that

$$\pi_*\mathrm{TP}(\mathbf{Z}_p[t]/T(1))\cong\mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!)\times\lim_{k\to\infty}\bigoplus_{n\geq 1}\Sigma\mathbf{Z}_p[\![\widetilde{p}]\!](\!(\hbar)\!)/\langle n\rangle_{\widetilde{p}}.$$

In a manner similar to Example 3.3.3, one calculates that if we write $\mathbf{Z}_p[\beta](\!(\hbar)\!)=\mathbf{Z}_p[\![q-1]\!](\!(\hbar)\!)$ by setting $q=1+\beta\hbar$, and $\langle n\rangle_{\mathbf{G}_m}(\hbar)=\frac{[n]_{\mathbf{G}_m}(\hbar)}{\hbar}$ is the divided n-series of the rescaled multiplicative formal group law $x+y+(q-1)xy$, then

(39)
$$\pi_*\mathrm{HP}(\mathrm{ku}_p^\wedge[t]/\mathrm{ku}_p^\wedge)\cong\mathbf{Z}_p[\![q-1]\!](\!(\hbar)\!)\times\lim_{k\to\infty}\bigoplus_{n\geq 1}\Sigma\mathbf{Z}_p[\![q-1]\!](\!(\hbar)\!)/((q-1)^{k+1},\langle n\rangle_{\mathbf{G}_m}(\hbar)).$$

Moreover, $\pi_*\mathrm{HP}(\mathrm{BP}\langle 1\rangle[t]/\mathrm{BP}\langle 1\rangle)\cong\pi_*\mathrm{HP}(\mathrm{ku}_p^\wedge[t]/\mathrm{ku}_p^\wedge)^{\mathbf{F}_p^\times}$, where $\mathbf{F}_p^\times$ acts on $\mathrm{HP}(\mathrm{ku}_p^\wedge[t]/\mathrm{ku}_p^\wedge)$ via its action by Adams operations on $\mathrm{ku}_p^\wedge$. Note that since $\mathbf{F}_p^\times$ has order coprime to p, taking $\mathbf{F}_p^\times$-invariants preserves small limits and colimits after p-localization. In particular, $\pi_*\mathrm{HP}(\mathrm{BP}\langle 1\rangle[t]/\mathrm{BP}\langle 1\rangle)$ is isomorphic to $\pi_*\mathrm{TP}(\mathbf{Z}_p[t]/T(1))$.

The following is also a consequence of the forthcoming work of Arpon Raksit ([**Rak20**]) mentioned above.

Lemma 3.3.4. *There is an $\mathbf{Z}_p^\times$-equivariant isomorphism*

$$\mathrm{H}^*(q\Omega_{\mathbf{Z}_p[t]})(\!(\hbar)\!) \cong \mathbf{Z}_p[\![q-1]\!](\!(\hbar)\!) \times \bigoplus_{n\geq 1} \Sigma\mathbf{Z}_p[\![q-1]\!](\!(\hbar)\!)/\langle n\rangle_{\mathbf{G}_m}(\hbar).$$

Proof. For the formal group law over $\mathrm{ku}_p^\wedge$, we have

$$\langle n\rangle_{\mathbf{G}_m}(\hbar) = \sum_{i=1}^n \binom{n}{i}\hbar^{i-1}\beta^{i-1} = [n]_q \in \mathbf{Z}_p[\beta](\!(\hbar)\!),$$

where $q := 1+\beta\hbar$. The claim now follows from the fact that the differential ∇_q in $q\Omega_{\mathbf{Z}_p[t]}$ sends $t^n \mapsto [n]_q t^{n-1}dt$. $\square$

In particular, $\pi_*\mathrm{HP}(\mathrm{BP}\langle 1\rangle[t]/\mathrm{BP}\langle 1\rangle) \cong \pi_*\mathrm{TP}(\mathbf{Z}_p[t]/T(1))$ is a 2-periodification of a completion of $\mathrm{H}^*(\widetilde{p}\Omega_{\mathbf{Z}_p[t]})$. This calculation leads to the following expectation related to Conjecture 3.1.14:

Conjecture 3.3.5. *Let R be an animated $\mathbf{Z}_p$-algebra. Then $\mathrm{TP}(R/X(p))$ admits a motivic filtration $\mathrm{F}^\star_{\mathrm{mot}}\mathrm{TP}(R/X(p))$ such that $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TP}(R/X(p)) \simeq \hat{\mathbb{\Delta}}_{R/\mathbf{Z}_p[\![\widetilde{p}]\!]}[2i]\otimes_R \epsilon^R$, where $\hat{\mathbb{\Delta}}_{R/\mathbf{Z}_p[\![\widetilde{p}]\!]}$ is the Nygaard completion of $\widetilde{p}\Omega_R$.*

We now turn to a higher chromatic analogue of (part of) this picture.

Definition 3.3.6. Let R be an $\mathbf{E}_2$-ring, and equip $\mathrm{HH}(R[t]/R) := \mathrm{THH}(S[t])\otimes R$ with the S^1-action inherited from $\mathrm{THH}(S[t])$ and the trivial action on R.

Warning 3.3.7. If R is only an $\mathbf{E}_2$-ring, one *cannot* define Hochschild homology relative to R; in particular, the notation $\mathrm{HH}(R[t]/R)$ is rather abusive. As explained in [**DHL$^+$23**, Corollary 2.9], if R' is an $\mathbf{E}_1$-R-algebra, then $\mathrm{HH}(R'/R)$ only exists (and has a natural S^1-action) when R is a *framed* $\mathbf{E}_2$-ring[14]. In other words, if R is merely an $\mathbf{E}_2$-ring, it would not be clear how to define $\mathrm{HH}(R[t]/R)$, had we not known that $R[t]$ admits a lift to the sphere spectrum. This leads to the following unfortunate warning: if R is an $\mathbf{E}_2^{\mathrm{fr}}$-ring with a nontrivial S^1-action, then the (more natural) circle action on $\mathrm{HH}(R[t]/R)$ arising via the S^1-action on R *cannot* be necessarily identified with the circle action from Definition 3.3.6. However, for this article, we will only use the circle action from Definition 3.3.6.

View $\mathrm{BP}\langle n-1\rangle[t_1,\cdots,t_j]$ as a $\mathbf{Z}^j_{\geq 0}$-graded ring, where t_i has weight $(0,\cdots,1,\cdots,0)$. Then, define $\mathrm{HP}^{\mathrm{gr}}(\mathrm{BP}\langle n\rangle[t_1,\cdots,t_j]/\mathrm{BP}\langle n\rangle)$ to be the S^1-Tate construction of $\mathrm{HH}(\mathrm{BP}\langle n\rangle[t_1,\cdots,t_j]/\mathrm{BP}\langle n\rangle)$ taken internally to $\mathbf{Z}^j_{\geq 0}$-graded $\mathrm{BP}\langle n\rangle$-modules. Similarly, define $\mathrm{TP}^{\mathrm{gr}}(\mathrm{BP}\langle n-1\rangle[t_1,\cdots,t_j]/X(p^n))$ to be the S^1-Tate construction of $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle[t_1,\cdots,t_j]/X(p^n))$ taken internally to $\mathbf{Z}^j_{\geq 0}$-graded $\mathrm{BP}\langle n-1\rangle$-modules. Then, related to Conjecture 3.3.5, we have the following result (which, when $n=0$, is a very special case of the main result of [**PV19**]):

[14]Suppose that R is an $\mathbf{E}_3$-algebra, and $\mathcal{C}$ is an R-linear ∞-category. The choice of a framed knot in $\mathbf{R}^3$ also defines an $\mathbf{E}_{n-3}$-map $\int_{S^1} R \to R$, and hence allows one to define relative Hochschild homology $\mathrm{HH}(\mathcal{C}/R)$. However, this does not define an S^1-action on $\mathrm{HH}(\mathcal{C}/R)$! Thanks to Robert Burklund for this point.

Proposition 3.3.8. *There is a p-complete isomorphism of $\mathbf{Z}^j_{\geq 0}$-graded modules equipped with a map from $\pi_* \mathrm{BP}\langle n\rangle^{tS^1}[B\Delta_n] \cong \pi_* \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$:*

$$\pi_* \mathrm{HP}^{\mathrm{gr}}(\mathrm{BP}\langle n\rangle[t_1, \cdots, t_j]/\mathrm{BP}\langle n\rangle)[B\Delta_n] \cong \pi_* \mathrm{TP}^{\mathrm{gr}}(\mathrm{BP}\langle n-1\rangle[t_1, \cdots, t_j]/X(p^n)).$$

The map $\mathrm{TP}^{\mathrm{gr}}(\mathrm{BP}\langle n-1\rangle[t]/X(p^n)) \to \mathrm{TP}(\mathrm{BP}\langle n-1\rangle/X(p^n))$ is an equivalence after $K(n)$-localization.

Proof. For simplicity, we assume that $j = 1$ and write t instead of t_1. In the graded setting, we may commute the S^1-Tate construction with the infinite direct sum (i.e., Lemma 3.3.2 is not necessary). It follows that there are *graded* equivalences

$$\mathrm{TP}^{\mathrm{gr}}(\mathrm{BP}\langle n-1\rangle[t]/X(p^n), (t)) \simeq \bigoplus_{m\geq 1} \Sigma\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/X(p^n))^{t\mathbf{Z}/m}(m),$$
$$\mathrm{HP}^{\mathrm{gr}}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle, (t)) \simeq \bigoplus_{m\geq 1} \Sigma\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/m}(m).$$

The desired result now follows from Theorem 2.2.4(a). The second statement follows from the above equivalences and the fact that $L_{K(n)}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}) = 0$ by Lemma 3.3.9. $\square$

The proof above used the following (well-known) fact.

Lemma 3.3.9. *Let $\mathrm{BP}\langle n\rangle$ denote any form of the truncated Brown-Peterson spectrum. Then we have $L_{K(n)}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}) = 0$.*

Proof. We first observe that $\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}$ depends only on the p-completion of $\mathrm{BP}\langle n\rangle$; indeed, the obvious variant of [**NS18**, Lemma I.2.9] shows that if X is a bounded-below spectrum with $\mathbf{Z}/p^m$-action, then $X^{t\mathbf{Z}/p^m}$ is p-complete, and the map $X^{t\mathbf{Z}/p^m} \to (X^\wedge_p)^{t\mathbf{Z}/p^m}$ is an equivalence. Since all forms of $\mathrm{BP}\langle n\rangle$ are equivalent after p-completion by [**AL17**], we may therefore reduce to proving the claim for a single form of $\mathrm{BP}\langle n\rangle$.

To show that $L_{K(n)}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}) = 0$, it suffices to show (since $\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}$ is an MU-module) that $(\pi_*\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}[\frac{1}{v_n}])/(p, \cdots, v_{n-1}) \cong \pi_* k(n)^{t\mathbf{Z}/p^m}[\frac{1}{v_n}] = 0$. Recall that $\pi_*\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m} \cong \mathrm{BP}\langle n\rangle_*(\!(\hbar)\!)/[p^m](\hbar)$. We will work with the form of $\mathrm{BP}\langle n\rangle$ such that the associated formal group law over $\pi_*\mathrm{BP}\langle n\rangle$ induces the Honda formal group law over $\pi_* k(n)$. Then, the p^m-series of the formal group law over $\pi_* k(n)$ satisfies $[p^m](\hbar) = v_n^{\frac{p^{nm}-1}{p^n-1}} \hbar^{p^{mn}}$; so $\pi_* k(n)^{t\mathbf{Z}/p^m} \cong \mathbf{F}_p[v_n](\!(\hbar)\!)/v_n^{\frac{p^{nm}-1}{p^n-1}}$. In particular, v_n is nilpotent in $k(n)^{t\mathbf{Z}/p^m}$, so that $\pi_* k(n)^{t\mathbf{Z}/p^m}[\frac{1}{v_n}] = 0$. $\square$

Remark 3.3.10. In general, $\pi_*\mathrm{HP}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle)$ looks like a completion of the 2-periodification of the cohomology of the following two-term complex:

$$\mathrm{BP}\langle n\rangle_*[\![\hbar]\!][t] \xrightarrow{\nabla} \mathrm{BP}\langle n\rangle_*[\![\hbar]\!][t]dt,\ \nabla : t^m \mapsto \tfrac{[m]_{\mathrm{BP}\langle n\rangle}(\hbar)}{\hbar} t^{m-1}dt. \tag{40}$$

This is a variant of the q-de Rham complex, and was first considered by Arpon Raksit (in forthcoming work). Note that an analogue of (40) can be defined for a formal group law $F(x, y)$ over any commutative ring A:

$$F\Omega_{A[t]/A} := \left(A[\![\hbar]\!][t] \xrightarrow{\nabla} A[\![\hbar]\!][t]dt\right),\ \nabla : t^m \mapsto \tfrac{[m](\hbar)}{\hbar} t^{m-1}dt; \tag{41}$$

we will study basic combinatorial properties of such complexes in [**DM23**]. After base-changing to $\mathbf{Q}$, the operator ∇ can be characterized by the formula $\hbar t\nabla = \exp_F(t\partial_t \log_F(\hbar))$.

We also have:

Proposition 3.3.11. *If $\mathcal{C}$ is a left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, and $\mathcal{C}[t]$ denotes $\mathcal{C}\otimes_{\mathrm{BP}\langle n-1\rangle}\mathrm{BP}\langle n-1\rangle[t]$, then Conjecture 2.2.18 implies that the map $L_{K(n)}\mathrm{TP}^{\mathrm{gr}}(\mathcal{C}[t]/X(p^n)) \to L_{K(n)}\mathrm{TP}(\mathcal{C}/X(p^n))$ is an equivalence.*

PROOF. Observe that

$$\mathrm{THH}(\mathcal{C}[t]/X(p^n)) \simeq \mathrm{THH}(\mathcal{C}/X(p^n))(0) \oplus \bigoplus_{m\geq 1} (S^1/\mu_m)_+ \otimes \mathrm{THH}(\mathcal{C}/X(p^n))(m),$$

so that

$$\mathrm{TP}^{\mathrm{gr}}(\mathcal{C}[t]/X(p^n)) \simeq \mathrm{TP}(\mathcal{C}/X(p^n))(0) \oplus \bigoplus_{m\geq 1} \Sigma\mathrm{THH}(\mathcal{C}/X(p^n))^{t\mathbf{Z}/p^m}(m).$$

Now, Conjecture 2.2.18 implies that $\mathrm{THH}(\mathcal{C}/X(p^n))^{t\mathbf{Z}/p^m}$ is a $\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}$-module. But $L_{K(n)}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p^m}) = 0$ by Lemma 3.3.9, so that $L_{K(n)}\mathrm{TP}^{\mathrm{gr}}(\mathcal{C}[t]/X(p^n)) \simeq L_{K(n)}\mathrm{TP}(\mathcal{C}/X(p^n))$, as desired. $\square$

Example 3.3.12. Let $n = 0$, and suppose $\mathcal{C}$ is the ∞-category of quasicoherent sheaves on an $\mathbf{F}_p$-scheme X. Then Proposition 3.3.11 says that the map $\mathrm{TP}^{\mathrm{gr}}(\mathbf{A}^1 \times X) \to \mathrm{TP}(X)$ is a rational equivalence. This is generally not true in the non-graded setting.

Remark 3.3.13. Note that the functor $L_{K(0)}\mathrm{TP}$ is *not* nil-invariant; the same is true of the functor $L_{K(n)}\mathrm{TP}(-/T(n))$ on $\mathrm{BP}\langle n-1\rangle$-algebras. Indeed, [**Hor20**, Theorem 1.1] says that the map $L_{K(0)}\mathrm{TP}(\mathbf{F}_p[t]/t^k) \to L_{K(0)}\mathrm{TP}(\mathbf{F}_p) \simeq \mathbf{Q}_p^{tS^1}$ is an isomorphism if and only if k is a power of p. We can also see this at the level of algebra by calculating the crystalline cohomology of $\mathbf{F}_p[t]/t^k$. If R denotes the p-completion of the PD-envelope of the quotient map $\mathbf{Z}_p[t] \to \mathbf{F}_p[t]/t^k$ (so that $R = \mathbf{Z}_p\left[t, \frac{t^{kj}}{j!}|j\geq 1\right]^\wedge_p$), then [**BO78**, Theorem 7.23] implies that $\Gamma_{\mathrm{crys}}((\mathbf{F}_p[t]/t^k)/\mathbf{Z}_p)$ is quasi-isomorphic to the de Rham complex $\Omega^\bullet_{R/\mathbf{Z}_p}$. Note that R is additively isomorphic to the p-completion of $\bigoplus_{0\leq i\leq k-1}\bigoplus_{j\geq 0}\mathbf{Z}_p\{\frac{t^{kj+i}}{j!}\}$.

Since the derivative of $\frac{t^{kj+i}}{j!}$ is $(kj+i)\frac{t^{kj+i-1}}{j!}$, which simplifies to $k\frac{t^{k(j-1)+k-1}}{(j-1)!}$ when $i = 0$, we find that $\pi_0\Gamma_{\mathrm{crys}}((\mathbf{F}_p[t]/t^k)/\mathbf{Z}_p) \cong \mathbf{Z}_p$, and

$$\begin{aligned}\pi_{-1}\Gamma_{\mathrm{crys}}((\mathbf{F}_p[t]/t^k)/\mathbf{Z}_p) \cong &\left(\bigoplus_{j\geq 0}\mathbf{Z}_p/k\cdot\left\{\frac{t^{k(j+1)-1}}{j!}\right\}\right)^\wedge_p \\ &\oplus\left(\bigoplus_{0\leq i<k-1}\bigoplus_{j\geq 0}\mathbf{Z}_p/(kj+i+1)\cdot\left\{\frac{t^{kj+i}}{j!}\right\}\right)^\wedge_p.\end{aligned}$$

For instance, suppose $k = p$. Then $pj+i+1 \equiv i+1 \pmod p$, which is never zero since $0 \leq i < p-1$. Therefore, the second summand is zero since $pj+i+1$ is a p-adic unit, and we find that $\pi_{-1}\Gamma_{\mathrm{crys}}((\mathbf{F}_p[t]/t^p)/\mathbf{Z}_p) \cong \left(\bigoplus_{j\geq 0}\mathbf{Z}/p\cdot\left\{\frac{t^{p(j+1)-1}}{j!}\right\}\right)^\wedge_p$.

However, if k is not a power of p, the second summand contains a non-torsion piece; for example, if $k = 2$ and p is odd, the second summand contains the p-completion of $\bigoplus_{m\geq 0} \mathbf{Z}/p^m$, which is non-torsion.

Example 3.3.14. Let $n = 1$; then, Proposition 3.3.11 says that Conjecture 2.2.18 implies that up to a Nygaard-type completion, $L_{K(1)}\mathrm{TP}^{\mathrm{gr}}(R[t]/X(p)) \xrightarrow{\sim} L_{K(1)}\mathrm{TP}(R/X(p))$ for R being an $\mathbf{E}_1$-$\mathbf{Z}_p$-algebra. In the non-graded setting, this is generally not true; this is in contrast to [**LMMT20**, Corollary 4.24] (for instance), which says that $K(1)$-local algebraic K-theory is $\mathbf{A}^1$-invariant on connective $K(1)$-acyclic ring spectra (in particular, on connective $\mathbf{E}_1$-$\mathbf{Z}_p$-algebras).

Let us now pivot somewhat to a slightly different topic, working at the famed prime $p = 2$. Then $\widetilde{p}\Omega_R = q\Omega_R$, and there is an interesting action of $\mathbf{Z}/2 \subseteq \mathbf{Z}_2^\times$ on $q\Omega_R$ sending $q \mapsto q^{-1}$. If we view q as the Chern class (in K-theory) of the tautological line bundle on $\mathbf{C}P^\infty$, this corresponds to the action of $\mathbf{Z}/2$ on $\mathbf{C}P^\infty$ given by complex conjugation. This motivates the following discussion:

Remark 3.3.15. We expect that most of the results and conjectures in this article continue to hold with $\mathbf{Z}/2$-equivariance, where "real" topological Hochschild homology $\mathrm{THH}_{\mathbf{R}}$ is interpreted to mean the construction described in [**DMPR17, HHK$^+$20**]. Recall that $\mathbf{Z}/2$ acts on $\mathrm{SU}(n)$ by complex conjugation; we will denote this $\mathbf{Z}/2$-space by $\mathrm{SU}(n)_{\mathbf{R}}$. Let σ (resp. $\rho = \sigma + 1$) denote the sign representation (resp. regular representation), and if X is a $\mathbf{Z}/2$-space, let $\Omega^\sigma X$ denote the space of maps $\mathrm{Map}(S^\sigma, X)$. There is a $\mathbf{Z}/2$-equivariant $\mathbf{E}_\rho$-map

$$\Omega^\sigma \mathrm{SU}(n)_{\mathbf{R}} \simeq \Omega^\rho \mathrm{BSU}(n)_{\mathbf{R}} \to \Omega^\rho \mathrm{BSU}_{\mathbf{R}} \simeq \mathrm{BU}_{\mathbf{R}},$$

which equips its Thom spectrum $X(n)_{\mathbf{R}}$ with the structure of an $\mathbf{E}_\rho$-ring. One can show that the equivariant Quillen idempotent on $\mathrm{MU}_{\mathbf{R}}$ restricts to an idempotent on $X(n)_{\mathbf{R}}$, and we will write $T(n)_{\mathbf{R}}$ to denote the resulting summand of $X(2^n)_{\mathbf{R}}$. Moreover, $\Phi^{\mathbf{Z}/2}T(n)_{\mathbf{R}} \simeq y(n)$ as 2-local $\mathbf{E}_1$-algebras.

We then expect:

Conjecture 3.3.16. *The following are true:*

(a) $T(n)_{\mathbf{R}}$ admits the structure of an $\mathbf{E}_\rho \rtimes \mathrm{U}(1)_{\mathbf{R}}$-algebra, and $X(2^n)_{\mathbf{R}}$ splits as a direct sum of shifts of $T(n)_{\mathbf{R}}$ such that the inclusion $T(n)_{\mathbf{R}} \to X(2^n)_{\mathbf{R}}$ of the unit summand is a map of $\mathbf{E}_\rho$-algebras. In particular, $\mathrm{THH}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}})$ exists and admits an $\mathrm{U}(1)_{\mathbf{R}}$-action[15].

(b) Let $\mathrm{BP}\langle n\rangle_{\mathbf{R}}$ denote the Real truncated Brown-Peterson spectrum. Then there are equivalences

$$\mathrm{THH}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}}) \simeq \mathrm{BP}\langle n-1\rangle_{\mathbf{R}}[\Omega S^{2^n\rho+1}], \tag{42}$$

$$\mathrm{THH}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n-1)_{\mathbf{R}}) \simeq \mathrm{BP}\langle n-1\rangle_{\mathbf{R}} \oplus \bigoplus_{j\geq 1} \Sigma^{2^{n-1}j\rho-1}\mathrm{BP}\langle n\rangle_{\mathbf{R}}/j \tag{43}$$

of $\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}$-modules. The second equivalence requires $n \geq 1$. Furthermore, the class in $\pi_{2^n\rho}\mathrm{THH}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}})$ induced by the map $E : S^{2^n\rho} \to \Omega S^{2^n\rho+1}$ detects $\sigma^\rho(\underline{v}_n)$.

[15]Note that $\mathrm{U}(1)_{\mathbf{R}} = S^\sigma$.

(c) *There is a* $\mathbf{Z}/2$*-equivariant space* $\widetilde{K}_n$ *and an equivariant fibration*

$$S^{2^n\rho-1} \to \widetilde{K}_n \to \Omega S^{2^n\rho+1}$$

such that $\widetilde{K}_0 = \Omega S^\rho$ *and* $\widetilde{K}_1 = \Omega S^{\rho+1}\langle \rho+1\rangle$*. For* $n=0$*, this is simply the EHP sequence for* S^σ*. The boundary map* $\Omega^2 S^{2^{n+1}+1} \to S^{2^{n+1}-1}$ *of the underlying fibration is degree* 2 *on the bottom cell of the source, and* $(\widetilde{K}_n)^{\mathbf{Z}/2} = K_{n-1}$ *as* $(S^{2^n\rho-1})^{\mathbf{Z}/2} = S^{2n-1}$*-fibrations over* $(\Omega S^{2^n\rho+1})^{\mathbf{Z}/2} = \Omega S^{2^n+1}$*.*

(d) *For any* $\mathbf{Z}/2$*-equivariant* $\mathbf{E}_\sigma$*-*$T(n)_{\mathbf{R}}$*-algebra* R*, there is an equivariant cofiber sequence*

$$\mathrm{THH}_{\mathbf{R}}(R/T(n-1)_{\mathbf{R}}) \to \mathrm{THH}_{\mathbf{R}}(R/T(n)_{\mathbf{R}}) \to \Sigma^{2^n\rho}\mathrm{THH}_{\mathbf{R}}(R/T(n)_{\mathbf{R}}),$$

where the second map is a $\mathbf{Z}/2$*-equivariant analogue of the topological Sen operator.*

(e) *Let*

$$\mathrm{TP}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}}) := \mathrm{THH}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}})^{t_{C_2}\mathrm{U}(1)_{\mathbf{R}}},$$

where the notation "$t_{C_2}\mathrm{U}(1)_{\mathbf{R}}$*" means the parametrized Tate construction from* [**QS21**, Remark 1.17]*. Then there is a* $\mathbf{Z}/2$*-equivariant equivalence*

$$\mathrm{TP}_{\mathbf{R}}(\mathrm{BP}\langle n-1\rangle_{\mathbf{R}}/T(n)_{\mathbf{R}}) \simeq \mathrm{BP}\langle n\rangle_{\mathbf{R}}^{t_{C_2}\mathrm{U}(1)_{\mathbf{R}}},$$

where $\mathrm{U}(1)_{\mathbf{R}}$ *acts trivially on* $\mathrm{BP}\langle n\rangle_{\mathbf{R}}$*.*

(f) *Let* R *be a* 2*-complete animated commutative ring, equipped with the trivial* $\mathbf{Z}/2$*-action. Then there is a* $\mathbf{Z}/2$*-equivariant filtration on* $\mathrm{TP}_{\mathbf{R}}(R/T(1)_{\mathbf{R}})$ *such that*

$$\mathrm{gr}^i_{\mathrm{mot}}\mathrm{TP}_{\mathbf{R}}(R/T(1)_{\mathbf{R}}) \simeq (\hat{\mathbb{\Delta}}_{R/\mathbf{Z}[\![q-1]\!]})^\wedge_2[2i] = (q\hat{\Omega}_R)^\wedge_2[2i],$$

where the $\mathbf{Z}/2$*-action on the right-hand side is obtained by viewing* $\mathbf{Z}/2 \subseteq \mathbf{Z}_2^\times \cong \mathbf{Z}/2 \times (1+4\mathbf{Z}_2)$ *and using the* $\mathbf{Z}_2^\times$*-action on the* 2*-completed* q*-de Rham complex.*

Example 3.3.17. Note that [**DMPR17**, Theorem 5.18] and [**HHK**$^+$**20**, Theorem A.1] prove (42) for $n=0$ and (43) for $n=1$, respectively.

Remark 3.3.18. We expect that $\mathrm{BP}\langle n\rangle_{\mathbf{R}}^{\mathbf{Z}/2} \otimes T(n)$ is concentrated in even degrees.

Proposition 3.3.19. *The equivalence* (42) *is true for* $n=1$*.*

Proof sketch. This can be proved analogously to Theorem 2.2.4(a) for $n=1$ using the equivariant Toda fiber sequence

$$S^{\rho+\sigma} \to \Omega S^{\rho+1}\langle \rho+1\rangle \to \Omega S^{2\rho+1}$$

of [**Dev23a**, Equation 7.1].[16] Indeed, recall from [**Dev23a**, Example 7.1.3] that $X(2)_{\mathbf{R}}$ is the Thom spectrum of the map $\Omega^\sigma S^{\rho+\sigma} \to \mathrm{BGL}_1(S)$ detecting $\widetilde{\eta} \in \pi_\sigma S$. By an argument similar to [**BCS10**], this implies that $\mathrm{THH}_{\mathbf{R}}(X(2)_{\mathbf{R}}) \simeq X(2)_{\mathbf{R}}[S^{\rho+\sigma}]$. Therefore:

$$\begin{aligned}\mathrm{THH}_{\mathbf{R}}(\underline{\mathbf{Z}}/X(2)_{\mathbf{R}}) &\simeq \underline{\mathbf{Z}}[\Omega S^{\rho+1}\langle\rho+1\rangle] \otimes_{X(2)_{\mathbf{R}}[S^{\rho+\sigma}]} X(2)_{\mathbf{R}} \\ &\simeq \underline{\mathbf{Z}}[\Omega S^{\rho+1}\langle\rho+1\rangle] \otimes_{\underline{\mathbf{Z}}[S^{\rho+\sigma}]} \underline{\mathbf{Z}} \simeq \mathbf{Z}[\Omega S^{2\rho+1}],\end{aligned}$$

where the last equivalence uses the equivariant Toda fiber sequence. □

[16] See also [**Dev23a**, Theorem 7.2.1], which says that $\underline{\mathbf{Z}}$ is the Thom spectrum of a map $\Omega^\rho S^{2\rho+1} \to \mathrm{BGL}_1(X(2)_{\mathbf{R}})$ whose bottom cell detects $\underline{v}_1 \in \pi_\rho X(2)_{\mathbf{R}}$.

Example 3.3.20. Let us note some additional evidence for Conjecture 3.3.16(a): if X is a $\mathbf{Z}/2$-space, then the cofiber sequence $(\mathbf{Z}/2)_+ \to S^0 \to S^\sigma$ of spaces implies that $(\Omega^\sigma X)^{\mathbf{Z}/2}$ is equivalent to the fiber of the canonical map $X^{\mathbf{Z}/2} \to X$. In particular, since $(\mathrm{SU}(n)_\mathbf{R})^{\mathbf{Z}/2} = \mathrm{SO}(n)$, we see that $(\Omega^\sigma \mathrm{SU}(n)_\mathbf{R})^{\mathbf{Z}/2} \simeq \Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$. Since geometric fixed points preserves colimits, this implies that $\Phi^{\mathbf{Z}/2} X(n)_\mathbf{R}$ is the Thom spectrum of the map

$$(\Omega^\sigma \mathrm{SU}(n)_\mathbf{R})^{\mathbf{Z}/2} \simeq \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to \Omega(\mathrm{SU}/\mathrm{SO}) \simeq \mathrm{BO} \simeq \mathrm{BU}_\mathbf{R}^{\mathbf{Z}/2}.$$

Since $\Phi^{\mathbf{Z}/2} T(n)_\mathbf{R} \simeq y(n)$ as 2-local $\mathbf{E}_1$-algebras, Conjecture 3.3.16(a) would imply that $\Phi^{\mathbf{Z}/2} X(2^n)_\mathbf{R}$ (i.e., the Thom spectrum of the map $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to \mathrm{BO}$) is a direct sum of shifts of $y(n)$ such that the inclusion $y(n) \to \Phi^{\mathbf{Z}/2} X(2^n)_\mathbf{R}$ of the unit summand is an $\mathbf{E}_1$-map. This is indeed true, and was proved in [**Yan92**].

Example 3.3.21. The strongest evidence for Conjecture 3.3.16(d) is the following. It follows from [**Dev23a**, Construction 7.1.1] that there is a map $\Omega S^{n\rho-1} \to \mathrm{BGL}_1(X(n-1)_\mathbf{R})$ whose Thom spectrum is $X(n)_\mathbf{R}$. The same construction used to prove Theorem 3.1.4 then shows that for any $\mathbf{Z}/2$-equivariant $\mathbf{E}_\sigma$-$X(n)_\mathbf{R}$-algebra R, there is an equivariant cofiber sequence

$$\mathrm{THH}_\mathbf{R}(R/X(n-1)_\mathbf{R}) \to \mathrm{THH}_\mathbf{R}(R/X(n)_\mathbf{R}) \to \Sigma^{n\rho}\mathrm{THH}_\mathbf{R}(R/X(n)_\mathbf{R}), \tag{44}$$

where the second map is a $\mathbf{Z}/2$-equivariant analogue of the topological Sen operator. It is not difficult to see that given the first half of Conjecture 3.3.16(a), Conjecture 3.3.16(d) can be easily proved using the construction of Theorem 3.1.4.

For example, we have $X(2)_\mathbf{R} = T(1)_\mathbf{R}$, and the cofiber sequence of (44) is precisely Conjecture 3.3.16(d). For $R = \underline{\mathbf{Z}}$, (44) becomes a cofiber sequence

$$\underline{\mathbf{Z}}[\Omega S^{\rho+1}\langle \rho+1\rangle] \simeq \mathrm{THH}_\mathbf{R}(\underline{\mathbf{Z}}) \to \mathrm{THH}_\mathbf{R}(\underline{\mathbf{Z}}/X(2)_\mathbf{R}) \simeq \underline{\mathbf{Z}}[\Omega S^{2\rho+1}] \simeq \bigoplus_{n\geq 0} \Sigma^{2n\rho}\underline{\mathbf{Z}} \to \bigoplus_{m\geq 1} \Sigma^{2m\rho}\underline{\mathbf{Z}}.$$

A version of this fiber sequence was in fact already studied in [**HHK$^+$20**, Lemma A.3].

Remark 3.3.22. Conjecture 3.3.16(a) and Conjecture 3.3.16(d) together imply that

$$\pi_*\Phi^{\mathbf{Z}/2}\mathrm{THH}_\mathbf{R}(\mathrm{BP}\langle n-1\rangle_\mathbf{R}) \simeq \mathbf{F}_2[t, \sigma^2(v_{n-1}), \sigma(v_{j-1}) | 1 \leq j \leq n]/(\sigma(v_j)^2),$$

where $|t| = 2^n = |\sigma^2(v_{n-1})|$ and $|\sigma(v_{j-1})| = 2^j - 1$. This can also be proved unconditionally using methods similar to that of [**DMPR17**, Theorem 5.23], by writing

$$\Phi^{\mathbf{Z}/2}\mathrm{THH}_\mathbf{R}(\mathrm{BP}\langle n-1\rangle_\mathbf{R}) \simeq \Phi^{\mathbf{Z}/2}\mathrm{BP}\langle n-1\rangle_\mathbf{R} \otimes_{\mathrm{BP}\langle n-1\rangle_\mathbf{R}} \Phi^{\mathbf{Z}/2}\mathrm{BP}\langle n-1\rangle_\mathbf{R},$$

and using that $\pi_*\Phi^{\mathbf{Z}/2}\mathrm{BP}\langle n-1\rangle_\mathbf{R} \cong \mathbf{F}_2[t]$.

Note that if we assume Conjecture 3.3.16(c), then $\mathrm{THH}_\mathbf{R}(\mathrm{BP}\langle n-1\rangle_\mathbf{R}/T(n-1)_\mathbf{R}) \simeq \mathrm{BP}\langle n-1\rangle_\mathbf{R}[\widetilde{K}_n]$; the conjectural equivalence $\widetilde{K}_n^{\mathbf{Z}/2} = K_{n-1}$ then gives an equivalence

$$\Phi^{\mathbf{Z}/2}\mathrm{THH}_\mathbf{R}(\mathrm{BP}\langle n-1\rangle_\mathbf{R}/T(n-1)_\mathbf{R}) \simeq \Phi^{\mathbf{Z}/2}\mathrm{BP}\langle n-1\rangle_\mathbf{R}[K_{n-1}]. \tag{45}$$

Observe that

$$\pi_*\Phi^{\mathbf{Z}/2}\mathrm{BP}\langle n-1\rangle_\mathbf{R}[K_{n-1}] \cong \mathbf{F}_2[t, x, e]/e^2,$$

where $|x| = 2^n$ and $|y| = 2^n - 1$. For instance, when $n = 1$, there is an equivalence $(\Omega S^{\rho+1}\langle\rho+1\rangle)^{\mathbf{Z}/2} = \Omega S^2$, and (45) reduces to the equivalence

$$\Phi^{\mathbf{Z}/2}\mathrm{THH}_{\mathbf{R}}(\underline{\mathbf{Z}}) \simeq \Phi^{\mathbf{Z}/2}\underline{\mathbf{Z}}[\Omega S^2] \simeq (\tau_{\geq 0}\mathbf{F}_2^{tS^1})[\Omega S^2].$$

3.4. Aside: the Segal conjecture. In this section, we make some brief remarks regarding the Segal conjecture; the reader is referred to [**HW20**, Section 4] and [**Mat21**, Section 5] for a discussion of its algebraic interpretation and a review of the literature on this topic.

Definition 3.4.1. An $\mathbf{E}_\infty$-ring R is said to satisfy the Segal conjecture if the cyclotomic Frobenius $\mathrm{THH}(R) \to \mathrm{THH}(R)^{t\mathbf{Z}/p}$ is an equivalence in large degrees.

Example 3.4.2. Let R be a commutative $\mathbf{F}_p$-algebra. If R is Cartier smooth in the sense of [**KM21**, Section 2] and $\Omega^n_{R/\mathbf{F}_p} = 0$ for $n \gg 0$, then R satisfies the Segal conjecture in the sense of Definition 3.4.1 (see [**Mat22**, Corollary 9.5]).

For instance, suppose $R = k$ is a field of characteristic $p > 0$. Then $\mathrm{THH}(k) \simeq \mathrm{HH}(k/\mathbf{F}_p)[\sigma]$ as a module over $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{F}_p[\sigma]$, and $\pi_i\mathrm{HH}(k/\mathbf{F}_p) = 0$ for $i > \log_p[k : k^p] = \dim_k \Omega^1_{k/\mathbf{F}_p}$. This implies that the localization map $\mathrm{THH}(k) \to \mathrm{THH}(k)[\frac{1}{\sigma}] \simeq_\varphi \mathrm{THH}(k)^{t\mathbf{Z}/p}$ is an equivalence in degrees $> \log_p[k : k^p] - 2$.

Example 3.4.3. The proof of [**HW20**, Theorem 4.3.1 and Corollary 4.2.3] can be used to show that the map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)\otimes_{\mathrm{BP}\langle n-1\rangle}\mathbf{F}_p \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle)^{t\mathbf{Z}/p}\otimes_{\mathrm{BP}\langle n-1\rangle} \mathbf{F}_p$ is an equivalence in degrees $> n + \sum_{i=0}^{n-1}|v_i| = \sum_{i=0}^{n-1}(2p^i - 1) = 2\frac{p^n-1}{p-1} - n$. Note that $2\frac{p^n-1}{p-1} - n$ is also precisely the shift appearing in Mahowald-Rezk duality for $\mathrm{BP}\langle n\rangle$ (see [**MR99**, Corollary 9.3]).

Remark 3.4.4. Assume Conjecture 2.1.9, so that we can define the THH of a left $T(n)$-linear ∞-category relative to $T(n)$. Since we do not know if THH relative to $T(n)$ admits the structure of a cyclotomic spectrum (presumably it does not), it does not seem possible to state a direct analogue of Definition 3.4.1 in this context. However, recall that if k is a perfect field of characteristic $p > 0$ and R is an animated k-algebra, the cyclotomic Frobenius $\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{t\mathbf{Z}/p}$ is the Frobenius-linear map given by inverting $\sigma \in \pi_2\mathrm{THH}(k)$: this is a consequence of the observation that the map $\varphi : \mathrm{THH}(k) \to \mathrm{THH}(k)^{t\mathbf{Z}/p}$ is given by composing the localization $\mathrm{THH}(k) \to \mathrm{THH}(k)[\sigma^{-1}]$ with a Frobenius-linear equivalence $\mathrm{THH}(k)[\sigma^{-1}] \simeq_{\mathrm{Frob}} \mathrm{THH}(k)^{t\mathbf{Z}/p}$.

This observation motivates the following terminology: we say that an $\mathbf{E}_1$-$\mathrm{BP}\langle n-1\rangle$-algebra R satisfies the "$T(n)$-Segal conjecture" if the base-change of the localization map $\mathrm{THH}(R/T(n)) \to \mathrm{THH}(R/T(n))[\theta_n^{-1}]$ along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p = \mathrm{BP}\langle n-1\rangle/(p,\cdots,v_{n-1})$ is an equivalence in large degrees. Note that if $n = 1$, this is equivalent to saying that the p-completion of the map $\mathrm{THH}(R/T(1)) \to \mathrm{THH}(R/T(1))[\theta^{-1}]$ is an equivalence in large degrees. One can similarly say that an $\mathbf{E}_1$-$\mathbf{Z}_p$-algebra R satisfies the "$J(p)$-Segal conjecture" if the map $\mathrm{THH}(R/J(p)) \to \mathrm{THH}(R/J(p))[x^{-1}]$ is an equivalence in large degrees.

Proposition 3.4.5. *If we assume Conjecture 2.1.9, the localization map*

$$\gamma : \mathrm{THH}(\mathrm{BP}\langle n-1\rangle[x_1,\cdots,x_d]/T(n)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle[x_1,\cdots,x_d]/T(n))[\theta_n^{-1}]$$

is an equivalence in degrees $> d - 2p^n$ after base-changing along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$. In particular, the flat polynomial algebra $\mathrm{BP}\langle n-1\rangle[x_1,\cdots,x_d]$ satisfies the $T(n)$-Segal conjecture.

Proof. Write $T := \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))$ for notational simplicity. Using (38), we have

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle[t]/T(n))[\theta^{-1}] \simeq T[\theta^{-1}] \oplus \bigoplus_{n\geq 1} T[\theta^{-1}] \otimes (S^1/\mu_n)_+.$$

Since the map $T \to T[\theta^{-1}]$ is an equivalence in degrees $> -2p^n$ after base-changing along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$, the map $T \otimes (S^1/\mu_n)_+ \to T[\theta^{-1}] \otimes (S^1/\mu_n)_+$ is an equivalence in degrees $> -2p^n+1$ after base-changing along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$. Because the map $\gamma : \mathrm{THH}(\mathrm{BP}\langle n-1\rangle[t]/T(n)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle[t]/T(n))[\theta^{-1}]$ preserves the summands, we see that γ is an equivalence in degrees $> -2p^n+1$ after base-changing along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$. Inducting on the number of variables, we find that the map γ is an equivalence in degrees $> d-2p^n$ after base-changing along $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$, as desired. □

Remark 3.4.6. When $d=0$, Proposition 3.4.5 should be compared to [**HW20**, Theorem 4.3.1]. In fact, we expect it is possible to recover their result using Proposition 3.4.5. We also note the following variant. Let $R := \mathrm{BP}\langle n-1\rangle[t_1,\cdots,t_d]$ denote the flat polynomial $\mathbf{E}_2$-$\mathrm{BP}\langle n-1\rangle$-algebra on classes t_i in even degrees (i.e., the base-change of the $\mathbf{E}_\infty$-MU-algebra $\mathrm{MU}[t_1,\cdots,t_d]$ along the $\mathbf{E}_3$-map $\mathrm{MU} \to \mathrm{BP}\langle n-1\rangle$). The argument of Proposition 3.4.5 then shows that after base-change along the composite $R \to \mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$, the localization map $\gamma : \mathrm{THH}(R/T(n)) \to \mathrm{THH}(R/T(n))[\theta_n^{-1}]$ is an equivalence in degrees $> -2p^n + \sum_{j=1}^d (|t_j|+1)$. When $n=0$, this is [**HW20**, Corollary 4.2.3].

Proposition 3.4.7. *Let R be a p-torsionfree discrete commutative ring such that R/p is regular Noetherian. Suppose $L\Omega^n_R = 0$ for $n > d$. Then Conjecture 3.1.14 implies that R satisfies the $J(p)$-Segal conjecture: in fact, the map* $\mathrm{THH}(R/J(p)) \to \mathrm{THH}(R/J(p))[x^{-1}]$ *is an equivalence in degrees $> d-2$.*

Proof. Recall that Conjecture 3.1.14 asserts that $\mathrm{THH}(R/J(p))$ has a filtration such that $\mathrm{gr}^i_{\mathrm{mot}}\mathrm{THH}(R/J(p)) \simeq (\mathrm{F}^{\mathrm{conj}}_i \widehat{\Omega}^{\not{D}}_R)[2i]$, and such that the map $\gamma : \mathrm{THH}(R/J(p)) \to \mathrm{THH}(R/J(p))[x^{-1}]$ induces the map $\mathrm{F}^{\mathrm{conj}}_i \widehat{\Omega}^{\not{D}}_R \to \widehat{\Omega}^{\not{D}}_R$ on $\mathrm{gr}^i_{\mathrm{mot}}[-2i]$. By [**BL22a**, Remark 4.7.4], $\widehat{\Omega}^{\not{D}}_R/\mathrm{F}^{\mathrm{conj}}_i \widehat{\Omega}^{\not{D}}_R$ is concentrated in cohomological degrees $\geq i+1$, so that the cofiber of $\mathrm{gr}^i(\gamma)$ is concentrated in degrees $\leq 2i-(i+1) = i-1$. Moreover, the hypothesis that $L\Omega^n_R = 0$ for $n > d$ implies that γ induces an equivalence on $\mathrm{gr}^i_{\mathrm{mot}}$ for $i \geq d$. Combining these observations gives the desired statement (see also the proof of [**Mat22**, Corollary 9.5]). □

3.5. Aside: Cartier isomorphism.

In this section, we study a topological analogue of the Cartier isomorphism for the two-term complexes from Remark 3.3.10; we will study basic algebraic properties of these complexes in future work. To avoid dealing with completion issues, we use the following (see Warning 3.3.7 for a remark about the notation $\mathrm{HH}(R[t]/R)$):

Definition 3.5.1. Let R be an $\mathbf{E}_2$-ring. The polynomial $\mathbf{E}_1$-R-algebra $R[t] = R[\mathbf{N}]$ acquires a natural $\mathbf{Z}$-grading, and we will write $\mathrm{HH}(R[t]/R)_{\leq m}$ to denote the graded left R-module given by truncating $\mathrm{HH}(R[t]/R) := R \otimes \mathrm{THH}(S[t])$ in weights $\geq m+1$. Explicitly, $\mathrm{HH}(R[t]/R)_{\leq m}$ is equivalent to $R \oplus \left(\bigoplus_{1\leq n\leq m} R \otimes (S^1/\mu_n)_+\right)$.

Lemma 3.5.2. *If $X \in \mathrm{Sp}^{BS^1}$, the following composite is an equivalence:*

$$X^{t\mathbf{Z}/p} \otimes (S^1/\mu_n)_+ \xrightarrow{\sim} (X \otimes (S^1/\mu_n)_+)^{t\mathbf{Z}/p} \xrightarrow{\psi} (X \otimes (S^1/\mu_{np})_+)^{t\mathbf{Z}/p}.$$

Moreover, if $p \nmid m$, then $(X \otimes (S^1/\mu_m)_+)^{t\mathbf{Z}/p} = 0$.

Proof. If $\varphi : (S^1/\mu_n)_+ \to ((S^1/\mu_{np})_+)^{h\mathbf{Z}/p}$ denotes the unstable Frobenius (sending $x \mapsto x^{1/p}$), the cofiber of the composite

$$\psi : (S^1/\mu_n)_+ \to ((S^1/\mu_{np})_+)^{h\mathbf{Z}/p} \to (S^1/\mu_{np})_+$$

has induced $\mathbf{Z}/p$-action, where $(S^1/\mu_n)_+$ and $((S^1/\mu_{np})_+)^{h\mathbf{Z}/p}$ are equipped with the trivial $\mathbf{Z}/p$-action. Therefore, the canonical map $X^{t\mathbf{Z}/p} \otimes (S^1/\mu_n)_+ \to (X \otimes (S^1/\mu_n)_+)^{t\mathbf{Z}/p}$ is an equivalence (since $(S^1/\mu_n)_+$ is a finite spectrum with trivial $\mathbf{Z}/p$-action). This gives the first claim. Finally, if $p \nmid m$, then the $\mathbf{Z}/p$-action on S^1/μ_m is free, so that $(X \otimes (S^1/\mu_m)_+)^{t\mathbf{Z}/p} = 0$, as desired. □

Proposition 3.5.3 (Cartier isomorphism)**.** *Let R be an $\mathbf{E}_2$-ring. Then:*

(a) There is an S^1-equivariant map $\mathfrak{C} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) \to \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$, where $\mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ is endowed with the residual S^1/μ_p-action and $\mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p})$ is endowed with the diagonal S^1-action arising from the S^1-action on HH and the residual S^1/μ_p-action on $R^{t\mathbf{Z}/p}$. Moreover, the map $\mathfrak{C}$ sends $t \mapsto t^p$.

(b) For each $m \geq 1$, the map $\mathfrak{C}$ induces an equivalence $\mathfrak{C}_{\leq m} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p})_{\leq m} \xrightarrow{\sim} (\mathrm{HH}(R[t]/R)_{\leq mp})^{t\mathbf{Z}/p}$.

Proof. Recall that there is an equivalence $\mathrm{HH}(R[t]/R) \simeq R \otimes \mathrm{THH}(S[t])$. Since the $\mathbf{Z}/p$-Tate construction is lax symmetric monoidal, we obtain the map $\mathfrak{C}$ via the composite

$$\begin{aligned}\mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) &\simeq R^{t\mathbf{Z}/p} \otimes \mathrm{THH}(S[t]) \\ &\xrightarrow{\mathrm{id}\otimes\varphi} R^{t\mathbf{Z}/p} \otimes \mathrm{THH}(S[t])^{t\mathbf{Z}/p} \\ &\to (R \otimes \mathrm{THH}(S[t]))^{t\mathbf{Z}/p} \simeq \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}.\end{aligned}$$

For each $m \geq 1$, there is an equivalence

$$(\mathrm{HH}(R[t]/R)_{\leq m})^{t\mathbf{Z}/p} \simeq R^{t\mathbf{Z}/p} \oplus \left(\bigoplus_{1\leq n\leq m} R \otimes (S^1/\mu_n)_+\right)^{t\mathbf{Z}/p}.$$

Since the maps $\varphi : (S^1/\mu_n)_+ \to ((S^1/\mu_{np})_+)^{h\mathbf{Z}/p}$ define the Frobenius on $\mathrm{THH}(S[t]) \simeq S \oplus \bigoplus_{n\geq 1}(S^1/\mu_n)_+$, we see from Lemma 3.5.2 that for each $m \geq 1$, the map $\mathfrak{C}_{\leq m}$ defines an equivalence

$$\bigoplus_{1\leq j\leq m} R^{t\mathbf{Z}/p} \otimes (S^1/\mu_j)_+ \xrightarrow{\sim} \left(\bigoplus_{1\leq n\leq mp} R \otimes (S^1/\mu_n)_+\right)^{t\mathbf{Z}/p}.$$

The left-hand side is $\mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p})_{\leq m}$, and the right-hand side is $(\mathrm{HH}(R[t]/R)_{\leq mp})^{t\mathbf{Z}/p}$. □

Remark 3.5.4. When R is an $\mathbf{E}_\infty$-ring, the map $\mathfrak{C} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) \to \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ of Proposition 3.5.3 can also be constructed using (a simple case of) [**Law21**, Theorem 1.3]. The cited result says the following. Suppose k is an $\mathbf{E}_\infty$-ring, so that the Tate-valued Frobenius $k \to k^{t\mathbf{Z}/p}$ admits an extension $\mathrm{THH}(k) \to k^{t\mathbf{Z}/p}$ to an S^1-equivariant map of $\mathbf{E}_\infty$-rings. If A is an $\mathbf{E}_1$-k-algebra, and M is an A-bimodule in Mod_k, then there is a relative Tate diagonal

$$k^{t\mathbf{Z}/p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A, M) \to \mathrm{THH}^k(A, M^{\otimes_A p})^{t\mathbf{Z}/p},$$

where THH^k denotes THH relative to k. To construct $\mathfrak{C}$, take $k = R$ and $A = M = k[t]$. Then

$$k^{t\mathbf{Z}/p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A, M) \simeq k^{t\mathbf{Z}/p} \otimes \mathrm{THH}(S^0[t]) \simeq \mathrm{HH}(k^{t\mathbf{Z}/p}[t]/k^{t\mathbf{Z}/p}),$$

since $\mathrm{THH}(A, M) \simeq \mathrm{THH}(A) \simeq \mathrm{THH}(S^0[t]) \otimes \mathrm{THH}(k)$. Similarly, $\mathrm{THH}^k(A, M^{\otimes_A p}) \simeq \mathrm{HH}(A/k)$, and it is straightforward to check that Lawson's relative Tate diagonal agrees with the map $\mathfrak{C}$.

One advantage of the construction of $\mathfrak{C}$ in Proposition 3.5.3 is that it is manifestly S^1-equivariant, and does not rely on R being an $\mathbf{E}_\infty$-ring. More generally, one finds that if $\mathcal{C}$ is a stable ∞-category and R is any $\mathbf{E}_2$-ring, the cyclotomic Frobenius on $\mathrm{THH}(\mathcal{C})$ defines an S^1-equivariant map $\mathfrak{C} : \mathrm{HH}(\mathcal{C} \otimes R^{t\mathbf{Z}/p}/R^{t\mathbf{Z}/p}) \to \mathrm{HH}(\mathcal{C} \otimes R/R)^{t\mathbf{Z}/p}$ which generalizes the map of Proposition 3.5.3. This map is furthermore an equivalence if $\mathcal{C}$ is smooth and proper.

Remark 3.5.5. In Proposition 3.5.3, the map $\mathfrak{C} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) \to \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ is itself almost an equivalence: the main issue is that the canonical map

$$\mathrm{colim}_m \left(\mathrm{HH}(R[t]/R)_{\leq mp}\right)^{t\mathbf{Z}/p} \to \left(\mathrm{colim}_m \mathrm{HH}(R[t]/R)_{\leq mp}\right)^{t\mathbf{Z}/p} \simeq \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$$

may not be an equivalence. However, Proposition 3.5.3 implies that the *graded* map $\mathfrak{C}^{\mathrm{gr}} : \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p} \to \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}$ is itself an equivalence.

Remark 3.5.6. If R is a complex-oriented $\mathbf{E}_2$-ring, let $[p](\hbar) \in \pi_{-2}R^{hS^1}$ denote the p-series of the formal group law over R. If $M \in \mathrm{LMod}_R^{BS^1}$, then it is not difficult to show that there is an equivalence $M^{tS^1}/\frac{[p](\hbar)}{\hbar} \xrightarrow{\sim} M^{t\mathbf{Z}/p}$. (Although certainly well-known, the only reference in the literature for a statement in this generality seems to be [**HRW22**, Lemma 6.2.2].) In particular, $\mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p} \simeq \mathrm{HP}(R[t]/R)/\frac{[p](\hbar)}{\hbar}$, so that Proposition 3.5.3 and Remark 3.5.5 imply that there is an S^1-equivariant *graded* equivalence

$$\mathfrak{C} : \mathrm{HH}(R^{t\mathbf{Z}/p}[t]/R^{t\mathbf{Z}/p}) \to \mathrm{HP}(R[t]/R)/\tfrac{[p](\hbar)}{\hbar} \simeq \mathrm{HH}(R[t]/R)^{t\mathbf{Z}/p}.$$

In future work, we will show that if R is further assumed to be an $\mathbf{E}_\infty$-ring and $\mathcal{C}$ is a R-linear ∞-category, then the $(R^{t\mathbf{Z}/p})^{hS^1} \simeq (R^{tS^1})^\wedge_p$-module $(\mathrm{THH}(\mathcal{C}) \otimes_{\mathrm{THH}(R)} R^{t\mathbf{Z}/p})^{hS^1}$ behaves as a noncommutative analogue of $L\eta_{[p](\hbar)/\hbar}$ applied to $\mathrm{HP}(\mathcal{C}/R)$. Here, $\hbar \in \pi_{-2}R^{hS^1}$ is the complex-orientation of R, and $[p](\hbar)/\hbar \in \pi_0 R^{tS^1}$ is the quotient of the p-series of the associated formal group law.

Remark 3.5.7. There is no reason to restrict to polynomial rings in a single variable in the equivalence of Proposition 3.5.3(b); we leave the details of the resulting statement to the reader.

Example 3.5.8. Let $R = \mathbf{Z}$. Then $\mathbf{Z}^{t\mathbf{Z}/p}$ is an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra, and has homotopy groups given by $\mathbf{F}_p(\!(\hbar)\!)$ with $|\hbar| = 2$. Therefore, $\mathbf{Z}^{t\mathbf{Z}/p} \simeq \mathbf{F}_p^{tS^1}$ as $\mathbf{E}_2$-rings[17], and

[17] In fact, they are equivalent as $\mathbf{E}_\infty$-rings. Although seemingly innocuous, even the weaker claim that $\mathbf{Z}^{t\mathbf{Z}/p}$ admits the structure of an $\mathbf{E}_\infty$-$\mathbf{F}_p$-algebra is surprisingly difficult to prove from first principles (see [**NS18**, Remark IV.4.17]). One might try to argue as follows: since

Proposition 3.5.3 (combined with Remark 3.5.5) specializes to the statement that there is a Frobenius-linear equivalence

$$\mathfrak{C} : \mathrm{HH}(\mathbf{F}_p[t]/\mathbf{F}_p)(\!(\hbar)\!)_{\leq m} \simeq \mathrm{HH}(\mathbf{F}_p^{tS^1}[t]/\mathbf{F}_p^{tS^1})_{\leq m} \xrightarrow{\sim} (\mathrm{HH}(\mathbf{Z}[t]/\mathbf{Z})_{\leq mp})^{t\mathbf{Z}/p}.$$

Note that $\mathrm{HH}(\mathbf{Z}[t]/\mathbf{Z})^{t\mathbf{Z}/p} \simeq \mathrm{HP}(\mathbf{Z}[t]/\mathbf{Z})/p \simeq \mathrm{TP}(\mathbf{F}_p[t])/p$. Since the HKR theorem implies that $\mathrm{HH}(\mathbf{F}_p[t]/\mathbf{F}_p)^{tS^1}$ is a 2-periodification of the Hodge cohomology of $\mathbf{A}^1_{\mathbf{F}_p}$, and $\mathrm{HP}(\mathbf{Z}[t]/\mathbf{Z})/p$ is a 2-periodification of the de Rham cohomology of $\mathbf{A}^1_{\mathbf{Z}_p}$ modulo p (which is the de Rham cohomology of $\mathbf{A}^1_{\mathbf{F}_p}$), one can view $\mathfrak{C}$ as a topological analogue of the Cartier isomorphism for the affine line. It reduces to the usual Cartier isomorphism on graded pieces. In this case, the statement of Proposition 3.5.3 should also be compared to [**Kal08, Kal17, Mat20**]. Taking homotopy fixed points for the S^1-equivariance of $\mathfrak{C}$ from Proposition 3.5.3(a), we obtain a Frobenius-linear equivalence

$$\mathfrak{C}^{hS^1} : (\mathrm{HH}(\mathbf{Z}^{t\mathbf{Z}/p}[t]/\mathbf{Z}^{t\mathbf{Z}/p})^{hS^1})_{\leq m} \xrightarrow{\sim} ((\mathrm{HH}(\mathbf{Z}[t]/\mathbf{Z})_{\leq mp})^{tS^1})^{\wedge}_p. \tag{46}$$

More succinctly, there is a *graded* equivalence

$$(\mathfrak{C}^{\mathrm{gr}})^{hS^1} : \mathrm{HH}(\mathbf{Z}^{t\mathbf{Z}/p}[t]/\mathbf{Z}^{t\mathbf{Z}/p})^{hS^1} \xrightarrow{\sim} \mathrm{HP}^{\mathrm{gr}}(\mathbf{Z}[t]/\mathbf{Z})^{\wedge}_p.$$

Using the HKR filtration on $\mathrm{HH}(\mathbf{Z}[t]/\mathbf{Z})$, one can prove that $\mathrm{HH}(\mathbf{Z}^{t\mathbf{Z}/p}[t]/\mathbf{Z}^{t\mathbf{Z}/p})^{hS^1}$ admits a filtration whose graded pieces are given by even shifts of $L\eta_p\Gamma_{\mathrm{dR}}(\mathbf{Z}_p[t]/\mathbf{Z}_p) \simeq L\eta_p\Gamma_{\mathrm{crys}}(\mathbf{F}_p[t]/\mathbf{Z}_p)$. We will explain this in greater detail in a future article. Since $\mathrm{HP}(\mathbf{Z}_p[t]/\mathbf{Z}_p) \simeq \mathrm{TP}(\mathbf{F}_p)$, (46) can be regarded as a 2-periodification of the "Cartier isomorphism" $L\eta_p\Gamma_{\mathrm{crys}}(\mathbf{F}_p[t]/\mathbf{Z}_p) \simeq \Gamma_{\mathrm{crys}}(\mathbf{F}_p[t]/\mathbf{Z}_p)$ for the crystalline cohomology of $\mathbf{F}_p[t]$ (see [**BO78**, Theorem 8.20] for the general case).

Example 3.5.9. Let $R = \mathrm{ku}$. Then $\pi_*\mathrm{ku}^{t\mathbf{Z}/p} \cong \mathbf{Z}[\zeta_p](\!(\hbar)\!)$, and it is expected that this lifts to an equivalence $\mathrm{ku}^{t\mathbf{Z}/p} \simeq \mathbf{Z}[\zeta_p]^{tS^1}$ of $\mathbf{E}_\infty$-rings (see also Example 3.5.11 below). Nevertheless, there *is* an equivalence $\mathrm{ku}^{t\mathbf{Z}/p} \simeq \mathbf{Z}[\zeta_p]^{tS^1}$ of $\mathbf{E}_2$-rings (one can show this by using [**Mao20**, Theorem 1.19]; thanks to Arpon Raksit for pointing this out). Therefore, Proposition 3.5.3 and Remark 3.5.5 give an equivalence

$$\mathfrak{C} : \mathrm{HH}(\mathbf{Z}[\zeta_p][t]/\mathbf{Z}[\zeta_p])(\!(\hbar)\!)_{\leq m} \simeq \mathrm{HH}(\mathbf{Z}[\zeta_p]^{tS^1}[t]/\mathbf{Z}[\zeta_p]^{tS^1})_{\leq m} \xrightarrow{\sim} (\mathrm{HH}(\mathrm{ku}[t]/\mathrm{ku})_{\leq mp})^{t\mathbf{Z}/p}.$$

Note that $\mathrm{HH}(\mathrm{ku}[t]/\mathrm{ku})^{t\mathbf{Z}/p} \simeq \mathrm{HP}(\mathrm{ku}[t]/\mathrm{ku})/[p]_q$. Here, we identify $\frac{[p]_{\mathbf{G}_m}(\hbar)}{\hbar} \in \pi_0\mathrm{ku}^{tS^1} \cong \mathbf{Z}[\![q-1]\!]$ with the q-integer $[p]_q$.

By the HKR theorem, one can view $\mathrm{HH}(\mathbf{Z}[\zeta_p][t]/\mathbf{Z}[\zeta_p])(\!(\hbar)\!)$ as a 2-periodification of the Hodge cohomology of $\mathbf{A}^1_{\mathbf{Z}}$ base-changed along the map $\mathbf{Z} \to \mathbf{Z}[\zeta_p]$. Similarly, the aforementioned work of Raksit (see [**Rak20**] and Remark 3.3.1, as well as Lemma 3.3.4) implies that $\mathrm{HP}(\mathrm{ku}[t]/\mathrm{ku})$ can be viewed as a 2-periodification of the q-de Rham complex of $\mathbf{Z}[t]$. Since killing $[p]_q \in \mathbf{Z}[\![q-1]\!]$ amounts to specializing q to a primitive pth root of unity, one can view Proposition 3.5.3 as a topological

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$p = 0 \in \pi_0\mathbf{Z}^{t\mathbf{Z}/p}$, there is a map from the free $\mathbf{E}_\infty$-algebra with $p = 0$ to $\mathbf{Z}^{t\mathbf{Z}/p}$. However, oddly enough, the free $\mathbf{E}_\infty$-algebra with $p = 0$ is *not* an $\mathbf{E}_\infty$-$\mathbf{F}_p$-algebra; this dashes any hopes of proving that $\mathbf{Z}^{t\mathbf{Z}/p}$ is an $\mathbf{E}_\infty$-$\mathbf{F}_p$-algebra through this argument.

More generally, the free $\mathbf{E}_n$-algebra R with $p = 0$ is not an $\mathbf{E}_n$-$\mathbf{F}_p$-algebra unless $n = 2$: indeed, applying the $\mathbf{E}_n$-cotangent complex to the composite $\mathbf{F}_p \to R \to \mathbf{F}_p$ of $\mathbf{E}_n$-algebra maps shows that $L^{\mathbf{E}_n}_{\mathbf{F}_p}$ is a retract of $\mathbf{F}_p \otimes_R L^{\mathbf{E}_n}_R \simeq \mathbf{F}_p$. This forces $L^{\mathbf{E}_n}_{\mathbf{F}_p} = \mathbf{F}_p$, i.e., $\mathbf{F}_p$ would be built from the sphere by attaching a single $\mathbf{E}_n$-cell in degree 1 – but this is impossible, since $\mathbf{F}_p \otimes \mathbf{F}_p \not\simeq \mathbf{F}_p[\Omega^n S^{n+1}]$ unless $n = 2$.

analogue of the Cartier isomorphism for the q-de Rham complex of the affine line (see, e.g., [**Sch17**, Proposition 3.4]).

Taking homotopy fixed points for the S^1-equivariance of $\mathfrak{C}$ from Proposition 3.5.3(a), we obtain an equivalence

$$\mathfrak{C}^{hS^1} : (\mathrm{HH}(\mathrm{ku}^{t\mathbf{Z}/p}[t]/\mathrm{ku}^{t\mathbf{Z}/p})^{hS^1})_{\leq m} \xrightarrow{\sim} ((\mathrm{HH}(\mathrm{ku}[t]/\mathrm{ku})_{\leq mp})^{tS^1})^{\wedge}_p. \tag{47}$$

More succinctly, there is a *graded* equivalence

$$(\mathfrak{C}^{\mathrm{gr}})^{hS^1} : \mathrm{HH}(\mathrm{ku}^{t\mathbf{Z}/p}[t]/\mathrm{ku}^{t\mathbf{Z}/p})^{hS^1} \xrightarrow{\sim} \mathrm{HP}^{\mathrm{gr}}(\mathrm{ku}[t]/\mathrm{ku})^{\wedge}_p.$$

In future work, we show that Raksit's filtration on $\mathrm{HP}(\mathrm{ku}[t]/\mathrm{ku})$ can be refined to construct a filtration on $\mathrm{HH}(\mathrm{ku}^{t\mathbf{Z}/p}[t]/\mathrm{ku}^{t\mathbf{Z}/p})^{hS^1}$ whose graded pieces are given by even shifts of $L\eta_{[p]_q} q\Omega_{\mathbf{Z}_p[t]}$. Then, (47) can be regarded as a 2-periodification of the "Cartier isomorphism" $\phi^*_{\mathbf{Z}_p[\![q-1]\!]} q\Omega_{\mathbf{Z}_p[t]} \simeq L\eta_{[p]_q} q\Omega_{\mathbf{Z}_p[t]}$ for the q-de Rham cohomology of $\mathbf{Z}_p[t]$. (See [**BS19**, Theorem 1.16(4)] applied to the q-crystalline prism $(\mathbf{Z}_p[\![q-1]\!], [p]_q)$.)

Remark 3.5.10. Example 3.5.9 admits a mild generalization. Namely, if $\mathrm{ku}^{\mathbf{Z}/p^{n-1}}$ denotes the strict fixed points (so $\mathrm{ku}^{\mathbf{Z}/p^{n-1}} = \tau_{\geq 0}(\mathrm{ku}^{h\mathbf{Z}/p^{n-1}})$), then one can calculate $\pi_*(\mathrm{ku}^{\mathbf{Z}/p^{n-1}})^{t\mathbf{Z}/p} \cong \pi_*\mathbf{Z}_p[\zeta_{p^n}]^{tS^1}$. One can show that this can be extended to an equivalence $(\mathrm{ku}^{\mathbf{Z}/p^{n-1}})^{t\mathbf{Z}/p} \simeq \mathbf{Z}_p[\zeta_{p^n}]^{tS^1}$ of $\mathbf{E}_2$-rings. Proposition 3.5.3 and Remark 3.5.5 give a graded equivalence

$$\mathfrak{C} : \mathrm{HH}(\mathbf{Z}[\zeta_{p^n}][t]/\mathbf{Z}[\zeta_{p^n}])(\!(\hbar)\!) \xrightarrow{\sim} \mathrm{HH}(\mathrm{ku}^{\mathbf{Z}/p^{n-1}}[t]/\mathrm{ku}^{\mathbf{Z}/p^{n-1}})^{t\mathbf{Z}/p}.$$

Here, the action of $\mathbf{Z}/p$ on $\mathrm{HH}(\mathrm{ku}^{\mathbf{Z}/p^{n-1}}[t]/\mathrm{ku}^{\mathbf{Z}/p^{n-1}}) = \mathrm{THH}(S[t]) \otimes \mathrm{ku}^{\mathbf{Z}/p^{n-1}}$ is via the *diagonal* action on THH and $\mathrm{ku}^{\mathbf{Z}/p^{n-1}}$. In this case, one can therefore view Proposition 3.5.3 as a topological analogue of the Cartier isomorphism for Hodge-Tate cohomology relative to the prism $(\mathbf{Z}_p[\![q^{1/p^{n-1}}-1]\!], [p]_q)$ of the affine line.

Example 3.5.11. More generally, let $R = \mathrm{BP}\langle n\rangle$. As recalled in Remark 2.2.16, [**AMS98**, Proposition 2.3] proved that there is an isomorphism $\pi_*\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p} \cong \pi_*\mathrm{BP}\langle n-1\rangle^{tS^1}$, and this was conjectured to lift to an equivalence of spectra in [**DJK**$^+$**86**, Conjecture 1.2]. If we assume that there is in fact an equivalence $\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p} \simeq \mathrm{BP}\langle n-1\rangle^{tS^1}$ of $\mathbf{E}_2$-rings, Proposition 3.5.3 and Remark 3.5.5 give an equivalence

$$\mathfrak{C} : \mathrm{HH}(\mathrm{BP}\langle n-1\rangle[t]/\mathrm{BP}\langle n-1\rangle)(\!(\hbar)\!)_{\leq m} \xrightarrow{\sim} (\mathrm{HH}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle)_{\leq mp})^{t\mathbf{Z}/p}.$$

Therefore, Proposition 3.5.3 in this case can be viewed as an analogue of the Cartier isomorphism for the affine line in the setting of "v_n-adic Hodge theory". Taking homotopy fixed points for the S^1-equivariance of $\mathfrak{C}$ from Proposition 3.5.3(a), we obtain an equivalence

$$\mathfrak{C}^{hS^1} : (\mathrm{HH}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p}[t]/\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p})^{hS^1})_{\leq m} \xrightarrow{\sim} ((\mathrm{HH}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle)_{\leq mp})^{tS^1})^{\wedge}_p. \tag{48}$$

More succinctly, there is a *graded* equivalence

$$(\mathfrak{C}^{\mathrm{gr}})^{hS^1} : \mathrm{HH}(\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p}[t]/\mathrm{BP}\langle n\rangle^{t\mathbf{Z}/p})^{hS^1} \xrightarrow{\sim} \mathrm{HP}^{\mathrm{gr}}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle)^{\wedge}_p.$$

Note that Conjecture 2.2.18 in particular implies that if $T(n)$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring, then $\mathrm{HP}(\mathrm{BP}\langle n\rangle[t]/\mathrm{BP}\langle n\rangle)$ is closely related to $\mathrm{TP}(\mathrm{BP}\langle n-1\rangle[t]/T(n))$

by Proposition 3.3.8. In this form, (48) holds when BP$\langle n\rangle$ is replaced by any complex-oriented $\mathbf{E}_2$-ring R. As in the preceding examples, we believe that when R is connective, this can be regarded as a 2-periodification of a "Cartier isomorphism" for the two-term complex (41). See [**DM23**] for further discussion.

4. Relationship to the moduli stack of formal groups

4.1. Incarnation of the topological Sen operator over $\mathcal{M}_{\mathrm{FG}}$. In Section 3, we showed that the descent spectral sequence of Remark 2.2.12 admits a generalization given by the topological Sen operator (Theorem 3.1.4). This has an incarnation over $\mathcal{M}_{\mathrm{FG}}$, as we now explain. The analogues of Theorem 2.2.4, Theorem 3.1.4, etc., that we discuss in this section are useful for making topological predictions since the calculations involved are easier.

Recollection 4.1.1 (Even filtration)**.** Let $\mathrm{F}^\star_{\mathrm{ev}} : \mathrm{CAlg} \to \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{fil}})$ be the *even filtration* of [**HRW22**]: if $\mathrm{CAlg}^{\mathrm{ev}}$ denotes the full subcategory of CAlg spanned by the $\mathbf{E}_\infty$-rings with even homotopy, then $\mathrm{F}^\star_{\mathrm{ev}}$ is the right Kan extension of the functor $\tau_{\geq 2\star} : \mathrm{CAlg}^{\mathrm{ev}} \to \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{fil}})$ along the inclusion $\mathrm{CAlg}^{\mathrm{ev}} \hookrightarrow \mathrm{CAlg}$. Note that since $\tau_{\geq 2\star}$ is lax symmetric monoidal and $\mathrm{F}^\star_{\mathrm{ev}}$ is defined by a right Kan extension, it is also a lax symmetric monoidal functor. We will need the following result from [**HRW22**]: if R is an $\mathbf{E}_\infty$-ring such that $\mathrm{MU} \otimes R \in \mathrm{CAlg}^{\mathrm{ev}}$, then $\mathrm{F}^\star_{\mathrm{ev}} R$ is p-completely equivalent to the underlying filtered $\mathbf{E}_\infty$-ring of its Adams-Novikov tower $\nu(R) \in \mathrm{Syn}^{\mathrm{ev}}_{\mathrm{MU}}(\mathrm{Sp}) = \mathrm{Mod}_{\mathrm{Tot}(\tau_{\geq 2\star}\mathrm{MU}^{\otimes \bullet+1})}(\mathrm{Sp}^{\mathrm{fil}})$. (Also see [**Pst18, GIKR18**].) In this case, the associated graded Hopf algebroid $(\mathrm{MU}_*(R), \mathrm{MU}_*(\mathrm{MU}\otimes R))$ defines a stack over $B\mathbf{G}_m$. If R is complex-oriented, then this stack is isomorphic to $\mathrm{Spec}(\pi_* R)/\mathbf{G}_m$, where the $\mathbf{G}_m$-action encodes the grading on $\pi_* R$.

Observation 4.1.2. In order to define the stack $\widetilde{\mathcal{M}}_R$ associated to the graded Hopf algebroid $(\mathrm{MU}_*(R), \mathrm{MU}_*(\mathrm{MU}\otimes R))$, one does not need R to be an $\mathbf{E}_\infty$-ring: it only needs to admit the structure of a homotopy commutative ring such that $\mathrm{MU}_*(R)$ is concentrated in even degrees. This perspective is explained in Hopkins' lecture in [**DFHH14**, Chapter 9]. In particular, one can define the stack associated to $X(n)$: this is the moduli stack of graded formal groups equipped with a coordinate of order $\leq n$, and strict isomorphisms between them. (See, e.g., [**Mil19**, Section 2].)

Variant 4.1.3. We will find it convenient to work with the p-typical variant of the graded Hopf algebroid $(\mathrm{MU}_*(R), \mathrm{MU}_*(\mathrm{MU} \otimes R))$. Namely, if R is a p-local homotopy commutative ring such that $\mathrm{BP}_*(R)$ is concentrated in even degrees, then we will write $\mathcal{M}_R$ to denote the graded stack associated to the graded Hopf algebroid $(\mathrm{BP}_*(R), \mathrm{BP}_*(\mathrm{BP} \otimes R))$. For example, $\mathcal{M}_{T(n)}$ is the moduli stack of p-typical graded formal groups equipped with a coordinate up to order $\leq p^{n+1} - 1$; by p-typicality, this is further isomorphic to the moduli stack of p-typical graded formal groups equipped with a coordinate up to order $\leq p^n$. In particular, $\mathcal{M}_{S^0}$ is isomorphic to the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of p-typical graded formal groups. Similarly, if R is a p-local complex-oriented homotopy commutative ring, then $\mathcal{M}_R$ is isomorphic to $\mathrm{Spec}(\pi_* R)/\mathbf{G}_m$.

Example 4.1.4. The unit map $S^0 \to \mathrm{MU}$ induces the map $\widetilde{\mathcal{M}}_{\mathrm{MU}} \cong \mathrm{Spec}(\mathrm{MU}_*)/\mathbf{G}_m \to \widetilde{\mathcal{M}}_{S^0}$ which describes the flat cover of the moduli stack of graded formal groups given by the graded Lazard ring. This map exhibits $\widetilde{\mathcal{M}}_{S^0}$ as the quotient of $\mathrm{Spec}(\mathrm{MU}_*)/\mathbf{G}_m$ by the group scheme $\mathrm{Spec}(\pi_*(\mathrm{MU} \otimes \mathrm{MU}))/\mathbf{G}_m$. Note that $\mathrm{MU} \otimes \mathrm{MU} \simeq \mathrm{MU}[\mathrm{BU}]$; since $\pi_* \mathbf{Z}[\mathrm{BU}]$ is the coordinate ring of the big Witt ring scheme, we see that $\mathrm{Spec}(\pi_*(\mathrm{MU} \otimes \mathrm{MU}))/\mathbf{G}_m$ is a lift of the big Witt ring scheme to $\mathrm{Spec}(\mathrm{MU}_*)/\mathbf{G}_m$. Similarly, $\mathcal{M}_{S^0} = \mathcal{M}_{\mathrm{FG}}$ is the quotient of $\mathcal{M}_{\mathrm{BP}} = \mathrm{Spec}(\mathrm{BP}_*)/\mathbf{G}_m$ by a lift of the p-typical Witt ring scheme W to $\mathrm{Spec}(\mathrm{BP}_*)/\mathbf{G}_m$.

Remark 4.1.5. If $A \to B$ is a map of p-local $\mathbf{E}_\infty$-rings such that $\mathrm{MU}_*(A)$ and $\mathrm{MU}_*(B)$ are even, then there is an induced map $\mathcal{M}_B \to \mathcal{M}_A$ of graded stacks. Recall that $\mathrm{THH}(B/A)$ is the geometric realization of the simplicial A-algebra $B^{\otimes_A \bullet+1}$. Applying $\mathrm{F}^\star_{\mathrm{ev}}$ levelwise to $B^{\otimes_A \bullet+1} \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{CAlg}_A)$ produces an Adams-Novikov analogue of the Bökstedt spectral sequence:

$$\pi_* \mathrm{HH}(\mathcal{M}_B/\mathcal{M}_A) \Rightarrow \pi_* \mathrm{gr}^\bullet_{\mathrm{ev}} \mathrm{THH}(B/A).$$

In particular, note that $\mathrm{HH}(\mathcal{M}_B/\mathcal{M}_{\mathrm{FG}})$ is an approximation to $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(B)$. For this spectral sequence to exist, it is not necessary that A and B be $\mathbf{E}_\infty$-rings: for example, it suffices that $A \to B$ be a map of p-local $\mathbf{E}_2$-rings such that $\mathrm{MU}_*(A)$ and $\mathrm{MU}_*(B)$ are even, and such that $\mathrm{THH}(B/A)$ is bounded below and has even MU-homology. Then, $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(B/A)$ must be interpreted as the associated graded of the Adams-Novikov filtration on $\mathrm{THH}(B/A)$; see [**HRW22**, Corollary 1.1.6].

Example 4.1.6. Let $\mathcal{M}^s_{\mathrm{FG}}$ denote the total space of the canonical line bundle over $\mathcal{M}_{\mathrm{FG}}$ (determined by the map $\mathcal{M}_{\mathrm{FG}} \to B\mathbf{G}_m$). If R is a p-quasisyntomic ring, then [**BMS19**, Theorem 1.12] and Remark 4.1.5 give a spectral sequence

$$\pi_* \mathrm{HH}(\mathrm{Spec}(R)/\mathcal{M}^s_{\mathrm{FG}}) \Rightarrow \pi_* \mathcal{N}^* \hat{\mathbb{\Delta}}_R[2*].$$

Indeed, $\mathcal{M}_R = \mathrm{Spec}(R)/\mathbf{G}_m \cong \mathrm{Spec}(R) \times B\mathbf{G}_m$, so that the underlying R-algebra of $\mathrm{HH}(\mathcal{M}_R/\mathcal{M}_{\mathrm{FG}})$ is $\mathrm{HH}(\mathrm{Spec}(R)/\mathcal{M}^s_{\mathrm{FG}})$.

Example 4.1.7. The complex orientation $\mathrm{BP} \to \mathrm{BP}\langle n\rangle$ induces a map $\mathcal{M}_{\mathrm{BP}\langle n\rangle} \to \mathcal{M}_{\mathrm{BP}}$ which factors the structure map $\mathcal{M}_{\mathrm{BP}\langle n\rangle} \to \mathcal{M}_{\mathrm{FG}}$. Explicitly, we have the following composite map of stacks over $B\mathbf{G}_m$:

$$\mathrm{Spec}(\mathrm{BP}\langle n\rangle_*)/\mathbf{G}_m \to \mathrm{Spec}(\mathrm{BP}_*)/\mathbf{G}_m \to \mathcal{M}_{\mathrm{FG}}.$$

Taking cotangent complexes gives the following transitivity cofiber sequence in $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{BP}\langle n\rangle_*}$:

$$\mathrm{BP}\langle n\rangle_* \otimes_{\mathrm{BP}_*} L_{\mathrm{Spec}(\mathrm{BP}_*)/\mathcal{M}_{\mathrm{FG}}} \to L_{\mathrm{BP}\langle n\rangle_*/\mathcal{M}_{\mathrm{FG}}} \to L_{\mathrm{BP}\langle n\rangle_*/\mathrm{BP}_*}.$$

Since $\mathrm{BP}_*/(v_{n+1}, v_{n+2}, \cdots) \cong \mathrm{BP}\langle n\rangle_*$, observe that $L_{\mathrm{BP}\langle n\rangle_*/\mathrm{BP}_*}$ is a free $\mathrm{BP}\langle n\rangle_*$-module generated by classes $\sigma(v_{n+1}), \sigma(v_{n+2}), \cdots$. Similarly, the discussion in Example 4.1.4 implies that $L_{\mathrm{Spec}(\mathrm{BP}_*)/\mathcal{M}_{\mathrm{FG}}}$ is a free BP_*-module generated by classes $d(t_i)$. From this, one can deduce that $L_{\mathrm{Spec}(\mathrm{BP}\langle n\rangle_*)/\mathbf{G}_m/\mathcal{M}_{\mathrm{FG}}}$ is a free $\mathrm{BP}\langle n\rangle_*$-module generated by classes $\sigma(v_j)$ with $j \geq n+1$ and $d(t_i)$ with $i \geq 1$. By the HKR theorem, $\pi_*\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle n\rangle_*)/\mathbf{G}_m/\mathcal{M}_{\mathrm{FG}})$ is isomorphic to $\mathrm{Sym}_{\mathrm{BP}\langle n\rangle_*}(L_{\mathrm{BP}\langle n\rangle_*/\mathcal{M}_{\mathrm{FG}}}[1])$, which can be identified as

$$\pi_*\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle n\rangle_*)/\mathbf{G}_m/\mathcal{M}_{\mathrm{FG}}) \cong \mathrm{BP}\langle n\rangle_*\langle \sigma^2 v_j | j \geq n+1\rangle \otimes_{\mathrm{BP}\langle n\rangle_*} \Lambda_{\mathrm{BP}\langle n\rangle_*}(dt_i | i \geq 1).$$

Since v_j lives in degree $2p^j - 2$ and weight $p^j - 1$, the class $\sigma^2 v_j$ lives in degree $2p^j = |v_j| + 2$ and weight p^j; similarly, since t_i lives in degree $2p^i - 2$ and weight $p^j - 1$, the class dt_i lives in degree $2p^i - 1$ and weight p^j.

Example 4.1.8. The same discussion for the following composite map of stacks over $B\mathbf{G}_m$

$$\mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m \to \mathrm{Spec}(\mathrm{BP}_*)/\mathbf{G}_m \to \mathcal{M}_{T(n)}$$

shows that $L_{\mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m/\mathcal{M}_{T(n)}}$ is a free $\mathrm{BP}\langle n-1\rangle_*$-module generated by classes $\sigma(v_j)$ with $j \geq n$ and $d(t_i)$ with $i \geq n+1$. Therefore, the HKR theorem implies that $\pi_*\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m/\mathcal{M}_{T(n)})$ is isomorphic to a symmetric algebra over $\mathrm{BP}\langle n-1\rangle_*$ on classes $\sigma^2(v_i)$ for $i \geq n$, and $d(t_i)$ for $i \geq n+1$. Explicitly,

$$\pi_*\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m/\mathcal{M}_{T(n)}) \cong \mathrm{BP}\langle n-1\rangle_*\langle \sigma^2 v_j | j \geq n\rangle \otimes_{\mathrm{BP}\langle n-1\rangle_*} \Lambda_{\mathrm{BP}\langle n-1\rangle_*}(dt_i | i \geq n+1).$$

The class $\sigma^2 v_j$ lives in degree $2p^j = |v_j| + 2$ and weight p^j, and the class dt_i lives in degree $2p^i - 1$ and weight p^j. This mirrors the calculation of the E^2-term of the Bökstedt spectral sequence in Proposition 2.2.14.

In fact, one can recover Theorem 2.2.4 in this way by running the Adams-Novikov-Bökstedt spectral sequence (Remark 4.1.5) and using the $\mathbf{E}_2$-Dyer-Lashof argument of Proposition 2.2.14 to resolve the extension problems on the E^∞-page. We use the term "recover" in a very weak sense here: the differentials in the Adams-Novikov-Bökstedt spectral sequence are forced by the differentials in the usual Bökstedt spectral sequence (Proposition 2.2.14). Explicitly, we have

$$d^{p-1}(\gamma_j(\sigma^2 v_m)) = \gamma_{j-p}(\sigma^2 v_m) dt_m$$

modulo decomposables, and the spectral sequence collapses on the E_p-page. There are topologically determined extensions $(\sigma^2 v_m)^p = \sigma^2 v_{m+1}$ modulo decomposables, which give an isomorphism (as implied by Theorem 2.2.4)

$$\pi_*\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \cong \mathrm{BP}\langle n-1\rangle_*[\sigma^2(v_n)].$$

Recollection 4.1.9. Let Y be a scheme, and let $q : X \to \mathbf{A}^1 \times Y$ be a morphism, so that X is a scheme over Y via the projection $\mathrm{pr} : \mathbf{A}^1 \times Y \to Y$. Then the transitivity cofiber sequence in $\mathrm{QCoh}(X)$ runs

$$q^* L_{\mathbf{A}^1 \times Y/Y} \to L_{X/Y} \to L_{X/\mathbf{A}^1 \times Y}.$$

Since $q^* L_{\mathbf{A}^1 \times Y/Y}$ is a free $\mathcal{O}_X$-module of rank 1 generated by dt (where t is a coordinate on $\mathbf{A}^1$), we obtain a cofiber sequence

$$\mathrm{dR}^*_{X/Y} \to \mathrm{dR}^*_{X/\mathbf{A}^1 \times Y} \xrightarrow{\nabla} \mathrm{dR}^*_{X/\mathbf{A}^1 \times Y} dt,$$

where $\mathrm{dR}^*_{X/Y} = \bigoplus_{i \geq 0} (\wedge^i L_{X/Y})[-i]$ denotes the underlying derived commutative algebra of the downwards-shearing of $\mathrm{Sym}_{\mathcal{O}_X}(L_{X/Y}[1](1))$. The map ∇ is the Gauss-Manin connection for the morphism q. Note that ∇ satisfies Griffiths transversality: it sends the nth piece of the Hodge filtration to the $(n-1)$st piece.

Remark 4.1.10. Observe that if q is taken to be the morphism $Y \to \mathbf{A}^1 \times Y$ given by the inclusion of the origin into $\mathbf{A}^1$, then $\mathrm{dR}_{Y/\mathbf{A}^1 \times Y}$ is p-completely isomorphic to the divided power algebra $\mathcal{O}_Y\langle t\rangle$. Using the fact that $\mathrm{dR}_{Y/Y} \cong \mathcal{O}_Y$, it is not difficult to see that the Gauss-Manin connection ∇ must send $\gamma_j(t) \mapsto \gamma_{j-1}(t) dt$. Here, we set $\gamma_{-1}(t) = 0$. In particular, ∇ is a PD-derivation.

Example 4.1.11 (The topological Sen operator and $\mathcal{M}_{\mathrm{FG}}$)**.** The map $T(n-1) \to T(n)$ of homotopy commutative rings induces a map $\mathcal{M}_{T(n)} \to \mathcal{M}_{T(n-1)}$ of graded stacks, which sends a p-typical graded formal group equipped with a coordinate up to order $\leq p^n$ to the underlying p-typical graded formal group equipped with a coordinate up to order $\leq p^n - 1$. The map $\mathcal{M}_{T(n)} \to \mathcal{M}_{T(n-1)}$ is an affine bundle: in other words, it exhibits $\mathcal{M}_{T(n-1)}$ as the quotient of $\mathcal{M}_{T(n)}$ by the group scheme $\mathbf{G}_a^{(p^n-1)}/\mathbf{G}_m$ over $B\mathbf{G}_m$, where $\mathbf{G}_a^{(p^n-1)}$ denotes the affine line with $\mathbf{G}_m$-action of weight $p^n - 1$. This follows, for instance, from [**Pet17**, Reduction of Lemma 3.2.3

to Lemma 3.2.7]. If $f : X \to \mathcal{M}_{T(n)}$ is a stack over $\mathcal{M}_{T(n)}$, the transitivity cofiber sequence in $\mathrm{QCoh}(X)$ is given by

$$f^* L_{\mathcal{M}_{T(n)}/\mathcal{M}_{T(n-1)}} \to L_{X/\mathcal{M}_{T(n-1)}} \to L_{X/\mathcal{M}_{T(n)}}.$$

Since $\mathcal{M}_{T(n)} \to \mathcal{M}_{T(n-1)}$ is a $\mathbf{G}_a$-bundle, we see that $L_{\mathcal{M}_{T(n)}/\mathcal{M}_{T(n-1)}}$ is a free $\mathcal{O}_{\mathcal{M}_{T(n)}}$-module of rank 1 generated by the class dt_i. It follows that there is a cofiber sequence

$$\mathrm{HH}(X/\mathcal{M}_{T(n-1)}) \to \mathrm{HH}(X/\mathcal{M}_{T(n)}) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{2p^n,p^n}\mathrm{HH}(X/\mathcal{M}_{T(n)}) \tag{49}$$

of quasicoherent sheaves on X, where $\Sigma^{n,w}$ denotes a shift by degree n and weight w. As indicated by the notation, the map Θ_{mot} behaves as an analogue on $\mathcal{M}_{\mathrm{FG}}$ of the topological Sen operator of Theorem 3.1.4; more precisely, it is the effect of the topological Sen operator at the level of the E^2-page of the Adams-Novikov-Bökstedt spectral sequence of Remark 4.1.5. Moreover, the discussion in Recollection 4.1.9 says that Θ_{mot} can be understood as an analogue of the Gauss-Manin connection.

Example 4.1.12. The topological Sen operator on $\mathrm{THH}(\mathbf{Z}_p/J(p)) \cong \mathbf{Z}_p[x]$ sends $x^j \mapsto jx^{j-1}$, so that the action of the Sen operator is precisely the action of $\mathbf{G}_a^\sharp$ on $\mathbf{G}_a = \mathrm{Spec}\,\mathbf{Z}_p[x]$ given by $\partial_x : \mathbf{Z}_p[x] \to \mathbf{Z}_p[x]$. Therefore, there is a p-complete graded isomorphism $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p) \cong \Gamma(\mathbf{G}_a/\mathbf{G}_a^\sharp; \mathcal{O})$. In the same way, one can argue that there is a p-complete isomorphism $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p} \cong \Gamma(\mathbf{G}_m/\mathbf{G}_m^\sharp; \mathcal{O})$.

This perspective is related to the stacky approach to Hodge-Tate cohomology à la [**Dri22, BL22a**] in the following way. By [**Dri22**, Proposition 3.5.1], there is an isomorphism $\mathbf{G}_a/\mathbf{G}_a^\sharp \cong \mathbf{G}_a^{\mathrm{dR}}$; similarly, $\mathbf{G}_m/\mathbf{G}_m^\sharp \cong \mathbf{G}_m^{\mathrm{dR}}$. Therefore:

$$\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p) \cong \Gamma(\mathbf{G}_a^{\mathrm{dR}}; \mathcal{O}), \tag{50}$$

$$\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p} \cong \Gamma(\mathbf{G}_m^{\mathrm{dR}}; \mathcal{O}). \tag{51}$$

Since $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}$ is supposed to arise as the cohomology of the total space $\mathrm{Tot}(\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\})$ of the Breuil-Kisin twisting line bundle $\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}$ over $\mathrm{WCart}^{\mathrm{HT}}$, the isomorphism (51) suggests that $\mathrm{Tot}(\mathcal{O}_{\mathrm{WCart}^{\mathrm{HT}}}\{1\}) \cong \mathbf{G}_m^{\mathrm{dR}}$. In turn, this suggests that $\mathrm{WCart}^{\mathrm{HT}}$ should be $\mathbf{G}_m^{\mathrm{dR}}/\mathbf{G}_m = B\mathbf{G}_m^\sharp$. This is indeed true: it is precisely [**BL22a**, Theorem 3.4.13].

Similarly, $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p)$ is supposed to arise as the cohomology of the total space of the Breuil-Kisin twisting line bundle over the "extended Hodge-Tate locus" Δ_0' in Drinfeld's Σ'. (The stack Δ_0' is defined in [**Dri22**, Section 5.10.1].) In [**Bha22**], the stack Σ' is denoted by $\mathrm{Spf}(\mathbf{Z}_p)^{\mathcal{N}}$, and one might therefore denote Δ_0' by $\mathrm{Spf}(\mathbf{Z}_p)^{\mathcal{N},\mathrm{HT}}$. The isomorphism (50) then suggests that the total space of the Breuil-Kisin line bundle over $\mathrm{Spf}(\mathbf{Z}_p)^{\mathcal{N},\mathrm{HT}}$ is $\mathbf{G}_a^{\mathrm{dR}}$, which in turn suggests that $\mathrm{Spf}(\mathbf{Z}_p)^{\mathcal{N},\mathrm{HT}}$ should be $\mathbf{G}_a^{\mathrm{dR}}/\mathbf{G}_m \cong \mathbf{G}_a/(\mathbf{G}_a^\sharp \rtimes \mathbf{G}_m)$. This is indeed true: it is precisely [**Dri22**, Lemma 5.12.4].

Had we worked with the evenly faithfully flat cover $\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p) \to \mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(\mathbf{Z}_p/S[t])$ (where $t \mapsto p$) instead, the stack associated to the even filtration on $\mathrm{THH}(\mathbf{Z}_p)$ would in fact be presented by (and is therefore isomorphic to) $\mathbf{G}_a^{\mathrm{dR}}/\mathbf{G}_m$.

Variant 4.1.13. One can also study the stack $\mathcal{M}_{J(p)}$ associated to the $\mathbf{E}_2^{\mathrm{fr}}$-ring $J(p)$. It is not difficult to show that the morphism $\mathcal{M}_{J(p)} \to \mathcal{M}_{\mathrm{FG}}$ exhibits $\mathcal{M}_{J(p)}$ as a $\mathbf{G}_m$-bundle over $\mathcal{M}_{\mathrm{FG}}$; for example, the fiber product $\mathrm{Spec}(\mathrm{MU}_*)/\mathbf{G}_m \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}_{J(p)}$ is isomorphic to $\mathrm{Spec}(\pi_*(\mathrm{MU} \otimes J(p)))/\mathbf{G}_m$, but there is an equivalence of $\mathbf{E}_2$-MU-algebras $\mathrm{MU} \otimes J(p) \simeq \mathrm{MU}[t^{\pm 1}]$ with $|t| = 0$.

Since $\mathcal{M}_{J(p)}$ is a $\mathbf{G}_m$-bundle over $\mathcal{M}_{\mathrm{FG}}$, descent in Hochschild homology is controlled by a Gauss-Manin connection. If Y is a scheme and $q : X \to \mathbf{G}_m \times Y$ is a morphism, then there is a cofiber sequence

$$\mathrm{dR}^*_{X/Y} \to \mathrm{dR}^*_{X/\mathbf{G}_m \times Y} \xrightarrow{\nabla} \mathrm{dR}^*_{X/\mathbf{G}_m \times Y} d\log(t).$$

If X is a stack over $\mathcal{M}_{J(p)}$, we then obtain a cofiber sequence

$$\mathrm{HH}(X/\mathcal{M}_{\mathrm{FG}}) \to \mathrm{HH}(X/\mathcal{M}_{J(p)}) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{2,1}\mathrm{HH}(X/\mathcal{M}_{J(p)})$$

of quasicoherent sheaves on X. This is an analogue on $\mathcal{M}_{\mathrm{FG}}$ of the topological Sen operator of (17).

Remark 4.1.14. Suppose that $T(1)$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring (this is true at $p = 2$). The unit map on $T(1)$ defines a map $\mathrm{TP}(\mathbf{Z}_p) \to \mathrm{TP}(\mathbf{Z}_p/T(1))$. Since $\mathrm{TP}(\mathbf{Z}_p/T(1))$ is concentrated in even degrees by Theorem 2.2.4, one can define the motivic filtration on $\mathrm{TP}(\mathbf{Z}_p/T(1))$ using the double-speed Postnikov filtration. Under the isomorphism $\pi_*\mathrm{TP}(\mathbf{Z}_p/T(1)) \cong \pi_*\mathrm{BP}\langle 1\rangle^{tS^1} \cong \mathbf{Z}_p[\![\widetilde{p}]\!]^{tS^1}$, one can view gr^0 of the motivic filtration $\mathrm{TP}(\mathbf{Z}_p/T(1))$ as $\mathbf{Z}_p[\![\widetilde{p}]\!]$. Recall that $\mathrm{TP}(\mathbf{Z}_p)$ is a homotopical analogue of the Cartier-Witt stack $\mathrm{WCart}_{\mathbf{Z}_p}$ from [**BL22b**]. One can then view the map $\mathrm{TP}(\mathbf{Z}_p) \to \mathrm{TP}(\mathbf{Z}_p/T(1))$ as an analogue of the following map induced by the q-de Rham point:

$$\rho_{\widetilde{p}\mathrm{dR}} : \mathrm{Spf}\, \mathbf{Z}_p[\![\widetilde{p}]\!] \cong (\mathrm{Spf}\, \mathbf{Z}_p[\![q-1]\!])/\mathbf{F}_p^\times \to (\mathrm{Spf}\, \mathbf{Z}_p[\![q-1]\!])/\mathbf{Z}_p^\times \xrightarrow{\rho_{q\mathrm{dR}}} \mathrm{WCart}_{\mathbf{Z}_p}.$$

This map classifies the prism $(\mathbf{Z}_p[\![\widetilde{p}]\!], (\widetilde{p}))$, and can reasonably be called the $\widetilde{p}$-de Rham point.

As explained in the end of the introduction to [**HRW22**], one hopes that the unit map $S^0 \to \mathrm{TP}(\mathbf{Z}_p)$ induces the map $\mathrm{WCart}_{\mathbf{Z}_p} \to \mathcal{M}_{\mathrm{FG}}$ classifying Drinfeld's formal group over $\mathrm{WCart}_{\mathbf{Z}_p} = \Sigma$ from [**Dri21**] on the associated graded of the motivic filtration. If Conjecture 2.2.18 were true (i.e., there is an equivalence $\mathrm{TP}(\mathbf{Z}_p/T(1)) \simeq \mathrm{BP}\langle 1\rangle^{tS^1}$ of spectra), the resulting unit map $S^0 \to \mathrm{TP}(\mathbf{Z}_p/T(1)) \to \mathrm{BP}\langle 1\rangle^{tS^1}$ would just be the unit of the $\mathbf{E}_\infty$-ring $\mathrm{BP}\langle 1\rangle^{tS^1}$. Since $\mathrm{BP}\langle 1\rangle$ is complex-oriented, the formal group over $\pi_0\mathrm{BP}\langle 1\rangle^{tS^1} \cong \mathbf{Z}_p[\![\widetilde{p}]\!]$ must be isomorphic to the formal group of $\mathrm{BP}\langle 1\rangle$, i.e., the p-typification of the multiplicative formal group. In particular, the aforementioned expectation about the formal group over $\mathrm{WCart}_{\mathbf{Z}_p}$ and its relation to $\mathrm{TP}(\mathbf{Z}_p)$ would predict that the pullback of Drinfeld's formal group over $\mathrm{WCart}_{\mathbf{Z}_p}$ along the map $\rho_{\widetilde{p}\mathrm{dR}}$ is the p-typification of the multiplicative formal group over $\mathbf{Z}_p[\![\widetilde{p}]\!]$. This is indeed true, and was proved in [**Dri21**, Section 2.10.6]. This lends further evidence to the idea that the map $\mathrm{TP}(\mathbf{Z}_p) \to \mathrm{TP}(\mathbf{Z}_p/T(1))$ is a homotopical analogue of the $\widetilde{p}$-de Rham point of $\mathrm{WCart}_{\mathbf{Z}_p}$.

4.2. Comparing THH relative to $T(n)$ and $T(n+1)$. Recall from Theorem 2.2.4 that $\pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/T(n+1)) \otimes_{\mathrm{BP}\langle n\rangle} \mathrm{BP}\langle n-1\rangle)$ is (additively) equivalent to the "subalgebra" $\mathrm{BP}\langle n-1\rangle_*[\theta_{n-1}^p]$ of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \cong \mathrm{BP}\langle n-1\rangle_*[\theta_{n-1}]$. This picture has an analogue over $\mathcal{M}_{\mathrm{FG}}$, as we now explain. We first need a simple calculation.

Remark 4.2.1. Let R be a commutative ring, and let $x \in R$ be a regular element. Then there is a p-completed equivalence $\mathrm{dR}^*_{R[t]/x/R} \simeq R[t]\langle x'\rangle/x \otimes_{R[t]/x} \Lambda_{R[t]/x}(dt)$ with $|x'| = 0$. Indeed, this follows from combining the observation that $R[t]/x \cong R[t] \otimes_R R/x$ with the following p-completed equivalences: $\mathrm{dR}^*_{R[t]/R} \simeq \Lambda_{R[t]}(dt)$,

$\mathrm{dR}^*_{R/x/R} \simeq R\langle x'\rangle/x$. Similarly, there is an equivalence $\mathrm{HH}(R[t]/x/R) \simeq R[t][S^1 \times \mathbf{C}P^\infty]/x$.

Example 4.2.2. Let $i_{n-1} : \mathcal{Z}(v_{n-1}) \hookrightarrow \mathcal{M}_{T(n)}$ denote the closed substack cut out by the global section $v_{n-1} \in \mathrm{H}^0(\mathcal{M}_{T(n)}; \mathcal{O}_{\mathcal{M}_{T(n)}})$. If $f : X \to \mathcal{M}_{T(n)}$ is a stack over $\mathcal{M}_{T(n)}$, let $X^{v_{n-1}=0}$ denote the pullback of X along i_{n-1}, and let $f : X^{v_{n-1}=0} \to \mathcal{Z}(v_{n-1})$ denote the structure morphism. Then $i_{n-1}^*\mathrm{HH}(X/\mathcal{M}_{T(n)}) = \mathrm{HH}(X^{v_{n-1}=0}/\mathcal{Z}(v_{n-1}))$. In the case $X = \mathrm{Spec}(\mathrm{BP}\langle n\rangle_*)/\mathbf{G}_m$, there is an isomorphism $X^{v_{n-1}=0} = \mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m$. We will now relate $\mathrm{HH}(X^{v_{n-1}=0}/\mathcal{Z}(v_{n-1}))$ to $\mathrm{HH}(X^{v_{n-1}=0}/\mathcal{M}_{T(n-1)})$ by calculating $\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)})$.

Recall from Example 4.1.11 that there is a $\mathbf{G}_a$-bundle $\mathcal{M}_{T(n)} \to \mathcal{M}_{T(n-1)}$. Note that $L_{\mathcal{M}_{T(n)}/\mathcal{M}_{T(n-1)}}$ is a free $\mathcal{O}_{\mathcal{M}_{T(n)}}$-module of rank 1 generated by a class $d(t_n)$, and that $L_{\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n)}}$ is a free $\mathcal{O}_{\mathcal{Z}(v_{n-1})}$-module of rank 1 generated by a class $\sigma^2(v_{n-1})$. Applying Remark 4.2.1, we find that

$$\pi_*\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)}) \cong \mathcal{O}_{\mathcal{Z}(v_{n-1})}\langle \sigma^2(v_{n-1})\rangle \otimes_{\mathcal{O}_{\mathcal{Z}(v_{n-1})}} \Lambda_{\mathcal{O}_{\mathcal{Z}(v_{n-1})}}(dt_n). \tag{52}$$

We therefore see that $\mathrm{HH}(X^{v_{n-1}=0}/\mathcal{M}_{T(n-1)})$ a subquotient of the tensor product of $\mathrm{HH}(X^{v_{n-1}=0}/\mathcal{Z}(v_{n-1}))$ and $f^*\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)}) \cong \mathcal{O}_{X^{v_{n-1}=0}}\langle\sigma(v_{n-1})\rangle[dt_n]/(dt_n)^2$. Let us now take f to be the morphism $\mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m \to \mathcal{M}_{T(n)}$. The E^2-page of the Adams-Novikov-Bökstedt spectral sequence for $\mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1))$ is given by

$$E^2_{*,*} = \mathrm{BP}\langle n-2\rangle_*\langle \sigma^2 v_j | j \geq n-1\rangle \otimes_{\mathrm{BP}\langle n-1\rangle_*} \Lambda_{\mathrm{BP}\langle n-1\rangle_*}(dt_j | j\geq n),$$

and the extensions on the E^∞-page are given by $(\sigma^2 v_j)^{p^{n-j}} = \sigma^2 v_n$. The above discussion therefore shows that $f^*\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)})$ precisely detects the "bottom piece" of this E^2-page, i.e., the subalgebra $\mathrm{BP}\langle n-2\rangle_*\langle\sigma^2 v_{n-1}\rangle \otimes_{\mathrm{BP}\langle n-2\rangle_*} \Lambda_{\mathrm{BP}\langle n-2\rangle_*}(dt_n)$. Therefore, the preceding calculation of $\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)})$ gives one explanation for why $\pi_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n)) \otimes_{\mathrm{BP}\langle n-1\rangle} \mathrm{BP}\langle n-2\rangle)$ is (additively) equivalent to the "subalgebra" $\mathrm{BP}\langle n-2\rangle_*[\theta_{n-2}^p]$ of $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1)) \cong \mathrm{BP}\langle n-2\rangle_*[\theta_{n-2}]$.

Remark 4.2.3. We can extend the analysis of Example 4.2.2 further. Let $0 \leq j \leq n-1$, and let $i_{j,\cdots,n-1} : \mathcal{Z}(v_{[j,n)}) \hookrightarrow \mathcal{M}_{T(n)}$ denote the closed substack cut out by the global sections $v_j, \cdots, v_{n-1} \in \mathrm{H}^0(\mathcal{M}_{T(n)}; \mathcal{O}_{\mathcal{M}_{T(n)}})$. If $f : X \to \mathcal{M}_{T(n)}$ is a stack over $\mathcal{M}_{T(n)}$, let $X^{v_j,\cdots,v_{n-1}=0}$ denote the pullback of X along i, and let $f : X^{v_j,\cdots,v_{n-1}=0} \to \mathcal{Z}(v_{[j,n)})$ denote the structure morphism. Then $i^*_{j,\cdots,n-1}\mathrm{HH}(X/\mathcal{M}_{T(n)})$ is equivalent to $\mathrm{HH}(X^{v_j,\cdots,v_{n-1}=0}/\mathcal{Z}(v_{[j,n)}))$. In the case $X = \mathrm{Spec}(\mathrm{BP}\langle n-1\rangle_*)/\mathbf{G}_m$, there is an isomorphism $X^{v_j,\cdots,v_{n-1}=0} = \mathrm{Spec}(\mathrm{BP}\langle j-1\rangle_*)/\mathbf{G}_m$. We can now relate $\mathrm{HH}(X^{v_j,\cdots,v_{n-1}=0}/\mathcal{Z}(v_{[j,n)}))$ to $\mathrm{HH}(X^{v_j,\cdots,v_{n-1}=0}/\mathcal{M}_{T(j)})$ by calculating $\mathrm{HH}(\mathcal{Z}(v_{[j,n)})/\mathcal{M}_{T(j)})$.

We claim that there is an isomorphism

$$\mathrm{HH}(\mathcal{Z}(v_{[j,n)})/\mathcal{M}_{T(j)}) \cong \mathcal{O}_{\mathcal{Z}(v_{[j,n)})}\langle\sigma(v_i) | j\leq i\leq n-1\rangle \otimes_{\mathcal{O}_{\mathcal{Z}(v_{[j,n)})}} \Lambda_{\mathcal{O}_{\mathcal{Z}(v_{[j,n)})}}(dt_i | j+1 \leq i \leq n)$$

To prove this, we will use descending induction on j; the base case $j = n-1$ was studied in Example 4.2.2. For the inductive step, suppose we know the result for $j+1$. Let $i_j : \mathcal{Z}(v_{[j,n)}) \to \mathcal{Z}(v_{j+1}, \cdots, v_{n-1})$ denote the closed substack cut out by

v_j. Then there are isomorphisms

$$\begin{aligned}\mathrm{HH}(\mathcal{Z}(v_{[j,n)})/\mathcal{M}_{T(j+1)}^{v_j=0}) &\cong i_j^*\mathrm{HH}(\mathcal{Z}(v_{j+1},\cdots,v_{n-1})/\mathcal{M}_{T(j+1)}) \\ &\cong \mathcal{O}_{\mathcal{Z}(v_{[j,n)})}\langle\sigma^2(v_i)|j+1\leq i\leq n-1\rangle \otimes_{\mathcal{O}_{\mathcal{Z}(v_{[j,n)})}} \Lambda_{\mathcal{O}_{\mathcal{Z}(v_{[j,n)})}}(dt_i|j+2\leq i\leq n)\end{aligned}$$

Recall that Example 4.2.2 gives an isomorphism between $\mathrm{HH}(\mathcal{M}_{T(j+1)}^{v_j=0}/\mathcal{M}_{T(j)})$ and $\mathcal{O}_{\mathcal{M}_{T(j+1)}^{v_j=0}}\langle\sigma^2(v_j)\rangle\otimes_{\mathcal{O}_{\mathcal{M}_{T(j+1)}^{v_j=0}}}\Lambda_{\mathcal{O}_{\mathcal{M}_{T(j+1)}^{v_j=0}}}(dt_{j+1})$. The desired calculation of $\mathrm{HH}(\mathcal{Z}(v_{[j,n)})/\mathcal{M}_{T(j)})$ is now a simple computation with the transitivity sequence for the composite

$$\mathcal{Z}(v_{[j,n)}) \to \mathcal{M}_{T(j+1)}^{v_j=0} \to \mathcal{M}_{T(j)}.$$

Let $X = \mathrm{Spec}(\mathrm{BP}\langle j-1\rangle_*)/\mathbf{G}_m$, and let $f : X \to \mathcal{Z}(v_{[j,n)})$ be the structure map. Then the above discussion implies that $\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle j-1\rangle_*)/\mathbf{G}_m/\mathcal{M}_{T(j)})$ is isomorphic to the tensor product of $f^*\mathrm{HH}(\mathcal{Z}(v_{[j,n)})/\mathcal{M}_{T(j)})$ and $\mathrm{HH}(\mathrm{Spec}(\mathrm{BP}\langle j-1\rangle_*)/\mathbf{G}_m/\mathcal{Z}(v_{[j,n)}))$. This gives the E^2-page of the Adams-Novikov-Bökstedt spectral sequence computing $\pi_*\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ (see Remark 4.1.5), and one can run this spectral sequence as in Proposition 2.2.14. If $\mathrm{BP}\langle j-1\rangle$ admits the structure of an $\mathbf{E}_3$-algebra, there are extensions $\sigma^2(v_i)^p = \sigma^2(v_{i+1})$ modulo decomposables on the E^∞-page of this spectral sequence.

Let $T(n)/v_{[j,n)}$ denote $T(n)/(v_j,\cdots,v_{n-1})$. Since $\theta_j \in \pi_*\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ is represented by $\sigma^2(v_j)$, we find that $\mathrm{THH}(T(n)/v_{[j,n)}/T(j))$ is (additively) equivalent to as $T(n)[\theta_j]/(v_{[j,n)},\theta_j^{p^{n-j}})$. (See Remark 4.2.5 for a more topological perspective on this observation.) This discussion provides an algebraic perspective on why $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{[j,n)}$ is (additively) equivalent to as the "subalgebra" of $\pi_*\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ generated by $\theta_j^{p^{n-j}}$.

Remark 4.2.4. In topology, Example 4.2.2 plays out as follows, if we assume[18] Conjecture 2.1.9. Let $n\geq 1$. We begin by observing that $T(n)/v_{n-1}$ is the Thom spectrum of an $\mathbf{E}_1$ map $\mu : \Omega J_{p-1}(S^{2p^{n-1}}) \to \mathrm{BGL}_1(T(n-1))$; in particular, $T(n)/v_{n-1}$ admits the structure of an $\mathbf{E}_1$-ring. To see this, we first define the map μ as follows. There is a map $S^{2p^{n-1}} \to B^2\mathrm{GL}_1(X(p^n-1))$ which detects the class $v_{n-1}\in\pi_{2p^{n-1}-2}X(p^n-1)$, which naturally extends to a map $J_{p-1}(S^{2p^{n-1}}) \to B^2\mathrm{GL}_1(X(p^n-1))$ since we are working p-locally. Therefore, we obtain an $\mathbf{E}_1$-map $\Omega J_{p-1}(S^{2p^{n-1}}) \to \mathrm{BGL}_1(X(p^n-1))$. The projection $X(p^n-1)\to T(n-1)$ is a map of $\mathbf{E}_2$-rings by Conjecture 2.1.9, and therefore induces an $\mathbf{E}_1$-map $\mathrm{BGL}_1(X(p^n-1)) \to \mathrm{BGL}_1(T(n-1))$. Composition with the $\mathbf{E}_1$-map $\Omega J_{p-1}(S^{2p^{n-1}}) \to \mathrm{BGL}_1(X(p^n-1))$ produces the desired map μ. The fact that the Thom spectrum of μ can be identified with $T(n)/v_{n-1}$ can be proved directly using (54) below. It follows from this discussion that there is an equivalence

$$\mathrm{THH}(T(n)/v_{n-1}/T(n-1)) \simeq T(n)[J_{p-1}(S^{2p^{n-1}})]/v_{n-1}.$$

Moreover, under the equivalence $\mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1)) \simeq \mathrm{BP}\langle n-2\rangle[\Omega S^{2p^{n-1}+1}]$ of Theorem 2.2.4(a), the map $\mathrm{THH}(T(n)/v_{n-1}/T(n-1)) \to \mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1))$ induced by the map $T(n)/v_{n-1}\to\mathrm{BP}\langle n-2\rangle$ is given by the skeletal inclusion of $J_{p-1}(S^{2p^{n-1}}) \to \Omega S^{2p^{n-1}+1}$. The projection $\mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1)) \to$

[18]There is an unconditional variant of the following discussion, obtained by replacing $T(n)$ with $X(p^{n+1}-1)$. However, this comes at the cost of adding the spaces Δ_n into the mix.

$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{n-1}$ can be identified with the effect on $\mathrm{BP}\langle n-2\rangle$-chains of the James-Hopf map $\Omega S^{2p^{n-1}+1} \to \Omega S^{2p^n+1}$. Therefore, the EHP sequence

$$J_{p-1}(S^{2p^{n-1}}) \to \Omega S^{2p^{n-1}+1} \to \Omega S^{2p^n+1}$$

shows that $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{n-1}$ is (additively) equivalent to precisely as the "subalgebra" of $\mathrm{THH}(\mathrm{BP}\langle n-2\rangle/T(n-1))$ generated by θ_{n-1}^p. The above calculation of $\mathrm{THH}(T(n)/v_{n-1}/T(n-1))$ is a topological incarnation of the calculation of $\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)})$ in Example 4.2.2. Indeed, the Adams-Novikov-Bökstedt spectral sequence (see Remark 4.1.5) runs

$$E^2_{*,*} = \pi_*\mathrm{HH}(\mathcal{Z}(v_{n-1})/\mathcal{M}_{T(n-1)}) \Rightarrow \pi_*\mathrm{gr}^\bullet_{\mathrm{ev}}\mathrm{THH}(T(n)/v_{n-1}/T(n-1)), \tag{53}$$

and the E^2-page is given by (52). Again, one can establish analogues of the Bökstedt differentials (6) in the Adams-Novikov-Bökstedt spectral sequence, and thereby obtain an alternative approach to the above calculation of $\mathrm{THH}(T(n)/v_{n-1}/T(n-1))$.

Remark 4.2.5. Let us continue to assume Conjecture 2.1.9, and let $0 \leq j \leq n-1$. Recall that $T(n)/v_{[j,n)}$ denote $T(n)/(v_j, \cdots, v_{n-1})$. Recall that

$$\mathrm{H}_*(T(n)/v_{[j,n)}; \mathbf{F}_p) = \begin{cases} \mathbf{F}_2[\zeta_1^2, \cdots, \zeta_j^2, \zeta_{j+1}, \cdots, \zeta_n] & p = 2, \\ \mathbf{F}_p[\zeta_1, \cdots, \zeta_n] \otimes \Lambda_{\mathbf{F}_p}(\tau_j, \cdots, \tau_{n-1}) & p > 2. \end{cases} \tag{54}$$

It is natural to ask if the discussion in Remark 4.2.4 extends to a description of $\mathrm{THH}(T(n)/v_{[j,n)}/T(j))$, paralleling Remark 4.2.3. This is an ill-posed question, since it is not clear that $T(n)/v_{[j,n)}$ admits the structure of an $\mathbf{E}_1$-algebra. Nevertheless, if $T(n)/v_{[j,n)}$ did admit the structure of an $\mathbf{E}_1$-$T(j)$-algebra, then an analysis similar to Theorem 2.2.4 shows that

$$\mathrm{THH}(T(n)/v_{[j,n)}/T(j)) \simeq T(n)[J_{p^{n-j}-1}(S^{2p^j})]/v_{[j,n)}.$$

This is the topological analogue of the calculation of Remark 4.2.3. Under the equivalence $\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j)) \simeq \mathrm{BP}\langle j-1\rangle[\Omega S^{2p^j+1}]$ of Theorem 2.2.4(a), the map $\mathrm{THH}(T(n)/v_{[j,n)}/T(j)) \to \mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ induced by the map $T(n)/v_{[j,n)} \to \mathrm{BP}\langle j-1\rangle$ is given by the skeletal inclusion of $J_{p^{n-j}-1}(S^{2p^j}) \to \Omega S^{2p^j+1}$. The projection

$$\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{[j,n)}$$

can be identified with the effect on $\mathrm{BP}\langle j-1\rangle$-chains of the James-Hopf map $\Omega S^{2p^j+1} \to \Omega S^{2p^n+1}$. Therefore, the EHP sequence

$$J_{p^{n-j}-1}(S^{2p^j}) \to \Omega S^{2p^j+1} \to \Omega S^{2p^n+1}$$

shows that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{[j,n)}$ is (additively) equivalent to precisely the "subalgebra" of $\pi_*\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ generated by $\theta_j^{p^{n-j}}$.

Since $\mathrm{THH}(T(n)/v_{[j,n)}/T(j)) \simeq T(n)[J_{p^{n-j}-1}(S^{2p^j})]/v_{[j,n)}$, one expects $T(n)/v_{[j,n)}$ to have an $\mathbf{E}_1$-cell structure over $T(j)$ described by the cell structure of $J_{p^{n-j}-1}(S^{2p^j})$. Although we do not know how to prove this unconditionally, it is not difficult to show if we further assume [**Dev23a**, Conjectures D and E]. In this case, [**Dev23a**, Corollary B] says that there is a map $f : \Omega^2 S^{2p^j+1} \to \mathrm{BGL}_1(T(j))$ which detects $v_j \in \pi_{2p^j-2}T(j)$ on the bottom cell of the source, such that the Thom spectrum of f is a form of $\mathrm{BP}\langle j-1\rangle$. Let[19] $f_{n,j} : \Omega J_{p^{n-j}-1}(S^{2p^j}) \to \mathrm{BGL}_1(T(j))$ denote the composite of f with the $\mathbf{E}_1$-map $\Omega J_{p^{n-j}-1}(S^{2p^j}) \to \Omega^2 S^{2p^j+1}$. Then the Thom spectrum of $f_{n,j}$ is equivalent to $T(n)/v_{[j,n)}$ as a $T(j)$-module. This is not quite an "$\mathbf{E}_1$-cell structure" for $T(n)/v_{[j,n)}$, since $f_{n,j}$ is not an $\mathbf{E}_1$-map; nevertheless, this construction of $T(n)/v_{[j,n)}$ suffices to calculate $\mathrm{THH}(T(n)/v_{[j,n)}/T(j))$.

Example 4.2.6. If R is an $\mathbf{E}_2$-algebra which is an $\mathbf{E}_1$-$T(n)$-algebra, one can loosely interpret the above discussion as saying that the square

(55)
$$\begin{tikzcd} R[J_{p^{n-j}-1}(S^{2p^j})] \simeq R \otimes_{T(n)} \mathrm{THH}(T(n)/v_{[j,n)}/T(j)) \arrow[r] \arrow[d] & \mathrm{THH}(R/v_{[j,n)}/T(j)) \arrow[d] \\ R \arrow[r] & \mathrm{THH}(R/T(n))/v_{[j,n)} \end{tikzcd}$$

exhibits the top-right corner as the "tensor product of the top-left and bottom-right corners". Note that the homotopy of the top-left corner is $R[\theta_j]/\theta_j^{p^{n-j}}$. The bottom-right corner should be thought of as $\mathrm{THH}(R/v_{[j,n)}/T(n)/v_{[j,n)})$, although it is difficult to make this picture precise (since $T(n)/v_{[j,n)}$ does not admit the structure of an $\mathbf{E}_2$-algebra).

For instance, if $R = \mathrm{BP}\langle n-1\rangle$, then the square (55) says that the square

(56)
$$\begin{tikzcd} \mathrm{BP}\langle n-1\rangle[J_{p^{n-j}-1}(S^{2p^j})] \arrow[r] \arrow[d] & \mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j)) \simeq \mathrm{BP}\langle j-1\rangle[\Omega S^{2p^j+1}] \arrow[d] \\ R \arrow[r] & \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{[j,n)} \simeq \mathrm{BP}\langle j-1\rangle[\Omega S^{2p^n+1}] \end{tikzcd}$$

exhibits the top-right corner as the tensor product of the top-left and bottom-right corners. This is essentially the observation that the map $\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j)) \to \mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/v_{[j,n)}$ sends $\theta_j^m \mapsto 0$ unless $p^{n-j}|m$, in which case $\theta_j^m \mapsto \theta_n^{m/p^{n-j}}$. This therefore explains the similarity between $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))$ and $\mathrm{THH}(\mathrm{BP}\langle j-1\rangle/T(j))$ given by Theorem 2.2.4(a).

Remark 4.2.7. As mentioned before, the lack of structure on the objects involved above make it difficult to use the above picture to understand the multiplicative structure on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$; but it does point to a plan of attack. Namely, one can attempt to understand the even filtration on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/v_{[0,n)}$ by considering the natural map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/v_{[0,n)} \to \mathrm{THH}(\mathbf{F}_p)$. It is not hard to see that this map is an eff cover, so that the stack associated to the even filtration on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/v_{[0,n)}$ is the quotient of the scheme associated to the even filtration on $\mathrm{THH}(\mathbf{F}_p)$ by a certain group scheme. The scheme associated to the even filtration on $\mathrm{THH}(\mathbf{F}_p)$ is precisely $\mathbf{G}_a$, and the above discussion suggests that the stack associated to the even filtration on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/v_{[0,n)}$ is isomorphic

[19] In Remark 4.2.4, we described the map $f_{n,n-1}$ without assuming [**Dev23a**, Conjectures D and E]. It is generally not possible to describe $f_{n,j}$ similarly if $j < n-1$: although there is a map $S^{2p^j} \to B^2\mathrm{GL}_1(T(j))$ which detects $v_j \in \pi_{2p^j-2}T(j)$, there are p-local obstructions to extending along $S^{2p^j} \to J_{p^{n-j}-1}(S^{2p^j})$ if $n-j > 1$. These obstructions can be viewed as the $\mathbf{E}_2$-Browder brackets on v_j; [**Dev23a**, Conjecture E] implies that these Browder brackets can be compatibly trivialized.

to $\mathbf{G}_a/W[F^n]$; this is also suggested by work of Lee in [**Lee22**]. We hope to study this in future work joint with Jeremy Hahn and Arpon Raksit. To this end, we set up some groundwork for future investigation of this stack in Appendix C, where we study some basic properties of $W[F^n]$.

Remark 4.2.8. The calculation of $\mathrm{THH}(T(n)/v_{[j,n)}/T(j))$ in Remark 4.2.5 shows that more is true: if $n \geq k-1$, the structure of $\mathrm{BP}\langle n\rangle$ as an $\mathbf{E}_1$-$X(p^k)$-algebra (i.e., $\mathrm{THH}(\mathrm{BP}\langle n\rangle/T(k))$) mirrors the structure of $\mathrm{BP}\langle n-k\rangle$ as an $\mathbf{E}_1$-algebra over the sphere (i.e., $\mathrm{THH}(\mathrm{BP}\langle n-k\rangle)$).

Remark 4.2.9. Note that the Thom spectrum of the map $f_{n,0}$ has been studied in [**MRS01**], where it was denoted $y(n)$. Just as the $y(n)$ describe a filtration of $y(\infty) = \mathbf{F}_p$ by $\mathbf{E}_1$-algebras, the spectra $T(n)/v_{[j,n)}$ describe a filtration of $\mathrm{BP}\langle j-1\rangle$. For instance, it is not difficult to show that for $j \leq k \leq n-1$, the spectrum $T(n)/v_{[j,n)}$ is $\mathrm{Tel}(k)$-acyclic. Therefore, if $T(n)/v_{[j,n)}$ admits the structure of an $\mathbf{E}_1$-ring, the same argument as [**LMMT20**, Corollary 4.15] implies that the map $K(T(n)/v_{[j,n)}) \to K(\mathrm{BP}\langle j-1\rangle)$ is an $\mathrm{Tel}(k)$-equivalence for $j \leq k \leq n-1$. Since $K(\mathrm{BP}\langle j-1\rangle)$ is $\mathrm{Tel}(k)$-locally contractible for $k \geq j+1$, the only interesting case is $k = j$; in this case, we find that the maps

$$K(T(j+1)/v_j) \to K(T(j+2)/(v_j, v_{j+1})) \to \cdots \to K(\mathrm{BP}\langle j-1\rangle)$$

are all $\mathrm{Tel}(j)$-equivalences.

Remark 4.2.10. Since $T(n)/v_{[j,n)}$ is closely related to $T(j)$ by Remark 4.2.5, it is natural to wonder if there is a relationship between $T(n)/v_{[j,n)}$ and $T(n+k)/v_{[j,n+k)}$, in a manner compatible with their relationship to $T(j)$. By Remark 4.2.4, $T(n+k)/v_{[n,n+k)}$ is the Thom spectrum of a map $\Omega J_{p^k-1}(S^{2p^n}) \to \mathrm{BGL}_1(T(n))$. It follows that if $T(n)/v_{[j,n)}$ admits the structure of an $\mathbf{E}_1$-ring, then $T(n+k)/v_{[j,n+k)}$ is the Thom spectrum of a map $\Omega J_{p^k-1}(S^{2p^n}) \to \mathrm{BGL}_1(T(n)/v_{[j,n)})$. As mentioned in Remark 4.2.5, if we further assume [**Dev23a**, Conjectures D and E], the spectrum $T(n)/v_{[j,n)}$ (resp. $T(n+k)/v_{[j,n+k)}$) is the Thom spectrum of a map $\Omega J_{p^{n-j}-1}(S^{2p^j}) \to \mathrm{BGL}_1(T(j))$ (resp. $\Omega J_{p^{n+k-j}-1}(S^{2p^j}) \to \mathrm{BGL}_1(T(j))$).

The relationship between the two presentations of $T(n+k)/v_{[j,n+k)}$ (as a Thom spectrum over $T(n)/v_{[j,n)}$ and over $T(j)$) is explained by the following observation in unstable homotopy theory: there is a fibration[20]

$$J_{p^m-1}(S^{2d}) \to J_{p^{m+k}-1}(S^{2d}) \xrightarrow{H} J_{p^k-1}(S^{2dp^m}). \tag{57}$$

Indeed, applying (57) when $m = n-j$ and $d = p^j$, we obtain a fibration of $\mathbf{E}_1$-spaces:

$$\Omega J_{p^{n-j}-1}(S^{2p^j}) \to \Omega J_{p^{n+k-j}-1}(S^{2p^j}) \xrightarrow{H} \Omega J_{p^k-1}(S^{2p^n}).$$

The composite of $f_{n+k,j} : \Omega J_{p^{n+k-j}-1}(S^{2p^j}) \to \mathrm{BGL}_1(T(j))$ with the map $\Omega J_{p^{n-j}-1}(S^{2p^j}) \to \Omega J_{p^{n+k-j}-1}(S^{2p^j})$ is $f_{n,j} : \Omega J_{p^{n-j}-1}(S^{2p^j}) \to \mathrm{BGL}_1(T(j))$. Therefore, [**Dev23a**, Proposition 2.1.6] implies that there is a map $f_{n+k,j} : \Omega J_{p^k-1}(S^{2p^n}) \to \mathrm{BGL}_1(T(n)/v_{[j,n)})$

[20] To construct the fibration (57), recall that there is an EHP sequence

$$J_{p^m-1}(S^{2d}) \to \Omega S^{2d+1} \xrightarrow{H} \Omega S^{2dp^m+1}.$$

By dimension considerations, the canonical map $J_{p^{m+k}-1}(S^{2d}) \to \Omega S^{2d+1}$ factors through $J_{p^{m+k}-1}(S^{2d}) \to J_{p^k-1}(S^{2dp^m}) \times_{\Omega S^{2dp^m+1}} \Omega S^{2d+1}$, and one can easily check that this map is an equivalence. This implies the desired fiber sequence (57).

whose Thom spectrum is $T(n+k)/v_{[j,n+k)}$. This is the desired relationship between the various presentations of $T(n+k)/v_{[j,n+k)}$.

Remark 4.2.11. Observe that the preceding discussion implies, in particular, that there is a map $q_k : \Omega J_{p^k-1}(S^{2p^n}) \to \mathrm{BGL}_1(y(n))$ such that the composite $S^{2p^n-1} \to \Omega J_{p^k-1}(S^{2p^n}) \to \mathrm{BGL}_1(y(n))$ detects $v_n \in \pi_{2p^n-2} y(n)$, and such that the Thom spectrum of the map q_k is $y(n+k)$. Taking $k \to \infty$, this implies that there is a map $q_\infty : \Omega^2 S^{2p^n+1} \to \mathrm{BGL}_1(y(n))$ whose Thom spectrum is $y(\infty) = \mathbf{F}_p$. The map q_∞ is adjoint to the $\mathbf{E}_1$-map $\Omega^3 S^{2p^n+1}_+ \to y(n)$ from [**MRS01**, Section 4.1] which detects v_n on the bottom cell of the source.

Appendix A. Analogues for ko and tmf

Many of the results from the body of this article extend to the case of ko and tmf. In this section, we will state these results; since the proofs are essentially the same, we will not give arguments unless the situation is substantially different. We will specialize to the case $p = 2$ for simplicity. One of the main observations in Theorem 2.2.4 is that the structure of $\mathrm{BP}\langle n\rangle$ as an $\mathbf{E}_1$-algebra over $X(p^n)$ (or rather, $T(n)$) mirrors the structure of $\mathbf{Z}_p$ as an $\mathbf{E}_1$-algebra over S^0. For ko and tmf, there are analogues of $X(p^n)$, which we studied in [**Dev23a**].

Recollection A.1. Let A denote the free $\mathbf{E}_1$-algebra $S/\!\!/\nu$ with a nullhomotopy of ν, i.e., the Thom spectrum of the $\mathbf{E}_1$-map $\Omega S^5 \to \mathrm{BGL}_1(S)$ which detects $\nu \in \pi_3(S)$ on the bottom cell[21]. This spectrum has the property that $\mathrm{H}_*(A; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4]$ (in fact, $\mathrm{BP}_*(A) \cong \mathrm{BP}_*[\frac{\eta_R(v_1^2)-v_1^2}{4}] \cong \mathrm{BP}_*[t_1^2 + v_1t_1]$). There is an $\mathbf{E}_1$-map $i : A \to \mathrm{ko}$ such that under the isomorphism $\mathrm{H}_*(\mathrm{ko}; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \cdots]$, the map i corresponds to the inclusion of $\mathbf{F}_2[\zeta_1^4]$. In particular, the map $A \to \mathrm{ko}$ is an equivalence in dimensions ≤ 4. There is in fact an $\mathbf{E}_1$-map $A \to \mathrm{MSpin}$, induced from an $\mathbf{E}_1$-map $\Omega S^5 \to \mathrm{BSpin}$. There is also an $\mathbf{E}_1$-map $A \to X(2) = T(1)$, such that $T(1) \simeq A \otimes C\eta$. We note that the "$Q_0$-Margolis homology" of $\mathrm{H}_*(\mathrm{ko}; \mathbf{F}_2)$ (i.e., the homology of Sq^1 viewed as a differential acting on $\mathrm{H}_*(\mathrm{ko}; \mathbf{F}_2)$) is precisely $\mathrm{H}_*(A; \mathbf{F}_2)$.

Similarly, let B denote the $\mathbf{E}_1$-algebra of [**Dev23a**, Definition 3.2.17][22], so that there is an $\mathbf{E}_1$-space N such that B is the Thom spectrum of an $\mathbf{E}_1$-map $N \to \mathrm{BString}$. We will not recall the construction of N here; we only say that B is obtained from the $\mathbf{E}_1$-quotient $S/\!\!/\sigma$ by further taking an "$\mathbf{E}_1$-quotient" by the class in $\pi_{11}(S/\!\!/\sigma)$ constructed from a nullhomotopy of $\nu\sigma \in \pi_{10}(S)$. This spectrum has the property that $\mathrm{H}_*(B; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^8, \zeta_2^4]$ (in fact, $\mathrm{BP}_*(B) \cong \mathrm{BP}_*[b_4, y_6]$, where $b_4 \equiv t_1^4$ and $y_6 \equiv t_2^2$ modulo $(2, v_1, \cdots)$)[23]. There is an $\mathbf{E}_1$-map $i : B \to \mathrm{tmf}$ such that under the isomorphism $\mathrm{H}_*(\mathrm{tmf}; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \cdots]$, the map i corresponds to the inclusion of $\mathbf{F}_2[\zeta_1^8, \zeta_2^4]$. In particular, the map $B \to \mathrm{tmf}$ is an equivalence in dimensions ≤ 12. There is in fact an $\mathbf{E}_1$-map $B \to \mathrm{MString}$. There is also an $\mathbf{E}_1$-map $B \to T(2)$ such that $T(2) \simeq B \otimes DA_1$, where DA_1 is an 8-cell complex whose mod 2 cohomology is isomorphic to the subalgebra of the Steenrod algebra generated by Sq^2 and Sq^4. We note that the "Q_0-Margolis homology" of $\mathrm{H}_*(\mathrm{tmf}; \mathbf{F}_2)$ (i.e., the homology of Sq^1 viewed as a differential acting on $\mathrm{H}_*(\mathrm{tmf}; \mathbf{F}_2)$) is precisely $\mathrm{H}_*(B; \mathbf{F}_2)$.

The following was implicitly stated in [**Dev23a**], but we make it explicit here:

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[21] The spectrum A has been studied before by Mahowald and his coauthors in [**Mah79, DM81, Mah81b, Mah81a, Mah82, MU77**], where it is often denoted X_5.

[22] The $\mathbf{E}_1$-ring B has been briefly studied under the name $\overline{X}$ in [**HM02**].

[23] For the sake of illustration, we remark that if $p = 2$, then b_4 can be taken to be the following cobar representative for $\sigma = \alpha_{4/4}$, where the v_is are Hazewinkel's generators:

$$\begin{aligned} b_4 &= \frac{1}{2}\left(\frac{\eta_R(v_1^4) - v_1^4}{8} - (\eta_R(v_1v_2) - v_1v_2)\right) \\ &= 5t_1^4 + 9t_1^3v_1 + 7t_1^2v_1^2 - 2t_1t_2 + 2t_1v_1^3 - t_1v_2 - t_2v_1. \end{aligned}$$

Here, we used the formula

$$\eta_R(v_2) = v_2 - 5v_1t_1^2 - 3v_1^2t_1 + 2t_2 - 4t_1^3.$$

Conjecture A.2. *The $\mathbf{E}_1$-algebra structures on A and B admit extensions to $\mathbf{E}_2^{\mathrm{fr}}$-algebra structures such that the maps $A \to X(2)$, $B \to X(4)_{(2)}$, $A \to \mathrm{ko}$, and $B \to \mathrm{tmf}$ admit the structure of $\mathbf{E}_2^{\mathrm{fr}}$-maps.*

A calculation paralleling Proposition 2.2.14 shows:

Proposition A.3. *Assume Conjecture A.2. There are isomorphisms*

$$\mathrm{H}_*(\mathrm{THH}(\mathrm{ko}/A);\mathbf{F}_2) \cong \mathrm{H}_*(\mathrm{ko};\mathbf{F}_2)[\sigma(\zeta_3)] \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma(\zeta_2^2)),$$
$$\mathrm{H}_*(\mathrm{THH}(\mathrm{tmf}/B);\mathbf{F}_2) \cong \mathrm{H}_*(\mathrm{tmf};\mathbf{F}_2)[\sigma(\zeta_4)] \otimes_{\mathbf{F}_2} \Lambda_{\mathbf{F}_2}(\sigma(\zeta_3^2)).$$

Here, $|\sigma(\zeta_3)| = 8$, $|\sigma(\zeta_2^2)| = 7$, $|\sigma(\zeta_4)| = 16$, and $|\sigma(\zeta_3^2)| = 15$.

Using the Adams spectral sequence for $\pi_*\mathrm{THH}(\mathrm{ko}/A)$ and $\pi_*\mathrm{THH}(\mathrm{tmf}/B)$ as in Theorem 2.2.4(b) (and using ko- and tmf-linearity), one finds:

Theorem A.4. *Assume Conjecture A.2. Upon 2-completion, there are equivalences*

$$\mathrm{THH}(\mathrm{ko}/A) \simeq \mathrm{ko} \oplus \bigoplus_{j\geq 1} \Sigma^{8j-1}\mathrm{ko}/2j,$$
$$\mathrm{THH}(\mathrm{tmf}/B) \simeq \mathrm{tmf} \oplus \bigoplus_{j\geq 1} \Sigma^{16j-1}\mathrm{tmf}/2j.$$

Remark A.5. Since $\mathrm{ko}\otimes C\eta \simeq \mathrm{ku}$, Theorem A.4 implies that $\mathrm{THH}(\mathrm{ko}/A)\otimes C\eta \simeq \mathrm{THH}(\mathrm{ku}/T(1))$. Relatedly, there is an equivalence $\mathrm{ko}\otimes T(1) \simeq \mathrm{ku}[\Omega S^5]$ of $\mathbf{E}_1$-$T(1)$-algebras, which implies that

$$\mathrm{THH}(\mathrm{ko}) \otimes T(1) \simeq \mathrm{THH}(\mathrm{ko}\otimes T(1)/T(1)) \simeq \mathrm{ku}[S^5] \oplus \bigoplus_{j\geq 1} \Sigma^{8j-1}\mathrm{ku}[S^5]/2j.$$

Along similar lines, Theorem A.4 implies that $\mathrm{THH}(\mathrm{tmf}/B)\otimes DA_1 \simeq \mathrm{THH}(\mathrm{BP}\langle 2\rangle/T(2))$. There is also a 2-local equivalence $\mathrm{tmf}\otimes T(2) \simeq \mathrm{BP}\langle 2\rangle[N]$ of $\mathbf{E}_1$-$T(2)$-algebras, so that

$$\mathrm{THH}(\mathrm{tmf}) \otimes T(2) \simeq \mathrm{BP}\langle 2\rangle[BN] \oplus \bigoplus_{j\geq 1} \Sigma^{16j-1}\mathrm{BP}\langle 2\rangle[BN]/2j.$$

Note that $\mathrm{BP}\langle 2\rangle[N] \simeq \mathrm{BP}\langle 2\rangle[\Omega S^9\times\Omega S^{13}]$, so that $\pi_*(\mathrm{tmf}\otimes T(2)) \cong \mathbf{Z}_{(2)}[v_1, v_2, x_8, y_{12}]$, where $|v_1| = 2$, $|v_2| = 6$, $|x_8| = 8$, and $|y_{12}| = 12$. This gives a potential approach to calculating THH(ko) (resp. THH(tmf)) via the $T(1)$-based (resp. $T(2)$-based) Adams-Novikov spectral sequence. Describing this spectral sequence is essentially equivalent to calculating the analogue of the topological Sen operator for $\mathrm{THH}(\mathrm{ko}/A)$, whose construction is described below in Construction A.9.

Remark A.6. Recall from Figure 1 that the structure of ko over A mirrors the structure of tmf over B, which in turn mirrors the structure of $\mathrm{BP}\langle n\rangle$ over $T(n)$; in other words, the calculation of Theorem A.4 is along the diagonal line (n, n) in Figure 1. It is natural to wonder whether there is an $\mathbf{E}_1$-ring $\widetilde{A}$ equipped with an $\mathbf{E}_1$-map $A \to \widetilde{A}$ and an $\mathbf{E}_1$-map $\widetilde{A} \to \mathrm{ko}$ such that the structure of ko over $\widetilde{A}$ mirrors the structure of $\mathrm{BP}\langle n-1\rangle$ over $T(n)$. (This is the "off-diagonal line" $(n, n-1)$ in Figure 1.) This question is only interesting when $p = 2$, since $\mathrm{ko}_{(p)}$ splits as a direct sum of even shifts of $\mathrm{BP}\langle 1\rangle$ if $p > 2$. Let us localize at 2 for the remainder of this discussion. Examining the argument establishing Theorem A.4 when $p = 2$, one finds that the mod 2 homology of $\widetilde{A}$ must be $\mathrm{H}_*(\widetilde{A};\mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4, \zeta_2^2]$.

If A admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring, then $\widetilde{A}$ can be constructed as follows. The class $\sigma_1 \in \pi_5(A)$ determined by a nullhomotopy of $\eta\nu$ (see [**Dev23a**, Remark

3.2.17]) defines a map $S^6 \to \mathrm{BGL}_1(A)$, which, thanks to our assumption on A, extends to an $\mathbf{E}_1$-map $\Omega S^7 \to \mathrm{BGL}_1(A)$. The desired $\mathbf{E}_1$-A-algebra $\widetilde{A}$ can be defined as Thom spectrum of this map. (According to [**Dev23a**, Remark 5.1.5], one should not expect $\widetilde{A}$ to admit a natural construction as a Thom spectrum over the sphere.) Note that $\widetilde{A} \otimes C\eta \simeq T(2)$.

The same argument as Theorem 2.2.4(a) shows:

Proposition A.7. *If both A and $\widetilde{A}$ admit the structure of $\mathbf{E}_2^{\mathrm{fr}}$-rings and* ko *admits the structure of an $\mathbf{E}_1$-$\widetilde{A}$-algebra, then there is a 2-complete equivalence*

$$\mathrm{THH}(\mathrm{ko}/\widetilde{A}) \simeq \mathrm{ko}[\Omega S^9],$$

where the generator in $\pi_8\mathrm{THH}(\mathrm{ko}/\widetilde{A})$ is $\sigma^2(v_2)$. Moreover,

$$\mathrm{H}^c_*(\mathrm{TP}(\mathrm{ko}/\widetilde{A}); \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3^2, \zeta_4, \cdots](\!(\hbar)\!). \tag{58}$$

Remark A.8. Similarly, if B admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring, the class $\sigma_2 \in \pi_{13}(B)$ from [**Dev23a**, Remark 3.2.24] defines a map $S^{14} \to \mathrm{BGL}_1(B)$. Thanks to our assumption on B, this extends to an $\mathbf{E}_1$-map $\Omega S^{15} \to \mathrm{BGL}_1(B)$. Define $\widetilde{B}$ to be Thom spectrum of this map, so that $\mathrm{H}_*(\widetilde{B}; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2]$. Note that $\widetilde{B} \otimes DA_1 \simeq T(3)$.

If $\widetilde{B}$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring and tmf admits the structure of an $\mathbf{E}_1$-$\widetilde{B}$-algebra, then the same argument as in Theorem 2.2.4(a) shows that there is a 2-complete equivalence

$$\mathrm{THH}(\mathrm{tmf}/\widetilde{B}) \simeq \mathrm{tmf}[\Omega S^{17}],$$

where the generator in $\pi_{16}\mathrm{THH}(\mathrm{tmf}/\widetilde{B})$ is $\sigma^2(v_3)$. Moreover,

$$\mathrm{H}^c_*(\mathrm{TP}(\mathrm{tmf}/\widetilde{B}); \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5, \cdots](\!(\hbar)\!). \tag{59}$$

Construction A.9 (Topological Sen operator for THH relative to A)**.** By [**BCS10**, Theorem 1], $\mathrm{THH}(A)$ is equivalent to the Thom spectrum of the composite

$$\mathcal{L}S^5 \xrightarrow{\mathcal{L}\nu} \mathcal{L}B^2\mathrm{GL}_1(S) \simeq B^2\mathrm{GL}_1(S) \times \mathrm{BGL}_1(S) \xrightarrow{\mathrm{id}\times\eta} \mathrm{BGL}_1(S).$$

There is a *nonsplit* fiber sequence

$$\Omega S^5 \to \mathcal{L}S^5 \to S^5, \tag{60}$$

and the restriction of the above composite along the map $\Omega S^5 \to \mathcal{L}S^5$ is the map $\Omega S^5 \to \mathrm{BGL}_1(S)$ which defines A. It follows from the fiber sequence (60) that $\mathrm{THH}(A)$ is the Thom spectrum of a map $S^5 \to \mathrm{BGL}_1(A)$ which detects a class $x \in \pi_4(A) \cong \pi_4(\mathrm{ko})$. In particular, $\mathrm{THH}(A)$ is an A-module with two cells. Now assume Conjecture A.2; then [**DHL**$^+$**23**, Corollary 2.8] gives a splitting $\mathrm{THH}(A) \to A$, which implies that the class $x \in \pi_4(A)$ must be trivial. In other words, $\mathrm{THH}(A) \simeq A[S^5]$. If $\mathcal{C}$ is an A-linear ∞-category, this implies the existence of a cofiber sequence

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C}/A) \xrightarrow{\Theta'} \Sigma^6\mathrm{THH}(\mathcal{C}/A). \tag{61}$$

One should be able to recover the calculation of THH(ko) from [**AHL10**] using (61) in the case $\mathcal{C} = \mathrm{Mod}_{\mathrm{ko}}$ and the calculation of Theorem A.4. Similarly to Remark 2.2.5, Theorem A.4 and (61) imply that

$$\mathrm{THH}(\mathrm{ko}/A)/2 \simeq \mathrm{ko}[S^7 \times \Omega S^9]/2,$$
$$\mathrm{THH}(\mathrm{ko}) \otimes_{\mathrm{ko}} \mathbf{F}_2 \simeq \mathbf{F}_2[S^5 \times S^7 \times \Omega S^9].$$

The latter of these has been proven by Angeltveit-Rognes in [**AR05**, Theorem 6.2].

Remark A.10. Recall from [**MR99**, Corollary 9.3] that Mahowald-Rezk duality gives an equivalence $W\mathrm{ko} \simeq \Sigma^6\mathrm{ko}$ (resp. $W\mathrm{BP}\langle 1\rangle \simeq \Sigma^{2p}\mathrm{BP}\langle 1\rangle$); the shift of 6 (resp. $2p$) in this equivalence arises for the same reason as in (61) (resp. (16) with $n = 1$): both correspond to the class $\sigma^2(t_1^2)$ (resp. $\sigma^2(t_1)$). We hope to explore this further in future work.

Remark A.11. There is also an analogue of the topological Sen operator for B. To describe it, one observes using an argument similar to Construction A.9 that $\mathrm{THH}(B/S/\!\!/\sigma) \simeq B[S^{13}]$ and that $\mathrm{THH}(S/\!\!/\sigma) \simeq (S/\!\!/\sigma)[S^9]$. This implies that if $\mathcal{C}$ is a B-linear ∞-category, there are cofiber sequences

$$\mathrm{THH}(\mathcal{C}/S/\!\!/\sigma) \to \mathrm{THH}(\mathcal{C}/B) \xrightarrow{\Theta_B} \Sigma^{14}\mathrm{THH}(\mathcal{C}/B),$$

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C}/S/\!\!/\sigma) \xrightarrow{\Theta'_B} \Sigma^{10}\mathrm{THH}(\mathcal{C}/S/\!\!/\sigma).$$

However, it is significantly more complicated to describe these cofiber sequences in almost any nontrivial example, so we omit further discussion. Nevertheless, one can use Theorem A.4 to show the following equivalences analogous to Remark 2.2.5:

$$\mathrm{THH}(\mathrm{tmf}/B)/2 \simeq \mathrm{tmf}[S^{15} \times \Omega S^{17}]/2,$$

$$\mathrm{THH}(\mathrm{tmf}) \otimes_{\mathrm{tmf}} \mathbf{F}_2 \simeq \mathbf{F}_2[S^9 \times S^{13} \times S^{15} \times \Omega S^{17}];$$

note that $\mathbf{F}_2[S^9 \times S^{13}] \simeq \mathbf{F}_2[BN]$. The latter of these has been proven by Angeltveit-Rognes in [**AR05**, Theorem 6.2].

Assume Conjecture A.2, and let $p = 2$. Then there is a map $\mathcal{M}_{T(1)} \to \mathcal{M}_A$ of stacks over $\mathcal{M}_{\mathrm{FG}}$, which exhibits $\mathcal{M}_{T(1)}$ as a 2-fold fppf cover of $\mathcal{M}_A$. Recall that $\mathcal{M}_{T(1)}$ is isomorphic to the moduli stack of graded formal groups equipped with a coordinate up to order ≤ 2 (equivalently, order ≤ 3 for 2-typical formal groups). Similarly, we have:

Proposition A.12. *The stack $\mathcal{M}_A$ is isomorphic to the moduli stack of graded formal groups equipped with an even coordinate up to order ≤ 5.*

PROOF. Recall that there is a fiber sequence

$$\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2 \to \mathrm{BU}(1) \simeq \mathbf{C}P^\infty \to \mathrm{BSU}(2) \simeq \mathbf{H}P^\infty.$$

Let $n \geq 1$. There is a homotopy equivalence $\mathbf{H}P^n \times_{\mathbf{H}P^\infty} \mathbf{C}P^\infty \simeq \mathbf{C}P^{2n+1}$ (since $S^{4n+3}/\mathrm{SU}(2) = \mathbf{H}P^n$ and $S^{4n+3}/\mathrm{U}(1) = \mathbf{C}P^{2n+1}$), which produces the "twistor fibration", i.e., the fiber sequence

$$S^2 \to \mathbf{C}P^{2n+1} \to \mathbf{H}P^n. \tag{62}$$

The map $\mathbf{C}P^{2n+1} \to \mathbf{H}P^n$ is given in coordinates by the map $[z_1 : \cdots : z_{2n+2}] \mapsto [z_1 + z_2\mathbf{j} : \cdots : z_{2n+1} + z_{2n+2}\mathbf{j}]$. Note that $\mathrm{SU}(2)/\mathrm{U}(1) = \mathbf{C}P^1$ is the unit sphere $S(\mathfrak{su}(2))$ in the adjoint representation of SU(2), so (62) equivalently says that $\mathbf{C}P^{2n+1}$ is the sphere bundle of the adjoint bundle of rank 3 over $\mathbf{H}P^n$.

Let R be a complex-oriented homotopy commutative ring with associated formal group $\mathbf{G}$ over $\pi_*(R)$; we will assume for simplicity that 2 is not a zero-divisor in π_*R. Then $R^*(\mathbf{C}P^5)$ is isomorphic to the ring of functions on $\mathbf{G}$ which vanish to order ≥ 6. The Serre spectral sequence associated to the fiber sequence (62) implies that $R^*(\mathbf{H}P^2)$ is isomorphic to $R^*(\mathbf{C}P^5)^{\mathbf{Z}/2}$, where $\mathbf{Z}/2$ acts by inversion on the formal group. This implies the desired claim, since there is an equivalence

$\mathbf{H}P^2 \simeq \Sigma^4 C\nu$ of spectra, and A is the free $\mathbf{E}_1$-ring whose unit factors through the inclusion $S^0 \to C\nu$. □

Remark A.13. The description of $\mathcal{M}_A$ in Proposition A.12 has concrete applications; for instance, in [**Dev22**], we show that $\mathcal{M}_{\mathrm{tmf}\otimes A} = \mathcal{M}_A \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}_{\mathrm{ell}}$ can be identified with the moduli stack of elliptic curves $\mathcal{E}$ equipped with a splitting of the Hodge filtration on $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E})$, and use this to describe an topological analogue of the integral ring of quasimodular forms.

Remark A.14. As explained in [**Dev23a**, Remark 7.1.7], there is a $\mathbf{Z}/2$-equivariant $\mathbf{E}_1$-algebra $A_{\mathbf{Z}/2}$ whose underlying $\mathbf{E}_1$-algebra is A, and such that $\Phi^{\mathbf{Z}/2} A_{\mathbf{Z}/2} = X(2)_{(2)}$ as $\mathbf{E}_1$-algebras. This is a topological interpretation of the following observation suggested by Proposition A.12: $\mathcal{M}_A$ is "half" of $\mathcal{M}_{X(2)}$; more precisely, there is a two-fold fppf cover $\mathcal{M}_{X(2)} \to \mathcal{M}_A$. This is an algebraic analogue of the equivalence $A \otimes C\eta \simeq X(2)$.

We also note that there is a $\mathbf{Z}/2$-equivariant analogue of the fiber sequence (62): namely, there is a $\mathbf{Z}/2$-equivariant twistor fibration

$$\begin{tikzcd} S^\rho \arrow[r] \arrow[d, "\cong"] & \mathbf{C}P^{2n-1}_{\mathbf{R}} \arrow[r] \arrow[d, "\cong"] & \mathbf{H}P^{n-1}_{\mathbf{R}} \arrow[d, "\cong"] \\ S^{2\rho-1}/S^\sigma \arrow[r] & S^{2n\rho-1}/S^\sigma \arrow[r] & S^{2n\rho-1}/S^{2\rho-1}. \end{tikzcd}$$

where $\mathbf{Z}/2$ acts on $\mathbf{H}P^n$ via the action of $\mathbf{Z}/2 \subseteq S^1 \subseteq \mathrm{SO}(3)$ on $\mathbf{H}$. The underlying fibration is (62), while the $\mathbf{Z}/2$-fixed points gives the fibration

$$S^1 \to \mathbf{R}P^{2n-1} \to \mathbf{C}P^{n-1}$$

which exhibits $\mathbf{R}P^{2n-1}$ as the sphere bundle of the complex line bundle $\mathcal{O}(2)$ on $\mathbf{C}P^{n-1}$.

Construction A.15. One consequence of the identification of $\mathcal{M}_A$ in Proposition A.12 is that $\mathcal{M}_A \to \mathcal{M}_{\mathrm{FG}}$ is an affine bundle, so that the pullback of the cotangent complex $L_{\mathcal{M}_A/\mathcal{M}_{\mathrm{FG}}}$ to $\mathrm{Spec}(\mathrm{BP}_*(A))/\mathbf{G}_m \cong \mathrm{Spec}(\mathrm{BP}_*[t_1^2 + v_1 t_1])/\mathbf{G}_m$ can be identified with a free $\mathrm{BP}_*[t_1^2 + v_1 t_1]$-module of rank 1 generated by the class $d(t_1^2 + v_1 t_1)$ in weight 2. Using Recollection 4.1.9, we obtain the algebraic analogue of (61): if X is a stack over $\mathcal{M}_A$, there is a cofiber sequence

$$\mathrm{HH}(X/\mathcal{M}_{\mathrm{FG}}) \to \mathrm{HH}(X/\mathcal{M}_A) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{6,3}\mathrm{HH}(X/\mathcal{M}_A).$$

The stack $\mathcal{M}_{\mathrm{ko}}$ can be identified with the moduli stack of curves of the form $y = x^2 + bx + c$ with change of coordinate $x \mapsto x + r$, and $\mathrm{HH}(\mathcal{M}_{\mathrm{ko}}/\mathcal{M}_{\mathrm{FG}})$ describes the E_1-page of the Adams-Novikov-Bökstedt spectral sequence calculating THH(ko) (see Remark 4.1.5). It would be interesting to explicitly describe $\mathrm{HH}(\mathcal{M}_{\mathrm{ko}}/\mathcal{M}_{\mathrm{FG}})$; note that

$$\pi_*\mathrm{HH}(\mathcal{M}_{\mathrm{ko}}/\mathcal{M}_A) \cong \mathcal{O}_{\mathcal{M}_{\mathrm{ko}}}\langle \sigma^2 v_j | j \geq 2\rangle \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{ko}}}} \Lambda_{\mathcal{O}_{\mathcal{M}_{\mathrm{ko}}}}(dt_i | i \geq 2),$$

where $\sigma^2(v_j)$ lives in degree 2^{j+1} and weight 2^j, and dt_i lives in degree $2^{i+1} - 1$ and weight 2^i. This can be proved exactly as in Example 4.1.8; weight considerations presumably allow one to fully describe $\Theta_{\mathrm{mot}} : \mathrm{HH}(X/\mathcal{M}_A) \to \Sigma^{6,3}\mathrm{HH}(X/\mathcal{M}_A)$, and hence $\mathrm{HH}(\mathcal{M}_{\mathrm{ko}}/\mathcal{M}_{\mathrm{FG}})$.

Remark A.16. Assume Conjecture A.2, and let $p = 2$. It is trickier to describe the stack $\mathcal{M}_B$ in a manner analogous to Proposition A.12. As a first approximation, if we assume that $S/\!\!/\sigma$ admits the structure of a homotopy commutative ring, one can attempt to describe the moduli stack $\mathcal{M}_{S/\!\!/\sigma}$. However, it is provably impossible to construct a Hurewicz fibration

$$S^4 \to \mathbf{H}P^5 \to \mathbf{O}P^2$$

in the point-set category. This is a consequence of [**Sch81**, Theorem 5.1], which states more generally that if $F \to E \to X$ is a Hurewicz fibration where E is homotopy equivalent to $\mathbf{H}P^{2n+1}$ and F and X are homotopy equivalent to finite CW-complexes, then either F or X must be contractible. Note that this result implies that there cannot even be a Hurewicz fibration

$$S^4 \to \mathbf{H}P^3 \to S^8.$$

Similarly, there cannot be Hurewicz fibrations

$$\mathbf{C}P^3 \to \mathbf{C}P^7 \to S^8,$$
$$\mathbf{C}P^3 \to \mathbf{C}P^{11} \to \mathbf{O}P^2;$$

see [**LV94**] for the impossibility of the first Hurewicz fibration (which implies the impossibility of the second Hurewicz fibration). These no-go results make it difficult to give a formal group-theoretic description of $R^*(\mathbf{O}P^2)$ (and hence of $\mathcal{M}_{S/\!\!/\sigma}$, since $\mathbf{O}P^2 \simeq \Sigma^8 C\sigma$) where R is a complex-oriented homotopy commutative ring.

The story for ko admits a slightly different generalization to higher heights.

Example A.17. Observe that $S^5 = \mathrm{SU}(4)/\mathrm{Sp}(2)$, and that the map $\Omega S^5 \to \mathrm{BU}$ (whose Thom spectrum is A) can be viewed as the composite

$$\Omega(\mathrm{SU}(4)/\mathrm{Sp}(2)) \to \Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp} \to \mathrm{BU}.$$

The equivalence $\Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp}$ is given by Bott periodicity, and the map $\mathrm{BSp} \to \mathrm{BU}$ takes a symplectic bundle to its underlying unitary bundle.

Motivated by Example A.17, we are led to the following definition:

Definition A.18. Define an $\mathbf{E}_1$-algebra $X_{\mathbf{H}}(n)$ via the Thom spectrum of the composite

$$\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)) \to \Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp} \to \mathrm{BU}.$$

There is a canonical $\mathbf{E}_1$-map $X_{\mathbf{H}}(n) \to X_{\mathbf{H}}(\infty) = \mathrm{MSp}$.

The spectrum $X_{\mathbf{H}}(n)$ has been studied by Andy Baker.

Remark A.19. See [**CM88**] for a detailed study of the space $\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n))$. Let us note that if $\mathrm{SU}(2n)_{\mathbf{H}}$ denotes the $\mathbf{Z}/2$-equivariant loop space with the $\mathbf{Z}/2$-action given by the symplectic involution

$$A \mapsto J\overline{A}J^{-1},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus n},$$

then $\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)) \simeq (\Omega^\sigma \mathrm{SU}(2n)_{\mathbf{H}})^{\mathbf{Z}/2}$. Indeed, the fixed points of $\mathrm{SU}(2n)_{\mathbf{H}}$ is $\mathrm{Sp}(n)$ (by definition), so we can apply the first sentence of Example 3.3.20 to conclude.

Normed division algebra	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$
$\mathbf{E}_1$-ring	$y(n) \subseteq \Phi^{\mathbf{Z}/2} X(2^n)_{\mathbf{R}}$	$T(n) \subseteq X(2^n)$	$X_{\mathbf{H}}(2^n)$
Limit as $n \to \infty$	$\mathbf{F}_2 \subseteq \Phi^{\mathbf{Z}/2}\mathrm{MU}_{\mathbf{R}} = \mathrm{MO}$	$\mathrm{BP} \subseteq \mathrm{MU}$	MSp
Mod 2 homology	$\mathbf{F}_2[\zeta_1, \cdots, \zeta_n]$	$\mathbf{F}_2[\zeta_1^2, \cdots, \zeta_n^2]$	$\mathbf{F}_2[\zeta_1^4, \cdots, \zeta_n^4]$

TABLE 2. The analogies between $y(n)$, $T(n)$, and $X_{\mathbf{H}}(n)$, where the implicit prime is $p = 2$. The inclusion $y(n) \subseteq \Phi^{\mathbf{Z}/2} X(2^n)_{\mathbf{R}}$ is discussed in Example 3.3.20; see [**Yan92**]. The final row is to be interpreted as follows: $\mathrm{H}_*(\Phi^{\mathbf{Z}/2} X(2^n)_{\mathbf{R}}; \mathbf{F}_2)$ is a direct sum of even shifts of $\mathbf{F}_2[\zeta_1, \cdots, \zeta_n]$; similarly for $\mathrm{H}_*(X(2^n); \mathbf{F}_2)$ and $\mathrm{H}_*(X_{\mathbf{H}}(2^n); \mathbf{F}_2)$.

Remark A.20. As the notation indicates, $X_{\mathbf{H}}(n)$ should be viewed as a quaternionic analogue of the $X(n)$ spectra from [**Rav84**]; see Table 2.

Note that there are isomorphisms of algebras

$$\mathrm{H}_*(\Phi^{\mathbf{Z}/2} X(n)_{\mathbf{R}}; \mathbf{F}_2) \cong \mathrm{H}_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)); \mathbf{F}_2) \cong \mathbf{F}_2[x_1, \cdots, x_{n-1}],$$
$$\mathrm{H}_*(X_{\mathbf{H}}(n); \mathbf{Z}) \cong \mathrm{H}_*(\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)); \mathbf{Z}) \cong \mathbf{Z}[y_1, \cdots, y_{n-1}],$$

where $|x_j| = j$ and $|y_j| = 4j$.

Example A.21. By construction, $X_{\mathbf{H}}(2) \simeq A = S /\!\!/ \nu$.

Construction A.22. Suppose that $X_{\mathbf{H}}(n)$ admits the structure of a homotopy commutative ring. One can then also ask for an interpretation of the stack $\mathcal{M}_{X_{\mathbf{H}}(n)}$ analogous to Proposition A.12. It turns out that the difficulties of Remark A.16 are no longer an issue for $X_{\mathbf{H}}(n)$. Indeed, the analogue of the map $S^4 \to \Omega S^5$ (whose Thomification is the map $C\nu \to A$ used in the proof of Proposition A.12) is given by a map $\iota : \mathbf{H}P^{n-1} \to \Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n))$ which exhibits $\mathbf{H}P^{n-1}$ as the generating complex of $\Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n))$. (See, e.g., [**CM88**, Proposition 1.4].) Moreover, the composite

$$\mathbf{H}P^{n-1} \xrightarrow{\iota} \Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)) \to \Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp}$$

factors as $\mathbf{H}P^{n-1} \hookrightarrow \mathbf{H}P^\infty \simeq \mathrm{BSp}(1) \to \mathrm{BSp}$. Since the Thom spectrum of the tautological quaternionic line bundle over $\mathbf{H}P^{n-1}$ is $\Sigma^{-4}\mathbf{H}P^n$, the map ι Thomifies to a map $\Sigma^{-4}\mathbf{H}P^n \to X_{\mathbf{H}}(n)$.

Using the twistor fibration (62) and the map $\Sigma^{-4}\mathbf{H}P^n \to X_{\mathbf{H}}(n)$ of Construction A.22, one can argue as in Proposition A.12 to show:

Proposition A.23. *The stack $\mathcal{M}_{X_{\mathbf{H}}(n)}$ is isomorphic to the moduli stack of graded formal groups equipped with an even coordinate up to order $\leq 2n+1$.*

Remark A.24. Suppose that $X_{\mathbf{H}}(n)$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring. There is also a canonical map $\mathcal{M}_{X_{\mathbf{H}}(n-1)} \to \mathcal{M}_{X_{\mathbf{H}}(n)}$ which exhibits $\mathcal{M}_{X_{\mathbf{H}}(n)}$ as the quotient of $\mathcal{M}_{X_{\mathbf{H}}(n-1)}$ by the group scheme $\mathbf{G}_a^{(2n-2)}/\mathbf{G}_m$ over $B\mathbf{G}_m$, where $\mathbf{G}_a^{(2n-2)}$ denotes the affine line with $\mathbf{G}_m$-action of weight $2n-2$. This is the algebraic analogue of the following:

Lemma A.25. *The spectrum $X_{\mathbf{H}}(n)$ is equivalent to the Thom spectrum of a map $\Omega S^{4n-3} \to \mathrm{BGL}_1(X_{\mathbf{H}}(n-1))$.*

Proof. By [**Dev23a**, Proposition 2.1.6] (see also [**Bea17**]), it suffices to establish that there is a fiber sequence of $\mathbf{E}_1$-spaces

$$\Omega(\mathrm{SU}(2n-2)/\mathrm{Sp}(n-1)) \to \Omega(\mathrm{SU}(2n)/\mathrm{Sp}(n)) \to \Omega S^{4n-3}.$$

To see this, observe that there is a diffeomorphism $\mathrm{SU}(2n)/\mathrm{Sp}(n) \cong \mathrm{SU}(2n-1)/\mathrm{Sp}(n-1)$, and hence a fibration

$$\begin{tikzcd}
\mathrm{SU}(2n-2)/\mathrm{Sp}(n-1) \arrow[r] \arrow[d, equal] & \mathrm{SU}(2n-1)/\mathrm{Sp}(n-1) \arrow[r] \arrow[d, "\cong"] & \mathrm{SU}(2n-1)/\mathrm{SU}(2n-2) \arrow[d, "\cong"] \\
\mathrm{SU}(2n-2)/\mathrm{Sp}(n-1) \arrow[r] & \mathrm{SU}(2n)/\mathrm{Sp}(n) \arrow[r] & S^{4n-3}.
\end{tikzcd}$$

The desired fiber sequence is obtained by looping the bottom row. □

Remark A.26. On the bottom cell of the source, the map $\Omega S^{4n-3} \to \mathrm{BGL}_1(X_{\mathbf{H}}(n-1))$ defines a class $\chi_n^{\mathbf{H}} \in \pi_{4n-5} X_{\mathbf{H}}(n-1)$, and $\chi_{2^n}^{\mathbf{H}}$ is detected in the E_2-page of the Adams-Novikov spectral sequence for $X_{\mathbf{H}}(2^n-1)$ by $[t_n^2]$. Moreover, if $X_{\mathbf{H}}(n-1)$ admits the structure of an $\mathbf{E}_2^{\mathrm{fr}}$-ring and $X_{\mathbf{H}}(n)$ admits the structure of an $\mathbf{E}_1$-$X_{\mathbf{H}}(n-1)$-algebra, then $\mathrm{THH}(X_{\mathbf{H}}(n)/X_{\mathbf{H}}(n-1)) \simeq X_{\mathbf{H}}(n)[\Omega S^{4n-3}]$.

We can then conclude (as in Theorem 3.1.4 and Example 4.1.11) that if $\mathcal{C}$ is an $X_{\mathbf{H}}(n)$-linear ∞-category and X is a stack over $\mathcal{M}_{X_{\mathbf{H}}(n)}$, then there are cofiber sequences

$$\mathrm{THH}(\mathcal{C}/X_{\mathbf{H}}(n-1)) \to \mathrm{THH}(\mathcal{C}/X_{\mathbf{H}}(n)) \xrightarrow{\Theta'} \Sigma^{4n-2}\mathrm{THH}(\mathcal{C}/X_{\mathbf{H}}(n)),$$

$$\mathrm{HH}(X/\mathcal{M}_{X_{\mathbf{H}}(n-1)}) \to \mathrm{HH}(X/\mathcal{M}_{X_{\mathbf{H}}(n)}) \xrightarrow{\Theta_{\mathrm{mot}}} \Sigma^{4n-2,2n-1}\mathrm{HH}(X/\mathcal{M}_{X_{\mathbf{H}}(n)}).$$

Only the first cofiber sequence requires that $X_{\mathbf{H}}(n-1)$ and $X_{\mathbf{H}}(n)$ admit the structure of $\mathbf{E}_2^{\mathrm{fr}}$-rings, and that $X_{\mathbf{H}}(n)$ admits the structure of an $\mathbf{E}_1$-$X_{\mathbf{H}}(n-1)$-algebra; the second cofiber sequence only requires that $X_{\mathbf{H}}(n)$ admit the structure of a homotopy commutative ring.

Appendix B. Alternative calculation of $\widehat{\Omega}^{\not{D}}_{\mathbf{Z}/p^n}$

In this brief section, we give an alternative algebraic argument for Corollary 3.2.15 following [**BL22b**, Example 5.15]. I am very grateful to Sasha Petrov for an illuminating discussion about this entire appendix; see also [**Pet23**, Lemma 6.13].

Alternative proof of Corollary 3.2.15. Let R be a (discrete) commutative $\mathbf{Z}/p^n$-algebra. Then [**BL22b**, Construction 3.8] implies that

$$\begin{aligned}\mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}}(R) &\simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathbf{Z}/p^n, W(R)/V(1))\\ &\simeq \{z \in W(R) | zV(1) = p^n\} = \{x \in W(R) | V(Fz) = p^n\}.\end{aligned}$$

Since V is injective, this is a torsor for $W[F](R) = \mathbf{G}_a^\sharp(R)$. Moreover, this torsor is trivializable, i.e., p^n is in the image of VF. In fact, we claim that

$$p^n = V(p^{n-1}) = VF(V(p^{n-2})) \in W(\mathbf{Z}/p^n). \tag{63}$$

To see this, let us compute in ghost coordinates. Recall that if $w(x) = (w_0(x), w_1(x), \cdots)$ are the ghost coordinates of $x \in W(R)$, then $w_{n+1}(Vx) = pw_n(x)$. Since $w(p^n) = (p^n, p^n, \cdots)$ and $w(V(p^{n-1})) = (0, p^n, p^n, \cdots)$, we see that

$$w(p^n - V(p^{n-1})) = (p^n, 0, 0, \cdots).$$

Since the map $\mathbf{G}_a^\sharp \cong W[F] \to W$ sends $x \in \mathbf{G}_a^\sharp$ to the Witt vector whose ghost coordinates are $(x, 0, 0, \cdots)$, the claim (63) follows from the observation that $p^n \in W[F](\mathbf{Z}_p)$ is sent to zero in $W[F](\mathbf{Z}/p^n)$. □

Remark B.1. As pointed out by Sasha Petrov, the preceding calculation also determines the $\mathbf{G}_m^\sharp$-action on $\mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}}$ as follows. The above discussion says that the isomorphism $\mathbf{G}_a^\sharp \xrightarrow{\sim} \mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}}$ sends $x \mapsto x + V(p^{n-2})$. Under this isomorphism, the action of $g \in \mathbf{G}_m^\sharp$ on $x + V(p^{n-2}) \in \mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}}$ is given by

$$g(x + V(p^{n-2})) = gx + gV(p^{n-2}) = gx + V(F(g)p^{n-2});$$

but $F(g) = 1$ since $\mathbf{G}_m^\sharp = W^\times[F]$, so that this can be identified with $gx + V(p^{n-2})$. In other words, the isomorphism $\mathbf{G}_a^\sharp \xrightarrow{\sim} \mathrm{Spec}(\mathbf{Z}/p^n)^{\not{D}}$ is equivariant for the scaling action of $\mathbf{G}_m^\sharp$ on $\mathbf{G}_a^\sharp$.

One can get a formula which is more "accurate" than (63) via the following (see also [**Ill22**, Page 56], where part of this statement is attributed to Gabber)[24].

Lemma B.2. *Let y denote the element of $W(\mathbf{Z}_p)$ associated to the ghost coordinates $(1 - p^{p-1}, 1 - p^{p^2-1}, \cdots)$. Then $[p] + V(y) = p$. Moreover, $y = Fx$ for some $x \in W(\mathbf{Z}_p)$ if and only if $p > 2$; in this case, $x \in W(\mathbf{Z}_p)^\times$ (and hence $y \in W(\mathbf{Z}_p)^\times$). If $p = 2$, then $y[2^m]$ is in the image of F for any $m \geq 2$.*

Remark B.3. Let us assume p is odd for simplicity. Then Lemma B.2 implies that $p - [p] \in W(\mathbf{Z}_p)$ is a unit multiple of $V(1)$, since $p - [p] = V(y) = xV(1)$ and $x \in W(\mathbf{Z}_p)^\times$.[25] It follows from [**BL22b**, Construction 3.8] that if $X = \mathrm{Spf}(R)$ is a bounded p-adic formal scheme, then the diffracted Hodge complex $X^{\not{D}}$ is given on p-nilpotent rings S by $X^{\not{D}}(S) = X(W(R)/(p - [p]))$.

[24]Our understanding is that this result is quite well-known; some form is heavily used in [**BMS18**].

[25]Analogously, $[2](2 - [2]) = [2]V(y) = V(y[4]) \in W(\mathbf{Z}_p)$ is divisible by $V(1)$ for $p = 2$.

Remark B.4. Applying F to the identity $[p]+V(y)=p$, we see that $[p^2]=p(1-y)$. In particular, the element $a\in W(\mathbf{Z}_p)$ of **[Dri22**, Lemma 4.7.3] can be identified with $1-y$.

Remark B.5. Using Lemma B.2, we can give an "alternative" formula for a preimage of p^n under VF. Indeed, we have $p=[p]+V(y)$ for some $y\in W(\mathbf{Z}_p)$, so that $p^n=[p^n]+\sum_{i=0}^{n-1}\binom{n}{i}[p^i]V(y)^{n-i}$ in $W(\mathbf{Z}_p)$. Because $V(a)b=V(aFb)$ and $FV=p$, we have $V(a)^n=V(p^{n-1}a^n)$ by an easy induction on n. Moreover, $[p^i]V(a)=V(aF[p^i])=V([p^{pi}]a)$. Since $[p^n]=0\in W(\mathbf{Z}/p^n)$ (and hence in $W(R)$), we have

$$p^n=\sum_{i=0}^{n-1}\binom{n}{i}[p^i]V(y)^{n-i}=\sum_{i=0}^{n-1}\binom{n}{i}V(p^{n-i-1}y^{n-i}F[p^i])\in W(\mathbf{Z}/p^n).$$

Assume $p>2$, so that Lemma B.2 implies that $y=Fx$ for some $x\in W(\mathbf{Z}_p)$. The multiplicativity of F now lets us conclude that

$$p^n=VF\left(\sum_{i=0}^{n-1}\binom{n}{i}p^{n-i-1}x^{n-i}[p^i]\right)\in W(\mathbf{Z}/p^n),$$

so that $p^n\in W(R)$ is in the image of VF, as desired.

One can check that

$$\sum_{i=0}^{n-1}\binom{n}{i}p^{n-i-1}y^{n-i}[p^{pi}]=p^{n-1}\in W(\mathbf{Z}/p^n).$$

This is essentially an elaboration on the proof of Lemma B.2. Indeed, applying w_j, we have

$$\begin{aligned} w_j\left(\sum_{i=0}^{n-1}\binom{n}{i}p^{n-i-1}y^{n-i}[p^{pi}]\right) &= \frac{1}{p}\sum_{i=0}^{n-1}\binom{n}{i}(p-p^{p^{j+1}})^{n-i}p^{p^{j+1}i} \\ &= p^{n-1}-p^{p^{j+1}n-1}. \end{aligned}$$

It therefore suffices to show that the Witt vector $a\in W(\mathbf{Z}_p)$ with coordinates $w_j(a)=p^{p^{j+1}n-1}$ vanishes in $W(\mathbf{Z}/p^n)$, which follows from a direct calculation.

Let us end with a proof of Lemma B.2; the explicit formulas below are unnecessary for any conceptual development, but we included it since the computation was rather fun.

Proof of Lemma B.2. First, it is easy to see that y is well-defined. Let us now check that $p=[p]+V(y)$. If $w(x)=(w_0(x),w_1(x),\cdots)$ are the ghost coordinates of $x\in W(R)$, then $w_{n+1}(Vx)=pw_n(x)$. It follows that $w_n(Vy)=p-p^{p^n}$. Since $w([p])=(p,p^p,p^{p^2},\cdots)$, we have

$$w([p]+Vy)=w([p])+w(Vy)=(p,p,\cdots)=w(p),$$

so that $p=[p]+V(y)$, as claimed.

To prove the claim about y being in the image of F, recall that if $x\in W(R)$, then the ghost coordinates of Fx are given by $w_n(Fx)=w_{n+1}(x)$. In particular, $y=Fx$ for some $x\in W(\mathbf{Z}_p)$ if and only if we can solve

$$1-p^{p^n-1}=x_0^{p^n}+px_1^{p^{n-1}}+\cdots+p^nx_n$$

for some $x_0, \cdots, x_n \in \mathbf{Z}_p$ and all $n \geq 1$. This is impossible for $p = 2$. Indeed, first note that we need $x_0^2 + 2x_1 = 1 - p^{p-1} = -1$, so that $x_0^2 \equiv 1 \pmod 2$ (and hence $x_0 \equiv 1 \pmod 2$). Write $x_0 = 1+2s$, so that $x_0^2+2x_1 = 1+4s(1+s)+2x_1$. In order for this to equal $1-p^{p-1} = -1$, we need $4s(1+s)+2x_1 = -2$, i.e., $x_1 \equiv 1 \pmod 2$. This implies that $x_0^4 \equiv 1 \pmod 8$ and $x_1^2 \equiv 1 \pmod 4$ (so $2x_1^2 \equiv 2 \pmod 8$). Since $1-p^{p^2-1} = -7 = x_0^4+2x_1^2+4x_2$, we can reduce modulo 8 to find that $1 \equiv 1+2+4x_2 \pmod 8$. But then x_2 would solve $4x_2 \equiv -2 \pmod 8$, which is impossible.

Now assume $p > 2$. Since $x_0^p + px_1 = 1 - p^{p-1}$, we have $x_0^p \equiv 1 \pmod p$; this implies that $x_0^{p^n} \equiv 1 \pmod{p^{n+1}}$. Writing $x_0^{p^n} = 1 - p^{n+1}s_n$ for some $s_n \in \mathbf{Z}_p$, we have $x_1 = ps_1 - p^{p-2}$. Since $p > 2$, we see that $x_1 = p(s_1 - p^{p-3}) \in p\mathbf{Z}_p$. We claim that x_n exists and is an element of $p\mathbf{Z}_p$ for $n \geq 1$. We established the base case $n = 1$ above, so assume that $x_1, \cdots, x_{n-1} \in p\mathbf{Z}_p$, and let $x_i = pt_i$. We then have

$$\begin{aligned} p^n x_n &= 1 - p^{p^n-1} - (x_0^{p^n} + px_1^{p^{n-1}} + \cdots + p^{n-1}x_{n-1}^p) \\ &= p^{n+1}s_n - p^{p^n-1} - p^{p^{n-1}+1}t_1^{p^{n-1}} - \cdots - p^{p+n-1}t_{n-1}^p, \end{aligned}$$

so that

$$x_n = ps_n - p^{p^n-1-n} - p^{p^{n-1}+1-n}t_1^{p^{n-1}} - \cdots - p^{p-1}t_{n-1}^p.$$

This is clearly divisible by p since $p > 2$ (so that $p^n - 1 - n \geq 1$ for $n \geq 1$). Therefore, x_n exists and lives in $p\mathbf{Z}_p$, as desired. (Note that if $p = 2$ and $n = 1$, then $p^n - 1 - n = 0$, so $x_1 \notin 2\mathbf{Z}_2$.) If one prefers an explicit formula, the above argument shows that once one writes $x_0 = 1 - ps_0$, then $x_j = pt_j$ for $j \geq 1$ can be defined inductively by

$$t_n = \sum_{i=1}^{p^n} \frac{(-1)^{i+1}}{p^{n+1-i}} \binom{p^n}{i} s_0^i - p^{p^n-2-n} - \sum_{k=1}^{n-1} p^{p^k-k-1} t_{n-k}^{p^k}.$$

The first term is s_n; note that $\frac{1}{p^{n+1-i}}\binom{p^n}{i} \in \mathbf{Z}$. Since $x_0 \equiv 1 \pmod p$ and $x_i \equiv 0 \pmod p$ for $i \geq 1$, it is easy to see that all the ghost components of x lie in $1 + p\mathbf{Z}_p \subseteq \mathbf{Z}_p^\times$; this implies that $x \in W(\mathbf{Z}_p)$ is invertible, as claimed.

Let us now assume that $p = 2$, and show that $y[2^m]$ is in the image of F for any $m \geq 2$. To see this, observe that the ghost components of $y[2^m]$ are given by

$$w_n(y[2^m]) = w_n(y)w_n([2^m]) = 2^{m2^n}(1 - 2^{2^{n+1}-1}).$$

We therefore need to solve

$$2^{m2^{n-1}}(1 - 2^{2^n-1}) = x_0^{2^n} + 2x_1^{2^{n-1}} + \cdots + 2^n x_n$$

for some $x_0, \cdots, x_n \in \mathbf{Z}_2$ and all $n \geq 1$. When $n = 1$, we have $x_0^2 + 2x_1 = -2^m$, so that $x_0^2 \equiv 0 \pmod 2$ since $m > 0$. It follows that $x_0 = 2t_0$ for some $t_0 \in \mathbf{Z}_2$. We now claim that x_n exists for $n \geq 0$ and lives in $2\mathbf{Z}_2$. We established the base case $n = 0$ above, so assume $x_0, x_1, \cdots, x_{n-1} \in 2\mathbf{Z}_2$, and write $x_i = 2t_i$. Then

$$\begin{aligned} 2^n x_n &= 2^{m2^{n-1}}(1 - 2^{2^n-1}) - (x_0^{2^n} + 2x_1^{2^{n-1}} + \cdots + 2^{n-1}x_{n-1}^2) \\ &= 2^{m2^{n-1}}(1 - 2^{2^n-1}) - (2^{2^n}t_0^{2^n} + 2^{2^{n-1}+1}t_1^{2^{n-1}} + \cdots + 2^{n+1}t_{n-1}^2), \end{aligned}$$

so that

$$x_n = 2^{m2^{n-1}-n}(1 - 2^{2^n-1}) - (2^{2^n-n}t_0^{2^n} + 2^{2^{n-1}+1-n}t_1^{2^{n-1}} + \cdots + 2t_{n-1}^2).$$

Because $m \geq 2$ and $2^j - j \geq 1$ for every $j \geq 0$, we see that $x_n \in 2\mathbf{Z}_2$, as desired. (Of course, the key case is $m = 2$; when $m = 1$ and $n = 1$, the term $2^{m2^{n-1}-n}(1 -$

$2^{2^n-1}) = -1 \notin 2\mathbf{Z}_2$.) If one prefers an explicit formula, note that the above argument shows that once one writes $x_0 = 2t_0$, then $x_j = 2t_j$ can be defined inductively by

$$t_n = 2^{m2^{n-1}-n-1}(1 - 2^{2^n-1}) - \sum_{i=1}^{n} 2^{2^i-i-1} t_{n-i}^{2^i}.$$

Note that x is *not* invertible in $W(\mathbf{Z}_2)$; instead, since $x_j \in 2\mathbf{Z}_2$, the nth ghost component $w_n(x) \in 2^{n+1}\mathbf{Z}_2$. $\square$

Appendix C. Cartier duals of $W[F^n]$ and $W^\times[F^n]$

This section was inspired by the results proved above, but it does not play an essential role in the body of this article. Corollary C.12 below can be viewed as an algebraic way to bookkeep the structure possessed by the topological Sen operators; and, as we hope to show in future work, it sits as an intermediary between the topological and algebraic Sen operators of Theorem 3.1.4 and Example 4.1.11 (see Remark C.19). We begin with the following (presumably well-known) result. I am (again) grateful to Sasha Petrov for a relevant discussion on it.

Proposition C.1. *There is an isomorphism of group schemes over $\mathbf{Z}_{(p)}$ between $W[F^n] := \ker(F^n : W \to W)$ and the Cartier dual of the completion of $W_n = W/V^n$ at the origin.*

Proof. Let us model W by the p-typical big Witt vectors. Given $f(t) \in W$, let $a_0, a_1, a_2, \cdots$ denote the ghost components of f, so that $t d\log(f(t)) = \sum_{m\geq 0} a_m t^{p^m}$. Then $f(t) \in W[F^n]$ if and only if $a_m = 0$ for $m \geq n$.

Let us first prove the claim of the proposition when $n = 1$. Then, $d\log(f(t))$ is a constant, and $f(0) = 1$; we claim that this is equivalent to the condition that f defines a homomorphism $\hat{\mathbf{G}}_a \to \mathbf{G}_m$, i.e., that $f(x+y) = f(x)f(y)$. To check this, first suppose that $f(x+y) = f(x)f(y)$. Then $\partial_x f(x+y) = f(y)f'(x)$, so that

$$\frac{\partial_x f(x+y)}{f(x+y)} = \frac{f'(x)}{f(x)} = d\log(f(x))$$

is independent of y. Taking $x = 0$, we see that $d\log(f(x))$ is constant, as desired. The reverse direction (that $d\log(f(x))$ being constant and $f(0) = 1$ implies that $f(x+y) = f(x)f(y)$) is similar.

In the general case, note that since the Frobenius on W shifts the ghost components by $F : (a_0, a_1, a_2, \cdots) \mapsto (a_1, a_2, a_3, \cdots)$, the Frobenius F applied to f satisfies:

$$d\log(F^j(f)(t)) = \sum_{m=0}^{n-j} a_{m+j} t^{p^m},$$

so that there is an equality of power series

$$F^j(f)(t) = \exp\left(\sum_{m=0}^{n-j} \frac{a_{m+j}}{p^m} t^{p^m}\right).$$

Note that this is a slight variant of the the Artin-Hasse exponential. Define a map $g : \hat{W}_n \to \mathbf{G}_m$ on Witt components $(x_0, \cdots, x_{n-1})$ (not ghost components!) as follows:

$$\begin{aligned} g(x_0, \cdots, x_{n-1}) = \prod_{j=0}^{n-1} F^j(f)(x_j) &= \exp\left(\sum_{j=0}^{n-1}\sum_{m=0}^{n-j} \frac{a_{m+j}}{p^m} x_j^{p^m}\right) \\ &= \exp\left(\sum_{m=0}^{n-1} \frac{a_m}{p^m}\left(\sum_{j=0}^{m} p^j x_j^{p^{m-j}}\right)\right). \end{aligned}$$

The coefficient of $\frac{a_m}{p^m}$ is precisely the mth Witt polynomial, so that the function g is indeed additive on $\hat{W}_n$. Moreover, the assignment $f \mapsto g$ indeed gives an isomorphism $W[F^n] \xrightarrow{\sim} \mathrm{Hom}(\hat{W}_n, \mathbf{G}_m)$, as one can check inductively using the case $n = 1$ and the fact that it induces an isomorphism over $\mathbf{Q}$. $\square$

Remark C.2 (Integral case)**.** One does not need p-typicality for the above statement to hold. Namely, if $\mathbf{W}$ denotes the big Witt ring scheme, it is a classical fact that the Cartier dual of $\mathbf{W}$ over $\mathbf{Z}$ is canonically identified with $\hat{\mathbf{W}}$. As in the p-typical case above, the pairing $\mathbf{W} \times \hat{\mathbf{W}} \to \mathbf{G}_m$ sends

$$(a, b) \mapsto \exp\left(\sum_{n \geq 1} \frac{w_n(a) w_n(b)}{n}\right).$$

One only needs to check that this expression is in fact defined over $\mathbf{Z}$. To see this, first observe that if the Witt components of b are $(b_1, b_2, \cdots)$, we have

$$\exp\left(\sum_{n \geq 1} \frac{w_n(a) w_n(b)}{n}\right) = \prod_{j \geq 1} \exp\left(\sum_{n \geq 1} \frac{w_{nj}(a) b_j^n}{n}\right).$$

Note that $w_{nj}(a) = w_n(F_j a)$, so that if $(F_j a)_d$ denote the Witt components of $F_j a$, we have

$$\exp\left(\sum_{n \geq 1} \frac{w_n(a) w_n(b)}{n}\right) = \prod_{j, d \geq 1} (1 - (F_j a)_d b_j^d),$$

giving the desired integral representation. In fact, the last step can be generalized via the following rephrasing of the Dwork lemma:

Lemma C.3. *Let R be a torsionfree ring equipped with ring maps $\phi_p : R \to R$ for each prime p such that $\phi_p(r) \equiv r^p \pmod{p}$ for all $r \in R$. Let $(x_n)_{n \geq 1}$ be a sequence of elements such that $x_n \equiv \phi_p(x_{n/p}) \pmod{p^{v_p(n)}}$ for each prime p and every $n \in p\mathbf{Z}_{\geq 0}$. Then $f(t) := \exp\left(\sum_{n \geq 1} \frac{x_n t^n}{n}\right)$ lies in $1 + tR[\![t]\!] \subseteq 1 + t(R \otimes \mathbf{Q})[\![t]\!]$.*

Proof. Let $g(t) = 1 - t \in R[\![t]\!]$, so that there is an identity

$$g(t) = \exp(\log(1 - t)) = \exp\left(-\sum_{n \geq 1} \frac{t^n}{n}\right).$$

Because $f(0) = 1$, we can write $f(t) = \prod_{j \geq 1}(1 - r_j t^j) = \prod_{j \geq 1} g(r_j t^j)$ for unique $r_j \in R \otimes \mathbf{Q}$. Since $g(t)$ is integral, it is sufficient to show that the elements r_j are also integral. Applying $d\log$, we find that

$$\sum_{n \geq 1} \frac{x_n t^n}{n} = d\log(f)(t) = \sum_{j \geq 1} d\log(g)(r_j t^j) = -\sum_{j, m \geq 1} \frac{r_j^m t^{jm}}{m}.$$

It follows that $x_n = -\sum_{j|n} j r_j^{n/j}$. One can now argue in exactly the same way as the usual Dwork lemma (i.e., by induction on $r_j \in R$ for $j|n$ with $j \neq n$) to argue that each r_j is integral. $\square$

Let us remark that the argument above can be used to show that if $\hat{\mathbf{W}}_n$ is the completion of the big Witt vectors of length n over $\mathbf{Z}$, the Cartier dual of $\hat{\mathbf{W}}_n$ can be identified with the subgroup of those $a \in \mathbf{W}$ such that $w_d(a) = 0$ if $d \nmid n$.

Corollary C.4. *Write the underlying scheme of W_n as $\prod_{i=0}^{n-1} \mathbf{G}_a$ (where the ith copy of $\mathbf{G}_a$ has coordinate Φ_i). There is a fully faithful functor $\mathrm{QCoh}(BW[F^n]) \hookrightarrow \mathrm{QCoh}(W_n)$ whose essential image consists of those p-complete $M \in \mathrm{QCoh}(W_n)$ such that Φ_i acts locally nilpotently on $\mathrm{H}^*(M/p)$ for each $0 \leq i \leq n-1$. Furthermore, this functor is symmetric monoidal for the convolution tensor product on $\mathrm{QCoh}(W_n)$.*

If $\mathcal{F} \in \mathrm{QCoh}(BW[F^n])$ is sent to $M \in \mathrm{QCoh}(W_n)$ under this functor, one obtains a cube[26] *$\Phi_\bullet : 2^{[n-1]} \to \mathrm{Mod}_{\mathbf{Z}_p}$ whose vertices are all M and such that the edge from the subset $\{i_1, \cdots, i_{j-1}\}$ to $\{i_1, \cdots, i_j\}$ is given by the operator $\Phi_j : M \to M$. Then, the global sections $\Gamma(BW[F^n]; \mathcal{F})$ can be identified with the total fiber of the cube $\Phi_\bullet$.*

Remark C.5. More generally, the argument of Proposition C.1 shows that there is an isomorphism of group schemes over $\mathbf{Z}_{(p)}$ between $W_m[F^n] := \ker(F^n : W_m \to W_m)$ and the Cartier dual of $W_n[F^m]$. One can give a simpler proof of this fact over a perfect field k of characteristic $p > 0$ using the theory of Dieudonné modules: the Dieudonné module of $W_m[F^n]$ over k is $W(k)[F, V]/(F^n, V^m)$, while the Dieudonné module of $W_n[F^m]$ over k is $W(k)[F, V]/(F^m, V^n)$.

The argument of Proposition C.1 also shows the following result; this also appears in [**Dri21**, Appendix D] and [**AHL22**, Section 2.2]:

Proposition C.6. *Let $\hat{\mathbf{G}}_\lambda$ be the degeneration of $\hat{\mathbf{G}}_m$ to $\hat{\mathbf{G}}_a$ given by $\mathrm{Spf}\, \mathbf{Z}_{(p)}[t, \lambda, \frac{1}{1+t\lambda}]^\wedge_t$ with group law $x + y + \lambda xy$. Then the $\mathbf{Z}_{(p)}[\lambda]$-linear Cartier dual of $\hat{\mathbf{G}}_\lambda$ is isomorphic*[27] *to the group scheme $\mathbf{D}(\hat{\mathbf{G}}_\lambda) = \mathrm{Spec}\, \mathbf{Z}_{(p)}[\lambda, z, \frac{\prod_{j=0}^{n-1}(z-j\lambda)}{n!}]$ over $\mathbf{A}^1_\lambda = \mathrm{Spec}\, \mathbf{Z}_{(p)}[\lambda]$ with coproduct $z \mapsto z \otimes 1 + 1 \otimes z$.*

Proof. A homomorphism $f : \hat{\mathbf{G}}_\lambda \to \mathbf{G}_m \times \mathbf{A}^1_\lambda$ is an element of $\mathbf{Z}_{(p)}[t, \lambda, \frac{1}{1+t\lambda}]^\wedge_t$ such that

$$f(x + y + \lambda xy) = f(x)f(y).$$

This condition implies that

$$(1 + \lambda y)f'(x + y + \lambda xy) = f(y)f'(x),$$

so that dividing both sides by $f(x + y + \lambda xy)$, we have

$$(1 + \lambda y) \cdot d\log(f)(x + y + \lambda xy) = d\log(f)(x).$$

Taking $x = 0$, we see that $d\log(f)(y)$ is a constant multiple of $\frac{1}{1+\lambda y}$ (where the constant is given by $\frac{f'(0)}{f(0)}$), and hence

$$f(y) = (1 + \lambda y)^{z/\lambda} = \sum_{n \geq 0} y^n \frac{\prod_{j=0}^{n-1}(z - j\lambda)}{n!}$$

---

[26] Recall that $[n]$ denotes the set $\{0, \cdots, n\}$.

[27] For convenience, write ku^{tS^1} to denote the p-completion of ku^{tS^1}. It is useful to note that if we set $\lambda = q - 1$ and complete at (p, λ), the formal group $\hat{\mathbf{G}}_\lambda$ over $\mathbf{Z}_p[\![q-1]\!]$ can be identified with the formal group over $\pi_0(\mathrm{ku}^{tS^1})$ induced by the canonical complex orientation of the 2-periodic $\mathbf{E}_\infty$-ring ku^{tS^1}. In other words, $\hat{\mathbf{G}}_\lambda = \mathrm{Spf}(\mathrm{ku}^{tS^1})^0(\mathbf{C}P^\infty)$, where $(\mathrm{ku}^{tS^1})^0(\mathbf{C}P^\infty)$ with equipped with the $(p, q-1, t)$-adic topology. This implies that its $\mathbf{Z}_p[\![q-1]\!]$-linear Cartier dual is $\mathrm{Spf}(\mathrm{ku}^{tS^1})_0(\mathbf{C}P^\infty)$, where $(\mathrm{ku}^{tS^1})_0(\mathbf{C}P^\infty)$ is equipped with the $(p, q-1)$-adic topology. The calculation of this proposition can be interpreted as calculating the algebra $(\mathrm{ku}^{tS^1})_0(\mathbf{C}P^\infty)$ equipped with its Pontryagin product.

for some fixed z. This gives the desired claim, similarly to Proposition C.1. Note that $f(y) = \exp(z\log_F(y))$, where $\log_F$ is the logarithm of the formal group law $x + y + \lambda xy$ over $\mathbf{A}^1_\lambda$. $\square$

Remark C.7. Observe that $\mathbf{D}(\hat{\mathbf{G}}_\lambda)$ is isomorphic to the subgroup $(W \times \mathbf{A}^1_\lambda)[F + [-\lambda]^{p-1}]$ of $W \times \mathbf{A}^1_\lambda$ cut out by $\{x|Fx = [-\lambda]^{p-1}x\}$; see [**MRT19**, Proposition 6.3.3] and [**Dri21**, Proposition D.4.10]. The key point is that if $f(x) \in W$ and $xd\log(f(x)) = \sum_{m\geq 1} a_m x^m$, then $f(t) \in (W \times \mathbf{A}^1_\lambda)[F + [-\lambda]^{p-1}]$ if and only if

$$a_{p^{n+1}} = ((-\lambda)^{p-1})^{p^n} a_{p^n} = (-\lambda)^{p^{n+1}-p^n} a_{p^n}.$$

To check this, note that

$$xd\log(f)(x) = \frac{zx}{1+\lambda x} = \sum_{n\geq 0} (-\lambda)^n z x^{n+1},$$

so that $a_m = (-\lambda)^{m-1}z$, and $a_m = (-\lambda)^{m-n}a_n$ if $m \geq n$.

Remark C.8. A similar argument shows that if $\mathbf{G}$ denotes the group scheme $\operatorname{Spec} \mathbf{Z}/p^N[\lambda]\langle x\rangle$ with group law $x + y + \lambda xy$ (so that when $\lambda = 0$, we get $\mathbf{G}_a^\sharp$), then the $\mathbf{Z}/p^N[\lambda]$-linear Cartier dual of $\mathbf{G}$ is isomorphic to the completion of $\operatorname{Spec} \mathbf{Z}/p^N[\lambda, z]$ at the locus $\prod_{j=0}^{p-1}(z - j\lambda) = z(z^{p-1} - \lambda^{p-1})$ (see, e.g., [**Dri21**, Section B.4]). It follows that the ∞-category of $\mathbf{G}$-representations is equivalent to the ∞-category of $\mathbf{Z}/p^N$-modules M equipped with an operator $z : M \to M$ such that $z(z^{p-1} - \lambda^{p-1})$ acts locally nilpotently on $\mathrm{H}^*(M \otimes_{\mathbf{Z}/p^N} \mathbf{F}_p)$.

Recollection C.9. In [**BL22a**, Lemma 3.5.18], Bhatt and Lurie show that the following is a Cartesian square of group schemes over $\mathbf{Z}/p^k$:

$$\begin{tikzcd} \mathbf{G}_m^\sharp \arrow[r, "\log"] \arrow[d] & \mathbf{G}_a^\sharp \arrow[d, "x\mapsto\exp(px)"] \\ \mathbf{G}_m \arrow[r, "x\mapsto x^p"'] & \mathbf{G}_m^{(1)}. \end{tikzcd} \tag{64}$$

We will generalize this below in Corollary C.10. In [**DM23**], we prove another generalization of this square, albeit in a different direction: $\mathbf{G}_a^\sharp$ is replaced by the Cartier dual of a formal group $\hat{\mathbf{G}}$, and $\mathbf{G}_m^\sharp$ is replaced by an appropriate $\hat{\mathbf{G}}$-analogue of the divided power completion.

Corollary C.10. *Let $k \geq 0$. There is an isomorphism of group schemes over $\mathbf{Z}/p^k$ between the Cartier dual of $W^\times[F^n] := \ker(F^n : W^\times \to W^\times)$ and the completion of W_n at its $\mathbf{F}_p$-rational points $W_n(\mathbf{F}_p) \cong \mathbf{Z}/p^n$.*

Proof. Following [**BL22a**, Remark 3.5.17], it suffices to prove the following analogue of [**BL22a**, Lemma 3.5.18]: there is a Cartesian diagram of flat group schemes over $\mathbf{Z}/p^k$ given by

$$\begin{tikzcd} W^\times[F^n] \arrow[r, "\log"] \arrow[d] & W[F^n] \arrow[d, "x\mapsto\exp(p^n x)"] \\ \mathbf{G}_m \arrow[r, "x\mapsto x^{p^n}"'] & \mathbf{G}_m^{(n)}. \end{tikzcd} \tag{65}$$

Here, the left vertical map $W^\times[F^n] \to \mathbf{G}_m$ is the composite

$$W^\times[F^n] \to W^\times \to W^\times/V \cong \mathbf{G}_m.$$

Indeed, taking the Cartier dual of (65) and using Proposition C.1, we obtain a pushout diagram of formal group schemes

$$\begin{tikzcd} p^n\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[d] \\ \hat{W}_n \arrow[r] & \mathbf{D}(W^\times[F^n]). \end{tikzcd}$$

This implies that $\mathbf{D}(W^\times[F^n])$ is the completion of W_n at its $\mathbf{F}_p$-rational points $W_n(\mathbf{F}_p) \cong \mathbf{Z}/p^n$, as desired.

The proof that the square (65) is Cartesian is in fact a consequence of [**BL22a**, Lemma 3.5.18]. As in [**BL22a**, Lemma 3.5.18], since all group schemes involved are flat over $\mathbf{Z}/p^k$, it suffices to prove that the diagram is Cartesian after base-changing to $\mathbf{F}_p$ (i.e., assume that $k = 1$). Indeed, there is an isomorphism $W^\times[F^n] \simeq W[F^n] \times \mu_{p^n}$ of group schemes over $\mathbf{F}_p$, sending $x \mapsto (\log(x), x \pmod{V})$: this can be seen by induction on n (with the base case being provided by [**BL22a**, Lemma 3.5.18]). □

Remark C.11. One could have alternatively/equivalently proved Corollary C.10 by observing that the square (65) for $n-1$ maps to (65) for n; all components of this map of squares are the canonical ones, except on the bottom-right $\mathbf{G}_m$ (where it is given by the pth power map $\mathbf{G}_m \to \mathbf{G}_m^{(1)}$). Diagramatically:

(66)
$$\begin{tikzcd} W^\times[F^{n-1}] \arrow[rr, "\log"] \arrow[dr] \arrow[dd] & & W[F^{n-1}] \arrow[dr] \arrow[dd, "x\mapsto\exp(p^{n-1}x)"' near start] & \\ & W^\times[F^n] \arrow[rr, "\log"' near start] \arrow[dd] & & W[F^n] \arrow[dd, "x\mapsto\exp(p^n x)"] \\ \mathbf{G}_m \arrow[rr, "x\mapsto x^{p^{n-1}}" near end] \arrow[dr, "\mathrm{id}"'] & & \mathbf{G}_m^{(n-1)} \arrow[dr, "x\mapsto x^p"] & \\ & \mathbf{G}_m \arrow[rr, "x\mapsto x^{p^n}"] & & \mathbf{G}_m^{(n)}. \end{tikzcd}$$

Taking Cartier duals, we obtain a map of pushout squares:

$$
\begin{tikzcd}
p^n\mathbf{Z} \arrow[rr] \arrow[dd] \arrow[dr] & & \hat{W}_n \arrow[dd] \arrow[dr] & \\
& \mathbf{Z} \arrow[rr] \arrow[dd] & & \mathbf{D}(W^\times[F^n]) \arrow[dd] \\
p^{n-1}\mathbf{Z} \arrow[rr] \arrow[dr] & & \hat{W}_{n-1} \arrow[dr] & \\
& \mathbf{Z} \arrow[rr] & & \mathbf{D}(W^\times[F^{n-1}]).
\end{tikzcd}
\tag{67}
$$

Again, Corollary C.10 follows by induction on n, using [**BL22a**, Lemma 3.5.18] for the base case.

Corollary C.12. *Write the underlying scheme of W_n as $\prod_{i=0}^{n-1}\mathbf{G}_a$ (where the ith copy of $\mathbf{G}_a$ has coordinate Ψ_i). There is a fully faithful functor $\mathrm{QCoh}(BW^\times[F^n]) \hookrightarrow \mathrm{QCoh}(W_n)$ whose essential image consists of those p-complete $M \in \mathrm{QCoh}(W_n)$ such that $\Psi_i^p - \Psi_i$ acts locally nilpotently on $\mathrm{H}^*(M/p)$ for each $0 \leq i \leq n-1$. Furthermore, this functor is symmetric monoidal for the convolution tensor product on $\mathrm{QCoh}(W_n)$.*

If $\mathcal{F} \in \mathrm{QCoh}(BW^\times[F^n])$ is sent to $M \in \mathrm{QCoh}(W_n)$ under this functor, one obtains a cube $\Psi_\bullet : 2^{[n-1]} \to \mathrm{Mod}_{\mathbf{Z}_p}$ whose vertices are all M and such that the edge from the subset $\{i_1, \cdots, i_{j-1}\}$ to $\{i_1, \cdots, i_j\}$ is given by the operator $\Psi_j : M \to M$. Then, the global sections $\Gamma(BW^\times[F^n]; \mathcal{F})$ can be identified with the total fiber of the cube $\Psi_\bullet$.

Example C.13. If $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(BW^\times[F^2])$ correspond to tuples (M, Ψ_0^M, Ψ_1^M) and $(M', \Psi_0^{M'}, \Psi_1^{M'})$, then the global sections $\Gamma(BW^\times[F^2]; \mathcal{F})$ can be identified with the total fiber of the square

$$
\begin{tikzcd}
M \arrow[r, "\Psi_0^M"] \arrow[d, "\Psi_1^M"'] & M \arrow[d, "\Psi_1^M"] \\
M \arrow[r, "\Psi_0^M"'] & M.
\end{tikzcd}
$$

Moreover, $\mathcal{F} \otimes \mathcal{G}$ corresponds to the module $M \otimes M'$, where

$$
\begin{aligned}
\Psi_0^{M\otimes M'} &= \Psi_0^M \otimes 1 + 1 \otimes \Psi_0^{M'}, \\
\Psi_1^{M\otimes M'} &= \Psi_1^{M'} \otimes 1 + 1 \otimes \Psi_1^{M'} - \frac{1}{p}\sum_{i=1}^{p-1} \binom{p}{i} (\Psi_0^M)^i \otimes (\Psi_0^{M'})^{p-i}.
\end{aligned}
$$

More generally, if $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(BW^\times[F^n])$ correspond to tuples $(M, \Psi_0^M, \cdots, \Psi_{n-1}^M)$ and $(M', \Psi_0^{M'}, \cdots, \Psi_{n-1}^{M'})$, let us write $\Psi := (\Psi_0, \Psi_0, \cdots)$. Let $w_j(\Psi) = \sum_{i=0}^j p^i \Psi_i^{p^{j-i}}$ denote the corresponding Witt polynomial; then

$$
w_j(\Psi^{M\otimes M'}) = w_j(\Psi^M) \otimes 1 + 1 \otimes w_j(\Psi^{M'}). \tag{68}
$$

Proposition C.14. *Define a homomorphism $W^\times[F^n] \to \mathbf{G}_m$ via the composite*
$$W^\times[F^n] \to W^\times \to (W/V)^\times \cong \mathbf{G}_m.$$
Let $\mathcal{O}\{1\}$ denote the line bundle over $BW^\times[F^n]$ determined by the resulting map $BW^\times[F^n] \to B\mathbf{G}_m$. Under the functor of Corollary C.12, the total space of the line bundle $\mathcal{O}\{1\}$ corresponds to the p-completion of $\mathbf{Z}_p[x^{\pm 1}]$ with the action of Ψ_j determined by the following requirement on Witt polynomials:

$$w(\Psi) = (w_0(\Psi), w_1(\Psi), w_2(\Psi), \cdots) = (x\partial_x, x\partial_x, x\partial_x, \cdots). \tag{69}$$

Proof. The map $BW^\times[F^n] \to B\mathbf{G}_m$ determines the stack $\mathbf{G}_m/W^\times[F^n]$ over $BW^\times[F^n]$, so that the corresponding object in $\mathrm{QCoh}(W_n)$ under the functor of Corollary C.12 has underlying module given by $\mathcal{O}_{\mathbf{G}_m} = \mathbf{Z}[x^{\pm 1}]$. It is not too hard to show from the definition of the map $BW^\times[F^n] \to B\mathbf{G}_m$ that under the functor of Corollary C.12, the line bundle $\mathcal{O}\{1\}$ over $BW^\times[F^n]$ corresponds to the p-complete module $\mathbf{Z}_p$ (with generator x) where Ψ_0 acts on x by 1, and Ψ_j acts on x by zero for $j \geq 1$. The action of $w_j(\Psi)$ on $\mathcal{O}\{m\} = \mathbf{Z}_p \cdot x^m$ then follows from (68). □

Example C.15. For instance, it follows from (69) that
$$\begin{aligned}
\Psi_0 &= x\partial_x,\\
\Psi_1 &= \frac{x\partial_x}{p}\left(1 - (x\partial_x)^{p-1}\right),\\
\Psi_2 &= \frac{x\partial_x}{p^2}\left(1 - (x\partial_x)^{p^2-1} - \frac{1}{p^{p-1}}\sum_{j=0}^{p}(-1)^j\binom{p}{j}(x\partial_x)^{(p-1)(j+1)}\right).
\end{aligned}$$

Remark C.16. Using Corollary C.4, a similar calculation can be used to describe the $\mathbf{G}_a$-bundle $\mathbf{G}_a/W[F^n] = F_*^n W/pF^{n-1}$ over $BW[F^n]$; and, in particular, the ∞-category $\mathrm{QCoh}((\mathbf{G}_a/W[F^n])/\mathbf{G}_m)$. Let us summarize this calculation as follows. Recall that $\mathbf{G}_a/W[F] \cong \mathbf{G}_a/\mathbf{G}_a^\sharp$ is isomorphic to $\mathbf{G}_a^{\mathrm{dR}}$, so that if $\mathcal{A}_1 := \mathbf{Z}_p\{x, \partial_x\}/([\partial_x, x] = 1)$ is the Weyl algebra, then $\mathrm{QCoh}(\mathbf{G}_a/W[F]) \simeq \mathrm{LMod}_{\mathcal{A}_1}^{\partial_x\text{-nilp}}$. Similarly, if we write $\frac{1}{m!}\partial_x^m = \partial_x^{[m]}$ (so that $\partial_x^{[m]}(x^k) = \binom{k}{m}x^{k-m}$), define $\mathcal{A}_1^{[n]}$ via
$$\mathcal{A}_1^{[n]} = \mathbf{Z}_p\left\{x, \partial_x, \cdots, \partial_x^{[p^{n-1}]}\right\}/([\partial_x^{[p^j]}, x] = \partial_x^{[p^j-1]}).$$
Note that $\partial_x^{[p^j-1]}$ is a p-adic unit multiple of $\prod_{k=0}^{j-1}(\partial_x^{[p^k]})^{p-1}$. Then, the action of $W[F^n]$ on $\mathbf{G}_a$ implies that there is an equivalence
$$\mathrm{QCoh}(\mathbf{G}_a/W[F^n]) \simeq \mathrm{LMod}_{\mathcal{A}_1^{[n]}}^{\partial_x^{[p^j]}\text{-nilp}};$$
this can be extended to an equivalence between $\mathrm{QCoh}((\mathbf{G}_a/W[F^n])/\mathbf{G}_m)$ and the ∞-category of graded $\mathcal{A}_1^{[n]}$-modules such that $\partial_x^{[p^j]}$ acts nilpotently for $0 \leq j \leq n-1$, where $x \in \mathcal{A}_1^{[n]}$ has weight 1 and $\partial_x^{[p^j]} \in \mathcal{A}_1^{[n]}$ has weight $-p^j$. Algebras of divided power differential operators such as $\mathcal{A}_1^{[n]}$ were initially studied by Berthelot in [**Ber96**].

Warning C.17. Note that $\mathbf{G}_a/W[F^n]$ is not a ring stack. Indeed, the map $W[F^n] \to \mathbf{G}_a$ is not a quasi-ideal: the W-module structure on $W[F^n]$ does not factor through $W \twoheadrightarrow W_1 = \mathbf{G}_a$ (if $F^n(x) = 0$, then $xV(y) = V(F(x)y)$ need not vanish). However, it *does* factor through $W \twoheadrightarrow W_n$ (if $F^n(x) = 0$, then

$xV^n(y) = V^n(F^n(x)y) = 0$); indeed, $W_n/W[F^n] \cong W/p^n$ admits the structure of a ring stack.

Remark C.18. The proof of Corollary C.10 showed that there is an isomorphism

$$W^\times[F^n] \cong W[F^n] \times \mu_{p^n}$$

over $\mathbf{F}_p$. Let $\mathfrak{X}$ be a smooth p-adic formal scheme over $\mathbf{Z}_p$, and let $X = \mathfrak{X} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$. Suppose that the $\mathbf{G}_m^\sharp$-action on $\widehat{\Omega}_{\mathfrak{X}}^{\not{D}} \otimes_{\mathbf{Z}_p} \mathbf{F}_p = F_{X,*}\Omega^\bullet_{X/\mathbf{F}_p}$ refines to a $W^\times[F^n]$-action. (For instance, let $(\widehat{\Omega}_{\mathfrak{X}}^{\not{D}})_0$ denote the weight 0 piece of the $\mathbf{Z}/p$-grading on $\widehat{\Omega}_{\mathfrak{X}}^{\not{D}}$ inherited from the $\mathbf{G}_m^\sharp$-action. The datum of a refinement to a $W^\times[F^2]$-action leads to an operator on $(\widehat{\Omega}_{\mathfrak{X}}^{\not{D}})_0$ which acts on $\mathrm{gr}^{pi}_{\mathrm{conj}}\widehat{\Omega}_{\mathfrak{X}}^{\not{D}}$ by multiplication by $-i$.) In this case, the $\mathbf{Z}/p$-grading on $F_{X,*}\Omega^\bullet_{X/\mathbf{F}_p}$ from [**BL22a**, Remark 4.7.20] would refine to a $\mathbf{Z}/p^n$-grading; this would imply a refinement of the Deligne-Illusie theorem [**DI87**], stating that $\tau_{\geq -p^n+1}F_{X,*}\Omega^\bullet_{X/\mathbf{F}_p}$ would be decomposable.

Remark C.19 ("Witty" interpretation of [**Dev23b**]). The work of [**Lee22**] suggests that the base-change along $\mathrm{BP}\langle n-1\rangle_* \to \mathbf{F}_p$ (even along $\mathrm{BP}\langle n-1\rangle_* \to \mathbf{Z}_p$) of the stack constructed from the associated graded of the motivic filtration [**HRW22**] on $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)^{t\mathbf{Z}/p}$ (resp. $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$) is isomorphic to the stack $(\mathbf{G}_m/W^\times[F^n])/\mathbf{G}_m \cong BW^\times[F^n]$ (resp. $(\mathbf{G}_a/W[F^n])/\mathbf{G}_m \cong (F^n_*W/pF^{n-1})/\mathbf{G}_m$). Note that over $\mathbf{F}_p$, the map $W[F^n] \to \mathbf{G}_a$ factors as

$$W[F^n] \twoheadrightarrow W[F^n]/V = \alpha_{p^n} \hookrightarrow \mathbf{G}_a,$$

which lets us identify $\mathbf{G}_a/W[F^n] \cong \mathbf{G}_a^{(p^n)} \times BF_*W[F^n]$. In topology, this corresponds to (4) when $j = 0$ (i.e., [**ACH21**, Proposition 2.9]). We are currently investigating this and its consequences with Jeremy Hahn and Arpon Raksit. In particular, this suggests that if a $\mathbf{Z}_p$-scheme "lifts to $\mathrm{BP}\langle n-1\rangle$", the $\mathbf{G}_m^\sharp$-action on $\widehat{\Omega}_{\mathfrak{X}}^{\not{D}}$ refines to a $W^\times[F^n]$-action. From this perspective, the operators Ψ_j from above are closely related to the topological Sen operators Θ_j from the body of this article: roughly, Θ_j can be understood as $w_{j-1}(\Psi)$.

Given Remark C.18, one is therefore naturally led to the following question: if X is a smooth and proper $\mathbf{F}_p$-scheme which "lifts to $\mathrm{BP}\langle n-1\rangle$" and $\dim(X) < p^n$, does the Hodge-de Rham spectral sequence for X degenerate at the E_1-page? This question need not make sense, since $\mathrm{BP}\langle n-1\rangle$ is generally not an $\mathbf{E}_\infty$-ring [**Law18, Sen17**]. However, since $\mathrm{BP}\langle n-1\rangle$ admits the structure of an $\mathbf{E}_3$-ring, one can nevertheless ask whether such a degeneration statement holds noncommutatively if $\mathrm{QCoh}(X)$ admits a lift to a left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category. This line of thinking was motivation for the following result (see [**Dev23b**]): if $\mathrm{QCoh}(X)$ lifts to a left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, and $\dim(X) < p^n$, then the Tate spectral sequence for $\mathrm{HP}(X/\mathbf{F}_p)$ degenerates at the E_2-page.

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