HODGE THEORY FOR ELLIPTIC CURVES AND THE HOPF ELEMENT ν

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Abstract. We show that the vector bundle on the moduli stack M_{ell} of elliptic curves associated to the 2-cell complex $C\nu$ is isomorphic to the de Rham cohomology sheaf $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$ of the universal elliptic curve $\mathcal{E} \to M_{\mathrm{ell}}$. We use this to calculate the homotopy groups of the $\mathbf{E}_1$-quotient $\mathrm{tmf}/\!\!/\nu$ of tmf by ν, called the spectrum of "topological quasimodular forms", by relating its Adams–Novikov spectral sequence to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration.

1. Introduction

In this article, we study the relationship between the Hopf invariant one element $\nu \in \pi_3\mathrm{tmf}$ and the Hodge filtration for elliptic curves. Namely, we show that the vector bundle on the moduli stack M_{ell} of elliptic curves associated to $C\nu$ is isomorphic to the (middle) de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$ of the universal elliptic curve $\mathcal{E} \to M_{\mathrm{ell}}$. A version of this relationship had been stated by Hopkins in [Hop02, Section 5]. Using this, we calculate the homotopy groups of the $\mathbf{E}_1$-quotient $\mathrm{tmf}/\!\!/\nu$ of tmf by $\nu \in \pi_3(\mathbb{S})$ by showing that the E_2-page of its Adams–Novikov spectral sequence is isomorphic to the cohomology of the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham filtration. The $\mathbf{E}_1$-ring $\mathrm{tmf}/\!\!/\nu$ is called the spectrum of *topological quasimodular forms* (see Remark 5.2). The results of this article have been known to Charles Rezk, and probably other experts.

The ring spectrum $\mathrm{tmf}/\!\!/\nu$ is interesting for several reasons. One motivation for studying it comes from the Ando-Hopkins-Rezk orientation $\mathrm{MU}\langle 6\rangle \to \mathrm{tmf}$ (see [AHR10]). As is made clear during the course of the proof, a key reason for why this orientation does not factor through the map $\mathrm{MU}\langle 6\rangle \to \mathrm{MSU}$ is because tmf detects the element $\nu \in \pi_3(\mathbb{S})$; this in turn is related to the fact that the weight 2 Eisenstein series is not a modular form. Since $\mathrm{tmf}/\!\!/\nu$ is the "smallest" coherently structured (i.e., $\mathbf{E}_1$-) tmf-algebra with a nullhomotopy of ν, one might expect the composite

$$\mathrm{MU}\langle 6\rangle \to \mathrm{tmf} \to \mathrm{tmf}/\!\!/\nu \tag{1}$$

to factor through MSU via an $\mathbf{E}_1$-map. Although we do not prove in this article that the composite (1) factors through MSU, we will use the results of this article to address this question in future work. The connection between ν and the weight 2 Eisenstein series is also discussed in Section 5. The relationship between ν and de Rham cohomology is also independently interesting, because the Hodge–de Rham filtration on the de Rham cohomology of an elliptic curve is related to many deep topics in arithmetic geometry (such as Grothendieck–Messing theory).

We begin in Section 2 by recalling some background on Hodge theory for cubic curves from algebraic geometry. In particular, we give a Hopf algebroid presentation for the moduli stack $M^{\mathrm{dR}}_{\mathrm{cub}}$ of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence. In Section 3, we prove our main technical result relating the Adams–Novikov spectral sequence of $\mathrm{tmf}/\!\!/\nu$ to the cohomology of the moduli stack $M^{\mathrm{dR}}_{\mathrm{cub}}$. Finally, in Section 5 we prove Theorem 5.1, which calculates this Adams–Novikov spectral sequence.

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It degenerates at the E_4-page, and $\mathrm{TMF}/\!\!/\nu$ is found to be 24-periodic. Moreover, $\mathrm{tmf}/\!\!/\nu$ is homotopy commutative, and we prove that $\mathrm{tmf}/\!\!/\nu \otimes \Sigma^\infty_+ \Omega S^3$ admits the structure of an $\mathbf{E}_2$-ring. These results were discovered independently by Rezk in unpublished work, and we give our own proof of his calculation of $\pi_*(\mathrm{tmf}/\!\!/\nu)$.

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2. Background on Hodge theory

In this section, we recall some background on Hodge theory for cubic curves over a general base scheme. Multiple sources (such as [Kat73, Appendix A1.2]) discuss Hodge theory for (smooth) elliptic curves.

Let $f : X \to Y$ be a morphism of schemes. One then has the $f^{-1}\mathcal{O}_Y$-linear relative de Rham complex $\Omega^\bullet_{X/Y}$.

Definition 2.1. The *ith relative de Rham cohomology* $\mathrm{H}^i_{\mathrm{dR}}(X/Y)$ of $f : X \to Y$ is defined to be the hypercohomology sheaf $\mathbf{R}^i f_*(\Omega^\bullet_{X/Y})$ on Y.

The hypercohomology spectral sequence defines the Hodge–de Rham spectral sequence of sheaves on Y:

$$E_1^{s,t} = R^t f_* \Omega^s_{X/Y} \Rightarrow \mathrm{H}^{s+t}_{\mathrm{dR}}(X/Y).$$

The following (easy) result is well-known; note that there are no assumptions on the characteristic of the base, since f is of relative dimension 1 (for maps of higher relative dimension, one would need to make additional assumptions on the characteristic).

Theorem 2.2. *If $f : X \to Y$ is a smooth, proper, and surjective morphism of relative dimension 1 with geometrically connected fibers, then the Hodge–de Rham spectral sequence degenerates at the E_1-page.*

Since f is of relative dimension 1, the only interesting de Rham cohomology is in the middle dimension, i.e., $\mathrm{H}^1_{\mathrm{dR}}(X/Y)$. In particular, for such f, there is an exact sequence

$$0 \to f_*\Omega^1_{X/Y} \to \mathrm{H}^1_{\mathrm{dR}}(X/Y) \to R^1 f_* \mathcal{O}_X \to 0$$

of quasicoherent sheaves on Y; this is called the *Hodge–de Rham exact sequence*. Moreover, the pairing $\mathrm{H}^1_{\mathrm{dR}}(X/Y) \otimes_{\mathcal{O}_Y} \mathrm{H}^1_{\mathrm{dR}}(X/Y) \to \mathcal{O}_Y$ is determined by the canonical perfect pairing

$$R^1 f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} f_*\Omega^1_{X/Y} \to R^1 f_* \Omega^1_{X/Y} \xrightarrow{\text{trace}} \mathcal{O}_Y.$$

We now specialize to the case when $f : X \to Y$ is an elliptic curve $f : E \to S$. Then $f_*\Omega^1_{E/S}$ is the line bundle $\omega_{E/S}$ of invariant differentials. The pairing $\omega_{E/S} \otimes_{\mathcal{O}_S} R^1 f_* \mathcal{O}_E \to \mathcal{O}_S$ is perfect, and so $R^1 f_* \mathcal{O}_E \cong \omega^{-1}_{E/S}$. In particular, the Hodge–de Rham exact sequence for $E \to S$ becomes

$$0 \to \omega_{E/S} \to \mathrm{H}^1_{\mathrm{dR}}(E/S) \to \omega^{-1}_{E/S} \to 0. \tag{2}$$

If M_{ell} denotes the moduli stack of elliptic curves, and $\mathcal{E} \to M_{\mathrm{ell}}$ is the universal elliptic curve, then (2) exhibits $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$ as an element of $\mathrm{Ext}^1_{M_{\mathrm{ell}}}(\omega^{-1}, \omega)$.

Remark 2.3. If S is a p-adic scheme, then (2) corresponds to the Hodge filtration of the Dieudonné module $\mathbf{D}(E[p^\infty]/S)$ of E under the isomorphism $\mathrm{H}^1_{\mathrm{dR}}(E/S) \cong \mathbf{D}(E[p^\infty]/S)$; see, for instance, [Kat81, Section V].

The following is an immediate consequence of [Kat73, Equation A1.2.3] (see also [Poo20]):

Proposition 2.4. *If $f : E \to S$ is an elliptic curve, then there is an isomorphism $\mathrm{H}^1_{\mathrm{dR}}(E/S) \otimes \omega_{E/S} \cong f_* \Omega^1_{E/S}(2\infty)$.*

The map

$$f_* \Omega^1_{E/S}(2\infty) \cong \mathrm{H}^1_{\mathrm{dR}}(E/S) \otimes \omega_{E/S} \to \omega_{E/S}^{-1} \otimes \omega_{E/S} = \mathcal{O}_S$$

induced by the Hodge–de Rham exact sequence sends a section of $f_* \Omega^1_{E/S}(2\infty)$ to its residue at ∞.

We can now generalize the above story to the non-smooth setting. First, we recall the definition of a cubic curve.

Definition 2.5. A *cubic curve* $f : E \to S$ over a scheme S is a flat and proper morphism of finite presentation whose fibers are reduced, irreducible curves of arithmetic genus 1, along with a section $\infty : S \to E$ whose image is contained in the smooth locus E^{sm} of f. Let M_{cub} denote the stack of cubic curves, and let $f : \mathcal{E} \to M_{\mathrm{cub}}$ denote the universal cubic curve.

Let ω denote the line bundle on M_{cub} assigning to a cubic curve $f : E \to S$ the cotangent bundle $\omega_{E/S}$ along the section ∞. There is an isomorphism $\mathrm{H}^1(M_{\mathrm{ell}}; \omega^{\otimes 2}) \cong \mathrm{H}^1(M_{\mathrm{cub}}; \omega^{\otimes 2})$ (which can be deduced, e.g., from the calculations in [Bau08]). The vector bundle $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$ over M_{ell} defines a class in $\mathrm{Ext}^1_{M_{\mathrm{ell}}}(\omega^{-1}, \omega) \cong \mathrm{H}^1(M_{\mathrm{ell}}; \omega^{\otimes 2})$; so this class extends uniquely to an element in $\mathrm{Ext}^1_{M_{\mathrm{cub}}}(\omega^{-1}, \omega) \cong \mathrm{H}^1(M_{\mathrm{cub}}; \omega^{\otimes 2})$. In a terrible abuse of notation, we will denote the resulting vector bundle on M_{cub} by $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}})$.

Definition 2.6. Let $M^{\mathrm{dR}}_{\mathrm{cub}}$ denote the moduli stack of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence, and let $M^{\mathrm{dR}}_{\mathrm{ell}} = M^{\mathrm{dR}}_{\mathrm{cub}} \times_{M_{\mathrm{cub}}} M_{\mathrm{ell}}$ denote the moduli stack of elliptic curves with a chosen splitting of the Hodge–de Rham exact sequence.

In order to do calculations with $M^{\mathrm{dR}}_{\mathrm{cub}}$, we would like to obtain a Hopf algebroid presentation of this stack. To do so, we recall a Hopf algebroid presentation of M_{cub}; see [Del75, Equation 1.6]. Zariski-locally on any base scheme S, a cubic curve is described by a Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \tag{3}$$

with other choices of coordinates (x, y) given by the transformations

$$x \mapsto x + r,\ y \mapsto y + sx + t.$$

The moduli stack M_{cub} of cubic curves is presented by the Hopf algebroid

$$(D, \Gamma) = (\mathbf{Z}[a_1, a_2, a_3, a_4, a_6], D[r, s, t]),$$

with gradings[1] $|a_i| = i$ and $|r| = 2$, $|s| = 1$, and $|t| = 3$. Studying how the coefficients a_i transform gives the right unit $\eta_R : D \to \Gamma$ of this Hopf algebroid:

$$\begin{aligned}
a_1 &\mapsto a_1 + 2s,\\
a_2 &\mapsto a_2 - a_1 s + 3r - s^2,\\
a_3 &\mapsto a_3 + a_1 r + 2t,\\
a_4 &\mapsto a_4 + a_3 s + 2a_2 r - a_1 t - a_1 rs - 2st + 3r^2,\\
a_6 &\mapsto a_6 + a_4 r - a_3 t + a_2 r^2 - a_1 rt - t^2 + r^3.
\end{aligned}$$

To determine a Hopf algebroid presentation of $M^{\mathrm{dR}}_{\mathrm{cub}}$, note that after choosing x, y, the coordinate x defines a function on the smooth locus of $\mathcal{E}$ with a double pole at ∞, and it is in fact the only such non-constant function on the smooth locus of $\mathcal{E}$ (this follows from the

[1] Recall that the topological grading is *double* the algebraic grading.

usual calculation [KM85, Section 2.2.5] with the Riemann-Roch formula). By Proposition 2.4, the choice of x determines a splitting of the Hodge–de Rham exact sequence: namely, a regular 1-form ν on the smooth locus of $\mathcal{E}$ determines an independent 1-form $x\nu$ such that ν and $x\nu$ span $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$. Since coordinate changes in x are given by $x \mapsto x+r$, the element $[r]$ in the cobar complex for the Hopf algebroid (D,Γ) must detect the extension in $\mathrm{Ext}^1_{M_{\mathrm{ell}}}(\mathcal{O},\omega^{\otimes 2}) \cong \mathrm{H}^1(M_{\mathrm{ell}};\omega^{\otimes 2})$ determined by the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}})$. Recalling that $\mathrm{H}^1(M_{\mathrm{ell}};\omega^{\otimes 2}) \cong \mathrm{H}^1(M_{\mathrm{cub}};\omega^{\otimes 2})$, we see that $[r]$ detects the extension in $\mathrm{Ext}^1_{M_{\mathrm{cub}}}(\mathcal{O},\omega^{\otimes 2}) = \mathrm{Ext}^{1,2}_\Gamma(D,D)$ determined by the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}})$. By the preceding discussion, a choice of Hodge–de Rham splitting on the universal cubic curve amounts to fixing a choice of x (although y is allowed to vary); this amounts to setting $r=0$ in the Hopf algebroid presenting M_{cub}. Consequently:

Proposition 2.7. *The moduli stack $M^{\mathrm{dR}}_{\mathrm{cub}}$ of cubic curves with a chosen splitting of the Hodge–de Rham exact sequence is presented by the Hopf algebroid $(D,\Sigma) = (\mathbf{Z}[a_1,a_2,a_3,a_4,a_6], D[s,t])$, with gradings $|a_i| = i$, $|s| = 1$, and $|t| = 3$. The right unit is the same as in that of the elliptic curve Hopf algebroid, except with $r=0$:*

$$\begin{aligned}
&a_1 \mapsto a_1 + 2s,\\
&a_2 \mapsto a_2 - a_1 s - s^2,\\
&a_3 \mapsto a_3 + 2t,\\
&a_4 \mapsto a_4 + a_3 s - a_1 t - 2st,\\
&a_6 \mapsto a_6 - a_3 t - t^2.
\end{aligned} \tag{4}$$

3. The relationship with $\mathrm{tmf}/\!\!/\nu$

In this section, we study the $\mathbf{E}_1$-quotient $\mathrm{tmf}/\!\!/\nu$ of tmf by ν, and relate its Adams–Novikov spectral sequence to the cohomology of $M^{\mathrm{dR}}_{\mathrm{cub}}$. The results of this section are well-known to some experts.

We begin by recalling one construction of the $\mathbf{E}_1$-quotient $\mathrm{tmf}/\!\!/\nu$. This satisfies the following universal property: if R is any $\mathbf{E}_1$-tmf-algebra such that $\nu = 0$, then there is a canonical $\mathbf{E}_1$-tmf-algebra map $\mathrm{tmf}/\!\!/\nu \to R$. Therefore,

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Mod(tmf)})}(\mathrm{tmf}/\!\!/\nu, R) = \begin{cases} \Omega^{\infty+4}R & \text{if } \nu = 0 \in \pi_3 R \\ \emptyset & \text{else.}\end{cases}$$

The following definition is justified by [AB19, Theorem 4.10]:

Definition 3.1. The $\mathbf{E}_1$-ring $\mathrm{tmf}/\!\!/\nu$, called *topological quasimodular forms* (see Remark 5.2 for a justification for the name), is the Thom spectrum of the dotted extension in the following diagram:

$$\begin{tikzcd}
S^4 \arrow[r, "\nu"] \arrow[d] & B\mathrm{GL}_1(\mathrm{tmf}) \\
\Omega S^5 \arrow[ur, dashed] &
\end{tikzcd}$$

This dotted extension exists since $B\mathrm{GL}_1(\mathrm{tmf})$ admits the structure of an $\mathbf{E}_1$-space (in fact, it is an $\mathbf{E}_\infty$-space, since tmf is an $\mathbf{E}_\infty$-ring).

Remark 3.2. The element $\nu \in \pi_3(\mathrm{tmf})$ is spherical, and so this diagram factors as

$$\begin{tikzcd}
S^4 \arrow[r, "2v_1^2"] \arrow[d] & \mathrm{BSpin} \arrow[r, "J"] & B\mathrm{GL}_1(\mathbb{S}) \arrow[d] \\
\Omega S^5 \arrow[ur, dashed] & & B\mathrm{GL}_1(\mathrm{tmf}).
\end{tikzcd}$$

Here, $2v_1^2$ is the generator of $\pi_4\mathrm{BSpin} \cong \mathbf{Z}$. Following the notation of [Dev19], we will write A to denote the Thom spectrum of the loop map $\Omega S^5 \to B\mathrm{GL}_1(\mathbb{S})$. Then there is a canonical equivalence $\mathrm{tmf}/\!\!/\nu \simeq \mathrm{tmf} \otimes A$ of $\mathbf{E}_1$-tmf-algebras, so there is in particular an $\mathbf{E}_1$-algebra map $A \to \mathrm{tmf}/\!\!/\nu$.

The $\mathbf{E}_1$-ring A will be useful below. In [Dev19], it is shown that there is an $\mathbf{E}_1$-map $A \to \mathrm{BP}$. Moreover, the BP-homology of A at the prime 2 is isomorphic to $\mathrm{BP}_*[y_2]$, where y_2 is sent to t_1^2 modulo decomposables under the map $\mathrm{BP}_*(A) \to \mathrm{BP}_*(\mathrm{BP})$. In particular, $\mathrm{H}_*(A; \mathbf{F}_2) \cong \mathbf{F}_2[\zeta_1^4]$. One then has (see [Dev20, Example 3.1.14 and Example 3.2.15]):

Proposition 3.3. *There is a nontrivial simple 2-torsion element $\sigma_1 \in \langle \eta, \nu, 1_A \rangle \subseteq \pi_5(A) \cong \pi_5(C\nu)$ specified up to indeterminacy by the relation $\eta\nu = 0$. One choice of this element is represented by $[t_2]$ in the Adams–Novikov spectral sequence for A, and by h_{21} in the (mod 2) Adams spectral sequence for A.*

The image of the class $\sigma_1 \in \pi_5(A)$ under the unit map $A \to \mathrm{tmf} \otimes A = \mathrm{tmf}/\!\!/\nu$ defines a torsion element in $\pi_5(\mathrm{tmf}/\!\!/\nu)$, which we will also denote by σ_1. We will study this element further in Theorem 5.1.

Remark 3.4. The element $\sigma_1^4 \in \pi_{20}(\mathbb{S}/\!\!/\nu)$ is the image of $\overline{\kappa} \in \pi_{20}(\mathbb{S})$ under the unit map $\mathbb{S} \to \mathbb{S}/\!\!/\nu$.

To connect $\mathrm{tmf}/\!\!/\nu$ and Hodge theory for cubic curves, we make the following observation. Recall that $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}}) \in \mathrm{Ext}^1_{M_{\mathrm{cub}}}(\omega^{-1}, \omega)$.

Proposition 3.5. *Let $f : \mathcal{E} \to M_{\mathrm{cub}}$ denote the universal cubic curve over the moduli stack of cubic curves. Then $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}}) \in \mathrm{Ext}^1_{M_{\mathrm{cub}}}(\omega^{-1}, \omega) \cong \mathrm{H}^1(M_{\mathrm{cub}}; \omega^2)$ detects ν in the E_2-page of the Adams-Novikov spectral sequence for* tmf*.*

Proof. This is essentially argued in [Hop02, Section 5.2]. We know that $\mathrm{H}^1(M_{\mathrm{cub}}; \omega^2) = \mathbf{Z}/12$ by the calculations in [Bau08]; the element $[r]$ in the cobar complex determined by the Hopf algebroid (D, Γ) is a representative for the generator. This element detects ν in the Adams–Novikov spectral sequence for tmf, and by the discussion before Proposition 2.7, also detects the extension class of the Hodge–de Rham exact sequence. $\square$

Any spectrum X defines a quasicoherent sheaf on the moduli stack M_{FG} of formal groups; see, e.g., [Mat16, Section 2.1]. Pulling back along the map $M_{\mathrm{cub}} \to M_{FG}$ defines a quasicoherent sheaf on M_{cub} which we will denote by $\mathcal{F}(X)$.

Corollary 3.6. *The rank two vector bundle $\mathcal{F}(C\nu)$ on the moduli stack of cubic curves corresponding to $C\nu$ is isomorphic to $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}})$.*

Remark 3.7. In [Rez13, Section 11.5], the Hodge–de Rham exact sequence appears in a different but related guise, as a class in the E_2-page of a spectral sequence $\{E_r^{s,t}\}$ converging to the homotopy groups of the space of $\mathbf{E}_\infty$-maps $\mathbf{Z}_+ \to \mathrm{TMF}$. The element $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{ell}}) \in E_2^{1,4}$ detects a nontrivial class in $\pi_3\mathrm{Map}_{\mathbf{E}_\infty}(\mathbf{Z}_+, \mathrm{TMF}) = \pi_3\mathbb{G}_m(\mathrm{TMF})$, i.e., an $\mathbf{E}_\infty$-map $K(\mathbf{Z}, 3)_+ \to \mathrm{TMF}$. This is related to the $\mathbf{E}_\infty$-twisting of TMF explored in [ABG10].

Since $\mathrm{tmf}/\!\!/\nu$ is the $\mathbf{E}_1$-quotient of tmf by ν by Remark 3.2, it is the universal $\mathbf{E}_1$-tmf-algebra with a nullhomotopy of ν. If $\mathrm{tmf}/\!\!/\nu$ is a homotopy commutative ring (which we will show is indeed the case in Corollary 5.9), then we would be able to consider the stack associated to $\mathrm{tmf}/\!\!/\nu$ (in the sense of [DFHH14, Chapter 9], [Mat16, Section 2.1]), and it would be reasonable to expect that Proposition 3.5 implies that this stack is the moduli of cubic curves with a choice of splitting of the Hodge–de Rham spectral sequence. We have:

Theorem 3.8. *Let $g : M^{\mathrm{dR}}_{\mathrm{cub}} \to M_{\mathrm{cub}}$ denote the structure morphism. Then the sheaf on M_{cub} associated to A is isomorphic as an algebra to the pushforward $g_*\mathcal{O}_{M^{\mathrm{dR}}_{\mathrm{cub}}}$.*

Proof. Let $\mathcal{C}$ be a presentable symmetric monoidal (∞-)category, and let T denote the functor $\mathcal{C}^{\mathrm{unital}} \to \mathrm{Alg}_{\mathbf{E}_1}(\mathcal{C})$ sending a unital object $i : \mathbf{1} \to X$ to the free $\mathbf{E}_1$-algebra in $\mathcal{C}$ whose unit factors through i: this may be defined via the homotopy pushout

$$\begin{tikzcd} \mathrm{Free}_{\mathbf{E}_1}(\mathbf{1}) \arrow[r] \arrow[d] & \mathbf{1} \arrow[d] \\ \mathrm{Free}_{\mathbf{E}_1}(X) \arrow[r] & T(X) \end{tikzcd}$$

in $\mathrm{Alg}_{\mathbf{E}_1}(\mathcal{C})$. The functor $\mathcal{F} : \mathrm{Sp} \to \mathrm{QCoh}(M_{FG})$ (and hence the functor $\mathcal{F} : \mathrm{Sp} \to \mathrm{QCoh}(M_{\mathrm{cub}})$) is lax symmetric monoidal. Recall that the functor $\mathcal{F} : \mathrm{Sp} \to \mathrm{QCoh}(M_{FG})$ can be identified with the functor of MU-homology $X \mapsto \mathrm{MU}_*(X)$, viewed as a $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$-comodule. Since bounded-below spectra of finite type with even cells have free MU-homology (by an easy inductive argument on skeleta and the fact that MU-homology preserves filtered colimits), this implies that $\mathcal{F}$ is in fact symmetric monoidal when restricted to bounded-below spectra of finite type with even cells. In particular, if X is a unital bounded-below spectrum of finite type with even cells, then $T(\mathcal{F}(X)) \simeq \mathcal{F}(T(X))$. It is easy to see by the universal property of T that $T(C\nu) \simeq A$, so it follows from Proposition 3.5 that $\mathcal{F}(A) \cong T(f_*\Omega^1_{\mathcal{E}/M_{\mathrm{cub}}}(2\infty))$. Here, motivated by Proposition 2.4, $f_*\Omega^1_{\mathcal{E}/M_{\mathrm{cub}}}(2\infty)$ denotes $\omega \otimes \mathrm{H}^1_{\mathrm{dR}}(\mathcal{E}/M_{\mathrm{cub}})$. It therefore suffices to show that $T(f_*\Omega^1_{\mathcal{E}/M_{\mathrm{cub}}}(2\infty)) \cong g_*\mathcal{O}_{M^{\mathrm{dR}}_{\mathrm{cub}}}$.

Its universal property defines an algebra map $\varphi : T(f_*\Omega^1_{\mathcal{E}/M_{\mathrm{cub}}}(2\infty)) \to g_*\mathcal{O}_{M^{\mathrm{dR}}_{\mathrm{cub}}}$ of sheaves on M_{cub}. To check that this is an isomorphism, it suffices to show that φ is an isomorphism upon pulling back to any affine $\mathrm{Spec}(R)$ on which ω is trivial (we thank the referee for a simplification of our original argument). In this case the claim is easy to see: the pullback of $g_*\mathcal{O}_{M^{\mathrm{dR}}_{\mathrm{cub}}}$ is isomorphic to a polynomial R-algebra on a single generator (given by the square of a trivialization of ω), while the pullback of $T(f_*\Omega^1_{\mathcal{E}/M_{\mathrm{cub}}}(2\infty))$ is isomorphic to a free associative R-algebra on the same generator. $\square$

Corollary 3.9. *There is an Adams-Novikov spectral sequence*

$$E_2^{s,2t} = \mathrm{H}^s(M^{\mathrm{dR}}_{\mathrm{cub}}; g^*\omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\mathrm{tmf}/\!\!/\nu).$$

Proof. By [Mat16, Corollary 5.3], the Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$ is given by

$$E_2^{s,2t} = \mathrm{H}^s(M_{\mathrm{cub}}; \mathcal{F}(A) \otimes_{\mathcal{O}_{M_{\mathrm{cub}}}} \omega^{\otimes t}) \Rightarrow \pi_{2t-s}(\mathrm{tmf}/\!\!/\nu).$$

Combining Theorem 3.8 with the projection isomorphism shows that $\mathcal{F}(A) \otimes_{\mathcal{O}_{M_{\mathrm{cub}}}} \omega^{\otimes t} \cong g_*(g^*\omega^{\otimes t})$. The morphism g is flat and affine, so $E_2^{s,2t} \cong \mathrm{H}^s(M^{\mathrm{dR}}_{\mathrm{cub}}; g^*\omega^{\otimes t})$, as desired. $\square$

Remark 3.10. Corollary 3.9 says that although $\mathrm{tmf}/\!\!/\nu$ is not *a priori* a homotopy commutative ring, there is a descent spectral sequence which would exist if there was a sheaf of structured ring spectra on $M^{\mathrm{dR}}_{\mathrm{cub}}$ whose global sections is $\mathrm{tmf}/\!\!/\nu$. We pose this as a conjecture:

Conjecture 3.11. *There is a sheaf of even-periodic $\mathbf{E}_2$-rings $\mathcal{O}^{\mathrm{der}}$ on the étale site of $M^{\mathrm{dR}}_{\mathrm{ell}}$ such that if $f : \mathrm{Spec}\, R \to M^{\mathrm{dR}}_{\mathrm{ell}}$ is an étale map, then $\mathcal{O}^{\mathrm{der}}(f)$ is the Landweber-exact theory corresponding to the composite $\mathrm{Spec}\, R \to M^{\mathrm{dR}}_{\mathrm{ell}} \to M_{\mathrm{ell}} \to M_{FG}$, and such that the global sections $\Gamma(M^{\mathrm{dR}}_{\mathrm{ell}}; \mathcal{O}^{\mathrm{der}})$ is equivalent as an $\mathbf{E}_1$-ring to $\mathrm{TMF}/\!\!/\nu$. Moreover, the resulting $\mathbf{E}_2$-ring structure on $\mathrm{TMF}/\!\!/\nu$ extends to an $\mathbf{E}_2$-ring structure on $\mathrm{tmf}/\!\!/\nu$.*

4. Multiplicative structure on $\mathrm{tmf}/\!\!/\nu$

In this section, we prove a result relating to Conjecture 3.11.

Definition 4.1. Let $S^0[\sigma]$ denote the $\mathbf{E}_2$-algebra given by $\Sigma^\infty_+ \Omega S^3$, where we regard ΩS^3 as the double loop space $\Omega^2 \mathbf{H}P^\infty$. By the James splitting, $S^0[\sigma] \simeq \bigoplus_{n\geq 0} S^{2n}$; one might therefore view $S^0[\sigma]$ as a polynomial ring over the sphere on a generator in degree 2. If R is an $\mathbf{E}_1$-ring, let $R[\sigma]$ denote the $\mathbf{E}_1$-ring $R \otimes_{S^0} S^0[\sigma]$.

Theorem 4.2. *The $\mathbf{E}_1$-algebra structure on $(\mathrm{tmf}/\!\!/\nu)[\sigma]$ admits a refinement to an $\mathbf{E}_2$-algebra structure.*

Proof. Recall from [ABG10, Section 8] (see also Remark 3.7) that there is an $\mathbf{E}_\infty$-map $K(\mathbf{Z},4) \to \mathrm{BGL}_1(\mathrm{tmf})$, which detects $\nu \in \pi_3(\mathrm{tmf})$ on π_4. The $\mathbf{E}_1$-map $\mu : \Omega S^5 \to \mathrm{BGL}_1(\mathrm{tmf})$ which defines $\mathrm{tmf}/\!\!/\nu$ factors as

$$\Omega S^5 \to \Omega K(\mathbf{Z},5) \simeq K(\mathbf{Z},4) \to \mathrm{BGL}_1(\mathrm{tmf}).$$

The quotient map $\mathrm{SU}(3) \to \mathrm{SU}(3)/\mathrm{SU}(2) \simeq S^5$ defines an $\mathbf{E}_1$-map $\Omega\mathrm{SU}(3) \to \Omega S^5$. The key observation is that the resulting composite

$$\Omega\mathrm{SU}(3) \to \Omega S^5 \to K(\mathbf{Z},4),$$

although *a priori* only an $\mathbf{E}_1$-map, admits the structure of an $\mathbf{E}_2$-map. Indeed, it is given by doubly looping the map $\mathrm{BSU}(3) \to K(\mathbf{Z},6)$ given by the Chern class $c_3 \in \mathrm{H}^6(\mathrm{BSU}(3);\mathbf{Z})$. Therefore, we have the following diagram (where the maps are labeled by their multiplicative structure):

$$\begin{tikzcd}
\Omega S^3 = \Omega\mathrm{SU}(2) \arrow[r, "\mathbf{E}_2"] & \Omega\mathrm{SU}(3) \arrow[r, "\mathbf{E}_1"] \arrow[dr, "\mathbf{E}_2"'] & \Omega S^5 \arrow[d, "\mathbf{E}_1"] \arrow[dr, "\mathbf{E}_1"] & \\
 & & K(\mathbf{Z},4) \arrow[r, "\mathbf{E}_\infty"'] & \mathrm{BGL}_1(\mathrm{tmf}).
\end{tikzcd}$$

By the main result of [AB19], we conclude that the Thom spectrum of the resulting map $\Omega\mathrm{SU}(3) \to \mathrm{BGL}_1(\mathrm{tmf})$ admits the structure of an $\mathbf{E}_2$-algebra. The top row in the above diagram is a fiber sequence, and the composite $\Omega\mathrm{SU}(2) \to \Omega\mathrm{SU}(3) \to K(\mathbf{Z},4)$ is null as an $\mathbf{E}_2$-map (indeed, its two-fold delooping defines the pullback of c_3 to $\mathrm{BSU}(2)$, which vanishes). Therefore, if $(\Omega S^5)^\mu$ denotes the Thom spectrum of the $\mathbf{E}_1$-map $\mu : \Omega S^5 \to \mathrm{BGL}_1(\mathrm{tmf})$, then the Thom spectrum of the map $\Omega\mathrm{SU}(3) \to \mathrm{BGL}_1(\mathrm{tmf})$ may be identified with $\Omega S^3_+ \otimes (\Omega S^5)^\mu \simeq (\mathrm{tmf}/\!\!/\nu)[\sigma]$, as desired. □

Remark 4.3. In general, the argument of Theorem 4.2 shows the following statement. Let R be an $\mathbf{E}_3$-ring, and let $x \in \pi_{2n-1}(R)$ be a homotopy class which is detected on π_{2n} by an $\mathbf{E}_2$-map $K(\mathbf{Z},2n) \to \mathrm{BGL}_1(R)$. Then $(R/\!\!/x) \otimes \Omega\mathrm{SU}(n-1)_+$ admits the structure of an $\mathbf{E}_2$-ring.

Remark 4.4. It is unclear whether one can "kill" the polynomial generator σ in Theorem 4.2 to conclude that $\mathrm{tmf}/\!\!/\nu$ itself admits the structure of an $\mathbf{E}_2$-ring, although we strongly believe this to be the case.

One might ask if $\mathrm{tmf}/\!\!/\nu$ admits the structure of an $\mathbf{E}_3$-algebra. We do not know how to prove this, but we suspect that the $\mathbf{E}_1$-algebra structure on $A = \mathbb{S}/\!\!/\nu$ does not refine to an $\mathbf{E}_3$-algebra structure.

5. The Adams-Novikov spectral sequence

Our goal in this section is to calculate the homotopy groups of $\mathrm{tmf}/\!\!/\nu$ via the Adams-Novikov spectral sequence of Corollary 3.9. To do this calculation, we will use the Hopf algebroid presentation in Proposition 2.7. The calculation of the Adams-Novikov spectral sequence was done independently by Charles Rezk; although he stated part of the result to the author in an email, the argument is the author's (so errors are the author's fault).

Theorem 5.1. *There is an isomorphism*

$$\mathrm{H}^*(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes *}) \cong \mathbf{Z}[b_2, b_4, b_6, b_8, h_1, h_{21}]/I, \tag{5}$$

where $b_i \in \mathrm{H}^0(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes i})$ *of total degree* $2i$*,* $h_1 \in \mathrm{H}^1(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega)$ *of total degree* 1*, and* $h_{21} \in \mathrm{H}^1(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes 3})$ *of total degree* 5*. If one defines*

$$c_4 = b_2^2 - 24b_4,\ \ c_6 = -b_2^3 + 36b_2b_4 - 216b_6,$$
$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6,$$

then the ideal I *is generated by the relations*

$$2h_1 = 0,\ \ 2h_{21} = 0,\ \ h_1^2b_6 = h_{21}^2b_2,\ \ 4b_8 = b_2b_6 - b_4^2,\ \ 1728\Delta = c_4^3 - c_6^2.$$

Moreover, the Adams–Novikov spectral sequence of Corollary 3.9 collapses on the E_4*-page, and* $\pi_*(\mathrm{tmf}/\!\!/\nu)$ *is determined by the differentials*

$$d_3(b_2) = h_1^3,\ \ d_3(b_4) = h_1^2h_{21},\ \ d_3(b_6) = h_1h_{21}^2,\ \ d_3(b_8) = h_{21}^3.$$

One has the relations

$$\eta^3 = 0,\ \ \eta^2\sigma_1 = 0,\ \ \eta\sigma_1^2 = 0,\ \ \sigma_1^3 = 0$$

in the homotopy of $\mathrm{tmf}/\!\!/\nu$*, in addition to the relations in* I*. Here* η *is represented by* h_1*, and* σ_1 *is represented by* h_{21}*. All the torsion in* $\mathrm{tmf}/\!\!/\nu$ *is concentrated in dimensions congruent to* $1, 2 \pmod 4$*.*

Before giving the proof, we discuss some consequences.

Remark 5.2. By Theorem 5.1, there is a ring isomorphism

$$\mathrm{H}^0(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes *}) \cong \mathbf{Z}[b_2, b_4, b_6, b_8]/(4b_8 = b_2b_6 - b_4^2,\ 1728\Delta = c_4^3 - c_6^2).$$

This ring has been studied before in characteristic zero (in which case $b_8 = (b_2b_6 - b_4^2)/4$), e.g., in [KZ95, Mov12], where it is referred to as the ring of *quasimodular forms*.

In fact, the Hopf algebroid (D, Σ) presenting $M_{\mathrm{cub}}^{\mathrm{dR}}$ (from Proposition 2.7) becomes discrete after inverting 2; indeed, the transformation $y \to y - a_1x/2 - a_3/2$ transforms the Weierstrass equation (3) into

$$y^2 = x^3 + a_2x^2 + a_4x + a_6,$$

and one cannot make any coordinate changes to x since it is fixed. We find that $(D[1/2], \Sigma[1/2])$ is isomorphic to the discrete Hopf algebroid $(D' = \mathbf{Z}[1/2][a_2, a_4, a_6], D')$. We therefore see that $M_{\mathrm{cub}}^{\mathrm{dR}}[1/\Delta] \otimes \mathbf{C}$ is precisely the scheme T from [Mov12, Section 5.5]. One can recover [Mov12, Proposition 5.4] from Proposition 2.7 by base-changing to an algebraically closed field of characteristic zero. Theorem 5.1 therefore provides a calculation of the ring of *integral* quasimodular forms, and also justifies calling the ring spectrum $\mathrm{tmf}/\!\!/\nu$ by the name "topological quasimodular forms". It would be interesting to understand a topological analogue of the Ramanujan θ-operator.

The following corollary is a calculation via Theorem 5.1.

Corollary 5.3. *In Adams–Novikov filtration zero,* b_i^2 *and twice any monomial in the* b_i*s survive to the* E_∞*-page for* $i = 2, 4, 6, 8$*, as do* b_2b_6 *and* $\lambda_1 b_8 b_2^2 + \lambda_2 b_2 b_4 b_6$ *for* $\lambda_1 \equiv \lambda_2 \pmod 2$*. In particular,* $\Delta \in \pi_{24}(\mathrm{tmf}/\!\!/\nu)$*, so the* $\mathbf{E}_1$*-ring* $\mathrm{TMF}/\!\!/\nu$ *is 24-periodic with periodicity generator* Δ*.*

Remark 5.4. Note that $\mathrm{tmf}/\!\!/\nu$ is complex orientable after inverting 2. This can be seen algebraically by noting (as in Remark 5.2) that the Hopf algebroid (D, Σ) presenting $M_{\mathrm{cub}}^{\mathrm{dR}}$ becomes discrete after inverting 2, and so $\pi_*(\mathrm{tmf}[1/2]/\!\!/\nu) \cong \mathbf{Z}[1/2][a_2, a_4, a_6]$ with $|a_i| = 2i$. In light of this, we only need to prove Theorem 5.1 after 2-localization.

Remark 5.5. The Hurewicz image of tmf in $\mathrm{tmf}/\!\!/\nu$ can be determined from Theorem 5.1. The subring generated by η, σ_1, $2b_2$, and b_2^2 is in the image of the map $\pi_*A \to \pi_*\mathrm{tmf}/\!\!/\nu$. The relationship between $\pi_*(\mathrm{tmf}/\!\!/\nu)$ and $\pi_*(\mathrm{tmf})$, however, is more interesting than merely the Hurewicz image. The (2-local) calculation in [Bau08] shows that $\overline{\kappa}\nu$ vanishes in $\pi_{23}(\mathrm{tmf})$; this is detected in the Adams–Novikov spectral sequence by a d_5-differential $d_5(\Delta) = \overline{\kappa}\nu$. This implies that the element $8\Delta \in \pi_{24}(\mathrm{tmf})$ can be expressed as an element of the Toda bracket $\langle 8, \nu, \overline{\kappa}\rangle$. Equivalently, the map $\overline{\kappa} : S^{20} \to \mathrm{tmf}$ extends to a map from $\Sigma^{20}C\nu$ (and hence from $\Sigma^{20}\mathrm{tmf} \wedge C\nu$); then, composition with the map $S^{24} \to \Sigma^{20}C\nu$ which is degree 8 on the top cell produces the element $8\Delta \in \pi_{24}\mathrm{tmf}$ (up to indeterminacy). In other words, 8Δ comes from an element of $\pi_{24}(\Sigma^{20}\mathrm{tmf} \wedge C\nu) \cong \pi_4(\mathrm{tmf} \wedge C\nu)$. Under the canonical map $\mathrm{tmf} \wedge C\nu \to \mathrm{tmf}/\!\!/\nu$, this element corresponds to $2b_2 \in \pi_4(\mathrm{tmf}/\!\!/\nu)$. Similarly, the element $\Delta\eta \in \pi_{24}(\mathrm{tmf})$ can be related to the element $\sigma_1 \in \pi_5(\mathrm{tmf}/\!\!/\nu)$. This is related to the approach taken in [Dev19] to show that the Ando-Hopkins-Rezk orientation MString $\to$ tmf from [AHR10] is surjective on homotopy.

Remark 5.6. After base-change to $\mathbf{F}_p$, there is a dotted map

$$\begin{tikzcd} & M_{\mathrm{ell}}^{\mathrm{dR}} \arrow[d] \\ M_{\mathrm{ell}}^{\mathrm{ord}} \arrow[r] \arrow[ur, dashed] & M_{\mathrm{ell}}, \end{tikzcd}$$

where $M_{\mathrm{ell}}^{\mathrm{ord}}$ denotes the moduli stack of ordinary elliptic curves. This existence of this dotted map is well-known in arithmetic geometry: it is the statement that the Frobenius (which exists for ordinary elliptic curves via quotienting out by the canonical subgroup) splits the Hodge filtration (see [Kat73, Section A2.3]).

Remark 5.7. The inclusion of the cusp on $\overline{M_{\mathrm{ell}}}$ defines an $\mathbf{E}_\infty$-map $c : \mathrm{tmf} \to \mathrm{ko}$ as in [LN14, Theorem 1.2]. Since $\nu = 0 \in \pi_3\mathrm{ko}$, the universal property of $\mathrm{tmf}/\!\!/\nu$ implies that there is a map $\mathrm{tmf}/\!\!/\nu \to \mathrm{ko}$ of $\mathbf{E}_1$-tmf-algebras. On homotopy, this map kills σ_1, b_4, b_6, b_8, and sends $\eta \mapsto \eta$, $2b_2 \mapsto 2v_1^2$, and $b_2^2 \mapsto v_1^2$.

Rezk pointed out that Theorem 5.1 can be used to show that $\mathrm{tmf}/\!\!/\nu$ admits the structure of a homotopy commutative ring; one can give an alternative proof using Theorem 4.2.

Remark 5.8. Let R be an $\mathbf{E}_2$-ring. Suppose that B is an R-module equipped with a multiplication $\mu : B \otimes_R B \to B$, such that B has R-module cells in dimensions $\equiv 0 \pmod n$. If $\tau : B \otimes_R B \to B \otimes_R B$ is the flip automorphism, then the obstruction to μ being homotopy commutative is the difference $\mu - \mu\tau : B \otimes_R B \to B$. Note that $B \otimes_R B$ also has R-module cells in dimensions $\equiv 0 \pmod n$. There is a cofiber sequence of R-modules

$$(B \otimes_R B)^{(n(j-1))} \to (B \otimes_R B)^{(nj)} \to \bigoplus \Sigma^{nj} R,$$

where the direct sum is over the top-dimensional R-module cells of $(B \otimes_R B)^{(nj)}$. Suppose that the restriction of $\mu - \mu\tau$ to the $n(j-1)$-R-module skeleton $(B \otimes_R B)^{(n(j-1))}$ of $B \otimes_R B$ is null. Then, the obstruction to the restriction of $\mu - \mu\tau$ to the nj-R-module skeleton $(B \otimes_R B)^{(nj)}$ of $B \otimes_R B$ also being null is given by an R-linear map $\bigoplus \Sigma^{nj} R \to B$. This is a collection of classes in $\pi_{nj}(B)$. In other words, obstructions to the R-linear multiplication on B being homotopy commutative live in $\pi_{nj}(B)$ for $j \geq 2$.

Corollary 5.9. *The $\mathbf{E}_1$-ring $\mathrm{tmf}/\!\!/\nu$ admits the structure of a homotopy commutative ring.*

Proof. Since A is the Thom spectrum of a bundle over ΩS^5, it has one cell in each nonnegative dimension divisible by 4; therefore, $\mathrm{tmf}/\!\!/\nu$ has tmf-module cells in dimensions divisible by 4. By Remark 5.8, the obstructions to its homotopy commutativity live in dimensions $\equiv 0 \pmod 4$. Since $\nu = 0$ in $\mathbf{Q}$, there is an equivalence $A_{\mathbf{Q}} \simeq \mathbf{Q}[\Omega S^5]$ of $\mathbf{E}_1$-$\mathbf{Q}$-algebras; and ΩS^5 is rationally equivalent to $K(\mathbf{Q}, 4)$, which is even an infinite loop space, so that $A_{\mathbf{Q}}$ is an $\mathbf{E}_\infty$-ring. In particular, the obstructions to the homotopy commutativity of $\mathrm{tmf}/\!\!/\nu$ vanish after rationalization. By Theorem 5.1, all the homotopy groups of $\mathrm{tmf}/\!\!/\nu$ in dimensions divisible by 4 are torsion-free, so the obstructions to the homotopy commutativity of $\mathrm{tmf}/\!\!/\nu$ must also vanish. □

An immediate consequence of Theorem 3.8 and Corollary 5.9 is:

Corollary 5.10. *The stack $M_{\mathrm{tmf}/\!\!/\nu}$ associated to the homotopy commutative ring $\mathrm{tmf}/\!\!/\nu$ is isomorphic to $M_{\mathrm{cub}}^{\mathrm{dR}}$.*

Remark 5.11. Corollary 5.10 implies, for instance, that the fact that ν is not detected by $L_{K(1)}\mathrm{tmf}$ is related to the existence of the map $M_{\mathrm{ell}}^{\mathrm{ord}} \to M_{\mathrm{ell}}^{\mathrm{dR}}$ from Remark 5.6.

Finally, we give the proof of Theorem 5.1.

Proof of Theorem 5.1. We will implicitly 2-localize everywhere; this is sufficient by Remark 5.4. We begin by calculating $\mathrm{H}^*(M_{\mathrm{cub}}^{\mathrm{dR}}; g^*\omega^{\otimes *}) = \mathrm{Ext}_\Sigma(D, D)$, where

$$(D, \Sigma) = (\mathbf{Z}[a_1, a_2, a_3, a_4, a_6], D[s, t]).$$

Following [Sil86, Chapter III], define quantities

$$\begin{aligned} b_2 &= a_1^2 + 4a_2, \\ b_4 &= 2a_4 + a_1 a_3, \\ b_6 &= a_3^2 + 4a_6, \\ b_8 &= a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2. \end{aligned}$$

Notice that $b_2 b_6 - b_4^2 = 4b_8$ and that $1728\Delta = c_4^3 - c_6^2$ where c_4, c_6, and Δ are as in the theorem statement. The classes $b_i \in D$ are invariant under the right unit of (D, Σ).

Let I denote the ideal $(2, a_1, a_3, a_4)$, and define a Hopf algebroid

$$(\overline{D}, \overline{\Sigma}) = (D/I, \Sigma/I) = (\mathbf{F}_2[a_2, a_6], \overline{D}[s, t]).$$

The right unit sends

$$a_2 \mapsto a_2 + s^2,\ a_6 \mapsto a_6 + t^2.$$

Then there is a Bockstein spectral sequence

$$E_1^{p,q,n} = \mathrm{Ext}_{\overline{\Sigma}}^{p,n}(\overline{D}, \mathrm{Sym}_{\overline{D}}^q(I/I^2)) \Rightarrow \mathrm{Ext}_\Sigma^{p,n}(D, D), \tag{6}$$

with $d_r : E_r^{p,q,n} \to E_r^{p+1,q+r,n}$. We will compute this similarly to [Rez07, Section 16.5]. First, observe that $I/I^2 = \overline{D} \otimes_{\mathbf{F}_2} V$, with $V = \mathbf{F}_2\{\overline{a}_0, \overline{a}_1, \overline{a}_3, \overline{a}_4\}$ where $\overline{a}_0$, $\overline{a}_1$, $\overline{a}_3$, and $\overline{a}_4$ represent 2, a_1, a_3, and a_4 respectively. The comodule structure $I/I^2 \to I/I^2 \otimes_{\overline{D}} \overline{\Sigma}$ sends

$$\begin{aligned} \overline{a}_0 &\mapsto \overline{a}_0, \\ \overline{a}_1 &\mapsto \overline{a}_1 + \overline{a}_0 s, \\ \overline{a}_3 &\mapsto \overline{a}_3 + \overline{a}_0 t, \\ \overline{a}_4 &\mapsto \overline{a}_4 + \overline{a}_3 s + \overline{a}_1 t + \overline{a}_0 st. \end{aligned} \tag{7}$$

There is a map $\overline{D} \to \mathbf{F}_2$ induced by sending a_2 and a_6 to zero, and so we obtain a Hopf algebroid $(\mathbf{F}_2, C)$ with

$$C = \mathbf{F}_2 \otimes_{\overline{D}} \overline{\Sigma} \otimes_{\overline{D}} \mathbf{F}_2 \cong \mathbf{F}_2[a_2, a_6, s, t]/(a_2, a_6, \eta_R(a_2), \eta_R(a_6)) \cong \mathbf{F}_2[s, t]/(s^2, t^2) = E(s, t).$$

To emphasize the connection to homotopy theory, we write h_1 for s and h_{21} for t. Now, the map $\overline{D} \to \mathbf{F}_2 \otimes_{\overline{D}} \overline{\Sigma} = \mathbf{F}_2[h_1, h_{21}]$ given by sending a_2 to h_1^2 and a_6 to h_{21}^2 is faithfully flat, and defines a Morita equivalence $(\overline{D}, \overline{\Sigma}) \to (\mathbf{F}_2, C)$ of Hopf algebroids. Moreover, the Hopf algebroid $(\mathbf{F}_2, C)$ presents the (graded) stack $B\alpha_2 \times B\alpha_2$ over $\mathrm{Spec}(\mathbf{F}_2)$, where

$\alpha_2 = \operatorname{Spec} \mathbf{F}_2[x]/x^2$ is the kernel of Frobenius on the additive group scheme over $\mathbf{F}_2$. It follows that

$$\operatorname{Ext}^{p,n}_{\overline{\Sigma}}(\overline{D}, \operatorname{Sym}^q_{\overline{D}}(I/I^2)) \cong \operatorname{Ext}^{p,n}_C(\mathbf{F}_2, \operatorname{Sym}^q_{\mathbf{F}_2}(V)) = \mathrm{H}^{p,n}(B\alpha_2 \times B\alpha_2; \operatorname{Sym}^q(V)).$$

To calculate the E_1-page, first observe that the comodule structure on I/I^2 appearing in Equation (7) is a representation of $\alpha_2 \times \alpha_2$ on $V^* = \operatorname{Spec} \operatorname{Sym}(V)$. Therefore:

$$\operatorname{Ext}^{0,*}_C(\mathbf{F}_2, \operatorname{Sym}^*_{\mathbf{F}_2}(V)) = \operatorname{Sym}^*(V)^{\alpha_2 \times \alpha_2} = \mathbf{F}_2[\overline{a}_0, \overline{a}_1^2, \overline{a}_0\overline{a}_4 + \overline{a}_1\overline{a}_3, \overline{a}_3^2, \overline{a}_4^2]; \tag{8}$$

indeed, these are the invariants under the $\alpha_2 \times \alpha_2$-action on V. The expressions for b_2, b_4, b_6, and b_8 show that they are represented in the Bockstein spectral sequence by $\overline{a}_1^2$, $\overline{a}_0\overline{a}_4 + \overline{a}_1\overline{a}_3$, $\overline{a}_3^2$, and $\overline{a}_4^2$, respectively; in particular, all of the generators of $V^{\alpha_2 \times \alpha_2}$ are permanent cycles in the Bockstein spectral sequence. Moreover, $\mathrm{H}^{*,*}(B\alpha_2 \times B\alpha_2; V) \cong \mathbf{F}_2$, since V is a cofree C-comodule. Since $\operatorname{Sym}^0(V) = \mathbf{F}_2$, we have $\mathrm{H}^{*,*}(B\alpha_2 \times B\alpha_2; \operatorname{Sym}^0(V)) \cong \mathbf{F}_2[h_1, h_{21}]$, where $h_1 = [s]$ and $h_{21} = [t]$. As a C-comodule, $\operatorname{Sym}^*(V)$ is a direct sum of shifts of $\mathbf{F}_2$ and V; using this decomposition together with Equation (8) gives the E_1-page of the Bockstein spectral sequence (6):

$$E_1^{*,*,*} = \mathbf{F}_2[\overline{a}_0, \overline{a}_1^2, \overline{a}_0\overline{a}_4 + \overline{a}_1\overline{a}_3, \overline{a}_3^2, \overline{a}_4^2, h_1, h_{21}].$$

Note that $|\overline{a}_0| = (0,0)$, $|\overline{a}_1^2| = (0,2)$, $|\overline{a}_3^2| = (0,6)$, $|\overline{a}_4^2| = (0,8)$, $|h_1| = (1,1)$, $|h_{21}| = (1,3)$, where the bidegree is (p,n); the q-degree is just the degree of the monomial, and h_1 and h_2 have q-degree 0.

Let us now calculate the differentials in the Bockstein spectral sequence. The right unit in Equation (4) gives Bockstein differentials $a_1 \mapsto \overline{a}_0 h_1$ and $a_3 \mapsto \overline{a}_0 h_{21}$ (these correspond to 2η and $2\sigma_1$ being null in $\pi_*(\mathrm{tmf}/\!\!/\nu)$, respectively). The following differentials in the cobar complex

$$d(a_2) = \eta_R(a_2) - a_2 = -(a_1 s + s^2),$$
$$d(a_6) = \eta_R(a_6) - a_6 = -(a_3 t + t^2),$$

imply the relations $a_1 s = -s^2$ and $a_3 t = -t^2$ on the E_2-page. By explicit calculation, we have the following differential in the cobar complex:

$$d(b_8) = \eta_R(b_8) - b_8 = 6a_1a_3st - (a_1^2t^2 + a_3^2s^2),$$

which, using the relations $a_1 s = -s^2$ and $a_3 t = -t^2$, becomes $6s^2t^2 - (a_1^2t^2 + a_3^2s^2)$. Now $6s^2t^2$ lives in higher filtration (its p-degree is 4, while the p-degree of $a_1^2t^2$ and $a_3^2s^2$ is 2), so this produces a differential with target $\overline{a}_1^2t^2 + \overline{a}_3^2s^2$. Since b_2 is detected by $\overline{a}_1^2$ and b_6 is detected by $\overline{a}_3^2$, this imposes the relation $h_1^2 b_6 = -h_{21}^2 b_2$ in the E_∞-page of the Bockstein spectral sequence. Combining together all of these facts gives Equation (5) as the cohomology of the moduli stack $M^{\mathrm{dR}}_{\mathrm{cub}}$. As in Corollary 3.9, this is the E_2-page of the Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$.

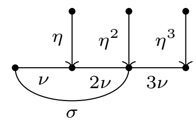

FIGURE 1. 15-skeleton of A at the prime 2 shown horizontally, with 0-cell on the left. The element σ_1 is shown by the arrow labeled η: this means that when restricted to the 4-skeleton $C\nu$ of A, the map $\sigma_1 : S^5 \to C\nu$ is given by η on the top cell of $C\nu$. In other words, the map $\pi_5(A) \cong \pi_5(C\nu) \to \pi_5(S^4)$ sends σ_1 to η. Similarly, the element σ_1^2 is shown by the arrow labeled η^2; therefore, when restricted to the 8-skeleton X_2 of A, the map $\sigma_1^2 : S^{10} \to X_2$ is given by η^2 on the top cell of X_2. In other words, the map $\pi_{10}(A) \cong \pi_{10}(X_2) \to \pi_{10}(S^8)$ sends σ_1^2 to η^2.

We now calculate the Adams–Novikov differentials. See Figures 2, 3, 4, and 5 for a depiction of the d_3-differentials on b_2 and b_4 (and h_1- and h_{21}-multiplications on these classes) in the E_3- and $E_4 = E_\infty$-pages of the Adams–Novikov spectral sequence (in Adams grading). Notice that Figures 2 and 3 are essentially given by overlaying two copies of the Adams–Novikov spectral sequence for ko, albeit with one copy shifted to the right by 4 units (compare to Remark 5.7). Similarly, Figures 4 and 5 can be obtained by shifting Figures 2 and 3 to the right by 4 units and relabeling the classes (for example, h_1 is relabeled by h_{21}, and b_2 is relabeled by b_4). In the same way, it is possible to draw the relevant portion of the Adams–Novikov spectral sequence for the d_3-differentials on b_6 and b_8 as well, by shifting Figures 2 and 3 to the right by 8 and 12 units (respectively) and relabeling. To obtain the full Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$, one can overlay these figures and identify eponymous classes. By the above prescription, the resulting picture will essentially look like eight copies of shifts of the Adams–Novikov spectral sequence for ko.

The 15-skeleton of A is shown in Figure 1. We know that $\eta^3 = 4\nu$ vanishes in $\pi_* A$, so h_1^3 must die in the Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$. There is only one possibility, namely the d_3-differential $d_3(b_2) = h_1^3$. (Note that this differential already exists in the Adams–Novikov spectral sequence for A, where b_2 is represented by v_1^2, i.e., the class $[y_2]$ in the cobar complex (via Proposition 3.3).) As a consequence, h_1 is a permanent cycle in the Adams–Novikov spectral sequence for A (and represents η).

Next, we know from Proposition 3.3 that σ_1 is detected in the Adams–Novikov spectral sequence for A by h_{21}. Since $\sigma_1 \in \langle \eta, \nu, 1_A \rangle$, one has that $\eta^2\sigma_1 = 0$ in $\pi_*(\mathrm{tmf}/\!\!/\nu)$. Explicitly, $\eta^2\sigma_1 \in \langle \eta, \nu, \eta \rangle \eta$. But $\langle \eta, \nu, \eta \rangle = \nu^2$ (no indeterminacy), and $\eta\nu^2 = 0$. Therefore, $h_1^2 h_{12}$ must die. There is no possibility other than $d_3(b_4)$ for a differential to kill $h_1^2 h_{21}$ (except for a d_3-differential on b_2^2, but $d_3(b_2^2) = 0$). Note that h_1 and $h_1 h_{12}$ are permanent cycles and represent σ_1 and $\eta\sigma_1$, respectively.

For the third differential, note that since there is a d_3-differential $d_3(b_2) = h_1^3$, we have $d_3(h_{21}^2 b_2) = h_1^3 h_{21}^2$. But there is a relation $h_1^2 b_6 = h_{21}^2 b_2$, so $d_3(h_1^2 b_6) = h_1^3 h_{21}^2$, which forces $d_3(b_6) = h_1 h_{21}^2$. Since there can be no nonzero classes in higher filtration (see Figures 2 and 3), we find that $\eta\sigma_1^2 = 0$.

Finally, $\eta\sigma_1^3 = 0$ in $\pi_*\mathrm{tmf}/\!\!/\nu$ (using $\eta\sigma_1^2 = 0$). It follows that the element $h_1 h_{21}^3$ must be the target of a differential in the Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$. The only possibilities are a d_3-differential on $h_1 b_4^2$, $h_1 b_2 b_6$, or $h_1 b_8$. Only $h_1 b_8$ can kill $h_1 h_{21}^3$, and this forces a d_3-differential $d_3(b_8) = h_{21}^3$. At this point, there are no more possibilities for differentials in the Adams–Novikov spectral sequence for $\mathrm{tmf}/\!\!/\nu$, and the spectral sequence collapses at the E_4-page. □

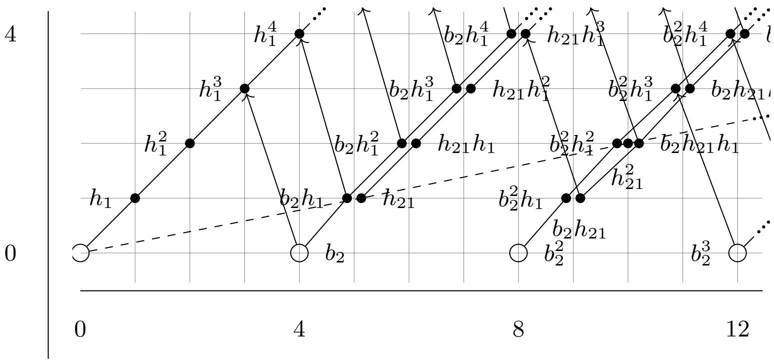

FIGURE 2. Part of the E_3-page of the spectral sequence, where the "primary" d_3-differential $d_3(b_2) = h_1^3$ is indicated. Although it is not hard to extend this drawing further, the spectral sequence starts to look a little cluttered. The sloped lines indicate η-multiplication. The dashed line indicates σ_1-multiplication, and we have only indicated it on the unit class to avoid cluttering; we have also not drawn in the classes given by h_1-multiplies of powers of h_{21}, etc.

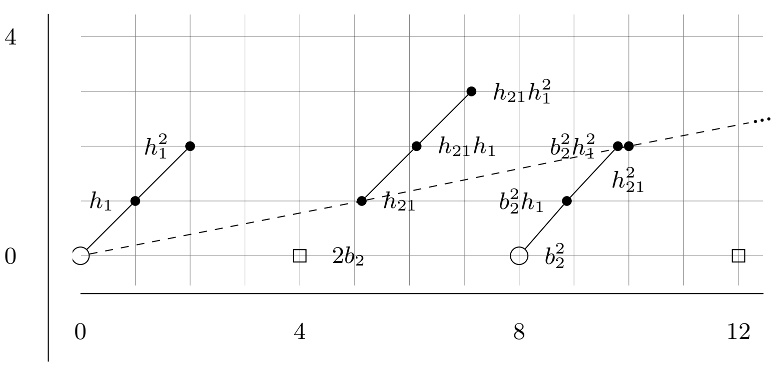

FIGURE 3. The part of the E_4-page of the spectral sequence which is relevant for b_2. Although $h_{21}h_1^2$ appears in this picture, it is only because we have not also drawn in the d_3-differential on b_4 (which kills $h_{21}h_1^2$, as shown in Figure 4).

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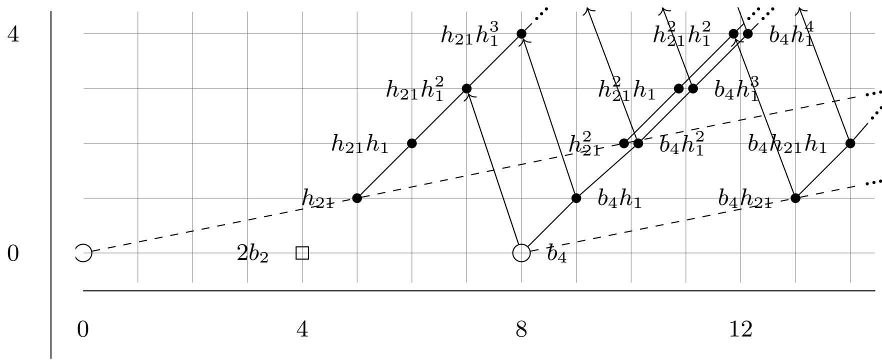

FIGURE 4. Part of the E_3-page of the spectral sequence where the "primary" d_3-differential $d_3(b_4) = h_{21}h_1^2$ is indicated. For reference, we have also drawn in $2b_2$ (but not b_2^2 or $2b_3^2$, etc.). The dashed line indicates σ_1-multiplication, and we have only indicated it on the unit class to avoid cluttering.

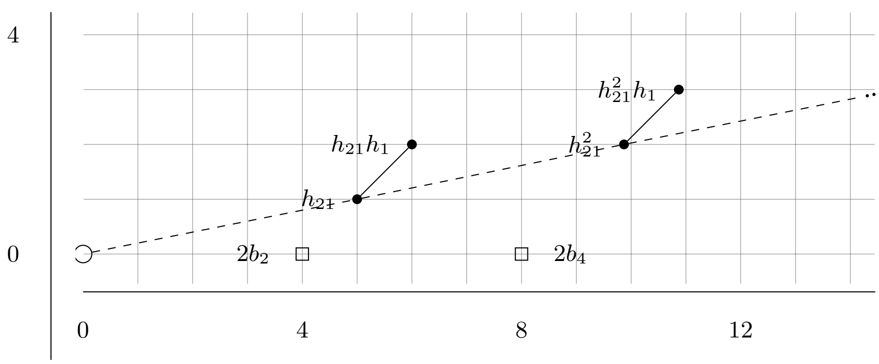

FIGURE 5. The part of the E_4-page of the spectral sequence which is relevant for b_4. Although $h_{21}^2h_1$ appears in this picture, it is only because we have not also drawn in the d_3-differential on b_6 (which kills $h_{21}^2h_1$).

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