Geometric Langlands duality with generalized coefficients

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Overview

Geometric Langlands duality

Motivated by the number field/function field/manifolds analogy, Beilinson and Drinfeld proposed a geometric variant of Langlands duality, where number rings are replaced by Riemann surfaces. This relates the *topology* of a split reductive group G over $\mathbf{Z}$ to the *algebraic geometry* of its "Langlands dual group" $\check{G}_k$. (E.g., $G = \mathrm{SL}_n$, $\check{G} = \mathrm{PGL}_n$.)

If Σ is a Riemann surface and k is a commutative ring, they proposed that there should be an equivalence

$$\mathrm{Shv}(\mathrm{Bun}_G(\Sigma); k) \simeq \mathrm{QCoh}(\mathrm{Loc}_{\check{G}_k}(\Sigma)).$$

Here, $\mathrm{Bun}_G(\Sigma)$ is the stack of (algebraic) G-bundles on Σ; $\check{G}_k$ is the *Langlands dual group* scheme, defined over k; and $\mathrm{Loc}_{\check{G}_k}(\Sigma)$ is the stack of $\check{G}_k$-local systems on Σ. (Not quite correct as stated...)

It is a very interesting conjecture which has generated a lot of deep and beautiful mathematics.

Geometric Satake

One way to approach the conjecture is to prove it "locally"; for example, replace Σ by a formal bubble, namely $\mathbb{B} := D \amalg_{D^\circ} D$ where D is a formal disk and D° is a formal punctured disk. Then

$$\mathrm{Bun}_G(\mathbb{B}) = G(\mathcal{O})\backslash G(F)/G(\mathcal{O}),$$

where $G(F) = G(\mathbf{C}(\!(t)\!))$ and $G(\mathcal{O}) = G(\mathbf{C}[\![t]\!])$. The quotient $G(\mathcal{O})\backslash G(F)$ is called the *affine Grassmannian*, and is denoted Gr_G.

In this case, the conjecture is a theorem of Bezrukavnikov-Finkelberg for $k = \mathbf{Q}$. (After using Koszul duality,) it states that there is an equivalence

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); \mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^*_{\mathbf{Q}}[2]/\check{G}_{\mathbf{Q}}),$$

where $\check{\mathfrak{g}}^*_{\mathbf{Q}}$ is the *coadjoint representation*. This is called the (derived) geometric Satake equivalence. It is essentially geometric Langlands for $\Sigma = \mathbf{P}^1$.

Remarks

Assume from now that G is *simply-laced* and $\pi_1(G) = 0$ (i.e., isogenous to $\mathrm{SL}_n, \mathrm{Spin}_{2n}$, E_6, E_7, or E_8). Then $\check{G}_k = G_k/Z(G_k)$, and one can identify $\check{\mathfrak{g}}_k^* \cong \mathfrak{g}_k$. So we can rewrite:

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); \mathbf{Q}) \simeq \mathrm{QCoh}(\mathfrak{g}_{\mathbf{Q}}[2]/\check{G}_{\mathbf{Q}}).$$

This is a *Fourier transform*: it sends the δ-sheaf at basepoint of Gr_G to the structure sheaf of $\mathfrak{g}_{\mathbf{Q}}[2]/\check{G}_{\mathbf{Q}}$. Taking endomorphisms, recover the well-known statement that $C^*(BG; \mathbf{Q}) \cong \mathrm{Sym}(\mathfrak{g}_{\mathbf{Q}}^*[-2])^{\check{G}_{\mathbf{Q}}}$. (But this is circular: this isomorphism is used in proving derived Satake.)

Quillen showed that there is a homotopy equivalence $\mathrm{Gr}_G \simeq \Omega G$, and in fact the Satake equivalence also captures a lot of classical calculations about the equivariant (co)homology of the based loop space of G.

Goal

Goal

Understand what happens if k is replaced by a commutative ring *spectrum*.

To understand the form that the answer might take, we will consider the case when G is a torus T. (You could take $T = \mathbf{G}_m$, but this obscures some of the combinatorics.) In this case:

- $\mathrm{Gr}_T = \Omega T = \pi_1(T)$ is just the lattice of cocharacters $\mathbf{G}_m \to T$, denoted $\mathbb{X}_*(T)$.
- The action of $T(\mathcal{O}) \simeq T$ on Gr_T is trivial.

Together, these facts tell us that $\mathrm{Shv}(\mathrm{Gr}_T/T(\mathcal{O}); k)$ is a rather simple category.

Torus

Let us unwind:

$$\mathrm{Shv}(\mathrm{Gr}_T/T(\mathcal{O}); k) \simeq \mathrm{Shv}(\mathbb{X}_*(T) \times BT; k) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Shv}(BT; k).$$

What do we mean by $\mathrm{Shv}(BT; k)$? This should be the category of T-equivariant k-modules. So, we could either work:

- *Borel-equivariantly*, so $\mathrm{Shv}(BT; k) = \mathrm{Mod}^{\wedge}_{C^*(BT;k)}$. Thus

$$\mathrm{Shv}(BT; k) = \mathrm{QCoh}(\mathsf{Hom}(\mathbb{X}^*(T), \widehat{\mathbf{G}}_k^Q)),$$

  where $\widehat{\mathbf{G}}_k^Q = \mathsf{Spf}\, C^*(BT; k)$ denotes the Quillen formal group over k.
- *genuine-equivariantly* (if k admits a genuine-equivariant refinement). So

$$\mathrm{Shv}(BT; k) = \mathrm{QCoh}(\mathsf{Hom}(\mathbb{X}^*(T), \mathbf{G}_k^Q)),$$

  where $\mathbf{G}_k^Q$ is a decompletion of the Quillen formal group.

Torus

If $\mathbf{H}_k^{\mathsf{Spec}} := \widehat{\mathbf{G}}_k^Q$ or $\mathbf{G}_k^Q$, and $T_{\mathbf{H}_k^{\mathsf{Spec}}} := \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{H}_k^{\mathsf{Spec}})$, we find

$$\mathrm{Shv}(\mathrm{Gr}_T/T(\mathcal{O}); k) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{QCoh}(T_{\mathbf{H}_k^{\mathsf{Spec}}}).$$

Notice that if $\check{T}_k := \mathrm{Spec}\, k[\mathbb{X}_*(T)]$, then $\mathrm{Rep}(\check{T}_k) = \bigoplus_{\mathbb{X}_*(T)} \mathrm{Mod}_k$. The group scheme $\check{T}_k$ is the *Langlands dual torus* defined over k. We find:

Satake equivalence for a torus

There is a k-linear equivalence

$$\mathrm{Shv}(\mathrm{Gr}_T/T(\mathcal{O}); k) \simeq \mathrm{QCoh}(T_{\mathbf{H}_k^{\mathsf{Spec}}} \times B\check{T}_k).$$

Works for any compact abelian T. If T is finite, $\check{T}_k$ is the Pontryagin dual, and the Satake equivalence becomes Hopkins-Kuhn-Ravenel character theory.

Other reductive groups

Given our success with tori, natural to wonder about the case of a general (split) reductive group G. Let $T \subseteq G$ be a maximal torus.

There is a theory of genuine-equivariant sheaves on topological stacks in development by Cnossen-Maegawa-Volpe and Konovalov-Perunov-Prikhodko. So one can make sense of $\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k)$.

We run into a problem on the Langlands dual side: what would replace $\check{T}_k$? If k is an ordinary commutative ring, it is replaced by the Langlands dual group $\check{G}_k$ defined over k: this is an algebraic group whose maximal torus is $\check{T}_k$.

If k is an arbitrary commutative ring spectrum, one needs to make sense of $\check{G}_k$ as a group scheme over k. Is this even possible?

No-go

One cannot naturally lift SL_2 to ku as an $\mathbf{E}_4$-scheme: power operations do not respect the relation $\det = 1$. (What about as an $\mathbf{E}_3$- or $\mathbf{E}_2$-scheme? I don't know.)

What to do?

Pretend that $\check{G}_k$ exists over k, and that there was a Satake equivalence

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k) \simeq \mathrm{QCoh}(\mathfrak{X}_k)$$

for some spectral k-stack $\mathfrak{X}_k$ having to do with $\check{G}_k$.

Suppose k is even. Any spectral k-stack X which is locally constructed from even affine k-schemes admits a degeneration to an ordinary graded $\pi_*(k)$-stack $X^\heartsuit$, given by degenerating $\mathcal{O}_X$ to $\pi_*\mathcal{O}_X$. (Just the even filtration.)

So, if there was a Satake equivalence as above, one would get a 1-parameter degeneration of $\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k)$ into QCoh of $\mathfrak{X}_k^\heartsuit$.

Revised goal

Try to construct the $\pi_*(k)$-stack $\mathfrak{X}_k^\heartsuit$ which $\mathfrak{X}_k$ degenerates to, and actually *prove* that there is a 1-parameter degeneration

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k) \rightsquigarrow \mathrm{QCoh}(\mathfrak{X}_k^\heartsuit).$$

Examples

We have two examples of the stack $\mathfrak{X}_k^\heartsuit$:

- k is an ordinary commutative ring: then Bezrukavnikov-Finkelberg tell us that

  $$\mathfrak{X}_k^\heartsuit = \mathfrak{g}_k(2)/\check{G}_k$$

  over $\pi_*(k) = k$.
- G is a torus T, and k arbitrary. Then $\mathfrak{X}_k = T_{\mathbf{H}_k^{\mathrm{Spec}}} \times B\check{T}_k$. So, if $\mathbf{H}$ is the group scheme over $\pi_*(k)$ given by $(\widehat{\mathbf{G}}_k^Q)^\heartsuit$ or $(\mathbf{G}_k^Q)^\heartsuit$, then

  $$\mathfrak{X}_k^\heartsuit = T_{\mathbf{H}} \times B\check{T}_{\pi_*(k)},$$

  where $T_{\mathbf{H}} = \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{H})$ and $\check{T}_{\pi_*(k)}$ denotes the *ordinary* group scheme given by the Langlands dual torus.

Note that $\mathbf{H} = \mathrm{Spf}\, \pi_*(k^{hT})$ in the Borel-equivariant case.

Adapting G to $\mathbf{H}$

We will write $\mathfrak{X}_k^{\heartsuit}$ as $G_{\mathbf{H}}/\check{G}_{\pi_*(k)}$ for some stack $G_{\mathbf{H}}$ such that $G_{\mathbf{G}_a(2)} = \mathfrak{g}_k(2)$, and $T_{\mathbf{H}} = \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{H})$. Here, $\check{G}_{\pi_*(k)}$ denotes the *ordinary* Langlands dual group, base-changed along $\mathbf{Z} \to \pi_*(k)$.

Definition (Fratila-Gunningham-Li, Moulinos-Robalo-Toen, Khan-Bouaziz, D., ...)

Let X be a $\pi_*(k)$-stack. The $\mathbf{H}$-loop space $\mathcal{L}_{\mathbf{H}}(X)$ is defined using the Tannakian formalism as

$$\mathcal{L}_{\mathbf{H}}(X) := \mathrm{Fun}^{\otimes, L}_{\pi_*(k)}(\mathrm{QCoh}(X)^{\otimes}, \mathrm{IndCoh}_0(\mathbf{H})^{\star}).$$

Here, $\mathrm{Coh}_0(\mathbf{H})^{\star}$ is the category of coherent sheaves on $\mathbf{H}$ of length zero, with symmetric monoidal structure given by convolution.

If $\mathbf{H}$ is a formal group, then $\mathcal{L}_{\mathbf{H}}(X) = \mathrm{Map}(B\mathbf{H}^{\vee}, X)$ where $\mathbf{H}^{\vee}$ is the Cartier dual of $\mathbf{H}$.

Examples

When $X = BG_{\pi_*(k)}$, there is a map $\mathcal{L}_{\mathbf{H}}(BG_{\pi_*(k)}) \to BG_{\pi_*(k)}$. The pullback along $\mathrm{Spec}(\pi_*(k)) \to BG_{\pi_*(k)}$ will be written $G_{\mathbf{H}}$. Here is a table of examples:

$\mathbf{H}$	$G_{\mathbf{H}}$
$\mathbf{G}_a(2)$	$\mathfrak{g}(2)$
$\widehat{\mathbf{G}_a}(2)$	$\mathfrak{g}^{\wedge}_{\mathcal{N}}(2)$
$\mathbf{G}_m$	G
$\widehat{\mathbf{G}_m}$	$G^{\wedge}_{\mathcal{U}}$
E elliptic curve	$\mathrm{Bun}^{\mathrm{ss}}_G(E)^{\mathrm{triv}}$

For notational simplicity, I have dropped the subscript $\pi_*(k)$; everything is defined over this base. Here, $\mathcal{N}$ is the cone of nilpotent elements, and $\mathcal{U}$ is the cone of unipotent elements.

General conjecture

Conjecture (D.)

If k is even, G is simply-laced and simply-connected, then there is a 1-parameter degeneration

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(G_{\mathbf{H}}/\check{G}),$$

where the right-hand side is defined over $\pi_*(k)$. Think of as a sheafy version of the even filtration. (If k is not even, then work even-locally on k.)

One also work non-G-equivariantly: then there should be a 1-parameter degeneration

$$\mathrm{Shv}^{G(\mathcal{O})-\mathrm{cbl}}(\mathrm{Gr}_G; k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\mathcal{N}_{\mathbf{H}}/\check{G}),$$

where $\mathcal{N}_{\mathbf{H}}$ is the "**H**-nilpotent cone", given by central fiber of the invariant-theoretic quotient map $G_{\mathbf{H}} \to G_{\mathbf{H}} /\!\!/ \check{G}$.

General conjecture

If k is an ordinary commutative ring, the conjecture says (in the genuine equivariant setting)

$$\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{g}(2)/\check{G}).$$

View as integral refinement of Bezrukavnikov-Finkelberg. In the Borel-equivariant setting, get $\mathfrak{g}^{\wedge}_{\mathcal{N}}(2)/\check{G}$; *renormalized* version (see Arinkin-Gaitsgory).

On the other extreme, suppose $G = 0$ and $k = \mathbb{S}$. Working even-locally on $\mathbb{S}$, one obtains the 1-parameter degeneration via Adams-Novikov:

$$\mathrm{Shv}(*; \mathbb{S}) = \mathrm{Sp} \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\mathcal{M}_{\mathrm{FG}}).$$

So one should think of the conjecture as mixing Langlands duality with Adams-Novikov phenomena.

A result

Here is a statement providing evidence for the conjecture (not quite correct as written).

Theorem (D.)

Suppose $k = \mathbf{Z}, \mathrm{ku}, \mathrm{KU}, \mathrm{ko}, j, \mathrm{KO}$, *or elliptic cohomology. Also suppose* G *is not of type* E_8. *Then there is a filtered category* $\mathcal{C}^{\mathrm{fil}}$ *over* $\mathrm{fil}^\star_{\mathrm{ev}}(k)$ *whose:*

- *underlying* k*-linear category* $\mathcal{C}$ *is* $\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k)$*;*
- *the associated graded* $\mathrm{gr}^\star_{\mathrm{ev}}(k)$*-linear category* $\mathcal{C}^{\mathrm{gr}}$ *is equivalent to* $\mathrm{QCoh}^{\mathrm{gr}}(G_{\mathbf{H}}/\check{G})$ *upon base-change to any algebraically closed field under* $\mathrm{gr}^\star_{\mathrm{ev}}(k)$*.*

When $G = \mathrm{GL}_n$, one does not need to do this base-change. This case was previously considered by Cautis-Kamnitzer when $k = \mathrm{KU}$.

Main tools: calculation of equivariant homology $\pi_* C^G_*(\Omega G; k)$ in terms of $\check{G}$; and purity arguments using cellularity of Gr_G (Schubert filtration).

Philosophy + remarks

How should one think about the 1-parameter degeneration

$$\mathrm{Shv}^{G(\mathcal{O})-\mathrm{cbl}}(\mathrm{Gr}_G/G(\mathcal{O}); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\mathcal{N}_{\mathbf{H}}/\check{G})?$$

(Working with the non-equivariant version of the conjecture for simplicity.) Recall when $G = 0$ and $k = \mathbb{S}$, this was supposed to be the degeneration of Sp to $\mathrm{QCoh}^{\mathrm{gr}}(\mathcal{M}_{\mathrm{FG}})$. This can be implemented through synthetic spectra, or equivalently (upon profinite completion) the category $\mathrm{SH}^{\mathrm{cell}}(\mathsf{Spec}(\mathbf{C}))$.

If X is a scheme over $\mathbf{C}$ equipped with a cellular stratification $\mathcal{S}$ (so each stratum is an affine space), let $\mathrm{SH}^{\mathcal{S}-\mathrm{cell}}(X)$ be the category of motivic spectra over X whose !- and $*$-restriction to each stratum is cellular. Then (upon profinite completion) one gets a 1-parameter degeneration

$$\mathrm{SH}^{\mathcal{S}-\mathrm{cell}}(X)[\tau^{-1}] \approx \mathrm{Shv}^{\mathcal{S}-\mathrm{cbl}}(X; \mathrm{Sp}) \rightsquigarrow \mathrm{SH}^{\mathcal{S}-\mathrm{cell}}(X)_{\tau=0},$$

and the right-hand side is sometimes $\mathrm{QCoh}^{\mathrm{gr}}$ on some algebraic stack. Can view as a "relative" version of synthetic spectra. The conjectural degeneration above roughly corresponds to the case $X = \mathrm{Gr}_G$ with the Schubert stratification.

Philosophy + remarks

Langlands duality with coefficients in an ordinary commutative ring k is of a "motivic nature", meaning roughly that the spectral side is ambivalent to the choice of k. If k is a ring spectrum, then the conjecture says instead that the spectral side depends on the choice of k essentially *only* through the corresponding 1-dimensional formal group $\mathbf{H}$ which controls Chern classes.

Note that in the stack $G_{\mathbf{H}}/\check{G}$, the "numerator" $G_{\mathbf{H}}$ depends on $\mathbf{H}$, so its fibers over $\mathrm{Spec}_{B\mathbf{G}_m}(\mathrm{gr}^{\star}_{\mathrm{ev}}(\mathbb{S})) \cong \mathcal{M}_{\mathrm{FG}}$ vary. But the "denominator" $B\check{G}$ is completely independent of the formal group $\mathbf{H}$: in fact, it is pulled back along the map $\mathcal{M}_{\mathrm{FG}} \to B\mathbf{G}_m$, so in a sense it is "defined over $\mathbf{F}_1$". This is in accordance with the motivic nature of Langlands duality.

Philosophy + remarks

Can also match objects under the degeneration: a G-space X defines a $\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k)$-module category; describing its degeneration in terms of $G_{\mathbf{H}}/\check{G}$ can often be very interesting. If k is an ordinary commutative ring, this is the content of *relative Langlands duality* (Ben-Zvi–Sakellaridis–Venkatesh). Here is an example:

Theorem (D.; here $X = \mathrm{PGL}_2/\mathbf{G}_m$)

There is a 1-parameter degeneration

$$\mathrm{Shv}(\mathrm{PGL}_2(\mathcal{O})\backslash\mathrm{PGL}_2(F)/\mathbf{G}_m(F); \mathrm{ku}) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(T^*_\beta(\mathbf{A}^2)/\mathrm{SL}_2),$$

where $T^*_\beta(\mathbf{A}^2)$ is the scheme of pairs $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$ such that $1 + \beta\langle u, v\rangle$ is a unit. The action of $\mathbf{Z}/2 = N_{\mathrm{PGL}_2}(\mathbf{G}_m)/\mathbf{G}_m$ on the left-hand side identifies with (a β-deformation of) the symplectic Fourier transform.

Upon base-change along $\mathrm{ku} \to \mathbf{Z}$, get a geometrization of spherical harmonics.

Loop rotation

The category $\mathrm{Shv}(\mathrm{Gr}_G/G(\mathcal{O}); k)$ is an $\mathbf{E}_3 \rtimes S^1$-monoidal category. I'll ignore the $\mathbf{E}_3$-structure, and focus on the S^1-action: this comes from *loop-rotation*. E.g., under the homotopy equivalence between Gr_G and $\Omega^2 BG = \mathrm{Map}_*(S^2, BG)$, the S^1-action rotates S^2. One can therefore consider the k^{hS^1}-linear category $\mathrm{Shv}_{S^1}(\mathrm{Gr}_G/G(\mathcal{O}); k)$.

Theorem (Bezrukavnikov-Finkelberg)

There is a $\mathbf{Q}^{hS^1} = \mathbf{Q}[\hbar]$-linear equivalence

$$\mathrm{Shv}_{S^1}(\mathrm{Gr}_G/G(\mathcal{O}); \mathbf{Q})[\hbar^{-1}] \simeq U(\check{\mathfrak{g}})\text{-mod}(\mathrm{Rep}(\check{G}))[\hbar^{\pm 1}].$$

Here, $U(\check{\mathfrak{g}})$ is the universal enveloping algebra of $\check{G}$.

Without loop rotation, the right-hand side was $\mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G})$. So, adding loop-rotation amounts to *deformation quantizing* $\check{\mathfrak{g}}^*$ to $U(\check{\mathfrak{g}})$. (There is a much more general story about $\mathbf{E}_3 \rtimes S^1$-algebras and deformation quantizations, via $\mathrm{fil}^\star_{\mathrm{ev}} C^*(\mathrm{Conf}_n(\mathbf{R}^3)_{hS^1}; \mathbb{S})$; for another time!)

Torus

What happens when we add in loop-rotation equivariance for other commutative ring spectra k? When $G = T$ is a torus, the T-action on $\mathrm{Gr}_T = \Omega T$ is trivial; but it is **not** *loop-rotation equivariantly* trivial. This is for the same reason that the S^1-action on Hochschild homology is interesting. In general (working Borel-equivariantly for simplicity), one finds:

Theorem (D.)

Suppose k is even, so that $\pi_*(k^{hS^1}) \cong \pi_*(k)[\hbar]^\wedge$. Let $T = \mathbf{G}_m$ for simplicity, so $\check{T} = \mathbf{G}_m$ too. Then there is a 1-parameter degeneration

$$\mathrm{Shv}_{S^1}(\mathrm{Gr}_T/T(\mathcal{O}); k) \rightsquigarrow \mathcal{D}^{\mathbf{H}}_{\check{T}}\text{-}\mathrm{mod}(\mathrm{Rep}(\check{T} \times \check{T})),$$

where $\mathcal{D}^{\mathbf{H}}_{\check{T}}$ is the associative ("$\mathbf{H}$-Weyl") $\pi_*(k)$-algebra defined by

$$\mathcal{D}^{\mathbf{H}}_{\check{T}} := \pi_*(k)[\hbar]\langle x^{\pm 1}, \nabla^{\mathbf{H}}_x \rangle^\wedge / (\nabla^{\mathbf{H}}_x x = (x \nabla^{\mathbf{H}}_x) +_{\mathbf{H}} \hbar).$$

Calculation is Koszul dual to an unpublished result of Arpon Raksit about the even filtration on $\mathrm{HC}^-((\mathbf{G}_m)_k/k)$. Can rephrase in terms of $\mathbf{E}_2$-Hochschild cohomology.

Torus

The algebra $\mathcal{D}^{\mathbf{H}}_{\check{T}}$ on the preceding slide is just the usual Weyl algebra of $\check{T}$ when k is an ordinary commutative ring; and it recovers the q-Weyl algebra when $k = \mathrm{ku}$. I will remark that the preceding result could be rewritten as

$$\mathrm{Shv}_{S^1}(\mathrm{Gr}_T/T(\mathcal{O}); k) \rightsquigarrow U_{\mathbf{H}}(\check{T})\text{-mod}(\mathrm{Rep}(\check{T})),$$

where $U_{\mathbf{H}}(\check{T}) = (\mathcal{D}^{\mathbf{H}}_{\check{T}})^{\check{T}}$ is isomorphic to $\pi_*(k)[\hbar, \nabla^{\mathbf{H}}_x]^\wedge$.

One can view $U_{\mathbf{H}}(\check{T})$ as an analogue of the enveloping algebra $U(\check{\mathfrak{t}})$.

What about other G? Let's for simplicity take $G = \mathrm{PGL}_2$, so $\check{G} = \mathrm{SL}_2$, and ask: what is the analogue of $U(\mathfrak{sl}_2)$ which deformation quantizes $(\mathrm{PGL}_2)_{\mathbf{H}}$?

$G = \mathrm{PGL}_2$

(Vague) conjecture (D.)

The category $\mathrm{Shv}_{S^1}(\mathrm{Gr}_{\mathrm{PGL}_2}/\mathrm{PGL}_2(\mathcal{O}); k)$ is related to modules over the associative algebra

$$U_{\mathbf{H}}(\mathrm{SL}_2) := \pi_*(k)[\hbar]\langle e, f, h\rangle^\wedge/I,$$

where I is given by the relations

$$\begin{aligned} eh &= (h -_{\mathbf{H}} \hbar)e, \\ fh &= (h +_{\mathbf{H}} \hbar)f, \\ ef - fe &= h(\overline{h} +_{\mathbf{H}} \hbar) - \overline{h}(h +_{\mathbf{H}} \hbar). \end{aligned}$$

Here, $\overline{h}$ is the inverse of h in $\mathbf{H}$.

I'm close to being able to prove such a statement, but cannot yet; relations above come from calculations with $\mathrm{Gr}_{\mathrm{PGL}_2}$. When $k = \mathbf{Z}[1/2]$, get $U(\mathfrak{sl}_2)$; when $k = \mathrm{ku}$, get essentially the quantum group $U_q(\mathrm{SL}_2)$ (where $q = 1 + \beta\hbar$).

Remarks

I find the algebra $U_{\mathbf{H}}(\mathrm{SL}_2)$ very beautiful. Its representation theory is similar to that of $U(\mathfrak{sl}_2)$ and of the quantum group. Also, it has a central "Casimir" element

$$c := fe - \overline{h}(h +_{\mathbf{H}} \hbar),$$

and there is an isomorphism

$$U_{\mathbf{H}}(\mathrm{SL}_2)/c \cong R\Gamma(\mathbf{P}^1; \mathcal{D}^{\mathbf{H}}_{\mathbf{P}^1}).$$

This is exactly like in Beilinson-Bernstein. One can also generalize $U_{\mathbf{H}}(\mathrm{SL}_2)$ to $U_{\mathbf{H}}(\check{G})$ for other $\check{G}$, via an **H**-deformation of the Serre relations in $U(\check{\mathfrak{g}})$.

I don't (yet?) know how to relate $U_{\mathbf{H}}(\check{G})$ to $\mathrm{Shv}_{S^1}(\mathrm{Gr}_G/G(\mathcal{O}); k)$. It should nevertheless be interesting to study $U_{\mathbf{H}}(\check{G})$ independently, e.g., in the context of Lusztig-Williamson's "philosophy of generations". In general, I think that there is a lot about representation theory that the combination of chromatic homotopy theory + geometry can be used to uncover.

Thank you!