

ROOTS OF UNITY IN $K(n)$ -LOCAL RINGS

ABSTRACT. The goal of this paper is to address the following question: if A is an \mathbf{E}_k -ring for some $k \geq 1$ and $f: \pi_0 A \rightarrow B$ is a map of commutative rings, when can we find an \mathbf{E}_k -ring R with an \mathbf{E}_k -ring map $g: A \rightarrow R$ such that $\pi_0 g = f$? A classical result in the theory of realizing \mathbf{E}_∞ -rings, due to Goerss–Hopkins, gives an affirmative answer to this question if f is étale. The goal of this paper is to provide answers to this question when f is ramified. We prove a non-realizability result in the $K(n)$ -local setting for every $n \geq 1$ for H_∞ -rings containing primitive p th roots of unity. As an application, we give a proof of the folk result that the Lubin–Tate tower from arithmetic geometry does not lift to a tower of H_∞ -rings over Morava E -theory.

1. INTRODUCTION

In this paper, we study some rigidity properties of structured ring spectra in $K(n)$ -local spectra (see [HS99] for a thorough introduction). The primary question we are concerned with in this paper is the following:

Question 1.1. Let A be an \mathbf{E}_k -ring for some $k \geq 1$, and let $f: \pi_0 A \rightarrow B$ be a map of commutative rings. Can we find an \mathbf{E}_k -ring R with an \mathbf{E}_k -ring map $g: A \rightarrow R$ such that $\pi_0 g = f$?

As one might expect, Question 1.1 is hard to resolve in general, particularly without any restrictions on the map f . Variants of this question with additional conditions imposed on the map f have been studied quite extensively. In the case $k = \infty$, Question 1.1 was originally raised in [SVW99], where they used topological Hochschild homology to provide a negative answer to the general form of the question: they showed that the extension $\mathbf{Z} \rightarrow \mathbf{Z}[i]$ does not lift to an \mathbf{E}_∞ -ring extension of the sphere spectrum. The question of adjoining a primitive root of unity to an \mathbf{E}_∞ -ring in the unramified case was studied in [SVW99, Theorem 3] and [BR07, Example 2.2.8]. We refer the reader to the lovely paper [Goe09] for further discussion of Question 1.1 from the perspective of chromatic homotopy theory.

One solution to Question 1.1 (which sets up a general framework for answering such questions) is provided by Goerss–Hopkins obstruction theory (see [GH04]); using their work, the following result was deduced in [BR07] (see also [Lur16, Theorem 7.5.0.6]):

Theorem 1.2. *Question 1.1 admits an affirmative answer if f is an étale map.*

For instance, if k is a finite field of characteristic p and $W(k)$ is the ring of Witt vectors, then the map $\mathbf{Z}_p \rightarrow W(k)$ is étale. As $\pi_0 \mathbf{S}_p = \mathbf{Z}_p$, Theorem 1.2 gives an \mathbf{E}_∞ -ring $\mathbf{S}_{W(k)}$, called the spherical Witt vectors, such that $\pi_0 \mathbf{S}_{W(k)} = W(k)$.

The goal of this paper is to address Question 1.1 in the world of $K(n)$ -local spectra when f is ramified. Our main result states that f in general cannot be lifted to a map of structured ring spectra, even if one restricts to studying H_∞ -rings. A H_∞ -ring structure is the analogue of an \mathbf{E}_∞ -ring structure in the stable homotopy category; in particular, \mathbf{E}_∞ -rings provide examples of H_∞ -rings. We show:

Theorem 1.3. *Let $n > 0$. There is no nontrivial $K(n)$ -local H_∞ -ring R at a prime $p > 2$ (resp. at $p = 2$) such that $\pi_0 R$ contains a primitive p th root of unity (resp. a 4th root of unity).*

Since \mathbf{E}_∞ -rings are, in particular, H_∞ -rings, Theorem 1.3 shows that no nontrivial $K(n)$ -local \mathbf{E}_∞ -ring R at a prime $p > 2$ (resp. at $p = 2$) such that $\pi_0 R$ contains a primitive p th root of unity (resp. a 4th root of unity).

One of the primary motivations for desiring a positive answer to Question 1.1 is because it often allows for the construction of sheaves of \mathbf{E}_∞ -rings on chromatically interesting moduli stacks M (such as the moduli stack of formal groups M_{FG} and the moduli stack of elliptic curves) which lift the structure sheaf of M . This is the perspective through which Question 1.1 is discussed in [Goe09]. For instance, Theorem 1.2 was used by Goerss–Hopkins–Miller to construct a sheaf of even-periodic \mathbf{E}_∞ -rings on the étale site of the moduli stack of elliptic curves; the global sections of this sheaf is the \mathbf{E}_∞ -ring TMF . A lot of large-scale phenomena in chromatic homotopy theory are shadows of the hope (discussed in [Goe09]) that there is some Grothendieck topology on the moduli stack M_{FG} of one-dimensional formal groups for which the structure sheaf $\mathcal{O}_{M_{FG}}$ admits a lift to a sheaf of \mathbf{E}_∞ -rings; the global sections of this \mathbf{E}_∞ -ring would be the sphere spectrum. Theorem 1.3 places severe restrictions on any Grothendieck topology on any moduli stack M (including M_{FG}) for which there is a sheaf of \mathbf{E}_∞ -rings on the associated site lifting the structure sheaf of M .

Our proof of Theorem 1.3 spans the whole of Section 2, and relies heavily on power operations obtained from the H_∞ -ring structure imposed on R . The proof also relies on a result of Hahn’s from [Hah16], which says that a $K(n)$ -acyclic H_∞ -ring is $K(n+1)$ -acyclic.

The following is a natural question motivated by Theorem 1.3:

Question 1.4. Consider the setup of Question 1.1, and let $k \geq 1$ be finite. Is there an analogue of Theorem 1.3 for $K(n)$ -local \mathbf{E}_k -rings? Namely, is there a $K(n)$ -local \mathbf{E}_k -ring R such that $\pi_0 R$ contains a primitive p th root of unity?

Remark 1.5. Other notions of ramification in the world of structured ring spectra have been studied. For instance, the map $\mathrm{bo} \rightarrow \mathrm{bu}$ is an example of a ramified extension in the sense of Dundas–Lindenstrauss–Richter [DLR18].

In Section 3, we study an application of Theorem 1.3 to lifting the Lubin–Tate tower from arithmetic geometry into the realm of spectral algebraic geometry. The Lubin–Tate tower, originally introduced in [Dri74], and further studied (for instance) in [RZ96, Chapter 3], plays an important role in the proof of the Jacquet–Langlands correspondence (see [Rog83]), and also carries information about power operations on Morava E -theory by a result of Strickland’s (see [Str98]). As the name suggests, it is a tower of moduli problems (which are in fact represented by affine formal schemes) living over the Lubin–Tate moduli space. The n th level of this tower parametrizes deformations of formal groups along with a level $\Gamma_1(p^n)$ -structure, i.e., a basis for the p^n -torsion in the deformed formal group. As a consequence of Theorem 1.3, we prove:

Theorem 1.6. *Let $M_k^h = \mathrm{Spf} A_k^h$ denote the Lubin–Tate space at level p^k and height h , and let E_h denote Morava E -theory at height h and the prime p . If $k > 0$, there is no H_∞ -ring over E_h whose underlying ring is A_k^h .*

It follows from the main result of [Str98] that there is an isomorphism $A_k^h \cong E_h^0(BC_{p^k})/\mathrm{Tr}$, where Tr is a certain ideal of $E_h^0(BC_{p^k})$, known as the transfer ideal. The ring $E_h^0(BC_{p^k})/\mathrm{Tr}$ parametrizes power operations on height h Morava E -theory. Consequently, Theorem 1.6 says that rings of power operations do not lift to derived algebraic geometry. A different proof of Theorem 1.6 was provided by Niko Naumann in private communication.

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2. RAMIFICATION AND STRUCTURED RING SPECTRA

We will use the following terminology.

Definition 2.1. A θ -ring is a ring A with an operator $\theta: A \rightarrow A$ such that the following identities are satisfied:

$$\begin{aligned}\theta(0) &= 0 \\ \theta(x+y) &= \theta(x) + \theta(y) + \frac{x^p + y^p - (x+y)^p}{p} \\ \theta(xy) &= \theta(x)y^p + \theta(y)x^p + p\theta(x)\theta(y).\end{aligned}$$

These conditions imply that $\psi^p(x) = x^p + p\theta(x)$ is a ring endomorphism of A .

The following result is described in [BMMS86, Chapter VIII], [Bou96], and [Hop14, Section 3] (the latter for \mathbf{E}_∞ -rings).

Theorem 2.2. *Let R be a $K(1)$ -local H_∞ -ring. Then, $\pi_0 R$ naturally has the structure of a θ -ring via power operations, denoted ψ^p and θ .*

Using the existence of these power operations, Mike Hopkins (in unpublished work) gave a proof of the following fact.

Lemma 2.3. *Let R be the spectrum $K_p[\zeta_p]$ obtained by adjoining a primitive p th root of unity to p -adic K -theory. Then R is not an H_∞ -ring.*

Proof. The spectrum R is $K(1)$ -local. If $\pi_0 R$ contained a primitive p th root of unity ζ , then the power operations would supply an equality

$$\psi^p(\zeta) = 1 + p\theta(\zeta).$$

Since ψ^p is a ring homomorphism, if Φ_p denotes the p th cyclotomic polynomial, we have

$$\Phi_p(\psi^p(\zeta)) = \psi^p(\Phi_p(\zeta)) = \psi^p(0) = 0,$$

so $\psi^p(\zeta)$ is another primitive p th root of unity, i.e., $\psi^p(\zeta) = \zeta^k$ for some $1 \leq k \leq p-1$. Since $\theta(\zeta) \neq 0$, the prime p must divide $1 - \zeta^k$, which is impossible. \square

In an email, Tyler Lawson gave a proof of the following generalization of Lemma 2.3 at the prime 2.

Proposition 2.4. *There is no $K(1)$ -local H_∞ -ring R such that $\pi_0 R$ contains a primitive 4th root of unity.*

Lawson asked if this result can be generalized to all heights and primes. Theorem 1.3 gives an affirmative answer to this question.

Proof. Denote by i the fourth root of unity in $\pi_0 R$. One can easily check that

$$\theta(xy) = \theta(x)y^2 + \theta(y)x^2 + 2\theta(x)\theta(y);$$

therefore, if $x = y = i$, then

$$-1 = \theta(-1) = -2(\theta(i) - \theta(i)^2).$$

This, however, implies that 2 is invertible, which is impossible by Lemma 2.5. \square

In the proof of Proposition 2.4, we utilized the following easy result.

Lemma 2.5. *Let E be a p -local ring spectrum (where p is any prime). If p is a unit in $\pi_0 E$, then E is $K(n)$ -locally trivial for any $n \geq 1$.*

Proof. Since p is a unit in $\pi_0 E$, the spectrum E is p -locally rational. This implies that $K(n)_* E$ is zero for any $n \geq 1$. \square

Proposition 2.6. *Let $p > 2$ be an odd prime. Suppose A is a θ -ring such that p is not a unit in A . Then A does not contain a primitive p th root of unity ζ .*

Proof. By definition, we have

$$\theta(x + y) = \theta(x) + \theta(y) + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i}.$$

By induction, we find that

$$(1) \quad \theta \left(\sum_{i=1}^k x_i \right) = \sum_{i=1}^k \theta(x_i) + \frac{\sum_{i=1}^k x_i^p - \left(\sum_{i=1}^k x_i \right)^p}{p}$$

Suppose A is a θ -ring containing a primitive p th root of unity. Since

$$(2) \quad 1 + \zeta + \cdots + \zeta^{p-1} = 0,$$

we learn that

$$\theta(1 + \zeta + \cdots + \zeta^{p-1}) = 0.$$

Applying Equation (1) to the left hand side, we get

$$(3) \quad 0 = \sum_{i=0}^{p-1} \theta(\zeta^i) + \frac{\sum_{i=0}^{p-1} \zeta^{ip} - \left(\sum_{i=0}^{p-1} \zeta^i \right)^p}{p}$$

Equation (2) implies that

$$(4) \quad \frac{\sum_{i=0}^{p-1} \zeta^{ip} - \left(\sum_{i=0}^{p-1} \zeta^i \right)^p}{p} = \frac{p - 0}{p} = 1.$$

It remains to compute $\sum_{i=0}^{p-1} \theta(\zeta^i)$. As ψ^p is multiplicative, we deduce that

$$\theta(x^n) = \frac{(x^p + p\theta(x))^n - x^{np}}{p};$$

this implies that the first term on the right hand side of Equation (3) is

$$\sum_{i=0}^{p-1} \theta(\zeta^i) = \sum_{i=1}^{p-1} \frac{(1 + p\theta(\zeta))^i - 1}{p} = \sum_{i=1}^{p-1} \sum_{k=1}^i \binom{i}{k} p^{k-1} \theta(\zeta)^k.$$

Splitting up terms, this sum becomes

$$\begin{aligned} \sum_{i=0}^{p-1} \theta(\zeta^i) &= \theta(\zeta) \sum_{i=1}^{p-1} i + \sum_{i=1}^{p-1} \sum_{k=2}^i \binom{i}{k} p^{k-1} \theta(\zeta)^k \\ &= p \left(\frac{p-1}{2} \theta(\zeta) + \sum_{i=1}^{p-1} \sum_{k=2}^i \binom{i}{k} p^{k-2} \theta(\zeta)^k \right). \end{aligned}$$

Let

$$Z = \frac{p-1}{2} \theta(\zeta) + \sum_{i=1}^{p-1} \sum_{k=2}^i \binom{i}{k} p^{k-2} \theta(\zeta)^k.$$

Equation (3) and Equation (4) show that

$$0 = pZ + 1,$$

which implies that p is invertible in A ; contradiction. □

Before proceeding, let us recall a theorem of Hahn's from [Hah16]:

Theorem 2.7 (Hahn). *A $K(n)$ -acyclic H_∞ -ring R is $K(n+1)$ -acyclic.*

Proof of Theorem 1.3. Proposition 2.4 proves the result when $p = 2$, so we can restrict to the case when p is odd. Suppose R is a nontrivial $K(1)$ -local ring such that $\pi_0 R$ contains a primitive p th root of unity. By Lemma 2.5, p is not a unit in $\pi_* R$. If $n = 1$, then $\pi_0 R$ is a θ -ring. It follows from Proposition 2.6 that $\pi_0 R$ is trivial, so R is contractible.

To prove the corollary at all heights, suppose $n \geq 2$, and let R be a $K(n)$ -local \mathbf{E}_∞ -ring such that $\pi_0 R$ contains a primitive p th root of unity ζ_p . As $\pi_0 R \xrightarrow{\pi_0 L_{K(1)}} \pi_0 L_{K(1)} R$ is a ring homomorphism, the element $\pi_0 L_{K(1)}(\zeta_p)$ is another primitive p th root of unity in $\pi_0 L_{K(1)} R$. By the arguments laid out above, $L_{K(1)} R \simeq 0$, i.e., R is $K(1)$ -acyclic. Theorem 2.7 therefore implies that R is $K(n)$ -acyclic for every $n \geq 2$, so $R \simeq L_{K(n)} R \simeq 0$, as desired. \square

Remark 2.8. By Theorem 1.3, any \mathbf{E}_∞ -ring R such that $p^{1/p} \in \pi_0 R$ must be $K(n)$ -acyclic for every $n \geq 1$, since $p^{1/p} + 1$ is, up to units, a primitive p th root of unity. This fact can also be proven directly, by applying $\theta(-)$ to the equation $(p^{1/p})^p - p = 0$ and using the multiplicativity of ψ^p as in the proof above.

3. THE LUBIN–TATE TOWER

In arithmetic geometry, the *Lubin–Tate tower* (originally introduced in [Dri74]; see also [RZ96, Chapter 3] and [Wei16, Section 2]) is a tower of finite flat extensions of the Lubin–Tate space, given by the rings representing deformations (of the Honda formal group) along with a chosen Drinfel’d level structure. These level structures play an important role in constructing power operations on Morava E -theory. They are also of immense arithmetic interest, as the Lubin–Tate tower simultaneously admits actions of three groups which form the main object of study in local Langlands.

Let R be a complete local Artinian $W(\overline{\mathbf{F}}_p)$ -algebra, and let (G, ι) be a deformation of the Honda formal group of height h to R . Let \mathfrak{m} denote the maximal ideal of R equipped with the group law given by G .

Definition 3.1. A *Drinfel’d level p^k -structure* is a homomorphism $\phi: (\mathbf{Z}/p^k)^h \rightarrow \mathfrak{m}$ such that there is an inequality of Cartier divisors

$$\sum_{a \in (\mathbf{Z}/p^k)^h[p]} [\phi(a)] \leq G[p].$$

If we pick a coordinate x on G , then this amounts to asking that

$$\left(\prod_{a \in (p^{k-1}\mathbf{Z}/p^k\mathbf{Z})^h} (x - \phi(a)) \right) |p|(x)$$

inside $R[[x]]$.

Drinfel’d proved the following result (see [Dri74, Proposition 4.3]).

Theorem 3.2 (Drinfel’d). *The functor that sends a complete local Artinian $W(\overline{\mathbf{F}}_p)$ -algebra (R, \mathfrak{m}) to the set of deformations of the Honda formal group (of height h) along with a level p^k -structure is corepresented by an $\pi_0 E_h$ -algebra A_k^h .*

Remark 3.3. These rings can be defined inductively, by adjoining the p^k -torsion in G ; for instance, if $k > 1$, then $A_k^h = A_{k-1}^h[x_1, \dots, x_h]/([p](x_i) - \phi(a_i))$, where $\{a_i\}$ is the canonical basis for $(\mathbf{Z}/p^k)^n$.

Example 3.4. At height $h = 1$, a level p^k -structure on \mathbb{G}_m is a choice of basis for the p^k -torsion; this is exactly the choice of a primitive p^k -torsion point. One might therefore expect that $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. This is indeed true: we have $A_1^1 \cong \mathbf{Z}_p[\zeta_p]$ (see, e.g., [Wei16, Section 2.3]; alternatively, this follows immediately from the explicit description given in [Dri74, Lemma in Proposition 4.3]). It follows from this that if $k > 1$, then $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. Indeed, recall that $[p](x) = (1+x)^p - 1$. Moreover, note that

if $\phi: (\mathbf{Z}/p^{k-1})^h \rightarrow A_{k-1}^1$ is the universal level p^{k-1} -level structure, then $\phi(1) = \zeta_{p^{k-1}} - 1$. Using Remark 3.3, we find that $A_k^1 \cong A_{k-1}^h[x]/([p](x) - \phi(1)) \cong \mathbf{Z}_p[\zeta_{p^k}]$, as desired.

The main result of this section is:

Theorem 3.5. *Let $M_k^h = \mathrm{Spf} A_k^h$ denote the Lubin–Tate space at level p^k and height h , and let E_h denote Morava E -theory at height h and the prime p . If $k > 0$, there is no H_∞ -ring over E_h whose underlying ring is A_k^h .*

Proof. There is a “determinant” map of formal schemes $M_k^h \rightarrow M_k^1$ constructed in [Wei16, Section 2.5]. We showed in Example 3.4 that $A_k^1 \cong \mathbf{Z}_p[\zeta_{p^k}]$. The image of ζ_{p^k} under the induced map $\mathbf{Z}_p[\zeta_{p^k}] \rightarrow A_k^h$ is a primitive p^k th root of unity in A_k^h . Applying Theorem 1.3, we conclude that A_k^h does not lift to a $K(h)$ -local H_∞ -ring; in particular, there is no $K(h)$ -local E_h -algebra whose homotopy is isomorphic to A_k^h . \square

Remark 3.6. The PEL Shimura varieties with Drinfel’d level structures considered in [HT01, §III.1] do not lift to derived stacks: if they did, then the completion at a height h point would give an \mathbf{E}_∞ -ring lifting A_k^h , contradicting Theorem 3.5.

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