ÉTALE COMPARISON

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1. The main results

In this talk, we will describe the étale comparison theorem in prismatic cohomology; since this is perhaps the most algebro-geometric of the planned talks, I will try to describe connections to homotopy theory wherever appropriate. The moral of this talk is that absolutely integrally closed valuation rings of rank ≤ 1 are very well behaved, and often many calculations reduce to the study of these rings.

The main result is:

Theorem 1.1 ([BS19, Theorem 9.1]). *Let (A, I) be a bounded prism, and let X be a smooth p-adic formal scheme over $\overline{A} = A/I$. If X_η denotes the generic fiber over $\mathbf{Q}_p$ viewed as an adic space, there is an isomorphism*

$$\Gamma_{\mathrm{et}}(X_\eta; \mathbf{Z}/p^n) \cong \left(\Gamma_{\Delta}(X/A)/p^n[\tfrac{1}{I}]\right)^{\phi=1}.$$

In particular, if $X = \mathrm{Spf}(S)$ is affine and $(A, (d))$ is a perfect prism, then

$$\Gamma_{\mathrm{et}}(\mathrm{Spec}(R[\tfrac{1}{p}]); \mathbf{Z}/p^n) \cong \left(\Delta_{S/A}[\tfrac{1}{d}]/p^n\right)^{\phi=1}.$$

We will only focus on the proof of Theorem 1.1 in the case when $X = \mathrm{Spf}(S)$ is affine. Recall that taking ϕ-fixed points only produces the untwisted syntomic cohomology $\mathbf{Z}_p(0)$: if R is a perfectoid ring, then $\mathbf{Z}_p(0)^{\mathrm{syn}}(R) = A_{\mathrm{inf}}(R)^{\phi=1}$. One can therefore ask for a variant of the étale comparison which compares the p-adic Tate twists $\mathbf{Z}_p(n)^{\mathrm{syn}}$ of syntomic cohomology with the usual étale sheaves $\mathbf{Z}_p(n)^{\mathrm{et}}$.

Recollection 1.2. If R is a quasisyntomic ring, then the syntomic complexes $\mathbf{Z}_p(n)^{\mathrm{syn}}(R)$ are defined as

$$\mathbf{Z}_p(n)^{\mathrm{syn}}(R) = \mathrm{fib}(\mathcal{N}^{\geq n}\hat{\Delta}_R\{n\} \xrightarrow{\phi-\mathrm{can}} \hat{\Delta}_R\{n\}).$$

If R is perfectoid and $d \in A_{\mathrm{inf}}(R)$ is a generator of the kernel of Fontaine's map $\theta : A_{\mathrm{inf}}(R) \to R$, then this is simply

$$\mathbf{Z}_p(n)^{\mathrm{syn}}(R) = \mathrm{fib}(\phi^{-1}(d^n)A_{\mathrm{inf}}(R) \xrightarrow{\frac{\phi}{d^n}-1} A_{\mathrm{inf}}(R)).$$

A definition which is more in line with this seminar might be that $\mathbf{Z}_p(n)^{\mathrm{syn}}(R)$ is the global sections of the structure sheaf of the syntomic stack $\mathrm{Spf}(R)^{\mathrm{syn}}$, which we haven't yet gotten to (and probably won't...).

Theorem 1.3 ([BL22, Theorem 8.5.1]). *Let R be an animated $\mathbf{Z}_p[\zeta_{p^\infty}]$-algebra. The sequence $(1, \zeta_p, \zeta_{p^2}, \cdots)$ defines an element $\epsilon \in \pi_0\mathbf{Z}_p(1)^{\mathrm{syn}}(\mathbf{Z}_p[\zeta_{p^\infty}])$. Then, there is a comparison map*

$$\left(\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{syn}}(R)\right)[\tfrac{1}{\epsilon}] \to \bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{et}}(R[\tfrac{1}{p}]),$$

which is an isomorphism after p-completion.

In this form, the étale comparison theorem looks like it might have a homotopy-theoretic analogue.

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Recollection 1.4. One of the main results of [BMS19] is that there is a filtration on $\mathrm{TC}(R)$ such that $\mathrm{gr}^n_{\mathrm{mot}} \mathrm{TC}(R) = \mathbf{Z}_p(n)^{\mathrm{syn}}(R)[2n]$. An older result, due to Thomason [Tho85], says that there is a filtration on $K(1)$-local K-theory $L_{K(1)}K(R[\frac{1}{p}])$ such that $\mathrm{gr}^n_{\mathrm{mot}} L_{K(1)}K(R[\frac{1}{p}]) = \mathbf{Z}_p(n)^{\mathrm{et}}(R[\frac{1}{p}])[2n]$. This is often stated as the fact that if p is invertible on a (nice enough) X, there is a Thomason spectral sequence

$$E_1^{i,j} = \mathrm{H}^i_{\mathrm{et}}(X; \mathbf{Z}_p(j)) \Rightarrow \pi_{2j-i} L_{K(1)}K(X).$$

A natural topological analogue of the comparison between étale and syntomic cohomology would be a comparison between the $K(1)$-localizations of TC and K-theory.

Theorem 1.5 ([BCM20, Theorem 2.17])**.** *If R is a commutative ring, there is a natural equivalence $L_{K(1)}\mathrm{TC}(R) \simeq L_{K(1)}K(R^\wedge_p[\frac{1}{p}])$.*

Example 1.6. A rather silly example of this result is the case when R is an $\mathbf{F}_p$-algebra. Then, $\mathrm{TC}(R)$ is a module over $\mathrm{TC}(\mathbf{F}_p) \cong C^*(S^1; \mathbf{Z}_p)$; since $L_{K(1)}\mathrm{TC}(\mathbf{F}_p) = 0$, we see that $L_{K(1)}\mathrm{TC}(R) = 0$. Similarly, $R^\wedge_p[\frac{1}{p}] = 0$, so Theorem 1.5 just asserts that $0 = 0$.

It would be interesting to know whether there is a motivically filtered analogue of Theorem 1.5, so that one recovers Theorem 1.1 on associated graded. We will not discuss Theorem 1.5 in this talk.

Remark 1.7. In the statement of Theorem 1.5, we did not make any assumption that R be an animated $\mathbf{Z}_p[\zeta_{p^\infty}]$-algebra, but such an assumption was present in Theorem 1.3. This assumption simply guarantees the existence of the element ϵ. To explain this, let us take R to be $\mathbf{Z}_p^{\mathrm{cycl}} := \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge_p$, so that R is in fact a p-complete perfectoid ring. Then

$$\pi_* \mathrm{TP}(\mathbf{Z}_p^{\mathrm{cycl}}) \cong A_{\mathrm{inf}}(\mathbf{Z}_p^{\mathrm{cycl}})[\hbar^{\pm 1}] \cong \mathbf{Z}_p[q^{\pm 1/p^\infty}]^\wedge_{(p,[p]_q)}[\hbar^{\pm 1}],$$

where $|\hbar| = -2$. This can be used to show that there is a class in $\pi_2 \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$, which represents $\epsilon \in \mathbf{Z}_p(1)^{\mathrm{syn}}(\mathbf{Z}_p^{\mathrm{cycl}})$.

Let us now turn to actually proving these results. As we already saw in Remark 1.7 with $\mathbf{Z}_p^{\mathrm{cycl}}$, things tend to simplify a lot for perfectoid rings. Consequently, one possible strategy towards proving these results might be to somehow reduce to the perfectoid case. This is indeed possible, through the use of the "arc-topology". We will therefore break this talk up into the following parts:

(a) Theorem 1.3, assuming Theorem 1.1.
(b) Defining the arc-topology.
(c) Theorem 1.1 in the case $n = 0$, and the reduction to perfectoid rings.

2. Weight ≥ 1 comparison

Throughout this section, we will assume Theorem 1.1. Let R be an animated $\mathbf{Z}_p[\zeta_{p^\infty}]$-algebra. To prove Theorem 1.3, we need to understand $\mathbf{Z}_p(n)^{\mathrm{syn}}(R)[\frac{1}{\epsilon}]$ and $\mathbf{Z}_p(n)^{\mathrm{et}}(R[\frac{1}{p}])$. Let us begin with étale cohomology.

Lemma 2.1. *Let R be an animated $\mathbf{Z}_p[\zeta_{p^\infty}]$-algebra. There is an isomorphism of graded $\mathbf{Z}_p$-algebras*

$$\bigoplus_{n \in \mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{et}}(R[\tfrac{1}{p}]) \xrightarrow{\sim} \mathbf{Z}_p(0)^{\mathrm{et}}(R[\tfrac{1}{p}])[\epsilon^{\pm 1}],$$

where ϵ lives in weight 1.

Proof. We begin by analyzing the case $R = \mathbf{Z}_p[\zeta_{p^\infty}]$ (so $R[\frac{1}{p}] \cong \mathbf{Q}_p(\zeta_{p^\infty})$). Then, $\mathbf{Z}_p(1)^{\mathrm{et}}(\mathbf{Q}_p(\zeta_{p^\infty}))$ is tautologically isomorphic to the Tate module $T_p(\mathbf{Q}_p(\zeta_{p^\infty})^\times)$. This has a canonical class ϵ defined by the system $(1, \zeta_p, \zeta_{p^2}, \cdots)$, and it furnishes a canonical isomorphism $\mathbf{Z}_p^{\mathrm{et}} \to \mathbf{Z}_p(1)^{\mathrm{et}}$ of étale sheaves on $\mathbf{Q}_p(\zeta_{p^\infty})$-schemes. This immediately implies the desired result. ∎

This reduces calculations on the étale side to weight zero, where can now use Theorem 1.1 to obtain an isomorphism

$$\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{et}}(R[\tfrac{1}{p}]) \xrightarrow{\sim} \mathbf{Z}_p(0)^{\mathrm{et}}(R[\tfrac{1}{p}])[\epsilon^{\pm 1}] \xrightarrow{\sim} (\mathbb{\Delta}_{R/A}[\tfrac{1}{d}]^\wedge_p)^{\phi=1}[\epsilon^{\pm 1}]. \tag{1}$$

Here, we work with the prism $A = A_{\mathrm{inf}}(\mathbf{Z}_p^{\mathrm{cycl}}) = \mathbf{Z}_p[q^{\pm 1/p^\infty}]^\wedge_{(p,[p]_q)}$, and $d = [p]_q$. There is a class in $\mathbf{Z}_p(1)^{\mathrm{syn}}(\mathbf{Z}_p^{\mathrm{cycl}})$, which will also be denoted ϵ, which may be understood from a topological perspective as follows.

Construction 2.2. The desired class $\epsilon \in \mathbf{Z}_p(1)^{\mathrm{syn}}(\mathbf{Z}_p^{\mathrm{cycl}})$ can be viewed as a class in $\pi_2\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$. There is a canonical map $B(\mathbf{Z}_p^{\mathrm{cycl}})^\times_+ \to K(\mathbf{Z}_p^{\mathrm{cycl}})$, which upon composition with the cyclotomic trace gives an $\mathbf{E}_\infty$-map

$$(B\mu_{p^\infty})_+ \simeq B(\mathbf{Z}_p^{\mathrm{cycl}})^\times_+ \to \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}}).$$

Taking p-completions, we obtain an $\mathbf{E}_\infty$-map $(\mathbf{C}P^\infty_+)^\wedge_p \to \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$. As mentioned, we will view ϵ as a class in $\mathbf{Z}_p(1)^{\mathrm{syn}}(\mathbf{Z}_p^{\mathrm{cycl}})$, and hence as a map $\epsilon : \mathbf{Z}_p(0)^{\mathrm{syn}} \to \mathbf{Z}_p(1)^{\mathrm{syn}}$. We therefore obtain an endomorphism ϵ of $\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{syn}}(R)$.

To prove Theorem 1.3, it suffices to show that the right-hand side of (1) can be identified with $\left(\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{syn}}(R)\right)[\frac{1}{\epsilon}]$. As in Lemma 2.1, the key case to understand is $R = \mathbf{Z}_p^{\mathrm{cycl}}$. In this case, $\mathbf{Z}_p(0)^{\mathrm{et}}(\mathbf{Q}_p(\zeta_{p^\infty})) \cong \mathbf{Z}_p$, so we need to show that there is a p-complete isomorphism

$$\left(\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{syn}}(\mathbf{Z}_p^{\mathrm{cycl}})\right)[\tfrac{1}{\epsilon}] \simeq \mathbf{Z}_p[\epsilon^{\pm 1}]. \tag{2}$$

We will only prove (2), and instead refer to [BL22, Proof of Theorem 8.5.1] for the argument that for a general $\mathbf{Z}_p^{\mathrm{cycl}}$-algebra R, there is a p-complete equivalence

$$(\mathbb{\Delta}_{R/\mathbf{Z}_p[q^{\pm 1/p^\infty}]^\wedge_{(p,[p]_q)}}[1/[p]_q]^\wedge_p)^{\phi=1}[\epsilon^{\pm 1}] \xrightarrow{\sim} \left(\bigoplus_{n\in\mathbf{Z}} \mathbf{Z}_p(n)^{\mathrm{syn}}(R)\right)[\tfrac{1}{\epsilon}].$$

One can prove the equivalence (2) algebraically, but a topological perspective goes as follows. The $\mathbf{E}_\infty$-map $(\mathbf{C}P^\infty_+)^\wedge_p \to \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}}) \to L_{K(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$ sends the Bott class in $\pi_2\mathbf{C}P^\infty$ to an invertible class in $\pi_2 L_{K(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$ by [BCM20, Proposition 3.5]. Therefore, we obtain an $\mathbf{E}_\infty$-map

$$(\mathbf{C}P^\infty_+)^\wedge_p[\tfrac{1}{\beta}] \simeq \mathrm{KU}^\wedge_p \to L_{K(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}}).$$

The equivalence (2) can be viewed as an algebraic manifestation of the following:

Proposition 2.3. *The map* $\mathrm{KU}^\wedge_p \to L_{K(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$ *is an equivalence of* $\mathbf{E}_\infty$*-rings.*

The argument that follows is sort of unsatisfactory: it uses heavy tools, and in particular, is essentially how Theorem 1.5 is proved. There's probably a more elementary proof.

Proof. Using the main result of [CMM18], there is a Cartesian square (everything is p-completed)

$$\begin{array}{ccc} K(\mathbf{Z}_p^{\mathrm{cycl}}) & \longrightarrow & \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}}) \\ \downarrow & & \downarrow \\ K(\mathbf{Z}_p^{\mathrm{cycl}}/p) & \longrightarrow & \mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}}/p). \end{array}$$

The bottom row is killed by $L_{K(1)}$ since they are algebras over $L_{K(1)}K(\mathbf{F}_p) \simeq 0$. Therefore, $L_{K(1)}K(\mathbf{Z}_p^{\mathrm{cycl}}) \xrightarrow{\sim} L_{K(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cycl}})$, so we need to show that $\mathrm{KU}^\wedge_p \xrightarrow{\sim} L_{K(1)}K(\mathbf{Z}_p^{\mathrm{cycl}})$.

Moreover, $L_{K(1)}K(\mathbf{Z}_p^{\mathrm{cycl}}) \simeq L_{K(1)}K(\mathbf{Q}_p(\zeta_{p^\infty}))$ by [BCM20, Theorem 1.1], or by [LMMT20, Corollary 4.22]. By [HN20, Lemma 1.3.7] and Suslin rigidity, there are equivalences

$$\mathrm{KU}_p^\wedge \xrightarrow{\sim} L_{K(1)}K(\mathcal{O}_{\mathbf{C}_p}) \xrightarrow{\sim} L_{K(1)}K(\mathbf{C}_p),$$

so it suffices to show that there is an equivalence $L_{K(1)}K(\mathbf{Q}_p(\zeta_{p^\infty})) \xrightarrow{\sim} L_{K(1)}K(\mathbf{C}_p)$. But this is a map of filtered $\mathbf{E}_\infty$-rings (where both objects are equipped with Thomason's filtration), and so it suffices to check that the étale cohomologies of $\mathbf{Q}_p(\zeta_{p^\infty})$ and $\mathbf{C}_p$ with coefficients in $\mathbf{Z}_p(j)$ agree for all j. This can be done by a direct calculation: by Lemma 2.1, $\mathbf{Z}_p(j)^{\mathrm{et}}(\mathbf{Q}_p(\zeta_{p^\infty})) \cong \mathbf{Z}_p(0)^{\mathrm{et}}(\mathbf{Q}_p(\zeta_{p^\infty}))$ and $\mathbf{Z}_p(j)^{\mathrm{et}}(\mathbf{C}_p) \cong \mathbf{Z}_p(0)^{\mathrm{et}}(\mathbf{C}_p)$ for every $j \in \mathbf{Z}$. It therefore suffices to observe that $\mathbf{Z}_p(0)^{\mathrm{et}}(\mathbf{Q}_p(\zeta_{p^\infty})) = \mathbf{Z}_p(0)^{\mathrm{et}}(\mathbf{C}_p) \cong \mathbf{Z}_p$. □

In the remainder of this talk, we will prove Theorem 1.1.

3. The arc-topology

The arc-topology was introduced by Bhatt and Mathew in [BM21]; in this talk, we will only use a slight variant, known as the arc_p-topology. I found [Ked22] to be a helpful resource.

Definition 3.1. A map $f : X \to Y$ of qcqs schemes is an arc-*cover* if for every valuation ring V of rank ≤ 1 equipped with a map $\mathrm{Spec}(V) \to X$, there is an extension[1] $V \to W$ of valuation rings of rank ≤ 1, and a map $\mathrm{Spec}(W) \to Y$, which fits into the diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spec}(V) \arrow[r] & X. \end{tikzcd}$$

This defines a Grothendieck topology in the usual way: one takes the coverings $\{Y_i \to X\}_{i \in I}$ to be those such that for any affine open $U \hookrightarrow X$, there is a finite set J, a map $e : J \to I$, and affine opens $U_j \subseteq f_{e(j)}^{-1}(U)$ for each $j \in J$, such that $\coprod_j U_j \to U$ is an arc-cover. This topology is known as the arc-*topology*.

The arc_p-*topology* is essentially the same thing; one now assumes that the schemes involved are derived p-complete, and the test valuation rings are p-complete, have $p \neq 0$, and are still of rank ≤ 1. Since we will always be p-complete, we will not bother distinguishing between the arc- and arc_p-topologies.

Remark 3.2. In practice, knowing that something is an arc-sheaf isn't enough to show that it is an arc_p-sheaf. To make the translation, one needs to use the observation that if $A \to B$ is an arc_p-cover, then $A \to B \times A/p$ is an arc-cover. For example, if F is an arc-sheaf on p-complete R-algebras, then F will be an arc_p-sheaf if the following condition is satisfied: the map $F(A \times B) \to F(A)$ is an isomorphism for every p-complete ring of the form $A \times B$ with $pB = 0$. We will not check this in any of our discussion below.

Remark 3.3. There is a variant of the arc-topology which is known as the *v-topology*: here, a map $f : X \to Y$ of qcqs schemes is a *v-cover* if for every valuation ring V equipped with a map $\mathrm{Spec}(V) \to X$, there is an extension $V \to W$ of valuation rings, and a map $\mathrm{Spec}(W) \to Y$, which fits into the diagram

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathrm{Spec}(V) \arrow[r] & X. \end{tikzcd}$$

[1] I.e., an injective local map.

In other words, V and W are no longer required to be of rank ≤ 1. The main difference between the v- and arc-topologies is the assumption on the rank of the test valuation rings. Moreover, all v-covers are arc-covers, and these are the same in the Noetherian situation.

The v-topology agrees with the h-topology (which is generated by étale covers and proper surjections) in the case of finite-type maps between Noetherian schemes. In general, though, v-covers are inverse limits of h-covers.

Remark 3.4 (Criterion for arc-descent)**.** Given a valuation ring V and a prime ideal $\mathfrak{p} \subseteq V$, we can "break up" V into $V/\mathfrak{p}$ and $V_{\mathfrak{p}}$. This leads to the following criterion for being an arc-sheaf ([BM21, Theorem 4.1]): a functor $\mathcal{F} : \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z},\leq 0}$ which is finitary is an arc-sheaf if and only if it is a v-sheaf, and for every valuation ring V with algebraically closed fraction field, and every prime ideal $\mathfrak{p} \subseteq V$, the following square is Cartesian:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(V/\mathfrak{p}) \arrow[d] \\ \mathcal{F}(V_{\mathfrak{p}}) \arrow[r] & \mathcal{F}(\kappa(\mathfrak{p})). \end{tikzcd}$$

This can be thought of as an excision property. This gives a convenient approach to checking that a functor $\mathcal{F}$ something is an arc-sheaf: one first checks that $\mathcal{F}$ satisfies h-descent; using finitary properties, one could then check that $\mathcal{F}$ satisfies v-descent; if $\mathcal{F}$ satisfies excision, then it satisfies arc-descent.

One of the main utilities of the arc-topology is that it has a very nice basis, given by perfectoid rings.

Recollection 3.5. A ring R is said to be perfectoid if R is p-complete, R/p is semiperfect, the kernel of Fontaine's map $\theta : W(R^\flat) \to R$ is principal, and there is some $\pi \in R$ such that π^p is a unit multiple of p. (If R is p-torsionfree, then the condition on $\ker(\theta)$ can be replaced by the following: if $x \in R[1/p]$ and $x^p \in R$, then $x \in R$.)

Lemma 3.6. *Any p-adic formal scheme X admits an* arc*-cover by* $\mathrm{Spf}(S)$ *for some perfectoid ring S.*

Proof. We can assume that $X = \mathrm{Spf}(R)$. Let I be a set of representatives of rank 1 valuations on R; then, there is a map $R \to \prod_{v \in I} R_v$, where R_v is the valuation ring associated to $v \in I$. Let R_v^+ denote the absolute integral closure of R_v, and let $(R_v^+)_p^\wedge$ denote its p-completion. We claim that R_v^+ is perfectoid; the resulting map $R \to \prod_{v \in I}(R_v^+)_p^\wedge$ is our desired arc-cover.

Let us now show that the p-completion of an absolutely integrally closed domain V is perfectoid. First, if V is an $\mathbf{F}_p$-algebra, then the assumptions on V imply that it must be perfect: indeed, V is reduced, and the Frobenius is surjective (since for any $v \in V$, the monic polynomial $x^p - v \in V[x]$ has a root), which implies the claim. Now, suppose that V is of mixed characteristic. Because V is not a DVR, we can choose some $x \in V$ such that $x^p | p$. Since the Frobenius on V/p is surjective, there is an element $y \in V$ whose image under the Frobenius map $V/p \to V/p$ is $\frac{p}{x^p}$. In other words, $y^p = \frac{p}{x^p} \pmod p$, i.e., $\frac{(xy)^p}{p} \equiv 1 \pmod p$. It follows that if $\pi = xy$, then π^p is a unit multiple of p. This gives the desired claim. □

This formally implies the following useful result (when using this below, we'll ignore the hypercompleteness issue):

Proposition 3.7 (Unfolding)**.** *There is an equivalence between the ∞-category of* arc*-hypersheaves of spectra on derived p-complete rings and the ∞-category of* arc*-hypersheaves of spectra on perfectoid rings.*

4. Weight zero comparison

Let us now show Theorem 1.1. Define the following two functors F and G on affine formal schemes:

$$F : R \mapsto \Gamma_{\mathrm{et}}(R[\tfrac{1}{p}]; \mathbf{Z}/p^n),$$
$$G : R \mapsto (\mathbb{\Delta}_{R/A}[\tfrac{1}{d}]/p^n)^{\phi=1}.$$

Lemma 4.1. *The functor F is an* arc*-sheaf.*

Proof. In fact, we will show more generally that if R is a commutative ring and $\mathcal{F}$ is a torsion étale sheaf on $\mathrm{Spec}(R)$, then the functor sending an R-scheme $c : X \to \mathrm{Spec}(R)$ to $\Gamma_{\mathrm{et}}(X; c^*\mathcal{F})$ is an arc-sheaf. This is [BM21, Theorem 5.4], and the argument below is taken from *loc. cit.*

By the discussion in Remark 3.4, it suffices to show that the assignment $X \mapsto \Gamma_{\mathrm{et}}(X; c^*\mathcal{F})$ is a v-sheaf, and that it satisfies excision. Using [Ryd10, Theorem 3.12], we can reduce to checking that $\Gamma_{\mathrm{et}}(-; \mathcal{F})$ satisfies descent for a proper surjection $f : X \to Y$ of finite type. Since the étale topology has enough points (given by strictly Henselian local rings). We can therefore assume that Y is of the form $\mathrm{Spec}(R)$ where R is a strictly Henselian local ring. Let y be the closed point of R, let $\kappa(y)$ be its residue field, and let X_y denote the fiber product $X \times_Y \{y\}$. Then proper base-change guarantees that $\Gamma_{\mathrm{et}}(X; \mathcal{F}) \simeq \Gamma_{\mathrm{et}}(X_y; \mathcal{F})$. Therefore, it suffices to show that $\Gamma_{\mathrm{et}}(-; \mathcal{F})$ satisfies descent for the map $X_y \to \{y\}$. However, the map $X_y \to \{y\}$ admits a section after base-changing along $\mathrm{Spec}(\overline{\kappa(y)}) \to \mathrm{Spec}(\kappa(y)) = \{y\}$. The topological invariance of étale cohomology lets us conclude that this base-change does not affect the value of $\Gamma_{\mathrm{et}}(-; \mathcal{F})$. We therefore obtain a splitting of the cosimplicial diagram $\Gamma_{\mathrm{et}}(X_y^{\times \bullet}; \mathcal{F})$, which implies that $\Gamma_{\mathrm{et}}(-; \mathcal{F})$ satisfies descent for the map $X_y \to \{y\}$.

It remains to show that the assignment $X \mapsto \Gamma_{\mathrm{et}}(X; \mathcal{F})$ satisfies excision. This again follows from the fact that the étale topology has enough points given by strictly Henselian local rings. First, observe that an absolutely integrally closed valuation ring V is strictly Henselian: it is Henselian and has algebraically closed residue field because any polynomial in $V[x]$ splits into a product of linear factors thanks to the absolute integral closedness of V. Some thought shows that $V/\mathfrak{p}$ and $V_\mathfrak{p}$ are therefore also both strictly Henselian. Since V and $V/\mathfrak{p}$ are both strictly Henselian, and both have the same residue field, we conclude that $\Gamma_{\mathrm{et}}(V; \mathcal{F}) \xrightarrow{\sim} \Gamma_{\mathrm{et}}(V/\mathfrak{p}; \mathcal{F})$. Similarly, $\Gamma_{\mathrm{et}}(V_\mathfrak{p}; \mathcal{F}) \xrightarrow{\sim} \Gamma_{\mathrm{et}}(\kappa(\mathfrak{p}); \mathcal{F})$. The desired excision square is therefore obviously Cartesian. □

Lemma 4.2. *The functor G is an* arc_p*-sheaf.*

Proof. Let us now show that G is an arc_p-sheaf. (We remind the reader that we will ignore the important difference between arc_p- and arc-sheaves.) Recall that $\mathbb{\Delta}_{R/A,\mathrm{perf}}$ denoted the (p, I)-completion of the filtered colimit of $\mathbb{\Delta}_{R/A}$ along its Frobenius. The canonical map $\mathbb{\Delta}_{R/A}/p \to \mathbb{\Delta}_{R/A,\mathrm{perf}}/p$ induces an equivalence after inverting d and extracting ϕ-fixed points. Therefore,

$$G(R) \xrightarrow{\sim} (\mathbb{\Delta}_{R/A,\mathrm{perf}}[\tfrac{1}{d}]/p^n)^{\phi=1},$$

and it suffices to show that $R \mapsto (\mathbb{\Delta}_{R/A,\mathrm{perf}}[\tfrac{1}{d}]/p^n)^{\phi=1}$ is an arc-sheaf. In fact, $R \mapsto \mathbb{\Delta}_{R/A,\mathrm{perf}}$ is itself an arc-sheaf.

Given a p-complete ring R as above, one can define its perfectoidization $R_{\mathrm{perf}} := \mathbb{\Delta}_{R/A,\mathrm{perf}} \otimes_A A/I$. (Despite its definition, this is independent of the choice of prism (A, I).) It suffices to show that the assignment $R \mapsto R_{\mathrm{perf}}$ is an arc-sheaf. Something significantly stronger turns out to be true: in fact, $R_{\mathrm{perf}} \simeq \Gamma_{\mathrm{arc}}(R; \mathcal{O})$! In some sense, this is the key ingredient in all arguments of this sort (note that we only need to know that $\mathcal{O}_{\mathrm{perf}}$ is an arc-sheaf, not the much stronger claim that arc-sheafification of $\mathcal{O}$ is $\mathcal{O}_{\mathrm{perf}}$). Instead of proving this important claim in the generality of p-complete rings, let us only explain a weaker (but closely related!) statement for $\mathbf{F}_p$-algebras.

If R is an $\mathbf{F}_p$-algebra, then R_{perf} is the usual perfection of R. Indeed, take $(A, I) = (\mathbf{Z}_p, (p))$, so that $\overline{\Delta}_{R/A} = \Delta_{R/A}/p$ inherits a Frobenius map ϕ. Since $\overline{\Delta}_{R/A}$ has a filtration whose graded pieces are $\mathrm{gr}_i\,\overline{\Delta}_{R/A} = \Omega^i_{R/\mathbf{F}_p}$, the direct limit $R_{\mathrm{perf}} = \overline{\Delta}_{R/A,\mathrm{perf}}$ also has a filtration whose graded pieces are given by the direct limit of $\Omega^i_{R/\mathbf{F}_p}$ along the Frobenius. However, since Frobenius kills differential forms, this direct limit vanishes unless $i = 0$ (in which case it is precisely R_{perf}).

We now claim that $R_{\mathrm{perf}} \simeq \Gamma_h(R; \mathcal{O})$, i.e., that the h-sheafification of the structure sheaf $\mathcal{O}$ is the perfection $\mathcal{O}_{\mathrm{perf}}$. This apparently goes back to Gabber (of course), and is the sort of result that the general isomorphism $R_{\mathrm{perf}} \simeq \Gamma_{\mathrm{arc}}(R; \mathcal{O})$ is modeled after.

Recall that a ring A is said to be *absolutely weakly normal* if the following two conditions are satisfied[2]:

(a) for all $x, y \in A$ such that $x^3 = y^2 \in A$, there is a unique $a \in A$ such that $x = a^2$ and $y = a^3$. In other words, there is a unique dotted arrow in the following diagram

$$\begin{tikzcd} & \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}[t] \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed] & \mathrm{cusp} = \operatorname{Spec} \mathbf{Z}[t^2, t^3]. \end{tikzcd}$$

(Why are the exponents 2 and 3 special? We'll see below that for our purposes, this is just because they're the smallest pair of coprime integers.)

(b) for all primes p and all $x, y \in A$ such that $p^p x = y^p \in A$, there is a unique $a \in A$ such that $x = a^p$ and $y = pa$. In other words, there is a unique dotted arrow in the following diagram

$$\begin{tikzcd} & (\mathbf{A}^1)^{(-1)} = \operatorname{Spec} \mathbf{Z}[t^{1/p}] \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbf{Z}[t, pt^{1/p}]. \end{tikzcd}$$

Remark 4.3. One could ask for the *a priori* stronger requirement that for any $n \geq 1$, there is a unique dotted arrow in the following diagram

$$\begin{tikzcd} & (\mathbf{A}^1)^{(-n)} = \operatorname{Spec} \mathbf{Z}[t^{1/p^n}] \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbf{Z}[t, pt^{1/p}, \cdots, p^n t^{1/p^n}]. \end{tikzcd}$$

However, this is equivalent to the condition for $n = 1$. One direction is easy, so suppose that the lifting condition is satisfied for $n = 1$, and consider a diagram as above. Let x_j denote the image of $p^j t^{1/p^j}$ in A, so that $x_j^p = p^{j(p-1)+1} x_{j-1}$. Suppose by induction that for $j \leq n-1$, there is a unique $a \in A$ such that $x_j = p^j a^{p^{n-1-j}}$ (so a represents $t^{1/p^{n-1}}$). Then,

$$x_n^p = p^{n(p-1)+1} x_{n-1} = p^{np} a = p^p (p^{p(n-1)} a).$$

Therefore, the lifting condition for $n = 1$ implies that there is a unique $b_0 \in A$ such that $x_n = pb_0$ and $p^p(p^{p(n-2)}a) = p^{p(n-1)}a = b_0^p$. The latter condition implies that there is a unique $b_1 \in A$ such that $b_0 = pb_1$ and $p^{p(n-2)}a = b_1^p$. Inductively, we conclude that there is some b_{n-1} such that $b_0 = p^{p(n-1)} b_{n-1}$, and $a = b_{n-1}^p$. This implies the desired lifting.

[2]If only the first condition is satisfied, then A is called *seminormal*.

If A is a ring, let A^{awn} denote its absolutely weak normal closure. It turns out ([Staty, Tag 0EUR]) that the canonical map $A \to A^{\mathrm{awn}}$ exhibits $\mathrm{Spec}(A^{\mathrm{awn}})$ as the initial object in the category of universal homeomorphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. The relevance of this notion in our context is that [Staty, Tag 0EVU] implies that the assignment $R \mapsto R^{\mathrm{awn}}$ is the sheafification of the structure sheaf in the h-topology (for arbitary R, not just $\mathbf{F}_p$-algebras).

It therefore suffices to show that if R is an $\mathbf{F}_p$-algebra, then $R_{\mathrm{perf}} \simeq R^{\mathrm{awn}}$. This is [Staty, Tag 0EVW], which we review here since the argument is fun. The map $R \to R_{\mathrm{perf}}$ induces a universal homeomorphism, so the universal property of R^{awn} gives a map $R_{\mathrm{perf}} \to R^{\mathrm{awn}}$. It suffices to show that R_{perf} is itself absolutely weakly normal; more generally, any perfect $\mathbf{F}_p$-algebra A is absolutely weakly normal. Let us check both of the conditions above:

(a) Suppose $x, y \in A$ such that $x^3 = y^2$. First suppose $p = 2$. Then $x = a^2$ for some $a \in A$, and therefore we can take $y = a^3$. Similarly, if $p = 3$, then $y = a^3$ for some $a \in A$, and therefore we can take $x = a^2$. Now assume that $p > 3$, so that $p = 2i + 3j$ for some $i, j > 0$. Let $a \in A$ be such that $x = a^p$, and let $b \in A$ be such that $y = b^p$. Then
$$(a^i b^j)^{2p} = a^{2ip} b^{2jp} = x^{2i} y^{2j} = x^{2i} x^{3j} = x^p.$$
Therefore, $x = (a^i b^j)^2$. Similarly,
$$(a^i b^j)^{3p} = a^{3ip} b^{3jp} = x^{3i} y^{3j} = y^{2i} y^{3j} = y^p,$$
so that $y = (a^i b^j)^3$.

(b) Let ℓ be a prime, and suppose that $x, y \in A$ is a pair such that $\ell^\ell x = y^\ell \in A$. If $\ell = p$, then $y^p = 0$, so $y = 0$; and $x = a^p$ for some $a \in A$. If $\ell \neq p$, then take $a = y/\ell$: clearly $x = a^\ell$, while $y = \ell a$.

$\square$

Construction 4.4. We will now construct a natural map $F \to G$ of arc_p-sheaves. In fact, we will construct a map $F \to \Gamma_{\mathrm{arc}_p}(-; \mathbf{Z}/p^n)$; the desired map will then be the composite with the canonical map $\mathbf{Z}/p^n \to G$ of arc_p-sheaves.

To construct the map $F \to \Gamma_{\mathrm{arc}_p}(-; \mathbf{Z}/p^n)$, we claim that F is the arc_p-sheafification of $\mathrm{H}^0(F)$. Then, there is a map $\mathrm{H}^0(F) \to \mathrm{H}^0_{\mathrm{arc}_p}(-; \mathbf{Z}/p^n)$, since the étale topology is coarser than the arc_p-topology. The claim about F implies that this refines uniquely to a map $F \to \Gamma_{\mathrm{arc}_p}(-; \mathbf{Z}/p^n)$ of arc_p-sheaves.

It remains to show that F is the arc_p-sheafification of $\mathrm{H}^0(F)$. Suppose V is a p-complete valuation ring with algebraically closed field; it suffices to show that $F(V) = \Gamma_{\mathrm{et}}(V[\frac{1}{p}]; \mathbf{Z}/p^n)$ is concentrated in degree zero. This is precisely [Staty, Tag 0GY7].

The weight zero étale comparison Theorem 1.1 is now:

Proposition 4.5. *The map $F \to G$ is an isomorphism.*

Proof. Using Proposition 3.7, it suffices to show that if R is a perfectoid ring, then the map $\Gamma_{\mathrm{et}}(R[\frac{1}{p}]; \mathbf{Z}/p^n) \to (\mathbb{\Delta}_{R/A}[\frac{1}{d}]/p^n)^{\phi=1}$ is an isomorphism. In this case, $\mathbb{\Delta}_{R/A} \cong A_{\mathrm{inf}}(R)$, so we need to show that there is a cofiber sequence
$$\Gamma_{\mathrm{et}}(R[\tfrac{1}{p}]; \mathbf{Z}/p^n) \to A_{\mathrm{inf}}(R)[\tfrac{1}{d}]/p^n \xrightarrow{\phi-1} A_{\mathrm{inf}}(R)[\tfrac{1}{d}]/p^n.$$
Writing $A_{\mathrm{inf}}(R) = W(R^\flat)$, and using the isomorphism $W(R^\flat)[\frac{1}{d}] \cong W(R^\flat[\frac{1}{d}])$, we obtain an exact sequence
$$\Gamma_{\mathrm{et}}(R^\flat[\tfrac{1}{d}]; \mathbf{Z}/p^n) \to W(R^\flat)[\tfrac{1}{d}]/p^n \xrightarrow{\phi-1} W(R^\flat)[\tfrac{1}{d}]/p^n$$
induced by the Artin-Schreier sequence
$$0 \to \mathbf{Z}/p^n \to W_n \xrightarrow{\phi-1} W_n \to 0 \tag{3}$$

It therefore suffices to show that $\Gamma_{\mathrm{et}}(R^\flat[\frac{1}{d}]; \mathbf{Z}/p^n) \cong \Gamma_{\mathrm{et}}(R[\frac{1}{p}]; \mathbf{Z}/p^n)$, which can be deduced using Scholze's tilting equivalence on étale sites.

A different approach is to still use the fact that both sides of the map $\Gamma_{\mathrm{et}}(R[\frac{1}{p}]; \mathbf{Z}/p^n) \to (\Delta_{R/A}[\frac{1}{d}]/p^n)^{\phi=1}$ satisfy arc-descent, to reduce to the case when $R = \prod_I V$ is a product of p-complete absolutely integrally closed valuation rings of rank ≤ 1. (Recall from Lemma 3.6 that every p-adic affine formal scheme admits an arc_p-cover of this form.) By the discussion in Construction 4.4, the étale cohomology $\Gamma_{\mathrm{et}}(R[\frac{1}{p}]; \mathbf{Z}/p^n)$ is discrete: in fact, it is isomorphic to $(\mathbf{Z}/p^n)^I$. On the other hand, since each V is perfectoid (see the proof of Lemma 3.6), we see that R is a product of perfectoid rings, so that $\Delta_{R/A} \cong A_{\mathrm{inf}}(R) = W(R^\flat)$. Moreover, $\Delta_{R/A}[\frac{1}{d}] \cong W(R^\flat)[\frac{1}{d}] \cong W(R^\flat[\frac{1}{d}])$. Therefore, the desired exact sequence follows from the Artin-Schreier sequence (3). $\square$

In both cases, one uses the arc_p-topology, and either Proposition 3.7 or explicit calculations for p-complete absolutely integrally closed valuation rings of rank ≤ 1, to reduce to calculations for perfectoids. Here, the perfectness of everything involved makes many things discrete, so that calculations become more concrete.

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