CHROMATIC ABERRATIONS OF GEOMETRIC SATAKE OVER THE REGULAR LOCUS

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> It is an established practice to take old theorems about ordinary homology, and generalise them so as to obtain theorems about generalised homology theories.
>
> — J. F. Adams, [Ada69]

Abstract. Let G be a connected, simply-laced, almost simple algebraic group over $\mathbf{C}$, let G_c be a maximal compact subgroup of $G(\mathbf{C})$, and let T be a maximal torus. The derived geometric Satake equivalence of Bezrukavnikov-Finkelberg localizes to an equivalence between a full subcategory of $\mathrm{Loc}_{G_c}(\Omega G_c; \mathbf{C})$ and $\mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}[2]/\check{G})$, which can be thought of as a version of the geometric Satake equivalence "over the regular locus". In this article, we study the analogous story when $\mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C})$ is replaced by the ∞-category of T-equivariant local systems of k-modules over $\mathrm{Gr}_G(\mathbf{C})$, where k is (2-periodic) rational cohomology, (complex) K-theory, or elliptic cohomology. We show that the ∞-category $\mathrm{Loc}_{T_c}(\Omega G_c; k)$ admits a 1-parameter degeneration to an ∞-category of quasicoherent sheaves built out of the geometry of various Langlands-dual stacks associated to k and the 1-dimensional group scheme computing S^1-equivariant k-cohomology. For example, when k is an elliptic cohomology theory with elliptic curve E, the ∞-category $\mathrm{Loc}_{T_c}(\Omega G_c; k)$ degenerates to the ∞-category of quasicoherent sheaves on the moduli stack of $\check{B}$-bundles of degree 0 on E. We also study several applications of these equivalences, including: proving multiplicative and elliptic versions of the Gelfand-Graev action on the affine closure of the cotangent bundle of the basic affine space; the interaction between power operations like Steenrod and Adams operations and geometric Langlands duality; relations to work of Brylinski-Zhang and stable splittings due to Miller; and the analogy to work of Hopkins-Kuhn-Ravenel.

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Part of this work was done when the author was supported by the PD Soros Fellowship and NSF DGE-2140743. The present article is a preliminary version, so any comments and suggestions for improving it are greatly appreciated! I'll post major updates to the arXiv, but I'll upload minor edits to my website at `https://sanathdevalapurkar.github.io/files/grG-regular.pdf`; so please see there for the most up-to-date version.

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1. Introduction

1.1. **The derived geometric Satake equivalence.** Let G be a semisimple algebraic group over $\mathbf{C}$, and let Gr_G denote the affine Grassmannian, defined as $G(\!(t)\!)/G[\![t]\!]$. The geometric Satake equivalence of Mirkovic-Vilonen states:

Theorem 1.1 (Mirkovic-Vilonen, [MV07]). *Let $\check{G}_{\mathbf{Z}}$ denote the smooth split reductive group scheme over $\mathbf{Z}$ whose root datum is dual to that of G. Then there is an equivalence of symmetric monoidal abelian categories*

$$\mathrm{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{Z}) \simeq \mathrm{Rep}(\check{G}_{\mathbf{Z}}).$$

The abelian category $\mathrm{Perv}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{Z})$ arises as the heart of a t-structure on the stable ∞-category $\mathrm{Shv}^c_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{Z})$ of $G[\![t]\!]$-equivariant constructible sheaves on Gr_G. However, Theorem 1.1 does not lift to an equivalence of stable ∞-categories between $\mathrm{Shv}^c_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{Z})$ and the derived ∞-category of $\mathrm{Rep}(\check{G}_{\mathbf{Z}})$. Nevertheless, one has:

Theorem 1.2 (Bezrukavnikov-Finkelberg, [BF08]). *Let $\check{G} = \check{G}_{\mathbf{C}}$. There is a $\mathbf{C}$-linear equivalence of monoidal stable ∞-categories*

$$\mathrm{Shv}^c_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

Here, $\check{\mathfrak{g}}^*[2]$ denotes the derived scheme given by $\check{\mathfrak{g}}^*$ placed in (homological) degree 2. Using Koszul duality, the right-hand side can be rewritten as

$$\mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}) \simeq \mathrm{Coh}((\{0\} \times_{\check{\mathfrak{g}}} \{0\})/\check{G}).$$

In [CR23, Theorem 6.6.1], it was shown that the above equivalence can be refined to an equivalence of monoidal factorization stable ∞-categories. (We will not need this refinement here, and only mention it for the sake of completeness.) There are several variants of Theorem 1.2 which have been proved in the literature; of relevance to us is a theorem from [ABG04] concerning the Iwahori subgroup I of $G[\![t]\!]$. Namely, fix a Borel subgroup $B \subseteq G$ with unipotent radical N, and let $I = G[\![t]\!] \times_G B$. Then:

Theorem 1.3 (Arkhipov-Bezrukavnikov-Ginzburg, [ABG04]). *There is an equivalence*

$$\mathrm{Shv}^c_I(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{Coh}((\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G}),$$

where $\tilde{\check{\mathcal{N}}} = T^*(\check{G}/\check{B})$ is the Springer resolution. Using Koszul duality, the right-hand side can be rewritten as

$$\mathrm{Coh}((\tilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G}) \simeq \mathrm{Perf}(\tilde{\check{\mathfrak{g}}}_{\mathbf{C}}[2]/\check{G}_{\mathbf{C}}),$$

where $\tilde{\check{\mathfrak{g}}}_{\mathbf{C}}[2] = \check{G} \times^{\check{B}} \check{\mathfrak{n}}^{\perp}[2]$ is a shifted analogue of the Grothendieck-Springer resolution.

The equivalences of Theorem 1.2 and Theorem 1.3 admit simpler analogues, where one considers the full subcategories of the ind-completions of $\mathrm{Shv}^c_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C})$ and $\mathrm{Shv}^c_I(\mathrm{Gr}_G; \mathbf{C})$ generated by the constant sheaf. Let us denote these ∞-categories by $\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C})$ and $\mathrm{Loc}_I(\mathrm{Gr}_G; \mathbf{C})$. In this case, as we will discuss in the body of the article, the above results restrict to equivalences

$$\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G}), \tag{1}$$

$$\mathrm{Loc}_I(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{QCoh}(\tilde{\check{\mathfrak{g}}}^{\mathrm{reg}}[2]/\check{G}). \tag{2}$$

Here, $\check{\mathfrak{g}}^{*,\mathrm{reg}}$ denotes the open subscheme of *regular* elements in $\check{\mathfrak{g}}^*$, i.e., those elements whose stabilizer under the coadjoint action of $\check{G}$ has dimension given by the rank of $\check{G}$; and similarly for $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}$. We will refer to these equivalences as the derived geometric Satake equivalence (resp. the ABG equivalence) over the regular locus. In fact, one can can give a proof of Theorem 1.2 using (1) combined with the fact that $\check{\mathfrak{g}}^{*,\mathrm{reg}}$ has complement of large codimension in $\check{\mathfrak{g}}^*$.

Our goal in this article is to study an analogue of the derived geometric Satake equivalence over the regular locus where $\mathbf{C}$ is replaced by other ring (spectra) of coefficients. The observation allowing us to do this is that the ∞-categories $\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G;\mathbf{C})$ and $\mathrm{Loc}_I(\mathrm{Gr}_G;\mathbf{C})$ are homotopy-theoretic in nature. Indeed, they depend only on the $G[\![t]\!]$-equivariant (resp. I-equivariant) homotopy type of Gr_G. This, in turn can be understood using a result of Quillen and Garland-Raghunathan (see [GR75, Mit88]), which gives a homeomorphism $\mathrm{Gr}_G \simeq \Omega G_c$; so all computations reduce to homotopy-theoretic statements about ΩG_c. Here, $G_c \subseteq G(\mathbf{C})$ is a maximal compact subgroup. Since $G[\![t]\!]$ is homotopy equivalent to G_c (and I is homotopy equivalent to a maximal torus $T_c \subseteq G_c$), the equivalences (1) and (2) can be restated as a pair of equivalences

$$\mathrm{Loc}_{G_c}(\mathrm{Gr}_G;\mathbf{C}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G}), \tag{3}$$

$$\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;\mathbf{C}) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}[2]/\check{G}). \tag{4}$$

1.2. **A K-theoretic and elliptic variant.** As mentioned above, our goal in this article is to generalize the equivalence (2) to the case of K-theoretic and elliptic cohomology coefficients. In order to motivate the discussion, we need to review some of the setup of equivariant generalized cohomology.

If G_c is a compact Lie group, Atiyah and Segal defined G_c-equivariant complex K-theory KU_{G_c} in [Seg68, AS69]: this is a generalized cohomology theory, viewed as a spectrum in the sense of homotopy theory, which classifies G_c-equivariant vector bundles on finite G_c-spaces. Direct sum and tensor products of G-equivariant vector bundles equips KU_{G_c} with the structure of a *ring* spectrum; in fact, it is an $\mathbf{E}_\infty$-ring, meaning (for instance) that the multiplication on cohomology can be refined by Adams operations. Despite its definition, the geometric interpretation of cocycles for equivariant K-theory as equivariant vector bundles will play *no* role below.

Two important examples are the following. When G_c is the trivial group, KU_{G_c} is simply periodic complex K-theory KU, and Bott periodicity gives a graded isomorphism $\pi_*\mathrm{KU} \cong \mathbf{Z}[\beta^{\pm 1}]$ with the Bott class β in weight 2. On the other hand, when G_c is a connected compact Lie group with complex representation ring $\mathrm{RU}(G_c)$, the coefficient ring $\pi_*\mathrm{KU}_{G_c}$ is the tensor product $\mathrm{RU}(G_c) \otimes_{\mathbf{Z}} \mathbf{Z}[\beta^{\pm 1}]$. In particular, if G_c is a compact torus T_c, then $\mathrm{Spec}\, \pi_*\mathrm{KU}_{T_c}$ is the corresponding algebraic torus $T_{\mathbf{Z}[\beta^{\pm 1}]}$ over $\mathbf{Z}[\beta^{\pm 1}]$.

In Section 3, we define a KU-linear ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$ of T_c-equivariant local systems of KU-modules on Gr_G; this should be viewed as a KU-theoretic analogue of the ∞-category $\mathrm{Loc}_I(\mathrm{Gr}_G;\mathrm{KU})$. The ideal analogue of (2) would identify $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$ with the ∞-category of perfect complexes on some stack over KU defined in terms of the Langlands dual group. Unfortunately, we do not know how to define this putative stack over KU; instead, we will study a particular *degeneration* of this ∞-category, denoted $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$. The reason for studying this degeneration is explained in the introduction to Section 4; we will also discuss its "philosophical" meaning momentarily. For the moment, let us just note the utility of this degeneration: while $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$ is a KU-linear ∞-category, $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$ is instead an ordinary ($\mathbf{Z}$-linear) ∞-category; so we do not have to be concerned with questions such as the definition of the dual group over KU.

Exactly the same setup works if k is a complex oriented 2-periodic $\mathbf{E}_\infty$-ring which is an elliptic cohomology theory (in the sense of [Lur09, GM20, GM23]) with associated elliptic curve E over $\pi_0(k)$. Namely, we construct a k-linear ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;k)$, as well as a degeneration $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)$ which is instead an ordinary ($\pi_0(k)$-linear) ∞-category. In both this case and the case of KU, an object $\mathcal{F}\in\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;k)$ defines a corresponding object $\mathcal{F}^{\mathrm{gr}}\in\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)$, and there is a spectral sequence

(5)
$$\pi_*(k)\otimes_{\pi_0(k)}\pi_*\mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)}(\underline{k}^{\mathrm{gr}},\mathcal{F}^{\mathrm{gr}})\Rightarrow\pi_*\mathrm{Map}_{\mathrm{Loc}_{T_c}(\mathrm{Gr}_G;k)}(\underline{k},\mathcal{F})=\pi_*\Gamma_{T_c}(\mathrm{Gr}_G;\mathcal{F}).$$

Here, $\underline{k}$ denotes the constant sheaf.

To state our main result, we need a small observation. Suppose G is a connected, almost simple, and *simply-laced* algebraic group over $\mathbf{C}$, and let $\check{G}_{\mathbf{C}}$ denote the Langlands dual over $\mathbf{C}$. Then $\check{G}_{\mathbf{C}}$ is centrally isogeneous to G. For instance, if G is simply-connected, $\check{G}_{\mathbf{C}}$ is the quotient of G by its center. In general (under the simply-laced hypothesis), the action of G on itself by conjugation descends/ascends to an action of $\check{G}_{\mathbf{C}}$. In particular, the action of $T_{\mathbf{C}}$ on G by conjugation descends/ascends to an action of $\check{T}_{\mathbf{C}}$ on G. Therefore, if T_c denote the maximal compact subgroup of $T_{\mathbf{C}}$ (and similarly for $\check{T}_c$), then the conjugation action of T_c on Gr_G descends/ascends to an action of $\check{T}_c$ on Gr_G.

The main result of this article is the following.

Theorem 1.4 (Corollary 6.8, Corollary 7.8, Corollary 8.11)**.** *Suppose G is a connected, almost simple, and simply-laced algebraic group over $\mathbf{C}$. Let $T\subseteq G$ be a maximal torus, and let $\check{T}_c$ denote a maximal compact subgroup of the Langlands dual torus over $\mathbf{C}$. Let k denote either 2-periodified rational cohomology $\mathbf{Q}[u^{\pm1}]$, complex K-theory* KU*, or elliptic cohomology with associated elliptic curve E, and let F be an algebraically closed field containing $\pi_0(k)$. Then there are equivalences*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;\mathbf{Q}[u^{\pm1}])\otimes_{\mathbf{Q}}F\simeq\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\prime,\mathrm{reg}}/\check{G}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;\mathrm{KU})\otimes_{\mathbf{Z}}F\simeq\mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;k)\otimes_{\pi_0(k)}F\simeq\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}).$$

Here, the dual groups on the right-hand side are defined over F; the final equivalence is assuming that k is an elliptic cohomology theory; $\widetilde{\check{\mathfrak{g}}}'$ denotes $\check{G}\times^{\check{B}}\check{\mathfrak{b}}$; $\widetilde{\check{G}}$ denotes $\check{G}\times^{\check{B}}\check{B}$ for the conjugation action of $\check{B}$ on itself[1]*; and $\mathrm{Bun}^0_{\check{B}}(E)$ denotes the moduli stack of degree zero $\check{B}$-bundles on E.*

Since T_c and $\check{T}_c$ are isogeneous by a finite group, there is no difference between $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathbf{Q}[u^{\pm1}])$ and $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;\mathbf{Q}[u^{\pm1}])$. Therefore, the first part of Theorem 1.4 is, of course, just (4) (once one identifies $\widetilde{\check{\mathfrak{g}}}'\cong\widetilde{\check{\mathfrak{g}}}$). However, we warn the reader that if k is not a $\mathbf{Q}$-algebra, the categories $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)$ and $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;k)$ are genuinely different, and do not agree even upon base-change along $\pi_0(k)\to F$.

[1]We warn the reader that the symbol $\widetilde{\check{G}}$ will mean $\check{G}\times^{\check{B}}\check{B}$ *only* in the introduction; it has a slightly different meaning in the body of the article, described in Definition 7.1.

Remark 1.5. The same argument used to prove Theorem 1.4 shows that if G is a connected, almost simple, and *simply-laced* algebraic group over $\mathbf{C}$ with torsion-free fundamental group, and k denotes either 2-periodified rational cohomology $\mathbf{Q}[u^{\pm 1}]$, complex K-theory KU, or elliptic cohomology with associated elliptic curve E, then there are equivalences

$$\begin{aligned}
\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathbf{Q}[u^{\pm 1}]) \otimes_{\mathbf{Q}} F &\simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}/\check{G}),\\
\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F &\simeq \mathrm{QCoh}(\check{G}^{\mathrm{reg}}/\check{G}),\\
\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} F &\simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)^{\mathrm{reg}}),
\end{aligned}$$

where $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)^{\mathrm{reg}} \subseteq \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)$ is a particular open substack of the moduli stack of semistable degree zero $\check{G}$-bundles on E. More generally, our arguments are easily modified to show that if L is the Levi quotient of a parabolic subgroup $P \subseteq G$, then there are equivalences

$$\begin{aligned}
\mathrm{Loc}^{\mathrm{gr}}_{\check{L}_c}(\mathrm{Gr}_G; \mathbf{Q}[u^{\pm 1}]) \otimes_{\mathbf{Q}} F &\simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\prime,\mathrm{reg}}_{\check{P}}/\check{P}),\\
\mathrm{Loc}^{\mathrm{gr}}_{\check{L}_c}(\mathrm{Gr}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F &\simeq \mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}_{\check{P}}/\check{P}),\\
\mathrm{Loc}^{\mathrm{gr}}_{\check{L}_c}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} F &\simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{P}}(E)^{\mathrm{reg}}),
\end{aligned}$$

where $\widetilde{\check{\mathfrak{g}}}^{\prime}_{\check{P}}$ denotes $\check{G} \times^{\check{P}} \check{\mathfrak{p}}$; $\widetilde{\check{G}}_{\check{P}}$ denotes $\check{G} \times^{\check{P}} \check{P}$ for the conjugation action of $\check{P}$ on itself; and $\mathrm{Bun}^{\mathrm{ss}}_{\check{P}}(E)$ denotes the moduli stack of degree zero semistable $\check{P}$-bundles on E. The equivalence concerning $\mathrm{Loc}^{\mathrm{gr}}_{\check{L}_c}(\mathrm{Gr}_G; \mathbf{Q}[u^{\pm 1}])$ above is closely related to the parabolic variant of Theorem 1.3 which can be deduced from [CD23].

The proofs of the equivalences with $\mathbf{Q}[u^{\pm 1}]$-coefficients follow from work of Bezrukavnikov-Finkelberg-Mirkovic [BFM05] and Yun-Zhu [YZ11]; and the proofs of the equivalences involving KU follow from the aforementioned work of Bezrukavnikov-Finkelberg-Mirkovic [BFM05]. However, we reprove these results in the present article so as to provide a *uniform* approach to all the equivalences of Theorem 1.4 and the preceding remark. In particular, the work of [BFM05] relies on knowing a geometric interpretation of classes in equivariant homology/K-theory; since such a geometric interpretation is unknown in the case of elliptic cohomology, the approach taken there is not amenable to generalization.

Remark 1.6. Let us mention some previous work towards analogues of the geometric Satake equivalence with other coefficients. For instance, a conjecture regarding the case of complex K-theory was proposed in [CK18]; in a similar vein, a discussion of the K-theoretic case is the content of the talk [Lon21a]. In [YZ21], Yang and Zhao study a higher chromatic analogue of quantum groups, and it would be interesting to study the relationship between the present article and their work. After the first version of this paper was written, the preprint [Zho23] was posted on the arXiv; it is concerned with ideas similar to the ones studied here.

Remark 1.7. In Section 10, we also study a degeneration $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k)$ of the category $\mathrm{Loc}_{G_c}(G_c; k)$ of *conjugation-equivariant* locally constant sheaves on G_c. Namely, we show that (at least if G_c has torsion-free fundamental group) $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k)$ is equivalent to the category of quasicoherent sheaves on the additive (resp. multiplicative; resp. elliptic) regular centralizer group scheme for $\check{G}$ if $k = \mathbf{Q}[u^{\pm 1}]$ (resp. $k = \mathrm{KU}$; resp. k is an elliptic cohomology theory). Motivated by [NZ09] and

[GPS18, Theorem 1.1], one can heuristically interpret our discussion as describing a version of mirror symmetry for the wrapped Fukaya category of the symplectic stack $T^*(G_c/G_c)$, albeit with coefficients in the complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k. Namely, the "k-theoretic" mirror to $T^*(G_c/G_c)$ is the appropriate variant of the regular centralizer group scheme for the Langlands dual group.

Remark 1.8. We also study the effect of power operations on k under Langlands duality. The reader is referred to Theorem 9.11 for a precise statement, but let us just mention here that using the theory of degree p isogenies on $\mathbf{G}_a$, $\mathbf{G}_m$, and the elliptic curve E, we show that power operations on k correspond under Theorem 1.4 to natural "Artin-Schreier" maps on $\widetilde{\check{\mathfrak{g}}}'/\check{G}$, $\widetilde{\check{G}}/\check{G}$, and $\mathrm{Bun}^0_{\check{B}}(E)$.

Let us now discuss further the degeneration of the KU-linear ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ to $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$. This can be explained in "two" ways[2]:

(a) One important lesson from chromatic homotopy theory is that the stable homotopy groups of spheres are closely connected to the coherent cohomology of the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of 1-dimensional formal groups. For instance, the Adams-Novikov spectral sequence can be restated as a spectral sequence
$$E_2^{*,*} \cong \mathrm{H}^*(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes *}) \Rightarrow \pi_*(S^0),$$
where S^0 is the sphere spectrum and ω is the line bundle of invariant differentials on the universal 1-dimensional formal group over $\mathcal{M}_{\mathrm{fg}}$. This picture can in fact be categorified: the ∞-category Sp of spectra behaves like the ∞-category $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ in a very precise sense[3]. Namely, the Adams-Novikov spectral sequence is categorified by a 1-parameter degeneration of Sp into $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$; see [Pst23b, GWX21]. Moreover, this degeneration can be constructed using the stable motivic category over $\mathbf{C}$. This gives a precise sense in which "topology is approximated by algebra". The degeneration of $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ into $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ is of exactly the same type. In fact, both degenerations ($\mathrm{Sp} \rightsquigarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ and $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k) \rightsquigarrow \mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$) can be constructed simultaneously using the even filtration of [HRW22, Pst23a], and we will address this point in a future article. (That is to say, the Adams-Novikov spectral sequence and (5) should both be regarded as special cases of a more general construction.)

(b) In geometric representation theory, one often considers "graded lifts" of categories of ($\mathbf{C}$-valued, say) sheaves on a scheme/stack X: this is generally defined as the category $\mathrm{Shv}^{\mathrm{mixed}}(X; \mathbf{C})$ of *mixed* sheaves on X. See [BBD82] and the more recent [HL22]. It is generally expected, for instance, that there is a mixed variant of Theorem 1.2, stating that there is an equivalence $\mathrm{Shv}^{\mathrm{mixed}}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*(2)/\check{G})$, where $\check{\mathfrak{g}}^*(2)/\check{G}$ is now a classical (not derived!) stack, except with a grading which places $\check{\mathfrak{g}}^*(2)$ in weight 2.

[2]The quotes are to indicate that these two approaches are really the same; since it would be too digressive to do so here, we will explain the meaning of this (admittedly cryptic) statement in a sequel to this article.

[3]This is not quite correct: namely, one has to instead replace $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ by the ind-completion of the thick subcategory of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ generated by tensor powers of ω. For brevity, we will simply denote this variant category by $\mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. In the homotopy theory literature, e.g., [Gre21] or [BHS20, Definition 5.14], this variant subcategory is often instead denoted by $\mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$. However, the use of the symbol $\mathrm{IndCoh}(\mathcal{M}_{\mathrm{fg}})$ does not agree with more established notion of ind-coherent sheaves from [GR17].

This grading can be ignored if we replace $\mathbf{C}$ by its 2-periodification. One can view $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ as a category of "mixed" k-valued local systems. The reason that no grading is visible on the right-hand sides of the equivalences in Theorem 1.4 is that the $\mathbf{E}_\infty$-ring k is assumed to be 2-periodic. A natural question, of course, is to define a full ∞-category $\mathrm{Shv}^{\mathrm{gr}}_I(\mathrm{Gr}_G; k)$ which specializes to the usual category of mixed sheaves when $k = \mathbf{C}[u^{\pm 1}]$; we hope to address this in the future article referred to in the preceding bullet point.

The perspective of the degeneration $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k) \rightsquigarrow \mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ being analogous to the degeneration $\mathrm{Sp} \rightsquigarrow \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$ – and furthermore that both are related to the even filtration of [HRW22, Pst23a] – is very helpful, because it gives us an indication of how to define $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ when k is not necessarily complex-oriented and 2-periodic. In particular, we also study the example of k being *real* K-theory KO. This is an $\mathbf{E}_\infty$-ring with somewhat complicated homotopy groups. Despite this, the $\mathbf{E}_\infty$-ring KO is itself easy to describe using KU: namely, there is an action of $\mathbf{Z}/2$ on KU by complex conjugation, and $\mathrm{KO} = \mathrm{KU}^{h\mathbf{Z}/2}$. Moreover, just like the standard Adams-Novikov spectral sequence for the homotopy of the sphere spectrum, there is a spectral sequence

$$E_2^{*,*} \cong \mathrm{H}^*(B\mathbf{Z}/2; \omega^*) \Rightarrow \pi_*(\mathrm{KO}),$$

where ω is the line bundle over $B\mathbf{Z}/2$ given by the sign action of $\mathbf{Z}/2$ on $\mathbf{Z}$. That is, the sphere spectrum is to $\mathcal{M}_{\mathrm{fg}}$ as KO is to $B\mathbf{Z}/2$. There is also a good notion of equivariant real K-theory.

Instead of constructing a degeneration of the KO-linear ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$ into a graded $\pi_*(\mathrm{KO})$-linear ∞-category, one can instead construct a degeneration of $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$ into a $\mathrm{QCoh}(B\mathbf{Z}/2)$-module ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$. The construction of $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$ is straightforward: the $\mathbf{Z}/2$-action via complex conjugation on $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; \mathrm{KU})$ defines a $\mathbf{Z}/2$-action on $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KU})$, and this defines the $\mathrm{QCoh}(B\mathbf{Z}/2)$-module category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$.

Proposition 1.9 (Proposition 7.23)**.** *Let θ denote the involution on $\tilde{\check{G}}/\check{G}$ given on $\tilde{\check{G}} = \check{G} \times^{\check{B}} \check{B}$ by $(g, x) \mapsto (g, x^{-1})$, so that the quotient $(\tilde{\check{G}}/\check{G})/\langle\theta\rangle$ defines a stack over $B\mathbf{Z}/2$. Then there is a $\mathrm{QCoh}(B\mathbf{Z}/2)$-linear equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\tilde{T}_c}(\mathrm{Gr}_G; \mathrm{KO}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}((\tilde{\check{G}}^{\mathrm{reg}}/\check{G})/\langle\theta\rangle).$$

For instance, if $G = \mathrm{PGL}_2$, so that $\tilde{\check{G}}$ is the moduli of pairs (g, ℓ) with $g \in \mathrm{SL}_2$ and $\ell = [x : y] \in \mathbf{P}^1$ is a line preserved by g, the involution θ simply inverts g.

Similarly, if one fixes a prime p and some integer $n \geq 0$, and sets $T_c[p^n]$ to be the p^n-torsion subgroup of T_c, one can also define a $\mathrm{QCoh}(B\mathbf{Z}_p^\times)$-module category $\mathrm{Loc}^{\mathrm{gr}}_{T_c[p^n]}(\mathrm{Gr}_G; L_{K(1)}S^0)$. This ∞-category is a degeneration of the ∞-category $\mathrm{Loc}_{T_c[p^n]}(\mathrm{Gr}_G; L_{K(1)}S^0)$ of $T_c[p^n]$-equivariant local systems of $L_{K(1)}S^0$-modules on Gr_G, where $L_{K(1)}S^0 = (\mathrm{KU}^\wedge_p)^{h\mathbf{Z}_p^\times}$ is the $K(1)$-*local sphere* (also known as the "image of J" spectrum) [Ada66, Rav84]. In this case, let $\tilde{\check{G}}_{p^n} \subseteq \tilde{\check{G}}$ denote the locus of pairs $(g, \check{B}') \in \tilde{\check{G}}$ where $\check{B}' \subseteq \check{G}$ is a Borel subgroup containing g such that the eigenvalues of g are all p^nth roots of unity. Then there is a $\mathbf{Z}_p^\times \times \check{G}$-action on $\tilde{\check{G}}_{p^n}$, and Proposition 7.26 similarly states that a $\mathrm{QCoh}(B\mathbf{Z}_p^\times)$-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c[p^n]}(\mathrm{Gr}_G; L_{K(1)}S^0) \otimes_{\mathbf{Z}_p} F \simeq \mathrm{QCoh}((\widetilde{\check{G}}^{\mathrm{reg}}_{p^n}/\check{G})/\mathbf{Z}_p^\times).$$

1.3. **Outline and other results.** We now give an overview of the content of this article. In Section 2, we briefly review the derived geometric Satake and the Arkhipov-Bezrukavnikov-Ginzburg equivalences, and show how to deduce the corresponding statements (1) and (2) over the regular loci.

Motivated by the issue of *decompleting* Borel-equivariant cohomology (which appears naturally in studying the derived geometric Satake and the Arkhipov-Bezrukavnikov-Ginzburg equivalences), we recall in Section 3 the setup of (genuine) equivariant generalized cohomology following [Lur09, GM20, GM23]. For a compact abelian group T_c and a finite T_c-space X, we also introduce the category $\mathrm{Loc}_{T_c}(X; k)$ of equivariant local systems of k-modules on X for an $\mathbf{E}_\infty$-ring k equipped with an "oriented" 1-dimensional commutative group scheme $\mathbf{G}$. This section also reviews/generalizes the all-important Atiyah-Bott localization theorem, and the corresponding results of Goresky-Kottwitz-MacPherson [GKM98].

In Section 4, we introduce the degeneration $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ of the ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ that we discussed in the preceding subsection, and also discuss the definition of this ∞-category for the non-complex-oriented $\mathbf{E}_\infty$-ring KO. The definition of $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ depends only on the equivariant homology $k_*^{T_c}(\mathrm{Gr}_G)$ (denoted by $\pi_*\mathcal{F}_{T_c}(\mathrm{Gr}_G)^\vee$ in the body of this article) as a coalgebra/Hopf algebroid over the equivariant cohomology $\pi_* k_{T_c}$ of a point.

The next Section 5 is essentially purely combinatorial: it studies the case when G is a torus T, and imposes the additional data of loop-rotation equivariance. In terms of the homeomorphism $\mathrm{Gr}_G \cong \Omega G_c$, this comes from the S^1-action on ΩG_c obtained by viewing it as $\Omega^2(BG_c) = \mathrm{Map}_*(S^2, BG_c)$ and using the S^1-action by rotation on S^2. Namely, we show that if k is an $\mathbf{E}_\infty$-ring equipped with an "oriented" 1-dimensional commutative group scheme $\mathbf{G}$, then $k_*^{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_T)$ (or really, its sheafification over $\mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G})$) can be identified with a $\mathbf{G}$-*analogue* of the Weyl algebra of the Langlands dual torus $\check{T}$. For instance, if $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$, then $k_*^{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_T)$ is the rescaled Weyl algebra of $\check{T}$; similarly, if $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, then $k_*^{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_T)$ is the q-Weyl algebra of the dual torus $\check{T}$. We also explain the relationship between this $\mathbf{G}$-analogue of the Weyl algebra of $\check{T}$ and the "F-de Rham complex" of [DM23].

In Section 6, we review the classical story concerning $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ when $k = \mathbf{Q}[u^{\pm 1}]$. The purpose of this section is to reprove the results of [BFM05, YZ11] using only techniques amenable to generalization to other equivariant cohomology theories. In particular, in Corollary 6.8, we reprove the equivalence between $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ and $\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G})$. The remainder of Section 6 is concerned with the question of loop-rotation equivariance. Using results of [GKV97] and [Gin18], we prove that $\mathrm{Loc}^{\mathrm{gr}}_{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; k)$ can be identified with a certain localization of the Harish-Chandra category $\mathrm{HC}^\hbar_{\check{G}} = U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{G},\mathrm{weak})}$. This line of argument is, in some sense, precisely the opposite of that of [Lon18].

We finally turn to the K-theoretic story in Section 7. Corollary 7.8 therein states that $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ is equivalent to $\mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}/\check{G})$ when G is connected, almost simple, and simply-laced; in this section, unlike in Theorem 1.4, the multiplicative

Grothendieck-Springer resolution $\widetilde{\check{G}}$ is defined to be $\check{G} \times^{\check{B}} B$ (instead of $\check{G} \times^{\check{B}} \check{B}$). This is to be understood as analogous to the usual Grothendieck-Springer resolution $\widetilde{\check{\mathfrak{g}}}$, which is defined to be $\check{G} \times^{\check{B}} \check{\mathfrak{n}}^{\perp}$ (as opposed to $\check{G} \times^{\check{B}} \check{\mathfrak{b}}$). We also briefly study the question of loop-rotation equivariance in Theorem 7.17 using degenerate affine nil-Hecke algebras, and phrase some precise expectations about the relationship to the representation theory of quantum groups; but we do not yet know how to prove these statements. In Proposition 7.23, we also study the effect under Langlands duality of complex conjugation on equivariant K-theory.

The elliptic story is studied in Section 8, where we use the important results of [Dav19] to show in Corollary 8.11 that $\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{T}_c}(\mathrm{Gr}_G; k)$ is equivalent to a localization of $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(E))$ when G is connected, almost simple, and simply-laced. We also briefly study the question of loop-rotation equivariance in Theorem 8.15, but do not even know how to describe the expected Langlands dual story. It should, however, be related to the representation theory of elliptic quantum groups [Fel95].

The remainder of this article is concerned with comparisons to (by now) classical constructions in equivariant algebraic topology. Section 9 studies the effect of "power operations" on k under the Langlands duality of Theorem 1.4. These are additional symmetries of the $\mathbf{E}_\infty$-ring k which yield the theory of Steenrod operations in ordinary cohomology and Adams operations in K-theory. (This is closely related to, but distinct from, work [Lon21b] of Lonergan.) We review how the theory of isogenies of $\mathbf{G}_0$ controls power operations for k, which is used to show that power operations for k identify with natural "Artin-Schreier" type maps on $\check{\mathfrak{g}}/\check{G}$, $\check{G}/\check{G}$, and $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)$.

In Section 10, we explain how the degeneration of $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ to $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ should be viewed as analogous to the Hochschild-Kostant-Rosenberg degeneration of Hochschild homology to differential forms. (See [Rak20] for a modern take on this degeneration.) Using this perspective, we show how (when G has torsion-free fundamental group) Theorem 1.4 recovers results of [BZ00] identifying the conjugation-equivariant cohomologies $\mathrm{H}^*_{G_c}(G_c; \mathbf{Q})$ (resp. $\mathrm{KU}^*_{G_c}(G_c)$) with the algebra of Kähler differentials on $\mathfrak{t}/\!\!/W$ (resp. $T/\!\!/W$); the same argument also describes the equivariant elliptic cohomology of G_c in terms of the algebra of Kähler differentials on the moduli *space* of semistable degree zero G-bundles on the elliptic curve.

The results of this article were motivated by the work of Hopkins-Kuhn-Ravenel [HKR00] describing the generalized equivariant cohomology of *finite* groups. In Appendix A, we briefly review their results (and the corresponding categorifications, due to Lurie [Lur19]). Despite the case of finite groups being the diametric opposite to the case of connected compact Lie groups studied in this article, we give a heuristic argument showing that Theorem 1.4 can be viewed as an analogue of (some of) the results of [HKR00, Lur19].

Another motivation for the results of this article came from physics. Namely, the equivariant homology of Gr_G describes the Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory [BFN18], and one expects that the generalized equivariant KU-homology (resp. elliptic homology) of Gr_G is related to the Coulomb branch of 4d $\mathcal{N} = 2$ (resp. 5d $\mathcal{N} = 1$) pure gauge theory. We briefly review this story in Appendix B, and give explicit generators and relations for the Coulomb branches of 3d $\mathcal{N} = 4$ and 4d $\mathcal{N} = 2$ pure gauge theories with gauge groups SL_2 and PGL_2. The 4-dimensional case is a q-analogue of the quantization of the Atiyah-Hitchin manifold [AH88] from [BDG17, Equation 5.51].

1.4. **Notation and terminology.** In most of this article, G will denote a connected, almost simple, and *simply-laced* algebraic group over $\mathbf{C}$, and B will denote a Borel subgroup therein. If H is a reductive algebraic group over $\mathbf{C}$, we will write H_c to denote the maximal compact subgroup of the complex Lie group $H(\mathbf{C})$, so H_c is a compact Lie group. We will use the terminology "finite H_c-space" to mean a finite H_c-CW complex. I have tried to be careful to add the subscript c where necessary, but some omissions have certainly inevitably crept in.

The symbol k will denote an $\mathbf{E}_\infty$-ring which will generally be fixed at the beginning of each section/theorem statement. The symbol F will denote an algebraically closed field containing $\pi_0(k)$. The Langlands dual group $\check{G}$ will generally be defined over F; if we wish to view it as defined over a commutative ring R, we will denote it by $\check{G}_R$. If T is a torus, we will write $\mathbb{X}^*(T)$ and $\mathbb{X}_*(T)$ to denote its lattice of characters and cocharacters. We will also write $\check{\Lambda}$ to denote the root lattice of G and Λ to denote the coroot lattice of G. The symbol $\widetilde{T}$ will denote the extended torus $T \times \mathbf{G}_m^{\mathrm{rot}}$. If $\check{N} \subseteq \check{G}$ is the unipotent radical of a Borel subgroup of $\check{G}$, we will write $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$ to denote a nondegenerate character of $\check{\mathfrak{n}}$, i.e., an additive character which is nonzero on each simple root space. If X is an $\check{G}$-scheme, we will write $T^*(X/_\psi \check{N})$ to denote the Hamiltonian reduction of T^*X (that is, if $\mu : T^*(X) \to \check{\mathfrak{n}}^*$ is the moment map, then $T^*(X/_\psi \check{N}) \cong \mu^{-1}(\psi)/\check{N}$).

Finally, some category-theoretic notation. We will write $\mathcal{S}$ to denote the ∞-category of spaces (also known as "anima", but we will not use this terminology here). If A is an $\mathbf{E}_\infty$-ring, $\mathcal{C}$ is an A-linear ∞-category (that is, a Mod_A-module in the ∞-category of presentable stable ∞-categories) and $A \to B$ is a map of $\mathbf{E}_\infty$-rings, then $\mathcal{C} \otimes_A B$ will denote the B-linear ∞-category $\mathcal{C} \otimes_{\mathrm{Mod}_A} \mathrm{Mod}_B$.

1.5. **Acknowledgements.** I am grateful to Lin Chen, Charles Fu, Tom Gannon, and Kevin Lin for helpful conversations and for entertaining my numerous silly questions. I am also grateful to David Ben-Zvi, Victor Ginzburg, Yiannis Sakellaridis, David Treumann, and Akshay Venkatesh for very enlightening discussions, and Pavel Safronov for a useful email. Thanks to Ben Gammage for discussions which helped shape my understanding of some of the topics in Appendix B, and to Hiraku Nakajima for a very informative email exchange on the same topic. My interest in this area started after I took a class taught by Roman Bezrukavnikov when I was an undergraduate; I am very grateful to him for suggesting that I read [CG10], and also for introducing me to [BFM05], which led me down the beautiful road to geometric representation theory. Since the first version of this article, my understanding of the subject has also been influenced by [BZSV23]. I would also like to thank anonymous referees for helpful suggestions on improving this article. Last, but certainly far from least, the influence, support, advice, and encouragement of my advisors Dennis Gaitsgory and Mike Hopkins is evident throughout this project; I cannot thank them enough.

2. The regular locus

In this section, we will quickly review the derived geometric Satake equivalence following [BF08] and [AG15]. Let k denote a commutative $\mathbf{Q}$-algebra; all Langlands dual objects will be assumed to live over k, and are base-changes of their "split forms" over $\mathbf{Q}$.

Setup 2.1. Let G be a connected reductive group (over $\mathbf{C}$, always), and let $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ denote the affine Grassmannian. There is a canonical left action of $G(\!(t)\!)$ on Gr_G, and hence an action of $G[\![t]\!] \subseteq G(\!(t)\!)$. The affine Grassmannian is a union of $G[\![t]\!]$-invariant closed subschemes X_α of finite type, and one defines $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k) = \mathrm{colim}_\alpha \mathrm{Shv}_{G[\![t]\!]}(X_\alpha; k)$. Inside $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)$ are two full subcategories:

- $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)^{\mathrm{lcc}}$ is the full subcategory of objects whose image under the forgetful functor $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k) \to \mathrm{Shv}(\mathrm{Gr}_G; k)$ is compact. Such objects are called "locally compact".
- $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)^\omega$ of compact objects in $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)$.

The ∞-category $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)$ admits a monoidal structure, which in fact restricts to a monoidal structure on each of the full subcategories above.

Setup 2.2. Let (e, f, h) denote a principal $\mathfrak{sl}_2$-triple in the Langlands dual Lie algebra $\check{\mathfrak{g}}$. The element f defines a nondegenerate character $\psi : \check{\mathfrak{n}} \to \mathbf{A}^1$. Let $\check{\mathfrak{g}}^{*,e}$ denote the orthogonal complement to the subspace $[e, \check{\mathfrak{g}}] \subseteq \check{\mathfrak{g}}$. This defines the *Kostant slice* $\psi + \check{\mathfrak{g}}^{*,e} \subseteq \check{\mathfrak{g}}^*$; we will denote this inclusion by κ. Composing the invariant-theoretic quotient map $\chi : \check{\mathfrak{g}}^* \to \check{\mathfrak{g}}^* /\!\!/ \check{G}$ with the Kostant slice defines an isomorphism. In other words, the following composite is an isomorphism:

$$\psi + \check{\mathfrak{g}}^{*,e} \xrightarrow{\kappa} \check{\mathfrak{g}}^* \xrightarrow{\chi} \check{\mathfrak{g}}^* /\!\!/ \check{G}.$$

It will be convenient to identify $\psi + \check{\mathfrak{g}}^{*,e}$ with $\check{\mathfrak{g}}^* /\!\!/ \check{G}$ under this isomorphism. If the vector space $\check{\mathfrak{g}}^*$ is placed in weight 2, the map κ can be checked to give a *graded* map

$$\kappa : \check{\mathfrak{g}}^*(2) /\!\!/ \check{G} \to \check{\mathfrak{g}}^*(2).$$

Shearing this graded map (in the sense of [Dev24, Section 2.1]) defines a map $\check{\mathfrak{g}}^*[2] /\!\!/ \check{G} \to \check{\mathfrak{g}}^*[2]$, which we will also denote by κ.

Lemma 2.3 (Chevalley restriction). *There is an isomorphism $\check{\mathfrak{g}}^* /\!\!/ \check{G} \cong \mathfrak{t} /\!\!/ W$, which refines to an isomorphism of graded schemes*

$$\check{\mathfrak{g}}^*(2) /\!\!/ \check{G} \cong \mathfrak{t}(2) /\!\!/ W \cong \mathrm{Spec}\, \mathrm{H}^*_G(*; \mathbf{C}).$$

The first part of the following result is [BF08, Theorem 5], and the second part is [AG15, Theorem 12.5.3].

Theorem 2.4 (Bezrukavnikov-Finkelberg, Arinkin-Gaitsgory)**.** *There is a monoidal equivalence*

$$\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)^{\mathrm{lcc}} \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}),$$

which restricts to a monoidal equivalence

$$\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; k)^\omega \simeq \mathrm{Perf}_{\check{\mathcal{N}}/\check{G}}(\check{\mathfrak{g}}^*[2]/\check{G}),$$

where the right-hand side is the full subcategory of those perfect complexes which are set-theoretically supported on the nilpotent cone of $\check{\mathfrak{g}}^*$. Furthermore, there is a *commutative diagram*

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G;k)^{\mathrm{lcc}}) \arrow[r, "\sim"] \arrow[d, "p_!"'] & \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G}) \arrow[d, "\kappa^*"] \\ \mathrm{Shv}_{G[\![t]\!]}(*;k) \arrow[r, "\sim"'] & \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\!\!/\check{G}), \end{tikzcd}$$

where $p : \mathrm{Gr}_G \to *$ is the canonical map to a point and κ^* is pullback along the (shifted) Kostant slice.

We will refer to the first equivalence of Theorem 2.4 as the *derived geometric Satake equivalence*, or more colloquially as "derived Satake".

Definition 2.5. A point $x \in \check{\mathfrak{g}}^*$ is called *regular* if its centralizer $Z_{\check{G}}(x) \subseteq \check{G}$ has dimension given by the rank of $\check{G}$. Let $\check{\mathfrak{g}}^{*,\mathrm{reg}}$ denote the locus of regular elements; this is an open subscheme whose complement is of codimension 3.

Theorem 2.6 (Kostant, [Kos63])**.** *The $\check{G}$-orbit of the Kostant slice $\kappa : \check{\mathfrak{g}}^*/\!\!/\check{G} \to \check{\mathfrak{g}}^*$ identifies with the regular locus $\check{\mathfrak{g}}^{*,\mathrm{reg}}$.*

Corollary 2.7. *Let $\underline{k}_{\mathrm{Gr}_G} \in \mathrm{Ind}(\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G;k)^{\mathrm{lcc}})$ denote the constant sheaf, and let $\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G;k)$ denote the full subcategory generated by $\underline{k}_{\mathrm{Gr}_G}$. Then there is an equivalence*

$$\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G;k) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{G}).$$

Proof. Observe that $\underline{k}_{\mathrm{Gr}_G}$ is the pullback $p^*\underline{k}$ of the (necessarily constant) sheaf $\underline{k} \in \mathrm{Shv}_{G[\![t]\!]}(*;k)$. Since p^* is the right adjoint to $p_!$ (and κ_* is the right adjoint to κ^*), the commutative diagram of Theorem 2.4 says that $\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G;k)$ is equivalent to the full subcategory of $\mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G})$ generated by $\kappa_*\mathcal{O}_{\check{\mathfrak{g}}^*[2]/\!\!/\check{G}}$. However, Theorem 2.6 implies that this full subcategory is equivalent to $\mathrm{QCoh}(\check{\mathfrak{g}}^{*,*}[2]/\check{G})$, as desired. □

A parallel story holds for the Arkhipov-Bezrukavnikov-Ginzburg (called "ABG" in this article) equivalence from [ABG04].

Recollection 2.8. Let $\widetilde{\check{\mathfrak{g}}}$ denote the Grothendieck-Springer resolution, so that $\widetilde{\check{\mathfrak{g}}} = T^*(\check{G}/\check{N})/\check{T}$. The action of $\check{G}$ on $T^*(\check{G}/\check{N})$ defines the moment map $\mu : \widetilde{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}^*$. Let $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}$ denote the preimage of the regular locus $\check{\mathfrak{g}}^{*,\mathrm{reg}} \subseteq \check{\mathfrak{g}}^*$ under the moment map μ.

Proposition 2.9. *There is an isomorphism $\widetilde{\check{\mathfrak{g}}} \cong \check{G} \times^{\check{B}} \check{\mathfrak{b}}^*$, as well as a map $\kappa : \psi + \check{\mathfrak{t}}^* \subseteq \check{\mathfrak{b}}^*$ which fits into a Cartesian square*

$$\begin{tikzcd} \psi + \check{\mathfrak{t}}^* \arrow[r] \arrow[d] & \check{\mathfrak{b}}^* \arrow[r] & \widetilde{\check{\mathfrak{g}}} \arrow[d, "\mu"] \\ \psi + \check{\mathfrak{g}}^{*,e} \arrow[rr] & & \check{\mathfrak{g}}^*. \end{tikzcd}$$

Proof. Let $\check{M}$ be a Hamiltonian $\check{G}$-scheme with moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$. Then the pullback $\check{M} \times_{\check{\mathfrak{g}}^*} (\psi + \check{\mathfrak{g}}^{*,e})$ can be identified with the Whittaker reduction $\check{M}/_{\psi}\check{N}$.

Indeed, a theorem of Kostant's from [Kos78] identifies $\psi+\check{\mathfrak{g}}^{*,e}$ with $(\psi+\check{\mathfrak{n}}^{-,\perp})/\check{N}^-$, so that there are isomorphisms

$$\begin{aligned}\check{M}\times_{\check{\mathfrak{g}}^*}(\psi+\check{\mathfrak{g}}^{*,e}) &\cong \check{M}/\check{G}\times_{\check{\mathfrak{g}}^*/\check{G}}(\psi+\check{\mathfrak{g}}^{*,e})\\ &\cong \check{M}/\check{G}\times_{\check{\mathfrak{g}}^*/\check{G}}(\psi+\check{\mathfrak{n}}^{-,\perp})/\check{N}^-\\ &\cong (\check{M}\times_{\check{\mathfrak{n}}^{-,*}}\{\psi\})/\check{N}^- = \check{M}/_\psi \check{N}^-.\end{aligned}$$

Therefore, the fiber product in the statement of the proposition identifies with the Whittaker reduction $\widetilde{\check{\mathfrak{g}}}/_\psi \check{N}^-$. Since $\widetilde{\check{\mathfrak{g}}}\cong T^*(\check{G}/\check{N})/\check{T}$, we may identify $\widetilde{\check{\mathfrak{g}}}/_\psi \check{N}^-$ with the quotient by $\check{T}$ of $T^*(\check{N}^-{}_\psi\backslash \check{G}/\check{N})$. Since Whittaker functions are supported on the big cell, this twisted cotangent bundle is in turn isomorphic to $T^*(\check{N}^-{}_\psi\backslash(\check{N}^-\times \check{T}\times \check{N})/\check{N})\cong \check{T}\times(\psi+\check{\mathfrak{t}}^*)$. The desired Cartesian square follows. □

Again, $\widetilde{\check{\mathfrak{g}}}$ admits a $\mathbf{G}_m$-action obtained by placing $\check{\mathfrak{b}}^*$ in weight 2, and the map $\kappa:\check{\mathfrak{t}}^*\to\check{\mathfrak{b}}^*$ is equivariant if $\check{\mathfrak{t}}^*$ is also placed in weight 2. Therefore, shearing (as in [Dev24, Section 2.1]) defines a map

$$\check{\mathfrak{t}}^*[2]\xrightarrow{\kappa}\check{\mathfrak{b}}^*[2]\to\widetilde{\check{\mathfrak{g}}}[2].$$

We will sometimes denote this composite also by κ.

The first part of the below equivalence was proved by Arkhipov-Bezrukavnikov-Ginzburg in [ABG04]; the commutative diagram below follows from Proposition 2.9 and Theorem 2.4.

Theorem 2.10. *Let $B\subseteq G$ be a Borel subgroup, and let $I=G[\![t]\!]\times_G B$ denote the associated Iwahori subgroup. Then there is an equivalence*

$$\mathrm{Shv}_I(\mathrm{Gr}_G;k)^{\mathrm{lcc}}\simeq \mathrm{Perf}(\widetilde{\check{\mathfrak{g}}}[2]/\check{G}),$$

which restricts to a monoidal equivalence

$$\mathrm{Shv}_I(\mathrm{Gr}_G;k)^{\omega}\simeq \mathrm{Perf}_{\check{\mathcal{N}}/\check{G}}(\widetilde{\check{\mathfrak{g}}}[2]/\check{G}).$$

Furthermore, there is a commutative diagram

$$\begin{tikzcd}\mathrm{Ind}(\mathrm{Shv}_I(\mathrm{Gr}_G;k)^{\mathrm{lcc}}) \arrow[r, "\sim"] \arrow[d, "p_!"'] & \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}[2]/\check{G}) \arrow[d, "\kappa^*"] \\ \mathrm{Shv}_I(*;k) \arrow[r, "\sim"'] & \mathrm{QCoh}(\check{\mathfrak{t}}^*[2]),\end{tikzcd}$$

where $p:\mathrm{Gr}_G\to *$ is the canonical map to a point and κ^* is pullback along the (shifted) Kostant slice.

As in Corollary 2.7, we find:

Corollary 2.11. *Let $\underline{k}_{\mathrm{Gr}_G}\in\mathrm{Shv}_I(\mathrm{Gr}_G;k)$ denote the constant sheaf, and let $\mathrm{Loc}_I(\mathrm{Gr}_G;k)$ denote the full subcategory generated by $\underline{k}_{\mathrm{Gr}_G}$. Then there is an equivalence*

$$\mathrm{Loc}_I(\mathrm{Gr}_G;k)\simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}[2]/\check{G}).$$

The constant sheaf has singular support given by the zero section. In fact, the ∞-categories $\mathrm{Loc}_{G[\![t]\!]}(\mathrm{Gr}_G;k)$ and $\mathrm{Loc}_I(\mathrm{Gr}_G;k)$ are the subcategories of *locally constant* (equivariant) sheaves on Gr_G. As such, they depend only on the underlying homotopy types of $G[\![t]\!]$, I, and Gr_G.

Notation 2.12. Let G_c be the maximal compact subgroup of $G(\mathbf{C})$, and let T_c be the maximal torus of G_c corresponding to the Borel B. It is not difficult to see that there are homotopy equivalences

$$G[\![t]\!] \simeq G(\mathbf{C}) \simeq G_c$$
$$I \simeq B(\mathbf{C}) \simeq T_c.$$

The homotopy type of Gr_G follows from the next result, due to Quillen and Garland-Raghunathan:

Theorem 2.13 (Quillen, Garland-Raghunathan, [GR75, Mit88])**.** *There is a homeomorphism* $\mathrm{Gr}_G \simeq \Omega G_c$ *which is equivariant for the left-action of* $G_c \subseteq G(\mathbf{C}) \subseteq G(\mathbf{C}[\![t]\!])$ *on the left-hand side and the action of* G_c *on the right-hand side given by conjugation.*

In our discussion below, we will mostly be concerned with the homology of Gr_G, in which case we may replace Gr_G by ΩG_c. To this extent, we will implicitly use Theorem 2.13 without further mention. We will describe analogues of the equivalences of Corollary 2.7 and Corollary 2.11 for equivariant K-theory and equivariant elliptic cohomology.

3. Equivariant cohomology and the case of tori

In order to study and prove analogues of the equivalences of Corollary 2.7 and Corollary 2.11 for other cohomology theories, we need to review some foundational aspects of the theory of equivariant cohomology. I have reviewed some of the basics of equivariant K-theory in [Dev24, Section 2.2]. The theory of equivariant elliptic cohomology is developed similarly, and we will now describe this story (in a somewhat leisurely fashion) following [Lur09, GM23, GM20]. At the end of this section, we describe the geometric Satake equivalence for tori.

The basic question we will address is giving a definition of the ∞-category $\mathrm{Loc}_{T_c}(X; k)$ for a T_c-space X for a sufficiently general $\mathbf{E}_\infty$-ring k. When k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, Theorem 2.10 requires that there is an equivalence

$$\mathrm{Loc}_{T_c}(*; k) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}^*[2]).$$

One often defines the ∞-category of k-modules on a space X as the ∞-category $\mathrm{Fun}(X, \mathrm{Mod}_k)$. However, when $X = BT_c$, the ∞-category $\mathrm{Fun}(BT_c, \mathrm{Mod}_k)$ does *not* agree with $\mathrm{QCoh}(\check{\mathfrak{t}}^*[2])$; instead, it only agrees with a certain completion of this ∞-category, as we will now explain.

Lemma 3.1. *Let k be an $\mathbf{E}_\infty$-algebra. Then there is an equivalence*

$$\mathrm{Fun}(BT_c, \mathrm{Mod}_k) \simeq \mathrm{IndCoh}(\{1\} \times_{\check{T}} \{1\}).$$

If, moreover, k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, this can be rewritten as an equivalence

$$\mathrm{Fun}(BT_c, \mathrm{Mod}_k) \simeq \mathrm{QCoh}(\widehat{\check{\mathfrak{t}}}^*[2]),$$

where $\widehat{\check{\mathfrak{t}}}^*$ denotes the completion of $\check{\mathfrak{t}}^*$ at the origin.

Proof. If X is a finite space, there is an equivalence $\mathrm{Fun}(X, \mathrm{Mod}_k) \simeq \mathrm{IndCoh}_{C_*(\Omega X; k)}$, where $C_*(\Omega X; k)$ is the $\mathbf{E}_1$-k-algebra of k-chains on the based loop space ΩX. When $X = BT_c$, we may identify $\Omega X = T_c$. Recall that T_c is the classifying space of the lattice $\mathbb{X}_*(T)$, so that there is an equivalence

$$C_*(T_c; k) \cong k \otimes_{C_*(\mathbb{X}_*(T); k)} k.$$

Of course, we may identify $C_*(\mathbb{X}_*(T); k) \cong k[\mathbb{X}_*(T)]$ with the ring of functions on $\check{T}$. Therefore, $\mathrm{Spec}\, C_*(T_c; k) \cong \{1\} \times_{\check{T}} \{1\}$, as desired.

Koszul duality gives an equivalence $\mathrm{IndCoh}(C_*(T_c; k)) \to \mathrm{QCoh}(C^*(BT_c; k))$ given by $M \mapsto \mathrm{Hom}_{C_*(T_c; k)}(k, M)$. If k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, then $C^*(BT_c; k)$ is formal, and so it can be identified with the shearing of $\mathrm{H}^*(BT_c; k)$. But

$$\mathrm{Spf}\, \mathrm{H}^*(BT_c; k) \cong \widehat{\mathfrak{t}}(2) \cong \widehat{\check{\mathfrak{t}}}^*(2),$$

so $\mathrm{IndCoh}(C_*(T_c; k))$ is equivalent to $\mathrm{QCoh}(\widehat{\check{\mathfrak{t}}}^*[2])$, as desired. $\square$

Example 3.2. Suppose $T_c = S^1$. Then Lemma 3.1 tells us that $\mathrm{Fun}(BS^1, \mathrm{Mod}_k) \simeq \mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$; the equivalence sends a functor $BS^1 \to \mathrm{Mod}_k$, regarded as a k-module M with S^1-action, to its homotopy invariants M^{hS^1}. Let $t \in \pi_{-2}(k^{hS^1})$ denote a generator. Observe that if $a_\lambda : k \to k[2]$ denotes the boundary map in the cofiber sequence $k[1] \to C_*(S^1; k) \to k$, the homotopy invariants of $k[a_\lambda^{-1}]$ are simply $k^{hS^1}[t^{-1}]$ (i.e., the Tate construction). In particular, $\pi_*(k[a_\lambda^{-1}])^{hS^1} \cong k(\!(t)\!)$. However, there is no (ind-)object in $\mathrm{Fun}(BS^1, \mathrm{Mod}_k)$ whose image in $\mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$

has homotopy given by $k[t^{\pm 1}]$: any object of $\mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$ must have t as a topologically nilpotent element in its homotopy.

We therefore need an alternative definition of $\mathrm{Loc}_{T_c}(*; k)$, so that it is equivalent to $\mathrm{QCoh}(\check{\mathfrak{t}}^*[2])$ when k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra. Motivated by methods from equivariant homotopy theory, as well as [Lur09, Lur18a, Lur18b, Lur19], we will simply *define* $\mathrm{Loc}_{T_c}(*; k)$ to be the category of quasicoherent sheaves on a (spectral) stack $\mathcal{M}_T$. That this category has any relation to topology will come from the requirement that the category of quasicoherent sheaves on the *completion* of $\mathcal{M}_T$ at a certain basepoint is equivalent to the ind-completion of $\mathrm{Fun}(BT_c, \mathrm{Mod}_k)$.

For this, we review some constructions from [Lur09] in a form suitable for our applications. This review will necessarily be brief, since a detailed exposition may be found in *loc. cit.*; there is also some discussion in the early sections of [GKV95] in the setting of ordinary (as opposed to spectral) algebraic geometry.

Setup 3.3. Fix an $\mathbf{E}_\infty$-ring k and a commutative k-group $\mathbf{G}$, so $\mathbf{G}$ defines a functor $\mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z},\geq 0}$ which is representable by a *flat* k-algebra; here, $\mathrm{Mod}_{\mathbf{Z},\geq 0}$ denotes the category of connective $\mathbf{Z}$-module spectra. We will write $\mathbf{G}_0$ to denote the resulting commutative group scheme over $\pi_0 k$. Note that taking zeroth spaces defines an equivalence between $\mathrm{Mod}_{\mathbf{Z},\geq 0}$ and topological abelian groups.

Definition 3.4. A *preorientation of* $\mathbf{G}$ is a pointed map $S^2 \to \Omega^\infty \mathbf{G}(k)$ of spaces, i.e., a map $\Sigma^2 \mathbf{Z} \to \mathbf{G}(k)$ of $\mathbf{Z}$-modules (by adjunction). This induces a map $\mathbf{C}P^\infty = \Omega^\infty \Sigma^2 \mathbf{Z} \to \Omega^\infty \mathbf{G}(k)$ of topological abelian groups, and hence a map $\mathrm{Spf}\, A^{\mathbf{C}P^\infty} \to \mathbf{G}$ of $\mathbf{E}_\infty$-k-group schemes. (Note that $\mathrm{Spf}\, A^{\mathbf{C}P^\infty}$ need not admit the structure of a commutative k-group scheme: for instance, $A^{\mathbf{C}P^\infty}$ need not be flat over k.)

Definition 3.5. Given a preorientation $S^2 \to \Omega^\infty \mathbf{G}(k)$, we obtain a map $\mathcal{O}_{\mathbf{G}} \to C^*(S^2; k)$ of $\mathbf{E}_\infty$-k-algebras. On π_0, this induces a map $\pi_0 \mathcal{O}_{\mathbf{G}} = \mathcal{O}_{\mathbf{G}_0} \to \pi_0 C^*(S^2; k)$. However, the target can be identified with the trivial square-zero extension $\pi_0 k \oplus \pi_{-2} k$, so that the preorientation defines a derivation $\mathcal{O}_{\mathbf{G}_0} \to \pi_{-2} k$. This defines a map $\beta : \omega = \Omega^1_{\mathbf{G}_0/\pi_0 k} \to \pi_{-2} k$. The preorientation is called an *orientation* if $\mathbf{G}_0$ is smooth of relative dimension 1 over $\pi_0 k$, and the composite

$$\pi_n k \otimes_{\pi_0 k} \omega \to \pi_n k \otimes_{\pi_0 k} \pi_{-2} k \xrightarrow{\beta} \pi_{n-2} k$$

is an isomorphism for each $n \in \mathbf{Z}$. This forces k to be 2-periodic (but does not force its homotopy to be concentrated in even degrees).

Warning 3.6. As discussed in [Lur09, Section 3.2], the universal $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra over which the additive group scheme $\mathbf{G}_a$ admits an orientation is given by $\mathbf{Z}[\mathbf{C}P^\infty][\frac{1}{\beta}] = \mathbf{Q}[\beta^{\pm 1}]$. Therefore, we are allowed to let $\mathbf{G} = \mathbf{G}_a$ in the story below only when k is a 2-periodic $\mathbf{E}_\infty$-$\mathbf{Q}$-*algebra*. (If k is not an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra, one cannot in general define $\mathbf{G}_a = \mathrm{Spec}\, k[t]$ as a commutative k-group: the coproduct $k[t] \to k[x, y]$ will in general not be a map of $\mathbf{E}_\infty$-k-algebras.)

We can now review the definition of T_c-equivariant k-cohomology when T_c is a compact torus. We will write T to denote the corresponding split torus over $\mathbf{Z}$.

Construction 3.7. Fix an $\mathbf{E}_\infty$-ring k as above and a commutative k-group $\mathbf{G}$. Given a compact abelian Lie group T_c, define a k-scheme $\mathcal{M}_T$ by the mapping stack $\mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G})$. The underlying $\pi_0(k)$-schemes will be denoted by $\mathbf{G}_0$ and $\mathcal{M}_{T,0}$. If we wish to emphasize the dependence on k, we will add a superscript (e.g., $\mathcal{M}_T^k$).

We will be particularly interested in the case when T_c is a torus. Let $\mathcal{T}$ be the full subcategory of $\mathcal{S}$ spanned by those spaces which are homotopy equivalent to BT_c with T_c being a compact abelian Lie group. By arguing as in [Lur19, Theorem 3.5.5], a preorientation of $\mathbf{G}$ is equivalent to the data of a functor $\mathcal{M} : \mathcal{T} \to \mathrm{Aff}_k$ along with compatible equivalences $\mathcal{M}(BT_c) \simeq \mathcal{M}_T$. The $\mathbf{E}_\infty$-k-algebra $\mathcal{O}_{\mathcal{M}_T}$ is the T_c-equivariant k-cochains of a point, and will occasionally be denoted by k_T.

We can now sketch the construction of the T_c-equivariant k-cochains of more general T_c-spaces; see [Lur09, Theorem 3.2]. Let T_c be a torus over $\mathbf{C}$ for the remainder of this discussion, and let $\mathbf{G}$ be an *oriented* commutative k-group. Let $\mathcal{S}(T_c)$ denote the ∞-category of finite T_c-spaces, i.e., the smallest subcategory of $\mathrm{Fun}(BT_c, \mathcal{S})$ which contains the quotients T_c/T_c' for closed subgroups $T_c' \subseteq T_c$, and which is closed under finite colimits. There is a functor $\mathcal{F}_T : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T)$ which is uniquely characterized by the requirement that it preserve finite limits and sends $T_c/T_c' \mapsto q_* \mathcal{O}_{\mathcal{M}_{T_c'}}$. Here, $q : \mathcal{M}_{T_c'} \to \mathcal{M}_T$ is the canonical map induced by the inclusion $T_c' \subseteq T_c$. If $X \in \mathcal{S}(T_c)$, then the *T_c-equivariant k-cochains of X* is the global sections $\Gamma(\mathcal{M}_T; \mathcal{F}_T(X))$; we will denote it by $C^*_{T_c}(X; k)$. This can be extended to define T_c-equivariant k-cochains of filtered colimits of finite T_c-spaces. If we wish to emphasize the dependence on k, we will denote $\mathcal{F}_T(X)$ by $\mathcal{F}_T(X; k)$.

Remark 3.8. If k is 2-periodic and $\mathbf{G}$ is a commutative k-group, then [Lur18b, Proposition 4.3.23] shows that the data of an orientation on k (in the sense of Definition 3.5) is equivalent to the formal completion of $\mathbf{G}$ at the origin being isomorphic to $\mathrm{Spf}\, C^*(BS^1; k)$. That is, when $\mathbf{G}$ is oriented, the formal completion of $\mathcal{M}_T$ at its basepoint is isomorphic to $\mathrm{Spf}\, C^*(BT_c; k)$.

We will denote the functor $\Gamma(\mathcal{M}_T; \mathcal{F}_T(-)) : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{Mod}(\Gamma(\mathcal{M}_T; \mathcal{O}_{\mathcal{M}_T}))$ by $C^*_{T_c}(-; k) : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{Mod}(k_T)$.

Definition 3.9. If $X \in \mathcal{S}(T_c)$, then the *T_c-equivariant k-chains of X* is the quasicoherent sheaf on $\mathcal{M}_T$ given by the $\mathcal{O}_{\mathcal{M}_T}$-linear dual $\mathcal{F}_T(X)^\vee$. We will denote its global sections by $C_*^{T_c}(X; k)$. Note that if X admits an $\mathbf{E}_n$-algebra structure (compatible with the T_c-action), then $\mathcal{F}_T(X)^\vee$ admits the structure of an $\mathbf{E}_n$-algebra[4] in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_T))$. Note that $C_*^{T_c}(*; k) \simeq k_T$, which completes to the k-cochains (*not* k-chains) of BT_c.

If X is a filtered colimit $\mathrm{colim}_\alpha X_\alpha$ of finite T_c-spaces, we will write $\mathcal{F}_T(X)^\vee$ to denote $\mathrm{colim}_\alpha(\mathcal{F}_T(X_\alpha)^\vee)$. Note that if we equip the presentation of X as a filtered colimit $\mathrm{colim}_\alpha X_\alpha$ with the structure of a filtered $\mathbf{E}_n$-algebra, then $\mathcal{F}_T(X)^\vee$ acquires the structure of an $\mathbf{E}_n$-algebra in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_T))$.

Notation 3.10. Let $\lambda : T \to \mathbf{G}_m$ be a character, and let $T_\lambda = \ker(\lambda)$. Then the map $q : \mathcal{M}_{T_\lambda} \to \mathcal{M}_T$ is a closed immersion, and we will denote the ideal in $\mathcal{O}_{\mathcal{M}_T}$ defined by this closed immersion by $\mathcal{I}_\lambda$. Equivalently, let V_λ denote the T_c-representation obtained by the projection $T \to T_\lambda$. Then $\mathcal{I}_\lambda$ is given by the line bundle $\mathcal{F}_T(S^{V_\lambda})$.

It is trickier to extend the definition of equivariant cochains to nonabelian groups, but a construction is sketched in [Lur09, Section 3.5], and a detailed construction is given in [GM23]. However, we will not recall this here, because we will only be concerned with torus-equivariance in the present article.

[4] If $\mathcal{C}$ is a symmetric monoidal ∞-category, [Lur18a, Corollary 3.3.4] can be used to show that there is an equivalence $\mathrm{coCAlg}(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbf{E}_n}(\mathrm{coCAlg}(\mathcal{C}))$.

We now take a moment to prove some foundational aspects of the theory of generalized equivariant cohomology.

Lemma 3.11 (Atiyah-Bott localization)**.** *Let X be a finite T_c-space, and let $\mathcal{U}_X \subseteq \mathcal{M}_T$ denote the complement of the union of the closed substacks $\mathcal{M}_{T'}$ over all stabilizers $T'_c \subseteq T_c$ of points in X. Then the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^{T_c})$ is an isomorphism after pulling back to $\mathcal{U}_X$.*

Proof. This follows from induction on the cell structure of X. Namely, the statement is true when the T-action on X is trivial, which gives the base case. For the inductive step, note that if X is the cofiber of a map $T/T' \to Y$, then there is a cofiber sequence $\mathcal{F}_T(X) \to \mathcal{F}_T(Y) \to \mathcal{F}_T(T/T')$; but $\mathcal{F}_T(T/T')$ is isomorphic to the pushforward of the structure sheaf along the map $\mathcal{M}_{T'} \to \mathcal{M}_T$, and so it vanishes upon pulling back to $\mathcal{U}_X$. This implies that the map $\mathcal{F}_T(X) \to \mathcal{F}_T(Y)$ is an isomorphism upon pulling back to $\mathcal{U}_X$, as desired. $\square$

One consequence of Lemma 3.11 which is worth restating is the following. Let $\mathring{\mathcal{M}}_T$ denote the complement of the union of the closed subschemes $\mathcal{M}_{T'}$ ranging over all closed *proper* subgroups $T' \subsetneq T$. Then the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^{T_c})$, and hence the map $\mathcal{F}_T(X^{T_c})^\vee \to \mathcal{F}_T(X)^\vee$, is an equivalence upon restriction to $\mathring{\mathcal{M}}_T$.

We will also need a version of the Goresky-Kottwitz-MacPherson approach [GKM98] to equivariant cohomology; in the setting of generalized equivariant cohomology, it has also been studied in [HHH05, GM23]. As such, we will only give a sketch of the relevant argument.

Definition 3.12. Let X be a finite T_c-space equipped with a chosen presentation in terms of T_c-cells. Say that X is a *GKM space* if the following conditions are satisfied:

(a) $\pi_0 \mathcal{F}_T(X)$ is a vector bundle over $\mathcal{M}_{T,0}$;
(b) if $X^{(1)}$ denotes the equivariant 1-skeleton of X, then $X^{(1)}$ consists of a finite number of spheres S^λ meeting only at the fixed points, where λ ranges over characters of T.

In this setup, let V denote the set X^{T_c} of fixed points, and let E denote the set of characters λ such that $S^\lambda \subseteq X^{(1)}$. There are two maps $E \rightrightarrows V$ sending λ to the points $0, \infty \in S^\lambda \subseteq X^{(1)}$. The resulting graph with set of vertices V and set of edges E will be referred to as the *GKM graph* of X.

The utility of the first condition in the above definition is due to the following.

Lemma 3.13. *Let X be a finite T_c-space. If $\pi_0\mathcal{F}_T(X)$ is a vector bundle over $\mathcal{M}_{T,0}$, the map $\pi_0\mathcal{F}_T(X) \to \pi_0\mathcal{F}_T(X^{T_c})$ is an injection.*

Proof. Since the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X^{T_c}) \to \mathcal{F}_T(X^{T_c})_{\mathring{\mathcal{M}}_T}$ factors as $\mathcal{F}_T(X) \to \mathcal{F}_T(X)|_{\mathring{\mathcal{M}}_T} \to \mathcal{F}_T(X^{T_c})|_{\mathring{\mathcal{M}}_T}$, and the map $\mathcal{F}_T(X)|_{\mathring{\mathcal{M}}_T} \to \mathcal{F}_T(X^{T_c})|_{\mathring{\mathcal{M}}_T}$ is an equivalence by Lemma 3.11, it suffices to show that the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X)|_{\mathring{\mathcal{M}}_T}$ induces an injection on π_0. But $\pi_0\mathcal{F}_T(X)$ was assumed to be a vector bundle over $\mathcal{M}_{T,0}$, so one is reduced to the case $X = *$, i.e., to showing that the map $\mathcal{O}_{\mathcal{M}_T} \to \mathcal{O}_{\mathcal{M}_T}|_{\mathring{\mathcal{M}}_T}$ induces an injection on π_0. This, however, is clear, since the closed subscheme $\mathcal{M}_{T',0} \hookrightarrow \mathcal{M}_{T,0}$ defined by each closed subgroup $T' \subseteq T$ is cut out by a regular sequence. $\square$

Proposition 3.14 (Goresky-Kottwitz-MacPherson)**.** *Let X be a finite GKM T_c-space, and choose a presentation in terms of T_c-cells. For each character $\lambda : T \to S^1$, let T_λ denote the kernel of T, let $q_\lambda : \mathcal{M}_{T_\lambda} \to \mathcal{M}_T$ denote the induced map, and let $S(\lambda)$ denote the unit representation sphere. Then there is an equalizer diagram*

$$\pi_0\mathcal{F}_T(X) \hookrightarrow \pi_0\mathcal{F}_T(X^{T_c}) \cong \mathrm{Map}(V, \mathcal{O}_{\mathcal{M}_{T,0}}) \rightrightarrows \prod_{\lambda\in E} q_{\lambda,*}\mathcal{O}_{\mathcal{M}_{T_\lambda,0}},$$

where the two maps in the equalizer are defined in the evident manner.

Proof sketch. First, we show that the maps $\pi_0\mathcal{F}_T(X) \to \pi_0\mathcal{F}_T(X^{T_c})$ and $\pi_0\mathcal{F}_T(X^{(1)}) \to \pi_0\mathcal{F}_T(X^{T_c})$ have the same image. There is an evident map from the image of $\pi_0\mathcal{F}_T(X) \to \pi_0\mathcal{F}_T(X^{T_c})$ to the image of $\pi_0\mathcal{F}_T(X^{(1)}) \to \pi_0\mathcal{F}_T(X^{T_c})$, which we will denote by f. The map f is an injection by Lemma 3.13. Let T' denote a proper closed subgroup of T of codimension 1, and let $U' \subseteq \mathcal{M}_{T',0}$ denote the complement of the union of the closed varieties $\mathcal{M}_{T'',0}$ ranging over the proper closed subgroups $T'' \subseteq T'$. By Lemma 3.11, the map f is an isomorphism upon restriction to $U' \subseteq \mathcal{M}_{T',0} \subseteq \mathcal{M}_{T,0}$ for each proper closed subgroup $T' \subseteq T$ of codimension 1. Therefore, the locus $Z \subseteq \mathcal{M}_{T,0}$ over which f fails to be an isomorphism is contained in the union of closed subvarieties $\mathcal{M}_{T',0}$ for finitely many $T' \subseteq T$ of codimension at least 2. However, the map $\pi_0\mathcal{F}_T(X) \to \pi_0\mathcal{F}_T(X)|_{\mathcal{M}_{T,0}-Z}$ is an isomorphism (by Hartogs). Since the same is true of the map $\pi_0\mathcal{F}_T(X^{T_c}) \to \pi_0\mathcal{F}_T(X^{T_c})|_{\mathcal{M}_{T,0}-Z}$, and the map $\pi_0\mathcal{F}_T(X) \to \pi_0\mathcal{F}_T(X^{T_c})$ factors through the map $\pi_0\mathcal{F}_T(X^{(1)}) \to \pi_0\mathcal{F}_T(X^{T_c})$, the desired result follows.

For the equalizer diagram, an easy induction on the cell structure of X reduces us to the case $X = S^\lambda$ for a character $\lambda : T \to S^1$. In this case, the isomorphism $T/T_\lambda \cong S^\lambda$ defines an isomorphism between $\pi_0\mathcal{F}_T(S(\lambda))$ and the pushforward of the structure sheaf along the map $\mathcal{M}_{T_\lambda,0} \to \mathcal{M}_{T,0}$. Since $S^\lambda \cong \Sigma S(\lambda)$, we obtain an equalizer diagram

$$\pi_0\mathcal{F}_T(S^\lambda) \to \mathcal{O}_{\mathcal{M}_{T,0}} \oplus \mathcal{O}_{\mathcal{M}_{T,0}} \cong \mathrm{Map}(\{0,\infty\}, \mathcal{O}_{\mathcal{M}_{T,0}}) \rightrightarrows q_{\lambda,*}\mathcal{O}_{\mathcal{M}_{T_\lambda,0}}.$$

This proves the desired claim. □

The same argument proves the following dual to Proposition 3.14 (see also [Bri00]).

Proposition 3.15. *Let X be a finite GKM T_c-space, and choose a presentation in terms of T_c-cells. Then $\pi_0\mathcal{F}_T(X)^\vee$ is isomorphic to the subset of $\pi_0\mathcal{F}_T(X^{T_c})^\vee \cong \mathcal{O}_{\mathcal{M}_{T,0}}[X^{T_c}]$ of those $\sum_{x\in X^{T_c}} f_x[x] \in \mathcal{O}_{\mathcal{M}_{T,0}}[X^{T_c}]$ such that:*

- *For each fixed point $x \in X^{T_c}$, the poles of f_x all have order ≤ 1, and these are contained in the ideal sheaf of $\mathcal{O}_{\mathcal{M}_{T_\lambda,0}}$ for each character $\lambda : T_c \to S^1$ such that the T_c-orbit S^λ meets x.*
- *For each character $\lambda : T_c \to S^1$ such that the T_c-orbit S^λ meets $x_0, x_\infty \in X^{T_c}$, we have*

$$\mathrm{Res}_{\mathcal{M}_{T_\lambda,0}}(f_{x_0}) + \mathrm{Res}_{\mathcal{M}_{T_\lambda,0}}(f_{x_\infty}) = 0.$$

These results can be extended without much trouble to ind-T_c-spaces X with isolated fixed points satisfying the conditions of Definition 3.12. (The first condition therein should be replaced by the condition that $\pi_0\mathcal{F}_T(X)$ is an ind-vector bundle over $\mathcal{M}_{T,0}$.)

The preceding discussion can be categorified, as we now explain. The following categorifies the T_c-equivariant k-cochains $C^*_{T_c}(X; k)$.

Construction 3.16. Let $\mathrm{Loc}_{T_c}(*; k)$ denote the ∞-category $\mathrm{QCoh}(\mathcal{M}_T)$. Let $T_c' \subseteq T_c$ be a closed subgroup, so that there is an associated morphism $q : \mathcal{M}_{T'} \to \mathcal{M}_T$. This defines a symmetric monoidal functor $\mathrm{QCoh}(\mathcal{M}_T) \to \mathrm{QCoh}(\mathcal{M}_{T'})$, which equips $\mathrm{QCoh}(\mathcal{M}_{T'})$ with the structure of a $\mathrm{QCoh}(\mathcal{M}_T)$-module.

Let $\mathcal{L}\mathrm{oc}_{T_c}(-; k) : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{ShvCat}(\mathcal{M}_T))$ be the functor uniquely characterized by the requirement that it preserve finite limits and send $T/T' \mapsto \mathrm{QCoh}(\mathcal{M}_{T'})$. If $X \in \mathcal{S}(T_c)$, then the ∞-category $\mathrm{Loc}_{T_c}(X; k)$ of *T_c-equivariant local systems of k-modules on X* is defined to be the global sections of the quasicoherent stack $\mathcal{L}\mathrm{oc}_{T_c}(X; k)$ on $\mathcal{M}_T$. If X is a T_c-space which is presented as a filtered colimit of finite T_c-spaces X_α, we will write $\mathrm{Loc}_{T_c}(X; k)$ to denote $\mathrm{colim}\, \mathrm{Loc}_{T_c}(X_\alpha; k)$.

If $f : X \to Y$ is a map in $\mathcal{S}(T_c)$, the associated symmetric monoidal functor $f^* : \mathrm{Loc}_{T_c}(Y; k) \to \mathrm{Loc}_{T_c}(X; k)$ (induced by taking global sections of the morphism $f^* : \mathcal{L}\mathrm{oc}_{T_c}(Y; k) \to \mathcal{L}\mathrm{oc}_{T_c}(X; k)$ of $\mathbf{E}_\infty$-algebras in quasicoherent stacks over $\mathcal{M}_T$) will be called the *pullback*. One can show that $\mathrm{Loc}_{T_c}(X; k)$ is a presentable stable ∞-category, and that f^* preserves small colimits (so it has a right adjoint f_*, which will be called *pushforward*).

For instance, if $T_c = \{1\}$, then $\mathrm{Loc}_{T_c}(X; k)$ is equivalent to the ∞-category $\mathrm{Loc}(X; k) := \mathrm{Fun}(X, \mathrm{Mod}_k)$ of local systems on X.

Remark 3.17. Let X be a finite T_c-space. The *constant local system $\underline{k}_T$* is defined to be the image of $\mathcal{O}_{\mathcal{M}_T}$ under the symmetric monoidal functor $\mathrm{Loc}_{T_c}(*; k) \simeq \mathrm{QCoh}(\mathcal{M}_T) \to \mathrm{Loc}_{T_c}(X; k)$ induced by pullback along $f : X \to *$. Observe that if $\underline{k}_T$ denotes the constant local system, then $\mathrm{End}_{\mathrm{Loc}_{T_c}(X;k)}(\underline{k}_T) \simeq C^*_{T_c}(X; k)$. Indeed, $\mathrm{End}_{\mathrm{Loc}_{T_c}(X;k)}(\underline{k}_T) \simeq \Gamma(\mathcal{M}_T; f_* f^* \mathcal{O}_{\mathcal{M}_T})$, but it is easy to see that $f_* f^* \mathcal{O}_{\mathcal{M}_T} = \mathcal{F}_T(X) \in \mathrm{QCoh}(\mathcal{M}_T)$. The desired claim then follows from Construction 3.7.

Remark 3.18. If the complexification of T_c were a *finite* diagonalizable group scheme (such as μ_n), the desired category $\mathrm{Loc}_{T_c}(X; k)$ is closely related to the ∞-category of *$\mathbf{G}$-tempered local systems* on the orbispace $X/\!\!/T$, as described in [Lur19]. Our understanding is that Lurie is planning to describe an extension of the work in [Lur19] and its connections to equivariant homotopy theory in a future article. We warn the reader that Construction 3.16 is somewhat *ad hoc*; so the resulting category of equivariant local systems may or may not agree with the output of forthcoming work of Lurie.

Remark 3.19. If X is a finite T_c-space, a more straightforward definition of the category of T_c-equivariant local systems on X is simply the category $\mathrm{Fun}(X/T_c, \mathrm{Mod}_k)$. Equivalently, it can be described as the functor $\mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ which is uniquely characterized by the requirement that it preserve finite limits and send $T_c/T_c' \mapsto \mathrm{Fun}(BT_c', \mathrm{Mod}_k)$. It follows from Lemma 3.1 that $\mathrm{Fun}(BT_c', \mathrm{Mod}_k)$ is equivalent to $\mathrm{Mod}(C^*(BT_c'; k))$. As discussed in Remark 3.8, if the group scheme $\mathbf{G}$ is oriented, then this is in turn equivalent to $\mathrm{QCoh}(\widehat{\mathcal{M}_T})$, where $\widehat{\mathcal{M}_T}$ is the completion of $\mathcal{M}_T$ at its basepoint. That is, $\mathrm{Fun}(BT_c', \mathrm{Mod}_k)$ can be viewed as a completion of $\mathrm{QCoh}(\mathcal{M}_{T'})$. This implies that $\mathrm{Fun}(X/T_c, \mathrm{Mod}_k)$ can be viewed as a completion of the subcategory of compact objects of $\mathrm{Loc}_{T_c}(X; k)$. Motivated by this, we will write $\mathrm{Loc}^\wedge_{T_c}(X; k)$ to denote $\mathrm{Fun}(X/T_c, \mathrm{Mod}_k)$; we will use the same notation to denote the extension of the assignment $X \mapsto \mathrm{Loc}^\wedge_{T_c}(X; k)$ to filtered colimits of finite T_c-spaces.

Using this discussion, let us now discuss geometric Satake with k-coefficients in the case of a torus.

Theorem 3.20. *Fix a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k and an oriented commutative k-group scheme $\mathbf{G}$. Let $\check{T} = \operatorname{Spec} k[\mathbb{X}^*(\check{T})]$ denote the dual torus over k. In the following statements, all actions of $\check{T}$ are trivial. Then there are equivalences*

$$\mathrm{Loc}^{\wedge}_{T_c}(\mathrm{Gr}_T; k) \simeq \mathrm{IndCoh}((\{1\} \times_{\check{T}} \{1\})/\check{T}),$$
$$\mathrm{Loc}_{T_c}(\mathrm{Gr}_T; k) \simeq \mathrm{QCoh}(\mathcal{M}_T/\check{T}).$$

Moreover, there is an isomorphism of spectral group k-schemes

$$\operatorname{Spec} \mathcal{F}_T(\mathrm{Gr}_T)^\vee \cong \mathcal{M}_T \times_{\mathrm{Spec}(k)} \check{T} \cong \mathcal{M}_T \times_{\mathcal{M}_T/\check{T}} \mathcal{M}_T.$$

Proof. Since the underlying topological space of Gr_T is simply the lattice $\mathbb{X}_*(T)$, it follows from Lemma 3.1 that

$$\mathrm{Loc}^{\wedge}_{T_c}(\mathrm{Gr}_T; k) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Loc}^{\wedge}_{T_c}(*; k) \simeq \mathrm{QCoh}(B\check{T}) \otimes_{\mathrm{Mod}_k} \mathrm{IndCoh}(\{1\} \times_{\check{T}} \{1\}).$$

For the trivial action of $\check{T}$ on $\{1\} \times_{\check{T}} \{1\}$, this is precisely $\mathrm{IndCoh}((\{1\} \times_{\check{T}} \{1\})/\check{T})$. Exactly the same discussion proves the second equivalence:

$$\mathrm{Loc}_{T_c}(\mathrm{Gr}_T; k) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Loc}_{T_c}(*; k) \simeq \mathrm{QCoh}(B\check{T}) \otimes_{\mathrm{Mod}_k} \mathrm{QCoh}(\mathcal{M}_T).$$

The claim about $\mathcal{F}_T(\mathrm{Gr}_T)^\vee$ can be proved similarly. □

Remark 3.21. Note that in Theorem 3.20, the "spectral"/algebro-geometric description of $\mathrm{Loc}^{\wedge}_{T_c}(\mathrm{Gr}_T; k)$ does not seem to depend on the choice of coefficient k (in particular, not on $\mathbf{G}$). This dependence, however, can be made more explicit by noting that $\mathrm{IndCoh}(\{1\} \times_{\check{T}} \{1\})$ is equivalent to $\mathrm{Mod}(k^{hT_c}) \simeq \mathrm{QCoh}(\widehat{\mathcal{M}_T})$. That is, there is an equivalence $\mathrm{Loc}^{\wedge}_{T_c}(\mathrm{Gr}_T; k) \simeq \mathrm{QCoh}(\widehat{\mathcal{M}_T}/\check{T})$.

Our basic goal is to find a replacement of Theorem 3.20 where Gr_T is replaced by Gr_G for a general connected reductive group G.

4. Degenerations

We begin this section by immediately amending the goal referred to at the end of the preceding section. Namely, instead of studying the ∞-category $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ for a connected reductive group G and a maximal torus $T \subseteq G$, we will study a particular *degeneration* of this ∞-category. Before discussing the construction of this degeneration, let us motivate *why* it is useful (see also the introduction for some "philosophy" regarding this degeneration).

Suppose that there was an equivalence of the form $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k) \simeq \mathrm{QCoh}(\mathfrak{X}_k)$ for some spectral k-stack $\mathfrak{X}_k$. In order for such an equivalence to be considered related to Langlands duality, the stack $\mathfrak{X}_k$ must have some relationship to the dual group $\check{G}$; for instance, one can wonder whether the underlying classical $\pi_0(k)$-stack of $\mathfrak{X}_k$ lives over the classifying stack $B\check{G}_{\pi_0(k)}$. Here, $\check{G}_{\pi_0(k)}$ is the base-change of the Chevalley split form of $\check{G}$ along the map $\mathbf{Z} \to \pi_0(k)$. (When k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, the stack $\mathfrak{X}_k$ is $\widetilde{\check{\mathfrak{g}}}[2]/\check{G}$, which does indeed live over $B\check{G}$.) It follows that the most satisfying description of $\mathfrak{X}_k$ would involve a lift of the dual group $\check{G}$ to a (flat) spectral group scheme over k. Unfortunately, this is far from clear: even giving a flat lift of SL_2 to complex K-theory KU seems difficult.

Instead, let us return to the general situation of a finite T_c-space X. One can then view $\mathrm{Loc}_{T_c}(X; k)$ as a categorification of the cochains $\mathcal{F}_T(X) \in \mathrm{QCoh}(\mathcal{M}_T)$; so for the moment, let us just describe a degeneration of $\mathcal{F}_T(X)$ and $\mathcal{M}_T$. There is a natural filtered lift of $\mathcal{M}_T = \mathrm{Spec}\, k_T$ to a filtered $\tau_{\geq \star}(k)$-scheme, given by $\mathrm{Spec}\, \tau_{\geq \star}(k_T)$. (This construction is, of course, closely related to the even filtration constructed in [HRW22, Pst23a].) In particular, one obtains a corresponding graded $\pi_*(k)$-scheme $\mathrm{Spec}\, \pi_*(k_T)$. Note that this is now a *classical* scheme, with no spectral algebro-geometric nature. If k is even-periodic, i.e., is equipped with an isomorphism $\pi_*(k) \cong k[u^{\pm 1}]$ with $u \in \pi_2(k)$, then this is equivalent to the data of the classical $\pi_0(k)$-scheme $\mathrm{Spec}\, \pi_0(k_T)$. (Recall that this is the affinization of the scheme $\mathcal{M}_{T,0}$; to get to the definition described below, one needs to replace $\mathrm{Spec}\, \pi_0(k_T)$ in the below discussion by $\mathcal{M}_{T,0}$.)

If the finite T_c-space X has even cells, then one can construct a well-behaved filtered lift of $\mathcal{F}_T(X)$ to a filtered quasicoherent sheaf over $\mathrm{Spec}\, \tau_{\geq \star}(k_T)$, given by $\tau_{\geq \star}\mathcal{F}_T(X)$. This defines a corresponding graded variant of $\mathcal{F}_T(X)$, given simply by the quasicoherent sheaf $\pi_0\mathcal{F}_T(X)$ over $\mathrm{Spec}\, \pi_0(k_T)$. Again, this is an object in the realm of *classical* algebraic geometry; so when applied to the affine Grassmannian Gr_G, it is something that could, in theory, be described in terms of the usual dual group $\check{G}$ base-changed to $\pi_0(k)$.

The idea for constructing the desired degeneration of $\mathrm{Loc}_{T_c}(X; k)$ is very similar; we now turn to its mechanics. Let us begin with a simple observation. If Y is a connected space, the ∞-category $\mathrm{Loc}(Y; k) = \mathrm{Fun}(Y, \mathrm{Mod}_k)$ of local systems on Y is equivalent, by Koszul duality, to $\mathrm{LMod}_{C_*(\Omega Y; k)}$. This is very useful, since it allows one to reduce the study of local systems to the study of a particular (derived) algebra. A similar property is true for $\mathrm{Loc}_{T_c}(X; k)$:

Proposition 4.1. *Let X be a connected finite T_c-space. Then there is an equivalence* $\mathrm{Loc}_{T_c}(X; k) \simeq \mathrm{LMod}_{\mathcal{F}_T(\Omega X)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$.

Proof. Let $s : * \to X$ denote the inclusion of a point. We claim that $s^* : \mathrm{Loc}_{T_c}(X; k) \to \mathrm{QCoh}(\mathcal{M}_T)$ admits a left adjoint $s_!$. Indeed, the statement for general X follows formally from the case of $X = T/T'$ for some closed subgroup $T' \subseteq T$ (so s is the inclusion of the trivial coset). In this case, s^* is the functor $\mathrm{QCoh}(\mathcal{M}_{T'}) \to \mathrm{QCoh}(\mathcal{M}_T)$ given by pushforward along the associated morphism $q : \mathcal{M}_{T'} \to \mathcal{M}_T$, so it has a left adjoint $s_!$ given by q^*. Note that s^* also has a right adjoint; in particular, it preserves small limits and colimits. Observe now that $s_! \mathcal{O}_{\mathcal{M}_T}$ is a compact generator of $\mathrm{Loc}_{T_c}(X; k)$: indeed, suppose $\mathcal{F} \in \mathrm{Loc}_{T_c}(X; k)$ such that $\mathrm{Map}_{\mathrm{Loc}_{T_c}(X;k)}(s_! \mathcal{O}_{\mathcal{M}_T}, \mathcal{F}) \simeq 0$ as an object of $\mathrm{QCoh}(\mathcal{M}_T)$. Because $s^* \mathcal{F} \simeq \mathrm{Map}_{\mathrm{Loc}_{T_c}(X;k)}(s_! \mathcal{O}_{\mathcal{M}_T}, \mathcal{F})$ in $\mathrm{QCoh}(\mathcal{M}_T)$, we see that $s^* \mathcal{F} \simeq 0$. Using the connectivity of X, we see that $\mathcal{F}$ itself must be zero, which implies that $s_! \mathcal{O}_{\mathcal{M}_T}$ is a compact generator of $\mathrm{Loc}_{T_c}(X; k)$. It follows from the Barr-Beck-Lurie theorem [Lur16, Theorem 4.7.3.5] that $\mathrm{Loc}_{T_c}(X; k)$ is equivalent to the ∞-category of left $\mathrm{End}_{\mathrm{Loc}_{T_c}(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T})$-modules in $\mathrm{QCoh}(\mathcal{M}_T)$. But $\mathrm{End}_{\mathrm{Loc}_{T_c}(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T}) \simeq s^* s_! \mathcal{O}_{\mathcal{M}_T}$, which identifies with $\mathcal{F}_T(\Omega X)^\vee$. □

Remark 4.2. Modifying the preceding argument shows that if X is a connected finite T_c-space, there is an equivalence

$$\mathrm{Loc}_{T_c}(X; k) \simeq \mathrm{coLMod}_{\mathcal{F}_T(X)^\vee}(\mathrm{QCoh}(\mathcal{M}_T)). \tag{6}$$

In particular, if X admits an $\mathbf{E}_n$-algebra structure (compatible with the T_c-action), then $\mathcal{F}_T(X)^\vee$ admits the structure of an $\mathbf{E}_n$-algebra[5] in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_T))$, and the equivalence (6) is $\mathbf{E}_n$-monoidal for the convolution tensor product on both sides.

Proposition 4.1 and Remark 4.2 continue to hold even when X is a filtered colimit of finite T_c-spaces. In order for the claim in Remark 4.2 about $\mathbf{E}_n$-algebra structures to hold, we need the filtered diagram $\{X_\lambda\}$ presenting X to admit the structure of an $\mathbf{E}_n$-algebra in filtered T_c-spaces. We will need to apply this in the case when X is the affine Grassmannian, in which case we can apply the following observation.

Lemma 4.3. *The $\mathbb{X}_*(T)^+$-indexed Schubert filtration $\{\mathrm{Gr}_G^{\leq\lambda}(\mathbf{C})\}$ naturally admits the structure of an $\mathbf{E}_2$-algebra in $\mathrm{Fun}(\mathbb{X}_*(T)^+, \mathcal{S}(T_c))$.*

Proof. This can be proved in essentially the same way as [HY19, Theorem 3.10]; let us sketch the argument. We will utilize [Lur16, Proposition 5.4.5.15], which states that if $\mathcal{C}$ is a symmetric monoidal ∞-category, then a nonunital $\mathbf{E}_2$-algebra object in $\mathcal{C}$ is equivalent to the datum of a locally constant $\mathrm{N}(\mathrm{Disk}(\mathbf{C}))_{\mathrm{nu}}$-algebra object in $\mathcal{C}$. Concretely, this amounts to specifying an object $A(D) \in \mathcal{C}$ for every disk $D \subseteq \mathbf{C}$ and coherent maps $\bigotimes_{i=1}^n A(D_i) \to A(D)$ for every inclusion $\coprod_{i=1}^n D_i \to D$ of disks, such that for every embedding $D \subseteq D'$ of disks, the induced map $A(D) \to A(D')$ is an equivalence.

In this case, $\mathcal{C} = \mathrm{Fun}(\mathbb{X}_*(T)^+, \mathcal{S}(T_c))$, and the object $A(D) \in \mathrm{Fun}(\mathbb{X}_*(T)^+, \mathcal{S}(T_c))$ assigned to a disk $D \subseteq \mathbf{C}$ may be defined via the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$. Namely, the Beilinson-Drinfeld Grassmannian admits (by construction) a morphism $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}_{\mathbf{A}^1}$; upon taking complex points, we obtain a map $\mathrm{Gr}_{G,\mathrm{Ran}}(\mathbf{C}) \to \mathrm{Ran}(\mathbf{C})$. If $S \subseteq \mathbf{C}$ is a subset, then the preimage of $\mathrm{Ran}(S) \subseteq \mathrm{Ran}(\mathbf{C})$ defines a subspace $\mathrm{Gr}_{G,\mathrm{Ran}}(S \subseteq \mathbf{C}) \subseteq \mathrm{Gr}_{G,\mathrm{Ran}}(\mathbf{C})$. The filtration of Gr_G via the Bruhat decomposition extends to a filtration $\mathrm{Gr}_{G,\mathrm{Ran},\leq\mu}$ of $\mathrm{Gr}_{G,\mathrm{Ran}}$ by dominant coweights $\mu \in \mathbb{X}_*(T)^+$; see [Zhu17, 3.1.11]. Finally, the

[5] If $\mathcal{C}$ is a symmetric monoidal ∞-category, [Lur18a, Corollary 3.3.4] can be used to show that there is an equivalence $\mathrm{coCAlg}(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbf{E}_n}(\mathrm{coCAlg}(\mathcal{C}))$.

object $A(D) \in \mathrm{Fun}(\mathbb{X}_*(T)^+, \mathcal{S}(T_c))$ associated to a disk $D \subseteq \mathbf{C}$ is the functor $\mathbb{X}_*(T)^+ \to \mathcal{S}(T_c)$ sending $\mu \in \mathbb{X}_*(T)^+$ to $\mathrm{Gr}_{G,\mathrm{Ran},\leq\mu}(D \subseteq \mathbf{C})$.

Suppose $\coprod_{i=1}^n D_i \to D$ is an inclusion of disks. The induced map $\bigotimes_{i=1}^n A(D_i) \to A(D)$ is defined as follows. Let $\mu \in \mathbb{X}_*(T)^+$; for every n-tuple $(\mu_1, \cdots, \mu_n)$ with $\sum_{i=1}^n \mu_i \leq \mu$, we need to exhibit maps $\bigotimes_{i=1}^n A(D_i)(\mu_i) \to A(D)(\mu)$ satisfying the obvious coherences. But

$$\bigotimes_{i=1}^n A(D_i)(\mu_i) = \prod_{i=1}^n \mathrm{Gr}_{G,\mathrm{Ran},\leq\mu_i}(D_i \subseteq \mathbf{C}),$$

so it suffices to show that if $\mu_1 + \mu_2 \leq \mu$, then there are maps $\mathrm{Gr}_{G,\mathrm{Ran},\leq\mu_1}(D_1 \subseteq \mathbf{C}) \times \mathrm{Gr}_{G,\mathrm{Ran},\leq\mu_2}(D_2 \subseteq \mathbf{C}) \to \mathrm{Gr}_{G,\mathrm{Ran},\leq\mu}(D \subseteq \mathbf{C})$. The argument for this is exactly as in [HY19, Construction 3.15].

We next need to show that the $\mathrm{N}(\mathrm{Disk}(\mathbf{C}))_{\mathrm{nu}}$-algebra A defined above is locally constant, i.e., that if $D \subseteq D'$ is an embedding of disks, then $A(D) \to A(D')$ is an equivalence of functors $\mathbb{X}_*(T)^+ \to \mathcal{S}(T_c)$. This follows from [HY19, Proposition 3.17]. To conclude, it suffices (by [Lur16, Theorem 5.4.4.5]) to establish the existence of a quasi-unit for the functor $A : \mathbb{X}_*(T)^+ \to \mathcal{S}(T_c)$, i.e., a map $1_{\mathrm{Fun}(\mathbb{X}_*(T)^+,\mathcal{S}(T_c))} \to A$ which is both a left and right unit up to homotopy. Since the unit in $\mathrm{Fun}(\mathbb{X}_*(T)^+, \mathcal{S}(T_c))$ is the functor sending $\mu \in \mathbb{X}_*(T)^+$ to the point $*$, a quasi-unit is the datum of a map $* \to \mathrm{Gr}_{G,\leq\mu}(\mathbf{C})$ for each $\mu \in \mathbb{X}_*(T)^+$. As in the proof of [HY19, Theorem 3.10], this can be taken to be the inclusion of the point corresponding to the trivial G-bundle over $\mathbf{A}^1$ with the canonical trivialization away from the origin. □

Using the preceding discussion, we can now define our desired degeneration.

Definition 4.4. Suppose that X is a (ind-)finite T_c-space with even cells (such as Gr_G). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X; k)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X; k) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_T(X)^\vee)}(\mathrm{QCoh}(\mathcal{M}_{T,0})).$$

The "constant sheaf" $\underline{k}^{\mathrm{gr}}$ in this category is the comodule $\pi_0(\mathcal{F}_T(X)^\vee)$ itself. Similarly, suppose Y is a finite T_c-space such that ΩY has even cells (such as G_c). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y; k)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y; k) = \mathrm{LMod}_{\pi_0(\mathcal{F}_T(\Omega Y)^\vee)}(\mathrm{QCoh}(\mathcal{M}_{T,0})).$$

The "constant sheaf" $\underline{k}^{\mathrm{gr}}$ in this category is the structure sheaf $\mathcal{O}_{\mathcal{M}_{T,0}}$ viewed as a $\pi_0(\mathcal{F}_T(\Omega Y)^\vee)$-module via the augmentation.

These should be viewed as "mixed" (in the sense of [BBD82]) variants of the full ∞-categories $\mathrm{Loc}_{T_c}(X; k)$ and $\mathrm{Loc}_{T_c}(Y; k)$.

Remark 4.5. There is an apparent asymmetry in Definition 4.4: why could we not have defined $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y; k)$ to be $\mathrm{coLMod}_{\pi_0(\mathcal{F}_T(Y)^\vee)}(\mathrm{QCoh}(\mathcal{M}_{T,0}))$? The issue is that since Y contains odd-dimensional cells, taking π_0 of $\mathcal{F}_T(Y)^\vee$ is a very destructive process. More generally, as in the discussion at the beginning of this section, $\pi_0\mathcal{F}_T(X)^\vee$ for a finite T_c-space X should only be regarded as a well-behaved reflection of $\mathcal{F}_T(X)^\vee$ itself when X has even cells.

Remark 4.6. If $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2, then (using the results of [ABG04]) $\mathrm{Loc}_{T_c}(\mathrm{Gr}_G; k)$ is equivalent to the shearing of the 2-periodification of the category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$. This can be understood as a statement about formality. If k is a more general $\mathbf{E}_\infty$-ring (like complex K-theory KU), then formality is generally impossible: for instance, a KU-module M is generally not equivalent (even as a spectrum!) to the shearing of $\pi_*(M)$, unless M is also a $\mathbf{Q}$-module.

Remark 4.7. We will not discuss G_c-equivariant cohomology much in this article, except for the end of Section 6. There, we will only consider the case $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2. In this case, the equivariant cohomology $\mathrm{H}^*_{G_c}(*; \mathbf{Q})$ is concentrated in even weights; in fact, we may identify $\operatorname{Spec} \mathrm{H}^0_{G_c}(*; k) \cong \mathfrak{t}/\!\!/W$. It is still reasonable to define $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)$ to be

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k) = \mathrm{coLMod}_{\mathrm{H}_0^{G_c}(\mathrm{Gr}_G;k)}(\mathrm{QCoh}(\mathfrak{t}/\!\!/W)).$$

Similarly, the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k)$ can be defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k) = \mathrm{LMod}_{\mathrm{H}_0^{G_c}(\mathrm{Gr}_G;k)}(\mathrm{QCoh}(\mathfrak{t}/\!\!/W)).$$

Example 4.8. If $G = T$ is a maximal torus, it follows from Theorem 3.20 that there are equivalences of $\pi_0(k)$-linear ∞-categories

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_T; k) \simeq \mathrm{QCoh}(\mathcal{M}_{T,0}/\check{T}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(T_c; k) \simeq \mathrm{QCoh}(\mathcal{M}_{T,0} \times_{\operatorname{Spec} \pi_0(k)} \check{T}).$$

Suppose X is a (ind-)finite T_c-space with even cells. Since $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X; k)$ is a degeneration of $\mathrm{Loc}_{T_c}(X; k)$, one should expect a spectral sequence computing the cohomology $\Gamma_{T_c}(X; \mathcal{F})$ for $\mathcal{F} \in \mathrm{Loc}_{T_c}(X; k)$ from corresponding objects $\mathcal{F}^{\mathrm{gr}} \in \mathrm{Loc}^{\mathrm{gr}}_{T_c}(X; k)$. Similarly, if Y is a finite T_c-space such that ΩY has even cells, one should expect a spectral sequence computing the cohomology $\Gamma_{T_c}(Y; \mathcal{F})$ for $\mathcal{F} \in \mathrm{Loc}_{T_c}(Y; k)$ from corresponding objects $\mathcal{F}^{\mathrm{gr}} \in \mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y; k)$. This is a special case of the following general setup.

Construction 4.9. Recall that if $\mathfrak{X}$ is a spectral stack and $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the truncation $\underline{\tau}_{\geq n}(\mathcal{F})$ is the quasicoherent $\mathcal{O}_{\mathfrak{X}}$-module given on an affine open U by $\tau_{\geq n}(\mathcal{F}(U))$; similarly for $\underline{\tau}_{\leq n}$ and $\underline{\tau}_{[n,m]}$ with $m \geq n$. There is a functor $\mathrm{QCoh}(\mathcal{M}_T) \to \mathrm{QCoh}(\mathcal{M}_{T,0})$ given by sending a quasicoherent sheaf $\mathcal{F}$ on $\mathcal{M}_T$ to the quasicoherent sheaf $\underline{\tau}_{[0,1]}(\mathcal{F})$ over $\mathcal{M}_{T,0}$. This functor can be expressed as the composite of two functors: the first sends the $\mathcal{O}_{\mathcal{M}_T}$-module $\mathcal{F}$ to the filtered $\underline{\tau}_{\geq 2\star}\mathcal{O}_{\mathcal{M}_T}$-module $\underline{\tau}_{\geq 2\star}(\mathcal{F})$; and the second is given by taking associated graded. Note that since the structure sheaf $\mathcal{O}_{\mathcal{M}_T}$ is 2-periodic, the data of the graded $\underline{\pi}_{2*}\mathcal{O}_{\mathcal{M}_T}$-module $\mathrm{gr}(\underline{\tau}_{\geq 2\star}(\mathcal{F}))$ is equivalent to the data of the (ungraded) $\mathcal{O}_{\mathcal{M}_{T,0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$.

Let $\mathcal{A}$ be an $\mathbf{E}_\infty$-coalgebra in $\mathrm{QCoh}(\mathcal{M}_T)$ whose homotopy sheaves are concentrated in even degrees (such as $\mathcal{F}_T(X)^\vee$). If $\mathcal{F} \in \mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_T))$, the comodule map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_T}} \mathcal{A}$ induces a comodule map

$$\tau_{\leq 2\star}\mathcal{F} \to \tau_{\leq 2\star}(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_T}} \mathcal{A}) \to \tau_{\leq 2\star}(\mathcal{F}) \otimes_{\tau_{\leq 2\star}(\mathcal{O}_{\mathcal{M}_T})} \tau_{\leq 2\star}(\mathcal{A})$$

due to the oplax symmetric monoidality of the truncation functor. Taking associated graded and using the 2-periodicity of $\mathcal{O}_{\mathcal{M}_T}$, we obtain a $\pi_0(\mathcal{A})$-comodule structure on the $\mathcal{O}_{\mathcal{M}_{T,0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$. This defines a functor $\mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_T)) \to \mathrm{coMod}_{\pi_0(\mathcal{A})}(\mathrm{QCoh}(\mathcal{M}_{T,0}))$, which we will denote by $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{gr}}$. For instance, if $\mathcal{A} = \mathcal{F}_T(X)^\vee$ and $\mathcal{F} \in \mathrm{Loc}_{T_c}(X; k) = \mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_T))$, then there is a spectral sequence

$$\pi_*(k) \otimes_{\pi_0(k)} \pi_*\mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X;k)}(\underline{k}^{\mathrm{gr}}, \mathcal{F}^{\mathrm{gr}}) \Rightarrow \pi_*\mathrm{Map}_{\mathrm{Loc}_{T_c}(X;k)}(\underline{k}, \mathcal{F}) = \pi_*\Gamma_{T_c}(X; \mathcal{F}).$$

Similarly, let $\mathcal{B}$ be an $\mathbf{E}_1$-algebra in $\mathrm{QCoh}(\mathcal{M}_T)$ whose homotopy sheaves are concentrated in even degrees (such as $\mathcal{F}_T(\Omega Y)^\vee$). If $\mathcal{F} \in \mathrm{LMod}_{\mathcal{B}}(\mathrm{QCoh}(\mathcal{M}_T))$, the module map $\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{M}_T}} \mathcal{F} \to \mathcal{F}$ induces a comodule map

$$\tau_{\geq 2\star}(\mathcal{B}) \otimes_{\tau_{\geq 2\star}(\mathcal{O}_{\mathcal{M}_T})} \tau_{\geq 2\star}(\mathcal{F}) \simeq \tau_{\geq 2\star}(\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{M}_T}} \mathcal{F}) \to \tau_{\geq 2\star}(\mathcal{F})$$

due to the lax symmetric monoidality of the cotruncation functor. Taking associated graded and using the 2-periodicity of $\mathcal{O}_{\mathcal{M}_T}$, we obtain a left $\pi_0(\mathcal{B})$-module structure on the $\mathcal{O}_{\mathcal{M}_{T,0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$. This defines a functor $\mathrm{LMod}_{\mathcal{B}}(\mathrm{QCoh}(\mathcal{M}_T)) \to \mathrm{LMod}_{\pi_0(\mathcal{B})}(\mathrm{QCoh}(\mathcal{M}_{T,0}))$, which we will denote by $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{gr}}$. For instance, if $\mathcal{B} = \mathcal{F}_T(\Omega Y)^\vee$ and $\mathcal{F} \in \mathrm{Loc}_{T_c}(Y; k) = \mathrm{LMod}_{\mathcal{B}}(\mathrm{QCoh}(\mathcal{M}_T))$, then there is a spectral sequence

$$\pi_*(k) \otimes_{\pi_0(k)} \pi_* \mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y;k)}(\underline{k}^{\mathrm{gr}}, \mathcal{F}^{\mathrm{gr}}) \Rightarrow \pi_* \mathrm{Map}_{\mathrm{Loc}_{T_c}(Y;k)}(\underline{k}, \mathcal{F}) = \pi_* \Gamma_{T_c}(Y; \mathcal{F}).$$

Let us now discuss how one might define analogous degenerations if k is not necessarily an even and 2-periodic $\mathbf{E}_\infty$-ring. Although this discussion can be generalized to some other $\mathbf{E}_\infty$-rings (such as TMF), we will focus only on the case when k is the $\mathbf{E}_\infty$-ring KO of *real K-theory*. Here is a brief summary of its relevant properties: KO can be defined from KU using the $\mathbf{Z}/2$-action on KU via complex conjugation. Namely, $\mathrm{KO} = \mathrm{KU}^{h\mathbf{Z}/2}$; in fact, as proved in [Rog08], the map $\mathrm{KO} \to \mathrm{KU}$ is a $\mathbf{Z}/2$-Galois extension, meaning that the base-change of any KO-module to KU acquires the structure of a $\mathbf{Z}/2$-equivariant KU-module. In the discussion below, we will not need to know much about KO, other than the following facts: the generator of $\mathbf{Z}/2$ sends $u \in \pi_2(\mathrm{KU})$ to $-u$; and the homotopy groups of KO are *not* even, nor are they 2-periodic[6]. Therefore, KO does not quite fit into the setup of Section 3 and Section 4. Nevertheless, the fact that KO is the homotopy fixed points $\mathrm{KU}^{h\mathbf{Z}/2}$ does admit a spectral algebro-geometric description: the global sections of the spectral stack $\mathrm{Spec}(\mathrm{KU})/(\mathbf{Z}/2)$ can be identified with KO. Moreover, any KO-module N defines a quasicoherent sheaf over this spectral stack given by the $\mathbf{Z}/2$-action on $\mathrm{KU} \otimes_{\mathrm{KO}} N$.

Therefore, a more reasonable analogue of the degeneration from a KU-module M to $\pi_*(M)$ for a KO-module N is given by considering the graded $\mathbf{Z}/2$-equivariant $\pi_*(\mathrm{KU})$-module $\pi_*(\mathrm{KU} \otimes_{\mathrm{KO}} N)$. If $\mathrm{KU} \otimes_{\mathrm{KO}} N$ is even, then (since $\pi_*(\mathrm{KU})$ is isomorphic to $\mathbf{Z}[u^{\pm 1}]$ with u in weight 2), we may simply view this as the data of the $\mathbf{Z}/2$-equivariant abelian group $\pi_0(\mathrm{KU} \otimes_{\mathrm{KO}} N)$. That is, studying (spectral) algebraic geometry over KO amounts simply to keeping track of $\mathbf{Z}/2$-equivariance for (spectral) algebraic geometry over KU. Moreover, the analogue of the degeneration of the spectral scheme $\mathrm{Spec}\,\mathrm{KU}$ to $\mathrm{Spec}(\pi_*(\mathrm{KU}))/\mathbf{G}_m \cong \mathrm{Spec}(\mathbf{Z})$ should be understood as a degeneration of the spectral scheme $\mathrm{Spec}\,\mathrm{KO}$ to the $\mathbf{G}_m$-quotient of $\mathrm{Spec}(\pi_*(\mathrm{KU}))/(\mathbf{Z}/2)$, i.e., to the classifying stack $B\mathbf{Z}/2$.

Motivated by this discussion, we may define KO_{T_c} for a compact torus T_c as the homotopy $\mathbf{Z}/2$-fixed points of KU_{T_c} for a particular $\mathbf{Z}/2$-action extending the action of $\mathbf{Z}/2$ on KU^{hT_c} by complex conjugation. To do so, we need the following simple observation.

---

[6]In fact, there is an isomorphism

$$\pi_*(\mathrm{KO}) \cong \mathbf{Z}[\eta, x, y^{\pm 1}]/(2\eta, \eta^3, \eta x, x^2 - 4y),$$

where η is in degree 1, x is in degree 4, and y is in degree 8. The map $\pi_*(\mathrm{KO}) \to \pi_*(\mathrm{KU}) \cong \mathbf{Z}[u^{\pm 1}]$ kills η, sends x to $2u^2$, and sends y to u^4.

Lemma 4.10. *Under the isomorphism $\pi_0(\mathrm{KU}^{hS^1}) \cong \mathbf{Z}[\![x-1]\!]$, the action of $\mathbf{Z}/2$ by complex conjugation sends $x \mapsto x^{-1}$.*

Motivated by Lemma 4.10, we make the following:

Construction 4.11. There is an action of $\mathbf{Z}/2$ on the multiplicative group $(\mathbf{G}_m)_{\mathrm{KU}}$ over KU given by inversion. If T_c is a compact torus, this extends to an action of $\mathbf{Z}/2$ on $\mathcal{M}_T^{\mathrm{KU}} = T_{\mathrm{KU}}$. Define $\mathcal{M}_T^{\mathrm{KO}}$ to be the spectral stack over $\mathrm{Spec}(\mathrm{KU})/(\mathbf{Z}/2)$ given by $\mathcal{M}_T^{\mathrm{KU}}/(\mathbf{Z}/2)$. Observe that the underlying stack of $\mathcal{M}_T^{\mathrm{KO}}$ is given by $\mathcal{M}_{T,0}/(\mathbf{Z}/2)$ over $B\mathbf{Z}/2$ (again, $\mathbf{Z}/2$ acts on $\mathcal{M}_{T,0} \cong T$ by inversion).

It is clear from Construction 3.7 that the functor $\mathcal{F}_T(-;\mathrm{KU}) : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T^{\mathrm{KU}})$ factors through a functor $\mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T^{\mathrm{KO}})$. We will denote this new functor by $\mathcal{F}_T(-;\mathrm{KO})$. In exactly the same way as in Construction 3.16, one can define a $\mathrm{QCoh}(\mathcal{M}_T^{\mathrm{KO}})$-linear ∞-category $\mathrm{Loc}_{T_c}(X;\mathrm{KO})$ for a finite T_c-space X. As in Remark 4.2, there will be an equivalence

$$\mathrm{Loc}_{T_c}(X;\mathrm{KO}) \simeq \mathrm{coMod}_{\mathcal{F}_T(X;\mathrm{KO})^\vee}(\mathrm{QCoh}(\mathcal{M}_T^{\mathrm{KO}}));$$

furthermore, the latter category is equivalent to the ∞-category of $\mathbf{Z}/2$-equivariant objects in $\mathrm{Loc}_{T_c}(X;\mathrm{KU})$.

Thus, following Definition 4.4, we are led to the following.

Definition 4.12. Suppose that X is a (ind-)finite T_c-space with even cells (such as Gr_G). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X;\mathrm{KO})$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(X;\mathrm{KO}) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_T(X;\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{T,0}/(\mathbf{Z}/2))).$$

Similarly, suppose Y is a finite T_c-space such that ΩY has even cells (such as G_c). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y;\mathrm{KO})$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(Y;\mathrm{KO}) = \mathrm{LMod}_{\pi_0(\mathcal{F}_T(\Omega Y;\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{T,0}/(\mathbf{Z}/2))).$$

Example 4.13. It follows from Example 4.8 that there are equivalences of $\mathrm{QCoh}(B\mathbf{Z}/2)$-linear ∞-categories

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_T;\mathrm{KO}) \simeq \mathrm{QCoh}(T/(\mathbf{Z}/2) \times B\check{T}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(T_c;\mathrm{KO}) \simeq \mathrm{QCoh}(T/(\mathbf{Z}/2) \times \check{T}).$$

Note that under Definition 4.12, understanding $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathrm{KO})$ is tantamount to understanding the action of complex conjugation on $\pi_0(\mathcal{F}_T(\mathrm{Gr}_G;\mathrm{KU})^\vee)$. The action of complex conjugation on KU is given simply by the action of the Adams operation ψ^{-1}. It is therefore natural to fix a prime p and contemplate a parallel story with KO replaced by the "image of J" spectrum $L_{K(1)}S^0 = (\mathrm{KU}^\wedge_p)^{h\mathbf{Z}_p^\times}$. Here, $\mathbf{Z}_p^\times$ acts continuously on $\mathrm{KU}^\wedge_p$ by Adams operations: there is a map $\mathbf{Z}_p^\times \to \mathrm{End}_{\mathbf{E}_\infty}(\mathrm{KU}^\wedge_p)$ sending $n \in \mathbf{Z}_p^\times$ to the Adams operation $\psi^n : \mathrm{KU}^\wedge_p \to \mathrm{KU}^\wedge_p$. In fact, the above map defines an equivalence $\mathbf{Z}_p^\times \xrightarrow{\sim} \mathrm{Aut}_{\mathbf{E}_\infty}(\mathrm{KU}^\wedge_p)$.

The homotopy groups of $L_{K(1)}S^0$ are somewhat complicated[7], but just as with KO, studying (spectral) algebraic geometry over $L_{K(1)}S^0$ amounts simply to keeping track of $\mathbf{Z}_p^\times$-equivariance for (spectral) algebraic geometry over $\mathrm{KU}^\wedge_p$. That is to say, $L_{K(1)}S^0$ is the global sections of the structure sheaf on the spectral stack

[7] Explicitly, if $p > 2$, then $\pi_i L_{K(1)}S^0$ is isomorphic to $\mathbf{Z}_p$ when $i = 0, -1$, and is isomorphic to $\mathbf{Z}/p^{v_p(j)+1}$ for $i = 2(p-1)j - 1$. The order of the latter subgroup is precisely the p-part of the denominator of $B_{2(i+1)}/(i+1)$, where B_{2j} is the $2j$th Bernoulli number.

$\mathrm{Spf}(\mathrm{KU}_p^\wedge)/\mathbf{Z}_p^\times$. Moreover, the analogue of the degeneration of the spectral scheme $\mathrm{Spf}\,\mathrm{KU}_p^\wedge$ to $\mathrm{Spf}(\pi_*(\mathrm{KU}_p^\wedge))/\mathbf{G}_m \cong \mathrm{Spf}(\mathbf{Z}_p)$ should be understood as a degeneration of the spectral scheme $\mathrm{Spf}\, L_{K(1)}S^0$ to the $\mathbf{G}_m$-quotient of $\mathrm{Spf}(\pi_*(\mathrm{KU}_p^\wedge))/\mathbf{Z}_p^\times$, i.e., to the classifying stack $B\mathbf{Z}_p^\times$.

To define an analogue of Definition 4.12 for $L_{K(1)}S^0$, we need to upgrade the $\mathbf{Z}_p^\times$-action on KU to an action on equivariant K-theory. Recall that if $\mathcal{T}$ denotes the full subcategory of $\mathcal{S}$ spanned by those spaces which are homotopy equivalent to BT_c with T_c being a compact abelian Lie group, the data of a preorientation of $\mathbf{G} = \mathbf{G}_m$ is equivalent to the data of a functor $\mathcal{M} : \mathcal{T} \to \mathrm{Aff}_{\mathrm{KU}}$ along with compatible equivalences $\mathcal{M}(BT_c) \simeq \mathcal{M}_T$. This can be composed with the functor $\mathrm{Aff}_{\mathrm{KU}} \to \mathrm{Aff}^{p\text{-cplt}}_{\mathrm{KU}_p^\wedge}$ of p-completion.

Unfortunately, even at the level of classical algebra, there is no natural action of $\mathbf{Z}_p^\times$ on $\mathbf{G}_m = \mathrm{Spf}\,\mathbf{Z}_p[x^{\pm 1}]$ where $n \in \mathbf{Z}_p^\times$ sends $x \mapsto x^n$: the power series $x^n = \sum_{i\geq 0} \binom{n}{i}(x-1)^n$ need not converge without a further completion. Nevertheless, such an action of $\mathbf{Z}_p^\times$ does exist if we restrict to the subgroups $\mu_{p^n} = \mathrm{Spf}\,\mathbf{Z}_p[\mathbf{Z}/p^n] \subseteq \mathbf{G}_m$; in fact, the action factors through the surjection $\mathbf{Z}_p^\times \twoheadrightarrow (\mathbf{Z}/p^n)^\times$. The subgroups μ_{p^n} naturally lift to KU (by $\mathrm{Spec}\,\mathrm{KU}[\mathbf{Z}/p^n]$), and each admits a natural $\mathbf{Z}_p^\times$-action. Of course, these $\mathbf{Z}_p^\times$-actions exist even before p-completion; but to get a well-behaved operation on equivariant $\mathbf{Z}/p^n$-cohomology, we need the $\mathbf{Z}_p^\times$-action to preserve the preorientation on μ_{p^n}, and this in turn happens once KU is p-completed.

Suppose, therefore, that we restrict to the full subcategory $\mathcal{T}_p \subseteq \mathcal{T}$ spanned by those spaces which are homotopy equivalent to BA with A being a p-power torsion compact abelian Lie group. Then the preceding paragraph implies that $\mathcal{M}|_{\mathcal{T}_p} : \mathcal{T}_p \to \mathrm{Aff}_{\mathrm{KU}}$ refines to a functor $\mathcal{T}_p \to (\mathrm{Aff}^{p\text{-cplt}}_{\mathrm{KU}_p^\wedge})^{h\mathbf{Z}_p^\times}$. Following Construction 3.7 verbatim defines an action of $\mathbf{Z}_p^\times$ on $\mathcal{M}_A$, and furthermore equips the quasicoherent sheaf $\mathcal{F}_A(X) \in \mathrm{QCoh}(\mathcal{M}_A)$ associated to a finite A-space X with a $\mathbf{Z}_p^\times$-equivariant structure. We will write $\mathcal{M}_A^{L_{K(1)}S^0} = \mathcal{M}_A/\mathbf{Z}_p^\times$, and let $\mathcal{F}_A(-; L_{K(1)}S^0)$ denote the corresponding functor $\mathcal{S}(A)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_A^{L_{K(1)}S^0})$. Again, following Definition 4.4, we are led to the following.

Definition 4.14. Suppose that A is a p-power torsion abelian group, and X is a (ind-)finite A-space with even cells (such as Gr_G). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_A(X; L_{K(1)}S^0)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_A(X; L_{K(1)}S^0) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_A(X;\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{A,0}/\mathbf{Z}_p^\times)).$$

Similarly, suppose Y is a finite A-space such that ΩY has even cells (such as G_c). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_A(Y; L_{K(1)}S^0)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_A(Y; L_{K(1)}S^0) = \mathrm{LMod}_{\pi_0(\mathcal{F}_A(\Omega Y;\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{A,0}/\mathbf{Z}_p^\times)).$$

5. Loop rotation equivariance

In this section, we describe an extension of Theorem 3.20 (or rather, of Example 4.8) which includes loop-rotation equivariance. Recall that Theorem 3.20 gives an isomorphism $\mathcal{F}_{T_c}(\mathrm{Gr}_T)^\vee \cong \mathcal{O}(\check{T}_k \times_{\mathrm{Spec}(k)} \mathcal{M}_T)$. The action of T on Gr_T refines to an action of $\widetilde{T} = T \times \mathbf{G}_m^{\mathrm{rot}}$, where $\mathbf{G}_m^{\mathrm{rot}}$ acts by loop rotation; we may therefore consider the *loop-rotation equivariant* homology $\mathcal{F}_{\widetilde{T}_c}(\mathrm{Gr}_T)^\vee$. There is an equivalence $\mathcal{M}_{\widetilde{T}} \simeq \mathcal{M}_T \times \mathbf{G}$, where the second factor is identified as $\mathcal{M}_{\mathbf{G}_m^{\mathrm{rot}}}$. Therefore, $\mathcal{F}_{\widetilde{T}_c}(\mathrm{Gr}_T)^\vee$ is a quasicoherent sheaf over $\mathcal{M}_T \times \mathbf{G}$ whose fiber over the zero section of $\mathbf{G}$ recovers $\mathcal{F}_{T_c}(\mathrm{Gr}_T)^\vee$.

Definition 5.1. Let $\mathbf{H}$ be a smooth 1-dimensional group scheme over a base commutative ring A, let T_c be a compact torus, and let $\mathbf{H}_T = \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{H})$. (When $\mathbf{G}$ is an oriented commutative k-group scheme, and $\mathbf{H} = \mathbf{G}_0$ is its underlying group scheme over $A = \pi_0(k)$, then $\mathbf{H}_T$ is precisely $\mathcal{M}_{T,0}$.) Let λ be a cocharacter of T_c, so that λ defines a homomorphism $\mathbb{X}^*(T) \to \mathbf{Z}$, and hence a homomorphism $\lambda^* : \mathbf{H} \to \mathbf{H}_T$. In turn, this defines a map

$$f^\lambda : \mathbf{H}_{\widetilde{T}} \simeq \mathbf{H}_T \times \mathbf{H} \xrightarrow{\mathrm{pr}\times\lambda^*} \mathbf{H}_T.$$

If y is a local section of $\mathcal{O}_{\mathbf{H}_T}$, we will write $\lambda^*(y)$ to denote the resulting local section of $\mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$.

Let $\mathcal{D}_{\check{T}}^{\mathbf{H}}$ denote the quotient of the associative $\mathcal{O}_{\mathbf{H}}$-algebra $\mathcal{O}_{\mathbf{H}_{\widetilde{T}}}\langle x_\lambda | \lambda \in \mathbb{X}_*(T)\rangle$ by the relations given locally by

$$x_\lambda \cdot x_\mu = x_{\lambda+\mu},\ y \cdot x_\lambda = x_\lambda \cdot \lambda^*(y).$$

Here, $\lambda, \mu \in \mathbb{X}_*(T)$, and y is a local section of $\mathcal{O}_{\mathbf{H}_T}$. We will call $\mathcal{D}_{\check{T}}^{\mathbf{H}}$ the *algebra of* $\mathbf{H}$*-differential operators* on $\check{T}$.

Remark 5.2. The algebra $\mathcal{D}_{\check{T}}^{\mathbf{H}}$ satisfies a Mellin transform: namely, it follows from unwinding the definition that there is an equivalence

$$\mathcal{D}_{\check{T}}^{\mathbf{H}}\text{-mod} \simeq \mathrm{IndCoh}(\mathbf{H}_{\widetilde{T}}/\mathbb{X}^*(\check{T})),$$

where $\lambda \in \mathbb{X}^*(\check{T}) \cong \mathbb{X}_*(T)$ acts on $\mathbf{H}_{\widetilde{T}}$ via $y \mapsto \lambda^* y$.

Notation 5.3. If k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}_0$ is the $\pi_0(k)$-group underlying a oriented commutative A-group $\mathbf{G}$, we will write $\mathcal{D}_{\check{T}}^{\mathbf{G}}$ to denote $\mathcal{D}_{\check{T}}^{\mathbf{G}_0}$, and refer to it as the *algebra of* $\mathbf{G}$*-differential operators* on $\check{T}$. We hope this does not cause any confusion.

Proposition 5.4. *There is an isomorphism*

$$\pi_0 \mathcal{F}_{\widetilde{T}}(\mathrm{Gr}_T)^\vee \cong \mathcal{D}_{\check{T}}^{\mathbf{G}}$$

of $\mathcal{O}_{\mathbf{G}_0}$*-algebras. In particular, there is an equivalence*

$$\mathrm{Loc}_{\widetilde{T}_c}(\mathrm{Gr}_T; k) \simeq \mathcal{D}_{\check{T}}^{\mathbf{G}}\text{-mod}^{(\check{T}\times\check{T},\mathrm{weak})},$$

where the right-hand side denotes the category of left $\mathcal{D}_{\check{T}}^{\mathbf{G}}$*-modules whose underlying quasicoherent sheaf over* $\check{T}$ *is equivariant for* $\check{T} \times \check{T}$*-action on* $\check{T}$ *given by left and right multiplication.*

Proof. Since $\mathrm{Gr}_T \cong \mathbb{X}_*(T)$, it is easy to see that $\pi_0 \mathcal{F}_{\widetilde{T}}(\mathrm{Gr}_T)^\vee \cong \bigoplus_{\lambda \in \mathbb{X}_*(T)} \pi_0 \mathcal{O}_{\mathcal{M}_{\widetilde{T}}}$; let x_λ be a generator of the summand indexed by $\lambda \in \mathbb{X}_*(T)$. If $\lambda \in \mathbb{X}_*(T) = \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{Z})$, the map $\Omega T_c \to \Omega T_c$ given by multiplication-by-λ is $T_c \times S^1_{\mathrm{rot}}$-equivariant for the homomorphism $T_c \times S^1_{\mathrm{rot}} \to T_c \times S^1_{\mathrm{rot}}$ given by

$$(t, \theta) \mapsto (t\lambda(\theta), \theta),$$

where λ is viewed as a homomorphism $S^1 \to T$. On weight lattices, this homomorphism induces the map $\mathbb{X}^*(T) \times \mathbf{Z} \to \mathbb{X}^*(T) \times \mathbf{Z}$ which sends $(\mu, n) \mapsto (\mu, n + \mathbb{X}_*(T)(\mu))$. In particular, the composite

$$\mathbb{X}^*(T) \to \mathbb{X}^*(T) \times \mathbf{Z} \to \mathbb{X}^*(T) \times \mathbf{Z}$$

sends $\mu \mapsto (\mu, \mathbb{X}_*(T)(\mu))$. Applying $\mathrm{Hom}(-, \mathbf{G})$ to this composite precisely produces the map $f^\lambda : \mathcal{M}_{\widetilde{T}} \to \mathcal{M}_T$ from Definition 5.1. This implies the desired identification of $\pi_0 \mathcal{F}_{\widetilde{T}}(\mathrm{Gr}_T)^\vee$. $\square$

Example 5.5. Let $T \cong S^1$ be a torus of rank 1 (for simplicity). Suppose $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2, so $\mathbf{G} = \mathbf{G}_a$ and $\mathcal{O}_{\mathbf{G}_0} \cong \mathbf{Q}[\hbar]$. Then the algebra of Definition 5.1 is the quotient of the $\mathbf{Q}[\hbar]$-algebra $\mathbf{Q}[\hbar]\langle y, x^{\pm 1}\rangle$ by the relation $yx = x(y + \hbar)$. In other words, y acts as the operator $\hbar x \partial_x$, so we simply have that

$$\mathrm{H}_0^{\widetilde{T}}(\mathrm{Gr}_T; \mathbf{Q}[u^{\pm 1}]) \cong \mathrm{H}_*^{\widetilde{T}}(\mathrm{Gr}_T; \mathbf{Q}) \cong \mathbf{Q}[\hbar]\langle \hbar x \partial_x, x^{\pm 1}\rangle.$$

This has been stated previously as [BFN18, Proposition 5.19(2)]. In particular, the localization $\mathrm{H}_0^{\widetilde{T}}(\mathrm{Gr}_T; \mathbf{Q}[u^{\pm 1}])[\hbar^{-1}]$ is isomorphic to the rescaled Weyl algebra $\mathcal{D}^\hbar_{\check{T}}$; this is the motivation behind the terminology in Definition 5.1. Note that for a general torus, Remark 5.2 simply reduces to the standard Mellin transform, which gives an equivalence between $\mathrm{DMod}_\hbar(\check{T})$ and $\mathrm{QCoh}(\mathfrak{t}_{\mathbf{Q}[\hbar]}/\mathbb{X}^*(\check{T}))$.

Example 5.6. Again, let $T \cong S^1$ be a torus of rank 1 (for simplicity). Suppose $k = \mathrm{KU}$, so $\mathbf{G} = \mathbf{G}_m$ and $\mathcal{O}_{\mathbf{G}_0} \cong \mathbf{Z}[q^{\pm 1}]$. Then the algebra of Definition 5.1 is the quotient of the $\mathbf{Z}[q^{\pm 1}]$-algebra $\mathbf{Z}[q^{\pm 1}]\langle y^{\pm 1}, x^{\pm 1}\rangle$ by the relation $yx = qxy$. (This is also known as the "quantum torus".) In other words, y acts as the operator $q^{x\partial_x}$ sending $f(x) \mapsto f(qx)$, so we simply have that

$$\mathrm{KU}_0^{\widetilde{T}}(\mathrm{Gr}_T) \cong \mathbf{Z}[q^{\pm 1}]\langle q^{x\partial_x}, x^{\pm 1}\rangle.$$

This is closely related to the q-Weyl algebra $\mathcal{D}_q = \mathbf{Z}[q^{\pm 1}]\langle \Theta, x^{\pm 1}\rangle/(\Theta x = x(q\Theta + 1))$ for $\check{T} = \mathbf{G}_m$: indeed, since the logarithmic q-derivative $\Theta = x\nabla_q$ is given by the fraction $\frac{q^{x\partial_x} - 1}{q-1}$, the pullback of $\mathcal{D}^{\mathbf{G}}_{\check{T}}$ along $\mathbf{G}_m - \{1\} \hookrightarrow \mathbf{G}_m$ is isomorphic to the algebra $\mathcal{D}_q[\frac{1}{q-1}]$. Note that Remark 5.2 gives a "q-Mellin transform", i.e., an equivalence between $\mathrm{LMod}_{\mathrm{KU}_0^{\widetilde{T}}(\mathrm{Gr}_T)}$ and $\mathrm{QCoh}((\mathbf{G}_m)_{\mathbf{Z}[q^{\pm 1}]}/\mathbf{Z})$, where $\mathbf{Z}$ acts on $(\mathbf{G}_m)_{\mathbf{Z}[q^{\pm 1}]}$ by sending $y \mapsto qy$.

Let us briefly outline the relationship between the algebra $\mathcal{D}^{\mathbf{H}}_{\check{T}}$ of Definition 5.1 and the F-de Rham complex of [DM23].

Notation 5.7. For the purpose of this discussion, we will assume that $T \cong S^1$ is a torus of rank 1, so that $\check{T} \cong \mathbf{G}_m$. We will also fix an invariant differential form on the formal completion $\hat{\mathbf{H}}$ of $\mathbf{H}$ at the zero section, so that there is an isomorphism

$\hat{\mathbf{H}} \cong \operatorname{Spf} A[\![t]\!]$ of formal A-schemes. Let $F(x,y)$ denote the resulting formal group law over A, and define the n-series of F by

$$[n]_F := \overbrace{F(t, F(t, F(t, \cdots F(t,t)\cdots)))}^{n}.$$

We will often write $x +_F y = x +_{\mathbf{G}} y$ to denote $F(x,y)$. Let $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}$ denote the completion of $\mathcal{D}^{\mathbf{H}}_{\check{T}}$ at the zero section of $\mathcal{M}_{0,\widetilde{T}} \cong \mathbf{H}_T \times \mathbf{H}$.

Lemma 5.8 (Cartier duality)**.** *Let $\hat{\mathbf{H}}$ be a 1-dimensional formal group over a commutative ring A, and let $\operatorname{Cart}(\hat{\mathbf{H}})$ denote its Cartier dual (see* [Dri21, Section 3.3] *for more on Cartier duals of formal groups). Then there is an equivalence of categories $\operatorname{QCoh}(\hat{\mathbf{H}}) \simeq \operatorname{QCoh}(B\operatorname{Cart}(\hat{\mathbf{H}}))$ sending the convolution tensor product on the left-hand side to the usual tensor product on the right-hand side. Under this equivalence, the functor $\operatorname{QCoh}(\hat{\mathbf{H}}) \to \operatorname{Mod}_A$ given by restriction to the zero section is identified with the functor $\operatorname{QCoh}(B\operatorname{Cart}(\hat{\mathbf{H}})) \to \operatorname{Mod}_A$ given by pullback along the map $\operatorname{Spec}(A) \to B\operatorname{Cart}(\hat{\mathbf{H}})$.*

Proposition 5.9. *There is a canonical action of $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}$ on $(\mathbf{G}_m)_{A[\![t]\!]} = \operatorname{Spf} A[\![t]\!][x^{\pm 1}]$ such that $A[\![t]\!][x^{\pm 1}] \otimes_{\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}} A[\![t]\!][x^{\pm 1}]$ is isomorphic to the two-term complex*

$$C^\bullet = (A[\![t]\!][x^{\pm 1}] \to A[\![t]\!][x^{\pm 1}]dx),\ x^n \mapsto [n]_F x^n dx$$

from [DM23, Remark 4.3.8].

Proof sketch. Since T is of rank 1, there is an isomorphism $\mathbf{H}_T \cong \mathbf{H}$, and hence an isomorphism $\hat{\mathbf{H}}_T \cong \hat{\mathbf{A}}^1$ of formal A-schemes, where $\hat{\mathbf{H}}_T$ denotes the completion of $\mathbf{H}_T$ at the zero section. Let y be a local coordinate on $\mathbf{H}_T$. Then, $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}$ is isomorphic to the quotient of the associative $\hat{\mathcal{O}}_{\mathbf{H}}$-algebra $\hat{\mathcal{O}}_{\mathbf{H}\times\mathbf{H}_T}\langle x^{\pm 1}\rangle$ subject to the relation $yx = x(y +_{\mathbf{G}} t)$. The t-adic filtration on $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}$ therefore has associated graded $\operatorname{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}) \cong \hat{\mathcal{O}}_{\mathbf{H}_T}[x^{\pm 1}][\![\bar{t}]\!]$, where $\bar{t}$ lives in weight 1. View A as a $\mathcal{O}_{\mathbf{H}_T}$-algebra via the zero section, i.e., the augmentation $\mathcal{O}_{\mathbf{H}_T} \to A$. Then, the action of $\operatorname{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}})$ on $A[x^{\pm 1}][\![\bar{t}]\!]$ is induced by the augmentation $\hat{\mathcal{O}}_{\mathbf{H}_T} \to A$. The isomorphism $\hat{\mathbf{H}}_T \cong \hat{\mathbf{A}}^1$ of formal A-schemes then implies an isomorphism $A \otimes_{\mathcal{O}_{\mathbf{H}_T}} A \cong A[\epsilon]/\epsilon^2$ with ϵ in homological degree 1. It follows that

$$A[\![\bar{t}]\!][x^{\pm 1}] \otimes_{\operatorname{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}})} A[\![\bar{t}]\!][x^{\pm 1}] \simeq A[\![\bar{t}]\!][x^{\pm 1}][\epsilon]/\epsilon^2,$$

where $\bar{t}$ is in weight 1 and degree 0, and ϵ is in weight 0 and degree 1.

By Lemma 5.8, the t-adic filtration on $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}}$ is equivalent to the data of a $\operatorname{Cart}(\hat{\mathbf{H}})$-action on $A[\![\bar{t}]\!][x^{\pm 1}] \otimes_{\operatorname{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{T}})} A[\![\bar{t}]\!][x^{\pm 1}] \simeq A[\![\bar{t}]\!][x^{\pm 1}][\epsilon]/\epsilon^2$. This in turn is equivalent to the data of a differential

$$\nabla : A[\![\bar{t}]\!][x^{\pm 1}] \to A[\![\bar{t}]\!][x^{\pm 1}] \cdot \epsilon$$

satisfying an $\hat{\mathbf{H}}$-analogue of the Leibniz rule: if[8] $\nabla(x^n) = f(n)x^n\epsilon$ for some $f(n) \in A[\![t]\!]$, then $f(n+m) = f(n) +_{\mathbf{G}} f(m)$. It therefore suffices to determine $\nabla(x)$; but the relation $yx = x(y +_{\mathbf{G}} t)$ forces $\nabla(x) = tx\epsilon$. This implies that

$$\nabla(x^n) = (\overbrace{t +_{\mathbf{G}} \cdots +_{\mathbf{G}} t}^{n})x^n\epsilon = [n]_F x^n \epsilon,$$

[8]Note that ∇ has to be homogeneous in the degree of the monomial in x, as can be seen by keeping track of the x-weight.

as desired. ☐

Example 5.10. When $\mathbf{H} = \mathbf{G}_a$ over[9] $\mathbf{Q}$, the complex $C^\bullet$ is

$$C^\bullet = (\mathbf{Q}[\![\hbar]\!][x^{\pm 1}] \to \mathbf{Q}[\![\hbar]\!][x^{\pm 1}]dx),\ x^n \mapsto n\hbar x^n dx.$$

Indeed, since $yx = x(y+\hbar)$, we have $yx^n = x^n(y+n\hbar)$; since $t = \hbar$ in this case, we have $x^n \mapsto n\hbar x^n \epsilon$. This is evidently a $\hbar$-rescaling of the classical de Rham complex of $\mathbf{G}_m$.

When $\mathbf{H} = \mathbf{G}_m$ over $\mathbf{Z}$, the complex $C^\bullet$ is

$$C^\bullet = (\mathbf{Z}[\![q-1]\!][x^{\pm 1}] \to \mathbf{Z}[\![q-1]\!][x^{\pm 1}]dx),\ x^n \mapsto (q^n-1)x^n dx.$$

Indeed, since $yx = x(qy)$, we have $yx^n = x^n(q^n y)$, and hence

$$(y-1)x^n = x^n(q^n y - 1) = x^n((y-1) +_F (q^n - 1)),$$

where $F(z,w) = z + w + zw$ is the multiplicative formal group law; since $t = q-1$ in this case, we have $x^n \mapsto (q^n-1)x^n\epsilon$. The complex $C^\bullet$ is a $(q-1)$-rescaling of the q-de Rham complex of $\mathbf{G}_m$ from [Sch17].

Remark 5.11. The complex of Proposition 5.9 is not quite the F-de Rham complex of [DM23, Definition 4.3.6]; rather, if η_t denotes the décalage functor of [BO78] with respect to the ideal $(t) \subseteq A[\![t]\!]$, the F-de Rham complex is given by the décalage $\eta_t C^\bullet$. In particular, the complex of Proposition 5.9 is isomorphic to the F-de Rham complex after inverting t. One can modify the algebra $\mathcal{D}_{\check{T}}^{\mathbf{H}}$ of Definition 5.1 (by performing a noncommutative analogue of an affine blowup/deformation to the normal cone[10]) such that the relative tensor product as in Proposition 5.9 is the F-de Rham complex itself. Since it is not needed for this article, we will not describe this modification here.

Remark 5.12. Suppose k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring equipped with an oriented commutative k-group scheme $\mathbf{G}$. Proposition 5.9 says that $\hat{\mathcal{D}}_{\check{T}}^{\mathbf{G}_0}$ is Koszul dual to the complex $C^\bullet$. Forthcoming work of Arpon Raksit shows that the décalage $\eta_t C^\bullet$ can be recovered from the "even filtration" (in the sense of [HRW22]) on the periodic cyclic homology $\mathrm{HP}(\tau_{\geq 0}k[x^{\pm 1}]/\tau_{\geq 0}k)$. See also the discussion in [Dev23, Section 3.3]. Using similar techniques, one can show that $C^\bullet$ can be recovered from the even filtration on the negative cyclic homology $\mathrm{HC}^-(k[x^{\pm 1}]/k) = \mathrm{HH}(k[x^{\pm 1}]/k)^{hS^1}$.

Recalling that $T = S^1$, this $\mathbf{E}_\infty$-k-algebra is simply $\mathrm{HC}^-(k[\Omega T]/k)$. The Hochschild homology $\mathrm{HH}(k[\Omega T]/k) \simeq k \otimes \mathrm{THH}(S[\Omega T])$ is S^1-equivariantly equivalent to the k-chains $C_*(\mathcal{L}T; k)$ on the free loop space of T. (For a reference, see [NS18, Corollary IV.3.3].) The k-chains $k[\mathcal{L}T]$ is S^1-equivariantly Koszul dual[11] to $k[\Omega T]^{hT}$; this can be identified as a completion of $\mathcal{F}_T(\Omega T)^\vee$ at the zero section of $\mathcal{M}_T$. In other words, $\mathrm{HC}^-(k[\Omega T]/k)$ is Koszul dual to the completion of $\mathcal{F}_{T\times S^1_{\mathrm{rot}}}(\Omega T)^\vee$ at the zero section of $\mathcal{M}_T \times \mathbf{G}$. This is the topological source of the Koszul duality of Proposition 5.9.

[9]Of course, one can work over $\mathbf{Z}$ too; we just chose $\mathbf{Q}$ to continue with Example 5.5.

[10]For instance, in the case of Example 5.5, this procedure simply adjoins the fraction $\frac{y}{\hbar}$; in the case of Example 5.6, this procedure simply adjoins the fraction $\frac{y-1}{q-1}$.

[11]This Koszul duality essentially stems from the (non-S^1_{rot}-equivariant) decomposition $\mathcal{L}T \simeq T \times \Omega T$.

6. Review of the classical case

To prepare ourselves for the calculations of $\pi_0(\mathcal{F}_T(\mathrm{Gr}_G)^\vee)$ for k being complex K-theory or elliptic cohomology, we begin with the simpler case of k being $\mathbf{Q}[u^{\pm 1}]$ with u in degree 2; recall that $\mathcal{M}_{T,0}$ is then isomorphic to $\mathfrak{t}$. In this case, the discussion in the present section follows from the work of Bezrukavnikov, Finkelberg, and Mirkovic in [BFM05], as well as the work of Yun and Zhu in [YZ11]. We will nevertheless go through this calculation (and discuss several applications) since the argument is different from that of the papers mentioned above, and also because it will serve as a useful template later. Our goal is specifically to *not* appeal to the derived geometric Satake equivalence of [BF08], but rather do the calculation in such a way that proof technique generalizes to the K-theoretic or elliptic setting, so as to apply it to prove an analogue of [BF08].

In the remainder of this article, we will assume the group G is connected, almost simple, and simply-laced. The assumption that G is simply-laced provides many simplifications; in particular, it implies that the Chevalley split forms of the groups G and $\check{G}$ are centrally isogenous (so that the adjoint action of G on $\mathfrak{g}$ descends to an action of $\check{G}$ on $\mathfrak{g}$), and that there is a $\check{G}$-equivariant isomorphism $\mathfrak{g} \cong \check{\mathfrak{g}}^*$ (even over $\mathbf{Z}$). However, we will *never* use an $\check{G}$-equivariant isomorphism $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$! The latter fails over $\mathbf{Z}$ (e.g., $\mathfrak{sl}_2 \not\cong \mathfrak{pgl}_2$ over $\mathbf{Z}$), and such failures become amplified in the settings of K-theory and elliptic cohomology.

In the following discussion, all dual groups are to be understood as defined over $\mathbf{Q}$ (although some of our discussion will work even over $\mathbf{Z}$, perhaps with some small primes inverted).

Definition 6.1 ((Additive) Kostant slice)**.** Fix a nondegenerate character $\psi \in \check{\mathfrak{n}}^*$; under the isomorphism $\check{\mathfrak{g}}^* \cong \mathfrak{g}$, there is an isomorphism $\check{\mathfrak{n}}^* \cong \mathfrak{n}$, and ψ corresponds to a principal nilpotent element $f \in \mathfrak{n}$. Let (e, f, h) be the associated $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and let $\psi_- : \check{\mathfrak{n}}_- \to \mathbf{A}^1$ denote the element corresponding to e. Let $\check{\mathfrak{g}}^{*,\psi_-} \cong \mathfrak{g}^e$ denote the centralizer (so $\mathfrak{g} = \mathfrak{g}^e \oplus [e, \mathfrak{g}]$), and let $\mathcal{S} := f + \mathfrak{g}^e \subseteq \mathfrak{g}^{\mathrm{reg}}$ be the Kostant slice. Note that $\mathcal{S} \cong \psi + \check{\mathfrak{g}}^{*,\psi_-} \subseteq \check{\mathfrak{g}}^{*,\mathrm{reg}}$. The composite $f + \mathfrak{g}^e \to \mathfrak{g} \to \mathfrak{g}/\!\!/G \cong \mathfrak{t}/\!\!/W$ is an isomorphism, by [Kos63].

Recall that the Grothendieck-Springer resolution is defined as

$$\widetilde{\check{\mathfrak{g}}} = \check{\mathfrak{n}}^\perp \times^{\check{B}} \check{G} \cong \mathfrak{b} \times^{\check{B}} \check{G},$$

so that $\widetilde{\check{\mathfrak{g}}}/\check{G} \simeq \mathfrak{b}/\check{B}$. A point of $\widetilde{\check{\mathfrak{g}}}$ can be regarded as a pair $(\check{\mathfrak{b}}', x \in (\check{\mathfrak{n}}')^\perp)$; here, $\check{\mathfrak{b}}'$ denotes a Borel subalgebra of $\check{\mathfrak{g}}$, and $\check{\mathfrak{n}}'$ denotes its nilpotent radical. There is a map $\widetilde{\chi} : \widetilde{\check{\mathfrak{g}}} \to \mathfrak{t}$ which sends a pair $(\check{\mathfrak{b}}', x)$ to the image of x modulo $(\check{\mathfrak{b}}')^\perp$. Let $\widetilde{\mathcal{S}}$ denote the fiber product $\mathcal{S} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}}$, so that

$$\widetilde{\mathcal{S}} \subseteq \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}} = \check{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}}.$$

Then, Kostant's result on the Kostant slice implies formally that the composite

$$\widetilde{\mathcal{S}} \to \widetilde{\check{\mathfrak{g}}} \xrightarrow{\widetilde{\chi}} \mathfrak{t}$$

is an isomorphism. We will often abusively write the inclusion of $\widetilde{\mathcal{S}}$ as a map $\kappa : \mathfrak{t} \to \widetilde{\check{\mathfrak{g}}}$.

In fact, we will only care about the composite $\mathfrak{t} \to \widetilde{\check{\mathfrak{g}}} \to \widetilde{\check{\mathfrak{g}}}/\check{G}$ below, so we will also denote it by κ. If we identify $\widetilde{\check{\mathfrak{g}}}/\check{G} \cong \mathfrak{b}/\check{B}$, then the map κ admits a simple description: it is the composite $f + \mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/\check{B}$. (See Proposition 2.9.) In our discussion below, we will often identify $f + \mathfrak{t}$ with $\psi + \check{\mathfrak{t}}^*$.

Definition 6.2. The stabilizer (inside $\check{G}$) of the Kostant slice $\mathcal{S} \subseteq \mathfrak{g}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{G} \times \mathcal{S}$, and will be denoted by $\check{J}$. It will be called the *regular centralizer group scheme*; if we wish to emphasize the dependence on G, we will denote it by $\check{J}(G)$. Note that since the composite $\mathcal{S} \to \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!\!/\check{G}$ is an isomorphism, we may identify

$$\check{J} \cong \mathcal{S} \times_{\mathfrak{g}/\check{G}} \mathcal{S}.$$

Similarly, the stabilizer (inside $\check{G}$) of the Kostant slice $\widetilde{\mathcal{S}} \subseteq \widetilde{\mathfrak{g}}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{G} \times \widetilde{\mathcal{S}}$, and will be denoted by $\widetilde{\check{J}}$. Since $\widetilde{\mathcal{S}} \cong \mathcal{S} \times_{\check{\mathfrak{g}}^*} \widetilde{\mathfrak{g}}$, we may identify

$$\widetilde{\check{J}} \cong \check{J} \times_{\mathcal{S}} \widetilde{\mathcal{S}} \cong (f + \mathfrak{t}) \times_{\mathfrak{b}/\check{B}} (f + \mathfrak{t}).$$

Theorem 6.3. *There is an isomorphism of group schemes over $f + \mathfrak{t} \cong \mathfrak{t} \cong \mathcal{M}_{T,0}$:*

$$\operatorname{Spec} \pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee \cong (f + \mathfrak{t}) \times_{\mathfrak{b}/\check{B}} (f + \mathfrak{t}).$$

Theorem 6.3 can be proved directly using Proposition 3.15, but I find the discussion below more enlightening (of course, it is essentially an elaboration of the application of Proposition 3.15). We first need a few lemmas.

Lemma 6.4. *The projection map $\widetilde{\check{J}} \to \psi + \check{\mathfrak{t}}^*$ (onto either factor) is flat.*

Proof. For this, we will follow [YZ11, Step II]. Consider the morphism $\check{B} \times \check{\mathfrak{t}}^* \to \check{\mathfrak{n}}^\perp$ sending $(g, x) \mapsto \mathrm{Ad}_g(\psi + x) - \psi$. Unwinding definitions shows that there is a Cartesian square

$$\begin{tikzcd}
\widetilde{\check{J}} \arrow[r] \arrow[d] & \check{\mathfrak{t}}^* \arrow[d] \\
\check{B} \times \check{\mathfrak{t}}^* \arrow[r] & \check{\mathfrak{n}}^\perp,
\end{tikzcd}$$

so $\widetilde{\check{J}}$ is a closed subscheme of $\check{B} \times \check{\mathfrak{t}}^*$ of codimension $\dim(\check{\mathfrak{b}}^\perp) = \dim(\check{N})$. This means that the fibers of the map $\widetilde{\check{J}} \to \check{\mathfrak{t}}^*$ are have dimension at least $\dim(\check{B}) - \dim(\check{N}) = \mathrm{rank}(\check{G})$. If all fibers had dimension *exactly* $\mathrm{rank}(\check{G})$, then miracle flatness would imply that the map $\widetilde{\check{J}} \to \check{\mathfrak{t}}^*$ is flat. To show that all fibers have dimension $\mathrm{rank}(\check{G})$, observe that there is a contracting $\mathbf{G}_m$-action on the vector space $\check{\mathfrak{t}}^*$ which pushes everything down to the origin; so it suffices to show that the fiber over $0 \in \check{\mathfrak{t}}^*$ is of the correct dimension.

That is, we need to see that the scheme

$$Y := \{(g, x) \in \check{B} \times \check{\mathfrak{t}}^* | \mathrm{Ad}_g(\psi) = \psi + x\}$$

is $\mathrm{rank}(\check{G})$-dimensional. First, observe that if $\mathrm{Ad}_g(\psi) = \psi + x \in \check{\mathfrak{n}}^\perp$ with $x \in \check{\mathfrak{t}}^*$, then actually $x = 0$. This is because the image of x under the map

$$\check{\mathfrak{n}}^\perp \to (\check{\mathfrak{n}} \oplus \check{\mathfrak{n}}^-)^\perp \cong \check{\mathfrak{t}}^*$$

is the same as the image of $\psi + x$, which is the same as the image of $\mathrm{Ad}_g(\psi)$. But the above map $\check{\mathfrak{n}}^\perp \to \check{\mathfrak{t}}^*$ is Ad-invariant, and so the image of $\mathrm{Ad}_g(\psi)$ is equal to the image of ψ, which is zero. This means that the image of x is also zero. But the inclusion $\check{\mathfrak{t}}^* \subseteq \check{\mathfrak{n}}^\perp$ splits the map $\check{\mathfrak{n}}^\perp \to \check{\mathfrak{t}}^*$, and so we see that $x = 0$. Therefore,

$$Y \cong \{g \in \check{B} | \mathrm{Ad}_g(\psi) = \psi\} = Z_{\check{B}}(\psi).$$

The centralizer of ψ is contained entirely in $\check{B}$, so $Z_{\check{B}}(\psi) \cong Z_{\check{G}}(\psi)$. This, in turn, has dimension given by $\mathrm{rank}(\check{G})$ since ψ (corresponding to $e \in \mathfrak{g}$) is a regular nilpotent. $\square$

Note that

$$\widetilde{\check{J}} \cong \{(x, y, g) \in \check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \check{B} | \mathrm{Ad}_g(\psi + x) = \psi + y\}.$$

The argument at the end of Lemma 6.4 allows us to identify $x = y \in \check{\mathfrak{t}}^*$, and so

$$\widetilde{\check{J}} \cong \{(x, g) \in \check{\mathfrak{t}}^* \times \check{B} | \mathrm{Ad}_g(\psi + x) = \psi + x\}.$$

Notation 6.5. If α is a root of $\check{G}$, let $\{e_\alpha, h_\alpha\}$ denote a pinning of $\check{G}$. Say that a point $x \in \check{\mathfrak{t}}^*$ is *α-generic* if $x(h_\beta) \neq 0$ for all roots $\beta \neq \alpha$. This implies that the centralizer $Z_{\check{G}}(x)$ has semisimple rank at most 1. Let $\check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ denote the α-regular locus. Observe that $\check{\mathfrak{t}}^*_{\mathrm{reg}} = \bigcup_{\alpha \in \Phi} \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \subseteq \check{\mathfrak{t}}^*$ is open, with complement of codimension 2.

Lemma 6.6. *There is an isomorphism*

$$\widetilde{\check{J}}(\check{G})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{J}}(Z_{\check{G}}(x)^\circ)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}, \tag{7}$$

where $Z_{\check{G}}(x)$ is the centralizer of some $x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ which lies on the α-hyperplane, and $Z_{\check{G}}(x)^\circ$ denotes the connected component of the identity.

Proof sketch. Let us, for simplicity, write $\check{H}$ to denote $Z_{\check{G}}(x)^\circ$. There is a map from the left-hand side to the right-hand side, which sends

$$\check{\mathfrak{t}}^* \times \check{B} \ni (x, g) \mapsto (x, g) \in \check{\mathfrak{t}}^* \times (\check{B} \cap \check{H}).$$

Note that $\check{B} \cap \check{H}$ is a Borel subgroup of $\check{H}$. To see that the above map gives an isomorphism, observe that if $y \in \check{\mathfrak{t}}^*$, we may identify the centralizer in $\check{G}$ of $\psi + y$ with the centralizer in $Z_{\check{G}}(y)^\circ$ of ψ. That (7) is an isomorphism is now a consequence of the observation that if $y \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$, then this centralizer $Z_{\check{G}}(y)^\circ$ is contained in $\check{H}$. That is, if $(x, g) \in \widetilde{\check{J}}(\check{G})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$, then g is already contained in H, and so $(x, g) \in \widetilde{\check{J}}(\check{H})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$. $\square$

Proof of Theorem 6.3. We begin by noting that Gr_G only has even cells; so $\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee = \pi_0 C^T_*(\mathrm{Gr}_G; \mathbf{Q}[u^{\pm 1}])$ can be identified with $\mathrm{H}^T_*(\mathrm{Gr}_G; \mathbf{Q})$, regarded now as an ungraded $\mathbf{Q}$-algebra. Similarly, $\pi_0(k_T) \cong \mathrm{H}^*_T(*; \mathbf{Q})$, again regarded as an ungraded $\mathbf{Q}$-algebra. The equivariant formality of Gr_G implies that $\mathrm{H}^T_*(\mathrm{Gr}_G; \mathbf{Q})$ is flat over $\mathrm{H}^*_T(*; \mathbf{Q})$. To prove Theorem 6.3, it therefore suffices to prove an isomorphism

$$\widetilde{\check{J}}|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathrm{Spec}\, \mathrm{H}^{T_c}_*(\Omega G; \mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$$

for each root α. By Atiyah-Bott localization, the right-hand side can be identified with

$$\mathrm{Spec}\, \mathrm{H}^{T_c}_*(\Omega G; \mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathrm{Spec}\, \mathrm{H}^{T_c}_*(\Omega Z_G(x); \mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}, \tag{8}$$

where $Z_G(x)$ is the centralizer of some $x \in \mathfrak{t}_{\alpha\text{-reg}}$ which lies on the α-hyperplane. Note that the right-hand side depends only on the connected component $Z_G(x)^\circ$

of the identity in $Z_G(x)$; so we might as well replace $Z_G(x)$ by $Z_G(x)^\circ$. Using Lemma 6.6, we are therefore reduced to showing that there is an isomorphism

$$\mathrm{H}_*^{T_c}(\Omega Z_G(x)^\circ; \mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \widetilde{\check{J}}(Z_{\check{G}}(x)^\circ)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

Since $Z_G(x)^\circ$ has semisimple rank 1, we are reduced to checking that Theorem 6.3 holds in this case.

That is, we may assume G is the product of a torus with GL_2, SL_2, or PGL_2. It is easy to match up the contribution from the toral factors, so we will assume that G is GL_2, SL_2, or PGL_2.

- For GL_2, we may identify $\mathfrak{gl}_2^* \cong \mathfrak{gl}_2$. Then, $\widetilde{\check{J}}$ is the centralizer (in $\check{B}$) of $\left(\begin{smallmatrix} x & 0 \\ 1 & y \end{smallmatrix}\right)$. It is not hard to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & y \end{smallmatrix}\right)$ if and only if $c = \frac{a-d}{x-y}$, meaning that

$$\widetilde{\check{J}} \cong \operatorname{Spec} \mathbf{Q}[x, y, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

The coproduct sends $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$.

Let us now calculate $\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})$ as an algebra over $\mathrm{H}^*_{T^2}(*; \mathbf{Q}) \cong \mathbf{Q}[x, y]$. There is a simple T^2-equivariant cell decomposition of $\Omega \mathrm{GL}_2$ with $\mathbb{X}_*(T^2) = \mathbf{Z}^2$ many 0-cells, and where there is a T^2-equivariant 1-cell connecting μ_1 to μ_2 if and only if $\mu_1 - \mu_2$ is a multiple of a root of GL_2. (There are higher equivariant cells, but they will not matter.) This implies, by Atiyah-Bott localization, that the fixed points of the T^2-action on $\Omega \mathrm{GL}_2$ are simply $\Omega T^2 = \mathbf{Z}^2$, and so

$$\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})[\tfrac{1}{x-y}] \cong \mathrm{H}_*^{T^2}(\Omega T^2; \mathbf{Q})[\tfrac{1}{x-y}] \cong \mathbf{Q}[x, y, \tfrac{1}{x-y}, a^{\pm 1}, d^{\pm 1}].$$

On the other hand, the *completion* $\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)}$ can be determined directly. After completing at $(x-y, y) = (x, y)$, the equivariant homology $\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})$ simply becomes the *Borel-equivariant* homology, and this can be computed directly via a spectral sequence

$$E_2 = \mathrm{H}^*(BT^2; \mathbf{Q}) \otimes_k \mathrm{H}_*(\Omega \mathrm{GL}_2; \mathbf{Q}) \Rightarrow \mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x,y)}.$$

Since $\mathrm{H}_*(\Omega \mathrm{GL}_2; \mathbf{Q}) = \mathbf{Q}[A^{\pm 1}, b]$ with A in weight 0 and b in weight 2, the E_2-page of this spectral sequence is concentrated entirely in even degrees, and hence collapses. This means that

$$\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x,y)} \cong k[\![x, y]\!][A^{\pm 1}, b].$$

This in fact comes from an isomorphism

$$\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)} \cong \mathbf{Q}[x, y, A^{\pm 1}, b]^\wedge_{(x-y)}.$$

We may therefore recover $\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})$ via the gluing square

$$\begin{tikzcd}
\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q}) \arrow[r] \arrow[d] & \mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})[\frac{1}{x-y}] \arrow[d] \\
\mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)} \arrow[r] & \mathrm{H}_*^{T^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)}[\frac{1}{x-y}].
\end{tikzcd}$$

Explicitly:

$$\begin{tikzcd}
\mathrm{H}_*^{T^2}(\Omega\mathrm{GL}_2;\mathbf{Q}) \arrow[r] \arrow[d] & \mathbf{Q}[x,y,\frac{1}{x-y},a^{\pm 1},d^{\pm 1}] \arrow[d] \\
\mathbf{Q}[x,y,A^{\pm 1},b]^\wedge_{(x-y)} \arrow[r] & \mathbf{Q}[x,y,A^{\pm 1},b]^\wedge_{(x-y)}[\frac{1}{x-y}].
\end{tikzcd}$$

The right vertical map sends $a-d \mapsto b(x-y)$; and $d \mapsto A$. Note that $b(x-y)$ is topologically nilpotent, so $A+b(x-y)$ is a unit, and this is what a maps to. This discussion implies that the fiber product above identifies with

$$\mathrm{H}_*^{T^2}(\Omega\mathrm{GL}_2;\mathbf{Q}) \cong \mathbf{Q}[x,y,a^{\pm 1},d^{\pm 1},\tfrac{a-d}{x-y}].$$

We need to determine the coproduct. Since this ring is flat over $\mathbf{Q}[x,y]$, it suffices to determine the coproduct after inverting $x-y$. As we have seen, $\mathrm{H}_*^{T^2}(\Omega\mathrm{GL}_2;\mathbf{Q})[\frac{1}{x-y}] \cong \mathrm{H}_*^{T^2}(\Omega T^2;\mathbf{Q})[\frac{1}{x-y}]$, and $\Omega T^2 = \mathbf{Z}^2$. The coproduct here simply comes from the *diagonal* on $\mathbf{Z}^2$, which obviously sends $a \mapsto a\otimes a$ and $d \mapsto d\otimes d$. It follows that

$$\operatorname{Spec}\mathrm{H}_*^{T^2}(\Omega\mathrm{GL}_2;\mathbf{Q}) \cong \widetilde{\check{J}}$$

as (graded) group schemes over $\mathbf{Q}[x,y]$, as desired.

- For $G=\mathrm{SL}_2$, one can similarly calculate that

$$\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q}) \cong \mathbf{Q}[x,a^{\pm 1},b]/(a=1+2xb) \cong \mathbf{Q}[x,a^{\pm 1},\tfrac{a-1}{2x}].$$

The coproduct is determined by the formula $a\mapsto a\otimes a$, so that

$$b \mapsto b\otimes 1 + 1\otimes b + 2xb\otimes b.$$

For completeness, let us quickly summarize the argument. The fixed points of S^1 acting on $\Omega\mathrm{SU}(2)$ is $\Omega S^1 = \mathbf{Z}$, and the action of S^1 on $\mathrm{SU}(2)\cong S^3$ exhibits it as the one-point compactification of $\mathbf{R}\oplus\mathbf{C}$, where $\mathbf{R}$ is the trivial representation and $\mathbf{C}$ is the *weight* 2 representation. Therefore, inverting the Chern class $2x$ of the weight 2 representation lets us identify

$$\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})[\tfrac{1}{2x}] \cong \mathrm{H}_*^{S^1}(\Omega S^1;\mathbf{Q})[\tfrac{1}{2x}] \cong \mathbf{Q}[x^{\pm 1},a^{\pm 1}].$$

On the other hand, the completion of $\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})$ at the class $2x$ is, via the same spectral sequence argument as in the preceding bullet point, given by

$$\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})^\wedge_{(2x)} \cong \mathbf{Q}[\![x]\!][b],$$

with b in weight 2. The ring $\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})$ can be recovered via the gluing square

$$\begin{tikzcd}
\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q}) \arrow[r] \arrow[d] & \mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})[\frac{1}{2x}] \arrow[d] \\
\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})^\wedge_{(2x)} \arrow[r] & \mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2;\mathbf{Q})^\wedge_{(2x)}[\frac{1}{2x}].
\end{tikzcd}$$

The right vertical map sends $a-1 \mapsto b \cdot 2x$, and so the above Cartesian square gives an isomorphism

$$\mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2; \mathbf{Q}) \cong \mathbf{Q}[x, a^{\pm 1}, b]/(a = 1 + 2xb),$$

as desired.

On the other hand, $\widetilde{\check{J}}$ is the centralizer in $\check{B} \subseteq \mathrm{PGL}_2$ of $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right) \subseteq \mathfrak{sl}_2 \cong \check{\mathfrak{g}}^*$. It is easy to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \check{B}$ (where we only care about this as an element of PGL_2!) stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right)$ if and only if $2xc = a - 1$. Therefore,

$$\widetilde{\check{J}} \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, c]/(a = 1 + 2xc),$$

and again the group law is determined by the formulae

$$a \mapsto a \otimes a, \;\; c \mapsto c \otimes 1 + 1 \otimes c + 2xc \otimes c.$$

Therefore,

$$\operatorname{Spec} \mathrm{H}_*^{S^1}(\Omega\mathrm{SL}_2; \mathbf{Q}) \cong \widetilde{\check{J}}$$

as (graded) group schemes over $\mathbf{Q}[x]$, as desired.

- In exactly the same way, for $G = \mathrm{PGL}_2$, one can similarly calculate that

$$\mathrm{H}_*^{S^1}(\Omega\mathrm{PGL}_2; \mathbf{Q}) \cong \mathbf{Q}[x, a^{\pm 1}, b]/(a^2 = 1 + xb) \cong \mathbf{Q}[x, a^{\pm 1}, \tfrac{a^2-1}{x}].$$

This is because the fixed points of S^1 acting on $\Omega\mathrm{PGL}_2 \simeq \mathbf{Z}/2 \times \Omega S^3$ is $\mathbf{Z}$, and the action of S^1 on PGL_2, which is homotopy equivalent to $\mathbf{R}P^3$, exhibits it as the $\mathbf{Z}/2$-quotient of the one-point compactification of $\mathbf{R} \oplus \mathbf{C}$, where $\mathbf{R}$ is the trivial representation and $\mathbf{C}$ is the *weight* 1 representation. The coproduct is determined by the formula $a \mapsto a \otimes a$, so that

$$b \mapsto b \otimes 1 + 1 \otimes b + xb \otimes b.$$

On the other hand, $\widetilde{\check{J}}$ is the centralizer in $\check{B} \subseteq \mathrm{SL}_2$ of the equivalence class of $\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathfrak{pgl}_2 \cong \check{\mathfrak{g}}^*$. It is easy to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right) \in \check{B}$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)$ if and only if $xc = a - a^{-1}$. Therefore,

$$\widetilde{\check{J}} \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, c]/(a = a^{-1} + xc) \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, \tfrac{a - a^{-1}}{x}].$$

Replacing c by $b := ca^{-1}$, we see that the group law is determined by the formulae

$$a \mapsto a \otimes a, \;\; b \mapsto b \otimes 1 + 1 \otimes b + xb \otimes b.$$

Therefore,

$$\operatorname{Spec} \mathrm{H}_*^{S^1}(\Omega\mathrm{PGL}_2; \mathbf{Q}) \cong \widetilde{\check{J}}$$

as (graded) group schemes over $\mathbf{Q}[x]$, as desired. □

Remark 6.7. Just for posterity, let us record a more canonical variant of the calculation above for $\check{G} = \mathrm{SL}_2$, which does not require picking a Borel subgroup (i.e., which does not involve identifying $\widetilde{\check{\mathfrak{g}}}/\check{G} \cong \mathfrak{b}/\check{B}$). For simplicity, we will use the fact that 2 is invertible in $\mathbf{Q}$ to identify $\mathfrak{sl}_2 \cong \mathfrak{pgl}_2$. In this case, the Grothendieck-Springer resolution $\widetilde{\check{\mathfrak{g}}} = T^*(\mathbf{A}^2 - \{0\})/\mathbf{G}_m$ is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1$; we will think of a point in $\widetilde{\mathfrak{g}}$ as a pair $(x \in \mathfrak{sl}_2, \ell \subseteq \mathbf{C}^2)$ such that x preserves ℓ. The Kostant slice $\kappa : \mathfrak{t} \cong \mathbf{A}^1 \to \widetilde{\check{\mathfrak{g}}}$ is the map sending $\lambda \in \mathbf{A}^1$ to the pair (x, ℓ)

with $x = \left(\begin{smallmatrix} 0 & \lambda^2 \\ 1 & 0 \end{smallmatrix}\right)$ and $\ell = [\lambda : 1]$. Indeed, this is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{A}^1 \cong \mathfrak{t} \arrow[r, "\kappa"] \arrow[d, "\lambda\mapsto\lambda^2"'] & \widetilde{\mathfrak{sl}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \mathfrak{t}/\!\!/W \arrow[r, "\kappa", "\lambda\mapsto\left(\begin{smallmatrix} 0 & \lambda \\ 1 & 0 \end{smallmatrix}\right)"'] & \mathfrak{sl}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\check{\mathfrak{g}}}$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we compute that

$$\mathrm{Ad}_g \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} bd - ac\lambda^2 & (a\lambda)^2 - b^2 \\ d^2 - (c\lambda)^2 & ac\lambda^2 - bd \end{pmatrix}, \; g \cdot [\lambda : 1] = [a\lambda + b : c\lambda + d].$$

From this, we see that $\mathrm{Ad}_g(x) = x$ if and only if $a = d$ and $b = c\lambda^2$, in which case g also fixes $[\lambda : 1]$. In other words, $g = \left(\begin{smallmatrix} a & c\lambda^2 \\ c & a \end{smallmatrix}\right)$ with $a, c \in k$; in order for $\det(g) = 1$, we need $a^2 - c^2\lambda^2 = 1$. When $\lambda \neq 0$, both x and g are diagonalized by the matrix $\frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -\lambda^{-1} & -\lambda^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} t & 0 \\ 0 & w \end{smallmatrix}\right)$ where $2a = t + w$ and $2\lambda c = t - w$. Since we have $\det(g) = a^2 - (c\lambda)^2 = 1$, we have $w = t^{-1}$. This shows that if k is not of characteristic 2, then $\mathfrak{t} \times_{\widetilde{\mathfrak{sl}}_2/\mathrm{SL}_2} \mathfrak{t} \cong \operatorname{Spec} \mathbf{Q}[\lambda, t^{\pm 1}, \frac{t - t^{-1}}{\lambda}]$.

Corollary 6.8. *There is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G}).$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k) \to \mathrm{Loc}^{\mathrm{gr}}_{T_c}(*; k)$ *identifies with the pullback functor* $\kappa^* : \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G}) \to \mathrm{QCoh}(\mathfrak{t})$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$ is equivalent to the category of comodules over $\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee = \mathrm{H}^T_*(\mathrm{Gr}_G; \mathbf{Q})$ in the category of $\pi_0 k_T \cong \mathrm{H}^*_T(*; \mathbf{Q})$-modules. By Theorem 6.3, it can be identified the category of quasicoherent sheaves on the quotient stack $(f + \mathfrak{t})/\widetilde{\check{J}}$. As discussed after Lemma 6.4, we may view $\widetilde{\check{J}}$ as a closed subgroup scheme of the constant group scheme $\check{B} \times (f + \mathfrak{t})$. This gives an isomorphism

$$(f + \mathfrak{t})/\widetilde{\check{J}} \cong \check{B}\backslash(\check{B} \times (f + \mathfrak{t}))/\widetilde{\check{J}}.$$

It follows from Kostant's work in [Kos63] that the $\check{B}$-orbit of $f + \mathfrak{t}$ inside $\mathfrak{b}$ is precisely the regular locus $\mathfrak{b}^{\mathrm{reg}}$. Since $\widetilde{\check{J}}$ is definitionally the stabilizer of $f + \mathfrak{t} \subseteq \mathfrak{b}$, the quotient $(\check{B} \times (f + \mathfrak{t}))/\widetilde{\check{J}}$ is isomorphic to $\mathfrak{b}^{\mathrm{reg}}$; so there is an isomorphism $(f + \mathfrak{t})/\widetilde{\check{J}} \cong \mathfrak{b}^{\mathrm{reg}}/\check{B}$. To finish, note that $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G} \cong \mathfrak{b}^{\mathrm{reg}}/\check{B}$. □

Remark 6.9. Note that the definition of the Kostant slice $f + \mathfrak{t} \subseteq \mathfrak{b}$ involved the choice of a regular nilpotent element $f \in \mathfrak{g}$. However, this choice does not materialize in Corollary 6.8. This is because two such slices obtained by choosing two different regular nilpotent elements in $\mathfrak{g}$ are *conjugate* to each other (by $\check{B}$). That is, while the specific inclusion $f + \mathfrak{t} \subseteq \mathfrak{b}$ depends on the choice of f, the composite $f + \mathfrak{t} \subseteq \mathfrak{b} \to \mathfrak{b}/\check{B}$ is independent of said choice.

Before doing so, let us make a few remarks regarding Theorem 6.3.

Example 6.10. Suppose $G = \mathrm{SL}_n$. In this case, $\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ is simply isomorphic to a polynomial algebra $\mathbf{Q}[b_1, \cdots, b_{n-1}]$ on $n-1$ generators. The coproduct is given by $b_j \mapsto \sum_i b_i \otimes b_{j-i}$, where b_0 is understood to be 1. This result is classical, and can be found, for instance, in [Bot58]. The proof there amounts to the following observation. Consider the map $\mathbf{C}P^{n-1} \to \Omega\mathrm{SL}_n$ given by sending $\ell \in \mathbf{C}P^{n-1}$ to (an appropriate rescaling of) the loop sending $\theta \in S^1$ to rotation by angle θ about the line ℓ. Then the image of the induced map $\mathrm{H}_*(\mathbf{C}P^{n-1}; \mathbf{Q}) \to \mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ generates $\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$; that is, $\mathbf{C}P^{n-1}$ is a generating complex for $\Omega\mathrm{SL}_n$. The formula for the coproduct comes from the coproduct on $\mathrm{H}_*(\mathbf{C}P^{n-1}; \mathbf{Q})$, which is determined easily by the cup product on $\mathrm{H}^*(\mathbf{C}P^{n-1}; \mathbf{Q})$. The above description of $\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ implies that $\mathrm{Spec}\,\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ is isomorphic to the group scheme $\mathbf{W}_{n-1}$ of big Witt vectors of length $n-1$.

On the other hand, Theorem 6.3 implies that $\mathrm{Spec}\,\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ is isomorphic to the centralizer inside PGL_n of of the regular nilpotent $f \in \mathfrak{sl}_n$. Indeed, if R is a $\mathbf{Q}$-algebra, then an element $g \in \mathrm{GL}_n(R)$ commutes with f if and only if g is an invertible polynomial in e. By the Cayley-Hamilton theorem, such a polynomial is divisible by the minimal polynomial t^n of e; that is, $g \in (R[t]/t^n)^\times$. For this to live in $\mathrm{PGL}_n(R)$, we need to quotient out by the scalars $R^\times$. The assignment $R \mapsto (1 + tR[t]/t^n)^\times$ is precisely the functor of points of $\mathbf{W}_{n-1}$. One can therefore understand the isomorphism between $\mathrm{Spec}\,\mathrm{H}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ and $Z_{\mathrm{PGL}_n}(e)$ as being a way to identify the two descriptions of the Witt vector group scheme (either via its functor of points, or via the explicit Witt addition law).

Example 6.11. Continuing the preceding example (so $G = \mathrm{SL}_n$), it is not hard to add in torus-equivariance (so $T_c = (S^1)^{n-1}$). In this case, we will identify $\mathrm{H}^*_{T_c}(*; \mathbf{Q}) \cong \mathbf{Q}[x_1, \cdots, x_n]$. One can write down an explicit T_c-equivariant cell structure on $\Omega\mathrm{SU}(n)$ to find that $\mathrm{Spec}\,\mathrm{H}^{T_c}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ is isomorphic to the deformation of $\mathbf{W}_{n-1}$ over $\mathrm{Spec}\,\mathrm{H}^*_{T_c}(*; \mathbf{Q}) \cong \mathbf{A}^n$ which sends a $\mathbf{Q}[x_1, \cdots, x_n]$-algebra R to the group of units $(1+tR[t]/(t-x_1)\cdots(t-x_n))^\times$. On the other hand, by the same argument using Cayley-Hamilton, the centralizer inside $\mathrm{GL}_n(R)$ of $f + x \in \mathfrak{sl}_n$ is isomorphic to the group $(R[t]/(t-x_1)\cdots(t-x_n))^\times$, since the characteristic polynomial of $f + x$ is precisely $(t-x_1)\cdots(t-x_n)$. Quotienting by the scalars $R^\times$, we obtain an isomorphism between $\mathrm{Spec}\,\mathrm{H}^{T_c}_*(\Omega\mathrm{SL}_n; \mathbf{Q})$ and $Z_{\mathrm{PGL}_n}(f + x)$.

Remark 6.12. For a general reductive group G, Kostant proved (in [Kos78]) an isomorphism $(f + \mathfrak{b})/\check{N} \cong \mathfrak{t}/\!\!/W$. In fact, the natural map $(f + \mathfrak{b})/\check{N} \to \mathfrak{g}/\check{G}$ identifies with the map $\mathcal{S} \to \mathfrak{g}/\check{G}$ given by the Kostant slice. Since $(f + \mathfrak{b})/\check{N}$ is isomorphic to the quotient $\check{G}\backslash T^*(\check{G}/_\psi \check{N})$ of the Whittaker reduction of $T^*(\check{G})$, it follows that there are isomorphisms

$$\begin{aligned}\check{J} &\cong (f + \mathfrak{b})/\check{N} \times_{\mathfrak{g}/\check{G}} (f + \mathfrak{b})/\check{N} \\ &\cong \check{G}\backslash T^*(\check{G}/_\psi \check{N}) \times_{\check{\mathfrak{g}}^*/\check{G}} T^*(\check{N}_\psi\backslash\check{G})/\check{G} \\ &\cong T^*(\check{N}_\psi\backslash\check{G}/_\psi\check{N}).\end{aligned}$$

That is, $\check{J}$ can be identified with the *bi-Whittaker reduction* of the cotangent bundle $T^*(\check{G})$. In particular, Theorem 6.3 gives an isomorphism

$$\mathrm{Spec}\,\mathrm{H}^{T_c}_*(\mathrm{Gr}_G; \mathbf{Q}) \cong \check{\mathfrak{t}}^* \times_{\check{\mathfrak{t}}^*/\!\!/W} T^*(\check{N}_\psi\backslash\check{G}/_\psi\check{N}).$$

In fact, this isomorphism can be checked to be W-equivariant (for the action of W on $\mathrm{H}^{T_c}_*(\mathrm{Gr}_G; \mathbf{Q})$ via the action on T, and for the action on the right-hand side coming from the cover $\check{\mathfrak{t}}^* \to \check{\mathfrak{t}}^* /\!\!/ W$). This implies that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^G(\mathrm{Gr}_G; \mathbf{Q}) \cong T^*(\check{N}_\psi \backslash \check{G} /_\psi \check{N}).$$

This isomorphism has been exploited heavily in [Tel14], among others.

The calculation of Theorem 6.3 is quite powerful. Here is a simple application, motivated by [GK22] and [GR15]; see also [Dev24, Example 3.6.13], where the same example is presented.

Proposition 6.13 (Gelfand-Graev action)**.** *The natural action of $\check{G} \times \check{T}$ on the affine closure $\overline{T^*(\check{G}/\check{N})}$ extends to an action of $\check{G} \times (W \rtimes \check{T})$, where W is the Weyl group.*

Proof. Let $T^*(\check{G}/\check{N})_{\mathrm{reg}} = \check{G} \times^{\check{N}} \check{\mathfrak{n}}^\perp_{\mathrm{reg}}$ denote the regular locus in $T^*(\check{G}/\check{N})$; then $\overline{T^*(\check{G}/\check{N})_{\mathrm{reg}}} \subseteq \overline{T^*(\check{G}/\check{N})}$ is open, with complement of codimension 2, so that $\overline{T^*(\check{G}/\check{N})} \cong \overline{T^*(\check{G}/\check{N})_{\mathrm{reg}}}$. Note that there is an isomorphism

$$\check{G} \backslash T^*(\check{G}/\check{N})_{\mathrm{reg}} / \check{T} \cong \check{\mathfrak{n}}^\perp_{\mathrm{reg}} / \check{B},$$

so (the proof of) Corollary 6.8 gives isomorphisms

$$T^*(\check{G}/\check{N})_{\mathrm{reg}} \cong (\check{G} \times \check{T}) \times^{\check{B}} \check{\mathfrak{n}}^\perp_{\mathrm{reg}} \cong (\check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*)) / \widetilde{\check{J}}. \tag{9}$$

There is a canonical W-action on $\check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*)$, given by the natural W-actions on $\check{T}$ and on $\psi + \check{\mathfrak{t}}^* \cong \check{\mathfrak{t}}^*$. Similarly, $\widetilde{\check{J}}$ also admits a natural W-action; it is given via Theorem 6.3 by the natural W-action on $\mathrm{H}_*^{T_c}(\mathrm{Gr}_G; \mathbf{Q})$. Moreover, the closed immersion

$$\widetilde{\check{J}} \to \check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*)$$

is W-equivariant (indeed, the map $\widetilde{\check{J}} \to \check{T} \times (\psi + \check{\mathfrak{t}}^*)$ is induced by the inclusion $\mathrm{H}_*^{T_c}(\mathrm{Gr}_T; \mathbf{Q}) \to \mathrm{H}_*^{T_c}(\mathrm{Gr}_G; \mathbf{Q})$ on equivariant homology). This implies that the quotient of (9) admits a W-action, which defines a W-action on the affine closure of $T^*(\check{G}/\check{N})_{\mathrm{reg}}$ as desired. □

Note that we assumed in Corollary 2.11 that G is simply-laced; but this is not necessary, because we know (by the discussion in Section 2) that the main result of [ABG04] implies Corollary 2.11 is true for any connected reductive G. Alternatively, one can observe that the proof of Corollary 2.11 itself never seriously appeals to G being simply-laced.

The W-action of Proposition 6.13 is known as the (semiclassical) *Gelfand-Graev action*. The moment map $\overline{T^*(\check{G}/\check{N})} \to \check{\mathfrak{g}}^*$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd}
\widetilde{\check{\mathfrak{g}}} \arrow[r, hook] \arrow[dr] & \overline{T^*(\check{G}/\check{N})}/\check{T} \arrow[d] \\
& \check{\mathfrak{g}}^*
\end{tikzcd}$$

which relates $\overline{T^*(\check{G}/\check{N})}$ to the Grothendieck-Springer resolution; and via this diagram, the Gelfand-Graev action is closely related to the Weyl action in Springer theory.

Example 6.14. When $\check{G} = \mathrm{SL}_2$, the affine closure $\overline{T^*(\check{G}/\check{N})}$ is simply $T^*(\mathbf{A}^2)$, and the $W = \mathbf{Z}/2$-action on it is given by the symplectic Fourier transform. To see this, let $\check{J}_X$ denote the kernel of the homomorphism $\widetilde{\check{J}} \to \check{T} \times (\psi + \check{\mathfrak{t}}^*)$ of group schemes over $\psi + \check{\mathfrak{t}}^*$. (This follows the notation from [Dev24].) Then (9) gives an isomorphism

$$(\check{G} \times (\psi + \check{\mathfrak{t}}^*))/\check{J}_X \xrightarrow{\cong} T^*(\check{G}/\check{N})_{\mathrm{reg}}.$$

In the case at hand, $\psi + \check{\mathfrak{t}}^* \cong \mathbf{A}^1$ with coordinate x, and the group scheme $\check{J}_X$ is just $\mathrm{Spec}\,\mathbf{Z}[x,b]/bx$ (where the group law sends $b \mapsto b \otimes 1 + 1 \otimes b$). The above isomorphism defines a map

$$q : \mathrm{SL}_2 \times \mathbf{A}^1 \to \overline{T^*(\check{G}/\check{N})} = T^*(\mathbf{A}^2),$$

and the affine closure of the image is all of $T^*(\mathbf{A}^2)$. The map q can be explicitly described as follows. View a point of $T^*(\mathbf{A}^2)$ as a pair $((\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}), (v_1, v_2))$ of a vector and a covector. Then q is the natural extension to $\mathrm{SL}_2 \times \mathbf{A}^1$ of the map $\kappa : \mathbf{A}^1 \to T^*(\mathbf{A}^2)$ which sends $x \mapsto ((\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (x, 0))$. In other words, q sends

$$(g, x) = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), x \right) \mapsto \left(g(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (g^T)^{-1}(x, 0) \right) = ((\begin{smallmatrix} a \\ c \end{smallmatrix}), (dx, -bx)).$$

Of course, one could also swap the roles of $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$ in $T^*(\mathbf{A}^2)$; the map κ would then send $x \mapsto ((\begin{smallmatrix} 0 \\ x \end{smallmatrix}), (0, 1))$, and q would send

$$(g, x) = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), x \right) \mapsto \left((\begin{smallmatrix} 0 \\ x \end{smallmatrix}) \cdot g^T, (0, 1) \cdot g^{-1} \right) = \left((\begin{smallmatrix} bx \\ dx \end{smallmatrix}), (-c, a) \right).$$

If we compose with the involution sending $x \mapsto -x$, the resulting involution

$$((\begin{smallmatrix} a \\ c \end{smallmatrix}), (dx, -bx)) \mapsto \left((\begin{smallmatrix} -bx \\ -dx \end{smallmatrix}), (-c, a) \right).$$

This, of course, is precisely the symplectic Fourier transform, which sends

$$((\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}), (v_1, v_2)) \mapsto ((\begin{smallmatrix} v_2 \\ -v_1 \end{smallmatrix}), (-u_2, u_1)).$$

We will now discuss a *deformation quantization* of Corollary 6.8 by adding loop-rotation equivariance. Write $\widetilde{T} = T \times \mathbf{G}_m^{\mathrm{rot}}$ to denote the corresponding affine torus. In the case when G is a torus, we have already discussed this in Section 5. For more general G, this turns out to be a bit tricky: while $\mathrm{H}_*^{T_c}(\mathrm{Gr}_G; \mathbf{Q})$ is a bicommutative Hopf algebra[12], the loop-rotation equivariant homology $\mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Gr}_G; \mathbf{Q})$ is only a cocommutative coalgebroid over $\mathrm{H}^*_{\widetilde{T}_c}(*; \mathbf{Q})$. That is, it does not admit an algebra structure. While this is not a mathematical issue, it does make the task of explicitly understanding $\mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Gr}_G; \mathbf{Q})$ in a satisfactory way more complicated. Instead, it turns out to be easier to describe $\mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Fl}_G; \mathbf{Q})$, where Fl_G is the *affine flag variety*, defined as the quotient $G(\!(t)\!)/I$ for the Iwahori subgroup $I \subseteq G[\![t]\!]$ associated to a Borel subgroup $B \subseteq G$. To state the result, we need a definition from [GKV97].

Definition 6.15. Let $(\Lambda, \Phi, \check{\Lambda}, \check{\Phi})$ be a root datum with associated Weyl group W and torus $T = \mathrm{Hom}(\Lambda, \mathbf{G}_m)$. Let Δ be a base of simple roots, let Φ^+ denote the corresponding set of positive roots, and let Φ' denote the subset $W \cdot \Delta \subseteq \Phi$. Let $\mathbf{H}$ be a 1-dimensional group scheme (over a commutative ring R). As in Definition 5.1, let $\mathbf{H}_T = \mathrm{Hom}(\Lambda, T)$, and for each character $\lambda \in \Lambda$, let $\mathbf{H}_{T_\lambda} \hookrightarrow \mathbf{H}_T$ denote the subgroup corresponding to the subtorus $T_\lambda = \ker(\lambda) \subseteq T$. Let $Q(\mathcal{O}_{\mathbf{H}_T})$ denote the sheaf of functions on the generic point of $\mathcal{O}_{\mathbf{H}_T}$. The twisted group

[12] To be more precise, the $\mathbf{E}_2$-space structure on Gr_G equips $C_*^{T_c}(\mathrm{Gr}_G; \mathbf{Q})$ with the structure of an $\mathbf{E}_2$-algebra in $\mathbf{E}_\infty$-coalgebras over $C^*_{T_c}(*; \mathbf{Q})$.

algebra $Q(\mathcal{O}_{\mathbf{H}_T})[W]$ is the algebra which is additively given by the tensor product $Q(\mathcal{O}_{\mathbf{H}_T}) \otimes_F F[W]$, and whose multiplication law is given by

$$(f_1 \otimes w_1) \cdot (f_2 \otimes w_2) = (f \cdot w_1 g) \otimes (w_1 w_2).$$

The algebra $\mathcal{H}(\mathbf{H}, T, W)$ is defined to be the subset of $Q(\mathcal{O}_{\mathbf{H}_T})[W]$ of those elements $\sum_{w \in W} f_w[w]$ such that:

- The poles of f_x all have order ≤ 1, and these are contained in the divisors $\mathbf{H}_{T_\alpha}$ for each $\alpha \in \Phi'$.
- For each $w \in W$ and $\alpha \in \Phi^+ \cap \Phi'$, we have
$$\mathrm{Res}_{\mathbf{H}_{T_\alpha}}(f_w) + \mathrm{Res}_{\mathbf{H}_{T_\alpha}}(f_{s_\alpha w}) = 0.$$

In [GKV97], this algebra is denoted $\widetilde{\mathbf{H}}$. It is proved in [GKV97, Theorem 1.4] that $\mathcal{H}(\mathbf{H}, T, W)$ is a *subalgebra* of $Q(\mathcal{O}_{\mathbf{H}_T})[W]$.

Remark 6.16. The pair $(Q(\mathcal{O}_{\mathbf{H}_T}), Q(\mathcal{O}_{\mathbf{H}_T})[W])$ admits the structure of a (cocommutative) Hopf algebroid; we will abusively say that $Q(\mathcal{O}_{\mathbf{H}_T})[W]$ admits the structure of a Hopf $Q(\mathcal{O}_{\mathbf{H}_T})$-algebroid. The coproduct comes from the diagonal on W; the left unit comes from the inclusion $Q(\mathcal{O}_{\mathbf{H}_T}) \subseteq Q(\mathcal{O}_{\mathbf{H}_T})[W]$; and the right unit comes from the action of W on $\mathbf{H}_T$ (which defines a coaction of W on $\mathcal{O}_{\mathbf{H}_T}$ that extends to a coaction on $Q(\mathcal{O}_{\mathbf{H}_T})$). The resulting Hopf $\mathcal{O}_{\mathbf{H}_T}$-algebroid structure on $Q(\mathcal{O}_{\mathbf{H}_T})[W]$ restricts to $\mathcal{H}(\mathbf{H}, T, W)$, so that $\mathcal{H}(\mathbf{H}, T, W)$ admits the structure of a (cocommutative) Hopf $\mathcal{O}_{\mathbf{H}_T}$-algebroid.

If Λ denotes the *co*root lattice of G, let $W^{\mathrm{aff}} = \Lambda \rtimes W$ denote the corresponding affine Weyl group, and let $\widetilde{W} = \mathbb{X}_*(T) \rtimes W$ denote the extended affine Weyl group.[13] For clarity, note that the action of $\widetilde{W}$ on $\mathbb{X}^*(T)$ (and hence on $\mathbf{H}_T$) is given as follows: if α is a coweight of T and $n \in \mathbf{Z}$, the generator $s_{\alpha,n}$ of W^{aff} acts on $\mathbb{X}^*(T)$ by reflection along the affine hyperplane $\{x \in \mathbb{X}^*(T) | \langle x, \alpha \rangle = n\}$. The *degenerate nil-Hecke algebra* $\mathcal{H}(\mathbf{H}, \widetilde{T}, \widetilde{W})$ is defined to be $\mathbb{X}_*(T) \ltimes_\Lambda \mathcal{H}(\mathbf{H}, \widetilde{T}, W^{\mathrm{aff}})$. In the following discussion, we will simply write $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$ to denote $\mathbb{X}_*(T) \ltimes_\Lambda Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[W^{\mathrm{aff}}]$, so that $\mathcal{H}(\mathbf{H}, \widetilde{T}, \widetilde{W})$ is contained in $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$.

There is a natural inclusion $\mathcal{D}^{\mathbf{H}}_{\check{T}}[W] \hookrightarrow \mathcal{H}(\mathbf{H}, \widetilde{T}, \widetilde{W})$ of (sheaves of) algebras. The following result can be proved exactly as in [Gin18, Proposition 7.2.4]; one only has to use [GKV97, Proposition 2.3] in place of [Gin18, Lemma 7.1.5], and also observe that the arguments of [Lon17] generalize to the setting of descent along the map $\mathbf{H}_T/W \to \mathbf{H}_T /\!\!/ W$.

Proposition 6.17. *Let $\mathcal{F}$ be $\mathcal{D}^{\mathbf{H}}_{\check{T}}[W]$-module*[14]*. Then the action of $\mathcal{D}^{\mathbf{H}}_{\check{T}}[W]$ extends (necessarily uniquely) along the inclusion $\mathcal{D}^{\mathbf{H}}_{\check{T}}[W] \hookrightarrow \mathcal{H}(\mathbf{H}, \widetilde{T}, \widetilde{W})$ if and only if the natural map $\mathcal{O}_{\mathbf{H}_T} \otimes_{\mathcal{O}^W_{\mathbf{H}_T}} \mathcal{F}^W \to \mathcal{F}$ is an isomorphism.*

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[13]The affine Weyl group W^{aff} introduced above is very slightly different from the affine Weyl group studied in [Gin18, Section 7.2] or [Gan22b]; the affine Weyl group there is the semidirect product $W'^{\mathrm{aff}} = \check{\Lambda} \rtimes W$, where $\check{\Lambda}$ is the *root* lattice of G. When G is simply-laced, these are, of course, isomorphic; but they differ otherwise.

[14]Here, we mean a module in the usual, underived, sense of the word; but it is easy to generalize the statement to the setting of perfect $\mathcal{D}^{\mathbf{H}}_{\check{T}}[W]$-modules by induction on the length of the bounded complex.

Remark 6.18. Let us remark on a relationship to [Gan22b]. Following *loc. cit.*, let $\Gamma_{W^{\mathrm{aff}}}$ denote the ind-scheme given by the union of graphs of the affine Weyl group W^{aff} acting on $\mathbf{H}_{\widetilde{T}}$, and let $\Gamma_{\widetilde{W}}$ denote $\widetilde{W} \times^{W^{\mathrm{aff}}} \mathbf{H}_{\widetilde{T}}$. Then there are two projections $\Gamma_{\widetilde{W}} \rightrightarrows \mathbf{H}_{\widetilde{T}}$. This can be extended to a simplicial diagram $\Gamma_\bullet$ of ind-schemes. Define the stack $\mathbf{H}_{\widetilde{T}} /\!\!/ \widetilde{W}$ to be the geometric realization of $\Gamma_\bullet$. (For instance, if W is trivial, this is the quotient $\mathbf{H}_{\widetilde{T}}/\mathbb{X}_*(T)$. Similarly, if $\mathbf{H} = \mathbf{G}_a$, so that $\mathbf{H}_{\widetilde{T}} \cong \mathfrak{t} \oplus \mathbf{A}^1_\hbar$, then the specialization of the quotient $\mathbf{H}_{\widetilde{T}} /\!\!/ \widetilde{W}$ to $\hbar = 1$ agrees with the quotient $\mathfrak{t} /\!\!/ \widetilde{W}$ from [Gan22b].) In general, there is a map of stacks $\phi : \mathbf{H}_{\widetilde{T}}/\widetilde{W} \to \mathbf{H}_{\widetilde{T}} /\!\!/ \widetilde{W}$. By arguing exactly as in [Gan22b, Theorem 4.23], one can show that the pullback functor $\phi^!$ is fully faithful; and furthermore, an object of $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{T}}/\widetilde{W})$ descends[15] along ϕ if and only if the corresponding object of $\mathrm{IndCoh}(\mathbf{H}_T/W)$ descends to $\mathbf{H}_T /\!\!/ W$. Since Remark 5.2 gives an equivalence between $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{T}}/\widetilde{W})$ and $\mathcal{D}^{\mathbf{H}}_{\widetilde{T}}[W]$-mod, Proposition 6.17 can be used to obtain an equivalence between $\mathcal{H}(\mathbf{H}, \widetilde{T}, \widetilde{W})$-mod and $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{T}} /\!\!/ \widetilde{W})$.

Proposition 3.15 yields the following result due to Kostant and Kumar [KK90, KK86, Kum02], which (as we will explain momentarily) could also be seen as a consequence of results from [Gin18, Lon18, BF08]. In the discussion below, $\mathbf{H} = \mathbf{G}_a$. Note that $\mathrm{H}^*_{\widetilde{T}_c}(*; \mathbf{Q})$ is isomorphic to $\mathcal{O}_{\widetilde{\mathfrak{t}}} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$.

Theorem 6.19. *There is an isomorphism of associative $\mathbf{Q}[\hbar]$-algebras*

$$\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W}). \tag{10}$$

Here, $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\mathrm{H}^*_{\widetilde{T}_c}(*; \mathbf{Q}) \cong \mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$-algebroids.

Proof. The affine flag variety Fl_G is an ind-finite GKM space in the sense of Definition 3.12, and so we may use Proposition 3.15 to describe $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$. The GKM graph of Fl_G has set of vertices given by $\mathrm{Fl}_G^{\widetilde{T}_c} = \mathbb{X}_*(T) \rtimes W \cong \widetilde{W}$, and an edge $w \to s_{\alpha,n} w$ for each affine reflection $s_{\alpha,n} \in \widetilde{W}$. In particular, if $\mathring{\widetilde{\mathfrak{t}}}$ denotes the complement of the union of affine hyperplanes in $\widetilde{\mathfrak{t}}$, then $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$ is a subalgebra of $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G^{\widetilde{T}_c}; \mathbf{Q})|_{\mathring{\widetilde{\mathfrak{t}}}}$. The latter is isomorphic to $\mathrm{H}^{\widetilde{T}_c}_*(\widetilde{W}; \mathbf{Q})|_{\mathring{\widetilde{\mathfrak{t}}}}$, which in turn can be identified (using Proposition 5.4, for instance) with a localization of $\mathcal{D}^\hbar_{\widetilde{T}}[W]$. This localization of $\mathcal{D}^\hbar_{\widetilde{T}}[W]$ is isomorphic to $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$, so $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$ is a subalgebra of $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$.

Proposition 3.15 now gives an isomorphism between the two subsets

$$\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q}) \subseteq Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}] \supseteq \mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W}).$$

To see that this is an isomorphism of sub*algebras*, simply observe that both $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$ and $\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})$ inherit their multiplicative structure from $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$. That this is an isomorphism of Hopf $\mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$-algebroids is also elementary: for instance, the coproduct on both $\mathrm{H}^{\widetilde{T}_c}_*(\mathrm{Fl}_G; \mathbf{Q})$ and $\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})$ are inherited from the $\mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$-linear coproduct on $Q(\mathcal{O}_{\mathbf{H}_{\widetilde{T}}})[\widetilde{W}]$ coming from the diagonal on $\mathrm{Fl}_G^{\widetilde{T}_c} = \widetilde{W}$. $\square$

[15] That is, it lies in the essential image of the left adjoint $\phi^!$ to ϕ_*^{IndCoh}.

Remark 6.20. The left-hand side of Theorem 6.19 admits an obvious grading; on the right-hand side, the resulting grading on $\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})$ can be identified with that inherited from $Q(\mathcal{O}_{(\mathbf{G}_a)_{\widetilde{T}}})[\widetilde{W}]$, where the coordinates of $(\mathbf{G}_a)_{\widetilde{T}}$ are placed in weight 2.

Moreover, Theorem 6.19 holds even if $\mathbf{Q}$ is replaced by $\mathbf{Z}$ (as long as, on the right-hand side, $\mathbf{G}_a$ is viewed as defined over $\mathbf{Z}$).

Remark 6.21. Suppose W is finite. Then [GKV97, Proposition 2.3] states that *upon rationalization*, the action of $\mathcal{H}(\mathbf{G}_a, T, W)$ on $\mathcal{O}_{(\mathbf{G}_a)_T} = \mathcal{O}_{\mathfrak{t}}$ gives an isomorphism between $\mathcal{H}(\mathbf{G}_a, T, W)$ and $\mathrm{End}_{\mathcal{O}_{\mathfrak{t}}^W}(\mathcal{O}_{\mathfrak{t}})$. (This result is false without rationalization, or at least without inverting enough primes.) Its $\mathcal{O}_{\mathfrak{t}}$-linear dual is therefore $\mathcal{O}_{\mathfrak{t}} \otimes_{\mathcal{O}_{\mathfrak{t}}^W} \mathcal{O}_{\mathfrak{t}} \cong \mathcal{O}_{\mathfrak{t} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}}$. Note that this naturally admits the structure a cocommutative Hopf $\mathcal{O}_{\mathfrak{t}}$-algebroid. The analogue of Theorem 6.19 states that there is an isomorphism $\mathrm{H}_*^{T_c}(G_c/T_c; \mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, T, W)$ of (cocommutative) Hopf $\mathrm{H}^*_{T_c}(*; \mathbf{Q}) \cong \mathcal{O}_{\mathfrak{t}}$-algebroids.

Let $\mathbf{e} = \frac{1}{|W|} \sum_{w \in W} [w]$ denote the symmetrizer, viewed as an element of $\mathbf{Q}[W]$. The spherical subalgebra $\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})^{\mathrm{sph}}$ is defined to be $\mathbf{e}\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})\mathbf{e}$. The following result is now an easy consequence of Theorem 6.19.

Corollary 6.22. *There is an isomorphism of associative* $\mathbf{Q}[\hbar]$*-algebras*

$$\mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})^{\mathrm{sph}}.$$

Here, $\mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q})$ *is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf* $\mathrm{H}^*_{G \times S^1_{\mathrm{rot}}}(*; \mathbf{Q}) \cong \mathcal{O}_{\mathfrak{t}/\!\!/W \times \mathbf{A}^1_\hbar}$*-algebroids.*

Let $\mathcal{D}^\hbar_{\check{G}}$ denote the algebra of (rescaled) differential operators on $\check{G}$, and let $\check{N}_\psi \backslash \mathcal{D}^\hbar_{\check{G}} /_\psi \check{N}$ denote its bi-Whittaker reduction (that is, its two-sided Hamiltonian reduction by the left and right actions of $\check{N}$ with respect to a nondegenerate character $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$). Corollary 6.22 and [Gin18, Theorem 1.2.1] yield:

Corollary 6.23 ([BF08, Theorem 3])**.** *There is an isomorphism of associative* $\mathbf{Q}[\hbar]$*-algebras*

$$\mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q}) \cong \check{N}_\psi \backslash \mathcal{D}^\hbar_{\check{G}} /_\psi \check{N}.$$

Note that via the diagonal on $\check{G}$, the bi-Whittaker reduction $\check{N}_\psi \backslash \mathcal{D}^\hbar_{\check{G}} /_\psi \check{N}$ admits the structure of a (cocommutative) Hopf algebroid over $U_\hbar(\check{\mathfrak{g}}) /_\psi \check{N}$; by [Kos78], the latter is isomorphic to $Z(U_\hbar(\check{\mathfrak{g}})) \cong \mathrm{Sym}(\mathfrak{t}^*)^W[\hbar]$. Again, one can verify that the isomorphism of Corollary 6.23 is one of cocommutative Hopf coalgebroids over $\mathrm{H}^*_{G_c \times S^1_{\mathrm{rot}}}(*; \mathbf{Q}) \cong \mathrm{Sym}(\mathfrak{t}^*)^W[\hbar]$.

Remark 6.24. Since $\mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q})$ is Morita equivalent to $\mathrm{H}_*^{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Fl}_G; \mathbf{Q})$, Remark 6.18, Theorem 6.19, Corollary 6.22, and Corollary 6.23 together tell us that there are equivalences of categories

$$\begin{aligned}\mathrm{H}_*^{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Fl}_G; \mathbf{Q})\text{-mod} &\simeq \mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q})\text{-mod} \\ &\simeq \check{N}_\psi \backslash \mathcal{D}^\hbar_{\check{G}} /_\psi \check{N}\text{-mod} \simeq \mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})\text{-mod} \simeq \mathrm{IndCoh}(\widetilde{\mathfrak{t}}/\!\!/\widetilde{W}).\end{aligned}$$

Definition 6.25. Denote by $\mathrm{HC}^{\hbar}_{\check{G}}$ the ∞-category $\mathcal{D}^{\hbar}_{\check{G}}\text{-mod}^{\check{G}\times\check{G},\mathrm{weak}} \simeq U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{G},\mathrm{weak}}$ of Harish-Chandra bimodules. Let $\kappa_\hbar : \mathrm{HC}^{\hbar}_{\check{G}} \to U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{N},\psi)}$ denote the Kostant functor of [BF08, Section 2.3], so that it is given by the composite

$$\mathrm{HC}^{\hbar}_{\check{G}} \xrightarrow{\text{forget}} U_\hbar(\check{\mathfrak{g}})\text{-mod} \xrightarrow{\mathrm{Av}_{\check{N},\psi}} U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{N},\psi)}.$$

Note that by Skryabin's theorem (see the appendix of [Pre02]), there is an equivalence $U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{N},\psi)} \simeq \mathrm{QCoh}(\mathfrak{t}/\!\!/W \times \mathbf{A}^1_\hbar)$. Define $(\mathrm{HC}^{\hbar}_{\check{G}})_{\mathrm{reg}}$ to denote the localizing subcategory of $\mathrm{HC}^{\hbar}_{\check{G}}$ on which $\kappa_\hbar$ is conservative.

One can check that upon "setting $\hbar = 1$", the category $(\mathrm{HC}^{\hbar}_{\check{G}})_{\mathrm{reg}}$ identifies with the category $\mathcal{HC}_{\mathrm{nondeg}}$ from [Gan22a, Remark 4.22].[16] Before proceeding, we need a category-theoretic result, which follows from [Lur16, Corollary 4.7.5.3].

Proposition 6.26. *Let $\mathcal{C}^\bullet$ be an augmented cosimplicial presentable stable ∞-category. Suppose that:*

(a) For every $[n] \in \mathbf{\Delta}^+$, the face map $d^0 : \mathcal{C}^i \to \mathcal{C}^{i+1}$ admits a left adjoint $(d^0)^L$.
(b) The "Beck-Chevalley conditions" hold. That is, for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}^+$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}^{m+1} \arrow[r, "([0]\star\alpha)^*"] \arrow[d, "(d^0)^L"'] & \mathcal{C}^{n+1} \arrow[d, "(d^0)^L"] \\ \mathcal{C}^m \arrow[r, "\alpha^*"] & \mathcal{C}^n. \end{tikzcd}$$

Then the functor $\mathcal{C}^{-1} \to \mathrm{Tot}(\mathcal{C}^\bullet|_{N(\mathbf{\Delta})})$ admits a fully faithful right adjoint; moreover, the essential image of this functor identifies with the full subcategory of $\mathcal{C}^{-1}$ on which the functor $\mathcal{C}^{-1} \to \mathcal{C}^0$ is conservative.

It is my understanding that the following result is closely related to work-in-progress of Gannon and Ginzburg.

Corollary 6.27. *Recall the category $\mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; k)$ from Remark 4.7. There is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; k) \simeq (\mathrm{HC}^{\hbar}_{\check{G}})_{\mathrm{reg}}.$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; k) \to \mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(*; k)$ identifies with the functor $\kappa_\hbar : (\mathrm{HC}^{\hbar}_{\check{G}})_{\mathrm{reg}} \to \mathrm{QCoh}(\mathfrak{t}/\!\!/W \times \mathbf{A}^1_\hbar)$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; k)$ is the ∞-category of left comodules over $\mathrm{H}^{G_c\times S^1_{\mathrm{rot}}}_*(\mathrm{Gr}_G; k)$ in $\mathrm{Loc}_{G_c\times S^1_{\mathrm{rot}}}(*; k)$. The latter category can be identified with

$$\mathrm{Loc}_{G_c\times S^1_{\mathrm{rot}}}(*; k) \simeq \mathrm{QCoh}(\mathfrak{t}/\!\!/W \times \mathbf{A}^1_\hbar) \simeq U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{N},\psi)}.$$

[16] Let me note here my strong opposition to "setting $\hbar = 1$". As we have seen above, $\hbar$ arises naturally as a generator of $\mathrm{H}^2_{S^1_{\mathrm{rot}}}(*; \mathbf{Q})$, and as such, it lives in *nonzero grading*. It is therefore not sensible to set $\hbar$ to be equal to a nonzero number. A better – and in some sense equivalent – way to "set $\hbar = 1$" in a graded $\mathbf{Q}[\hbar]$-module/category $M_\hbar$ is to extract the weight zero piece of the localization $M_\hbar[\hbar^{-1}]$. Doing this procedure to $(\mathrm{HC}^{\hbar}_{\check{G}})_{\mathrm{reg}}$ will product $\mathcal{HC}_{\mathrm{nondeg}}$.

Let us denote this category by $\mathcal{C}^0$. Just as in Skryabin's theorem, there is an equivalence

$$\check{N}_\psi\backslash\mathcal{D}^\hbar_{\check{G}}/_\psi\check{N}\text{-mod} \simeq \mathcal{D}^\hbar_{\check{G}}\text{-mod}^{(\check{N}\times\check{N},\psi\times\psi)}.$$

The Hopf algebroid structure on the pair $(U_\hbar(\check{\mathfrak{g}})/_\psi\check{N}, \check{N}_\psi\backslash\mathcal{D}^\hbar_{\check{G}}/_\psi\check{N})$ defines a cosimplicial diagram

$$\mathcal{C}^0 \rightrightarrows \mathcal{C}^1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{C}^1\otimes_{\mathcal{C}^0}\mathcal{C}^1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

The preceding discussion implies that its totalization computes the ∞-category of comodules over the cocommutative Hopf algebroid $(U_\hbar(\check{\mathfrak{g}})/_\psi\check{N}, \check{N}_\psi\backslash\mathcal{D}^\hbar_{\check{G}}/_\psi\check{N})$. Corollary 6.23 gives an isomorphism $\mathrm{H}_*^{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G;k)\cong\check{N}_\psi\backslash\mathcal{D}^\hbar_{\check{G}}/_\psi\check{N}$ of cocommutative Hopf algebroids over $U_\hbar(\check{\mathfrak{g}})/_\psi\check{N}\cong\mathrm{H}^*_{G_c\times S^1_{\mathrm{rot}}}(*;k)$, and so the totalization of the above cosimplicial diagram is equivalent to $\mathrm{Loc}^{\mathrm{gr}}_{G_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G;k)$.

There are equivalences

$$\mathcal{C}^0 = U_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{N},\psi)} \simeq \mathcal{D}^\hbar_{\check{G}}\text{-mod}^{(\check{G},\mathrm{weak}),(\check{N},\psi)},$$
$$\mathcal{C}^1 = \mathcal{D}^\hbar_{\check{G}}\text{-mod}^{(\check{N}\times\check{N},\psi\times\psi)} \simeq \mathcal{C}^0\otimes_{\mathrm{HC}^\hbar_{\check{G}}}\mathcal{C}^0 \simeq \mathrm{End}_{\mathrm{HC}^\hbar_{\check{G}}}(\mathcal{C}^0),$$

which refine to give an equivalence of cosimplicial ∞-categories

$$\mathcal{C}^\bullet \simeq (\mathcal{C}^0)^{\otimes_{\mathrm{HC}^\hbar_{\check{G}}}\bullet+1}.$$

Observe that $\mathcal{C}^\bullet$ extends to an augmented cosimplicial ∞-category $\widetilde{\mathcal{C}^\bullet}$ by setting $\mathcal{C}^{-1}=\mathrm{HC}^\hbar_{\check{G}}$, where the functor $\mathcal{C}^{-1}\to\mathcal{C}^0$ induced by the unique morphism $[-1]\to[0]$ in $\mathbf{\Delta}^+$ is given by the Kostant functor $\kappa_\hbar$. It is straightforward to check that both conditions in Proposition 6.26 hold for $\widetilde{\mathcal{C}^\bullet}$, so we find that $\mathrm{Tot}(\mathcal{C}^\bullet)$ is equivalent the localizing subcategory $(\mathrm{HC}^\hbar_{\check{G}})_{\mathrm{reg}}$ of $\mathcal{C}^{-1}=\mathrm{HC}^\hbar_{\check{G}}$ spanned those objects on which the Kostant functor is conservative. ∎

Remark 6.28. One can also deduce Corollary 6.27 from [BF08], as discussed in [Lon18]. This, combined with [Gin18, Theorem 1.2.1], gives an alternative proof of Theorem 6.19 assuming the results of [BF08]. However, as mentioned in the introduction to this section, we specifically do *not* want to appeal to [BF08], since it does not have analogues in the K-theoretic or elliptic settings.

Remark 6.29. There is a Kostant functor

$$\kappa_\hbar : \mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})}\to U_\hbar(\check{\mathfrak{t}})\text{-mod}\simeq\mathrm{QCoh}(\mathfrak{t}\times\mathbf{A}^1_\hbar)$$

given by the composite

$$\begin{aligned}\mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})} &\xrightarrow{\text{forget}} \mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{T},\mathrm{weak})}\\ &\xrightarrow{\mathrm{Av}_{\check{N},\psi}} \mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{T},\mathrm{weak}),(\check{N},\psi)}\\ &\simeq \mathrm{DMod}_\hbar(\check{T})^{(\check{T},\mathrm{weak})}\simeq U_\hbar(\check{\mathfrak{t}})\text{-mod}.\end{aligned}$$

Using $\kappa_\hbar$, one can define an ∞-category $\mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})}_{\mathrm{reg}}$. Just as in Corollary 6.27, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G;k)\simeq\mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})}_{\mathrm{reg}}. \tag{11}$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{T_c\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G;k)\to\mathrm{Loc}^{\mathrm{gr}}_{T_c\times S^1_{\mathrm{rot}}}(*;k)$ identifies with the Kostant functor $\mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})}_{\mathrm{reg}}\to\mathrm{QCoh}(\mathfrak{t}\times\mathbf{A}^1_\hbar)$. The arguments in this case are slightly more subtle, though: the equivariant homology $\mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Gr}_G; \mathbf{Q})$ no longer admits an algebra structure, but it still does admit the structure of a cocommutative coalgebra over $\mathrm{H}^*_{\widetilde{T}_c}(*; \mathbf{Q})$. In fact, $\mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Gr}_G; \mathbf{Q})$ is isomorphic as a $(\mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{Q}), \mathrm{H}_*^{\widetilde{T}_c}(\mathrm{Fl}_G; \mathbf{Q}))$-bimodule to the $(\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W}), \mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W}))$-bimodule $\mathcal{H}(\mathbf{G}_a, \widetilde{T}, \widetilde{W})\mathbf{e}$. (This bimodule is denoted $\mathbb{M}_\hbar$ in [Gin18, Theorem 8.1.2].)

Remark 6.30. Just as Corollary 2.11 can be viewed as a "generic" version of the Arkhipov-Bezrukavnikov-Ginzburg [ABG04] equivalence

$$\mathrm{Shv}^c_I(\mathrm{Gr}_G; k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}/\check{G}),$$

the equivalence Corollary 6.27 can be viewed as a "generic" version of the Bezrukavnikov-Finkelberg [BF08] equivalence

$$\mathrm{Shv}^c_{G(\mathcal{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G; k) \simeq \mathrm{HC}^\hbar_{\check{G}}.$$

Similarly, the equivalence of (11) can be viewed as a "generic" version of the quantized Arkhipov-Bezrukavnikov-Ginzburg equivalence

$$\mathrm{Shv}^c_{I \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G; k) \simeq \mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G} \times \check{T}, \mathrm{weak})}.$$

Unfortunately, I am not aware of a reference for this final statement, but it can be deduced easily from the work of Ginzburg-Riche in [GR15].

7. The K-theoretic story

Our goal in this section is to prove an analogue of Corollary 6.8, albeit with coefficients in $k = \mathrm{KU}$. Note that in this case, $\mathcal{M}_{T,0} \cong T$. To do so, we need an analogue of Definition 6.1 and constructions surrounding it. Recall that the group G (over $\mathbf{C}$) is connected, almost simple, and simply-laced. We will also fix an algebraically closed field F,over which the Langlands dual group $\check{G}$ will live. When dealing with the algebraic geometry (as opposed to the topology) of G, we will also view it as living over F; since G is simply-laced, it is isogenous to $\check{G}$.

Definition 7.1. Let G^{sc} denote the simply-connected cover of G, and let $f \in G^{\mathrm{sc}}$ be a principal nilpotent element as defined in [Ste65, Theorem 4.6]. We will denote its image under the map $G^{\mathrm{sc}} \to G$ also by f. Then the map $\mathbf{G}_a \to G$ corresponding to e factors through the map $\mathbf{G}_a = B \to \mathrm{SL}_2$; we will denote the image of the standard generator $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ under the map $\mathrm{SL}_2 \to G$ by $e \in G$. Let $Z_G(e)^\circ$ be the connected component of the identity in the centralizer of e in G. Define the *multiplicative Kostant slice* $\mathcal{S}_\mu$ by $f \cdot Z_G(e)^\circ \subseteq G$. Since G is assumed to be simply-connected, the composite

$$\mathcal{S}_\mu \to G \to G /\!\!/ G \cong G /\!\!/ \check{G} \cong T /\!\!/ W$$

is an isomorphism. We will often denote the inclusion of the Kostant slice by $\kappa : T /\!\!/ W \to G$.

The *multiplicative Grothendieck-Springer resolution* $\widetilde{\check{G}}$ is defined as

$$\widetilde{\check{G}} = B \times^{\check{B}} \check{G},$$

where $\check{B}$ acts on B by conjugation. (This makes sense thanks to the assumption that G is simply-laced.) There is a natural map $\widetilde{\check{G}} \to G$, given by the conjugation action of $\check{G}$ on B. Let $\widetilde{\mathcal{S}}_\mu$ denote the fiber product $\widetilde{\mathcal{S}}_\mu \times_G \widetilde{\check{G}}$, so that the composite

$$\widetilde{\mathcal{S}}_\mu \to \widetilde{\check{G}} \to T$$

is an isomorphism; we will denote the inclusion of $\widetilde{\mathcal{S}}_\mu$ as a map $\kappa : \widetilde{\mathcal{S}}_\mu \cong T \to \widetilde{\check{G}}$.

As with the additive Kostant slice, we will only care about the composite $T \to \widetilde{\check{G}} \to \widetilde{\check{G}}/\check{G}$ below, so we will also denote it by κ. If we identify $\widetilde{\check{G}}/\check{G} \cong B/\check{B}$, then the map κ admits a simple description: it is the composite $f \cdot T \to B \to B/\check{B}$.

Definition 7.2. The stabilizer (inside $\check{G}$) of the multiplicative Kostant slice $\mathcal{S}_\mu \subseteq G^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{G} \times \mathcal{S}_\mu$, and will be denoted by $\check{J}_\mu$. It will be called the *multiplicative regular centralizer group scheme*; if we wish to emphasize the dependence on G, we will denote it by $\check{J}_\mu(G)$. Note that since the composite $\mathcal{S}_\mu \to G^{\mathrm{reg}} \to G /\!\!/ \check{G}$ is an isomorphism, we may identify

$$\check{J}_\mu \cong \mathcal{S}_\mu \times_{G/\check{G}} \mathcal{S}_\mu.$$

Similarly, the stabilizer (inside $\check{G}$) of the multiplicative Kostant slice $\widetilde{\mathcal{S}}_\mu \subseteq \widetilde{\check{G}}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{G} \times \widetilde{\mathcal{S}}_\mu$, and will be denoted by $\widetilde{\check{J}}_\mu$. Since $\widetilde{\mathcal{S}}_\mu \cong \mathcal{S}_\mu \times_G \widetilde{\check{G}}$, we may identify

$$\widetilde{\check{J}}_\mu \cong \check{J}_\mu \times_{\mathcal{S}_\mu} \widetilde{\mathcal{S}}_\mu \cong (f \cdot T) \times_{B/\check{B}} (f \cdot T).$$

Theorem 7.3. *There is an isomorphism of group schemes over $f \cdot T \cong T \cong \mathcal{M}_{T,0}$:*

$$\mathrm{Spec}(\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee \otimes_{\mathbf{Z}} F) \cong (f \cdot T) \times_{B/\check{B}} (f \cdot T).$$

Just as in Theorem 6.3, the proof of Theorem 7.3 will rely on two lemmas.

Lemma 7.4. *The projection map $\widetilde{\check{J}}_\mu \to f \cdot T$ (onto either factor) is flat.*

Proof. Like in the proof of Lemma 6.4, it suffices, by miracle flatness, to show that the fibers of the map $\widetilde{\check{J}}_\mu \to f \cdot T$ have dimension exactly $\mathrm{rank}(\check{G})$. The fiber of this map over $f \cdot x \in f \cdot T$ is the scheme

$$Y = \{(g, y) \in \check{B} \times T | \mathrm{Ad}_g(fy) = fx\}.$$

Observe that the image of $\mathrm{Ad}_g(fy)$ and fx (viewed as elements of B) under the map $B \to T$ are y and x; so $y = x$ in T, which means that Y is isomorphic to the centralizer $Z_{\check{B}}(fx)$. The dimension estimate is equivalent to the claim that fx is a regular element of G, since this means that its centralizer has minimal dimension (namely, the rank of G, which is also the rank of $\check{G}$). The desired regularity of fx follows from the discussion in [Ste65, Remark 4.7]. (Note that, as mentioned in *loc. cit.*, the specific choice of the regular unipotent f is crucial for the regularity of fx.) $\square$

Notation 7.5. Let α be a root of $\check{G}$. Say that a point $x \in T$ is *α-generic* if $x(h_\beta) \neq 1$ for all roots $\beta \neq \alpha$. This implies that the centralizer $Z_{\check{G}}(x)$ has semisimple rank at most 1. Let $T_{\alpha\text{-reg}}$ denote the α-regular locus. Observe that $T_{\mathrm{reg}} = \bigcup_{\alpha \in \Phi} T_{\alpha\text{-reg}} \subseteq T$ is open, with complement of codimension 2.

The proof of the next result is exactly as in Lemma 6.6.

Lemma 7.6. *There is an isomorphism*

$$\widetilde{\check{J}}_\mu(\check{G})|_{T_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{J}}_\mu(Z_{\check{G}}(x)^\circ)|_{T_{\alpha\text{-reg}}}, \tag{12}$$

where $Z_{\check{G}}(x)$ is the centralizer of some $x \in T_{\alpha\text{-reg}}$ which lies on the α-hyperplane, and $Z_{\check{G}}(x)^\circ$ denotes the connected component of the identity.

Proof of Theorem 7.3. The argument of Theorem 6.3 reduces us to checking that the isomorphism of Theorem 7.3 holds if G has semisimple rank 1, i.e., is the product of a torus with one of GL_2, SL_2, or PGL_2. Again, it is easy to match up the contributions from the toral factors, so we will assume that G is either GL_2, SL_2, or PGL_2. In this case, we can even replace F by $\mathbf{Z}$.

- When $G = \mathrm{GL}_2$, we may identify $\widetilde{\check{J}}_\mu$ with the centralizer (in $\check{B}$) of $\left(\begin{smallmatrix} x & 0 \\ x & y \end{smallmatrix}\right)$. It is easy to compute that $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & y \end{smallmatrix}\right)$ if and only if $c = \frac{a-d}{x-y} \cdot x$, meaning that

$$\widetilde{\check{J}}_\mu \cong \mathrm{Spec}\, \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

  The coproduct sends $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$. The same argument as in Theorem 6.3 implies that

$$\mathrm{KU}_*^{T_c}(\Omega \mathrm{GL}_2) \cong \mathbf{Z}[\beta^{\pm 1}, x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

  The map induced on T-equivariant KU-homology by the inclusion $T^2 \to \mathrm{GL}_2$ is simply given by the inclusion of the subalgebra $\mathbf{Z}[\beta^{\pm 1}, x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}]$.

The coproduct on this subalgebra (and hence, on $\mathrm{KU}_*^{T_c}(\Omega\mathrm{GL}_2)$) is determined by the formulas $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{T_c}(\Omega\mathrm{GL}_2)$ is isomorphic to $\widetilde{J}_\mu$ as group schemes over $\operatorname{Spec} \pi_0\mathrm{KU}_{T_c} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$, as desired.

- When $G = \mathrm{SL}_2$, we may identify $\widetilde{\check{J}}_\mu$ with the centralizer (in $\check{B} \subseteq \mathrm{PGL}_2$) of $\left(\begin{smallmatrix} x & 0 \\ x & x^{-1} \end{smallmatrix}\right)$. An element $\left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \check{B} \subseteq \mathrm{PGL}_2$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & x^{-1} \end{smallmatrix}\right)$ if and only if $c = \frac{a-1}{x-x^{-1}} \cdot x$. Therefore,

$$\widetilde{\check{J}}_\mu \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-1}{x-x^{-1}}];$$

the coproduct sends $a \mapsto a \otimes a$.

Next, there is an isomorphism

$$\mathrm{KU}_*^{S^1}(\Omega\mathrm{SL}_2) \cong \mathbf{Z}[\beta^{\pm 1}, x^{\pm 1}, a^{\pm 1}, \tfrac{a-1}{x^2-1}].$$

This is proved exactly as in Theorem 6.3; the role of the class $2x$ is now played by the Chern class $x^2 - 1 \in \pi_0\mathrm{KU}_{S^1}$ of the weight 2 representation of S^1. (Recall that the action of S^1 on $G_c \cong \mathrm{SU}(2) \cong S^3$ exhibits it as the one-point compactification of the trivial 1-dimensional representation summed with the weight 2 representation of S^1 on $\mathbf{C}$.) The map induced on T-equivariant KU-homology by the inclusion $S^1 \to \mathrm{SU}(2)$ of the maximal torus is simply given by the inclusion of the subalgebra $\mathbf{Z}[\beta^{\pm 1}, x^{\pm 1}, a^{\pm 1}]$. The coproduct on this subalgebra (and hence, on $\mathrm{KU}_*^{S^1}(\Omega\mathrm{SL}_2)$) is determined by the formula $a \mapsto a \otimes a$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{S^1}(\Omega\mathrm{SL}_2)$ is isomorphic to $\widetilde{\check{J}}_\mu$ as group schemes over $\operatorname{Spec} \pi_0\mathrm{KU}_{S^1} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}]$, as desired.

- When $G = \mathrm{PGL}_2$, we may identify $\widetilde{\check{J}}_\mu$ with the centralizer (in $\check{B} \subseteq \mathrm{SL}_2$) of $\left(\begin{smallmatrix} x & 0 \\ x & 1 \end{smallmatrix}\right)$. An element $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right) \in \check{B} \subseteq \mathrm{SL}_2$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & 1 \end{smallmatrix}\right)$ if and only if $c = \frac{a-a^{-1}}{x-1} \cdot x$. Therefore,

$$\widetilde{\check{J}}_\mu \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-1}];$$

the coproduct sends $a \mapsto a \otimes a$. Again, as in the preceding cases, there is an isomorphism

$$\mathrm{KU}_*^{S^1}(\Omega\mathrm{PGL}_2) \cong \mathbf{Z}[\beta^{\pm 1}, x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-1}],$$

where the coproduct sends $a \mapsto a \otimes a$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{S^1}(\Omega\mathrm{PGL}_2)$ is isomorphic to $\widetilde{\check{J}}_\mu$ as group schemes over $\operatorname{Spec} \pi_0\mathrm{KU}_{S^1} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}]$, as desired. □

Remark 7.7. Just for posterity, let us record a more canonical variant of the calculation above for $\check{G} = \mathrm{SL}_2$, which does not require picking a Borel subgroup (i.e., which does not involve identifying $\widetilde{\check{G}}/\check{G} \cong B/\check{B}$). If $\lambda \in \mathbf{G}_m$, we denote $\lambda + \lambda^{-1} \in \mathbf{A}^1$ by $f(\lambda)$. The Kostant slice $\kappa : \check{T} \cong \mathbf{G}_m \to \widetilde{\mathrm{SL}}_2$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \begin{pmatrix} f(\lambda) - 1 & f(\lambda) - 2 \\ 1 & 1 \end{pmatrix}, \ \ell = [\lambda - 1 : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ: the key point is the conic relation

$$2\lambda = f(\lambda) - \sqrt{f(\lambda)^2 - 4}.$$

Indeed, this calculation of $\kappa(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}_m \cong \check{T} \arrow[r, "\kappa"] \arrow[d, "\lambda\mapsto f(\lambda)"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \check{T}/\!\!/W \arrow[r, "\kappa", "\lambda\mapsto\left(\begin{smallmatrix} \lambda-1 & \lambda-2 \\ 1 & 1 \end{smallmatrix}\right)"'] & \mathrm{SL}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we directly compute that $\mathrm{Ad}_g(x) = x$ if and only if $b = c(f(\lambda)-2)$ and $a-d = (f(\lambda)-2)c$, in which case g also preserves ℓ. Therefore, $g = \left(\begin{smallmatrix} (f(\lambda)-2)c+d & (f(\lambda)-2)c \\ c & d \end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$d^2 + c(f(\lambda) - 2)(d - c) = 1.$$

Both x and g can be simultaneously diagonalized (if $f(\lambda) \neq \pm 2$); note that $\lambda + \lambda^{-1}$ is an eigenvalue of x. If t is an eigenvalue of g, then we have $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$ and $d = \frac{t^2\lambda+1}{t(\lambda+1)}$. When k is not of characteristic 2, this shows that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong k[\lambda^{\pm 1}, t^{\pm 1}, \tfrac{t-t^{-1}}{\lambda-\lambda^{-1}}].$$

This in turn implies that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{PGL}_2} \mathbf{G}_m \cong k[\lambda^{\pm 1}, t^{\pm 2}, \tfrac{t^2-1}{\lambda-\lambda^{-1}}],$$

as desired.

Corollary 7.8. *There is an F-linear equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}).$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KU}) \to \mathrm{Loc}^{\mathrm{gr}}_{T_c}(*; \mathrm{KU})$ *identifies with the pullback functor* $\kappa^* : \mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}) \to \mathrm{QCoh}(T)$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KU})$ is equivalent to the category of comodules over $\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee = \mathrm{KU}^T_0(\mathrm{Gr}_G)$ in the category of $\pi_0\mathrm{KU}_{T_c}$-modules. By Theorem 7.3, it can be identified the category of quasicoherent sheaves on the quotient stack $(f \cdot T)/\widetilde{\check{J}}_\mu$. We may view $\widetilde{\check{J}}_\mu$ as a closed subgroup scheme of the constant group scheme $\check{B} \times (f \cdot T)$. This gives an isomorphism

$$(f \cdot T)/\widetilde{\check{J}}_\mu \cong \check{B}\backslash(\check{B} \times (f \cdot T))/\widetilde{\check{J}}_\mu.$$

It follows from Steinberg's work in [Ste65] that the $\check{B}$-orbit of $f \cdot T$ inside B is precisely the regular locus B^{reg}. Since $\widetilde{\check{J}}_\mu$ is definitionally the stabilizer of $f \cdot T \subseteq B$, the quotient $(\check{B} \times (f \cdot T))/\widetilde{\check{J}}_\mu$ is isomorphic to B^{reg}; so there is an isomorphism $(f \cdot T)/\widetilde{\check{J}}_\mu \cong B^{\mathrm{reg}}/\check{B}$. To finish, note that $\widetilde{\check{G}}^{\mathrm{reg}}/\check{G} \cong B^{\mathrm{reg}}/\check{B}$. $\square$

Remark 7.9. It can be shown that if G is has torsion-free fundamental group, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G;\mathrm{KU})\otimes_{\mathbf{Z}} F\simeq \mathrm{QCoh}(G^{\mathrm{reg}}/\check{G}).$$

Here, the left-hand side is defined (just as in Section 4) to be the ∞-category $\mathrm{coLMod}_{\pi_0(\mathcal{F}_G(\mathrm{Gr}_G)^\vee)}(\mathrm{QCoh}(T/\!\!/W))$. Note that this is a sensible definition since $\pi_*\mathcal{F}_G(\mathrm{Gr}_G)^\vee$ is concentrated in even degrees. Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G;\mathrm{KU})\to\mathrm{Loc}^{\mathrm{gr}}_{G_c}(*;\mathrm{KU})$ identifies with the pullback functor κ^* : $\mathrm{QCoh}(G^{\mathrm{reg}}/\check{G})\to\mathrm{QCoh}(T/\!\!/W)$. The proof of the displayed equivalence is quite similar to that of Corollary 7.8, and in fact can be deduced from it using the observation that $\pi_0(\mathcal{F}_G(\mathrm{Gr}_G)^\vee)=\pi_0(\mathcal{F}_T(\mathrm{Gr}_G)^\vee)^W$ and that the natural map $\widetilde{\check{G}}^{\mathrm{reg}}\to G^{\mathrm{reg}}$ is a (ramified) W-cover. The first statement uses that G has torsion-free fundamental group, and the second is a multiplicative version of Grothendieck-Springer theory.

Remark 7.10. In [Dev24, Section 3.7], we study a variant of Corollary 7.8, where KU is replaced by *connective* complex K-theory ku; that is, $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathrm{KU})$ is replaced by $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;\mathrm{ku})$. On the Langlands dual side, this has the effect of replacing $\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}$ by the 1-parameter family over $\mathrm{Spec}(\pi_*(\mathrm{ku}))/\mathbf{G}_m\cong\mathbf{A}^1/\mathbf{G}_m$ whose generic fiber is $\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}$, and whose special fiber is $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G}$.

Remark 7.11. There is a variant of Corollary 7.8 if G is not simply-laced, but it is more complicated to state. Let us just give the analogue of Theorem 7.3. Suppose G is not simply-laced, and let T be a maximal torus of G; then $\check{\mathfrak{g}}$ is the fixed point subalgebra $\check{\mathfrak{h}}^\tau$ of an finite-order outer automorphism τ of a simply-laced Lie algebra $\check{\mathfrak{h}}$. Let H denote the simply-connected simply-laced group corresponding to the Langlands dual $\mathfrak{h}$, and let T_H denote its maximal torus. Then we may identify the fixed subset $\mathbb{X}^*(T')^\tau$ with $\mathbb{X}^*(T)$. If n denotes the order of τ, there is an action of $\mathbf{Z}/n$ on $T[\![t]\!]$, $G[\![t]\!]$, and $G(\!(t)\!)$, given by τ on T and G, and $t\mapsto\zeta_n\tau$ for a primitive nth root of unity ζ_n. The appropriate replacement of $\pi_0\mathcal{F}_T(\mathrm{Gr}_G)^\vee$ in this case is $\pi_0\mathcal{F}_{T[\![t]\!]^{\mathbf{Z}/n}}(G_{\mathrm{ad}}(\!(t)\!)^{\mathbf{Z}/n}/G_{\mathrm{ad}}[\![t]\!]^{\mathbf{Z}/n})^\vee$. The analogue of Theorem 7.3 (see [FT19, Theorem 3.9]) states that this algebra is isomorphic to the stabilizer $\mathcal{S}_\mu\times_{\check{G}/\check{G}}\mathcal{S}_\mu$.

There is *another* choice of slice when G is simply-connected; the calculation of Theorem 7.3 continues to hold for it, too, as we now illustrate in the example of SL_2.

Definition 7.12 (Steinberg slice)**.** Let G be a simply-connected semisimple algebraic group. Given $w\in W$, let $N_w=N\cap w^{-1}N^-w$, so that $N_w=\prod_{\alpha\in\Phi_w}U_\alpha$, where Φ_w is the set of roots made negative by w. Let $w=\prod_{\alpha\in\Delta}s_\alpha\in W$ be a Coxeter element, and let $\dot{w}$ be a lift of w to $N_G(T)$. Define the Steinberg slice $\Sigma=\dot{w}N_w\subseteq G$. Then [Ste65] proved/stated that the composite $\Sigma\to G\to G/\!\!/G\cong T/\!\!/W$ is an isomorphism. Let $\widetilde{\Sigma}$ denote the fiber product $\Sigma\times_G\widetilde{\check{G}}$, so that the composite $\widetilde{\Sigma}\to\widetilde{\check{G}}\to T$ is an isomorphism. We will denote the inclusion of $\widetilde{\Sigma}$ by $\sigma:T\to\widetilde{\check{G}}$.

Observation 7.13. We will illustrate the calculation of $T\times_{\widetilde{\check{G}}/\check{G}}T$ (with T mapping to $\widetilde{\check{G}}$ by σ) when $G=\mathrm{SL}_2$. View a point in $\widetilde{\check{G}}$ as a pair $(x\in\mathrm{SL}_2,\ell\subseteq\mathbf{C}^2)$ such that x preserves ℓ. The Steinberg slice $\sigma : \mathbf{G}_m \to \widetilde{\mathrm{SL}}_2$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \begin{pmatrix} \lambda + \lambda^{-1} & -1 \\ 1 & 0 \end{pmatrix},\ \ell = [\lambda : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ. This calculation of $\sigma(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{G}_m \cong \check{T} \arrow[r, "\sigma"] \arrow[d, "\lambda\mapsto\lambda+\lambda^{-1}"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\
\mathbf{A}^1 \cong \check{T}/\!\!/W \arrow[r, "\sigma", "\lambda\mapsto\left(\begin{smallmatrix}\lambda & -1 \\ 1 & 0\end{smallmatrix}\right)"'] & \mathrm{SL}_2.
\end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, one can directly compute that g commutes with $\left(\begin{smallmatrix} \lambda+\lambda^{-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$ if and only if $a = c(\lambda + \lambda^{-1}) + d$ and $b = -c$. Therefore, $g = \left(\begin{smallmatrix} c(\lambda+\lambda^{-1})+d & -c \\ c & d \end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$c^2 + d^2 + cd(\lambda + \lambda^{-1}) = 1.$$

As long as $\lambda \neq \pm 1$, both x and g can be simultaneously diagonalized by $\left(\begin{smallmatrix} \lambda & \lambda^{-1} \\ 1 & 1 \end{smallmatrix}\right)$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} c\lambda+d & 0 \\ 0 & c\lambda^{-1}+d \end{smallmatrix}\right)$. If $t = c\lambda + d$, then $c\lambda^{-1} + d = t^{-1}$ by the above determinant relation. We also have that $a = t - \frac{\lambda(t-t^{-1})}{\lambda-\lambda^{-1}}$ and $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$. This shows that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 1}, \tfrac{t-t^{-1}}{\lambda-\lambda^{-1}}],$$

and hence that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{PGL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 2}, \tfrac{t^2-1}{\lambda-\lambda^{-1}}],$$

as desired.

Theorem 7.3 has several applications. Here is one, following the same proof as in Proposition 6.13; it gives a *multiplicative* version of the Gelfand-Graev action on the affine closure $\overline{T^*(\check{G}/\check{N})}$:

Proposition 7.14 (Multiplicative Gelfand-Graev action)**.** *The natural action of $\check{G} \times \check{T}$ on the affine closure $\overline{\check{G} \times^{\check{N}} B}$ extends to an action of $\check{G} \times (W \rtimes \check{T})$, where W is the Weyl group.*

Unlike with Proposition 6.13, Proposition 7.14 does require G to be simply-laced; otherwise $\overline{\check{G} \times^{\check{N}} B}$ would not even be well-defined. The moment map $\overline{\check{G} \times^{\check{N}} B} \to G$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd}
\tilde{\check{G}} \arrow[r, hook] \arrow[dr] & \overline{\check{G} \times^{\check{N}} B}/\check{T} \arrow[d] \\
& G
\end{tikzcd}$$

which relates $\overline{\check{G} \times^{\check{N}} B}$ to the multiplicative Grothendieck-Springer resolution; and via this diagram, the multiplicative Gelfand-Graev action is closely related to the Weyl action in trigonometric/multiplicative Springer theory.

Example 7.15. Let us make the above action explicit in the example of $\check{G} = \mathrm{SL}_2$ (so $W = \mathbf{Z}/2$). The group B in this case is contained in PGL_2, and can be chosen to be represented by matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$. The action of $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \check{N}$ on $\check{G} \times B$ sends

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & an+b \\ c & cn+d \end{smallmatrix}\right), \ \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x & y-n(x-1) \\ 0 & 1 \end{smallmatrix}\right).$$

As explained in [Dev24, Remark 5.1.19], this means that the $\mathbf{G}_a$-action fixes a, c, x, $B := ay + (x-1)b$, and $D = cy + (x-1)d$. There is a single relation between these classes, given by

$$aD - cB = x - 1.$$

Let us relabel these variables so that $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $v = (v_1, v_2) = (D, -B)$. Since x must be invertible, it follows that the affine closure $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B}$ is given by the complement of the hypersurface $1 + \langle u, v\rangle$ in $T^*(\mathbf{A}^2)$. This is Van den Bergh's multiplicative quiver variety $\mathcal{B}(U, V)$ from [Van08], specialized to the case when the vector spaces U, V are $\mathbf{A}^2, \mathbf{A}^1$. An elementary analysis as in Example 6.14 shows that the $\mathbf{Z}/2$-action of Proposition 7.14 is given on $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B} \subseteq T^*(\mathbf{A}^2)$ by the formula

$$(\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right), (v_1, v_2)) \mapsto \left(\tfrac{1}{1+\langle u,v\rangle}\left(\begin{smallmatrix} -v_2 \\ v_1 \end{smallmatrix}\right), (u_2, -u_1)\right).$$

In particular, it can be viewed as a multiplicative version of the symplectic Fourier transform.

Remark 7.16. The multiplicative symplectic Fourier transform of Example 7.15 is related to another, more geometric, Fourier-type transform, as we now describe. Let ℓ be a (complex) line. Recall from [Bei87] that the (1-)category $\mathrm{Perv}(\ell)$ of perverse sheaves on ℓ with respect to the stratification by $0 \in \ell$ and its complement is equivalent to the category of diagrams of the form

$$X \underset{u}{\overset{v}{\rightleftarrows}} Y \tag{13}$$

with X and Y being vector spaces, such that $\mathrm{id}_Y + uv$ (and therefore $\mathrm{id}_X + vu$) is invertible. This equivalence sends $\mathcal{F} \in \mathrm{Perv}(\ell)$ to its spaces of nearby and vanishing cycles at $0 \in \ell$ (and the maps u, v arise via monodromy). The Fourier-Sato transform (see [KS90, Definition 3.7.8]) gives an equivalence $\mathrm{Perv}(\ell) \to \mathrm{Perv}(\ell^*)$, and one can check that it sends an object (13) to the object

$$Y \underset{-v}{\overset{u(\mathrm{id}+vu)^{-1}}{\rightleftarrows}} X.$$

Example 7.15 defines a morphism from $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} B}$ to the moduli of isomorphism classes of objects of $\mathrm{Perv}(\ell)$ (where $X = \mathbf{A}^2$ and $Y = \mathbf{A}^1$); this morphism intertwines the multiplicative symplectic Fourier transform with the Fourier-Sato transform.

We will now discuss the question of loop-rotation equivariance. Recall from Definition 6.15 the algebra $\mathcal{H}(\mathbf{H}, T, W)$ associated to a 1-dimensional group scheme $\mathbf{H}$ over a field F and a root system with torus T and Weyl group W. In the following discussion, we will set $\mathbf{H} = \mathbf{G}_m$, so that $\mathbf{H}_T = T$; we will also write q to denote the standard character of S^1_{rot}, so that $\pi_0\mathrm{KU}_{S^1_{\mathrm{rot}}} \cong \mathbf{Z}[q^{\pm 1}]$. Exactly the same argument as in Theorem 6.19 shows the following result; here, G does not need to be simply-laced.

Theorem 7.17. *There is an isomorphism of associative $\mathbf{Z}[q^{\pm 1}]$-algebras*

$$\pi_0\mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee \cong \mathcal{H}(\mathbf{G}_m, \widetilde{T}, \widetilde{W}). \tag{14}$$

Here, $\pi_0\mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\pi_0\mathrm{KU}_{\widetilde{T}_c} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$-algebroids.

Remark 7.18. Recall the quotient $\widetilde{T}/\!\!/\widetilde{W}$ from Remark 6.18. The discussion therein combined with Theorem 7.17 gives an equivalence of categories

$$\pi_0\mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee\text{-mod} \simeq \mathcal{H}(\mathbf{G}_m, \widetilde{T}, \widetilde{W})\text{-mod} \simeq \mathrm{IndCoh}(\widetilde{T}/\!\!/\widetilde{W}).$$

It follows, via the argument of Corollary 6.27, that $\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{T}_c}(\mathrm{Fl}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F$ is equivalent to the quotient of $\mathrm{QCoh}(\widetilde{T})$ by the action of $\mathrm{IndCoh}(\widetilde{T}/\!\!/\widetilde{W})$.

Assume, again, that G is simply-laced. Just as in Section 6, one would like to use Theorem 7.17 to prove analogues of Corollary 6.27 and (11). Namely, we expect that $\mathrm{Loc}^{\mathrm{gr}}_{T_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathrm{KU})$ can be identified with a certain localization of the *quantum* version $\mathcal{O}^{\mathrm{univ},q}_{\check{G}}$ of $\mathrm{DMod}_\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},\mathrm{weak})}$; the category $\mathcal{O}^{\mathrm{univ},q}_{\check{G}}$ itself is described in [KS20, Definition 4.24]. Similarly, we expect that $\mathrm{Loc}^{\mathrm{gr}}_{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathrm{KU})$ can be identified with a certain localization of the *quantum* version $\mathrm{HC}^q_{\check{G}}$ of $\mathrm{HC}^\hbar_{\check{G}}$; the category $\mathrm{HC}^q_{\check{G}}$ is described in [KS20, Definition 2.24]. The main sticking point to proving these results is that we do not have a q-analogue of [Gin18, Theorem 1.2.1] (which would give a q-analogue of Corollary 6.23); I am currently exploring this direction of research. This is related to [FT19, Conjecture 3.17].

We now turn to the question of the analogue of Corollary 7.8 if KU is replaced by *real* K-theory KO. (Recall the definition of $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KO})$ from Definition 4.12.) We begin by constructing a $\mathbf{Z}/2$-action on $\widetilde{\check{G}}/\check{G} \cong B/\check{B}$.

Lemma 7.19. *There is a map $\gamma : T \to \check{B}$ such that if $x \in T$, then $\mathrm{Ad}_{\gamma(x)}$ sends $(fx)^{-1}$ to fx^{-1}; moreover, $\gamma(x)$ squares to the identity.*

Proof. This follows from the fact that $(fx)^{-1}$ and fx^{-1} in B both have image x^{-1} under the map $B \to B/\!\!/\check{B} \cong T$. $\square$

Definition 7.20. Denote by χ the map $B \to B/\!\!/\check{B} \cong T$. There is an involution θ of B sending $x \mapsto \mathrm{Ad}_{\gamma(\chi(x))}(x^{-1})$, and similarly an involution θ of the constant group scheme $\check{B} \times T$ over T sending $(g, y) \mapsto (\mathrm{Ad}_{\gamma(y)}(g), y^{-1})$. This defines an involution θ of $B/\check{B}$, and hence a $\mathbf{Z}/2$-action on it.

Example 7.21. Suppose $G = \mathrm{GL}_2$ or SL_2. Then one can take for γ the constant map $T \cong \mathbf{G}_m^2 \to \check{B}$ sending $(x, y) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. If $G = \mathrm{PGL}_2$, one can simply multiply γ by a primitive fourth root of unity to get an element of $\check{G} = \mathrm{SL}_2$. If $G = \mathrm{GL}_3$, then one can take for γ the map $T \cong \mathbf{G}_m^3 \to \check{B}$ sending $(x, y, z) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & zy^{-1} \end{pmatrix}$.

It is easy to show:

Lemma 7.22. *The involution $\theta : B/\check{B} \to B/\check{B}$ is isomorphic to the map induced by inversion on B.*

Proposition 7.23. *The $\mathbf{Z}/2$-action by θ on $\widetilde{\check{G}}/\check{G} \cong B/\check{B}$ restricts to an action on $\widetilde{\check{G}}^{\mathrm{reg}}/\check{G}$, and under the equivalence of Corollary 7.8, it identifies with the $\mathbf{Z}/2$-action via complex conjugation on equivariant* KU*. In particular, there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KO}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}((\widetilde{\check{G}}^{\mathrm{reg}}/\check{G})/\langle\theta\rangle).$$

Proof. It follows from Definition 7.20 that there is a commutative diagram

$$\begin{tikzcd} T \arrow[r, "x\mapsto x^{-1}"] \arrow[d, "\kappa"'] & T \arrow[d, "\kappa"] \\ B \arrow[r, "\theta"'] & B. \end{tikzcd}$$

Therefore, θ induces an automorphism of $T \times_{B^{\mathrm{reg}}/\check{B}} T$, and it suffices (by the proof of Corollary 7.8) to show that under the isomorphism

$$\operatorname{Spec} \pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee \cong T \times_{B^{\mathrm{reg}}/\check{B}} T \tag{15}$$

of Theorem 7.3, the action of θ corresponds to the action of complex conjugation on equivariant K-theory. Let $T^{\mathrm{gen}} \subseteq T$ denote the complement of the union of all hypertori cut out by the coroots of G. Since both sides of (15) are flat and affine over T, their rings of functions inject into the corresponding localizations along the map $T^{\mathrm{gen}} \to T$. Furthermore, these localizations are $\mathbf{Z}/2$-equivariant (for complex conjugation and θ, respectively), and so it suffices to show that these localizations are $\mathbf{Z}/2$-equivariantly isomorphic.

By Lemma 3.11, there is an isomorphism

$$\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee|_{T^{\mathrm{gen}}} \cong \pi_0 \mathcal{F}_T(\mathrm{Gr}_T)^\vee|_{T^{\mathrm{gen}}} \cong \mathcal{O}_{T^{\mathrm{gen}}}[\mathbb{X}_*(T)].$$

Under this isomorphism, the action via complex conjugation on KU is given simply by inversion on T^{gen}, and acts trivially on $\mathbb{X}_*(T)$. Similarly, since $fx \in B$ is regular *semisimple* if $x \in T^{\mathrm{gen}}$, and the centralizers of regular semisimple elements are tori, there is an isomorphism

$$(T \times_{B^{\mathrm{reg}}/\check{B}} T) \times_T T^{\mathrm{gen}} \cong T^{\mathrm{gen}} \times \check{T}.$$

Under this isomorphism, the action of θ is given simply by inversion on T^{gen}, and acts trivially on $\check{T}$. This clearly matches with the action on $\pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee|_{T^{\mathrm{gen}}}$ via complex conjugation on KU, as desired. $\square$

Proposition 7.23 says that, up to replacing $B/\check{B}$ by $\check{B}/\check{B}$ (that is, replacing $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; \mathrm{KU})$ by $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; \mathrm{KU})$), the $\mathbf{Z}/2$-action via complex conjugation on equivariant KU identifies under Corollary 7.8 with the $\mathbf{Z}/2$-action on $\check{B}/\check{B} = \mathrm{Map}(B\mathbf{Z}, B\check{B})$ coming from inversion on $\mathbf{Z}$.

Remark 7.24. Assume G has torsion-free fundamental group. One can similarly compute the effect of complex conjugation for G_c-equivariant local systems. Namely, as in Lemma 7.19, there is a map $\delta : T/\!\!/W \to \check{B}$ such that if $x \in T/\!\!/W$, then $\mathrm{Ad}_{\delta(x)}$ sends $(fx)^{-1}$ to fx. Just as in Definition 7.20, we obtain an involution Θ on $G/\check{G}$ which can be identified with the effect of inversion on G, and the resulting $\mathbf{Z}/2$-action on $\mathrm{QCoh}(G^{\mathrm{reg}}/\check{G})$ identifies, under the equivalence of Remark 7.9, with the $\mathbf{Z}/2$-action on $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; \mathrm{KU})$ coming from complex conjugation on equivariant KU. This gives an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; \mathrm{KO}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}((G^{\mathrm{reg}}/\check{G})/\langle\Theta\rangle).$$

It is also possible to describe an analogue of Corollary 7.8 with coefficients in the $K(1)$-local sphere $L_{K(1)}S^0$ (for some fixed prime p). Recall from Definition 4.14 that if A is a p-power torsion abelian group and X is a (ind-)finite A-space with even cells, then the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_A(X; L_{K(1)}S^0)$ is obtained from $\mathrm{Loc}^{\mathrm{gr}}_A(X; \mathrm{KU})$ by taking homotopy $\mathbf{Z}_p^\times$-invariants.

Definition 7.25. For $n \geq 0$, let $\tilde{\check{G}}_{p^n}$ denote the (derived) fiber product

$$\tilde{\check{G}}_{p^n} := \tilde{\check{G}} \times_T T[p^n].$$

That is, $\tilde{\check{G}}_{p^n}/\check{G} \cong B_{p^n}/\check{B}$, where B_{p^n} is the subgroup of those elements of B whose eigenvalues are all p^nth roots of unity. Similarly, let $\tilde{\check{G}}^{\mathrm{reg}}_{p^n}$ denote the fiber product

$$\tilde{\check{G}}^{\mathrm{reg}}_{p^n} := \tilde{\check{G}}^{\mathrm{reg}} \times_T T[p^n],$$

There is an action of $\mathbf{Z}_p^\times$ (which factors through an action of $(\mathbf{Z}/p^n)^\times$) on B_{p^n} given by exponentiation; the $\mathbf{Z}_p^\times$-action commutes with the $\check{B}$-action by conjugation, and hence defines an action on the quotient stack $B_{p^n}/\check{B} \cong \tilde{\check{G}}_{p^n}/\check{G}$.

Proposition 7.26. *Let $n \geq 0$. The $\mathbf{Z}_p^\times$-action on $\tilde{\check{G}}_{p^n}/\check{G}$ restricts to an action on $\tilde{\check{G}}^{\mathrm{reg}}_{p^n}/\check{G}$, and there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c[p^n]}(\mathrm{Gr}_G; L_{K(1)}S^0) \otimes_{\mathbf{Z}_p} F \simeq \mathrm{QCoh}((\tilde{\check{G}}^{\mathrm{reg}}_{p^n}/\check{G})/\mathbf{Z}_p^\times).$$

Proof. Base-changing the $\mathrm{QCoh}(T)$-linear equivalence Corollary 7.8 along $\mathrm{QCoh}(T) \to \mathrm{QCoh}(T[p^n])$ gives an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{T_c[p^n]}(\mathrm{Gr}_G; \mathrm{KU}^\wedge_p) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}(\tilde{\check{G}}^{\mathrm{reg}}_{p^n}/\check{G}).$$

Since the $\mathbf{Z}_p^\times$-action on $T[p^n]$ is given by exponentiation, the strategy of Proposition 7.23 shows that the $\mathbf{Z}_p^\times$-action on the left-hand side of the above equivalence via Adams operations on p-completed KU identifies with the $\mathbf{Z}_p^\times$-action on $\tilde{\check{G}}^{\mathrm{reg}}_{p^n}/\check{G}$ described in Definition 7.25. Taking homotopy $\mathbf{Z}_p^\times$-invariants of the displayed equivalence then yields the desired statement. $\square$

8. The elliptic story

In this section, we will work over a given algebraically closed field F. For the moment, G will be a (split) almost-simple and simply-connected group over F; when we talk about the topology of G later, we will assume that G lives over $\mathbf{C}$. Let E be a (smooth) elliptic curve over k, let $\mathrm{Bun}_B^0(E)$ denote the moduli stack of B-bundles on E of degree 0, and let $\mathrm{Bun}_T^0(E)$ denote the scheme of T-bundles on E of degree 0. We will also make use of the stack $\mathrm{Bun}_G^{\mathrm{ss}}(E)$ of semistable G-bundles on E.

Definition 8.1. Say that a B-bundle $\mathcal{P}_B$ on E is *regular* if $\dim \mathrm{Aut}(\mathcal{P}_B) = \mathrm{rank}(G)$. Let $\mathrm{Bun}_B^0(E)^{\mathrm{reg}}$ denote the open substack of $\mathrm{Bun}_B^0(E)$ defined by the regular B-bundles. Similarly, if $\mathcal{P} \in \mathrm{Bun}_G^{\mathrm{ss}}(E)$ is a semistable G-bundle on E, we say that $\mathcal{P}$ is *regular* if $\dim \mathrm{Aut}(\mathcal{P}) = \mathrm{rank}(G)$. Let $\mathrm{Bun}_G^{\mathrm{ss}}(E)^{\mathrm{reg}} \subseteq \mathrm{Bun}_G^{\mathrm{ss}}(E)$ denote the open substack of regular semistable G-bundles.

Proposition 8.2. *The map $\mathrm{Bun}_B^0(E) \to \mathrm{Bun}_T^0(E)$ admits a canonical unique section $\kappa : \mathrm{Bun}_T^0(E) \to \mathrm{Bun}_B^0(E)$ landing in $\mathrm{Bun}_B^0(E)^{\mathrm{reg}}$.*

Proof. Let $\mathcal{P}$ be a semistable G-bundle on E. By [Dav19, Proposition 4.4.5], the regularity of $\mathcal{P}$ is equivalent to the condition that for any (or some) B-reduction $\mathcal{P}_B$ of $\mathcal{P}$ of degree 0, the associated N-bundle $\mathcal{P}_B/T$ is induced from an $N_{\mathcal{P}}$-bundle with nontrivial associated N_α-bundle for each $\alpha \in \Delta_{\mathcal{P}}$. Moreover, every geometric fiber of the map $\mathrm{Bun}_G^{\mathrm{ss}}(E) \to \mathrm{Hom}(\mathbb{X}^*(T), E)/\!\!/W$ to the coarse moduli space of $\mathrm{Bun}_G^{\mathrm{ss}}(E)$ contains a unique regular semistable G-bundle. Also see [FMW98, Proposition 3.9], where a similar result is stated.

Following [Dav19, Definition 3.1.7], set

$$\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E)^{\mathrm{reg}} \cong \mathrm{Bun}_G^{\mathrm{ss}}(E)^{\mathrm{reg}} \times_{\mathrm{Hom}(\mathbb{X}^*(T),E)/\!\!/W} \mathrm{Hom}(\mathbb{X}^*(T), E).$$

Let $\mathrm{Bun}_B^0(E)^{\mathrm{reg}}$ denote the moduli stack of B-bundles on E of degree 0. It then follows from the isomorphism $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E) \cong \mathrm{Bun}_B^0(E)$ of [Dav19, Proposition 2.1.11] and the equality $\dim \mathrm{Aut}(\mathcal{P}) = \dim \mathrm{Aut}(\mathcal{P}_B)$ that there is an isomorphism $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E)^{\mathrm{reg}} \cong \mathrm{Bun}_B^0(E)^{\mathrm{reg}}$. In particular, every geometric fiber of the map $\mathrm{Bun}_B^0(E) \to \mathrm{Hom}(\mathbb{X}^*(T), E) = \mathrm{Bun}_T^0(E)$ contains a unique regular B-bundle of degree 0.

The existence of κ is a consequence of [Dav19, Theorem 4.3.2], which is a refinement of [FM00, Theorem 5.1.1]. Since we will not need the full strength of [Dav19, Theorem 4.3.2] outside of this proof, we will only briefly recall the necessary notation and statements. In *loc. cit.*, the scheme $\mathrm{Bun}_T^0(E)$ is denoted by Y. Let $\widetilde{\mathrm{Bun}}_G(E)$ denote the Kontsevich-Mori compactification of $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E) \cong \mathrm{Bun}_B^0(E)$; see [Dav19, Definition 2.1.2]. Let Θ denote the theta-line bundle over $\mathrm{Bun}_T^0(E)$ of [Dav19, Corollary 3.2.10], and let $\widetilde{\chi} : \widetilde{\mathrm{Bun}}_G(E) \to \Theta^{-1}/\mathbf{G}_m$ denote the map constructed in [Dav19, Corollary 3.3.2]. Then, [Dav19, Theorem 4.3.2] shows that there is a map $\Theta^{-1} \to \widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E)$ landing in $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E)^{\mathrm{reg}}$ such that the composite

$$\Theta^{-1} \to \widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E) \xrightarrow{\widetilde{\chi}} \Theta^{-1}/\mathbf{G}_m$$

is the canonical map. Composing with the zero section of Θ^{-1}, we obtain a map

$$\mathrm{Bun}_T^0(E) \cong 0_{\Theta^{-1}} \to \Theta^{-1} \to \widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E)^{\mathrm{reg}} \cong \mathrm{Bun}_B^0(E).$$

This is the desired map κ. $\square$

Definition 8.3. The map $\kappa : \mathrm{Bun}_T^0(E) \to \mathrm{Bun}_B^0(E)$ from Proposition 8.2 will be called the *elliptic Kostant slice.*

If E is replaced by the constant stack S^1 or by $B\mathbf{G}_a$, the stack $\mathrm{Bun}_B^0(E)$ is to be interpreted as B/B and $\mathfrak{b}/B$, respectively. The analogue of the elliptic Kostant section is given by the maps $f \cdot T \to B/B$ and $f + \mathfrak{t} \to \mathfrak{b}/B$, respectively. Note that one does *not* see the stacks $B/\check{B}$ and $\mathfrak{b}/\check{B}$ (after all, no mention of $\check{G}$ has been made so far!). The following is [Dav19, Lemma 3.1.11].

Lemma 8.4. *Let $I \subseteq \Phi^-$ be a subset, and let $\mathrm{Bun}_T^0(E)_I$ denote the subscheme of $\mathrm{Bun}_T^0(E)$ defined by those bundles $\mathcal{P}_T$ whose α-component is trivial precisely for $\alpha \in I$. Let $N_I \subseteq N$ be the smallest unipotent subgroup which is invariant under T-conjugation and which contains N_α for every $\alpha \in I$. Then the natural map*

$$\mathrm{Bun}_{TN_I}^0(E) \times_{\mathrm{Bun}_T^0(E)} \mathrm{Bun}_T^0(E)_I \to \mathrm{Bun}_B^0(E) \times_{\mathrm{Bun}_T^0(E)} \mathrm{Bun}_T^0(E)_I$$

is an isomorphism.

Example 8.5. Suppose that $I = \emptyset$, so that $\mathrm{Bun}_T^0(E)_\emptyset$ denotes the open subscheme of T-bundles of degree zero whose α-component is nontrivial for every negative root α. The isomorphism $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E) \cong \mathrm{Bun}_B^0(E)$ implies that the map $\widetilde{\mathrm{Bun}}_G^{\mathrm{ss}}(E) \to \mathrm{Bun}_T^0(E)$ is an isomorphism over $\mathrm{Bun}_T^0(E)_\emptyset$. In particular, every point of $\mathrm{Bun}_T^0(E)_\emptyset$ has a canonical associated (regular) semistable G-bundle.

The above results continue to hold if E is replaced by the constant stack S^1 or by $B\mathbf{G}_a$ (in which case $\mathrm{Bun}_B^0(E)$ is to be interpreted as B/B and $\mathfrak{b}/B$, respectively). In the case of S^1, for instance, the semistable G-bundles obtained in this way from $\mathrm{Bun}_T^0(E)_\emptyset$ are precisely those which lie in the regular *semisimple* locus G^{rss}/G; similarly for the case of $B\mathbf{G}_a$.

We now turn to the topology of G, so it is connected, almost simple, and simply-laced over $\mathbf{C}$. In this setting, k will be an even 2-periodic $\mathbf{E}_\infty$-ring equipped with an oriented group scheme $\mathbf{G}$ whose underlying classical scheme $\mathbf{G}_0$ over $\pi_0(k)$ is an elliptic curve E. We will continue to fix an algebraically closed field F containing $\pi_0(k)$, over which the Langlands dual group $\check{G}$ will live. As usual, when dealing with the algebraic geometry (as opposed to the topology) of G, we will also view it as living over F; since G is simply-laced, it is isogenous to $\check{G}$.

Definition 8.6. The *elliptic regular centralizer group scheme* $\widetilde{\check{J}}_{\mathrm{ell}}$ is defined to be the group scheme over $\mathrm{Bun}_{\check{T}}^0(E)$ given by the fiber product

$$\widetilde{\check{J}}_{\mathrm{ell}} \cong \mathrm{Bun}_{\check{T}}^0(E) \times_{\mathrm{Bun}_{\check{B}}^0(E)} \mathrm{Bun}_{\check{T}}^0(E).$$

Note that this is very slightly (but importantly) different from the definition of $\widetilde{\check{J}}_\mu$ and $\widetilde{\check{J}}$; the analogues of the fiber product above would instead be $(f \cdot \check{T}) \times_{\check{B}/\check{B}} (f \cdot \check{T})$ and $(f + \check{\mathfrak{t}}) \times_{\check{\mathfrak{b}}/\check{B}} (f + \check{\mathfrak{t}})$.

In the following discussion, we will consider the $\check{T}$-equivariant elliptic homology of Gr_G (instead of the T-equivariant elliptic homology); this will capture the minor difference between the definitions of $\widetilde{\check{J}}_{\mathrm{ell}}$ and $\widetilde{\check{J}}$ mentioned above.

Theorem 8.7. *There is an isomorphism of group schemes over $\mathrm{Bun}_{\check{T}}^0(E) \cong \mathcal{M}_{\check{T},0}$:*

$$\mathrm{Spec}_{\mathrm{Bun}_{\check{T}}^0(E)}(\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee) \otimes_{\pi_0(k)} F \cong \mathrm{Bun}_{\check{T}}^0(E) \times_{\mathrm{Bun}_{\check{B}}^0(E)} \mathrm{Bun}_{\check{T}}^0(E).$$

Here, $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{T}}(E)}(\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee)$ *denotes the relative* Spec *of* $\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee$ *over* $\mathrm{Bun}^0_{\check{T}}(E)$*.*

As with Theorem 6.3 and Theorem 7.3, the proof of Theorem 8.7 relies on two lemmas.

Lemma 8.8. *The projection map* $\widetilde{\check{J}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{T}}(E)$ *(onto either factor) is flat.*

Proof. Like in the proof of Lemma 6.4, it suffices, by miracle flatness, to show that the fibers of the map $\widetilde{\check{J}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{T}}(E)$ have dimension exactly $\mathrm{rank}(\check{G})$. But this follows from the fact that the map $\mathrm{Bun}^0_{\check{T}}(E) \to \mathrm{Bun}^0_{\check{B}}(E)$ lands in $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}$ (see Proposition 8.2). □

For a root α, let $\mathrm{Bun}^0_{\check{T}}(E)_{\alpha\text{-reg}} \subseteq \mathrm{Bun}^0_{\check{T}}(E)$ denote the union of the substacks $\mathrm{Bun}^0_{\check{T}}(E)_{\{\alpha\}}$ and $\mathrm{Bun}^0_{\check{T}}(E)_{\emptyset}$. The next result follows exactly as in Lemma 6.6 (using Lemma 8.4).

Lemma 8.9. *There is an isomorphism*

$$\widetilde{\check{J}}_{\mathrm{ell}}(\check{G})|_{\mathrm{Bun}^0_{\check{T}}(E)_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{J}}_{\mathrm{ell}}(Z_{\check{G}}(x)^\circ)|_{\mathrm{Bun}^0_{\check{T}}(E)_{\alpha\text{-reg}}}, \tag{16}$$

where $Z_{\check{G}}(x)$ *is the centralizer of some* $x \in \mathrm{Bun}^0_{\check{T}}(E)_{\alpha\text{-reg}}$ *which lies in* $\mathrm{Bun}^0_{\check{T}}(E)_{\{\alpha\}}$*, and* $Z_{\check{G}}(x)^\circ$ *denotes the connected component of the identity.*

Recall that if X is a scheme with subschemes $V = V(\mathcal{I}) \subseteq D = V(\mathcal{J})$ (so that $\mathcal{J} \subseteq \mathcal{I}$) where D is locally principal, the affine blowup $\mathrm{Bl}^D_V(X)$ is defined to be the complement of $V_+(\mathcal{J})$ in the blowup $\mathrm{Bl}_V(X)$. That is, it is the relative Spec of the algebra $\mathcal{O}_X[\frac{\mathcal{J}}{\mathcal{J}}]$ of weight zero elements in $\mathrm{Bl}_{\mathcal{I}}(\mathcal{O}_X)[\frac{1}{\mathcal{J}}]$, where $\mathrm{Bl}_{\mathcal{I}}(\mathcal{O}_X) = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$ is the Rees algebra.

Proof of Theorem 8.7. The argument of Theorem 6.3 reduces us to checking that the isomorphism of Theorem 8.7 holds if G has semisimple rank 1, i.e., is the product of a torus with one of GL_2, SL_2, or PGL_2. Again, it is easy to match up the contributions from the toral factors, so we will assume that G is either GL_2, SL_2, or PGL_2. In this case, we can even replace F by $\pi_0(k)$. The proofs are all rather uniform (as we have seen in Theorem 6.3 and Theorem 7.3), so we will simply illustrate the argument when $G = \mathrm{SL}_2$ and $G = \mathrm{PGL}_2$.

We begin with the case $G = \mathrm{SL}_2$. Since $\check{T} = \mathbf{G}_m$, we may identify $\mathrm{Bun}^0_{\check{T}}(E) \cong E$; to emphasize that it plays the role of the base of S^1-equivariant elliptic cohomology, we will denote it by $\mathcal{M}$. Let $\infty \in \mathcal{M} = E$ denote the identity section. Consider the closed subschemes

$$V = \{(\infty, 1)\} \subseteq D = \{\infty\} \times \mathbf{G}_m \subseteq \mathcal{M} \times \mathbf{G}_m.$$

Then, as in Theorem 6.3 and Theorem 7.3, $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{T}}(E)}(\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee)$ identifies with the affine blowup $\mathrm{Bl}^D_V(\mathcal{M} \times \mathbf{G}_m)$.

Since $\check{G} = \mathrm{PGL}_2$, an S-point of the stack $\mathrm{Bun}^0_{\check{B}}(E)$ is the data of a degree zero rank 2 vector bundle $\mathcal{V}$ over $S \times E$ along with a line subbundle $\mathcal{L} \subseteq \mathcal{V}$ and an isomorphism $\mathcal{V}/\mathcal{L} \cong \mathcal{O}_{S\times E}$. In this language, the elliptic Kostant section $\mathcal{M} = E \to \mathrm{Bun}^0_{\check{B}}(E)$ classifies the unique indecomposable extension $\mathcal{V}$ of $\mathcal{O}_{\mathcal{M}\times E}$ by the Poincaré line bundle $\mathcal{P}$. (Recall that $\mathcal{P}$ can be identified, for instance, with the line bundle corresponding to the divisor $\Delta - E \times \{\infty\} - \{\infty\} \times E$.) This extension is classified by a nonzero section of $\underline{\mathrm{Ext}}^1_{\mathcal{M}\times E}(\mathcal{O}_{\mathcal{M}\times E}, \mathcal{P})$.

Let us now compute $\widetilde{\check{J}}_{\mathrm{ell}}$. The fiber product $\mathcal{M} \times_{\mathrm{Bun}^0_{\check{B}}(E)} \mathcal{M}$ is isomorphic (as a group scheme over $\mathcal{M}$) to the subgroup of the constant group scheme $\underline{\check{B}} := \mathcal{M} \times \check{B}$ of those $b \in \underline{\check{B}}$ such that $b \cdot \mathcal{V} = \mathcal{V}$. First, let $U = (\mathcal{M} - \{\infty\}) \times E$; then $\mathcal{V}|_U$ splits as $\mathcal{O}_U \oplus \mathcal{P}|_U$. Indeed, the restriction $\mathcal{P}|_U$ is a nontrivial line bundle on U, so its pushforward to $\mathcal{M} - \{\infty\}$ has no cohomology (and hence the extension class is trivial). It follows that $\mathrm{Aut}_{\check{B}}(\mathcal{V})|_U = \mathcal{M} \times_{\mathrm{Bun}^0_{\check{B}}(E)} U$ can be identified with $U \times \mathbf{G}_m$.

On the other hand, let $Z = \{\infty\} \times E$ denote the complement of U, so that the formal neighborhood $\hat{Z}$ of Z is isomorphic to $\mathcal{M}^\wedge_\infty \times E = \hat{\mathbf{A}}^1 \times E$. Let t denote a coordinate on $\hat{\mathbf{A}}^1$. Then, the restriction of $\mathcal{P}$ to $\hat{Z}$ is given by the 1-parameter family of line bundles $\mathcal{O}_{\hat{Z}}(t-\infty)$ over $\hat{\mathbf{A}}^1 \times E$. The restriction of $\mathcal{V}$ to $\hat{Z}$ is classified by a map $\mathcal{O}_{\hat{Z}} \to \mathcal{O}_{\hat{Z}}(t-\infty)[1]$ which vanishes except at the origin of $\hat{\mathbf{A}}^1$, where it is given by the unique (up to nonzero scalar) nontrivial map $\mathcal{O}_E \to \mathcal{O}_E[1]$.

For instance, $\mathcal{V}|_Z$ is isomorphic to the Atiyah bundle over E from [Ati57] (i.e., the unique indecomposable rank 2 extension of the structure sheaf by itself), so that it can be realized away from $\infty \in E$ by pairs (f_1, f_2) of regular functions on E; and near ∞ by pairs (f_1, f_2) such that f_1 and $f_1 - zf_2$ are regular, where z is a local coordinate of E. Under this description, $\mathrm{End}(\mathcal{V}|_Z) = \mathrm{End}(\mathcal{V})|_Z$ is spanned by the identity and the map $(f_1, f_2) \mapsto (0, f_1)$. That is, $\mathrm{End}(\mathcal{V})|_Z$ is isomorphic to the group of matrices $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, and so $\mathrm{Aut}_{\check{B}}(\mathcal{V})|_Z$ is isomorphic to $Z \times \mathbf{G}_a$. It is easy to extend this description to the formal neighborhood of Z, and thereby find that $\mathrm{Aut}_{\check{B}}(\mathcal{V})|_{\hat{Z}}$ is isomorphic to the canonical degeneration of $\mathbf{G}_m$ into $\mathbf{G}_a$. In other words, there is an isomorphism

$$\mathrm{Aut}_{\check{B}}(\mathcal{V})|_{\hat{Z}} \cong \mathrm{Spec}\, \pi_0(k)[\![t]\!][a^{\pm 1}, \tfrac{a-1}{t}].$$

Gluing this with the description of $\mathrm{Aut}_{\check{B}}(\mathcal{V})|_U$ from the preceding paragraph, we find that $\mathrm{Aut}_{\check{B}}(\mathcal{V}) \cong \mathcal{M} \times_{\mathrm{Bun}^0_{\check{B}}(E)} \mathcal{M}$ is isomorphic to the affine blowup $\mathrm{Bl}^D_V(\mathcal{M} \times \mathbf{G}_m)$. We will leave it to the reader to verify that the resulting sequence of isomorphisms

$$\mathrm{Aut}_{\check{B}}(\mathcal{V}) \cong \mathcal{M} \times_{\mathrm{Bun}^0_{\check{B}}(E)} \mathcal{M} \cong \mathrm{Bl}^D_V(\mathcal{M} \times \mathbf{G}_m) \cong \mathrm{Spec}_{\mathrm{Bun}^0_{\check{T}}(E)}(\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee)$$

is one of group schemes over $\mathcal{M}$.

The case when $G = \mathrm{PGL}_2$ is very similar; we only indicate the necessary changes. Let $E[2] \subseteq E$ denote the 2-torsion subgroup, and consider the closed subschemes

$$V = E[2] \times \mu_2 \subseteq D = E[2] \times \mathbf{G}_m \subseteq \mathcal{M} \times \mathbf{G}_m.$$

By arguing as in Theorem 6.3 and Theorem 7.3, we find that $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{T}}(E)}(\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee)$ identifies with the affine blowup $\mathrm{Bl}^D_V(\mathcal{M} \times \mathbf{G}_m)$. In this case, $\check{G} = \mathrm{SL}_2$, and the elliptic Kostant section $\mathcal{M} = E \to \mathrm{Bun}^0_{\check{B}}(E)$ sends a line bundle $\mathcal{L}$ to the trivially filtered SL_2-bundle $\mathcal{O}_E \subseteq \mathcal{O}_E \oplus \mathcal{L}$ if $\mathcal{L}^2 \not\cong \mathcal{O}_E$; and to the Atiyah extension of $\mathcal{L}$ by itself if $\mathcal{L}^2 \cong \mathcal{O}_E$. This extension is defined by a nontrivial element of $\mathrm{Ext}^1_E(\mathcal{L}, \mathcal{L}^{-1}) \cong \mathrm{H}^1(E; \mathcal{L}^{-2})$. The calculation of $\mathcal{M} \times_{\mathrm{Bun}^0_{\check{B}}(E)} \mathcal{M}$ follows exactly the same path as in the case $G = \mathrm{SL}_2$ studied above. $\square$

Remark 8.10. The most classical instantiation of the Atiyah bundle $\mathcal{A}$ is via the Weierstrass functions. The $\mathbf{G}_a$-torsor over E associated to $\mathcal{A}$ is the complement of the section at ∞ of the projective line $\mathbf{P}(\mathcal{A})$. If we work complex-analytically, E^{an} can be identified as the quotient $\mathbf{C}/\Lambda$ for some rank 2 lattice $\Lambda \subseteq \mathbf{C}$. Associated to Λ are two Weierstrass functions defined on $\mathbf{C}$:

$$\wp(z;\Lambda) = \tfrac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left(\tfrac{1}{(z-\lambda)^2} - \tfrac{1}{\lambda^2}\right),$$
$$\zeta(z;\Lambda) = \tfrac{1}{z} + \sum_{\lambda\in\Lambda-\{0\}} \left(\tfrac{1}{z-\lambda} + \tfrac{1}{\lambda} + \tfrac{z}{\lambda^2}\right).$$

Note that $\wp(z;\Lambda)$ is doubly-periodic, i.e., $\wp(z+\lambda;\Lambda) = \wp(z;\Lambda)$ for any $\lambda \in \Lambda$. Alternatively, $\wp$ defines a map $\mathbf{C} \to \mathbf{C}$ which factors through a map $\mathbf{C}/\Lambda = E^{\mathrm{an}} \to \mathbf{C}$.

Although $\zeta(z;\Lambda)$ is not doubly-periodic, an easy calculation shows that $\wp(z;\Lambda) = -\partial_z\zeta(z;\Lambda)$; so if $\lambda \in \Lambda$, then $\zeta(z+\lambda;\Lambda) - \zeta(z;\Lambda) = c(\lambda)$ for some constant $c(\lambda)$. The function $\lambda \mapsto c(\lambda)$ is evidently additive, and defines a homomorphism $\Lambda \to \mathbf{C}$, which defines a $\mathbf{C}$-bundle over $E^{\mathrm{an}} = \mathbf{C}/\Lambda$. This $\mathbf{C}$-bundle is precisely the analytification of the $\mathbf{G}_a$-torsor associated to the Atiyah bundle. It follows that although ζ is not defined on E^{an}, this analytification is the universal space over E^{an} on which ζ is well-defined.

This discussion also describes the total space of the rank 2-bundle $\mathcal{A}^{\mathrm{an}}$ purely analytically. For instance, if $q \in \mathbf{C}^\times$ is a unit complex number of modulus < 1, we can identify $\mathrm{Tot}(\mathcal{A}^{\mathrm{an}})$ over the Tate curve $\mathbf{C}^\times/q^{\mathbf{Z}}$ with the quotient

$$\mathrm{Tot}(\mathcal{A}^{\mathrm{an}}) = \left(\mathbf{C}^\times \times \mathbf{C}^2\right) / \left((z,x) \sim (qz, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)x)\right).$$

The appearance of the Jordan block $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ is the basic reason why the Atiyah bundle plays the role of the principal nilpotent element f in the proof of Theorem 8.7.

Corollary 8.11. *There is an F-linear equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}).$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;k) \to \mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(*;k)$ identifies with the pullback functor $\kappa^* : \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(E)) \to \mathrm{QCoh}(\mathrm{Bun}^0_{\check{T}}(E))$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G;k)$ is equivalent to the category of comodules over $\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee$ in $\mathrm{QCoh}(\mathcal{M}_{\check{T},0}) = \mathrm{QCoh}(\mathrm{Bun}^0_{\check{T}}(E))$. By Theorem 8.7, it can be identified the category of quasicoherent sheaves on the quotient stack $\mathrm{Bun}^0_{\check{T}}(E)/\widetilde{\check{J}}_{\mathrm{ell}}$. We may view $\widetilde{\check{J}}_{\mathrm{ell}}$ as a closed subgroup scheme of the constant group scheme $\check{B} \times \mathrm{Bun}^0_{\check{T}}(E)$. This gives an isomorphism

$$\mathrm{Bun}^0_{\check{T}}(E)/\widetilde{\check{J}}_{\mathrm{ell}} \cong \check{B}\backslash(\check{B} \times \mathrm{Bun}^0_{\check{T}}(E))/\widetilde{\check{J}}_{\mathrm{ell}}.$$

Let $\mathrm{Bun}^0_{\check{B}}(E)_{\mathrm{triv}}$ denote the scheme whose S-points are of $\check{B}$-bundles over $S \times E$ of degree 0 equipped with a trivialization at $S \times \{\infty\}$, so that there is a natural map $\mathrm{Bun}^0_{\check{B}}(E)_{\mathrm{triv}} \to \mathrm{Bun}^0_{\check{B}}(E)$. Let $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}_{\mathrm{triv}}$ denote the restriction of $\mathrm{Bun}^0_{\check{B}}(E)_{\mathrm{triv}}$ to the regular locus $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}} \subseteq \mathrm{Bun}^0_{\check{B}}(E)$. It follows from Davis' work in [Dav19] that the $\check{B}$-orbit of $\mathrm{Bun}^0_{\check{T}}(E)$ inside $\mathrm{Bun}^0_{\check{B}}(E)_{\mathrm{triv}}$ is precisely the regular locus $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}_{\mathrm{triv}}$. Since $\widetilde{\check{J}}_{\mathrm{ell}}$ is by definition the stabilizer of $\kappa : \mathrm{Bun}^0_{\check{T}}(E) \to \mathrm{Bun}^0_{\check{B}}(E)$, the quotient $\check{B}\backslash(\check{B} \times \mathrm{Bun}^0_{\check{T}}(E))/\widetilde{\check{J}}_{\mathrm{ell}}$ is isomorphic to $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}$; so there is an isomorphism $\mathrm{Bun}^0_{\check{T}}(E)/\widetilde{\check{J}}_{\mu} \cong \mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}}$. ∎

Remark 8.12. The work of Gepner and Meier in [GM23, GM20] sets up the theory of G_c-equivariant elliptic cohomology for compact Lie groups G_c. In particular, they describe a scheme $\mathcal{M}_G$ over k with underlying scheme $\mathcal{M}_{G,0}$ over $\pi_0(k)$, such that the global sections of the structure sheaf of $\mathcal{M}_G$ computes G_c-equivariant k-cohomology. Using this setup (and assuming a slight extension of the results of [Dav19] replacing the simply-connectedness assumption with the condition of having torsion-free fundamental group), it can be shown that if G is almost simple and simply-laced, and has torsion-free fundamental group, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)^{\mathrm{reg}}).$$

Here, the left-hand side is defined (just as in Section 4) to be the ∞-category $\mathrm{coLMod}_{\pi_0(\mathcal{F}_G(\mathrm{Gr}_G)^\vee)}(\mathrm{QCoh}(\mathcal{M}_{G,0}))$. The proof of the displayed equivalence is quite similar to that of Corollary 8.11, and in fact can be deduced from it using the observation that $\pi_0(\mathcal{F}_G(\mathrm{Gr}_G)^\vee) = \pi_0(\mathcal{F}_T(\mathrm{Gr}_G)^\vee)^W$ and that the natural map $\mathrm{Bun}^0_{\check{B}}(E)^{\mathrm{reg}} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)^{\mathrm{reg}}$ is a (ramified) W-cover. The first statement uses that G is simply-connected, and the second is the elliptic version of Grothendieck-Springer theory studied in [Dav19, Proposition 3.1.14].

Remark 8.13. The statements of Corollary 6.8, Corollary 7.8, and Corollary 8.11 can be packaged into a single statement as follows. Suppose k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring, and let $\mathbf{G}$ be an oriented commutative k-group scheme. Let $\mathbf{G}_0$ denote the underlying commutative group scheme over $\pi_0(k)$, and let $\mathbf{G}_0^\vee = \mathrm{Hom}(\mathbf{G}_0, B\mathbf{G}_m)$ denote its 1-shifted Cartier dual. Let F be an algebraically closed field containing $\pi_0(k)$; then there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}).$$

Similarly, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}).$$

In fact, the arguments of Corollary 6.8, Corollary 7.8, and Corollary 8.11 show that these equivalences are monoidal for the convolution tensor products on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k)$ and $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k)$ coming from the $\mathbf{E}_2$-structure on Gr_G, and the ordinary tensor product of quasicoherent sheaves. Moreover, a simple adaptation of the discussion at the end of Section 7 shows that the above equivalences are canonical: they respect natural symmetries of k.

To see that these equivalences do indeed package Corollary 6.8, Corollary 7.8, and Corollary 8.11, note that if $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$, then the 1-shifted Cartier dual of $\mathbf{G}_0$ is $B\hat{\mathbf{G}}_a$, and $\mathrm{Map}(B\hat{\mathbf{G}}_a, B\check{B}) \cong \check{\mathfrak{b}}/\check{B}$. Similarly, if $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, then the 1-shifted Cartier dual of $\mathbf{G}_0$ is $B\mathbf{Z}$, and $\mathrm{Map}(B\mathbf{Z}, B\check{B}) \cong \check{B}/\check{B}$. Finally, if $\mathbf{G}_0$ is an elliptic curve E, then its 1-shifted Cartier dual is $\mathrm{Pic}^0(E) = E$, so $\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) = \mathrm{Bun}^0_{\check{B}}(E)$. In fact, in this language, the calculations of [Dev24] show that the stated equivalence continues to hold if $k = \mathrm{ku}$ (now one must replace $\pi_0(k)$ by $\mathbf{Z}[\beta]$, and F by $F[\beta]$) and $\mathbf{G}$ is the group scheme $\mathrm{Spec}\,\mathbf{Z}[\beta, x, \frac{1}{1+\beta x}]$ with group law $x + y + \beta xy$.

Observe that if $\mathcal{L}$ is a degree zero line bundle on $\mathbf{G}_0^\vee$, then $\mathrm{H}^*(\mathbf{G}_0^\vee; \mathcal{L})$ vanishes unless $\mathcal{L}$ is trivial, in which case it is isomorphic to an exterior algebra over k on a class in degree 1. Using this, the Kostant slice is straightforward to describe in the semisimple rank 1 cases. For instance, if $\check{G} = \mathrm{PGL}_2$, the map $\kappa : \mathbf{G}_0 \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)$

can be understood as follows. Since $\mathbf{G}_0 = \mathrm{Hom}(\mathbf{G}_0^\vee, B\mathbf{G}_m)$, a point of $\mathbf{G}_0$ can be viewed as a degree zero line bundle on $\mathbf{G}_0^\vee$. Given such a line bundle $\mathcal{L}$, the map κ sends it to the trivial $\check{B}$-bundle $\mathcal{L} \subseteq \mathcal{L} \oplus \mathcal{O}_{\mathbf{G}_0^\vee} \twoheadrightarrow \mathcal{O}_{\mathbf{G}_0^\vee}$ if $\mathcal{L}$ is nontrivial, and to the unique nontrivial extension $\mathcal{O}_{\mathbf{G}_0^\vee} \subseteq \mathcal{A} \twoheadrightarrow \mathcal{O}_{\mathbf{G}_0^\vee}$ if $\mathcal{L}$ is trivial. This nontrivial extension comes from a nonzero section of $\mathrm{H}^1(\mathbf{G}_0^\vee; \mathcal{O})$.

Just as with Proposition 6.13 and Proposition 7.14, the calculation of Theorem 8.7 gives an *elliptic* version of the Gelfand-Graev action on the affine closure $\overline{T^*(\check{G}/\check{N})}$. Taking the affine closure in the naive sense is very destructive in the case of elliptic cohomology. Nevertheless, one can define $\overline{T^*_E(\check{G}/\check{N})}$ to be the relative spectrum over $\mathrm{Bun}^0_{\check{T}}(E)$ of π_0 of the pushforward of the structure sheaf along the quotient morphism

$$(\check{G} \times \check{T} \times \mathrm{Bun}^0_{\check{T}}(E))/\widetilde{\check{J}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{T}}(E).$$

Proposition 8.14 (Elliptic Gelfand-Graev action). *The natural action of $\check{G} \times \check{T}$ on $\overline{T^*_E(\check{G}/\check{N})}$ extends to an action of $\check{G} \times (W \rtimes \check{T})$, where W is the Weyl group.*

It is possible to make this action explicit in the case when $\check{G} = \mathrm{SL}_2$ (as in Example 6.14 and Example 7.15). The moment map $\overline{T^*_E(\check{G}/\check{N})}/\check{G} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Bun}^0_{\check{B}}(E) \arrow[r, hook] \arrow[dr] & \overline{T^*_E(\check{G}/\check{N})}/\check{T} \arrow[d] \\
& \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)
\end{tikzcd}$$

which relates $\overline{T^*_E(\check{G}/\check{N})}$ to the elliptic Grothendieck-Springer resolution [BN15]; and via this diagram, the elliptic Gelfand-Graev action is closely related to the Weyl action in elliptic Springer theory.

Recall from Definition 6.15 the algebra $\mathcal{H}(\mathbf{H}, T, W)$ associated to a 1-dimensional group scheme $\mathbf{H}$ over a field F and a root system with torus T and Weyl group W. In the following discussion, we will set $\mathbf{H} = E$, so that $\mathbf{H}_T = \mathrm{Bun}^0_T(E) = \mathcal{M}_T$. Exactly the same argument as in Theorem 6.19 shows the following result; here, G does not need to be simply-laced.

Theorem 8.15. *There is an isomorphism of sheaves of associative algebras over $\mathbf{H}_{\mathbf{G}_m^{\mathrm{rot}}} = E$:*

$$\pi_0 \mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee \cong \mathcal{H}(E, \widetilde{T}, \widetilde{W}). \tag{17}$$

Here, $\pi_0 \mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\mathcal{O}_{\mathcal{M}_{\widetilde{T},0}} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{T}}}$-algebroids.

Remark 8.16. Recall the quotient $\mathrm{Bun}^0_{\widetilde{T}}(E)/\!\!/\widetilde{W}$ from Remark 6.18. The discussion therein combined with Theorem 8.15 gives an equivalence of categories

$$\pi_0 \mathcal{F}_{\widetilde{T}_c}(\mathrm{Fl}_G)^\vee\text{-mod} \simeq \mathcal{H}(E, \widetilde{T}, \widetilde{W})\text{-mod} \simeq \mathrm{IndCoh}(\mathrm{Bun}^0_{\widetilde{T}}(E)/\!\!/\widetilde{W}).$$

It follows, via the argument of Corollary 6.27, that $\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{T}_c}(\mathrm{Fl}_G; k) \otimes_{\pi_0(k)} F$ is equivalent to the quotient of $\mathrm{QCoh}(\mathrm{Bun}^0_{\widetilde{T}}(E))$ by the action of $\mathrm{IndCoh}(\mathrm{Bun}^0_{\widetilde{T}}(E)/\!\!/\widetilde{W})$.

Assume, again, that G is simply-laced. Just as in Section 6, one would like to use Theorem 8.15 to prove analogues of Corollary 6.27 and (11). However, unlike with Theorem 7.17, we do not even have a putative description for the Langlands dual side. By analogy with the K-theoretic case, it is natural to expect that the dual side will be related to elliptic quantum groups à la [Fel95]; I am currently exploring this direction of research.

9. Power operations under Langlands duality

We will momentarily review some of the rich theory of power operations in homotopy theory; these force the existence of additional structures on the Langlands dual side of Theorem 1.4. Our goal in this section is to describe these structures explicitly. This section is motivated by an idea suggested by David Treumann.

Before proceeding, we warn the reader of a terminological mismatch. In [Lon21b], Lonergan uses "Steenrod operators" to construct new structures on Coulomb branches (and in particular, on $\check{J}$). These operators, as we will explain in future work, are better viewed as $\mathbf{E}_3$-*power operations* coming from an $\mathbf{E}_3$-algebra structure on $C_*^{G_c}(\mathrm{Gr}_G; \mathbf{F}_p)$. While these are related to Steenrod operations in the usual sense of the word (as used by algebraic topologists), they are not the same. More generally, $\mathbf{E}_3$-power operations on $\mathcal{F}_G(\mathrm{Gr}_G)^\vee$ are closely related to, but distinct from, the power operations we will describe below.

Let k be an $\mathbf{E}_\infty$-ring; we will momentarily specialize to the case when k is 2-periodic *integral* cohomology, complex K-theory, or elliptic cohomology. The theory of power operations describes the additional structure acquired by k-cohomology from the $\mathbf{E}_\infty$-structure on k. As we will see below, it is closely related to the structure of isogenies on the associated 1-dimensional group scheme. This relationship is not new; we refer the reader to [Str98, And00, Rez14] for some sources.

Construction 9.1. Any $\mathbf{E}_\infty$-ring k admits a *Tate-valued Frobenius* $k \to k^{t\mathbf{Z}/p}$, which is given by the composite of the Tate diagonal $k \to (k^{\otimes p})^{t\mathbf{Z}/p}$ with the $\mathbf{Z}/p$-Tate construction of the multiplication map $k^{\otimes p} \to k$. See, e.g., [NS18, Definition IV.1.1] for a modern reference.

If k admits additional structure, then this structure can be refined: namely, if k admits a refinement to a normed algebra in the ∞-category of genuine $\mathbf{Z}/p$-spectra (which will be true in the examples we will study), and $\Phi^{\mathbf{Z}/p}k$ is its geometric fixed points, then the Tate-valued Frobenius $k \to k^{t\mathbf{Z}/p}$ lifts to an $\mathbf{E}_\infty$-map $\varphi : k \to \Phi^{\mathbf{Z}/p}k$. This map is given by taking geometric fixed points of the $\mathbf{Z}/p$-equivariant norm-multiplication map $N^{\mathbf{Z}/p}k \to k$, where $N^{\mathbf{Z}/p}k$ is the Hill-Hopkins-Ravenel norm from [HHR16].

If X is any (finite) space, let $\mathcal{F}_k(X)$ denote the $\mathbf{E}_\infty$-k-algebra of k-cochains on X, and let $\mathcal{F}_k(X)^\vee$ denote the $\mathbf{E}_\infty$-k-coalgebra of k-chains on X. Then φ induces maps

$$\mathcal{F}_k(X) \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(X),\ \mathcal{F}_k(X)^\vee \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(X)^\vee.$$

We will denote either of these maps by φ_X, and call them the *decompleted Frobenius*. Sometimes, we will consider the further composites to $\mathcal{F}_{k^{t\mathbf{Z}/p}}(X)$ and $\mathcal{F}_{k^{t\mathbf{Z}/p}}(X)^\vee$; these composites exist for any $\mathbf{E}_\infty$-ring k, even if it does not lift to a normed algebra in genuine $\mathbf{Z}/p$-spectra.

In the above context, one should view $\Phi^{\mathbf{Z}/p}k$ as a decompletion of $k^{t\mathbf{Z}/p}$; we will see this in Example 9.4 below.

Remark 9.2. Let I_{tr} denote the transfer ideal in $\pi_0\mathcal{F}_k(X \times B\mathbf{Z}/p)$, given by the image of the map $\pi_0\mathcal{F}_k(X) \to \pi_0\mathcal{F}_k(X \times B\mathbf{Z}/p)$ induced by the transfer. On π_0, the map $\varphi_X : \mathcal{F}_k(X) \to \mathcal{F}_{k^{t\mathbf{Z}/p}}(X)$ then factors as a composite

$$\pi_0\mathcal{F}_k(X) \to \pi_0\mathcal{F}_k(X \times B\mathbf{Z}/p)/I_{\mathrm{tr}} \to \pi_0\mathcal{F}_{k^{t\mathbf{Z}/p}}(X).$$

The first map in this composite is often referred to as the *total power operation*. We will denote it by φ_X^{tr}. It will not be used below in any serious way; we have mentioned it only for completeness.

Remark 9.3. Construction 9.1 might seem somewhat abstract, but it has very concrete consequences. Suppose, for simplicity, that k is even and 2-periodic, and that $\pi_0\mathcal{F}_k(X \times B\mathbf{Z}/p) \cong \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)$. Under the assumption on k, this happens if, for instance, either X is a finite space with even cells, or $\pi_0\mathcal{F}_k(B\mathbf{Z}/p)$ is flat over $\pi_0(k)$. The total power operation is then a ring map

$$\varphi_X^{\mathrm{tr}} : \pi_0\mathcal{F}(X) \to \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)/I_{\mathrm{tr}}.$$

Moreover, under the hypothesis on k, there is an isomorphism $\pi_0\mathcal{F}_k(B\mathbf{Z}/p) \cong \pi_0(k)[\![t]\!]/[p](t)$, where $[p](t)$ is the p-series of the formal group law over $\pi_0\mathcal{F}_k(\mathbf{C}P^\infty) \cong \pi_0(k)[\![t]\!]$. The composite of Remark 9.2 can be identified with the map

$$\pi_0\mathcal{F}_k(X) \xrightarrow{\varphi_X^{\mathrm{tr}}} \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)/I_{\mathrm{tr}} \to \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)[1/t].$$

If k admits the structure of a normed algebra in genuine $\mathbf{Z}/p$-spectra, then this composite factors through

$$\pi_0\mathcal{F}_k(X) \xrightarrow{\varphi_X} \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\Phi^{\mathbf{Z}/p}(k) \to \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)[1/t].$$

It follows, in particular, that φ_X^{tr} and φ_X together define a map

$$\pi_0\mathcal{F}_k(X) \to \pi_0\mathcal{F}_k(X) \otimes_{\pi_0(k)} \left(\pi_0\Phi^{\mathbf{Z}/p}(k) \times_{\pi_0\mathcal{F}_k(B\mathbf{Z}/p)[1/t]} \pi_0\mathcal{F}_k(B\mathbf{Z}/p)/I_{\mathrm{tr}}\right).$$

The fiber product on the right-hand side does not have any denominators in t, and we will see this explicitly in the examples below.

Example 9.4. Let us explicate the preceding remark in two examples.

- Suppose $k = \mathbf{Z}[u^{\pm 1}]$ with u in degree 2. Then $\pi_0\mathcal{F}(B\mathbf{Z}/p) \cong \mathbf{Z}[\![t]\!]/pt$, and the transfer ideal is simply generated by t. Therefore, $\pi_0\mathcal{F}(B\mathbf{Z}/p)/I_{\mathrm{tr}} \cong \mathbf{F}_p[\![t]\!]$. If X is a finite space with even cells, then the map of Remark 9.2 can be viewed as an (ungraded) map

  $$\mathrm{H}^*(X; \mathbf{Z}) \xrightarrow{\varphi_X^{\mathrm{tr}}} \mathrm{H}^*(X; \mathbf{F}_p[\![t]\!]) \to \mathrm{H}^*(X; \mathbf{F}_p(\!(t)\!)).$$

  The decompleted Frobenius is given by an (ungraded) map

  $$\varphi_X : \mathrm{H}^*(X; \mathbf{Z}) \to \mathrm{H}^*(X; \mathbf{F}_p[t^{\pm 1}]).$$

  Explicitly, these maps are given on a class $\alpha \in \mathrm{H}^*(X; \mathbf{Z})$ by the formula

  $$\alpha \mapsto \sum_{i \geq 0} (-1)^i P^i(\alpha) t^{(p-1)i}.$$

  Here, P^i is the ith Steenrod operation. That is to say, φ_X encodes the action of the Steenrod algebra on $\mathrm{H}^*(X; \mathbf{Z})$. As expected by Remark 9.3, there are no denominators in t in the above formula. For instance, if $X = \mathbf{C}P^n$ for any finite n, this map sends $x \in \mathrm{H}^2(\mathbf{C}P^n; \mathbf{Z})$ to $x - t^{p-1}x^p$.
- Suppose $k = \mathrm{KU}$. Then $\pi_0\mathcal{F}(B\mathbf{Z}/p) \cong \mathbf{Z}[\![t]\!]/((1+t)^p - 1)$, and the transfer ideal is simply generated by t. Therefore,

  $$\pi_0\mathcal{F}(B\mathbf{Z}/p)/I_{\mathrm{tr}} \cong \mathbf{Z}[\![t]\!]/\tfrac{(1+t)^p - 1}{t} \cong \mathbf{Z}[\zeta_p]^\wedge_t.$$

Here, ζ_p is a primitive pth root of unity and $t = \zeta_p - 1$. Note that since t^{p-1} is a unit multiple of p in $\mathbf{Z}_p[\zeta_p]$, the t-completion above is equivalent to p-completion. The ring $\mathbf{Z}_p[\zeta_p]$ is flat over $\pi_0(k)^\wedge_p = \mathbf{Z}_p$, and so the composite of Remark 9.2 can be viewed as a ring map

$$\mathrm{KU}^0(X) \xrightarrow{\varphi_X^{\mathrm{tr}}} \mathrm{KU}^0(X)[\zeta_p]^\wedge_p \to \mathrm{KU}^0(X)[\zeta_p]^\wedge_p[1/p] \tag{18}$$

The geometric fixed points $\Phi^{\mathbf{Z}/p}\mathrm{KU}$, on the other hand, has homotopy groups given by

$$\pi_*\Phi^{\mathbf{Z}/p}\mathrm{KU} \cong \mathbf{Z}[q^{\pm 1}, \beta^{\pm 1}][\tfrac{1}{(q-1)\cdots(q^{p-1}-1)}]/(q^p - 1) \cong \mathbf{Z}[\zeta_p, \beta^{\pm 1}][1/p];$$

the final isomorphism comes from noticing that $(\zeta_p - 1)\cdots(\zeta_p^{p-1} - 1)$ is p if $p > 2$, and is -2 if $p = 2$. The decompleted Frobenius is given by a ring map

$$\varphi_X : \mathrm{KU}^0(X) \to \mathrm{KU}^0(X)[\zeta_p][1/p].$$

Note that this map is, indeed, a de-p-adic completion of (18). Both φ_X^{tr} and φ_X send a vector bundle V to the pth Adams operation $\psi^p(V) \in \mathrm{KU}^0(X)$, viewed as a subalgebra of $\mathrm{KU}^0(X)[\zeta_p]^\wedge_p$ and of $\mathrm{KU}^0(X)[\zeta_p][1/p]$. As expected by Remark 9.3, there are no denominators in $t = \zeta_p - 1$ in this formula.

In order to understand the interaction between these power operations and Theorem 1.4, we will need to port Construction 9.1 to the setting of genuine equivariant (co)homology. Namely, we need a decompletion of the map

$$\varphi_{BS^1} : \mathcal{F}_k(BS^1) \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(BS^1) \simeq \lim_n \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathbf{C}P^n).$$

First, observe that this map factors through an $\mathbf{E}_\infty$-map

$$\varphi'_{BS^1} : \mathcal{F}_k(BS^1) \to \Phi^{\mathbf{Z}/p}k \otimes_k \mathcal{F}_k(BS^1) \simeq \Phi^{\mathbf{Z}/p}k \otimes_k \lim_n \mathcal{F}_k(\mathbf{C}P^n);$$

the map from the target to $\mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(BS^1)$ generally induces a strict inclusion on homotopy[17]. Note that φ'_{BS^1} can be viewed as a homomorphism

$$\hat{\mathbf{G}} \times_{\operatorname{Spec} k} \operatorname{Spec} \Phi^{\mathbf{Z}/p}k \to \hat{\mathbf{G}},$$

where $\hat{\mathbf{G}} = \operatorname{Spf} \mathcal{F}_k(BS^1)$.

We will now specialize to the case when k is 2-periodic integral cohomology, complex K-theory, or elliptic cohomology, and let $\mathbf{G}$ denote $\mathbf{G}_a$, $\mathbf{G}_m$, or the spectral elliptic curve E over k (respectively). The choice of $\mathbf{G}$ equips k with a lift to the ∞-category of normed rings in genuine $\mathbf{Z}/p$-spectra. As usual, let $\mathbf{G}_0$ denote the underlying group scheme over $\pi_0(k)$. Our desired decompletion will then be given by a particular homomorphism

$$\varphi : \mathbf{G} \times_{\operatorname{Spec} k} \operatorname{Spec} \Phi^{\mathbf{Z}/p}k \to \mathbf{G}. \tag{19}$$

To describe it, we need to give a moduli-theoretic interpretation of $\Phi^{\mathbf{Z}/p}k$. Let $\mathbf{G}[p]$ denote the p-torsion subgroup of $\mathbf{G}$, so that $\mathbf{G}[p] = \operatorname{Hom}(\mathbf{Z}/p, \mathbf{G})$.

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[17]For instance, take $k = \mathrm{KU}$. Then the map φ_{BS^1} is given on homotopy by the map $\mathbf{Z}[\![t]\!] \to \mathbf{Z}[\zeta_p][1/p][\![t]\!]$ which sends $t \mapsto (1+t)^p - 1$. This factors through a map $\mathbf{Z}[\![t]\!] \to \mathbf{Z}[\zeta_p][\![t]\!][1/p]$; this is the effect of the map φ'_{BS^1} on homotopy. Note that there is a strict inclusion $\mathbf{Z}[\zeta_p][\![t]\!][1/p] \subseteq \mathbf{Z}[\zeta_p][1/p][\![t]\!]$.

There is a natural action of $\mathbf{F}_p^\times$ on $\mathbf{G}[p]$ given by sending $i \in \mathbf{F}_p^\times$ to the multiplication-by-i map $[i]$. Let $U \subseteq \mathbf{G}[p]$ denote the open subscheme given by the complement of the closed subscheme

$$\bigcup_{i\in\mathbf{F}_p^\times} \ker(\mathbf{G}_0[p] \xrightarrow{[i]} \mathbf{G}_0[p]) \subseteq \mathbf{G}_0[p].$$

The following is a straightforward consequence of [HM23, Proposition 2.25].

Lemma 9.5. *The spectral scheme* $\operatorname{Spec}\Phi^{\mathbf{Z}/p}k$ *is isomorphic to* U *over* k.

The spectral scheme $U \subseteq \mathbf{G}[p]$ is specified by its underlying (classical) scheme $U_0 \subseteq \mathbf{G}_0[p]$ over $\pi_0(k)$. If Y is a $\pi_0(k)$-scheme, a map $Y \to U_0$ is equivalent to the data of a homomorphism $f : \mathbf{Z}/p \to \mathbf{G}_Y = \mathbf{G} \times_{\operatorname{Spec}\pi_0(k)} Y$ such that $f(i)$ is not the identity section for $i \in \mathbf{Z}/p - \{0\}$. This implies that f exhibits $\mathbf{Z}/p$ as a closed subgroup scheme of $\mathbf{G}_Y$ which is isomorphic to the Cartier divisor $\sum_{j\in\mathbf{F}_p} f(j)$.

Construction 9.6. Over U_0, there is a universal isogeny $q_0 : \mathbf{G}_{0,U_0} \to \mathbf{G}_{0,U_0}$ given by quotienting by the subgroup scheme $\mathbf{Z}/p \cong \sum_{j\in\mathbf{F}_p} f(j)$. This isogeny defines an *étale* morphism $\mathcal{O}_{\mathbf{G}_{0,U_0}} \to \mathcal{O}_{\mathbf{G}_{0,U_0}}$; so [Lur16, Theorem 7.5.0.6] implies that the isogeny q_0 lifts to a map $q : \mathbf{G}_U \to \mathbf{G}_U$ over $\operatorname{Spec}\Phi^{\mathbf{Z}/p}k$. (In general, q_0 is to be understood as an analogue for $\mathbf{G}_0$ of the Artin-Schreier map on $\mathbf{G}_a$.) The map (19) is then given by the composite

$$\mathbf{G}_U \xrightarrow{q} \mathbf{G}_U \simeq \mathbf{G} \times_{\operatorname{Spec}k} \operatorname{Spec}\Phi^{\mathbf{Z}/p}k \xrightarrow{\mathrm{pr}} \mathbf{G}.$$

We will denote its underlying map by

$$\varphi_0 : \mathbf{G}_0 \times_{\operatorname{Spec}\pi_0(k)} \operatorname{Spec}\pi_0(\Phi^{\mathbf{Z}/p}k) \to \mathbf{G}_0.$$

Example 9.7. Let us explicate Construction 9.6 in two examples.

(a) Let $k = \mathbf{Z}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$. Then $U_0 = \operatorname{Spec}\mathbf{F}_p[t^{\pm 1}]$, and the isogeny $q : \mathbf{G}_{0,U_0} \to \mathbf{G}_{0,U_0}$ is given by the Artin-Schreier map

$$x \mapsto x - t^{p-1}x^p.$$

(b) Let $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$ with coordinate y. Then $U_0 = \operatorname{Spec}\mathbf{Z}[\zeta_p][1/p]$, and $q : \mathbf{G}_{0,U_0} \to \mathbf{G}_{0,U_0}$ is given by the map

$$y \mapsto 1 + \prod_{j\in\mathbf{F}_p} (y - \zeta_p^j) = y^p.$$

Remark 9.8. Let us mention for the sake of completeness that one can interpolate between the two cases in Example 9.7, using the group scheme $\mathbf{G} = \mathbf{G}_\beta$ over connective complex K-theory ku studied in [Dev24]. (Using this, the results discussed below can be extended to the case $k = \mathrm{ku}$, too, but we will not address this here.) Explicitly, $\mathbf{G}_0$ is the group scheme over $\mathbf{Z}[\beta]$ given by $\mathbf{Z}[\beta, v^{\pm 1}][\frac{v-1}{\beta}]$, where the group law is determined by $v \mapsto v \otimes v$. We will (perhaps unexpectedly) define $t^{-1} := \frac{v-1}{\beta}$. Define the scheme

$$U_0 = \operatorname{Spec}\mathbf{Z}[\beta, v^{\pm 1}][\tfrac{v-1}{\beta}, \tfrac{\beta^{p-1}}{(v-1)\cdots(v^{p-1}-1)}]/\tfrac{v^p-1}{\beta}.$$

Note that $v = \zeta_p$ is a primitive pth root of unity, and $\beta = \frac{\zeta_p - 1}{t^{-1}}$. The scheme U_0 is rather remarkable: its fiber the locus where β is a unit is precisely $\operatorname{Spec}\mathbf{Z}[\zeta_p, \beta^{\pm 1}][1/p]$, while its fiber over $\beta = 0$ is given by $\operatorname{Spec} \mathbf{F}_p[t^{\pm 1}]$. (In homotopy theory, U_0 arises as $\operatorname{Spec} \pi_*(\Phi^{\mathbf{Z}/p}\mathrm{ku})$, where ku is connective complex K-theory.)

Let y denote the invertible coordinate on $\mathbf{G}_{0,U_0}$, and let $x = \frac{y-1}{\beta}$. Then the map $q : \mathbf{G}_{0,U_0} \to \mathbf{G}_{0,U_0}$ is given by the map $y \mapsto y^p$ and $\beta \mapsto p\beta$, so that it sends

$$q : x = \tfrac{y-1}{\beta} \mapsto \tfrac{y^p-1}{p\beta} = \tfrac{(1+\beta x)^p - 1}{p\beta}.$$

We claim that, as a morphism over $\operatorname{Spec} \mathbf{Z}[\beta]$, this map interpolates between the isogenies of Cases a and b in Example 9.7. First, it is obvious that when β is a unit, we simply recover Case b. Next, let us consider the fiber over $\beta = 0$. Recall that $\beta = (\zeta_p - 1)t$, so the binomial theorem gives

$$q(x) = \tfrac{1}{p} \sum_{i=1}^{p} \binom{p}{i} \beta^{i-1} x^i = \sum_{i=1}^{p} \tfrac{(\zeta_p - 1)^{i-1}}{p} \binom{p}{i} t^{i-1} x^i.$$

Almost all terms vanish modulo β, except for the terms $i = 1, p$; one is left with

$$q(x) \equiv x + \tfrac{(\zeta_p-1)^{p-1}}{p} t^{p-1} x^p = x + \tfrac{t^{p-1}x^p}{[1]_{\zeta_p} \cdots [p-1]_{\zeta_p}} = x - t^{p-1}x^p \pmod{\beta},$$

as desired. (Here, $[j]_q = \frac{q^j - 1}{q-1}$ is the q-integer corresponding to $j \in \mathbf{Z}$; we are using the fact that $[1]_{\zeta_p} \cdots [p-1]_{\zeta_p} \equiv -1 \pmod{\zeta_p - 1}$, which amounts to the fact that $(p-1)! \equiv -1 \pmod{p}$.) In general, ku gives a degeneration of power operations/the pth Adams operation on KU to power operations/the Steenrod algebra action on ordinary cohomology; this is very classical, and goes back, e.g., to [Ati66, Proposition 6.4 and Theorem 6.5].

For any compact torus T_c, we obtain a map

$$\varphi_T : \mathcal{M}_T \times_{\operatorname{Spec} k} \operatorname{Spec} \Phi^{\mathbf{Z}/p} k \to \mathcal{M}_T,$$

whose underlying map on classical $\pi_0(k)$-schemes will be denoted by $\varphi_{T,0}$. If X is any (ind-)finite T_c-space X, we then obtain maps

$$\mathcal{F}_T(X) \to \varphi_{T,*}\varphi_T^* \mathcal{F}_T(X),\ \mathcal{F}_T(X)^\vee \to \varphi_{T,*}\varphi_T^*(\mathcal{F}_T(X)^\vee).$$

We will denote these maps by $\varphi_{T,X}$, and call them the *T_c-equivariant decompleted Frobenius*. Note that $\varphi_{T,X}$ on $\mathcal{F}_T(X)$ is a map of $\mathbf{E}_\infty$-algebras in $\mathrm{QCoh}(\mathcal{M}_T)$, and similarly $\varphi_{T,X}$ on $\mathcal{F}_T(X)^\vee$ is a map of $\mathbf{E}_\infty$-coalgebras in $\mathrm{QCoh}(\mathcal{M}_T)$.

Remark 9.9. It is easy to see that if $\mathcal{H}(\mathbf{G}_0, T, W)$ denotes the nil-Hecke algebra from Definition 6.15 associated to a root system with torus T and Weyl group W, then the map $\varphi_{T,0}$ induces a map $\mathcal{H}(\mathbf{G}_0, T, W) \to \mathcal{H}(\mathbf{G}_0, T, W) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p}k)$. This map is very interesting, but we will postpone a detailed study of its combinatorial implications to a future article. When $\mathbf{G}_0 = \mathbf{G}_a$, for instance, this map describes the total Steenrod operation on the nil-Hecke algebra; similar ideas are explored in [Kit13, BC18].

We will now study the case $X = \mathrm{Gr}_G$, where G is connected, almost simple, and simply-laced over $\mathbf{C}$. For notational simplicity, we will write $\mathrm{Loc}_T^{\mathrm{gr}}(\mathrm{Gr}_G; \Phi^{\mathbf{Z}/p}k)$ to denote the tensor product $\mathrm{Loc}_T^{\mathrm{gr}}(\mathrm{Gr}_G; k) \otimes_{\pi_0 k} \pi_0(\Phi^{\mathbf{Z}/p}k)$. The T_c-equivariant decompleted Frobenius on $\mathcal{F}_T(\mathrm{Gr}_G)^\vee$ induces a functor

$$(\varphi_{T,\mathrm{Gr}_G})_* : \mathrm{Loc}_{T_c}^{\mathrm{gr}}(\mathrm{Gr}_G; k) \to \mathrm{Loc}_{T_c}^{\mathrm{gr}}(\mathrm{Gr}_G; \Phi^{\mathbf{Z}/p}k). \tag{20}$$

Moreover, the homomorphism $\varphi_{\check{T},0}$ induces a map in the opposite direction on Cartier duals, and hence a morphism

$$\varphi_{T,0} : \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) \times_{\mathrm{Spec}\,\pi_0(k)} \mathrm{Spec}\,\pi_0(\Phi^{\mathbf{Z}/p}k) \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee). \tag{21}$$

Remark 9.10. In fact, it follows from Remark 9.3 that one can replace $\pi_0(\Phi^{\mathbf{Z}/p}k)$ above by the fiber product $\pi_0(\Phi^{\mathbf{Z}/p}k) \times_{\pi_0\mathcal{F}_k(B\mathbf{Z}/p)[1/t]} \pi_0(\mathcal{F}_k(B\mathbf{Z}/p))/I_{\mathrm{tr}}$, which has the effect of working in a "t-lattice" inside $\pi_0(\Phi^{\mathbf{Z}/p}k)$. For simplicity, we will ignore this point below, and just work with $\pi_0(\Phi^{\mathbf{Z}/p}k)$.

The following says that the map (21) is precisely the effect of the T_c-equivariant decompleted Frobenius under Langlands duality.

Theorem 9.11. *Under the equivalence of Theorem 1.4 as rephrased in Remark 8.13 (which continues to hold true in the case $k = \mathbf{Z}[u^{\pm 1}]$, at least upon inverting enough primes), the functor $(\varphi_{\check{T},\mathrm{Gr}_G})_*$ of* (20) *identifies with the functor given by pullback along the map* (21)*. That is, the following diagram commutes:*

$$\begin{tikzcd} F \otimes_{\pi_0(k)} \mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k) \arrow[r, "(\varphi_{\check{T},\mathrm{Gr}_G})_*"] \arrow[d, "\sim"'] & F \otimes_{\pi_0(k)} \mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; \Phi^{\mathbf{Z}/p}k) \arrow[d, "\sim"] \\ \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}) \arrow[r, "\varphi^*_{\check{T},0}"'] & \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p}k). \end{tikzcd}$$

Proof. The argument is essentially that of Proposition 7.23, so we only give a sketch. Let us begin by observing that if $\kappa : \mathcal{M}_{\check{T},0} \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)$ denotes the Kostant section, there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\check{T},0} \times_{\mathrm{Spec}\,\pi_0(k)} \mathrm{Spec}\,\pi_0(\Phi^{\mathbf{Z}/p}k) \arrow[r, "\varphi_{\check{T},0}"] \arrow[d, "\kappa"'] & \mathcal{M}_{\check{T},0} \arrow[d, "\kappa"] \\ \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) \times_{\mathrm{Spec}\,\pi_0(k)} \mathrm{Spec}\,\pi_0(\Phi^{\mathbf{Z}/p}k) \arrow[r, "\varphi_{\check{T},0}"'] & \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee). \end{tikzcd}$$

The proof of Theorem 1.4 shows that it suffices to prove that under the isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\check{T},0}}(\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee) \cong \mathcal{M}_{\check{T},0} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{T},0}, \tag{22}$$

the $\check{T}_c$-equivariant decompleted Frobenius on $\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee$ identifies with the effect of the map $\varphi_{\check{T},0}$ on the right-hand side. For brevity, we will phrase this condition as the "Frobenius-equivariance" of (22).

Let $\mathcal{M}^{\mathrm{gen}}_{\check{T},0} \subseteq \mathcal{M}_{\check{T},0}$ denote the complement of $\bigcup_\alpha \mathcal{M}_{\check{T}_\alpha,0}$ as α ranges over the roots of $\check{G}$, and $\check{T}_\alpha$ denotes the kernel of the map $\alpha : \check{T} \to \mathbf{G}_m$. Since both sides of (22) are flat over $\mathcal{M}_{\check{T},0}$, their sheaves of functions inject into the corresponding localizations along the map $\mathcal{M}^{\mathrm{gen}}_{\check{T},0} \subseteq \mathcal{M}_{\check{T},0}$. It therefore suffices to show that when restricted to $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, the isomorphism of (22) is Frobenius-equivariant.

By Lemma 3.11, there is an isomorphism

$$\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}} \cong \pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_T)^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}} \cong \mathcal{O}_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}}[\mathbb{X}_*(T)].$$

Under this isomorphism, the $\check{T}_c$-equivariant decompleted Frobenius is given simply by the Frobenius on $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, and acts trivially on $\mathbb{X}_*(T)$. Similarly, there is an isomorphism

$$(\mathcal{M}_{\check{T},0} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{T},0}) \times_{\mathcal{M}_{\check{T},0}} \mathcal{M}^{\mathrm{gen}}_{\check{T},0} \cong \mathcal{M}^{\mathrm{gen}}_{\check{T},0} \times \check{T}.$$

Under this isomorphism, the action of $\varphi_{\check{T},0}$ is given simply by the Frobenius on $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, and acts trivially on $\check{T}$. It is clear that this matches with the Frobenius on $\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}}$, as desired. $\square$

The entire discussion above can be adapted without much difficulty to the setting of G_c-equivariant local systems. If G is almost simple and simply-laced, and has torsion-free fundamental group, then the analogue of Theorem 9.11 states the following. Under the equivalence of Remark 8.13 (which continues to hold true in the case $k = \mathbf{Z}[u^{\pm 1}]$, at least upon inverting enough primes), the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k) \arrow[r, "(\varphi_{\check{G},\mathrm{Gr}_G})_*"] \arrow[d, "\sim"'] & \mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \Phi^{\mathbf{Z}/p} k) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}) \arrow[r, "\varphi^*_{\check{G},0}"'] & \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p} k).
\end{tikzcd}$$

The top and bottom maps are defined just as in (20) and (21).

Let us now explicate Theorem 9.11 in two examples; since the description in the case of elliptic cohomology is not much more explicit than the statement of Theorem 9.11 (that is, that the decompleted Frobenius on $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(E)$ is induced by the degree p étale isogeny $E \to E$ over $\mathrm{Spec}\, \pi_0(\Phi^{\mathbf{Z}/p} k)$), we will only focus on the cases of ordinary cohomology and complex K-theory below for simplicity.

Example 9.12. Let $k = \mathbf{Z}[u^{\pm 1}]$, $\mathbf{G} = \mathbf{G}_a$, and invert $N \gg 0$ so that the equivalence of Corollary 6.8 continues to hold: that is, so that there is an equivalence $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k) \simeq \mathrm{QCoh}(\check{\mathfrak{b}}^{\mathrm{reg}}/\check{B})$. Such an integer necessarily exists, by spreading out the isomorphism of Theorem 6.3 over $\mathrm{Spec}\, \mathbf{Z}[1/N]$.[18] Under the identification $\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) \cong \check{\mathfrak{b}}/\check{B}$, the map (21) is given (for $p \nmid N$) by the map

$$\varphi_{\check{T},0} : (\check{\mathfrak{b}} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{F}_p[t^{\pm 1}])/\check{B} \to \check{\mathfrak{b}}/\check{B}$$

which is the $\check{B}$-quotient of the map

$$\check{\mathfrak{b}} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{F}_p[t^{\pm 1}] \to \check{\mathfrak{b}}, \ (x, t) \mapsto x - t^{p-1} x^{[p]}.$$

Here, $x^{[p]}$ denotes the restricted Lie operation on $\check{\mathfrak{b}}$. It follows from Theorem 9.11 that this map implements the action of the decompleted Frobenius/Steenrod operations on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; \mathbf{Z}[u^{\pm 1}])$ (upon inverting $N \gg 0$).

Similarly, under the identification $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee) \cong \check{\mathfrak{g}}/\check{G}$, the analogue of the map (21) is given (for $p \nmid N$) by the map

$$\varphi_{\check{G},0} : (\check{\mathfrak{g}} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{F}_p[t^{\pm 1}])/\check{G} \to \check{\mathfrak{g}}/\check{G}$$

which is the $\check{G}$-quotient of the map

$$\check{\mathfrak{g}} \times_{\mathrm{Spec}\, \mathbf{Z}} \mathrm{Spec}\, \mathbf{F}_p[t^{\pm 1}] \to \check{\mathfrak{g}}, \ (x, t) \mapsto x - t^{p-1} x^{[p]}.$$

[18] In future work, we will show that one can take $N = 1$.

Again, this map implements the action of the decompleted Frobenius/Steenrod operations on $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathbf{Z}[u^{\pm 1}])$ (upon inverting $N \gg 0$) under the equivalence between $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathbf{Z}[u^{\pm 1}])$ and $\mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}/\check{G})$.

For instance, suppose $G = \mathrm{SL}_2$, and assume $p > 2$. When restricted to the Kostant slice $f + \check{\mathfrak{g}}^e = \{(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix})\} \subseteq \check{\mathfrak{g}} = \mathfrak{pgl}_2$, the map $\varphi_{\check{G},0}$ sends

$$((\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}), t) \mapsto \begin{pmatrix} 0 & x - t^{p-1}x^{(p+1)/2} \\ 1 - t^{p-1}x^{(p-1)/2} & 0 \end{pmatrix}.$$

This is conjugate to the matrix $\begin{pmatrix} 0 & x(1-t^{p-1}x^{(p-1)/2})^2 \\ 1 & 0 \end{pmatrix}$, so we find that $\varphi_{\check{G},0}$ is given in coordinates by the map

$$\varphi_{\check{G},0} : x \mapsto x - 2t^{p-1}x^{(p+1)/2} + t^{2(p-1)}x^p = \prod_{j \in \mathbf{F}_p} (x - j^2 t^2)$$

on $f + \check{\mathfrak{g}}^e$. Under the isomorphism $f + \check{\mathfrak{g}}^e \cong \operatorname{Spec} \mathrm{H}^*_{\mathrm{SU}(2)}(*; \mathbf{Z})$, the coordinate x identifies with the first Pontryagin class p_1; and $\varphi_{\check{G},0}(x)$ is exactly the total Steenrod operation on this class, as expected.

Example 9.13. Let $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$. Under the identification $\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) \cong \check{B}/\check{B}$, the map (21) is given by the $\check{B}$-quotient of the pth power map on $\check{B}$. That is, if F is an algebraically closed field, then under the equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}(\check{B}^{\mathrm{reg}}/\check{B})$$

of Corollary 7.8, the decompleted Frobenius on the left-hand side (which encodes the pth Adams operation on KU) identifies with the pth power map on $\check{B}^{\mathrm{reg}}$. Similarly, under the equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathrm{KU}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}(\check{G}^{\mathrm{reg}}/\check{G}),$$

the decompleted Frobenius on the left-hand side (which encodes the pth Adams operation on KU) identifies with the pth power map on $\check{G}^{\mathrm{reg}}$.

For instance, suppose $\check{G} = \mathrm{SL}_2$, and assume $p > 2$. When restricted to the Kostant slice inside $\check{G} = \mathrm{SL}_2$ of matrices of the form $\left(\begin{smallmatrix} x-1 & x-2 \\ 1 & 1 \end{smallmatrix}\right)$, the map $\varphi_{\check{G},0}$ is given by raising to the pth power. It turns out that

$$\left(\begin{smallmatrix} x-1 & x-2 \\ 1 & 0 \end{smallmatrix}\right)^p \text{ is conjugate to } \kappa(x) = \left(\begin{smallmatrix} L_p(x)-1 & L_p(x)-2 \\ 1 & 1 \end{smallmatrix}\right),$$

where $L_n(x)$ is (a slight modification of) the "Lucas polynomial": namely,

$$L_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \tfrac{n}{n-j} \binom{n-j}{j} x^{n-2j}.$$

We therefore find that $\varphi_{\check{G},0}$ is given in coordinates by the map

$$\varphi_{\check{G},0}(x) = L_p(x).$$

Under the isomorphism between the Kostant slice for SL_2 and $\operatorname{Spec} \pi_0 \mathrm{KU}_{\mathrm{SU}(2)}$, the coordinate x identifies with the KU-theoretic Pontryagin class; and $\varphi_{\check{G},0}(x)$ is exactly the pth Adams operation on this class.

In fact, one can interpolate between Example 9.12 and Example 9.13 using the results of [Dev24] and Remark 9.8; we leave the formulation of the relevant statement to the reader. Using an argument similar to Remark 9.8, one can show that the pth power map on $\check{G}$ degenerates to the Artin-Schreier map on $\check{\mathfrak{g}}$ from Example 9.12. In the case $\check{G} = \mathrm{SL}_2$, for instance, the decompleted Frobenius on the Kostant slice/$\operatorname{Spec} \pi_0 \mathrm{ku}_{\mathrm{SU}(2)}$ is closely related to the "Dickson polynomial" [Dic97] of degree p with parameter given by the Bott class β.

The structures imposed by Theorem 9.11 are quite rigid. For instance, there is an action of $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k)$ by convolution, which, under the equivalences of Theorem 1.4 as rephrased in Remark 8.13, defines an action of $\mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}})$ on $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)^{\mathrm{reg}})$. This action is given by pullback along the map $\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee) \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)$, and it is compatible with power operations.

Example 9.14. When $k = \mathbf{Z}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$ (where we again invert some $N \gg 0$ so that the equivalence of Corollary 6.8 holds), the action of $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k)$ by convolution identifies with the action of $\mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{G})$ on $\mathrm{QCoh}(\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{B})$ via pullback along the map $\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{B} \to \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{G}$. It follows from Example 9.12 that this map is compatible with the decompleted Frobenius/Steenrod operations.

The composite map

$$\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{N} \to \check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{B} \to \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{G}$$

can be realized as the $\check{G}$-quotient of the restriction to regular loci of the moment map $\mu : T^*(\check{G}/\check{N}) \to \check{\mathfrak{g}}^*$. The action of the decompleted Frobenius/Steenrod operations on the regular locus of $T^*(\check{G}/\check{N})$ in fact extends to all of $T^*(\check{G}/\check{N})$ itself (and hence on its affine closure $\overline{T^*(\check{G}/\check{N})}$), and the moment map μ is equivariant for this action. The action of the decompleted Frobenius on $\overline{T^*(\check{G}/\check{N})}$ commutes with the Gelfand-Graev action of the Weyl group from Proposition 6.13; this can be seen by reducing to the rank 1 case described below (with a bit of care in keeping track of the difference between $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$).

An explicit description of this action when $\check{G} = \mathrm{SL}_2$ is as follows. If we identify $\overline{T^*(\check{G}/\check{N})} = T^*(\mathbf{A}^2)$ with coordinates $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$, then the total power operation is given by the map

$$\varphi : (u, v) \mapsto (u, v - t^{p-1} v \langle u, v \rangle^{p-1}).$$

Since the moment map $T^*(\mathbf{A}^2) \to \mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$ sends $(u, v) \mapsto \left(\begin{smallmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{smallmatrix}\right)$, it is easy to check that this map is compatible with the action of the decompleted Frobenius on $\mathfrak{sl}_2^*$ as described in Example 9.12.

Example 9.15. When $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, the action of $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k)$ by convolution identifies with the action of $\mathrm{QCoh}(G^{\mathrm{reg}}/\check{G})$ on $\mathrm{QCoh}(B^{\mathrm{reg}}/\check{B})$ via pullback along the map $B^{\mathrm{reg}}/\check{B} \to G^{\mathrm{reg}}/\check{G}$. It follows from Example 9.12 that this map is compatible with the decompleted Frobenius/pth Adams operation. The composite map

$$B^{\mathrm{reg}}/\check{N} \to B^{\mathrm{reg}}/\check{B} \to G^{\mathrm{reg}}/\check{G}$$

can be realized as the $\check{G}$-quotient of the restriction to regular loci of the multiplicative moment map $\mu : \check{G} \times^{\check{N}} B \to G$. The action of the decompleted Frobenius/pth Adams operation on the regular locus of $\check{G} \times^{\check{N}} B$ in fact extends to all of $\check{G} \times^{\check{N}} B$ itself (and hence on its affine closure $\overline{\check{G} \times^{\check{N}} B}$), and the moment map μ is equivariant for this action. The action of the decompleted Frobenius on $\overline{\check{G} \times^{\check{N}} B}$ commutes with the Gelfand-Graev action of the Weyl group from Proposition 7.14; this can be seen by reducing to the rank 1 case described below (with a bit of care in keeping track of the difference between $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$).

An explicit description of the action of the decompleted Frobenius when $\check{G} = \mathrm{SL}_2$ is as follows. As in Example 7.15, we may identify $\overline{\check{G} \times^{\check{N}} B}$ with an open subset of $T^*(\mathbf{A}^2)$ with coordinates $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$. The total power operation is then given by the map

$$\varphi : (u, v) \mapsto \left(u, v\tfrac{(1+\langle u,v\rangle)^p - 1}{\langle u,v\rangle}\right).$$

Since the moment map $\overline{\check{G} \times^{\check{N}} B} \to \mathrm{PGL}_2$ sends $(u, v) \mapsto \left(\begin{smallmatrix} 1+u_1v_1 & u_1v_2 \\ u_2v_1 & 1+u_2v_2 \end{smallmatrix}\right)$, it is easy to check that this map is compatible with the action of the pth power map on PGL_2 as described in Example 9.13. In checking that the total power operation is compatible with the Gelfand-Graev action as described in Example 7.15, the basic input is the identity $q^{-1}[p]_{q^{-1}} = q^{-p}[p]_q$ applied to $q = 1 + \langle u, v\rangle$.

More generally, (a mild variant of) the relative Langlands program from [BZSV23] predicts that if X is an affine spherical G-variety, there exists a graded affine Hamiltonian $\check{G}$-variety $\check{M}$ over $\mathbf{Z}$ (possibly with an integer $N \gg 0$ inverted) with moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$ such that there is an equivalence

$$\mathrm{Shv}^c_{G[\![t]\!]}(X(\!(t)\!); \mathbf{Z}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G}).$$

Here, $\mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G})$ denotes the ∞-category of perfect complexes on the shearing of $\check{M}$ with respect to its grading. Moreover, under a $\mathbf{Z}$-linear analogue of the derived geometric Satake equivalence, the natural action of $\mathrm{Shv}^c_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{Z})$ on the left-hand side by convolution should identify with the action of $\mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G})$ on $\mathrm{Perf}^{\mathrm{sh}}(\check{M}/\check{G})$ via pullback along the moment map. This equivalence will restrict (and degenerate) to an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{G[\![t]\!]}(X(\!(t)\!); \mathbf{Z}) \simeq \mathrm{Perf}(\check{M}^{\mathrm{reg}}/\check{G})$$

for some open $\check{M}^{\mathrm{reg}} \subseteq \check{M}$, which again satisfies a form of Hecke compatibility. Following the discussion above, the left-hand side will admit an action of the decompleted Frobenius/Steenrod operations, and so one expects the right-hand side to also admit such a structure. That is to say, $\check{M}$ (or more canonically, the stack $\check{M}/\check{G}$) should admit an action of the decompleted Frobenius, and the moment map $\mu : \check{M} \to \check{\mathfrak{g}}^*$ should be compatible with this action; here $\check{\mathfrak{g}}^*$ is equipped with the action of the decompleted Frobenius described in Example 9.12. These extra symmetries on $\check{M}$ are very interesting, and we expect them to play an important role in positive-characteristic analogues of the relative Langlands program.

In [Dev24], we propose a version of this picture for sheaves with coefficients in KU (and more generally in ku): the main difference is that $\check{M}$ must be replaced by a *quasi-Hamiltonian* $\check{G}$-variety in the sense of [AMM98], so that its moment map goes from $\check{M}$ to $\check{G}$. Again, $\check{M}$ should admit an action of the decompleted Frobenius/pth Adams operation on KU, and the multiplicative moment map $\check{M} \to \check{G}$ should be compatible with this action, where the action of the decompleted Frobenius on $\check{G}$ is as described in Example 9.13. Outside of simple cases like Example 9.15, the quasi-Hamiltonian varieties can be quite complicated; so we will not discuss this case below.

Let us present two explicit and nontrivial examples of "Frobenius compatibility" in the context of relative Langlands.

Example 9.16 (Symplectic period)**.** The "quaternionic" Satake equivalence of [CMNO22] says that there is an equivalence

$$\mathrm{Shv}^c_{\mathrm{GL}_{2n}[\![t]\!]}(\mathrm{GL}_{2n}(\!(t)\!)/\mathrm{Sp}_{2n}(\!(t)\!); \mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{M}/\mathrm{GL}_{2n}),$$

where $\check{M} \cong \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n^*$ is equipped with a particular Hamiltonian structure. (Here, GL_n sits diagonally inside GL_{2n}.) Such an equivalence will continue to hold over $\mathbf{Z}[1/N]$ for some $N \gg 0$, so we may consider the decompleted Frobenius for $p \nmid N$. In particular, we will assume $p > 2$. The moment map $\check{M} \to \mathfrak{gl}_{2n}^*$ is induced by the inclusion $\mathfrak{gl}_n^* \to \mathfrak{gl}_{2n}^*$ sending $\mu : x \mapsto \left(\begin{smallmatrix} 0 & x \\ \mathrm{id}_n & 0 \end{smallmatrix}\right)$. Unwinding the proof of [CMNO22] shows that the decompleted Frobenius/Steenrod algebra acts on $\check{M}$ via the map

$$\varphi : x \mapsto x - 2t^{p-1}x^{(p+1)/2} + t^{2(p-1)}x^p = \prod_{j \in \mathbf{F}_p} (x - j^2 t^2 \mathrm{id}_n)$$

on $\mathfrak{gl}_n^*$. (Observe that the formula for φ is a matrix version of the total Steenrod operation on $\mathrm{H}^*_{\mathrm{SU}(2)}(*; \mathbf{F}_p)$.) If $x \in \mathfrak{gl}_n^*$, it is *not* true that $\varphi(\mu(x)) = \mu(\varphi(x))$; but these two elements of $\mathfrak{gl}_{2n}^*$ are conjugate, from which it follows that the moment map $\check{M}/\mathrm{GL}_{2n} \to \mathfrak{gl}_{2n}^*/\mathrm{GL}_{2n}$ is equivariant for the action of the decompleted Frobenius.

The following example generalizes Example 9.14 and Example 9.15.

Example 9.17 (Mirabolic Satake)**.** In [BFGT21], it was shown that there is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{GL}_{n-1}[\![t]\!]}(\mathrm{Gr}_{\mathrm{GL}_n}; \mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(T^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})),$$

where, if we identify $T^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$ with $\mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) \oplus \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$, the moment map $T^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1}) \to \mathfrak{gl}_n^* \times \mathfrak{gl}_{n-1}^*$ sends $(f, g) \mapsto (fg, gf)$. Such an equivalence will continue to hold over $\mathbf{Z}[1/N]$ for some $N \gg 0$, so we may consider the decompleted Frobenius for $p \nmid N$. Unwinding the proof of the above equivalence shows that the decompleted Frobenius/Steenrod algebra acts on $T^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$ via

$$\varphi : (f, g) \mapsto (f, g - t^{p-1} g (fg)^{p-1}).$$

It is easy to check that the moment map is indeed Frobenius-equivariant.

There is also a multiplicative version of this picture. Namely, it follows from [Dev24, Remark 4.3.4] that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{GL}_{n-1}[\![t]\!]}(\mathrm{Gr}_{\mathrm{GL}_n}; \mathrm{KU}) \simeq \mathrm{Perf}(\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1})^{\mathrm{reg}}/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})),$$

where $\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1})^{\mathrm{reg}}$ is a particular open subset inside Van den Bergh's variety from [Van08]:

$$\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1}) = \{(f, g) \in \mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) \oplus \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1}) | \mathrm{id} + fg \in \mathrm{GL}_n\}.$$

There is a multiplicative moment map $\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1}) \to \mathrm{GL}_n \times \mathrm{GL}_{n-1}$ which sends $(f, g) \mapsto (\mathrm{id} + fg, \mathrm{id} + gf)$. The decompleted Frobenius/pth Adams operation acts on $\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1})$ via

$$\varphi : (f, g) \mapsto (f, f^{-1}((\mathrm{id} + fg)^p - \mathrm{id})),$$

and again, the multiplicative moment map is Frobenius-equivariant.

10. Comparison to Brylinski-Zhang

In [BZ00], Brylinski-Zhang compute the G_c-equivariant complex K-theory of G_c for a connected compact Lie group G_c with torsion-free fundamental group as the ring $\Omega^*_{\mathrm{RU}(G)/\mathbf{Z}} = \Omega^*_{T/\!\!/W/\mathbf{Z}}$ of Kähler differentials on the complex representation ring of G. Our goal in this section is to describe the relationship between this calculation and (the proof of) Theorem 1.4.

We begin by stating an obvious corollary of Theorem 1.4. Recall that if $\mathbf{G}_0$ be either $\mathbf{G}_a$, $\mathbf{G}_m$, or an elliptic curve E, and $\mathcal{M}_{T,0} = \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G}_0)$, there is a Kostant section $\kappa : \mathcal{M}_{T,0} \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)$ as described in Definition 6.1, Definition 7.1, and Proposition 8.2. Recall that F is an algebraically closed field of characteristic zero containing $\pi_0(k)$.

Theorem 10.1. *Let G be a connected almost simple simply-laced group. Let k denote either $\mathbf{Q}[u^{\pm 1}]$, KU, or elliptic cohomology, and let $\mathbf{G}_0$ be either $\mathbf{G}_a$, $\mathbf{G}_m$, or an elliptic curve E over $\pi_0(k)$, respectively. Then there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(G_c; k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathcal{M}_{\check{T},0} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{T},0}),$$

where the right-hand side denotes the self-intersection of the Kostant slice.

Proof. Recall from Definition 4.4 that

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(G_c; k) = \mathrm{LMod}_{\pi_0(\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\check{T},0})).$$

In Theorem 6.3, Theorem 7.3, and Theorem 8.7, we showed that $\mathrm{Spec}_{\mathcal{M}_{\check{T},0}}(\pi_0(\mathcal{F}_{\check{T}}(\mathrm{Gr}_G)^\vee))$ is isomorphic to the self-intersection $\mathcal{M}_{\check{T},0} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{T},0}$, so the desired equivalence follows. □

In the same way, if G is further assumed to have torsion-free fundamental group, and $\mathcal{M}_{G,0}$ denotes the moduli *space* of semistable G-bundles on $\mathbf{G}_0^\vee$, there is a Kostant section $\kappa : \mathcal{M}_{G,0} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)$. In the additive and multiplicative cases, this follows from Definition 6.1, Definition 7.1, and in the elliptic case, it can be deduced from [Dav19] as in Proposition 8.2. Just as in Theorem 10.1, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(G_c; k) \otimes_{\pi_0(k)} F \simeq \mathrm{QCoh}(\mathcal{M}_{\check{G},0} \times_{\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{G},0}) \tag{23}$$

where the right-hand side denotes the self-intersection of the Kostant slice. Under this equivalence, the "constant sheaf" in $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(G_c; k)$ is sent to the pushforward of the structure sheaf under the relative diagonal

$$\delta : \mathcal{M}_{\check{G},0} \to \mathcal{M}_{\check{G},0} \times_{\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)} \mathcal{M}_{\check{G},0}.$$

In the remainder of this section, we will explain how (23) implies the calculation of [BZ00], as well as the relationship to the Hochschild-Kostant-Rosenberg theorem. (This, of course, is a triple of authors distinct from Hopkins-Kuhn-Ravenel with initials "HKR"!) For simplicity, we will only focus on the case when k is $\mathbf{Q}[u^{\pm 1}]$ or KU (so $\mathbf{G}_0$ is either $\mathbf{G}_a$ or $\mathbf{G}_m$, and $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)$ is either $\check{\mathfrak{g}}/\check{G}$ or $\check{G}/\check{G}$). With a little bit of elbow grease, one can show that most of the results below continue to work for elliptic cohomology, too.

Recall that $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(G_c; k)$ is intended to be an approximation to a k-linear ∞-category of G_c-equivariant local systems on G_c. The algebra of endomorphisms of the constant sheaf in this ∞-category is given by the equivariant cochains $\mathcal{F}_G(G_c)$. This is a quasicoherent sheaf over the spectral k-scheme $\mathcal{M}_G$, and it can be described explicitly as follows. If $\mathfrak{X}_k$ is a spectral prestack over k, let $\mathcal{L}\mathfrak{X}_k$ denote the free loop space of $\mathfrak{X}_k$, i.e., the mapping prestack $\mathrm{Map}(B\mathbf{Z}, \mathfrak{X}_k)$. Here, $\mathbf{Z}$ is viewed as a constant stack over k. The global sections of the structure sheaf of $\mathcal{L}\mathfrak{X}_k$ computes the Hochschild homology $\mathrm{HH}(\mathfrak{X}_k/k)$.

Proposition 10.2. *Assume (for simplicity) that k is either $\mathbf{Q}[u^{\pm 1}]$ or KU. If G is connected, then there is an isomorphism of spectral k-schemes*

$$\mathrm{Spec}_{\mathcal{M}_G}(\mathcal{F}_G(G_c)) \cong \mathcal{L}\mathcal{M}_G.$$

In particular, there is an isomorphism of $\mathbf{E}_\infty$-k_{G_c}-algebras

$$\mathcal{F}_G(G_c) \cong \mathrm{HH}(\mathcal{M}_G/k).$$

Proof. Recall that $B\mathbf{Z}$ is isomorphic to the constant k-stack S^1, which can be written as the pushout $* \amalg_{*\amalg *} *$. Therefore, since $\mathcal{M}_G = \mathrm{Spec}\, k_{G_c}$ is affine (because k is either $\mathbf{Q}[u^{\pm 1}]$ or KU), we may wite $\mathcal{L}\mathcal{M}_G = \mathrm{Spec}(k_{G_c} \otimes_{k_{G_c} \otimes_k k_{G_c}} k_{G_c})$. Since the functor $\mathcal{F}_G : \mathcal{S}(G_c)^{\mathrm{op}} \to \mathrm{Mod}_{k_{G_c}}$ sends finite products of connected finite G-spaces to tensor products, we find that

$$k_{G_c} \otimes_{k_{G_c} \otimes_k k_{G_c}} k_{G_c} \cong \mathcal{F}_G(*) \otimes_{\mathcal{F}_{G\times G}(*)} \mathcal{F}_G(*) \cong \mathcal{F}_G(G_c),$$

since there is an isomorphism of orbispaces

$$*/G_c \times_{*/(G_c\times G_c)} */G_c \cong G_c/G_c. \quad \square$$

Remark 10.3. The approach of Proposition 10.2 can be used to compute the equivariant cohomology $\mathcal{F}_G(\Omega G_c)$, too. Namely, observe that there is an isomorphism of orbispaces

$$*/G_c \times_{*/G_c \times_{*/(G_c\times G_c)} */G_c} */G_c \cong (\Omega G_c)/G_c,$$

so that there is an isomorphism

$$\mathcal{F}_G(\Omega G_c) = k_{G_c} \otimes_{k_{G_c} \otimes_{k_{G_c} \otimes_k k_{G_c}} k_{G_c}} k_{G_c}.$$

The right-hand side can be expressed more succinctly as the factorization homology $\int_{S^2}(k_{G_c}/k)$.

More generally, observe that if $K_c \subseteq G_c$ is a closed subgroup such that G_c/K_c is a finite K_c-space (where K_c acts on the left by multiplication), and $L(G_c/K_c)$ denotes the (topological) free loop space of G_c/K_c, then

$$G_c\backslash L(G_c/K_c) \simeq K_c\backslash\Omega(G_c/K_c) \simeq (* \times_{* \times_{*/G_c} */K_c} *)/K_c \simeq */K_c \times_{*/K_c \times_{*/G_c} */K_c} */K_c.$$

It follows that there is an isomorphism

$$\mathcal{F}_G(\mathcal{L}(G_c/K_c)) = k_{K_c} \otimes_{k_{K_c} \otimes_{k_{G_c}} k_{K_c}} k_{K_c}.$$

The right-hand side can be expressed more succinctly as the relative Hochschild homology $\mathrm{HH}(\mathcal{M}_K/\mathcal{M}_G)$. The discussion above computing $\mathcal{F}_K(\Omega K_c)$ is the special case of the above calculation when $G_c = K_c \times K_c$, with K_c embedded diagonally.

Example 10.4. Let $k = \mathbf{Q}[u^{\pm 1}]$. Then the preceding discussion shows that $C^*_{G_c}(\Omega G_c; \mathbf{Q}[u^{\pm 1}])$ is isomorphic to $\int_{S^2}(k_{G_c}/k) = \mathrm{HH}(k_{G_c}/k_{G_c} \otimes_k k_{G_c})$. The latter has a Hochschild-Kostant-Rosenberg filtration whose associated graded is given by the 2-periodification $L\Omega^*_{\mathfrak{t}/\!\!/W/(\mathfrak{t}/\!\!/W \times_{\mathrm{Spec}\,\mathbf{Q}} \mathfrak{t}/\!\!/W)}[u^{\pm 1}]$ of the derived Hodge complex of $\mathfrak{t}/\!\!/W$ embedded diagonally into $\mathfrak{t}/\!\!/W \times_{\mathrm{Spec}\,\mathbf{Q}} \mathfrak{t}/\!\!/W$. Since we are working rationally, the Hochschild-Kostant-Rosenberg filtration splits, and so there is an isomorphism

$$\int_{S^2}(k_{G_c}/k) \cong L\Omega^*_{\mathfrak{t}/\!\!/W/(\mathfrak{t}/\!\!/W \times_{\mathrm{Spec}\,\mathbf{Q}} \mathfrak{t}/\!\!/W)}[u^{\pm 1}].$$

Note that if X (like $\mathfrak{t}/\!\!/W$) is an affine space over a commutative ring R, then $L\Omega^*_{X/(X\times_{\mathrm{Spec}(R)}X)} \cong \Gamma^*(\Omega^1_{X/R})$; so the above isomorphism could instead be stated as

$$\int_{S^2}(k_{G_c}/k) \cong \mathrm{Sym}_{\mathcal{O}_{\mathfrak{t}/\!\!/W}}(\Omega^1_{\mathfrak{t}/\!\!/W})[u^{\pm 1}] = \mathcal{O}_{T(\mathfrak{t}/\!\!/W)}[u^{\pm 1}],$$

where $T(\mathfrak{t}/\!\!/W)$ is the tangent bundle of $\mathfrak{t}/\!\!/W$. It follows that there is an isomorphism

$$\mathrm{Spec}\, C^*_{G_c}(\Omega G_c; \mathbf{Q}[u^{\pm 1}]) \cong T(\mathfrak{t}/\!\!/W) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\mathbf{Q}[u^{\pm 1}]).$$

This recovers the $\hbar = 0$ case of [BF08, Theorem 1]. By keeping track of loop-rotation equivariance (namely, S^1 acting on S^2 by rotation about a fixed axis), the general result can also be established by similar arguments.

Let us now discuss the relationship between Proposition 10.2 and (23). Although the cases $k = \mathbf{Q}[u^{\pm 1}]$ and $k = \mathrm{KU}$ can be treated simultaneously, we will present the discussion separately for both for the sake of clarity. The upshot of this discussion is that the approximation to $\mathcal{F}_G(G_c)$ afforded by the degeneration of $\mathrm{Loc}_{G_c}(G_c; k)$ to $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k)$ identifies, under Proposition 10.2 and (23), with the Hochschild-Kostant-Rosenberg spectral approximation of $\pi_*\mathrm{HH}(\mathcal{M}_G/k)$ by $\Omega^*_{\mathcal{M}_{G,0}/\pi_0(k)}$.

Lemma 10.5. *Let H be a smooth affine group scheme over an affine scheme $S = \mathrm{Spec}(R)$, let $\delta : S \to H$ denote the zero section, and let $\mathfrak{h}$ denote its Lie algebra (viewed as a vector bundle over S). Then $\mathrm{End}_{\mathrm{QCoh}(H)}(\delta_*\mathcal{O}_S)$ has a filtration whose associated graded is isomorphic to $\mathcal{O}_{\mathfrak{h}^*[1]}$. If R is a $\mathbf{Q}$-algebra, this filtration splits.*

Proof. The endomorphism algebra $\mathrm{End}_{\mathrm{QCoh}(H)}(\delta_*\mathcal{O}_S)$ is isomorphic to the R-linear dual of $\mathcal{O}_{S\times_H S}$. The derived scheme $S \times_H S$ depends only on the formal completion $\hat{H}$. Note that $\hat{H}$ admits a filtration (coming from powers of the ideal sheaf of the zero section of $\hat{H}$) whose associated graded is isomorphic to $\hat{\mathfrak{h}}$; furthermore, the exponential map defines a splitting of this filtration when R is a $\mathbf{Q}$-algebra. This defines a filtration on $S \times_H S$ whose associated graded is isomorphic to $S \times_{\mathfrak{h}} S = \mathfrak{h}[-1]$. Therefore, the R-linear dual of $\mathcal{O}_{S\times_H S}$ is isomorphic to $\mathcal{O}_{\mathfrak{h}^*[1]}$. □

Example 10.6. Suppose $k = \mathbf{Q}[u^{\pm 1}]$, and let $\check{J} = \mathfrak{t}/\!\!/W \times_{\check{\mathfrak{g}}^*/\check{G}} \mathfrak{t}/\!\!/W$. Then (23) states that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c; k) \otimes_{\mathbf{Q}} F \simeq \mathrm{QCoh}(\check{J}),$$

and the "constant sheaf" $\underline{k}^{\mathrm{gr}}$ in $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(G_c; k)$ is sent to the pushforward of the structure sheaf under the identity section $\delta : \mathfrak{t}/\!\!/W \to \check{J}$. Taking endomorphisms, we find that

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Q}} F \cong \mathrm{End}_{\mathrm{QCoh}(\check{J})}(\delta_*\mathcal{O}_{\mathfrak{t}/\!\!/W}).$$

By Lemma 10.5, the right-hand side admits a (split) filtration whose associated graded is isomorphic to the algebra of functions on $\mathrm{Lie}_{\mathfrak{t}/\!\!/W}(\check{J})^*[1]$. By [Ric17, Theorem 3.4.2], one finds that the Lie algebra $\mathrm{Lie}_{\mathfrak{t}/\!\!/W}(\check{J})$ is isomorphic to the cotangent bundle $T^*(\mathfrak{t}/\!\!/W)$, so that $\mathrm{Lie}_{\mathfrak{t}/\!\!/W}(\check{J})^*[1]$ is isomorphic to $T[1](\mathfrak{t}/\!\!/W)$. Its ring of functions is precisely the Hodge cohomology $\Omega^*_{\mathfrak{t}/\!\!/W/F} = \bigoplus(\Omega^i_{\mathfrak{t}/\!\!/W/F})[-i]$ of $\mathfrak{t}/\!\!/W$. Summarizing, we have found that there is an isomorphism

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Q}} F \cong \Omega^*_{\mathfrak{t}/\!\!/W/F}.$$

On the other hand, it follows from the constructions in Section 4 that there is a filtration on $\mathcal{F}_G(G_c) = \mathrm{End}_{\mathrm{Loc}_{G_c}(G_c;k)}(\underline{k})$ whose associated graded is $\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;k)}(\underline{k}^{\mathrm{gr}})[u^{\pm 1}]$. By the above discussion, the latter is $\Omega^*_{\mathfrak{t}/\!\!/W/F}$. Proposition 10.2 shows that $\mathcal{F}_G(G_c)\otimes_k F[u^{\pm 1}]$ is isomorphic to the Hochschild homology $\mathrm{HH}(\mathfrak{t}/\!\!/W/F)[u^{\pm 1}]$. There is therefore a filtration on $\mathrm{HH}(\mathfrak{t}/\!\!/W/F)[u^{\pm 1}]$ whose associated graded is $\Omega^*_{\mathfrak{t}/\!\!/W/F}[u^{\pm 1}]$. This filtration is precisely the Hochschild-Kostant-Rosenberg filtration on Hochschild homology (see, e.g., [Ant18, Rak20]).

Example 10.7. Suppose $k = \mathrm{KU}$, and assume G is simply-laced and has torsion-free fundamental group. Let $\check{J}_\mu = T/\!\!/W \times_{G/\check{G}} T/\!\!/W$. Then (23) states that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;\mathrm{KU}) \otimes_{\mathbf{Z}} F \simeq \mathrm{QCoh}(\check{J}_\mu),$$

and the "constant sheaf" $\underline{\mathrm{KU}}^{\mathrm{gr}}$ in $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(G_c;\mathrm{KU})$ is sent to the pushforward of the structure sheaf under the identity section $\delta : T/\!\!/W \to \check{J}_\mu$. Taking endomorphisms, we find that

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;\mathrm{KU})}(\underline{\mathrm{KU}}^{\mathrm{gr}}) \otimes_{\mathbf{Z}} F \cong \mathrm{End}_{\mathrm{QCoh}(\check{J}_\mu)}(\delta_* \mathcal{O}_{T/\!\!/W}).$$

By Lemma 10.5, the right-hand side admits a (split) filtration whose associated graded is isomorphic to the algebra of functions on $\mathrm{Lie}_{T/\!\!/W}(\check{J}_\mu)^*[1]$. There is a multiplicative analogue of [Ric17, Theorem 3.4.2] which states the Lie algebra $\mathrm{Lie}_{T/\!\!/W}(\check{J}_\mu)$ is isomorphic to the cotangent bundle $T^*(T/\!\!/W)$. In particular, $\mathrm{Lie}_{T/\!\!/W}(\check{J}_\mu)^*[1]$ is isomorphic to $T[1](T/\!\!/W)$. Its ring of functions is precisely the Hodge cohomology $\Omega^*_{T/\!\!/W/F} = \bigoplus(\Omega^i_{T/\!\!/W/F})[-i]$ of $T/\!\!/W$. Summarizing, we have found that there is an isomorphism

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Z}} F \cong \Omega^*_{T/\!\!/W/F}.$$

On the other hand, it follows from the constructions in Section 4 that there is a filtration on $\mathcal{F}_G(G_c) = \mathrm{End}_{\mathrm{Loc}_{G_c}(G_c;\mathrm{KU})}(\mathrm{KU})$ whose associated graded is $\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{G_c}(G_c;\mathrm{KU})}(\underline{\mathrm{KU}}^{\mathrm{gr}})[u^{\pm 1}]$. By the above discussion, the latter is $\Omega^*_{T/\!\!/W/F}$. Proposition 10.2 shows that $\mathcal{F}_G(G_c)\otimes_{\mathrm{KU}} F[u^{\pm 1}]$ is isomorphic to the Hochschild homology $\mathrm{HH}(T/\!\!/W/F)[u^{\pm 1}]$. There is therefore a filtration on $\mathrm{HH}(T/\!\!/W/F)[u^{\pm 1}]$ whose associated graded is $\Omega^*_{T/\!\!/W/F}[u^{\pm 1}]$. Again, this filtration is precisely the Hochschild-Kostant-Rosenberg filtration on Hochschild homology.

Remark 10.8. While we are on the topic of the equivariant K-theory of G_c, let us note the relationship between (23) and the work of Freed-Hopkins-Teleman [FHT11a, FHT13, FHT11b, FHT08, FT15].[19] We will be brief, since we will not use these results below. Associated to a class $\tau \in \mathrm{H}^4(BG_c;\mathbf{Z})$ is the "twisted equivariant K-homology" $\mathrm{KU}^G_\tau(G_c)$. When τ is sufficiently nondegenerete, Freed-Hopkins-Teleman computed that $\pi_*\mathrm{KU}^G_\tau(G_c)$ is isomorphic to $\mathrm{RU}(G)/I^\tau$ for a particular ideal I^τ (called the "Verlinde ideal"). The categorification of this isomorphism from [FT15] shows that, associated to τ, there is a map $W : T/\!\!/W \cong \mathrm{Spec}\,\pi_0\mathrm{KU}_G \to \mathbf{A}^1$ such that (under certain hypotheses on τ), there is an isomorphism between $\pi_*\mathrm{KU}^G_\tau(G_c) \otimes_{\mathbf{Z}} F$ and the Jacobian ring of W.

This is related to (23) in the following manner. Below, we will implicitly base-change all rings from $\mathbf{Z}$ to F, to avoid cumbersome notation. Recall from [FHT11a,

[19]Nearly the same perspective can also be found in some of Teleman's talks; e.g., [Tel18].

Equation 3] that there is a spectral sequence

$$E_1^{*,*} \cong \pi_*\mathrm{KU}_G \otimes_{\pi_*\mathcal{F}_G(\mathrm{Gr}_G)^\vee} \pi_*\mathrm{KU}_G \Rightarrow \pi_*\mathrm{KU}_\tau^G(G_c). \tag{24}$$

The tensor product is derived; moreover, the class τ defines a particular $\pi_*\mathcal{F}_G(\mathrm{Gr}_G)^\vee$-module structure, and one of the tensor factors is given this module structure. (The other tensor factor is given the module structure coming from the augmentation.) Using Theorem 7.3, let us view $\mathrm{Spec}\,\pi_*\mathcal{F}_G(\mathrm{Gr}_G)^\vee$ as the (2-periodification of) $\check{J}_\mu$. Then τ defines a particular closed subscheme $T/\!\!/W \cong L_\tau \hookrightarrow \check{J}_\mu$ (which is in fact a Lagrangian), and the E_1-page of this spectral sequence can be identified with (the 2-periodification of) the ring of functions on $L_\tau \times_{\check{J}_\mu} T/\!\!/W$. If L_τ lies in the formal neighborhood of $\check{J}_\mu$, then we may replace $\check{J}_\mu$ in this fiber product by its formal completion $\hat{\check{J}}_\mu$ at the zero section. Since we have implicitly base-changed everything to the characteristic zero field F, the argument of Lemma 10.5 further lets us replace $\hat{\check{J}}_\mu$ by its Lie algebra, which (as mentioned in Example 10.7) is given by $T^*(T/\!\!/W)$. Under this replacement, the map $T/\!\!/W \cong L_\tau \to \hat{\check{J}}_\mu$ becomes identified with the map $dW : T/\!\!/W \to T^*(T/\!\!/W)$, where $W : T/\!\!/W \to \mathbf{A}^1$ is the map from [FT15]. The derived fiber product $L_\tau \times_{T^*(T/\!\!/W)} T/\!\!/W$ is precisely the Jacobian ring of W; that is to say, the E_1-page of the spectral sequence (24) identifies with the Jacobian ring of W. If the spectral sequence (24) degenerates at the E_1-page, then we conclude that $\pi_*\mathrm{KU}_\tau^G(G_c)$ is isomorphic to the Jacobian ring of W, as desired.

In fact, the Hochschild-Kostant-Rosenberg filtrations on $\mathrm{HH}(\mathfrak{t}/\!\!/W/F)$ and $\mathrm{HH}(T/\!\!/W/F)$ from Example 10.6 and Example 10.7 both split, since F is of characteristic zero and $\mathfrak{t}/\!\!/W$ and $T/\!\!/W$ are smooth schemes. We therefore conclude that there are isomorphisms

$$\mathrm{H}^*_{G_c}(G_c; F[u^{\pm 1}]) \cong \Omega^*_{\mathfrak{t}/\!\!/W/F}[u^{\pm 1}],$$
$$\mathrm{KU}^*_{G_c}(G_c) \otimes_{\mathbf{Z}} F \cong \Omega^*_{T/\!\!/W/F}[u^{\pm 1}],$$

the latter for G_c being simply-laced. (This assumption can be removed with further work.) The final isomorphism above recovers (the base-change to F of) the isomorphism of Brylinski-Zhang. Arguing as above, one also finds that if k is an elliptic cohomology theory, G_c is simply-laced and has torsion-free fundamental group, and $i : \mathrm{Spec}(F[u^{\pm 1}]) \to \mathcal{M}_G$ is a map with F being an algebraically closed field of characteristic zero, there is an isomorphism of quasicoherent sheaves over $\mathrm{Spec}(F[u^{\pm 1}])$:

$$\pi_* i^*\mathcal{F}_G(G_c) \cong \Omega^*_{\mathcal{M}_{G,0}/F}[u^{\pm 1}]. \tag{25}$$

As stated, (25) holds if k is $\mathbf{Q}[u^{\pm 1}]$, complex K-theory, or elliptic cohomology.

Remark 10.9. At least in the case of classical groups, additive isomorphisms of the form discussed in this section follow from stronger statements about splittings of the suspension spectrum $(G_c)_+$. Such statements were proved in [Mil85]; let us illustrate this when $G = \mathrm{GL}_n$. For $j \leq n$, let $\mathrm{Gr}_j(\mathbf{C}^n) = \mathrm{U}(n)/(\mathrm{U}(j)\times\mathrm{U}(n-j))$, and let $\mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}$ denote the Thom spectrum of the vector bundle over $\mathrm{Gr}_j(\mathbf{C}^n)$ given by the pulling back the adjoint representation of $\mathrm{U}(j)$ along the map $\mathrm{Gr}_j(\mathbf{C}^n) \to \mathrm{BU}(j)$. Then there is a $\mathrm{U}(n)$-equivariant splitting

$$(G_c)_+ \simeq \bigoplus_{j=0}^n \mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}.$$

This induces a splitting of $\mathcal{F}_G(G_c)$, and hence of $i^*\mathcal{F}_G(G_c)$ for any map $i : \mathrm{Spec}(F[u^{\pm 1}]) \to \mathcal{M}_G$ with F being an algebraically closed field of characteristic zero. One can show that there is an isomorphism

$$\pi_* i^* \mathcal{F}_G(\mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}) \cong \Omega^j_{\mathcal{M}_{\mathrm{GL}_n,0}/F}[u^{\pm 1}],$$

so taking the direct sum over $j = 0, \cdots, n$ gives an additive equivalence of the form (25).

Although such splittings of $(G_c)_+$ were proved in [Mil85] only for classical groups, they can also be extended with some work to the exceptional groups, too. For instance, there is a G_2-equivariant splitting

$$(\mathrm{G}_2)_+ \simeq S^0 \oplus \Sigma(\mathrm{G}_2/\mathrm{U}(2)) \oplus S^{\mathfrak{g}_2},$$

where $\mathrm{U}(2)$ is embedded inside G_2 via the short root, and $S^{\mathfrak{g}_2}$ is the one-point compactification of the adjoint representation. The map $\Sigma\mathrm{G}_2/\mathrm{U}(2) \to \mathrm{G}_2$ is adjoint to the map $\mathrm{G}_2/\mathrm{U}(2) \to \Omega\mathrm{G}_2$ which exhibits $\mathrm{G}_2/\mathrm{U}(2)$ as a generating variety for $\Omega\mathrm{G}_2$ (see [Bot58]). Again, the terms in this splitting match the splitting of $\Omega^*_{\mathcal{M}_{G,0}/F}$ into $\Omega^1_{\mathcal{M}_{G,0}/F}$, $\Omega^1_{\mathcal{M}_{G,0}/F}$, and $\Omega^2_{\mathcal{M}_{G,0}/F}$.

Appendix A. Comparison to Hopkins-Kuhn-Ravenel

The calculations of this article (more precisely, the perspective of Remark 8.13) were motivated by the work of Hopkins-Kuhn-Ravenel [HKR00], who study the case of finite groups. In this section, we will describe a relationship to their work. Our discussion will be rather heuristic, and we will sweep a few details under the rug to keep the exposition readable.

Before proceeding, the first thing to note is that while the present article only discusses *connected* compact Lie groups, Hopkins-Kuhn-Ravenel only study *discrete* compact Lie groups (that is, finite groups). Next, the work of [HKR00] only deals with Borel-equivariant cohomology. This means that one does *not* need to assume that the complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k is equipped with an oriented commutative k-group $\mathbf{G}$; recall from Section 3 that the purpose of $\mathbf{G}$ is to provide a decompletion of Borel-equivariant cohomology for compact abelian Lie groups. All that is needed is the formal completion $\hat{\mathbf{G}}$ of $\mathbf{G}$ at the identity section. Note that this is not extra data associated to k, since $\hat{\mathbf{G}} = \mathrm{Spf}\, k^{\mathbf{C}P^\infty_+}$. Let $\hat{\mathbf{G}}_0$ denote the underlying 1-dimensional formal group over $\pi_0(k)$.

In fact, an even more stringent condition is required of k in [HKR00]: not only is it required to be complex-oriented and 2-periodic, but $\pi_0(k)$ is required to be a complete local Noetherian domain with maximal ideal $\mathfrak{m}$ whose residue field $\pi_0(k)/\mathfrak{m}$ is of characteristic $p > 0$, such that p is not nilpotent in $\pi_0(k)$. Let n denote the height of the formal group $\hat{\mathbf{G}}_0$ base-changed along $\pi_0(k) \to \pi_0(k)/\mathfrak{m}$. In the following discussion, we will simply write $k^0(X)$ to denote π_0 of the the k-cochains on X (instead of the more cumbersome notation $\pi_0\mathcal{F}(X)$).

Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$, and choose a continuous embedding $\pi_0(k) \to \mathcal{O}_{\mathbf{C}_p}$. The base-change of $\hat{\mathbf{G}}_0$ to $\mathcal{O}_{\mathbf{C}_p}$ defines a formal group law on the maximal ideal of $\mathcal{O}_{\mathbf{C}_p}$; assume that the base-change of $\hat{\mathbf{G}}_0$ along the map $\pi_0(k) \to \pi_0(k)/\mathfrak{m}$ has finite height. Then, there exists an exponential isomorphism

$$e : (\mathbf{Q}_p/\mathbf{Z}_p)^n \xrightarrow{\sim} (\mathfrak{m}_{\mathcal{O}_{\mathbf{C}_p}}, +_{\hat{\mathbf{G}}_0}), \tag{26}$$

where n is the height of $\hat{\mathbf{G}}_0$. The basic calculation driving the results of [HKR00] is the following.

Proposition A.1. *There is an isomorphism*

$$k^0(B\mathbf{Z}/p^j) \cong \pi_0(k)[\![t]\!]/[p^j](t),$$

where $[p^j](t) \in \pi_0(k)[\![t]\!]$ is the p^j-series of the formal group law $\hat{\mathbf{G}}_0$, and t is the first Chern class of the standard character $\mathbf{Z}/p^j \cong \mu_{p^j} \subseteq S^1$. That is, there is an isomorphism $\mathrm{Spf}\, k^0(B\mathbf{Z}/p^j) \cong \hat{\mathbf{G}}_0[p^j]$.

Construction A.2. Proposition A.1 and the discussion preceding it gives an isomorphism

$$\mathrm{Spf}(k^0(B\mathbf{Z}/p^j)) \otimes_{\mathrm{Spf}\,\pi_0(k)} \mathrm{Spec}\,\mathbf{C}_p \cong \tfrac{1}{p^j}\mathbf{Z}/\mathbf{Z},$$

where the right-hand side denotes the constant group scheme over $\mathbf{C}_p$. A choice of generator (e.g., $\frac{1}{p^j}$) of this group therefore gives a map $k^0(B\mathbf{Z}/p^j) \to \mathbf{C}_p$. Now let F be a finite group, and let $f : \mathbf{Z}_p^n \to F$ be a homomorphism. Then f factors as a map $\mathbf{Z}_p^n \to (\mathbf{Z}/p^j)^n \to F$ for some j, so there is a map $k^0(BF) \to k^0(B(\mathbf{Z}/p^j)^n)$.

Taking the product of the maps $k^0(B\mathbf{Z}/p^j) \to \mathbf{C}_p$ described above gives a map $k^0(B(\mathbf{Z}/p^j)^n) \to \mathbf{C}_p$, which finally defines a composite map

$$k^0(BF) \to k^0(B(\mathbf{Z}/p^j)^n) \to \mathbf{C}_p.$$

This composite depends only on the conjugacy class of f, and so this construction defines a map $\mathrm{Hom}(\mathbf{Z}_p^n, F)/\!\!/F \to \mathrm{Map}(k^0(BF), \mathbf{C}_p)$, whose adjoint is a map $k^0(BF) \to \mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, F)/\!\!/F, \mathbf{C}_p)$. Here, F acts on $\mathrm{Hom}(\mathbf{Z}_p^n, F)$ by conjugation.

In the discussion below, F will be a finite group. For simplicity, we will further assume that $k^*(BF)$ is concentrated in even degrees (so, by the 2-periodicity of k, it is completely determined by $k^0(BF)$). If X is an F-space, the homotopy orbits of X will be denoted X_{hF}, while the ordinary quotient of X by the F-action will be denoted $X/\!\!/F$.

Theorem A.3 (Hopkins-Kuhn-Ravenel)**.** *The map from Construction A.2 defines an isomorphism*

$$k^0(BF) \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, F)/\!\!/F, \mathbf{C}_p).$$

The quotient $\mathrm{Hom}(\mathbf{Z}_p^n, F)/\!\!/F$ *can be replaced by the homotopy orbits* $\mathrm{Hom}(\mathbf{Z}_p^n, F)_{hF}$*, since F is a finite group and its order is invertible in $\mathbf{C}_p$.*

Note that the homotopy orbits $\mathrm{Hom}(\mathbf{Z}_p^n, F)_{hF}$ can be identified with $\mathrm{Map}(BT_p^n, BF)$, where $T_p^n = (\mathbf{Q}_p/\mathbf{Z}_p)^n$ is the p-adic n-torus. One can use a ring smaller than $\mathbf{C}_p$ in Theorem A.3; essentially, one only needs to extend scalars to the rationalization of the smallest ring containing $\pi_0(k)$ over which the exponential isomorphism (26) holds.

In [Lur19], Lurie observes that the isomorphism of Theorem A.3 can be categorified, at least if one assumes the data of a decompletion $\mathbf{G}$ of $\hat{\mathbf{G}}$. (We refer the reader to [Lur19] for further details, since the specific setup will not concern us much below.) Namely, if F is a finite group, Lurie defines an ∞-category $\mathrm{Loc}_F(*; k)$ (denoted by $\mathrm{LocSys}_{\mathbf{G}}(BF)$ in *loc. cit.*), and proves the following as (a consequence of) [Lur19, Theorem 6.4.1]:

Theorem A.4 (Lurie)**.** *Fix an embedding $\pi_0(k) \to \mathbf{C}_p$, so it defines an $\mathbf{E}_\infty$-map $k \to \mathbf{C}_p[u^{\pm 1}]$. There is a symmetric monoidal fully faithful embedding*

$$\mathrm{Loc}_F(*; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{Loc}(\mathrm{Map}(BT_p^n, BF); \mathbf{C}_p[u^{\pm 1}]).$$

The essential image of the above embedding is described in [Lur19, Theorem 6.5.13].

Let us examine the isomorphism Theorem A.3 and the embedding Theorem A.4 further; we will rephrase the right-hand sides of both results as algebro-geometric objects. To do this, note that the exponential isomorphism between $\hat{\mathbf{G}}_0 \otimes_{\pi_0(k)} \mathbf{C}_p$ and $(\mathbf{Q}_p/\mathbf{Z}_p)^n$ defines an isomorphism between $\mathbf{D}(\hat{\mathbf{G}}_0) \otimes_{\pi_0(k)} \mathbf{C}_p$ and $\mathbf{Z}_p^n$. Here, $\mathbf{D}(\hat{\mathbf{G}}_0) = \mathrm{Hom}(\hat{\mathbf{G}}_0, \mathbf{G}_m)$ is the Cartier dual of $\hat{\mathbf{G}}_0$. Note that the 1-shifted Cartier dual $\hat{\mathbf{G}}_0^\vee$ can be identified with the classifying stack of $\mathbf{D}(\hat{\mathbf{G}}_0)$.

View the finite group F as defining a constant group scheme $\underline{F}$ over $\mathbf{C}_p$. Since $\hat{\mathbf{G}}_0^\vee \otimes_{\pi_0(k)} \mathbf{C} \cong \mathbf{Z}_p^n$, the mapping stack $\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})$ is the quotient of the discrete scheme $\underline{\mathrm{Hom}(\mathbf{Z}_p^n, F)}$ by the constant group scheme $\underline{F}$ acting by conjugation.

It follows that the $\mathbf{C}_p$-algebra $\mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, F)/\!\!/F, \mathbf{C}_p)$ is the algebra of functions on the mapping stack $\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})$. (Not that since the order of F is invertible in $\mathbf{C}_p$, the derived and classical algebras of functions agree.) Similarly, $\mathrm{Loc}(\mathrm{Map}(BT_p^n, BF); \mathbf{C}_p)$ can be viewed as the category of quasicoherent sheaves on the mapping stack $\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})$. Therefore, Theorem A.3 and Theorem A.4 can be restated as:

$$\pi_0 k_F \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \Gamma(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}); \mathcal{O}), \tag{27}$$

$$\mathrm{Loc}_F(*; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})) \otimes_{\mathrm{Mod}_{\mathbf{C}_p}} \mathrm{Mod}_{\mathbf{C}_p[u^{\pm 1}]}. \tag{28}$$

One can even replace $\hat{\mathbf{G}}_0$ in the above by $\mathbf{G}_0$. Observe, now, that $\mathrm{Map}(\mathbf{G}_0^\vee, B\underline{F})$ is simply the stack $\mathrm{Bun}_{\underline{F}}(\mathbf{G}_0^\vee)$.

We can now compare (27) and (28) to the discussion in the body of this article. Assume now that k is either $\mathbf{Q}[u^{\pm 1}]$, KU, or elliptic cohomology. If G_c was instead a connected compact Lie group, the analogue of (27) states that $\pi_0 k_{G_c} \otimes_{\pi_0(k)} \mathbf{C}$ is the ring of (classical, not derived!) global sections on $\mathrm{Bun}_G^{\mathrm{ss}}(\mathbf{G}_0^\vee)$, where G is the complex reductive group corresponding to G_c. This is clear when $\mathbf{G}_0$ is $\mathbf{G}_a$ (and $k = \mathbf{Q}[u^{\pm 1}]$) or $\mathbf{G}_m$ (and $k = \mathrm{KU}$). In the case when $\mathbf{G}_0$ is an elliptic curve, this is essentially part of the definition of equivariant elliptic cohomology as sketched in [Lur09] and constructed in [GM23, GM20].

Let us continue to assume that G_c is a connected compact Lie group, and further impose that it is simply-laced and almost simple. We will now a give a heuristic argument suggesting that Theorem 1.4 – or rather, its variant from Remark 8.12 describing $\mathrm{Loc}_{G_c}^{\mathrm{gr}}(\mathrm{Gr}_G; k)$ – can be viewed as an analogue of (28).

Indeed, the rephrasing of Remark 8.12 from Remark 8.13 states that there is an equivalence

$$\mathrm{Loc}_{\check{G}_c}^{\mathrm{gr}}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} \mathbf{C} \simeq \mathrm{QCoh}(\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}). \tag{29}$$

The regular locus $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}$ is an open substack of $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)$ (whose complement has codimension ≥ 2, as proved in [Dav19, Proposition 3.1.16]), and so there is a fully faithful embedding $\mathrm{QCoh}(\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}) \hookrightarrow \mathrm{QCoh}(\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee))$. That is, there is a fully faithful embedding

$$\mathrm{Loc}_{\check{G}_c}^{\mathrm{gr}}(\mathrm{Gr}_G; k) \otimes_{\pi_0(k)} \mathbf{C} \hookrightarrow \mathrm{QCoh}(\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)). \tag{30}$$

Assume for the moment that (30) holds if $\check{G}$ is a finite group $\check{F}$ (and replace $\mathbf{C}$ above by $\mathbf{C}_p$). Of course, it is not clear what the Langlands dual F of $\check{F}$ should mean; but it is reasonable to believe that, whatever it is, F should be a finite group (or perhaps a finite group scheme). In any case, Gr_F will just be a point, so the left-hand side of (30) is simply $\mathrm{Loc}_{\check{F}}^{\mathrm{gr}}(*; k) \otimes_{\pi_0(k)} \mathbf{C}$. It is reasonable to expect that, thanks to a formality-type statement, the 2-periodification of the category $\mathrm{Loc}_{\check{F}}^{\mathrm{gr}}(*; k) \otimes_{\pi_0(k)} \mathbf{C}$ is equivalent to $\mathrm{Loc}_{\check{F}}(*; k) \otimes_k \mathbf{C}[u^{\pm 1}]$.

Turning to the right-hand side of (30), note that because $\check{F}$ is a finite group, there is no meaningful notion of semistability, and so $\mathrm{Bun}_{\check{F}}^{\mathrm{ss}}(\mathbf{G}_0^\vee) = \mathrm{Bun}_{\check{F}}(\mathbf{G}_0^\vee)$. With these translations made (so the left-hand side of (30) is replaced by $\mathrm{Loc}_{\check{F}}(*; k) \otimes_k \mathbf{C}[u^{\pm 1}]$, and the right-hand side by the 2-periodification of $\mathrm{QCoh}(\mathrm{Bun}_{\check{F}}(\mathbf{G}_0^\vee))$), (30) is precisely of the form (28), as claimed.

Remark A.5. The above comparison between the quotient $\mathrm{Gr}_G/G[\![t]\!]$ for a connected compact Lie group G_c and the classifying space BF for a finite group F

can be made more precise by noting that $\mathrm{Gr}_G/G[\![t]\!]$ is homotopy equivalent to the mapping space $\mathrm{Map}(S^2, BG_c) = \mathrm{Bun}_{G_c}(S^2)$, and that if F is a finite group, then $\mathrm{Bun}_F(S^2) = BF$.

The work of Hopkins-Kuhn-Ravenel in fact proves a statement which is much more general than Theorem A.3 (and similarly, Lurie's work in [Lur19] yields a much stronger statement than Theorem A.4). Namely, they prove the following.

Theorem A.6 (Hopkins-Kuhn-Ravenel)**.** *Let F be a finite group, and let X be a finite F-space. For each homomorphism $\alpha : \mathbf{Z}_p^n \to F$, let X^α denote the fixed locus of* $\mathrm{im}(\alpha)$*. Then there is an isomorphism*

$$k^*(X_{hF}) \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \mathrm{H}^*\left(\left(\amalg_{\alpha \in \mathrm{Hom}(\mathbf{Z}_p^n, F)} X^\alpha\right) /\!\!/ F; \mathbf{C}_p[u^{\pm 1}]\right).$$

The isomorphism of Theorem A.3 is the special case when X is a point. In [Lur19], Lurie shows that Theorem A.6 is a consequence of a more general statement. If X is a finite F-space, Lurie defines an ∞-category $\mathrm{Loc}_F(X; k)$ (denoted by $\mathrm{LocSys}_{\mathbf{G}}(X//F)$ in *loc. cit.*), and proves the following as [Lur19, Theorem 6.4.1]:

Theorem A.7 (Lurie)**.** *There is a symmetric monoidal fully faithful embedding*

$$\mathrm{Loc}_F(X; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{Loc}(\mathrm{Map}(BT_p^n, X_{hF}); \mathbf{C}_p[u^{\pm 1}]).$$

The essential image of the above embedding is described in [Lur19, Theorem 6.5.13]. For the reader interested in chasing down references: specifically, Theorem A.7 generalizes [Lur19, Theorem 4.3.2]; the latter implies Theorem A.6 by [Lur19, Corollary 4.3.4]. The basic observation is that the mapping space $\mathrm{Map}(BT_p^n, X_{hF})$ is equivalent to $\left(\amalg_{\alpha \in \mathrm{Hom}(\mathbf{Z}_p^n, F)} X^\alpha\right)_{hF}$. Note that the homotopy quotient $\mathrm{Hom}(\mathbf{Z}_p^n, F)_{hF}$ can be written as a disjoint union $\amalg_{[\alpha]} BZ(\alpha)$ ranging over conjugacy classes of homomorphisms $\alpha : \mathbf{Z}_p^n \to F$; here $Z(\alpha)$ denotes the centralizer of the image of α. Similarly, the homotopy orbits $\left(\amalg_{\alpha \in \mathrm{Hom}(\mathbf{Z}_p^n, F)} X^\alpha\right)_{hF}$ can be rewritten as the disjoint union $\amalg_{[\alpha]} X^\alpha_{hZ(\alpha)}$.

Remark A.8. One could contemplate a variant of Theorem A.6 and Theorem A.7 which replaces $\mathbf{C}_p$ by other $\mathbf{E}_\infty$-k-algebras (e.g., over which the base-change of $\hat{\mathbf{G}}_0$ is not necessarily isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^n$, but over which it has $(\mathbf{Q}_p/\mathbf{Z}_p)^j$ as a summand for some $j < n$). The analogues of Theorem A.6 and Theorem A.7 in this generality were proved in [Sta13, Sta15] and [Lur19].

Given the analogy between Theorem A.4 and Theorem 1.4, it is natural to ask for an analogue of Theorem A.7 for connected compact Lie groups. In the following discussion, we suggest an analogy: namely, one could view the k-theoretic variant (described for $k = \mathrm{ku}$ in [Dev24]) of the local unramified relative Langlands conjecture of [BZSV23] as an analogue of the aforementioned results.

To understand this, let us again massage Theorem A.6 and Theorem A.7 to a form more suited to algebro-geometric considerations. We will continue to assume for simplicity that $k^*(BF)$ is concentrated in even degrees. Theorem A.6 describes how, under the isomorphism of Theorem A.3, the $k^0(BF) \otimes_{\pi_0(k)} \mathbf{C}_p$-module $k^*(X_{hF}) \otimes_{\pi_0(k)} \mathbf{C}_p$ decomposes as a module over $\Gamma(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}); \mathcal{O})$. Similarly, Theorem A.7 says that there is an explicit $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}))$-module category $\widetilde{\mathcal{C}}_X$ and a fully faithful $\mathrm{Loc}_F(*; k) \otimes_k \mathbf{C}_p[u^{\pm 1}]$-linear embedding

$$\mathrm{Loc}_F(X; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \widetilde{\mathcal{C}}_X \otimes_{\mathbf{C}_p} \mathbf{C}_p[u^{\pm 1}].$$

Note that one source of $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}))$-module categories are maps $\check{L} \to \mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})$: namely, $\mathrm{QCoh}(\check{L})$ is a $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}))$-module category. That is, one could imagine that $\widetilde{\mathcal{C}}_X$ is of the form $\mathrm{QCoh}(\check{L})$ for some such $\check{L}$ as above which is associated to X. (While one can give a somewhat *ad hoc* definition of $\check{L}$ in terms of the fixed point spaces X^α and their (co)homology[20], it should be rather interesting to intrinsically understand the algebro-geometric properties of $\check{L}$ directly.)

More generally, recall that the data of a k-linear ∞-category with F-action is just a $\mathrm{Fun}(BF, \mathrm{Mod}_k)$-module category. Since $\mathrm{Fun}(BF, \mathrm{Mod}_k)$ is a completion of the ∞-category $\mathrm{Loc}_F(*; k)$, one might view the data of a $\mathrm{Loc}_F(*; k)$-module category $\mathcal{C}$ as a decompletion of the notion of a k-linear ∞-category with F-action. One example of such a category is $\mathrm{Loc}_F(X; k)$ for a finite F-space X. If $\mathbf{1}_\mathcal{C}$ is a distinguished object of $\mathcal{C}$, then $\mathrm{End}_\mathcal{C}(\mathbf{1}_\mathcal{C}) \otimes_{\pi_0(k)} \mathbf{C}_p$ is a $k^0(BF) \otimes_{\pi_0(k)} \mathbf{C}_p$-module, and hence a $\Gamma(\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F}); \mathcal{O})$-module. One could now ask for a description of this module structure; when $\mathcal{C} = \mathrm{Loc}_F(X; k)$ and $\mathbf{1}_\mathcal{C}$ is the constant sheaf therein, this is precisely answered by Theorem A.6. Similarly, one could ask for an analogue of Theorem A.7 in this generalized context. Summarizing, both Theorem A.6 and Theorem A.7 can be understood as describing how a $\mathrm{Loc}_F(*; k)$-module category decomposes over the mapping stack $\mathrm{Map}(\hat{\mathbf{G}}_0^\vee, B\underline{F})$.

Let G_c be a connected, almost simple, simply-laced compact Lie group. Then, as discussed above, the analogue of $\mathrm{Loc}_F(*; k)$ is the ∞-category $\mathrm{Loc}_{\check{G}_c}(\mathrm{Gr}_G; k)$. Moreover, the analogue of the tensor product on $\mathrm{Loc}_F(*; k)$ is the *convolution* tensor product on $\mathrm{Loc}_{\check{G}_c}(\mathrm{Gr}_G; k)$ coming, for instance, from the $\check{G}_c$-equivariant $\mathbf{E}_2$-space structure on $\mathrm{Gr}_G \cong \Omega G_c$. As mentioned in Remark 8.13, the equivalence of (29) is monoidal for the convolution tensor product on $\mathrm{Loc}_{\check{G}_c}(\mathrm{Gr}_G; k)$ and the ordinary tensor product of quasicoherent sheaves on $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}$.

Based on the discussion above, one can interpret the following question as an analogue of Theorem A.6 and Theorem A.7: how does a $\mathrm{Loc}_F(*; k)$-module category decompose over $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}$? More precisely, any finite G_c-space X should:

(a) define a $\mathrm{Loc}_{\check{G}_c}(\mathrm{Gr}_G; k)$-module category $\mathcal{C}_X$; this is the analogue of the $\mathrm{Loc}_F(*; k)$-module category $\mathrm{Loc}_F(X; k)$.
(b) define a fully faithful embedding $\mathcal{C}_X \hookrightarrow \widetilde{\mathcal{C}}_X$ into an explicit $\mathrm{QCoh}(\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee))$-module category $\widetilde{\mathcal{C}}_X$; this is the analogue of the fully faithful embedding $\mathrm{Loc}_F(X; k) \hookrightarrow \bigoplus_{[\alpha]} \mathrm{Loc}(X^\alpha_{hZ(\alpha)}; \mathbf{C}_p)$ from Theorem A.7.

In the following discussion, we will quietly replace $\mathrm{Loc}_{\check{G}_c}(\mathrm{Gr}_G; k)$ by $\mathrm{Loc}_{G_c}(\mathrm{Gr}_G; k)$ for conceptual simplicity; this, of course, changes the quasicoherent side, but to avoid getting into more detail than is necessary, we will pretend that the dual side remains unchanged[21]. To describe a candidate for $\mathcal{C}_X$, recall that the quotient $\mathrm{Gr}_G/G[\![t]\!]$ is homotopy equivalent to the mapping space $\mathrm{Map}(S^2, BG_c) = \mathrm{Bun}_{G_c}(S^2)$. This, in turn, can be described as the double coset stack $G_c\backslash(LG_c)/G_c$, where LG_c denotes the (topological) free loop space of G_c. Any G_c-space X defines an LG_c-space LX, and the stack $G_c\backslash(LG_c)/G_c$ acts on $(LX)/G_c$ by convolution. That is, the ∞-category $\mathrm{Loc}_{G_c}(\mathrm{Gr}_G; k)$ with its convolution tensor product acts on

[20]For instance, take $\check{L}$ to be the stack $\amalg_{[\alpha]} \mathrm{Spec}(\mathrm{H}^*(X^\alpha; \mathbf{C}_p))/\underline{Z(\alpha)}$.

[21]If $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$, the object $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee) = \check{\mathfrak{g}}/\check{G}$ must be replaced by $\check{\mathfrak{g}}^*/\check{G} = \mathfrak{g}/\check{G}$; and similarly, if $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, the object $\mathrm{Bun}_{\check{G}}^{\mathrm{ss}}(\mathbf{G}_0^\vee) = \check{G}/\check{G}$ must be replaced by $G/\check{G}$.

$\mathrm{Loc}_{G_c}(LX;k)$. One could therefore regard the latter category as a candidate for $\mathcal{C}_X$, and further ask for the following strengthening of (a) and (b) above:

- there should be a stack $\check{L}^{\mathrm{reg}}$ equipped with a map $\check{L}^{\mathrm{reg}} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}$ such that there is an equivalence
$$\mathcal{C}_X = \mathrm{Loc}_{G_c}(LX;k) \simeq \mathrm{QCoh}(\check{L}^{\mathrm{reg}}).$$
- the stack $\check{L}^{\mathrm{reg}}$ should be an open substack of a larger stack $\check{L}$, and the map $\check{L}^{\mathrm{reg}} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)^{\mathrm{reg}}$ extends to a map $\check{L} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)$. This gives a fully faithful embedding
$$\mathcal{C}_X \hookrightarrow \widetilde{\mathcal{C}}_X := \mathrm{QCoh}(\check{L}).$$

Note that $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)$ is the quotient of $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)_{\mathrm{triv}}$ by $\check{G}$, so one could equivalently view $\check{L}$ as the data of a $\check{G}$-stack $\check{M}$ equipped with a $\check{G}$-equivariant map
$$\mu : \check{M} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}_0^\vee)_{\mathrm{triv}}.$$
The relation between $\check{L}$ and $\check{M}$ is that $\check{L} = \check{M}/\check{G}$.

Example A.9. If $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G}_0 = \mathbf{G}_a$, then $\check{M}$ is simply a $\check{G}$-stack equipped with a $\check{G}$-equivariant map $\mu : \check{M} \to \check{\mathfrak{g}}^*$. Similarly, if $k = \mathrm{KU}$ and $\mathbf{G}_0 = \mathbf{G}_m$, then $\check{M}$ is simply a $\check{G}$-stack equipped with a $\check{G}$-equivariant map $\mu : \check{M} \to G$.

Suppose X is the analytification of an affine G-variety $X_{\mathbf{C}}$. In [BZSV23], Ben-Zvi–Sakellaridis–Venkatesh study (under certain additional conditions on $X_{\mathbf{C}}$) the full ∞-category $\mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!);\mathbf{C})$ as a module over $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G;\mathbf{C})$. The local unramified geometric conjecture of [BZSV23] (see [BZSV23, Conjecture 7.5.1]) says – up to the issue of shearing, which we will ignore here – that associated to $X_{\mathbf{C}}$ is a Hamiltonian $\check{G}$-stack $\check{M}$ such that there is an equivalence of categories $\mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!);\mathbf{C}) \simeq \mathrm{QCoh}(\check{M}/\check{G})$. The data of a Hamiltonian $\check{G}$-structure on $\check{M}$ gives, in particular, an $\check{G}$-equivariant moment map $\check{M} \to \check{\mathfrak{g}}^*$ which makes $\mathrm{QCoh}(\check{M}/\check{G})$ into a $\mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$-module category. Moreover, under certain assumptions on $X_{\mathbf{C}}$, there is a fully faithful embedding $\mathrm{Loc}_{G_c}(LX;\mathbf{C}) \hookrightarrow \mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!);\mathbf{C})$. Putting this together, we find a picture exactly like the one described in the preceding paragraph: namely, assuming [BZSV23, Conjecture 7.5.1], there is a fully faithful embedding
$$\mathrm{Loc}_{G_c}(LX;\mathbf{C}) \hookrightarrow \mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!);\mathbf{C}) \simeq \mathrm{QCoh}(\check{M}/\check{G})$$
of $\mathrm{Loc}_{G_c}(LX;\mathbf{C})$ into an explicit $\mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$-module category. Therefore, one could view (the 2-periodification of) [BZSV23, Conjecture 7.5.1] as a conjectural analogue for connected compact Lie groups and $k = \mathbf{C}[u^{\pm 1}]$ of Theorem A.6 and Theorem A.7.[22] Motivated by this discussion, we propose in [Dev24] that there should be a variant of [BZSV23, Conjecture 7.5.1] for sheaves with coefficients in other $\mathbf{E}_\infty$-rings (like connective complex K-theory ku or elliptic cohomology).

[22]Of course, since F is a finite group, Theorem A.6 and Theorem A.7 are contentless if $k = \mathbf{C}[u^{\pm 1}]$; so what we mean by the analogy between [BZSV23, Conjecture 7.5.1] and Theorem A.7 is that the latter admits a conjectural generalization to connected compact Lie groups, and that the resulting statement specialized to $k = \mathbf{C}[u^{\pm 1}]$ is still interesting.

Appendix B. Coulomb branches of pure supersymmetric gauge theories

In this brief appendix, we explain some motivation for the results of this article from the perspective of Coulomb branches of 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories with a generic choice of complex structure. The goal here is not to be precise, but instead explain some motivation for the ideas in this article. While reading this appendix, the reader should keep in mind that I know very little physics!

Recollection B.1. In [BFN18, Nak16] (see also [Nak17]), it is argued that the Coulomb branch of 3d $\mathcal{N}=4$ pure gauge theory on $\mathbf{R}^3$ can be modeled by the algebraic symplectic variety $\mathcal{M}_C := \operatorname{Spec} \mathrm{H}_*^{G_c}(\mathrm{Gr}_G; \mathbf{C})$ over $\mathbf{C}$. This is in turn isomorphic by [BF08, Theorem 3] (reproved here as Corollary 6.23) to the phase space of the Toda lattice for $\check{G}$, as well as (by [BFN18, Theorem A.1]) to the moduli space of solutions of Nahm's equations on $[-1,1]$ for a compact form of $\check{G}$ with an appropriate boundary condition. The *quantized* Coulomb branch of 3d $\mathcal{N}=4$ pure gauge theory on $\mathbf{R}^3$ is then modeled by $\mathcal{A}_\epsilon := \mathrm{H}_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{C})$. Note that $\mathcal{A}_\epsilon$ is isomorphic to the algebra of operators of the quantized Toda lattice for $\check{G}$.

Remark B.2. The physical reason for the definition of $\mathcal{A}_\epsilon$ is the "Ω-background" (introduced in [NS09]); we refer the reader to [BBB$^+$20, Tel14] for helpful expositions on this topic. The essential idea is as follows: the equivariant homology $C_*^G(\mathrm{Gr}_G; \mathbf{C})$ admits the structure of an $\mathbf{E}_3^{\mathrm{fr}}$-algebra. In particular, the $\mathbf{E}_3$-algebra structure on $C_*^G(\mathrm{Gr}_G; \mathbf{C})$ is equivariant for the action of S^1 on $C_*^G(\mathrm{Gr}_G; \mathbf{C})$ via loop rotation, and the action of S^1 on $\mathbf{E}_3$ via rotation about a line $\ell \subseteq \mathbf{R}^3$. Using the fact that the fixed points of the S^1-action on $\mathbf{R}^3$ are given by the line ℓ, it is argued in [BBB$^+$20] that the homotopy fixed points of $C_*^G(\mathrm{Gr}_G; \mathbf{C})$ admits the structure of an $\mathbf{E}_1$-$C^*_{S^1}(*; \mathbf{C})$-algebra. Furthermore, the associative multiplication on $C_*^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{C})$ degenerates to the 2-shifted Poisson bracket on $\mathrm{H}_*^{G_c}(\mathrm{Gr}_G; \mathbf{C})$ obtained from the $\mathbf{E}_3$-algebra structure. The "Ω-background" is supposed to refer to the compatibility of the S^1-action on $C_*^G(\mathrm{Gr}_G; \mathbf{C})$ with the S^1-action on the $\mathbf{E}_3$-operad.

From the mathematical perspective, the idea that S^1-actions can be viewed as deformation quantizations has been made precise by [Pre15, Toe14], and more recently in [But20a, But20b], at least in characteristic zero. Although often not said explicitly, the idea has been a cornerstone of the development of Hochschild homology and its relatives. (The reader can skip the following discussion, since it will not be necessary in the remainder of this section; we only include it for completeness.)

Consider a smooth $\mathbf{C}$-scheme X, so that the Hochschild-Kostant-Rosenberg theorem gives an isomorphism $\mathrm{HH}(X/\mathbf{C}) \simeq \mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$. There is an isomorphism $\mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1]) \simeq \bigoplus_{n\geq 0} (\wedge^n \Omega^1_{X/\mathbf{C}})[n]$, so $\mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$ can be understood as a shearing of the algebra $\Omega^*_{X/\mathbf{C}} = \bigoplus_{n\geq 0} (\wedge^n \Omega^1_{X/\mathbf{C}})[-n]$ of differential forms. The Hochschild-Kostant-Rosenberg theorem further states that the S^1-action on $\mathrm{HH}(X/\mathbf{C})$ is a shearing of the de Rham differential on $\Omega^*_{X/\mathbf{C}}$.

The Koszul dual of the algebra $\mathrm{HH}(X/\mathbf{C}) \simeq \mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$ is $\mathrm{Sym}(T_{X/\mathbf{C}}[-2]) \simeq \mathcal{O}_{T^*[2]X}$; in the same way, the sheaf of differential operators on X is Koszul dual to the de Rham complex of X. This can be drawn pictorially as follows:

$$\begin{tikzcd}
\mathrm{Sym}(T_{X/\mathbf{C}}[-2]) \simeq \mathcal{O}_{T^*[2]X} \arrow[r, rightsquigarrow, "\text{def. quant}"] \arrow[d, rightsquigarrow, "\text{Koszul dual}"'] & \mathcal{D}^{\hbar}_{X/\mathbf{C}} \arrow[d, rightsquigarrow, "\text{Koszul dual}"] \\
\mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/\mathbf{C}}[1]) \simeq \mathrm{HH}(X/\mathbf{C}) \arrow[r, rightsquigarrow, "S^1\text{-action}"'] & \text{shearing of } (\Omega^*_{X/k}, d_{\mathrm{dR}}).
\end{tikzcd}$$

Since the algebra $\mathcal{D}^{\hbar}_X$ of differential operators is a quantization of $T^*[2]X$, this diagram illustrates the idea that the S^1-action on Hochschild homology plays the role of a Koszul dual to deformation quantization.

Example B.3. We will keep $G = \mathrm{PGL}_2$ as a running example in discussing Coulomb branches (see also [SW97, Section 2]), so that $\check{G} = \mathrm{SL}_2$. In this case,

$$\mathcal{M}_C \cong \mathrm{Spec}\, \mathbf{C}[x, a^{\pm 1}, \tfrac{a-a^{-1}}{x}]^{\mathbf{Z}/2} \cong \mathrm{Spec}\, \mathbf{C}[x^2, a+a^{-1}, \tfrac{a-a^{-1}}{x}]$$

by Theorem 6.3, where $\mathbf{Z}/2$ acts on $\mathbf{C}[x, a^{\pm 1}, \frac{a-a^{-1}}{x}]$ by $x \mapsto -x$ and $a \mapsto a^{-1}$. This is the regular centralizer group scheme of SL_2. Let us denote by

$$\begin{aligned}
\Phi &= x^2, \\
U &= a + a^{-1}, \\
V &= \tfrac{a-a^{-1}}{x}.
\end{aligned}$$

Then

$$U^2 - \Phi V^2 = (a+a^{-1})^2 - (a-a^{-1})^2 = 4,$$

so $\mathcal{M}_C$ is isomorphic to the subvariety of $\mathbf{A}^3_{\mathbf{C}}$ cut out by the above equation. This is known as the *Atiyah-Hitchin manifold*, and was studied in great detail in [AH88] (see [AH88, Page 20] for the definition). In [BFN18, Theorem A.1], it was shown that the Atiyah-Hitchin manifold is isomorphic to the moduli space of solutions of Nahm's equations on $[-1, 1]$ for $\mathrm{SU}(2)$ with an appropriate boundary condition.

Since a normal vector to the defining equation of $\mathcal{M}_C$ is $2U\partial_U - V^2\partial_\phi - 2V\Phi\partial_V$, the standard holomorphic 3-form $dU \wedge d\Phi \wedge dV$ on $\mathbf{A}^3_{\mathbf{C}}$ induces a holomorphic symplectic form $\frac{d\Phi \wedge dV}{2U}$ on $\mathcal{M}_C$. (This can also be written as $\frac{dU \wedge dV}{V^2}$ or as $\frac{d\Phi \wedge dU}{2\Phi V}$.) The associated Poisson bracket on $\mathcal{O}_{\mathcal{M}_C} \cong \mathrm{H}^{G_c}_*(\mathrm{Gr}_G; \mathbf{C})$ agrees with the 2-shifted Poisson bracket arising from the $\mathbf{E}_3$-structure on $C^{G_c}_*(\mathrm{Gr}_G; \mathbf{C})$.

The quantized algebra $\mathcal{A}_\epsilon$ can be described explicitly as follows. Let us write $\theta = \frac{1}{x}(s-1)$, where s is the simple reflection generating the Weyl group of SL_2. Then $\mathcal{A}_\epsilon$ is generated as an algebra over $\mathbf{C}[\![\hbar]\!]$ by $\mathbf{Z}/2$-invariant polynomials in x, $a^{\pm 1}$, and θ, where x is to be viewed as $a\partial_a$. Moreover, under the isomorphism $\mathcal{A}_\epsilon/\hbar \cong \mathcal{O}_{\mathcal{M}_C}$, the class x is sent to x, and θ is sent to $\frac{a-1}{x}$. We then have the commutation relation $[x, a^{\pm 1}] = \pm\hbar a^{-1}$, induced by $[\partial_a, a] = \hbar$; see Example 5.5. This implies that

$$[x^2, a^{\pm 1}] = \hbar^2 a^{\pm 1} \pm 2\hbar a^{\pm 1} x,$$

which in turn implies that $\mathcal{A}_\epsilon$ is the quotient of the free associative $\mathbf{C}[\![\hbar]\!]$-algebra on Φ, U, and $V = \frac{1}{x}(a - a^{-1})$ subject to the relations

$$\begin{aligned}[\Phi, V] &= 2\hbar U - \hbar^2 V,\\ [\Phi, U] &= 2\hbar\Phi V - \hbar^2 U,\\ [U, V] &= \hbar V^2,\\ U^2 - 4 &= \Phi V^2 - \hbar UV.\end{aligned}$$

Note that the commutation relations for $[\Phi, U]$ and $[U, V]$ in [DG19, Equation B.3] have typos, but it is stated correctly in [BDG17, Equation 5.51].

Example B.4. When $G = \mathrm{SL}_2$, we can identify $\mathcal{M}_C$ with the quotient of the scheme of Example B.3 by the free $\mathbf{Z}/2$-action sending $U \mapsto -U$ and $V \mapsto -V$; so

$$\mathcal{M}_C \operatorname{Spec} \mathbf{C}[x^2, (a + a^{-1})^2, \left(\tfrac{a-a^{-1}}{2x}\right)^2, \tfrac{(a+a^{-1})(a-a^{-1})}{2x}].$$

This is the regular centralizer group scheme for PGL_2. Note that if we denote

$$\begin{aligned}&\Phi = x^2,\\ &A = (a + a^{-1})^2,\\ &B = 4\left(\tfrac{a-a^{-1}}{2x}\right)^2 = \tfrac{(a-a^{-1})^2}{x^2},\\ &C = 2\tfrac{(a+a^{-1})(a-a^{-1})}{2x} = \tfrac{(a+a^{-1})(a-a^{-1})}{x},\end{aligned}$$

then we have relations

$$\begin{aligned}AB &= C^2,\\ A - \Phi B &= 4.\end{aligned}$$

In particular, $\mathcal{M}_C$ is cut out in $\mathbf{A}^3_{\mathbf{C}}$ (with coordinates Φ, B, and C) via the equation

$$C^2 - \Phi B^2 = 4B.$$

Note the similarity to the manifold from Example B.3: in fact, it is the quotient of the aforementioned manifold by the free $\mathbf{Z}/2$-action sending $U \mapsto -U$ and $V \mapsto -V$. In terms of these coordinates, $B = V^2$ and $C = UV$. (Sometimes, this quotient is also referred to as the Atiyah-Hitchin manifold.) It is also possible to describe $\mathcal{A}_\epsilon$; we leave this to the reader, since it is rather tedious.

Heuristic B.5. An unpublished conjecture of Gaiotto (which I learned about from Nakajima) says that the Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory over $\mathbf{R}^3 \times S^1$ with a generic choice of complex structure can be modeled by $\mathcal{M}^{4\mathrm{d}}_C := \operatorname{Spec} \mathrm{KU}_0^{G_c}(\mathrm{Gr}_G) \otimes_{\mathbf{Z}} \mathbf{C}$. Although I do not know Gaiotto's motivation for this conjecture (it is probably inspired by [SW97]), my attempt at heuristically justifying it goes as follows. (In [Dev24, Appendix C(b)], I suggest that it might be slightly better to consider $\operatorname{Spec} \mathrm{ku}_*^{G_c}(\mathrm{Gr}_G) \otimes_{\mathbf{Z}} \mathbf{C}$ instead, where ku denotes *connective* complex K-theory. The Bott class generating $\pi_2(\mathrm{ku})$ plays the role of the radius of the circle S^1.)

Recall that $\mathrm{Gr}_G/G[\![t]\!]$ can be viewed as $\mathrm{Bun}_G(S^2)$. It is reasonable to view $\mathrm{KU}_0(\mathrm{Bun}_G(S^2)) \otimes \mathbf{C}$ as closely related to $\mathrm{H}_*(L\mathrm{Bun}_G(S^2); \mathbf{C})$, where $L\mathrm{Bun}_G(S^2)$ denotes the topological free loop space of $\mathrm{Bun}_G(S^2)$. Since $LBG \simeq BLG$, we have $L\mathrm{Bun}_G(S^2) \simeq \mathrm{Bun}_{LG}(S^2)$, so one might view $\mathrm{H}_*(L\mathrm{Bun}_G(S^2); \mathbf{C})$ as the ring of functions on the "Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory on $\mathbf{R}^3$ with gauge group LG".

Making precise sense of this phrase seems difficult, but one possible workaround could be the following. It is often useful to view gauge theory with gauge group LG as "finite temperature" gauge theory with gauge group G. Recall that Wick rotation relates $(3+1)$-dimensional quantum field theory at a finite temperature T to statistical mechanics over $\mathbf{R}^3 \times S^1$ where the circle has radius $\frac{1}{2\pi T}$. This suggests that $\mathrm{H}_*(L\mathrm{Bun}_G(S^2); \mathbf{C})$ (which is more precisely to be replaced by $\mathrm{KU}_0^{G_c}(\mathrm{Gr}_G) \otimes \mathbf{C}$) can be viewed as the ring of functions on the "Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory on $\mathbf{R}^3 \times S^1$ with gauge group G". See [BFN18, Remark 3.14]. In [BFM05], $\mathrm{Spec}\,\mathrm{KU}_0^{G_c}(\mathrm{Gr}_G) \otimes \mathbf{C}$ was identified with the phase space of the relativistic Toda lattice for $\check{G}$.

One can also define a quantization of $\mathcal{M}_C^{4\mathrm{d}}$ via $\mathcal{A}_\epsilon^{4\mathrm{d}} := \mathrm{KU}_0^{G_c \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G) \otimes \mathbf{C}$; this can be viewed as a model for the quantized Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory on $\mathbf{R}^3 \times S^1$. The algebra $\mathcal{A}_\epsilon^{4\mathrm{d}}$ can be identified with the algebra of operators of the quantized relativistic Toda lattice for $\check{G}$.

Example B.6. When $G = \mathrm{PGL}_2$, Theorem 7.3 tells us that

$$\mathcal{M}_C^{4\mathrm{d}} \cong \mathrm{Spec}\,\mathbf{C}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-1}]^{\mathbf{Z}/2} \cong \mathrm{Spec}\,\mathbf{C}[x + x^{-1}, a + a^{-1}, \tfrac{(a-a^{-1})(x+1)}{x-1}],$$

where $\mathbf{Z}/2$ acts by $x \mapsto x^{-1}$ and $a \mapsto a^{-1}$. For simplicity, let us consider instead a slight variant of $\mathcal{M}_C^{4\mathrm{d}} = \mathrm{Spec}\,\mathrm{KU}_0^{\mathrm{PSU}(2)}(\mathrm{Gr}_{\mathrm{PGL}_2}) \otimes_{\mathbf{Z}} \mathbf{C}$, given by $\mathcal{M}'^{4\mathrm{d}}_C = \mathrm{Spec}\,\mathrm{KU}_0^{\mathrm{SU}(2)}(\mathrm{Gr}_{\mathrm{PGL}_2}) \otimes_{\mathbf{Z}} \mathbf{C}$. Then

$$\mathcal{M}'^{4\mathrm{d}}_C \cong \mathrm{Spec}\,\mathbf{C}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-x^{-1}}]^{\mathbf{Z}/2} \cong \mathrm{Spec}\,\mathbf{C}[x + x^{-1}, a + a^{-1}, \tfrac{a-a^{-1}}{x-x^{-1}}].$$

Let us write $\Psi = x + x^{-1}$, $W = a + a^{-1}$, and $Z = \frac{a-a^{-1}}{x-x^{-1}}$. Then, one easily verifies that $\mathcal{M}'^{4\mathrm{d}}_C$ is the subvariety of $\mathbf{A}^3_{\mathbf{C}}$ cut out by the equation

$$W^2 - (\Psi^2 - 4)Z^2 = 4.$$

This may be regarded as a multiplicative analogue of the Atiyah-Hitchin manifold. It would be very interesting to understand a relationship between this manifold and the moduli space of solutions to some analogue of Nahm's equations for $\mathrm{PSU}(2)$ with an appropriate boundary condition. The complex manifold $\mathcal{M}'^{4\mathrm{d}}_C$ has a holomorphic symplectic form given by $\frac{d\Psi \wedge dZ}{W}$, which can also be written as $\frac{d\Psi \wedge dW}{(\Psi^2-4)Z}$ or as $\frac{dZ \wedge dW}{\Psi Z^2}$.

It is also possible to explicitly describe the quantized algebra $\mathcal{A}_\epsilon^{4\mathrm{d}}$. The resulting description is not very enlightening, so we will only indicate how one reaches the answer. In this case, instead of the relation $[\partial_a, a] = \hbar$ which appeared in Example B.3, we have the relation $xa = qax$ (i.e., $xax^{-1}a^{-1} = q$); see Example 5.6. In particular, $xa^{-1} = q^{-1}a^{-1}x$, $x^{-1}a = q^{-1}ax^{-1}$, and $x^{-1}a^{-1} = qa^{-1}x^{-1}$. It follows after some tedious calculation that $\mathcal{A}_\epsilon^{4\mathrm{d}}$ is the quotient of the free associative $\mathbf{C}[q^{\pm 1}]$-algebra on Ψ, W, and $\frac{x+1}{x-1}(a - a^{-1})$ subject to four relations.

Suppose we consider instead the variant of $\mathcal{A}_\epsilon^{4\mathrm{d}}$ defined by $\mathcal{A}'^{4\mathrm{d}}_\epsilon = \mathrm{KU}_0^{\mathrm{SU}(2) \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{PGL}_2}) \otimes \mathbf{C}$. Then $\mathcal{A}'^{4\mathrm{d}}_\epsilon$ is the quotient of the free associative $\mathbf{C}[q^{\pm 1}]$-algebra on Ψ, W, and

$Z = \frac{1}{x-x^{-1}}(a - a^{-1})$ subject to the relations

$$\begin{aligned}
[\Psi, W] &= (q-1)(\Psi^2 - 4)Z - \tfrac{(q-1)^2}{2q}((\Psi^2-4)Z + \Psi W),\\
[\Psi, Z] &= (q-1)W - \tfrac{(q-1)^2}{2q}(\Psi Z + W),\\
[Z, W] &= (q-1)\Psi Z^2 - \tfrac{(q-1)^2}{2q}(\Psi Z + W)Z,\\
W^2 - 4 &= (\Psi^2-4)Z^2 - \tfrac{(q-1)^2}{2q}(\Psi^2-4)Z^2 + \tfrac{q^2-1}{2q}\Psi W Z.
\end{aligned}$$

Example B.7. When $G = \mathrm{SL}_2$, one can view $\mathcal{M}_C^{4\mathrm{d}}$ with the quotient of the scheme $\mathcal{M}'^{4\mathrm{d}}_C$ of Example B.6 by the free $\mathbf{Z}/2$-action sending $W \mapsto -W$ and $Z \mapsto -Z$; so

$$\mathcal{M}_C^{4\mathrm{d}} \cong \operatorname{Spec} \mathbf{C}[x + x^{-1}, (a+a^{-1})^2, \left(\tfrac{a-a^{-1}}{x-x^{-1}}\right)^2, \tfrac{(a-a^{-1})(a+a^{-1})}{x-x^{-1}}].$$

This is the regular centralizer group scheme for PGL_2. Note that if we denote

$$\begin{aligned}
\Psi &= x + x^{-1},\\
A &= (a+a^{-1})^2 = a^2 + a^{-2} + 2,\\
B &= \left(\tfrac{a-a^{-1}}{x-x^{-1}}\right)^2 = \tfrac{a^2+a^{-2}-2}{x^2+x^{-2}-2},\\
C &= \tfrac{(a+a^{-1})(a-a^{-1})}{x-x^{-1}} = \tfrac{a^2-a^{-2}}{x-x^{-1}},
\end{aligned}$$

then we have relations

$$\begin{aligned}
AB &= C^2,\\
A - (\Psi^2-4)B &= 4.
\end{aligned}$$

In particular, $\mathcal{M}_C^{4\mathrm{d}}$ is cut out in $\mathbf{A}^3_{\mathbf{C}}$ (with coordinates Ψ, B, and C) via the equation

$$C^2 - (\Phi^2 - 4)B^2 = 4B.$$

Note the similarity to Example B.6. It is also possible to describe $\mathcal{A}_\epsilon^{4\mathrm{d}}$; again, we leave this to the reader, since it is rather tedious.

Consider an elliptic curve $E(\mathbf{C})$ over $\mathbf{C}$. Motivated by Heuristic B.5 and [NY05], one might expect that (in some specific complex structure) the Coulomb branch of 5d $\mathcal{N} = 1$ pure gauge theory over $\mathbf{R}^3 \times E(\mathbf{C})$ can be modeled by the complexification of the G_c-equivariant k-homology of Gr_G, where k is an elliptic cohomology theory associated to a putative integral lift of E. Unfortunately, a classical result of Tate says that there are no smooth elliptic curves over $\mathbf{Z}$, so $E(\mathbf{C})$ cannot literally lift to $\mathbf{Z}$ (i.e., $\pi_0(k)$ cannot be $\mathbf{Z}$).

As a fix, one can more generally simultaneously consider all possible "Coulomb branches" $\mathcal{M}_C^{5\mathrm{d}} := \operatorname{Spec} \pi_0 \mathcal{F}_G(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$ associated to every complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k equipped with an oriented elliptic curve (this is almost equivalent to considering the universal example $\operatorname{Spec} \mathrm{tmf}_0^{G_c}(\mathrm{Gr}_G) \otimes \mathbf{C}$). We have described $\operatorname{Spec} \pi_0 \mathcal{F}_T(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$ in Theorem 8.7, from which one can calculate $\mathcal{M}_C^{5\mathrm{d}}$. Similarly, one can even use Theorem 8.15 to calculate $\pi_0 \mathcal{F}_{T \times \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$ and $\mathcal{A}_\epsilon^{5\mathrm{d}} := \pi_0 \mathcal{F}_{G \times \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$, but this is already incredibly complicated for $G = \mathrm{SL}_2$.

It would be very interesting to give a physical interpretation to $\pi_0 \mathcal{F}_G(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$ and $\pi_0 \mathcal{F}_{G \times \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G)^\vee \otimes \mathbf{C}$ for other 2-periodic $\mathbf{E}_\infty$-rings k, although we expect this to be very difficult (since most other chromatically interesting generalized cohomology theories only exist after profinite or p-adic completion, and do not admit transcendental analogues). It would also be very interesting to describe the analogue of our calculations for the ind-schemes $\mathcal{R}_{G,\mathbf{N}}$ introduced in [BFN18]. By adapting the methods of [BFN18, Section 4], this is approachable when G is a torus. We expect it to lead to interesting geometry for nonabelian G. In [Dev24], we extend the discussion of this paper (at least, the parts concerning ordinary cohomology and K-theory) to connective K-theory, and suggest an analogue of the relative Langlands program of [BZSV23] in this setting; as mentioned in the introduction of [Dev24], the story therein also admits an elliptic variant.

References

[ABG04] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg. Quantum groups, the loop Grassmannian, and the Springer resolution. *J. Amer. Math. Soc.*, 17(3):595–678, 2004.

[Ada66] F. Adams. On the groups $J(X)$. IV. *Topology*, 5:21–71, 1966.

[Ada69] J. F. Adams. Lectures on generalised cohomology. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 1–138. Springer, Berlin, 1969.

[AG15] D. Arinkin and D. Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Math. (N.S.)*, 21(1):1–199, 2015.

[AH88] M. Atiyah and N. Hitchin. *The geometry and dynamics of magnetic monopoles.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.

[AMM98] A. Alekseev, A. Malkin, and E. Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445–495, 1998.

[And00] M. Ando. Power operations in elliptic cohomology and representations of loop groups. *Trans. Amer. Math. Soc.*, 352(12):5619–5666, 2000.

[Ant18] B. Antieau. Periodic cyclic homology and derived de Rham cohomology. `https://arxiv.org/abs/1808.05246`, 2018.

[AS69] M. Atiyah and G. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.

[Ati57] M. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Ati66] M. Atiyah. Power operations in K-theory. *Quart. J. Math. Oxford Ser. (2)*, 17:165–193, 1966.

[BBB$^+$20] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte, and A. Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincare*, 21(4):1235–1310, 2020.

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BC18] A. Beliakova and B. Cooper. Steenrod structures on categorified quantum groups. *Fund. Math.*, 241(2):179–207, 2018.

[BDG17] M. Bullimore, T. Dimofte, and D. Gaiotto. The Coulomb branch of 3d $\mathcal{N} = 4$ theories. *Comm. Math. Phys.*, 354(2):671–751, 2017.

[Bei87] A. Beilinson. How to glue perverse sheaves. In *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 42–51. Springer, Berlin, 1987.

[BF08] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[BFGT21] A. Braverman, M. Finkelberg, V. Ginzburg, and R. Travkin. Mirabolic Satake equivalence and supergroups. *Compos. Math.*, 157(8):1724–1765, 2021.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005.

[BFN18] A. Braverman, M. Finkelberg, and H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Adv. Theor. Math. Phys.*, 22:1071–1147, 2018.

[BHS20] R. Burklund, J. Hahn, and A. Senger. Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra. `https://arxiv.org/abs/2010.10325`, 2020.

[BN15] D. Ben-Zvi and D. Nadler. Elliptic Springer theory. *Compos. Math.*, 151(8):1568–1584, 2015.

[BO78] P. Berthelot and A. Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[Bot58] R. Bott. The space of loops on a Lie group. *Michigan Math. J.*, 5:35–61, 1958.

[Bri00] M. Brion. Poincaré duality and equivariant (co)homology. *Michigan Math. J.*, 48:77–92, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[But20a] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Foundations. `https://arxiv.org/abs/2011.14988`, 2020.

[But20b] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Gauge theory applications. https://arxiv.org/abs/2011.14978, 2020.

[BZ00] J.-L. Brylinski and B. Zhang. Equivariant K-theory of compact connected Lie groups. *K-Theory*, 20(1):23–36, 2000. Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part I.

[BZSV23] D. Ben Zvi, Y. Sakellaridis, and A. Venkatesh. Relative Langlands duality. https://www.math.ias.edu/~akshay/research/BZSVpaperV1.pdf, 2023.

[CD23] H. Chen and G. Dhillon. A Langlands dual realization of coherent sheaves on the nilpotent cone. https://arxiv.org/abs/2310.10539, 2023.

[CG10] N. Chriss and V. Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1997 edition.

[CK18] S. Cautis and J. Kamnitzer. Quantum K-theoretic geometric Satake: the SL_n case. *Compos. Math.*, 154(2):275–327, 2018.

[CMNO22] T.-H. Chen, M. Macerato, D. Nadler, and J. O'Brien. Quaternionic Satake equivalence. https://arxiv.org/abs/2207.04078, 2022.

[CR23] J. Campbell and S. Raskin. Langlands duality on the Beilinson-Drinfeld Grassmannian. Available at https://gauss.math.yale.edu/~sr2532/, 2023.

[Dav19] D. Davis. The elliptic Grothendieck-Springer resolution as a simultaneous log resolution of algebraic stacks. https://arxiv.org/abs/1908.04140, 2019.

[Dev23] S. Devalapurkar. Topological Hochschild homology, truncated Brown-Peterson spectra, and a topological Sen operator. https://sanathdevalapurkar.github.io/files/thh-Xn.pdf, 2023.

[Dev24] S. Devalapurkar. ku-theoretic spectral decompositions for spheres and projective spaces. https://sanathdevalapurkar.github.io/files/hyperboloid_spectral_decomp.pdf, 2024.

[DG19] T. Dimofte and N. Garner. Coulomb branches of star-shaped quivers. *J. High Energy Phys.*, (2):004, front matter+87, 2019.

[Dic97] L. E. Dickson. The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group. *Ann. of Math.*, 11(1-6):65–120, 1896/97.

[DM23] S. Devalapurkar and M. Misterka. Generalized n-series and de Rham complexes. https://sanathdevalapurkar.github.io/files/fgls-and-dR-complexes.pdf, 2023.

[Dri21] V. Drinfeld. A 1-dimensional formal group over the prismatization of $\mathrm{Spf}\,\mathbf{Z}_p$. http://arxiv.org/abs/2107.11466, 2021.

[Fel95] G. Felder. Elliptic quantum groups. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pages 211–218. Int. Press, Cambridge, MA, 1995.

[FHT08] D. Freed, M. Hopkins, and C. Teleman. Twisted equivariant K-theory with complex coefficients. *J. Topol.*, 1(1):16–44, 2008.

[FHT11a] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[FHT11b] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory III. *Ann. of Math. (2)*, 174(2):947–1007, 2011.

[FHT13] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory II. *J. Amer. Math. Soc.*, 26(3):595–644, 2013.

[FM00] R. Friedman and J. Morgan. Holomorphic principal bundles over elliptic curves. II. The parabolic construction. *J. Differential Geom.*, 56(2):301–379, 2000.

[FMW98] R. Friedman, J. Morgan, and E. Witten. Principal G-bundles over elliptic curves. *Math. Res. Lett.*, 5(1-2):97–118, 1998.

[FT15] D. Freed and C. Teleman. Dirac families for loop groups as matrix factorizations. *C. R. Math. Acad. Sci. Paris*, 353(5):415–419, 2015.

[FT19] M. Finkelberg and A. Tsymbaliuk. Multiplicative slices, relativistic Toda and shifted quantum affine algebras. In *Representations and nilpotent orbits of Lie algebraic systems*, volume 330 of *Progr. Math.*, pages 133–304. Birkhäuser/Springer, Cham, 2019.

[Gan22a] T. Gannon. Classification of nondegenerate G-categories. https://arxiv.org/abs/2206.11247, 2022.

[Gan22b] T. Gannon. The coarse quotient for affine Weyl groups and pseudo-reflection groups. https://arxiv.org/abs/2206.00175, 2022.

[Gin18] V. Ginzburg. Nil-Hecke algebras and Whittaker $\mathcal{D}$-modules. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 137–184. Birkhäuser/Springer, Cham, 2018.

[GK22] V. Ginzburg and D. Kazhdan. Differential operators on G/U and the Gelfand-Graev action. *Adv. Math.*, 403:Paper No. 108368, 48, 2022.

[GKM98] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[GKV95] V. Ginzburg, M. Kapranov, and E. Vasserot. Elliptic algebras and equivariant elliptic cohomology. `https://arxiv.org/abs/q-alg/9505012`, 1995.

[GKV97] V. Ginzburg, M. Kapranov, and E. Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128(1):1–19, 1997.

[GM20] D. Gepner and L. Meier. On equivariant topological modular forms. `https://arxiv.org/abs/2004.10254`, 2020.

[GM23] D. Gepner and L. Meier. Equivariant elliptic cohomology with integral coefficients. Forthcoming, 2023.

[GPS18] S. Ganatra, J. Pardon, and V. Shende. Microlocal Morse theory of wrapped Fukaya categories. `https://arxiv.org/abs/1809.08807`, 2018.

[GR75] H. Garland and M. S. Raghunathan. A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott. *Proc. Nat. Acad. Sci. U.S.A.*, 72(12):4716–4717, 1975.

[GR15] V. Ginzburg and S. Riche. Differential operators on G/U and the affine Grassmannian. *J. Inst. Math. Jussieu*, 14(3):493–575, 2015.

[GR17] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[Gre21] R. Gregoric. Moduli stack of oriented formal groups and cellular motivic spectra over $\mathbf{C}$. `https://arxiv.org/abs/2111.15212`, 2021.

[GWX21] B. Gheorghe, G. Wang, and Z. Xu. The special fiber of the motivic deformation of the stable homotopy category is algebraic. *Acta Math.*, 226(2):319–407, 2021.

[HHH05] M. Harada, A. Henriques, and T. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197(1):198–221, 2005.

[HHR16] M. Hill, M. Hopkins, and D. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[HKR00] M. Hopkins, N. Kuhn, and D. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[HL22] Q. Ho and P. Li. Revisiting mixed geometry. `https://arxiv.org/abs/2202.04833`, 2022.

[HM23] M. Hausmann and L. Meier. Invariant prime ideals in equivariant Lazard rings. `https://arxiv.org/abs/2309.00850v1`, 2023.

[HRW22] J. Hahn, A. Raksit, and D. Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. `https://arxiv.org/abs/2206.11208`, 2022.

[HY19] J. Hahn and A. Yuan. Multiplicative structure in the stable splitting of $\Omega SL_n(\mathbf{C})$. *Adv. Math.*, 348:412–455, 2019.

[Kit13] N. Kitchloo. Cohomology operations and the Nil-Hecke ring. `https://math.jhu.edu/~nitu/papers/NH.pdf`, 2013.

[KK86] B. Kostant and S. Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Proc. Nat. Acad. Sci. U.S.A.*, 83(6):1543–1545, 1986.

[KK90] B. Kostant and S. Kumar. T-equivariant K-theory of generalized flag varieties. *J. Differential Geom.*, 32(2):549–603, 1990.

[Kos63] B. Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963.

[Kos78] B. Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

[KS90] M. Kashiwara and P. Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel.

[KS20] A. Kalmykov and P. Safronov. A categorical approach to dynamical quantum groups. `https://arxiv.org/abs/2008.09081`, 2020.

[Kum02] S. Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[Lon17] G. Lonergan. A remark on descent for Coxeter groups. `https://arxiv.org/abs/1707.01156v2`, 2017.

[Lon18] G. Lonergan. A Fourier transform for the quantum Toda lattice. *Selecta Math. (N.S.)*, 24(5):4577–4615, 2018.

[Lon21a] G. Lonergan. Geometric Satake over KU. Online talk, available at `https://www.youtube.com/watch?v=Aazmxfb2i_A`, 2021.

[Lon21b] G. Lonergan. Steenrod operators, the Coulomb branch and the Frobenius twist. *Compos. Math.*, 157(11):2494–2552, 2021.

[Lur09] J. Lurie. A survey of elliptic cohomology. In *Algebraic Topology*, volume 4 of *Abel. Symp.*, pages 219–277. Springer, 2009.

[Lur16] J. Lurie. Higher Algebra. `http://www.math.harvard.edu/~lurie/papers/HA.pdf`, 2016.

[Lur18a] J. Lurie. Elliptic Cohomology I: Spectral Abelian Varieties. `http://www.math.harvard.edu/~lurie/papers/Elliptic-I.pdf`, 2018.

[Lur18b] J. Lurie. Elliptic Cohomology II: Orientations. `http://www.math.harvard.edu/~lurie/papers/Elliptic-II.pdf`, 2018.

[Lur19] J. Lurie. Elliptic Cohomology III: Tempered Cohomology. `https://www.math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf`, 2019.

[Mil85] H. Miller. Stable splittings of Stiefel manifolds. *Topology*, 24(4):411–419, 1985.

[Mit88] S. Mitchell. Quillen's theorem on buildings and the loops on a symmetric space. *Enseign. Math. (2)*, 34(1-2):123–166, 1988.

[MV07] I. Mirkovic and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[Nak16] H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I. *Adv. Theor. Math. Phys.*, 20(3):595–669, 2016.

[Nak17] H. Nakajima. Introduction to a provisional mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories. `https://arxiv.org/abs/1706.05154`, 2017.

[NS09] N. Nekrasov and S. Shatashvili. Quantization of Integrable Systems and Four Dimensional Gauge Theories. In *16th International Congress on Mathematical Physics*, pages 265–289, 8 2009.

[NS18] T. Nikolaus and P. Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[NY05] H. Nakajima and K. Yoshioka. Instanton counting on blowup. II. K-theoretic partition function. *Transform. Groups*, 10(3-4):489–519, 2005.

[NZ09] D. Nadler and E. Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.*, 22(1):233–286, 2009.

[Pre02] A. Premet. Special transverse slices and their enveloping algebras. *Adv. Math.*, 170(1):1–55, 2002. With an appendix by Serge Skryabin.

[Pre15] A. Preygel. Ind-coherent complexes on loop spaces and connections. In *Stacks and categories in geometry, topology, and algebra*, volume 643 of *Contemp. Math.*, pages 289–323. Amer. Math. Soc., Providence, RI, 2015.

[Pst23a] P. Pstragowski. Perfect even modules and the even filtration. `https://arxiv.org/abs/2304.04685`, 2023.

[Pst23b] P. Pstragowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023.

[Rak20] A. Raksit. Hochschild homology and the derived de Rham complex revisited. `https://arxiv.org/abs/2007.02576`, 2020.

[Rav84] D. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.

[Rez14] C. Rezk. Isogenies, power operations, and homotopy theory. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1125–1145. Kyung Moon Sa, Seoul, 2014.

[Ric17] S. Riche. Kostant section, universal centralizer, and a modular derived Satake equivalence. *Math. Z.*, 286(1):223–261, 2017.

[Rog08] J. Rognes. *Galois Extensions of Structured Ring Spectra/Stably Dualizable Groups*, volume 192 of *Mem. Amer. Math. Soc.* American Mathematical Society, 2008.

[Sch17] P. Scholze. Canonical q-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(5):1163–1192, 2017.

[Seg68] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[Sta13] N. Stapleton. Transchromatic generalized character maps. *Algebr. Geom. Topol.*, 13(1):171–203, 2013.

[Sta15] N. Stapleton. Transchromatic twisted character maps. *J. Homotopy Relat. Struct.*, 10(1):29–61, 2015.

[Ste65] R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[Str98] N. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

[SW97] N. Seiberg and E. Witten. Gauge dynamics and compactification to three dimensions. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 333–366. World Sci. Publ., River Edge, NJ, 1997.

[Tel14] C. Teleman. Gauge theory and mirror symmetry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1309–1332. Kyung Moon Sa, Seoul, 2014.

[Tel18] C. Teleman. Topological Gauge Theory in low dimensions. `https://math.berkeley.edu/~teleman/math/Auckland.pdf`, 2018.

[Toe14] B. Toen. Derived algebraic geometry and deformation quantization. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 769–792. Kyung Moon Sa, Seoul, 2014.

[Van08] M. Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008.

[YZ11] Z. Yun and X. Zhu. Integral homology of loop groups via Langlands dual groups. *Represent. Theory*, 15:347–369, 2011.

[YZ21] Y. Yang and G. Zhao. Frobenii on Morava E-theoretical quantum groups. `https://arxiv.org/abs/2105.14681`, 2021.

[Zho23] C. Zhong. Equivariant oriented homology of the affine Grassmannian. `https://arxiv.org/abs/2301.13056`, 2023.

[Zhu17] X. Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017.

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