Integrable systems

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Lecture 3: Symplectic manifolds

The 2-form $dq \wedge dp$ on the cotangent bundle played an important role in Noether's theorem: if $\mu : T^*X \to \mathfrak{g}^*$ is the moment map, and $\xi \in \mathfrak{g}$, then the 1-form $\langle dq \wedge dp, \xi \rangle$ identifies with $d\langle \mu, \xi \rangle$. This 2-form $dq \wedge dp$ also appeared in some other ways:

- When we talked about the simple harmonic oscillator, its integral over of the ellipse in $T^*\mathbf{R}$ traced out by the motion $t \mapsto (q(t), p(t))$ of the simple harmonic oscillator was the "action", denoted I. We saw that this integral was invariant under time (i.e., $\dot{I} = 0$).
- When we rephrased the Euler-Lagrange equations in the Hamiltonian formalism, we associated a vector field X_H to the Hamiltonian $H : T^*X \to \mathbf{R}$ by the formula
$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}.$$
This formula can be alternatively interpreted as follows. One can pair a vector field with a 2-form to get a 1-form; for example, $\langle \frac{\partial}{\partial q}, dq \wedge dp \rangle = dp$, and $\langle \frac{\partial}{\partial p}, dq \wedge dp \rangle = -dq$ by antisymmetry. If we pair X_H with the 2-form $dq \wedge dp$, we get
$$\langle X_H, \omega \rangle = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq,$$
and this is precisely the 1-form dH on T^*X. One should therefore view the assignment $H \rightsquigarrow X_H$ as an "inverse" (under pairing with $dq \wedge dp$) of the assignment $H \rightsquigarrow dH$.

It is therefore clear that the 2-form $dq \wedge dp$ plays a very important role in the study of T^*X. Before we proceed, let me just describe a more general version of the first bullet above (to further motivate the story).

Proposition 1 (Liouville's theorem)**.** *Let V be a vector space. Suppose that $H : T^*V \to \mathbf{R}$ is a smooth function, and let $f_t : T^*V \to T^*V$ denote the flow of H (i.e., $\frac{\partial f_t}{\partial t} = X_H(f_t)$). If $U \subseteq T^*V$ is a subset, the volume* $\mathrm{vol}(U)$ *is preserved by the flow f_t.*

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Proof. This is not strictly necessary, but choose coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ on T^*V, so that

$$\mathrm{vol}(U) = \int_U dq_1 \wedge \cdots \wedge dq_n \wedge dp_1 \wedge \cdots \wedge dp_n.$$

Notice that if we write $\omega = \sum_i dq_i \wedge dp_i$, then the $2n$-form inside the integral is the n-fold wedge product $\frac{\omega^{\wedge n}}{n!}$; so we just need to see that the flow preserves ω. Let me assume $V = \mathbf{R}$ for simplicity (the general case just has more notation), and let $q_t = f_t(q)$ and $p_t = f_t(p)$. Then the change of coordinates for the transformation $dq \wedge dp \rightsquigarrow dq_t \wedge dp_t$ is given by the Jacobian

$$J = \det \begin{pmatrix} \frac{\partial q_t}{\partial q} & \frac{\partial p_t}{\partial q} \\ \frac{\partial q_t}{\partial p} & \frac{\partial p_t}{\partial p} \end{pmatrix}$$

But if we expand $q_t = q + \dot{q}dt + \cdots$ and $p_t = p + \dot{p}dt + \cdots$ and use Hamilton's equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$, we see that (ignoring higher order terms)

$$\begin{pmatrix} \frac{\partial q_t}{\partial q} & \frac{\partial p_t}{\partial q} \\ \frac{\partial q_t}{\partial p} & \frac{\partial p_t}{\partial p} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\partial^2 H}{\partial q \partial p} dt & -\frac{\partial^2 H}{\partial q^2} \\ \frac{\partial^2 H}{\partial p^2} & 1 - \frac{\partial^2 H}{\partial q \partial p} dt \end{pmatrix},$$

which has determinant equal to 1. □

Let us try to formalize the things we have observed so far:

- The 2-form $\omega := dq \wedge dp$ describes a map between tangent and cotangent fibers, which is an *isomorphism*.
- The flow associated to the Hamiltonian vector field X_H preserves the 2-form ω, and in particular, preserves volumes. Note that a 2×2-matrix A preserves areas if and only if it is *symplectic*, i.e., $A^T \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

This leads us to:

Definition 2. A *symplectic manifold* is a smooth n-manifold M equipped with a 2-form $\omega \in \Omega^2_M$ (that is, it defines for each $x \in M$ an alternating multilinear map $T_x M \otimes T_x M \to \mathbf{R}$, compatibly in x) such that:

- ω is closed, i.e., $d\omega = 0$.
- ω is nondegenerate, i.e., the map $T_x M \to T^*_x M$ given by $v \mapsto [w \mapsto \omega(v, w)]$ is an isomorphism.

Example 3. If X is a smooth manifold, the cotangent bundle T^*X admits a canonical symplectic form. Here is one way of thinking about it. The cotangent bundle admits a canonical 1-form θ, given as follows. Let $(x, \xi) \in T^*X$. The canonical map $\pi : T^*X \to X$ then defines a map

$$T_{(x,\xi)} T^*X \xrightarrow{\pi_*} T_x X \xrightarrow{\xi} \mathbf{R},$$

i.e., a 1-form θ on T^*X. The desired symplectic form is then given by $\omega := d\theta$. For instance, if $X = \mathbf{R}^n$ with coordinates $(q_1, \cdots, q_n)$ and cotangent fiber coordinates $(p_1, \cdots, p_n)$, the 1-form θ is precisely $\sum_j p_j dq_j$.

Before trying to understand the geometry of nonlinear symplectic manifolds, let us try to understand *linear* symplectic manifolds.

Proposition 4 (Darboux)**.** *Let V be a symplectic vector space, i.e., a vector space equipped with an alternating nondegenerate form $\omega : \wedge^2 V \to \mathbf{R}$. Then V is even-dimensional, and there is a symplectic basis of V (i.e., there is a basis $\{q_1, \cdots, q_n, p_1, \cdots, p_n\}$ such that $\omega(q_i, p_j) = 0$ for $i \neq j$, and $\omega(q_i, p_i) = -\omega(p_i, q_i) = 1$).*

Proof. This is a standard change-of-basis argument: take some nonzero element $q_1 \in V$, let $p_1 \in V$ denote some element such that $\omega(p_1, q_1) = 1$ (this means that p_1 is not in the line generated by q_1, because any such element would pair with q_1 to zero). Look at the symplectic complement to the subspace spanned by p_1, q_1, and induct. □

Corollary 5. *Let (M, ω) be a symplectic manifold, and let $x \in M$. Then there is a neighborhood U of x such $(U, \omega|_U)$ is isomorphic to $(\mathbf{R}^{2n}, \omega_{\mathrm{std}})$ as symplectic vector spaces. In particular, symplectic manifolds are even-dimensional.*

Suppose M is $2n$-dimensional, and let $x \in M$. Note that ω, being nondegenerate, gives an isomorphism $\wedge^j T_x M \xrightarrow{\sim} \wedge^{2n-j} T_x^* M$. In particular, $\omega^{\wedge n}$ defines an isomorphism $\wedge^{2n} T_x^* M \cong \mathbf{R}$, and so it defines a volume form on M. (Actually, it is better to consider $\frac{\omega^{\wedge n}}{n!}$, as one can see by computing in the case of $\mathbf{R}^{2n}$ equipped with its standard symplectic form.)

Remark 6. Suppose $A : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is a linear transformation preserving the standard symplectic form ω_{std}. Then its matrix in the basis $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ satisfies $A^T \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)^{\oplus n} A = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)^{\oplus n}$. Indeed, $\omega(v, w) = v^T \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) w$, and so

$$v^T \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) w = \omega(v, w) = \omega(Av, Aw) = v^T A^T \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) A w,$$

which implies the desired claim. The group of such linear transformations is called Sp_{2n} (it can be topologized as a subgroup of $\mathrm{GL}_{2n}(\mathbf{R})$), and is called the *symplectic group*.

Exercise 7. Let A be a symplectic matrix acting on $\mathbf{R}^{2n}$, and let $f(t)$ denote its characteristic polynomial. Then $f(t) = t^{2n} f(1/t)$.

Example 8. Here is a nonlinear example. Let $M = S^2 \subseteq \mathbf{R}^3$ with cylindrical coordinates θ, r, z. Then we can restrict the standard 3-form on $\mathbf{R}^3$ to S^2; in Cartesian coordinates x, y, z, the normal to S^2 is $x\partial_x + y\partial_y + z\partial_z$, and so the pullback of $dx \wedge dy \wedge dz$ is

$$x dy \wedge dz - y dx \wedge dz + z dx \wedge dy = dz \wedge d\theta$$

This is a symplectic form on S^2. Notice that this is *not* a cotangent bundle. In fact, non-cotangent bundles arise naturally in physics (we will see how later when talking about Hamiltonian/symplectic reduction).

One should think that symplectic manifolds are "locally cotangent bundles". Let us try to develop some of the results in preceding lectures in the setting of symplectic geometry. There were really three key things: Hamilton's equations, Liouville's theorem, and Noether's theorem. Let us begin with Hamilton's equations.

Construction 9. Suppose M is a symplectic manifold, and we are given a function $H : M \to \mathbf{R}$. Following the observation at the beginning of this lecture, we may define a *vector field* X_H on M by demanding that it pair with ω to gigve the 1-form dH, i.e., $\langle X_H, \omega \rangle = dH$. Because ω is nondegenerate, this uniquely pins down X_H.

Following the second lecture, we may now write down the flow equation for a curve in T^*M:

$$\dot{f} = X_H(f).$$

This is the analogue of Hamilton's equation in the setting of a general symplectic manifold.

Of course, it would be sort of pointless to write down such an equation if it didn't have the same nice properties as it did on cotangent bundles. Before giving the generalization of Liouville's theorem, let us quickly review the Lie derivative. Suppose X is a smooth manifold, and let V be a vector field on X with flow $f_t : X \to X$. If α is a j-form on X, the Lie derivative $\mathcal{L}_V \alpha$ is the j-form given by $\left.\frac{df_t^*(\alpha)}{dt}\right|_{t=0}$. If W is another vector field, the Lie bracket $[V, W]$ is the vector field $\left.\frac{df_{t,*}(W)}{dt}\right|_{t=0}$.

Exercise 10. Show that $[V, W]$ makes T_X into a Lie algebra.

Lemma 11. *There is an equality of j-forms*

$$\mathcal{L}_V \alpha = d\langle V, \alpha\rangle + \langle V, d\alpha\rangle.$$

This is sometimes called Cartan's "magic formula".

Theorem 12 (Liouville)**.** *Let $f_t : M \to M$ denote the flow of the Hamiltonian H. Then $f_t^*\omega = \omega$. That is, it is a **symplectomorphism**. In particular, f_t preserves the volume form $\frac{\omega^{\wedge n}}{n!}$ on M.*

PROOF. We need to see that $\mathcal{L}_{X_H}\omega = 0$. By Lemma 11,

$$\mathcal{L}_{X_H}\omega = d\langle X_H, \omega\rangle + \langle X_H, d\omega\rangle;$$

but the first term is $d(dH) = 0$, and the second term is zero because ω is closed. $\square$

Let us now turn to Noether's theorem. In this case, it will *not* be true that every 1-parameter family of symplectomorphisms of a symplectic manifold M gives rise to a conserved quantity. If ξ is a vector field on M (the 1-parameter family in question being the flow of this vector field), then one *can* obtain a 1-form on M given by $\langle \xi, \omega\rangle$; but in some cases, as we saw last time, this 1-form will actually be the exterior derivative of a function on M, and *this* function will be conserved. To explain this, let us make the following construction (for the umpteenth time).

Construction 13. Let $F : M \to \mathbf{R}$ be a function. Then one obtains a vector field X_F on M, characterized by the property that $\langle \omega, X_F\rangle = dF$. Now, given *two* functions $F, G : M \to \mathbf{R}$, we can define a third function $\{F, G\}$ on M given by the formula

$$\{F, G\} = \omega(X_F, X_G).$$

This is called the *Poisson bracket*. Note that

$$\omega(X_F, X_G) = \langle X_F, dG\rangle = X_F(G) = -\langle dF, X_G\rangle = -X_G(F).$$

Example 14. If $M = T^*\mathbf{R}$ with coordinates (q, p), then $X_F = \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}$. But now, since $\omega = dq \wedge dp$, we find that

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p}.$$

Notice that if $F = p$ and $G = q$, then this says that $\{p, q\} = 1$. In particular, if we keep track of this Poisson bracket, then position and momentum don't commute.

Exercise 15. Let M be a symplectic manifold. Show that $C^\infty(M)$ equipped with the Poisson bracket forms a Lie algebra, and that $X_{\{F,G\}} = [X_F, X_G]$. That is, the map $C^\infty(M) \to T_M$ sending $F \mapsto X_F$ is one of Lie algebras.

Exercise 16. A map $f : M \to M$ is a symplectomorphism if and only if $f : C^\infty(M) \to C^\infty(M)$ preserves Poisson brackets.

Theorem 17 (Restatement of Noether)**.** *Suppose $F : M \to \mathbf{R}$ is a function. Then along flow lines of a Hamiltonian $H : M \to \mathbf{R}$, one has*

$$\dot{F} = \{F, H\}.$$

In particular, if F "Poisson-commutes" with the Hamiltonian $H : M \to \mathbf{R}$, i.e., $\{F, H\} = 0$, then F is constant on flow lines of H.

Proof. Indeed, the flow equation gives that

$$\frac{dF(f(t))}{dt} = \langle dF, \dot{f}(t)\rangle = \langle dF, X_H\rangle = \omega(X_F, X_H) = \{F, H\},$$

as desired. □

The Jacobi identity for the Poisson bracket implies the following result (which was one of the first motivations for introducing Poisson brackets).

Corollary 18 (Poisson)**.** *Suppose F, G are two functions which Poisson-commute with H. Then so does $\{F, G\}$; in particular, $\{F, G\}$ is a new conserved quantity on flow lines of H.*

For instance, H is constant along its trajectories; this can be thought of as a statement of the conservation of energy. Now, our formulation of Noether's theorem from last time gave us that if $\mathfrak{g}$ was a Lie algebra acting on X by vector fields, there was a moment map $\mu : T^*X \to \mathfrak{g}^*$ characterized by the formula that if $\xi \in \mathfrak{g}$ (with the same symbol used to denote the corresponding vector field on X), then

$$d\langle \mu, \xi\rangle = \langle \omega, \xi\rangle. \tag{1}$$

So, what Noether's theorem really produces is a map $\mathfrak{g} \to C^\infty(T^*X)$ sending $\xi \mapsto \langle \mu, \xi\rangle$. Let us just turn this into a definition:

Definition 19. Let G be a Lie group. An action of G on a symplectic manifold M (so $\mathfrak{g}$ acts by vector fields, i.e., by a map $\mathfrak{g} \to T_M$) is called *Hamiltonian* if the map $\mathfrak{g} \to T_M$ lifts to an *G-equivariant* map of Lie algebras $\mathfrak{g} \to C^\infty(M)$. This gives an equivariant map $M \to \mathfrak{g}^*$, called the *moment map*, and it precisely satisfies (1).

References

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