CALCULUS AND COHOMOLOGY (OR, NONLINEAR NUMBERS)

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Abstract. Recent years have seen a proliferation of applications of homotopy theory to other branches of mathematics. In this survey, I will describe a story relating *chromatic* homotopy theory, which builds on the insight of Quillen, Morava, and many others connecting homotopy theory to the theory of 1-dimensional formal groups, to q-deformed mathematics (and "formal group" generalizations thereof), p-adic Hodge theory, geometric representation theory, and symplectic topology. The driving idea is to replace the integer n and the q-integer $[n]_q = \frac{q^n-1}{q-1}$ by the n-series of a 1-dimensional formal group law; this leads to an analogue of (q-)calculus which can be understood through invariants like Hochschild (co)homology. We explain some of the principles behind this generalization of (q-)calculus, like a stacky approach to the corresponding generalization of de Rham cohomology, as well as applications to representation theory, like formal group analogues of $\mathrm{U}(\mathfrak{gl}_n)$ and $\mathrm{U}_q(\mathrm{GL}_n)$; these can all be approached using ideas from homotopy theory.

1. Introduction

In the mid 1700's, mathematicians were studying "basic" (in the sense of "base-q") analogues of classical functions, like the logarithm [Eul53]. Euler and Gauss [Gau11] soon defined "basic" analogues of hypergeometric functions, and around the mid 1800's, Heine [Hei46] defined a q-analogue of the hypergeometric series and proved analogues of several results of Gauss; the basic premise is to replace the number $n \in \mathbf{Z}$ by the polynomial $[n]_q = \frac{q^n-1}{q-1} \in \mathbf{Z}[\![q-1]\!]$. Unfortunately, this work went somewhat unnoticed until Jackson and Rogers in the 1900's, who systematically developed q-analogue theory; see, e.g., [Jac09] where Jackson introduced the q-derivative. This soon blossomed into a rich subject (see, e.g., [GR04]), leading to many developments that have greatly changed the face of mathematics, like the theory of quantum groups [Dri87] and prismatic cohomology [Sch17, BS22] among several others.

The integer n and the q-integer $[n]_q$ are each the n-fold sum of the number 1 in the formal group laws $x + y$ and $x + y + (q-1)xy$. If $\widetilde{\mathrm{F}}$ is any (algebraizable) 1-dimensional formal group law over a commutative ring R, we are therefore led to consider elements $\langle n \rangle \in \mathrm{R}$ given by the n-fold sum of 1 under $\widetilde{\mathrm{F}}$. One of the theses of this survey article is that nearly everything in q-analogue theory admits an "$\widetilde{\mathrm{F}}$-analogue" obtained by replacing n or $[n]_q$ by the elements $\langle n \rangle$. The other thesis of this article is that via the aforementioned connection between homotopy theory and formal group laws discovered by Quillen [Qui69], these $\widetilde{\mathrm{F}}$-analogues admit proofs through homotopical/geometric methods which are *uniform* in $\widetilde{\mathrm{F}}$.

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Furthermore, this "translation" from geometry to algebra is a manifestation of the general principles of geometric Langlands theory.

The foundation of this whole story is a reinterpretation of the classical Weyl algebra of a smooth algebra A over a commutative ring k in terms of S^1-equivariant higher Hochschild cohomology relative to k (which we do in Section 2). This, in turn, is just Koszul dual to the relationship between Hochschild homology and differential forms discovered by Hochschild-Kostant-Rosenberg [HKR62] and recast in many different ways [BZN12, Ant19, Rak20, BMS19, HRW22] over the ensuing decades. Underlying this entire picture is the philosophy that S^1-equivariance captures a notion of deformation; we also observe that S^1-equivariantly framed $\mathbf{E}_3$-algebras are homotopical analogues of the notion of (Frobenius-constant; see [BK08]) deformation quantizations.

These definitions work equally well when k is taken to be a commutative ring *spectrum*. Chromatic homotopy theory (which we briefly survey in the beginning of Section 3) kicks in to produce a 1-dimensional formal group $\widetilde{\mathrm{F}}$ associated to k; then, S^1-equivariant higher Hochschild cohomology relative to k produces a "$\widetilde{\mathrm{F}}$-analogue" $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ of the Weyl algebra of the affine line. Here, there is an element ∂_x^k which acts on a monomial x^n by $\langle n\rangle x^{n-1}$; more generally it satisfies the commutation rule $\partial_x^k x = x\partial_x^k +_{\widetilde{\mathrm{F}}} 1$. We explore these ideas in Section 3. When k is the simplest example of a chromatically interesting ring spectrum, namely connective complex K-theory, $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ is just the q-Weyl algebra of the affine line. (In a Koszul dual form, this relationship had been independently discovered by Arpon Raksit.) We give homotopy-theoretic constructions of many $\widetilde{\mathrm{F}}$-analogues of classical facts about Weyl algebras, like the "large center" phenomena in characteristic $p > 0$ [BMR08].

Recently, it has been realized (see [Sim97, Dri18, Dri24, Bha24, BL22]) that the "stacky approach" to (q-)de Rham cohomology via de Rham stacks and its ilk yields a very rich theory with many applications. Motivated by this, we discuss a calculation of the "$\widetilde{\mathrm{F}}$-de Rham stacks" of $\mathbf{A}^1$ and $\mathbf{G}_m$ in Section 4; this builds on joint work [DHRY26, DM23] of myself with Jeremy Hahn, Arpon Raksit, and Allen Yuan, and separately with Max Misterka. Along the way, we describe some rather pretty identities involving an $\widetilde{\mathrm{F}}$-analogue of the (q-)logarithm which play a crucial role in nearly every calculation that I have encountered. These are used, for example, to prove properties of $\widetilde{\mathrm{F}}$-analogues of divided powers and of the polynomials $(x-y)^n$ and $(x-y)(x-qy)\cdots(x-q^{n-1}y)$ which play an important role in (q-deformed) calculus.

Armed with an $\widetilde{\mathrm{F}}$-analogue $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ of the Weyl algebra, one is led to wonder if there are analogues of the Fourier and Mellin transforms. These do indeed exist: it turns out that there is an equivalence $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1) \simeq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1)$ which behaves like the Fourier transform, and an equivalence $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{G}_m) \simeq \mathrm{QCoh}(\widetilde{\mathrm{F}}/\mathbf{Z})$ which behaves like the Mellin transform, where $\mathbf{Z}$ acts on $\widetilde{\mathrm{F}}$ by translation by 1. See Section 5. The Mellin transform of an $\widetilde{\mathrm{F}}$-analogue of the exponential function defines an $\widetilde{\mathrm{F}}$-analogue of the Γ-function, which satisfies many of the same properties as the usual (q-)Γ-function.

In Section 6, we turn to some applications to geometric representation theory. Since the work of Beilinson-Bernstein [BB81], the connection between D-modules and representation theory has led to great advances. Applying the theory of $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules defined above suggests that there is an $\widetilde{\mathrm{F}}$-analogue of much of the representation theory of reductive Lie algebras (which, when $\widetilde{\mathrm{F}}$ is the multiplicative formal group, specializes essentially to the theory of quantum groups). For instance, we give a definition of an $\widetilde{\mathrm{F}}$-analogue of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ and check that there is a version of Beilinson-Bernstein localization relating representations of $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_2)$ with $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules on the flag variety $\mathbf{P}^1$. Unlike with the quantum group, I do not know whether there is a compatible coproduct on $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)$ making it into a Hopf algebra. I also sketch some ideas surrounding the famous "Koszul duality" discovered by Beilinson-Ginzburg-Soergel [BGS96], which relates the category $\mathrm{Shv}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G}; k^{t\mathrm{S}^1})$ of $\mathrm{B}\times\mathrm{B}$-equivariant sheaves of $k^{t\mathrm{S}^1}$-modules on G with the category of (roughly) $\check{\mathrm{B}}\times\check{\mathrm{B}}$-monodromic $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules on $\check{\mathrm{G}}$.

It follows from the construction of the $\widetilde{\mathrm{F}}$-Weyl algebra that if T is a complex torus with dual $\check{\mathrm{T}}$, then $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$ is the loop-rotation equivariant "semi-infinite k-cohomology" $\pi_0\mathrm{R}\Gamma_{(\mathrm{T}_{\mathcal{O}}\times\mathrm{T}_{\mathcal{O}})\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{T}_{\mathcal{K}};\omega^{\mathrm{ren}})$, where $\mathrm{T}_{\mathcal{O}}=\mathrm{T}(\mathbf{C}[\![t]\!])$ (resp. $\mathrm{T}_{\mathcal{K}}=\mathrm{T}(\mathbf{C}(\!(t)\!))$) is the arc (resp. loop) group of T. If X is a (suitable) T-space, it then follows that the "semi-infinite k-cohomology" $\pi_0\mathrm{R}\Gamma_{\mathrm{T}_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$ defines a natural $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-module, where again $\mathrm{X}_{\mathcal{K}}=\mathrm{X}(\mathbf{C}(\!(t)\!))$. This is in turn closely related to Coulomb branches [BFN18], and in Section 7, we sketch some calculations of these $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-modules, which include $\widetilde{\mathrm{F}}$-analogues of hypergeometric functions. (Although I have tried to keep the exposition relatively accessible, this document unfortunately starts to get a bit technical at this point.) We also explain how the $\Gamma_{\widetilde{\mathrm{F}}}$-function of Section 5 can be viewed as a (regularized) Euler class of the normal bundle to $\mathrm{X}_{\mathcal{O}}\subseteq\mathrm{X}_{\mathcal{K}}$, and use it to sketch an $\widetilde{\mathrm{F}}$-analogue of the Gauss and Legendre multiplication formulas.

In Section 8, we suggest a "bigger picture" which aims to neatly wrap up the discussion of the preceding sections in the language of (relative) local geometric Langlands duality. We briefly describe some results from [Dev23, Dev25b] and explain their relation to $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules, and explain how an extension of the relative Langlands duality of [BZSV23] recovers some calculations from the preceding sections. This story is still very much in flux, so unfortunately our discussion will sometimes rest on "squishy" ground.

I hope this document illustrates some of the pretty mathematics that results from trying to do "calculus" with ring spectra. It seems to me that the resulting story can be viewed as a natural continuation of the rich tale of q-deformations and special function theory. As will be clear from the discussion below, there is clearly much that remains to be done and discovered, and I am excited to learn other ways in which this theory connects to other parts of mathematics! Personally, one lingering question I have is whether, just as the numbers n and $[n]_q$ count the number of points of $\mathbf{P}^{n-1}(\mathbf{F}_1)=\{1,\cdots,n-1,\infty\}$ and $\mathbf{P}^{n-1}(\mathbf{F}_q)$ respectively, do the generalized numbers $\langle n\rangle$ also count something? It is confusing that in this document, q (or rather $q-1$) plays the role of a formal variable, whereas when counting, it is the size of the finite field $\mathbf{F}_q$.

In the document below, *all* constructions will be taken in the derived sense unless I specify otherwise: every category (both Shv and QCoh will mean the corresponding derived categories), quotient, completion, tensor product, and fiber product will be derived. This is in part because of my belief that the derived world is the natural home for many constructions, but also because the realm of spectra is implicitly derived, with no natural notion of being "underived". I will also write k to be a commutative ring spectrum, often assumed to be connective, even, and admitting a Bott class (so $\pi_*(k)\cong\pi_0(k)[u]$ with $u\in\pi_2(k)$). If G is a topological group, I will write $k^{h\mathrm{G}}$ to denote the homotopy fixed points for the trivial (unless otherwise specified) action of G, so that $k^{h\mathrm{G}}=\mathrm{C}^*(\mathrm{BG};k)$. If X is a space ("anima") then I will write $k[\mathrm{X}]$ to denote the k-chains $\mathrm{C}_*(\mathrm{X};k)=k\otimes\Sigma^\infty_+\mathrm{X}$. I have also reserved the variable x for a coordinate on $\mathbf{A}^1$ or $\mathbf{G}_m$; the variable s for a coordinate on a formal group (except when writing the formula for the group law, which we write as $x+_{\widetilde{\mathrm{F}}}y$); the variable t for a "deformation parameter"; and the variable $\hbar$ for the Euler class of $\mathcal{O}(1)$ in $\pi_{-2}k^{h\mathrm{S}^1}=\mathrm{H}^2(\mathbf{C}\mathrm{P}^\infty;k)$, with $\hbar u=t$. Some other oft-used notation is reviewed in Construction 3.2.

2. Hochschild cohomology

Let k be an ordinary commutative ring, and let A be a smooth commutative k-algebra. Then the *Hochschild homology* $\mathrm{HH}(\mathrm{A}/k)$ is given by the derived tensor product $\mathrm{A} \otimes_{\mathrm{A}\otimes_k \mathrm{A}} \mathrm{A}$; geometrically, if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, then $\mathrm{Spec}\,\mathrm{HH}(\mathrm{A}/k)$ is the self-intersection of the diagonal $\mathrm{X} \to \mathrm{X} \times_k \mathrm{X}$. Since the circle S^1 can be written as the homotopy pushout $* \amalg_{*\amalg *} *$, one can rewrite $\mathrm{Spec}\,\mathrm{HH}(\mathrm{A}/k) = \mathrm{X} \times_{\mathrm{X}\times_k \mathrm{X}} \mathrm{X}$ as the mapping stack $\mathrm{Map}_k(\mathrm{S}^1, \mathrm{X})$. Equivalently, $\mathrm{Spec}\,\mathrm{HH}(\mathrm{A}/k)$ is the free loop space of X. This description shows that $\mathrm{HH}(\mathrm{A}/k)$ admits an action of the circle S^1. Despite $\mathrm{HH}(\mathrm{A}/k)$ being a (derived) commutative A-algebra, the S^1-action on $\mathrm{HH}(\mathrm{A}/k)$ is only k-linear. From this S^1-action, one can extract several other invariants: *negative cyclic homology* $\mathrm{HC}^-(\mathrm{A}/k) = \mathrm{HH}(\mathrm{A}/k)^{h\mathrm{S}^1}$, and *periodic cyclic homology* $\mathrm{HP}(\mathrm{A}/k) = \mathrm{HH}(\mathrm{A}/k)^{t\mathrm{S}^1}$.[1]

One of the most important results about Hochschild homology is a theorem of Hochschild-Kostant-Rosenberg [HKR62], which has been refined in recent years [BZN12, Ant19, Rak20, MRT22] to the following:

Theorem 2.1. *There is a (complete, multiplicative, decreasing) filtration on* $\mathrm{HH}(\mathrm{A}/k)$ *with* $\mathrm{gr}^n\mathrm{HH}(\mathrm{A}/k) \cong \Omega^n_{\mathrm{A}/k}[n]$. *Moreover, the* S^1*-action on* $\mathrm{HH}(\mathrm{A}/k)$ *admits a filtered refinement, so that it is given on associated graded pieces by the de Rham differential. This implies that there is a filtration on* $\mathrm{HC}^-(\mathrm{A}/k)$ *(resp.* $\mathrm{HP}(\mathrm{A}/k)$*) with* $\mathrm{gr}^n\mathrm{HC}^-(\mathrm{A}/k) \cong \Omega^{\bullet\geq n}_{\mathrm{A}/k}[2n]$ *(resp.* $\mathrm{gr}^n\mathrm{HP}(\mathrm{A}/k) \cong \Omega^{\bullet}_{\mathrm{A}/k}[2n]$*).*

The statement of Theorem 2.1 hides several subtleties: for instance, making precise the "filtered refinement" of the S^1-action on $\mathrm{HH}(\mathrm{A}/k)$ has been the subject of a lot of recent work [Rak20, MRT22, AR24, HM25]. One can also extend Theorem 2.1 to the case when A is not smooth as a k-algebra. In this case, $\Omega^n_{\mathrm{A}/k}$ must be replaced by its derived variant $\wedge^n_{\mathrm{A}}\mathrm{L}_{\mathrm{A}/k}$, where $\mathrm{L}_{\mathrm{A}/k}$ is the cotangent complex.

In this section, we will reinterpret Theorem 2.1 through Hochschild *co*homology. The main definition is:

Definition 2.2. Let k be a commutative ring, and let A be a (possibly derived) commutative k-algebra. The $\mathbf{E}_2$-*Hochschild cohomology*, also called the $\mathbf{E}_2$-*center*, $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ is defined as $\mathrm{End}_{\mathrm{A}\otimes_{\mathrm{A}\otimes_k\mathrm{A}}\mathrm{A}}(\mathrm{A})$. Note the parallel to the Hochschild cohomology of an associative k-algebra B, which is defined as $\mathrm{End}_{\mathrm{B}\otimes_k\mathrm{B}^{\mathrm{op}}}(\mathrm{B})$.

The above definition works perfectly well if A is replaced by an $\mathbf{E}_2$-algebra object in a symmetric monoidal ∞-category $\mathcal{C}$ (taking $\mathcal{C} = \mathrm{Mod}_k$ recovers the above definition). The above definition makes it clear that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ is an *associative* algebra (more precisely, it is an $\mathbf{E}_1$-k-algebra). But just as the Deligne conjecture (proved by many people) guarantees that Hochschild cohomology is an $\mathbf{E}_2$-k-algebra, a "higher" version of the Deligne conjecture [Lur16, Fra13] guarantees that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ is an $\mathbf{E}_3$-k-algebra. This means that one has compatible maps

$$\mathrm{Conf}_n(\mathbf{R}^3) \to \mathrm{Map}_{\mathrm{Mod}_k}(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{\otimes_k n}, \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)) \tag{1}$$

which define a k-linear multiplication $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{\otimes_k n} \to \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ associated to each ordered configuration of n points in $\mathbf{R}^3$.

This already implies the existence of a large amount of structure on the homotopy groups $\pi_*\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$. Note that when $n = 2$, the space $\mathrm{Conf}_2(\mathbf{R}^3)$ is just homotopy equivalent to S^2. The assignment (1) therefore defines a k-linear map

$$\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) \otimes_k \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)[2] \to \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k),$$

[1]Here, if M is a k-module with S^1-action, then $\mathrm{M}^{h\mathrm{S}^1}$ denotes the cochains $\mathrm{R}\Gamma(\mathrm{BS}^1; \mathrm{M})$, and $\mathrm{M}^{t\mathrm{S}^1}$ denotes the corresponding Tate cohomology.

which on homotopy groups induces a bilinear map

$$\{-,-\} : \pi_i \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) \times \pi_j \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) \to \pi_{i+j+2} \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k). \tag{2}$$

Some elementary analysis shows that this equips $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ with the structure of a graded commutative *Poisson* algebra, where the Poisson bracket has weight 2. (This structure exists on the homotopy groups of *any* $\mathbf{E}_3$-k-algebra.)

In fact, a little more is true: since $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) = \mathrm{End}_{\mathrm{HH}(\mathrm{A}/k)}(\mathrm{A})$, and the augmentation $\mathrm{HH}(\mathrm{A}/k) \to \mathrm{A}$ exhibits A as an S^1-equivariant $\mathrm{HH}(\mathrm{A}/k)$-algebra, it follows that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ admits an S^1-action too. However, this action does not commute with the $\mathbf{E}_3$-k-algebra structure on $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$. Rather, if one chooses a maximal torus $\mathrm{S}^1 \subseteq \mathrm{SO}(3)$, then the action of $\mathrm{SO}(3)$ on the $\mathbf{E}_3$-operad restricts to an S^1-action.[2] As such, one can apply [Lur16, Definition 5.4.2.10] to the map $\mathrm{BS}^1 \to \mathrm{BSO}(3) \to \mathrm{BTop}(3)$ to form an operad $\mathbf{E}_{3,\mathrm{BS}^1}$. It can be shown that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ admits the structure of an $\mathbf{E}_{3,\mathrm{BS}^1}$-algebra. Informally, this means that the maps (1) exhibiting $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ as an $\mathbf{E}_3$-k-algebra are (compatibly) S^1-equivariant, where S^1 acts on $\mathrm{Conf}_n(\mathbf{R}^3)$ via its action on $\mathbf{R}^3$, and acts on $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ as described above.

This S^1-action is extremely powerful. To understand why, let us consider the structure that exists on the homotopy fixed points $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1}$. Since the map (1) is S^1-equivariant, the object which naturally parametrizes multiplications $(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1})^{\otimes_{k^{h\mathrm{S}^1}} n} \to \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1}$ is not $\mathrm{Conf}_n(\mathbf{R}^3)$ itself, but rather the subspace $\mathrm{Conf}_n(\mathbf{R}^3)^{\mathrm{S}^1}$. This is just $\mathrm{Conf}_n(\mathbf{R})$, where $\mathbf{R} \subseteq \mathbf{R}^3$ is the line fixed by the $\mathrm{S}^1 \subseteq \mathrm{SO}(3)$-action. In particular, $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1}$ is only an $\mathbf{E}_1$-$k^{h\mathrm{S}^1}$*-algebra.* At the level of homotopy groups, if one identifies $\pi_* k^{h\mathrm{S}^1} = \mathrm{H}^{-*}(\mathrm{BS}^1; k)$ with $k[\hbar]$ (where $\hbar$ lives in weight -2), then

> $\pi_*(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1})$ is a graded associative $k[\hbar]$-algebra. Its reduction modulo $\hbar$ is $\pi_*(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k))$, which is a graded commutative Poisson k-algebra. Moreover, the Poisson bracket is the image modulo $\hbar$ of the commutator.

Said differently, $\pi_*(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1})$ is a deformation quantization of the Poisson algebra $\pi_*(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k))$. In fact, this is true of any even $\mathbf{E}_{3,\mathrm{BS}^1}$-algebra, so $\mathbf{E}_{3,\mathrm{BS}^1}$-algebras can be viewed as giving a homotopy-theoretic generalization of the notion of deformation quantization.

Given the rich amount of structure available on $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$, one may want an identification of it in more classical terms. This is provided by the following analogue of Theorem 2.1:

Theorem 2.3. *Let k be a commutative ring, let A be a smooth k-algebra, and let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$. Then:*

(a) $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ *is isomorphic to* $\mathrm{Sym}_{\mathrm{A}}(\mathrm{T}_{\mathrm{A}/k}(-2)) \cong \mathcal{O}_{\mathrm{T}^*(2)(\mathrm{X}/k)}$ *as graded commutative Poisson k-algebras, where the Poisson structure on* $\mathcal{O}_{\mathrm{T}^*(2)(\mathrm{X}/k)}$ *comes from the natural symplectic form.*

(b) $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1}$ *is isomorphic to the rescaled Weyl algebra* $\mathrm{D}^{\hbar}_{\mathrm{A}/k}$*, namely the associative $k[\hbar]$-algebra generated by $f \in \mathrm{A}$ and $s \in \mathrm{T}_{\mathrm{A}/k}$ (the latter placed in weight -2) subject to the relation*

$$sf - fs = \hbar s(f).$$

(c) $\pi_0 \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{t\mathrm{S}^1}$ *is isomorphic to the Weyl algebra* $\mathrm{D}_{\mathrm{A}/k}$.

Theorem 2.3 is in fact *equivalent* to Theorem 2.1 by Koszul duality (see the remarks below).

[2] Note that all such actions are conjugate. In any case, the proof of the Deligne conjecture implicitly involves choosing a linear embedding $\mathbf{R}^2 \subseteq \mathbf{R}^3$. If $\mathbf{R} \subseteq \mathbf{R}^3$ denotes the inclusion of its orthogonal complement, then rotation about this line in $\mathbf{R}^3$ defines the desired maximal torus in $\mathrm{SO}(3)$. The statement below that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ is an $\mathbf{E}_{3,\mathrm{BS}^1}$-$k$-algebra exploits the choice of linear embedding $\mathbf{R}^2 \subseteq \mathbf{R}^3$.

Proof sketch. Instead of proving Theorem 2.3 in general, let me sketch the argument when $\mathrm{A} = k[x]$ is a polynomial ring. Then

$$\pi_*(\mathrm{A} \otimes_{\mathrm{A}\otimes_k \mathrm{A}} \mathrm{A}) \cong \pi_*(k[x] \otimes_{k[x,y]} k[x]) \cong k[x] \otimes_k \Lambda_k(\sigma(x-y)).$$

Here, Λ_k denotes an exterior algebra, and $\sigma(x-y)$ denotes the class in weight 1 represented by the difference $x - y \in k[x,y]$. It follows that

$$\pi_*(\mathrm{A} \otimes_{\mathrm{A}\otimes_{\mathrm{A}\otimes_k \mathrm{A}} \mathrm{A}} \mathrm{A}) \cong k[x] \otimes_k \Gamma_k(\sigma^2(x-y)),$$

where Γ_k is the divided power algebra, and $\sigma^2(x-y)$ lives in weight 2. Taking the A-linear dual of $\mathrm{A} \otimes_{\mathrm{A}\otimes_{\mathrm{A}\otimes_k \mathrm{A}} \mathrm{A}} \mathrm{A}$ produces $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$; so $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) \cong k[x,s]$, where s lives in weight -2 and is dual to the class $\sigma^2(x-y)$. This is clearly isomorphic to $\mathrm{Sym}_{k[x]}(\mathrm{T}_{k[x]/k})$. To see that $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{h\mathrm{S}^1}$ is as claimed, we may replace $\mathrm{A} = k[x]$ by $k[x^{\pm 1}] = k[\mathbf{Z}]$. Then, the computation follows from Theorem 3.1. $\square$

Let me make a few remarks about Theorem 2.3.

(a) Many basic constructions in (algebraic) symplectic geometry can be understood using $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$. For example, it is well-known that cotangent bundles are only functorial in the category of Lagrangian correspondences. The same is true for $\mathbf{E}_2$-centers: if $\mathrm{A} \to \mathrm{B}$ is a map of commutative k-algebras (or more generally $\mathbf{E}_2$-k-algebras), one generally does not acquire a map $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) \to \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{B}/k)$.

(b) Theorem 2.3 also indicates that $\mathbf{E}_3$-schemes which do not arise as $\mathbf{E}_2$-centers of smooth k-algebras can be viewed as homotopy-theoretic analogues of non-cotangent bundles. In fact, $\mathbf{E}_3$-algebras of the form $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ for $\mathbf{E}_2$-k-algebras A which are *not* concentrated in degree zero can sometimes be "close" to being cotangent bundles. For example, when $\mathrm{A} = \mathrm{C}^*(\mathrm{BG}; k) = k^{h\mathrm{G}}$, one can identify $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ with a completion of $k[\Omega\mathrm{G}]^{h\mathrm{G}}$, whose homotopy (when $k^{h\mathrm{G}}$ is concentrated in even degrees) was computed in [BFM05, YZ11] to be the ring of functions on a twisted two-sided Hamiltonian reduction $\mathrm{T}^*(\check{\mathrm{N}}_k \backslash_\psi \check{\mathrm{G}}_k /_\psi \check{\mathrm{N}}_k)$ of the cotangent bundle of the Langlands dual group $\check{\mathrm{G}}_k$ over k (see also [Dev23]).

(c) By definition, $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ is the endomorphism algebra $\mathrm{End}_{\mathrm{HH}(\mathrm{A}/k)}(\mathrm{A})$; in particular, it can be viewed as the Koszul dual of $\mathrm{HH}(\mathrm{A}/k)$ with respect to the augmentation $\mathrm{HH}(\mathrm{A}/k) \to \mathrm{A}$. This is reflected in Theorem 2.3 as follows. The scheme $\mathrm{Spec}\, \pi_*(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)) \cong \mathrm{T}^*(2)(\mathrm{X}/k)$ is Koszul dual to $\mathrm{Spec}\, \mathrm{gr}^*(\mathrm{HH}(\mathrm{A}/k))[-2*] \cong \widehat{\mathrm{BT}^\sharp}(-2)(\mathrm{X}/k)$. Here, $\widehat{\mathrm{BT}^\sharp}$ denotes the delooping (over X) of the divided power completion of the tangent bundle of X over k at the zero section (where the tangent bundle is equipped with the $\mathbf{G}_m$-action of weight -2). (That $\mathrm{T}^*(2)(\mathrm{X}/k)$ is Koszul dual to $\widehat{\mathrm{BT}^\sharp}(-2)(\mathrm{X}/k)$ is a special case of a more general Koszul duality between $\mathrm{B}\widehat{\mathrm{V}^\sharp}$ and V^* where V is a perfect complex over X.) Taking loop-rotation equivariance, one recovers the Koszul duality between $\pi_0(\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{t\mathrm{S}^1}) \cong \mathrm{D}_{\mathrm{A}/k}$ and $\mathrm{gr}^0(\mathrm{HP}(\mathrm{A}/k)) \cong \Omega^\bullet_{\mathrm{A}/k}$.

(d) The calculation of Theorem 2.3 gives a very simple construction of the Getzler-Gauss-Manin connection. Namely, if Y is an (affine, say) scheme over A, then $\mathrm{HH}(\mathrm{Y}/\mathrm{A}) = \mathrm{HH}(\mathrm{Y}/k) \otimes_{\mathrm{HH}(\mathrm{A}/k)} \mathrm{A}$, so there is an S^1-equivariant action of $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k) = \mathrm{End}_{\mathrm{HH}(\mathrm{A}/k)}(\mathrm{A})$ on $\mathrm{HH}(\mathrm{Y}/\mathrm{A})$. One can check that this defines an action of $\tau_{\geq 2\star} \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{t\mathrm{S}^1} \cong \mathrm{D}_{\mathrm{A}/k}$ on $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HP}(\mathrm{Y}/\mathrm{A})$. When $\mathrm{A} = k[x]$, the action of $\partial_x \in \pi_0 \mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)^{t\mathrm{S}^1} = \mathrm{D}_{k[x]/k}$ defines an endomorphism of $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HP}(\mathrm{Y}/k[x])$ which is precisely a motivically-filtered refinement of the (Getzler-)Gauss-Manin connection.

3. Generalized differential operators

We can now finally turn to the main topic of interest in this document: generalized differential operators. To motivate this story, I need to recall a deep relationship between homotopy theory and formal groups, initially observed by Quillen [Qui69] and then ballooned into a rich area of mathematics by the work of various people like Morava [Mor85], Ravenel [Rav84], Hopkins [DHS88, HS98, Hop87], and others. Let k be an $\mathbf{E}_\infty$-ring (i.e., homotopy-coherently commutative algebra object in spectra) for which there is an isomorphism $\mathrm{H}^{-*}(\mathrm{BS}^1; k) \cong \pi_*(k)[\hbar]^\wedge$, where $\hbar$ lives in weight -2. Such an $\mathbf{E}_\infty$-ring is called *complex-oriented*, and there are many examples of such: ordinary cohomology, complex K-theory KU, connective complex K-theory ku, and complex cobordism MU are some of the most prominent ones.

These examples all share a common property, namely that they are *even*. This means that the homotopy groups $\pi_*(k)$ vanish in odd degrees. This automatically implies that k is complex-oriented, but not canonically so: evenness guarantees that an element $\hbar$ as above *exists*, but there is still freedom in choosing such a class.[3] In recent years, it has become clear that while complex orientations are geometrically important, merely requiring the existence of a complex orientation without making a specific choice offers greater flexibility in certain constructions. For instance, evenness is a *property* of an $\mathbf{E}_\infty$-ring spectrum, while complex orientations are additional *data*.[4]

So, let k be an even $\mathbf{E}_\infty$-ring. The tensor product of line bundles defines a map $\mathrm{BS}^1 \times \mathrm{BS}^1 \to \mathrm{BS}^1$. Since $\mathrm{H}^*(\mathrm{BS}^1; k)$ is flat over $\pi_*(k)$, the Künneth formula gives a map $\mathrm{H}^*(\mathrm{BS}^1; k) \to \mathrm{H}^*(\mathrm{BS}^1; k) \widehat{\otimes}_{\pi_* k} \mathrm{H}^*(\mathrm{BS}^1; k)$, which equips $\mathrm{Spf}\, \mathrm{H}^*(\mathrm{BS}^1; k)$ with the structure of a graded 1-dimensional formal group over $\pi_*(k)$, i.e., a 1-dimensional formal group F over $\mathrm{Spec}(\pi_*(k))/\mathbf{G}_m$. The choice of a complex orientation $\hbar$ amounts to the choice of a coordinate on F. If k is not even, then $\mathrm{H}^*(\mathrm{BS}^1; k)$ may fail to be flat over $\pi_*(k)$, but one can always work even-locally: following [HRW22] (see also [Gre25]), if one defines $\mathrm{Spev}(k) = \mathrm{colim}_{k \to \mathrm{A}} \mathrm{Spec}(\pi_*(\mathrm{A}))/\mathbf{G}_m$ as the colimit ranges over all $\mathbf{E}_\infty$-maps $k \to \mathrm{A}$ with A even, then $\mathrm{Spev}(k^{h\mathrm{S}^1})$ defines a 1-dimensional formal group over $\mathrm{Spev}(k)$, which we will continue to denote by F (or F_k, to emphasize dependence on k).

The work of Quillen and Landweber-Novikov can now be rephrased as follows: if $\mathcal{M}_{\mathrm{FG}}$ denotes the moduli stack of 1-dimensional formal groups, then the map $\mathrm{Spev}(\mathbb{S}) \to \mathcal{M}_{\mathrm{FG}}$ classifying $\mathrm{F}_\mathbb{S}$ over $\mathrm{Spev}(\mathbb{S})$ is an isomorphism, and moreover the natural map $\mathrm{Spev}(\mathrm{MU}) \to \mathcal{M}_{\mathrm{FG}}$ identifies with the fpqc covering of $\mathcal{M}_{\mathrm{FG}}$ given by the moduli stack of 1-dimensional formal groups *equipped with a coordinate*. In particular, $\mathrm{F}_\mathbb{S}$ *is* the universal 1-dimensional formal group. It is this result that breathes life into the connection between homotopy theory and arithmetic geometry. For instance, if X is a spectrum and A is an even $\mathbf{E}_\infty$-ring, then $\mathrm{H}^*(\mathrm{X}; \mathrm{A})$ defines a (graded) module over $\pi_*(\mathrm{A})$, so working even-locally, one obtains a quasicoherent sheaf $\mathcal{H}^*(\mathrm{X}; k)$ over $\mathrm{Spev}(k)$. This defines a functor $\mathrm{Sp} \to \mathrm{QCoh}(\mathrm{Spev}(k))$, which can be thought of as a mild refinement of the functor of k-cohomology. In the universal case when

[3] In fact, the ideal generated by $\hbar$ is well-defined – as the kernel of the canonical map $\mathrm{H}^*(\mathrm{BS}^1; k) \to \pi_*(k)$ – and the choice of a generator of this ideal is the data of a complex orientation.

[4] There are several interesting $\mathbf{E}_\infty$-rings which are not complex-oriented (hence not even), like real K-theory KO, connective real K-theory ko, Adams' J-theory j, and stable cohomotopy $\mathbb{S}$. Each of these examples admits a "cover" $k \to \mathrm{A}$ by an even ring A; here, the word "cover" is taken in the sense of [HRW22], and it means that for each even $\mathbf{E}_\infty$-k-algebra B, the homotopy groups of the tensor product $\mathrm{B} \otimes_k \mathrm{A}$ is faithfully flat over $\pi_*(\mathrm{B})$. In other words, many $\mathbf{E}_\infty$-ring spectra are "locally" even, and this is often good enough for many purposes. This perspective will be embedded in our discussion below: we will mainly discuss the case of even $\mathbf{E}_\infty$-rings, and sometimes indicate how it generalizes to the locally even case.

$k = \mathbb{S}$, one obtains a functor $\mathrm{Sp} \to \mathrm{IndPerf}(\mathcal{M}_{\mathrm{FG}})$ refining stable cohomotopy, which is in a sense the best approximation to the category of spectra by ordinary algebra.

Let us now see how this perspective is useful in our context of generalized differential operators. Our starting point is the following:

Theorem 3.1. *Let k be an even $\mathbf{E}_\infty$-ring, and let x be a class in degree zero. Given a complex orientation of k, there is an isomorphism*

$$\pi_* \mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{h\mathrm{S}^1} \cong \pi_*(k)[\hbar]\{x, \widetilde{\partial_x^k}\}^\wedge/(\widetilde{\partial_x^k} x = x\widetilde{\partial_x^k} +_{\mathrm{F}} \hbar).$$

Here, $\hbar$ and $\widetilde{\partial_x^k}$ live in weight -2, and x lives in weight zero.[5] *In particular, the right-hand side is canonically independent of the choice of complex orientation of k.*

Proof. We begin by doing the calculation with $k[x]$ replaced by $k[x^{\pm 1}] = k[\mathbf{Z}]$. Let us write $\mathrm{T} = \mathrm{B}\mathbf{Z}$ (so $\mathrm{T} = \mathrm{S}^1$; but we want to distinguish it from the S^1 acting naturally on $\mathfrak{Z}_{\mathbf{E}_2}$). There is an S^1-equivariant equivalence

$$\mathfrak{Z}_{\mathbf{E}_2}(k[\mathbf{Z}]/k) \simeq \mathrm{End}_{\mathrm{HH}(k[\mathbf{Z}]/k)}(k[\mathbf{Z}]) \simeq \mathrm{End}_{k[\mathcal{L}\mathrm{T}]}(k[\mathbf{Z}]) \simeq k[\Omega\mathrm{T}]^{h\mathrm{T}}.$$

The homotopy groups of $(k[\Omega\mathrm{T}]^{h\mathrm{T}})^{h\mathrm{S}^1}$ were computed in [Dev25b, Section 3.5], and one finds:

$$\pi_* \mathfrak{Z}_{\mathbf{E}_2}(k[\mathbf{Z}]/k)^{h\mathrm{S}^1} \cong \pi_*(k)[\hbar]\{x^{\pm 1}, \widetilde{\theta_x^k}\}^\wedge/(\widetilde{\theta_x^k} x = x(\widetilde{\theta_x^k} +_{\mathrm{F}} \hbar)).$$

Here, $\widetilde{\partial_x^k}$ is the Euler class in $\pi_{-2}(k^{h\mathrm{T}})$. The commutation relation appearing above comes from the following simple observation: if $\lambda \in \Omega\mathrm{T} = \mathbb{X}_*(\mathrm{T})$, the map $\Omega\mathrm{T} \to \Omega\mathrm{T}$ given by λ-multiplication is $\mathrm{T} \times \mathrm{S}^1$-equivariant for the map

$$\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) : \mathrm{T} \times \mathrm{S}^1 \to \mathrm{T} \times \mathrm{S}^1, \ (t, \theta) \mapsto (t\lambda(\theta), \theta),$$

and when $\lambda = 1 \in \mathbb{X}_*(\mathrm{T}) = \mathbf{Z}$, the effect of this map on equivariant cohomology is the map $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) : \mathrm{F} \times \mathrm{F} \to \mathrm{F} \times \mathrm{F}$ which sends $(\widetilde{\partial_x^k}, \hbar) \mapsto (\widetilde{\partial_x^k} +_{\mathrm{F}} \hbar, \hbar)$.

Since $k[x] \to k[x^{\pm 1}]$ is a localization, there is an S^1-equivariant map $\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k) \to \mathfrak{Z}_{\mathbf{E}_2}(k[x^{\pm 1}]/k)$. A calculation with factorization homology shows that this map is an injection on homotopy, and it is given by the map $\pi_*(k)[x, s] \to \pi_*(k)[x^{\pm 1}, \widetilde{\theta_x^k}]$ sending $s \mapsto x^{-1}\widetilde{\theta_x^k}$. Together with the above calculation of $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(k[\mathbf{Z}]/k)^{h\mathrm{S}^1}$, this computes $\pi_* \mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{h\mathrm{S}^1}$ as desired. $\square$

For simplicity, I will assume from now that k is connective, even, and admits a Bott class, i.e., that k is an $\mathbf{E}_\infty$-ring with homotopy groups given by $\pi_*(k) \cong \pi_0(k)[u]$ where u lives in degree 2.

Construction 3.2. Let $k^{t\mathrm{S}^1}$ denote the $\mathbf{E}_\infty$-ring obtained by inverting the Euler class of the standard representation of the circle in $\mathrm{C}^*(\mathrm{BS}^1; k)$. Since k is assumed to be even, $k^{t\mathrm{S}^1}$ is 2-periodic, and $\mathrm{Spev}(k^{t\mathrm{S}^1}) \cong \mathrm{Spf}(\pi_0(k^{t\mathrm{S}^1}))$. There is a unit map $k \to k^{t\mathrm{S}^1}$, which induces a map $\mathrm{Spf}(\pi_0(k^{t\mathrm{S}^1})) \to \mathrm{Spev}(k)$. Pulling back F along this map defines a 1-dimensional formal group $\widetilde{\mathrm{F}}$ over $\mathrm{Spf}(\pi_0(k^{t\mathrm{S}^1}))$ which is in fact an *algebraic* group. Explicitly, if one writes

$$\pi_*(k^{t\mathrm{S}^1}) \cong \pi_0(k)[\![t]\!][u, \hbar^{\pm 1}]/(u\hbar = t) \cong \pi_0(k)[\![t]\!][\hbar^{\pm 1}],$$

the group law $\widetilde{\mathrm{F}}$ is given by

$$x +_{\widetilde{\mathrm{F}}} y = \tfrac{1}{\hbar}((\hbar x) +_{\mathrm{F}} (\hbar y)).$$

[5]Below, we will think of $\widetilde{\partial_x^k}$ as being canonically associated to $\widetilde{\mathrm{F}}$ instead of k, so perhaps it would be better denoted by $\widetilde{\partial_x^{\widetilde{\mathrm{F}}}}$, but this notation feels too heavy.

Modulo any fixed power of t, the power series $x +_{\widetilde{\mathrm{F}}} y$ is just a polynomial, so $\widetilde{\mathrm{F}}$ is in fact an algebraic group over $\mathrm{Spf}(\pi_0(k)[\![t]\!])$. We will write $\langle n\rangle \in \pi_0(k)[\![t]\!]$ to denote the n-fold sum $1 +_{\widetilde{\mathrm{F}}} \cdots +_{\widetilde{\mathrm{F}}} 1$.

The dual of the Lie algebra of $\widetilde{\mathrm{F}}$ defines a line bundle denoted ω over $\mathrm{Spf}(\pi_0(k)[\![t]\!])$; sometimes, tensoring by ω will be denoted with $\{1\}$. (When there is a global coordinate on $\widetilde{\mathrm{F}}$, as will be in the examples discussed below, ω is trivial; but if we move away from the case when k is assumed to be even, then ω may be nontrivial.)

Example 3.3. Suppose k is connective complex K-theory ku, so that F is the formal group over $\pi_*(k) \cong \mathbf{Z}[u]$ given by $x + y + uxy$. If we write $t = q - 1 = u\hbar$, then $\widetilde{\mathrm{F}}$ is the group law over $\pi_0(k^{t\mathrm{S}^1}) = \mathbf{Z}[\![q-1]\!]$ given by $x + y + (q-1)xy$, and $\langle n\rangle = [n]_q = \frac{q^n-1}{q-1}$.

Theorem 3.1 $(+\epsilon)$ implies:

Corollary 3.4. *Fix k as above. Given a complex orientation of k, there is an isomorphism*

$$\pi_0(\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{t\mathrm{S}^1}) \cong \pi_0(k)[\![t]\!]\{x, \partial_x^k\}^\wedge/(\partial_x^k x = x\partial_x^k +_{\widetilde{\mathrm{F}}} 1).$$

Here, all elements live in weight zero, and the completion is at ∂_x^k. In particular, the right-hand side is canonically independent of the choice of complex orientation of k. Moreover, the $k^{t\mathrm{S}^1}$-linear action of $\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{t\mathrm{S}^1}$ on $k[x]^{t\mathrm{S}^1}$ is given on π_0 as follows: x acts by x-multiplication, and ∂_x^k acts by $x^n \mapsto \langle n\rangle x^{n-1}$.

Motivated by Theorem 2.3, we are led to:

Definition 3.5. The algebra $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ of $\widetilde{\mathrm{F}}$-*differential operators* on $\mathbf{A}^1$ is the $\pi_0(k^{t\mathrm{S}^1}) = \pi_0(k)[\![t]\!]$-algebra given by $\pi_0(\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{t\mathrm{S}^1})$. The algebra $\mathrm{D}_{\mathbf{A}^1}^{\mathrm{F}}$ of *rescaled* $\widetilde{\mathrm{F}}$-differential operators on $\mathbf{A}^1$ is the graded $\pi_*(k^{h\mathrm{S}^1})$-algebra given by $\pi_*(\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{h\mathrm{S}^1})$; we will focus mainly on $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$. Note that $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ is a bialgebroid over $\pi_0(k[x]^{t\mathrm{S}^1}) \cong \pi_0(k)[\![t]\!][x]$, where the coproduct on $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ is calculated by

$$\Delta(x\partial_x^k) = (x\partial_x^k \otimes 1) +_{\widetilde{\mathrm{F}}} (1 \otimes x\partial_x^k).$$

This encodes an $\widetilde{\mathrm{F}}$-analogue of the (q-)Leibniz rule.

The $\widetilde{\mathrm{F}}$-*cotangent bundle* of $\mathbf{A}^1$ is the scheme over $\mathrm{Spev}(k) = \mathrm{Spec}(\pi_*(k))/\mathbf{G}_m$ given by

$$\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{A}^1 := \mathrm{Spec}(\pi_*\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k))/\mathbf{G}_m.$$

Note that $\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{A}^1 \cong \mathbf{A}^1 \times \mathrm{F}$. This admits a canonical ($\pi_0(k)$-linear) symplectic structure given by $dx \wedge \omega$, where ω is a nonzero invariant differential of F. These definitions can be extended in the obvious way to any affine space $\mathbf{A}^n$.

Unfortunately, it does not seem possible to define an algebra of $\widetilde{\mathrm{F}}$-differential operators (or even the $\widetilde{\mathrm{F}}$-cotangent bundle) on an *arbitrary* scheme over $\pi_0(k)$. However, if X is a scheme over $\pi_0(k)$ equipped with a formally étale map $\mathrm{X} \to \mathbf{A}^n$, then X admits a (unique) lift X_k to a scheme over k itself, and thus one can define $\mathrm{D}_{\mathrm{X}}^{\widetilde{\mathrm{F}}}$ as $\pi_0\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{X}_k/k)^{t\mathrm{S}^1}$. Similarly, one can define $\mathrm{D}_{\mathrm{X}}^{\widetilde{\mathrm{F}}}$ for any toric variety X over $\pi_0(k)$, and more generally for any δ-scheme (although existence in the latter case is far from obvious).

In general, the question of defining $\mathrm{D}_{\mathrm{X}}^{\widetilde{\mathrm{F}}}$ for a $\pi_0(k)$-scheme X is closely related to the question of lifting X to a scheme over k itself. This does *not* mean that one asks for a lift of X to k as a "spectral scheme" in the sense of Lurie [Lur17]: this is far too strong an assumption, which fails to be satisfied in most examples. Instead, to define $\mathrm{D}_{\mathrm{X}}^{\widetilde{\mathrm{F}}}$, one only needs that the sheaf $\mathcal{O}_{\mathrm{X}}$ of commutative $\pi_0(k)$-algebras admits a lift to a sheaf $\widetilde{\mathcal{O}_{\mathrm{X}}}$ of $\mathbf{E}_2$-k-*algebras* such that

$\widetilde{\mathcal{O}_{\mathrm{X}}} \otimes_k \pi_0(k) \cong \mathcal{O}_{\mathrm{X}}$. This is because the $\mathbf{E}_2$-center $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ makes sense as soon as A is an $\mathbf{E}_2$-k-algebra (and $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{A}/k)$ acquires an S^1-action as soon as A is an $\mathbf{E}_{2,\mathrm{BS}^1}$-$k$-algebra).

Example 3.6. Let R be an ordinary commutative ring, and let $k = \mathrm{R}[u]$ denote the $\mathbf{E}_\infty$-R-algebra with a generator in degree 2. Then $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$ is the $\pi_0(k) = \mathrm{R}$-algebra given by

$$\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1} = \mathrm{R}[\![t]\!]\{x, \partial^{\mathrm{R}}_x\}^\wedge/(\partial^{\mathrm{R}}_x x = x\partial^{\mathrm{R}}_x + 1),$$

so that it is just the extension of the usual Weyl algebra of $\mathbf{A}^1$ over R along the map $\mathrm{R} \to \mathrm{R}[\![t]\!]$. In particular, ∂^{R}_x is the usual derivative.

Example 3.7. Let $k = \mathrm{ku}$. Then $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$ is the $\mathbf{Z}[\![q-1]\!]$-algebra given by

$$\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1} = \mathbf{Z}[\![q-1]\!]\{x, \partial^{\mathrm{ku}}_x\}^\wedge/(\partial^{\mathrm{ku}}_x x = qx\partial^{\mathrm{ku}}_x + 1).$$

It follows that ∂^{ku}_x satisfies the q-Leibniz rule, and hence can be identified with the q-derivative ∂^q_x sending $f(x) \mapsto \frac{f(qx)-f(x)}{(q-1)x}$. The resulting theory of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-modules is therefore the theory of q-differential calculus in a single variable. The modern theory of prismatic cohomology features q-calculus in center stage [Sch17, BMS19, BS22, BL22]; this in turn is explained by various recent results on the relationship between ku (and variants thereof, like the image of J spectrum) and topological Hochschild homology. See [DR25, Dev25b, Wag25] for more on this connection.

Suppose X is a scheme over $\pi_0(k)$ which lifts to k; for instance, X could be a torus. By construction, $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$ is a variant of the algebra of differential operators on X where the tangent directions are "adapted" to the formal group F over $\mathrm{Spev}(k)$. This is perhaps seen most clearly in the semiclassical limit:

Example 3.8. Suppose $\mathrm{X} = \mathbf{G}_m$. Then $\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{G}_m \cong \mathbf{G}_m \times \mathrm{F}$, and this admits a symplectic form given by $d\log(x) \wedge \omega$, where ω is a nonzero invariant differential of F. More generally, if X is a torus T with Langlands dual torus $\check{\mathrm{T}}$, then $\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathrm{T} \cong \mathrm{T} \times \check{\mathrm{T}}_{\mathrm{F}}$, where $\check{\mathrm{T}}_{\mathrm{F}} = \mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}), \mathrm{F})$. When $\mathrm{F} = \widehat{\mathbf{G}_a}$, this recovers the completed cotangent bundle $(\mathrm{T}^*\mathrm{T})^\wedge_0 \cong \mathrm{T} \times \widehat{\mathfrak{t}^*}$.

The thesis of this article is that many aspects of classical and q-deformed calculus admit generalizations to "$\widetilde{\mathrm{F}}$-calculus", and that these uniform generalizations can often be explained through topological methods (since, after all, $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$ is defined homotopically!).

Just as with usual D-modules, a *solution* to a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$-module $\mathcal{F}$ is a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$-module map $\mathcal{F} \to \pi_0(k)[\![t]\!] \otimes_{\pi_0(k)} \mathcal{O}_{\mathrm{X}}$. Similarly, the (derived) $\widetilde{\mathrm{F}}$-*de Rham complex* of a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$-module $\mathcal{F}$ is defined to be $\mathrm{RHom}_{\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}\text{-mod}}(\pi_0(k)[\![t]\!] \otimes_{\pi_0(k)} \mathcal{O}_{\mathrm{X}}, \mathcal{F})$. We will denote the $\widetilde{\mathrm{F}}$-de Rham complex of $\pi_0(k)[\![t]\!] \otimes_{\pi_0(k)} \mathcal{O}_{\mathrm{X}}$ by $\widetilde{\mathrm{F}}\mathrm{dR}_{\mathrm{X}}$.

Also, the standard constructions of functors between categories of D-modules goes through in exactly the same way to define $*$-pushforward, !-pullback, and external tensor product (hence also !-tensor product). Verdier duality, however, is more subtle: already in characteristic $p > 0$, it is *not* true that smooth schemes satisfy Poincaré duality in algebraic de Rham cohomology. (For example, this fails for $\mathbf{A}^1$.) However, Poincaré duality in algebraic de Rham cohomology does hold for smooth and proper schemes. A key example is $\mathbf{P}^1$:

Example 3.9. The $\widetilde{\mathrm{F}}$-de Rham cohomology $\widetilde{\mathrm{F}}\mathrm{dR}_{\mathbf{A}^1}$ of $\mathbf{A}^1$ is given by the two-term complex

$$\pi_0(k)[\![t]\!][x] \to \pi_0(k)[\![t]\!][x]d_kx, \ x^n \mapsto \partial^k_x(x^n)d_kx = \langle n\rangle x^{n-1}d_kx.$$

The term $\pi_0(k)[\![t]\!][x]d_kx$ should more precisely be understood as $\pi_0(k)[\![t]\!][x][-1]\{-1\}$, where we recall that the symbol $\{-1\}$ means tensoring by the Lie algebra of $\widetilde{\mathrm{F}}$ over $\mathrm{Spf}(\pi_0(k)[\![t]\!])$.

A direct calculation using the presentation $\mathbf{P}^1 = \mathbf{A}^1 \amalg_{\mathbf{G}_m} \mathbf{A}^1$ shows that $\widetilde{\mathrm{F}}\mathrm{dR}_{\mathbf{P}^1} \cong \pi_0(k)[\![t]\!] \oplus \pi_0(k)[\![t]\!][-2]\{-1\}$, which does indeed satisfy Poincaré duality.

Let us give a couple of interesting examples of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-modules.

Example 3.10. The *exponential* $\mathrm{D}^{\widetilde{\mathrm{F}}}$*-module* is the cyclic left $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-module generated by the relation $\partial_x^k = 1$. A solution is a function $\exp_{\widetilde{\mathrm{F}}}(x)$ such that $\partial_x^k \exp_{\widetilde{\mathrm{F}}}(x) = \exp_{\widetilde{\mathrm{F}}}(x)$; normalizing so that $\exp_{\widetilde{\mathrm{F}}}(0) = 1$ gives $\exp_{\widetilde{\mathrm{F}}}(x) = \sum_{n\geq 0} \frac{x^n}{\langle n\rangle!}$, which is well-defined in $\pi_0(k)[\![t]\!][x, \frac{x^n}{\langle n\rangle!}]_{n\geq 0}$.

Example 3.11. Let s be a $\pi_0(k)[\![t]\!]$-point of $\widetilde{\mathrm{F}}$. Then the cyclic left $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}$-module generated by the relation $x\partial_x^k = \alpha$ is an $\widetilde{\mathrm{F}}$-analogue of the usual D-module on $\mathbf{G}_m$ corresponding (in characteristic zero) to a local system on $\mathbf{C}^\times$ with monodromy $\exp(2\pi i s)$. There is a "universal" solution $\nu_{\widetilde{\mathrm{F}}}(x,s)$ to this $\mathrm{D}^{\widetilde{\mathrm{F}}}$-module where s ranges over all of $\widetilde{\mathrm{F}}$, instead of fixing a particular $\pi_0(k)[\![t]\!]$-point of $\widetilde{\mathrm{F}}$; this solution is an $\widetilde{\mathrm{F}}$-analogue of the function x^s. We will see in Definition 4.11 that if we normalize so that $\nu(x,0) = 1$, this universal function is given by

$$\nu_{\widetilde{\mathrm{F}}}(x,s) := \sum_{n\geq 0} s(s -_{\widetilde{\mathrm{F}}} 1)\cdots(s -_{\widetilde{\mathrm{F}}} \langle n-1\rangle)\frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}$$

for a particular polynomial $(x-1)^n_{\widetilde{\mathrm{F}}}$ of degree n in x such that $\partial_x^k(x-1)^n_{\widetilde{\mathrm{F}}} = \langle n\rangle(x-1)^{n-1}_{\widetilde{\mathrm{F}}}$. For example, in the case of the additive formal group, $\nu_{\widetilde{\mathrm{F}}}(x,s)$ is $(1+(x-1))^s = x^s$ (by the binomial theorem); and in the case of the multiplicative formal group, $\nu_{\widetilde{\mathrm{F}}}(x,s)$ is

$$\sum_{n\geq 0} q^{-\binom{n}{2}} s(s-[1]_q)\cdots(s-[n-1]_q)\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!} = x^{\log(1+(q-1)s)/\log(q)}.$$

Example 3.12. Let $j : \mathbf{G}_m \subseteq \mathbf{A}^1$ and $i : \{0\} \subseteq \mathbf{A}^1$. Then $j_*j^*\mathcal{O}_{\mathbf{A}^1} = \pi_0(k)[\![t]\!][x^{\pm 1}]$ with the standard action of $\mathrm{D}^{\mathrm{F}}_{\mathbf{A}^1}$. Alternatively, since $(\partial_x^k x)x^{-1} = 0$, $j_*j^*\mathcal{O}_{\mathbf{A}^1} = \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}/(\partial_x^k x)$. Now, $i_*i^!\mathcal{O}_{\mathbf{A}^1}$ is the fiber

$$i_*i^!\mathcal{O}_{\mathbf{A}^1} \to \mathcal{O}_{\mathbf{A}^1} = \pi_0(k)[\![t]\!][x] \to j_*j^*\mathcal{O}_{\mathbf{A}^1} = \pi_0(k)[\![t]\!][x^{\pm 1}],$$

so that $i_*i^!\mathcal{O}_{\mathbf{A}^1} = (\pi_0(k)[\![t]\!][x^{\pm 1}]/\pi_0(k)[\![t]\!][x])[-1]$. Let $\pi_0(k)[\![t]\!]\langle\delta\rangle_{\widetilde{\mathrm{F}}}$ denote the $\pi_0(k)[\![t]\!]$-module generated by $\frac{(\partial_x^{\mathrm{F}})^n}{\langle -n\rangle\langle -n+1\rangle\cdots\langle -1\rangle}\delta$ for $n \geq 0$. There is an action of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$ on $\pi_0(k)[\![t]\!]\langle\delta\rangle_{\widetilde{\mathrm{F}}}$ via the obvious action of ∂_x^k, and x acts by $x\delta = 0$. Then, there is an isomorphism of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-modules

$$\pi_0(k)[\![t]\!][x^{\pm 1}]/\pi_0(k)[\![t]\!][x] \cong \pi_0(k)[\![t]\!]\langle\delta\rangle_{\widetilde{\mathrm{F}}},\ x^{-n-1} \mapsto \frac{(\partial_x^{\mathrm{F}})^n}{\langle -n\rangle\langle -n+1\rangle\cdots\langle -1\rangle}\delta,$$

so that $i_*i^!\mathcal{O}_{\mathbf{A}^1} = \pi_0(k)[\![t]\!]\langle\delta\rangle_{\widetilde{\mathrm{F}}}[-1]$.

Although there is no Verdier duality for $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules on $\mathbf{A}^1$ (since, for instance, the $\widetilde{\mathrm{F}}$-de Rham cohomology of $\mathbf{A}^1$ does not generally satisfy Poincaré duality), one can still define a "dualizing sheaf" by restricting the dualizing $\mathrm{D}^{\widetilde{\mathrm{F}}}$-module on $\mathbf{P}^1$ to $\mathbf{A}^1$. This suggests that it is reasonable to set

$$i_*i^*\mathcal{O}_{\mathbf{A}^1} := i_*i^!\mathcal{O}_{\mathbf{A}^1}[2]\{1\} \cong (\pi_0(k)[\![t]\!][x^{\pm 1}]/\pi_0(k)[\![t]\!][x])[1]\{1\}.$$

There is a map $\mathcal{O}_{\mathbf{A}^1} \to i_*i^*\mathcal{O}_{\mathbf{A}^1}$, which classifies a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-module $j_!j^*\mathcal{O}_{\mathbf{A}^1}$ given by an extension

$$(\pi_0(k)[\![t]\!][x^{\pm 1}]/\pi_0(k)[\![t]\!][x])\{1\} \to j_!j^*\mathcal{O}_{\mathbf{A}^1} \to \mathcal{O}_{\mathbf{A}^1} \cong \pi_0(k)[\![t]\!][x].$$

It is not hard to see that any such extension is split as $\pi_0(k)[\![t]\!][x]$-modules, so

$$j_!j^*\mathcal{O}_{\mathbf{A}^1} \cong \pi_0(k)[\![t]\!][x] \oplus (\pi_0(k)[\![t]\!][x^{\pm 1}]/\pi_0(k)[\![t]\!][x])\{1\}.$$

The action of ∂_x^k is more interesting: on $(x^n, 0)$ or $(x^{-n}, 0)$ with $n \geq 1$, ∂_x^k acts in the usual way; but now $\partial_x^k(x^0, 0) = (0, x^{-1})$. In this way, one can check that $j_! j^* \mathcal{O}_{\mathbf{A}^1} \cong \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}/(x\partial_x^k)$; in particular, it can be viewed as an $\widetilde{\mathrm{F}}$-analogue of the Heaviside step distribution H, which is defined to have the property that $\partial_x \mathrm{H} = \delta$, so $x\partial_x \mathrm{H} = 0$.

Example 3.13. There is a map $j_! j^* \mathcal{O}_{\mathbf{A}^1} \to j_* j^* \mathcal{O}_{\mathbf{A}^1}$ given by the composite

$$j_! j^* \mathcal{O}_{\mathbf{A}^1} \to \pi_0(k)[\![t]\!][x] \to \pi_0(k)[\![t]\!][x^{\pm 1}] \cong j_* j^* \mathcal{O}_{\mathbf{A}^1}.$$

It is not hard to show that there is a cofiber sequence

$$j_! j^* \mathcal{O}_{\mathbf{A}^1} \to j_* j^* \mathcal{O}_{\mathbf{A}^1} \to i_* i^* \mathcal{O}_{\mathbf{A}^1} \oplus i_* i^* \mathcal{O}_{\mathbf{A}^1}[-1]\{-1\}.$$

This leads to an important example of a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-module, given by the fiber product

$$\begin{tikzcd} \Xi_{\mathbf{A}^1} \arrow[r] \arrow[d] & i_* i^* \mathcal{O}_{\mathbf{A}^1}\{-1\} \arrow[d] \\ j_* j^* \mathcal{O}_{\mathbf{A}^1}[1] \arrow[r] & i_* i^* \mathcal{O}_{\mathbf{A}^1}[1] \oplus i_* i^* \mathcal{O}_{\mathbf{A}^1}\{-1\}. \end{tikzcd}$$

By construction, there are cofiber sequences

$$j_! j^* \mathcal{O}_{\mathbf{A}^1}[1] \to \Xi_{\mathbf{A}^1} \to i_* i^* \mathcal{O}_{\mathbf{A}^1}\{-1\},$$
$$i_* i^* \mathcal{O}_{\mathbf{A}^1} \to \Xi_{\mathbf{A}^1} \to j_* j^* \mathcal{O}_{\mathbf{A}^1}[1].$$

We will refer to $\Xi_{\mathbf{A}^1}$ as the *tilting* $\mathrm{D}^{\widetilde{\mathrm{F}}}$-module on $\mathbf{A}^1$, since it has a filtration (up to twists by $\{1\}$) by "standards" (!-extensions) and "costandards" ($*$-extensions). If all $\langle n \rangle$ are inverted, then this can be viewed as the cyclic left $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-module generated by the relation $x\partial_x^k x = 0$.

Let us finally make some observations about structural properties of "modular" reductions of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$, sticking to the case $\mathrm{X} = \mathbf{A}^1$ and $\mathbf{G}_m$ for simplicity. If X is a smooth scheme over a (perfect) field of characteristic $p > 0$, the Weyl algebra D_{X} admits a large center: in fact, there is an isomorphism $\mathcal{O}_{\mathrm{T}^*(\mathrm{X}/k)} \xrightarrow{\cong} \mathrm{Z}(\mathrm{D}_{\mathrm{X}})$ given by the p-curvature map sending a derivation ξ to $\xi^p - \xi^{[p]}$. This "large-center" phenomenon persists for the algebra $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$ introduced above, and in fact the p-curvature map itself admits an elementary homotopy-theoretic construction:

Construction 3.14. Let R be an $\mathbf{E}_{3,\mathrm{BS}^1}$-$k$-algebra (in fact, an $\mathbf{E}_{3,\mathrm{B}\mathbf{Z}/p}$-$k$-algebra structure is enough). Then the multiplication map $\mathrm{R}^{\otimes_k p} = \int_{\mathbf{Z}/p} \mathrm{R}/k \to \mathrm{R}$ is $\mathbf{Z}/p$-equivariant and exhibits R as a $\mathbf{Z}/p$-equivariant $\mathbf{E}_1$-$\mathrm{R}^{\otimes_k p}$-algebra; it can even be factored as a map

$$\int_{\mathbf{Z}/p} \mathrm{R}/k = \mathrm{R}^{\otimes_k p} \to \mathrm{HH}(\mathrm{R}/k) = \int_{\mathrm{S}^1} \mathrm{R}/k \to \int_{\mathbf{R}^2} \mathrm{R}/k = \mathrm{R},$$

where the final map is S^1-equivariant and exhibits R as an S^1-equivariant $\mathbf{E}_1$-$\mathrm{HH}(\mathrm{R}/k)$-algebra. (see [DHL+23]). Since the $\mathbf{Z}/p$-Tate construction is lax symmetric monoidal, there is therefore a $k^{t\mathbf{Z}/p}$-linear map $(\mathrm{R}^{\otimes_k p})^{t\mathbf{Z}/p} \to \mathrm{R}^{t\mathbf{Z}/p}$ which exhibits $\mathrm{R}^{t\mathbf{Z}/p}$ as an $\mathbf{E}_1$-$(\mathrm{R}^{\otimes_k p})^{t\mathbf{Z}/p}$-algebra. Composition with the Tate diagonal $\mathrm{R} \to (\mathrm{R}^{\otimes_{\mathbb{S}} p})^{t\mathbf{Z}/p} \to (\mathrm{R}^{\otimes_k p})^{t\mathbf{Z}/p}$ then defines a map $\varphi_{\mathrm{R}} : \mathrm{R} \to \mathrm{R}^{t\mathbf{Z}/p}$ which exhibits $\mathrm{R}^{t\mathbf{Z}/p}$ as an $\mathbf{E}_1$-R-algebra, and which is linear for the $\mathbf{E}_\infty$-Frobenius $\varphi_k : k \to k^{t\mathbf{Z}/p}$; in particular, it induces a map $\mathrm{R} \otimes_k {}^{\varphi} k^{t\mathbf{Z}/p} \to \mathrm{R}^{t\mathbf{Z}/p}$. When R is an $\mathbf{E}_\infty$-ring, φ_{R} is the $\mathbf{E}_\infty$-Frobenius of R.

If $k = \mathbf{Z}_p$ (so that $\pi_*(k^{t\mathbf{Z}/p}) \cong \mathbf{F}_p[\hbar^{\pm 1}]$) and R is p-torsionfree, then the map $\varphi_{\mathrm{R}} : \mathrm{R} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p^{t\mathbf{Z}/p} \to \mathrm{R}^{t\mathbf{Z}/p}$ on homotopy equips $\pi_0(\mathrm{R}^{t\mathrm{S}^1})$ with the structure of a *Frobenius-constant quantization* of $\pi_*(\mathrm{R})$ (over the base $\pi_0(k^{t\mathrm{S}^1}) \cong \pi_0(k)[\![t]\!]$) in the sense of [BK08]. In the special case when $\mathrm{R} = k[\Omega \mathrm{G}]^{h\mathrm{G}}$ for a connected compact Lie group G, this Frobenius-constant quantization structure on the loop-rotation equivariant homology $k[\Omega G]^{h(G\times S^1_{\mathrm{rot}})}$ was proved in [Lon18].

When k is a (p-torsionfree) commutative ring, $k_0 = k/p$, A is a smooth k-algebra, $X_0 = \mathrm{Spec}(A/p)$, and $R = \mathfrak{Z}_{\mathbf{E}_2}(A/k)$, one can extend the proof of Theorem 2.3 to show that the map $R \otimes_k {}^{\varphi}k^{t\mathbf{Z}/p} \to R^{t\mathbf{Z}/p}$ is given on homotopy by the p-curvature map:

$$\pi_0(R \otimes_k {}^{\varphi}k^{t\mathbf{Z}/p}) \cong \mathcal{O}_{T^*(X_0/k_0)} \to \pi_0(R^{t\mathbf{Z}/p}) \cong D_{X_0/k_0}.$$

Since $R^{t\mathbf{Z}/p}$ is an $\mathbf{E}_1$-$R \otimes_k {}^{\varphi}k^{t\mathbf{Z}/p}$-algebra, it follows that this map is central, which is one of the key properties of the classical p-curvature map.

If k is connective, even, and admits a Bott class, then the map $\varphi : \mathfrak{Z}_{\mathbf{E}_2}(k[x]/k) \to \mathfrak{Z}_{\mathbf{E}_2}(k[x]/k)^{t\mathbf{Z}/p}$ is easy to describe on homotopy: it is the map $k[x^p, \widetilde{\partial^k_{x^p}}] \to D^{\widetilde{F}}_{\mathbf{A}^1}[\hbar^{\pm 1}]/\langle p\rangle$ sending $x^p \mapsto x^p$ and $\widetilde{\partial_{x^p}} \mapsto \hbar^{-1}(\partial^k_x)^p$. The map is much more interesting for $k[x^{\pm 1}]$: then, the map $\varphi : \mathfrak{Z}_{\mathbf{E}_2}(k[x^{\pm 1}]/k) \to \mathfrak{Z}_{\mathbf{E}_2}(k[x^{\pm 1}]/k)^{t\mathbf{Z}/p}$ is given on homotopy by the map

$$k[x^{\pm p}, \widetilde{\theta^k_{x^p}}] \to D^{\widetilde{F}}_{\mathbf{G}_m}[\hbar^{\pm 1}]/\langle p\rangle,\ x^p \mapsto x^p,\ \widetilde{\theta^k_{x^p}} \mapsto \hbar^{-1}\prod_{i=0}^{p-1}(\theta^k_x -_{\widetilde{F}} \langle i\rangle). \tag{3}$$

Again, this map is central, and in fact its linearization along $\varphi : \pi_*(k) \to \pi_*(k^{t\mathbf{Z}/p})$ defines an isomorphism onto the center of $D^{\widetilde{F}}_{\mathbf{G}_m}[\hbar^{\pm 1}]/\langle p\rangle$. Moreover, it is an Azumaya algebra over $k[x^{\pm p}, \widetilde{\theta^k_{x^p}}]$ of rank p^2 which splits when base-changed along $k[x^{\pm p}, \widetilde{\theta^k_{x^p}}] \to k[x^{\pm p}]$. Note that the image of $\widetilde{\theta^k_{x^p}}$ under φ is in fact the total power operation/$\mathbf{E}_\infty$-Frobenius on $k^{\mathbf{C}P^\infty_+}$; this is no surprise, since as indicated in the proof of Theorem 3.1, $\mathfrak{Z}_{\mathbf{E}_2}(k[x^{\pm 1}]/k) \cong k[x^{\pm 1}] \otimes_k k^{\mathbf{C}P^\infty_+}$ (albeit not S^1-equivariantly), and the map φ is roughly the tensor product of the $\mathbf{E}_\infty$-Frobenii of each individual tensor factor.

4. A stacky approach

The classical theory of differential operators on smooth schemes in characteristic zero can be captured using *quasi-coherent* information via the de Rham space. Explicitly, if X is a smooth scheme over a $\mathbf{Q}$-algebra R, one can define the de Rham space X^{dR} as the functor on commutative R-algebras B of finite type by $X^{\mathrm{dR}}(B) = X(R_{\mathrm{red}})$. Then, Simpson showed that $R\Gamma(X^{\mathrm{dR}}; \mathcal{O})$ is naturally (quasi-)isomorphic to the de Rham complex $R\Gamma_{\mathrm{dR}}(X/k)$. Moreover, $\mathrm{QCoh}(X^{\mathrm{dR}})$ is equivalent to the category of D-modules on X. When X is a group scheme G, there is an isomorphism $G^{\mathrm{dR}} \cong G/\widehat{G}$, where $\widehat{G}$ is the completion of G at the identity. When $G = \mathbf{G}_m$, an analogous picture for prismatic cohomology was discovered by Drinfeld in [Dri21].

Naturally, one is led to hope for an analogous picture when X is a scheme over $\pi_0(k)$ for which one can define the sheaf $D^{\widetilde{F}}_X$, and the category of D-modules on X is replaced by the category of $\widetilde{F}$-D-modules. I do not know how to do this for arbitrary (k-liftable) $\pi_0(k)$-schemes X, but a rather beautiful picture emerges if one specializes to the case when X is an affine space or a torus.

The following definition is motivated by (and will likely be contained in) joint work [DHRY26] with Jeremy Hahn, Arpon Raksit, and Allen Yuan. I will fix a prime p, and implicitly p-complete below.

Definition 4.1. Let X_k be an affine scheme over k (we will only study this in the case when X_k is an affine space or a torus), and let X be the corresponding scheme defined over $\pi_0(k)$. Define $X^{\widetilde{F}\mathrm{dR}}$ as

$$X^{\widetilde{F}\mathrm{dR}} = \mathrm{colim}_{\mathrm{HH}(X_k/k)^{t\mathbf{Z}/p} \to A}\, \mathrm{Spf}(\pi_*(A^{hS^1}), (\hbar))/\mathbf{G}_m,$$

where the colimit runs over even $\mathbf{E}_\infty$-$\mathrm{HH}(\mathrm{X}_k/k)^{t\mathbf{Z}/p}$-algebras A.

Often, there is a suitable even cover $\mathrm{HH}(\mathrm{X}_k/k)^{t\mathbf{Z}/p} \to \mathrm{A}$, and then

$$\mathrm{X}^{\widetilde{\mathrm{F}}\mathrm{dR}} = \mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Spf}(\pi_*((\mathrm{A}^{\otimes_{\mathrm{HH}(\mathrm{X}_k/k)^{t\mathbf{Z}/p}} \bullet+1})^{h\mathrm{S}^1}), (\hbar))/\mathbf{G}_m.$$

For instance, when $\mathrm{X}_k = \mathrm{Spec}\, k[x]$, the map $\mathrm{HH}(k[x]/k)^{t\mathbf{Z}/p} \to k[x]^{t\mathbf{Z}/p}$ is a cover (similarly for $\mathrm{X}_k = \mathrm{Spec}\, k[x^{\pm 1}]$). To avoid getting into technicalities, I will not be very careful with completions (of the type "sheared" de Rham vs. de Rham) below. One can show:

Proposition 4.2. *Suppose X_k is an affine scheme over k such that $\mathfrak{Z}_{\mathbf{E}_2}(\mathrm{X}_k/k)$ is concentrated in even degrees. Then there is an equivalence of categories $\mathrm{QCoh}(\mathrm{X}^{\widetilde{\mathrm{F}}\mathrm{dR}}) \simeq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})$.*

Remark 4.3. Let F be the multiplicative formal group, so that for a scheme X over $\mathbf{Z}$ with a lift X_{ku} to ku, the category $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})$ describes q-differential operators on X. It follows from Proposition 4.2 and [DR25, Dev25b, DHRY26] that upon p-completion for $p > 2$, the *category* $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})$ itself (rather miraculously) makes sense for *any* formal $\mathbf{Z}_p$-scheme X, independently of whether it lifts to ku (and if it lifts, the choice of lift): namely, one can define $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})$ to be the category of (q-)prismatic crystals on $\mathrm{X} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta_p]$ in the sense of [Bha24, BL22]. However, in this generality, one generally does not have an analogue of the sheaf $\mathrm{D}_{\mathrm{X}}^{\widetilde{\mathrm{F}}}$.

In the case of usual de Rham cohomology, one has the following results, which we will now aim to generalize:

Theorem 4.4. *Let R be a commutative ring; assume for simplicity that R is p-nilpotent.*

(a) *There is an isomorphism*

$$\mathbf{A}^1 \times_{\mathbf{A}^1_{\mathrm{dR}}} \mathbf{A}^1 \cong \mathrm{Spec}\, \mathrm{R}[x, y, \tfrac{(x-y)^n}{n!}] \cong \mathbf{A}^1 \times \mathbf{G}_a^\sharp,$$

which gives an isomorphism $\mathbf{A}^1_{\mathrm{dR}} \cong \mathbf{A}^1/\mathbf{G}_a^\sharp$.

(b) *There is an isomorphism*

$$\mathbf{G}_m \times_{\mathbf{G}_m^{\mathrm{dR}}} \mathbf{G}_m \cong \mathbf{G}_m \times \mathbf{G}_m^\sharp$$

of group schemes. Moreover, there is a Cartesian square

$$\begin{tikzcd} \mathbf{G}_m^\sharp \arrow[r, "\log"] \arrow[d] & \mathbf{G}_a^\sharp \arrow[d, "x \mapsto \exp(px)"] \\ \mathbf{G}_m \arrow[r, "y \mapsto y^p"'] & \mathbf{G}_m^{(1)}. \end{tikzcd}$$

In particular, the map $\mathbf{G}_m \to \mathbf{G}_m^{\mathrm{dR}}$ factors through the Frobenius $\mathbf{G}_m \to \mathbf{G}_m^{(1)}$, and exhibits an isomorphism $\mathbf{G}_m^{\mathrm{dR}} \cong \mathbf{G}_m^{(1)}/\mathbf{G}_a^\sharp$.[6]

In order to generalize Theorem 4.4, we need an $\widetilde{\mathrm{F}}$-analogue of divided powers. There are two candidates, and studying their interplay will be the heart of our generalization of Theorem 4.4(b). Recall that $\mathbf{G}_a^\sharp = \mathrm{Spec}\, \mathrm{R}[z, \frac{z^n}{n!}]$ is the Cartier dual to $\widehat{\mathbf{G}}_a$. Motivated by this, we are led to:

[6]That the map $\mathbf{G}_m \to \mathbf{G}_m^{\mathrm{dR}}$ factors through the Frobenius can be viewed as one instantiation of the (stacky) theory of prismatic cohomology in characteristic p, which gives a canonical Frobenius untwist of crystalline cohomology.

Definition 4.5. Let $\mathrm{F}_{\mathrm{univ}}$ denote the universal formal group law, defined over the Lazard ring L, and let $\widetilde{\mathrm{F}}_{\mathrm{univ}}$ denote its rescaling, defined over $\mathrm{L}[\![t]\!]$. Since L is torsion-free, the ring of functions $\mathcal{O}_{\widetilde{\mathrm{F}}^\vee_{\mathrm{univ}}}$ on the Cartier dual $\widetilde{\mathrm{F}}^\vee_{\mathrm{univ}}$ satisfies

$$\mathcal{O}_{\widetilde{\mathrm{F}}^\vee_{\mathrm{univ}}} \subseteq \mathcal{O}_{\widetilde{\mathrm{F}}^\vee_{\mathbf{Q},\mathrm{univ}}} \cong (\mathrm{L} \otimes \mathbf{Q})[\![t]\!][y],$$

where $\widetilde{\mathrm{F}}^\vee_{\mathbf{Q},\mathrm{univ}}$ is the Cartier dual of the base-change of the rescaled universal formal group along $\mathrm{L}[\![t]\!] \to (\mathrm{L}\otimes\mathbf{Q})[\![t]\!]$. This allows one to write

$$\mathcal{O}_{\widetilde{\mathrm{F}}^\vee_{\mathrm{univ}}} \cong \mathrm{L}[\![t]\!][y, \widetilde{\beta}^{\mathrm{univ}}_n(y)]_{n\geq 0}$$

for some polynomials $\widetilde{\beta}^{\mathrm{univ}}_n(y) \in (\mathrm{L}\otimes\mathbf{Q})[\![t]\!][y]$. The element y defines a *homomorphism* $\widetilde{\mathrm{F}}^\vee_{\mathrm{univ}} \to \mathbf{G}_a$.

For a general formal group law F, one can similarly write $\widetilde{\mathrm{F}}^\vee \cong \mathrm{Spf}(\pi_0(k)[\![t]\!][y, \widetilde{\beta}_n(y)]_{n\geq 0})$ for some classes $\widetilde{\beta}_n(y) \in (\pi_0(k)\otimes\mathbf{Q})[\![t]\!][y]$. These classes might be more appropriately denoted $\widetilde{\beta}^{\mathrm{F}}_n(y)$, but we will just write $\widetilde{\beta}_n(y)$ for notational simplicity.

Remark 4.6. The polynomials $\widetilde{\beta}^{\mathrm{univ}}_n(y) \in (\mathrm{L}\otimes\mathbf{Q})[\![t]\!][y]$ can be computed explicitly as follows. If $\ell_{\mathrm{univ}}(s) \in (\mathrm{L}\otimes\mathbf{Q})[\![s]\!]$ is the logarithm of the universal formal group law over L, then one has

$$\exp(\ell_{\mathrm{univ}}(s)y) = \sum_{n\geq 0} \widetilde{\beta}^{\mathrm{univ}}_n(y)s^n \in (\mathrm{L}\otimes\mathbf{Q})[\![t,s]\!][y].$$

The same formula therefore determines $\widetilde{\beta}_n(y) \in (\pi_0(k)\otimes\mathbf{Q})[\![t]\!][y]$. For instance, when F is the additive formal group, $\ell(s) = s$, so that $\widetilde{\beta}_n(y) = \frac{y^n}{n!}$. Similarly, when $k = \mathrm{ku}$, so that one has $\widetilde{\ell}(s) = \frac{\log(1+(q-1)s)}{q-1}$, it follows that

$$\exp\left(\tfrac{\log(1+(q-1)s)}{q-1}y\right) = (1+(q-1)s)^{y/(q-1)} = \sum_{n\geq 0} \tfrac{y(y-(q-1))\cdots(y-(n-1)(q-1))}{n!}s^n,$$

so that $\widetilde{\beta}_n(y) = \frac{y(y-(q-1))\cdots(y-(n-1)(q-1))}{n!} \in \mathbf{Q}[\![q-1]\!][y]$.

The polynomials $\widetilde{\beta}_n(y)$ do *not* let us describe $\mathbf{A}^1 \times_{\mathbf{A}^1_{\widetilde{\mathrm{F}}\mathrm{dR}}} \mathbf{A}^1$. Instead, we need some other polynomials, which were first defined in joint work with Max Misterka [DM23]:

Definition 4.7. Let $(x+y)^n_{\widetilde{\mathrm{F}}}$ denote the unique sequence of polynomials (defined for $n \geq 0$) characterized by the following:

(a) $(x+y)^0_{\widetilde{\mathrm{F}}} = 1$;
(b) $(x+y)^n_{\widetilde{\mathrm{F}}} = 0$ for $y = -x$ and $n > 0$;
(c) $\partial^k_x (x+y)^n_{\widetilde{\mathrm{F}}} = \langle n\rangle (x+y)^{n-1}_{\widetilde{\mathrm{F}}}$.

We will write $(x-y)^n_{\widetilde{\mathrm{F}}}$ to denote $(x+(-y))^n_{\widetilde{\mathrm{F}}}$. The polynomial $(x+y)^n_{\widetilde{\mathrm{F}}}$ is homogeneous of degree n in x and y, and can be expanded as

$$(x+y)^n_{\widetilde{\mathrm{F}}} = \sum_{j=0}^n (0+1)^n_{\widetilde{\mathrm{F}}} \binom{n}{j}_{\widetilde{\mathrm{F}}} x^{n-j}y^j;$$

this is an analogue of the (q-)binomial theorem.

Let $\mathbf{G}^{\widetilde{\mathrm{F}}\sharp}_m$ denote the scheme $\mathrm{Spf}\left(\pi_0(k)[\![t]\!]\left[x^{\pm 1}, \frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]\right)$; later, we will argue that $\mathbf{G}^{\widetilde{\mathrm{F}}\sharp}_m$ is in fact a group scheme over $\mathrm{Spf}(\pi_0(k)[\![t]\!])$ where the coproduct on x is $x\otimes x$, so that there is a homomorphism $\mathbf{G}^{\widetilde{\mathrm{F}}\sharp}_m \to \mathbf{G}_m$.

For instance, when F is the additive formal group, $(x+y)^n_{\widetilde{\mathrm{F}}} = (x+y)^n$. When F is the multiplicative formal group, one has

$$(x+y)^n_{\widetilde{\mathrm{F}}} = (x+y)(x+qy)\cdots(x+q^{n-1}y),$$

where $q = 1+t$. Using the abstract characterization of the polynomials $(x+y)^n_{\widetilde{\mathrm{F}}}$, it is not hard to prove the following analogue of Theorem 4.4(a):

Proposition 4.8. *There is an isomorphism*

$$\mathbf{A}^1 \times_{\mathbf{A}^1_{\widetilde{\mathrm{F}}\mathrm{dR}}} \mathbf{A}^1 \cong \mathrm{Spf}\left(\pi_0(k)[\![t]\!]\left[x, y, \tfrac{(x-y)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]\right).$$

Moreover, there is an isomorphism $\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}} \cong \mathbf{G}_m/\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp}$ *of group stacks over* $\mathrm{Spf}(\pi_0(k)[\![t]\!])$[7].

In particular, it follows that

$$(\mathrm{Spec}(\pi_0(k))/\mathbf{A}^1)^{\widetilde{\mathrm{F}}\mathrm{dR}} := \mathbf{A}^1 \times_{\mathbf{A}^1_{\widetilde{\mathrm{F}}\mathrm{dR}}} \mathrm{Spec}(\pi_0(k))^{\widetilde{\mathrm{F}}\mathrm{dR}} \cong \mathrm{Spf}\left(\pi_0(k)[\![t]\!]\left[x, \tfrac{x^n}{\langle n\rangle!}\right]\right), \tag{4}$$

where $(\mathrm{Spec}(\pi_0(k))/\mathbf{A}^1)^{\widetilde{\mathrm{F}}\mathrm{dR}}$ denotes the relative $\widetilde{\mathrm{F}}$-de Rham stack of the inclusion of the origin in $\mathbf{A}^1$.

The analogue of Theorem 4.4(b) is trickier. First, we need an analogue of the homomorphism $\log : \mathbf{G}_m^\sharp \to \mathbf{G}_a^\sharp$.

Lemma 4.9. *Let* $\mathrm{F}_{\mathrm{univ}}$ *denote the universal formal group law, defined over the Lazard ring* L*, and let* $\widetilde{\mathrm{F}}_{\mathrm{univ}}$ *denote its rescaling, defined over* $\mathrm{L}[\![t]\!]$*. Then the function* $\log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x) \in (\mathrm{L}\otimes\mathbf{Q})[\![t, x-1]\!]$ *defined by* $\frac{\log(x)}{\ell(1)}$ *satisfies:*

(a) $\partial_x^{\widetilde{\mathrm{F}}_{\mathrm{univ}}} \log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x) = x^{-1}$;
(b) $\log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(xy) = \log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x) + \log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(y)$;
(c) *There is a series expansion*

$$\log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x) = \sum_{n\geq 1} \frac{\langle -n+1\rangle_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}\cdots\langle -1\rangle_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}}{\langle n\rangle_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}!}(x-1)^n_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}.$$

In particular, $\log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x)$ *lies in the subring* $\mathrm{L}[\![t]\!]\left[x, \frac{(x-1)^n_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}}{\langle n\rangle_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}!}\right] \subseteq (\mathrm{L}\otimes\mathbf{Q})[\![t, x-1]\!]$.

The image of the power series $\log_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}(x)$ under the map $\mathrm{L}[\![t]\!]\left[x, \frac{(x-1)^n_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}}{\langle n\rangle_{\widetilde{\mathrm{F}}_{\mathrm{univ}}}!}\right] \to \pi_0(k)[\![t]\!]\left[x, \frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]$ is called the $\widetilde{\mathrm{F}}$-*logarithm*. When F is the additive formal group, $\log_{\widetilde{\mathrm{F}}}(x) = \log(x)$, and when F is the multiplicative formal group, one has

$$\log_{\widetilde{\mathrm{F}}}(x) = \sum_{n\geq 1}(-1)^{n+1}q^{-\binom{n}{2}}\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q}.$$

This is Euler's q-logarithm $\log_q(x)$. The function $\log_{\widetilde{\mathrm{F}}}(x)$ will be the replacement for the logarithm in our analogue of Theorem 4.4(b). The technical heart of this analogue is the following, whose importance (at least, for me) is hard to overestimate:

Proposition 4.10. *There is an equality*

$$\sum_{n\geq 0}\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x))s^n = \sum_{n\geq 0} s(s -_{\widetilde{\mathrm{F}}} 1)\cdots(s -_{\widetilde{\mathrm{F}}} \langle n-1\rangle)\frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}$$

[7]At this point in this exposition, I have not yet shown that $\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp}$ is a *group* scheme! This will be shown below.

in $\pi_0(k)[\![t,s]\!]\left[x, \frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]$*. In particular, the coefficient of* y^n *on the right-hand side expresses* $\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x))$ *as an element of* $\pi_0(k)[\![t]\!]\left[x, \frac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]$.

Proof sketch. By definition of $\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x))$, one has

$$\sum_{n\geq 0} \widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x))s^n = \exp(\ell_{\widetilde{\mathrm{F}}}(y)\log_{\widetilde{\mathrm{F}}}(x)) = \exp\left(\log(x)\frac{\ell_{\mathrm{F}}(\hbar s)}{\ell_{\mathrm{F}}(\hbar)}\right) = x^{\ell_{\mathrm{F}}(\hbar s)/\ell_{\mathrm{F}}(\hbar)} = x^{\ell_{\widetilde{\mathrm{F}}}(s)/\ell_{\widetilde{\mathrm{F}}}(1)}.$$

One can now F-Taylor expand this around $x = 1$, by checking that $(\partial_x^k)^n x^{\ell_{\widetilde{\mathrm{F}}}(s)/\ell_{\widetilde{\mathrm{F}}}(1)} = s(s -_{\widetilde{\mathrm{F}}} 1)\cdots(s -_{\widetilde{\mathrm{F}}} \langle n-1\rangle)x^{\ell_{\widetilde{\mathrm{F}}}(s -_{\widetilde{\mathrm{F}}}\langle n\rangle)/\ell_{\widetilde{\mathrm{F}}}(1)}$, leading to the right-hand side of Proposition 4.10. □

When F is the additive formal group, Proposition 4.10 asserts that

$$\sum_{n\geq 0} \frac{\log(x)^n}{n!}s^n = \sum_{n\geq 0} s(s-1)\cdots(s-(n-1))\frac{(x-1)^n}{n!},$$

which is clear by writing the left-hand side as $\exp(\log(x)s) = x^s = (1+(x-1))^s$ and taking the binomial expansion. Already when F is the multiplicative formal group, Proposition 4.10 is a very nontrivial statement: it asserts that

$$\sum_{n\geq 0} \frac{\log_q(x)\cdots(\log_q(x)-(n-1)(q-1))}{n!}s^n = \sum_{n\geq 0} q^{-\binom{n}{2}} s(s-[1]_q)\cdots(s-[n-1]_q)\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}.$$

(This is in turn equal to $x^{\log(1+(q-1)s)/\log(q)}$.) This particular identity was discovered in a discussion with Michael Kural, and was motivation for Proposition 4.10.

Also, Proposition 4.10 lets us see that $\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp}$ is a group scheme: indeed, we need to see that $\frac{(x_1x_2-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}$ is well-defined in the ring $\pi_0(k)[\![t]\!]\left[x_1^{\pm 1}, x_2^{\pm 1}, \frac{(x_1-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}, \frac{(x_2-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]$. It follows from the definition of $\log_{\widetilde{\mathrm{F}}}(x)$ that $\log_{\widetilde{\mathrm{F}}}(x_1x_2) = \log_{\widetilde{\mathrm{F}}}(x_1)+\log_{\widetilde{\mathrm{F}}}(x_2)$. Moreover, $\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x_1)+\log_{\widetilde{\mathrm{F}}}(x_2))$ is a polynomial in $\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x_1))$ and $\widetilde{\beta}_n(\log_{\widetilde{\mathrm{F}}}(x_2))$. The identity from Proposition 4.10 then lets us conclude that $\frac{(x_1x_2-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}$ is indeed well-defined. In fact, Proposition 4.10 shows more: $\log_{\widetilde{\mathrm{F}}}(x)$ admits a lifting

$$\begin{tikzcd} & \widetilde{\mathrm{F}}^\vee \arrow[d] \\ \mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \arrow[r, "\log_{\widetilde{\mathrm{F}}}"'] \arrow[ur, dashed] & \mathbf{G}_a. \end{tikzcd}$$

That is, $\log_{\widetilde{\mathrm{F}}}$ defines a homomorphism $\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \to \widetilde{\mathrm{F}}^\vee$.[8]

Definition 4.11. Motivated by the case when F is the additive formal group, we will define $\nu_{\widetilde{\mathrm{F}}}(x,s)$ to denote the power series in Proposition 4.10. It should be viewed as an $\widetilde{\mathrm{F}}$-analogue of the function x^s. More precisely, $\nu_{\widetilde{\mathrm{F}}}(x,s)$ is the homomorphism

$$\nu_{\widetilde{\mathrm{F}}} : \mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \times_{\mathrm{Spf}(\pi_0(k)[\![t]\!])} \widetilde{\mathrm{F}} \xrightarrow{\log_{\widetilde{\mathrm{F}}}\times\mathrm{id}} \widetilde{\mathrm{F}}^\vee \times_{\mathrm{Spf}(\pi_0(k)[\![t]\!])} \widetilde{\mathrm{F}} \to \mathbf{G}_m,$$

where the final map is the Cartier duality pairing.

We can now finally state the analogue of Theorem 4.4(b):

[8]In the case when k is not necessarily even, $\log_{\widetilde{\mathrm{F}}}$ still defines a homomorphism $\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \to \widetilde{\mathrm{F}}^\vee$; but now, there will only be a homomorphism from $\widetilde{\mathrm{F}}^\vee$ to the line bundle $\mathrm{Lie}(\widetilde{\mathrm{F}})$, instead of to $\mathbf{G}_a$.

Theorem 4.12. *There is a Cartesian square*

$$\begin{tikzcd} \mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \arrow[r, "\log_{\widetilde{\mathrm{F}}}"] \arrow[d] & \widetilde{\mathrm{F}}^\vee \arrow[d, "\langle p\rangle^*"] \\ \mathbf{G}_m \arrow[r, "y\mapsto y^p"'] & \mathbf{G}_m^{(1)} \end{tikzcd}$$

over $\mathrm{Spf}(\pi_0(k)[\![t]\!],(t,p))$*, where* $\langle p\rangle^*:\widetilde{\mathrm{F}}^\vee\to\mathbf{G}_m^{(1)}$ *is Cartier dual to the homomorphism* $\underline{p\mathbf{Z}}\to\widetilde{\mathrm{F}}$ *which sends* $p\in\underline{p\mathbf{Z}}$ *to* $\langle p\rangle$*. In particular, the map* $\mathbf{G}_m\to\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}$ *factors through the Frobenius* $\mathbf{G}_m\to\mathbf{G}_m^{(1)}$*, and exhibits an isomorphism* $\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}\cong\mathbf{G}_m^{(1)}/\widetilde{\mathrm{F}}^\vee$*.*

Proof sketch. That the diagram commutes is the claim that $y^p=\langle p\rangle^*(\log_{\widetilde{\mathrm{F}}}(y))$. But $\langle p\rangle^*(x)=\exp(px\widetilde{\ell}(1))$, so

$$\langle p\rangle^*(\log_{\widetilde{\mathrm{F}}}(y))=\exp(p\widetilde{\ell}(1)\tfrac{\log(y)}{\widetilde{\ell}(1)})=y^p$$

as desired. Since all objects involved are t-complete, one can check that the square is Cartesian by checking that it is Cartesian when $t=0$. Then, it reduces to the analogous claim for the additive formal group, i.e., that there is a Cartesian square

$$\begin{tikzcd} \mathbf{G}_m^\sharp \arrow[r, "\log"] \arrow[d] & \mathbf{G}_a^\sharp \arrow[d, "x\mapsto\exp(px)"] \\ \mathbf{G}_m \arrow[r, "y\mapsto y^p"'] & \mathbf{G}_m^{(1)} \end{tikzcd}$$

over a p-nilpotent ring. In fact one can reduce to checking this over $\mathbf{F}_p$, namely that there is an exact sequence

$$0\to\mu_p\to\mathbf{G}_m^\sharp\xrightarrow{\log}\mathbf{G}_a^\sharp\to 0.$$

Then, the desired result was proved in [BL22, Lemma 3.5.18], but could also be argued more directly as follows: the homomorphism $\log:\mathbf{G}_m^\sharp\to\mathbf{G}_a^\sharp$ admits a splitting, given by the homomorphism $\mathbf{G}_a^\sharp\to\mathbf{G}_m^\sharp$ sending $x\mapsto\exp(x):=\sum_{n\geq 0}\frac{x^n}{n!}$. Note that this is well-defined:

$$\sum_{n\geq 0}\frac{(xs)^n}{n!}=\sum_{n\geq 0}s(s-1)\cdots(s-(n-1))\frac{(\exp(x)-1)^n}{n!},$$

so extracting the coefficient of $s(s-1)\cdots(s-(n-1))$ on the left-hand side exhibits $\frac{(\exp(x)-1)^n}{n!}$ as an element of $\mathcal{O}_{\mathbf{G}_a^\sharp}$. It is also clear that $\mu_p\subseteq\mathbf{G}_m^\sharp$ is contained in the kernel of $\log:\mathbf{G}_m^\sharp\to\mathbf{G}_a^\sharp$, and looking at coordinate rings one finds that $\mathbf{G}_m^\sharp\cong\mathbf{G}_a^\sharp\times\mu_p$ as desired. □

In particular, it follows that

$$(\mathrm{Spec}(\pi_0(k))/\mathbf{G}_m)^{\widetilde{\mathrm{F}}\mathrm{dR}}:=\mathbf{G}_m\times_{\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}}\mathrm{Spec}(\pi_0(k))^{\widetilde{\mathrm{F}}\mathrm{dR}}\cong\mathrm{Spf}\left(\pi_0(k)[\![t]\!]\left[x,\tfrac{(x-1)^n_{\widetilde{\mathrm{F}}}}{\langle n\rangle!}\right]\right),$$

where $(\mathrm{Spec}(\pi_0(k))/\mathbf{G}_m)^{\widetilde{\mathrm{F}}\mathrm{dR}}$ denotes the relative $\widetilde{\mathrm{F}}$-de Rham stack of the inclusion of the identity $1\in\mathbf{G}_m$. Note the contrast to (4): although one can identify $(\mathrm{Spec}(\pi_0(k))/\mathbf{G}_m)^{\widetilde{\mathrm{F}}\mathrm{dR}}\cong(\mathrm{Spec}(\pi_0(k))/\widehat{\mathbf{G}_m})^{\widetilde{\mathrm{F}}\mathrm{dR}}$ and $(\mathrm{Spec}(\pi_0(k))/\mathbf{A}^1)^{\widetilde{\mathrm{F}}\mathrm{dR}}\cong(\mathrm{Spec}(\pi_0(k))/\widehat{\mathbf{A}^1})^{\widetilde{\mathrm{F}}\mathrm{dR}}$, one generally does *not* have an isomorphism $(\mathrm{Spec}(\pi_0(k))/\widehat{\mathbf{G}_m})^{\widetilde{\mathrm{F}}\mathrm{dR}}\cong(\mathrm{Spec}(\pi_0(k))/\widehat{\mathbf{A}^1})^{\widetilde{\mathrm{F}}\mathrm{dR}}$ (even at the level of rings of functions viewed as associative $\pi_0(k)[\![t]\!]$-algebras), because there is generally not an isomorphism $(\widehat{\mathbf{G}_m})_k\cong(\widehat{\mathbf{A}^1})_k$ of *pointed* $\mathbf{E}_3$-k-*schemes* unless k is an ordinary commutative ring.

Theorem 4.12 also gives a notion of Chern classes in $\widetilde{\mathrm{F}}$-de Rham cohomology for "strict" line bundles: if $\mathrm{X}_k \to \mathrm{B}\mathbf{G}_m$ is a map classifying a "strict" line bundle over a k-scheme X_k, then the composite

$$\mathrm{X}^{\widetilde{\mathrm{F}}\mathrm{dR}} \to \mathrm{B}\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}} \cong \mathrm{B}\mathbf{G}_m^{(1)}/\mathrm{B}\widetilde{\mathrm{F}}^\vee \to \mathrm{B}^2\widetilde{\mathrm{F}}^\vee \to \mathrm{B}^2\mathrm{Lie}(\widetilde{\mathrm{F}})^{-1}$$

defines a class in $\mathrm{H}^2(\mathrm{X}^{\widetilde{\mathrm{F}}\mathrm{dR}}; \mathrm{Lie}(\widetilde{\mathrm{F}})^{-1}) = \mathrm{H}^2_{\widetilde{\mathrm{F}}\mathrm{dR}}(\mathrm{X}; \mathcal{O}\{1\})$. If $\widetilde{\mathrm{F}}$ admits a global coordinate, so $\mathrm{Lie}(\widetilde{\mathrm{F}}) \cong \mathbf{G}_a$, then this is a class in $\mathrm{H}^2_{\widetilde{\mathrm{F}}\mathrm{dR}}(\mathrm{X})$.

Example 4.13. When F is the multiplicative formal group, there are isomorphisms

$$\mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \cong \mathrm{Spf}\left(\mathbf{Z}[\![q-1]\!]\left[x, \tfrac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}\right]_{n\geq 0}\right),$$
$$\widetilde{\mathrm{F}}^\vee \cong \mathrm{Spf}\left(\mathbf{Z}[\![q-1]\!]\left[y, \tfrac{y(y-(q-1))\cdots(y-(n-1)(q-1))}{n!}\right]\right).$$

The map $\langle p\rangle^* : \widetilde{\mathrm{F}}^\vee \to \mathbf{G}_m$ sends $y \mapsto q^{py/(q-1)}$. In this case, the square of Theorem 4.12 was implicitly proved in [Dri21].

Remark 4.14. Theorem 4.12 can be interpreted in homotopy theory as follows. Recall that the map $\mathrm{HH}(k[x^{\pm 1}]/k)^{t\mathbf{Z}/p} \to k[x^{\pm 1}]^{t\mathbf{Z}/p} \cong k^{t\mathbf{Z}/p}[x^{\pm 1/p}]$ is an even cover. This implies that

$$\mathbf{G}_m \times_{\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}} \mathbf{G}_m \cong \mathrm{Spf}(\pi_0(k[x^{\pm 1}]^{t\mathbf{Z}/p} \otimes_{\mathrm{HH}(k[x^{\pm 1}]/k)^{t\mathbf{Z}/p}} k[x^{\pm 1}]^{t\mathbf{Z}/p})^{h\mathrm{S}^1}).$$

Let us describe how this can be identified with $\mathbf{G}_m \times \mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp}$. There is an S^1-equivariant map $k[\mathrm{B}\mathbf{Z}]^{\mathrm{triv}} \to \mathrm{HH}(k[x^{\pm 1}]/k)$ which detects the class $d\mathrm{log}_k(x) := x^{-1}dx$ on π_1 (here, the superscript triv denotes that $k[\mathrm{B}\mathbf{Z}]$ is equipped with the trivial S^1-action), and this map defines an equivalence

$$k[x^{\pm 1}] \cong \mathrm{HH}(k[x^{\pm 1}]/k) \otimes_{k[\mathrm{B}\mathbf{Z}]^{\mathrm{triv}}} k.$$

It follows that there is an S^1-equivariant equivalence

$$k[x^{\pm 1}] \otimes_{\mathrm{HH}(k[x^{\pm 1}]/k)} k[x^{\pm 1}] \cong k[x^{\pm 1}] \otimes_k k \otimes_{k[\mathrm{B}\mathbf{Z}]^{\mathrm{triv}}} k \cong k[x^{\pm 1}][\mathbf{C}\mathrm{P}^\infty]^{\mathrm{triv}}.$$

The class $d\mathrm{log}_k(x) \in \pi_1 k[\mathrm{B}\mathbf{Z}]^{\mathrm{triv}}$ suspends to a class $y = \sigma^2\mathrm{log}_k(x)$ in degree 2. If we write $\pi_*(k[\mathbf{C}\mathrm{P}^\infty]) \cong \pi_0(k)[u][y, \beta_n(y)]_{n\geq 1}$ for $|\beta_n(y)| = 2n$, then

$$\pi_*(k[x^{\pm 1}] \otimes_{\mathrm{HH}(k[x^{\pm 1}]/k)} k[x^{\pm 1}]) \cong \pi_0(k)[u, x^{\pm 1}][\sigma^2\mathrm{log}_k(x), \beta_n(\sigma^2\mathrm{log}_k(x))]_{n\geq 1}.$$

In the same way, one can compute that

$$\pi_*(k[x^{\pm 1}]^{t\mathbf{Z}/p} \otimes_{\mathrm{HH}(k[x^{\pm 1}]/k)^{t\mathbf{Z}/p}} k[x^{\pm 1}]^{t\mathbf{Z}/p}) \cong \pi_0(k)[\![t]\!][u^{\pm 1}, x^{\pm 1/p}]\left[\tfrac{\sigma^2\mathrm{log}_k(x)}{u}, \beta_n\left(\tfrac{\sigma^2\mathrm{log}_k(x)}{u}\right)\right]_{n\geq 1}/\langle p\rangle,$$

where u lives in degree 2 and $\frac{\beta_n(\sigma^2\mathrm{log}_k(x))}{u^n}$ is in degree zero. The S^1-homotopy fixed points spectral sequence collapses immediately (by evenness); if $\hbar$ is the Euler class of the S^1-action, then there is a single relation $\hbar u = \langle p\rangle$ on the E_∞-page. If we write $\mathrm{log}_k(x) = \hbar\sigma^2\mathrm{log}_k(x)$, a diagram chase shows that $x\partial_x^k \mathrm{log}_k(x) = 1$; so, $\mathrm{log}_k(x)$ is indeed $\mathrm{log}_{\widetilde{\mathrm{F}}}(x)$. Moreover,

$$\frac{\sigma^2\mathrm{log}_k(x)}{u} = \frac{\hbar\sigma^2\mathrm{log}_k(x)}{\hbar u} = \frac{\mathrm{log}_{\widetilde{\mathrm{F}}}(x)}{\langle p\rangle}.$$

Extracting π_0 of the homotopy S^1-fixed points, one therefore finds that

$$\pi_0(k[x^{\pm 1}]^{t\mathbf{Z}/p} \otimes_{\mathrm{HH}(k[x^{\pm 1}]/k)^{t\mathbf{Z}/p}} k[x^{\pm 1}]^{t\mathbf{Z}/p})^{h\mathrm{S}^1} \cong \left(\pi_0(k)[\![t]\!][x^{\pm 1/p}]\left[\tfrac{\mathrm{log}_{\widetilde{\mathrm{F}}}(x)}{\langle p\rangle}, \beta_n\left(\tfrac{\mathrm{log}_{\widetilde{\mathrm{F}}}(x)}{\langle p\rangle}\right)\right]_{n\geq 1}\right)^\wedge_{(p,\langle p\rangle)}.$$

Using Theorem 4.12, one can identify the ring on the right-hand side with $\mathcal{O}_{\mathbf{G}_m \times \mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}}$ as desired.

One can alternatively give a direct identification of $\mathbf{G}_m \times_{\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}} \mathbf{G}_m$ with $\mathbf{G}_m \times \mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}$ using homotopy theory; by running the above discussion backwards, this then gives an alternative proof of Theorem 4.12. Namely, it follows from our forthcoming work [DHRY26] that the kernel of the homomorphism $\mathbf{G}_m \to \mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}$ identifies with the Cartier dual of the pushout $\mathbf{Z} \amalg_{p\mathbf{Z}} \widetilde{\mathrm{F}}$, where the homomorphism $p\mathbf{Z} \to \widetilde{\mathrm{F}}$ sends $p \mapsto \langle p \rangle$. This pushout $\mathbf{Z} \amalg_{p\mathbf{Z}} \widetilde{\mathrm{F}}$ is an example of an S^1-*equivariant formal group* [CGK00, Hau22], a notion which plays a crucial role in our forthcoming work [DHRY26].

5. Fourier and Mellin transform

The algebra $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$ satisfies a Fourier transform. To describe it, let me introduce some notation: let $\iota(y)$ denote the unique power series such that $y\iota(y) = \overline{y}$, where $\overline{y}$ is the inverse of y in the group law $\widetilde{\mathrm{F}}$. Note that $\iota(\overline{y}) = \iota(y)^{-1}$.

Proposition 5.1. *There is an isomorphism of associative $\pi_0(k)[\![t]\!]$-algebras*[9]

$$\Phi : \mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}} \xrightarrow{\cong} \mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}, \ x \mapsto \partial_x^k, \ \partial_x^k \mapsto \iota(x\partial_x^k)x,$$

which in particular gives rise an equivalence of categories $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1) \simeq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1)$.

Proof. For notational simplicity, let $\theta_x^k = x\partial_x^k$. Then,

$$\Phi : x\partial_x^k \mapsto \partial_x^k \iota(x\partial_x^k)x = \partial_x^k x\iota(x\partial_x^k +_{\widetilde{\mathrm{F}}} 1) = (\theta_x^k +_{\widetilde{\mathrm{F}}} 1)\iota(\theta_x^k +_{\widetilde{\mathrm{F}}} 1) = \overline{\theta_x^k +_{\widetilde{\mathrm{F}}} 1},$$

so that $x\partial_x^k +_{\widetilde{\mathrm{F}}} 1 \mapsto \overline{\theta_x^k}$. On the other hand,

$$\Phi : \partial_x^k x \mapsto \iota(x\partial_x^k)x\partial_x^k = \overline{\theta_x^k},$$

so Φ is indeed respects the defining relations of $\mathrm{D}_{\mathbf{A}^1}^{\widetilde{\mathrm{F}}}$. Just like the usual Fourier transform, Φ does not square to the identity; instead,

$$\Phi^2 : x \mapsto \iota(x\partial_x^k)x, \ \partial_x^k \mapsto \iota(\partial_x^k \iota(x\partial_x^k)x)\partial_x^k = \iota(\overline{x\partial_x^k +_{\widetilde{\mathrm{F}}} 1})\partial_x^k = \iota(x\partial_x^k +_{\widetilde{\mathrm{F}}} 1)^{-1}\partial_x^k,$$

so Φ^2 is an isomorphism with inverse given by

$$(\Phi^2)^{-1} : x \mapsto \iota(x\partial_x^k)^{-1}x, \ \partial_x^k \mapsto \iota(x\partial_x^k +_{\widetilde{\mathrm{F}}} 1)\partial_x^k.$$

(Also, *unlike* the usual Fourier transform, Φ^4 is generally not the identity.) $\square$

When F is the additive formal group, Φ just sends $x \mapsto \partial_x$ and $\partial_x \mapsto -x$, so it is the usual Fourier transform. When F is the multiplicative formal group, Φ sends

$$x \mapsto \partial_x^q, \ \partial_x^q \mapsto -(1 + (q-1)x\partial_x^q)^{-1}x.$$

Remark 5.2. Proposition 5.1 has a homotopy-theoretic explanation: it amounts to the observation that there is an S^1-equivariant equivalence $\mathfrak{Z}_{\mathbf{E}_2}(k[x]/k) \cong \mathfrak{Z}_{\mathbf{E}_2}(k[u]/k)$ of $\mathbf{E}_3$-k-algebras, where $k[u]$ is the polynomial $\mathbf{E}_2$-k-algebra on a class in degree -2. Here, it is crucial that k is complex oriented.

In the setting of usual D-modules, the Fourier transform defines an equivalence $\mathrm{DMod}(\mathbf{A}^1) \cong \mathrm{DMod}(\mathbf{A}^1)$ which exchanges the pointwise tensor product and the convolution symmetric monoidal structure. However, it is not even obvious that $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1)$ admits a convolution symmetric monoidal structure!

A mild variant nevertheless turns out to be true: $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1)$ admits a convolution *monoidal* structure, and the Fourier transform of Proposition 5.1 defines a monoidal self-equivalence of

[9]This is technically not quite correct because of completion issues (x is not a formal variable). Appropriately modifying the statement would unfortunately require too much of a digression, so I apologize to the reader!

$\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{A}^1)$ which exchanges the pointwise tensor product and the convolution monoidal structure. This monoidal structure comes from the structure of an $\mathbf{E}_2$-monoid structure on $\mathbf{A}^1_k = \operatorname{Spec} k[x]$ viewed as an $\mathbf{E}_2$-scheme over k. In fact, this structure exists even when $k = \mathbb{S}$, and was essentially described in [Lur15].

Construction 5.3. Let $\mathbb{S}[y] = \mathbb{S}[\mathbf{Z}_{\geq 0}]$ denote the flat polynomial algebra over the sphere spectrum on a class y in degree 0 and weight 1. There is a filtration on monoid $\mathbf{Z}_{\geq 0}$ given by $\{0, \cdots, n\}$; this refines $\mathbf{Z}_{\geq 0}$ to a filtered monoid, so $\mathbb{S}[y]$ is equipped with the structure of a filtered augmented $\mathbf{E}_\infty$-ring. Taking the 2-bar construction with respect to this augmentation produces the $\mathbf{E}_\infty$-algebra in filtered $\mathbf{E}_2$-coalgebras given by $\{\mathbb{S}[\mathbf{CP}^n]\}_{n\geq 0}$. Dualizing produces a $\mathbf{E}_\infty$-coalgebra structure in filtered $\mathbf{E}_2$-algebras on $\mathbb{S}^{\mathbf{CP}^\infty_+}$. The associated graded $\mathbb{S}[u]$ is an $\mathbf{E}_\infty$-coalgebra in graded $\mathbf{E}_2$-algebras, where u is a class in weight 1 and degree -2. We can now apply the endofunctor of graded spectra called *shearing*, which sends a spectrum $\mathrm{X}(n)$ in weight n to $\mathrm{X}[2n](n)$; in the derived category of graded $\mathbf{Z}$-modules, this functor is symmetric monoidal, but it is only a (framed) $\mathbf{E}_2$-monoidal functor on graded spectra. The shearing of $\mathbb{S}[u]$ is $\mathbb{S}[x]$ with x in weight 1 and degree 0, so $\mathbb{S}[x]$ acquires the structure of an $\mathbf{E}_2$-coalgebra in graded $\mathbf{E}_2$-algebras.

Remark 5.4. Although shearing is canonically symmetric monoidal in graded MU-modules, so $\mathrm{MU}[x]$ is an $\mathbf{E}_\infty$-coalgebra in graded $\mathbf{E}_2$-algebras, this cannot be improved to saying that $\mathrm{MU}[x]$ is an $\mathbf{E}_\infty$-coalgebra in graded $\mathbf{E}_\infty$-algebras. (If this were true, then upon base-change to KU, the structure of power operations on KU-algebras implies that $\mathbf{G}_a$ would be a δ-group scheme, which is false.)

Also, although $\operatorname{Spec} \mathbb{S}[x]$ is an $\mathbf{E}_2$-monoid in $\mathbf{E}_2$-schemes over $\operatorname{Spec}(\mathbb{S})$, it is not a *group*! In other words, there is no $\mathbf{E}_2$-map $\mathbb{S}[x] \to \mathbb{S}[x]$ sending $x \mapsto -x$.[10]

In the setting of classical D-modules, the Fourier equivalence $\mathrm{DMod}(\mathbf{A}^1) \simeq \mathrm{DMod}(\mathbf{A}^1)$ is computed (up to shift) by the functor $\mathcal{F} \mapsto \mathrm{pr}^\vee_*(\mathrm{pr}^!(\mathcal{F}) \otimes^! \mu^!(\exp))$, where $\mathrm{pr}, \mathrm{pr}^\vee : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ are the two projections, $\mu : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ is the multiplication map, and exp is the exponential D-module. This is a reflection of the classical formula for the Fourier-Laplace transform:

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{2\pi i x \xi} dx.$$

The same formula is true for $\mathrm{DMod}_{\widetilde{\mathrm{F}}}$: now, the replacement of the exponential D-module is given by the cyclic $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1}$-module $\exp_{\widetilde{\mathrm{F}}}$ with relation $\partial_x^k = 1$. In other words, the kernel of the Fourier transform of Proposition 5.1 is $\mu^!(\exp_{\widetilde{\mathrm{F}}})$.

It is even easier to describe the *Mellin transform*: a $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}$-module lives over $\widetilde{\mathrm{F}}$ via the action of $s = x\partial_x^k$; if $\mathrm{T} = x$, then since $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m} \cong \pi_0(k)[\![t]\!]\{s, \mathrm{T}^{\pm 1}\}/(s\mathrm{T} = \mathrm{T}(s+1))$, one has:

Proposition 5.5. *There is an equivalence* $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{G}_m) \cong \mathrm{QCoh}(\widetilde{\mathrm{F}}/\underline{\mathbf{Z}})$*, where the constant group scheme* $\underline{\mathbf{Z}}$ *acts on* $\widetilde{\mathrm{F}}$ *by translation (in the group law) by* 1*. This equivalence is symmetric monoidal and exchanges convolution (on either side) with the pointwise tensor product (on either side).*

[10] If this were true, then there would be an $\mathbf{E}_2$-map $\mathbb{S}[x] \to \mathbb{S}$ sending $x \mapsto -1$. However, this is impossible. To see this, first note that such a map necessarily defines an $\mathbf{E}_2$-map $\mathbb{S}[x^{\pm 1}] \to \mathbb{S}$, which can be viewed as an $\mathbf{E}_2$-map $\mathbf{Z} \to \mathrm{GL}_1(\mathbb{S})$ sending $1 \mapsto -1$. Since $\tau_{\leq 1}\mathrm{GL}_1(\mathbb{S})$ is equivalent as an infinite loop space to the fiber of the $\mathbf{E}_\infty$-map $\mathbf{Z}/2 \to \mathrm{K}(\mathbf{F}_2, 2)$ given by Sq^2, it would follow in particular that the composite

$$\mathbf{Z} \to \mathrm{GL}_1(\mathbb{S}) \to \tau_{\leq 1}\mathrm{GL}_1(\mathbb{S}) \to \mathbf{Z}/2 \xrightarrow{\mathrm{Sq}^2} \mathrm{K}(\mathbf{F}_2, 2)$$

is null as an $\mathbf{E}_2$-map. Delooping twice, this amounts to the assertion that Sq^2 acts trivially on the canonical generator of $\mathrm{H}^2(\mathrm{B}^2\mathbf{Z}; \mathbf{F}_2)$, which is false.

Under the Mellin transform, pushforward along the p-curvature map $\mathcal{O}_{\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{G}_m} \to \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}/\langle p\rangle$ from Construction 3.14 identifies with the functor $\mathrm{QCoh}(\widetilde{\mathrm{F}}/\underline{\mathbf{Z}}) \to \mathrm{QCoh}(\widetilde{\mathrm{F}} \times \mathrm{B}\underline{p\mathbf{Z}})$ induced by pullback along the maps $\underline{p\mathbf{Z}} \to \underline{\mathbf{Z}}$ and $\widetilde{\mathrm{F}} \to \widetilde{\mathrm{F}}$ given by the formulas (3). (Note that when $\langle p\rangle = 0$, the action of $\underline{p\mathbf{Z}}$ on $\widetilde{\mathrm{F}}$ is trivial!)

Remark 5.6. If T is a torus, there is a symmetric monoidal equivalence $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{T}}) \cong \mathrm{QCoh}(\mathrm{T}_{\widetilde{\mathrm{F}}}/\mathbb{X}_*(\mathrm{T}))$. In fact, this can be generalized even further: as in Theorem 3.1, one can identify $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}} \cong \pi_0(k[\Omega\mathrm{T}]^{h\mathrm{T}})^{t\mathrm{S}^1_{\mathrm{rot}}}$. Note that $k[\Omega\mathrm{T}]^{h\mathrm{T}} \cong k[\mathcal{L}\mathrm{T}]^{h(\mathrm{T}\times\mathrm{T})}$, where $\mathcal{L}\mathrm{T}$ denotes the free loop space of T. Replacing T by a general connected reductive group G, one can compute [Dev23] that the category of (left) modules over $\pi_0(k[\mathcal{L}\mathrm{G}]^{h(\mathrm{T}\times\mathrm{T})})^{t\mathrm{S}^1_{\mathrm{rot}}}$ is equivalent to the category of ind-coherent sheaves on a certain quotient stack $\mathrm{T}_{\widetilde{\mathrm{F}}}/\!\!/\widetilde{\mathrm{W}}$ defined as the (stacky) quotient of $\mathrm{T}_{\widetilde{\mathrm{F}}}$ by the union of graphs of the action of the (extended) affine Weyl group $\widetilde{\mathrm{W}}$ on $\mathrm{T}_{\widetilde{\mathrm{F}}}$. When k is an ordinary commutative ring (so F is the additive formal group), this stack was studied in [Gan22] under the name $\mathfrak{t}/\!\!/\widetilde{\mathrm{W}}$.

Remark 5.7. The equivalence of Proposition 5.5 can be viewed as 1-shifted Cartier duality between $\mathbf{G}_m^{\widetilde{\mathrm{F}}\mathrm{dR}}$ and $\widetilde{\mathrm{F}}/\underline{\mathbf{Z}}$. As such, this transform is a categorification of the usual formula for the Mellin transform for a $\mathbf{C}$-valued function f on $(0,\infty)$:

$$\mathcal{M}f(s) = \int_{(0,\infty)} f(x)x^s \tfrac{dx}{x}.$$

The kernel of the Mellin transform is the function x^s. In the context of $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules, s is the coordinate on $\widetilde{\mathrm{F}}$, and the replacement of the kernel x^s is the function $\nu(x,s) : \mathbf{G}_m^{\widetilde{\mathrm{F}}\sharp} \times \widetilde{\mathrm{F}}^\vee \to \mathbf{G}_m$ from Definition 4.11. (This is why I chose s for the coordinate on $\widetilde{\mathrm{F}}$, to maintain consistency with complex analysis).

Remark 5.8. Let $\mathrm{T} = \mathrm{S}^1$ (to distinguish it from the loop-rotation S^1), so that $\pi_0(k^{t\mathrm{S}^1})^{h\mathrm{T}} \cong \mathcal{O}_{\widetilde{\mathrm{F}}}$. Just like Theorem 4.12 had a homotopy-theoretic interpretation via (a decompleted variant of) the periodic cyclic homology $\mathrm{HP}(k/k[\mathbf{Z}])$ (as explained in Remark 4.14), the Mellin transform above can also be interpreted homotopy-theoretically as an S^1-equivariant isomorphism between $\mathfrak{Z}_{\mathbf{E}_2}(k[\![\mathbf{Z}]\!]/k)$ and $\mathfrak{Z}_{\mathbf{E}_2}(k/k^{h\mathrm{T}})$.

Unwinding, this implies a computation of the periodic cyclic homology $\mathrm{HP}(k/k^{h\mathrm{T}})$: if $\binom{s}{n}_{\widetilde{\mathrm{F}}} := \frac{s(s-_{\widetilde{\mathrm{F}}}1)\cdots(s-_{\widetilde{\mathrm{F}}}\langle n-1\rangle)}{\langle n\rangle!}$, then there is an isomorphism

$$\pi_0\mathrm{HP}(k/k^{h\mathrm{T}}) \cong \pi_0(k)[\![t]\!]\left[s, \binom{s}{n}_{\widetilde{\mathrm{F}}}\right]_{n\geq 0}.$$

We remark that when $s = x\partial_x^k$, the F-binomial coefficient $\binom{s}{n}_{\widetilde{\mathrm{F}}}$ is precisely $\frac{x^n(\partial_x^k)^n}{\langle n\rangle!}$.

When all of the elements $\langle n\rangle \in \pi_0(k)[\![t]\!]$ are non-zerodivisors for $n \geq 1$, the uncompleted ring $\pi_0(k)[\![t]\!]\left[s, \binom{s}{n}_{\widetilde{\mathrm{F}}}\right]_{n\geq 0}$ is isomorphic to a ring of "$\widetilde{\mathrm{F}}$-integer valued polynomials": namely, it is isomorphic to the subring of $\pi_0(k)(\!(t)\!)[s]$ spanned by those polynomials $f(s)$ such that $f(\langle n\rangle) \in \pi_0(k)[\![t]\!]$ for all $n \geq 0$. Proposition 4.10 can be used to show that this ring is also a Hopf algebra over $\pi_0(k)[\![t]\!]$, where the coproduct sends $s \mapsto s +_{\widetilde{\mathrm{F}}} s'$; concretely, this asserts that $\binom{s+_{\widetilde{\mathrm{F}}}s'}{n}_{\widetilde{\mathrm{F}}}$ is a polynomial in $\binom{s}{i}_{\widetilde{\mathrm{F}}}$ and $\binom{s'}{j}_{\widetilde{\mathrm{F}}}$ for $i+j \leq n$. Note that the Künneth formula implies that the $\mathbf{E}_\infty$-k-algebra $\mathrm{HH}(k/k^{h\mathrm{T}})$ is a decompletion of the algebra $\mathrm{C}^*(\Omega\mathrm{T};k) = \mathrm{C}^*(\mathbf{Z};k)$, so it is not surprising that $\pi_0\mathrm{HP}(k/k^{h\mathrm{T}})$ is an $\widetilde{\mathrm{F}}$-analogue of the ring of integer-valued polynomials. For instance, when F is the multiplicative formal group, the fact that

$\pi_0\mathrm{HP}(\mathrm{ku}/\mathrm{ku}^{h\mathrm{T}}) \cong \mathbf{Z}[\![q-1]\!]\left[s, \binom{s}{n}_{\widetilde{\mathrm{F}}}\right]_{n\geq 0}$ computes a q-deformation of the ring of integer-valued polynomials was observed previously in [HH17]. [11]

More generally, it is very interesting to compute $\mathrm{HP}(k/k^{h\mathrm{G}})$ and $\mathrm{HP}(k^{h\mathrm{G}}/k^{h(\mathrm{G}\times\mathrm{G})})$ when G is a connected compact Lie group; if G is furthermore simply-connected, these can be identified with $\mathrm{C}^*_{\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{G}; k)[\hbar^{-1}]$ and $\mathrm{C}^*_{\mathrm{S}^1\times\mathrm{G}}(\Omega\mathrm{G}; k)[\hbar^{-1}]$, respectively. We will return to this in the future. It is closely related to the discussion in Section 8.

Proposition 5.5 can be used to give one construction of monodromic $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules. Suppose $\mathbf{G}_m$ acts on X, and let α be an $\pi_0(k)[\![t]\!]$-point of $\widetilde{\mathrm{F}}/\underline{\mathbf{Z}}$; this corresponds to a "character D^{F}-module" on $\mathbf{G}_m$. Then α defines a $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{G}_m) \cong \mathrm{QCoh}(\widetilde{\mathrm{F}}/\underline{\mathbf{Z}})$-module structure on $\mathrm{Mod}_{\pi_0(k)[\![t]\!]}$, so one can define

$$\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X}/(\mathbf{G}_m, \alpha)) = \mathrm{Fun}^{\mathrm{L}}_{\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{G}_m)}(\alpha^*\mathrm{Mod}_{\pi_0(k)[\![t]\!]}, \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})). \tag{5}$$

Note that if $\alpha = 0 \in \widetilde{\mathrm{F}}/\underline{\mathbf{Z}}$, then $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X}/(\mathbf{G}_m, \alpha)) = \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X}/\mathbf{G}_m)$ is a category of *strongly* $\mathbf{G}_m$-equivariant $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathrm{X}}$-modules. One can produce many interesting examples of twisted D^{F}-modules through Beilinson-Bernstein localization (we will discuss the untwisted version below in Proposition 6.5).

As in usual function theory, the interaction between the Fourier and Mellin transforms is very fruitful.

Example 5.9. Let $\exp_{\widetilde{\mathrm{F}}}$ denote the exponential $\mathrm{D}^{\widetilde{\mathrm{F}}}$-module on $\mathbf{A}^1$, and let $\exp_{\widetilde{\mathrm{F}}}|_{\mathbf{G}_m}$ denote its restriction to $\mathbf{G}_m$, so that $\exp_{\widetilde{\mathrm{F}}}|_{\mathbf{G}_m} \cong \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}/(x\partial_x^k = x)$. The Mellin transform of $\exp_{\widetilde{\mathrm{F}}}|_{\mathbf{G}_m}$ is the $\mathbf{Z}$-equivariant quasicoherent sheaf $\gamma_{\widetilde{\mathrm{F}}}$ on $\widetilde{\mathrm{F}}$ given by $\mathcal{O}_{\widetilde{\mathrm{F}}-\underline{\mathbf{Z}}} = \mathcal{O}_{\widetilde{\mathrm{F}}}[(s \pm_{\widetilde{\mathrm{F}}} 1)^{-1}, (s \pm_{\widetilde{\mathrm{F}}} \langle 2\rangle)^{-1}, \cdots]$, where the translation T acts by s-multiplication. Since the Mellin transform of the exponential function is the Γ-function (which satisfies $s\Gamma(s) = \Gamma(s+1)$), $\gamma_{\widetilde{\mathrm{F}}}$ can be viewed as an $\widetilde{\mathrm{F}}$-variant of the Γ-function.

A formal solution to the difference equation $sf = \mathrm{T}f$ in the ring $\pi_0(k)[\![t]\!]\left[s, \binom{s}{n}_{\widetilde{\mathrm{F}}}\right]_{n\geq 0}$ can be given as follows. By Proposition 4.10, the expression $\nu(x, s)$ makes sense for any $x \in \mathbf{G}_m$, and so one can define[12]

$$\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} 1) := \prod_{n\geq 1} \tfrac{\langle n\rangle}{s +_{\widetilde{\mathrm{F}}} \langle n\rangle} \tfrac{\nu_{\widetilde{\mathrm{F}}}(\langle n+1\rangle, s)}{\nu_{\widetilde{\mathrm{F}}}(\langle n\rangle, s)};$$

heuristically, this should be thought of as the (ill-defined) infinite product $\prod_{n\geq 1} \frac{1}{s +_{\widetilde{\mathrm{F}}} \langle n\rangle}$. It can be checked that $\Gamma_{\widetilde{\mathrm{F}}}(s)$ is well-defined and that $\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} 1) = s\Gamma_{\widetilde{\mathrm{F}}}(s)$. Moreover, when F is the additive formal group, this is Euler's famous product expansion for the Γ-function; when

[11] Since it is rather satisfying, let us observe the following neat consistency between algebra and topology: when $k = \mathrm{KU}$, there is an isomorphism $\mathrm{KU}^{h\mathrm{T}} \cong \mathrm{KU}[\![\mathbf{Z}]\!]$ of augmented $\mathbf{E}_\infty$-KU-algebras (see e.g., the reinterpretation of Snaith's theorem in [Lur18, Section 6.5]), so $\mathrm{HP}(\mathrm{KU}/\mathrm{KU}^{h\mathrm{T}}) \cong \mathrm{HP}(\mathrm{KU}/\mathrm{KU}[\![\mathbf{Z}]\!])$. This is reflected algebraically in the observation that there is an isomorphism

$$\mathbf{Z}(\!(q-1)\!)\left[s, \binom{s}{n}_{\widetilde{\mathrm{F}}}\right]_{n\geq 0} \cong \mathbf{Z}(\!(q-1)\!)\left[x, \tfrac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}\right]_{n\geq 0}$$

sending $s \mapsto \frac{x-1}{q-1}$: indeed, it follows that $s -_{\widetilde{\mathrm{F}}} \langle i\rangle = q^{-i}(s - [i]_q) \mapsto q^{-i}\frac{x-q^i}{q-1}$, so $\binom{s}{n}_{\widetilde{\mathrm{F}}} \mapsto q^{-\binom{n}{2}}(q-1)^{-n}\frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}$. Note that since there is no equivalence $\mathrm{E}^{h\mathrm{T}} \cong \mathrm{E}[\![\mathbf{Z}]\!]$ outside of the case E has height 1, one should not expect any relationship between the expressions $(x-1)^n_{\widetilde{\mathrm{F}}}$ and $\binom{s}{n}_{\widetilde{\mathrm{F}}}$ outside of the case $\mathrm{E} = \mathrm{KU}$.

[12] To make sense of this infinite product, one *a priori* needs to invert each $\langle n\rangle$, so that $\nu(\langle n\rangle, s)$ is well-defined; but consecutive terms "cancel" out these factors, so in fact one does not need to invert any $\langle n\rangle$ for the product defining $\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} 1)$ to be well-defined.

F is the multiplicative formal group, upon replacing s by the variable $y = \frac{\log(1+(q-1)s)}{\log(q)}$, the infinite product becomes $\prod_{n\geq 1} \frac{q^n-1}{q^{n+y}-1} \left(\frac{q^{n+1}-1}{q^n-1}\right)^y$, which is Heine's definition of the q-Gamma function $\Gamma_q(y)$.

Example 5.10. Let $a \in \mathbf{G}_m$. The Mellin transform of the δ-sheaf at $x = a$ (i.e., the $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}$-module given by $\bigoplus_{n\geq 0} \pi_0(k)[\![t]\!] \cdot \frac{(\partial_x^k)^n}{\langle -n\rangle\langle -n+1\rangle\cdots\langle -1\rangle}\delta_a$, where $(x-a)\delta_a = 0$) is the quasicoherent sheaf on $\widetilde{\mathrm{F}}/\underline{\mathbf{Z}}$ whose solution is the function $\nu_{\widetilde{\mathrm{F}}}(a, s -_{\widetilde{\mathrm{F}}} 1)$, which satisfies $s\nu(a, s -_{\widetilde{\mathrm{F}}} 1) = \partial_a^{\widetilde{\mathrm{F}}} a\nu_{\widetilde{\mathrm{F}}}(a, s -_{\widetilde{\mathrm{F}}} 1)$. This is an analogue of the classical fact that the Mellin transform of $\delta(x-a)$ is a^{s-1}.

6. An F-analogue of $\mathrm{U}(\mathfrak{gl}_n)$

The sheaf $\mathrm{D}_{\mathbf{P}^1}$ of global differential operators on $\mathbf{P}^1$ (over a commutative ring R) plays an important role in geometric representation theory. For instance, one basic component of Beilinson-Bernstein localization is that $\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{D}_{\mathbf{P}^1})$ is concentrated in degree zero, and furthermore that there is an isomorphism

$$\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{D}_{\mathbf{P}^1}) \cong \mathrm{U}(\mathfrak{sl}_2)/\mathrm{C} \tag{6}$$

where C is the Casimir element. Note that this isomorphism only holds if 2 is a unit in R! Our goal in this section is to explore the analogous story when $\mathrm{D}_{\mathbf{P}^1}$ is replaced by $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1}$.

Recall that k is a connective even $\mathbf{E}_\infty$-ring which admits a Bott class, and that if X is a scheme over $\pi_0(k)$ which admits a lift to a scheme X_k over k (in the sense that the structure sheaf $\mathcal{O}_\mathrm{X}$ admits a lifting to a sheaf of $\mathbf{E}_2$-k-algebras), then one can define the sheaf $\mathrm{D}^{\widetilde{\mathrm{F}}}_\mathrm{X}$ of $\pi_0(k^{t\mathrm{S}^1}) \cong \pi_0(k)[\![t]\!]$-algebras over X. As with any toric variety, the scheme $\mathbf{P}^1$ over $\pi_0(k)$ lifts to a scheme $\mathbf{P}^1_k$ over k (this is the "flat" projective space from [Lur17, Section 5.4]), so that one can define a sheaf $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1}$ over $\mathbf{P}^1$. Motivated by (6), one is led to ask: what is $\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1})$?

First, let us note that $\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1})$ is concentrated in degree zero. This is because $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1}/t$ is isomorphic to the usual sheaf of differential operators $\mathrm{D}_{\mathbf{P}^1}$, whose global sections are in degree zero; and $\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{P}^1})$ is t-complete. So, we are really just studying the ordinary $\pi_0(k)[\![t]\!]$-algebra of global $\widetilde{\mathrm{F}}$-differential operators on $\mathbf{P}^1$.

Let us fix the coordinate z on $\mathbf{P}^1$. On the patches $\mathbf{P}^1 - \{0\} \cong \mathbf{A}^1$ and $\mathbf{P}^1 - \{\infty\} \cong \mathbf{A}^1$ with coordinates z and z^{-1}, one has the operators ∂_z^k and $\partial_{z^{-1}}^k$; so we need to compute the relation between them. If $\overline{x}$ denotes the power series in x given by the inverse of x under the group law $\widetilde{\mathrm{F}}$, then $\overline{z\partial_z^k} = z^{-1}\partial_{z^{-1}}^k$.[13] It follows that $\partial_{z^{-1}}^k = z\overline{z\partial_z^k}$. (When k is a $\mathbf{Z}$-algebra, this is the statement that $\partial_{z^{-1}} = -z^2\partial_z$.) It follows easily from this that

$$\mathrm{R}\Gamma(\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{P}^1; \mathcal{O}) \cong \pi_0(k)[\![t]\!][\partial_z^k, z\partial_z^k, \partial_{z^{-1}}^k];$$

in fact one can also identify $\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{P}^1 = \mathrm{PGL}_2 \times^{\mathrm{B}} \mathrm{Lie}(\widetilde{\mathrm{F}})$, where B acts by the quotient $\mathrm{B} \to \mathbf{G}_m$ and the scaling action of $\mathbf{G}_m$ on $\mathrm{Hom}(\widetilde{\mathrm{F}}^\vee, \mathbf{G}_a) \cong \mathrm{Lie}(\widetilde{\mathrm{F}})$.

[13] Indeed, recall that $(z^{-1}\partial_{z^{-1}}^k)z^{-1} = z^{-1}((z^{-1}\partial_{z^{-1}}^k) +_{\widetilde{\mathrm{F}}} 1)$, so that $z(z^{-1}\partial_{z^{-1}}^k) = ((z^{-1}\partial_{z^{-1}}^k) +_{\widetilde{\mathrm{F}}} 1)z$. This implies the desired relation.

Motivated by the case when k is an ordinary commutative ring, let us write $f = \partial_z^k$, $h = z\partial_z^k$, and $e = \partial_{z^{-1}}^k$. Then one has the following relations in $\mathrm{H}^0(\mathbf{P}^1; \mathrm{D}_{\mathbf{P}^1}^{\widetilde{\mathrm{F}}})$:

$$fh = (h +_{\widetilde{\mathrm{F}}} 1)f, \tag{7}$$

$$eh = (h -_{\widetilde{\mathrm{F}}} 1)e, \tag{8}$$

$$[e, f] = (\overline{h} +_{\widetilde{\mathrm{F}}} 1)h - (h +_{\widetilde{\mathrm{F}}} 1)\overline{h}, \tag{9}$$

$$fe = (h +_{\widetilde{\mathrm{F}}} 1)\overline{h}. \tag{10}$$

To illustrate this, let us verify (10) (which, with the analogous identity for ef, implies (9)):

$$fe = \partial_z^k \partial_{z^{-1}}^k = \partial_z^k (z\overline{z\partial_z^k}) = ((z\partial_z^k) +_{\widetilde{\mathrm{F}}} 1)\overline{z\partial_z^k} = (h +_{\widetilde{\mathrm{F}}} 1)\overline{h}.$$

Motivated by this discussion, we are led to the following:

Definition 6.1. Let $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ denote the $\pi_0(k)[\![t]\!]$-algebra generated by three elements e, f, h subject to the relations (7), (8), and (9). Let $\mathrm{C}_{\widetilde{\mathrm{F}}}$ denote the "$\widetilde{\mathrm{F}}$-Casimir element", defined as $fe - (h +_{\widetilde{\mathrm{F}}} 1)\overline{h} \in \mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$. One can check that $\mathrm{C}_{\widetilde{\mathrm{F}}}$ is central in $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$.

The expression on the right-hand side of (9) appears rather complicated, but in fact (up to units) it is quite simple: it is just a unit multiple of $h +_{\widetilde{\mathrm{F}}} h = [2]_{\widetilde{\mathrm{F}}}(h)$.

The reason for the notation $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ is that when $k = \mathbf{Z}[u]$ (so F is the additive formal group), then $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ is the algebra over $\mathbf{Z}[\![t]\!]$ generated by e, f, h subject to the relations

$$[h, f] = -f, \ [h, e] = e, \ [e, f] = 2h.$$

Note that this is isomorphic to the universal enveloping algebra of $(\mathfrak{pgl}_2)_{\mathbf{Z}}$, base-changed from $\mathbf{Z}$ to $\mathbf{Z}[\![t]\!]$. In particular, upon inverting 2, one can identify $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)[1/2]$ with $\mathrm{U}(\mathfrak{sl}_2)[\![t]\!][1/2]$ where the standard h identifies with our $2h$.

Remark 6.2. It is easy to define $\widetilde{\mathrm{F}}$-analogues of $\mathrm{U}(\mathfrak{pgl}_n)$ and $\mathrm{U}(\mathfrak{gl}_n)$ (hence also of $\mathrm{U}(\mathfrak{sl}_n)$) by studying homogeneous $\widetilde{\mathrm{F}}$-differential operators in more variables. Namely, let $x_1, \cdots, x_n$ be an ordered list of variables, let $\partial_i^k = \partial_{x_i}^k$, and $u(s)$ denote the power series $\frac{(s +_{\widetilde{\mathrm{F}}} 1) - 1}{s}$ (this is a unit in $\pi_0(k)[\![t, s]\!]$). By computing the relations between $h_i := x_i\partial_i^k$, $e_i := x_i\partial_{i+1}^k$, and $f_i := x_{i+1}\partial_i^k$, one is led to define $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)$ as the $\pi_0(k)[\![t]\!]$-algebra generated by elements h_j, e_i, and f_i for $1 \leq j \leq n$ and $1 \leq i \leq n-1$ subject to the following relations:

$$\begin{aligned}
h_i h_j &= h_j h_i, \\
e_i h_j &= (h_j -_{\widetilde{\mathrm{F}}} \langle \alpha_i, \epsilon_j \rangle) e_i, \\
f_i h_j &= (h_j +_{\widetilde{\mathrm{F}}} \langle \alpha_i, \epsilon_j \rangle) f_i, \\
[e_i, f_j] &= \delta_{ij}((h_{i+1} +_{\widetilde{\mathrm{F}}} 1)h_i - (h_i +_{\widetilde{\mathrm{F}}} 1)h_{i+1}), \\
e_i e_j &= e_j e_i \text{ if } |i - j| > 1, \\
f_i f_j &= f_j f_i \text{ if } |i - j| > 1, \\
0 &= e_{j-1}e_j^2 - e_j e_{j-1} e_j (u(h_j +_{\widetilde{\mathrm{F}}} 1) + 1) + e_j^2 e_{j-1} u(h_j), \\
0 &= e_{j-1}^2 e_j - e_{j-1} e_j e_{j-1} (u(h_j) + 1) + e_j e_{j-1}^2 u(h_j -_{\widetilde{\mathrm{F}}} 1), \\
0 &= f_j^2 f_{j-1} - f_j f_{j-1} f_j (u(h_j) + 1) + f_{j-1} f_j^2 u(h_j -_{\widetilde{\mathrm{F}}} 1), \\
0 &= f_j f_{j-1}^2 - f_{j-1} f_j f_{j-1} (u(h_j +_{\widetilde{\mathrm{F}}} 1) + 1) + f_{j-1}^2 f_j u(h_j).
\end{aligned}$$

Here, we are using standard notation for the roots α_i of GL_n. Note that the Serre relations for a general formal group F are rather complicated, since $u(s)$ generally depends on s. However, when F is the additive formal group, $u(s) = 1$; and when F is the multiplicative formal group,

$u(s) = q$; so the terms $u(h)+1$ appearing in the Serre relations specialize to 2 and $q+1 = [2]_q$, respectively, hence recovering the usual Serre relations and a mild modification of the q-Serre relations, respectively. It can be shown that $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)$ satisfies the PBW theorem. Motivated by the above formulas, one can also construct "by hand" an associative $\pi_0(k)[\![t]\!]$-algebra $\mathrm{U}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})$ associated to a simply-laced[14] root datum, but I do not know if this is the "right" object, in that it is rather *ad hoc* so I do not know how/whether it is related to the theory of $\widetilde{\mathrm{F}}$-differential operators.

One can also define "Lusztig"/"divided power" variants of $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)$, by equipping each h_i with "$\widetilde{\mathrm{F}}$-binomial coefficients" $\binom{h_i}{n}_{\widetilde{\mathrm{F}}}$ as in Remark 5.8, and each e_i and f_i with "$\widetilde{\mathrm{F}}$-divided powers" $\frac{e_i^n}{\langle n\rangle!}$ and $\frac{f_i^n}{\langle n\rangle!}$. Using Proposition 4.10, one can also construct a quantum Frobenius $\mathrm{U}^{\mathrm{pd}}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)/\langle p\rangle \to \mathrm{U}^{\mathrm{pd}}(\mathfrak{gl}_n)$. (There are many other classical statements about $\mathrm{U}(\mathfrak{gl}_n)$ which admit pretty generalizations to $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{GL}_n)$, like Gelfand-Tsetlin theory; but I will not discuss them here.)

Example 6.3. Let $k = \mathrm{ku}$, so that $\widetilde{\mathrm{F}}$ is the group law $x+y+(q-1)xy$ over $\pi_0(k^{t\mathrm{S}^1}) \cong \mathbf{Z}[\![q-1]\!]$. Then the relations defining $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ become

$$fh = qhf + f,\ he = qeh + e,\ [e,f] = \tfrac{h(2+(q-1)h)}{1+(q-1)h}.$$

Note that if $\mathrm{K} = 1 + (q-1)h$, then these relations can in turn be stated as

$$\mathrm{K}f = q^{-1}f\mathrm{K},\ \mathrm{K}e = qe\mathrm{K},\ [e,f] = \tfrac{\mathrm{K}-\mathrm{K}^{-1}}{q-1},$$

and the Casimir element is

$$\mathrm{C}_{\widetilde{\mathrm{F}}} = ef + \frac{\mathrm{K} + q\mathrm{K}^{-1} - (q-1)}{(q-1)^2}.$$

In other words, $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ is essentially the quantum enveloping algebra of PGL_2, up to the issue of replacing $q-1$ by $q-q^{-1}$.

Remark 6.4. Unfortunately, outside of the case $k = \mathbf{Z}[u]$ and $k = \mathrm{ku}$ (corresponding to the additive and multiplicative formal groups, respectively), it is not clear to me whether $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ admits the structure of a Hopf algebra, i.e., if there is a compatible coproduct. This coproduct, if it exists, would satisfy $\Delta(h) = (h\otimes 1) +_{\widetilde{\mathrm{F}}} (1\otimes h)$.

Once $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ is defined in this way, it is not hard to adapt the argument for Beilinson-Bernstein localization to show (an analogous result holds for $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_n)$, as well as for monodromically twisted $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules as in (5)):

Proposition 6.5. *The functor of global sections defines an equivalence*

$$\mathrm{R}\Gamma(\mathbf{P}^1; -) : \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{P}^1) \xrightarrow{\cong} \mathrm{LMod}_{\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)/\mathrm{C}_{\widetilde{\mathrm{F}}}}.$$

Let us illustrate several examples of Proposition 6.5.

- Let $\mathrm{L}(s) = \mathcal{O}_{\mathbf{P}^1}[1]$. Then $\mathrm{R}\Gamma(\mathbf{P}^1; \mathrm{L}(s)) \cong \pi_0(k^{t\mathrm{S}^1})[1]$. This is the trivial $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$-module.
- Let $i : \{\infty\} \hookrightarrow \mathbf{P}^1$ denote the inclusion, and let $\Delta_0 = i_* i^* \mathcal{O}_{\mathbf{P}^1}$. This has no sections over $\mathbf{A}^1_\infty = \mathbf{P}^1 - \infty$, and over $\mathbf{A}^1_0 = \mathbf{P}^1 - 0$ its sections can be identified with

$$(\pi_0(k^{t\mathrm{S}^1})[x^{\pm 1}]/\pi_0(k^{t\mathrm{S}^1})[x])[1]\{1\} \cong \pi_0(k^{t\mathrm{S}^1})\langle\delta\rangle_{\widetilde{\mathrm{F}}}[1]\{1\}.$$

[14] In the non simply-laced case, I am not sure what the appropriate replacement of $u(h)$ should be.

Here, $\pi_0(k^{t\mathrm{S}^1})\langle\delta\rangle_{\widetilde{\mathrm{F}}}$ is the $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{A}^1_0}$-module which is free on $\frac{(\partial_x^k)^n}{\langle -n\rangle\langle -n+1\rangle\cdots\langle -1\rangle}\delta$, where $x = z^{-1}$, and δ is the δ-function at $\infty \in \mathbf{A}^1_0$ (so $x\delta = 0$). It is easy to compute the action of e, f, and h; this is depicted in the following:

$$\begin{tikzcd} \cdots \arrow[r, bend right, "\langle 4\rangle"'] & v_2 \arrow[l, bend right, "\langle -3\rangle"'] \arrow[loop above, "\langle 3\rangle"] \arrow[r, bend right, "\langle 3\rangle"'] & v_1 \arrow[l, bend right, "\langle -2\rangle"'] \arrow[loop above, "\langle 2\rangle"] \arrow[r, bend right, "\langle 2\rangle"'] & v_0 \arrow[l, bend right, "\langle -1\rangle"'] \arrow[loop above, "1"] \end{tikzcd}$$

The element e acts by leftward arrows; f acts by rightward arrows; and h acts by loops. We will call this $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$-module M_{-1}, so that $\mathrm{R}\Gamma(\mathbf{P}^1;\Delta_0) \cong \mathrm{M}_{-1}$.

- Let Δ_s denote $j_!\mathcal{O}_{\mathbf{A}^1}[1]$; this is a version of the Heaviside step function/distribution (see Example 3.12). To compute $\mathrm{R}\Gamma(\mathbf{P}^1;\Delta_s)$, observe that the $\mathcal{D}^{\mathrm{F}}_{\mathbf{P}^1}$-module corresponding to Δ_s has sections over $\mathbf{A}^1_0$ given by $\pi_0(k^{t\mathrm{S}^1})[z^{\pm 1}]$, and over $\mathbf{A}^1_\infty$ given by $\pi_0(k^{t\mathrm{S}^1})[z]$. It follows that $\mathrm{R}\Gamma(\mathbf{P}^1;\Delta_s) \cong \pi_0(k^{t\mathrm{S}^1})[z] = \bigoplus_{n\geq 0}\pi_0(k^{t\mathrm{S}^1})w_n$ where w_n is represented by z^n. The actions of e, f, and h are easily computed, and can be depicted as follows:

$$\begin{tikzcd} \cdots \arrow[r, bend right, "\langle 4\rangle"'] & w_3 \arrow[l, bend right, "\langle -3\rangle"'] \arrow[loop above, "\langle 3\rangle"] \arrow[r, bend right, "\langle 3\rangle"'] & w_2 \arrow[l, bend right, "\langle -2\rangle"'] \arrow[loop above, "\langle 2\rangle"] \arrow[r, bend right, "\langle 2\rangle"'] & w_1 \arrow[l, bend right, "\langle -1\rangle"'] \arrow[loop above, "1"] \arrow[r, bend right, "1"'] & w_0 \arrow[l, bend right, "0"'] \arrow[loop above, "0"] \end{tikzcd}$$

We will call this $\mathrm{U}_{\mathrm{F}}(\mathrm{PGL}_2)$-module M_0, so that $\mathrm{R}\Gamma(\mathbf{P}^1;\Delta_s) \cong \mathrm{M}_0$. Note that there is a map $\iota : \mathrm{M}_{-1} \hookrightarrow \mathrm{M}_0$ which sends $\iota(v_n) = w_{n+1}$. The cofiber of this map is just the $\pi_0(k^{t\mathrm{S}^1})$-module generated by w_0; so we obtain a cofiber sequence

$$\mathrm{M}_{-1} \xrightarrow{\iota} \mathrm{M}_0 \to \pi_0(k^{t\mathrm{S}^1}) \cdot w_0,$$

which is in fact a short exact sequence of $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$-modules. This is the cofiber sequence given by applying $\mathrm{R}\Gamma(\mathbf{P}^1; q^*-)$ to the cofiber sequence

$$\Delta_0 \to \Delta_s \to \mathrm{L}(s).$$

Although interesting by itself, my motivation for defining $\mathrm{U}_{\widetilde{\mathrm{F}}}(\mathrm{PGL}_2)$ was its relation to Langlands duality. To explain this, I need to briefly review some background on *Koszul duality* à la Beilinson-Ginzburg-Soergel [BGS96]. It relates categories $\mathcal{O}$ for a complex reductive group G and its dual group $\check{\mathrm{G}}$ (also defined over $\mathbf{C}$). Nowadays [BY13], this duality is often phrased geometrically as an equivalence between *mixed*/graded versions of the categories $\mathrm{Shv}_{\mathrm{B}\text{-cbl}}(\mathrm{G}/\mathrm{B};\mathbf{Q})$ and $\mathrm{Shv}_{\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}}/\check{\mathrm{B}};\mathbf{Q})$ of constructible sheaves of $\mathbf{Q}$-vector spaces on the flag varieties for G and $\check{\mathrm{G}}$. This equivalence exchanges standard and costandard sheaves, and IC-sheaves and tiltings, and also interleaves the Tate twist and homological shift.

The two sides change somewhat if one instead considers the category $\mathrm{Shv}_{\mathrm{B}}(\mathrm{G}/\mathrm{B};\mathbf{Q})$ of B-*equivariant* sheaves on G/B, i.e., the category of $\mathrm{B}\times\mathrm{B}$-equivariant sheaves on G. Then, the dual side gets modified to a completion $\widehat{\mathrm{Shv}}_{\check{\mathrm{B}}\times\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}};\mathbf{Q})$ of the category of $\check{\mathrm{B}}\times\check{\mathrm{B}}$-constructible sheaves on $\check{\mathrm{G}}$ which have unipotent monodromy along the fibers of the map $\check{\mathrm{U}}\backslash\check{\mathrm{G}}/\check{\mathrm{U}} \to \check{\mathrm{B}}\backslash\check{\mathrm{G}}/\check{\mathrm{B}}$. In other words, there is an equivalence

$$\mathrm{Shv}^{\mathrm{mixed}}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};\mathbf{Q}) \simeq \widehat{\mathrm{Shv}}^{\mathrm{mixed}}_{\check{\mathrm{B}}\times\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}};\mathbf{Q}).$$

See [BY13] for a proof. This equivalence is furthermore monoidal for the convolution monoidal structures on both sides. Already when G is a torus T, this equivalence involves making a choice which has no analogue if $\mathbf{Q}$ is replaced by a general commutative ring k (we will soon allow k to be a commutative ring spectrum too). In this case,

$$\mathrm{Shv}^{\mathrm{mixed}}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k) \simeq \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{H}^{2*}(\mathrm{BT};k)} = \mathrm{QCoh}^{\mathrm{gr}}(\widehat{\mathfrak{t}}_k(1)),$$
$$\mathrm{Shv}_{\check{\mathrm{B}}\times\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}};k) \simeq \mathrm{Mod}_{k[\pi_1\check{\mathrm{T}}]} = \mathrm{QCoh}(\mathrm{T}_k).$$

To identify $\mathrm{Shv}^{\mathrm{mixed}}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k)$ with $\widehat{\mathrm{Shv}}_{\check{\mathrm{B}}\times\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}};k)$ as a monoidal category, we therefore need to fix an isomorphism $\widehat{\mathrm{T}}_k \cong \widehat{\mathfrak{t}}_k$ of formal groups, i.e., an isomorphism $\widehat{\mathbf{G}}_m \cong \widehat{\mathbf{G}}_a$. Such an isomorphism is only possible if k is a $\mathbf{Q}$-algebra. But this calculation indicates a fix: if we replace $\mathrm{Shv}_{\check{\mathrm{B}}\times\check{\mathrm{B}}\text{-cbl}}(\check{\mathrm{G}};k)$ by $\mathrm{DMod}(\check{\mathrm{G}}_k)^{\check{\mathrm{B}}_k\times\check{\mathrm{B}}_k\text{-mon}}$, then it would still be true that $\mathrm{DMod}(\check{\mathrm{T}}_k)^{\check{\mathrm{T}}_k\times\check{\mathrm{T}}_k\text{-mon}} \simeq \mathrm{QCoh}(\mathfrak{t}_k)$. If k is a general commutative ring, one is therefore led to conjecture that there is an equivalence between "mixed" versions of $\mathrm{Shv}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k)$ and $\mathrm{DMod}(\check{\mathrm{G}}_k)^{\check{\mathrm{B}}_k\times\check{\mathrm{B}}_k\text{-mon}}$, and similarly between "mixed" versions of $\mathrm{Shv}_{\mathrm{B}\text{-cbl}}(\mathrm{G}/\mathrm{B};k)$ and $\mathrm{DMod}(\check{\mathrm{G}}_k/\check{\mathrm{B}}_k)^{\check{\mathrm{B}}_k\text{-mon}}$, which will in turn be subcategories of the torus-monodromic categories $\mathrm{DMod}(\check{\mathrm{G}}_k)^{\check{\mathrm{T}}_k\times\check{\mathrm{T}}_k\text{-mon}}$ and $\mathrm{DMod}(\check{\mathrm{G}}_k/\check{\mathrm{B}}_k)^{\check{\mathrm{T}}_k\text{-mon}}$.

What if k is allowed to be a ring spectrum? It turns out that it is much more natural to replace k by its S^1-Tate construction $k^{t\mathrm{S}^1}$ (this is because Koszul duality can be viewed as an $\mathrm{S}^1_{\mathrm{rot}}$-equivariant localization of Bezrukavnikov's equivalence [Bez16]). As usual, we will assume that k is connective, even, and admits a Bott class. Then, when G is a torus, $\mathrm{Shv}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k^{t\mathrm{S}^1}) = \mathrm{Mod}^{\wedge}_{(k^{t\mathrm{S}^1})^{h\mathrm{T}}}$ (see, e.g., [MNN17]), and this admits a 1-parameter degeneration (given by $\mathrm{Mod}^{\wedge,\mathrm{fil}}_{\tau_{\geq 2\star}(k^{t\mathrm{S}^1})^{h\mathrm{T}}}$) into $\mathrm{Mod}^{\wedge,\mathrm{gr}}_{\pi_{2*}(k^{t\mathrm{S}^1})^{h\mathrm{T}}}$. By the 2-periodicity of $k^{t\mathrm{S}^1}$, this category is in turn equivalent to $\mathrm{Mod}^{\wedge}_{\pi_0(k^{t\mathrm{S}^1})^{h\mathrm{T}}}$. If $\widetilde{\mathrm{F}}$ denotes the algebraic group over $\mathrm{Spf}(\pi_0(k^{t\mathrm{S}^1}))$ from Construction 3.2, and $\mathrm{T}_{\widetilde{\mathrm{F}}} = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}),\widetilde{\mathrm{F}})$, then there is an equivalence $\mathrm{Mod}^{\wedge}_{\pi_0(k^{t\mathrm{S}^1})^{h\mathrm{T}}} \simeq \mathrm{QCoh}(\mathrm{T}_{\widetilde{\mathrm{F}}})$. By Cartier duality, this is in turn $\mathrm{QCoh}(\mathrm{B}\check{\mathrm{T}}_{\widetilde{\mathrm{F}}^\vee})$, where $\check{\mathrm{T}}_{\widetilde{\mathrm{F}}^\vee} = \mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}),\widetilde{\mathrm{F}}^\vee)$. Proposition 4.8 in turn identifies this with $\mathrm{QCoh}(\check{\mathrm{T}}^{\widetilde{\mathrm{F}}\mathrm{dR}}/\check{\mathrm{T}}) =: \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{T}})^{\check{\mathrm{T}}\text{-mon}}$. This, in turn, is a *full subcategory* (which I will denote by $\check{\mathcal{O}}^{\widetilde{\mathrm{F}}}$) of $\mathrm{QCoh}(\check{\mathrm{T}}\backslash\check{\mathrm{T}}^{\widetilde{\mathrm{F}}\mathrm{dR}}/\check{\mathrm{T}}) =: \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{T}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}\text{-mon}}$, consisting of those objects supported in weight 0 for the action of one of the (left, say) copies of $\check{\mathrm{T}}$.[15] In other words, when G is a torus, there is a 1-parameter degeneration

$$\mathrm{Shv}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k^{t\mathrm{S}^1}) \rightsquigarrow \check{\mathcal{O}}^{\widetilde{\mathrm{F}}} \subseteq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}\text{-mon}},$$

which categorifies the degeneration of $\mathrm{C}^*_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k^{t\mathrm{S}^1})$ to $\mathrm{H}^*_{\mathrm{B}\times\mathrm{B}}(\mathrm{G};k^{t\mathrm{S}^1})$.

Motivated by this, it is natural to ask whether this can be extended to all connected reductive groups G. I would certainly like to believe this is true, but at this moment, I do not know how to define $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}\text{-mon}}$ or $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}}/\check{\mathrm{B}})^{\check{\mathrm{T}}\text{-mon}}$ in general. However, the latter *does* make sense if G has semisimple rank 1, and can be identified (by replacing $\check{\mathrm{T}}$ by its maximal quotient which acts faithfully on $\check{\mathrm{G}}/\check{\mathrm{B}} = \mathbf{P}^1$) with $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{P}^1)^{\mathbf{G}_m\text{-mon}}$. Then:

Theorem 6.6. *There is a 1-parameter degeneration*

$$\mathrm{Shv}_{\mathrm{B}\text{-cbl}}(\mathbf{C}\mathrm{P}^1;k^{t\mathrm{S}^1}) \rightsquigarrow \check{\mathcal{O}}^{\widetilde{\mathrm{F}}} \subseteq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{P}^1)^{\mathbf{G}_m\text{-mon}},$$

where $\check{\mathcal{O}}^{\widetilde{\mathrm{F}}}$ denotes the full subcategory of $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathbf{P}^1)^{\mathbf{G}_m\text{-mon}}$ compactly generated by the δ-$\mathrm{D}_{\widetilde{\mathrm{F}}}$-module ∇_0 at $\infty \in \mathbf{P}^1$ and the structure sheaf $\mathrm{L}(s)[-1] := \mathcal{O}_{\mathbf{P}^1}$. Moreover, this degeneration

[15]This is one of the major differences between D-modules and constructible sheaves: if $\mathrm{T}' \to \mathrm{T}$ is a homomorphism and T acts on X, then $\mathrm{Shv}_{\mathrm{T}'\text{-cbl}}(\mathrm{X};k) \simeq \mathrm{Shv}_{\mathrm{T}\text{-cbl}}(\mathrm{X};k)$; but $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})^{\mathrm{T}\text{-mon}} \not\simeq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})^{\mathrm{T}'\text{-mon}}$. However, if $\mathrm{T}' \to \mathrm{T}$ is surjective, then there is a fully faithful functor $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})^{\mathrm{T}\text{-mon}} \hookrightarrow \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})^{\mathrm{T}'\text{-mon}}$ consisting of those $\mathrm{D}_{\widetilde{\mathrm{F}}}$-modules with T'-action for which $\ker(\mathrm{T}' \to \mathrm{T})$ acts trivially.

sends

$$\delta_\infty \rightsquigarrow \nabla_0\{-1\},\ \underline{k}^{t\mathrm{S}^1}_{\mathbf{C}\mathrm{P}^1} \rightsquigarrow \Xi_s,\ j_!\underline{k}^{t\mathrm{S}^1}_{\mathbf{C}} \rightsquigarrow \Delta_s,\ j_*\underline{k}^{t\mathrm{S}^1}_{\mathbf{C}} \rightsquigarrow \nabla_s,\ [2] \rightsquigarrow \{-1\}.$$

Here, Ξ_s is a "tilting $\mathrm{D}_{\widetilde{\mathrm{F}}}$-module", whose restriction to $\mathbf{A}^1 \cong \mathbf{P}^1 - \{0\} \subseteq \mathbf{P}^1$ was described in Example 3.13.

When F is the additive formal group, $\mathrm{DMod}_{\widetilde{\mathrm{F}}}$ is just the usual category of D-modules; and when F is the multiplicative formal group, Remark 4.3 tells us that (at least upon p-completion for $p > 2$) $\mathrm{DMod}_{\widetilde{\mathrm{F}}}$ is the category of (q-)prismatic crystals. In these two cases, one *can* therefore make sense of the categories $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}\text{-mon}}$ and $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}}/\check{\mathrm{B}})^{\check{\mathrm{T}}\text{-mon}}$. In particular, when $k = \mathbf{Z}[u]$ or ku, one can ask if for a general connected complex reductive group G, the category $\mathrm{Shv}_{\mathrm{B}\times\mathrm{B}}(\mathrm{G}; k^{t\mathrm{S}^1})$ admits a 1-parameter degeneration to a certain subcategory of $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}\text{-mon}}$; this is work-in-progress.

7. Semi-infinite cohomology

If T is a torus, Theorem 3.1 shows that $\pi_0(k[\Omega\mathrm{T}]^{h\mathrm{T}})^{t\mathrm{S}^1} \cong \pi_0(k[\mathcal{L}\mathrm{T}]^{h(\mathrm{T}\times\mathrm{T})})^{t\mathrm{S}^1}$ is isomorphic to the $\widetilde{\mathrm{F}}$-Weyl algebra $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$ of the Langlands dual torus. One can construct many interesting examples of $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules over $\check{\mathrm{T}}$ through $\mathcal{L}$T-actions on the free loop spaces of various T-spaces. In the literature, such actions of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$ are often known as "shift operators" or "γ-sheaves (on tori)" [BK03]. Most of this section is primarily a straightforward adaptation of works of Givental and Iritani [Giv17, Giv95, Iri25, Iri20], from which I learned a lot. In particular, although we never talk about invariants like symplectic cohomology below, our discussion could certainly be couched in the language of Floer homotopy theory (but I will not do so, for lack of knowledge of this subject).

The simplest way to construct these $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-modules is via *semi-infinite cohomology*, whose construction we will now briefly sketch following [Ras17]; roughly speaking, it is cohomology in the pro-direction and homology in the ind-direction. In the discussion below, k will be an arbitrary $\mathbf{E}_\infty$-ring (one can just take it to be an $\mathbf{E}_2$-ring).

Construction 7.1. Suppose X is a filtered colimit $\mathrm{colim}_\lambda \mathrm{X}^\lambda$ of pro-locally compact Hausdorff spaces $\mathrm{X}^\lambda = \lim_\alpha \mathrm{X}^\lambda_\alpha$ (one can also allow topological stacks). Assume that the transition maps $f^\lambda_{\alpha\beta} : \mathrm{X}^\lambda_\alpha \to \mathrm{X}^\lambda_\beta$ are locally trivial fibrations whose fibers are affine spaces, and that the transition maps $\mathrm{X}^\lambda \to \mathrm{X}^\mu$ are closed embeddings which are pulled back from a finite stage, i.e., are pulled back from $\mathrm{X}^\lambda_\alpha \to \mathrm{X}^\mu_\alpha$ for some α. Such a presentation of X will be called *placid*.

For each λ, an assignment $\alpha \mapsto \mathcal{L}^\lambda_\alpha \in \mathrm{Shv}(\mathrm{X}^\lambda_\alpha; k)$ of $\otimes$-invertible objects will be called a *local dimension theory* if there are compatible isomorphisms $\mathcal{L}_\alpha \cong f^{\lambda,!}_{\alpha\beta}(\mathcal{L}_\beta)$. If $\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda_\alpha} := \omega_{\mathrm{X}^\lambda_\alpha} \otimes \mathcal{L}_\alpha^{-1}$, then $f^{\lambda,*}_{\alpha\beta}(\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda_\beta}) \cong \omega^{\mathrm{ren}}_{\mathrm{X}^\lambda_\alpha}$, so one can define $\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda}$ to be the $*$-pullback along the natural map $\mathrm{X}^\lambda \to \mathrm{X}^\lambda_\alpha$ of $\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda_\alpha}$ for any α. This should be thought of as cohomology in the pro-direction: if $\mathrm{X}^\lambda_\alpha$ is a smooth manifold, then one can take $\mathcal{L}_\alpha = \omega_{\mathrm{X}^\lambda_\alpha}$, and then $\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda}$ is just the constant sheaf on X^λ.

Now let $\lambda \leq \mu$, let $i : \mathrm{X}^\lambda \hookrightarrow \mathrm{X}^\mu$, and suppose one has local dimension theories $\mathcal{L}^\lambda$ and $\mathcal{L}^\mu$ on X^λ and X^μ, respectively. An assignment $\lambda \mapsto \tau^\lambda \in \mathrm{Shv}(\mathrm{X}^\lambda; k)$ (often invertible) will be called an *$\mathcal{L}$-compatible ind-dimension theory* if one has compatible isomorphisms $\tau^\lambda \otimes (\mathcal{L}^\lambda)^{-1} \cong i^*(\tau^\mu \otimes (\mathcal{L}^\mu)^{-1})$. One can then check that $i^!(\omega^{\mathrm{ren}}_{\mathrm{X}^\mu} \otimes \tau^\mu) \cong \omega^{\mathrm{ren}}_{\mathrm{X}^\lambda} \otimes \tau^\lambda$, so there are maps $i_!(\omega^{\mathrm{ren}}_{\mathrm{X}^\lambda} \otimes \tau^\lambda) \to \omega^{\mathrm{ren}}_{\mathrm{X}^\mu} \otimes \tau^\mu$. The colimit over λ of these transition maps defines an object $\omega^{\mathrm{ren}}_{\mathrm{X}} \in \mathrm{Shv}(\mathrm{X}; k)$[16]. This should be thought of as homology in the ind-direction: if each X^λ is

[16]This should really be understood as the $*$-variant of the category of sheaves, as described in the D-module setting in [Ras17].

already locally compact Hausdorff, one can take $\tau^\lambda = \mathcal{L}^\lambda$, and then $\omega_{\mathrm{X}}^{\mathrm{ren}}$ is the colimit along the natural maps $i_!(\omega_{\mathrm{X}^\lambda}) \to \omega_{\mathrm{X}^\mu}$.

The choices of $\mathcal{L}$ and τ are "semi-infinite" choices, analogous to semi-infinite indices between critical points in Floer homotopy theory.

We now discuss a few examples. All of them are topological quotient stacks of the form $\mathrm{X}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}$, where $\mathrm{H}_{\mathcal{O}} := \mathrm{H}(\mathbf{C}[\![t]\!])$ for a connected complex reductive group H, and $\mathrm{X}_{\mathcal{K}} = \mathrm{X}(\mathbf{C}(\!(t)\!))$ for a smooth affine H-space X. They will all have natural choices of ind-dimension theories, but I will not specify it in every example. Below, we will equip $\mathbf{C}\cdot t$ with the weight -1 action of $\mathrm{S}^1_{\mathrm{rot}}$.

Example 7.2. Let G be a connected complex reductive group. Presenting $\mathrm{G}_{\mathcal{K}}$ as the colimit of the preimages $\mathrm{G}_{\mathcal{K}}^{\leq\lambda}$ of the Schubert closures $\mathrm{Gr}_{\mathrm{G}}^{\leq\lambda} \subseteq \mathrm{Gr}_{\mathrm{G}}$ under the projection $\mathrm{G}_{\mathcal{K}} \to \mathrm{Gr}_{\mathrm{G}}$, and in turn viewing each $\mathrm{G}_{\mathcal{K}}^{\leq\lambda}$ as $\lim_\alpha \mathrm{G}_{\mathcal{K}}^{\leq\lambda}/\ker(\mathrm{G}_{\mathcal{O}} \to \mathrm{G}(\mathbf{C}[\![t]\!]/t^\alpha))$, one finds that $\mathrm{G}_{\mathcal{K}}$ (and in fact the topological stack $\mathrm{G}_{\mathcal{O}}\backslash\mathrm{G}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}$) is placid.

This example essentially reduces to equipping $\mathrm{Gr}_{\mathrm{G}} = \mathrm{G}_{\mathcal{O}}\backslash\mathrm{G}_{\mathcal{K}}$ (and in fact the topological stack $\mathrm{Gr}_{\mathrm{G}}/\mathrm{G}_{\mathcal{O}}$) with a placid presentation. This is easy, since $\mathrm{Gr}_{\mathrm{G}} = \mathrm{colim}_\lambda \mathrm{Gr}_{\mathrm{G}}^{\leq\lambda}$. There is a local dimension theory given by $\mathcal{L}^\lambda = \underline{k}_{\mathrm{Gr}_{\mathrm{G}}^{\leq\lambda}}$ (and the $\mathcal{L}$-compatible dimension theory τ also just consists of constant sheaves). Then

$$\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{Gr}_{\mathrm{G}};\omega^{\mathrm{ren}}) \cong k[\mathcal{L}\mathrm{G}]^{h(\mathrm{G}\times\mathrm{G})} \cong k[\Omega\mathrm{G}]^{h\mathrm{G}},$$
$$\mathrm{R}\Gamma_{(\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}})\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{G}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};\omega^{\mathrm{ren}}) \cong k[\mathcal{L}\mathrm{G}]^{h(\mathrm{G}\times\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})} \cong k[\Omega\mathrm{G}]^{h\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}.$$

One can check that, in fact, $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};\omega^{\mathrm{ren}})$ is an $\mathbf{E}_{3,\mathrm{BS}^1}$-$k$-algebra, and the identification with $k[\mathcal{L}\mathrm{G}]^{h(\mathrm{G}\times\mathrm{G})} \cong \mathfrak{Z}_{\mathbf{E}_2}(k[\Omega\mathrm{G}]/k)$ is one of $\mathbf{E}_{3,\mathrm{BS}^1}$-$k$-algebras. The discussion surrounding (2) implies that $\mathrm{Spev}(\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};\omega^{\mathrm{ren}}))$ is a Poisson $\mathrm{Spev}(k)$-scheme; in fact one can check that it is a *symplectic* $\mathrm{Spev}(k)$-scheme, but this is not a formal consequence of the $\mathbf{E}_{3,\mathrm{BS}^1}$-$k$-algebra structure.

Example 7.3. Let V be a complex G-representation; then $\mathrm{V}_{\mathcal{K}} = \mathrm{colim}_n\, t^{-n}\mathrm{V}_{\mathcal{O}}$. Since $t^{-n}\mathrm{V}_{\mathcal{O}} = \lim_j t^{-n}\mathrm{V}_{\mathcal{O}}/t^j$, this provides a placid presentation of $\mathrm{V}_{\mathcal{K}}$. Also, $\mathcal{L}_j^n = \omega_{t^{-n}\mathrm{V}_{\mathcal{O}}/t^j}$ is a local dimension theory, and if $f^n : t^{-n}\mathrm{V}_{\mathcal{O}} \to t^{-n}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}$ is the canonical map, then $\tau^n = (f^n)^*\omega_{t^{-n}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}}$ is an $\mathcal{L}$-compatible ind-dimension theory. It follows that

$$\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{V}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong \mathrm{colim}\left(k^{h\mathrm{G}} \xrightarrow{a_{\mathrm{V}}} (\Sigma^{\mathrm{V}}k)^{h\mathrm{G}} \xrightarrow{a_{\mathrm{V}}} (\Sigma^{2\mathrm{V}}k)^{h\mathrm{G}} \to \cdots\right);$$

here $\Sigma^{n\mathrm{V}}k = k \otimes \mathrm{S}^{\mathrm{V}^{\oplus n}}$, where $\mathrm{S}^{\mathrm{V}^{\oplus n}}$ is the one-point compactification of the n-fold direct sum $\mathrm{V}^{\oplus n}$, and the map a_{V} is induced by the inclusion $\mathrm{S}^0 \to \mathrm{S}^{\mathrm{V}}$. If k is (G,V)-orientable, i.e., the map $\mathrm{BG} \xrightarrow{\mathrm{V}} \mathrm{BGL}(\mathrm{V}) \simeq \mathrm{BO}(\dim(\mathrm{V})) \xrightarrow{\mathrm{J}} \mathrm{Pic}(k)$ is null, then V admits an Euler class $e_{\mathrm{V}} \in \mathrm{H}^{\dim(\mathrm{V})}(\mathrm{BG};k)$, and $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{V}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong k^{h\mathrm{G}}[e_{\mathrm{V}}^{-1}]$. This also admits an $\mathrm{S}^1_{\mathrm{rot}}$-equivariant analogue: $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{V}_{\mathcal{K}};\omega^{\mathrm{ren}})$ can be identified with

$$\mathrm{colim}\left(k^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})} \xrightarrow{a_{t^{-1}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}}} (\Sigma^{t^{-1}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}}k)^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})} \xrightarrow{a_{t^{-2}\mathrm{V}_{\mathcal{O}}/t^{-1}\mathrm{V}_{\mathcal{O}}}} (\Sigma^{t^{-2}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}}k)^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})} \to \cdots\right).$$

If k is $(\mathrm{G}\times\mathrm{S}^1,\mathrm{V})$-orientable (equivalently, separately (G,V)-orientable and complex orientable; the latter is guaranteed by our assumptions on k, but may not hold in general!), then the map $a_{t^{-n}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}} : k^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})} \to (\Sigma^{t^{-n}\mathrm{V}_{\mathcal{O}}/\mathrm{V}_{\mathcal{O}}}k)^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})}$ detects a class formally denoted $\Gamma_{\mathrm{G}}^{(n)}(\mathrm{V};\mathrm{F})^{-1} \in \mathrm{H}^{n\dim(\mathrm{V})}(\mathrm{B}(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}});k)$. Here, $\Gamma_{\mathrm{G}}^{(n)}(\mathrm{V};\mathrm{F})^{-1}$ is the product over the Chern roots $\mathcal{L}_i$ of V of the classes $\Gamma_{\mathrm{T}}^{(n)}(\mathcal{L}_i;\mathrm{F})^{-1}$, so it is specified by the case $\mathrm{V} = \mathbf{C}$ with the weight 1 action of $\mathrm{G} = \mathbf{G}_m$; if $\pi_* k^{h(\mathrm{G}\times \mathrm{S}^1_{\mathrm{rot}})} \cong \pi_*(k)[y,\hbar]^\wedge$, then an easy computation shows that

$$\Gamma^{(n)}_{\mathbf{G}_m}(\mathbf{C};\mathrm{F})^{-1} = \prod_{i=1}^n (y +_{\mathrm{F}} [i](\hbar)).$$

In particular, if we pass to $\pi_*(k^{h\mathrm{G}})^{t\mathrm{S}^1_{\mathrm{rot}}}$ and set $s = y\hbar^{-1}$, then inverting $\Gamma^{(n)}_{\mathbf{G}_m}(\mathbf{C};\mathrm{F})^{-1}$ is the same as inverting

$$\Gamma^{(n)}_{\mathbf{G}_m}(\mathbf{C};\widetilde{\mathrm{F}})^{-1} := \hbar^{-n}\Gamma^{(n)}_{\mathbf{G}_m}(\mathbf{C};\mathrm{F}) = \prod_{i=1}^n (s +_{\widetilde{\mathrm{F}}} \langle i\rangle).$$

As $n \to \infty$, the elements $\Gamma^{(n)}_{\mathbf{G}_m}(\mathbf{C};\widetilde{\mathrm{F}})^{-1}$ should be understood as converging to an element $\Gamma_{\mathbf{G}_m}(\mathbf{C};\widetilde{\mathrm{F}})^{-1}$ which is the (multiplicative) inverse of $\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} 1)$ from Example 5.9. In other words, $\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} 1)$ should be viewed as the multiplicative inverse of the Euler class of the normal bundle of $\mathrm{V}_{\mathcal{O}} \subseteq \mathrm{V}_{\mathcal{K}}$ (which has infinite codimension!).

Remark 7.4. Armed with the perspective of Example 7.3, one can (re)prove several identities about the $\Gamma_{\widetilde{\mathrm{F}}}$-function. For instance, if we p-complete k away from $p = 2$, so as to make $\widetilde{\mathrm{F}}$ into a formal $\mathbf{Z}[1/2]$-module, and we write $\langle 2\rangle^*\widetilde{\mathrm{F}}$ to denote the formal group obtained by pulling back $\widetilde{\mathrm{F}}$ along the map $\pi_0(k)[\![t]\!] \to \pi_0(k)[\![t]\!]$ sending $t \mapsto [2]_{\mathrm{F}}(t)$, one has an analogue of Legendre's duplication formula:

$$\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} s)\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(\langle 1/2\rangle) = \nu(\langle 2\rangle, s +_{\widetilde{\mathrm{F}}} s -_{\widetilde{\mathrm{F}}} 1)\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(s)\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} \langle 1/2\rangle);$$

there is also an analogue of the Gauss multiplication formula. Up to the factors of $\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(\langle 1/2\rangle)$ and $\nu(\langle 2\rangle, s +_{\widetilde{\mathrm{F}}} s -_{\widetilde{\mathrm{F}}} 1)$, which come from the normalization adopted in Example 5.9, these formulas can be explained very simply (and heuristically) as follows. Since $\Gamma_{\widetilde{\mathrm{F}}}(s)$ is the multiplicative inverse of the Euler class of the normal bundle of $t\mathcal{O} \subseteq \mathcal{K}$, and the Euler class of a direct sum of vector bundles is the product of Euler classes, it follows that the product $\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(s)^{-1}\Gamma_{\langle 2\rangle^*\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} \langle 1/2\rangle)^{-1}$ is the Euler class of the normal bundle of the embedding $t^2\mathbf{C}[\![t^2]\!] \times t\mathbf{C}[\![t^2]\!] \subseteq \mathbf{C}(\!(t^2)\!) \times t\mathbf{C}(\!(t^2)\!)$, where $\mathbf{C}$ is equipped with the weight 2 action of $\mathbf{G}_m$. But this embedding is just $t\mathbf{C}[\![t]\!] \subseteq \mathbf{C}(\!(t)\!)$, so the Euler class of its normal bundle is precisely $\Gamma_{\widetilde{\mathrm{F}}}(s +_{\widetilde{\mathrm{F}}} s)^{-1}$, as desired.

Remark 7.5. As explained in [Tel21], one can use the computation of Example 7.3 to compute the (loop-rotation equivariant) Coulomb branch [BFN18, BFN19] associated to the G-representation V. Let us briefly sketch an argument for this, ignoring loop-rotation equivariance and assuming (G,V)-orientability for simplicity; the argument below is only intended to be illustrative, and clearly requires care to be made precise (e.g., we need to assume, as in [Tel21], that there is a homomorphism $\mathbf{C}^\times \to \mathrm{G}$ whose induced $\mathbf{C}^\times$-action on V is by scaling to the origin).

Let $\mathcal{V}_{\mathcal{O}}$ and $\mathcal{V}_{\mathcal{K}}$ denote the (pro and ind-pro) constant vector bundles over $\mathrm{G}_{\mathcal{O}}\backslash\mathrm{G}_{\mathcal{K}} = \mathrm{Gr}_{\mathrm{G}}$ given by $\mathrm{V}_{\mathcal{O}} \times^{\mathrm{G}_{\mathcal{O}}} \mathrm{G}_{\mathcal{K}}$ and $\mathrm{V}_{\mathcal{K}} \times^{\mathrm{G}_{\mathcal{O}}} \mathrm{G}_{\mathcal{K}}$. Then the Coulomb branch $\mathcal{A}_{\mathrm{G},\mathrm{V}}$ is the semi-infinite $\mathrm{G}_{\mathcal{O}}$-equivariant cohomology of the equalizer $\mathcal{R}_{\mathrm{G},\mathrm{V}}$ of the two maps $\mathcal{V}_{\mathcal{O}} \to \mathcal{V}_{\mathcal{K}}$ given by the inclusion and the action map. In other words, there is an isomorphism

$$\mathcal{R}_{\mathrm{G},\mathrm{V}}/\mathrm{G}_{\mathcal{O}} \xrightarrow{\sim} \mathcal{V}_{\mathcal{O}}/\mathrm{G}_{\mathcal{O}} \times_{\mathcal{V}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}} \mathcal{V}_{\mathcal{O}}/\mathrm{G}_{\mathcal{O}}. \tag{11}$$

An analogue of the Serre spectral sequence in semi-infinite $\mathrm{G}_{\mathcal{O}}$-equivariant cohomology says that $\mathrm{R\Gamma}_{\mathrm{G}_{\mathcal{O}}}(\mathcal{R}_{\mathrm{G},\mathrm{V}};\omega^{\mathrm{ren}})$ is the coinvariants of the $\mathrm{R\Gamma}_{\mathrm{G}_{\mathcal{O}}}(\mathcal{V}_{\mathcal{K}};\omega^{\mathrm{ren}})$-coaction of $\mathrm{R\Gamma}_{\mathrm{G}_{\mathcal{O}}}(\mathcal{V}_{\mathcal{O}};\omega^{\mathrm{ren}})$ coming from the action map. Note that since $\mathrm{G}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}$ is homotopy equivalent to $\Omega\mathrm{G}$ by [Mit88, GR75], $\mathrm{R\Gamma}_{\mathrm{G}_{\mathcal{O}}}(\mathcal{V}_{\mathcal{O}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{G}]^{h\mathrm{G}}$. Similarly, Example 7.3 can be used to show that

$\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{V}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{G}]^{h\mathrm{G}}[e_{\mathrm{V}}^{-1}]$. It follows that $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{R}_{\mathrm{G,V}};\omega^{\mathrm{ren}})$ is the equalizer of the diagram

$$k[\Omega\mathrm{G}]^{h\mathrm{G}} \rightrightarrows k[\Omega\mathrm{G}]^{h\mathrm{G}}[e_{\mathrm{V}}^{-1}];$$

one of the maps is the unit, and the other can be computed to be multiplication by e_{V}. The resulting description of $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{R}_{\mathrm{G,V}};\omega^{\mathrm{ren}})$ is precisely that of [Tel21, Theorems 1 and 2].

Just like Example 7.3, one can check:

Example 7.6. Let $\mathrm{X} = \mathrm{G}/\mathrm{H}$ where $\mathrm{H} \subseteq \mathrm{G}$ is a connected reductive subgroup (so X is smooth and affine), and let $\mathcal{R}_{\mathrm{G,X}}$ denote the "relative Grassmannian" of [BZSV23]. Under mild assumptions on X, the loop space $\mathrm{X}_{\mathcal{K}}$ (as well as the quotient $\mathrm{X}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}$) admits a placid presentation (see [CL23, Theorem 35] and [Dri06]). There is a suitable local dimension theory $\mathcal{L}$ and $\mathcal{L}$-compatible ind-dimension theory τ such that $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{X}]^{h\mathrm{H}}$.

Just as in Remark 7.5, this computation can be used to recover (in a perhaps overly complicated way) the calculation that $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{R}_{\mathrm{G,X}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{H}]^{h\mathrm{H}}$. Namely, let $\mathcal{X}_{\mathcal{O}}$ and $\mathcal{X}_{\mathcal{K}}$ denote the constant (pro and ind-pro) schemes over $\mathrm{G}_{\mathcal{O}}\backslash\mathrm{G}_{\mathcal{K}} = \mathrm{Gr}_{\mathrm{G}}$ given by $\mathrm{X}_{\mathcal{O}} \times^{\mathrm{G}_{\mathcal{O}}} \mathrm{G}_{\mathcal{K}}$ and $\mathrm{X}_{\mathcal{K}} \times^{\mathrm{G}_{\mathcal{O}}} \mathrm{G}_{\mathcal{K}}$. Using a semi-infinite variant of the Serre spectral sequence, one can check as in Remark 7.5 that $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{R}_{\mathrm{G,X}};\omega^{\mathrm{ren}})$ is given by the (derived) coinvariants of the $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$-coaction on $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{O}};\omega^{\mathrm{ren}})$. But $\mathcal{X}_{\mathcal{O}} \cong \mathrm{H}_{\mathcal{O}}\backslash\mathrm{G}_{\mathcal{K}}$, so $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{O}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{G}]^{h\mathrm{H}}$; and similarly $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{K}};\omega^{\mathrm{ren}}) \cong k[\Omega\mathrm{G} \times \Omega\mathrm{X}]^{h\mathrm{H}}$. Under these identifications, the two maps $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{O}};\omega^{\mathrm{ren}}) \rightrightarrows \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$ are induced by applying $k[-]^{h\mathrm{H}}$ to the two maps $\Omega\mathrm{G} \rightrightarrows \Omega\mathrm{G} \times \Omega\mathrm{X}$ given by $g \mapsto (g,*)$ and $g \mapsto (g,\overline{g})$, where $\overline{g}$ is the image of g under the canonical map $\Omega\mathrm{G} \to \Omega\mathrm{X}$. Since there is a homotopy equivalence (analogous to (11))

$$\Omega\mathrm{H} \xrightarrow{\sim} \Omega\mathrm{G} \times_{\Omega\mathrm{G}\times\Omega\mathrm{X}} \Omega\mathrm{G},$$

where the fiber product is along the two maps described above, it follows that the (derived) coinvariants of the $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$-coaction on $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathcal{X}_{\mathcal{O}};\omega^{\mathrm{ren}})$ is precisely $k[\Omega\mathrm{H}]^{h\mathrm{H}}$, as desired.

If X is a G-space which is suitably nice, so that one can define $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$, then the $\mathrm{G}_{\mathcal{K}}$-action on $\mathrm{X}_{\mathcal{K}}$ defines a $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};\omega^{\mathrm{ren}}) = k[\Omega\mathrm{G}]^{h\mathrm{G}}$-module structure on $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$. This action is S^1-equivariant; in fact, $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$ typically admits the structure of an $\mathbf{E}_{2,\mathrm{BS}^1}$-$k$-algebra, and the above action exhibits it as an S^1-equivariant $\mathbf{E}_2$-$k[\Omega\mathrm{G}]^{h\mathrm{G}}$-algebra. Note that the loop-rotation equivariant $\mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{X}_{\mathcal{K}};\omega^{\mathrm{ren}})$ is now only a pointed $k[\Omega\mathrm{G}]^{h(\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}})}$-module; there is generally *no* ring structure.

When $\mathrm{X} = \mathrm{G}/\mathrm{H}$, for instance, we obtain the $k[\Omega\mathrm{G}]^{h\mathrm{G}}$-action on $k[\Omega\mathrm{X}]^{h\mathrm{H}}$ induced by the $\mathbf{E}_2$-ring structure on $k[\Omega\mathrm{X}]^{h\mathrm{H}}$ and the $\mathbf{E}_2$-map

$$k[\Omega\mathrm{G}]^{h\mathrm{G}} \to k[\Omega\mathrm{G}]^{h\mathrm{H}} \to k[\Omega\mathrm{X}]^{h\mathrm{H}}.$$

Let us now focus on the case when G is a torus T and V is a complex G-representation.

Example 7.7. Suppose that $\mathrm{V} = \mathbf{C}^n$ where each copy of $\mathbf{C}$ has the weight 1 action of $\mathrm{G} = \mathbf{G}_m$ (one could more generally take $\mathrm{G} = \mathrm{GL}_n$, but we stick to the case $\mathrm{G} = \mathbf{G}_m$ for simplicity). In the above setup, $e_{\mathrm{V}} \in \pi_0(k^{h\mathrm{T}})^{t\mathrm{S}^1} \cong \pi_0(k)[\![t,s]\!]$ can be identified with s^n. The $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}$-module structure on

$$\pi_0\mathrm{R}\Gamma_{(\mathbf{G}_m)_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathcal{K}^n;\omega^{\mathrm{ren}})[\hbar^{-1}] \cong \pi_0(k)[\![t,s]\!][(s \pm_{\widetilde{\mathrm{F}}} 1)^{-n}, (s \pm_{\widetilde{\mathrm{F}}} \langle 2\rangle)^{-n}, \cdots]$$

is given by $x\partial_x^{\widetilde{\mathrm{F}}}$ acting by multiplication by s, and x acts by $(x\partial_x^{\widetilde{\mathrm{F}}})^n$. The resulting $\widetilde{\mathrm{F}}$-differential equation $(x\partial_x^{\widetilde{\mathrm{F}}})^n = x$ is solved by the "$\widetilde{\mathrm{F}}$-n-Bessel function" $\mathrm{J}^{(n)}_{\widetilde{\mathrm{F}}}(x) := \sum_{i\geq 0} \frac{x^i}{\langle i\rangle!^n}$; when $n = 1$, this is the $\widetilde{\mathrm{F}}$-exponential function, and when $n = 2$, this is an $\widetilde{\mathrm{F}}$-analogue of the Bessel function.

When viewed in this way, many formulas with Bessel functions (like the Sonine-Gegenbauer multiplication formula [Wat44, Page 411]) can be given geometric proofs.

The Mellin transform of the above $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\mathbf{G}_m}$-module is the difference equation $s^n = \mathrm{T}$, which is solved by $\Gamma_{\widetilde{\mathrm{F}}}(s)^n$; in other words, the $\widetilde{\mathrm{F}}$-Mellin transform of $\mathrm{J}^{(n)}_{\widetilde{\mathrm{F}}}(x)$ is $\Gamma_{\widetilde{\mathrm{F}}}(s)^n$ (up to normalization). The non-S^1-equivariant cohomology $\mathrm{Spf}\, \pi_0 \mathrm{R}\Gamma_{(\mathbf{G}_m)_{\mathcal{O}}}(\mathcal{K}^n; \omega^{\mathrm{ren}})$ defines the Lagrangian subvariety $\{\mathrm{T} = s^n\}$ of $\mathrm{T}^*_{\widetilde{\mathrm{F}}}\mathbf{G}_m \cong \mathbf{G}_m \times \widetilde{\mathrm{F}}$.

It is easy to generalize the above example to show that for a more general T-representation V with associated homomorphism $\mathrm{T} \to \mathrm{T_V} \subseteq \mathrm{GL_V}$ (where $\mathrm{T_V}$ is the maximal torus of $\mathrm{GL_V}$), the $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-module structure on $\pi_0 \mathrm{R}\Gamma_{\mathrm{T}_{\mathcal{O}} \rtimes \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{V}_{\mathcal{K}}; \omega^{\mathrm{ren}})[\hbar^{-1}]$ is the pull-push of $\exp_{\widetilde{\mathrm{F}}}$ along the diagram

$$\check{\mathrm{T}} \leftarrow \check{\mathrm{T}}_{\mathrm{V}} \cong \mathbf{G}_m^{\dim(\mathrm{V})} \xrightarrow{\sum} \mathbf{A}^1.$$

The resulting $\widetilde{\mathrm{F}}$-differential equation on $\check{\mathrm{T}}$ is an $\widetilde{\mathrm{F}}$-analogue of the GKZ hypergeometric differential system [GZK89].

Let us return to Example 7.7. Fix a continuous embedding $\pi_0(k)[\![t]\!] \subseteq \mathbf{C}$ (so t is sent to a complex number with modulus ≤ 1). For $x \in \mathbf{G}_m$, let $\check{\mathrm{X}} := \{(y_1, \cdots, y_n) \in \mathbf{G}_m^n | y_1 \cdots y_n = x\}$. Then (up to factors of 2π) one can write $\mathrm{J}_{\widetilde{\mathrm{F}}}(x)$ as the contour integral

$$\mathrm{J}^{(n)}_{\widetilde{\mathrm{F}}}(x) = \int_\gamma \exp_{\widetilde{\mathrm{F}}}(y_1) \cdots \exp_{\widetilde{\mathrm{F}}}(y_n) d\mu_{\check{\mathrm{X}}}, \tag{12}$$

where γ is a simple torus inside $\check{\mathrm{X}}$ enclosing the origin inside $\mathbf{A}^n \supseteq \check{\mathrm{X}}$ and $d\mu_{\check{\mathrm{X}}}$ is the natural measure on $\check{\mathrm{X}}$ inherited from the inclusion $\check{\mathrm{X}} \subseteq \mathbf{G}_m^n$. For instance, for the additive formal group, this is the classical integral representation $\int \exp(y + x/y) \frac{dy}{y}$ of the Bessel function. There is a similar integral representation for solutions of the $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-module associated to any T-representation V (which are $\widetilde{\mathrm{F}}$-analogues of hypergeometric functions); via the Mellin transform of Proposition 5.5, these can also be rewritten as $\widetilde{\mathrm{F}}$-variants of Mellin-Barnes integrals. Just like the discussion following Example 7.3, many identities with these $\widetilde{\mathrm{F}}$-hypergeometric functions can be proved through simple geometric considerations.

The semiclassical limit of the $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-module $\pi_0 \mathrm{R}\Gamma_{\mathrm{T}_{\mathcal{O}} \rtimes \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{V}_{\mathcal{K}}; \omega^{\mathrm{ren}})[\hbar^{-1}]$ is given by the non-$\mathrm{S}^1_{\mathrm{rot}}$-equivariant cohomology, and hence is given by the Lagrangian subvariety $\mathrm{Spf}\, \pi_0 \mathrm{R}\Gamma_{\mathrm{T}_{\mathcal{O}}}(\mathrm{V}_{\mathcal{K}}; \omega^{\mathrm{ren}}) \subseteq \mathrm{T}^*_{\widetilde{\mathrm{F}}}\check{\mathrm{T}}$. At the level of solutions of $\mathrm{D}^{\widetilde{\mathrm{F}}}_{\check{\mathrm{T}}}$-modules, this amounts to taking the limit $\hbar \to 0$ of the oscillatory integral (12). Recall that the stationary phase approximation of an oscillatory integral $\int_{\mathbf{R}} \exp(f(y)) dy$ is given by the sum over the critical points y_0 of $f(y)$ of terms of the form $\frac{\exp(f(y_0))}{\sqrt{f^{(2)}(y_0)}}$, up to factors of 2π. In order for the critical locus of the logarithm of the integrand in (12) to agree with the Lagrangian $\mathrm{Spf}\, \pi_0 \mathrm{R}\Gamma_{\mathrm{T}_{\mathcal{O}}}(\mathrm{V}_{\mathcal{K}}; \omega^{\mathrm{ren}}) \subseteq \mathrm{T}^*_{\widetilde{\mathrm{F}}}\check{\mathrm{T}}$, as it must, some algebraic manipulations with Lagrange multipliers (adapted from the K-theoretic case in [Giv17]) show that one is forced to have:

Proposition 7.8. *There is an equality* $x\partial_x^k(\log(\exp_{\widetilde{\mathrm{F}}}(x))) = \widetilde{\ell}(x)$. *In other words, if* $\widetilde{\ell}(x) = \sum_{n \geq 1} a_n x^n$, *then there is a "Weierstrass product" expansion* $\exp_{\widetilde{\mathrm{F}}}(x) = \prod_{n\geq 1} \exp\left(\frac{a_n}{\langle n \rangle} x^n\right)$.

Example 7.9. When F is the additive formal group, Proposition 7.8 is just the obvious assertion that $\log(\exp(x)) = x$. When F is the multiplicative formal group, Proposition 7.8 asserts that

$$\log(\exp_q(x)) = \sum_{n\geq 1} (-1)^{n-1} \tfrac{(q-1)^{n-1}}{[n]_q} \tfrac{x^n}{n},$$

or in other words (using the expansion $\log(1+y) = \sum_{n\geq 1}(-1)^{n-1}\frac{y^n}{n}$ to collect terms) that

$$\exp_q(x) = \prod_{j\geq 0}(1+q^j(q-1)x)^{-1}.$$

This is the famous product expansion of the q-exponential (see [GR04, Section 1.3]).

Proposition 7.8 is fascinating for several reasons. First, it can be derived using only geometric considerations, and in fact as of this moment I do not have a purely algebraic proof! Second, it describes the relationship between $\log_{\widetilde{\mathrm{F}}}$ and $\exp_{\widetilde{\mathrm{F}}}$; outside of the case when F is the additive formal group, they are not inverses to each other, but Proposition 7.8 tells us that if $\widetilde{\ell}(x) = \sum_{n\geq 1} a_n x^n$, then $\log_{\widetilde{\mathrm{F}}}(\exp_{\widetilde{\mathrm{F}}}(x))$ is the series $\frac{1}{\widetilde{\ell}(1)}\sum_{n\geq 1}\frac{a_n}{\langle n\rangle}x^n$.[17]

8. Geometric Satake and variants

Many of the phenomena and calculations discussed above can be wrapped up neatly in the language of local geometric Langlands duality, by which we mean (variants of) the (derived) geometric Satake equivalence [BF08]; most of our discussion above is then a special case of this theory for a torus (!). Generalizing this picture to arbitrary connected reductive groups is work-in-progress; let me briefly sketch the resulting picture and explain what has been proved so far. Unfortunately, lack of space prevents us from giving more "leisurely" introduction to these ideas, but we refer the reader to [Dev25b, Section 1.1] for some discussion.

The derived geometric Satake equivalence of [BF08] (generalizing the abelian Satake equivalence of [MV07]) says that if G is a connected complex reductive group and k is a field of characteristic zero, there is a monoidal equivalence $\mathrm{Shv}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};k) \simeq \mathrm{IndPerf}(\check{\mathrm{G}}_k\backslash\mathrm{T}^*[2](\check{\mathrm{G}}_k)/\check{\mathrm{G}}_k)$; this category admits a Koszul dual description as $\mathrm{IndCoh}(\check{\mathrm{G}}_k\backslash\mathrm{T}[-1](\check{\mathrm{G}}_k)/\check{\mathrm{G}}_k)$. There is also a loop-rotation equivariant version, giving a monoidal equivalence $\mathrm{Shv}_{(\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}})\rtimes\mathrm{S}^1}(\mathrm{G}_{\mathcal{K}};k) \simeq \mathrm{IndPerf}^{\check{\mathrm{G}}_k}_{\mathrm{U}_\hbar(\check{\mathfrak{g}}_k)}$, the latter being a sheared version of the derived category of Harish-Chandra bimodules (which also admits a Koszul dual description). If k is not a field of characteristic zero, but is a more general commutative ring, a folklore expectation (but see [CR23, Tay25]) is that one should instead have an S^1-equivariant equivalence of categories $\mathrm{Shv}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};k) \simeq \mathrm{IndCoh}(\check{\mathrm{G}}_k\backslash\mathcal{L}(\check{\mathrm{G}}_k)/\check{\mathrm{G}}_k)$, where $\check{\mathrm{G}}_k$ denotes the Chevalley split form of the dual group defined over k, and $\mathcal{L}(\check{\mathrm{G}}_k) = \mathrm{Map}(\mathrm{B}\mathbf{Z},\check{\mathrm{G}}_k)$. This category admits a Koszul dual description as $\mathrm{IndPerf}^{\check{\mathrm{G}}_k\times\check{\mathrm{G}}_k}_{\mathfrak{Z}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k)}$; one recovers the characteristic zero statement above using Theorem 2.3 and formality to identify $\mathfrak{Z}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k)$ with the shearing of $\mathcal{O}_{\check{\mathrm{G}}_k}\otimes\mathrm{Sym}_k(\check{\mathfrak{g}}_k(-2)) \cong \mathcal{O}_{\mathrm{T}^*(2)\check{\mathrm{G}}_k}$. (See also [BZN13, BZN12].)

As explained in [Dev25b, Section 1.1], one might still expect this statement to remain true if k is a commutative ring *spectrum* for a suitable notion of "IndCoh", and for a suitable definition of $\check{\mathrm{G}}_k$. Neither of these are currently defined, but following the philosophy of this article, one could still hope to prove consequences of this expected statement on the level of algebra (i.e., upon extracting π_0): that is, one could still hope to prove that there are 1-parameter degenerations (like in Theorem 6.6)

$$\mathrm{Shv}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}};k) \rightsquigarrow \text{“}\mathrm{IndPerf}(\check{\mathrm{G}}\backslash(\mathrm{T}^*_{\widetilde{\mathrm{F}}}\check{\mathrm{G}})/\check{\mathrm{G}})\text{”} \tag{13}$$

$$\mathrm{Shv}_{(\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}})\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{G}_{\mathcal{K}};k)[\hbar^{-1}] \rightsquigarrow \text{“}\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{G}}\times\check{\mathrm{G}}}\text{”}, \tag{14}$$

where now $\check{\mathrm{G}}$ is viewed as living over $\pi_0(k)$. Unfortunately, unless $\check{\mathrm{G}}$ is a torus, I cannot (yet?) use Definition 3.5 to define the categories on the right-hand side, since I do not have

[17] For example, when F is the multiplicative formal group, $\log_q(\exp_q(x)) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{[n]_q\log(q)}\frac{((q-1)x)^n}{n}$.

a lifting of $\check{\mathrm{G}}$ to k (although I do believe this should be possible, at least as a group object in $\mathbf{E}_2$-k-schemes). Nevertheless, one can still compute many things about the categories on the left-hand side, which in turn lends evidence to the claim/belief that the categories on the right-hand side are well-defined for general connected reductive $\check{\mathrm{G}}$. For instance, the following is shown in [Dev23, Dev25b]:

Theorem 8.1. *Let k be $\mathbf{Z}[\beta]$, ku, or the connective cover of a 2-periodic elliptic cohomology theory*[18]*. Let G be a simply-laced connected complex reductive group with torsionfree fundamental group, and assume that G does not have any simple factors of type E_8 (so that G admits a faithful minuscule representation). Let F denote the associated 1-dimensional formal group over $\pi_0(k)$, let $\check{\mathrm{G}}^{\mathrm{dsc}}$ denote the Chevalley split form of the dual group defined over $\pi_0(k)$ with simply-connected derived subgroup, and let $\check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}} = \mathrm{Hom}(\mathrm{F}^\vee, \check{\mathrm{G}}^{\mathrm{dsc}})$.*

Then there is a monoidal filtered $\tau_{\geq 2\star}k$-linear category $\mathrm{Shv}^{\mathrm{fil}}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$ whose underlying k-linear category is $\mathrm{Shv}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$, and whose associated graded $\pi_{2*}(k)$-linear category $\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$ satisfies the following property: if C is an algebraically closed field of suitably large characteristic (or zero), there is a monoidal equivalence of graded $\pi_{2*}(k)$-linear categories[19]

$$\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k) \otimes_{\pi_0(k)} \mathrm{C} \simeq \mathrm{IndPerf}(\check{\mathrm{G}}\backslash(\check{\mathrm{G}} \times \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}})/\check{\mathrm{G}}) \otimes_{\pi_0(k)} \mathrm{C}.$$

Here, the monoidal structure on the left-hand side is convolution, and on the right-hand side is the standard tensor product.

For instance, when $k = \mathrm{ku}$, $\check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}$ is the deformation to the normal cone of the identity $1 \in (\check{\mathrm{G}}^{\mathrm{dsc}})^\wedge_{\mathcal{U}}$, where $\mathcal{U}$ is the unipotent cone. I expect Theorem 8.1 to hold with exactly the same conclusion for a general $\mathbf{E}_\infty$-ring k, except that $\mathrm{Shv}^{\mathrm{fil}}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$ will be a monoidal filtered $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear category.

Here are two special cases of Theorem 8.1:

(a) When G is a torus T, Theorem 8.1 just says that

$$\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{T}_{\mathcal{O}}\times\mathrm{T}_{\mathcal{O}}}(\mathrm{T}_{\mathcal{K}}; k) \simeq \mathrm{IndPerf}(\check{\mathrm{T}}\backslash(\check{\mathrm{T}} \times \check{\mathrm{T}}_{\mathrm{F}})/\check{\mathrm{T}});$$

it also admits a loop-rotation equivariant analogue

$$\mathrm{Shv}^{\mathrm{gr}}_{(\mathrm{T}_{\mathcal{O}}\times\mathrm{T}_{\mathcal{O}})\rtimes S^1}(\mathrm{T}_{\mathcal{K}}; k)[\hbar^{-1}] \simeq \mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{T}})^{\check{\mathrm{T}}\times\check{\mathrm{T}}}.$$

These calculations essentially amount to the computation of Theorem 3.1, using that $\mathfrak{Z}_{\mathbf{E}_2}(k[\Omega\mathrm{T}]/k) \cong k[\mathcal{L}\mathrm{T}]^{h(\mathrm{T}\times\mathrm{T})}$.

(b) When F is the additive formal group, corresponding to the case $k = \mathbf{Z}[u]$, the formal stack $\check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}$ is the completion $\check{\mathfrak{g}}^{\mathrm{dsc}}(2)^\wedge_{\mathcal{N}}/\mathbf{G}_m$ at the nilpotent cone of the Lie algebra of $\check{\mathrm{G}}^{\mathrm{dsc}}$; the grading shift by 2 comes from the coordinate of F lying in weight -2. One can then show that the minimal bilinear form defines a $\check{\mathrm{G}}$-equivariant isomorphism $\check{\mathfrak{g}}^{\mathrm{dsc}} \cong \check{\mathfrak{g}}^*$, so $\check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}} \cong \check{\mathfrak{g}}^*(2)^\wedge_{\mathcal{N}}/\mathbf{G}_m$, and Theorem 8.1 is then the renormalized form of the derived geometric Satake equivalence [BF08, AG15].

Theorem 8.1 tells us that at least when G is simply-laced, $\check{\mathrm{G}} \times \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}$ should be thought of as "$\mathrm{T}^*_{\widetilde{\mathrm{F}}}\check{\mathrm{G}} = \mathrm{Spec}(\pi_*\mathfrak{Z}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k))/\mathbf{G}_m$"; in fact, one can equip $\check{\mathrm{G}} \times \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}$ with the structure of a symplectic scheme over $\mathrm{Spev}(k)$, which in turn equips $\check{\mathrm{G}}\backslash(\check{\mathrm{G}} \times \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}})/\check{\mathrm{G}} = \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}/\check{\mathrm{G}}$ with a

[18] One can extend these results slightly to include cases like real K-theory ko, the image of J spectrum, or topological modular forms tmf. Ideally, k could be any connective $\mathbf{E}_\infty$-ring.

[19] Ideally, the base-change to C would not be required!

1-*shifted symplectic structure* in the sense of [PTVV13].[20] I do not yet have a loop-rotation equivariant analogue of Theorem 8.1, but this is work-in-progress. (When G is not simply-laced, the appropriate analogue of Theorem 8.1 involves folding Dynkin diagrams; I do not wish to discuss this here.)

When F is the multiplicative formal group and $k = \mathrm{ku}_p^\wedge$ (so $\pi_0(k) = \mathbf{Z}_p$) for $p > 2$, Remark 4.3 tells us that one can define the category $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\mathrm{X})$ for *any* p-adic formal scheme over $\mathbf{Z}_p$ (regardless of the existence of a lift to $\mathrm{ku}_p^\wedge$) as the category $\mathrm{QCoh}((\mathrm{X}[\zeta_p]/\mathbf{Z}_p[\![q-1]\!])^{\triangle})$ of q-prismatic crystals, in the language of [Dri24, BL22, Bha24]. In particular, the category $\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})$ is well-defined; but still, one cannot yet define the spectral side of (14), since the construction of prismatic crystals does not make it clear at all that the left and right actions of $\check{\mathrm{G}}$ on itself define an action of $\check{\mathrm{G}} \times \check{\mathrm{G}}$ on $\mathrm{QCoh}((\mathrm{X}[\zeta_p]/\mathbf{Z}_p[\![q-1]\!])^{\triangle})$.

Remark 8.2. There are also analogues of the results of [Bez16, ABG04]: for instance (under the assumptions of Theorem 8.1), if $\mathrm{I} \subseteq \mathrm{G}_{\mathcal{O}}$ is the Iwahori subgroup of $\mathrm{G}_{\mathcal{O}}$ corresponding to a chosen Borel subgroup $\mathrm{B} \subseteq \mathrm{G}$, and $\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}} := \check{\mathrm{B}}_{\mathrm{F}}^{\mathrm{dsc}} \times^{\check{\mathrm{B}}} \check{\mathrm{G}}$, there is an equivalence of graded $\pi_{2*}(k)$-linear categories[21]

$$\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{I}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k) \otimes_{\pi_0(k)} \mathrm{C} \simeq \mathrm{IndPerf}(\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}}/\check{\mathrm{G}}) \otimes_{\pi_0(k)} \mathrm{C}, \tag{15}$$

where $\check{\mathrm{B}}_{\mathrm{F}}^{\mathrm{dsc}} = \mathrm{Hom}(\mathrm{F}^\vee, \check{\mathrm{B}}^{\mathrm{dsc}})$ and $\check{\mathrm{B}}^{\mathrm{dsc}} \subseteq \check{\mathrm{G}}^{\mathrm{dsc}}$ is the dual Borel subgroup. The scheme $\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}}$ is an $\widetilde{\mathrm{F}}$-variant of the Grothendieck-Springer resolution. It specializes to the (completion at the nilpotent, resp. unipotent cone) of the usual Grothendieck-Springer resolution (resp. its multiplicative version) when F is the additive (resp. multiplicative) formal group. Comparing this equivalence with that of [ABG04] tells us that at least when G is simply-laced, the $\check{\mathrm{T}}$-torsor $\check{\mathrm{B}}_{\mathrm{F}}^{\mathrm{dsc}} \times^{\check{\mathrm{N}}} \check{\mathrm{G}}$ over $\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}}$ should be viewed as "$\mathrm{T}^*_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}}/\check{\mathrm{N}})$". The equivalence of (15) is compatible with the equivalence of Theorem 8.1 in the following sense: there is an action of $\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}_{\mathcal{O}}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$ on $\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{I}\times\mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; k)$ by convolution, and under the equivalences above, it identifies with the action of $\mathrm{IndPerf}(\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}})$ on $\mathrm{IndPerf}(\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}}/\check{\mathrm{G}})$ by pullback and tensoring along the action map $\widetilde{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}} \to \check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}$.

In fact, Remark 8.2 is a special case of an $\widetilde{\mathrm{F}}$-generalization of (and mild reinterpretation of) the conjectures of [BZSV23]: if X is a (suitable) smooth G-space, there should be a dual 1-shifted Lagrangian (in the sense of [PTVV13]) $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\check{\mathrm{G}} \to \check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}$ such that there is an equivalence

$$\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}}; k) \simeq \mathrm{IndPerf}(\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\check{\mathrm{G}}) \tag{16}$$

which is compatible with Theorem 8.1 in the same sense as in Remark 8.2. When F is the additive formal group, $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}$ is precisely a Hamiltonian $\check{\mathrm{G}}$-space (see [Saf16]), and (16) is then the local geometric conjecture of [BZSV23]. There are several examples of (16); see [Dev25a] and (for many more cases when F is the additive formal group) the survey in [BZSV23, Section 7.6]. Again, we expect a loop-rotation equivariant analogue. For instance, if the dual of X in the sense of [BZSV23] is $\check{\mathrm{M}}_{\widehat{\mathbf{G}}_a} = \mathrm{T}^*(\check{\mathrm{X}})$, then $\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}_{\mathcal{O}}\rtimes\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{X}_{\mathcal{K}}; k)$ should be equivalent to the (still undefined) category of (weakly) $\check{\mathrm{G}}$-equivariant (twisted) $\mathrm{D}^{\widetilde{\mathrm{F}}}$-modules on $\check{\mathrm{X}}$. Of course,

[20] When k is not complex oriented, $\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}$ does not quite admit a 1-shifted symplectic structure, but it is very close: the tangent complex of $\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}$ is given by $\underline{\check{\mathfrak{g}}} \xrightarrow{\mathrm{ad}} \underline{\check{\mathfrak{g}}^{\mathrm{dsc}}}\{1\} \cong \underline{\check{\mathfrak{g}}}^*\{1\}$, so $\mathrm{T}_{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}} \cong \mathrm{L}_{\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}}[1]\{1\}$. That is, $\check{\mathrm{G}}_{\mathrm{F}}^{\mathrm{dsc}}/\check{\mathrm{G}}$ is 1-shifted symplectic up to Tate-twisting by $\mathcal{O}\{1\}$.

[21] Again, ideally, the base-change to C would not be required!

implicit in this statement is the claim that the latter category can in fact be defined for any F!

The equivalence of (16) should also swap various objects: for instance, the constant sheaf $\underline{k}_{\mathrm{X}_{\mathcal{O}}}$ on $\mathrm{X}_{\mathcal{O}} \subseteq \mathrm{X}_{\mathcal{K}}$ should be sent to the structure sheaf $\mathcal{O}_{\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\check{\mathrm{G}}}$. A mild extension[22] of the expectations of [BZSV23] says that the (renormalized) dualizing sheaf $\omega^{\mathrm{ren}}_{\mathrm{X}_{\mathcal{K}}/\mathrm{G}_{\mathcal{O}}}$ should also be sent to the pushforward of the structure sheaf of a "Kostant section"[23] of the invariant-theoretic quotient map $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\check{\mathrm{G}} \to \check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\!\!/\check{\mathrm{G}}$. In particular, (16) implies that (up to a "transpose" twist)

$$\operatorname{Spec} \pi_* \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}}}(\mathrm{X}_{\mathcal{K}}; \omega^{\mathrm{ren}}) \cong \check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\!\!/\check{\mathrm{G}} \times_{\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\check{\mathrm{G}}} \check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\!\!/\check{\mathrm{G}}.$$

Moreover, this is an isomorphism of Lagrangians inside the corresponding isomorphism (which is baked into the proof of Theorem 8.1!) of symplectic stacks

$$\operatorname{Spec} \pi_* \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}} \times \mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; \omega^{\mathrm{ren}}) \cong \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}/\!\!/\check{\mathrm{G}} \times_{\check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}/\check{\mathrm{G}}} \check{\mathrm{G}}^{\mathrm{dsc}}_{\mathrm{F}}/\!\!/\check{\mathrm{G}}.$$

(When F is the additive formal group, this is part of the picture sketched in Teleman's ICM address [Tel14]: $\operatorname{Spec} \pi_* \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}} \times \mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; \omega^{\mathrm{ren}})$ is the group scheme $\mathrm{J}_{\check{\mathrm{G}}}$ of regular centralizers; see [BFM05, BF08, Ngo06].) This discussion can be used to recover the calculations of Section 7; let us illustrate this with Example 7.7 in the case $n = 2$.

Example 8.3. Consider the action of $\mathrm{G} = \mathrm{GL}_2$ on $\mathrm{X} = \mathbf{C}^2$, and let us for simplicity take k to be an ordinary commutative ring (so F is the additive formal group). Then $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}} \cong \mathrm{T}^*(\mathrm{GL}_2/\mathrm{GL}_1)$, so $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\mathrm{GL}_2 \cong \{(\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix})\}/\mathrm{GL}_1$, where x, y, and z have GL_1-weights 0, 1, and -1 respectively. The map $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\mathrm{GL}_2 \to \mathbf{A}^2/\mathrm{GL}_1 \cong \operatorname{Spec}(\pi_* k^{h\mathrm{GL}_2})$ sending $(x, y, z) \mapsto (x, yz)$ exhibits $\mathbf{A}^2$ as the invariant-theoretic quotient $\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\!\!/\mathrm{GL}_2$. The "Kostant section" mentioned above is the map $\mathbf{A}^2 \to \{(\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix})\}/\mathrm{GL}_1$ sending $(c_1, c_2) \mapsto (\begin{smallmatrix} c_1 & c_2 \\ 1 & 0 \end{smallmatrix})$. Since $a \in \mathrm{GL}_1$ sends $(\begin{smallmatrix} c_1 & c_2 \\ 1 & 0 \end{smallmatrix})$ to $(\begin{smallmatrix} c_1 & ac_2 \\ a^{-1} & 0 \end{smallmatrix})$, and this is equal to the transpose of $(\begin{smallmatrix} c_1 & c_2 \\ 1 & 0 \end{smallmatrix})$ exactly when $c_2 = a^{-1}$, it follows that

$$\mathbf{A}^2 \times_{\check{\mathrm{M}}_{\widetilde{\mathrm{F}}}/\mathrm{GL}_2} \mathbf{A}^2 \cong \operatorname{Spec} k[c_1, c_2^{\pm 1}],$$

which is indeed $\pi_* k^{h\mathrm{GL}_2}[c_2^{-1}]$ as predicted by Example 7.7. It is also not hard to check that the Lagrangian morphism to $\operatorname{Spec} \pi_* \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}} \times \mathrm{G}_{\mathcal{O}}}(\mathrm{G}_{\mathcal{K}}; \omega^{\mathrm{ren}})$ agrees with the action described in Example 7.7. One can also perform this computation with loop-rotation equivariance to recover the calculation that $\pi_0 \mathrm{R}\Gamma_{\mathrm{G}_{\mathcal{O}} \rtimes \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{X}_{\mathcal{K}}; \omega^{\mathrm{ren}})[\hbar^{-1}]$ is precisely obtained by localizing $\pi_0(k^{h\mathrm{GL}_2})^{t\mathrm{S}^1}$ at the Γ-function.

It would be very nice to make the discussion of this section totally unconditional – or even just make the expectations voiced above into precise conjectures! – e.g., by giving a definition of the category "$\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{G}})^{\check{\mathrm{G}} \times \check{\mathrm{G}}}$" (or more generally, of "$\mathrm{DMod}_{\widetilde{\mathrm{F}}}(\check{\mathrm{X}})^{\check{\mathrm{G}}}$" for suitable (spherical?) $\check{\mathrm{G}}$-varieties $\check{\mathrm{X}}$). I plan to return to this in the future.

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[22] Namely, duality not just of Hamiltonian G- and Ǧ-spaces, but also of Lagrangian correspondences between them; in this case, when F is the additive formal group, we are using a duality between the zero section of $\mathrm{T}^*\mathrm{X}$ and a particular Lagrangian correspondence between $\check{\mathrm{M}}_{\widehat{\mathbf{G}}_a}$ and the twisted cotangent bundle $\mathrm{T}^*(\check{\mathrm{G}}/_{\psi}\check{\mathrm{N}})$.

[23] When F is the additive formal group, for instance, this means that there should be a commutative diagram

$$\begin{tikzcd} \check{\mathrm{M}}_{\widehat{\mathbf{G}}_a}/\!\!/\check{\mathrm{G}} \arrow[r] \arrow[d] & \check{\mathrm{M}}_{\widehat{\mathbf{G}}_a}/\check{\mathrm{G}} \arrow[d] \\ \widehat{\check{\mathfrak{g}}}^*(2)/\!\!/\check{\mathrm{G}} \arrow[r] & \check{\mathfrak{g}}^*(2)^{\wedge}_{\mathcal{N}}/\check{\mathrm{G}}, \end{tikzcd}$$

where the bottom horizontal arrow is the Kostant slice. In particular, the composite $\check{\mathrm{M}}_{\widehat{\mathbf{G}}_a}/\!\!/\check{\mathrm{G}} \to \check{\mathrm{M}}_{\widehat{\mathbf{G}}_a}/\check{\mathrm{G}} \to \check{\mathfrak{g}}^*(2)^{\wedge}_{\mathcal{N}}/\check{\mathrm{G}}$ must hit the conjugacy class of a regular nilpotent element.

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