Spherochromatism in representation theory and arithmetic geometry

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Sanath Devalapurkar

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Dissertation Advisors: Michael Hopkins and Dennis Gaitsgory Sanath Devalapurkar

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Abstract

The goal of this thesis is to explain some applications of the perspective of chromatic homotopy theory to geometric representation theory and to arithmetic geometry. In the first half of this thesis, we study how the derived geometric Satake equivalence (due to Bezrukavnikov-Finkelberg, building on work of Ginzburg and Mirkovic-Vilonen) changes when one considers the category of constructible equivariant sheaves of k-modules on the affine Grassmannian of a complex (simply-laced) reductive group G, where k is a commutative ring *spectrum*. We state a conjecture describing the "spectral side" in terms of the Langlands dual group $\check{\mathrm{G}}$ and the 1-dimensional formal group associated to k via chromatic homotopy theory, and we make progress towards proving this conjecture. We also explore consequences of our conjecture in relation to the relative Langlands program recently elucidated by Ben-Zvi–Sakellaridis–Venkatesh.

In the second half of this thesis, we describe some joint work with Arpon Raksit, in which we refine work of Bökstedt-Madsen to provide a complete description of the topological Hochschild homology of the ring $\mathbf{Z}_p$ of p-adic integers in terms of the classical image of J spectrum. This result has several applications, both to homotopy theory and to arithmetic geometry, which we outline. We also describe some joint work with Jeremy Hahn, Arpon Raksit, and Allen Yuan, which aims to extend the theory of prismatization recently developed by Bhatt-Lurie-Drinfeld to the setting of ring spectra. This is tightly related to the theory of equivariant formal groups, and we provide some explicit calculations of these objects by generalizing rudiments of q-deformed calculus. The constructions we describe also have applications to classical arithmetic geometry; for example, we explain how our work can be used to provide a higher-dimensional refinement of Drinfeld's recent reinterpretation of Deligne-Illusie's work on Hodge theory in characteristic $p > 0$.

Contents

For my Tata, Suresh Devalapurkar

I love and miss you very much.

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Chapter 1

Introduction

It is well-acknowledged that geometry is hard. The goal of an algebraic topologist is to mitigate this by "linearizing" geometry as much as possible, so as to apply the more formulaic methods of linear algebra to the study of geometry. The meaning of the word "linear" has come to take on a life of its own, especially in recent years. Classically, dating to the time of Poincaré and Noether, this was taken to mean the usual realm of abstract algebra: the linearization construction in this case is the theory of *(co)homology*, which is a functor from the category of topological spaces to the category of (graded) abelian groups, vector spaces, etc. Today, this is still the interpretation of the word "linear" used by most mathematicians. There is, however, a much richer world which is much closer to geometry, and which specializes to the previous notion of linear algebra: this is the world of *spectra*, which are the central objects of study in stable homotopy theory. The broad goal of this thesis is to explore some interesting phenomena in representation theory and algebraic geometry which arise from adopting this more universal notion of linearization.[1]

More specifically, we will explore two directions of research.[2] The first we will pursue is, in some sense, motivated by Waldhausen's proposal (and Lurie's later realization [Lur5]) of spectral algebraic geometry, where one builds a theory of algebraic geometry whose basic building blocks are commutative ring *spectra* (as opposed to mere commutative rings). A mild variant of this theory was introduced recently by Hahn, Raksit, and Wilson [HRW], and expanded upon by myself, Hahn, Raksit, and Yuan [DHRY]: this is the theory of the *even filtration*, which takes as input an $\mathbf{E}_\infty$-ring k and produces as output a classical stack $\mathrm{Spev}(k)$ over $\mathrm{B}\mathbf{G}_m$. In some sense, everything in this thesis is a contemplation of this powerful construction. When applied to simple "homological" constructions like (topological) Hochschild homology, this theory reproduces and generalizes the recent breakthrough of prismatic and syntomic cohomology as pioneered by Bhatt-Scholze [BS1], Bhatt-Lurie [BL, Bha3], and Drinfeld [Dri2]. Our first goal in this thesis is to give an exposition of this construction and calculate some examples.

The second direction of research is essentially a generalization of the original reason for introducing spectra: they can be used to linearize topological spaces. If k is a ring spectrum and X is a topological space (or, more generally, a topological stack), one can construct a

[1] Hence the word "spherochromatism" in the title of this thesis: this is a term from optics, which refers to the phenomenon of a lens failing to focus all the wavelengths of light to the same point. (See https://richardbarron.net/cameras/2021/07/25/spherochromatism/ for a vivid example in photography.) In this case, the analogy is that the notion of linearization through stable homotopy theory "fails" (in a very interesting way) to produce the same answer for all the interesting ("chromatic") strata of the category of spectra (where spectra are viewed as cohomology theories); and studying the "fringes of color" thus obtained can often lead to very interesting mathematics.

[2] The presentation in this thesis will actually be reversed: we will first explore geometric representation theory, and then turn to the theory of prismatization.

k-linear category $\mathrm{Shv}(X; k)$ of sheaves of k-modules on X (perhaps constructible with respect to a certain stratification). If k is an ordinary commutative ring, then it is sometimes the case that $\mathrm{Shv}(X; k)$ is equivalent to the category of quasicoherent sheaves on an algebraic stack over k. Such equivalences tend to behave like Fourier transforms, and are ubiquitous in mirror symmetry and the geometric Langlands program. When k is a more general $\mathbf{E}_\infty$-ring, such equivalences do exist, but are harder to prove, in part because it is hard to explicitly construct spectral k-stacks.

However, a basic observation is that if X is sufficiently nice (for instance, is the analytification of a cellular complex variety), then one can use the even filtration to construct a 1-*parameter degeneration* of $\mathrm{Shv}(X; k)$ into the category of quasicoherent sheaves on a (classical!) stack $\check{M}_k$ defined over $\mathrm{Spev}(k)$. If X is a point and $k = S$ is the sphere spectrum, for instance, this is the 1-parameter degeneration of the category of spectra into the category of quasicoherent sheaves on the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of 1-dimensional formal groups afforded by the Adams-Novikov filtration. (*Chromatic* homotopy theory aims to study the category of spectra through this relationship between the sphere spectrum and $\mathcal{M}_{\mathrm{fg}}$.) Our goal will be to explore such degenerations in the context of geometric representation theory, where X is the flag variety or the affine Grassmannian of a complex reductive group G, and the degenerated stack $\check{M}_k$ is a hybrid object built from the Langlands dual group $\check{G}$ and a canonical 1-dimensional formal group defined using the stack $\mathrm{Spev}(k)$. This suggests a generalization of the (Betti) geometric Langlands program, as well as a relative analogue thereof generalizing [BZSV], which we will study in this thesis.

Before proceeding to a more detailed overview of the content of this thesis, let us say a brief word about the "even filtration" mentioned above (which was introduced in [HRW]). It can be viewed as a "smarter" version of the Postnikov filtration, and provides a canonical way to degenerate constructions in spectral algebraic geometry to constructions in ordinary algebraic geometry. The definition itself is very simple: for an $\mathbf{E}_\infty$-ring A, one defines $\mathrm{Spev}(A)$ to be the colimit (in fpqc stacks) of the stacks $\mathrm{Spec}(\pi_* B)/\mathbf{G}_m$, where the colimit ranges over all $\mathbf{E}_\infty$-maps $A \to B$ where B has even homotopy groups.

Many seemingly innocuous constructions in the setting of $\mathbf{E}_\infty$-rings/spectral algebraic geometry turn out to produce extremely interesting mathematics upon applying the even filtration. For instance, the even filtration was used in [HRW] to define a motivic filtration on the topological Hochschild homology of commutative ring spectra, whose associated graded pieces were taken to be the *definition* of (Nygaard-completed) prismatic cohomology for commutative ring spectra. (Here, the notion of topological Hochschild homology is the "innocuous construction" used as input.) One of our goals in this document is to explore the effect of other simple homotopical constructions under the even filtration.

We now turn to a more detailed overview of this thesis.

1.1 Spherochromatism in representation theory

1.1.1 Background

Let G be a fixed reductive group (over $\mathbf{Z}$, say), and let F be a (nonarchimedean, for simplicity) local field with ring of integers $\mathcal{O}$. The local arithmetic Langlands program posits a relationship between suitable representations of $G(F)$ on $\mathbf{C}$-vector spaces and (suitable) homomorphisms from the absolute Galois group of F into the Langlands dual group $\check{G}(\mathbf{C})$. One of the earliest and most important results in this area is the *Satake isomorphism*, which gives an identification between the vector space $\mathcal{H}_G$ of $\mathbf{C}$-valued L^2-functions on the double coset space $G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$, i.e., L^2-functions on $G(F)$ which are bi-$G(\mathcal{O})$-invariant, and the complexification of the Grothendieck group $\mathrm{K}_0(\mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}}))$ of representations of the Langlands dual group $\check{\mathrm{G}}_{\mathbf{C}}$. This identification is an isomorphism of rings, where $\mathcal{H}_{\mathrm{G}}$ is equipped with the algebra structure given by convolution, and $\mathrm{K}_0(\mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}}))$ is equipped with the algebra structure given by tensor product.

This isomorphism suffers from an interesting deficit. Namely, both sides have a natural inner product: on $\mathcal{H}_{\mathrm{G}}$, this comes from the embedding $\mathcal{H}_{\mathrm{G}} \subseteq \mathrm{L}^2(\mathrm{G}(\mathrm{F}); \mathbf{C})$; and on $\mathrm{K}_0(\mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}}))$, this comes from the dimension of Hom-spaces. However, the isomorphism $\mathcal{H}_{\mathrm{G}} \cong \mathrm{K}_0(\mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}})) \otimes_{\mathbf{Z}} \mathbf{C}$ does *not* preserve this inner product. One can, however, modify the inner product on $\mathrm{K}_0(\mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}}))$ appropriately (via the "Macdonald formula") to ensure that it matches with the inner product on $\mathcal{H}_{\mathrm{G}}$ under the Satake isomorphism. The requisite modification is explained through the *derived* Satake isomorphism, which we will now explain.

First, it turns out to be significantly easier to construct and study the requisite modification in the setting of *geometric* Langlands. Here, the local field F is replaced by a "geometric" local field (by which we mean $\mathbf{C}(\!(t)\!)$ or $\overline{\mathbf{F}_p}(\!(t)\!)$), and the ring of integers $\mathcal{O}$ is replaced an appropriate ring of integers therein (meaning $\mathbf{C}[\![t]\!]$ or $\overline{\mathbf{F}_p}(\!(t)\!)$, respectively); we will still denote these by F and $\mathcal{O}$ below. Most importantly, the vector space of *functions* is replaced by the category of (constructible) *sheaves*. This procedure is also known as categorification. If $k = \overline{\mathbf{F}_p}$, then the categories involved acquire a canonical action given by Frobenius, and taking its trace recovers the function spaces described previously.

For instance, fix $\mathrm{F} = \mathbf{C}(\!(t)\!)$ and $\mathcal{O} = \mathbf{C}[\![t]\!]$. The Satake isomorphism categorifies to the geometric Satake equivalence (proved by Ginzburg [Gin2] and Mirkovic-Vilonen [MV]), which states that there is an equivalence of abelian categories[3]

$$\mathrm{Perv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{C}) \simeq \mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{C}})^{\heartsuit}.$$

In fact, one can now prove something stronger: if $\check{\mathrm{G}}_{\mathbf{Z}}$ denotes Chevalley's split form of $\check{\mathrm{G}}_{\mathbf{C}}$ which is defined over $\mathbf{Z}$, then there is an equivalence of abelian categories

$$\mathrm{Perv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{Z}) \simeq \mathrm{Rep}(\check{\mathrm{G}}_{\mathbf{Z}})^{\heartsuit}.$$

The geometric Satake equivalence therefore exhibits a "motivic" character, which (at least for now) we interpret to mean a certain agnosticity from the point of view of spectral decomposition (i.e., the right-hand side) towards the choice of coefficients on the automorphic side (i.e., the left-hand side). However, the problem of inner products mentioned above shows up again with the geometric Satake equivalence: here, it manifests as the claim that the *derived* category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{C})$ is not equivalent to the derived category of representations of $\check{\mathrm{G}}_{\mathbf{C}}$. Rather, a famous theorem of Bezrukavnikov-Finkelberg [BF] states that there is a (monoidal) equivalence of categories

$$\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); \mathbf{C}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^*_{\mathbf{C}}[2]/\check{\mathrm{G}}_{\mathbf{C}}),$$

called the *derived Satake equivalence*. Here, $\check{\mathfrak{g}}^*_{\mathbf{C}}[2]$ denotes the shift of the coadjoint representation of $\check{\mathrm{G}}_{\mathbf{C}}$ to homological degree 2. The above equivalence is furthermore t-exact for the perverse t-structure on the left-hand side and the standard (homological truncation) t-structure on the right-hand side. Taking hearts of this t-structure therefore recovers the "abelian" geometric Satake equivalence as stated previously.

Given the "motivic" nature of the abelian geometric Satake equivalence, it is natural to wonder if the derived Satake equivalence could also work with $\mathbf{Z}$-coefficients – or, even simpler,

[3] The heart is to remind the "derived reader" that the object under consideration is an abelian category.

with coefficients in an arbitrary algebraically closed field k. This turns out to be very subtle, and in fact the obvious modification of the right-hand side (namely, $\mathrm{QCoh}(\check{\mathfrak{g}}_k^*[2]/\check{\mathrm{G}}_k)$) turns out to be false as stated. There are proposed fixes: for instance, using Koszul duality, the right-hand side of the derived Satake equivalence can be rewritten as $\mathrm{IndCoh}((\{0\} \times_{\check{\mathfrak{g}}_{\mathbf{C}}} \{0\})/\check{\mathrm{G}}_{\mathbf{C}})$[4]; although the naïve replacement of $\mathbf{C}$ by k does not do the trick, the picture of local geometric Langlands nevertheless suggests that there should be a (monoidal) equivalence of categories[5]

$$\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k) \simeq \mathrm{IndCoh}((\{1\} \times_{\check{\mathrm{G}}_k} \{1\})/\check{\mathrm{G}}_k). \tag{1.1.1}$$

Notice that the Lie algebra $\check{\mathfrak{g}}_{\mathbf{C}}$ has been replaced by the group $\check{\mathrm{G}}_k$. It might be possible to approach the equivalence (1.1.1) following the methods of [CR1]; however, one of the difficulties with this approach is that it is quite inexplicit, in that it is hard to calculate the images of objects on the left-hand side under this equivalence. (Below, we will also study the question of determining the images of objects on the left-hand side, in the guise of the "relative Langlands program" [BZSV].) The expectation (1.1.1) is closely related to the recent work [Ric, RW, BR, Bez] of Bezrukavnikov, Riche, and their collaborators.

There is a similar conjectural picture for the global (unramified, say) Betti geometric Langlands equivalence [BZN]. Namely, if C is a smooth projective curve over $\mathbf{C}$, one might expect an equivalence

$$\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_{\mathrm{G}}(\mathrm{C}); k) \simeq \mathrm{IndCoh}_{\check{\mathcal{N}}}(\mathrm{Loc}_{\check{\mathrm{G}}_k}(\mathrm{C})). \tag{1.1.2}$$

The right-hand side here is (a derived version of) the stack of $\check{\mathrm{G}}_k$-valued local systems on the curve C. (The reader uninitiated in geometric Langlands is encouraged to ignore the subscripts $\mathcal{N}$ and $\check{\mathcal{N}}$: these have to do with technical conditions of nilpotent singular support [AG]; IYKYK!) We will not focus on (1.1.2) here (but we will touch on it later in the introduction), but mention it nonetheless since it fits well into the theme discussed above.

One of the goals of this thesis is to suggest that although the derived Satake equivalence does not refine to an equivalence between $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ and the derived category $\mathrm{QCoh}(\check{\mathfrak{g}}_k^*[2]/\check{\mathrm{G}}_k)$, there nevertheless should be a 1-*parameter degeneration* of the category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ into the graded category $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathfrak{g}}_k^*(2)/\check{\mathrm{G}}_k)$. Here, $\check{\mathfrak{g}}_k^*(2)$ denotes the coadjoint representation of $\check{\mathrm{G}}_k$ placed in *weight* 2: note that the stack $\check{\mathfrak{g}}_k^*(2)/\check{\mathrm{G}}_k$ is no longer derived! Furthermore, and perhaps most crucially, this degeneration should be easier to prove. (To illustrate this, we take some steps towards proving such a degeneration in § 3.7.) The degeneration in question is quite easy to see from the spectral side: the derived stack $\{1\} \times_{\check{\mathrm{G}}_k} \{1\}$ admits a 1-parameter degeneration into the graded derived stack $\{0\} \times_{\check{\mathfrak{g}}_k(-2)} \{0\}$, which is Koszul dual (upon "shearing") to $\check{\mathfrak{g}}_k^*(2)$, and this gives a degeneration of $\mathrm{IndCoh}((\{1\} \times_{\check{\mathrm{G}}_k} \{1\})/\check{\mathrm{G}}_k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathfrak{g}}_k^*(2)/\check{\mathrm{G}}_k)$.

1.1.2 Elaborating on the phrase "motivic nature"

Above, we used the term "motivic nature" with regards to the geometric Satake equivalence to refer to a certain agnosticity with respect to the choice of coefficients. The philosophy of degeneration suggested at the end of the preceding subsection is a mild refinement: namely, the 1-parameter degeneration of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ is agnostic with respect to the choice of coefficients k. (Of course, this is also borne out in (1.1.1) without regards to any degenerations.)

[4] Here, the symbol $\{0\} \times_{\check{\mathfrak{g}}_{\mathbf{C}}} \{0\}$ denotes the derived fiber product.

[5] Here, as above, the symbol $\{1\} \times_{\check{\mathrm{G}}_k} \{1\}$ denotes the derived fiber product, and not a "balanced product".

Homotopy theory expands our notion of algebra: namely, there is a well-behaved theory of commutative ring spectra, which includes the usual theory of commutative rings as a very special case. For instance, the ring $\mathbf{Z}$ is no longer initial in the category of commutative ring spectra. Instead, the initial object in the category of commutative ring spectra is called the "sphere spectrum", and is denoted (here, at least) by the symbol S. As such, it is natural to wonder if there is a Langlands dual description of the ∞-category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k)$ with k being a commutative ring spectrum.[6] More generally, is there a version of the (local, say) geometric Langlands program (and perhaps more ambitiously, the arithmetic Langlands program!) which studies suitable representations of $\mathrm{G}(\mathrm{F})$ on modules over commutative ring spectra? In particular, is the resulting equivalence of "motivic nature"?

Fix a connective $\mathbf{E}_\infty$-ring spectrum k, so that there is an augmentation $k \to \pi_0(k)$. If one could define a lift $\check{\mathrm{G}}_k$ of $\check{\mathrm{G}}_{\pi_0(k)}$ to k, then, following (1.1.1), one might expect (via the "motivic philosophy") that there is a monoidal k-linear equivalence

$$\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k) \simeq \mathrm{IndCoh}((\{1\}\times_{\check{\mathrm{G}}_k}\{1\})/\check{\mathrm{G}}_k). \tag{1.1.3}$$

Here, the right-hand side denotes some category of "ind-coherent" sheaves on the putative spectral stack $(\{1\}\times_{\check{\mathrm{G}}_k}\{1\})/\check{\mathrm{G}}_k$. However, it seems difficult to directly prove such an equivalence following the arguments of [BF]; and in any case, one would need a refined notion of IndCoh, which is not merely built from homologically renormalizing the category QCoh, but rather – in the particular case of IndCoh of the stack $(\{1\}\times_{\check{\mathrm{G}}_k}\{1\})/\check{\mathrm{G}}_k$ – incorporates the myriad of subtleties having to do with the homotopy-theoretic notion of *genuine* equivariance. (That this is necessary will hopefully be clearer after Conjecture 1.1.1.)

Even if one could adapt alternative arguments for the derived Satake equivalence (like the one provided in [CR1]), it seems very hard to get a more concrete handle on the putative spectral stack $\{1\}\times_{\check{\mathrm{G}}_k}\{1\}$. Nevertheless, following the philosophy of degenerations suggested above, it should be possible to provide – and actually prove, without dealing with spectral algebro-geometric issues surrounding even defining $\check{\mathrm{G}}_k$! – a Langlands dual description of a 1-*parameter degeneration* of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k)$.

To explain the resulting picture, consider a somewhat silly case: take G to be the *trivial* group, and take k to be the sphere spectrum. In this case, $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k)$ is just the ∞-category Sp of spectra. This is tautologically equivalent to the category of quasicoherent sheaves on the spectral scheme $\mathrm{Spec}(\mathrm{S})$; but a more interesting picture (as mentioned earlier in the introduction) comes from the even filtration, which gives a degeneration of Sp into the (derived) category of quasicoherent sheaves on the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of (1-dimensional) formal groups. The latter is an object in the realm of classical algebraic geometry. In fact, this degeneration has been realized as coming from motivic homotopy theory, in a sense on which we will not elaborate here. The grading coming from the theory of weights is reflected under this degeneration via the canonical structure map $\mathcal{M}_{\mathrm{fg}} \to B\mathbf{G}_m$ sending a formal group to (the inverse of) its Lie algebra.

The preceding discussion suggests that for a general reductive group, the desired 1-parameter degeneration of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k)$ should naturally be linear over the moduli stack of formal groups. One of the goals of this thesis is to conjecture, and provide some evidence for, the desired degeneration of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F});k)$. This comes from a mild variant of the theory of "F-loop spaces" [MRT] (while this thesis was being written, the paper [BK2] appeared on the arXiv with the same construction).

[6] In the "genuine equivariant" setting, the theory of such sheaves has not yet been developed in the literature, but I have been informed that it is work-in-progress by Konovalov-Perunov-Prikhodko and Cnossen-Maegawa-Volpe. We will treat this as a black box below.

Namely, suppose $\mathbf{H}$ is the canonical 1-dimensional formal group over $\mathrm{Spev}(k)$ associated to the map $\mathrm{Spev}(k) \to \mathrm{Spev}(\mathrm{S}) \cong \mathcal{M}_{\mathrm{fg}}$. Then the category $\mathrm{Tors}_{\mathbf{H}}$ of coherent sheaves on $\mathbf{H}$ of finite length acquires a symmetric monoidal structure coming from convolution (i.e., pushforward along the group operation $\mathbf{H} \times_{\mathrm{Spev}(k)} \mathbf{H} \to \mathbf{H}$). If X is any (Tannakian) stack over $\mathrm{Spev}(k)$, one can then define the $\mathbf{H}$-loop space $\mathcal{L}_{\mathbf{H}}(\mathrm{X})$ of X using the Tannakian formalism as classifying exact symmetric monoidal functors $\mathrm{QCoh}(\mathrm{X})^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}$. In particular, if G is a group scheme over $\mathrm{Spev}(k)$ and $\mathrm{X} = \mathrm{BG}$, then there is a canonical map $\mathcal{L}_{\mathbf{H}}(\mathrm{BG}) \to \mathrm{BG}$ which exhibits $\mathcal{L}_{\mathbf{H}}(\mathrm{BG})$ as the quotient $\mathrm{G}_{\mathbf{H}}/\mathrm{G}$ for some stack $\mathrm{G}_{\mathbf{H}}$ over $\mathrm{Spev}(k)$ with an action of G. Given this setup, we can finally state:

Conjecture 1.1.1 (Conjecture 4.3.20)**.** *Suppose* G *is a connected simply-laced reductive group over* $\mathbf{C}$ *with torsion-free fundamental group. Let* $\mathrm{G}_{\mathbf{Z}}$ *denote its Chevalley split form, and let* $\check{\mathrm{G}}_{\mathbf{Z}}$ *denote the Chevalley split form of its Langlands dual group. Since* G *is assumed to be simply-laced, the conjugation action of* $\mathrm{G}_{\mathbf{Z}}$ *on itself defines an action of* $\check{\mathrm{G}}_{\mathbf{Z}}$ *on* $\mathrm{G}_{\mathbf{Z}}$ *(again by conjugation). The group schemes* $\mathrm{G}_{\mathbf{Z}}$ *and* $\check{\mathrm{G}}_{\mathbf{Z}}$ *over* $\mathrm{Spec}(\mathbf{Z})$ *pull back (along the canonical map* $\mathrm{Spev}(k) \to \mathrm{Spec}(\mathbf{Z})$*) to group schemes over* $\mathrm{Spev}(k)$ *which we will abusively denote by* G *and* $\check{\mathrm{G}}$*. In particular, one can define the group scheme* $\mathrm{G}_{\mathbf{H}}$ *over* $\mathrm{Spev}(k)$*, along with an action of* $\check{\mathrm{G}}$ *on it. Then, there is a* 1*-parameter degeneration of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ *into the graded category* $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$*, which is linear over the* 1*-parameter degeneration of* Mod_k *into* $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{Spev}(k))$*.*

As stated above, the conjecture is rather imprecise, because it does not specify the mechanism through which this degeneration occurs. This mechanism is given in Conjecture 4.3.20, and in particular in § 4.2: it is essentially the even filtration, and so can be viewed as a refined analogue of the Postnikov degeneration from cochains to cohomology. Of course, one might also expect analogues of Conjecture 1.1.1 for other variants of the derived Satake equivalence (like the work of Arkhipov-Bezrukavnikov-Ginzburg [ABG]). We will address this interesting question in § 1.1.3. One might also ask about the non simply-laced case. It is possible to formulate a similar conjecture by folding Dynkin diagrams, but the resulting statement is not as simple.

Conjecture 1.1.1 displays what I believe to be one of the tenets of the "motivic nature" of the geometric Satake equivalence. Namely, it is not that the degeneration of the category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ is agnostic with respect to the choice of $\mathbf{E}_\infty$-ring k of coefficients, but rather that this degeneration depends only on k *through* $\mathrm{Spev}(k)$ *and its corresponding* 1*-dimensional formal group* (i.e., its corresponding theory of Chern classes). Conjecture 1.1.1 in fact makes an even stronger claim: only the "numerator" of the stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ changes as k varies, but the Langlands dual group itself $\check{\mathrm{G}}$ remains the same! In some sense, this could be viewed as a statement about the "combinatorial" nature of the Langlands dual group.

Note that if k is an ordinary commutative ring, then $\mathbf{H} = \widehat{\mathbf{G}_a}(2)$ (or $\mathbf{G}_a(2)$, if one imposes genuine equivariance), and then $\mathrm{G}_{\mathbf{H}}$ is isomorphic to the completion of the Lie algebra $\mathfrak{g}(2)$ at the nilpotent cone $\mathcal{N}$. (In the genuine equivariant case, $\mathrm{G}_{\mathbf{H}}$ is just isomorphic to $\mathfrak{g}(2)$.) Since G is simply-laced, there is a natural isomorphism $\mathfrak{g} \cong \check{\mathfrak{g}}^*$. The stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ from Conjecture 1.1.1 then identifies with $(\check{\mathfrak{g}}^*)^\wedge_{\mathcal{N}}(2)/\check{\mathrm{G}}$. The resulting degeneration from $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ into $\mathrm{QCoh}^{\mathrm{gr}}((\check{\mathfrak{g}}^*)^\wedge_{\mathcal{N}}(2)/\check{\mathrm{G}})$ is essentially (the "renormalized form" of) the derived geometric Satake equivalence from [AG].

Similarly, if k is complex K-theory, then $\mathbf{H} = \widehat{\mathbf{G}_m}$ (or $\mathbf{G}_m$, if one imposes genuine equivariance). The scheme $\mathrm{G}_{\mathbf{H}}$ is then isomorphic to the completion of the group G at the unipotent cone $\mathcal{U}$ (and $\mathrm{G}_{\mathbf{H}}$ is isomorphic to G itself in the genuine equivariant case). The stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ from Conjecture 1.1.1 therefore identifies with $\mathrm{G}^\wedge_{\mathcal{U}}/\check{\mathrm{G}}$.

Note that Conjecture 1.1.1 says that the Adams-Novikov filtration on the sphere spectrum and derived geometric Satake with coefficients in an algebraically closed field are *compatible* degenerations! Namely, as discussed previously, the Adams-Novikov filtration/theory of synthetic spectra is the special case of Conjecture 1.1.1 when G is the trivial group and k is an interesting $\mathbf{E}_\infty$-ring (the sphere spectrum); and similarly, one obtains derived geometric Satake when G is interesting (a general connected reductive group) and k is homotopy-theoretically/arithmetically uninteresting (an algebraically closed field). These cases correspond to two different ways of specializing the stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathcal{M}_{\mathrm{fg}}$.

It is an elementary combinatorial exercise to check that Conjecture 1.1.1 holds when G is a torus (see Theorem 3.2.20). In this case, (1.1.3) already holds, and the basic observation is that Spev of the ring $k[\mathrm{B}\mathbb{X}_*(\mathrm{T})]$ of functions on the spectral stack $\{1\} \times_{\check{\mathrm{T}}_k} \{1\}$ is isomorphic to the 1-shifted Cartier dual of the stack $\mathrm{T}_{\mathbf{H}}$. The general theory of Cartier duality/the Fourier-Mukai transform then implies that the even filtration defines a 1-parameter degeneration of $\mathrm{IndCoh}((\{1\} \times_{\check{\mathrm{T}}_k} \{1\})/\check{\mathrm{T}}_k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{T}_{\mathbf{H}}/\check{\mathrm{T}})$; see Proposition 4.6.3. In a similar way, it may be the case that the spectral/Langlands dual side of the degeneration from Conjecture 1.1.1 for a general reductive group G over $\mathbf{C}$ can be calculated using by applying the even filtration to the conjectural object $\{1\} \times_{\check{\mathrm{G}}_k} \{1\}$ appearing in (1.1.3). However, even in the Borel-equivariant setting, Spev of the ring of functions on $\{1\} \times_{\check{\mathrm{G}}_k} \{1\}$ would only compute the completion of $\mathrm{G}_{\mathbf{H}}$ at the identity section[7], and one generally cannot recover $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}})$ from $\mathrm{QCoh}^{\mathrm{gr}}(\widehat{\mathrm{G}_{\mathbf{H}}})$ through "homological tricks". For example, if $\mathbf{H} = \mathbf{G}_m$, then $\mathrm{G}_{\mathbf{H}} \cong \mathrm{G}$, while $\widehat{\mathrm{G}_{\mathbf{H}}} \cong \widehat{\mathrm{G}}$ (the problem persists even in the Borel-equivariant setting, when $\mathbf{H} = \widehat{\mathbf{G}_m}$, in which case $\mathrm{G}_{\mathbf{H}} \cong \mathrm{G}^{\wedge}_{\mathcal{U}}$).

A first goal in this thesis is to prove the following result towards Conjecture 1.1.1:

Theorem 1.1.2 (Theorem 4.3.13, Corollary 4.3.17)**.** *Suppose* G *is a connected simply-laced reductive group over* $\mathbf{C}$ *with torsion-free fundamental group. Let k denote either rational cohomology, (complex or real) K-theory, or an elliptic cohomology theory (in the sense of [Lur1, Lur7], so that* $\mathbf{H}$ *is* $\mathbf{G}_a(2)$, $\mathbf{G}_m$, *or an elliptic curve.*[8] *Let* F *denote an algebraically closed field equipped with a map* $\pi_0(k) \to \mathrm{F}$. *Then, there is a filtered category* $\mathcal{C}^{\mathrm{reg,fil}}$ *over* $\mathrm{Spev}(k)$ *such that:*

- *its underlying category* $\mathcal{C}^{\mathrm{reg}}$ *is the full subcategory of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ *spanned by the locally constant sheaves, and*
- *if* $\mathcal{C}^{\mathrm{reg,gr}}$ *denotes the associated graded category, then* $\mathcal{C}^{\mathrm{reg,gr}} \otimes_{\pi_0(k)} \mathrm{F}$ *is equivalent to* $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}})$ *for some open locus* $\mathrm{G}^{\mathrm{reg}}_{\mathbf{H}} \subseteq \mathrm{G}_{\mathbf{H}}$ *with complement of codimension* 2.

Similarly, if G *is of type* A *or type* D *(and in the latter case, assume that* 2 *is a unit in k), then there is a filtered category* $\mathcal{C}^{\mathrm{fil}}$ *over* $\mathrm{Spev}(k)$ *such that:*

- *its underlying category* $\mathcal{C}$ *is the full subcategory of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ *generated by convolutions of sequences of minuscule IC-sheaves, and*
- *if* $\mathcal{C}^{\mathrm{gr}}$ *denotes the associated graded category, then* $\mathcal{C}^{\mathrm{gr}} \otimes_{\pi_0(k)} \mathrm{F}$ *is equivalent to* $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$.

[7] This is the difference between the "tempered" and "true" versions of the derived geometric Satake equivalence.

[8] Here, we have switched to using algebraizations of the formal group over $\mathrm{Spev}(k)$, because we will work with genuine equivariant sheaves below. We encourage the reader to ignore this issue; it is closely related to issues of "renormalization" that play an important role in the geometric Langlands correspondence [AG]. These issues are exacerbated for general $\mathbf{E}_\infty$-rings k.

We expect (an appropriate analogue of) the second part of Theorem 1.1.2 to hold for exceptional types, as well as for algebraically closed fields of *arbitrary* characteristic. As a first step towards this, we show in Theorem 3.7.7 that for any connected reductive group G over $\mathbf{C}$, the stack $\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k))$ with k being an algebraically closed field of arbitrary characteristic can be identified with the "Whittaker stack" $\check{\mathrm{G}}\backslash\mathrm{T}^*(\check{\mathrm{G}}/_\psi\check{\mathrm{N}})$ associated to the Langlands dual group, where k is subject only to a constraint on the existence of a nondegenerate Whittaker character.

Unfortunately, our proof of Theorem 1.1.2 is a bit unsatisfying, in that it does not work for the exceptional groups; it also does not use the definition of $\mathrm{G}_{\mathbf{H}}$ as the $\mathbf{H}$-loop space of BG (rather, it uses an explicit model for $\mathrm{G}_{\mathbf{H}}$ for each of the three choices of $\mathbf{H}$). In particular, it is unclear how Conjecture 1.1.1 might be proved for other $\mathbf{E}_\infty$-rings (like the sphere spectrum). Nevertheless, our argument for Theorem 1.1.2 shows that Conjecture 1.1.1 is natural in the $\mathbf{E}_\infty$-ring k, in that the sense that power operations (like the Tate-valued Frobenius [NS]) acting on $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ degenerate to operations on $\mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$ determined entirely by their effect on the formal group $\mathbf{H}$. See § 4.4 for more on this subject.

Let us mention some previous work towards analogues of the geometric Satake equivalence with generalized coefficients. For instance, an early paper in the context of geometric representation theory is [GKV1]. A conjecture about derived geometric Satake with coefficients in KU was proposed in [CK]; in a similar vein, a discussion of the case of KU is the content of the talk [Lon3]. In [YZ1], Yang and Zhao study a higher chromatic analogue of quantum groups, and it would be interesting to study the relationship between Conjecture 4.3.20 and their work.

Before proceeding to relative Langlands duality, let us mention some possible future directions of research which build on Conjecture 1.1.1 and Theorem 1.1.2. First, the derived Satake equivalence of Bezrukavnikov-Finkelberg admits a *deformation quantization*, given by keeping track of a canonical action of $\mathbf{G}_m$ on G(F) and G($\mathcal{O}$) by loop rotation. It is natural to ask if there is an analogue of Conjecture 1.1.1 which incorporates loop rotation equivariance. The corresponding degeneration of $\mathrm{Shv}_{(\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O}))\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{G}(\mathrm{F}); k)$ would be a deformation quantization of $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$. The quantization takes place along the parameter $\hbar$ in $\pi_*(k^{h\mathbf{G}_m^{\mathrm{rot}}}) \cong \pi_*(k)[\hbar]^\wedge$[9]. We discuss this question in § 4.6 (see, in particular, Conjecture 4.6.5 and Conjecture 4.6.6). The question of understanding the degeneration of $\mathrm{Shv}_{(\mathrm{G}(\mathcal{O})\times\mathrm{G}(\mathcal{O}))\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{G}(\mathrm{F}); k)$ becomes even more interesting when k itself has an interesting S^1-action. For example, if $k = \mathrm{THH}(\mathbf{F}_p)$ or $\mathrm{THH}(\mathbf{Z}_p)$, then the degeneration in question seems to be closely related to the theory of prismatic stacks à la [BL, Bha3, Dri2] and the Drinfeld formal group from [Dri1]. We will explore this in future work (it is, in some sense, a synthesis of our work in Part I and Part II).

Second, one could ask about variants of the above discussion for *global* geometric Langlands. I have done much less work on this, save for some calculations in genus 1, but I hope to study this topic in later work. If C is a curve of genus g, it is closely related to understanding the classical stack obtained by applying the even filtration to (the ring of functions of) the putative character variety $\check{\mathrm{G}}_k^{2g} \times_{\check{\mathrm{G}}_k} \{1\}$ over the $\mathbf{E}_\infty$-ring k. (Here, the map $\check{\mathrm{G}}_k^{2g} \to \check{\mathrm{G}}_k$ is given by taking commutators: $(a_1, b_1, \cdots, a_g, b_g) \mapsto [a_1, b_1]\cdots[a_g, b_g]$.)

Thirdly, it is natural to ask whether generalizing the choice of coefficients k to allow general $\mathbf{E}_\infty$-rings might produce interesting results in the *arithmetic* Langlands program. For example, it is conceivable that one could study the mod p cohomology of arithmetic groups by descending information about their k-cohomology, where k is a $\mathrm{K}(n)$- (or $\mathrm{T}(n)$-)local $\mathbf{E}_\infty$-ring

[9] This denotes a Nygaard completion; so if $\pi_*(k) = \pi_0(k)[\beta]$ with β in weight 2, for example, then $\pi_*(k)[\hbar]^\wedge \cong \pi_0(k)[\![t]\!][\beta, \hbar]/(\beta\hbar = t)$.

spectrum, and that the latter behaves similarly to the case when k is a $\mathbf{Q}$-algebra thanks to the phenomenon of ambidexterity in chromatic homotopy theory [HS, HL, CSY]. I find this an extremely interesting avenue of research, and I hope to explore it further in the future.

1.1.3 Relative Langlands duality

Relative Langlands duality, whose basic principles have been recorded recently in [BZSV], essentially aims to match objects under the Langlands correspondence. For instance, if F is a local field, and X is a G-space (over $\mathbf{Z}$, say), then one might try to describe the structure of the Galois representation associated to the representation $\mathrm{L}^2(\mathrm{X}(\mathrm{F}); \mathbf{C})$ of $\mathrm{G}(\mathrm{F})$. Similarly, if $\mathrm{F} = \mathbf{C}(\!(t)\!)$ is a "geometric" local field, one might try to provide a spectral decomposition of the category $\mathrm{Shv}(\mathrm{X}(\mathrm{F}); \mathbf{C})$ by describing it as the category of (quasi-)coherent sheaves on an object living over the stack $\mathrm{Loc}_{\check{\mathrm{G}}}(\mathrm{D}^\circ)$ of Langlands parameters.

We will focus in particular on the consequences of relative Langlands duality at the level of spherical[10] invariants. Namely, suppose X is a G-space over $\mathbf{C}$, and let $\mathcal{O} \subseteq \mathrm{F}$ denote $\mathbf{C}[\![t]\!] \subseteq \mathbf{C}(\!(t)\!)$. Then the category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ admits an action of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ by convolution. In particular, following the degenerations philosophy suggested above, one might expect that there is a filtered lift $\mathcal{C}^{\mathrm{fil}}$ of the category $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ along with an action of the filtered lift of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ from Conjecture 1.1.1. Assuming this conjecture, the corresponding graded category $\mathcal{C}^{\mathrm{gr}}$ should have an action of the graded category $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$. It is therefore reasonable to expect that $\mathcal{C}^{\mathrm{gr}} \simeq \mathrm{QCoh}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$ for some $\check{\mathrm{G}}$-space $\check{\mathrm{M}}_{\mathbf{H}}$ over $\mathrm{Spev}(k)$ equipped with an $\check{\mathrm{G}}$-equivariant map $\mu : \check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}}$.

The $\check{\mathrm{G}}$-space $\check{\mathrm{M}}_{\mathbf{H}}$ admits further structure. Namely, the convolution monoidal structure on $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ refines to an $\mathbf{E}_3$-monoidal structure. On the spectral side, this equips the stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ with a 1-*shifted symplectic structure* in the sense of [PTVV]. Similarly, if $\mathrm{Shv}^{\mathrm{Sat}}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ denotes the full subcategory of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ generated by the delta sheaf at $\mathrm{X}(\mathcal{O}) \subseteq \mathrm{X}(\mathrm{F})$ under the Hecke action of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$, then $\mathrm{Shv}^{\mathrm{Sat}}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ becomes an $\mathbf{E}_2$-monoidal category with an $\mathbf{E}_2$-monoidal action of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$. This structure degenerates to equip the map $\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ with the structure of a *Lagrangian morphism* in the sense of [PTVV]. In summary, we are led to conjecture:

Conjecture 1.1.3 (Conjecture 5.2.20)**.** *Suppose* G *is a connected reductive group over* $\mathbf{C}$ *and* X *is an affine* G*-space. Then there is a* $\check{\mathrm{G}}$*-space* $\check{\mathrm{M}}_{\mathbf{H}}$ *over* $\mathrm{Spev}(k)$ *and a* 1*-parameter degeneration of* $\mathrm{Shv}^{\mathrm{Sat}}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ *into the graded category* $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$*. If* G *is simply-laced with torsion-free fundamental group, then assuming Conjecture 1.1.1, the action of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O}) \times \mathrm{G}(\mathcal{O})}(\mathrm{G}(\mathrm{F}); k)$ *on* $\mathrm{Shv}^{\mathrm{Sat}}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ *degenerates to an action of* $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$ *via a Lagrangian morphism* $\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$*.*

When k is an ordinary commutative ring, so $\mathbf{H} = \mathbf{G}_a(2)$ (where we work with genuine equivariance for simplicity of presentation), then there is an identification $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}} \cong \check{\mathfrak{g}}^*(2)/\check{\mathrm{G}}$, and the data of a Lagrangian morphism $\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ is simply the datum of a graded Hamiltonian $\check{\mathrm{G}}$-space $\check{\mathrm{M}}$ along with its moment map $\check{\mathrm{M}} \to \check{\mathfrak{g}}^*(2)$. Conjecture 1.1.3 then becomes the local form of the relative Langlands conjecture from [BZSV, Conjecture 7.5.1]. The Hamiltonian $\check{\mathrm{G}}$-space $\check{\mathrm{M}}$ is sometimes called the "relative Langlands dual" of X (or really, of $\mathrm{T}^*\mathrm{X}$). (Already in this case, Conjecture 1.1.3 "explains" some subtleties in the proposal for the relative Langlands program from [BZSV] related to issues of spectral quantization; see Remark 5.2.24.) Similarly, if $k = \mathrm{KU}$ with $\mathbf{H} = \mathbf{G}_m$ (again, working with genuine equiv-

[10]Not my choice of terminology!

ariance for simplicity), then $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}} \cong \mathrm{G}/\check{\mathrm{G}}$, and a Lagrangian morphism $\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ is essentially the data of a "quasi-Hamiltonian $\check{\mathrm{G}}$-space" in the sense of [AMM].

Building on several works (like [BFGT, CMNO, CO]), we study some examples of relative Langlands duality. We will assume $k = \mathrm{ku}$, so that $\mathrm{Spev}(\mathrm{ku}) \cong \mathrm{Spec}(\mathbf{Z}[\beta])/\mathbf{G}_m$ and $\mathbf{H} = \mathbf{G}_\beta = \mathrm{Spec}(\mathbf{Z}[\beta, x, \frac{1}{1+\beta x}])/\mathbf{G}_m$ with group law $x + y + \beta xy$. One can then identify $(\mathrm{GL}_n)_{\mathbf{H}}$ with the space of those $n \times n$-matrices A such that $\det(\mathrm{id}_n + \beta \mathrm{A})$ is a unit. We show:

Theorem 1.1.4 (Corollary 5.5.8, Theorem 5.5.11, Theorem 5.5.16)**.** *Let* F *be an algebraically closed field, and let* $k = \mathrm{ku}$*. Then:*

- *Let* $\mathrm{G} = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$*, and let* $\mathrm{X} = \mathrm{GL}_n$ *with the left and right actions of* GL_n *and* GL_{n-1}*. Let* $\check{\mathrm{M}}_{\mathbf{H}}$ *denote the scheme over* $\mathrm{F}[\beta]$ *defined as the open locus of those* $(u, v) \in \mathrm{T}^* \mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n)$ *such that* $\det(\mathrm{id} + \beta uv)$ *is a unit. Then there is a filtered category* $\mathcal{C}^{\mathrm{fil}}$ *over* $\mathrm{Spev}(k)$ *such that its underlying category* $\mathcal{C}$ *is the full subcategory of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ *generated by convolutions of sequences of minuscule IC-sheaves, and if* $\mathcal{C}^{\mathrm{gr}}$ *denotes the associated graded category, then there is an equivalence*

$$\mathcal{C}^{\mathrm{gr}} \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}}).$$

  Here, the Lagrangian morphism $\mu : \check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ *sends* $(u, v) \mapsto (uv, vu)$*.*

- *Let* $\mathrm{G} = \mathrm{GL}_{2n}$*, and let* $\mathrm{X} = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$*. Embed* GL_n *into* GL_{2n} *via* $\mathrm{A} \mapsto \mathrm{diag}(\mathrm{A}, \mathrm{A})$*, and define* $\check{\mathrm{M}}_{\mathbf{H}} := \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n(4)$ *as a scheme over* $\mathrm{F}[\beta]$*. Then there is a filtered category* $\mathcal{C}^{\mathrm{fil}}$ *over* $\mathrm{Spev}(k)$ *such that its underlying category* $\mathcal{C}$ *is the full subcategory of* $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k)$ *generated by convolutions of sequences of minuscule IC-sheaves, and if* $\mathcal{C}^{\mathrm{gr}}$ *denotes the associated graded category, then there is an equivalence*

$$\mathcal{C}^{\mathrm{gr}} \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}}).$$

  Here, the Lagrangian morphism $\mu : \check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ *sends* $x \in \mathfrak{gl}_n$ *to the matrix* $\left(\begin{smallmatrix} \beta x & \mathrm{id}_n \\ x & 0 \end{smallmatrix}\right) \in (\mathrm{GL}_{2n})_{\mathbf{H}}$*.*

For the final example, suppose k *is an algebraically closed field, and let* $n \geq 2$ *be an even integer. Let* $\mathfrak{C}_n$ *denote the affine cone on the secant variety of lines on the Segre embedding* $(\mathbf{P}^1)^{n+1} \to \mathbf{P}^{2^{n+1}-1}$*. There is a* $\mathbf{G}_m$*-action on* $\mathbf{P}^1$ *given by* $[x : y] \mapsto [\lambda^2 x : y]$*, hence a* $\mathbf{G}_m$*-action on* $(\mathbf{P}^1)^{n+1}$*, and thus on* $\mathfrak{C}_n$*. Then, there is a filtered category* $\mathcal{C}^{\mathrm{fil}}$ *over* $\mathrm{Spev}(k) \cong (\mathrm{B}\mathbf{G}_m)_k$ *such that its underlying category* $\mathcal{C}$ *is the full subcategory of* $\mathrm{Shv}_{\mathrm{SO}_3(\mathcal{O})^{n+1}}(\mathrm{SO}_3(\mathrm{F})^n; k)$ *generated by convolutions of sequences of minuscule IC-sheaves, and if* $\mathcal{C}^{\mathrm{gr}}$ *denotes the associated graded category, then there is an equivalence*

$$\mathcal{C}^{\mathrm{gr}} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{C}_n/\mathrm{SL}_2^{n+1}).$$

When $n = 2$, the scheme $\mathfrak{C}_n$ can be identified with the triple tensor product $\mathbf{A}^2 \otimes \mathbf{A}^2 \otimes \mathbf{A}^2$ as an SL_2^3-space, and the final part of the preceding theorem then recovers work of Bhargava [Bha1]. It implies, for instance, that under the degeneration of $\mathrm{Shv}_{\mathrm{SO}_3(\mathcal{O}) \times \mathrm{SO}_3(\mathcal{O})}(\mathrm{SO}_3(\mathrm{F}); k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{sl}_2^*(2)/\mathrm{SL}_2)$ and the identification of $\mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$ with the space $(\mathbf{A}^2 \otimes \mathbf{A}^2)_{\mathbf{Z}/2} = \mathrm{Sym}^2(\mathbf{A}^2)$ of binary quadratic forms, the standard tensor product (which is not convolution!) on $\mathrm{Shv}_{\mathrm{SO}_3(\mathcal{O}) \times \mathrm{SO}_3(\mathcal{O})}(\mathrm{SO}_3(\mathrm{F}); k)$ degenerates to the symmetric monoidal structure on $\mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{sl}_2^*(2)/\mathrm{SL}_2)$ coming from Gauss composition of binary quadratic forms!

We also study some mild extensions of Theorem 1.1.4, such as the cases when X is $\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}$, $\mathrm{PGL}_{n+1}/\mathrm{GL}_n$, $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$, $\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$, $\mathrm{F}_4/\mathrm{Spin}_9$, or $\mathrm{G}_2/\mathrm{SL}_3$.

There are many variants of Theorem 1.1.4 which we do not touch on here. For instance, one can ask about proving a version of these results incorporating loop rotation equivariance. Following the discussion at the end of § 1.1.2, one expects that the corresponding degeneration of $\mathrm{Shv}_{\mathrm{G}(\mathcal{O})\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{X}(\mathrm{F}); k)$ would be a deformation quantization of $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$. Again, the quantization takes place along the parameter $\hbar \in \pi_{-2}(k^{h\mathbf{G}_m^{\mathrm{rot}}})$. We hope to show that when $k = \mathrm{ku}$ and the (usual) relative Langlands dual (in the sense of [BZSV]) to X is a Hamiltonian $\check{\mathrm{G}}$-space of the form $\mathrm{T}^*\check{\mathrm{X}}$ for some $\check{\mathrm{G}}$-space $\check{\mathrm{X}}$, this deformation is closely related to the q-de Rham cohomology of $\check{\mathrm{X}}$. One can also ask about extending the picture above to the setting of *global* relative Betti geometric Langlands (with coefficients in a general $\mathbf{E}_\infty$-ring), as in [BZSV, Part 3]. I hope to explore these avenues of research in the future.

1.2 Spherochromatism in arithmetic geometry

1.2.1 Background

One of the oldest results relating noncommutative geometry to commutative algebra is the Hochschild-Kostant-Rosenberg theorem [HKR1]. In modern language (see [Ant, Rak]), it states that if k is an ordinary commutative ring and R is a commutative k-algebra, then there is a filtration on $\mathrm{HH}(\mathrm{R}/k)$ whose associated graded pieces are given by $\bigoplus_i \mathrm{L}\Omega^i_{\mathrm{R}/k}[i](i) = \mathrm{Sym}_{\mathrm{R}}(\mathrm{L}_{\mathrm{R}/k}[1](1))$. The process of shearing, which takes a graded module $\mathrm{M}(n)$ in weight n and produces the graded module $\mathrm{M}[2n](n)$, identifies this associated graded with the shearing of the (derived) Hodge cohomology $\bigoplus_i \mathrm{L}\Omega^i_{\mathrm{R}/k}[-i](i) =: \mathrm{L}\Omega^*_{\mathrm{R}/k}$.

Under this degeneration of $\mathrm{HH}(\mathrm{R}/k)$ into the shearing of $\mathrm{L}\Omega^*_{\mathrm{R}/k}$, the k-linear S^1-action on $\mathrm{HH}(\mathrm{R}/k)$ gets identified with the shearing of the de Rham differential. This implies, for instance, that there is a filtration on $\mathrm{HP}(\mathrm{R}/k) := \mathrm{HH}(\mathrm{R}/k)^{t\mathrm{S}^1}$ whose associated graded is given by the shearing of (the Hodge completion of) the derived de Rham complex $\mathrm{L}\Omega_{\mathrm{R}/k}$ (see [Bha2]) placed in weight zero.

The unit map from the sphere spectrum to k allows us to view R as an $\mathbf{E}_\infty$-algebra over the sphere; in particular, one can consider the Hochschild homology $\mathrm{HH}(\mathrm{R}/\mathrm{S})$ of R relative to S, defined in the usual way as the tensor product $\mathrm{R} \otimes_{\mathrm{R}\otimes\mathrm{R}} \mathrm{R}$ (where now $\otimes$ is taken in the ∞-category of spectra). This object is denoted $\mathrm{THH}(\mathrm{R})$, and is called the *topological Hochschild homology* of R. Note that if $k = \mathbf{Q}$, then the canonical map $\mathrm{THH}(\mathrm{R}) \to \mathrm{HH}(\mathrm{R}/k)$ is an equivalence (because $\mathbf{Q}$ is an idempotent S-algebra), so we do not gain anything new by passing to THH.

However, the map $\mathrm{THH}(\mathrm{R}) \to \mathrm{HH}(\mathrm{R}/k)$ is *not* an equivalence if $k \neq \mathbf{Q}$, and in fact $\mathrm{THH}(\mathrm{R})$ posesses further structure which is invisible to $\mathrm{HH}(\mathrm{R}/k)$. Namely, it admits a *cyclotomic Frobenius*, which is an S^1-equivariant map $\varphi : \mathrm{THH}(\mathrm{R}) \to \mathrm{THH}(\mathrm{R})^{t\mathbf{Z}/p}$ from $\mathrm{THH}(\mathrm{R})$ into its $\mathbf{Z}/p$-Tate construction[11] (equipped with the residual action of $\mathrm{S}^1/(\mathbf{Z}/p) \cong \mathrm{S}^1$). The structure of an S^1-equivariant spectrum X along with an S^1-equivariant map $\varphi : \mathrm{X} \to \mathrm{X}^{t\mathbf{Z}/p}$ is called a (p-)*cyclotomic spectrum*, and was studied in detail by Nikolaus-Scholze [NS].

Just as with the Hochschild-Kostant-Rosenberg theorem, one can again construct a "motivic" filtration on $\mathrm{THH}(\mathrm{R})$ whose associated graded pieces are interesting arithmetic invariants of R. Namely, fix a prime p and assume that R is a p-complete discrete commutative ring with bounded p-power torsion whose cotangent complex $\mathrm{L}_{\mathrm{R}/\mathbf{Z}_p}$ has Tor-amplitude in $[0, 1]$, Building on the work of many others, like Hesselholt and Madsen, Bhatt-Morrow-Scholze [BMS, BS1] showed that there is a filtration on $\mathrm{THH}(\mathrm{R})$ – and hence related invariants like the $\mathbf{Z}/p$-Tate construction $\mathrm{THH}(\mathrm{R})^{t\mathbf{Z}/p}$, the topological negative cyclic homology

[11] Sometimes also called the "Tate cohomology".

$\mathrm{TC}^-(\mathrm{R}) = \mathrm{THH}(\mathrm{R})^{h\mathrm{S}^1}$, the topological periodic cyclic homology $\mathrm{TP}(\mathrm{R}) = \mathrm{THH}(\mathrm{R})^{t\mathrm{S}^1}$, and the topological cyclic homology $\mathrm{TC}(\mathrm{R})$ defined as the equalizer of the canonical and Frobenius maps $\mathrm{can}, \varphi : \mathrm{TC}^-(\mathrm{R}) \rightrightarrows \mathrm{TP}(\mathrm{R})$ – whose associated graded pieces are given by the shearing of certain natural invariants associated to R by the theory of prismatic cohomology. More precisely, they showed that the natural commutative diagram

$$\begin{tikzcd}
\mathrm{TC}^-(\mathrm{R}) \arrow[r, "\varphi"'] \arrow[d] & \mathrm{TP}(\mathrm{R}) \arrow[d] \\
\mathrm{THH}(\mathrm{R}) \arrow[r, "\varphi"'] & \mathrm{THH}(\mathrm{R})^{t\mathbf{Z}/p}
\end{tikzcd}$$

admits a lift to a diagram in filtered $\mathbf{E}_\infty$-rings such that the corresponding diagram in graded derived commutative rings can be identified with the shearing of the diagram

$$\begin{tikzcd}
\mathcal{N}^{\geq\star}\widehat{\mathbb{\Delta}}_\mathrm{R} \arrow[r, "\varphi"'] \arrow[d] & \widehat{\mathbb{\Delta}}_\mathrm{R}\{\star\} \arrow[d] \\
\mathcal{N}^{\star}\widehat{\overline{\mathbb{\Delta}}}_\mathrm{R} \arrow[r, "\varphi"'] & \widehat{\overline{\mathbb{\Delta}}}_\mathrm{R}\{\star\}.
\end{tikzcd}$$

Here, $\widehat{\mathbb{\Delta}}_\mathrm{R}$ denotes the Nygaard-completion of the *absolute prismatic cohomology* $\mathbb{\Delta}_\mathrm{R}$ of R, the symbol $\{\star\}$ denotes the Breuil-Kisin twist, $\mathcal{N}^{\geq\star}\widehat{\mathbb{\Delta}}_\mathrm{R}$ denotes the Nygaard filtration on prismatic cohomology, $\widehat{\overline{\mathbb{\Delta}}}_\mathrm{R}$ denotes the Nygaard-completed Hodge-Tate cohomology of R, and $\mathcal{N}^\star\widehat{\mathbb{\Delta}}_\mathrm{R}$ denotes the associated graded of the Nygaard filtration.

The (absolute) prismatic cohomology of R mentioned above is a very interesting 1-parameter "arithmetic" deformation[12] of the de Rham cohomology of R relative to $\mathbf{Z}_p$. For instance, if R is an $\mathbf{F}_p$-algebra, then $\mathbb{\Delta}_\mathrm{R}$ is (a Frobenius untwist of) the crystalline cohomology of R (which is a deformation of the de Rham cohomology of R relative to $\mathbf{F}_p$ along the parameter $p \in \mathbf{Z}_p$), the Hodge-Tate cohomology $\overline{\mathbb{\Delta}}_\mathrm{R}$ is (a Frobenius untwist of) the de Rham cohomology of R relative to $\mathbf{F}_p$, and the nth associated graded piece of the Nygaard filtration identifies with the nth step of the conjugate filtration on the (derived) de Rham cohomology of R relative to $\mathbf{F}_p$. Similarly, if R is an algebra over $\mathbf{Z}_p^{\mathrm{cyc}} := \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge_p$, then $\mathbb{\Delta}_\mathrm{R}$ can be identified with the "q-de Rham cohomology" of R, which is an interesting deformation of the de Rham cohomology of R relative to $\mathbf{Z}_p^{\mathrm{cyc}}$ along the parameter $[p]_q = \frac{q^p-1}{q-1} \in \mathbf{Z}_p[q^{\pm 1/p^\infty}]^\wedge_{(p,q-1)}$.

1.2.2 The topological Hochschild homology of $\mathbf{Z}_p$

Hahn-Raksit-Wilson [HRW] recently gave a purely homotopy-theoretic construction of these motivic filtrations on THH and its friends, via the even filtration. For instance, the even filtration on THH(R) is precisely Bhatt-Morrow-Scholze's motivic filtration. The key input into identifying the associated graded pieces of these filtrations is the theorem of Bökstedt which identifies the homotopy groups of $\mathrm{THH}(\mathbf{F}_p)$ with the polynomial ring $\mathbf{F}_p[\mu_0]$ with μ_0 in weight 2.

In fact, most of the "large-scale" structure of the prismatic cohomology/topological Hochschild homology of $\mathbf{F}_p$-algebras is determined by the knowledge of $\mathrm{THH}(\mathbf{F}_p)$ as a cyclotomic spectrum. This was completely determined by Nikolaus and Scholze [NS]: they showed that

[12] This degeneration is "orthogonal" to the degenerations discussed in the previous section.

$\mathrm{THH}(\mathbf{F}_p)$ is a Frobenius twist of $\mathbf{Z}_p$, in the following sense. Equip $\mathbf{Z}_p$ with the trivial S^1-action, so that there is a natural map $\mathbf{Z}_p \to \mathbf{Z}_p^{h\mathbf{Z}/p}$, which induces a map $\mathbf{Z}_p \to \mathbf{Z}_p^{h\mathbf{Z}/p} \to \mathbf{Z}_p^{t\mathbf{Z}/p}$. Since $\mathbf{Z}_p$ is connective, we obtain a map $f : \mathbf{Z}_p \to \tau_{\geq 0}(\mathbf{Z}_p^{t\mathbf{Z}/p})$. Now, let $\mathbf{Z}_p^{(-1)}$ denote the cyclotomic spectrum whose underlying spectrum with S^1-action is given by $\tau_{\geq 0}(\mathbf{Z}_p^{t\mathbf{Z}/p})$, and whose cyclotomic Frobenius is given by the composite

$$\tau_{\geq 0}(\mathbf{Z}_p^{t\mathbf{Z}/p}) \to \mathbf{Z}_p^{t\mathbf{Z}/p} \xrightarrow{f^{t\mathbf{Z}/p}} (\tau_{\geq 0}(\mathbf{Z}_p^{t\mathbf{Z}/p}))^{t\mathbf{Z}/p}.$$

There is a natural map of cyclotomic $\mathbf{E}_\infty$-rings $\mathbf{Z}_p \to \mathbf{Z}_p^{(-1)}$. Nikolaus and Scholze then showed that there is a unique map $\mathbf{Z}_p \to \mathrm{THH}(\mathbf{F}_p)$ of cyclotomic $\mathbf{E}_\infty$-rings which identifies $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{Z}_p^{(-1)}$ (as cyclotomic $\mathbf{E}_\infty$-rings). Furthermore, the map $\mathbf{Z}_p \to \mathrm{THH}(\mathbf{F}_p)$ induces an equivalence $\mathbf{Z}_p^{t\mathrm{S}^1} \xrightarrow{\sim} \mathrm{TP}(\mathbf{F}_p)$.

Note that this result is a quantitative form of the philosophy of *chromatic redshift* [Rog1, Rog2], which states that the topological periodic cyclic homology of a ring of chromatic height n should have chromatic height $n+1$. Here, the ring $\mathbf{F}_p$ has chromatic height -1, and the ring $\mathbf{Z}_p$ has chromatic height 0; so the equivalence $\mathrm{TP}(\mathbf{F}_p) \simeq \mathbf{Z}_p^{t\mathrm{S}^1}$ confirms chromatic redshift in this case.

The theory of absolute prismatic cohomology becomes particularly interesting when the input R has mixed characteristic. Again, the "large-scale" structure of the prismatic cohomology/topological Hochschild homology of $\mathbf{Z}_p$-algebras is determined by the knowledge of $\mathrm{THH}(\mathbf{Z}_p)$ as a cyclotomic spectrum. In fact, $\mathrm{TC}(\mathbf{Z}_p)$ had already been computed by Bökstedt-Madsen [BM] when p is odd (but only as a spectrum!), but this analysis does not describe $\mathrm{THH}(\mathbf{Z}_p)$ itself. In joint work with Arpon Raksit, we refine their work to completely describe $\mathrm{THH}(\mathbf{Z}_p)$ as a cyclotomic spectrum.

To state the result, let p be an odd prime, and let j_p denote the *image of J* spectrum, defined as the connective cover of the K(1)-local sphere $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}$. Equivalently, by the Adams conjecture (which is a theorem), it is the connective cover of the homotopy fixed points $\mathrm{KU}_p^{h\mathbf{Z}_p^\times}$ of p-complete complex K-theory by the continuous action of the p-adic units $\mathbf{Z}_p^\times$ via Adams operations. As before, let $j_p^{(-1)}$ denote the cyclotomic spectrum whose underlying spectrum with S^1-action is given by $\tau_{\geq 0}(j_p^{t\mathbf{Z}/p})$, and whose cyclotomic Frobenius is given by the composite

$$\tau_{\geq 0}(j_p^{t\mathbf{Z}/p}) \to j_p^{t\mathbf{Z}/p} \xrightarrow{f^{t\mathbf{Z}/p}} (\tau_{\geq 0}(j_p^{t\mathbf{Z}/p}))^{t\mathbf{Z}/p}.$$

The map $f : j_p \to \tau_{\geq 0}(j_p^{t\mathbf{Z}/p})$ is again induced by the composite $j_p \to j_p^{h\mathbf{Z}/p} \to j_p^{t\mathbf{Z}/p}$. There is a natural map of cyclotomic $\mathbf{E}_\infty$-rings $j_p \to j_p^{(-1)}$.

Theorem 1.2.1 (Joint with A. Raksit; Theorem 6.1.4)**.** *Fix $p > 2$. There is a unique map $j_p \to \mathrm{THH}(\mathbf{Z}_p)$ of cyclotomic $\mathbf{E}_\infty$-rings which identifies $\mathrm{THH}(\mathbf{Z}_p) \simeq j_p^{(-1)}$ (as cyclotomic $\mathbf{E}_\infty$-rings). Furthermore, the map $j_p \to \mathrm{THH}(\mathbf{Z}_p)$ induces an equivalence $j_p^{t\mathrm{S}^1} \xrightarrow{\sim} \mathrm{TP}(\mathbf{Z}_p)$.*

Again, this result gives a quantitative form of the chromatic redshift philosophy: the ring $\mathbf{Z}_p$ has chromatic height 0, and the ring j_p has chromatic height 1; so the equivalence $\mathrm{TP}(\mathbf{Z}_p) \simeq j_p^{t\mathrm{S}^1}$ confirms chromatic redshift in this case. Theorem 1.2.1 has numerous applications and extensions. Let us briefly mention some of them:

a. We refine a result of Petrov-Vologodsky [PV] to show that if $\mathcal{C}$ is a dualizable $\mathbf{Z}_p$-linear ∞-category, there is a natural lax symmetric monoidal equivalence

$$\mathrm{TP}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \simeq \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p).$$

In particular, this equivalence is $\mathrm{TP}(\mathbf{F}_p) \simeq \mathbf{Z}_p^{t\mathrm{S}^1}$-linear (this is an improvement on the result of [PV]). This can be viewed as a noncommutative analogue of the comparison between the (Hodge-completed) derived de Rham cohomology of a p-adic formal scheme X and the (Nygaard-completed) crystalline cohomology of its special fiber $\mathrm{X} \times_{\mathrm{Spf}(\mathbf{Z}_p)} \mathrm{Spec}(\mathbf{F}_p)$. Because of issues with Nygaard completion, the displayed equivalence cannot extend to the case $p = 2$ (which also implies that Theorem 1.2.1 cannot hold if $p = 2$).

b. We use Theorem 1.2.1 to give a canonical description of $\mathrm{TC}(\mathbf{Z}_p)$. We also show that if R is a connective $\mathbf{E}_1$-ring, and $\mathrm{F}(-)$ denotes either p-complete TC or algebraic K-theory, then there is a Cartesian square

$$\begin{tikzcd} \mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\pi_0\mathrm{R}) \arrow[d] \\ \mathrm{TC}^-(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}) \arrow[r] & \mathrm{TP}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}). \end{tikzcd}$$

In particular, the fiber of the top horizontal map is $\Sigma\mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})_{h\mathrm{S}^1}$, so we can completely describe the failure of K(1)-local algebraic K-theory/TC to be a truncating invariant on connective $\mathbf{E}_1$-rings.

c. Further analysis using the preceding result gives a complete calculation of the K(1)-local algebraic K-theory of several interesting ring spectra. For instance, the maps $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{S}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(j_p) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S})$ are all equivalences, and are all canonically equivalent (also see [BHM]) to

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{S}) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{S} \oplus \Sigma\mathrm{L}_{\mathrm{K}(1)}\mathrm{S} \oplus \mathrm{L}_{\mathrm{K}(1)}\, \mathrm{fib}(\mathrm{tr} : \Sigma\mathbf{C}\mathrm{P}^\infty \to \mathrm{S}).$$

Similarly, there is an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Q}_p) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{S} \oplus \Sigma\mathrm{L}_{\mathrm{K}(1)}\mathrm{S} \oplus \mathrm{KU}_p,$$

as well as an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{KU}_p) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{KU}_p[\mathbf{C}\mathrm{P}^\infty].$$

It is quite interesting that in all of these cases, $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R})$ splits into a direct sum of the form $\mathrm{A} \oplus \Sigma\mathrm{A} \oplus \Sigma\mathrm{B}$. I expect that, at least in the latter two cases, this phenomenon is closely related to an extension of the ideas of "arithmetic topology" to the setting of ring spectra.

There is a variant of Theorem 1.2.1 which describes $\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ instead of $\mathrm{THH}(\mathbf{Z}_p)$: one needs to replace j_p by $j_{p,0} := \tau_{\geq 0}(\mathrm{KU}_p^{h(1+p\mathbf{Z}_p^\times)})$. Using this variant, a result suggested to me by Jacob Lurie states:

Theorem 1.2.2 (Theorem 6.4.1)**.** *Let $p > 2$, and view $\mathbf{Z}_p[\zeta_p]$ as an $\mathrm{S}[\![q^{1/p}-1]\!]$-algebra via the map $q^{1/p} \mapsto \zeta_p$. Then there is a $\mathbf{Z}_p^\times$-equivariant equivalence of cyclotomic $\mathbf{E}_\infty$-$\mathrm{S}[\![q-1]\!]$-algebras*

$$\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!]).$$

Furthermore, these are equivalent to $\mathrm{ku}_p^{(-1)}$ as $\mathrm{S}^1 \times \mathbf{Z}_p^\times$-equivariant $\mathbf{E}_\infty$-$\mathrm{S}[\![q-1]\!]$-algebras.

A. Raksit communicated to me that Nikolaus had previously proved an equivalence of S^1-equivariant $\mathbf{E}_1$-rings between $\mathrm{ku}_p^{(-1)}$ and $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/S[\![q^{1/p}-1]\!])$; see [MW, Theorem 3.18] for an argument. Theorem 1.2.2 has several interesting applications. For instance, it implies easily that if R is an $\mathbf{E}_1$-ring, then a Frobenius twist of $\mathrm{HP}((\mathrm{R}\otimes\mathrm{ku}_p)/\mathrm{ku}_p)$ identifies ($\mathbf{Z}_p^\times$-equivariantly) with the periodic cyclic homology $\mathrm{TP}((\mathrm{R}\otimes\mathbf{Z}_p[\zeta_p])/S[\![q^{1/p}-1]\!])$. This can be viewed as a version of the comparison between q-de Rham cohomology and prismatic cohomology relative to the q-de Rham prism.

Here is another application of Theorem 1.2.2 (see Corollary 6.4.5). Let $\mathcal{C}$ be a $\mathbf{Z}_p[\zeta_p]$-linear ∞-category (such as $\mathrm{QCoh}(\mathrm{X})$ for a p-adic formal scheme over $\mathrm{Spf}(\mathbf{Z}_p[\zeta_p])$), and let $\mathcal{C}_0=\mathcal{C}\otimes_{\mathbf{Z}_p[\zeta_p]}\mathbf{F}_p$ denote its special fiber. Let $\mathrm{F}(\mathcal{C})$ denote the total fiber of the following commutative square:

$$\begin{tikzcd} \mathrm{TC}(\mathcal{C}_0)\times\mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \arrow[r] \arrow[d] & \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \arrow[d] \\ \mathrm{HC}^-(\mathcal{C}_0/\mathbf{F}_p) \arrow[r] & \mathrm{HP}(\mathcal{C}_0/\mathbf{F}_p). \end{tikzcd} \tag{1.2.1}$$

The only nontrivial piece of this square is the map $\mathrm{TC}(\mathcal{C}_0)\to\mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])$, for which we refer the reader to Construction 6.4.4. Then, Theorem 1.2.2 implies that there is a natural lax symmetric monoidal equivalence

$$\mathrm{F}(\mathcal{C})\simeq\mathrm{TC}(\mathcal{C})\otimes_{j_{p,0}}\mathbf{Z}_p.$$

In particular, the natural map from $\mathrm{TC}(\mathcal{C})$ to $\mathrm{F}(\mathcal{C})$ has a filtration

$$\mathrm{TC}(\mathcal{C})\to\cdots\to\mathrm{TC}(\mathcal{C})\otimes_{j_{p,0}}\tau_{\leq 4}(j_{p,0})\to\mathrm{TC}(\mathcal{C})\otimes_{j_{p,0}}\tau_{\leq 2}(j_{p,0})\to\mathrm{TC}(\mathcal{C})\otimes_{j_{p,0}}\tau_{\leq 0}(j_{p,0})=\mathrm{F}(\mathcal{C}),$$

where the fiber of each map $\mathrm{TC}(\mathcal{C})\otimes_{j_{p,0}}\tau_{\leq 2n}(j_{p,0})\to\mathrm{F}(\mathcal{C})$ is killed by $p^n n!$. The analogue of the square (1.2.1) with $\mathrm{TC}(\mathcal{C}_0)$ replaced by the syntomic cohomology of a p-adic formal scheme over $\mathrm{Spf}(\mathbf{Z}_p[\zeta_p])$ was recently announced by Lurie [Lur9]; this result can be proved by adapting our arguments for (1.2.1) to the "synthetic" setting of [AR]. Let us note that upon inverting p, the commutative square (1.2.1) leads to a cofiber sequence

$$\mathrm{F}(\mathcal{C})_{\mathbf{Q}}\to\mathrm{TC}(\mathcal{C}_0)_{\mathbf{Q}}\times\mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}}\to\mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}},$$

and thus the preceding discussion implies that there is a Cartesian square

$$\begin{tikzcd} \mathrm{TC}(\mathcal{C})_{\mathbf{Q}} \arrow[r] \arrow[d] & \mathrm{TC}(\mathcal{C}_0)_{\mathbf{Q}} \arrow[d] \\ \mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}} \arrow[r] & \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}}; \end{tikzcd}$$

thus one recovers the Beilinson fiber square of [AMMN].

1.2.3 Prismatic stacks for ring spectra

Since the work of Bhatt-Morrow-Scholze exposited in § 1.2.1, the picture of prismatic cohomology has taken off in two ways: the even filtration on the THH of $\mathbf{E}_\infty$-rings following [HRW], the theory of *prismatic stacks* à la Bhatt-Lurie-Drinfeld [BL, Bha3, Dri2]. The motivation for the latter came from an observation of Simpson that if X is a smooth scheme over a field of characteristic zero, then there is a stack X^{dR}, known as the *de Rham stack* of X, such that $\mathrm{R}\Gamma(\mathrm{X}^{\mathrm{dR}}; \mathcal{O})$ is equivalent to the de Rham complex of X. More generally, $\mathrm{QCoh}(\mathrm{X}^{\mathrm{dR}})$ is equivalent to the category of D-modules on X.

In [BL, Bha3, Dri2], Bhatt-Lurie-Drinfeld constructed a theory of prismatic stacks: if R is a p-complete commutative ring, then one can construct stacks R^{Δ}, $\mathrm{R}^{\mathrm{Nyg}}$, $\mathrm{R}^{\mathrm{conj}}$, X^{HT} such that its global sections produce the (absolute) prismatic cohomology, the Nygaard-filtered prismatic cohomology, the associated graded pieces of the Nygaard filtration, and the Hodge-Tate cohomology of R when R has bounded p-power torsion and its cotangent complex $\mathrm{L}_{\mathrm{R}/\mathbf{Z}_p}$ has Tor-amplitude in $[0,1]$. There are also two maps $\varphi : \mathrm{R}^{\Delta} \to \mathrm{R}^{\mathrm{Nyg}}$ and $\mathrm{can} : \mathrm{R}^{\Delta} \to \mathrm{R}^{\mathrm{Nyg}}$ which, on global sections, compute the Frobenius and the inclusion of the Nygaard filtration into prismatic cohomology.

On the other hand, Hahn-Raksit-Wilson's homotopy-theoretic construction of the Bhatt-Morrow-Scholze motivic filtration on THH allows one to define similar motivic filtrations on THH (and friends) of an $\mathbf{E}_\infty$*-ring*. In particular, the associated graded pieces of these filtrations define the (Nygaard-completed) prismatic, Hodge-Tate, de Rham, ... cohomology of $\mathbf{E}_\infty$-rings. (Among other developments, we extend these definitions to obtain a non-Nygaard-completed variant of prismatic cohomology in joint work with Jeremy Hahn, Arpon Raksit, and Allen Yuan.) Note that just as Conjecture 1.1.1 puts derived geometric Satake with coefficients in an algebraically closed field on the same footing as the Adams-Novikov filtration on the sphere spectrum, Hahn-Raksit-Wilson's (and subsequently, our) construction of the prismatic cohomology of $\mathbf{E}_\infty$-rings is also placed on the same footing as the Adams-Novikov filtration! Namely, the motivic filtration on THH (and friends) is defined to be the even filtration, just as the Adams-Novikov filtration can be identified with the even filtration on the sphere spectrum.

The situation described above can be summarized by the following cartoon:

$$\begin{tikzcd} & \text{Prismatic, Nygaard, ... \textit{stacks}} \arrow[d, "\text{Bhatt-Lurie-Drinfeld}"] \\ \begin{array}{l}\text{THH as a (decompleted)}\\ \text{cyclotomic spectrum}\end{array} \arrow[ur, dashed, "\text{D.-Hahn-Raksit-Yuan}"] \arrow[r, "\begin{array}{c}\text{Bhatt-Morrow-Scholze}\\ \text{Hahn-Raksit-Wilson}\end{array}"'] & \begin{array}{l}\text{Prismatic, syntomic, ... cohomology}\\ \text{à la Bhatt-Morrow-Scholze}\end{array} \end{tikzcd} \tag{1.2.2}$$

In joint work with Jeremy Hahn, Arpon Raksit, and Allen Yuan, we construct the dashed arrow in the preceding diagram (so that it commutes). Namely, we use the theory of even stacks described previously to construct the stacks R^{Δ}, $\mathrm{R}^{\mathrm{Nyg}}$, $\mathrm{R}^{\mathrm{conj}}$, R^{HT} for a p-complete commutative ring spectrum R using THH(R) (along with some extra data). This, in particular, allows us to define a theory of "coefficients" for the prismatic cohomology of $\mathbf{E}_\infty$-rings.

As mentioned above, [BMS, HRW] have shown that one can recover *Nygaard-completed* versions of (Nygaard-filtered) prismatic cohomology $\hat{\Delta}_{\mathrm{X}}$ from the data of THH(X) viewed as a *cyclotomic spectrum* in the sense of Nikolaus-Scholze [NS]. To sidestep the issue of Nygaard-completion, we construct a refinement of cyclotomic spectra, termed *decompleted cyclotomic spectra*: this is essentially the category of cyclotomic spectra along with a *genuine equivariant* refinement of the underlying spectrum with naïve $\mathbf{Z}/p$-action. We then show that the choice of a decompleted cyclotomic structure on THH(X) can be used to construct not just the (Nygaard-uncompleted!) prismatic cohomology of R, but also the stacks R^{Δ}, $\mathrm{R}^{\mathrm{Nyg}}$, $\mathrm{R}^{\mathrm{conj}}$, R^{HT}. One of the main advantages of our construction, refining that of [HRW], is that it allows one to define the stacks R^{Δ}, $\mathrm{R}^{\mathrm{Nyg}}$, $\mathrm{R}^{\mathrm{conj}}$, R^{HT} for an $\mathbf{E}_\infty$*-ring* R. (In fact, one can also define these stacks if R is only an $\mathbf{E}_3$-ring, by work of Pstragowski [Pst] and forthcoming work of Pstragowski-Raksit.)

One can, for instance, study the stacks S^{Δ}, $\mathrm{S}^{\mathrm{Nyg}}$, $\mathrm{S}^{\mathrm{conj}}$, S^{HT} associated to the sphere spectrum S. We show:

Theorem 1.2.3 (Joint with J. Hahn, A. Raksit, and A. Yuan; Theorem 7.1.11, Theorem 7.2.4)**.** *The various stacks associated to the sphere spectrum can be identified as follows:*

a. The stack $\mathrm{S}^{\mathrm{conj}}$ is isomorphic to $\mathcal{M}_{\mathrm{fg}}$;

b. The stack S^{HT} is also isomorphic to $\mathcal{M}_{\mathrm{fg}}$;

c. The stack S^{Δ} can be identified with the universal 1-dimensional formal group $\widehat{\mathbf{G}}_{\mathrm{univ}}$ over $\mathcal{M}_{\mathrm{fg}}$ (so it classifies 1-dimensional formal groups equipped with a section);

d. The stack $\mathrm{S}^{\mathrm{Nyg}}$ is isomorphic to a certain completion of the moduli stack of "S^1-equivariant formal groups" (see [CGK, Str2, HM1]): these are formal group schemes C equipped with a homomorphism $\alpha : \mathbf{Z} \to \mathrm{C}$ such that C is complete at the image of α and such that the completion of C at $\alpha(1)$ is a 1-dimensional formal group;

e. The canonical map $\mathrm{can} : \mathrm{S}^{\Delta} \to \mathrm{S}^{\mathrm{Nyg}}$ sends a 1-dimensional formal group $\mathbf{H}$ with a section s to the S^1-equivariant formal group (C, α) given by the pushout

$$\begin{tikzcd} p\underline{\mathbf{Z}} \arrow[r, "p\mapsto s"] \arrow[d] & \mathbf{H} \arrow[d] \\ \underline{\mathbf{Z}} \arrow[r, "\alpha"'] & \mathrm{C}. \end{tikzcd}$$

f. The Frobenius map $\varphi : \mathrm{S}^{\Delta} \to \mathrm{S}^{\mathrm{Nyg}}$ sends a 1-dimensional formal group $\mathbf{H}$ with a section s to the S^1-equivariant formal group $\underline{\mathbf{Z}} \to \mathbf{H}$ (sending $1 \mapsto s$).

Furthermore, the canonical map $\mathbf{Z}_p^{\Delta} \to \mathrm{S}^{\Delta}$ coming from the unit map $\mathrm{S} \to \mathbf{Z}_p$ classifies the Drinfeld formal group over $\mathbf{Z}_p^{\Delta}$ constructed in [Dri1], so that the canonical section $v_1 \in \mathrm{H}^0((\mathbf{Z}_p^{\Delta})_{p=0}; \mathcal{O}\{p-1\})$ from [Bha3, Construction 6.2.1] is equal to the Hasse invariant of the Drinfeld formal group.

It is also possible to study the various stacks associated to the $\mathbf{E}_\infty$-ring MU; they turn out to be related to symmetric 2-cocycles valued in $\mathrm{B}\mathbf{G}_m$. However, because these stacks are not as well-understood as the corresponding stacks for the sphere spectrum, we defer discussion of them to our forthcoming paper [DHRY].

My joint work with A. Raksit on Theorem 1.2.1 also provides a description of the prismatic stacks $\mathbf{Z}_p^{\Delta}$, $\mathbf{Z}_p^{\mathrm{Nyg}}$, $\mathbf{Z}_p^{\mathrm{conj}}$, and $\mathbf{Z}_p^{\mathrm{HT}}$ in terms of a canonical decompletion $\mathbf{G}_j$ of the 1-dimensional formal group over $\mathrm{Spev}(j_p)$. For instance, $\mathbf{Z}_p^{\Delta}$ is the complement of the zero section in $\mathbf{G}_j$, and the pullback of the canonical map $\mathbf{Z}_p^{\Delta} \to \mathrm{Spev}(j_p)$ along the covering map $\mathrm{Spev}(\mathrm{ku}_p) \cong \mathbf{A}^1(-1)/\mathbf{G}_m \to \mathrm{Spev}(j_p)$ identifies with the q-de Rham prism $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!])$. We explicitly calculate $\mathrm{Spev}(j_p)$ in Example 2.2.6.

Our construction of the prismatic stacks R^{Δ}, $\mathrm{R}^{\mathrm{Nyg}}$, $\mathrm{R}^{\mathrm{conj}}$, R^{HT} takes as input a decompleted cyclotomic structure on the $\mathbf{E}_\infty$-ring THH(R). In particular, if A is another decompleted cyclotomic $\mathbf{E}_\infty$-ring, then one can similarly define various stacks A^{Δ}, $\mathrm{A}^{\mathrm{Nyg}}$, $\mathrm{A}^{\mathrm{conj}}$, and A^{HT}. For instance, if A is an $\mathbf{E}_\infty$-ring, then one can construct a decompleted cyclotomic $\mathbf{E}_\infty$-ring $\beta\mathrm{A}^{\mathrm{triv}}$ whose underlying cyclotomic $\mathbf{E}_\infty$-ring is just A with the trivial S^1-action; or if B is a connective cyclotomic $\mathbf{E}_\infty$-ring, then one can construct a decompleted cyclotomic $\mathbf{E}_\infty$-ring $\mathrm{B}^{(-1)}$ whose underlying cyclotomic $\mathbf{E}_\infty$-ring is the "Frobenius twist" of B (just as in Theorem 1.2.1).

In § 7.3, we study the stacks $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}}$, $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathrm{Nyg}}$, $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathrm{conj}}$, and $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathrm{HT}}$ associated to $\beta\mathrm{A}^{\mathrm{triv}}$ where A is the connective cover of an even-periodic $\mathbf{E}_\infty$-ring. It turns out that if $\mathrm{A}^{t\mathbf{Z}/p}$ has even homotopy groups, then $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}}$ is isomorphic to the associated 1-dimensional formal group $\pi : \widehat{\mathbf{G}} \to \mathrm{Spec}\,\pi_0(\mathrm{A})$. Furthermore, the map $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}} \to \mathrm{S}^{\mathbb{\Delta}}$ classifies the formal group $\pi^*\widehat{\mathbf{G}}$ equipped with its canonical section.

The composite map

$$(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}} \to \mathrm{S}^{\mathbb{\Delta}} \xrightarrow{\mathrm{can}} \mathrm{S}^{\mathrm{Nyg}}$$

is much more interesting. Theorem 1.2.3 says that this map classifies an S^1-equivariant formal group C_θ over $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}}$, and it is a natural question to ask for a more explicit description of C_θ. In joint work with Max Misterka, we describe the Cartier dual $\mathrm{C}_\theta^\vee$ of C_θ explicitly; see Theorem 7.4.26. For instance, in the case when A = ku, the stack $(\beta\mathrm{A}^{\mathrm{triv}})^{\mathbb{\Delta}}$ identifies with $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!])$. We show that if $\mathbf{G}_m^{\sharp,q}$ denotes the q-deformed version of the divided power hull of $1 \in \mathbf{G}_m$, given by

$$\mathbf{G}_m^{\sharp,q} = \mathrm{Spf}\left(\mathbf{Z}_p[\![q-1]\!]\left[y^{\pm 1}, \tfrac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q!}\right]\right),$$

where $[n]_q! = [1]_q \cdots [n]_q$ with $[n]_q = \frac{q^n-1}{q-1}$, then there is an isomorphism $\mathrm{C}_\theta^\vee \cong \mathbf{G}_m^{\sharp,q}$ of group schemes over $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!])$. Moreover, the pushout square of Theorem 1.2.3(e) is Cartier dual to a pullback square

$$\begin{tikzcd} \mathrm{C}_\theta^\vee \arrow[r, "\sim"] & \mathbf{G}_m^{\sharp,q} \arrow[r, "y\mapsto \log_q(y)"] \arrow[d, "\mathrm{can}"'] & \hat{\mathbf{G}}_{m,q-1}^\vee \arrow[d, "z\mapsto q^{pz/(q-1)}"] \\ & (\mathbf{G}_m)_{\mathbf{Z}_p[\![q-1]\!]} \arrow[r, "y\mapsto y^p"'] & (\mathbf{G}_m^{(1)})_{\mathbf{Z}_p[\![q-1]\!]}. \end{tikzcd} \tag{1.2.3}$$

Here, $\mathbf{G}_{m,q-1}$ is the formal group $\mathrm{Spf}(\mathbf{Z}_p[\![q-1,x]\!])$ with group law $x+y+(q-1)xy$; and $\log_q(y)$ denotes Euler's q-logarithm [Eul] given by the power series

$$\log_q(y) = \sum_{n\geq 1}(-1)^{n+1}q^{-\binom{n}{2}}\tfrac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q}.$$

Our proof of the square (1.2.3) involves a remarkable identity discovered in conversation with Michael Kural, which states that

$$\sum_{n\geq 0}\tfrac{\log_q(y)(\log_q(y)-(q-1))\cdots(\log_q(y)-(n-1)(q-1))}{n!}x^n = \sum_{n\geq 0}q^{-\binom{n}{2}}x(x-[1]_q)\cdots(x-[n-1]_q)\tfrac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q!}.$$

The pullback square (1.2.3) was also observed by Drinfeld in [Dri1]. Note that by taking vertical quotients, it defines an isomorphism of stacks

$$(\mathbf{G}_m)_{\mathbf{Z}_p[\![q-1]\!]}/\mathbf{G}_m^{\sharp,q} \cong (\mathbf{G}_m^{(1)})_{\mathbf{Z}_p[\![q-1]\!]}/\hat{\mathbf{G}}_{m,q-1}^\vee; \tag{1.2.4}$$

the cohomology of the structure sheaf on the left-hand side computes the q-de Rham cohomology of $\mathbf{G}_m$, while (by definition of the Drinfeld formal group [Dri1]) the cohomology of the structure sheaf on the right-hand side computes its prismatic cohomology relative to the q-de Rham prism. The square (1.2.3) can therefore be viewed as a stacky version of the comparison between q-de Rham cohomology and prismatic cohomology relative to the q-de Rham prism.

As we have mentioned above, Theorem 1.2.2 provides a homotopy-theoretic explanation for this comparison of cohomologies. In fact, our setup of the theory of prismatic stacks shows that Theorem 1.2.2 also implies the isomorphism of stacks (1.2.4), and hence the pullback square (1.2.3). I find it truly amazing that the $\mathbf{E}_\infty$-ring ku somehow "knows" about q-divided powers of the form $\frac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q}$, as well as the preceding identity with the q-logarithm! Even more fascinating is the observation that we obtain variants of $\log_q$ and the preceding identity by proving an analogue of the pullback square (1.2.3) for $\mathrm{C}_\theta^\vee$ over $(\beta\mathrm{A}^{\mathrm{triv}})^{\Delta}$ for other choices of A. Along the way, we build the rudiments of a formal group law analogue of the theory of q-calculus (building upon an unpublished observation of A. Raksit, or Koszul dually, upon our calculations in § 3.5).

In future work, we will use these explicit calculations (and tools from homotopy theory) to construct a well-behaved theory of hypergeometric functions adapted to any 1-dimensional formal group law (such that for the multiplicative formal group law, it reduces to the usual q-hypergeometric theory). We hope to show that it arises as a Picard-Fuchs equation for certain *spectral* schemes; see Remark 7.4.28.

We finally give some applications of the theory of prismatic stacks for ring spectra to the question of Hodge-de Rham degeneration in characteristic $p > 0$. For instance, we compute that if $\mathrm{BP}\langle n\rangle$ is the truncated Brown-Peterson spectrum (equipped with the $\mathbf{E}_3$-algebra structure of [HW]), the stack $\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0}$ is isomorphic to the classifying stack $\mathrm{BW}^\times[\mathrm{F}^n]$ of the kernel of n-fold Frobenius on the p-typical Witt vectors W. When $n = 0$, for instance, $\mathrm{BP}\langle n\rangle$ is just $\mathbf{Z}_p$, and the resulting identification of $(\mathbf{Z}_p^{\mathrm{HT}})_{p=0}$ with $\mathrm{BW}^\times[\mathrm{F}] \cong \mathrm{B}\mathbf{G}_m^\sharp$ was proved by Bhatt-Lurie-Drinfeld [BL, Dri2]. This was in turn used to provide a refinement of the Deligne-Illusie theorem [DI] on Hodge-de Rham degeneration. In the same way, our general result about $\mathrm{BP}\langle n\rangle$ implies:

Theorem 1.2.4 (Corollary 7.5.4)**.** *Suppose* X *is a smooth scheme over* $\mathbf{F}_p$ *which admits a choice of lift of the sheaf* $\mathcal{O}_\mathrm{X}$ *of commutative* $\mathbf{F}_p$*-algebras to a sheaf of* $\mathbf{E}_2$-$\mathrm{BP}\langle n\rangle$*-algebras. Then for any integer* i*, there is a natural decomposition:*

$$\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \tau^{[i,i+p^{n+1}-1]}\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}) \cong \bigoplus_{j=0}^{p^{n+1}-1} \mathrm{R\Gamma}(\mathrm{X}^{(1)}; \Omega^{i+j}_{\mathrm{X}^{(1)}/\mathbf{F}_p}[-(i+j)]).$$

In particular, if X *is furthermore proper and of dimension* $< p^{n+1}$*, then the Hodge-de Rham spectral sequence*

$$\mathrm{E}_1^{*,*} = \mathrm{H}^*(\mathrm{X}; \Omega^*_{\mathrm{X}/\mathbf{F}_p}) \Rightarrow \mathrm{H}^*_{\mathrm{dR}}(\mathrm{X}/\mathbf{F}_p)$$

degenerates at the E_1*-page.*

Following Mathew [Mat2], we also prove a noncommutative version of this statement, namely that if $\mathcal{C}$ is a smooth and proper $\mathbf{F}_p$-linear ∞-category such that $\pi_j\mathrm{HH}(\mathcal{C}/\mathbf{F}_p) = 0$ for $j \notin [-p^n, p^n]$, and such that $\mathcal{C}$ lifts to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, then the Tate spectral sequence

$$\mathrm{E}_2^{*,*} = \hat{\mathrm{H}}^*(\mathrm{BS}^1; \pi_*\mathrm{HH}(\mathcal{C}/\mathbf{F}_p)) \Rightarrow \pi_*\mathrm{HP}(\mathcal{C}/\mathbf{F}_p)$$

collapses at the E_2-page. In Theorem 7.5.17, we sketch a description of THH(MU) as an S^1-equivariant $\mathbf{E}_\infty$-ring and use it to provide a refinement of Theorem 1.2.4: in the setting of Theorem 1.2.4, if $\mathfrak{X}$ denotes the corresponding p-adic formal scheme over $\mathbf{Z}_p$ which lifts X (obtained by base-changing along the map $\mathrm{BP}\langle n\rangle \to \mathbf{Z}_p$), then there is a filtered isomorphism

$$\mathcal{N}^{\geq\star}\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathbf{Z}_p) \cong \mathrm{F}^{\geq\star}_{\mathrm{Hdg}}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} (p^\star)$$

in weights $\leq p^n - 1$. Here, $(p^\star)$ denotes the p-adic filtration on $\mathbf{Z}_p$. Some basic considerations involving the geometric Casselman-Shalika equivalence [FGKV, FGV] then show that there is such a filtered isomorphism (with $n = \infty$!) if $\mathrm{X} = \mathrm{BG}$ for any (split) reductive group G over $\mathbf{F}_p$; this refines a result of Petrov [Pet2]. Finally, we give an informal discussion of the relationship between the theory of prismatic stacks for chromatically interesting ring spectra (like $\mathrm{BP}\langle n\rangle$) and the dual Steenrod algebra in § 7.6.

Chapter 2

Even stacks

2.1 Constructing even stacks

The even filtration, introduced in [HRW] and expanded upon in [DHRY], is a construction which produces an fpqc stack from a ring spectrum. In this section, we will review this construction. It admits several variants: for instance, one could work with $\mathbf{E}_\infty$-rings (as we will below) or $\mathbf{E}_1$-rings following Pstragowski [Pst]; and if the $\mathbf{E}_\infty$-ring in question has an S^1-action, then one can consider an S^1-equivariant variant of the even filtration. For simplicity, we will stick to defining the even filtration for $\mathbf{E}_\infty$-rings. This technically excludes many important examples like the truncated Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$, but using Pstragowski's definition instead allows one to extend the constructions below to include them; we will sweep this under the rug when discussing examples later.

Let us begin by setting up some conventions. A prestack will be an accessible[1] presheaf of spaces on the opposite of the category $\mathrm{CAlg}^\heartsuit$ of ordinary commutative rings, and such an object will be called a stack if it is a sheaf for the fpqc topology; the categories of such objects are denoted $\mathcal{P}(\mathrm{Aff})$ and $\mathrm{Shv}(\mathrm{Aff})$, respectively. A graded (pre)stack is defined similarly, starting with the category $\mathrm{CAlg}^{\mathrm{gr},\heartsuit}$ of graded ordinary commutative rings. The category of quasicoherent sheaves on a prestack is defined by right Kan extending the functor $\mathrm{Mod} : \mathrm{CAlg}^\heartsuit \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$. If X is a prestack, then $\mathrm{QCoh}(\mathrm{X})$ is equipped with a t-structure whose heart is denoted $\mathrm{QCoh}(\mathrm{X})^\heartsuit$; for instance, if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, then $\mathrm{QCoh}(\mathrm{X})^\heartsuit$ is the abelian category of A-modules.

Definition 2.1.1. An $\mathbf{E}_\infty$-ring R is said to be *even* if $\pi_{2n+1}(\mathrm{R}) = 0$ for all $n \in \mathbf{Z}$. Let $\mathrm{CAlg}_{\mathrm{ev}}$ denote the full subcategory of CAlg spanned by the even $\mathbf{E}_\infty$-rings. This category admits a Grothendieck topology given by declaring a map to be a cover if it induces a faithfully flat map on homotopy groups.

Let A be an $\mathbf{E}_\infty$-ring, so that it defines a functor $\mathrm{CAlg}_{\mathrm{ev}} \to \mathcal{S}$ sending $\mathrm{R} \mapsto \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{A}, \mathrm{R})$. The *even stack* $\mathrm{Spev}(\mathrm{A}) : \mathrm{CAlg}^{\mathrm{gr},\heartsuit} \to \mathcal{S}$ of A is defined to be the left Kan extension of this functor along the functor $\pi_* : \mathrm{CAlg}_{\mathrm{ev}} \to \mathrm{CAlg}^{\mathrm{gr},\heartsuit}$. Alternatively, $\mathrm{Spev}(\mathrm{A})$ is the left Kan extension of the functor $\mathrm{CAlg}_{\mathrm{ev}} \to \mathrm{Shv}(\mathrm{Aff})_{/\mathrm{B}\mathbf{G}_m}$ sending $\mathrm{R} \mapsto \mathrm{Spec}(\pi_*\mathrm{R})/\mathbf{G}_m$. There is a canonical map $\mathrm{Spev}(\mathrm{A}) \to \mathrm{B}\mathbf{G}_m$ classifying a line bundle over $\mathrm{Spev}(\mathrm{A})$, which we will denote by $\mathcal{O}\{1\}$.

Remark 2.1.2. This definition is slightly in conflict with the notation used in [DHRY]: there, the symbol $\mathrm{Spev}(\mathrm{A})$ is used to denote the functor $\mathrm{CAlg}_{\mathrm{ev}} \to \mathcal{S}$ corepresented by A. There is also a slight mismatch with gradings. Namely, in my forthcoming work with Jeremy Hahn,

[1]The accessibility condition is required here essentially because of the well-known issue that fpqc sheafification is not a well-defined functor.

Arpon Raksit, and Allen Yuan, we consider instead the left Kan extension of the functor $\mathrm{CAlg}_{\mathrm{ev}} \to \mathrm{Shv}(\mathrm{Aff})_{/\mathrm{B}\mathbf{G}_m}$ sending $\mathrm{R} \mapsto \mathrm{Spec}(\pi_{2*}\mathrm{R})/\mathbf{G}_m$. We will sometimes use this variant below (especially in § 7.1), and will mention when we do so.

There is also a p-complete variant of the assignment $\mathrm{A} \rightsquigarrow \mathrm{Spev}(\mathrm{A})$, whose construction we leave to the reader.

Definition 2.1.3. If A is an $\mathbf{E}_\infty$-ring, there is a functor $\mathcal{F}_- : \mathrm{Mod}_\mathrm{A} \to \mathrm{QCoh}(\mathrm{Spev}(\mathrm{A}))$ which sends an A-module M to the quasicoherent sheaf $\mathcal{F}_\mathrm{M}$ on $\mathrm{Spev}(\mathrm{A})$ whose restriction to $\mathrm{Spec}(\pi_*\mathrm{R})/\mathbf{G}_m$ for each even $\mathbf{E}_\infty$-A-algebra $\mathrm{A} \to \mathrm{R}$ is given by the graded $\pi_*(\mathrm{R})$-module $\pi_*(\mathrm{M} \otimes_\mathrm{A} \mathrm{R})$.

The construction of Definition 2.1.1 behaves particularly well for certain maps of $\mathbf{E}_\infty$-rings.

Definition 2.1.4. A map $\mathrm{A} \to \mathrm{B}$ of $\mathbf{E}_\infty$-rings is called *evenly faithfully flat* (or *eff* for short) if for any map $\mathrm{A} \to \mathrm{C}$ of $\mathbf{E}_\infty$-rings with C being an even $\mathbf{E}_\infty$-ring, the map $\mathrm{C} \to \mathrm{C} \otimes_\mathrm{A} \mathrm{B}$ is an even cover (i.e., the map $\pi_*(\mathrm{C}) \to \pi_*(\mathrm{C} \otimes_\mathrm{A} \mathrm{B})$ is faithfully flat).

A large class of such maps are in fact *evenly free* (or *evenly projective*), in the sense that the tensor product $\mathrm{C} \otimes_\mathrm{A} \mathrm{B}$ is a direct sum of even shifts of C (or a retract thereof). This property is satisfied if B has an even cell structure as an A-module; in this case, we will say that the map $\mathrm{A} \to \mathrm{B}$ is *even cellular*.

The following is an easy consequence of the definition:

Lemma 2.1.5. *If* $\mathrm{A} \to \mathrm{B}$ *is an eff map, then there is an isomorphism*

$$\mathrm{colim}_{\mathbf{\Delta}} \mathrm{Spev}(\mathrm{B}^{\otimes_\mathrm{A} \bullet+1}) \to \mathrm{Spev}(\mathrm{A}).$$

Remark 2.1.6. Let A be an $\mathbf{E}_\infty$-ring. It is not true in general that the natural map from A to the global sections of the prestack $\mathrm{CAlg}_{\mathrm{ev}} \to \mathcal{S}$ sending $\mathrm{R} \mapsto \mathrm{Map}_{\mathrm{CAlg}}(\mathrm{A}, \mathrm{R})$ is an equivalence. However, it is an equivalence if there is an even $\mathbf{E}_\infty$-ring B along with an eff map $\mathrm{A} \to \mathrm{B}$ such that the map $\mathrm{A} \to \mathrm{Tot}(\mathrm{B}^{\otimes_\mathrm{A} \bullet+1})$ is an equivalence; see [HRW, Proposition 2.3.4]. We will refer to such an $\mathbf{E}_\infty$-ring A as being *evenly descendable*, and the choice of such a B as an *even eff cover*.

Similarly, the functor $\mathrm{Mod}_\mathrm{A} \to \mathrm{QCoh}(\mathrm{Spev}(\mathrm{A}))$ is well-behaved on an A-module M if there is an even $\mathbf{E}_\infty$-ring B along with an eff map $\mathrm{A} \to \mathrm{B}$ such that the map $\mathrm{M} \to \mathrm{Tot}(\mathrm{M} \otimes_\mathrm{A} \mathrm{B}^{\otimes_\mathrm{A} \bullet+1})$ is an equivalence. We will refer to such an A-module as being *evenly descendable*, and the choice of such a B as an M-*even eff cover*.

In the case that A is an evenly descendable $\mathbf{E}_\infty$-ring, there is a multiplicative spectral sequence

$$\mathrm{E}_2^{s,t} \cong \mathrm{H}^s(\mathrm{Spev}(\mathrm{A}); \mathcal{O}\{t\}) \Rightarrow \pi_{t-s}(\mathrm{A}). \tag{2.1.1}$$

More generally, if M is an evenly descendable module over $\mathbf{E}_\infty$-ring, there is a spectral sequence

$$\mathrm{E}_2^{s,t} \cong \mathrm{H}^s(\mathrm{Spev}(\mathrm{A}); \mathcal{F}_\mathrm{M} \otimes \mathcal{O}\{t\}) \Rightarrow \pi_{t-s}(\mathrm{M}). \tag{2.1.2}$$

We now turn to studying a slight variant of Definition 2.1.1 for $\mathbf{E}_\infty$-rings equipped with a Borel S^1-action. First, we need a construction.

Construction 2.1.7. Suppose B_* is a graded commutative ring concentrated in even weights (for simplicity) and $\mathrm{I} \subseteq \mathrm{B}_{-2}$ is an invertible B_0-module which is a sub-B_0-module of B_{-2}. Then we will write $\mathrm{Spf}(\mathrm{B}_*, \mathrm{I})/\mathbf{G}_m$ to denote the colimit $\mathrm{colim}_n \mathrm{Spec}(\mathrm{B}_*/\mathrm{I}^n)/\mathbf{G}_m$ taken in stacks over $\mathrm{B}\mathbf{G}_m$. There is a natural map $\mathrm{Spf}(\mathrm{B}_*, \mathrm{I})/\mathbf{G}_m \to \mathrm{Spec}(\mathrm{B}_*)/\mathbf{G}_m$.

Let $\mathrm{B}_* \otimes \mathrm{I}^*$ denote the graded commutative ring which is $\mathrm{B}_{2n} \otimes_{\mathrm{B}_0} \mathrm{I}^{\otimes_{\mathrm{B}_0} n}$ in weight $2n$. Then there is an isomorphism $\mathrm{Spec}(\mathrm{B}_*)/\mathbf{G}_m \cong \mathrm{Spec}(\mathrm{B}_* \otimes \mathrm{I}^*)/\mathbf{G}_m$, albeit not as stacks over $\mathrm{B}\mathbf{G}_m$. There is a canonical element in $(\mathrm{B}_* \otimes \mathrm{I}^*)_{-2}$ coming from the inclusion of $\mathrm{B}_0 \cong \mathrm{I} \otimes \mathrm{I}^{-1}$, which defines a graded map $\mathrm{Spec}(\mathrm{B}_* \otimes \mathrm{I}^*) \to \mathbf{A}^1(2)$. It therefore defines a canonical map $\mathrm{Spec}(\mathrm{B}_*)/\mathbf{G}_m \to \mathbf{A}^1(2)/\mathbf{G}_m$; importantly, the resulting line bundle $\mathcal{O}\langle 1\rangle$ over $\mathrm{Spec}(\mathrm{B}_*)/\mathbf{G}_m$ is *not* isomorphic to the canonical line bundle.

Definition 2.1.8. Let A be an $\mathbf{E}_\infty$-ring equipped with a (Borel) S^1-action. One then obtains a functor $\mathrm{Fun}(\mathrm{BS}^1, \mathrm{CAlg}_{\mathrm{ev}}) \to \mathcal{S}$ sending $\mathrm{R} \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{BS}^1, \mathrm{CAlg})}(\mathrm{A}, \mathrm{R})$. The S^1-*equivariant even stack* $\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{A})$ is the left Kan extension of the functor

$$\begin{aligned} \mathrm{CAlg}_{\mathrm{ev}} &\to \mathrm{Shv}(\mathrm{Aff}^\heartsuit)_{/(\mathbf{A}^1/\mathbf{G}_m \times \mathrm{B}\mathbf{G}_m)} \\ \mathrm{R} &\mapsto \mathrm{Spf}(\pi_*(\mathrm{R}^{h\mathrm{S}^1}), (t))/\mathbf{G}_m. \end{aligned}$$

Here, $(t) \subseteq \pi_{-2}(\mathrm{R}^{h\mathrm{S}^1})$ denotes the ideal generated by an Euler class for the S^1-action on R.

In exactly the same way as Lemma 2.1.5, one finds:

Lemma 2.1.9. *If* $\mathrm{A} \to \mathrm{B}$ *is an eff map of* S^1*-equivariant* $\mathbf{E}_\infty$*-rings, there is an isomorphism*

$$\mathrm{colim}_{\mathbf{\Delta}} \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{B}^{\otimes_{\mathrm{A}} \bullet+1}) \to \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{A}).$$

Remark 2.1.10. Let A be an S^1-equivariant $\mathbf{E}_\infty$-ring. If there is an even S^1-equivariant $\mathbf{E}_\infty$-ring B along with an S^1-equivariant eff map $\mathrm{A} \to \mathrm{B}$ such that the map $\mathrm{A} \to \mathrm{Tot}(\mathrm{B}^{\otimes_{\mathrm{A}} \bullet+1})$ is an S^1-equivariant equivalence, then the natural map from $\mathrm{A}^{h\mathrm{S}^1}$ to the global sections of the prestack $\mathrm{CAlg}_{\mathrm{ev}} \to \mathcal{S}$ sending $\mathrm{R} \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{BS}^1, \mathrm{CAlg})}(\mathrm{A}, \mathrm{R})$ is an equivalence. As before, we will refer to this property as being S^1-*equivariantly evenly descendable.*

In the case that A is an S^1-equivariantly evenly descendable $\mathbf{E}_\infty$-ring, there is a multiplicative spectral sequence

$$\mathrm{E}_2^{s,t} \cong \mathrm{H}^s(\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{A}); \mathcal{O}\{t\}) \Rightarrow \pi_{t-s}(\mathrm{A}^{h\mathrm{S}^1}). \tag{2.1.3}$$

Definition 2.1.8 is not quite sufficient for some of our applications to prismatic cohomology. We will now explain the necessary variant, developed in joint work with J. Hahn, A. Raksit, and A. Yuan.

Definition 2.1.11. Let G be a compact abelian Lie group. A G-equivariant spectrum X will be called *even* if for all closed subgroups $\mathrm{H} \subseteq \mathrm{G}$ and all virtual complex representations V of H, the group $\pi^{\mathrm{H}}_{\mathrm{V}-1}(\mathrm{X})$ vanishes. Note that this implies, in particular, that the genuine fixed points X^{H} as well as the geometric fixed points $\Phi^{\mathrm{H}}\mathrm{X}$ are even. This notion of evenness can be used to formulate an analogue of the even filtration: namely, if A is a G-equivariant $\mathbf{E}_\infty$-ring, declare $\mathrm{fil}^\star_{\mathrm{ev}}(\mathrm{A}^{\mathrm{G}}) = \lim_{\mathrm{A}\to\mathrm{B}} \tau_{\geq 2\star}(\mathrm{B}^{\mathrm{G}})$ and $\mathrm{fil}^\star_{\mathrm{ev}}(\Phi^{\mathrm{G}}\mathrm{A}) = \lim_{\mathrm{A}\to\mathrm{B}} \tau_{\geq 2\star}(\Phi^{\mathrm{G}}\mathrm{B})$, where the indexing diagram runs over all G-equivariant $\mathbf{E}_\infty$-maps from A to even G-equivariant $\mathbf{E}_\infty$-rings B.

This notion of evenness satisfies many desirable properties. In particular, the above notion of genuine equivariant even filtration satisfies descent with respect to the following type of maps.

Definition 2.1.12. Let G be a compact abelian Lie group, and let A be a G-equivariant $\mathbf{E}_\infty$-ring. If M is a unital A-module, say that A is *faithfully free* if the unit map $A \to M$ splits, and its cofiber is a direct sum of suspensions of A by virtual complex G-representations. Similarly, a map $A \to B$ of G-equivariant $\mathbf{E}_\infty$-rings will be called *faithfully evenly projective* if for every map $A \to C$ into an even G-equivariant $\mathbf{E}_\infty$-ring, the pushout $C \otimes_A B$ is a retract of a faithfully free unital C-module.

One can show that the faithfully evenly projective maps assemble into the covering maps of a Grothendieck topology on the category of even A-equivariant $\mathbf{E}_\infty$-rings. Most faithfully evenly projective maps in nature arise from the following class of examples:

Example 2.1.13. A *based even cell decomposition* of a G-equivariant A-module M is the data of a sequential diagram $R \to M_1 \to M_2 \to \cdots$ of G-equivariant R-modules whose colimit is M such that the cofiber of each map $M_i \to M_{i+1}$ is a direct sum of suspensions of A by virtual complex G-representations. If $A \to B$ is a map of G-equivariant $\mathbf{E}_\infty$-rings such that B is a retract of an A-module with a based even cell decomposition, then $A \to B$ is faithfully evenly projective.

Given this notion of faithfully evenly projective maps and the genuine equivariant analogue of the even filtration, one can define even stacks, etc.

2.2 Examples

We will now compile some basic examples of the constructions explained above.

Example 2.2.1. The sphere spectrum S is evenly descendable: the unit map from S to the complex cobordism spectrum MU is an even eff cover. In fact, the map $S \to MU$ is evenly cellular. There is an isomorphism between $\mathrm{Spev}(MU^{\otimes \bullet+1})$ and the moduli stack of 1-dimensional formal groups equipped with $\bullet$ coordinates, so that Spev(S) identifies with the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of 1-dimensional formal groups. The spectral sequence (2.1.1) in this case is the Adams-Novikov spectral sequence computing the homotopy groups of spheres.

Remark 2.2.2. Let A be an $\mathbf{E}_\infty$-ring. The unit map $S \to A$ defines a map $\mathrm{Spev}(A) \to \mathrm{Spev}(S) \cong \mathcal{M}_{\mathrm{fg}}$ which classifies a canonical 1-dimensional formal group $\mathbf{G}_A$ over Spev(A). If A is evenly descendable, this 1-dimensional formal group is just $\mathrm{Spev}(C^*(\mathbf{C}P^\infty; A))$.

Example 2.2.3. Let KO denote real K-theory, so that $KO = KU^{h\mathbf{Z}/2}$. The map $KO \to KU$ is an evenly descendable, because KU is a KO-module with even cells. In fact, $KU \simeq KO \otimes C\eta$. This implies that $\mathrm{Spev}(KO) \simeq B\mathbf{Z}/2$, where the structure map $\mathrm{Spev}(KO) \to B\mathbf{G}_m$ classifies the sign representation of $\mathbf{Z}/2$ on $\mathbf{Z}$. Note that if we identify $\mathbf{Z}/2$ with $\mathrm{Spec}\,\mathrm{Map}(\mathbf{Z}/2, \mathbf{Z}) = \mathrm{Spec}\,\mathbf{Z}[a]/(a^2 - a)$, where a is the delta function at the non-identity element of $\mathbf{Z}/2$, then the action of $\mathbf{Z}/2$ on $\pi_*(KU)$ is given by the coaction

$$\mathbf{Z}[\beta^{\pm 1}] \to \mathbf{Z}[\beta^{\pm 1}, a]/(a^2 - a),\ \beta \mapsto (1 - 2a)\beta. \tag{2.2.1}$$

Example 2.2.4. Let ko denote connective real K-theory, so that it is equivalent to $\tau_{\geq 0}(\mathrm{ku}^{h\mathbf{Z}/2})$ as an $\mathbf{E}_\infty$-ring. The map $\mathrm{ko} \to \mathrm{ku}$ is evenly descendable, because ku is a ko-module with even cells: in fact, $\mathrm{ku} \simeq \mathrm{ko} \otimes C\eta$. (The map $\mathrm{ko} \to MU \otimes \mathrm{ko}$ is also evenly descendable: since MU has even cells, $MU \otimes \mathrm{ko}$ has even cells as a ko-module; and one can compute independently that $MU \otimes \mathrm{ko}$ has even homotopy groups.) The stack Spev(ko) can be explicitly computed as the quotient by $\mathbf{G}_m$ of the geometric realization of the simplicial stack

$$\cdots \to \mathrm{Spec}(\pi_*(\mathrm{ku}^{\otimes_{\mathrm{ko}} 3})) \rightrightarrows \mathrm{Spec}(\pi_*(\mathrm{ku}^{\otimes_{\mathrm{ko}} 2})) \rightrightarrows \mathrm{Spec}(\pi_*(\mathrm{ku})).$$

A standard calculation says that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}) \cong \mathbf{Z}[\beta, r]/(r^2 - r\beta)$ with r in weight 2, and that the two maps $\eta_{\mathrm{L}}, \eta_{\mathrm{R}} : \mathrm{ku} \rightrightarrows \mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}$ send $\eta_{\mathrm{L}} : \beta \mapsto \beta$ and $\eta_{\mathrm{R}} : \beta \mapsto \beta - 2r$. Upon inverting β, we may identify $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku})[\beta^{-1}]$ with $\mathbf{Z}[\beta^{\pm 1}, a]/(a^2 - a)$ where $a = r\beta^{-1}$, and then η_{R} is precisely the coaction from (2.2.1). As described in [DFHH, Section 9], Spev(ko) classifies isomorphism classes of possibly singular quadratic curves (which are locally of the form $y = x^2 + \beta x$). Note that

$$\eta_{\mathrm{R}}(\beta^n) = \beta^n + ((-1)^n - 1)r\beta^{n-1},$$

so β^{2n} is a well-defined function on Spev(ko) for any $n \geq 0$; the complement of its vanishing locus is precisely $\mathrm{B}\mathbf{Z}/2$.

Example 2.2.5. Let tmf denote the connective $\mathbf{E}_\infty$-ring of topological modular forms. Then, the map $\mathrm{tmf} \to \mathrm{tmf} \otimes \mathrm{MU}$ is evenly descendable: since MU has even cells, $\mathrm{MU} \otimes \mathrm{tmf}$ has even cells as a tmf-module; and one can compute independently that $\mathrm{MU} \otimes \mathrm{tmf}$ has even homotopy groups. In fact, if $\mathrm{X}(4)$ denotes the Thom spectrum of the virtual bundle over $\Omega\mathrm{SU}(4)$ classified by the map $\Omega\mathrm{SU}(4) \to \Omega\mathrm{SU} \simeq \mathrm{BU}$, then $\mathrm{X}(4) \otimes \mathrm{tmf}$ already has even homotopy groups (see [DFHH, Mat1]). There is an isomorphism $\mathrm{Spec}\, \pi_{2*}(\mathrm{X}(4) \otimes \mathrm{tmf}) \cong \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ (where a_i lives in weight i) which identifies $\mathrm{Spec}\, \pi_{2*}(\mathrm{X}(4) \otimes \mathrm{tmf})$ with the moduli space of cubic curves in Weierstrass form. Similarly, $\mathrm{Spec}\, \pi_{2*}(\mathrm{X}(4)^{\otimes 2} \otimes \mathrm{tmf})$ identifies with the moduli space of cubic curves in Weierstrass form and coordinate changes. This leads to an isomorphism between Spev(tmf) and the moduli stack of cubic curves. Similarly, if TMF denotes the nonconnective 576-periodic variant of tmf, then Spev(TMF) is isomorphic to the moduli stack of (smooth) elliptic curves.

In the example below, we will use the "double-step" variant of the Spev construction from Remark 2.1.2.

Example 2.2.6. Let $p > 2$ be an odd prime, and let $j_p = \tau_{\geq 0}\mathrm{L}_{\mathrm{K}(1)}\mathbb{S}$ denote the connective image of J spectrum at p. Then j_p is evenly descendable; in fact, the map $j_p \to \mathrm{ku}_p$ is even cellular: if $g \in \mathbf{Z}_p^\times$ is a topological generator and ψ^g denotes the corresponding Adams operation on ku, there is a cofiber sequence

$$j_p \to \mathrm{ku}_p \xrightarrow{\psi^g - 1} \Sigma^2 \mathrm{ku}_p,$$

which implies even cellularity by induction.

One can therefore compute $\mathrm{Spev}(j_p)$ as the geometric realization of the simplicial stack $\mathrm{Spec}(\pi_{2*}(\mathrm{ku}_p^{\otimes_{j_p} \bullet + 1}))/\mathbf{G}_m$. We note that the stack $\mathrm{Spev}(j_p)$ has also been studied by Lurie under the moniker "$\mathbf{F}_1^{\mathrm{Syn}}$". To describe $\mathrm{Spev}(j_p)$ explicitly, we need some preliminaries.

Construction 2.2.7. Let $\mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{Z}_p)$ denote the algebra of continuous $\mathbf{Z}_p$-valued functions on $\mathbf{Z}_p^\times$, where the algebra structure arises via the diagonal on $\mathbf{Z}_p^\times$. This is a Banach algebra via the supremum norm $\|f\| = \max_{x \in \mathbf{Z}_p^\times} |f(x)|_p$, and it contains a dense subalgebra $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ consisting of those polynomials $f(x) \in \mathbf{Q}_p[x]$ such that $f(\mathbf{Z}_p^\times) \subseteq \mathbf{Z}_p$. This subalgebra $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ has the (increasing) *degree* filtration (given by the degree of a polynomial), so one can form the Rees algebra $\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ with respect to the Rees parameter β. Note that $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ becomes a Hopf algebra over $\mathbf{Z}_p$ via the multiplication on $\mathbf{Z}_p^\times$. In fact, the Hopf algebra structure respects the degree filtration on $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$, and so $\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ defines a Hopf algebroid over $\mathbf{Z}_p[\beta]$.

Proposition 2.2.8. *There is a p-complete isomorphism of Hopf algebroids over $\pi_*(\mathrm{ku})$:*

$$\pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p) \cong \mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p).$$

Proof. We will implicitly p-complete all objects below. Recall that $j_p = \tau_{\geq 0}(\mathrm{ku}_p^{h\mathbf{Z}_p^\times})$, so there is a canonical map $j_p \to \mathrm{ku}_p^{h\mathbf{Z}_p^\times}$, which induces a map

$$\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p \to \mathrm{ku}_p \otimes_{\mathrm{ku}_p^{h\mathbf{Z}_p^\times}} \mathrm{ku}_p \cong \mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathrm{ku}_p).$$

Upon K(1)-localizing (which, since we are p-complete, amounts simply to inverting the class β coming, say, from the first factor of ku_p), the map $j_p \to \mathrm{ku}_p^{h\mathbf{Z}_p^\times}$ becomes an isomorphism; so

$$\pi_*(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p)[1/\beta] \cong \mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \pi_*\mathrm{ku}_p)[1/\beta] \cong \mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{Z}_p)[\beta^{\pm 1}].$$

Since the map $j_p \to \mathrm{ku}_p$ is evenly descendable, the $\mathbf{E}_\infty$-ring $\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p$ has homotopy groups which are β-torsionfree and even. We therefore only need to determine the image of the map

$$\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p \to (\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p)[1/\beta]$$

on homotopy. If β' denotes the Bott class from the second factor of ku_p, and f_0 denotes the continuous function given by the inclusion $\mathbf{Z}_p^\times \hookrightarrow \mathbf{Z}_p$, then the above map sends $\beta' \mapsto \beta f_0$ on homotopy.

Moreover, $\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p$ is a flat ku_p-module, hence also has p-torsionfree even homotopy; so there is an injection

$$\pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p) \hookrightarrow \pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p)[1/p].$$

Upon inverting p, one can simply identify j_p with the rationalization of the p-complete sphere spectrum, so $\pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p)[1/p] \cong \mathbf{Q}_p[\beta, \beta']$, where β' denotes the Bott class from the second factor of ku_p. In summary, one has a commutative diagram where all maps are injections:

$$\begin{tikzcd}
\pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p) \arrow[rr] \arrow[d] & & \mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{Z}_p)[\beta^{\pm 1}] \arrow[d] \\
\mathbf{Q}_p[\beta, \beta'] \arrow[r] & \mathbf{Q}_p[\beta^{\pm 1}, \beta'] \arrow[r] & \mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{Q}_p)[\beta^{\pm 1}].
\end{tikzcd}$$

The intersection of $\mathbf{Q}_p[\beta, \beta']$ with $\mathrm{C}^0_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{Z}_p)[\beta^{\pm 1}]$ is precisely the algebra $\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$, so one obtains a map $\pi_{2*}(\mathrm{ku}_p \otimes_{j_p} \mathrm{ku}_p) \to \mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$. To check that it is a p-complete isomorphism, note that both sides are flat $\mathbf{Z}_p[\beta]$-algebras. Since the map is an isomorphism upon inverting β, it suffices to check that the map is an isomorphism modulo β; and even further, modulo (p, β).

One can identify $\mathrm{ku}/(p, \beta) = \mathbf{F}_p$, so we obtain the map

$$\pi_{2*}(\mathbf{F}_p \otimes_{j_p} \mathrm{ku}_p) \to \mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)/(p, \beta) \cong \mathrm{gr}(\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p))/p. \tag{2.2.2}$$

On the one hand, one can write any $a \in \mathbf{Z}_p^\times$ as $\sum_{i \geq 0} [f_i(a)] p^i$, where $f_i(a) \in \mathbf{F}_p$ and $f_0(a) \in \mathbf{F}_p^\times$. The function f_i is locally constant on $\mathbf{Z}_p^\times$ and constant on the $(1 + p^{i+1}\mathbf{Z}_p)$-cosets; it gives a bijection $\vec{f}: \mathbf{Z}_p^\times/(1 + p^{i+1}\mathbf{Z}_p) \xrightarrow{\cong} \mathbf{F}_p^\times \times \mathbf{F}_p^{\times i}$. Since $\mathrm{C}^0_{\mathrm{cts}}(\mathbf{F}_p; \mathbf{F}_p) \cong \mathbf{F}_p[x]/(x^p - x)$, we find that

$$\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)/p \cong \mathbf{F}_p[f_0, f_1, \cdots]/(f_0^{p-1} - 1, f_i^p - f_i).$$

Taking the associated graded for the degree filtration, we find

$$\mathrm{gr}(\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p))/p \cong \mathbf{F}_p[f_0, f_1, \cdots]/(f_0^{p-1}, f_i^p),$$

where f_0 lives in weight 1 and f_i lives in weight $(p-1)p^{i-1}$ for $i \geq 1$. On the one hand, one computes easily that

$$\pi_{2*}(\mathbf{F}_p \otimes_{j_p} \mathrm{ku}_p) \cong \mathbf{F}_p[\beta]/\beta^{p-1} \otimes_{\mathbf{F}_p} \mathbf{F}_p\langle\sigma\alpha_1\rangle \cong \mathbf{F}_p[\beta, \sigma(\alpha_1), \gamma_p(\sigma\alpha_1), \cdots]/(\beta^{p-1}, \gamma_{p^i}(\sigma\alpha_1)^p),$$

where β lives in weight 1 and $\sigma\alpha_1$ lives in weight $p-1$. Moreover, the map (2.2.2) sends $\beta \mapsto f_0$ and $\gamma_{p^i}(\sigma\alpha_1)$ to f_{i-1}, so it is an isomorphism, as desired. □

The following is now a consequence of Proposition 2.2.8:

Corollary 2.2.9. *The stack* $\mathrm{Spev}(j_p)$ *can be identified with the quotient of* $\mathbf{A}^1(-1)/\mathbf{G}_m$ *by the group scheme* $\mathrm{Spf}(\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p), (p))/\mathbf{G}_m$. *In particular, the nonvanishing locus of* β *identifies with the open substack* $\mathrm{Spev}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}) = \mathrm{B}\mathbf{Z}_p^\times \subseteq \mathrm{Spev}(j_p)$.

This implies, for instance, that one can identify $\mathrm{QCoh}(\mathrm{Spev}(j_p))$ with the category of decreasingly filtered $\mathbf{Z}_p$-modules $\mathrm{fil}^\star\mathrm{M}$ with a continuous action of $\mathbf{Z}_p^\times$ such that the $\mathbf{Z}_p^\times$-action on $\mathrm{gr}^n\mathrm{M}$ is via the nth power of the cyclotomic character. To see this, note that by using Corollary 2.2.9, it suffices to describe the Cartier dual of $\mathrm{Spf}(\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p), (p))/\mathbf{G}_m$ over $\mathbf{A}^1(-1)/\mathbf{G}_m$. First, note that the topological $\mathbf{Z}_p$-linear dual of the Iwasawa/completed group algebra $\mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!] = \lim_n \mathbf{Z}_p[\mathbf{Z}_p^\times/(1+p^n\mathbf{Z}_p)^\times]$ is $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$. Said differently, the Cartier dual of $\mathrm{Spf}(\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p))$ over $\mathrm{Spf}(\mathbf{Z}_p)$ is given by $\mathrm{Spf}(\mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!])$. Recall that $\mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!] \cong \mathbf{Z}_p[\![\mathrm{T}]\!]$, where $\mathrm{T}+1$ corresponds to the topological generator $g \in \mathbf{Z}_p^\times$. Under this identification, the Hopf algebra structure on $\mathbf{Z}_p[\![\mathrm{T}]\!]$ is determined by the coproduct $\mathrm{T} \mapsto \mathrm{T}\otimes 1 + 1 \otimes \mathrm{T} + \mathrm{T}\otimes\mathrm{T}$. In particular, $\mathrm{QCoh}(\mathrm{Spf}(\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p), (p)))$ can be identified with the category of $\mathbf{Z}_p[\![\mathrm{T}]\!]$-comodules, or equivalently the category of $\mathbf{Z}_p$-modules with a continuous $\mathbf{Z}_p^\times$-action.

The degree filtration on $\mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ is dual to the T-adic filtration on $\mathbf{Z}_p[\![\mathrm{T}]\!]$, so that $\mathrm{Rees}_u \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p)$ is dual (over $\mathbf{Z}_p[\beta]$) to $\mathrm{Rees}_\beta \mathbf{Z}_p[\![\mathrm{T}]\!] = \mathbf{Z}_p[\![\mathrm{T}]\!][\beta, t]/(\beta t = \mathrm{T})$. The $\mathbf{Z}_p[\beta]$-linear Hopf algebra structure on this ring is determined by the coproduct $t \mapsto t \otimes 1 + 1 \otimes t + \beta t \otimes t$. Let $\widetilde{\mathrm{K}} = \mathrm{Spf}(\mathrm{Rees}_u \mathbf{Z}_p[\![\mathrm{T}]\!], (p, \mathrm{T}))$, and let $\mathrm{K} = \widetilde{\mathrm{K}}/\mathbf{G}_m$. The graded K-action on $\mathbf{Z}_p[\beta]$ is given by the coaction

$$\mathbf{Z}_p[\beta] \to \mathbf{Z}_p[\beta] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![\mathrm{T}]\!][\beta', t]/(\beta' t = \mathrm{T}),\ \beta \mapsto \beta' \otimes \mathrm{T}.$$

This discussion implies that there is an equivalence

$$\mathrm{QCoh}(\mathrm{Spf}(\mathrm{Rees}_\beta \mathrm{P}(\mathbf{Z}_p^\times; \mathbf{Z}_p), (p))/\mathbf{G}_m) \simeq \mathrm{QCoh}((\mathbf{A}^1(-1)/\mathbf{G}_m)/\widetilde{\mathrm{K}}).$$

Via the Rees equivalence, the category $\mathrm{QCoh}(\mathbf{A}^1(-1)/\mathbf{G}_m)$ is that of decreasingly filtered $\mathbf{Z}_p$-modules $\mathrm{fil}^\star\mathrm{M}$. Unwinding the Rees equivalence, one finds the data of a $\widetilde{\mathrm{K}}$-action on $\mathrm{fil}^\star\mathrm{M}$ is precisely the data of a continuous $\mathbf{Z}_p^\times$-action on $\mathrm{fil}^\star\mathrm{M}$ such that the $\mathbf{Z}_p^\times$-action on $\mathrm{gr}^n(\mathrm{M})$ is via the nth power of the cyclotomic character.

There are also some examples using genuine equivariance.

Example 2.2.10. The map $\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}_{\mathrm{S}^1}$ is faithfully evenly projective. This example is crucial in understanding the "prismatization of MU" in the sense of § 7.1. Similarly, the canonical map $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}]$ is faithfully evenly projective; this, too, will be discussed briefly below.

Example 2.2.11. Let A be a compact abelian Lie group, let S_A denote the unit object in A-equivariant spectra, and let MU_A denote A-equivariant complex cobordism. Then the map $\mathrm{S}_\mathrm{A} \to \mathrm{MU}_\mathrm{A}$ is faithfully evenly projective in the sense of Definition 2.1.12, because the space BU_A classifying A-equivariant complex vector bundles of virtual dimension zero admits a based even cell decomposition. The corresponding even stack $\mathrm{Spev}_\mathrm{A}(\mathrm{S}_\mathrm{A})$ is then isomorphic to the moduli stack of A-*equivariant formal groups* (see [HM1]).

This notion will play a crucial role in our discussion on prismatization, so let us recall its definition here. Let $\underline{\mathrm{A}^\vee}$ denote the constant group scheme given by the Pontryagin dual of A. Then an A-*equivariant formal group* over a commutative ring k is the data of a commutative formal group C over k along with a homomorphism $\alpha : \underline{\mathrm{A}^\vee} \to \mathbf{H}$ such that C is complete at the image of α, and furthermore such that the completion of C at $\alpha(1)$ is a 1-dimensional formal group over k. We will write $\mathcal{M}_\mathrm{fg}^\mathrm{A}$ to denote the moduli stack of A-equivariant formal groups. Note that a continuous homomorphism $\mathrm{A} \to \mathrm{B}$ induces a map $\mathcal{M}_\mathrm{fg}^\mathrm{A} \to \mathcal{M}_\mathrm{fg}^\mathrm{B}$.

A *coordinate* on an A-equivariant formal group is the data of a regular function on C whose vanishing locus is precisely $\{\alpha(1)\} \subseteq \mathrm{C}$. In [HM1] (building on the work of several previous authors, like [Hau]), it is shown that $\mathrm{Spev}_\mathrm{A}(\mathrm{MU}_\mathrm{A})$ identifies with the moduli stack of A-equivariant formal groups equipped with a coordinate (and similarly for $\mathrm{MU}_\mathrm{A}^{\otimes n}$). This implies that $\mathrm{Spev}_\mathrm{A}(\mathrm{S}_\mathrm{A}) = \mathcal{M}_\mathrm{fg}^\mathrm{A}$, as desired.

Example 2.2.12. Let A be an $\mathbf{E}_\infty$-ring, and let G be a connected compact Lie group. Then the map $\mathrm{C}^*(\mathrm{BG};\mathrm{A}) \to \mathrm{C}^*(\mathrm{BT};\mathrm{A})$ is an eff cover, and $\mathrm{C}^*(\mathrm{BG};\mathrm{A})$ is evenly descendable if the same is true of A. Indeed, the map $\mathrm{C}^*(\mathrm{BG};\mathrm{A}) \to \mathrm{C}^*(\mathrm{BT};\mathrm{A})$ is in fact even cellular, because $\mathrm{BT} = (\mathrm{G}/\mathrm{T})_{h\mathrm{G}}$, so the cell structure of G/T defines an $\mathrm{C}^*(\mathrm{BG};\mathrm{A})$-module cell structure on $\mathrm{C}^*(\mathrm{BT};\mathrm{A})$; but G/T admits an even cell structure by the Bruhat decomposition. The fact that $\mathrm{C}^*(\mathrm{BG};\mathrm{A})$ is evenly descendable if the same is true of A follows from the fact that BT admits an even cell structure.

If A is itself even, for instance, it follows from Lemma 2.1.5 that there is an isomorphism

$$\mathrm{Spev}\,\mathrm{C}^*(\mathrm{BG};\mathrm{A}) \cong \mathrm{colim}_{\boldsymbol{\Delta}}\,\mathrm{Spec}\,\pi_*(\mathrm{C}^*(\mathrm{T}\backslash(\mathrm{G}/\mathrm{T})^{\times\bullet};\mathrm{A}))/\mathbf{G}_m.$$

We explore this further in Theorem 3.7.7. The idea of resolving by tori is also useful in mild variants of the above context: for instance, if $\Omega\mathrm{G}$ denotes the based loop space of G, we will write $\mathrm{Spev}\,\mathrm{C}_*^\mathrm{G}(\Omega\mathrm{G};k)$ to denote the geometric realization

$$\mathrm{Spev}\,\mathrm{C}_*^\mathrm{G}(\Omega\mathrm{G};k) = \mathrm{Spec}(\pi_*(\mathrm{C}_\mathrm{T}^*((\mathrm{G}/\mathrm{T})^{\times\bullet};k) \otimes_{\mathrm{C}_\mathrm{T}^*(*;k)} \mathrm{C}_*^\mathrm{T}(\Omega\mathrm{G};k)))/\mathbf{G}_m.$$

Part I

Spherochromatism in representation theory

Chapter 3

The homology of the affine Grassmannian

3.1 The regular locus

In this section, we will quickly review the derived geometric Satake equivalence following [BF] and [AG]. Let k denote a commutative $\mathbf{Q}$-algebra; all Langlands dual objects will be assumed to live over k, and are base-changes of their "split forms" over $\mathbf{Q}$.

Setup 3.1.1. Let G be a connected reductive group (over $\mathbf{C}$, always), and let $\mathrm{Gr}_{\mathrm{G}} = \mathrm{G}(\!(t)\!)/\mathrm{G}[\![t]\!]$ denote the affine Grassmannian. There is a canonical left action of $\mathrm{G}(\!(t)\!)$ on Gr_{G}, and hence an action of $\mathrm{G}[\![t]\!] \subseteq \mathrm{G}(\!(t)\!)$. The affine Grassmannian is a union of $\mathrm{G}[\![t]\!]$-invariant closed subschemes X_α of finite type, and one defines $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{colim}_\alpha \mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{X}_\alpha; k)$. Inside $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)$ are two full subcategories:

- $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}}$ is the full subcategory of objects whose image under the forgetful functor $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k) \to \mathrm{Shv}(\mathrm{Gr}_{\mathrm{G}}; k)$ is compact. Such objects are called "locally compact".
- $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\omega}$ of compact objects in $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)$.

The ∞-category $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)$ admits a monoidal structure, which in fact restricts to a monoidal structure on each of the full subcategories above.

Setup 3.1.2. Let (e, f, h) denote a principal $\mathfrak{sl}_2$-triple in the Langlands dual Lie algebra $\check{\mathfrak{g}}$. The element f defines a nondegenerate character $\psi : \check{\mathfrak{n}} \to \mathbf{A}^1$. Let $\check{\mathfrak{g}}^{*,e}$ denote the orthogonal complement to the subspace $[e, \check{\mathfrak{g}}] \subseteq \check{\mathfrak{g}}$. This defines the *Kostant slice* $\psi + \check{\mathfrak{g}}^{*,e} \subseteq \check{\mathfrak{g}}^*$; we will denote this inclusion by κ. Composing the invariant-theoretic quotient map $\chi : \check{\mathfrak{g}}^* \to \check{\mathfrak{g}}^* /\!\!/ \check{\mathrm{G}}$ with the Kostant slice defines an isomorphism. In other words, the following composite is an isomorphism:

$$\psi + \check{\mathfrak{g}}^{*,e} \xrightarrow{\kappa} \check{\mathfrak{g}}^* \xrightarrow{\chi} \check{\mathfrak{g}}^* /\!\!/ \check{\mathrm{G}}.$$

It will be convenient to identify $\psi + \check{\mathfrak{g}}^{*,e}$ with $\check{\mathfrak{g}}^* /\!\!/ \check{\mathrm{G}}$ under this isomorphism. If the vector space $\check{\mathfrak{g}}^*$ is placed in weight 2, the map κ can be checked to give a *graded* map

$$\kappa : \check{\mathfrak{g}}^*(2) /\!\!/ \check{\mathrm{G}} \to \check{\mathfrak{g}}^*(2).$$

Shearing this graded map (in the sense of [Dev3, Section 2.1]) defines a map $\check{\mathfrak{g}}^*[2] /\!\!/ \check{\mathrm{G}} \to \check{\mathfrak{g}}^*[2]$, which we will also denote by κ.

Lemma 3.1.3 (Chevalley restriction)**.** *There is an isomorphism $\check{\mathfrak{g}}^* /\!\!/ \check{\mathrm{G}} \cong \mathfrak{t} /\!\!/ \mathrm{W}$, which refines to an isomorphism of graded schemes*

$$\check{\mathfrak{g}}^*(2) /\!\!/ \check{\mathrm{G}} \cong \mathfrak{t}(2) /\!\!/ \mathrm{W} \cong \mathrm{Spec}\, \mathrm{H}^*_{\mathrm{G}}(*; \mathbf{C}).$$

The first part of the following result is [BF, Theorem 5], and the second part is [AG, Theorem 12.5.3].

Theorem 3.1.4 (Bezrukavnikov-Finkelberg, Arinkin-Gaitsgory)**.** *There is a monoidal equivalence*

$$\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}} \simeq \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{\mathrm{G}}),$$

which restricts to a monoidal equivalence

$$\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\omega} \simeq \mathrm{Perf}_{\check{\mathcal{N}}/\check{\mathrm{G}}}(\check{\mathfrak{g}}^*[2]/\check{\mathrm{G}}),$$

where the right-hand side is the full subcategory of those perfect complexes which are set-theoretically supported on the nilpotent cone of $\check{\mathfrak{g}}^*$. Furthermore, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}}) \arrow[r, "\sim"] \arrow[d, "p_!"'] & \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{\mathrm{G}}) \arrow[d, "\kappa^*"] \\ \mathrm{Shv}_{\mathrm{G}[\![t]\!]}(*; k) \arrow[r, "\sim"'] & \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\!\!/\check{\mathrm{G}}), \end{tikzcd}$$

where $p : \mathrm{Gr}_{\mathrm{G}} \to *$ is the canonical map to a point and κ^* is pullback along the (shifted) Kostant slice.

We will refer to the first equivalence of Theorem 3.1.4 as the *derived geometric Satake equivalence*, or more colloquially as "derived Satake".

Definition 3.1.5. A point $x \in \check{\mathfrak{g}}^*$ is called *regular* if its centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(x) \subseteq \check{\mathrm{G}}$ has dimension given by the rank of $\check{\mathrm{G}}$. Let $\check{\mathfrak{g}}^{*,\mathrm{reg}}$ denote the locus of regular elements; this is an open subscheme whose complement is of codimension 3.

Theorem 3.1.6 (Kostant, [Kos1])**.** *The $\check{\mathrm{G}}$-orbit of the Kostant slice $\kappa : \check{\mathfrak{g}}^*/\!\!/\check{\mathrm{G}} \to \check{\mathfrak{g}}^*$ identifies with the regular locus $\check{\mathfrak{g}}^{*,\mathrm{reg}}$.*

Corollary 3.1.7. *Let $\underline{k}_{\mathrm{Gr}_{\mathrm{G}}} \in \mathrm{Ind}(\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}})$ denote the constant sheaf, and let $\mathrm{Loc}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)$ denote the full subcategory generated by $\underline{k}_{\mathrm{Gr}_{\mathrm{G}}}$. Then there is an equivalence*

$$\mathrm{Loc}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{\mathrm{G}}).$$

Proof. Observe that $\underline{k}_{\mathrm{Gr}_{\mathrm{G}}}$ is the pullback $p^*\underline{k}$ of the (necessarily constant) sheaf $\underline{k} \in \mathrm{Shv}_{\mathrm{G}[\![t]\!]}(*; k)$. Since p^* is the right adjoint to $p_!$ (and κ_* is the right adjoint to κ^*), the commutative diagram of Theorem 3.1.4 says that $\mathrm{Loc}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_{\mathrm{G}}; k)$ is equivalent to the full subcategory of $\mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{\mathrm{G}})$ generated by $\kappa_*\mathcal{O}_{\check{\mathfrak{g}}^*[2]/\!\!/\check{\mathrm{G}}}$. However, Theorem 3.1.6 implies that this full subcategory is equivalent to $\mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}[2]/\check{\mathrm{G}})$, as desired. □

A parallel story holds for the Arkhipov-Bezrukavnikov-Ginzburg (called "ABG" in this article) equivalence from [ABG].

Recollection 3.1.8. Let $\widetilde{\check{\mathfrak{g}}}$ denote the Grothendieck-Springer resolution, so that $\widetilde{\check{\mathfrak{g}}} = \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}}$. The action of $\check{\mathrm{G}}$ on $\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})$ defines the moment map $\mu : \widetilde{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}^*$. Let $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}$ denote the preimage of the regular locus $\check{\mathfrak{g}}^{*,\mathrm{reg}} \subseteq \check{\mathfrak{g}}^*$ under the moment map μ.

Proposition 3.1.9. *There is an isomorphism $\widetilde{\check{\mathfrak{g}}} \cong \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \check{\mathfrak{n}}^{\perp}$, as well as a map $\kappa : \psi + \check{\mathfrak{t}}^* \subseteq \check{\mathfrak{n}}^{\perp}$ which fits into a Cartesian square*

$$\begin{tikzcd} \psi + \check{\mathfrak{t}}^* \arrow[r] \arrow[d] & \check{\mathfrak{n}}^{\perp} \arrow[r] & \widetilde{\check{\mathfrak{g}}} \arrow[d, "\mu"] \\ \psi + \check{\mathfrak{g}}^{*,e} \arrow[rr] & & \check{\mathfrak{g}}^*. \end{tikzcd}$$

Proof. Let $\check{\mathrm{M}}$ be a Hamiltonian $\check{\mathrm{G}}$-scheme with moment map $\mu : \check{\mathrm{M}} \to \check{\mathfrak{g}}^*$. Then the pullback $\check{\mathrm{M}} \times_{\check{\mathfrak{g}}^*} (\psi + \check{\mathfrak{g}}^{*,e})$ can be identified with the Whittaker reduction $\check{\mathrm{M}}/_{\psi}\check{\mathrm{N}}$. Indeed, a theorem of Kostant's from [Kos2] identifies $\psi + \check{\mathfrak{g}}^{*,e}$ with $(\psi + \check{\mathfrak{n}}^{-,\perp})/\check{\mathrm{N}}^-$, so that there are isomorphisms

$$\begin{aligned} \check{\mathrm{M}} \times_{\check{\mathfrak{g}}^*} (\psi + \check{\mathfrak{g}}^{*,e}) &\cong \check{\mathrm{M}}/\check{\mathrm{G}} \times_{\check{\mathfrak{g}}^*/\check{\mathrm{G}}} (\psi + \check{\mathfrak{g}}^{*,e}) \\ &\cong \check{\mathrm{M}}/\check{\mathrm{G}} \times_{\check{\mathfrak{g}}^*/\check{\mathrm{G}}} (\psi + \check{\mathfrak{n}}^{-,\perp})/\check{\mathrm{N}}^- \\ &\cong (\check{\mathrm{M}} \times_{\check{\mathfrak{n}}^{-,*}} \{\psi\})/\check{\mathrm{N}}^- = \check{\mathrm{M}}/_{\psi}\check{\mathrm{N}}^-. \end{aligned}$$

Therefore, the fiber product in the statement of the proposition identifies with the Whittaker reduction $\widetilde{\check{\mathfrak{g}}}/_{\psi}\check{\mathrm{N}}^-$. Since $\widetilde{\check{\mathfrak{g}}} \cong \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}}$, we may identify $\widetilde{\check{\mathfrak{g}}}/_{\psi}\check{\mathrm{N}}^-$ with the quotient by $\check{\mathrm{T}}$ of $\mathrm{T}^*(\check{\mathrm{N}}^-{}_{\psi}\backslash\check{\mathrm{G}}/\check{\mathrm{N}})$. Since Whittaker functions are supported on the big cell, this twisted cotangent bundle is in turn isomorphic to $\mathrm{T}^*(\check{\mathrm{N}}^-{}_{\psi}\backslash(\check{\mathrm{N}}^- \times \check{\mathrm{T}} \times \check{\mathrm{N}})/\check{\mathrm{N}}) \cong \check{\mathrm{T}} \times (\psi + \check{\mathfrak{t}}^*)$. The desired Cartesian square follows. □

Again, $\widetilde{\check{\mathfrak{g}}}$ admits a $\mathbf{G}_m$-action obtained by placing $\check{\mathfrak{n}}^{\perp}$ in weight 2, and the map $\kappa : \check{\mathfrak{t}}^* \to \check{\mathfrak{n}}^{\perp}$ is equivariant if $\check{\mathfrak{t}}^*$ is also placed in weight 2. Therefore, shearing (as in [Dev3, Section 2.1]) defines a map

$$\check{\mathfrak{t}}^*[2] \xrightarrow{\kappa} \check{\mathfrak{n}}^{\perp}[2] \to \widetilde{\check{\mathfrak{g}}}[2].$$

We will sometimes denote this composite also by κ.

The first part of the below equivalence was proved by Arkhipov-Bezrukavnikov-Ginzburg in [ABG]; the commutative diagram below follows from Proposition 3.1.9 and Theorem 3.1.4.

Theorem 3.1.10. *Let $\mathrm{B} \subseteq \mathrm{G}$ be a Borel subgroup, and let $\mathrm{I} = \mathrm{G}[\![t]\!] \times_{\mathrm{G}} \mathrm{B}$ denote the associated Iwahori subgroup. Then there is an equivalence*

$$\mathrm{Shv}_{\mathrm{I}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}} \simeq \mathrm{Perf}(\widetilde{\check{\mathfrak{g}}}[2]/\check{\mathrm{G}}),$$

which restricts to a monoidal equivalence

$$\mathrm{Shv}_{\mathrm{I}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\omega} \simeq \mathrm{Perf}_{\check{\mathcal{N}}/\check{\mathrm{G}}}(\widetilde{\check{\mathfrak{g}}}[2]/\check{\mathrm{G}}).$$

Furthermore, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Ind}(\mathrm{Shv}_{\mathrm{I}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{lcc}}) \arrow[r, "\sim"] \arrow[d, "p_!"'] & \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}[2]/\check{\mathrm{G}}) \arrow[d, "\kappa^*"] \\ \mathrm{Shv}_{\mathrm{I}}(*; k) \arrow[r, "\sim"'] & \mathrm{QCoh}(\check{\mathfrak{t}}^*[2]), \end{tikzcd}$$

where $p : \mathrm{Gr}_{\mathrm{G}} \to *$ is the canonical map to a point and κ^* is pullback along the (shifted) Kostant slice.

As in Corollary 3.1.7, we find:

Corollary 3.1.11. *Let* $\underline{k}_{\mathrm{Gr_G}} \in \mathrm{Shv_I}(\mathrm{Gr_G}; k)$ *denote the constant sheaf, and let* $\mathrm{Loc_I}(\mathrm{Gr_G}; k)$ *denote the full subcategory generated by* $\underline{k}_{\mathrm{Gr_G}}$*. Then there is an equivalence*

$$\mathrm{Loc_I}(\mathrm{Gr_G}; k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}[2]/\check{\mathrm{G}}).$$

The constant sheaf has singular support given by the zero section. In fact, the ∞-categories $\mathrm{Loc}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr_G}; k)$ and $\mathrm{Loc_I}(\mathrm{Gr_G}; k)$ are the subcategories of *locally constant* (equivariant) sheaves on $\mathrm{Gr_G}$. As such, they depend only on the underlying homotopy types of $\mathrm{G}[\![t]\!]$, I, and $\mathrm{Gr_G}$.

Notation 3.1.12. Let G_c be the maximal compact subgroup of $\mathrm{G}(\mathbf{C})$, and let T_c be the maximal torus of G_c corresponding to the Borel B. It is not difficult to see that there are homotopy equivalences

$$\begin{aligned} \mathrm{G}[\![t]\!] &\simeq \mathrm{G}(\mathbf{C}) \simeq \mathrm{G}_c \\ \mathrm{I} &\simeq \mathrm{B}(\mathbf{C}) \simeq \mathrm{T}_c. \end{aligned}$$

The homotopy type of $\mathrm{Gr_G}$ follows from the next result, due to Quillen and Garland-Raghunathan:

Theorem 3.1.13 (Quillen, Garland-Raghunathan, [GR2, Mit])**.** *There is a homotopy equivalence* $\mathrm{Gr_G} \simeq \Omega\mathrm{G}_c$ *(and a homeomorphism onto the subspace of* $\Omega\mathrm{G}_c$ *on those based loops with polynomial Fourier expansion) which is equivariant for the left-action of* $\mathrm{G}_c \subseteq \mathrm{G}(\mathbf{C}) \subseteq \mathrm{G}(\mathbf{C}[\![t]\!])$ *on the left-hand side and the action of* G_c *on the right-hand side by conjugation.*

In our discussion below, we will mostly be concerned with the homology of $\mathrm{Gr_G}$, in which case we may replace $\mathrm{Gr_G}$ by $\Omega\mathrm{G}_c$. To this extent, we will implicitly use Theorem 3.1.13 without further mention. We will describe analogues of the equivalences of Corollary 3.1.7 and Corollary 3.1.11 for equivariant K-theory and equivariant elliptic cohomology.

3.2 Equivariant cohomology and the case of tori

In order to study and prove analogues of the equivalences of Corollary 3.1.7 and Corollary 3.1.11 for other cohomology theories, we need to review some foundational aspects of the theory of equivariant cohomology. I have reviewed some of the basics of equivariant K-theory in [Dev3, Section 2.2]. The theory of equivariant elliptic cohomology is developed similarly, and we will now describe this story (in a somewhat leisurely fashion) following [Lur1, GM2, GM1]. At the end of this section, we describe the geometric Satake equivalence for tori.

The basic question we will address is giving a definition of the ∞-category $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ for a T_c-space X for a sufficiently general $\mathbf{E}_\infty$-ring k. When k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, Theorem 3.1.10 requires that there is an equivalence

$$\mathrm{Loc}_{\mathrm{T}_c}(*; k) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}^*[2]).$$

One often defines the ∞-category of k-modules on a space X as the ∞-category $\mathrm{Fun}(\mathrm{X}, \mathrm{Mod}_k)$. However, when $\mathrm{X} = \mathrm{BT}_c$, the ∞-category $\mathrm{Fun}(\mathrm{BT}_c, \mathrm{Mod}_k)$ does *not* agree with $\mathrm{QCoh}(\check{\mathfrak{t}}^*[2])$; instead, it only agrees with a certain completion of this ∞-category, as we will now explain.

Lemma 3.2.1. *Let k be an $\mathbf{E}_\infty$-algebra. Then there is an equivalence*

$$\mathrm{Fun}(\mathrm{BT}_c, \mathrm{Mod}_k) \simeq \mathrm{IndCoh}(\{1\} \times_{\check{\mathrm{T}}} \{1\}).$$

If, moreover, k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, this can be rewritten as an equivalence

$$\mathrm{Fun}(\mathrm{BT}_c, \mathrm{Mod}_k) \simeq \mathrm{QCoh}(\widehat{\check{\mathfrak{t}}^*}[2]),$$

where $\widehat{\check{\mathfrak{t}}^*}$ denotes the completion of $\check{\mathfrak{t}}^*$ at the origin.

Proof. If X is a finite space, there is an equivalence $\mathrm{Fun}(\mathrm{X}, \mathrm{Mod}_k) \simeq \mathrm{IndCoh}_{\mathrm{C}_*(\Omega\mathrm{X};k)}$, where $\mathrm{C}_*(\Omega\mathrm{X};k)$ is the $\mathbf{E}_1$-k-algebra of k-chains on the based loop space $\Omega\mathrm{X}$. When $\mathrm{X} = \mathrm{BT}_c$, we may identify $\Omega\mathrm{X} = \mathrm{T}_c$. Recall that T_c is the classifying space of the lattice $\mathbb{X}_*(\mathrm{T})$, so that there is an equivalence

$$\mathrm{C}_*(\mathrm{T}_c;k) \cong k \otimes_{\mathrm{C}_*(\mathbb{X}_*(\mathrm{T});k)} k.$$

Of course, we may identify $\mathrm{C}_*(\mathbb{X}_*(\mathrm{T});k) \cong k[\mathbb{X}_*(\mathrm{T})]$ with the ring of functions on $\check{\mathrm{T}}$. Therefore, $\mathrm{Spec}\, \mathrm{C}_*(\mathrm{T}_c;k) \cong \{1\} \times_{\check{\mathrm{T}}} \{1\}$, as desired.

Koszul duality gives an equivalence $\mathrm{IndCoh}(\mathrm{C}_*(\mathrm{T}_c;k)) \to \mathrm{QCoh}(\mathrm{C}^*(\mathrm{BT}_c;k))$ given by $\mathrm{M} \mapsto \mathrm{Hom}_{\mathrm{C}_*(\mathrm{T}_c;k)}(k, \mathrm{M})$. If k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, then $\mathrm{C}^*(\mathrm{BT}_c;k)$ is formal, and so it can be identified with the shearing of $\mathrm{H}^*(\mathrm{BT}_c;k)$. But

$$\mathrm{Spf}\, \mathrm{H}^*(\mathrm{BT}_c;k) \cong \widehat{\mathfrak{t}}(2) \cong \widehat{\check{\mathfrak{t}}^*}(2),$$

so $\mathrm{IndCoh}(\mathrm{C}_*(\mathrm{T}_c;k))$ is equivalent to $\mathrm{QCoh}(\widehat{\check{\mathfrak{t}}^*}[2])$, as desired. □

Example 3.2.2. Suppose $\mathrm{T}_c = \mathrm{S}^1$. Then Lemma 3.2.1 tells us that $\mathrm{Fun}(\mathrm{BS}^1, \mathrm{Mod}_k) \simeq \mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$; the equivalence sends a functor $\mathrm{BS}^1 \to \mathrm{Mod}_k$, regarded as a k-module M with S^1-action, to its homotopy invariants $\mathrm{M}^{h\mathrm{S}^1}$. Let $t \in \pi_{-2}(k^{h\mathrm{S}^1})$ denote a generator. Observe that if $a_\lambda : k \to k[2]$ denotes the boundary map in the cofiber sequence $k[1] \to \mathrm{C}_*(\mathrm{S}^1;k) \to k$, the homotopy invariants of $k[a_\lambda^{-1}]$ are simply $k^{h\mathrm{S}^1}[t^{-1}]$ (i.e., the Tate construction). In particular, $\pi_*(k[a_\lambda^{-1}])^{h\mathrm{S}^1} \cong \pi_*(k)(\!(t)\!)$. However (even if k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra), there is no (ind-)object in $\mathrm{Fun}(\mathrm{BS}^1, \mathrm{Mod}_k)$ whose image in $\mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$ has homotopy given by $\pi_*(k)[t^{\pm 1}]$: any object of $\mathrm{QCoh}(\widehat{\mathbf{A}^1}[2])$ must have t as a topologically nilpotent element in its homotopy.

We therefore need an alternative definition of $\mathrm{Loc}_{\mathrm{T}_c}(*;k)$, so that it is equivalent to $\mathrm{QCoh}(\check{\mathfrak{t}}^*[2])$ when k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra. Motivated by methods from equivariant homotopy theory, as well as [Lur1, Lur6, Lur7, Lur8], we will simply *define* $\mathrm{Loc}_{\mathrm{T}_c}(*;k)$ to be the category of quasicoherent sheaves on a (spectral) stack $\mathcal{M}_\mathrm{T}$. That this category has any relation to topology will come from the requirement that the category of quasicoherent sheaves on the *completion* of $\mathcal{M}_\mathrm{T}$ at a certain basepoint is equivalent to the ind-completion of $\mathrm{Fun}(\mathrm{BT}_c, \mathrm{Mod}_k)$.

For this, we review some constructions from [Lur1] in a form suitable for our applications. This review will necessarily be brief, since a detailed exposition may be found in *loc. cit.*; there is also some discussion in the early sections of [GKV1] in the setting of ordinary (as opposed to spectral) algebraic geometry.

Setup 3.2.3. Fix an $\mathbf{E}_\infty$-ring k and a commutative k-group $\mathbf{G}$, so $\mathbf{G}$ defines a functor $\mathrm{CAlg}_\mathrm{A} \to \mathrm{Mod}_{\mathbf{Z},\geq 0}$ which is representable by a *flat* k-algebra; here, $\mathrm{Mod}_{\mathbf{Z},\geq 0}$ denotes the category of connective $\mathbf{Z}$-module spectra. We will write $\mathbf{G}_0$ to denote the resulting commutative group scheme over $\pi_0 k$. Note that taking zeroth spaces defines an equivalence between $\mathrm{Mod}_{\mathbf{Z},\geq 0}$ and topological abelian groups.

Definition 3.2.4. A *preorientation of* $\mathbf{G}$ is a pointed map $S^2 \to \Omega^\infty \mathbf{G}(k)$ of spaces, i.e., a map $\Sigma^2 \mathbf{Z} \to \mathbf{G}(k)$ of $\mathbf{Z}$-modules (by adjunction). This induces a map $\mathbf{C}P^\infty = \Omega^\infty \Sigma^2 \mathbf{Z} \to \Omega^\infty \mathbf{G}(k)$ of topological abelian groups, and hence a map $\mathrm{Spf}\, A^{\mathbf{C}P^\infty} \to \mathbf{G}$ of $\mathbf{E}_\infty$-k-group schemes. (Note that $\mathrm{Spf}\, A^{\mathbf{C}P^\infty}$ need not admit the structure of a commutative k-group scheme: for instance, $A^{\mathbf{C}P^\infty}$ need not be flat over k.)

Definition 3.2.5. Given a preorientation $S^2 \to \Omega^\infty \mathbf{G}(k)$, we obtain a map $\mathcal{O}_{\mathbf{G}} \to C^*(S^2; k)$ of $\mathbf{E}_\infty$-k-algebras. On π_0, this induces a map $\pi_0 \mathcal{O}_{\mathbf{G}} = \mathcal{O}_{\mathbf{G}_0} \to \pi_0 C^*(S^2; k)$. However, the target can be identified with the trivial square-zero extension $\pi_0 k \oplus \pi_{-2} k$, so that the preorientation defines a derivation $\mathcal{O}_{\mathbf{G}_0} \to \pi_{-2} k$. This defines a map $\beta : \omega = \Omega^1_{\mathbf{G}_0/\pi_0 k} \to \pi_{-2} k$. The preorientation is called an *orientation* if $\mathbf{G}_0$ is smooth of relative dimension 1 over $\pi_0 k$, and the composite

$$\pi_n k \otimes_{\pi_0 k} \omega \to \pi_n k \otimes_{\pi_0 k} \pi_{-2} k \xrightarrow{\beta} \pi_{n-2} k$$

is an isomorphism for each $n \in \mathbf{Z}$. This forces k to be 2-periodic (but does not force its homotopy to be concentrated in even degrees).

Warning 3.2.6. As discussed in [Lur1, Section 3.2], the universal $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra over which the additive group scheme $\mathbf{G}_a$ admits an orientation is given by $\mathbf{Z}[\mathbf{C}P^\infty][\frac{1}{\beta}] = \mathbf{Q}[\beta^{\pm 1}]$. Therefore, we are allowed to let $\mathbf{G} = \mathbf{G}_a$ in the story below only when k is a 2-periodic $\mathbf{E}_\infty$-$\mathbf{Q}$-*algebra*. (If k is not an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra, one cannot in general define $\mathbf{G}_a = \mathrm{Spec}\, k[t]$ as a commutative k-group: the coproduct $k[t] \to k[x, y]$ will in general not be a map of $\mathbf{E}_\infty$-k-algebras.)

We can now review the definition of T_c-equivariant k-cohomology when T_c is a compact torus. We will write T to denote the corresponding split torus over $\mathbf{Z}$.

Construction 3.2.7. Fix an $\mathbf{E}_\infty$-ring k as above and a commutative k-group $\mathbf{G}$. Given a compact abelian Lie group T_c, define a k-scheme $\mathcal{M}_T$ by the mapping stack $\mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G})$. The underlying $\pi_0(k)$-schemes will be denoted by $\mathbf{G}_0$ and $\mathcal{M}_{T,0}$. If we wish to emphasize the dependence on k, we will add a superscript (e.g., $\mathcal{M}_T^k$).

We will be particularly interested in the case when T_c is a torus. Let $\mathcal{T}$ be the full subcategory of $\mathcal{S}$ spanned by those spaces which are homotopy equivalent to BT_c with T_c being a compact abelian Lie group. By arguing as in [Lur8, Theorem 3.5.5], a preorientation of $\mathbf{G}$ is equivalent to the data of a functor $\mathcal{M} : \mathcal{T} \to \mathrm{Aff}_k$ along with compatible equivalences $\mathcal{M}(BT_c) \simeq \mathcal{M}_T$. The $\mathbf{E}_\infty$-k-algebra $\mathcal{O}_{\mathcal{M}_T}$ is the T_c-equivariant k-cochains of a point, and will occasionally be denoted by k_T.

We can now sketch the construction of the T_c-equivariant k-cochains of more general T_c-spaces; see [Lur1, Theorem 3.2]. Let T_c be a torus over $\mathbf{C}$ for the remainder of this discussion, and let $\mathbf{G}$ be an *oriented* commutative k-group. Let $\mathcal{S}(T_c)$ denote the ∞-category of finite T_c-spaces, i.e., the smallest subcategory of $\mathrm{Fun}(BT_c, \mathcal{S})$ which contains the quotients T_c/T_c' for closed subgroups $T_c' \subseteq T_c$, and which is closed under finite colimits. There is a functor $\mathcal{F}_T : \mathcal{S}(T_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T)$ which is uniquely characterized by the requirement that it preserve finite limits and sends $T_c/T_c' \mapsto q_* \mathcal{O}_{\mathcal{M}_{T_c'}}$. Here, $q : \mathcal{M}_{T_c'} \to \mathcal{M}_T$ is the canonical map induced by the inclusion $T_c' \subseteq T_c$. If $X \in \mathcal{S}(T_c)$, then the T_c-*equivariant* k-*cochains of* X is the global sections $\Gamma(\mathcal{M}_T; \mathcal{F}_T(X))$; we will denote it by $C^*_{T_c}(X; k)$. This can be extended to define T_c-equivariant k-cochains of filtered colimits of finite T_c-spaces. If we wish to emphasize the dependence on k, we will denote $\mathcal{F}_T(X)$ by $\mathcal{F}_T(X; k)$.

Remark 3.2.8. If k is 2-periodic and $\mathbf{G}$ is a commutative k-group, then [Lur7, Proposition 4.3.23] shows that the data of an orientation on k (in the sense of Definition 3.2.5) is equivalent to the formal completion of $\mathbf{G}$ at the origin being isomorphic to $\mathrm{Spf}\, \mathrm{C}^*(\mathrm{BS}^1; k)$. That is, when $\mathbf{G}$ is oriented, the formal completion of $\mathcal{M}_\mathrm{T}$ at its basepoint is isomorphic to $\mathrm{Spf}\, \mathrm{C}^*(\mathrm{BT}_c; k)$.

We will denote the functor $\Gamma(\mathcal{M}_\mathrm{T}; \mathcal{F}_\mathrm{T}(-)) : \mathcal{S}(\mathrm{T}_c)^\mathrm{op} \to \mathrm{Mod}(\Gamma(\mathcal{M}_\mathrm{T}; \mathcal{O}_{\mathcal{M}_\mathrm{T}}))$ by $\mathrm{C}^*_{\mathrm{T}_c}(-; k) : \mathcal{S}(\mathrm{T}_c)^\mathrm{op} \to \mathrm{Mod}(k_\mathrm{T})$.

Definition 3.2.9. If $\mathrm{X} \in \mathcal{S}(\mathrm{T}_c)$, then the *$\mathrm{T}_c$-equivariant k-chains of* X is the quasicoherent sheaf on $\mathcal{M}_\mathrm{T}$ given by the $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-linear dual $\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$. We will denote its global sections by $\mathrm{C}^{\mathrm{T}_c}_*(\mathrm{X}; k)$. Note that if X admits an $\mathbf{E}_n$-algebra structure (compatible with the T_c-action), then $\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$ admits the structure of an $\mathbf{E}_n$-algebra[1] in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T}))$. Note that $\mathrm{C}^{\mathrm{T}_c}_*(*; k) \simeq k_\mathrm{T}$, which completes to the k-cochains (*not* k-chains) of BT_c.

If X is a filtered colimit $\mathrm{colim}_\alpha \mathrm{X}_\alpha$ of finite T_c-spaces, we will write $\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$ to denote $\mathrm{colim}_\alpha(\mathcal{F}_\mathrm{T}(\mathrm{X}_\alpha)^\vee)$. Note that if we equip the presentation of X as a filtered colimit $\mathrm{colim}_\alpha \mathrm{X}_\alpha$ with the structure of a filtered $\mathbf{E}_n$-algebra, then $\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$ acquires the structure of an $\mathbf{E}_n$-algebra in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T}))$.

Notation 3.2.10. Let $\lambda : \mathrm{T} \to \mathbf{G}_m$ be a character, and let $\mathrm{T}_\lambda = \ker(\lambda)$. Then the map $q : \mathcal{M}_{\mathrm{T}_\lambda} \to \mathcal{M}_\mathrm{T}$ is a closed immersion, and we will denote the ideal in $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$ defined by this closed immersion by $\mathcal{I}_\lambda$. Equivalently, let V_λ denote the T_c-representation obtained by the projection $\mathrm{T} \to \mathrm{T}_\lambda$. Then $\mathcal{I}_\lambda$ is given by the line bundle $\mathcal{F}_\mathrm{T}(\mathrm{S}^{\mathrm{V}_\lambda})$.

It is trickier to extend the definition of equivariant cochains to nonabelian groups, but a construction is sketched in [Lur1, Section 3.5], and a detailed construction is given in [GM2]. However, we will not recall this here, because we will only be concerned with torus-equivariance in the present article.

We now take a moment to prove some foundational aspects of the theory of generalized equivariant cohomology.

Lemma 3.2.11 (Atiyah-Bott localization [AB2])**.** *Let* X *be a finite* T_c*-space, and let* $\mathcal{U}_\mathrm{X} \subseteq \mathcal{M}_\mathrm{T}$ *denote the complement of the union of the closed substacks* $\mathcal{M}_{\mathrm{T}'}$ *over all stabilizers* $\mathrm{T}'_c \subseteq \mathrm{T}_c$ *of points in* X*. Then the map* $\mathcal{F}_\mathrm{T}(\mathrm{X}) \to \mathcal{F}_\mathrm{T}(\mathrm{X}^{\mathrm{T}_c})$ *is an isomorphism after pulling back to* $\mathcal{U}_\mathrm{X}$*.*

Proof. This follows from induction on the cell structure of X. Namely, the statement is true when the T-action on X is trivial, which gives the base case. For the inductive step, note that if X is the cofiber of a map $\mathrm{T}/\mathrm{T}' \to \mathrm{Y}$, then there is a cofiber sequence $\mathcal{F}_\mathrm{T}(\mathrm{X}) \to \mathcal{F}_\mathrm{T}(\mathrm{Y}) \to \mathcal{F}_\mathrm{T}(\mathrm{T}/\mathrm{T}')$; but $\mathcal{F}_\mathrm{T}(\mathrm{T}/\mathrm{T}')$ is isomorphic to the pushforward of the structure sheaf along the map $\mathcal{M}_{\mathrm{T}'} \to \mathcal{M}_\mathrm{T}$, and so it vanishes upon pulling back to $\mathcal{U}_\mathrm{X}$. This implies that the map $\mathcal{F}_\mathrm{T}(\mathrm{X}) \to \mathcal{F}_\mathrm{T}(\mathrm{Y})$ is an isomorphism upon pulling back to $\mathcal{U}_\mathrm{X}$, as desired. ∎

One consequence of Lemma 3.2.11 which is worth restating is the following. Let $\mathring{\mathcal{M}}_\mathrm{T}$ denote the complement of the union of the closed subschemes $\mathcal{M}_{\mathrm{T}'}$ ranging over all closed *proper* subgroups $\mathrm{T}' \subsetneq \mathrm{T}$. Then the map $\mathcal{F}_\mathrm{T}(\mathrm{X}) \to \mathcal{F}_\mathrm{T}(\mathrm{X}^{\mathrm{T}_c})$, and hence the map $\mathcal{F}_\mathrm{T}(\mathrm{X}^{\mathrm{T}_c})^\vee \to \mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$, is an equivalence upon restriction to $\mathring{\mathcal{M}}_\mathrm{T}$.

We will also need a version of the Goresky-Kottwitz-MacPherson approach [GKM] to equivariant cohomology; in the setting of generalized equivariant cohomology, it has also been studied in [HHH, GM2]. As such, we will only give a sketch of the relevant argument.

Definition 3.2.12. Let X be a finite T_c-space equipped with a chosen presentation in terms of T_c-cells. Say that X is a *GKM space* if the following conditions are satisfied:

[1] If $\mathcal{C}$ is a symmetric monoidal ∞-category, [Lur6, Corollary 3.3.4] can be used to show that there is an equivalence $\mathrm{coCAlg}(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbf{E}_n}(\mathrm{coCAlg}(\mathcal{C}))$.

a. $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ is a vector bundle over $\mathcal{M}_{\mathrm{T},0}$;

b. if $\mathrm{X}^{(1)}$ denotes the equivariant 1-skeleton of X, then $\mathrm{X}^{(1)}$ consists of a finite number of spheres S^λ meeting only at the fixed points, where λ ranges over characters of T.

In this setup, let V denote the set $\mathrm{X}^{\mathrm{T}_c}$ of fixed points, and let E denote the set of characters λ such that $\mathrm{S}^\lambda \subseteq \mathrm{X}^{(1)}$. There are two maps $\mathrm{E} \rightrightarrows \mathrm{V}$ sending λ to the points $0, \infty \in \mathrm{S}^\lambda \subseteq \mathrm{X}^{(1)}$. The resulting graph with set of vertices V and set of edges E will be referred to as the *GKM graph* of X.

The utility of the first condition in the above definition is due to the following.

Lemma 3.2.13. *Let* X *be a finite* T_c*-space. If* $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ *is a vector bundle over* $\mathcal{M}_{\mathrm{T},0}$*, the map* $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$ *is an injection.*

Proof. Since the map $\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c}) \to \mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})_{\mathring{\mathcal{M}}_{\mathrm{T}}}$ factors as $\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \mathcal{F}_{\mathrm{T}}(\mathrm{X})|_{\mathring{\mathcal{M}}_{\mathrm{T}}} \to \mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})|_{\mathring{\mathcal{M}}_{\mathrm{T}}}$, and the map $\mathcal{F}_{\mathrm{T}}(\mathrm{X})|_{\mathring{\mathcal{M}}_{\mathrm{T}}} \to \mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})|_{\mathring{\mathcal{M}}_{\mathrm{T}}}$ is an equivalence by Lemma 3.2.11, it suffices to show that the map $\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \mathcal{F}_{\mathrm{T}}(\mathrm{X})|_{\mathring{\mathcal{M}}_{\mathrm{T}}}$ induces an injection on π_0. But $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ was assumed to be a vector bundle over $\mathcal{M}_{\mathrm{T},0}$, so one is reduced to the case $\mathrm{X} = *$, i.e., to showing that the map $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}} \to \mathcal{O}_{\mathcal{M}_{\mathrm{T}}}|_{\mathring{\mathcal{M}}_{\mathrm{T}}}$ induces an injection on π_0. This, however, is clear, since the closed subscheme $\mathcal{M}_{\mathrm{T}',0} \hookrightarrow \mathcal{M}_{\mathrm{T},0}$ defined by each closed subgroup $\mathrm{T}' \subseteq \mathrm{T}$ is cut out by a regular sequence. □

Proposition 3.2.14 (Goresky-Kottwitz-MacPherson)**.** *Let* X *be a finite GKM* T_c*-space, and choose a presentation in terms of* T_c*-cells. For each character* $\lambda : \mathrm{T} \to \mathrm{S}^1$*, let* T_λ *denote the kernel of* T*, let* $q_\lambda : \mathcal{M}_{\mathrm{T}_\lambda} \to \mathcal{M}_{\mathrm{T}}$ *denote the induced map, and let* $\mathrm{S}(\lambda)$ *denote the unit representation sphere. Then there is an equalizer diagram*

$$\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \hookrightarrow \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c}) \cong \mathrm{Map}(\mathrm{V}, \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}) \rightrightarrows \prod_{\lambda \in \mathrm{E}} q_{\lambda,*}\mathcal{O}_{\mathcal{M}_{\mathrm{T}_\lambda,0}},$$

where the two maps in the equalizer are defined in the evident manner.

Proof sketch. First, we show that the maps $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$ and $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{(1)}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$ have the same image. There is an evident map from the image of $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$ to the image of $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{(1)}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$, which we will denote by f. The map f is an injection by Lemma 3.2.13. Let T' denote a proper closed subgroup of T of codimension 1, and let $\mathrm{U}' \subseteq \mathcal{M}_{\mathrm{T}',0}$ denote the complement of the union of the closed varieties $\mathcal{M}_{\mathrm{T}'',0}$ ranging over the proper closed subgroups $\mathrm{T}'' \subseteq \mathrm{T}'$. By Lemma 3.2.11, the map f is an isomorphism upon restriction to $\mathrm{U}' \subseteq \mathcal{M}_{\mathrm{T}',0} \subseteq \mathcal{M}_{\mathrm{T},0}$ for each proper closed subgroup $\mathrm{T}' \subseteq \mathrm{T}$ of codimension 1. Therefore, the locus $\mathrm{Z} \subseteq \mathcal{M}_{\mathrm{T},0}$ over which f fails to be an isomorphism is contained in the union of closed subvarieties $\mathcal{M}_{\mathrm{T}',0}$ for finitely many $\mathrm{T}' \subseteq \mathrm{T}$ of codimension at least 2. However, the map $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})|_{\mathcal{M}_{\mathrm{T},0}-\mathrm{Z}}$ is an isomorphism (by Hartogs). Since the same is true of the map $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})|_{\mathcal{M}_{\mathrm{T},0}-\mathrm{Z}}$, and the map $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$ factors through the map $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{(1)}) \to \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}_c})$, the desired result follows.

For the equalizer diagram, an easy induction on the cell structure of X reduces us to the case $\mathrm{X} = \mathrm{S}^\lambda$ for a character $\lambda : \mathrm{T} \to \mathrm{S}^1$. In this case, the isomorphism $\mathrm{T}/\mathrm{T}_\lambda \cong \mathrm{S}^\lambda$ defines an isomorphism between $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{S}(\lambda))$ and the pushforward of the structure sheaf along the map $\mathcal{M}_{\mathrm{T}_\lambda,0} \to \mathcal{M}_{\mathrm{T},0}$. Since $\mathrm{S}^\lambda \cong \Sigma\mathrm{S}(\lambda)$, we obtain an equalizer diagram

$$\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{S}^\lambda) \to \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}} \oplus \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}} \cong \mathrm{Map}(\{0, \infty\}, \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}) \rightrightarrows q_{\lambda,*}\mathcal{O}_{\mathcal{M}_{\mathrm{T}_\lambda,0}}.$$

This proves the desired claim. □

The same argument proves the following dual to Proposition 3.2.14 (see also [Bri]).

Proposition 3.2.15. *Let* X *be a finite GKM* T_c*-space, and choose a presentation in terms of* T_c*-cells. Then* $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$ *is isomorphic to the subset of* $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{X}^{\mathrm{T}_c})^\vee \cong \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}[\mathrm{X}^{\mathrm{T}_c}]$ *of those* $\sum_{x\in\mathrm{X}^{\mathrm{T}_c}} f_x[x] \in \mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}[\mathrm{X}^{\mathrm{T}_c}]$ *such that:*

- *For each fixed point* $x \in \mathrm{X}^{\mathrm{T}_c}$*, the poles of* f_x *all have order* ≤ 1*, and these are contained in the ideal sheaf of* $\mathcal{O}_{\mathcal{M}_{\mathrm{T}_\lambda,0}}$ *for each character* $\lambda : \mathrm{T}_c \to \mathrm{S}^1$ *such that the* T_c*-orbit* S^λ *meets* x*.*
- *For each character* $\lambda : \mathrm{T}_c \to \mathrm{S}^1$ *such that the* T_c*-orbit* S^λ *meets* $x_0, x_\infty \in \mathrm{X}^{\mathrm{T}_c}$*, we have*
$$\mathrm{Res}_{\mathcal{M}_{\mathrm{T}_\lambda,0}}(f_{x_0}) + \mathrm{Res}_{\mathcal{M}_{\mathrm{T}_\lambda,0}}(f_{x_\infty}) = 0.$$

These results can be extended without much trouble to ind-T_c-spaces X with isolated fixed points satisfying the conditions of Definition 3.2.12. (The first condition therein should be replaced by the condition that $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{X})$ is an ind-vector bundle over $\mathcal{M}_{\mathrm{T},0}$.)

The preceding discussion can be categorified, as we now explain. The following categorifies the T_c-equivariant k-cochains $\mathrm{C}^*_{\mathrm{T}_c}(\mathrm{X}; k)$.

Construction 3.2.16. Let $\mathrm{Loc}_{\mathrm{T}_c}(*; k)$ denote the ∞-category $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$. Let $\mathrm{T}'_c \subseteq \mathrm{T}_c$ be a closed subgroup, so that there is an associated morphism $q : \mathcal{M}_{\mathrm{T}'} \to \mathcal{M}_\mathrm{T}$. This defines a symmetric monoidal functor $\mathrm{QCoh}(\mathcal{M}_\mathrm{T}) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}'})$, which equips $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}'})$ with the structure of a $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$-module.

Let $\mathcal{L}oc_{\mathrm{T}_c}(-; k) : \mathcal{S}(\mathrm{T}_c)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{ShvCat}(\mathcal{M}_\mathrm{T}))$ be the functor uniquely characterized by the requirement that it preserve finite limits and send $\mathrm{T}/\mathrm{T}' \mapsto \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}'})$. If $\mathrm{X} \in \mathcal{S}(\mathrm{T}_c)$, then the ∞-category $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ of T_c*-equivariant local systems of* k*-modules on* X is defined to be the global sections of the quasicoherent stack $\mathcal{L}oc_{\mathrm{T}_c}(\mathrm{X}; k)$ on $\mathcal{M}_\mathrm{T}$. If X is a T_c-space which is presented as a filtered colimit of finite T_c-spaces X_α, we will write $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ to denote $\mathrm{colim}\,\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}_\alpha; k)$.

If $f : \mathrm{X} \to \mathrm{Y}$ is a map in $\mathcal{S}(\mathrm{T}_c)$, the associated symmetric monoidal functor $f^* : \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Y}; k) \to \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ (induced by taking global sections of the morphism $f^* : \mathcal{L}oc_{\mathrm{T}_c}(\mathrm{Y}; k) \to \mathcal{L}oc_{\mathrm{T}_c}(\mathrm{X}; k)$ of $\mathbf{E}_\infty$-algebras in quasicoherent stacks over $\mathcal{M}_\mathrm{T}$) will be called the *pullback*. One can show that $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ is a presentable stable ∞-category, and that f^* preserves small colimits (so it has a right adjoint f_*, which will be called *pushforward*).

For instance, if $\mathrm{T}_c = \{1\}$, then $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ is equivalent to the ∞-category $\mathrm{Loc}(\mathrm{X}; k) := \mathrm{Fun}(\mathrm{X}, \mathrm{Mod}_k)$ of local systems on X.

Remark 3.2.17. Let X be a finite T_c-space. The *constant local system* $\underline{k}$ is defined to be the image of $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$ under the symmetric monoidal functor $\mathrm{Loc}_{\mathrm{T}_c}(*; k) \simeq \mathrm{QCoh}(\mathcal{M}_\mathrm{T}) \to \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ induced by pullback along $f : \mathrm{X} \to *$. Observe that if $\underline{k}$ denotes the constant local system, then $\mathrm{End}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(\underline{k}) \simeq \mathrm{C}^*_{\mathrm{T}_c}(\mathrm{X}; k)$. Indeed, $\mathrm{End}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(\underline{k}) \simeq \Gamma(\mathcal{M}_\mathrm{T}; f_*f^*\mathcal{O}_{\mathcal{M}_\mathrm{T}})$, but it is easy to see that $f_*f^*\mathcal{O}_{\mathcal{M}_\mathrm{T}} = \mathcal{F}_\mathrm{T}(\mathrm{X}) \in \mathrm{QCoh}(\mathcal{M}_\mathrm{T})$. The desired claim then follows from Construction 3.2.7.

Remark 3.2.18. If the complexification of T_c were a *finite* diagonalizable group scheme (such as μ_n), the desired category $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ is closely related to the ∞-category of $\mathbf{G}$*-tempered local systems* on the orbispace $\mathrm{X}/\!\!/\mathrm{T}$, as described in [Lur8]. Our understanding is that Lurie is planning to describe an extension of the work in [Lur8] and its connections to equivariant homotopy theory in a future article. We warn the reader that Construction 3.2.16 is somewhat *ad hoc*; so the resulting category of equivariant local systems may or may not agree with the output of forthcoming work of Lurie.

Remark 3.2.19. If X is a finite T_c-space, a more straightforward definition of the category of T_c-equivariant local systems on X is simply the category $\mathrm{Fun}(\mathrm{X}/\mathrm{T}_c, \mathrm{Mod}_k)$. Equivalently, it can be described as the functor $\mathcal{S}(\mathrm{T}_c)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ which is uniquely characterized by the requirement that it preserve finite limits and send $\mathrm{T}_c/\mathrm{T}'_c \mapsto \mathrm{Fun}(\mathrm{BT}'_c, \mathrm{Mod}_k)$. It follows from Lemma 3.2.1 that $\mathrm{Fun}(\mathrm{BT}'_c, \mathrm{Mod}_k)$ is equivalent to $\mathrm{Mod}(\mathrm{C}^*(\mathrm{BT}'_c; k))$. As discussed in Remark 3.2.8, if the group scheme $\mathbf{G}$ is oriented, then this is in turn equivalent to $\mathrm{QCoh}(\widehat{\mathcal{M}_{\mathrm{T}}})$, where $\widehat{\mathcal{M}_{\mathrm{T}}}$ is the completion of $\mathcal{M}_{\mathrm{T}}$ at its basepoint. That is, $\mathrm{Fun}(\mathrm{BT}'_c, \mathrm{Mod}_k)$ can be viewed as a completion of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}'})$. This implies that $\mathrm{Fun}(\mathrm{X}/\mathrm{T}_c, \mathrm{Mod}_k)$ can be viewed as a completion of the subcategory of compact objects of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$. Motivated by this, we will write $\mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{X}; k)$ to denote $\mathrm{Fun}(\mathrm{X}/\mathrm{T}_c, \mathrm{Mod}_k)$; we will use the same notation to denote the extension of the assignment $\mathrm{X} \mapsto \mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{X}; k)$ to filtered colimits of finite T_c-spaces.

Using this discussion, let us now discuss geometric Satake with k-coefficients in the case of a torus.

Theorem 3.2.20. *Fix a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k and an oriented commutative k-group scheme $\mathbf{G}$. Let $\check{\mathrm{T}} = \mathrm{Spec}\, k[\mathbb{X}^*(\check{\mathrm{T}})]$ denote the dual torus over k. In the following statements, all actions of $\check{\mathrm{T}}$ are trivial. Then there are equivalences*

$$\mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \mathrm{IndCoh}((\{1\} \times_{\check{\mathrm{T}}} \{1\})/\check{\mathrm{T}}),$$
$$\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}/\check{\mathrm{T}}).$$

Moreover, there is an isomorphism of spectral group k-schemes

$$\mathrm{Spec}\, \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}})^\vee \cong \mathcal{M}_{\mathrm{T}} \times_{\mathrm{Spec}(k)} \check{\mathrm{T}} \cong \mathcal{M}_{\mathrm{T}} \times_{\mathcal{M}_{\mathrm{T}}/\check{\mathrm{T}}} \mathcal{M}_{\mathrm{T}}.$$

Proof. Since the underlying topological space of Gr_{T} is simply the lattice $\mathbb{X}_*(\mathrm{T})$, it follows from Lemma 3.2.1 that

$$\mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \bigoplus_{\mathbb{X}_*(\mathrm{T})} \mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(*; k) \simeq \mathrm{QCoh}(\mathrm{B}\check{\mathrm{T}}) \otimes_{\mathrm{Mod}_k} \mathrm{IndCoh}(\{1\} \times_{\check{\mathrm{T}}} \{1\}).$$

For the trivial action of $\check{\mathrm{T}}$ on $\{1\} \times_{\check{\mathrm{T}}} \{1\}$, this is precisely $\mathrm{IndCoh}((\{1\} \times_{\check{\mathrm{T}}} \{1\})/\check{\mathrm{T}})$. Exactly the same discussion proves the second equivalence:

$$\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \bigoplus_{\mathbb{X}_*(\mathrm{T})} \mathrm{Loc}_{\mathrm{T}_c}(*; k) \simeq \mathrm{QCoh}(\mathrm{B}\check{\mathrm{T}}) \otimes_{\mathrm{Mod}_k} \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}).$$

The claim about $\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}})^\vee$ can be proved similarly. □

Remark 3.2.21. Note that in Theorem 3.2.20, the "spectral"/algebro-geometric description of $\mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k)$ does not seem to depend on the choice of coefficient k (in particular, not on $\mathbf{G}$). This dependence, however, can be made more explicit by noting that $\mathrm{IndCoh}(\{1\} \times_{\check{\mathrm{T}}} \{1\})$ is equivalent to $\mathrm{Mod}(k^{h\mathrm{T}_c}) \simeq \mathrm{QCoh}(\widehat{\mathcal{M}_{\mathrm{T}}})$. That is, there is an equivalence $\mathrm{Loc}^{\wedge}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \mathrm{QCoh}(\widehat{\mathcal{M}_{\mathrm{T}}}/\check{\mathrm{T}})$. See Proposition 4.6.3 for more on the relationship between the two equivalences in Theorem 3.2.20.

Our basic goal is to find a replacement of Theorem 3.2.20 where Gr_{T} is replaced by Gr_{G} for a general connected reductive group G.

3.3 Degenerations

We begin this section by immediately amending the goal referred to at the end of the preceding section. Namely, instead of studying the ∞-category $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ for a connected reductive group G and a maximal torus $\mathrm{T} \subseteq \mathrm{G}$, we will study a particular *degeneration* of this ∞-category. Before discussing the construction of this degeneration, let us motivate *why* it is useful (see also the introduction for some "philosophy" regarding this degeneration).

Suppose that there was an equivalence of the form $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}(\mathfrak{X}_k)$ for some spectral k-stack $\mathfrak{X}_k$. In order for such an equivalence to be considered related to Langlands duality, the stack $\mathfrak{X}_k$ must have some relationship to the dual group $\check{\mathrm{G}}$; for instance, one can wonder whether the underlying classical $\pi_0(k)$-stack of $\mathfrak{X}_k$ lives over the classifying stack $\mathrm{B}\check{\mathrm{G}}_{\pi_0(k)}$. Here, $\check{\mathrm{G}}_{\pi_0(k)}$ is the base-change of the Chevalley split form of $\check{\mathrm{G}}$ along the map $\mathbf{Z} \to \pi_0(k)$. (When k is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, the stack $\mathfrak{X}_k$ is $\widetilde{\check{\mathfrak{g}}}[2]/\check{\mathrm{G}}$, which does indeed live over $\mathrm{B}\check{\mathrm{G}}$.) The most satisfying description of $\mathfrak{X}_k$ must therefore involve a lift of the dual group $\check{\mathrm{G}}$ to a (flat) spectral group scheme over k. Unfortunately, the existence of such a lift is far from clear: giving a flat lift of SL_2 (even just as a *scheme*!) to complex K-theory leads to very subtle questions; see § 3.4.

Instead, let us return to the general situation of a finite T_c-space X. One can then view $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ as a categorification of the cochains $\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$; so for the moment, let us just describe a degeneration of $\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ and $\mathcal{M}_{\mathrm{T}}$. There is a natural filtered lift of $\mathcal{M}_{\mathrm{T}} = \mathrm{Spec}\, k_{\mathrm{T}}$ to a filtered $\tau_{\geq \star}(k)$-scheme, given by $\mathrm{Spec}\, \tau_{\geq\star}(k_{\mathrm{T}})$. (This construction is, of course, closely related to the even filtration constructed in [HRW, Pst].) In particular, one obtains a corresponding graded $\pi_*(k)$-scheme $\mathrm{Spec}\, \pi_*(k_{\mathrm{T}})$. Note that this is now a *classical* scheme, with no spectral algebro-geometric nature. If k is even-periodic, i.e., is equipped with an isomorphism $\pi_*(k) \cong \pi_0(k)[u^{\pm 1}]$ with $u \in \pi_2(k)$, then this is equivalent to the data of the classical $\pi_0(k)$-scheme $\mathrm{Spec}\, \pi_0(k_{\mathrm{T}})$. (Recall that this is the affinization of the scheme $\mathcal{M}_{\mathrm{T},0}$; to get to the definition described below, one needs to replace $\mathrm{Spec}\, \pi_0(k_{\mathrm{T}})$ in the below discussion by $\mathcal{M}_{\mathrm{T},0}$.)

If the finite T_c-space X has even cells, then one can construct a well-behaved filtered lift of $\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ to a filtered quasicoherent sheaf over $\mathrm{Spec}\, \tau_{\geq\star}(k_{\mathrm{T}})$, given by $\tau_{\geq\star}\mathcal{F}_{\mathrm{T}}(\mathrm{X})$. This defines a corresponding graded variant of $\mathcal{F}_{\mathrm{T}}(\mathrm{X})$, given simply by the quasicoherent sheaf $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ over $\mathrm{Spec}\, \pi_0(k_{\mathrm{T}})$. Again, this is an object in the realm of *classical* algebraic geometry; so when applied to the affine Grassmannian Gr_{G}, it is something that could, in theory, be described in terms of the usual dual group $\check{\mathrm{G}}$ base-changed to $\pi_0(k)$.

The idea for constructing the desired degeneration of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ is very similar; we now turn to its mechanics. Let us begin with a simple observation. If Y is a connected space, the ∞-category $\mathrm{Loc}(\mathrm{Y}; k) = \mathrm{Fun}(\mathrm{Y}, \mathrm{Mod}_k)$ of local systems on Y is equivalent, by Koszul duality, to $\mathrm{LMod}_{\mathrm{C}_*(\Omega\mathrm{Y}; k)}$. This is very useful, since it allows one to reduce the study of local systems to the study of a particular (derived) algebra. A similar property is true for $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$:

Proposition 3.3.1. *Let* X *be a connected finite* T_c*-space. Then there is an equivalence* $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k) \simeq \mathrm{LMod}_{\mathcal{F}_{\mathrm{T}}(\Omega\mathrm{X})^\vee}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}))$.

Proof. Let $s : * \to \mathrm{X}$ denote the inclusion of a point. We claim that $s^* : \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$ admits a left adjoint $s_!$. Indeed, the statement for general X follows formally from the case of $\mathrm{X} = \mathrm{T}/\mathrm{T}'$ for some closed subgroup $\mathrm{T}' \subseteq \mathrm{T}$ (so s is the inclusion of the trivial coset). In this case, s^* is the functor $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}'}) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$ given by pushforward along the associated morphism $q : \mathcal{M}_{\mathrm{T}'} \to \mathcal{M}_{\mathrm{T}}$, so it has a left adjoint $s_!$ given by q^*. Note that s^* also has a right adjoint; in particular, it preserves small limits and colimits. Observe now that $s_!\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$ is a compact generator of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$: indeed, suppose

$\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ such that $\mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}}, \mathcal{F}) \simeq 0$ as an object of $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$. Because $s^*\mathcal{F} \simeq \mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}}, \mathcal{F})$ in $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$, we see that $s^*\mathcal{F} \simeq 0$. Using the connectivity of X, we see that $\mathcal{F}$ itself must be zero, which implies that $s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}}$ is a compact generator of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$. It follows from the Barr-Beck-Lurie theorem [Lur4, Theorem 4.7.3.5] that $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ is equivalent to the ∞-category of left $\mathrm{End}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}})$-modules in $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$. But $\mathrm{End}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}}) \simeq s^* s_! \mathcal{O}_{\mathcal{M}_\mathrm{T}}$, which identifies with $\mathcal{F}_\mathrm{T}(\Omega \mathrm{X})^\vee$. □

Remark 3.3.2. Modifying the preceding argument shows that if X is a connected finite T_c-space, there is an equivalence

$$\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k) \simeq \mathrm{coLMod}_{\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T})). \tag{3.3.1}$$

In particular, if X admits an $\mathbf{E}_n$-algebra structure (compatible with the T_c-action), then $\mathcal{F}_\mathrm{T}(\mathrm{X})^\vee$ admits the structure of an $\mathbf{E}_n$-algebra[2] in $\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T}))$, and the equivalence (3.3.1) is $\mathbf{E}_n$-monoidal for the convolution tensor product on both sides.

Proposition 3.3.1 and Remark 3.3.2 continue to hold even when X is a filtered colimit of finite T_c-spaces. In order for the claim in Remark 3.3.2 about $\mathbf{E}_n$-algebra structures to hold, we need the filtered diagram $\{\mathrm{X}_\lambda\}$ presenting X to admit the structure of an $\mathbf{E}_n$-algebra in filtered T_c-spaces. We will need to apply this in the case when X is the affine Grassmannian, in which case we can apply the following observation.

Lemma 3.3.3. *The $\mathbb{X}_*(\mathrm{T})^+$-indexed Schubert filtration $\{\mathrm{Gr}_\mathrm{G}^{\leq\lambda}(\mathbf{C})\}$ naturally admits the structure of an $\mathbf{E}_2$-algebra in $\mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+, \mathcal{S}(\mathrm{T}_c))$.*

Proof. This can be proved in essentially the same way as [HY, Theorem 3.10]; let us sketch the argument. We will utilize [Lur4, Proposition 5.4.5.15], which states that if $\mathcal{C}$ is a symmetric monoidal ∞-category, then a nonunital $\mathbf{E}_2$-algebra object in $\mathcal{C}$ is equivalent to the datum of a locally constant $\mathrm{N}(\mathrm{Disk}(\mathbf{C}))_{\mathrm{nu}}$-algebra object in $\mathcal{C}$. Concretely, this amounts to specifying an object $\mathrm{A}(\mathrm{D}) \in \mathcal{C}$ for every disk $\mathrm{D} \subseteq \mathbf{C}$ and coherent maps $\bigotimes_{i=1}^n \mathrm{A}(\mathrm{D}_i) \to \mathrm{A}(\mathrm{D})$ for every inclusion $\coprod_{i=1}^n \mathrm{D}_i \to \mathrm{D}$ of disks, such that for every embedding $\mathrm{D} \subseteq \mathrm{D}'$ of disks, the induced map $\mathrm{A}(\mathrm{D}) \to \mathrm{A}(\mathrm{D}')$ is an equivalence.

In this case, $\mathcal{C} = \mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+, \mathcal{S}(\mathrm{T}_c))$, and the object $\mathrm{A}(\mathrm{D}) \in \mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+, \mathcal{S}(\mathrm{T}_c))$ assigned to a disk $\mathrm{D} \subseteq \mathbf{C}$ may be defined via the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran}}$. Namely, the Beilinson-Drinfeld Grassmannian admits (by construction) a morphism $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran}} \to \mathrm{Ran}_{\mathbf{A}^1}$; upon taking complex points, we obtain a map $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran}}(\mathbf{C}) \to \mathrm{Ran}(\mathbf{C})$. If $\mathrm{S} \subseteq \mathbf{C}$ is a subset, then the preimage of $\mathrm{Ran}(\mathrm{S}) \subseteq \mathrm{Ran}(\mathbf{C})$ defines a subspace $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran}}(\mathrm{S} \subseteq \mathbf{C}) \subseteq \mathrm{Gr}_{\mathrm{G},\mathrm{Ran}}(\mathbf{C})$. The filtration of Gr_G via the Bruhat decomposition extends to a filtration $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu}$ of $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran}}$ by dominant coweights $\mu \in \mathbb{X}_*(\mathrm{T})^+$; see [Zhu, 3.1.11]. Finally, the object $\mathrm{A}(\mathrm{D}) \in \mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+, \mathcal{S}(\mathrm{T}_c))$ associated to a disk $\mathrm{D} \subseteq \mathbf{C}$ is the functor $\mathbb{X}_*(\mathrm{T})^+ \to \mathcal{S}(\mathrm{T}_c)$ sending $\mu \in \mathbb{X}_*(\mathrm{T})^+$ to $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu}(\mathrm{D} \subseteq \mathbf{C})$.

Suppose $\coprod_{i=1}^n \mathrm{D}_i \to \mathrm{D}$ is an inclusion of disks. The induced map $\bigotimes_{i=1}^n \mathrm{A}(\mathrm{D}_i) \to \mathrm{A}(\mathrm{D})$ is defined as follows. Let $\mu \in \mathbb{X}_*(\mathrm{T})^+$; for every n-tuple $(\mu_1, \cdots, \mu_n)$ with $\sum_{i=1}^n \mu_i \leq \mu$, we need to exhibit maps $\bigotimes_{i=1}^n \mathrm{A}(\mathrm{D}_i)(\mu_i) \to \mathrm{A}(\mathrm{D})(\mu)$ satisfying the obvious coherences. But

$$\bigotimes_{i=1}^n \mathrm{A}(\mathrm{D}_i)(\mu_i) = \prod_{i=1}^n \mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu_i}(\mathrm{D}_i \subseteq \mathbf{C}),$$

[2] If $\mathcal{C}$ is a symmetric monoidal ∞-category, [Lur6, Corollary 3.3.4] can be used to show that there is an equivalence $\mathrm{coCAlg}(\mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbf{E}_n}(\mathrm{coCAlg}(\mathcal{C}))$.

so it suffices to show that if $\mu_1 + \mu_2 \leq \mu$, then there are maps $\mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu_1}(\mathrm{D}_1 \subseteq \mathbf{C}) \times \mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu_2}(\mathrm{D}_2 \subseteq \mathbf{C}) \to \mathrm{Gr}_{\mathrm{G},\mathrm{Ran},\leq\mu}(\mathrm{D} \subseteq \mathbf{C})$. The argument for this is exactly as in [HY, Construction 3.15].

We next need to show that the $\mathrm{N}(\mathrm{Disk}(\mathbf{C}))_{\mathrm{nu}}$-algebra A defined above is locally constant, i.e., that if $\mathrm{D} \subseteq \mathrm{D}'$ is an embedding of disks, then $\mathrm{A}(\mathrm{D}) \to \mathrm{A}(\mathrm{D}')$ is an equivalence of functors $\mathbb{X}_*(\mathrm{T})^+ \to \mathcal{S}(\mathrm{T}_c)$. This follows from [HY, Proposition 3.17]. To conclude, it suffices (by [Lur4, Theorem 5.4.4.5]) to establish the existence of a quasi-unit for the functor $\mathrm{A} : \mathbb{X}_*(\mathrm{T})^+ \to \mathcal{S}(\mathrm{T}_c)$, i.e., a map $1_{\mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+,\mathcal{S}(\mathrm{T}_c))} \to \mathrm{A}$ which is both a left and right unit up to homotopy. Since the unit in $\mathrm{Fun}(\mathbb{X}_*(\mathrm{T})^+, \mathcal{S}(\mathrm{T}_c))$ is the functor sending $\mu \in \mathbb{X}_*(\mathrm{T})^+$ to the point $*$, a quasi-unit is the datum of a map $* \to \mathrm{Gr}_{\mathrm{G},\leq\mu}(\mathbf{C})$ for each $\mu \in \mathbb{X}_*(\mathrm{T})^+$. As in the proof of [HY, Theorem 3.10], this can be taken to be the inclusion of the point corresponding to the trivial G-bundle over $\mathbf{A}^1$ with the canonical trivialization away from the origin. □

Suppose, now, that A is an $\mathbf{E}_1$-ring with even homotopy. Any left A-module M then defines a filtered left $\tau_{\geq 2\star}(\mathrm{A})$-module $\tau_{\geq 2\star}(\mathrm{M})$; we will denote the corresponding associated graded left $\pi_{2*}(\mathrm{A})$-module by $\mathrm{gr}_{\mathrm{ev}}(\mathrm{M})$. If $\mathrm{M}, \mathrm{N} \in \mathrm{LMod}_{\mathrm{A}}$, there is then a canonical (complete and exhaustive) filtration on the A-module $\mathrm{Map}_{\mathrm{A}}(\mathrm{M}, \mathrm{N})$ whose associated graded is given by the shearing of $\mathrm{Map}_{\pi_{2*}(\mathrm{A})}(\mathrm{gr}_{\mathrm{ev}}(\mathrm{M}), \mathrm{gr}_{\mathrm{ev}}(\mathrm{N}))$. Informally, this means that there is a 1-parameter degeneration (constructed using the double-speed Postnikov filtration) from $\mathrm{LMod}_{\mathrm{A}}$ to the category $\mathrm{LMod}^{\mathrm{gr}}_{\pi_{2*}(\mathrm{A})}$, given by the filtered category $\mathrm{LMod}_{\tau_{\geq 2\star}\mathrm{A}}$. Motivated by the preceding discussion, we can now define our desired degeneration of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$.

Definition 3.3.4. Suppose that X is a (ind-)finite T_c-space with even cells (such as Gr_{G}). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X}; k)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X}; k) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})).$$

The “constant sheaf” $\underline{k}^{\mathrm{gr}}$ in this category is the comodule $\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee)$ itself. Similarly, suppose Y is a finite T_c-space such that $\Omega\mathrm{Y}$ has even cells (such as G_c). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Y}; k)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Y}; k) = \mathrm{LMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\Omega\mathrm{Y})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})).$$

The “constant sheaf” $\underline{k}^{\mathrm{gr}}$ in this category is the structure sheaf $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}$ viewed as a $\pi_0(\mathcal{F}_{\mathrm{T}}(\Omega\mathrm{Y})^\vee)$-module via the augmentation.

These should be viewed as “mixed” (in the sense of [BBD]) variants of the full ∞-categories $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ and $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Y}; k)$.

Remark 3.3.5. If X admits an $\mathbf{E}_n$-algebra structure (compatible with the T_c-action), then the $\mathbf{E}_n$-algebra structure on $\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$ equips $\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee)$ with the structure of a commutative algebra object in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})$. In particular, $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X}; k)$ acquires a symmetric monoidal structure, which we will refer to as the “convolution tensor structure” and denote by $\star$.

Remark 3.3.6. There is an apparent asymmetry in Definition 3.3.4: why could we not have defined $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Y}; k)$ to be $\mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{Y})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}))$? The issue is that since Y contains odd-dimensional cells, taking π_0 of $\mathcal{F}_{\mathrm{T}}(\mathrm{Y})^\vee$ is a very destructive process. More generally, as in the discussion at the beginning of this section, $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$ for a finite T_c-space X should only be regarded as a well-behaved reflection of $\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$ itself when X has even cells.

However, if Y was the total space of an iterated fibration of odd-dimensional spheres (which happens when $\mathrm{Y} = \mathrm{U}(n)$ or $\mathrm{Sp}(n)$), then one could alternatively consider the category of comodules in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})$ over the truncation $\tau_{[-1,0]}(\mathcal{F}_{\mathrm{T}}(\mathrm{Y})^\vee)$. If the cobar construction on $\tau_{[-1,0]}(\mathcal{F}_{\mathrm{T}}(\mathrm{Y})^\vee)$ is given by $\pi_0(\mathcal{F}_{\mathrm{T}}(\Omega\mathrm{Y})^\vee)$, it then follows from Koszul duality that (up to finiteness questions) this new category would be equivalent to the definition of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Y}; k)$ from Definition 3.3.4.

Remark 3.3.7. If $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2, then (using the results of [ABG]) $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ is equivalent to the shearing of the 2-periodification of the category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$. This can be understood as a statement about formality. If k is a more general $\mathbf{E}_\infty$-ring (like complex K-theory KU), then formality is generally impossible: for instance, a KU-module M is generally not equivalent (even as a spectrum!) to the shearing of $\pi_*(\mathrm{M})$, unless M is also a $\mathbf{Q}$-module.

Remark 3.3.8. We will not discuss G_c-equivariant cohomology much in this article, except for the end of § 3.6. There, we will only consider the case $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2. In this case, the equivariant cohomology $\mathrm{H}^*_{\mathrm{G}_c}(*; \mathbf{Q})$ is concentrated in even weights; in fact, we may identify $\mathrm{Spec}\, \mathrm{H}^0_{\mathrm{G}_c}(*; k) \cong \mathfrak{t}/\!\!/\mathrm{W}$. It is still reasonable to define $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ to be

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{coLMod}_{\mathrm{H}^{\mathrm{G}_c}_0(\mathrm{Gr}_{\mathrm{G}}; k)}(\mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W})).$$

Similarly, the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c; k)$ can be defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c; k) = \mathrm{LMod}_{\mathrm{H}^{\mathrm{G}_c}_0(\mathrm{Gr}_{\mathrm{G}}; k)}(\mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W})).$$

Example 3.3.9. If $\mathrm{G} = \mathrm{T}$ is a maximal torus, it follows from Theorem 3.2.20 that there are equivalences of $\pi_0(k)$-linear ∞-categories

$$\begin{aligned}\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}}; k) &\simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}/\check{\mathrm{T}}),\\ \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{T}_c; k) &\simeq \mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0} \times_{\mathrm{Spec}\, \pi_0(k)} \check{\mathrm{T}}).\end{aligned}$$

Suppose X is a (ind-)finite T_c-space with even cells. Since $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X}; k)$ is a degeneration of $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$, one should expect a spectral sequence computing the cohomology $\Gamma_{\mathrm{T}_c}(\mathrm{X}; \mathcal{F})$ for $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k)$ from corresponding objects $\mathcal{F}^{\mathrm{gr}} \in \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X}; k)$. Similarly, if Y is a finite T_c-space such that $\Omega \mathrm{Y}$ has even cells, one should expect a spectral sequence computing the cohomology $\Gamma_{\mathrm{T}_c}(\mathrm{Y}; \mathcal{F})$ for $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Y}; k)$ from corresponding objects $\mathcal{F}^{\mathrm{gr}} \in \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Y}; k)$. This is a special case of the following general setup.

Construction 3.3.10. Recall that if $\mathfrak{X}$ is a spectral stack and $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the truncation $\underline{\tau}_{\geq n}(\mathcal{F})$ is the quasicoherent $\mathcal{O}_{\mathfrak{X}}$-module given on an affine open U by $\tau_{\geq n}(\mathcal{F}(\mathrm{U}))$; similarly for $\underline{\tau}_{\leq n}$ and $\underline{\tau}_{[n,m]}$ with $m \geq n$. There is a functor $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}) \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})$ given by sending a quasicoherent sheaf $\mathcal{F}$ on $\mathcal{M}_{\mathrm{T}}$ to the quasicoherent sheaf $\underline{\tau}_{[0,1]}(\mathcal{F})$ over $\mathcal{M}_{\mathrm{T},0}$. This functor can be expressed as the composite of two functors: the first sends the $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-module $\mathcal{F}$ to the filtered $\underline{\tau}_{\geq 2\star}\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-module $\underline{\tau}_{\geq 2\star}(\mathcal{F})$; and the second is given by taking associated graded. Note that since the structure sheaf $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$ is 2-periodic, the data of the graded $\underline{\pi}_{2*}\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-module $\mathrm{gr}(\underline{\tau}_{\geq 2\star}(\mathcal{F}))$ is equivalent to the data of the (ungraded) $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$.

Let $\mathcal{A}$ be an $\mathbf{E}_\infty$-coalgebra in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$ whose homotopy sheaves are concentrated in even degrees (such as $\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$), and assume that $\mathcal{A}$ is flat over $\mathcal{M}_{\mathrm{T}}$. If $\mathcal{F} \in \mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}))$, the comodule map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}} \mathcal{A}$ then induces a graded comodule map

$$\mathrm{gr}(\tau_{\leq 2\star}\mathcal{F}) \to \mathrm{gr}(\tau_{\leq 2\star}(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}} \mathcal{A})) \cong \mathrm{gr}(\tau_{\leq 2\star}\mathcal{F}) \otimes_{\pi_\star(\mathcal{O}_{\mathcal{M}_{\mathrm{T}}})} \pi_\star(\mathcal{A}).$$

Using the 2-periodicity of $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$, we obtain a $\pi_0(\mathcal{A})$-comodule structure on the $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$. This defines a functor $\mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})) \to \mathrm{coMod}_{\pi_0(\mathcal{A})}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}))$, which we will denote by $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{gr}}$. For instance, if $\mathcal{A} = \mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$ is flat over $\mathcal{M}_{\mathrm{T}}$ and $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X}; k) = \mathrm{coMod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}))$, then there is a spectral sequence

$$\pi_*(k) \otimes_{\pi_0(k)} \pi_*\mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};k)}(\underline{k}^{\mathrm{gr}}, \mathcal{F}^{\mathrm{gr}}) \Rightarrow \pi_*\mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};k)}(\underline{k}, \mathcal{F}) = \pi_*\Gamma_{\mathrm{T}_c}(\mathrm{X}; \mathcal{F}). \tag{3.3.2}$$

Similarly, let $\mathcal{B}$ be an $\mathbf{E}_1$-algebra in $\mathrm{QCoh}(\mathcal{M}_\mathrm{T})$ whose homotopy sheaves are concentrated in even degrees (such as $\mathcal{F}_\mathrm{T}(\Omega\mathrm{Y})^\vee$). If $\mathcal{F} \in \mathrm{LMod}_\mathcal{B}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T}))$, the module map $\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{M}_\mathrm{T}}} \mathcal{F} \to \mathcal{F}$ induces a comodule map

$$\tau_{\geq 2\star}(\mathcal{B}) \otimes_{\tau_{\geq 2\star}(\mathcal{O}_{\mathcal{M}_\mathrm{T}})} \tau_{\geq 2\star}(\mathcal{F}) \simeq \tau_{\geq 2\star}(\mathcal{B} \otimes_{\mathcal{O}_{\mathcal{M}_\mathrm{T}}} \mathcal{F}) \to \tau_{\geq 2\star}(\mathcal{F})$$

due to the lax symmetric monoidality of the cotruncation functor. Taking associated graded and using the 2-periodicity of $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$, we obtain a left $\pi_0(\mathcal{B})$-module structure on the $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}$-module $\underline{\tau}_{[0,1]}(\mathcal{F})$. This defines a functor $\mathrm{LMod}_\mathcal{B}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T})) \to \mathrm{LMod}_{\pi_0(\mathcal{B})}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}))$, which we will denote by $\mathcal{F} \mapsto \mathcal{F}^\mathrm{gr}$. For instance, if $\mathcal{B} = \mathcal{F}_\mathrm{T}(\Omega\mathrm{Y})^\vee$ and $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Y}; k) = \mathrm{LMod}_\mathcal{B}(\mathrm{QCoh}(\mathcal{M}_\mathrm{T}))$, then there is a spectral sequence

$$\pi_*(k) \otimes_{\pi_0(k)} \pi_* \mathrm{Map}_{\mathrm{Loc}^\mathrm{gr}_{\mathrm{T}_c}(\mathrm{Y};k)}(\underline{k}^\mathrm{gr}, \mathcal{F}^\mathrm{gr}) \Rightarrow \pi_* \mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Y};k)}(\underline{k}, \mathcal{F}) = \pi_* \Gamma_{\mathrm{T}_c}(\mathrm{Y}; \mathcal{F}).$$

Let us now discuss how one might define analogous degenerations if k is not necessarily an even and 2-periodic $\mathbf{E}_\infty$-ring. Our discussion below can be regarded as a "first take"; a cleaner perspective will be provided below in Definition 4.2.4.

Although this discussion can be generalized to some other $\mathbf{E}_\infty$-rings (such as TMF), we will focus only on the case when k is the $\mathbf{E}_\infty$-ring KO of *real K-theory*. Here is a brief summary of its relevant properties: KO can be defined from KU using the $\mathbf{Z}/2$-action on KU via complex conjugation. Namely, $\mathrm{KO} = \mathrm{KU}^{h\mathbf{Z}/2}$; in fact, as proved in [Rog2], the map $\mathrm{KO} \to \mathrm{KU}$ is a $\mathbf{Z}/2$-Galois extension, meaning that the base-change of any KO-module to KU acquires the structure of a $\mathbf{Z}/2$-equivariant KU-module. In the discussion below, we will not need to know much about KO, other than the following facts: the generator of $\mathbf{Z}/2$ sends the Bott class $\beta \in \pi_2(\mathrm{KU})$ to $-\beta$; and the homotopy groups of KO are *not* even, nor are they 2-periodic[3]. Therefore, KO does not quite fit into the setup of § 3.2 and § 3.3. Nevertheless, the fact that KO is the homotopy fixed points $\mathrm{KU}^{h\mathbf{Z}/2}$ does admit a spectral algebro-geometric description: the global sections of the spectral stack $\mathrm{Spec}(\mathrm{KU})/(\mathbf{Z}/2)$ can be identified with KO. Moreover, any KO-module N defines a quasicoherent sheaf over this spectral stack given by the $\mathbf{Z}/2$-action on $\mathrm{KU} \otimes_\mathrm{KO} \mathrm{N}$.

Therefore, a more reasonable analogue of the degeneration from a KU-module M to $\pi_*(\mathrm{M})$ for a KO-module N is given by considering the graded $\mathbf{Z}/2$-equivariant $\pi_*(\mathrm{KU})$-module $\pi_*(\mathrm{KU} \otimes_\mathrm{KO} \mathrm{N})$. If $\mathrm{KU} \otimes_\mathrm{KO} \mathrm{N}$ is even, then (since $\pi_*(\mathrm{KU})$ is isomorphic to $\mathbf{Z}[\beta^{\pm 1}]$ with β in weight 2), we may simply view this as the data of the $\mathbf{Z}/2$-equivariant abelian group $\pi_0(\mathrm{KU} \otimes_\mathrm{KO} \mathrm{N})$. That is, studying (spectral) algebraic geometry over KO amounts simply to keeping track of $\mathbf{Z}/2$-equivariance for (spectral) algebraic geometry over KU. Moreover, the analogue of the degeneration of the spectral scheme Spec KU to $\mathrm{Spec}(\pi_*(\mathrm{KU}))/\mathbf{G}_m \cong \mathrm{Spec}(\mathbf{Z})$ should be understood as a degeneration of the spectral scheme Spec KO to the $\mathbf{G}_m$-quotient of $\mathrm{Spec}(\pi_*(\mathrm{KU}))/(\mathbf{Z}/2) \cong \mathbf{G}_m/(\mathbf{Z}/2)$, i.e., to the classifying stack $\mathrm{B}\mathbf{Z}/2$. This stack is exactly Spev(KO) from Example 2.2.3.

Motivated by this discussion, we may define $\mathrm{KO}_{\mathrm{T}_c}$ for a compact torus T_c as the homotopy $\mathbf{Z}/2$-fixed points of $\mathrm{KU}_{\mathrm{T}_c}$ for a particular $\mathbf{Z}/2$-action extending the action of $\mathbf{Z}/2$ on $\mathrm{KU}^{h\mathrm{T}_c}$ by complex conjugation. To do so, we need the following simple observation.

[3] In fact, there is an isomorphism

$$\pi_*(\mathrm{KO}) \cong \mathbf{Z}[\eta, 2\beta^2, \beta^{\pm 4}]/(2\eta, \eta^3, \eta \cdot (2\beta^2), (2\beta)^2 - 4\beta^4),$$

where η is in degree 1, $2\beta^2$ is in degree 4, and β^4 is in degree 8. The map $\pi_*(\mathrm{KO}) \to \pi_*(\mathrm{KU}) \cong \mathbf{Z}[\beta^{\pm 1}]$ kills η, and sends the other classes to their eponyms.

Lemma 3.3.11. *Under the isomorphism $\pi_0(\mathrm{KU}^{h\mathrm{S}^1}) \cong \mathbf{Z}[\![q-1]\!]$, the action of $\mathbf{Z}/2$ by complex conjugation sends $q \mapsto q^{-1}$. In other words, the action of $\mathbf{Z}/2$ on $\pi_0(\mathrm{KU}^{h\mathrm{S}^1})$ is given by the coaction*

$$\mathbf{Z}[\![q-1]\!] \to \mathbf{Z}[\![q-1]\!][a]/(a^2-a),\ q \mapsto q^{1-2a}.$$

Motivated by Lemma 3.3.11, we make the following:

Construction 3.3.12. There is an action of $\mathbf{Z}/2$ on the multiplicative group $(\mathbf{G}_m)_{\mathrm{KU}}$ over KU given by inversion. If T_c is a compact torus, this extends to an action of $\mathbf{Z}/2$ on $\mathcal{M}_{\mathrm{T}}^{\mathrm{KU}} = \mathrm{T}_{\mathrm{KU}}$. Define $\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}}$ to be the spectral stack over $\mathrm{Spec}(\mathrm{KU})/(\mathbf{Z}/2)$ given by $\mathcal{M}_{\mathrm{T}}^{\mathrm{KU}}/(\mathbf{Z}/2)$. Observe that the underlying stack of $\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}}$ is given by $\mathcal{M}_{\mathrm{T},0}/(\mathbf{Z}/2)$ over $\mathrm{B}\mathbf{Z}/2$ (again, $\mathbf{Z}/2$ acts on $\mathcal{M}_{\mathrm{T},0} \cong \mathrm{T}$ by inversion).

It is clear from Construction 3.2.7 that the functor $\mathcal{F}_{\mathrm{T}}(-;\mathrm{KU}) : \mathcal{S}(\mathrm{T}_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}^{\mathrm{KU}})$ factors through a functor $\mathcal{S}(\mathrm{T}_c)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}})$. We will denote this new functor by $\mathcal{F}_{\mathrm{T}}(-;\mathrm{KO})$. In exactly the same way as in Construction 3.2.16, one can define a $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}})$-linear ∞-category $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};\mathrm{KO})$ for a finite T_c-space X. As in Remark 3.3.2, there will be an equivalence

$$\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};\mathrm{KO}) \simeq \mathrm{coMod}_{\mathcal{F}_{\mathrm{T}}(\mathrm{X};\mathrm{KO})^\vee}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}}));$$

furthermore, the latter category is equivalent to the ∞-category of $\mathbf{Z}/2$-equivariant objects in $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};\mathrm{KU})$.

Thus, following Definition 3.3.4, we are led to the following.

Definition 3.3.13. Suppose that X is a (ind-)finite T_c-space with even cells (such as Gr_{G}). The ∞-category $\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{X};\mathrm{KO})$ is defined as

$$\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{X};\mathrm{KO}) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{X};\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}/(\mathbf{Z}/2))).$$

Similarly, suppose Y is a finite T_c-space such that $\Omega\mathrm{Y}$ has even cells (such as G_c). The ∞-category $\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Y};\mathrm{KO})$ is defined as

$$\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Y};\mathrm{KO}) = \mathrm{LMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\Omega\mathrm{Y};\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0}/(\mathbf{Z}/2))).$$

These categories admit an interesting grading (unlike the analogues with KU-coefficients): the stack $\mathcal{M}_{\mathrm{T},0}/(\mathbf{Z}/2) = \mathrm{T}/(\mathbf{Z}/2)$ lives over $\mathrm{B}\mathbf{G}_m$ via the composite $\mathrm{T}/(\mathbf{Z}/2) \to \mathrm{B}\mathbf{Z}/2 \to \mathrm{B}\mathbf{G}_m$ where the final map classifies the sign representation of $\mathbf{Z}/2$. We will denote the resulting line bundle over $\mathrm{T}/(\mathbf{Z}/2)$ by ω.

Just as in (3.3.2), if $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};\mathrm{KO})$, there is a spectral sequence

$$\mathrm{E}_2^{*,*} \cong \pi_*\mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{X};\mathrm{KO})}(\underline{\mathrm{KO}}^{\mathrm{gr}}, \mathcal{F}^{\mathrm{gr}} \otimes \omega^{\otimes *}) \Rightarrow \pi_*\mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{X};\mathrm{KO})}(\underline{k}, \mathcal{F}) = \pi_*\Gamma_{\mathrm{T}_c}(\mathrm{X};\mathcal{F}). \tag{3.3.3}$$

There is an isomorphism

$$\mathrm{E}_2^{*,*} \cong \mathrm{H}^*(\mathrm{B}\mathbf{Z}/2, \pi_*\mathrm{Map}_{\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{X};\mathrm{KU})}(\underline{\mathrm{KU}}^{\mathrm{gr}}, \mathcal{F}^{\mathrm{gr}})[\beta^{\pm 1}]),$$

where $\mathbf{Z}/2$ acts on β by negation.

Example 3.3.14. It follows from Example 3.3.9 that there are equivalences of $\mathrm{QCoh}(\mathrm{B}\mathbf{Z}/2)$-linear ∞-categories

$$\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{T}};\mathrm{KO}) \simeq \mathrm{QCoh}(\mathrm{T}/(\mathbf{Z}/2) \times \mathrm{B}\check{\mathrm{T}}),$$
$$\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{T}_c;\mathrm{KO}) \simeq \mathrm{QCoh}(\mathrm{T}/(\mathbf{Z}/2) \times \check{\mathrm{T}}).$$

Before proceeding to describing an analogue of the above picture with KO replaced by the K(1)-local sphere, we will describe $\mathrm{KO_T} = \Gamma(\mathcal{M}_\mathrm{T}^{\mathrm{KO}}; \mathcal{O})$ for the sake of completeness. There is a spectral sequence

$$\mathrm{E}_2^{s,*} \cong \mathrm{H}^s(\mathbf{Z}/2; \mathcal{O}_\mathrm{T}[\beta^{\pm 1}]) \cong \mathrm{H}^s(\mathbf{V}(\omega^{-1})^\times; \mathcal{O}) \Rightarrow \pi_{*-s}(\mathrm{KO_T}), \tag{3.3.4}$$

where $*$ denotes the grading on $\mathcal{O}_\mathrm{T}[\beta^{\pm 1}]$ (so β is in weight 2). Here, $\mathbf{V}(\omega^{-1})^\times$ is the complement of the zero section in the total space of the line bundle ω^{-1} over $\mathrm{T}/(\mathbf{Z}/2)$. The action of $\mathbf{Z}/2$ on $\mathcal{O}_\mathrm{T}[\beta^{\pm 1}]$ is given by inversion on T, and sends $\beta \mapsto -\beta$. One can view (3.3.4) as the spectral sequence (3.3.3) computing the cohomology of the constant sheaf on a point. As we will explain below, this spectral sequence has nontrivial differentials, so it does not immediately collapse.

For simplicity, we will focus on the case $\mathrm{T} = \mathrm{S}^1$, so $\mathcal{O}_\mathrm{T} = \mathbf{Z}[x^{\pm 1}]$. Then an elementary calculation in group cohomology shows that the E_2-page of (3.3.4) is given by

$$\mathrm{E}_2^{*,*} \cong \mathbf{Z}[\eta, \beta^{\pm 2}, x + x^{-1}, \tfrac{x^n - x^{-n}}{\beta}]_{n \geq 1}/2\eta,$$

where all classes except for η lie in $\mathrm{E}_2^{0,*}$, and $\eta \in \mathrm{E}_2^{1,2}$. A standard calculation in homotopy theory (coming from the analysis of the Adams-Novikov spectral sequence) says that there is a differential $d_3(\beta^2) = \eta^3$. There are no further differentials past this point, and propagating this differential shows:

Proposition 3.3.15. *There is an isomorphism*

$$\pi_*(\mathrm{KO}_{\mathrm{S}^1}) \cong \mathbf{Z}[\eta, 2\beta^2, \beta^{\pm 4}, x + x^{-1}, \tfrac{x^n - x^{-n}}{\beta}]_{n \geq 1}/(2\eta, \eta^3, \eta \cdot (2\beta^2), (2\beta^2)^2 = 4\beta^4),$$

where the terms $\eta, 2\beta^2, \beta^{\pm 4}$ simply contribute a copy of $\pi_*(\mathrm{KO})$, the term $x + x^{-1}$ contributes a class to $\pi_0(\mathrm{KO}_{\mathrm{S}^1})$, and the terms $\frac{x^n - x^{-n}}{\beta}$ contribute infinitely many classes to $\pi_2(\mathrm{KO}_{\mathrm{S}^1})$.

It is hard to extract concrete implications[4] for Langlands duality from the structure of $\pi_*(\mathrm{KO}_{\mathrm{S}^1})$; so we will not compute the homotopy groups of $\pi_0(\mathcal{F}_\mathrm{T}(\mathrm{Gr_G}; \mathrm{KU})^\vee)$ below, and content ourselves with just describing the $\mathbf{Z}/2$-action on $\pi_0(\mathcal{F}_\mathrm{T}(\mathrm{Gr_G}; \mathrm{KU})^\vee)$.

Remark 3.3.16. The above story can be extended to include the case of *connective* real K-theory $\mathrm{ko} = \tau_{\geq 0}(\mathrm{KO})$, too. Since we will only return to this picture occasionally in this article, we will be scant on details. Recall that connective complex K-theory ku is an $\mathbf{E}_\infty$-ring such that $\pi_*(\mathrm{ku}) = \mathbf{Z}[\beta]$ with β in degree 2 (so that $\mathrm{ku}/\beta = \mathbf{Z}$ and $\mathrm{ku}[\beta^{-1}] = \mathrm{KU}$). Its S^1-equivariant version $\mathrm{ku}_{\mathrm{S}^1}$ has homotopy groups given by $\pi_*(\mathrm{ku}_{\mathrm{S}^1}) \cong \mathbf{Z}[\beta, x, \frac{1}{1+\beta x}]$ with x in weight -2. Let $\mathbf{G}_\beta = \operatorname{Spec} \pi_*(\mathrm{ku}_{\mathrm{S}^1})/\mathbf{G}_m$, where the group law is given by $x + y + \beta xy$. If T is a torus, let $\mathrm{T}_\beta = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{G}_\beta)$.

Since ku is the connective cover $\tau_{\geq 0}(\mathrm{KU})$ of KU, the action of $\mathbf{Z}/2$ on KU by complex conjugation lifts to an action of $\mathbf{Z}/2$ on ku. While there is a map $\mathrm{ko} \to \mathrm{ku}^{h\mathbf{Z}/2}$, this map is *not* an equivalence; rather, it exhibits ko as the connective cover $\tau_{\geq 0}(\mathrm{ku}^{h\mathbf{Z}/2})$. In particular, while the $\mathbf{E}_\infty$-ring $\mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}$ is not equivalent to $\mathrm{Map}(\mathbf{Z}/2, \mathrm{ku})$, it is still a finite free ku-algebra with even homotopy. Therefore, the appropriate degeneration of the spectral scheme Spec(ko) is no longer the algebraic stack $(\mathrm{Spec}(\pi_*(\mathrm{ku}))/\mathbf{G}_m)/(\mathbf{Z}/2)$, but is rather given by the stack Spev(ko) computed in Example 2.2.4.

[4] This is not to say that computing KO-(co)homology groups is a worthless endeavor: in [Ada1], Adams famously computed the KO-cohomology of real projective spaces to solve the question of counting linearly independent vector fields on spheres.

Just as $\mathrm{B}\mathbf{Z}/2$ is an open substack in $\mathrm{Spev(ko)}$, the stack $\mathcal{M}_{\mathrm{T}}^{\mathrm{KO}}$ is also open in a certain stack $\mathcal{M}_{\mathrm{T}}^{\mathrm{ko}}$, which can be defined as the stack of homomorphisms from $\mathbb{X}^*(\mathrm{T})$ to the quotient of $\mathbf{G}_\beta$ (viewed as a scheme over $\mathrm{Spec}(\mathbf{Z}[\beta])/\mathbf{G}_m$) by the coaction of $\pi_*(\mathrm{ku}\otimes_{\mathrm{ko}}\mathrm{ku})$ given by

$$\mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}] \to \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}, r]/(r^2-\beta r),\ x \mapsto x - \tfrac{rx^2}{1+\beta x}. \tag{3.3.5}$$

This might look a bit strange, but it is a pleasant exercise to verify (using the binomial formula) that upon inverting β, it identifies with the map

$$\mathbf{Z}[\beta^{\pm 1}, x, \tfrac{1}{1+\beta x}] \to \mathbf{Z}[\beta^{\pm 1}, x, \tfrac{1}{1+\beta x}, a]/(a^2-a),\ (1+\beta x) \mapsto (1+\beta x)^{1-2a}$$

as forced by Lemma 3.3.11. Given the stack $\mathcal{M}_{\mathrm{T}}^{\mathrm{ko}}$, one can define $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}}^{\mathrm{ko}})$-linear ∞-categories $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{ko})$ exactly as in Definition 3.3.13. We will return to this below in § 3.8.

Remark 3.3.17. Just as in Remark 3.3.16, one can use the even filtration of [HRW] to define a category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};k)^{\mathrm{Bor}}$ of *Borel-equivariant* sheaves of k-modules, where k is any $\mathbf{E}_\infty$-ring. For instance, suppose that X is a (ind-)finite T_c-space with even cells (such as Gr_{G}). If $k = \mathrm{MU}$ is complex cobordism and $\widehat{\mathbf{G}_0}$ is the corresponding universal 1-dimensional formal group law over $\mathrm{Spec}\,\pi_*(\mathrm{MU})/\mathbf{G}_m$, then one can define $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{MU})^{\mathrm{Bor}}$ to be

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{MU})^{\mathrm{Bor}} = \mathrm{coLMod}_{\pi_0\mathrm{MU}[\mathrm{X}]^{h\mathrm{T}}}(\mathrm{QCoh}(\widehat{\mathcal{M}_{\mathrm{T},0}})),$$

where $\widehat{\mathcal{M}_{\mathrm{T},0}} = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \widehat{\mathbf{G}_0})$. This is a category which is linear over $\mathrm{Spec}\,\pi_*(\mathrm{MU})/\mathbf{G}_m$. Using this and Adams-Novikov descent, one can define the category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{S}^0)^{\mathrm{Bor}}$ if $k = \mathrm{S}^0$ is the sphere spectrum as the totalization of the cosimplicial diagram $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{MU}^{\otimes\bullet})$. Using the work of Quillen, Landweber, and Novikov [Qui, Lan2, Nov], one finds that $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{X};\mathrm{S}^0)^{\mathrm{Bor}}$ is linear over the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of 1-dimensional formal groups.

Finally, we turn to the $\mathrm{K}(1)$-local sphere. To motivate it, note that the action of complex conjugation on KU is given simply by the action of the Adams operation ψ^{-1}. It is therefore natural to wonder about the action of other Adams operations. To this end, we will fix a prime p and contemplate a parallel story with KO replaced by the "image of J"/K(1)-local sphere spectrum $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0 = (\mathrm{KU}_p^\wedge)^{h\mathbf{Z}_p^\times}$, where $\mathbf{Z}_p^\times$ acts continuously on $\mathrm{KU}_p^\wedge$ by Adams operations: there is a map $\mathbf{Z}_p^\times \to \mathrm{Aut}_{\mathbf{E}_\infty}(\mathrm{KU}_p^\wedge)$ sending $n \in \mathbf{Z}_p^\times$ to the Adams operation $\psi^n : \mathrm{KU}_p^\wedge \to \mathrm{KU}_p^\wedge$. (In fact, this map is an equivalence!)

The homotopy groups of $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ are somewhat complicated[5], but just as with KO, studying (spectral) algebraic geometry over $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ amounts simply to keeping track of $\mathbf{Z}_p^\times$-equivariance for (spectral) algebraic geometry over $\mathrm{KU}_p^\wedge$. That is to say, $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ is the global sections of the structure sheaf on the spectral stack $\mathrm{Spf}(\mathrm{KU}_p^\wedge)/\mathbf{Z}_p^\times$. Moreover, the analogue of the degeneration of the spectral scheme $\mathrm{Spf}\,\mathrm{KU}_p^\wedge$ to $\mathrm{Spf}(\pi_*(\mathrm{KU}_p^\wedge))/\mathbf{G}_m \cong \mathrm{Spf}(\mathbf{Z}_p)$ should be understood as a degeneration of the spectral scheme $\mathrm{Spf}\,\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ to the $\mathbf{G}_m$-quotient of $\mathrm{Spf}(\pi_*(\mathrm{KU}_p^\wedge))/\mathbf{Z}_p^\times$, i.e., to the classifying stack $\mathrm{B}\mathbf{Z}_p^\times$.

To define an analogue of Definition 3.3.13 for $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$, we need to upgrade the $\mathbf{Z}_p^\times$-action on KU to an action on equivariant K-theory. Recall that if $\mathcal{T}$ denotes the full subcategory of $\mathcal{S}$ spanned by those spaces which are homotopy equivalent to BT_c with T_c being a compact abelian Lie group, the data of a preorientation of $\mathbf{G} = \mathbf{G}_m$ is equivalent to the data of a

[5]Explicitly, if $p > 2$, then $\pi_i\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ is isomorphic to $\mathbf{Z}_p$ when $i = 0, -1$, and is isomorphic to $\mathbf{Z}/p^{v_p(j)+1}$ for $i = 2(p-1)j-1$. The order of the latter subgroup is precisely the p-part of the denominator of $\mathrm{B}_{2(i+1)}/(i+1)$, where B_{2j} is the $2j$th Bernoulli number.

functor $\mathcal{M} : \mathcal{T} \to \mathrm{Aff}_{\mathrm{KU}}$ along with compatible equivalences $\mathcal{M}(\mathrm{BT}_c) \simeq \mathcal{M}_{\mathrm{T}}$. This can be composed with the functor $\mathrm{Aff}_{\mathrm{KU}} \to \mathrm{Aff}^{p\text{-cpl}}_{\mathrm{KU}^\wedge_p}$ of p-completion.

Unfortunately, even at the level of classical algebra, there is no natural action of $\mathbf{Z}_p^\times$ on $\mathbf{G}_m = \mathrm{Spf}\, \mathbf{Z}_p[x^{\pm 1}]$ where $n \in \mathbf{Z}_p^\times$ sends $x \mapsto x^n$: the power series $x^n = \sum_{i\geq 0} \binom{n}{i}(x-1)^n$ need not converge without a further completion. Nevertheless, such an action of $\mathbf{Z}_p^\times$ does exist if we restrict to the subgroups $\mu_{p^n} = \mathrm{Spf}\, \mathbf{Z}_p[\mathbf{Z}/p^n] \subseteq \mathbf{G}_m$; in fact, the action factors through the surjection $\mathbf{Z}_p^\times \twoheadrightarrow (\mathbf{Z}/p^n)^\times$. The subgroups μ_{p^n} naturally lift to KU (by $\mathrm{Spec}\, \mathrm{KU}[\mathbf{Z}/p^n]$), and each admits a natural $\mathbf{Z}_p^\times$-action. Of course, these $\mathbf{Z}_p^\times$-actions exist even before p-completion; but to get a well-behaved operation on $\mathbf{Z}/p^n$-equivariant KU-cohomology, we need the $\mathbf{Z}_p^\times$-action to preserve the preorientation on μ_{p^n}, and this in turn happens once KU is p-completed.

Suppose, therefore, that we restrict to the full subcategory $\mathcal{T}_p \subseteq \mathcal{T}$ spanned by those spaces which are homotopy equivalent to BA with A being a p-power torsion compact abelian Lie group. Then the preceding paragraph implies that $\mathcal{M}|_{\mathcal{T}_p} : \mathcal{T}_p \to \mathrm{Aff}_{\mathrm{KU}}$ refines to a functor $\mathcal{T}_p \to (\mathrm{Aff}^{p\text{-cpl}}_{\mathrm{KU}^\wedge_p})^{h\mathbf{Z}_p^\times}$. Following Construction 3.2.7 verbatim defines an action of $\mathbf{Z}_p^\times$ on $\mathcal{M}_{\mathrm{A}}$, and furthermore equips the quasicoherent sheaf $\mathcal{F}_{\mathrm{A}}(\mathrm{X}) \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{A}})$ associated to a finite A-space X with a $\mathbf{Z}_p^\times$-equivariant structure. We will write $\mathcal{M}_{\mathrm{A}}^{\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0} = \mathcal{M}_{\mathrm{A}}/\mathbf{Z}_p^\times$, and let $\mathcal{F}_{\mathrm{A}}(-; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)$ denote the corresponding functor $\mathcal{S}(\mathrm{A})^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{A}}^{\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0})$. Again, following Definition 3.3.4, we are led to[6]:

Definition 3.3.18. Suppose that A is a p-power torsion abelian group, and X is a (ind-)finite A-space with even cells (such as Gr_{G}). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{X}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{X}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{A}}(\mathrm{X};\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{A},0}/\mathbf{Z}_p^\times)).$$

Similarly, suppose Y is a finite A-space such that $\Omega\mathrm{Y}$ has even cells (such as G_c). The ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{Y}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)$ is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{Y}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) = \mathrm{LMod}_{\pi_0(\mathcal{F}_{\mathrm{A}}(\Omega\mathrm{Y};\mathrm{KU})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{A},0}/\mathbf{Z}_p^\times)).$$

These categories admit an interesting grading (just like the analogue with KO-coefficients): the stack $\mathcal{M}_{\mathrm{A},0}/(\mathbf{Z}/2) = \mathrm{A}/\mathbf{Z}_p^\times$ lives over $\mathrm{B}\mathbf{G}_m$ via the composite $\mathrm{A}/\mathbf{Z}_p^\times \to \mathrm{B}\mathbf{Z}_p^\times \to \mathrm{B}\mathbf{G}_m$ where the final map classifies the standard (cyclotomic) representation of $\mathbf{Z}_p^\times$ on $\mathbf{Z}_p$. We will denote the resulting line bundle over $\mathrm{A}/\mathbf{Z}_p^\times$ by ω.

For the sake of completness (and partly because it is a pleasant calculation), let us describe $(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mathrm{T}[p^\infty]} = \Gamma(\mathcal{M}_{\mathrm{T}[p^\infty]}^{\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0}; \mathcal{O})$ when p is odd. Since this is built as a limit of the spectra $(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mathrm{T}[p^n]}$, we will just compute each of these individually. There is a spectral sequence

$$\mathrm{E}_2^{s,*} \cong \mathrm{H}^s_{\mathrm{cts}}(\mathbf{Z}_p^\times; \mathcal{O}_{\mathrm{T}[p^n]}[\beta^{\pm 1}]) \cong \mathrm{H}^s(\mathbf{V}(\omega^{-1})^\times; \mathcal{O}) \Rightarrow \pi_{*-s}((\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mathrm{T}[p^n]}), \tag{3.3.6}$$

where $*$ denotes the grading on $\mathcal{O}_{\mathrm{T}[p^n]}[\beta^{\pm 1}]$ (so β is in weight 2). Here, $\mathbf{V}(\omega^{-1})^\times$ is the complement of the zero section in the total space of the line bundle ω^{-1} over $\mathrm{A}/\mathbf{Z}_p^\times$. Fix a topological generator $g \in \mathbf{Z}_p^\times$ such that $g^{p-1} = 1 + p$, so that its action (denoted ψ^g) on $\mathcal{O}_{\mathrm{T}[p^n]}[\beta^{\pm 1}]$ is given by exponentiation on $\mathrm{T}[p^n]$, and sends $\beta \mapsto g\beta$. One can view (3.3.6) as the spectral sequence (3.3.2) computing the cohomology of the constant sheaf on a point. This spectral sequence has no nontrivial differentials, so it collapses; this, however, is no longer true if $p = 2$.

[6] Just as with connective ko, one can also define a variant of Definition 3.3.18 for the *connective* image of J spectrum j. We leave this to the interested reader.

For simplicity, we will focus on the case $\mathrm{T} = \mathrm{S}^1$, so $\mathcal{O}_{\mathrm{T}[p^n]} = \mathcal{O}_{\mu_{p^n}} = \mathbf{Z}_p[x^{\pm 1}]/(x^{p^n} - 1)$ (recall that we have p-completed!). We then have:

Proposition 3.3.19. *If $p > 2$, there are isomorphisms*

$$\pi_j((\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mu_{p^n}}) \cong \begin{cases} \pi_j(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \oplus \mathbf{Z}_p^{\oplus n} & j = 0, -1 \\ \pi_j(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \oplus \bigoplus_{i=0}^{n-1} \mathbf{Z}_p/kp^{n-i} & j = 2k-1, k \in \mathbf{Z} \\ 0 & \textit{else.} \end{cases}$$

Proof. The E_2-page of (3.3.6) is given by the group cohomology of $\mathbf{Z}_p^\times$ acting on $\mathbf{Z}_p[x^{\pm 1}, \beta^{\pm 1}]/(x^{p^n} - 1)$, so $\mathrm{E}_2^{*,2k}$ is given by the cohomology of the two-term complex

$$\mathbf{Z}_p[x^{\pm 1}]/(x^{p^n} - 1) \xrightarrow{\psi^g - 1} \mathbf{Z}_p[x^{\pm 1}]/(x^{p^n} - 1),$$
$$f(x) \mapsto g^k f(x^g) - f(x).$$

Let us sketch the calculation of the cohomology of this complex, which we will denote by $\mathrm{C}^\bullet$ below. Write $\mathbf{Z}_p[x^{\pm 1}]/(x^{p^n} - 1) = \mathbf{Z}_p[\mathbf{Z}/p^n]$, so it is a free $\mathbf{Z}_p$-module on the classes $\{1, x, \cdots, x^{p^n - 1}\}$. The action of $\mathbf{Z}_p^\times$ on $\mathbf{Z}/p^n$ (which factors through the quotient map $\mathbf{Z}_p^\times \twoheadrightarrow (\mathbf{Z}/p^n)^\times$) has $n+1$ orbits, with representatives given by $\{p^i\}_{0 \leq i \leq n-1} \cup \{0\}$. The orbit of 0 is a singleton, and the orbit of p^i has size $p^{n-i-1}(p-1)$. It follows that the $p^n \times p^n$-matrix $\psi^g - 1$ can be written as the block sum $(g^k - 1) \oplus \bigoplus_{i=0}^{n-1} \mathrm{A}_i$, where A_i is an $p^{n-i-1}(p-1) \times p^{n-i-1}(p-1)$-matrix. For consistency, we will write A_{-1} to denote the scalar $g^k - 1$.

Let $0 \leq i \leq n-1$. Then the matrix A_i acts on the submodule

$$\mathbf{Z}_p^{\oplus p^{n-i-1}(p-1)} = \mathbf{Z}_p\{x^{p^i}, x^{p^i g}, \cdots, x^{p^i g^{p^{n-i-1}(p-1)-1}}\},$$

and each row and column of A_i has exactly two entries (namely, -1 on the diagonal entry, and g^k elsewhere). Computing the Smith normal form of this matrix shows that A_i has no kernel unless $k = 0$, in which case its kernel is free of rank 1. If $k = 0$, then the cokernel of A_i is also free of rank 1, and if $k \neq 0$, then the cokernel of A_i is $\mathbf{Z}_p/(g^{kp^{n-i-1}(p-1)} - 1)$. Since $g \in \mathbf{Z}_p^\times$ was chosen to satisfy $g^{p-1} = 1 + p$, it follows that $\mathbf{Z}_p/(g^{kp^{n-i-1}(p-1)} - 1) \cong \mathbf{Z}_p/kp^{n-i}$.

We only need to take care of the block A_{-1}. If $k = 0$, then A_{-1} is the zero matrix; but if k is nonzero, then A_{-1} has no kernel, and has cokernel given by $\mathbf{Z}_p/(g^k - 1)$. It follows that if $k = 0$, then

$$\mathrm{H}^s(\mathrm{C}^\bullet) \cong \mathbf{Z}_p^{\oplus n+1} \text{ for } s = 0, 1.$$

If $k \neq 0$, then

$$\mathrm{H}^s(\mathrm{C}^\bullet) \cong \begin{cases} 0 & s = 0 \\ \mathbf{Z}_p/(g^k - 1) \oplus \bigoplus_{i=0}^{n-1} \mathbf{Z}_p/kp^{n-i} & s = 1. \end{cases}$$

The groups $\mathrm{E}_2^{s,*}$ vanish for $s > 1$, so there cannot be any differentials in the spectral sequence (3.3.6). Using the calculation of the homotopy groups of $\mathrm{K}(1)$-local sphere, we obtain the desired answer for $\pi_*((\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mu_{p^n}})$. ☐

Just as with $\mathrm{KO}_{\mathrm{S}^1}$, the groups $\pi_*((\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)_{\mu_{p^n}})$ are interesting but form a rather unpleasant ring to do algebraic geometry with; so we will content ourselves with just understanding the category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c[p^\infty]}(\mathrm{Gr}_\mathrm{G}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)$ below (and not calculate actual homotopy groups).

3.4 (Not) lifting SL_2

In this brief section, we study the question of lifting SL_2 as a group scheme over $\mathbf{Z}$ to other $\mathbf{E}_\infty$-rings like K-theory KU or the sphere spectrum S^0. To set up the question, let us first make the notion of "lifting" precise: if X is a scheme over $\mathbf{Z}$ with structure sheaf $\mathcal{O}_\mathrm{X}$ and k is an $\mathbf{E}_\infty$-ring equipped with an $\mathbf{E}_\infty$-map $k \to \mathbf{Z}$, a flat lifting of X to k as an $\mathbf{E}_n$-scheme (see [Fra]) will mean the data of a sheaf $\mathcal{O}_\mathrm{X}^\mathrm{top}$ of $\mathbf{E}_n$-k-algebras on X along with an isomorphism $\mathcal{O}_\mathrm{X}^\mathrm{top} \otimes_k \mathbf{Z} \xrightarrow{\simeq} \mathcal{O}_\mathrm{X}$. It is easy to lift GL_n to the sphere as an $\mathbf{E}_\infty$-scheme (i.e., a spectral scheme in the sense of [Lur5]), because it is an open subset in $\mathbf{A}^{n^2}$. However, we will see in a moment that a simple calculation proves that SL_2 itself cannot be lifted in a natural way (and slight variants of this question lead to very subtle issues that we have been unable to resolve).

Observe that many schemes associated to SL_2 lift all the way to the sphere spectrum. For instance, each choice of Borel subgroup $\mathrm{B} \subseteq \mathrm{SL}_2$ (with unipotent radical U) defines a surjection $\mathrm{SL}_2 \to \mathbf{A}^2 - \{0\}$, given by quotienting on the left or the right by U. The scheme $\mathbf{A}^2$ admits a flat lift to a spectral scheme $(\mathbf{A}^2)_{\mathrm{S}^0}$ over S^0, given by $\operatorname{Spec} \mathrm{S}^0[\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}]$. The simple observation is the following:

Proposition 3.4.1. *There is no flat lifting* $(\mathrm{SL}_2)_{\mathrm{S}^0}$ *of* SL_2 *to* S^0 *(or even to connective complex K-theory* ku*) as an* $\mathbf{E}_4$*-scheme along with a lifting* $(\mathrm{SL}_2)_{\mathrm{S}^0} \to (\mathbf{A}^2)_{\mathrm{S}^0}$ *of the maps* $\mathrm{SL}_2 \to \mathbf{A}^2 - \{0\} \subseteq \mathbf{A}^2$.

Proof. Fix a prime p, and let $n \geq 1$. A flat lifting to ku of an affine (say) scheme $\mathrm{X} = \operatorname{Spec}(\mathrm{R})$ over $\mathbf{Z}$ defines power operations on R. Indeed, if $\widetilde{\mathrm{R}}$ is the $\mathbf{E}_n$-ku-algebra lifting R, then $\mathrm{R}_p^\wedge \cong \pi_0(\mathrm{L}_{\mathrm{K}(1)}\widetilde{\mathrm{R}})$. If A is a K(1)-local $\mathbf{E}_{2n+1}$-KU-algebra, then $\pi_0(\mathrm{A})$ admits the structure of a "weak δ_n-ring", in the sense that there is a map $\delta : \pi_0(\mathrm{A}) \to \pi_0(\mathrm{A}/p^n)$ of sets (where A/p^n denotes the derived quotient) such that

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) - \tfrac{1}{p}((x+y)^p - x^p - y^p) \pmod{p^{n-1}}, \\ \delta(xy) &= \delta(x)y^p + \delta(y)x^p + p\delta(x)\delta(y) \pmod{p^{n-1}}. \end{aligned}$$

If A refines to an $\mathbf{E}_{2n+2}$-KU-algebra, then $\pi_0(\mathrm{A})$ further admits the structure of a "δ_n-ring", meaning that the above relations hold modulo p^n. (I am grateful to Ishan Levy for a discussion about this.) In this case, the map $\psi : \pi_0(\mathrm{A}) \to \pi_0(\mathrm{A}/p^{n+1})$ sending $x \mapsto x^p + p\delta(x)$ is a ring map lifting the Frobenius. Observe that

$$\delta(-x) = \begin{cases} -\delta(x) - x^2 & p = 2, \\ -\delta(x) & p > 2. \end{cases}$$

The operation δ is furthermore natural in maps of K(1)-local $\mathbf{E}_{2n+1}$-KU-algebras. When $n = \infty$, the power operation δ is constructed in [Hop], and its construction for finite n is nearly identical.

The $\mathbf{Z}$-algebra $\mathrm{R} = \mathbf{Z}[x]$ admits a canonical lifting to S^0 as an $\mathbf{E}_\infty$-ring (via $\mathrm{S}^0[\mathbf{Z}_{\geq 0}] = \mathrm{S}^0[x]$). The corresponding δ-operation on R is simply given by $\delta(x) = 0$. By choosing $\mathrm{U} \subseteq \mathrm{SL}_2$ to be the subgroup of upper or lower triangular matrices, one obtains two maps $\mathrm{SL}_2 \to \mathbf{A}^2$ which send a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to (a, b) and (d, b). The resulting map $f : \mathcal{O}_{\mathbf{A}^2} \otimes_\mathbf{Z} \mathcal{O}_{\mathbf{A}^2} \to \mathcal{O}_{\mathrm{SL}_2}$ is a surjection, with kernel given by the determinant ideal $(ad - bc - 1)$. If SL_2 admits a lift to an $\mathbf{E}_4$-ku-algebra compatibly with the two maps $\mathrm{SL}_2 \to \mathbf{A}^2$, then the map f must be one of δ_1-rings. It follows that δ vanishes on the generators $a, b, c, d \in \mathcal{O}_{\mathrm{SL}_2}$. In particular, $\delta(ad) = \delta(a)d^p + \delta(d)a^p + p\delta(a)\delta(d)$ must vanish in $\mathcal{O}_{\mathrm{SL}_2}/p$; similarly for $\delta(bc)$.

If $p = 2$, then

$$\begin{aligned}\delta(ad - bc) &= \delta(ad) + \delta(-bc) + adbc \\ &= \delta(ad) - \delta(bc) - (-bc)^2 + adbc = bc,\end{aligned}$$

where the final equality is because $\delta(ad) = \delta(bc) = 0$ and $ad - bc = 1$. Similarly, if $p > 2$, then

$$\begin{aligned}\delta(ad - bc) &= \delta(ad) + \delta(-bc) - \tfrac{1}{p}((ad - bc)^p - (ad)^p - (-bc)^p) \\ &= \tfrac{1}{p}((bc + 1)^p - b^p c^p - 1),\end{aligned}$$

again because $\delta(ad) = \delta(bc) = 0$. The fact that $\delta(ad - bc) = \delta(1) = 0$ implies that for any commutative $\mathbf{F}_p$-algebra R and a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathrm{R})$, the polynomial $\frac{1}{p}((bc+1)^p - b^p c^p - 1)$ must vanish in R. This is clearly false: take $\mathrm{R} = \mathbf{F}_p[x]$ and the matrix $\left(\begin{smallmatrix} x+1 & x \\ 1 & 1 \end{smallmatrix}\right)$. (One could of course use any prime p to obtain this contradiction; but we allow flexibility in the choice of p to assuage any worries about SL_2 being liftable upon localization at some primes but not others.) $\square$

Remark 3.4.2. Since $\delta(x)$ behaves like the pth divided power $-\gamma_p(x)$, the argument of Proposition 3.4.1 can alternatively be interpreted as showing that the ideal which cuts out $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$ does not have a divided power structure, even over $\mathbf{Z}/p^2$.

Since a weak δ_1-ring structure on a commutative ring R is just a map of sets $\delta : \mathrm{R} \to \mathrm{R}/p$ satisfying no relations, the above argument does not prove the analogue of Proposition 3.4.1 with $\mathbf{E}_4$ replaced by $\mathbf{E}_3$ or $\mathbf{E}_2$.

Remark 3.4.3. Using the geometric Casselman-Shalika formula [FGKV, FGV], the theory of (locally constant) factorization categories, and the Kirillov model for Whittaker invariants from [GL], it can be shown that if $\check{\mathrm{G}}$ is a (split) reductive group over $\mathbf{Z}$, then $\mathrm{QCoh}(\mathrm{B}\check{\mathrm{G}})$ admits a lift (as an $\mathbf{E}_2^{\mathrm{fr}}$-monoidal category) to the sphere spectrum, via the Whittaker category $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S})$ with coefficients in the ∞-category of spectra. (*A priori*, one only has a factorizable structure on $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S})$, but the local constancy/constructibility condition required by [Lur4, Theorem 5.5.4.10] to upgrade this to an $\mathbf{E}_2^{\mathrm{fr}}$-monoidal structure on $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S})$ can be proved using the main results of [NP2], just as in [Noc, Theorem 4.6].)

However, there is an important subtlety regarding the fiber functor: already when S is replaced by an ordinary commutative ring k, the global sections functor $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; k) \to \mathrm{Mod}_k$ does *not* identify with the standard forgetful functor $\mathrm{Rep}(\check{\mathrm{G}}_k) \to \mathrm{Mod}_k$. Rather, the functor sends an $\check{\mathrm{G}}_k$-representation V (in the derived category of k-modules) to its *shearing* under the $\mathbf{G}_m$-action on V induced from the cocharacter $2\rho : \mathbf{G}_m \to \check{\mathrm{T}} \subseteq \check{\mathrm{G}}$. In particular, the endomorphisms of the global sections functor $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; k) \to \mathrm{Mod}_k$ identify with the $\mathbf{E}_\infty$-k-algebra $\mathcal{O}_{\check{\mathrm{G}}[2\rho]}$ obtained by shearing $\mathcal{O}_{\check{\mathrm{G}}}$ with respect to the grading coming from $2\rho : \mathbf{G}_m \to \check{\mathrm{T}}$. Using the global sections functor $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S}) \to \mathrm{Sp}$, it follows that the Hopf algebra $\mathcal{O}_{\check{\mathrm{G}}[2\rho]}$ lifts to the sphere as an $\mathbf{E}_2^{\mathrm{fr}}$-*algebra* in $\mathbf{E}_1$-coalgebras in Sp. (See [Lur2] for related discussion.) We hope to show in future work that $\mathcal{O}_{\check{\mathrm{G}}[2\rho](2\rho)}$ lifts to the sphere as a *graded* $\mathbf{E}_2^{\mathrm{fr}}$-algebra in $\mathbf{E}_1$-coalgebras in Sp, so that by shearing (which is an $\mathbf{E}_2^{\mathrm{fr}}$-monoidal functor on graded spectra [DHL+]), one would obtain a lift of $\check{\mathrm{G}}$ itself as a group object in $\mathbf{E}_2^{\mathrm{fr}}$-schemes over the sphere spectrum.

Note that the same argument in Proposition 3.4.1 shows that SL_n also cannot be lifted as an $\mathbf{E}_4$-scheme to to S^0 (or even to connective complex K-theory ku) for any $n \geq 2$ compatibly with its natural actions on $\mathbf{A}^n$.

Remark 3.4.4. The argument of Proposition 3.4.1 is very robust. It can be used to show, for instance, that if $1 < k < n-1$, then the Grassmannian $\mathrm{Gr}_k(\mathbf{A}^n)$ over $\mathbf{Z}$ cannot be lifted as an $\mathbf{E}_4$-scheme to S^0, or even to ku, compatibly with its Plücker embedding into $\mathbf{P}(\wedge^k \mathbf{A}^n) \cong \mathbf{P}^{\binom{n}{k}-1}$ (which is lifted via the flat projective space of [Lur5, Construction 5.4.1.3]). In fact, an even stronger statement is true: [BTLM, Theorem 6] implies that for a semisimple algebraic group G, the flag variety G/P over $\mathbf{Z}/p^2$ does not have a lift of Frobenius as long as the parabolic subgroup P is contained in one of the maximal parabolics enumerated in [BTLM, Examples 4.3.1-4.3.7]. This, in particular, recovers the statement about Grassmannians above. The proof of the general claim uses Bott vanishing, which is more sophisticated than the hands-on approach of Proposition 3.4.1.

For concreteness, let us demonstrate this non-liftability for $\mathrm{Gr}_2(\mathbf{A}^4)$, which is cut out inside $\mathbf{P}^5$ (with coordinates $[x_0 : x_1 : x_2 : y_0 : y_1 : y_2]$) via the formula $x_0y_0 - x_1y_1 + x_2y_2 = 0$. Let $a = x_0y_0$, $b = x_1y_1$, and $c = x_2y_2$, so that $b = a + c$. Again, $\delta(a) = \delta(b) = \delta(c) = 0$. When $p = 2$, for instance, this implies that

$$\begin{aligned}\delta(a-b+c) &= \delta(a) + \delta(-b) + \delta(c) + ab + bc - ac \\ &= -b^2 + ab + bc - ac = -ac.\end{aligned}$$

For a general prime, one has $\delta(a-b+c) = \frac{1}{p}(a^p + c^p - (a+c)^p)$. Since $\delta(a-b+c) = \delta(0) = 0$, this implies that $\frac{1}{p}(a^p + c^p - (a+c)^p) = 0$; since this function is not identically zero on $\mathrm{Gr}_2(\mathbf{A}^4)_{\mathbf{F}_p}$, we obtain the desired contradiction. Note that, just as in Remark 3.4.2, this argument says that the ideal cut out by $x_0y_0 - x_1y_1 + x_2y_2$ in $\mathbf{Z}[x_0, \cdots, y_2]/p^2$ does not admit a divided power structure; from this perspective, the above observation should be attributed to Koblitz [BO, Section 3.3(4)].

Corollary 3.4.5. *Let* $(\mathrm{GL}_2)_{\mathrm{S}^0}$ *denote the spectral scheme* $(\mathbf{A}^4)_{\mathrm{S}^0}[\frac{1}{ad-bc}]$, *and let* $(\mathrm{GL}_1)_{\mathrm{S}^0} = \mathrm{Spec}\,\mathrm{S}^0[x^{\pm 1}]$. *Then the map* $\det : \mathrm{GL}_2 \to \mathbf{G}_m$ *over* $\mathbf{Z}$ *does not lift to a map* $(\mathrm{GL}_2)_{\mathrm{S}^0} \to (\mathrm{GL}_1)_{\mathrm{S}^0}$ *exhibiting* $(\mathrm{GL}_2)_{\mathrm{S}^0}$ *as an* $\mathbf{E}_4$*-scheme over* $(\mathrm{GL}_1)_{\mathrm{S}^0}$*; in fact, such a lifting is prohibited even over* ku.

Proof. If there was a lifting $(\mathrm{GL}_2)_{\mathrm{ku}} \to (\mathrm{GL}_1)_{\mathrm{ku}}$ which exhibits $(\mathrm{GL}_2)_{\mathrm{ku}}$ as an $\mathbf{E}_4$-scheme over $(\mathrm{GL}_1)_{\mathrm{ku}}$, then there would be a map $\mathrm{ku}[x^{\pm 1}] \to \mathcal{O}_{(\mathrm{GL}_2)_{\mathrm{ku}}} = \mathrm{ku}[a, b, c, d, \frac{1}{ad-bc}]$ sending $x \mapsto ad - bc$ which exhibits $\mathcal{O}_{(\mathrm{GL}_2)_{\mathrm{ku}}}$ as an $\mathbf{E}_4$-$\mathrm{ku}[x^{\pm 1}]$-algebra. Base-changing along the map $\mathrm{ku}[x^{\pm 1}] \to \mathrm{ku}$ sending $x \mapsto 1$ would then produce an $\mathbf{E}_4$-ku-algebra $\mathcal{O}_{(\mathrm{GL}_2)_{\mathrm{ku}}} \otimes_{\mathrm{ku}[x^{\pm 1}]} \mathrm{ku}$ which lifts $\mathcal{O}_{\mathrm{SL}_2}$ to ku. Such a lifting is prohibited by Proposition 3.4.1. □

If $(\mathrm{GL}_1^{\mathrm{free}})_{\mathrm{S}^0} = \mathrm{Spec}\,\mathrm{S}^0\{x\}[1/x]$ where $\mathrm{S}^0\{x\}$ denotes the free $\mathbf{E}_\infty$-ring on one generator, then there *is* a map $(\mathrm{GL}_2)_{\mathrm{S}^0} \to (\mathrm{GL}_1^{\mathrm{free}})_{\mathrm{S}^0}$ exhibiting $(\mathrm{GL}_2)_{\mathrm{S}^0}$ as an $\mathbf{E}_\infty$-scheme over $(\mathrm{GL}_1^{\mathrm{free}})_{\mathrm{S}^0}$. However, $(\mathrm{GL}_1^{\mathrm{free}})_{\mathrm{S}^0}$ is not a flat lift of GL_1 to S^0. In other words, there is no reasonable way to construct "strict" determinants over the sphere spectrum (or even over ku), at least in the setting of spectral algebraic geometry of $\mathbf{E}_4$-schemes.

Remark 3.4.6. There *is* a lifting of det to a map $(\mathrm{GL}_2)_{\mathrm{MU}} \to (\mathrm{GL}_1)_{\mathrm{MU}}$ of $\mathbf{E}_2$-MU-schemes[7]. In fact, any $\mathbf{E}_1$-MU-algebra map from $\mathrm{MU}[x^{\pm 1}]$ to an even $\mathbf{E}_2$-MU-algebra A can be refined to an $\mathbf{E}_2$-MU-algebra map. Indeed, an $\mathbf{E}_1$-MU-algebra map $f : \mathrm{MU}[x^{\pm 1}] \to \mathrm{A}$ can be viewed as

[7] However, it does *not* necessarily exhibit $(\mathrm{GL}_2)_{\mathrm{MU}}$ as an $\mathbf{E}_2$-scheme over $(\mathrm{GL}_1)_{\mathrm{MU}}$; so the fiber product $(\mathrm{GL}_2)_{\mathrm{MU}} \times_{(\mathrm{GL}_1)_{\mathrm{MU}}} \mathrm{Spec}(\mathrm{MU})$ may not exist in $\mathbf{E}_2$-MU-schemes. This is one of the quirks of spectral algebraic geometry with $\mathbf{E}_n$-rings for finite n: even if X and Y are spectral schemes with sheaves of $\mathbf{E}_\infty$-rings, a map $f : \mathrm{X} \to \mathrm{Y}$ of $\mathbf{E}_n$-schemes may not exhibit X as an $\mathbf{E}_n$-Y-scheme. This is because an $\mathbf{E}_n$-map $\mathrm{A} \to \mathrm{B}$ between $\mathbf{E}_\infty$-rings need not exhibit B as an $\mathbf{E}_n$-A-algebra.

the map $g : \mathrm{S}^1 \to \mathrm{BGL}_1(\mathrm{A})$ which detects $f(x) \in \pi_1(\mathrm{BGL}_1(\mathrm{A})) = \pi_0(\mathrm{A})^\times$. The map f can be upgraded to an $\mathbf{E}_2$-MU-algebra map if and only if g can be delooped once. Since $\mathrm{BS}^1 = \mathbf{C}\mathrm{P}^\infty$ has an even cell structure and $\mathrm{B}^2\mathrm{GL}_1(\mathrm{A})$ has even homotopy, there are no obstructions to extending the map $\mathrm{S}^2 \to \mathrm{B}^2\mathrm{GL}_1(\mathrm{A})$ detecting $f(x)$ along the inclusion $\mathrm{S}^2 \to \mathrm{BS}^1$. It follows from this discussion that there is an $\mathbf{E}_2$-MU-algebra map $\det_{\mathrm{MU}} : \mathrm{MU}[x^{\pm 1}] \to \mathcal{O}_{(\mathrm{GL}_2)_{\mathrm{MU}}} = \mathrm{MU}[a, b, c, d, \frac{1}{ad-bc}]$ which sends $x \mapsto ad - bc$. This, however, only exhibits $\mathcal{O}_{(\mathrm{GL}_2)_{\mathrm{MU}}}$ as an $\mathbf{E}_1$-$\mathrm{MU}[x^{\pm 1}]$-algebra. To exhibit it as an $\mathbf{E}_2$-$\mathrm{MU}[x^{\pm 1}]$-algebra, one would need to upgrade $\det_{\mathrm{MU}}$ to an $\mathbf{E}_3$-MU-algebra map, but this seems quite difficult.

The argument for Proposition 3.4.1 does not require any knowledge of the coalgebra structure on $\mathcal{O}_{\mathrm{SL}_2}$, so it is possible that SL_2 lifts as an $\mathbf{E}_3$-scheme to S^0 or ku, but does not lift as an $\mathbf{E}_1$-group object therein. The group scheme $\mathbf{G}_a$ provides a simple example of this phenomenon of lifting as a spectral scheme, but not as a group scheme:

Proposition 3.4.7. *There is no flat lifting $(\mathbf{G}_a)_{\mathrm{S}^0}$ of $\mathbf{G}_a$ to S^0 (or even to connective complex K-theory* ku*) as an $\mathbf{E}_1$-group object in $\mathbf{E}_4$-schemes.*

Proof. The $\mathbf{E}_4$-ku-algebra of functions on the flat lifting $(\mathbf{G}_a)_{\mathrm{ku}}$ must be given by $\mathrm{ku}[x]$. The group law on $\mathbf{G}_a$ is given by the coproduct $\mathbf{Z}[x] \to \mathbf{Z}[x, y]$ sending $x \mapsto x + y$. We therefore need to show that there is no $\mathbf{E}_4$-ku-algebra map $\Delta : \mathrm{ku}[x] \to \mathrm{ku}[x, y]$ given by $\Delta(x) = x + y$ on π_0. This follows from the δ_1-ring structure: we need $\Delta(\delta(x)) = \delta(\Delta(x))$ in $\mathbf{F}_p[x]$. But $\delta(x) = \delta(y) = 0$, so $\delta(\Delta(x)) = \frac{1}{p}(x^p + y^p - (x + y)^p)$ must vanish in $\mathbf{F}_p[x, y]$, which is a contradiction. $\square$

It was already shown in [Lur7, Proposition 1.6.20] that there is no flat lifting $(\mathbf{G}_a)_{\mathrm{S}^0}$ of $\mathbf{G}_a$ to the truncation $\tau_{\leq 1}(\mathrm{S}^0)$ as a group scheme; this of course prohibits such a lifting to S^0, too. The proof in *loc. cit.* uses the nontriviality of the Hopf element $\eta \in \pi_1(\mathrm{S}^0)$. Since this element vanishes in ku, the proof therein cannot be directly adapted to prove Proposition 3.4.7. However, let us note that using [Lur3, Proposition 5.4.9], one can show that the additive group over $\mathbf{Z}$ admits a flat lifting to a group object in $\mathbf{E}_2$-schemes over S^0.

3.5 Loop rotation equivariance

In this section, we describe an extension of Theorem 3.2.20 (or rather, of Example 3.3.9) which includes loop-rotation equivariance. Recall that Theorem 3.2.20 gives an isomorphism $\mathcal{F}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}})^\vee \cong \mathcal{O}(\check{\mathrm{T}}_k \times_{\mathrm{Spec}(k)} \mathcal{M}_{\mathrm{T}})$. The action of T on Gr_{T} refines to an action of $\widetilde{\mathrm{T}} = \mathrm{T} \times \mathbf{G}_m^{\mathrm{rot}}$, where $\mathbf{G}_m^{\mathrm{rot}}$ acts by loop rotation; we may therefore consider the *loop-rotation equivariant* homology $\mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{T}})^\vee$. There is an equivalence $\mathcal{M}_{\widetilde{\mathrm{T}}} \simeq \mathcal{M}_{\mathrm{T}} \times \mathbf{G}$, where the second factor is identified as $\mathcal{M}_{\mathbf{G}_m^{\mathrm{rot}}}$. Therefore, $\mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{T}})^\vee$ is a quasicoherent sheaf over $\mathcal{M}_{\mathrm{T}} \times \mathbf{G}$ whose fiber over the zero section of $\mathbf{G}$ recovers $\mathcal{F}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{T}})^\vee$.

Definition 3.5.1. Let $\mathbf{H}$ be a smooth 1-dimensional group scheme over a base commutative ring A, let T_c be a compact torus, and let $\mathbf{H}_{\mathrm{T}} = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{H})$. (When $\mathbf{G}$ is an oriented commutative k-group scheme, and $\mathbf{H} = \mathbf{G}_0$ is its underlying group scheme over $\mathrm{A} = \pi_0(k)$, then $\mathbf{H}_{\mathrm{T}}$ is precisely $\mathcal{M}_{\mathrm{T},0}$.) Let λ be a cocharacter of T_c, so that λ defines a homomorphism $\mathbb{X}^*(\mathrm{T}) \to \mathbf{Z}$, and hence a homomorphism $\lambda^* : \mathbf{H} \to \mathbf{H}_{\mathrm{T}}$. In turn, this defines a map

$$f^\lambda : \mathbf{H}_{\widetilde{\mathrm{T}}} \simeq \mathbf{H}_{\mathrm{T}} \times \mathbf{H} \xrightarrow{\mathrm{pr} \times \lambda^*} \mathbf{H}_{\mathrm{T}}.$$

If y is a local section of $\mathcal{O}_{\mathbf{H}_{\mathrm{T}}}$, we will write $\lambda^*(y)$ to denote the resulting local section of $\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$.

Let $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}$ denote the quotient of the associative $\mathcal{O}_{\mathbf{H}}$-algebra $\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}\langle x_\lambda | \lambda \in \mathbb{X}_*(\mathrm{T})\rangle$ by the relations given locally by

$$x_\lambda \cdot x_\mu = x_{\lambda+\mu},\ y \cdot x_\lambda = x_\lambda \cdot \lambda^*(y).$$

Here, $\lambda, \mu \in \mathbb{X}_*(\mathrm{T})$, and y is a local section of $\mathcal{O}_{\mathbf{H}_{\mathrm{T}}}$. We will call $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}$ the *algebra of* $\mathbf{H}$-*differential operators* on $\check{\mathrm{T}}$.

Remark 3.5.2. The algebra $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}$ satisfies a Mellin transform: namely, it follows from unwinding the definition that there is an equivalence

$$\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}\text{-mod} \simeq \mathrm{IndCoh}(\mathbf{H}_{\widetilde{\mathrm{T}}}/\mathbb{X}^*(\check{\mathrm{T}})),$$

where $\lambda \in \mathbb{X}^*(\check{\mathrm{T}}) \cong \mathbb{X}_*(\mathrm{T})$ acts on $\mathbf{H}_{\widetilde{\mathrm{T}}}$ via $y \mapsto \lambda^* y$.

Notation 3.5.3. If k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}_0$ is the $\pi_0(k)$-group underlying a oriented commutative A-group $\mathbf{G}$, we will write $\mathcal{D}^{\mathbf{G}}_{\check{\mathrm{T}}}$ to denote $\mathcal{D}^{\mathbf{G}_0}_{\check{\mathrm{T}}}$, and refer to it as the *algebra of* $\mathbf{G}$*-differential operators* on $\check{\mathrm{T}}$. We hope this does not cause any confusion.

Proposition 3.5.4. *There is an isomorphism*

$$\pi_0 \mathcal{F}_{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}})^\vee \cong \mathcal{D}^{\mathbf{G}}_{\check{\mathrm{T}}}$$

of $\mathcal{O}_{\mathbf{G}_0}$*-algebras. In particular, there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{T}}; k) \simeq \mathcal{D}^{\mathbf{G}}_{\check{\mathrm{T}}}\text{-mod}^{(\check{\mathrm{T}}\times\check{\mathrm{T}},\mathrm{weak})},$$

where the right-hand side denotes the category of left $\mathcal{D}^{\mathbf{G}}_{\check{\mathrm{T}}}$*-modules whose underlying quasicoherent sheaf over* $\check{\mathrm{T}}$ *is equivariant for* $\check{\mathrm{T}}\times\check{\mathrm{T}}$*-action on* $\check{\mathrm{T}}$ *given by left and right multiplication.*

Proof. Since $\mathrm{Gr}_{\mathrm{T}} \cong \mathbb{X}_*(\mathrm{T})$, it is easy to see that $\pi_0\mathcal{F}_{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}})^\vee \cong \bigoplus_{\lambda\in\mathbb{X}_*(\mathrm{T})} \pi_0 \mathcal{O}_{\mathcal{M}_{\widetilde{\mathrm{T}}}}$; let x_λ be a generator of the summand indexed by $\lambda \in \mathbb{X}_*(\mathrm{T})$. If $\lambda \in \mathbb{X}_*(\mathrm{T}) = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{Z})$, the map $\Omega\mathrm{T}_c \to \Omega\mathrm{T}_c$ given by multiplication-by-λ is $\mathrm{T}_c \times S^1_{\mathrm{rot}}$-equivariant for the homomorphism $\mathrm{T}_c \times S^1_{\mathrm{rot}} \to \mathrm{T}_c \times S^1_{\mathrm{rot}}$ given by

$$(t, \theta) \mapsto (t\lambda(\theta), \theta),$$

where λ is viewed as a homomorphism $S^1 \to \mathrm{T}$. On weight lattices, this homomorphism induces the map $\mathbb{X}^*(\mathrm{T}) \times \mathbf{Z} \to \mathbb{X}^*(\mathrm{T}) \times \mathbf{Z}$ which sends $(\mu, n) \mapsto (\mu, n + \lambda(\mu))$. In particular, the composite

$$\mathbb{X}^*(\mathrm{T}) \to \mathbb{X}^*(\mathrm{T}) \times \mathbf{Z} \to \mathbb{X}^*(\mathrm{T}) \times \mathbf{Z}$$

sends $\mu \mapsto (\mu, \lambda(\mu))$. Applying $\mathrm{Hom}(-, \mathbf{G})$ to this composite precisely produces the map $f^\lambda : \mathcal{M}_{\widetilde{\mathrm{T}}} \to \mathcal{M}_{\mathrm{T}}$ from Definition 3.5.1. This implies the desired identification of $\pi_0\mathcal{F}_{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}})^\vee$. □

Example 3.5.5. Let $\mathrm{T} \cong S^1$ be a torus of rank 1 (for simplicity). Suppose $k = \mathbf{Q}[u^{\pm 1}]$ with u in degree 2, so $\mathbf{G} = \mathbf{G}_a$ and $\mathcal{O}_{\mathbf{G}_0} \cong \mathbf{Q}[\hbar]$. Then the algebra of Definition 3.5.1 is the quotient of the $\mathbf{Q}[\hbar]$-algebra $\mathbf{Q}[\hbar]\langle y, x^{\pm 1}\rangle$ by the relation $yx = x(y + \hbar)$. In other words, y acts as the operator $\hbar x\partial_x$, so we simply have that

$$\mathrm{H}_0^{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}}; \mathbf{Q}[u^{\pm 1}]) \cong \mathrm{H}_*^{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}}; \mathbf{Q}) \cong \mathbf{Q}[\hbar]\langle \hbar x\partial_x, x^{\pm 1}\rangle.$$

This has been stated previously as [BFN, Proposition 5.19(2)]. In particular, the localization $\mathrm{H}_0^{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}}; \mathbf{Q}[u^{\pm 1}])[\hbar^{-1}]$ is isomorphic to the rescaled Weyl algebra $\mathcal{D}^{\hbar}_{\check{\mathrm{T}}}$; this is the motivation behind the terminology in Definition 3.5.1. Note that for a general torus, Remark 3.5.2 simply reduces to the standard Mellin transform, which gives an equivalence between $\mathrm{DMod}_\hbar(\check{\mathrm{T}})$ and $\mathrm{IndCoh}(\mathfrak{t}_{\mathbf{Q}[\hbar]}/\mathbb{X}^*(\check{\mathrm{T}}))$; here, $\lambda \in \mathbb{X}^*(\check{\mathrm{T}})$ acts on $\mathfrak{t}_{\mathbf{Q}[\hbar]}$ by $x \mapsto x + (d\lambda)(\hbar)$.

Example 3.5.6. Again, let $\mathrm{T} \cong \mathrm{S}^1$ be a torus of rank 1 (for simplicity). Suppose $k = \mathrm{KU}$, so $\mathbf{G} = \mathbf{G}_m$ and $\mathcal{O}_{\mathbf{G}_0} \cong \mathbf{Z}[q^{\pm 1}]$. Then the algebra of Definition 3.5.1 is the quotient of the $\mathbf{Z}[q^{\pm 1}]$-algebra $\mathbf{Z}[q^{\pm 1}]\langle y^{\pm 1}, x^{\pm 1}\rangle$ by the relation $yx = qxy$. (This is also known as the "quantum torus".) In other words, y acts as the operator $q^{x\partial_x}$ sending $f(x) \mapsto f(qx)$, so we simply have that

$$\mathrm{KU}_0^{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}}) \cong \mathbf{Z}[q^{\pm 1}]\langle q^{x\partial_x}, x^{\pm 1}\rangle.$$

This is closely related to the q-Weyl algebra $\mathcal{D}_q = \mathbf{Z}[q^{\pm 1}]\langle \Theta, x^{\pm 1}\rangle / (\Theta x = x(q\Theta + 1))$ for $\check{\mathrm{T}} = \mathbf{G}_m$: indeed, since the logarithmic q-derivative $\Theta = x\nabla_q$ is given by the fraction $\frac{q^{x\partial_x}-1}{q-1}$, the pullback of $\mathcal{D}_{\check{\mathrm{T}}}^{\mathbf{G}}$ along $\mathbf{G}_m - \{1\} \hookrightarrow \mathbf{G}_m$ is isomorphic to the algebra $\mathcal{D}_q[\frac{1}{q-1}]$. Note that Remark 3.5.2 gives a "q-Mellin transform", i.e., an equivalence between $\mathrm{LMod}_{\mathrm{KU}_0^{\widetilde{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{T}})}$ and $\mathrm{IndCoh}(\mathrm{T}_{\mathbf{Z}[q^{\pm 1}]}/\mathbb{X}^*(\check{\mathrm{T}}))$, where $\lambda \in \mathbb{X}^*(\check{\mathrm{T}}) = \mathbb{X}_*(\mathrm{T})$ acts on $\mathrm{T}_{\mathbf{Z}[q^{\pm 1}]}$ by sending $y \mapsto \lambda(q)y$.

Let us briefly outline the relationship between the algebra $\mathcal{D}_{\check{\mathrm{T}}}^{\mathbf{H}}$ of Definition 3.5.1 and the F-de Rham complex of Definition 7.4.3.

Notation 3.5.7. For the purpose of this discussion, we will assume that $\mathrm{T} \cong \mathrm{S}^1$ is a torus of rank 1, so that $\check{\mathrm{T}} \cong \mathbf{G}_m$. We will also fix an invariant differential form on the formal completion $\hat{\mathbf{H}}$ of $\mathbf{H}$ at the zero section, so that there is an isomorphism $\hat{\mathbf{H}} \cong \mathrm{Spf}\, \mathrm{A}[\![t]\!]$ of formal A-schemes. Let $\mathrm{F}(x, y)$ denote the resulting formal group law over A, and define the n-series of F by

$$[n]_{\mathrm{F}} := \overbrace{\mathrm{F}(t, \mathrm{F}(t, \mathrm{F}(t, \cdots \mathrm{F}(t, t) \cdots)))}^{n}.$$

We will often write $x +_{\mathrm{F}} y = x +_{\mathbf{G}} y$ to denote $\mathrm{F}(x, y)$. Let $\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}$ denote the completion of $\mathcal{D}_{\check{\mathrm{T}}}^{\mathbf{H}}$ at the zero section of $\mathbf{H}$.

Lemma 3.5.8 (Cartier duality)**.** *Let $\hat{\mathbf{H}}$ be a 1-dimensional formal group over a commutative ring A, and let $\mathrm{Cart}(\hat{\mathbf{H}})$ denote its Cartier dual (see [Dri1, Section 3.3] for more on Cartier duals of formal groups). Then there is an equivalence of categories $\mathrm{QCoh}(\hat{\mathbf{H}}) \simeq \mathrm{QCoh}(\mathrm{BCart}(\hat{\mathbf{H}}))$ sending the convolution tensor product on the left-hand side to the usual tensor product on the right-hand side. Under this equivalence, the functor $\mathrm{QCoh}(\hat{\mathbf{H}}) \to \mathrm{Mod}_{\mathrm{A}}$ given by restriction to the zero section is identified with the functor $\mathrm{QCoh}(\mathrm{BCart}(\hat{\mathbf{H}})) \to \mathrm{Mod}_{\mathrm{A}}$ given by pullback along the map $\mathrm{Spec}(\mathrm{A}) \to \mathrm{BCart}(\hat{\mathbf{H}})$.*

Proposition 3.5.9. *There is a canonical action of $\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}$ on $(\mathbf{G}_m)_{\mathrm{A}[\![t]\!]} = \mathrm{Spf}\, \mathrm{A}[\![t]\!][x^{\pm 1}]$ such that $\mathrm{A}[\![t]\!][x^{\pm 1}] \otimes_{\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}} \mathrm{A}[\![t]\!][x^{\pm 1}]$ is isomorphic to the two-term complex*

$$\mathrm{C}^\bullet = (\mathrm{A}[\![t]\!][x^{\pm 1}] \to \mathrm{A}[\![t]\!][x^{\pm 1}]dx), \ x^n \mapsto [n]_{\mathrm{F}} x^n dx$$

from [DM, Remark 4.3.8].

Proof sketch. Since T is of rank 1, there is an isomorphism $\mathbf{H}_{\mathrm{T}} \cong \mathbf{H}$, and hence an isomorphism $\hat{\mathbf{H}}_{\mathrm{T}} \cong \hat{\mathbf{A}}^1$ of formal A-schemes, where $\hat{\mathbf{H}}_{\mathrm{T}}$ denotes the completion of $\mathbf{H}_{\mathrm{T}}$ at the zero section. Let y be a local coordinate on $\mathbf{H}_{\mathrm{T}}$. Then, $\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}$ is isomorphic to the quotient of the associative $\hat{\mathcal{O}}_{\mathbf{H}}$-algebra $\hat{\mathcal{O}}_{\mathbf{H}\times\mathbf{H}_{\mathrm{T}}}\langle x^{\pm 1}\rangle$ subject to the relation $yx = x(y +_{\mathbf{G}} t)$. The t-adic filtration on $\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}$ therefore has associated graded $\mathrm{gr}(\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}}) \cong \hat{\mathcal{O}}_{\mathbf{H}_{\mathrm{T}}}[x^{\pm 1}][\![\bar{t}]\!]$, where $\bar{t}$ lives in weight 1. View A as a $\mathcal{O}_{\mathbf{H}_{\mathrm{T}}}$-algebra via the zero section, i.e., the augmentation $\mathcal{O}_{\mathbf{H}_{\mathrm{T}}} \to \mathrm{A}$. Then, the action of $\mathrm{gr}(\hat{\mathcal{D}}_{\check{\mathrm{T}}}^{\mathbf{H}})$ on $\mathrm{A}[x^{\pm 1}][\![\bar{t}]\!]$ is induced by the augmentation $\hat{\mathcal{O}}_{\mathbf{H}_{\mathrm{T}}} \to \mathrm{A}$. The isomorphism $\hat{\mathbf{H}}_{\mathrm{T}} \cong \hat{\mathbf{A}}^1$

of formal A-schemes then implies an isomorphism $\mathrm{A} \otimes_{\mathcal{O}_{\mathbf{H}_{\mathrm{T}}}} \mathrm{A} \cong \mathrm{A}[\epsilon]/\epsilon^2$ with ϵ in homological degree 1. It follows that

$$\mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \otimes_{\mathrm{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{\mathrm{T}}})} \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \simeq \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}][\epsilon]/\epsilon^2,$$

where $\bar{t}$ is in weight 1 and degree 0, and ϵ is in weight 0 and degree 1.

By Lemma 3.5.8, the t-adic filtration on $\hat{\mathcal{D}}^{\mathbf{H}}_{\check{\mathrm{T}}}$ is equivalent to the data of a $\mathrm{Cart}(\hat{\mathbf{H}})$-action on $\mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \otimes_{\mathrm{gr}(\hat{\mathcal{D}}^{\mathbf{H}}_{\check{\mathrm{T}}})} \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \simeq \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}][\epsilon]/\epsilon^2$. This in turn is equivalent to the data of a differential

$$\nabla : \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \to \mathrm{A}[\![\bar{t}]\!][x^{\pm 1}] \cdot \epsilon$$

satisfying an $\hat{\mathbf{H}}$-analogue of the Leibniz rule: if[8] $\nabla(x^n) = f(n)x^n\epsilon$ for some $f(n) \in \mathrm{A}[\![t]\!]$, then $f(n+m) = f(n) +_{\mathbf{G}} f(m)$. It therefore suffices to determine $\nabla(x)$; but the relation $yx = x(y +_{\mathbf{G}} t)$ forces $\nabla(x) = tx\epsilon$. This implies that

$$\nabla(x^n) = (\overbrace{t +_{\mathbf{G}} \cdots +_{\mathbf{G}} t}^{n})x^n\epsilon = [n]_{\mathrm{F}} x^n \epsilon,$$

as desired. □

Example 3.5.10. When $\mathbf{H} = \mathbf{G}_a$ over[9] $\mathbf{Q}$, the complex $\mathrm{C}^\bullet$ is

$$\mathrm{C}^\bullet = (\mathbf{Q}[\![\hbar]\!][x^{\pm 1}] \to \mathbf{Q}[\![\hbar]\!][x^{\pm 1}]dx), \ x^n \mapsto n\hbar x^n dx.$$

Indeed, since $yx = x(y+\hbar)$, we have $yx^n = x^n(y + n\hbar)$; since $t = \hbar$ in this case, we have $x^n \mapsto n\hbar x^n \epsilon$. This is evidently a $\hbar$-rescaling of the classical de Rham complex of $\mathbf{G}_m$.

When $\mathbf{H} = \mathbf{G}_m$ over $\mathbf{Z}$, the complex $\mathrm{C}^\bullet$ is

$$\mathrm{C}^\bullet = (\mathbf{Z}[\![q-1]\!][x^{\pm 1}] \to \mathbf{Z}[\![q-1]\!][x^{\pm 1}]dx), \ x^n \mapsto (q^n - 1)x^n dx.$$

Indeed, since $yx = x(qy)$, we have $yx^n = x^n(q^n y)$, and hence

$$(y-1)x^n = x^n(q^n y - 1) = x^n((y-1) +_{\mathrm{F}} (q^n - 1)),$$

where $\mathrm{F}(z,w) = z + w + zw$ is the multiplicative formal group law; since $t = q-1$ in this case, we have $x^n \mapsto (q^n-1)x^n\epsilon$. The complex $\mathrm{C}^\bullet$ is a $(q-1)$-rescaling of the q-de Rham complex of $\mathbf{G}_m$ from [Sch].

Remark 3.5.11. The complex of Proposition 3.5.9 is not quite the F-de Rham complex of Definition 7.4.3 (see [DM, Definition 4.3.6]); rather, if η_t denotes the décalage functor of [BO] with respect to the ideal $(t) \subseteq \mathrm{A}[\![t]\!]$, the F-de Rham complex is given by the décalage $\eta_t \mathrm{C}^\bullet$. In particular, the complex of Proposition 3.5.9 is isomorphic to the F-de Rham complex after inverting t. One can modify the algebra $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}$ of Definition 3.5.1 (by performing a noncommutative analogue of an affine blowup/deformation to the normal cone[10]) such that the relative tensor product as in Proposition 3.5.9 is the F-de Rham complex itself. Since it is not needed for this article, we will not describe this modification here.

[8] Note that ∇ has to be homogeneous in the degree of the monomial in x, as can be seen by keeping track of the x-weight.

[9] Of course, one can work over $\mathbf{Z}$ too; we just chose $\mathbf{Q}$ to continue with Example 3.5.5.

[10] For instance, in the case of Example 3.5.5, this procedure simply adjoins the fraction $\frac{y}{\hbar}$; in the case of Example 3.5.6, this procedure simply adjoins the fraction $\frac{y-1}{q-1}$.

Remark 3.5.12. Suppose k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring equipped with an oriented commutative k-group scheme $\mathbf{G}$. Proposition 3.5.9 says that $\hat{\mathcal{D}}^{\mathbf{G}_0}_{\check{\mathrm{T}}}$ is Koszul dual to the complex $\mathrm{C}^\bullet$. Forthcoming work of Arpon Raksit shows that the décalage $\eta_t \mathrm{C}^\bullet$ can be recovered from the "even filtration" (in the sense of [HRW]) on the periodic cyclic homology $\mathrm{HP}(\tau_{\geq 0}k[x^{\pm 1}]/\tau_{\geq 0}k)$. See also the discussion in [Dev2, Section 3.3]. Using similar techniques, one can show that $\mathrm{C}^\bullet$ can be recovered from the even filtration on the negative cyclic homology $\mathrm{HC}^-(k[x^{\pm 1}]/k) = \mathrm{HH}(k[x^{\pm 1}]/k)^{h\mathrm{S}^1}$.

Recalling that $\mathrm{T} = \mathrm{S}^1$, this $\mathbf{E}_\infty$-k-algebra is simply $\mathrm{HC}^-(k[\Omega\mathrm{T}]/k)$. The Hochschild homology $\mathrm{HH}(k[\Omega\mathrm{T}]/k) \simeq k \otimes \mathrm{THH}(\mathrm{S}[\Omega\mathrm{T}])$ is S^1-equivariantly equivalent to the k-chains $\mathrm{C}_*(\mathcal{L}\mathrm{T}; k)$ on the free loop space of T. (For a reference, see [NS, Corollary IV.3.3].) The k-chains $k[\mathcal{L}\mathrm{T}]$ is S^1-equivariantly Koszul dual[11] to $k[\Omega\mathrm{T}]^{h\mathrm{T}}$; this can be identified as a completion of $\mathcal{F}_\mathrm{T}(\Omega\mathrm{T})^\vee$ at the zero section of $\mathcal{M}_\mathrm{T}$. In other words, $\mathrm{HC}^-(k[\Omega\mathrm{T}]/k)$ is Koszul dual to the completion of $\mathcal{F}_{\mathrm{T}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{T})^\vee$ at the zero section of $\mathcal{M}_\mathrm{T} \times \mathbf{G}$. This is the topological source of the Koszul duality of Proposition 3.5.9.

3.6 Review of the classical case

To prepare ourselves for the calculations of $\pi_0(\mathcal{F}_\mathrm{T}(\mathrm{Gr}_\mathrm{G})^\vee)$ for k being complex K-theory or elliptic cohomology, we begin with the simpler case of k being $\mathbf{Q}[u^{\pm 1}]$ with u in degree 2; recall that $\mathcal{M}_{\mathrm{T},0}$ is then isomorphic to $\mathfrak{t}$. In this case, the discussion in the present section follows from the work of Bezrukavnikov, Finkelberg, and Mirkovic in [BFM], as well as the work of Yun and Zhu in [YZ2]. We will nevertheless go through this calculation (and discuss several applications) since the argument is different from that of the papers mentioned above, and also because it will serve as a useful template later. Our goal is specifically to *not* appeal to the derived geometric Satake equivalence of [BF], but rather do the calculation in such a way that proof technique generalizes to the K-theoretic or elliptic setting, so as to apply it to prove an analogue of [BF].

In the remainder of this article, we will assume the group G *is connected, almost simple, and simply-laced.* The assumption that G is simply-laced provides many simplifications; in particular, it implies that the Chevalley split forms of the groups G and $\check{\mathrm{G}}$ are centrally isogenous (so that the adjoint action of G on $\mathfrak{g}$ descends to an action of $\check{\mathrm{G}}$ on $\mathfrak{g}$), and that there is a $\check{\mathrm{G}}$-equivariant isomorphism $\mathfrak{g} \cong \check{\mathfrak{g}}^*$ (even over $\mathbf{Z}$). However, we will *never* use an $\check{\mathrm{G}}$-equivariant isomorphism $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$! The latter fails over $\mathbf{Z}$ (e.g., $\mathfrak{sl}_2 \not\cong \mathfrak{pgl}_2$ over $\mathbf{Z}$), and the effect of reliance on such failures becomes amplified in the settings of K-theory and elliptic cohomology.

In the following discussion, all dual groups are to be understood as defined over $\mathbf{Q}$ (although some of our discussion will work even over $\mathbf{Z}$, perhaps with some small primes inverted).

Definition 3.6.1 ((Additive) Kostant slice)**.** Fix a nondegenerate character $\psi \in \check{\mathfrak{n}}^*$; under the isomorphism $\check{\mathfrak{g}}^* \cong \mathfrak{g}$, there is an isomorphism $\check{\mathfrak{n}}^* \cong \mathfrak{n}$, and ψ corresponds to a principal nilpotent element $f \in \mathfrak{n}$. Let (e, f, h) be the associated $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, and let $\psi_- : \check{\mathfrak{n}}_- \to \mathbf{A}^1$ denote the element corresponding to e. Let $\check{\mathfrak{g}}^{*,\psi_-} \cong \mathfrak{g}^e$ denote the centralizer (so $\mathfrak{g} = \mathfrak{g}^e \oplus [e, \mathfrak{g}]$), and let $\mathcal{S} := f + \mathfrak{g}^e \subseteq \mathfrak{g}^{\mathrm{reg}}$ be the Kostant slice. Note that $\mathcal{S} \cong \psi + \check{\mathfrak{g}}^{*,\psi_-} \subseteq \check{\mathfrak{g}}^{*,\mathrm{reg}}$. The composite $f + \mathfrak{g}^e \to \mathfrak{g} \to \mathfrak{g}/\!\!/\mathrm{G} \cong \mathfrak{t}/\!\!/\mathrm{W}$ is an isomorphism, by [Kos1].

Recall that the Grothendieck-Springer resolution is defined as

$$\widetilde{\check{\mathfrak{g}}} = \check{\mathfrak{n}}^\perp \times^{\check{\mathrm{B}}} \check{\mathrm{G}} \cong \mathfrak{b} \times^{\check{\mathrm{B}}} \check{\mathrm{G}},$$

[11] This Koszul duality essentially stems from the (non-$\mathrm{S}^1_{\mathrm{rot}}$-equivariant) decomposition $\mathcal{L}\mathrm{T} \simeq \mathrm{T} \times \Omega\mathrm{T}$.

so that $\widetilde{\check{\mathfrak{g}}}/\check{\mathrm{G}} \simeq \mathfrak{b}/\check{\mathrm{B}}$. A point of $\widetilde{\check{\mathfrak{g}}}$ can be regarded as a pair $(\check{\mathfrak{b}}', x \in (\check{\mathfrak{n}}')^\perp)$; here, $\check{\mathfrak{b}}'$ denotes a Borel subalgebra of $\check{\mathfrak{g}}$, and $\check{\mathfrak{n}}'$ denotes its nilpotent radical. There is a map $\widetilde{\chi} : \widetilde{\check{\mathfrak{g}}} \to \mathfrak{t}$ which sends a pair $(\check{\mathfrak{b}}', x)$ to the image of x modulo $(\check{\mathfrak{b}}')^\perp$. Let $\widetilde{\mathcal{S}}$ denote the fiber product $\mathcal{S} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}}$, so that

$$\widetilde{\mathcal{S}} \subseteq \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}} = \check{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}}.$$

Then, Kostant's result on the Kostant slice implies formally that the composite

$$\widetilde{\mathcal{S}} \to \widetilde{\check{\mathfrak{g}}} \xrightarrow{\widetilde{\chi}} \mathfrak{t}$$

is an isomorphism. We will often abusively write the inclusion of $\widetilde{\mathcal{S}}$ as a map $\kappa : \mathfrak{t} \to \widetilde{\check{\mathfrak{g}}}$.

In fact, we will only care about the composite $\mathfrak{t} \to \widetilde{\check{\mathfrak{g}}} \to \widetilde{\check{\mathfrak{g}}}/\check{\mathrm{G}}$ below, so we will also denote it by κ. If we identify $\widetilde{\check{\mathfrak{g}}}/\check{\mathrm{G}} \cong \mathfrak{b}/\check{\mathrm{B}}$, then the map κ admits a simple description: it is the composite $f + \mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/\check{\mathrm{B}}$. (See Proposition 3.1.9.) In our discussion below, we will often identify $f + \mathfrak{t}$ with $\psi + \check{\mathfrak{t}}^*$.

Definition 3.6.2. The stabilizer (inside $\check{\mathrm{G}}$) of the Kostant slice $\mathcal{S} \subseteq \mathfrak{g}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{\mathrm{G}} \times \mathcal{S}$, and will be denoted by $\check{\mathrm{J}}$. It will be called the *regular centralizer group scheme*; if we wish to emphasize the dependence on G, we will denote it by $\check{\mathrm{J}}(\mathrm{G})$. Note that since the composite $\mathcal{S} \to \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!\!/\check{\mathrm{G}}$ is an isomorphism, we may identify

$$\check{\mathrm{J}} \cong \mathcal{S} \times_{\mathfrak{g}/\check{\mathrm{G}}} \mathcal{S}.$$

Similarly, the stabilizer (inside $\check{\mathrm{G}}$) of the Kostant slice $\widetilde{\mathcal{S}} \subseteq \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{\mathrm{G}} \times \widetilde{\mathcal{S}}$, and will be denoted by $\widetilde{\check{\mathrm{J}}}$. Since $\widetilde{\mathcal{S}} \cong \mathcal{S} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}}$, we may identify

$$\widetilde{\check{\mathrm{J}}} \cong \check{\mathrm{J}} \times_{\mathcal{S}} \widetilde{\mathcal{S}} \cong (f + \mathfrak{t}) \times_{\mathfrak{b}/\check{\mathrm{B}}} (f + \mathfrak{t}).$$

Theorem 3.6.3. *There is an isomorphism of group schemes over $f + \mathfrak{t} \cong \mathfrak{t} \cong \mathcal{M}_{\mathrm{T},0}$:*

$$\operatorname{Spec} \pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong (f + \mathfrak{t}) \times_{\mathfrak{b}/\check{\mathrm{B}}} (f + \mathfrak{t}).$$

Theorem 3.6.3 can be proved directly using Proposition 3.2.15, but I find the discussion below more enlightening (of course, it is essentially an elaboration of the application of Proposition 3.2.15). We first need a few lemmas.

Lemma 3.6.4. *The projection map $\widetilde{\check{\mathrm{J}}} \to \psi + \check{\mathfrak{t}}^*$ (onto either factor) is flat.*

Proof. For this, we will follow [YZ2, Step II]. Consider the morphism $\check{\mathrm{B}} \times \check{\mathfrak{t}}^* \to \check{\mathfrak{n}}^\perp$ sending $(g, x) \mapsto \mathrm{Ad}_g(\psi + x) - \psi$. Unwinding definitions shows that there is a Cartesian square

$$\begin{tikzcd}
\widetilde{\check{\mathrm{J}}} \arrow[r] \arrow[d] & \check{\mathfrak{t}}^* \arrow[d] \\
\check{\mathrm{B}} \times \check{\mathfrak{t}}^* \arrow[r] & \check{\mathfrak{n}}^\perp,
\end{tikzcd}$$

so $\widetilde{\check{\mathrm{J}}}$ is a closed subscheme of $\check{\mathrm{B}} \times \check{\mathfrak{t}}^*$ of codimension $\dim(\check{\mathfrak{b}}^\perp) = \dim(\check{\mathrm{N}})$. This means that the fibers of the map $\widetilde{\check{\mathrm{J}}} \to \check{\mathfrak{t}}^*$ are have dimension at least $\dim(\check{\mathrm{B}}) - \dim(\check{\mathrm{N}}) = \mathrm{rank}(\check{\mathrm{G}})$. If all fibers had dimension *exactly* $\mathrm{rank}(\check{\mathrm{G}})$, then miracle flatness would imply that the map $\widetilde{\check{\mathrm{J}}} \to \check{\mathfrak{t}}^*$ is flat.

To show that all fibers have dimension $\mathrm{rank}(\check{\mathrm{G}})$, observe that there is a contracting $\mathbf{G}_m$-action on the vector space $\check{\mathfrak{t}}^*$ which pushes everything down to the origin; so it suffices to show that the fiber over $0 \in \check{\mathfrak{t}}^*$ is of the correct dimension.

That is, we need to see that the scheme

$$\mathrm{Y} := \{(g, x) \in \check{\mathrm{B}} \times \check{\mathfrak{t}}^* | \mathrm{Ad}_g(\psi) = \psi + x\}$$

is $\mathrm{rank}(\check{\mathrm{G}})$-dimensional. First, observe that if $\mathrm{Ad}_g(\psi) = \psi + x \in \check{\mathfrak{n}}^\perp$ with $x \in \check{\mathfrak{t}}^*$, then actually $x = 0$. This is because the image of x under the map

$$\check{\mathfrak{n}}^\perp \to (\check{\mathfrak{n}} \oplus \check{\mathfrak{n}}^-)^\perp \cong \check{\mathfrak{t}}^*$$

is the same as the image of $\psi + x$, which is the same as the image of $\mathrm{Ad}_g(\psi)$. But the above map $\check{\mathfrak{n}}^\perp \to \check{\mathfrak{t}}^*$ is Ad-invariant, and so the image of $\mathrm{Ad}_g(\psi)$ is equal to the image of ψ, which is zero. This means that the image of x is also zero. But the inclusion $\check{\mathfrak{t}}^* \subseteq \check{\mathfrak{n}}^\perp$ splits the map $\check{\mathfrak{n}}^\perp \to \check{\mathfrak{t}}^*$, and so we see that $x = 0$. Therefore,

$$\mathrm{Y} \cong \{g \in \check{\mathrm{B}} | \mathrm{Ad}_g(\psi) = \psi\} = \mathrm{Z}_{\check{\mathrm{B}}}(\psi).$$

The centralizer of ψ is contained entirely in $\check{\mathrm{B}}$, so $\mathrm{Z}_{\check{\mathrm{B}}}(\psi) \cong \mathrm{Z}_{\check{\mathrm{G}}}(\psi)$. This, in turn, has dimension given by $\mathrm{rank}(\check{\mathrm{G}})$ since ψ (corresponding to $e \in \mathfrak{g}$) is a regular nilpotent. $\square$

Note that

$$\widetilde{\check{\mathrm{J}}} \cong \{(x, y, g) \in \check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \check{\mathrm{B}} | \mathrm{Ad}_g(\psi + x) = \psi + y\}.$$

The argument at the end of Lemma 3.6.4 allows us to identify $x = y \in \check{\mathfrak{t}}^*$, and so

$$\widetilde{\check{\mathrm{J}}} \cong \{(x, g) \in \check{\mathfrak{t}}^* \times \check{\mathrm{B}} | \mathrm{Ad}_g(\psi + x) = \psi + x\}.$$

Notation 3.6.5. If α is a root of $\check{\mathrm{G}}$, let $\{e_\alpha, h_\alpha\}$ denote a pinning of $\check{\mathrm{G}}$. Say that a point $x \in \check{\mathfrak{t}}^*$ is *α-generic* if $x(h_\beta) \neq 0$ for all roots $\beta \neq \alpha$. This implies that the centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(x)$ has semisimple rank at most 1. Let $\check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ denote the α-regular locus. Observe that $\check{\mathfrak{t}}^*_{\mathrm{reg}} = \bigcup_{\alpha \in \Phi} \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \subseteq \check{\mathfrak{t}}^*$ is open, with complement of codimension 2.

Lemma 3.6.6. *There is an isomorphism*

$$\widetilde{\check{\mathrm{J}}}(\check{\mathrm{G}})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{\mathrm{J}}}(\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}, \tag{3.6.1}$$

where $\mathrm{Z}_{\check{\mathrm{G}}}(x)$ is the centralizer of some $x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ which lies on the α-hyperplane, and $\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ$ denotes the connected component of the identity.

Proof sketch. Let us, for simplicity, write $\check{\mathrm{H}}$ to denote $\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ$. There is a map from the left-hand side to the right-hand side, which sends

$$\check{\mathfrak{t}}^* \times \check{\mathrm{B}} \ni (x, g) \mapsto (x, g) \in \check{\mathfrak{t}}^* \times (\check{\mathrm{B}} \cap \check{\mathrm{H}}).$$

Note that $\check{\mathrm{B}} \cap \check{\mathrm{H}}$ is a Borel subgroup of $\check{\mathrm{H}}$. To see that the above map gives an isomorphism, observe that if $y \in \check{\mathfrak{t}}^*$, we may identify the centralizer in $\check{\mathrm{G}}$ of $\psi + y$ with the centralizer in $\mathrm{Z}_{\check{\mathrm{G}}}(y)^\circ$ of ψ. That (3.6.1) is an isomorphism is now a consequence of the observation that if $y \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$, then this centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(y)^\circ$ is contained in $\check{\mathrm{H}}$. That is, if $(x, g) \in \widetilde{\check{\mathrm{J}}}(\check{\mathrm{G}})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$, then g is already contained in H, and so $(x, g) \in \widetilde{\check{\mathrm{J}}}(\check{\mathrm{H}})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$. $\square$

Proof of Theorem 3.6.3. We begin by noting that Gr_{G} only has even cells; so $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee = \pi_0\mathrm{C}^{\mathrm{T}}_*(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q}[u^{\pm1}])$ can be identified with $\mathrm{H}^{\mathrm{T}}_*(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q})$, regarded now as an ungraded $\mathbf{Q}$-algebra. Similarly, $\pi_0(k_{\mathrm{T}}) \cong \mathrm{H}^*_{\mathrm{T}}(*;\mathbf{Q})$, again regarded as an ungraded $\mathbf{Q}$-algebra. The equivariant formality of Gr_{G} implies that $\mathrm{H}^{\mathrm{T}}_*(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q})$ is flat over $\mathrm{H}^*_{\mathrm{T}}(*;\mathbf{Q})$. To prove Theorem 3.6.3, it therefore suffices to prove an isomorphism

$$\widetilde{\check{\mathrm{J}}}|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \operatorname{Spec}\mathrm{H}^{\mathrm{T}_c}_*(\Omega\mathrm{G};\mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$$

for each root α. By Atiyah-Bott localization, the right-hand side can be identified with

$$\operatorname{Spec}\mathrm{H}^{\mathrm{T}_c}_*(\Omega\mathrm{G};\mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \operatorname{Spec}\mathrm{H}^{\mathrm{T}_c}_*(\Omega\mathrm{Z}_{\mathrm{G}}(x);\mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}, \tag{3.6.2}$$

where $\mathrm{Z}_{\mathrm{G}}(x)$ is the centralizer of some $x \in \mathfrak{t}_{\alpha\text{-reg}}$ which lies on the α-hyperplane. Note that the right-hand side depends only on the connected component $\mathrm{Z}_{\mathrm{G}}(x)^\circ$ of the identity in $\mathrm{Z}_{\mathrm{G}}(x)$; so we might as well replace $\mathrm{Z}_{\mathrm{G}}(x)$ by $\mathrm{Z}_{\mathrm{G}}(x)^\circ$. Using Lemma 3.6.6, we are therefore reduced to showing that there is an isomorphism

$$\mathrm{H}^{\mathrm{T}_c}_*(\Omega\mathrm{Z}_{\mathrm{G}}(x)^\circ;\mathbf{Q})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \widetilde{\check{\mathrm{J}}}(\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

Since $\mathrm{Z}_{\mathrm{G}}(x)^\circ$ has semisimple rank 1, we are reduced to checking that Theorem 3.6.3 holds in this case.

That is, we may assume G is the product of a torus with GL_2, SL_2, or PGL_2. It is easy to match up the contribution from the toral factors, so we will assume that G is GL_2, SL_2, or PGL_2.

- For GL_2, we may identify $\mathfrak{gl}_2^* \cong \mathfrak{gl}_2$. Then, $\widetilde{\check{\mathrm{J}}}$ is the centralizer (in $\check{\mathrm{B}}$) of $\left(\begin{smallmatrix} x & 0 \\ 1 & y\end{smallmatrix}\right)$. It is not hard to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & d\end{smallmatrix}\right)$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & y\end{smallmatrix}\right)$ if and only if $c = \frac{a-d}{x-y}$, meaning that
$$\widetilde{\check{\mathrm{J}}} \cong \operatorname{Spec}\mathbf{Q}[x, y, a^{\pm1}, d^{\pm1}, \tfrac{a-d}{x-y}].$$
The coproduct sends $a \mapsto a\otimes a$ and $d \mapsto d\otimes d$.

  Let us now calculate $\mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})$ as an algebra over $\mathrm{H}^*_{\mathrm{T}^2}(*;\mathbf{Q}) \cong \mathbf{Q}[x,y]$. There is a simple T^2-equivariant cell decomposition of $\Omega\mathrm{GL}_2$ with $\mathbb{X}_*(\mathrm{T}^2) = \mathbf{Z}^2$ many 0-cells, and where there is a T^2-equivariant 1-cell connecting μ_1 to μ_2 if and only if $\mu_1 - \mu_2$ is a multiple of a root of GL_2. (There are higher equivariant cells, but they will not matter.) This implies, by Atiyah-Bott localization, that the fixed points of the T^2-action on $\Omega\mathrm{GL}_2$ are simply $\Omega\mathrm{T}^2 = \mathbf{Z}^2$, and so
$$\mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})[\tfrac{1}{x-y}] \cong \mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{T}^2;\mathbf{Q})[\tfrac{1}{x-y}] \cong \mathbf{Q}[x, y, \tfrac{1}{x-y}, a^{\pm1}, d^{\pm1}].$$
  On the other hand, the *completion* $\mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})^\wedge_{(x-y)}$ can be determined directly. After completing at $(x-y, y) = (x,y)$, the equivariant homology $\mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})$ simply becomes the *Borel-equivariant* homology, and this can be computed directly via a spectral sequence
$$\mathrm{E}_2 = \mathrm{H}^*(\mathrm{BT}^2;\mathbf{Q})\otimes_k \mathrm{H}_*(\Omega\mathrm{GL}_2;\mathbf{Q}) \Rightarrow \mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})^\wedge_{(x,y)}.$$
  Since $\mathrm{H}_*(\Omega\mathrm{GL}_2;\mathbf{Q}) = \mathbf{Q}[\mathrm{A}^{\pm1}, b]$ with A in weight 0 and b in weight 2, the E_2-page of this spectral sequence is concentrated entirely in even degrees, and hence collapses. This means that
$$\mathrm{H}^{\mathrm{T}^2}_*(\Omega\mathrm{GL}_2;\mathbf{Q})^\wedge_{(x,y)} \cong k[\![x,y]\!][\mathrm{A}^{\pm1}, b].$$

This in fact comes from an isomorphism

$$\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)} \cong \mathbf{Q}[x, y, \mathrm{A}^{\pm 1}, b]^\wedge_{(x-y)}.$$

We may therefore recover $\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})$ via the gluing square

$$\begin{tikzcd}
\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q}) \arrow[r] \arrow[d] & \mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})[\frac{1}{x-y}] \arrow[d] \\
\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)} \arrow[r] & \mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})^\wedge_{(x-y)}[\frac{1}{x-y}].
\end{tikzcd}$$

Explicitly:

$$\begin{tikzcd}
\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q}) \arrow[r] \arrow[d] & \mathbf{Q}[x, y, \frac{1}{x-y}, a^{\pm 1}, d^{\pm 1}] \arrow[d] \\
\mathbf{Q}[x, y, \mathrm{A}^{\pm 1}, b]^\wedge_{(x-y)} \arrow[r] & \mathbf{Q}[x, y, \mathrm{A}^{\pm 1}, b]^\wedge_{(x-y)}[\frac{1}{x-y}].
\end{tikzcd}$$

The right vertical map sends $a - d \mapsto b(x-y)$; and $d \mapsto \mathrm{A}$. Note that $b(x-y)$ is topologically nilpotent, so $\mathrm{A} + b(x-y)$ is a unit, and this is what a maps to. This discussion implies that the fiber product above identifies with

$$\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q}) \cong \mathbf{Q}[x, y, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

We need to determine the coproduct. Since this ring is flat over $\mathbf{Q}[x, y]$, it suffices to determine the coproduct after inverting $x - y$. As we have seen, $\mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q})[\frac{1}{x-y}] \cong \mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{T}^2; \mathbf{Q})[\frac{1}{x-y}]$, and $\Omega \mathrm{T}^2 = \mathbf{Z}^2$. The coproduct here simply comes from the *diagonal* on $\mathbf{Z}^2$, which obviously sends $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$. It follows that

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{T}^2}(\Omega \mathrm{GL}_2; \mathbf{Q}) \cong \widetilde{\widetilde{\mathrm{J}}}$$

as (graded) group schemes over $\mathbf{Q}[x, y]$, as desired.

- For $\mathrm{G} = \mathrm{SL}_2$, one can similarly calculate that

$$\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2}; \mathbf{Q}) \cong \mathbf{Q}[x, a^{\pm 1}, b]/(a = 1 + 2xb) \cong \mathbf{Q}[x, a^{\pm 1}, \tfrac{a-1}{2x}].$$

The coproduct is determined by the formula $a \mapsto a \otimes a$, so that

$$b \mapsto b \otimes 1 + 1 \otimes b + 2xb \otimes b.$$

For completeness, let us quickly summarize the argument. The fixed points of S^1 acting on $\Omega \mathrm{SU}(2)$ is $\Omega \mathrm{S}^1 = \mathbf{Z}$, and the action of S^1 on $\mathrm{SU}(2) \cong \mathrm{S}^3$ exhibits it as the one-point compactification of $\mathbf{R} \oplus \mathbf{C}$, where $\mathbf{R}$ is the trivial representation and $\mathbf{C}$ is the *weight* 2 representation. Therefore, inverting the Chern class $2x$ of the weight 2 representation lets us identify

$$\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2}; \mathbf{Q})[\tfrac{1}{2x}] \cong \mathrm{H}_*^{\mathrm{S}^1}(\Omega \mathrm{S}^1; \mathbf{Q})[\tfrac{1}{2x}] \cong \mathbf{Q}[x^{\pm 1}, a^{\pm 1}].$$

On the other hand, the completion of $\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})$ at the class $2x$ is, via the same spectral sequence argument as in the preceding bullet point, given by

$$\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})^\wedge_{(2x)} \cong \mathbf{Q}[\![x]\!][b],$$

with b in weight 2. The ring $\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})$ can be recovered via the gluing square

$$\begin{tikzcd}
\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q}) \arrow[r] \arrow[d] & \mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})[\frac{1}{2x}] \arrow[d] \\
\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})^\wedge_{(2x)} \arrow[r] & \mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q})^\wedge_{(2x)}[\frac{1}{2x}].
\end{tikzcd}$$

The right vertical map sends $a-1 \mapsto b\cdot 2x$, and so the above Cartesian square gives an isomorphism

$$\mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q}) \cong \mathbf{Q}[x, a^{\pm 1}, b]/(a = 1+2xb),$$

as desired.

On the other hand, $\tilde{\check{\mathrm{J}}}$ is the centralizer in $\check{\mathrm{B}} \subseteq \mathrm{PGL}_2$ of $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right) \subseteq \mathfrak{sl}_2 \cong \check{\mathfrak{g}}^*$. It is easy to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \check{\mathrm{B}}$ (where we only care about this as an element of PGL_2!) stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right)$ if and only if $2xc = a-1$. Therefore,

$$\tilde{\check{\mathrm{J}}} \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, c]/(a = 1+2xc),$$

and again the group law is determined by the formulae

$$a \mapsto a\otimes a,\ c \mapsto c\otimes 1 + 1\otimes c + 2xc\otimes c.$$

Therefore,

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{S}^1}(\mathrm{Gr}_{\mathrm{SL}_2};\mathbf{Q}) \cong \tilde{\check{\mathrm{J}}}$$

as (graded) group schemes over $\mathbf{Q}[x]$, as desired.

- In exactly the same way, for $\mathrm{G} = \mathrm{PGL}_2$, one can similarly calculate that

$$\mathrm{H}_*^{\mathrm{S}^1}(\Omega\mathrm{PGL}_2;\mathbf{Q}) \cong \mathbf{Q}[x, a^{\pm 1}, b]/(a^2 = 1+xb) \cong \mathbf{Q}[x, a^{\pm 1}, \tfrac{a^2-1}{x}].$$

This is because the fixed points of S^1 acting on $\Omega\mathrm{PGL}_2 \simeq \mathbf{Z}/2 \times \Omega\mathrm{S}^3$ is $\mathbf{Z}$, and the action of S^1 on PGL_2, which is homotopy equivalent to $\mathbf{R}\mathrm{P}^3$, exhibits it as the $\mathbf{Z}/2$-quotient of the one-point compactification of $\mathbf{R}\oplus\mathbf{C}$, where $\mathbf{R}$ is the trivial representation and $\mathbf{C}$ is the *weight* 1 representation. The coproduct is determined by the formula $a \mapsto a\otimes a$, so that

$$b \mapsto b\otimes 1 + 1\otimes b + xb\otimes b.$$

On the other hand, $\tilde{\check{\mathrm{J}}}$ is the centralizer in $\check{\mathrm{B}} \subseteq \mathrm{SL}_2$ of the equivalence class of $\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in $\mathfrak{pgl}_2 \cong \check{\mathfrak{g}}^*$. It is easy to compute directly that $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right) \in \check{\mathrm{B}}$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)$ if and only if $xc = a - a^{-1}$. Therefore,

$$\tilde{\check{\mathrm{J}}} \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, c]/(a = a^{-1} + xc) \cong \operatorname{Spec} \mathbf{Q}[x, a^{\pm 1}, \tfrac{a-a^{-1}}{x}].$$

Replacing c by $b := ca^{-1}$, we see that the group law is determined by the formulae

$$a \mapsto a \otimes a, \ b \mapsto b \otimes 1 + 1 \otimes b + xb \otimes b.$$

Therefore,

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{S}^1}(\Omega \mathrm{PGL}_2; \mathbf{Q}) \cong \widetilde{\check{\mathrm{J}}}$$

as (graded) group schemes over $\mathbf{Q}[x]$, as desired. □

Remark 3.6.7. Just for posterity, let us record a more canonical variant of the calculation above for $\check{\mathrm{G}} = \mathrm{SL}_2$, which does not require picking a Borel subgroup (i.e., which does not involve identifying $\widetilde{\check{\mathfrak{g}}}/\check{\mathrm{G}} \cong \mathfrak{b}/\check{\mathrm{B}}$). For simplicity, we will use the fact that 2 is invertible in $\mathbf{Q}$ to identify $\mathfrak{sl}_2 \cong \mathfrak{pgl}_2$. In this case, the Grothendieck-Springer resolution $\widetilde{\check{\mathfrak{g}}} = \mathrm{T}^*(\mathbf{A}^2 - \{0\})/\mathbf{G}_m$ is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1$; we will think of a point in $\widetilde{\mathfrak{g}}$ as a pair $(x \in \mathfrak{sl}_2, \ell \subseteq \mathbf{C}^2)$ such that x preserves ℓ. The Kostant slice $\kappa : \mathfrak{t} \cong \mathbf{A}^1 \to \widetilde{\check{\mathfrak{g}}}$ is the map sending $\lambda \in \mathbf{A}^1$ to the pair (x, ℓ) with $x = \left(\begin{smallmatrix} 0 & \lambda^2 \\ 1 & 0 \end{smallmatrix}\right)$ and $\ell = [\lambda : 1]$. Indeed, this is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A}^1 \cong \mathfrak{t} \arrow[r, "\kappa"] \arrow[d, "\lambda \mapsto \lambda^2"'] & \widetilde{\mathfrak{sl}}_2 \arrow[d] \\
\mathbf{A}^1 \cong \mathfrak{t}/\!\!/\mathrm{W} \arrow[r, "\kappa", "\lambda \mapsto \left(\begin{smallmatrix} 0 & \lambda \\ 1 & 0 \end{smallmatrix}\right)"'] & \mathfrak{sl}_2.
\end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\check{\mathfrak{g}}}$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we compute that

$$\mathrm{Ad}_g \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} bd - ac\lambda^2 & (a\lambda)^2 - b^2 \\ d^2 - (c\lambda)^2 & ac\lambda^2 - bd \end{pmatrix}, \ g \cdot [\lambda : 1] = [a\lambda + b : c\lambda + d].$$

From this, we see that $\mathrm{Ad}_g(x) = x$ if and only if $a = d$ and $b = c\lambda^2$, in which case g also fixes $[\lambda : 1]$. In other words, $g = \left(\begin{smallmatrix} a & c\lambda^2 \\ c & a \end{smallmatrix}\right)$ with $a, c \in k$; in order for $\det(g) = 1$, we need $a^2 - c^2\lambda^2 = 1$. When $\lambda \neq 0$, both x and g are diagonalized by the matrix $\frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -\lambda^{-1} & -\lambda^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} t & 0 \\ 0 & w \end{smallmatrix}\right)$ where $2a = t + w$ and $2\lambda c = t - w$. Since we have $\det(g) = a^2 - (c\lambda)^2 = 1$, we have $w = t^{-1}$. This shows that if k is not of characteristic 2, then $\mathfrak{t} \times_{\widetilde{\mathfrak{sl}}_2/\mathrm{SL}_2} \mathfrak{t} \cong \operatorname{Spec} \mathbf{Q}[\lambda, t^{\pm 1}, \frac{t - t^{-1}}{\lambda}]$.

Corollary 3.6.8. *There is an equivalence*

$$\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}).$$

Furthermore, the pushforward functor $\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \to \mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(*; k)$ *identifies with the pullback functor* $\kappa^* : \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathfrak{t})$.

Proof. By definition, $\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k)$ is equivalent to the category of comodules over $\pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee = \mathrm{H}_*^{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{Q})$ in the category of $\pi_0 k_{\mathrm{T}} \cong \mathrm{H}_{\mathrm{T}}^*(*; \mathbf{Q})$-modules. By Theorem 3.6.3, it can be identified the category of quasicoherent sheaves on the quotient stack $(f + \mathfrak{t})/\widetilde{\check{\mathrm{J}}}$. As discussed after Lemma 3.6.4, we may view $\widetilde{\check{\mathrm{J}}}$ as a closed subgroup scheme of the constant group scheme $\check{\mathrm{B}} \times (f + \mathfrak{t})$. This gives an isomorphism

$$(f + \mathfrak{t})/\widetilde{\check{\mathrm{J}}} \cong \check{\mathrm{B}} \backslash (\check{\mathrm{B}} \times (f + \mathfrak{t}))/\widetilde{\check{\mathrm{J}}}.$$

It follows from Kostant's work in [Kos1] that the $\check{\mathrm{B}}$-orbit of $f+\mathfrak{t}$ inside $\mathfrak{b}$ is precisely the regular locus $\mathfrak{b}^{\mathrm{reg}}$. Since $\tilde{\check{\mathrm{J}}}$ is definitionally the stabilizer of $f+\mathfrak{t} \subseteq \mathfrak{b}$, the quotient $(\check{\mathrm{B}} \times (f+\mathfrak{t}))/\tilde{\check{\mathrm{J}}}$ is isomorphic to $\mathfrak{b}^{\mathrm{reg}}$; so there is an isomorphism $(f+\mathfrak{t})/\tilde{\check{\mathrm{J}}} \cong \mathfrak{b}^{\mathrm{reg}}/\check{\mathrm{B}}$. To finish, note that $\tilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}} \cong \mathfrak{b}^{\mathrm{reg}}/\check{\mathrm{B}}$. □

The equivalence of Corollary 3.6.8 is in fact symmetric monoidal for the convolution tensor structure on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ (described in Remark 3.3.5) and the standard tensor product on $\mathrm{QCoh}(\tilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}})$.

Remark 3.6.9. Note that the definition of the Kostant slice $f+\mathfrak{t} \subseteq \mathfrak{b}$ involved the choice of a regular nilpotent element $f \in \mathfrak{g}$. However, this choice does not materialize in Corollary 3.6.8. This is because two such slices obtained by choosing two different regular nilpotent elements in $\mathfrak{g}$ are *conjugate* to each other (by $\check{\mathrm{B}}$). That is, while the specific inclusion $f+\mathfrak{t} \subseteq \mathfrak{b}$ depends on the choice of f, the composite $f+\mathfrak{t} \subseteq \mathfrak{b} \to \mathfrak{b}/\check{\mathrm{B}}$ is independent of said choice.

Example 3.6.10. Suppose $\mathrm{G} = \mathrm{SL}_n$. In this case, $\mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ is simply isomorphic to a polynomial algebra $\mathbf{Q}[b_1, \cdots, b_{n-1}]$ on $n-1$ generators. The coproduct is given by $b_j \mapsto \sum_i b_i \otimes b_{j-i}$, where b_0 is understood to be 1. This result is classical, and can be found, for instance, in [Bot]. The proof there amounts to the following observation. Consider the map $\mathbf{C}\mathrm{P}^{n-1} \to \mathrm{Gr}_{\mathrm{SL}_n}$ given by sending $\ell \in \mathbf{C}\mathrm{P}^{n-1}$ to (an appropriate rescaling of) the loop sending $\theta \in \mathrm{S}^1$ to rotation by angle θ about the line ℓ. Then the image of the induced map $\mathrm{H}_*(\mathbf{C}\mathrm{P}^{n-1}; \mathbf{Q}) \to \mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ generates $\mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$; that is, $\mathbf{C}\mathrm{P}^{n-1}$ is a generating complex for $\mathrm{Gr}_{\mathrm{SL}_n}$. The formula for the coproduct comes from the coproduct on $\mathrm{H}_*(\mathbf{C}\mathrm{P}^{n-1}; \mathbf{Q})$, which is determined easily by the cup product on $\mathrm{H}^*(\mathbf{C}\mathrm{P}^{n-1}; \mathbf{Q})$. The above description of $\mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ implies that $\mathrm{Spec}\, \mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ is isomorphic to the group scheme $\mathbf{W}_{n-1}$ of big Witt vectors of length $n-1$.

On the other hand, Theorem 3.6.3 implies that $\mathrm{Spec}\, \mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ is isomorphic to the centralizer inside PGL_n of of the regular nilpotent $f \in \mathfrak{sl}_n$. Indeed, if R is a $\mathbf{Q}$-algebra, then an element $g \in \mathrm{GL}_n(\mathrm{R})$ commutes with f if and only if g is an invertible polynomial in e. By the Cayley-Hamilton theorem, such a polynomial is divisible by the minimal polynomial t^n of e; that is, $g \in (\mathrm{R}[t]/t^n)^\times$. For this to live in $\mathrm{PGL}_n(\mathrm{R})$, we need to quotient out by the scalars $\mathrm{R}^\times$. The assignment $\mathrm{R} \mapsto (1+t\mathrm{R}[t]/t^n)^\times$ is precisely the functor of points of $\mathbf{W}_{n-1}$. One can therefore understand the isomorphism between $\mathrm{Spec}\, \mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ and $\mathrm{Z}_{\mathrm{PGL}_n}(e)$ as being a way to identify the two descriptions of the Witt vector group scheme (either via its functor of points, or via the explicit Witt addition law).

Example 3.6.11. Continuing the preceding example (so $\mathrm{G} = \mathrm{SL}_n$), it is not hard to add in torus-equivariance (so $\mathrm{T}_c = (\mathrm{S}^1)^{n-1}$). In this case, we will identify $\mathrm{H}^*_{\mathrm{T}_c}(*; \mathbf{Q}) \cong \mathbf{Q}[x_1, \cdots, x_n]$. One can write down an explicit T_c-equivariant cell structure on $\Omega\mathrm{SU}(n)$ to find that $\mathrm{Spec}\, \mathrm{H}^{\mathrm{T}_c}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ is isomorphic to the deformation of $\mathbf{W}_{n-1}$ over $\mathrm{Spec}\, \mathrm{H}^*_{\mathrm{T}_c}(*; \mathbf{Q}) \cong \mathbf{A}^n$ which sends a $\mathbf{Q}[x_1, \cdots, x_n]$-algebra R to the group of units $(1+t\mathrm{R}[t]/(t-x_1)\cdots(t-x_n))^\times$. On the other hand, by the same argument using Cayley-Hamilton, the centralizer inside $\mathrm{GL}_n(\mathrm{R})$ of $f+x \in \mathfrak{sl}_n$ is isomorphic to the group $(\mathrm{R}[t]/(t-x_1)\cdots(t-x_n))^\times$, since the characteristic polynomial of $f+x$ is precisely $(t-x_1)\cdots(t-x_n)$. Quotienting by the scalars $\mathrm{R}^\times$, we obtain an isomorphism between $\mathrm{Spec}\, \mathrm{H}^{\mathrm{T}_c}_*(\mathrm{Gr}_{\mathrm{SL}_n}; \mathbf{Q})$ and $\mathrm{Z}_{\mathrm{PGL}_n}(f+x)$.

Remark 3.6.12. For a general reductive group G, Kostant proved (in [Kos2]) an isomorphism $(f+\mathfrak{b})/\check{\mathrm{N}} \cong \mathfrak{t}/\!\!/\mathrm{W}$. In fact, the natural map $(f+\mathfrak{b})/\check{\mathrm{N}} \to \mathfrak{g}/\check{\mathrm{G}}$ identifies with the map $\mathcal{S} \to \mathfrak{g}/\check{\mathrm{G}}$ given by the Kostant slice. Since $(f+\mathfrak{b})/\check{\mathrm{N}}$ is isomorphic to the quotient $\check{\mathrm{G}}\backslash \mathrm{T}^*(\check{\mathrm{G}}/_\psi \check{\mathrm{N}})$ of the

Whittaker reduction of $\mathrm{T}^*(\check{\mathrm{G}})$, it follows that there are isomorphisms

$$\begin{aligned}\check{\mathrm{J}} &\cong (f+\mathfrak{b})/\check{\mathrm{N}} \times_{\mathfrak{g}/\check{\mathrm{G}}} (f+\mathfrak{b})/\check{\mathrm{N}} \\ &\cong \check{\mathrm{G}}\backslash \mathrm{T}^*(\check{\mathrm{G}}/_\psi \check{\mathrm{N}}) \times_{\check{\mathfrak{g}}^*/\check{\mathrm{G}}} \mathrm{T}^*(\check{\mathrm{N}}_\psi\backslash\check{\mathrm{G}})/\check{\mathrm{G}} \\ &\cong \mathrm{T}^*(\check{\mathrm{N}}_\psi\backslash\check{\mathrm{G}}/_\psi\check{\mathrm{N}}).\end{aligned}$$

That is, $\check{\mathrm{J}}$ can be identified with the *bi-Whittaker reduction* of the cotangent bundle $\mathrm{T}^*(\check{\mathrm{G}})$. In particular, Theorem 3.6.3 gives an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; \mathbf{Q}) \cong \check{\mathfrak{t}}^* \times_{\check{\mathfrak{t}}^*/\!\!/\mathrm{W}} \mathrm{T}^*(\check{\mathrm{N}}_\psi\backslash\check{\mathrm{G}}/_\psi\check{\mathrm{N}}).$$

In fact, this isomorphism can be checked to be W-equivariant (for the action of W on $\mathrm{H}_*^{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; \mathbf{Q})$ via the action on T, and for the action on the right-hand side coming from the cover $\check{\mathfrak{t}}^* \to \check{\mathfrak{t}}^*/\!\!/\mathrm{W}$). This implies that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{G}}(\mathrm{Gr}_\mathrm{G}; \mathbf{Q}) \cong \mathrm{T}^*(\check{\mathrm{N}}_\psi\backslash\check{\mathrm{G}}/_\psi\check{\mathrm{N}}).$$

This isomorphism has been exploited heavily in [Tel1], among others.

The map $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}} \to \mathrm{B}\check{\mathrm{G}}$ defines a functor

$$\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k). \tag{3.6.3}$$

More generally, the map $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}} \to \mathrm{B}\check{\mathrm{T}} \times \mathrm{B}\check{\mathrm{G}}$ defines a functor

$$\mathrm{Rep}(\check{\mathrm{T}} \times \check{\mathrm{G}}) \to \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k). \tag{3.6.4}$$

If $\mathrm{V} \in \mathrm{Rep}(\check{\mathrm{G}})$, let $\mathcal{S}_k(\mathrm{V})$ denote the corresponding object of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k)$. It is natural to ask whether $\mathcal{S}_k(\mathrm{V}) \in \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k)$ is given by $\mathcal{F}^{\mathrm{gr}}$ for some $\mathcal{F} \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k)$. Of course, Corollary 3.1.11 says that the answer is yes; but it is not clear how to answer this question in a manner that will generalize to other $\mathbf{E}_\infty$-rings k. However, it is possible to give a positive (and generalizable) answer to this question in the case when V is a direct sum of tensor products of irreducible representations with *minuscule* highest weights.

Proposition 3.6.13. *Let $\lambda_\bullet = (\lambda_1, \cdots, \lambda_n)$ be a tuple of dominant minuscule weights of $\check{\mathrm{G}}$, let $|\lambda_\bullet| = \sum_i \lambda_i$, and let $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ denote the corresponding convolution variety [MV, NP1]. Let $\mathcal{F}_{\lambda_\bullet}$ denote the pushforward of the constant sheaf along the canonical map $q : \overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}} \to \overline{\mathrm{Gr}_\mathrm{G}^{|\lambda|}} \subseteq \mathrm{Gr}_\mathrm{G}$. If V_{λ_i} denotes the irreducible representation of $\check{\mathrm{G}}$ with highest weight λ_i, then there is an isomorphism $\mathcal{S}_k(\bigotimes_i \mathrm{V}_{\lambda_i}) \cong \mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$.*

Proof. First, suppose that $\lambda_\bullet = \lambda$ consists of single element. Let $\mathrm{P}_\lambda \subseteq \mathrm{G}$ denote the corresponding maximal parabolic subgroup, so that $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda}} \cong \mathrm{G}/\mathrm{P}_\lambda$, and let $\mathcal{F}_\lambda \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G}; k)$ denote the pushforward of the constant sheaf along the inclusion $\mathrm{G}/\mathrm{P}_\lambda \hookrightarrow \mathrm{Gr}_\mathrm{G}$. We then need to show that there is an isomorphism $\mathcal{S}_k(\mathrm{V}_\lambda) \cong \mathcal{F}^{\mathrm{gr}}_\lambda$.

Since V_λ is an $\check{\mathrm{G}}$-representation, the tensor product $\mathrm{V}_\lambda \otimes_\mathbf{Q} \mathcal{O}_\mathfrak{t}$ is a comodule over $\mathcal{O}_{\check{\mathrm{G}}\times\mathfrak{t}}$. In particular, it is a comodule over $\mathcal{O}_{\tilde{\check{\mathrm{J}}}}$ via the closed immersion $\tilde{\check{\mathrm{J}}} \hookrightarrow \check{\mathrm{G}} \times \mathfrak{t}$. It follows from Corollary 3.6.8 that we need to show that $\mathrm{V}_\lambda \otimes_\mathbf{Q} \mathcal{O}_\mathfrak{t}$ is isomorphic to $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G}/\mathrm{P}_\lambda)$ as $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{Gr}_\mathrm{G})^\vee \cong \mathcal{O}_{\tilde{\check{\mathrm{J}}}}$-comodules.

Let $\mathfrak{t}^{\mathrm{gen}}$ denote the complement of $\bigcup_\alpha \mathfrak{t}_\alpha$ as α ranges over the roots of $\check{\mathrm{G}}$, and $\mathfrak{t}_\alpha$ denotes the hyperplane cut out by α. Since $\mathrm{V}_\lambda \otimes_{\mathbf{Q}} \mathcal{O}_{\mathfrak{t}}$, $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$, and $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ are all flat over $\mathfrak{t}$, it suffices to prove that there is an isomorphism $\mathrm{V}_\lambda \otimes_{\mathbf{Q}} \mathcal{O}_{\mathfrak{t}} \cong \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$ of quasicoherent sheaves over $\mathfrak{t}$, and further show that they are isomorphic as $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong \mathcal{O}_{\tilde{\mathrm{J}}}$-comodules when restricted to $\mathfrak{t}^{\mathrm{gen}}$.

Let W_λ denote the Weyl group of P_λ, so that W_λ is the stabilizer of the weight λ. Since $\mathrm{G}/\mathrm{P}_\lambda$ has even cells, $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$ is a vector bundle over $\mathcal{O}_{\mathfrak{t}}$, and its rank can be determined by its restriction to $\mathfrak{t}^{\mathrm{gen}}$. By Atiyah-Bott localization, $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)|_{\mathfrak{t}^{\mathrm{gen}}} \cong \pi_0\mathcal{F}_{\mathrm{T}}((\mathrm{G}/\mathrm{P}_\lambda)^{\mathrm{T}})|_{\mathfrak{t}^{\mathrm{gen}}}$; but $(\mathrm{G}/\mathrm{P}_\lambda)^{\mathrm{T}} = \mathrm{W}/\mathrm{W}_\lambda$, so we conclude that $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$ is a free $\mathcal{O}_{\mathfrak{t}}$-module of rank $|\mathrm{W}/\mathrm{W}_\lambda|$. Since λ is minuscule, there is an isomorphism $\mathrm{V}_\lambda \cong \mathrm{Map}(\mathrm{W}/\mathrm{W}_\lambda, \mathbf{Q})$ (see, e.g., [Gro, Proposition 5.1]). We therefore conclude that there is an isomorphism $\mathrm{V}_\lambda \otimes_{\mathbf{Q}} \mathcal{O}_{\mathfrak{t}} \cong \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$ of quasicoherent sheaves over $\mathfrak{t}$.

To see that they are isomorphic as $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong \mathcal{O}_{\tilde{\mathrm{J}}}$-comodules when restricted to $\mathfrak{t}^{\mathrm{gen}}$, note that $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee|_{\mathfrak{t}^{\mathrm{gen}}} \cong \mathcal{O}_{\check{\mathrm{T}}\times\mathfrak{t}^{\mathrm{gen}}}$. We therefore need to check that the weights of $\check{\mathrm{T}}$ acting on V_λ and $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)|_{\mathfrak{t}^{\mathrm{gen}}}$ agree. The T-fixed points of the map $\mathrm{G}/\mathrm{P}_\lambda \to \mathrm{Gr}_{\mathrm{G}}$ is given by the map $\mathrm{W}/\mathrm{W}_\lambda \to \mathrm{Gr}_{\mathrm{T}} \cong \mathbb{X}_*(\mathrm{T})$ which is the inclusion of the W-orbit of λ; these are the weights of $\check{\mathrm{T}}$ acting on $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)|_{\mathfrak{t}^{\mathrm{gen}}}$. The desired isomorphism of $\check{\mathrm{T}}$-representations between V_λ and $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)|_{\mathfrak{t}^{\mathrm{gen}}}$ now follows from the fact that the weights of $\check{\mathrm{T}}$ on V_λ are also precisely the elements in the W-orbit of λ.

Suppose that $\lambda_\bullet$ has more than one element. Since the equivalence of Corollary 3.6.8 is symmetric monoidal, we find that $\mathcal{S}_k(\bigotimes_i \mathrm{V}_{\lambda_i})$ is equivalent to the convolution tensor product $\mathcal{F}^{\mathrm{gr}}_{\lambda_1} \star \cdots \star \mathcal{F}^{\mathrm{gr}}_{\lambda_n}$. We therefore need to show that there is an isomorphism $\mathcal{F}^{\mathrm{gr}}_{\lambda_1} \star \cdots \star \mathcal{F}^{\mathrm{gr}}_{\lambda_n} \cong \mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$. For this, note that the following diagram of homotopy types commutes:

$$\begin{tikzcd}
\overline{\mathrm{Gr}_{\mathrm{G}}^{\lambda_\bullet}} \arrow[r] \arrow[d] \arrow[dr, "q"] & \overline{\mathrm{Gr}_{\mathrm{G}}^{|\lambda|}} \arrow[d] \\
\mathrm{Gr}_{\mathrm{G}}^{\times n} \arrow[r] & \mathrm{Gr}_{\mathrm{G}};
\end{tikzcd}$$

here, the bottom horizontal map is the $\mathbf{E}_2$-multiplication on $\mathrm{Gr}_{\mathrm{G}} \simeq \Omega\mathrm{G}_c$. This implies that $\mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$ is isomorphic to the pushforward of $(\mathcal{F}_{\lambda_1} \boxtimes \cdots \boxtimes \mathcal{F}_{\lambda_n})^{\mathrm{gr}} \cong \mathcal{F}^{\mathrm{gr}}_{\lambda_1} \boxtimes \cdots \boxtimes \mathcal{F}^{\mathrm{gr}}_{\lambda_n}$ along the map $\mathrm{Gr}_{\mathrm{G}}^{\times n} \to \mathrm{Gr}_{\mathrm{G}}$; but this is precisely the definition of $\mathcal{F}^{\mathrm{gr}}_{\lambda_1} \star \cdots \star \mathcal{F}^{\mathrm{gr}}_{\lambda_n}$, as desired. $\square$

To convince homotopy theorists that the flag varieties appearing in Proposition 3.6.13 are in fact (relatively) familiar objects, we have recorded the list of such $\mathrm{G}/\mathrm{P}_\lambda$ for dominant minuscule λ (even in the non-simply-laced cases) in Table 3.1.[12]

Remark 3.6.14. Let λ be a dominant minuscule weight of $\check{\mathrm{G}}$. The coaction of $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong \mathrm{H}^{\mathrm{T}}_*(\mathrm{Gr}_{\mathrm{G}}; \mathbf{Q})$ on $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda) \cong \mathrm{H}^*_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda; \mathbf{Q})$ defines a homomorphism

$$\mathrm{Spec}\, \pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \to \mathrm{GL}(\mathrm{H}^*_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda; \mathbf{Q})) \tag{3.6.5}$$

of group schemes over $\mathfrak{t}$, where $\mathrm{GL}(\mathrm{H}^*_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda; \mathbf{Q}))$ denotes the group scheme of $\mathcal{O}_{\mathfrak{t}}$-linear automorphisms of the vector bundle $\mathrm{H}^*_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda; \mathbf{Q})$. By Theorem 3.6.3 and Proposition 3.6.13, this homomorphism factors as the composite

$$\tilde{\tilde{\mathrm{J}}} \to \check{\mathrm{G}} \times \mathfrak{t} \to \mathrm{GL}(\mathrm{V}_\lambda) \times \mathfrak{t}, \tag{3.6.6}$$

[12] Those homotopy theorists who have reached this far in the article may not need this table to be convinced!

G	G/P$_\lambda$	$\check{\mathrm{G}} \circlearrowleft \mathrm{V}_\lambda$	$\dim_{\mathbf{C}}(\mathrm{G}/\mathrm{P}_\lambda)$	$\lvert \mathrm{W}/\mathrm{W}_\lambda \rvert$
A_n	$\mathrm{Gr}_j(\mathbf{C}^{n+1})$, $1 \leq j \leq n$	$\wedge^j \mathrm{std}_{n+1}$	$j(n+1-j)$	$\binom{n+1}{j}$
B_n	Smooth quadric in $\mathbf{C}\mathrm{P}^n$	std_{2n}	$2n-1$	$2n$
C_n	Lagrangian Grassmannian $\mathrm{LGr}_n(\mathbf{C}^{2n})$	Spin	$\binom{n+1}{2}$	2^n
D_n	Smooth quadric in $\mathbf{C}\mathrm{P}^{n-1}$	std_{2n}	$2n-2$	$2n$
D_n	Orthogonal Grassmannian $\mathrm{OGr}_n(\mathbf{C}^{2n})$	Half-spin (both)	$\binom{n}{2}$	2^{n-1}
E_6	$\mathrm{EIII} \simeq (\mathrm{E}_6)_c/\mathrm{Spin}(10)\cdot \mathrm{U}(1)$	$\mathrm{std}_{27}, \mathrm{std}_{27}^*$	16	27
E_7	$\mathrm{EVII} \simeq (\mathrm{E}_7)_c/(\mathrm{E}_6)_c\cdot \mathrm{U}(1)$	std_{56}	27	56

Table 3.1: Minuscule homogeneous varieties for G of adjoint type. Here, $(\mathrm{E}_6)_c$ and $(\mathrm{E}_7)_c$ denote the compact forms of E_6 and E_7, respectively. The labelings EIII and EVII denote the labelings of these symmetric spaces in É. Cartan's classification. In the example of D_n acting on the orthogonal Grassmannian, there are two realizations as a homogeneous variety, which correspond to the two half-spin representations: namely, $\mathrm{SO}_{2n}/\mathrm{P}_{\alpha_{n-1}} \cong \mathrm{SO}_{2n}/\mathrm{P}_{\alpha_n}$. Note, also, that $\lvert \mathrm{W}/\mathrm{W}_\lambda \rvert$ is equal to the dimension of V_λ and also to the number of cells in a minimal (T-equivariant) cell structure on $\mathrm{G}/\mathrm{P}_\lambda$, while $\dim_{\mathbf{C}}(\mathrm{G}/\mathrm{P}_\lambda)$ is the highest weight of the restriction of $\check{\mathrm{G}} \to \mathrm{GL}(\mathrm{V}_\lambda)$ to the principal SL_2 inside $\check{\mathrm{G}}$.

where the second map describes the $\check{\mathrm{G}}$-action on V_λ. Similarly, the coaction of $\pi_0\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong \mathrm{H}_*^{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q})$ on $\pi_0\mathcal{F}_{\mathrm{G}}(\mathrm{G}/\mathrm{P}_\lambda) \cong \mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Q})$ defines a homomorphism

$$\operatorname{Spec} \pi_0\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee \to \mathrm{GL}(\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Q})) \tag{3.6.7}$$

of group schemes over $\operatorname{Spec} \mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Q}) \cong \mathfrak{t}/\!\!/\mathrm{W}$. As an $\mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}}$-module, $\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Q})$ is isomorphic to $\mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}} \otimes \mathrm{V}_\lambda$, and (3.6.7) factors as the composite

$$\check{\mathrm{J}} \to \check{\mathrm{G}} \times \mathfrak{t}/\!\!/\mathrm{W} \to \mathrm{GL}(\mathrm{V}_\lambda) \times \mathfrak{t}/\!\!/\mathrm{W}. \tag{3.6.8}$$

In fact, all of these maps already exist *integrally* (i.e., using cohomology with integral coefficients, and working with group schemes over $\mathbf{Z}$).

For instance, suppose $\mathrm{G} = \mathrm{SO}_{2n}$ is of type D_n, and let us take coefficients in $\mathbf{Z}' = \mathbf{Z}[1/2]$; otherwise, the cohomology of BG is not isomorphic to $\mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}}$. (See [Dev3, Example 3.2.14] for other classical types.) Then the Levi quotient of P_λ is $\mathrm{SO}_2 \times \mathrm{SO}_{2n-2}$, so that $\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}') \cong \mathbf{Z}'[x, p_1', \cdots, p_{n-2}', c_{n-1}']$ with x in weight 2, p_i' in weight $-4i$, and c_i' in weight $-2i$. A simple calculation with symmetric polynomials shows that as an algebra over $\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}') \cong \mathbf{Z}'[p_1, \cdots, p_{n-1}, c_n]$, there is an isomorphism

$$\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}') \cong \mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')[x, c_{n-1}']/(xc_{n-1}' = c_n,\ x^{2n-2} - p_1 x^{2n-4} - \cdots - p_{n-1} + {c_{n-1}'}^2).$$

As an $\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')$-module, this is indeed isomorphic to $\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}') \otimes \mathrm{std}_{2n}$. Building on Example 3.6.11 shows that as a group scheme over $\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')$, the functor of points of $\check{\mathrm{J}}$ sends an $\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')$-algebra R to the subgroup of those units $f(x, c_{n-1}') \in (\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}') \otimes_{\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')} \mathrm{R})^\times$ such that $f(x, c_{n-1}')^{-1} = f(-x, -c_{n-1}')$. The action of $\check{\mathrm{J}}$ on $\mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}')$ preserves the symmetric bilinear form given by

$$\begin{aligned} \mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}') \otimes_{\mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}')} \mathrm{H}^*_{\mathrm{P}_\lambda}(*;\mathbf{Z}') &\xrightarrow{\langle -,- \rangle} \mathrm{H}^*_{\mathrm{G}}(*;\mathbf{Z}') \\ f, g &\mapsto \text{coefficient of } x^{2n-2} \text{ in } f(x, c_{n-1}')g(-x, -c_{n-1}'). \end{aligned}$$

Geometrically, this bilinear form comes from G-equivariant Poincaré duality on G/P_λ, twisted by the natural action of $\mathbf{Z}/2$ on G/P_λ. (This $\mathbf{Z}/2$ acts on $H^*_{P_\lambda}(*;\mathbf{Z}')$ by sending $x \mapsto -x$ and $c'_{n-1} \mapsto -c'_{n-1}$.) The bilinear form $\langle -, - \rangle$ on $H^*_{P_\lambda}(*;\mathbf{Z}')$ gives the desired factorization (3.6.8) of the map $\check{J} \to GL_{2n} \times \mathfrak{t}/\!\!/W$ through the inclusion $SO_{2n} \times \mathfrak{t}/\!\!/W \hookrightarrow GL_{2n} \times \mathfrak{t}/\!\!/W$.

As we have seen, the calculation of Theorem 3.6.3 is quite powerful. Here is another simple application, motivated by [GK] and [GR3]; see also [Dev3, Example 3.6.13], where the same example is presented.

Proposition 3.6.15 (Gelfand-Graev action)**.** *The natural action of $\check{G}\times\check{T}$ on the affine closure $\overline{T^*(\check{G}/\check{N})}$ extends to an action of $\check{G} \times (W \ltimes \check{T})$, where W is the Weyl group.*

Proof. Let $T^*(\check{G}/\check{N})_{\mathrm{reg}} = \check{G} \times^{\check{N}} \check{\mathfrak{n}}^\perp_{\mathrm{reg}}$ denote the regular locus in $\overline{T^*(\check{G}/\check{N})}$; then $T^*(\check{G}/\check{N})_{\mathrm{reg}} \subseteq T^*(\check{G}/\check{N})$ is open, with complement of codimension 2, so that $\overline{T^*(\check{G}/\check{N})} \cong \overline{T^*(\check{G}/\check{N})_{\mathrm{reg}}}$. Note that there is an isomorphism

$$\check{G}\backslash T^*(\check{G}/\check{N})_{\mathrm{reg}}/\check{T} \cong \check{\mathfrak{n}}^\perp_{\mathrm{reg}}/\check{B},$$

so (the proof of) Corollary 3.6.8 gives isomorphisms

$$T^*(\check{G}/\check{N})_{\mathrm{reg}} \cong (\check{G} \times \check{T}) \times^{\check{B}} \check{\mathfrak{n}}^\perp_{\mathrm{reg}} \cong (\check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*))/\tilde{\check{J}}. \tag{3.6.9}$$

There is a canonical W-action on $\check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*)$, given by the natural W-actions on $\check{T}$ and on $\psi + \check{\mathfrak{t}}^* \cong \check{\mathfrak{t}}^*$. Similarly, $\tilde{\check{J}}$ also admits a natural W-action; it is given via Theorem 3.6.3 by the natural W-action on $H^{T_c}_*(\mathrm{Gr}_G;\mathbf{Q})$. Moreover, the closed immersion

$$\tilde{\check{J}} \to \check{G} \times \check{T} \times (\psi + \check{\mathfrak{t}}^*)$$

is W-equivariant (indeed, the map $\tilde{\check{J}} \to \check{T} \times (\psi + \check{\mathfrak{t}}^*)$ is induced by the inclusion $H^{T_c}_*(\mathrm{Gr}_T;\mathbf{Q}) \to H^{T_c}_*(\mathrm{Gr}_G;\mathbf{Q})$ on equivariant homology). This implies that the quotient of (3.6.9) admits a W-action, which defines a W-action on the affine closure of $T^*(\check{G}/\check{N})_{\mathrm{reg}}$ as desired. □

Note that we assumed in Corollary 3.6.8 that G is simply-laced; but this is not necessary, because we know (by the discussion in § 3.1) that the main result of [ABG] implies Corollary 3.6.8 is true for any connected reductive G. Alternatively, one can observe that the proof of Corollary 3.6.8 itself never seriously appeals to G being simply-laced.

Remark 3.6.16. The proof of Proposition 3.6.15 generalizes to show that if $\check{P} \subseteq \check{G}$ is a parabolic subgroup with Levi quotient $\check{L}$ and unipotent radical $U_{\check{P}}$, then the natural action of $\check{G} \times \check{L}$ on the affine closure $\overline{T^*(\check{G}/U_{\check{P}})}$ extends to an action of $\check{G} \times (W_L \ltimes \check{L})$, where $W_L = N_{\check{G}}(\check{L})/\check{L}$ is the Weyl group. (Also see [Gan3].)

The W-action of Proposition 3.6.15 is known as the (semiclassical) *Gelfand-Graev action.* The moment map $\overline{T^*(\check{G}/\check{N})} \to \check{\mathfrak{g}}^*$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd} \tilde{\check{\mathfrak{g}}} \arrow[r, hook] \arrow[dr] & \overline{T^*(\check{G}/\check{N})}/\check{T} \arrow[d] \\ & \check{\mathfrak{g}}^* \end{tikzcd}$$

which relates $\overline{T^*(\check{G}/\check{N})}$ to the Grothendieck-Springer resolution; and via this diagram, the Gelfand-Graev action is closely related to the Weyl action in Springer theory.

Example 3.6.17. When $\check{\mathrm{G}} = \mathrm{SL}_2$, the affine closure $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$ is simply $\mathrm{T}^*(\mathbf{A}^2)$, and the $\mathrm{W} = \mathbf{Z}/2$-action on it is given by the symplectic Fourier transform. To see this, let $\check{\mathrm{J}}_{\mathrm{X}}$ denote the kernel of the homomorphism $\widetilde{\check{\mathrm{J}}} \to \check{\mathrm{T}} \times (\psi + \check{\mathfrak{t}}^*)$ of group schemes over $\psi + \check{\mathfrak{t}}^*$. (This follows the notation from [Dev3].) Then (3.6.9) gives an isomorphism

$$(\check{\mathrm{G}} \times (\psi + \check{\mathfrak{t}}^*))/\check{\mathrm{J}}_{\mathrm{X}} \xrightarrow{\cong} \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})_{\mathrm{reg}}.$$

In the case at hand, $\psi + \check{\mathfrak{t}}^* \cong \mathbf{A}^1$ with coordinate x, and the group scheme $\check{\mathrm{J}}_{\mathrm{X}}$ is just $\operatorname{Spec} \mathbf{Z}[x, b]/bx$ (where the group law sends $b \mapsto b \otimes 1 + 1 \otimes b$). The above isomorphism defines a map

$$q : \mathrm{SL}_2 \times \mathbf{A}^1 \to \overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})} = \mathrm{T}^*(\mathbf{A}^2),$$

and the affine closure of the image is all of $\mathrm{T}^*(\mathbf{A}^2)$. The map q can be explicitly described as follows. View a point of $\mathrm{T}^*(\mathbf{A}^2)$ as a pair $((\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}), (v_1, v_2))$ of a vector and a covector. Then q is the natural extension to $\mathrm{SL}_2 \times \mathbf{A}^1$ of the map $\kappa : \mathbf{A}^1 \to \mathrm{T}^*(\mathbf{A}^2)$ which sends $x \mapsto ((\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (x, 0))$. In other words, q sends

$$(g, x) = \left((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), x\right) \mapsto \left(g(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (g^{\mathrm{T}})^{-1}(x, 0)\right) = ((\begin{smallmatrix} a \\ c \end{smallmatrix}), (dx, -bx)).$$

Of course, one could also swap the roles of $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$ in $\mathrm{T}^*(\mathbf{A}^2)$; the map κ would then send $x \mapsto ((\begin{smallmatrix} 0 \\ x \end{smallmatrix}), (0, 1))$, and q would send

$$(g, x) = \left((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), x\right) \mapsto \left((\begin{smallmatrix} 0 \\ x \end{smallmatrix}) \cdot g^{\mathrm{T}}, (0, 1) \cdot g^{-1}\right) = \left((\begin{smallmatrix} bx \\ dx \end{smallmatrix}), (-c, a)\right).$$

If we compose with the involution sending $x \mapsto -x$, the resulting involution

$$((\begin{smallmatrix} a \\ c \end{smallmatrix}), (dx, -bx)) \mapsto \left((\begin{smallmatrix} -bx \\ -dx \end{smallmatrix}), (-c, a)\right).$$

This, of course, is precisely the symplectic Fourier transform, which sends

$$((\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}), (v_1, v_2)) \mapsto ((\begin{smallmatrix} v_2 \\ -v_1 \end{smallmatrix}), (-u_2, u_1)).$$

We also have the following, which is an obvious consequence of Corollary 3.1.7 and Corollary 3.1.11 (but can be reproved using Corollary 3.6.8).

Proposition 3.6.18. *Let* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^{\heartsuit}$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathfrak{t}))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathfrak{t})$*. Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\check{\mathrm{G}} \backslash \overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of the functor (3.6.4)*. Furthermore, this functor is* W*-equivariant for the natural action of* $\mathrm{W} = \mathrm{N}_{\mathrm{G}_c}(\mathrm{T}_c)/\mathrm{T}_c$ *on the left-hand side and the Gelfand-Graev action of Proposition 3.6.15 on the right-hand side.*

Similarly, let $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^{\heartsuit}$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W}))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W})$*. Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{\mathrm{G}})$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of the functor $\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ *(analogous to* (3.6.3)*).*

Proof. If $V \in \mathrm{Rep}(\check{G})$, the object $\mathcal{S}_k(V)$ lies in $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)^{\heartsuit}$, and we need to show that there is an isomorphism

$$\mathrm{Map}_{\mathrm{QCoh}(\check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T})^{\heartsuit}}(V_1 \otimes_{\mathbf{Q}} \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}, V_2 \otimes_{\mathbf{Q}} \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}) \xrightarrow{\cong} \mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)^{\heartsuit}}(\mathcal{S}_k(V_1), \mathcal{S}_k(W_1))$$

of $\mathbf{Q}$-vector spaces for any two representations $V_1, V_2 \in \mathrm{Rep}(\check{G})$. In other words, By Corollary 3.6.8, there is an isomorphism

$$\mathrm{Map}_{\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G;k)^{\heartsuit}}(\mathcal{S}_k(V), \mathcal{S}_k(W)) \cong \mathrm{Map}_{\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G})^{\heartsuit}}(V \otimes_{\mathbf{Q}} \mathcal{O}_{\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}}, W \otimes_{\mathbf{Q}} \mathcal{O}_{\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}}).$$

Since $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G} \hookrightarrow \check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T}$ has complement of codimension 2 and $\overline{T^*(\check{G}/\check{N})}$ is affine and normal, the algebraic Hartogs lemma implies that the restriction map

$$\mathrm{Map}_{\mathrm{QCoh}(\check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T})^{\heartsuit}}(V \otimes_{\mathbf{Q}} \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}, W \otimes_{\mathbf{Q}} \mathcal{O}_{\overline{T^*(\check{G}/\check{N})}}) \to \mathrm{Map}_{\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G})^{\heartsuit}}(V \otimes_{\mathbf{Q}} \mathcal{O}_{\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}}, W \otimes_{\mathbf{Q}} \mathcal{O}_{\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}})$$

is an isomorphism, as desired. (This is where it is crucial that we work at the level of abelian categories.) The same argument works for $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)$; in this case, $\check{\mathfrak{g}}^{*,\mathrm{reg}} \hookrightarrow \check{\mathfrak{g}}^*$ even has complement of codimension 3. □

Proposition 3.6.18 gives an analogue of [BF, Theorem 4]: namely, if $\mathrm{QCoh}_{\mathrm{free}}(\check{\mathfrak{g}}^*/\check{G})$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\check{\mathfrak{g}}^*/\check{G})^{\heartsuit} \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)^{\heartsuit}.$$

Similarly, if $\mathrm{QCoh}_{\mathrm{free}}(\check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T})$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{G} \times \check{T}) \to \mathrm{QCoh}(\check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T})$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\check{G}\backslash\overline{T^*(\check{G}/\check{N})}/\check{T})^{\heartsuit} \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)^{\heartsuit}.$$

Let $\mathrm{Rep}_{\min}(\check{G})$ denote the idempotent completion of the subcategory of $\mathrm{Rep}(\check{G})$ spanned by tensor products of irreducible $\check{G}$-representations with minuscule highest weights. In general, if $\check{G}$ is simple (but not necessarily simply-laced) and not of types G_2, F_4, or E_8, then any representation is a summand of a tensor product of irreducible $\check{G}$-representations with minuscule highest weights, so that $\mathrm{Rep}_{\min}(\check{G}) \simeq \mathrm{Rep}(\check{G})$.[13]

Corollary 3.6.19. *Let* $\mathrm{QCoh}_{\mathrm{free}}(\check{\mathfrak{g}}^*/\check{G})^{\min,\heartsuit}$ *denote the essential image of* $\mathrm{Rep}_{\min}(\check{G})$ *under the pullback functor* $\mathrm{Rep}(\check{G})^{\heartsuit} \to \mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})^{\heartsuit}$*. Similarly, let* $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)^{\min,\heartsuit}$ *denote the idempotent completion of the subcategory of* $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)^{\heartsuit}$ *spanned by* $\mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$ *ranging over sequences* $\lambda_\bullet$ *of minuscule highest weights. Then there is an equivalence*

$$\mathrm{QCoh}_{\mathrm{free}}(\check{\mathfrak{g}}^*/\check{G})^{\min,\heartsuit} \simeq \mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)^{\min,\heartsuit}.$$

[13]When $\check{G}$ is of types G_2, F_4, or E_8, there are no minuscule weights at all. In general, $\mathrm{Rep}(\check{G})$ is the idempotent completion of its full subcategory spanned by tensor products of irreducible $\check{G}$-representations with minuscule and quasi-minuscule highest weights.

There is a similar equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)^{\mathrm{min},\heartsuit} \simeq \mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})^{\mathrm{min},\heartsuit},$$

where these categories are defined analogously by idempotent completion.

Note that the category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr_G}; k)^{\mathrm{min},\heartsuit}$ is the heart of a degeneration, in the sense of § 3.3, of the similarly-defined category $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr_G}; k)^{\mathrm{min}}$. (In particular, Corollary 3.6.19 gives an equivalence between the purely algebraically defined category $\mathrm{QCoh}_{\mathrm{free}}(\check{\mathfrak{g}}^*/\check{\mathrm{G}})^{\mathrm{min},\heartsuit}$ and a degeneration of the purely topologically defined category $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr_G}; k)^{\mathrm{min}}$.) If $\lambda_\bullet$ and $\mu_\bullet$ are two sequences of dominant minuscule weights of $\check{\mathrm{G}}$, there is an equivalence of k-modules

$$\begin{aligned}\mathrm{Map}_{\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr_G};k)^{\mathrm{min}}}(\mathcal{F}_{\lambda_\bullet}, \mathcal{F}_{\mu_\bullet}) &\simeq \mathcal{F}_{\mathrm{G}}(\overline{\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}}) \otimes_{\mathrm{coMod}_{\mathcal{F}_{\mathrm{G}}(\mathrm{Gr_G})^\vee}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{G}}))} \mathcal{F}_{\mathrm{G}}(\overline{\mathrm{Gr}^{\mu_\bullet}_{\mathrm{G}}}) \\ &\simeq \mathcal{F}_{\mathrm{G}}(\overline{\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}} \times_{\mathrm{Gr_G}} \overline{\mathrm{Gr}^{\mu_\bullet}_{\mathrm{G}}}),\end{aligned}$$

where the final equivalence uses the Künneth formula at the level of k-cochains. The category $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr_G}; k)^{\mathrm{min}}$ therefore compares to (the k-analogue of) the category from [CK, Section 3.4].

At first glance, the existence of the t-structure on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ from Proposition 3.6.18 may perhaps be a bit surprising, since k is a 2-*periodic* $\mathbf{E}_\infty$-ring. In fact, this periodicity prohibits $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ itself from having a t-structure. However, the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ "flattens" out the homological periodicity in $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ to a weight periodicity, but it is itself also a stable ∞-category. In particular, it has a *homological* shift operation, which is distinct from the operation of shifting *weights* (just as with the usual category of mixed sheaves). (The 2-periodicity of k implies that the weight-shifting operation on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ is an equivalence, which is why we do not see gradings/weights when discussing $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$; but the weight-shifting operation will be nontrivial on, say, $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathbf{Q})$.) The resulting homological shift on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; k)$ is no longer periodic, and it is therefore reasonable to equip this ∞-category with a t-structure. (This t-structure is unrelated to the perverse t-structure on $\mathrm{Shv_I}(\mathrm{Gr_G}; \mathbf{Q})$.)

We will now discuss a *deformation quantization* of Corollary 3.6.8 by adding loop-rotation equivariance. Write $\widetilde{\mathrm{T}} = \mathrm{T} \times \mathbf{G}_m^{\mathrm{rot}}$ to denote the corresponding affine torus. In the case when G is a torus, we have already discussed this in § 3.5. For more general G, this turns out to be a bit tricky: while $\mathrm{H}_*^{\mathrm{T}_c}(\mathrm{Gr_G}; \mathbf{Q})$ is a bicommutative Hopf algebra[14], the loop-rotation equivariant homology $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Gr_G}; \mathbf{Q})$ is only a cocommutative coalgebroid over $\mathrm{H}^*_{\widetilde{\mathrm{T}}_c}(*; \mathbf{Q})$. That is, it does not admit an algebra structure. While this is not a mathematical issue, it does make the task of explicitly understanding $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Gr_G}; \mathbf{Q})$ in a satisfactory way more complicated. Instead, it turns out to be easier to describe $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Fl_G}; \mathbf{Q})$, where $\mathrm{Fl_G}$ is the *affine flag variety*, defined as the quotient $\mathrm{G}(\!(t)\!)/\mathrm{I}$ for the Iwahori subgroup $\mathrm{I} \subseteq \mathrm{G}[\![t]\!]$ associated to a Borel subgroup $\mathrm{B} \subseteq \mathrm{G}$. To state the result, we need a definition from [GKV2].

Definition 3.6.20. Let $(\Lambda, \Phi, \check{\Lambda}, \check{\Phi})$ be a root datum with associated Weyl group W and torus $\mathrm{T} = \mathrm{Hom}(\Lambda, \mathbf{G}_m)$. Let Δ be a base of simple roots, let Φ^+ denote the corresponding set of positive roots, and let Φ' denote the subset $\mathrm{W} \cdot \Delta \subseteq \Phi$. Let $\mathbf{H}$ be a 1-dimensional group scheme (over a commutative ring R). As in Definition 3.5.1, let $\mathbf{H}_{\mathrm{T}} = \mathrm{Hom}(\Lambda, \mathrm{T})$, and for each character $\lambda \in \Lambda$, let $\mathbf{H}_{\mathrm{T}_\lambda} \hookrightarrow \mathbf{H}_{\mathrm{T}}$ denote the subgroup corresponding to the subtorus $\mathrm{T}_\lambda = \ker(\lambda) \subseteq \mathrm{T}$. Let $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\mathrm{T}}})$ denote the sheaf of functions on the generic point of $\mathcal{O}_{\mathbf{H}_{\mathrm{T}}}$.

[14] To be more precise, the $\mathbf{E}_2$-space structure on $\mathrm{Gr_G}$ equips $\mathrm{C}_*^{\mathrm{T}_c}(\mathrm{Gr_G}; \mathbf{Q})$ with the structure of an $\mathbf{E}_2$-algebra in $\mathbf{E}_\infty$-coalgebras over $\mathrm{C}^*_{\mathrm{T}_c}(*; \mathbf{Q})$.

The twisted group algebra $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$ is the algebra which is additively given by the tensor product $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}}) \otimes_\mathrm{F} \mathrm{F}[\mathrm{W}]$, and whose multiplication law is given by

$$(f_1 \otimes w_1) \cdot (f_2 \otimes w_2) = (f \cdot w_1 g) \otimes (w_1 w_2).$$

The algebra $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ is defined to be the subset of $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$ of those elements $\sum_{w\in\mathrm{W}} f_w[w]$ such that:

- The poles of f_x all have order ≤ 1, and these are contained in the divisors $\mathbf{H}_{\mathrm{T}_\alpha}$ for each $\alpha \in \Phi'$.
- For each $w \in \mathrm{W}$ and $\alpha \in \Phi^+ \cap \Phi'$, we have

$$\mathrm{Res}_{\mathbf{H}_{\mathrm{T}_\alpha}}(f_w) + \mathrm{Res}_{\mathbf{H}_{\mathrm{T}_\alpha}}(f_{s_\alpha w}) = 0.$$

In [GKV2], this algebra is denoted $\widetilde{\mathbf{H}}$. It is proved in [GKV2, Theorem 1.4] that $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ is a *subalgebra* of $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$.

Remark 3.6.21. The pair $(\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}}), \mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}])$ admits the structure of a (cocommutative) Hopf algebroid; we will abusively say that $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$ admits the structure of a Hopf $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})$-algebroid. The coproduct comes from the diagonal on W; the left unit comes from the inclusion $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}}) \subseteq \mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$; and the right unit comes from the action of W on $\mathbf{H}_\mathrm{T}$ (which defines a coaction of W on $\mathcal{O}_{\mathbf{H}_\mathrm{T}}$ that extends to a coaction on $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})$). The resulting Hopf $\mathcal{O}_{\mathbf{H}_\mathrm{T}}$-algebroid structure on $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})[\mathrm{W}]$ restricts to $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$, so that $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ admits the structure of a (cocommutative) Hopf $\mathcal{O}_{\mathbf{H}_\mathrm{T}}$-algebroid. (See [Woj, Theorem 4.11] for the case $\mathbf{H} = \mathbf{G}_a$.)

When W is finite, [GKV2, Proposition 2.3] states that upon rationalization, the action of $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ on $\mathcal{O}_{\mathbf{H}_\mathrm{T}}$ gives an isomorphism between $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ and $\mathrm{End}_{\mathcal{O}_{\mathbf{H}_\mathrm{T}}^\mathrm{W}}(\mathcal{O}_{\mathbf{H}_\mathrm{T}})$. This gives a Morita equivalence between the category of $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$-modules and the category of $\mathcal{O}_{\mathbf{H}_\mathrm{T}}^\mathrm{W}$-modules. Under this equivalence, the symmetric monoidal structure on the category of $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$-modules from the cocommutative Hopf algebroid structure on $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ identifies with the standard symmetric monoidal structure on the category of $\mathcal{O}_{\mathbf{H}_\mathrm{T}}^\mathrm{W}$-modules.

If Λ denotes the *co*root lattice of G, let $\mathrm{W}^\mathrm{aff} = \Lambda \rtimes \mathrm{W}$ denote the corresponding affine Weyl group, and let $\widetilde{\mathrm{W}} = \mathbb{X}_*(\mathrm{T}) \rtimes \mathrm{W}$ denote the extended affine Weyl group.[15] For clarity, note that the action of $\widetilde{\mathrm{W}}$ on $\mathbb{X}^*(\mathrm{T})$ (and hence on $\mathbf{H}_\mathrm{T}$) is given as follows: if α is a coweight of T and $n \in \mathbf{Z}$, the generator $s_{\alpha,n}$ of W^aff acts on $\mathbb{X}^*(\mathrm{T})$ by reflection along the affine hyperplane $\{x \in \mathbb{X}^*(\mathrm{T}) | \langle x, \alpha\rangle = n\}$. The *degenerate nil-Hecke algebra* $\mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ is defined to be $\mathbb{X}_*(\mathrm{T}) \ltimes_\Lambda \mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \mathrm{W}^\mathrm{aff})$. In the following discussion, we will simply write $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$ to denote $\mathbb{X}_*(\mathrm{T}) \ltimes_\Lambda \mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\mathrm{W}^\mathrm{aff}]$, so that $\mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ is contained in $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$.

There is a natural inclusion $\mathcal{D}_{\widetilde{\mathrm{T}}}^{\mathbf{H}}[\mathrm{W}] \hookrightarrow \mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ of (sheaves of) algebras. The following result can be proved exactly as in [Gin3, Proposition 7.2.4]; one only has to use [GKV2, Proposition 2.3] in place of [Gin3, Lemma 7.1.5], and also observe that the arguments of [Lon1] generalize to the setting of descent along the map $\mathbf{H}_\mathrm{T}/\mathrm{W} \to \mathbf{H}_\mathrm{T}/\!\!/\mathrm{W}$.

[15] The affine Weyl group W^aff introduced above is very slightly different from the affine Weyl group studied in [Gin3, Section 7.2] or [Gan2]; the affine Weyl group there is the semidirect product $\mathrm{W}'^\mathrm{aff} = \check{\Lambda} \rtimes \mathrm{W}$, where $\check{\Lambda}$ is the *root* lattice of G. When G is simply-laced, these are, of course, isomorphic; but they differ otherwise.

Proposition 3.6.22. *Let $\mathcal{F}$ be $\mathcal{D}^{\mathbf{H}}_{\widetilde{\mathrm{T}}}[\mathrm{W}]$-module*[16]*. Then the action of $\mathcal{D}^{\mathbf{H}}_{\widetilde{\mathrm{T}}}[\mathrm{W}]$ extends (necessarily uniquely) along the inclusion $\mathcal{D}^{\mathbf{H}}_{\widetilde{\mathrm{T}}}[\mathrm{W}] \hookrightarrow \mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ if and only if the natural map $\mathcal{O}_{\mathbf{H}_\mathrm{T}} \otimes_{\mathcal{O}^{\mathrm{W}}_{\mathbf{H}_\mathrm{T}}} \mathcal{F}^{\mathrm{W}} \to \mathcal{F}$ is an isomorphism.*

Remark 3.6.23. Let us remark on a relationship to [Gan2]. Following *loc. cit.*, let $\Gamma_{\mathrm{W}^{\mathrm{aff}}}$ denote the ind-scheme given by the union of graphs of the affine Weyl group $\mathrm{W}^{\mathrm{aff}}$ acting on $\mathbf{H}_{\widetilde{\mathrm{T}}}$, and let $\Gamma_{\widetilde{\mathrm{W}}}$ denote $\widetilde{\mathrm{W}} \times^{\mathrm{W}^{\mathrm{aff}}} \mathbf{H}_{\widetilde{\mathrm{T}}}$. Then there are two projections $\Gamma_{\widetilde{\mathrm{W}}} \rightrightarrows \mathbf{H}_{\widetilde{\mathrm{T}}}$. This can be extended to a simplicial diagram $\Gamma_\bullet$ of ind-schemes. Define the stack $\mathbf{H}_{\widetilde{\mathrm{T}}} /\!\!/ \widetilde{\mathrm{W}}$ to be the geometric realization of $\Gamma_\bullet$. (For instance, if W is trivial, this is the quotient $\mathbf{H}_{\widetilde{\mathrm{T}}}/\mathbb{X}_*(\mathrm{T})$. Similarly, if $\mathbf{H} = \mathbf{G}_a$, so that $\mathbf{H}_{\widetilde{\mathrm{T}}} \cong \mathfrak{t} \oplus \mathbf{A}^1_\hbar$, then the specialization of the quotient $\mathbf{H}_{\widetilde{\mathrm{T}}} /\!\!/ \widetilde{\mathrm{W}}$ to $\hbar = 1$ agrees with the quotient $\mathfrak{t} /\!\!/ \widetilde{\mathrm{W}}$ from [Gan2].) In general, there is a map of stacks $\phi : \mathbf{H}_{\widetilde{\mathrm{T}}}/\widetilde{\mathrm{W}} \to \mathbf{H}_{\widetilde{\mathrm{T}}} /\!\!/ \widetilde{\mathrm{W}}$. By arguing exactly as in [Gan2, Theorem 4.23], one can show that the pullback functor $\phi^!$ is fully faithful; and furthermore, an object of $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{\mathrm{T}}}/\widetilde{\mathrm{W}})$ descends[17] along ϕ if and only if the corresponding object of $\mathrm{IndCoh}(\mathbf{H}_\mathrm{T}/\mathrm{W})$ descends to $\mathbf{H}_\mathrm{T} /\!\!/ \mathrm{W}$. Since Remark 3.5.2 gives an equivalence between $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{\mathrm{T}}}/\widetilde{\mathrm{W}})$ and $\mathcal{D}^{\mathbf{H}}_{\widetilde{\mathrm{T}}}[\mathrm{W}]$-mod, Proposition 3.6.22 can be used to obtain an equivalence between $\mathcal{H}(\mathbf{H}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$-mod and $\mathrm{IndCoh}(\mathbf{H}_{\widetilde{\mathrm{T}}} /\!\!/ \widetilde{\mathrm{W}})$.

Proposition 3.2.15 yields the following result due to Kostant and Kumar [KK2, KK1, Kum], which (as we will explain momentarily) could also be seen as a consequence of results from [Gin3, Lon2, BF]. In the discussion below, $\mathbf{H} = \mathbf{G}_a$. Note that $\mathrm{H}^*_{\widetilde{\mathrm{T}}_c}(*; \mathbf{Q})$ is isomorphic to $\mathcal{O}_{\widetilde{\mathfrak{t}}} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$.

Theorem 3.6.24. *There is an isomorphism of associative $\mathbf{Q}[\hbar]$-algebras*

$$\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}}). \tag{3.6.10}$$

Here, $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q})$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\mathrm{H}^*_{\widetilde{\mathrm{T}}_c}(*; \mathbf{Q}) \cong \mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$-algebroids.

Proof. The affine flag variety Fl_G is an ind-finite GKM space in the sense of Definition 3.2.12, and so we may use Proposition 3.2.15 to describe $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q})$. The GKM graph of Fl_G has set of vertices given by $\mathrm{Fl}_\mathrm{G}^{\widetilde{\mathrm{T}}_c} = \mathbb{X}_*(\mathrm{T}) \rtimes \mathrm{W} \cong \widetilde{\mathrm{W}}$, and an edge $w \to s_{\alpha,n} w$ for each affine reflection $s_{\alpha,n} \in \widetilde{\mathrm{W}}$. In particular, if $\mathring{\widetilde{\mathfrak{t}}}$ denotes the complement of the union of affine hyperplanes in $\widetilde{\mathfrak{t}}$, then $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q})$ is a subalgebra of $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}^{\widetilde{\mathrm{T}}_c}; \mathbf{Q})|_{\mathring{\widetilde{\mathfrak{t}}}}$. The latter is isomorphic to $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\widetilde{\mathrm{W}}; \mathbf{Q})|_{\mathring{\widetilde{\mathfrak{t}}}}$, which in turn can be identified (using Proposition 3.5.4, for instance) with a localization of $\mathcal{D}^\hbar_{\widetilde{\mathrm{T}}}[\mathrm{W}]$. This localization of $\mathcal{D}^\hbar_{\widetilde{\mathrm{T}}}[\mathrm{W}]$ is isomorphic to $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$, so $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q})$ is a subalgebra of $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$.

Proposition 3.2.15 now gives an isomorphism between the two subsets

$$\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q}) \subseteq \mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}] \supseteq \mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}}).$$

To see that this is an isomorphism of sub*algebras*, simply observe that both $\mathrm{H}^{\widetilde{\mathrm{T}}_c}_*(\mathrm{Fl}_\mathrm{G}; \mathbf{Q})$ and $\mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ inherit their multiplicative structure from $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$. That this is an

[16] Here, we mean a module in the usual, underived, sense of the word; but it is easy to generalize the statement to the setting of perfect $\mathcal{D}^{\mathbf{H}}_{\widetilde{\mathrm{T}}}[\mathrm{W}]$-modules by induction on the length of the bounded complex.

[17] That is, it lies in the essential image of the left adjoint $\phi^!$ to ϕ_*^{IndCoh}.

isomorphism of Hopf $\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$-algebroids is also elementary: for instance, the coproduct on both $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_{\mathrm{G}};\mathbf{Q})$ and $\mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ are inherited from the $\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$-linear coproduct on $\mathrm{Q}(\mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$ coming from the diagonal on $\mathrm{Fl}_{\mathrm{G}}^{\widetilde{\mathrm{T}}_c} = \widetilde{\mathrm{W}}$. □

Remark 3.6.25. The left-hand side of Theorem 3.6.24 admits an obvious grading; on the right-hand side, the resulting grading on $\mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})$ can be identified with that inherited from $\mathrm{Q}(\mathcal{O}_{(\mathbf{G}_a)_{\widetilde{\mathrm{T}}}})[\widetilde{\mathrm{W}}]$, where the coordinates of $(\mathbf{G}_a)_{\widetilde{\mathrm{T}}}$ are placed in weight 2.

Moreover, Theorem 3.6.24 holds even if $\mathbf{Q}$ is replaced by $\mathbf{Z}$ (as long as, on the right-hand side, $\mathbf{G}_a$ is viewed as defined over $\mathbf{Z}$).

Remark 3.6.26. Suppose W is finite. Then [GKV2, Proposition 2.3] states that *upon rationalization*, the action of $\mathcal{H}(\mathbf{G}_a, \mathrm{T}, \mathrm{W})$ on $\mathcal{O}_{(\mathbf{G}_a)_{\mathrm{T}}} = \mathcal{O}_{\mathfrak{t}}$ gives an isomorphism between $\mathcal{H}(\mathbf{G}_a, \mathrm{T}, \mathrm{W})$ and $\mathrm{End}_{\mathcal{O}_{\mathfrak{t}}^{\mathrm{W}}}(\mathcal{O}_{\mathfrak{t}})$. (This result is false without rationalization, or at least without inverting enough primes.) Its $\mathcal{O}_{\mathfrak{t}}$-linear dual is therefore $\mathcal{O}_{\mathfrak{t}} \otimes_{\mathcal{O}_{\mathfrak{t}}^{\mathrm{W}}} \mathcal{O}_{\mathfrak{t}} \cong \mathcal{O}_{\mathfrak{t} \times_{\mathfrak{t}/\!\!/\mathrm{W}} \mathfrak{t}}$. Note that this naturally admits the structure a cocommutative Hopf $\mathcal{O}_{\mathfrak{t}}$-algebroid. The analogue of Theorem 3.6.24 states that there is an isomorphism $\mathrm{H}_*^{\mathrm{T}_c}(\mathrm{G}_c/\mathrm{T}_c;\mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, \mathrm{T}, \mathrm{W})$ of (cocommutative) Hopf $\mathrm{H}^*_{\mathrm{T}_c}(*;\mathbf{Q}) \cong \mathcal{O}_{\mathfrak{t}}$-algebroids.

Let $\mathbf{e} = \frac{1}{|\mathrm{W}|}\sum_{w\in \mathrm{W}}[w]$ denote the symmetrizer, viewed as an element of $\mathbf{Q}[\mathrm{W}]$. The spherical subalgebra $\mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})^{\mathrm{sph}}$ is defined to be $\mathbf{e}\mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})\mathbf{e}$. The following result is now an easy consequence of Theorem 3.6.24.

Corollary 3.6.27. *There is an isomorphism of associative $\mathbf{Q}[\hbar]$-algebras*

$$\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q}) \cong \mathcal{H}(\mathbf{G}_a, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})^{\mathrm{sph}}.$$

Here, $\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q})$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\mathrm{H}^*_{\mathrm{G}\times \mathrm{S}^1_{\mathrm{rot}}}(*;\mathbf{Q}) \cong \mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}\times \mathbf{A}^1_\hbar}$-algebroids.

Let $\mathcal{D}^\hbar_{\check{\mathrm{G}}}$ denote the algebra of (rescaled) differential operators on $\check{\mathrm{G}}$, and let $\check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi \check{\mathrm{N}}$ denote its bi-Whittaker reduction (that is, its two-sided Hamiltonian reduction by the left and right actions of $\check{\mathrm{N}}$ with respect to a nondegenerate character $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$). Corollary 3.6.27 and [Gin3, Theorem 1.2.1] yield:

Corollary 3.6.28 ([BF, Theorem 3])**.** *There is an isomorphism of associative $\mathbf{Q}[\hbar]$-algebras*

$$\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q}) \cong \check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi \check{\mathrm{N}}.$$

Note that the diagonal on $\check{\mathrm{G}}$ equips $\mathcal{D}^\hbar_{\check{\mathrm{G}}}$ with the structure of a coalgebra in the category of $\mathrm{U}_\hbar(\check{\mathfrak{g}})$-bimodules. This in turn equips the bi-Whittaker reduction $\check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi \check{\mathrm{N}}$ with the structure of a (cocommutative) Hopf algebroid over $\mathrm{U}_\hbar(\check{\mathfrak{g}})/_\psi\check{\mathrm{N}}$; by [Kos2], the latter is isomorphic to $\mathrm{Z}(\mathrm{U}_\hbar(\check{\mathfrak{g}})) \cong \mathrm{Sym}(\mathfrak{t}^*)^{\mathrm{W}}[\hbar]$. Again, one can verify (by reduction to the case of a torus) that the isomorphism of Corollary 3.6.28 is one of cocommutative Hopf coalgebroids over $\mathrm{H}^*_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(*;\mathbf{Q}) \cong \mathrm{Sym}(\mathfrak{t}^*)^{\mathrm{W}}[\hbar]$.

Remark 3.6.29. Since $\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Q})$ is Morita equivalent to $\mathrm{H}_*^{\mathrm{T}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Fl}_{\mathrm{G}};\mathbf{Q})$, Remark 3.6.23, Theorem 3.6.24, Corollary 3.6.27, and Corollary 3.6.28 together tell us that there are equivalences of categories

$$\begin{aligned}\mathrm{H}_*^{\mathrm{T}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Fl}_\mathrm{G};\mathbf{Q})\text{-mod} &\simeq \mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};\mathbf{Q})\text{-mod}\\ &\simeq \check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}}\text{-mod} \simeq \mathcal{H}(\mathbf{G}_a,\widetilde{\mathrm{T}},\widetilde{\mathrm{W}})\text{-mod}\simeq \mathrm{IndCoh}(\widetilde{\mathfrak{t}}/\!\!/\widetilde{\mathrm{W}}).\end{aligned}$$

Definition 3.6.30. Denote by $\mathrm{HC}^\hbar_{\check{\mathrm{G}}}$ the ∞-category $\mathcal{D}^\hbar_{\check{\mathrm{G}}}\text{-mod}^{\check{\mathrm{G}}\times\check{\mathrm{G}},\mathrm{weak}}\simeq \mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{\mathrm{G}},\mathrm{weak}}$ of Harish-Chandra bimodules. Let $\kappa_\hbar : \mathrm{HC}^\hbar_{\check{\mathrm{G}}}\to \mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{\mathrm{N}},\psi)}$ denote the Kostant functor of [BF, Section 2.3], so that it is given by the composite

$$\mathrm{HC}^\hbar_{\check{\mathrm{G}}}\xrightarrow{\text{forget}}\mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}\xrightarrow{\mathrm{Av}_{\check{\mathrm{N}},\psi}}\mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{\mathrm{N}},\psi)}.$$

Note that by Skryabin's theorem (see the appendix of [Pre]), there is an equivalence $\mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{\mathrm{N}},\psi)}\simeq \mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W}\times\mathbf{A}^1_\hbar)$. Define $(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}$ to denote the localizing subcategory of $\mathrm{HC}^\hbar_{\check{\mathrm{G}}}$ on which $\kappa_\hbar$ is conservative.

One can check that upon "setting $\hbar = 1$", the category $(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}$ identifies with the category $\mathcal{HC}_{\mathrm{nondeg}}$ from [Gan1, Remark 4.22].[18] Before proceeding, we need a category-theoretic result, which follows from [Lur4, Corollary 4.7.5.3].

Proposition 3.6.31. *Let $\mathcal{C}^\bullet$ be an augmented cosimplicial presentable stable ∞-category. Suppose that:*

a. For every $[n]\in\mathbf{\Delta}^+$, the face map $d^0:\mathcal{C}^i\to\mathcal{C}^{i+1}$ admits a left adjoint $(d^0)^{\mathrm{L}}$.

b. The "Beck-Chevalley conditions" hold. That is, for every morphism $\alpha:[m]\to[n]$ in $\mathbf{\Delta}^+$, the following diagram commutes:

$$\begin{tikzcd}\mathcal{C}^{m+1} \arrow[r, "([0]\star\alpha)^*"] \arrow[d, "(d^0)^{\mathrm{L}}"'] & \mathcal{C}^{n+1} \arrow[d, "(d^0)^{\mathrm{L}}"] \\ \mathcal{C}^m \arrow[r, "\alpha^*"] & \mathcal{C}^n.\end{tikzcd}$$

Then the functor $\mathcal{C}^{-1}\to\mathrm{Tot}(\mathcal{C}^\bullet|_{\mathrm{N}(\mathbf{\Delta})})$ admits a fully faithful right adjoint; moreover, the essential image of this functor identifies with the full subcategory of $\mathcal{C}^{-1}$ on which the functor $\mathcal{C}^{-1}\to\mathcal{C}^0$ is conservative.

It is my understanding that the following result is closely related to recent work of Gannon and Ginzburg [GG2] (but I have not made a comparison).

Corollary 3.6.32. *Recall the category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k)$ from Remark 3.3.8. There is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k)\simeq(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}.$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k)\to\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times\mathrm{S}^1_{\mathrm{rot}}}(*;k)$ identifies with the functor $\kappa_\hbar:(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}\to\mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W}\times\mathbf{A}^1_\hbar)$.

[18] Let me note here my aversion to the phrase "setting $\hbar = 1$". As we have seen above, $\hbar$ arises naturally as a generator of $\mathrm{H}^2_{\mathrm{S}^1_{\mathrm{rot}}}(*;\mathbf{Q})$, and as such, it lives in *nonzero grading*. It is therefore not sensible to set $\hbar$ to be equal to a nonzero number. A better – and in some sense equivalent – way to "set $\hbar = 1$" in a graded $\mathbf{Q}[\hbar]$-module/category $\mathrm{M}_\hbar$ is to extract the weight zero piece of the localization $\mathrm{M}_\hbar[\hbar^{-1}]$. Doing this procedure to $(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}$ will product $\mathcal{HC}_{\mathrm{nondeg}}$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)$ is the ∞-category of left comodules over $\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)$ in $\mathrm{Loc}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(*;k)$. The latter category can be identified with

$$\mathrm{Loc}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(*;k) \simeq \mathrm{QCoh}(\mathfrak{t}/\!\!/\mathrm{W}\times \mathbf{A}^1_\hbar) \simeq \mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{\mathrm{N}},\psi)}.$$

Let us denote this category by $\mathcal{C}^0$. Just as in Skryabin's theorem, there is an equivalence

$$\check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}}\text{-mod} \simeq \mathcal{D}^\hbar_{\check{\mathrm{G}}}\text{-mod}^{(\check{\mathrm{N}}\times\check{\mathrm{N}},\psi\times\psi)}.$$

The Hopf algebroid structure on the pair $(\mathrm{U}_\hbar(\check{\mathfrak{g}})/_\psi\check{\mathrm{N}}, \check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}})$ defines a cosimplicial diagram

$$\mathcal{C}^0 \rightrightarrows \mathcal{C}^1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{C}^1\otimes_{\mathcal{C}^0}\mathcal{C}^1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

The preceding discussion implies that its totalization computes the ∞-category of comodules over the cocommutative Hopf algebroid $(\mathrm{U}_\hbar(\check{\mathfrak{g}})/_\psi\check{\mathrm{N}}, \check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}})$. Corollary 3.6.28 gives an isomorphism $\mathrm{H}_*^{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k) \cong \check{\mathrm{N}}_\psi\backslash \mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}}$ of cocommutative Hopf algebroids over $\mathrm{U}_\hbar(\check{\mathfrak{g}})/_\psi\check{\mathrm{N}} \cong \mathrm{H}^*_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(*;k)$, and so the totalization of the above cosimplicial diagram is equivalent to $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)$.

There are equivalences

$$\begin{aligned}
\mathcal{C}^0 &= \mathrm{U}_\hbar(\check{\mathfrak{g}})\text{-mod}^{(\check{\mathrm{N}},\psi)} \simeq \mathcal{D}^\hbar_{\check{\mathrm{G}}}\text{-mod}^{(\check{\mathrm{G}},\mathrm{weak}),(\check{\mathrm{N}},\psi)},\\
\mathcal{C}^1 &= \mathcal{D}^\hbar_{\check{\mathrm{G}}}\text{-mod}^{(\check{\mathrm{N}}\times\check{\mathrm{N}},\psi\times\psi)} \simeq \mathcal{C}^0\otimes_{\mathrm{HC}^\hbar_{\check{\mathrm{G}}}}\mathcal{C}^0 \simeq \mathrm{End}_{\mathrm{HC}^\hbar_{\check{\mathrm{G}}}}(\mathcal{C}^0),
\end{aligned}$$

which refine to give an equivalence of cosimplicial ∞-categories

$$\mathcal{C}^\bullet \simeq (\mathcal{C}^0)^{\otimes_{\mathrm{HC}^\hbar_{\check{\mathrm{G}}}}\bullet+1}.$$

Observe that $\mathcal{C}^\bullet$ extends to an augmented cosimplicial ∞-category $\widetilde{\mathcal{C}^\bullet}$ by setting $\mathcal{C}^{-1} = \mathrm{HC}^\hbar_{\check{\mathrm{G}}}$, where the functor $\mathcal{C}^{-1}\to\mathcal{C}^0$ induced by the unique morphism $[-1]\to[0]$ in $\mathbf{\Delta}^+$ is given by the Kostant functor $\kappa_\hbar$. It is straightforward to check that both conditions in Proposition 3.6.31 hold for $\widetilde{\mathcal{C}^\bullet}$, so we find that $\mathrm{Tot}(\mathcal{C}^\bullet)$ is equivalent the localizing subcategory $(\mathrm{HC}^\hbar_{\check{\mathrm{G}}})_{\mathrm{reg}}$ of $\mathcal{C}^{-1} = \mathrm{HC}^\hbar_{\check{\mathrm{G}}}$ spanned those objects on which the Kostant functor is conservative. $\square$

Remark 3.6.33. One can also deduce Corollary 3.6.32 from [BF], as discussed in [Lon2]. This, combined with [Gin3, Theorem 1.2.1], gives an alternative proof of Theorem 3.6.24 assuming the results of [BF]. However, as mentioned in the introduction to this section, we specifically do *not* want to appeal to [BF], since it does not have analogues in the K-theoretic or elliptic settings.

Remark 3.6.34. Just as with Proposition 3.6.18, if $\mathrm{HC}^{\hbar,\mathrm{free}}_{\check{\mathrm{G}}}$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{\mathrm{G}})\to\mathrm{HC}^\hbar_{\check{\mathrm{G}}}$, then there is a fully faithful embedding

$$(\mathrm{HC}^{\hbar,\mathrm{free}}_{\check{\mathrm{G}}})^\heartsuit \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)^\heartsuit.$$

This can be understood as an analogue of [BF, Theorem 2].

Remark 3.6.35. There is a Kostant functor

$$\kappa_\hbar : \mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})} \to \mathrm{U}_\hbar(\check{\mathfrak{t}})\text{-mod} \simeq \mathrm{QCoh}(\mathfrak{t}\times\mathbf{A}^1_\hbar)$$

given by the composite

$$\begin{aligned}\mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})} &\xrightarrow{\text{forget}} \mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{T}},\mathrm{weak})}\\ &\xrightarrow{\mathrm{Av}_{\check{\mathrm{N}},\psi}} \mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{T}},\mathrm{weak}),(\check{\mathrm{N}},\psi)}\\ &\simeq \mathrm{DMod}_\hbar(\check{\mathrm{T}})^{(\check{\mathrm{T}},\mathrm{weak})} \simeq \mathrm{U}_\hbar(\check{\mathfrak{t}})\text{-mod}.\end{aligned}$$

Using $\kappa_\hbar$, one can define an ∞-category $\mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})}_{\mathrm{reg}}$. Just as in Corollary 3.6.32, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k) \simeq \mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})}_{\mathrm{reg}}. \tag{3.6.11}$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k) \to \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c\times\mathrm{S}^1_{\mathrm{rot}}}(*;k)$ identifies with the Kostant functor $\mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})}_{\mathrm{reg}} \to \mathrm{QCoh}(\mathfrak{t}\times\mathbf{A}^1_\hbar)$. The arguments in this case are slightly more subtle, though: the equivariant homology $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G};\mathbf{Q})$ no longer admits an algebra structure, but it still does admit the structure of a cocommutative coalgebra over $\mathrm{H}^*_{\widetilde{\mathrm{T}}_c}(*;\mathbf{Q})$. In fact, $\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G};\mathbf{Q})$ is isomorphic as a $(\mathrm{H}_*^{\mathrm{G}_c\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};\mathbf{Q}),\mathrm{H}_*^{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_\mathrm{G};\mathbf{Q}))$-bimodule to the $(\mathcal{H}(\mathbf{G}_a,\widetilde{\mathrm{T}},\widetilde{\mathrm{W}}),\mathbf{e}\mathcal{H}(\mathbf{G}_a,\widetilde{\mathrm{T}},\widetilde{\mathrm{W}})\mathbf{e})$-bimodule

$$\mathcal{H}(\mathbf{G}_a,\widetilde{\mathrm{T}},\widetilde{\mathrm{W}})\mathbf{e} \cong \check{\mathrm{N}}_\psi\backslash\mathcal{D}^\hbar_{\check{\mathrm{G}}}/_\psi\check{\mathrm{N}} \otimes_{\mathrm{Z}(\mathrm{U}_\hbar(\check{\mathfrak{g}}))} \mathrm{Sym}(\check{\mathfrak{t}})[\hbar].$$

This bimodule is denoted $\mathbb{M}_\hbar$ in [Gin3, Theorem 8.1.2].

Remark 3.6.36. Just as Corollary 3.1.11 can be viewed as a "generic" version of the Arkhipov-Bezrukavnikov-Ginzburg [ABG] equivalence

$$\mathrm{Shv}^c_\mathrm{I}(\mathrm{Gr}_\mathrm{G};k) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}/\check{\mathrm{G}}),$$

the equivalence Corollary 3.6.32 can be viewed as a "generic" version of the Bezrukavnikov-Finkelberg [BF] equivalence

$$\mathrm{Shv}^c_{\mathrm{G}(\mathcal{O})\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k) \simeq \mathrm{HC}^\hbar_{\check{\mathrm{G}}}.$$

Similarly, the equivalence of (3.6.11) can be viewed as a "generic" version of the quantized Arkhipov-Bezrukavnikov-Ginzburg equivalence

$$\mathrm{Shv}^c_{\mathrm{I}\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};k) \simeq \mathrm{DMod}_\hbar(\check{\mathrm{G}}/\check{\mathrm{N}})^{(\check{\mathrm{G}}\times\check{\mathrm{T}},\mathrm{weak})}.$$

Unfortunately, I am not aware of a reference for this final statement, but it can be deduced from the work of Ginzburg-Riche in [GR3].

3.7 Variant: algebraically closed fields

In this section, we will indicate variants of some of the arguments in the preceding section required to study a generalization to cohomology with coefficients in arbitrary algebraically closed fields. We first need some terminology.

Definition 3.7.1. Let $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$ denote an additive character, viewed as a point $\psi \in \check{\mathfrak{n}}^*$; say that ψ is *nondegenerate* if the centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(\psi)$ has dimension given by the rank of $\check{\mathrm{G}}$. If we identify $\check{\mathfrak{t}}^*$ with the subspace $(\check{\mathfrak{n}} \oplus \check{\mathfrak{n}}^-)^\perp \subseteq \check{\mathfrak{g}}^*$, the choice of ψ defines an inclusion $\kappa : \psi + \check{\mathfrak{t}}^* \hookrightarrow \check{\mathfrak{n}}^{-,\perp}$. Similarly, the preimage of ψ under the map $\check{\mathfrak{g}}^* \to \check{\mathfrak{n}}^*$ gives a map $\psi + \check{\mathfrak{n}}^\perp \to \check{\mathfrak{g}}^*$ which we will also denote by κ. We will say that ψ is *strongly nondegenerate* if it is nondegenerate and the $\check{\mathrm{G}}$-orbit of ψ in $\check{\mathfrak{g}}^*$ is the regular locus in the nilpotent cone $\mathcal{N} \subseteq \check{\mathfrak{g}}^*$.

There is a grading on $\mathfrak{n}^\perp$ and $\check{\mathfrak{g}}^*$ given by the cocharacter $2 - 2\rho$; this places ψ in weight 0, and $\check{\mathfrak{t}}^*$ in weight 2. For notational simplicity, we will hide this grading below, but it will be ever-present.

Theorem 3.7.2. *Let k be an algebraically closed field, and let G be a connected reductive group over $\mathbf{C}$. If a strongly nondegenerate character ψ exists, then there are equivalences*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}(2)/\check{\mathrm{G}}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathfrak{g}}^{*,\mathrm{reg}}(2)/\check{\mathrm{G}}),$$

where $\widetilde{\check{\mathfrak{g}}}(2) \cong \mathrm{T}^*(2)(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}}$.

Throughout this section, we will *assume* that a strongly nondegenerate character ψ exists. The proof of Theorem 3.7.2 follows the proof of Corollary 3.6.8; one just needs to justify a few additional steps. We will begin by arguing the first equivalence of Theorem 3.7.2.

Lemma 3.7.3. *If $\psi : \check{\mathfrak{n}} \to \mathbf{G}_a$ is a nondegenerate character, then $\mathrm{Z}_\psi(\check{\mathrm{G}}) \cap \check{\mathrm{N}}^- = \{1\}$.*

Proof. If $u \in \check{\mathrm{N}}^-$ fixes ψ, then it will normalize any Borel subalgebra which is annihilated by ψ. But the unique such Borel subalgebra is $\check{\mathfrak{b}}$, so u will normalize $\check{\mathfrak{b}}$. Since the normalizer of a Borel is itself, $u \in \check{\mathrm{B}} \cap \check{\mathrm{N}}^-$, and hence is the identity. □

Define

$$\widetilde{\check{\mathrm{J}}} := (\psi + \check{\mathfrak{t}}^*) \times_{\check{\mathfrak{n}}^\perp/\check{\mathrm{B}}} (\psi + \check{\mathfrak{t}}^*).$$

The same proof as in Lemma 3.6.4 shows:

Lemma 3.7.4. *The projection map $\widetilde{\check{\mathrm{J}}} \to \psi + \check{\mathfrak{t}}^*$ (onto either factor) is flat, and there is an isomorphism*

$$\widetilde{\check{\mathrm{J}}} \cong \{(x, g) \in \check{\mathfrak{t}}^* \times \check{\mathrm{B}} | \mathrm{Ad}_g(\psi + x) = \psi + x\}.$$

The following generalizes [BFM, YZ2].

Theorem 3.7.5. *There is an isomorphism*

$$\widetilde{\check{\mathrm{J}}} \cong \mathrm{Spec}\, \mathrm{H}^{\mathrm{T}}_*(\Omega \mathrm{G}; k)$$

of graded group schemes over $\mathrm{Spec}\, \mathrm{H}^*_{\mathrm{T}}(*; k) \cong \check{\mathfrak{t}}^*$, where T acts on $\Omega \mathrm{G}$ via conjugation on G.

Proof. Let us write $\widetilde{\check{\mathrm{J}}}(\check{\mathrm{G}})$ to denote the dependence on $\check{\mathrm{G}}$. As in Theorem 3.6.3, we will reduce to a semisimple rank 1 calculation. Namely, observe that $\mathrm{Gr}_{\mathrm{G}}(\mathbf{C}) \simeq \Omega \mathrm{G}$ has a cell structure with cells only in even dimensions; so $\mathrm{H}^{\mathrm{T}}_*(\Omega \mathrm{G}; k)$ is a flat $\mathrm{H}^*_{\mathrm{T}}(*; k)$-algebra, and it therefore suffices to prove an isomorphism

$$\widetilde{\check{\mathrm{J}}}|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathrm{Spec}\, \mathrm{H}^{\mathrm{T}}_*(\Omega \mathrm{G}; k)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$$

for each root α. By Atiyah-Bott localization, the right-hand side can be identified with

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{T}}(\Omega \mathrm{G}; k)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \operatorname{Spec} \mathrm{H}_*^{\mathrm{T}}(\Omega \mathrm{Z}_{\mathrm{G}}(\alpha); k)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \tag{3.7.1}$$

Note that the right-hand side depends only on the connected component of the identity in $\mathrm{Z}_{\mathrm{G}}(\alpha)$. For the other side, we claim that there is an isomorphism

$$\widetilde{\check{\mathrm{J}}}(\check{\mathrm{G}})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{\mathrm{J}}}(\mathrm{H})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}, \tag{3.7.2}$$

where as before, H is the connected component of the centralizer $\mathrm{Z}_x(\check{\mathrm{G}})$ of some $x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ which lies on the α-hyperplane. There is a map from the left-hand side to the right-hand side, which sends

$$\check{\mathfrak{t}}^* \times \check{\mathrm{B}} \ni (x, g) \mapsto (x, g) \in \check{\mathfrak{t}}^* \times (\check{\mathrm{B}} \cap \mathrm{H}).$$

Note that $\check{\mathrm{B}} \cap \mathrm{H}$ is a Borel subgroup of H. To see that the above map gives an isomorphism, observe that if $y \in \check{\mathfrak{t}}^*$, we may identify the centralizer in $\check{\mathrm{G}}$ of $\psi + y$ with the centralizer in $\mathrm{Z}_y(\check{\mathrm{G}})^\circ$ of ψ. That (3.7.2) is an isomorphism is now a consequence of the observation that if $y \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$, then this centralizer $\mathrm{Z}_y(\check{\mathrm{G}})^\circ$ is contained in H. That is, if $(x, g) \in \widetilde{\check{\mathrm{J}}}(\check{\mathrm{G}})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$, then g is already contained in H, and so $(x, g) \in \widetilde{\check{\mathrm{J}}}(\mathrm{H})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$.

Based on (3.7.1) and (3.7.2), we are reduced to showing that there is an isomorphism of group schemes

$$\widetilde{\check{\mathrm{J}}}(\mathrm{H})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \operatorname{Spec} \mathrm{H}_*^{\mathrm{T}}(\Omega \mathrm{Z}_{\mathrm{G}}(\alpha); k)|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

Note that H is connected of semisimple rank 1. So, it suffices to verify that the isomorphism of the theorem holds for connected reductive groups of semisimple rank 1. This calculation was already done in Theorem 3.6.3: one can see easily that the calculations there work with $\mathbf{Q}$ replaced even by $\mathbf{Z}$. ☐

Proposition 3.7.6. *There is an isomorphism of stacks* $\check{\mathfrak{t}}^*/\widetilde{\check{\mathrm{J}}} \cong \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}$.

Proof sketch. Note that there is an isomorphism $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}} \cong (\check{\mathfrak{n}}^\perp)^{\mathrm{reg}}/\check{\mathrm{B}}$. Since $\widetilde{\check{\mathrm{J}}}$ is the stabilizer of $\psi + \check{\mathfrak{t}}^* \to \check{\mathfrak{n}}^\perp$, it suffices to show that the $\check{\mathrm{B}}$-orbit of $\psi + \check{\mathfrak{t}}^* \subseteq \check{\mathfrak{n}}^\perp$ is exactly $(\check{\mathfrak{n}}^\perp)^{\mathrm{reg}}$. In turn, it suffices to show that the $\check{\mathrm{B}}$-orbit of $\psi \in \check{\mathfrak{b}}^\perp$ is exactly $(\check{\mathfrak{b}}^\perp)^{\mathrm{reg}}$. This requires some work, but can be proved in a manner similar to the case of characteristic zero. ☐

We now need to discuss the $\mathbf{E}_\infty$-ring of cochains on BG. The main result necessary for our discussion below is the following, whose proof will need several preliminaries. I am very grateful to Akshay Venkatesh for several discussions about and surrounding this statement.

Theorem 3.7.7. *There is an isomorphism of stacks* $\operatorname{Spev}(\mathrm{C}^*(\mathrm{BG}; k)) \cong (\psi + \check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}$ *(with the grading on the latter coming from* $2 - 2\rho$ *on* $\psi + \check{\mathfrak{n}}^\perp$ *and the* 2ρ*-grading on* $\check{\mathrm{N}}$*). In particular, there is a spectral sequence*

$$\mathrm{E}_2^{*,*} \cong \mathrm{H}^*((\psi + \check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}; \mathcal{O}\{*\}) \Rightarrow \mathrm{H}^*(\mathrm{BG}; k).$$

If $\mathrm{C}^*(\mathrm{BG}; k)$ was even (i.e., the characteristic of k is not a torsion prime for G), then $\operatorname{Spev}(\mathrm{C}^*(\mathrm{BG}; k))$ identifies with the graded scheme $\operatorname{Spec}(\mathrm{H}^*(\mathrm{BG}; k))$. In this case, Theorem 3.7.7 implies that the $\check{\mathrm{N}}$-action on $\psi + \check{\mathfrak{n}}^\perp$ is free. This is a refinement of a result of Kostant's (from [Kos2]) in characteristic zero to the case of non-torsion characteristics.

We note that if k is of characteristic zero, then Theorem 3.7.7 is usually proved by observing that $\operatorname{Spec}(\mathrm{H}^*(\mathrm{BG}; k)) = \operatorname{Spec}(\mathrm{H}^*(\mathrm{BT}; k))/\!\!/\mathrm{W}$ identifies with $\mathfrak{t}/\!\!/\mathrm{W}$, and similarly using

Kostant's theorem that $(\psi+\check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}} \cong \mathfrak{t}/\!\!/\mathrm{W}$. *Both* of these isomorphisms fail in small characteristics, but Theorem 3.7.7 asserts that there is nevertheless an isomorphism $\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k)) \cong (\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}$. On the topological side, the failure of the isomorphism $\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k)) \cong \mathfrak{t}(2)/\!\!/\mathrm{W}$ is reflected by the observation that $\mathfrak{t}(2)/\!\!/\mathrm{W} \cong \mathrm{Spev}(\mathrm{C}^*(\mathrm{BN_G(T)}; k))$, where $\mathrm{N_G(T)}$ is the normalizer of the maximal torus $\mathrm{T} \subseteq \mathrm{G}$, and the natural map $\mathrm{BN_G(T)} \to \mathrm{BG}$ generally does not an isomorphism on k-cohomology. Note that the map $\mathrm{BN_G(T)} \to \mathrm{BG}$ does induce a comparison map

$$\mathfrak{t}(2)/\!\!/\mathrm{W} \cong \mathrm{Spev}(\mathrm{C}^*(\mathrm{BN_G(T)}; k)) \to \mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k)) \cong (\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}},$$

and the difference between the two is related to the cohomology of the quotient $\mathrm{G}/\mathrm{N_G(T)}$.

Example 3.7.8. Let us illustrate Theorem 3.7.7 in the first case where cochains on BG is not even: namely, take $k = \mathbf{F}_2$ and $\mathrm{G} = \mathrm{SO}_3$. In this case, $\check{\mathrm{G}} = \mathrm{SL}_2$, so that

$$\psi + \check{\mathfrak{n}}^{\perp} = \left(\begin{smallmatrix} x & y \\ 1 & 0 \end{smallmatrix}\right) \subseteq \check{\mathfrak{g}}^* \cong \mathfrak{pgl}_2.$$

The group $\check{\mathrm{N}} \cong \mathbf{G}_a$ acts by conjugation:

$$\left(\begin{smallmatrix} x & y \\ 1 & 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} x+b & y-b(x+b) \\ 1 & -b \end{smallmatrix}\right).$$

Note that this matrix is equivalent in $\check{\mathfrak{g}}^* \cong \mathfrak{pgl}_2$ to $\left(\begin{smallmatrix} x+2b & y-b(x+b) \\ 1 & 0 \end{smallmatrix}\right)$, so that the action of $\mathbf{G}_a$ on $\psi + \check{\mathfrak{n}}^{\perp}$ is given by

$$b : (x, y) \mapsto (x + 2b, y - b(x + b)) = (x, y + b(x + b)).$$

Note that since $k = \mathbf{F}_2$, there is a homomorphism $\varphi : \mathbf{G}_a \times \mathbf{A}^1_x \to \mathbf{G}_a \times \mathbf{A}^1_x$ sending $(b, x) \mapsto (b(x+b), x)$, and the action of $\mathbf{G}_a$ on $\psi+\check{\mathfrak{n}}^{\perp}$ factors through the homomorphism φ. It follows that there is an isomorphism of stacks

$$(\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}} \cong \mathbf{A}^1_x/\ker(\varphi).$$

But $\ker(\varphi) \cong \mathrm{Spec}\, k[x, b]/b(x + b)$, whose cohomology as a group scheme over $\mathbf{A}^1_x$ is easily computed to be $k[x, \eta]$ with x in degree 0 and η in cohomological degree 1. In other words, there is an isomorphism

$$\mathrm{H}^*((\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}; \mathcal{O}\{*\}) \cong k[x, \eta]$$

with x in weight -2 and degree 0, and η in weight -2 and degree -1. The spectral sequence of Theorem 3.7.7 degenerates at the E_2-page, and one recovers the well-known fact that $\mathrm{H}^*(\mathrm{BSO}_3; \mathbf{F}_2) \cong \mathbf{F}_2[w_2, w_3]$: the Stiefel-Whitney class w_2 is represented by x, and the Stiefel-Whitney class w_3 is represented by η.

The preceding calculation illustrates our claim that $\mathrm{R\Gamma}((\psi+\check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}; \mathcal{O}\{*\})$ is generally *not* isomorphic to the affinization $\mathfrak{t}(2)/\!\!/\mathrm{W} := \mathrm{Spec}\,\mathrm{R\Gamma}(\mathrm{W}; \mathrm{Sym}(\mathfrak{t}^*(-2)))$. Indeed, the homotopy groups of the latter is isomorphic to the group cohomology $\mathrm{H}^*(\mathbf{Z}/2; k[x])$. Since k is of characteristic 2, the sign action of $\mathbf{Z}/2$ on x is trivial, so that $\mathrm{H}^*(\mathbf{Z}/2; k[x])$ is isomorphic to $k[x, \eta']$. Here, however, x is in weight -2 and degree 0, while η' is in weight 0 and degree -1. Although there is a map $k[x, \eta] \to k[x, \eta']$ sending $\eta \mapsto (\eta')^3$, it is not an isomorphism. This map is induced by the map $\mathrm{BN_G(T)} \to \mathrm{BG}$ on the level of global sections of $\mathrm{Spev}(\mathrm{C}^*(-; k))$.

Remark 3.7.9. The relative Langlands program (in the form suggested in Conjecture 5.2.20) suggests that the stack $\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k))$ should also be isomorphic to the (derived) affinization $\check{\mathfrak{g}}^*(2)/\!\!/\check{\mathrm{G}} := \mathrm{Spec}(\mathrm{R\Gamma}(\check{\mathrm{G}}; \mathrm{Sym}(\check{\mathfrak{g}}(-2)))$ of the graded stack $\check{\mathfrak{g}}^*(2)/\check{\mathrm{G}}$. I thank A. Venkatesh for suggesting this and some basic consistency checks in small ranks and characteristics. (As one sanity check, note that the quotient stack $\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG}; k)) \cong (\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}$ is indeed an affine stack.)

Let us now begin the proof of Theorem 3.7.7.

Lemma 3.7.10. *The tensor product* $\mathrm{Sym}(\check{\mathfrak{t}}) \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}} k$ *is a finite k-module.*

Proof. If $p > 2$, or $p = 2$ and $\check{G}$ has no SO_{2n+1}-factor, then [KW, Theorem 4] says that the map $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}} \to \mathcal{O}_{\check{\mathfrak{t}}^*}^{W}$ is an isomorphism (using that $\check{G}$ has no factors of type C). This means that

$$\check{\mathfrak{t}}^* \times_{\mathrm{Spec}\, \mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}} \{0\} \cong \check{\mathfrak{t}}^* \times_{\check{\mathfrak{t}}^*/\!\!/W} \{0\},$$

which is a finite scheme, so the lemma is true.

The only remaining case to check is $p = 2$ and $\check{G} = \mathrm{SO}_{2n+1}$. In this case, it is no longer true that the map $\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}} \to \mathcal{O}_{\check{\mathfrak{t}}^*}^{W}$ is an isomorphism. Indeed, the W-invariants on $\mathcal{O}_{\check{\mathfrak{t}}^*} = k[x_1, \cdots, x_n]$ are the same as the Σ_n-invariants, i.e., $\mathcal{O}_{\check{\mathfrak{t}}^*}^{W}$ is the polynomial algebra on the n elementary symmetric polynomials. To understand $\mathcal{O}_{\mathfrak{so}_{2n+1}^*}^{\mathrm{SO}_{2n+1}}$, we will use the exceptional isogeny $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$.[19] This isogeny can be used to obtain an SO_{2n+1}-equivariant isomorphism $\mathfrak{so}_{2n+1}^* \cong \mathfrak{sp}_{2n}$, and so there are isomorphisms

$$\mathcal{O}_{\mathfrak{so}_{2n+1}^*}^{\mathrm{SO}_{2n+1}} \cong \mathcal{O}_{\mathfrak{sp}_{2n}}^{\mathrm{SO}_{2n+1}} \cong \mathcal{O}_{\mathfrak{sp}_{2n}}^{\mathrm{Sp}_{2n}}.$$

Now this invariant ring was computed in [CR2, Theorem 6.2.2] (over any base!) to be $k[p_1, \cdots, p_n]$, where p_j is the jth elementary symmetric polynomial in the *squares* of $x_1, \cdots, x_n \in \mathcal{O}_{\check{\mathfrak{t}}^*} = k[x_1, \cdots, x_n]$. Even in this case, therefore, the tensor product

$$\mathrm{Sym}(\check{\mathfrak{t}}) \otimes_{\mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}} k \cong k[x_1, \cdots, x_n]/(e_j(x_1^2, \cdots, x_n^2) | 1 \leq j \leq n)$$

is a finite k-module, as desired. □

Proposition 3.7.11. *The map* $q : \psi + \check{\mathfrak{t}}^* \to (\psi + \check{\mathfrak{n}}^\perp)/\check{N}$ *is flat.*

Proof. We need to show that the adjoint action map

$$\check{N} \times (\psi + \check{\mathfrak{t}}^*) \to \psi + \check{\mathfrak{n}}^\perp$$

is flat. The fiberwise criterion for flatness reduces us to showing that the above map has zero-dimensional fibers. Since the above map is one of varieties with *contracting* $\mathbf{G}_m$-action, it suffices to show that the fiber over $\psi \in \psi + \check{\mathfrak{n}}^\perp$ is finite, i.e., that the scheme $Z := \{u \in \check{N} | \mathrm{Ad}_u(\psi) \in \psi + \check{\mathfrak{t}}^*\}$ has finitely many k-points.

By Lemma 3.7.3, it suffices to show that the image of the map $Z \to \check{\mathfrak{t}}^*$ sending $u \mapsto \mathrm{Ad}_u(\psi) - \psi$ is a finite k-scheme. If $\mathrm{Ad}_u(\psi) - \psi = x$, then since ψ is nilpotent, any invariant polynomial with vanishing constant term is zero on $\psi + x$, and hence on x. This means that $x \in \check{\mathfrak{t}}^* \times_{\mathrm{Spec}\, \mathcal{O}_{\check{\mathfrak{g}}^*}^{\check{G}}} \{0\}$, which is a finite scheme by Lemma 3.7.10. □

Proposition 3.7.12. *There is a Cartesian square*

$$\begin{tikzcd}
\psi + \check{\mathfrak{t}}^* \arrow[r] \arrow[d] & \widetilde{\check{\mathfrak{g}}}/\check{G} \arrow[d] \\
(\psi + \check{\mathfrak{n}}^\perp)/\check{N} \arrow[r] & \check{\mathfrak{g}}^*/\check{G}.
\end{tikzcd}$$

[19]Just for me to recall: this map comes from restricting an automorphism of (k^{2n+1}, q) to the set of null vectors for q, which forms a $2n$-dimensional hyperplane on which the symmetric form associated to q restricts to a symplectic form.

Proof. Define $\mathcal{Y}$ by the Cartesian square

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] & \widetilde{\check{\mathfrak{g}}} \arrow[d, "\mu"] \\ \psi + \check{\mathfrak{n}}^{\perp} \arrow[r] & \check{\mathfrak{g}}^*, \end{tikzcd}$$

so that $\mathcal{Y}$ is the moduli of pairs $(\check{\mathfrak{b}}', x)$ with $x \in (\psi + \check{\mathfrak{n}}^{\perp}) \cap \check{\mathfrak{n}}'^{\perp}$, where $\check{\mathfrak{n}}' = [\check{\mathfrak{b}}', \check{\mathfrak{b}}']$. We need to see that $\mathcal{Y}$ is a $\check{\mathrm{N}}$-torsor over $\check{\mathfrak{t}}^*$. Define a map

$$\pi : \mathcal{Y} \to \check{\mathfrak{t}}^*,\ (\check{\mathfrak{b}}', x) \mapsto x \pmod{\check{\mathfrak{b}}'^{\perp}},$$

so we need to see that $\mathcal{Y}_x := \pi^{-1}(x) \cong \check{\mathrm{N}}$. By translating x, we may assume $x = 0$, so we need to see that

$$\mathcal{Y}_0 = \{(\check{\mathfrak{b}}', x) | x \in (\psi + \check{\mathfrak{n}}^{\perp}) \cap \check{\mathfrak{b}}'^{\perp}\} \cong \check{\mathrm{N}}.$$

We claim:

($\star$) $\check{\mathfrak{b}}'$ and $\check{\mathfrak{b}}$ are in opposite position.

Using ($\star$), we can finish the argument: if $\check{\mathfrak{b}}'$ and $\check{\mathfrak{b}}$ are in opposite position, there is a unique element $u \in \mathrm{U}$ such that $\mathrm{Ad}_u(\check{\mathfrak{b}}^-) = \check{\mathfrak{b}}'$ (in other words, the variety of Borels in opposite position to $\check{\mathfrak{b}}$ forms the open U-orbit in the flag variety G/B^-). This also implies that there is a *unique* $x \in \check{\mathfrak{b}}'^{\perp} \cap (\psi + \check{\mathfrak{n}})^{\perp}$. Therefore, the map $\check{\mathrm{N}} \to \mathcal{Y}_0$ given by sending u to the Ad_u-translate of $(\check{\mathfrak{b}}^-, \psi)$ defines an isomorphism, as desired.

Let us now address ($\star$). Consider the grading $\check{\mathfrak{g}}^* = \bigoplus_{i \in \mathbf{Z}} \check{\mathfrak{g}}^*(i)$ of $\check{\mathfrak{g}}^*$ by the principal $\mathfrak{sl}_2$-triple, and let $\check{\mathfrak{g}}^{*, \leq j} = \bigoplus_{i \leq j} \check{\mathfrak{g}}^*(i)$. Similarly, define $\check{\mathfrak{g}}^{\leq j}$. This defines a filtration on $\check{\mathfrak{b}}'$ and on $\check{\mathfrak{b}}'^{\perp}$, by $\check{\mathfrak{b}}' \cap \check{\mathfrak{g}}^{\leq j}$ and $\check{\mathfrak{b}}'^{\perp} \cap \check{\mathfrak{g}}^{*, \leq j}$. We claim:

($\star\star$) One has $\mathrm{gr}(\check{\mathfrak{b}}') = \check{\mathfrak{b}}^-$.

This is sufficient to establish ($\star$), because any $x \in \check{\mathfrak{b}}' \cap \check{\mathfrak{b}}$ will be an element of $\check{\mathfrak{b}}'$ of filtration ≥ 0, and the only such elements are in $\check{\mathfrak{b}}' \cap \check{\mathfrak{g}}(0) = \check{\mathfrak{t}}$. In fact, ($\star\star$) is equivalent to ($\star$).

To prove ($\star\star$), will be convenient to identify $\mathrm{gr}(\check{\mathfrak{g}}^*) = \check{\mathfrak{g}}^*$. If y is any element of $\check{\mathfrak{g}}^*$, write $\overline{y}$ to denote its associated graded. Since $\check{\mathfrak{n}}^{\perp} = \check{\mathfrak{g}}^{*, \leq 0}$ and $\psi \in \check{\mathfrak{g}}^*(2)$, any element $x \in \psi + \check{\mathfrak{n}}^{\perp}$ has associated graded $\overline{x} = \overline{\psi}$. (Under the identification $\mathrm{gr}(\check{\mathfrak{g}}^*) = \check{\mathfrak{g}}^*$, this is just the element ψ.) Therefore, $\mathrm{gr}(\check{\mathfrak{b}}'^{\perp}) = \mathrm{gr}(\check{\mathfrak{b}}')^{\perp}$ contains ψ. Now, $\mathrm{gr}(\check{\mathfrak{b}}')$ is a solvable subalgebra of $\mathrm{gr}(\check{\mathfrak{g}}) = \check{\mathfrak{g}}$ of maximal dimension, so it is itself a Borel subalgebra of $\check{\mathfrak{g}}$. Again, because $\check{\mathfrak{b}}^-$ is the unique Borel subalgebra annihilated by ψ, we see that $\mathrm{gr}(\check{\mathfrak{b}}') = \check{\mathfrak{b}}^-$, as desired. $\square$

Remark 3.7.13. The preceding result is, in characteristic zero, essentially due to Kostant [Kos2] (but with a very different proof). What he showed is that there is a Cartesian square

$$\begin{tikzcd} \widetilde{\check{\mathfrak{g}}}_{\mathrm{reg}}/\check{\mathrm{G}} \arrow[r] \arrow[d, "\mu"] & \check{\mathfrak{t}}^* \arrow[d] \\ \check{\mathfrak{g}}^*_{\mathrm{reg}}/\check{\mathrm{G}} \arrow[r] & \check{\mathfrak{t}}^* /\!\!/ \mathrm{W}, \end{tikzcd}$$

and that the composite

$$(\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}} \to \check{\mathfrak{g}}^*_{\mathrm{reg}}/\check{\mathrm{G}} \to \check{\mathfrak{t}}^* /\!\!/ \mathrm{W}$$

is an isomorphism. Note that this implies Proposition 3.7.12: it follows from the work of Kostant that there is a commutative square

$$\begin{tikzcd}
\psi + \check{\mathfrak{t}}^* \arrow[r] \arrow[d] & \widetilde{\check{\mathfrak{g}}}_{\mathrm{reg}}/\check{\mathrm{G}} \arrow[r] \arrow[d, "\mu"] & \check{\mathfrak{t}}^* \arrow[d] \\
(\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}} \arrow[r] & \check{\mathfrak{g}}^*_{\mathrm{reg}}/\check{\mathrm{G}} \arrow[r] & \check{\mathfrak{t}}^* /\!\!/ \mathrm{W},
\end{tikzcd}$$

where the outer and right squares are Cartesian; so the left square is Cartesian, too, as desired.

Let us define $\mathcal{S}$ as the following groupoid scheme:

$$\mathcal{S} = (\psi + \check{\mathfrak{t}}^*) \times_{(\psi+\check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}} \psi + \check{\mathfrak{t}}^*.$$

Recall that $\mathcal{S}$ is flat over $\psi + \check{\mathfrak{t}}^*$. Since $\psi + \check{\mathfrak{t}}^*$ is normal, we can study $\mathcal{S}$ by restricting to an open subset of $\psi + \check{\mathfrak{t}}^*$ with complement of codimension 2. The whole point of Proposition 3.7.12 is that it allows us to induct on the rank of $\check{\mathrm{G}}$.

Definition 3.7.14. Say that a point $x \in \check{\mathfrak{t}}^*$ is *generic* if the identity component of the centralizer $\mathrm{Z}_x(\check{\mathrm{G}})$ is a Cartan subgroup of $\check{\mathrm{G}}$. In other words, x is a regular semisimple element of $\check{\mathfrak{g}}^*$. Similarly, if α is a root, say that a point $x \in \check{\mathfrak{t}}^*$ is *α-generic* if $x(h_\beta) \neq 0$ for all roots $\beta \neq \alpha$. This implies that the centralizer $\mathrm{Z}_x(\check{\mathrm{G}})$ has semisimple rank at most 1. Let $\check{\mathfrak{t}}^*_{\mathrm{gen}}$ denote the generic locus, and let $\check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ denote the α-regular locus.

Lemma 3.7.15.

- *One can identify $\check{\mathfrak{t}}^*_{\mathrm{gen}}$ with the open subset of those $x \in \check{\mathfrak{t}}^*$ such that $\langle x, h_{\pm\alpha}\rangle \neq 0$ for all roots $\alpha \in \Phi$. In particular, $\check{\mathfrak{t}}^*_{\mathrm{gen}} \subseteq \check{\mathfrak{t}}^*$ is open with complement of codimension 1.*
- *$\check{\mathfrak{t}}^*_{\mathrm{reg}} = \bigcup_{\alpha\in\Phi} \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \subseteq \check{\mathfrak{t}}^*$ is open with complement of codimension 2.*

Proof. The identity component of $\mathrm{Z}_x(\check{\mathrm{G}})$ is generated by $\check{\mathrm{T}}$ and the root subgroups $\check{\mathrm{U}}_\alpha$ which centralize x. These are exactly those α such that $\langle x, h_\alpha\rangle = 0$; the claim follows. □

Fix a root α, and define $\mathcal{R}_\alpha(\check{\mathrm{G}})$ by the Cartesian square

$$\begin{tikzcd}
\mathcal{R}_\alpha(\check{\mathrm{G}}) \arrow[r] \arrow[d] & \psi + \check{\mathfrak{t}}^* \arrow[d, "q"] \\
\psi + \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \arrow[r] & (\psi + \check{\mathfrak{n}}^{\perp})/\check{\mathrm{N}}.
\end{tikzcd} \tag{3.7.3}$$

We would like to describe $\mathcal{R}_\alpha(\check{\mathrm{G}})$ explicitly. Thanks to Proposition 3.7.12, $\mathcal{R}_\alpha(\check{\mathrm{G}})$ sits in a Cartesian square

$$\begin{tikzcd}
\mathcal{R}_\alpha(\check{\mathrm{G}}) \arrow[r] \arrow[d] & \widetilde{\check{\mathfrak{g}}} \arrow[d, "\mu"] \\
\psi + \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \arrow[r] & \check{\mathfrak{g}}^*,
\end{tikzcd}$$

and so it can be described as

$$\mathcal{R}_\alpha(\check{\mathrm{G}}) = \{(\check{\mathfrak{b}} \subseteq \check{\mathfrak{g}}, x) | x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}},\ \psi + x \text{ annhilates } \check{\mathfrak{b}}\}.$$

We will use this description of $\mathcal{R}_\alpha(\check{G})$ to reduce its calculation to the semisimple rank 1 case. [20]

Fix an $x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ which lies on the α-hyperplane, let $H = Z_x(\check{G})^\circ$ denote the connected component of the identity in $Z_x(\check{G})$, and let $\mathfrak{h}$ denote its Lie algebra. Note that H is reductive, and let W_H denote its Weyl group (so it is just generated by the reflection across the α-hyperplane). Let $\check{T}$ be a maximal torus of H containing x, and let $\check{\mathfrak{t}}$ denote its Lie algebra. Note that $\mathfrak{h} = \check{\mathfrak{t}} \oplus \check{\mathfrak{n}}_\alpha \oplus \check{\mathfrak{n}}_{-\alpha}$.

Lemma 3.7.16. *The intersection $\check{\mathfrak{b}} \cap \mathfrak{h}$ is a Borel subalgebra of $\mathfrak{h}$.*

Proof. The Borel $\check{\mathfrak{b}}$ determines a set of positive roots for $\check{\mathfrak{g}}$. Now H is generated by $\check{T}$ and the root subgroups $\check{N}_\beta$ such that $\langle x, h_\beta\rangle = 0$. (That is, $\beta = \pm\alpha$ since x is α-regular.) The intersection $\check{\mathfrak{b}} \cap \mathfrak{h}$ is the product of $\check{\mathfrak{t}}$ and $\check{\mathfrak{n}}_\beta$ with $\beta > 0$, and hence is a Borel subalgebra of $\mathfrak{h}$. ☐

Fix a Borel subalgebra $\check{\mathfrak{b}}' \subseteq \mathfrak{h}$ containing $\check{\mathfrak{t}}$. Then, W acts transitively on the variety of Borels $\check{\mathfrak{b}} \subseteq \check{\mathfrak{g}}$ such that $\check{\mathfrak{b}} \cap \mathfrak{h} = \check{\mathfrak{b}}'$ (because such a Borel $\check{\mathfrak{b}}$ must contain $\check{\mathfrak{t}}$, and W acts transtively on the variety of such Borels). Also, the stabilizer inside W of any point is W_H. This means that

$$\mathcal{R}_\alpha(\check{G}) \cong \underline{W} \times^{\underline{W_H}} \{(\check{\mathfrak{b}}' \subseteq \mathfrak{h}, x) | x \in \check{\mathfrak{t}}^*_{\alpha\text{-reg}},\ \psi + x \text{ annhilates } \check{\mathfrak{b}}'\} = W \times^{W_H} \mathcal{R}_\alpha(H).$$

Note that $\mathcal{R}_\alpha(H)$ sits in a Cartesian square

$$\begin{tikzcd} \mathcal{R}_\alpha(H) \arrow[r] \arrow[d] & \widetilde{\mathfrak{h}} \arrow[d, "\mu"] \\ \psi + \check{\mathfrak{t}}^*_{\alpha\text{-reg}} \arrow[r] & \mathfrak{h}^*, \end{tikzcd}$$

but from the perspective of H, every element of $\check{\mathfrak{t}}^*$ is in $\check{\mathfrak{t}}^*_{\alpha\text{-reg}}$ (because the only roots of H are $\pm\alpha$!). So $\mathcal{R}_\alpha(H)$ can be described in terms of the Soergel-Kostant scheme $\mathcal{S}(H)$ for H:

$$\mathcal{R}_\alpha(H) \cong \mathcal{S}(H)|_{\psi + \check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

But now this is easy to calculate, because H is a connected reductive group with semisimple rank 1. Every such group is either $SL_2 \times A$, $PGL_2 \times A$, or $GL_2 \times A$ for some torus A. The torus component just goes along for the ride, and so we will just do the calculation for H being SL_2, PGL_2, or GL_2. In fact, instead of computing with the Grothendieck-Springer resolution, we will now revert back to directly computing with the Cartesian square (3.7.3). (The whole point of using the Grothendieck-Springer resolution was essentially to allow us to reduce to the semisimple rank 1 case.)

[20] As a baby case, let me summarize what the argument below is doing when x is *generic*, i.e., $\langle x, h_\alpha\rangle \neq 0$ for all α. In this case, if $\check{\mathfrak{b}}$ is a Borel annhilated by $\psi + x$, let $H = Z_x(\check{G})^\circ$ denote the connected component of the identity in $Z_x(\check{G})$. Since x is generic, H is a (maximal) torus of $\check{G}$. Now, $\check{\mathfrak{b}}$ contains $\mathfrak{h}$. Said differently, $\check{\mathfrak{b}} \cap \mathfrak{h}$ intersects $\mathfrak{h}$ in a Borel subalgebra of $\mathfrak{h}$ (of which there is a unique one, namely $\mathfrak{h}$). The conjugacy theorem for Borel subalgebras now says that for a given Borel subalgebra of $\mathfrak{h}$ (that is, $\mathfrak{h}$ itself), the set of Borels of $\check{\mathfrak{g}}$ which intersect $\mathfrak{h}$ in this given Borel is a W-torsor. This implies that $\mathcal{R}_\alpha(\check{G})|_{\psi+\check{\mathfrak{t}}^*_{\mathrm{gen}}}$ is isomorphic to the constant scheme $\underline{W}$.

Example 3.7.17. • Let $\mathrm{H} = \mathrm{GL}_2$, and identify $\mathfrak{gl}_2^* \cong \mathfrak{gl}_2$. In this case, we will just directly compute $\mathcal{S}$ itself. Namely, $\mathcal{S}$ consists of tuples $\big(\left(\begin{smallmatrix} x & 0 \\ 1 & y \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 0 \\ 1 & w \end{smallmatrix}\right), b\big)$ such that

$$\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 1 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z & 0 \\ 1 & w \end{smallmatrix}\right).$$

The left-hand side is $\left(\begin{smallmatrix} b+x & -b(b+x-y) \\ 1 & y-b \end{smallmatrix}\right)$, and so $b+x = z$, $y-b = w$, and the only relation imposed is that $b(b+x-y) = 0$. That is, $\mathcal{S}$ is isomorphic to

$$\mathcal{S} \cong \operatorname{Spec} k[b, x, y]/b(b+x-y).$$

- When $\check{\mathrm{G}} = \mathrm{SL}_2$, the same reasoning shows that

$$\mathcal{S} \cong \operatorname{Spec} k[b, x]/b(b+x).$$

Indeed, one can identify $\mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$, and $\mathcal{S}$ consists of (equivalence classes of) tuples $(\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 0 \\ 1 & 0 \end{smallmatrix}\right), b)$ such that

$$\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

But the left-hand side is $\left(\begin{smallmatrix} b+x & -b(b+x) \\ 1 & -b \end{smallmatrix}\right)$, and so $x + 2b = z$ and $b(b+x) = 0$.

- When $\check{\mathrm{G}} = \mathrm{PGL}_2$, the same reasoning shows that

$$\mathcal{S} \cong \operatorname{Spec} k[b, x]/b(b+2x).$$

Let us now bring some homotopy theory into the mix.

Lemma 3.7.18. *Let S^1 act on the complex plane $\mathbf{C}$ by the weight n representation, so that S^1 acts on the one-point compactification $\mathbf{C} \cup \{\infty\} \cong \mathrm{S}^2$. Then*

$$\mathrm{H}^*_{\mathrm{S}^1}(\mathrm{S}^2; k) \cong k[x, b]/(b^2 = bnx)$$

as an algebra over $\mathrm{H}^*_{\mathrm{S}^1}(*; k) \cong k[x]$.

This comes from the fact that the Chern class of the line bundle associated to the weight n representation $\mathbf{C}$ over BS^1 is nx. The idea, now, is to show that:

Proposition 3.7.19. *There is a graded isomorphism*

$$\operatorname{Spec} \mathrm{H}^*_{\mathrm{B}_\mathbf{C} \times \mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}; k) \cong \mathcal{S},$$

such that the two unit maps

$$\operatorname{Spec} \mathrm{H}^*_{\mathrm{B}_\mathbf{C} \times \mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}; k) \rightrightarrows \operatorname{Spec} \mathrm{H}^*_{\mathrm{T}}(*; k) \cong \check{\mathfrak{t}}^*$$

get identified with the two maps $\mathcal{S} \rightrightarrows \check{\mathfrak{t}}^* \cong \psi + \check{\mathfrak{t}}^*$.

Proof. Let us identify $\mathrm{H}^*_{\mathrm{B}_\mathbf{C} \times \mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}; k) \cong \mathrm{H}^*_{\mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}/\mathrm{B}_\mathbf{C}; k)$ (so we are isolating one of the Borels $\mathrm{B}_\mathbf{C}$). Then, since $\mathrm{G}_\mathbf{C}/\mathrm{B}_\mathbf{C}$ has an even cell structure, the cohomology $\mathrm{H}^*_{\mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}/\mathrm{B}_\mathbf{C}; k)$ is a flat $\mathrm{H}^*_{\mathrm{B}_\mathbf{C}}(*; k)$-algebra. Since the structure map $\mathcal{S} \to \psi + \check{\mathfrak{t}}^*$ is flat by assumption, it suffices to give an isomorphism

$$\mathcal{S}|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \operatorname{Spec}(\mathrm{H}^*_{\mathrm{B}_\mathbf{C}}(\mathrm{G}_\mathbf{C}/\mathrm{B}_\mathbf{C}; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}$$

for each root α. The left-hand side can be identified with

$$\mathcal{S}|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathcal{R}_\alpha(\check{\mathrm{G}}) \cong \underline{\mathrm{W}} \times^{\mathrm{W}_\mathrm{H}} \mathcal{R}_\alpha(\mathrm{H}).$$

I will now use the homotopy equivalence $\mathrm{G}_{\mathbf{C}}/\mathrm{B}_{\mathbf{C}} \cong \mathrm{G}/\mathrm{T}$; note that G/T is a compact manifold, so the Atiyah-Bott localization theorem similarly lets us identify

$$\mathrm{Spec}(\mathrm{H}^*_\mathrm{T}(\mathrm{G}/\mathrm{T}; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathrm{Spec}(\mathrm{H}^*_\mathrm{T}((\mathrm{G}/\mathrm{T})^\alpha; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

Here, $(\mathrm{G}/\mathrm{T})^\alpha$ is the fixed point set of $\exp(\mathbf{R}\alpha)$ on G/T. In other words, it's the zero locus of the vector field generated by α. One can identify $(\mathrm{G}/\mathrm{T})^\alpha$ with $\mathrm{Z}_\mathrm{G}(\alpha)/\mathrm{T}$, where $\mathrm{Z}_\mathrm{G}(\alpha)$ is the centralizer of α in G. Therefore,

$$\mathrm{Spec}(\mathrm{H}^*_\mathrm{T}(\mathrm{G}/\mathrm{T}; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathrm{Spec}(\mathrm{H}^*_\mathrm{T}(\mathrm{Z}_\mathrm{G}(\alpha)/\mathrm{T}; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}}.$$

This means we need to identify

$$\mathrm{Spec}(\mathrm{H}^*_\mathrm{T}(\mathrm{Z}_\mathrm{G}(\alpha)/\mathrm{T}; k))|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \underline{\mathrm{W}} \times^{\mathrm{W}_{\mathrm{Z}_\mathrm{G}(\alpha)}} \mathcal{R}_\alpha(\mathrm{H}).$$

Now, the group $\mathrm{Z}_\mathrm{G}(\alpha)$ is disconnected: one can identify $\pi_0\mathrm{Z}_\mathrm{G}(\alpha) \cong \mathrm{W}/\mathrm{W}_{\mathrm{Z}_\mathrm{G}(\alpha)}$. Also, $\mathcal{S}(\mathrm{H})|_{\check{\mathfrak{t}}^*_{\alpha\text{-reg}}} \cong \mathcal{R}_\alpha(\mathrm{H})$. So if $\mathrm{Z}_\mathrm{G}(\alpha)^\circ$ denote the connected component of the identity in $\mathrm{Z}_\mathrm{G}(\alpha)$, it suffices to identify

$$\mathrm{Spec}(\mathrm{H}^*_\mathrm{T}(\mathrm{Z}_\mathrm{G}(\alpha)^\circ/\mathrm{T}; k)) \cong \mathcal{S}(\mathrm{H}).$$

But the complexification of $\mathrm{Z}_\mathrm{G}(\alpha)^\circ$ is a connected reductive group with semisimple rank 1, and so it is isomorphic to $\mathrm{GL}_2 \times \mathrm{A}$, $\mathrm{PGL}_2 \times \mathrm{A}$, or $\mathrm{SL}_2 \times \mathrm{A}$ where A is a torus. We will ignore the torus A, since it just goes along for the ride.

The goal, therefore, is to describe $\mathrm{H}^*_{\mathrm{B}_\mathbf{C}}(\mathrm{H}_\mathbf{C}/\mathrm{B}_\mathbf{C}; k)$ where $\mathrm{H} = \mathrm{GL}_2, \mathrm{PGL}_2$, or SL_2. But this is very explicit:

- Suppose $\mathrm{H}_\mathbf{C} = \mathrm{GL}_2$, so $\mathrm{T} = \mathrm{T}^2$. The quotient H/T can be identified with the one-point compactification of the representation $\mathrm{T}^2 \to \mathrm{S}^1$ sending $(\lambda_1, \lambda_2) \mapsto \lambda_2\lambda_1^{-1}$. But now, Lemma 3.7.18 implies that

$$\mathrm{H}^*_{\mathrm{T}^2}(\mathrm{GL}_2/\mathrm{T}^2; k) \cong k[x, y, b]/(b^2 - b(y - x)).$$

- Suppose $\mathrm{H}_\mathbf{C} = \mathrm{PGL}_2$, so $\mathrm{T} = \mathrm{S}^1$. Then the action of S^1 on $\mathrm{H}/\mathrm{T} \cong \mathrm{S}^2$ can be identified with the one-point compactification of the standard representation of S^1 on $\mathbf{C}$. Lemma 3.7.18 gives an isomorphism

$$\mathrm{H}^*_{\mathrm{S}^1}(\mathrm{PGL}_2/\mathrm{S}^1; k) \cong k[x, b]/(b^2 - bx).$$

- Suppose $\mathrm{H}_\mathbf{C} = \mathrm{SL}_2$, so $\mathrm{T} = \mathrm{S}^1$. Then the action of S^1 on $\mathrm{H}/\mathrm{T} \cong \mathrm{S}^2$ can be identified with the one-point compactification of the weight 2 representation of S^1 on $\mathbf{C}$. Lemma 3.7.18 gives an isomorphism

$$\mathrm{H}^*_{\mathrm{S}^1}(\mathrm{SL}_2/\mathrm{S}^1; k) \cong k[x, b]/(b^2 - 2bx).$$

Each case identifies with $\mathcal{S}(\mathrm{H})$ (as a scheme over $\check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^*$) as computed in Example 3.7.17. □

We can identify

$$\mathrm{H}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k) \cong \pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k) \otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{C}^*_{\mathrm{T}}(*;k)\right),$$

and so one can restate Proposition 3.7.19 as an isomorphism

$$\operatorname{Spec}\pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k) \otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{C}^*_{\mathrm{T}}(*;k)\right) \cong (\psi + \check{\mathfrak{t}}^*) \times_{(\psi+\check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}} (\psi + \check{\mathfrak{t}}^*)$$

of schemes over

$$\operatorname{Spec}\pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k) \otimes \mathrm{C}^*_{\mathrm{T}}(*;k)\right) \cong \check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \cong (\psi + \check{\mathfrak{t}}^*) \times (\psi + \check{\mathfrak{t}}^*).$$

Corollary 3.7.20. *There is an isomorphism of simplicial schemes*

$$\operatorname{Spec}\pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k)^{\otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)}\bullet}\right) \cong (\psi + \check{\mathfrak{t}}^*)^{\times_{(\psi+\check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}}\bullet} =: \mathcal{S}^\bullet.$$

Proof. Let me describe the termwise identification (it works also to describe the cosimplicial schemes). By Proposition 3.7.19, we have

$$(\psi + \check{\mathfrak{t}}^*)^{\times_{(\psi+\check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}}n} \cong \mathcal{S}^{\times_{\psi+\check{\mathfrak{t}}^*}n-1} \cong \operatorname{Spec}\mathrm{H}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k)^{\otimes_{\mathrm{H}^*_{\mathrm{T}}(*;k)}n-1}.$$

To win, we need to see that

$$\mathrm{H}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k)^{\otimes_{\mathrm{H}^*_{\mathrm{T}}(*;k)}n-1} \cong \pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k)^{\otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)}n}\right).$$

Because

$$\mathrm{C}^*_{\mathrm{T}}(*;k)^{\otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)}n} \simeq \mathrm{C}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k)^{\otimes_{\mathrm{C}^*_{\mathrm{T}}(*;k)}n-1},$$

there is a “Tor-spectral sequence”

$$\mathrm{E}_1^{*,*} = \pi_{-*}\mathrm{H}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k)^{\otimes^{\mathrm{L}}_{\mathrm{H}^*_{\mathrm{T}}(*;k)}n-1} \Rightarrow \pi_{-*}\left(\mathrm{C}^*_{\mathrm{T}}(*;k)^{\otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)}n}\right).$$

Here, $\otimes^{\mathrm{L}}$ means the derived tensor product. But this spectral sequence degenerates at the E_1-page because $\mathrm{H}^*_{\mathrm{T}\times\mathrm{T}}(\mathrm{G};k)$ is flat over $\mathrm{H}^*_{\mathrm{T}}(*;k)$ (so $\otimes^{\mathrm{L}} = \otimes$), and the whole E_1-page is concentrated in even degrees (and differentials have odd degree). $\square$

We can finally turn to:

Proof of Theorem 3.7.7. The map $\mathrm{C}^*_{\mathrm{G}}(*;k) \to \mathrm{C}^*_{\mathrm{T}}(*;k)$ is evenly faithfully flat. Indeed, suppose $\mathrm{C}^*_{\mathrm{G}}(*;k) \to \mathrm{A}$ is a map to an even $\mathbf{E}_\infty$-ring. Then:

- We can identify
$$\mathrm{C}^*_{\mathrm{T}}(*;k) \otimes_{\mathrm{C}^*_{\mathrm{T}}(*;k)} \mathrm{A} \cong \mathrm{C}^*(\mathrm{G}/\mathrm{T};\mathrm{A}).$$
But G/T has an even cell structure, and so $\mathrm{C}^*(\mathrm{G}/\mathrm{T};\mathrm{A})$ is concentrated in even degrees because A has even homotopy.
- Again G/T has an even cell structure, and so since A has even homotopy, $\mathrm{C}^*(\mathrm{G}/\mathrm{T};\mathrm{A})$ is a flat A-algebra.

By Lemma 2.1.5, we may identify

$$\operatorname{Spev}(\mathrm{C}^*_{\mathrm{G}}(*;k)) \cong \left|\operatorname{Spec}\pi_*\left(\mathrm{C}^*_{\mathrm{T}}(*;k)^{\otimes_{\mathrm{C}^*_{\mathrm{G}}(*;k)}\bullet}\right)\right|.$$

Corollary 3.7.20 identifies this with $|\mathcal{S}^\bullet|$. But $|\mathcal{S}^\bullet| \cong (\psi+\check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}$, because the map $q : \psi+\check{\mathfrak{t}}^* \to (\psi+\check{\mathfrak{n}}^\perp)/\check{\mathrm{N}}$ is assumed to be faithfully flat. $\square$

Proof of Theorem 3.7.2. Proposition 3.7.6 and Theorem 3.7.5 together imply the first equivalence of Theorem 3.7.2. The case G-equivariant local systems follows once we show that there is an isomorphism of stacks

$$\mathrm{Spev}(\mathrm{C}^*(\mathrm{BG};k))/\,\mathrm{Spev}(\mathrm{C}_*^{\mathrm{G}}(\Omega\mathrm{G};k)) \cong \check{\mathfrak{g}}^{*,\mathrm{reg}}(2)/\check{\mathrm{G}}. \tag{3.7.4}$$

note that Proposition 3.7.12 implies that there is a Cartesian square

$$\begin{tikzcd} \psi + \check{\mathfrak{t}}^* \arrow[r] \arrow[d] & \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}} \arrow[d] \\ (\psi + \check{\mathfrak{n}}^\perp)/\check{\mathrm{N}} \arrow[r] & \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}}. \end{tikzcd}$$

We have already calculated that there is an isomorphism

$$\mathrm{Spev}(\mathrm{C}^*(\mathrm{BT};k))/\,\mathrm{Spev}(\mathrm{C}_*^{\mathrm{T}}(\Omega\mathrm{G};k)) \cong \check{\mathfrak{t}}^*(2)/\widetilde{\check{\mathrm{J}}} \cong \widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}(2)/\check{\mathrm{G}},$$

so that Theorem 3.7.7 and the above Cartesian square imply the desired isomorphism (3.7.4). □

3.8 The K-theoretic story

Our goal in this section is to prove an analogue of Corollary 3.6.8, albeit with coefficients in $k = \mathrm{KU}$. Note that in this case, $\mathcal{M}_{\mathrm{T},0} \cong \mathrm{T}$. To do so, we need an analogue of Definition 3.6.1 and constructions surrounding it. Recall that the group G (over $\mathbf{C}$) is connected, almost simple, and simply-laced. We will also fix an algebraically closed field F,over which the Langlands dual group $\check{\mathrm{G}}$ will live. When dealing with the algebraic geometry (as opposed to the topology) of G, we will also view it as living over F; since G is simply-laced, it is isogenous to $\check{\mathrm{G}}$.

Definition 3.8.1. Let G^{sc} denote the simply-connected cover of G, and let $f \in \mathrm{G}^{\mathrm{sc}}$ be a principal nilpotent element as defined in [Ste, Theorem 4.6]. We will denote its image under the map $\mathrm{G}^{\mathrm{sc}} \to \mathrm{G}$ also by f. Then the map $\mathbf{G}_a \to \mathrm{G}$ corresponding to f factors through the map $\mathbf{G}_a = \mathrm{B} \to \mathrm{SL}_2$; we will denote the image of the standard generator $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ under the map $\mathrm{SL}_2 \to \mathrm{G}$ by $e \in \mathrm{G}$. Let $\mathrm{Z}_{\mathrm{G}}(e)^\circ$ be the connected component of the identity in the centralizer of e in G. Define the *multiplicative Kostant slice* $\mathcal{S}_\mu$ by $f \cdot \mathrm{Z}_{\mathrm{G}}(e)^\circ \subseteq \mathrm{G}$. Since G is assumed to be simply-connected, the composite

$$\mathcal{S}_\mu \to \mathrm{G} \to \mathrm{G}/\!\!/\mathrm{G} \cong \mathrm{G}/\!\!/\check{\mathrm{G}} \cong \mathrm{T}/\!\!/\mathrm{W}$$

is an isomorphism. We will often denote the inclusion of the Kostant slice by $\kappa : \mathrm{T}/\!\!/\mathrm{W} \to \mathrm{G}$.

The *multiplicative Grothendieck-Springer resolution* $\widetilde{\check{\mathrm{G}}}$ is defined as

$$\widetilde{\check{\mathrm{G}}} = \mathrm{B} \times^{\check{\mathrm{B}}} \check{\mathrm{G}},$$

where $\check{\mathrm{B}}$ acts on B by conjugation. (This makes sense thanks to the assumption that G is simply-laced.) There is a natural map $\widetilde{\check{\mathrm{G}}} \to \mathrm{G}$, given by the conjugation action of $\check{\mathrm{G}}$ on B. Let $\widetilde{\mathcal{S}}_\mu$ denote the fiber product $\widetilde{\mathcal{S}}_\mu \times_{\mathrm{G}} \widetilde{\check{\mathrm{G}}}$, so that the composite

$$\widetilde{\mathcal{S}}_\mu \to \widetilde{\check{\mathrm{G}}} \to \mathrm{T}$$

is an isomorphism; we will denote the inclusion of $\widetilde{\mathcal{S}}_\mu$ as a map $\kappa : \widetilde{\mathcal{S}}_\mu \cong \mathrm{T} \to \widetilde{\check{\mathrm{G}}}$.

As with the additive Kostant slice, we will only care about the composite $\mathrm{T} \to \widetilde{\check{\mathrm{G}}} \to \widetilde{\check{\mathrm{G}}}/\check{\mathrm{G}}$ below, so we will also denote it by κ. If we identify $\widetilde{\check{\mathrm{G}}}/\check{\mathrm{G}} \cong \mathrm{B}/\check{\mathrm{B}}$, then the map κ admits a simple description: it is the composite $f \cdot \mathrm{T} \to \mathrm{B} \to \mathrm{B}/\check{\mathrm{B}}$.

Definition 3.8.2. The stabilizer (inside $\check{\mathrm{G}}$) of the multiplicative Kostant slice $\mathcal{S}_\mu \subseteq \mathrm{G}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{\mathrm{G}} \times \mathcal{S}_\mu$, and will be denoted by $\check{\mathrm{J}}_\mu$. It will be called the *multiplicative regular centralizer group scheme*; if we wish to emphasize the dependence on G, we will denote it by $\check{\mathrm{J}}_\mu(\mathrm{G})$. Note that since the composite $\mathcal{S}_\mu \to \mathrm{G}^{\mathrm{reg}} \to \mathrm{G}/\!\!/\check{\mathrm{G}}$ is an isomorphism, we may identify

$$\check{\mathrm{J}}_\mu \cong \mathcal{S}_\mu \times_{\mathrm{G}/\check{\mathrm{G}}} \mathcal{S}_\mu.$$

Similarly, the stabilizer (inside $\check{\mathrm{G}}$) of the multiplicative Kostant slice $\widetilde{\mathcal{S}}_\mu \subseteq \widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}$ is a closed subgroup scheme of the constant group scheme $\check{\mathrm{G}} \times \widetilde{\mathcal{S}}_\mu$, and will be denoted by $\widetilde{\check{\mathrm{J}}}_\mu$. Since $\widetilde{\mathcal{S}}_\mu \cong \mathcal{S}_\mu \times_{\mathrm{G}} \widetilde{\check{\mathrm{G}}}$, we may identify

$$\widetilde{\check{\mathrm{J}}}_\mu \cong \check{\mathrm{J}}_\mu \times_{\mathcal{S}_\mu} \widetilde{\mathcal{S}}_\mu \cong (f \cdot \mathrm{T}) \times_{\mathrm{B}/\check{\mathrm{B}}} (f \cdot \mathrm{T}).$$

The following calculation also appears in [BFM], albeit using different techniques.

Theorem 3.8.3. *There is an isomorphism of group schemes over $f \cdot \mathrm{T} \cong \mathrm{T} \cong \mathcal{M}_{\mathrm{T},0}$:*

$$\mathrm{Spec}(\pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \otimes_{\mathbf{Z}} \mathrm{F}) \cong (f \cdot \mathrm{T}) \times_{\mathrm{B}/\check{\mathrm{B}}} (f \cdot \mathrm{T}).$$

Just as in Theorem 3.6.3, the proof of Theorem 3.8.3 will rely on two lemmas.

Lemma 3.8.4. *The projection map $\widetilde{\check{\mathrm{J}}}_\mu \to f \cdot \mathrm{T}$ (onto either factor) is flat.*

Proof. Like in the proof of Lemma 3.6.4, it suffices, by miracle flatness, to show that the fibers of the map $\widetilde{\check{\mathrm{J}}}_\mu \to f \cdot \mathrm{T}$ have dimension exactly $\mathrm{rank}(\check{\mathrm{G}})$. The fiber of this map over $f \cdot x \in f \cdot \mathrm{T}$ is the scheme

$$\mathrm{Y} = \{(g, y) \in \check{\mathrm{B}} \times \mathrm{T} | \mathrm{Ad}_g(fy) = fx\}.$$

Observe that the image of $\mathrm{Ad}_g(fy)$ and fx (viewed as elements of B) under the map $\mathrm{B} \to \mathrm{T}$ are y and x; so $y = x$ in T, which means that Y is isomorphic to the centralizer $\mathrm{Z}_{\check{\mathrm{B}}}(fx)$. The dimension estimate is equivalent to the claim that fx is a regular element of G, since this means that its centralizer has minimal dimension (namely, the rank of G, which is also the rank of $\check{\mathrm{G}}$). The desired regularity of fx follows from the discussion in [Ste, Remark 4.7]. (Note that, as mentioned in *loc. cit.*, the specific choice of the regular unipotent f is crucial for the regularity of fx.) □

Notation 3.8.5. Let α be a root of $\check{\mathrm{G}}$. Say that a point $x \in \mathrm{T}$ is *α-generic* if $x(h_\beta) \neq 1$ for all roots $\beta \neq \alpha$. This implies that the centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(x)$ has semisimple rank at most 1. Let $\mathrm{T}_{\alpha\text{-reg}}$ denote the α-regular locus. Observe that $\mathrm{T}_{\mathrm{reg}} = \bigcup_{\alpha \in \Phi} \mathrm{T}_{\alpha\text{-reg}} \subseteq \mathrm{T}$ is open, with complement of codimension 2.

The proof of the next result is exactly as in Lemma 3.6.6.

Lemma 3.8.6. *There is an isomorphism*

$$\widetilde{\check{\mathrm{J}}}_\mu(\check{\mathrm{G}})|_{\mathrm{T}_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{\mathrm{J}}}_\mu(\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ)|_{\mathrm{T}_{\alpha\text{-reg}}}, \tag{3.8.1}$$

where $\mathrm{Z}_{\check{\mathrm{G}}}(x)$ is the centralizer of some $x \in \mathrm{T}_{\alpha\text{-reg}}$ which lies on the α-hyperplane, and $\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ$ denotes the connected component of the identity.

Proof of Theorem 3.8.3. The argument of Theorem 3.6.3 reduces us to checking that the isomorphism of Theorem 3.8.3 holds if G has semisimple rank 1, i.e., is the product of a torus with one of GL_2, SL_2, or PGL_2. Again, it is easy to match up the contributions from the toral factors, so we will assume that G is either GL_2, SL_2, or PGL_2. In this case, we can even replace F by $\mathbf{Z}$.

- When $\mathrm{G} = \mathrm{GL}_2$, we may identify $\widetilde{\check{\mathrm{J}}}_\mu$ with the centralizer (in $\check{\mathrm{B}}$) of $\left(\begin{smallmatrix} x & 0 \\ x & y \end{smallmatrix}\right)$. It is easy to compute that $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & y \end{smallmatrix}\right)$ if and only if $c = \frac{a-d}{x-y} \cdot x$, meaning that

  $$\widetilde{\check{\mathrm{J}}}_\mu \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

  The coproduct sends $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$. The same argument as in Theorem 3.6.3 implies that

  $$\mathrm{KU}_*^{\mathrm{T}_c}(\Omega \mathrm{GL}_2) \cong \mathbf{Z}[u^{\pm 1}, x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}, \tfrac{a-d}{x-y}].$$

  The map induced on T-equivariant KU-homology by the inclusion $\mathrm{T}^2 \to \mathrm{GL}_2$ is simply given by the inclusion of the subalgebra $\mathbf{Z}[u^{\pm 1}, x^{\pm 1}, y^{\pm 1}, a^{\pm 1}, d^{\pm 1}]$. The coproduct on this subalgebra (and hence, on $\mathrm{KU}_*^{\mathrm{T}_c}(\Omega \mathrm{GL}_2)$) is determined by the formulas $a \mapsto a \otimes a$ and $d \mapsto d \otimes d$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{\mathrm{T}_c}(\Omega \mathrm{GL}_2)$ is isomorphic to $\widetilde{\check{\mathrm{J}}}_\mu$ as group schemes over $\operatorname{Spec} \pi_0 \mathrm{KU}_{\mathrm{T}_c} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$, as desired.

- When $\mathrm{G} = \mathrm{SL}_2$, we may identify $\widetilde{\check{\mathrm{J}}}_\mu$ with the centralizer (in $\check{\mathrm{B}} \subseteq \mathrm{PGL}_2$) of $\left(\begin{smallmatrix} x & 0 \\ x & x^{-1} \end{smallmatrix}\right)$. An element $\left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \check{\mathrm{B}} \subseteq \mathrm{PGL}_2$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & x^{-1} \end{smallmatrix}\right)$ if and only if $c = \frac{a-1}{x-x^{-1}} \cdot x$. Therefore,

  $$\widetilde{\check{\mathrm{J}}}_\mu \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-1}{x-x^{-1}}];$$

  the coproduct sends $a \mapsto a \otimes a$.

  Next, there is an isomorphism

  $$\mathrm{KU}_*^{\mathrm{S}^1}(\Omega \mathrm{SL}_2) \cong \mathbf{Z}[u^{\pm 1}, x^{\pm 1}, a^{\pm 1}, \tfrac{a-1}{x^2-1}].$$

  This is proved exactly as in Theorem 3.6.3; the role of the class $2x$ is now played by the Chern class $x^2 - 1 \in \pi_0 \mathrm{KU}_{\mathrm{S}^1}$ of the weight 2 representation of S^1. (Recall that the action of S^1 on $\mathrm{G}_c \cong \mathrm{SU}(2) \cong \mathrm{S}^3$ exhibits it as the one-point compactification of the trivial 1-dimensional representation summed with the weight 2 representation of S^1 on $\mathbf{C}$.) The map induced on T-equivariant KU-homology by the inclusion $\mathrm{S}^1 \to \mathrm{SU}(2)$ of the maximal torus is simply given by the inclusion of the subalgebra $\mathbf{Z}[u^{\pm 1}, x^{\pm 1}, a^{\pm 1}]$. The coproduct on this subalgebra (and hence, on $\mathrm{KU}_*^{\mathrm{S}^1}(\Omega \mathrm{SL}_2)$) is determined by the formula $a \mapsto a \otimes a$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{\mathrm{S}^1}(\Omega \mathrm{SL}_2)$ is isomorphic to $\widetilde{\check{\mathrm{J}}}_\mu$ as group schemes over $\operatorname{Spec} \pi_0 \mathrm{KU}_{\mathrm{S}^1} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}]$, as desired.

- When $\mathrm{G} = \mathrm{PGL}_2$, we may identify $\widetilde{\check{\mathrm{J}}}_\mu$ with the centralizer (in $\check{\mathrm{B}} \subseteq \mathrm{SL}_2$) of $\left(\begin{smallmatrix} x & 0 \\ x & 1 \end{smallmatrix}\right)$. An element $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right) \in \check{\mathrm{B}} \subseteq \mathrm{SL}_2$ stabilizes $\left(\begin{smallmatrix} x & 0 \\ x & 1 \end{smallmatrix}\right)$ if and only if $c = \frac{a-a^{-1}}{x-1} \cdot x$. Therefore,

$$\widetilde{\check{\mathrm{J}}}_\mu \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-1}];$$

the coproduct sends $a \mapsto a \otimes a$. Again, as in the preceding cases, there is an isomorphism

$$\mathrm{KU}_*^{\mathrm{S}^1}(\Omega \mathrm{PGL}_2) \cong \mathbf{Z}[u^{\pm 1}, x^{\pm 1}, a^{\pm 1}, \tfrac{a-a^{-1}}{x-1}],$$

where the coproduct sends $a \mapsto a \otimes a$. It follows that $\operatorname{Spec} \mathrm{KU}_0^{\mathrm{S}^1}(\Omega \mathrm{PGL}_2)$ is isomorphic to $\widetilde{\check{\mathrm{J}}}_\mu$ as group schemes over $\operatorname{Spec} \pi_0 \mathrm{KU}_{\mathrm{S}^1} \cong \operatorname{Spec} \mathbf{Z}[x^{\pm 1}]$, as desired. $\square$

Remark 3.8.7. Just for posterity, let us record a more canonical variant of the calculation above for $\check{\mathrm{G}} = \mathrm{SL}_2$, which does not require picking a Borel subgroup (i.e., which does not involve identifying $\widetilde{\check{\mathrm{G}}}/\check{\mathrm{G}} \cong \mathrm{B}/\check{\mathrm{B}}$). If $\lambda \in \mathbf{G}_m$, we denote $\lambda + \lambda^{-1} \in \mathbf{A}^1$ by $f(\lambda)$. The Kostant slice $\kappa : \check{\mathrm{T}} \cong \mathbf{G}_m \to \widetilde{\mathrm{SL}}_2$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \begin{pmatrix} f(\lambda) - 1 & f(\lambda) - 2 \\ 1 & 1 \end{pmatrix}, \ \ell = [\lambda - 1 : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ: the key point is the conic relation

$$2\lambda = f(\lambda) - \sqrt{f(\lambda)^2 - 4}.$$

Indeed, this calculation of $\kappa(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}_m \cong \check{\mathrm{T}} \arrow[r, "\kappa"] \arrow[d, "\lambda \mapsto f(\lambda)"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \check{\mathrm{T}}/\!\!/\mathrm{W} \arrow[r, "\kappa", "\lambda \mapsto \left(\begin{smallmatrix} \lambda-1 & \lambda-2 \\ 1 & 1 \end{smallmatrix}\right)"'] & \mathrm{SL}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we directly compute that $\mathrm{Ad}_g(x) = x$ if and only if $b = c(f(\lambda) - 2)$ and $a - d = (f(\lambda) - 2)c$, in which case g also preserves ℓ. Therefore, $g = \left(\begin{smallmatrix} (f(\lambda)-2)c+d & (f(\lambda)-2)c \\ c & d \end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$d^2 + c(f(\lambda) - 2)(d - c) = 1.$$

Both x and g can be simultaneously diagonalized (if $f(\lambda) \neq \pm 2$); note that $\lambda + \lambda^{-1}$ is an eigenvalue of x. If t is an eigenvalue of g, then we have $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$ and $d = \frac{t^2\lambda+1}{t(\lambda+1)}$. When k is not of characteristic 2, this shows that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong k[\lambda^{\pm 1}, t^{\pm 1}, \tfrac{t-t^{-1}}{\lambda-\lambda^{-1}}].$$

This in turn implies that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{PGL}_2} \mathbf{G}_m \cong k[\lambda^{\pm 1}, t^{\pm 2}, \tfrac{t^2-1}{\lambda-\lambda^{-1}}],$$

as desired.

There is *another* choice of slice when G is simply-connected; the calculation of Theorem 3.8.3 continues to hold for it, too, as we now illustrate in the example of SL_2.

Definition 3.8.8 (Steinberg slice)**.** Let G be a simply-connected semisimple algebraic group. Given $w \in \mathrm{W}$, let $\mathrm{N}_w = \mathrm{N} \cap w^{-1}\mathrm{N}^- w$, so that $\mathrm{N}_w = \prod_{\alpha \in \Phi_w} \mathrm{U}_\alpha$, where Φ_w is the set of roots made negative by w. Let $w = \prod_{\alpha \in \Delta} s_\alpha \in \mathrm{W}$ be a Coxeter element, and let $\dot{w}$ be a lift of w to $\mathrm{N_G(T)}$. Define the Steinberg slice $\Sigma = \dot{w}\mathrm{N}_w \subseteq \mathrm{G}$. Then [Ste] proved/stated that the composite $\Sigma \to \mathrm{G} \to \mathrm{G}/\!\!/\mathrm{G} \cong \mathrm{T}/\!\!/\mathrm{W}$ is an isomorphism. Let $\widetilde{\Sigma}$ denote the fiber product $\Sigma \times_{\mathrm{G}} \widetilde{\check{\mathrm{G}}}$, so that the composite $\widetilde{\Sigma} \to \widetilde{\check{\mathrm{G}}} \to \mathrm{T}$ is an isomorphism. We will denote the inclusion of $\widetilde{\Sigma}$ by $\sigma : \mathrm{T} \to \widetilde{\check{\mathrm{G}}}$.

Observation 3.8.9. We will illustrate the calculation of $\mathrm{T} \times_{\widetilde{\check{\mathrm{G}}}/\check{\mathrm{G}}} \mathrm{T}$ (with T mapping to $\widetilde{\check{\mathrm{G}}}$ by σ) when $\mathrm{G} = \mathrm{SL}_2$. View a point in $\widetilde{\check{\mathrm{G}}}$ as a pair $(x \in \mathrm{SL}_2, \ell \subseteq \mathbf{C}^2)$ such that x preserves ℓ. The Steinberg slice $\sigma : \mathbf{G}_m \to \widetilde{\mathrm{SL}}_2$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \begin{pmatrix} \lambda + \lambda^{-1} & -1 \\ 1 & 0 \end{pmatrix}, \ \ell = [\lambda : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ. This calculation of $\sigma(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}_m \cong \check{\mathrm{T}} \arrow[r, "\sigma"] \arrow[d, "\lambda \mapsto \lambda + \lambda^{-1}"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \check{\mathrm{T}}/\!\!/\mathrm{W} \arrow[r, "\sigma", "\lambda \mapsto \left(\begin{smallmatrix} \lambda & -1 \\ 1 & 0 \end{smallmatrix}\right)"'] & \mathrm{SL}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, one can directly compute that g commutes with $\left(\begin{smallmatrix} \lambda+\lambda^{-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$ if and only if $a = c(\lambda + \lambda^{-1}) + d$ and $b = -c$. Therefore, $g = \left(\begin{smallmatrix} c(\lambda+\lambda^{-1})+d & -c \\ c & d \end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$c^2 + d^2 + cd(\lambda + \lambda^{-1}) = 1.$$

As long as $\lambda \neq \pm 1$, both x and g can be simultaneously diagonalized by $\left(\begin{smallmatrix} \lambda & \lambda^{-1} \\ 1 & 1 \end{smallmatrix}\right)$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} c\lambda+d & 0 \\ 0 & c\lambda^{-1}+d \end{smallmatrix}\right)$. If $t = c\lambda + d$, then $c\lambda^{-1} + d = t^{-1}$ by the above determinant relation. We also have that $a = t - \frac{\lambda(t-t^{-1})}{\lambda-\lambda^{-1}}$ and $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$. This shows that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 1}, \tfrac{t-t^{-1}}{\lambda-\lambda^{-1}}],$$

and hence that

$$\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{PGL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 2}, \tfrac{t^2-1}{\lambda-\lambda^{-1}}],$$

as desired.

Corollary 3.8.10. *There is an* F*-linear equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}).$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU}) \to \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(*;\mathrm{KU})$ *identifies with the pullback functor* $\kappa^* : \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{T})$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ is equivalent to the category of comodules over $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee = \mathrm{KU}^{\mathrm{T}}_0(\mathrm{Gr}_{\mathrm{G}})$ in the category of $\pi_0\mathrm{KU}_{\mathrm{T}_c}$-modules. By Theorem 3.8.3, it can be identified the category of quasicoherent sheaves on the quotient stack $(f\cdot\mathrm{T})/\widetilde{\check{\mathrm{J}}}_\mu$. We may view $\widetilde{\check{\mathrm{J}}}_\mu$ as a closed subgroup scheme of the constant group scheme $\check{\mathrm{B}}\times(f\cdot\mathrm{T})$. This gives an isomorphism

$$(f\cdot\mathrm{T})/\widetilde{\check{\mathrm{J}}}_\mu \cong \check{\mathrm{B}}\backslash(\check{\mathrm{B}}\times(f\cdot\mathrm{T}))/\widetilde{\check{\mathrm{J}}}_\mu.$$

It follows from Steinberg's work in [Ste] that the $\check{\mathrm{B}}$-orbit of $f\cdot\mathrm{T}$ inside B is precisely the regular locus $\mathrm{B}^{\mathrm{reg}}$. Since $\widetilde{\check{\mathrm{J}}}_\mu$ is definitionally the stabilizer of $f\cdot\mathrm{T}\subsetneq\mathrm{B}$, the quotient $(\check{\mathrm{B}}\times(f\cdot\mathrm{T}))/\widetilde{\check{\mathrm{J}}}_\mu$ is isomorphic to $\mathrm{B}^{\mathrm{reg}}$; so there is an isomorphism $(f\cdot\mathrm{T})/\widetilde{\check{\mathrm{J}}}_\mu \cong \mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}}$. To finish, note that $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}} \cong \mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}}$. ∎

Similarly, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathbf{Z}}\mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\prime,\mathrm{reg}}/\check{\mathrm{G}}),$$

where $\widetilde{\check{\mathrm{G}}}^{\prime}$ is $\check{\mathrm{G}}\times^{\check{\mathrm{B}}}\check{\mathrm{B}}$, with $\check{\mathrm{B}}$ acting on itself by conjugation. Note that $\widetilde{\check{\mathrm{G}}}^{\prime,\mathrm{reg}}/\check{\mathrm{G}} \cong \check{\mathrm{B}}^{\mathrm{reg}}/\check{\mathrm{B}}$ is an open substack of the stack $\check{\mathrm{B}}/\check{\mathrm{B}} \cong \mathrm{Map}(\mathrm{B}\mathbf{Z},\mathrm{B}\check{\mathrm{B}})$ of $\check{\mathrm{B}}$-bundles on the circle $\mathrm{S}^1 = \mathrm{B}\mathbf{Z}$.

The equivalence of Corollary 3.8.10 is in fact symmetric monoidal for the convolution tensor structure on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ (described in Remark 3.3.5) and the standard tensor product on $\mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}})$.

Remark 3.8.11. It can be shown that if G has torsion-free fundamental group, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathbf{Z}}\mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}}).$$

Just as in § 3.3, the left-hand side is defined as

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU}) = \mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee)}(\mathrm{QCoh}(\mathrm{T}/\!\!/\mathrm{W})).$$

Note that this is a sensible definition since $\pi_*\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ is concentrated in even degrees. Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU}) \to \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(*;\mathrm{KU})$ identifies with the pullback functor $\kappa^* : \mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{T}/\!\!/\mathrm{W})$. The proof of the displayed equivalence is quite similar to that of Corollary 3.8.10, and in fact can be deduced from it using the observation that $\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee) = \pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee)^{\mathrm{W}}$ and that the natural map $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}} \to \mathrm{G}^{\mathrm{reg}}$ is a (ramified) W-cover. The first statement uses that G has torsion-free fundamental group, and the second is a multiplicative version of Grothendieck-Springer theory.

Remark 3.8.12. In [Dev3, Section 3.7], we study a variant of Corollary 3.8.10, where KU is replaced by *connective* complex K-theory ku; that is, $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ is replaced by $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{ku})$. On the Langlands dual side, this has the effect of replacing $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}$ by the 1-parameter family over $\mathrm{Spec}(\pi_*(\mathrm{ku}))/\mathbf{G}_m \cong \mathbf{A}^1/\mathbf{G}_m$ whose generic fiber is $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}$, and whose special fiber is $\widetilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{\mathrm{G}}$.

Remark 3.8.13. There is a variant of Corollary 3.8.10 if G is not simply-laced, but it is more complicated to state. Let us just give the analogue of Theorem 3.8.3. Suppose G is not simply-laced, and let T be a maximal torus of G; then $\check{\mathfrak{g}}$ is the fixed point subalgebra $\check{\mathfrak{h}}^\tau$ of an finite-order outer automorphism τ of a simply-laced Lie algebra $\check{\mathfrak{h}}$. Let H denote the simply-connected simply-laced group corresponding to the Langlands dual $\mathfrak{h}$, and let $\mathrm{T_H}$ denote its maximal torus. Then we may identify the fixed subset $\mathbb{X}^*(\mathrm{T}')^\tau$ with $\mathbb{X}^*(\mathrm{T})$. If n denotes the order of τ, there is an action of $\mathbf{Z}/n$ on $\mathrm{T}[\![t]\!]$, $\mathrm{G}[\![t]\!]$, and $\mathrm{G}(\!(t)\!)$, given by τ on T and G, and $t \mapsto \zeta_n \tau$ for a primitive nth root of unity ζ_n. The appropriate replacement of $\pi_0 \mathcal{F}_\mathrm{T}(\mathrm{Gr_G})^\vee$ in this case is $\pi_0 \mathcal{F}_{\mathrm{T}[\![t]\!]^{\mathbf{Z}/n}}(\mathrm{G_{ad}}(\!(t)\!)^{\mathbf{Z}/n}/\mathrm{G_{ad}}[\![t]\!]^{\mathbf{Z}/n})^\vee$. The analogue of Theorem 3.8.3 (see [FT1, Theorem 3.9]) states that this algebra is isomorphic to the stabilizer $\mathcal{S}_\mu \times_{\check{\mathrm{G}}/\check{\mathrm{G}}} \mathcal{S}_\mu$.

The map $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}} \to \mathrm{B}\check{\mathrm{G}}$ defines a functor

$$\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F}. \tag{3.8.2}$$

More generally, the map $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}} \to \mathrm{B}\check{\mathrm{G}} \times \mathrm{B}\check{\mathrm{T}}$ defines a functor

$$\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{T}}) \to \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F}. \tag{3.8.3}$$

If $\mathrm{V} \in \mathrm{Rep}(\check{\mathrm{G}})$, let $\mathcal{S}_{\mathrm{KU}}(\mathrm{V})$ denote the corresponding object of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F}$. The same argument as in Proposition 3.6.13 shows the following, which says that $\mathcal{S}_{\mathrm{KU}}(\mathrm{V}) \in \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU})$ is the associated graded of a particular object $\mathcal{F}_\lambda \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr_G}; \mathrm{KU})$ if V is a minuscule $\check{\mathrm{G}}$-representation.

Proposition 3.8.14. *Let $\lambda_\bullet = (\lambda_1, \cdots, \lambda_n)$ be a tuple of dominant minuscule weights of $\check{\mathrm{G}}$, let $|\lambda_\bullet| = \sum_i \lambda_i$, and let $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ denote the corresponding convolution variety. Let $\mathcal{F}_{\lambda_\bullet}$ denote the pushforward of the constant sheaf along the canonical map $q : \overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}} \to \overline{\mathrm{Gr}_\mathrm{G}^{|\lambda|}} \subseteq \mathrm{Gr_G}$. If V_{λ_i} denotes the irreducible representation of $\check{\mathrm{G}}$ with highest weight λ_i, then there is an isomorphism $\mathcal{S}_{\mathrm{KU}}(\bigotimes_i \mathrm{V}_{\lambda_i}) \cong \mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$.*

It would be very interesting to understand whether Proposition 3.8.14 can be extended to other non-minuscule irreducible representations. As in Remark 3.6.14, if λ is a dominant minuscule weight of $\check{\mathrm{G}}$, then the coaction of $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{Gr_G})^\vee$ on $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G/P}_\lambda)$ defines a homomorphism

$$\mathrm{Spec}\, \pi_0\mathcal{F}_\mathrm{T}(\mathrm{Gr_G})^\vee \to \mathrm{GL}(\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G/P}_\lambda)) \tag{3.8.4}$$

of group schemes over T, where $\mathrm{GL}(\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G/P}_\lambda))$ denotes the group scheme of $\mathcal{O}_\mathrm{T}$-linear automorphisms of the vector bundle $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G/P}_\lambda)$. Under the isomorphisms of Theorem 3.8.3 and Proposition 3.8.14, this homomorphism factors as the composite

$$\widetilde{\check{\mathrm{J}}}_\mu \to \check{\mathrm{G}} \times \mathrm{T} \to \mathrm{GL}(\mathrm{V}_\lambda) \times \mathrm{T}, \tag{3.8.5}$$

where the second map describes the $\check{\mathrm{G}}$-action on V_λ. Similar statements hold with $\widetilde{\check{\mathrm{J}}}_\mu$ replaced by $\check{\mathrm{J}}_\mu$ and $\pi_0\mathcal{F}_\mathrm{T}(\mathrm{G/P}_\lambda)$ replaced by $\pi_0\mathcal{F}_\mathrm{G}(\mathrm{G/P}_\lambda) \cong \pi_0\mathrm{KU}_{\mathrm{L}_\lambda}$ (where L_λ is the Levi quotient of P_λ).

Theorem 3.8.3 has several applications. Here is one, following the same proof as in Proposition 3.6.15; it gives a *multiplicative* version of the Gelfand-Graev action on the affine closure $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$:

Proposition 3.8.15 (Multiplicative Gelfand-Graev action)**.** *The natural action of* $\check{\mathrm{G}} \times \check{\mathrm{T}}$ *on the affine closure* $\overline{\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}}$ *extends to an action of* $\check{\mathrm{G}} \times (\mathrm{W} \rtimes \check{\mathrm{T}})$*, where* W *is the Weyl group.*

In the following, we will write $\overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ to denote the affine closure of the "multiplicative" cotangent bundle $\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}$. Unlike with Proposition 3.6.15, Proposition 3.8.15 does require G to be simply-laced; otherwise $\overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ would not even be well-defined. The moment map $\overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})} \to \mathrm{G}$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd}
\widetilde{\check{\mathrm{G}}} \arrow[r, hook] \arrow[dr] & \overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}} \arrow[d] \\
& \mathrm{G}
\end{tikzcd}$$

which relates $\overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ to the multiplicative Grothendieck-Springer resolution; and via this diagram, the multiplicative Gelfand-Graev action is closely related to the Weyl action in trigonometric/multiplicative Springer theory.

Remark 3.8.16. The proof of Proposition 3.8.15 generalizes to show that if $\check{\mathrm{P}} \subseteq \check{\mathrm{G}}$ is a parabolic subgroup with Levi quotient $\check{\mathrm{L}}$ and unipotent radical $\mathrm{U}_{\check{\mathrm{P}}}$, then the natural action of $\check{\mathrm{G}} \times \check{\mathrm{L}}$ on the affine closure $\overline{\check{\mathrm{G}} \times^{\mathrm{U}_{\check{\mathrm{P}}}} \mathrm{P}}$ extends to an action of $\check{\mathrm{G}} \times (\mathrm{W}_\mathrm{L} \rtimes \check{\mathrm{L}})$, where $\mathrm{W}_\mathrm{L} = \mathrm{N}_{\check{\mathrm{G}}}(\check{\mathrm{L}})/\check{\mathrm{L}}$ is the Weyl group.

Example 3.8.17. Let us make the above action explicit in the example of $\check{\mathrm{G}} = \mathrm{SL}_2$ (so $\mathrm{W} = \mathbf{Z}/2$). The group B in this case is contained in PGL_2, and can be chosen to be represented by matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$. The action of $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \check{\mathrm{N}}$ on $\check{\mathrm{G}} \times \mathrm{B}$ sends

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & an+b \\ c & cn+d \end{smallmatrix}\right), \ \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x & y-n(x-1) \\ 0 & 1 \end{smallmatrix}\right).$$

As explained in [Dev3, Remark 5.1.19], this means that the $\mathbf{G}_a$-action fixes a, c, x, $\mathrm{B} := ay + (x-1)b$, and $\mathrm{D} = cy + (x-1)d$. There is a single relation between these classes, given by

$$a\mathrm{D} - c\mathrm{B} = x - 1.$$

Let us relabel these variables so that $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $v = (v_1, v_2) = (\mathrm{D}, -\mathrm{B})$. Since x must be invertible, it follows that the affine closure $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} \mathrm{B}}$ is given by the complement of the hypersurface $1 + \langle u, v\rangle$ in $\mathrm{T}^*(\mathbf{A}^2)$. This is Van den Bergh's multiplicative quiver variety $\mathcal{B}(\mathrm{U}, \mathrm{V})$ from [Van], specialized to the case when the vector spaces U, V are $\mathbf{A}^2, \mathbf{A}^1$. An elementary analysis as in Example 3.6.17 shows that the $\mathbf{Z}/2$-action of Proposition 3.8.15 is given on $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} \mathrm{B}} \subseteq \mathrm{T}^*(\mathbf{A}^2)$ by the formula

$$(\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right), (v_1, v_2)) \mapsto \left(\tfrac{1}{1+\langle u,v\rangle}\left(\begin{smallmatrix} -v_2 \\ v_1 \end{smallmatrix}\right), (u_2, -u_1)\right).$$

In particular, it can be viewed as a multiplicative version of the symplectic Fourier transform.

Remark 3.8.18. The multiplicative symplectic Fourier transform of Example 3.8.17 is related to another, more geometric, Fourier-type transform, as we now describe. Let ℓ be a (complex) line. Recall from [Bei] that the (1-)category $\mathrm{Perv}(\ell)$ of perverse sheaves on ℓ with respect to the stratification by $0 \in \ell$ and its complement is equivalent to the category of diagrams of the form

$$\mathrm{X} \underset{u}{\overset{v}{\rightleftarrows}} \mathrm{Y} \tag{3.8.6}$$

with X and Y being vector spaces, such that $\mathrm{id}_{\mathrm{Y}} + uv$ (and therefore $\mathrm{id}_{\mathrm{X}} + vu$) is invertible. This equivalence sends $\mathcal{F} \in \mathrm{Perv}(\ell)$ to its spaces of nearby and vanishing cycles at $0 \in \ell$ (and the maps u, v arise via monodromy). The Fourier-Sato transform (see [KS1, Definition 3.7.8]) gives an equivalence $\mathrm{Perv}(\ell) \to \mathrm{Perv}(\ell^*)$, and one can check that it sends an object (3.8.6) to the object

$$\mathrm{Y} \underset{-v}{\overset{u(\mathrm{id}+vu)^{-1}}{\leftrightarrows}} \mathrm{X}.$$

Example 3.8.17 defines a morphism from $\overline{\mathrm{SL}_2 \times^{\mathbf{G}_a} \mathrm{B}}$ to the moduli of isomorphism classes of objects of $\mathrm{Perv}(\ell)$ (where $\mathrm{X} = \mathbf{A}^2$ and $\mathrm{Y} = \mathbf{A}^1$); this morphism intertwines the multiplicative symplectic Fourier transform with the Fourier-Sato transform.

We also have the following analogue of Proposition 3.6.18, whose proof is exactly the same (one only needs to note that $\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}} \hookrightarrow \widetilde{\check{\mathrm{G}}}$ has complement of codimension 2, and similarly for $\mathrm{G}^{\mathrm{reg}} \hookrightarrow \mathrm{G}$).

Proposition 3.8.19. *Let* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathrm{T}))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathrm{T})$. *Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\check{\mathrm{G}} \backslash \overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of (3.8.3). *Furthermore, this functor is* W*-equivariant for the natural action of* $\mathrm{W} = \mathrm{N}_{\mathrm{G}_c}(\mathrm{T}_c)/\mathrm{T}_c$ *on the left-hand side and the Gelfand-Graev action of Proposition 3.8.15 on the right-hand side.*

Similarly, suppose G *has torsion-free fundamental group, and let* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathrm{T}/\!\!/\mathrm{W}))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathrm{T}/\!\!/\mathrm{W})$. *Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{G}/\check{\mathrm{G}})$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of the functor $\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F}$ *(analogous to* (3.8.2)*).*

Proposition 3.8.19 gives an analogue of [BF, Theorem 4]: namely, if $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{G}/\check{\mathrm{G}})$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{G}/\check{\mathrm{G}})$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\mathrm{G}/\check{\mathrm{G}})^\heartsuit \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\mathbf{Z}} \mathrm{F}.$$

Similarly, if $\mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}} \backslash \overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{T}}) \to \mathrm{QCoh}(\check{\mathrm{G}} \backslash \overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}} \backslash \overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})^\heartsuit \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\mathbf{Z}} \mathrm{F}.$$

This implies the following result.

Corollary 3.8.20. *Let* $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{G}/\check{\mathrm{G}})^{\mathrm{min},\heartsuit}$ *denote the essential image of* $\mathrm{Rep}_{\mathrm{min}}(\check{\mathrm{G}})$ *under the pullback functor* $\mathrm{Rep}(\check{\mathrm{G}})^\heartsuit \to \mathrm{QCoh}(\mathrm{G}/\check{\mathrm{G}})^\heartsuit$. *Similarly, let* $(\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\mathbf{Z}} \mathrm{F})^{\mathrm{min}}$ *denote the idempotent completion of the subcategory of* $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\mathbf{Z}} \mathrm{F}$ *spanned by* $\mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$ *ranging over sequences* $\lambda_\bullet$ *of minuscule highest weights. Then there is an equivalence*

$$\mathrm{QCoh}_{\mathrm{free}}(\mathrm{G}/\check{\mathrm{G}})^{\mathrm{min},\heartsuit} \simeq (\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\mathbf{Z}} \mathrm{F})^{\mathrm{min}}.$$

There is a similar equivalence

$$(\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})^{\heartsuit}\otimes_{\mathbf{Z}}\mathrm{F})^{\mathrm{min}}\simeq \mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathbf{G}_m}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})^{\mathrm{min},\heartsuit},$$

where these categories are defined analogously by idempotent completion.

Note, again, that the category $(\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})^{\heartsuit}\otimes_{\mathbf{Z}}\mathrm{F})^{\mathrm{min}}$ is the heart of a degeneration, in the sense of § 3.3, of the similarly-defined category $(\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathrm{KU}}\mathrm{F}[u^{\pm 1}])^{\mathrm{min}}$. (In particular, Corollary 3.8.20 gives an equivalence between the purely algebraically defined category $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{G}/\check{\mathrm{G}})^{\mathrm{min},\heartsuit}$ and a degeneration of the purely topologically defined category $(\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathrm{KU}}\mathrm{F}[u^{\pm 1}])^{\mathrm{min}}$.) If $\lambda_\bullet$ and $\mu_\bullet$ are two sequences of dominant minuscule weights of $\check{\mathrm{G}}$, there is an equivalence of KU-modules

$$\mathrm{Map}_{(\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathrm{KU}}\mathrm{F}[u^{\pm 1}])^{\mathrm{min}}}(\mathcal{F}_{\lambda_\bullet},\mathcal{F}_{\mu_\bullet})\simeq \mathcal{F}_{\mathrm{G}_c}(\overline{\mathrm{Gr}_{\mathrm{G}}^{\lambda_\bullet}}\times_{\mathrm{Gr}_{\mathrm{G}}}\overline{\mathrm{Gr}_{\mathrm{G}}^{\mu_\bullet}}),$$

so that the category $(\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})\otimes_{\mathrm{KU}}\mathrm{F}[u^{\pm 1}])^{\mathrm{min}}$ compares to the category from [CK, Section 3.5].

As with Proposition 3.6.18, the existence of the t-structure on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ from Proposition 3.8.19 may at first glance perhaps be a bit surprising, since KU is a 2-*periodic* $\mathbf{E}_\infty$-ring. Again, this periodicity prohibits $\mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ itself from having a t-structure; but the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};k)$ itself has both a *homological* shift operation and a (periodic) *weight* shifting operation. The homological shift on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KU})$ is no longer periodic, and it is therefore reasonable to equip this ∞-category with a t-structure.

We now turn to the question of the analogue of Corollary 3.8.10 if KU is replaced by *real* K-theory KO. (Recall the definition of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KO})$ from Definition 3.3.13.) We begin by constructing a $\mathbf{Z}/2$-action on $\tilde{\check{\mathrm{G}}}/\check{\mathrm{G}}\cong\mathrm{B}/\check{\mathrm{B}}$.

Lemma 3.8.21. *There is a map $\gamma:\mathrm{T}\to\check{\mathrm{B}}$ such that if $x\in\mathrm{T}$, then $\mathrm{Ad}_{\gamma(x)}$ sends $(fx)^{-1}$ to fx^{-1}; moreover, $\gamma(x)$ squares to the identity.*

Proof. This follows from the fact that $(fx)^{-1}$ and fx^{-1} in B both have image x^{-1} under the map $\mathrm{B}\to\mathrm{B}/\!\!/\check{\mathrm{B}}\cong\mathrm{T}$. □

Definition 3.8.22. Denote by χ the map $\mathrm{B}\to\mathrm{B}/\!\!/\check{\mathrm{B}}\cong\mathrm{T}$. There is an involution θ of B sending $x\mapsto\mathrm{Ad}_{\gamma(\chi(x))}(x^{-1})$, and similarly an involution θ of the constant group scheme $\check{\mathrm{B}}\times\mathrm{T}$ over T sending $(g,y)\mapsto(\mathrm{Ad}_{\gamma(y)}(g),y^{-1})$. This defines an involution θ of $\mathrm{B}/\check{\mathrm{B}}$, and hence a $\mathbf{Z}/2$-action on it.

Example 3.8.23. Suppose $\mathrm{G}=\mathrm{GL}_2$ or SL_2. Then one can take for γ the constant map $\mathrm{T}\cong\mathbf{G}_m^2\to\check{\mathrm{B}}$ sending $(x,y)\mapsto\left(\begin{smallmatrix}1&0\\1&-1\end{smallmatrix}\right)$. If $\mathrm{G}=\mathrm{PGL}_2$, one can simply multiply γ by a primitive fourth root of unity to get an element of $\check{\mathrm{G}}=\mathrm{SL}_2$. If $\mathrm{G}=\mathrm{GL}_3$, then one can take for γ the map $\mathrm{T}\cong\mathbf{G}_m^3\to\check{\mathrm{B}}$ sending $(x,y,z)\mapsto\begin{pmatrix}1&0&0\\1&-1&0\\0&0&zy^{-1}\end{pmatrix}$.

It is easy to show:

Lemma 3.8.24. *The involution $\theta:\mathrm{B}/\check{\mathrm{B}}\to\mathrm{B}/\check{\mathrm{B}}$ is isomorphic to the map induced by inversion on* B.

Proposition 3.8.25. *The $\mathbf{Z}/2$-action by θ on $\tilde{\check{\mathrm{G}}}/\check{\mathrm{G}}\cong\mathrm{B}/\check{\mathrm{B}}$ restricts to an action on $\tilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}}$, and under the equivalence of Corollary 3.8.10, it identifies with the $\mathbf{Z}/2$-action via complex conjugation on equivariant* KU. *In particular, there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}};\mathrm{KO})\otimes_{\mathbf{Z}}\mathrm{F}\simeq\mathrm{QCoh}((\tilde{\check{\mathrm{G}}}^{\mathrm{reg}}/\check{\mathrm{G}})/\langle\theta\rangle).$$

Proof. It follows from Definition 3.8.22 that there is a commutative diagram

$$\begin{tikzcd} \mathrm{T} \arrow[r, "x\mapsto x^{-1}"] \arrow[d, "\kappa"'] & \mathrm{T} \arrow[d, "\kappa"] \\ \mathrm{B} \arrow[r, "\theta"'] & \mathrm{B}. \end{tikzcd}$$

Therefore, θ induces an automorphism of $\mathrm{T} \times_{\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}}} \mathrm{T}$, and it suffices (by the proof of Corollary 3.8.10) to show that under the isomorphism

$$\operatorname{Spec} \pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \cong \mathrm{T} \times_{\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}}} \mathrm{T} \tag{3.8.7}$$

of Theorem 3.8.3, the action of θ corresponds to the action of complex conjugation on equivariant K-theory. Let $\mathrm{T}^{\mathrm{gen}} \subseteq \mathrm{T}$ denote the complement of the union of all hypertori cut out by the coroots of G. Since both sides of (3.8.7) are flat and affine over T, their rings of functions inject into the corresponding localizations along the map $\mathrm{T}^{\mathrm{gen}} \to \mathrm{T}$. Furthermore, these localizations are $\mathbf{Z}/2$-equivariant (for complex conjugation and θ, respectively), and so it suffices to show that these localizations are $\mathbf{Z}/2$-equivariantly isomorphic.

By Lemma 3.2.11, there is an isomorphism

$$\pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee|_{\mathrm{T}^{\mathrm{gen}}} \cong \pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}})^\vee|_{\mathrm{T}^{\mathrm{gen}}} \cong \mathcal{O}_{\mathrm{T}^{\mathrm{gen}}}[\mathbb{X}_*(\mathrm{T})].$$

Under this isomorphism, the action via complex conjugation on KU is given simply by inversion on $\mathrm{T}^{\mathrm{gen}}$, and acts trivially on $\mathbb{X}_*(\mathrm{T})$. Similarly, since $fx \in \mathrm{B}$ is regular *semisimple* if $x \in \mathrm{T}^{\mathrm{gen}}$, and the centralizers of regular semisimple elements are tori, there is an isomorphism

$$(\mathrm{T} \times_{\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}}} \mathrm{T}) \times_{\mathrm{T}} \mathrm{T}^{\mathrm{gen}} \cong \mathrm{T}^{\mathrm{gen}} \times \check{\mathrm{T}}.$$

Under this isomorphism, the action of θ is given simply by inversion on $\mathrm{T}^{\mathrm{gen}}$, and acts trivially on $\check{\mathrm{T}}$. This clearly matches with the action on $\pi_0 \mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee|_{\mathrm{T}^{\mathrm{gen}}}$ via complex conjugation on KU, as desired. □

Proposition 3.8.25 says that, up to replacing $\mathrm{B}/\check{\mathrm{B}}$ by $\check{\mathrm{B}}/\check{\mathrm{B}}$ (that is, replacing $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})$ by $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})$), the $\mathbf{Z}/2$-action via complex conjugation on equivariant KU identifies under Corollary 3.8.10 with the $\mathbf{Z}/2$-action on $\check{\mathrm{B}}/\check{\mathrm{B}} = \mathrm{Map}(\mathrm{B}\mathbf{Z}, \mathrm{B}\check{\mathrm{B}})$ coming from inversion on $\mathbf{Z}$.

Remark 3.8.26. Assume G has torsion-free fundamental group. One can similarly compute the effect of complex conjugation for G_c-equivariant local systems. Namely, as in Lemma 3.8.21, there is a map $\delta : \mathrm{T}/\!\!/\mathrm{W} \to \check{\mathrm{B}}$ such that if $x \in \mathrm{T}/\!\!/\mathrm{W}$, then $\mathrm{Ad}_{\delta(x)}$ sends $(fx)^{-1}$ to fx. Just as in Definition 3.8.22, we obtain an involution Θ on $\mathrm{G}/\check{\mathrm{G}}$ which can be identified with the effect of inversion on G, and the resulting $\mathbf{Z}/2$-action on $\mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}})$ identifies, under the equivalence of Remark 3.8.11, with the $\mathbf{Z}/2$-action on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})$ coming from complex conjugation on equivariant KU. This gives an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KO}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}((\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}})/\langle\Theta\rangle).$$

Applied to the constant sheaf, the spectral sequence (3.3.3) becomes

$$\mathrm{E}_2^{*,*} \cong \mathrm{H}^*(\mathbf{Z}/2; \mathcal{O}_{\mathrm{T}/\!\!/\mathrm{W} \times_{\mathrm{G}/\check{\mathrm{G}}} \mathrm{T}/\!\!/\mathrm{W}}[u^{\pm 1}]) \Rightarrow \mathrm{KO}_*^{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}) \otimes_{\mathbf{Z}} \mathrm{F}.$$

Let us now make a brief comment about the case of *connective* real K-theory ko, discussed in Remark 3.3.16. For this, recall from [Dev3, Section 3.7] that if $\mathbf{G}_\beta$ denotes the group scheme over $\mathrm{Spec}(\mathbf{Z}[\beta])/\mathbf{G}_m$ given by $\mathrm{Spec}\,\mathbf{Z}[\beta, x, \frac{1}{1+\beta x}]$ with group law $x+y+\beta xy$, $\mathfrak{D}(\mathbf{G}_\beta)$ denotes its Cartier dual, and H_β denotes $\mathrm{Hom}(\mathfrak{D}(\mathbf{G}_\beta), \mathrm{H})$ for any group scheme H, then there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{ku}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{B}^{\mathrm{reg}}_\beta/\check{\mathrm{B}}),$$

where $\mathrm{B}^{\mathrm{reg}}_\beta$ is the regular locus in B_β. Similarly, if G has torsion-free fundamental group, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{ku}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}_\beta/\check{\mathrm{G}}),$$

where $\mathrm{G}^{\mathrm{reg}}_\beta$ is the regular locus in G_β. To descend these equivalences to ko-coefficients, we need to describe an action of $\mathrm{Spec}(\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}))/\mathbf{G}_m$ on B_β and G_β. This is provided by (3.3.5): if we write $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}) \cong \mathbf{Z}[\beta, r]/(r^2 - \beta r)$, then the action is given by the map

$$\mathrm{Spec}(\mathbf{Z}[\beta, r]/(r^2 - \beta r)) \times_{\mathrm{Spec}(\mathbf{Z}[\beta])} \mathrm{G}_\beta \to \mathrm{G}_\beta, (r, x) \mapsto x - \tfrac{rx^2}{1+\beta x}.$$

When $\mathrm{G} = \mathrm{SL}_n$, this can be expressed in terms of $g = \mathrm{id} + \beta x$:

$$(r, g) \mapsto 1 + \tfrac{(\beta - 2r)(g-1)}{\beta} + r\tfrac{g - g^{-1}}{\beta}.$$

The action of $\mathrm{Spec}(\pi_*(\mathrm{ku}\otimes_{\mathrm{ko}}\mathrm{ku}))/\mathbf{G}_m$ on B_β and G_β defines stacks $\mathrm{B}^{\mathrm{ko}}_\beta$ and $\mathrm{G}^{\mathrm{ko}}_\beta$ over Spev(ko). For any closed point $\mathrm{Spec}(\mathrm{F}) \to \mathrm{Spev}(\mathrm{ko})$, we obtain F-linear equivalences

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{ko}) \otimes_{\mathrm{Spev}(\mathrm{ko})} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{B}^{\mathrm{ko,reg}}_\beta/\check{\mathrm{B}}),$$
$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{ko}) \otimes_{\mathrm{Spev}(\mathrm{ko})} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}^{\mathrm{ko,reg}}_\beta/\check{\mathrm{G}});$$

upon inverting β, these are the equivalences of Proposition 3.8.25 and Remark 3.8.26.

Let us note that the calculation in Proposition 3.3.15 tells us that the actual homotopy groups of $\mathrm{KO}^{\mathrm{G}_c}_*(\mathrm{Gr}_{\mathrm{G}})$ could differ from a calculation at the level of $\mathrm{QCoh}((\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}})/\langle\Theta\rangle)$, and furthermore that the resulting answer could be somewhat complicated. In general, the groups $\mathrm{KO}^{\mathrm{G}_c}_*(\mathrm{Gr}_{\mathrm{G}})$ will not necessarily be concentrated in even degrees, and the differentials in the preceding spectral sequence will capture some of the (equivariant) attaching maps of the (equivariant) cells in Gr_{G}. Let us now illustrate this by describing $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{G}})$ for $\mathrm{G} = \mathrm{SL}_3$.

Example 3.8.27. If $\mathrm{G} = \mathrm{SL}_2$, then the James splitting says that stable homotopy type of Gr_{G} splits as the direct sum $\bigoplus_{n\geq 0} \mathrm{S}^{2n}$. This implies that $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{G}}) \simeq \bigoplus_{n\geq 0} \mathrm{KO}_{*-2n}$, and in fact there is a ring isomorphism $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{G}}) \cong \mathrm{KO}_*[a]$ with a in weight 2. However, already in the case $\mathrm{G} = \mathrm{SL}_3$, the analogous ring isomorphism $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{G}}) \cong \mathrm{KO}_*[a, b]$ (with a in weight 2 and b in weight 4) fails. Let us indicate the topological reason for this failure: there is a map $\mathbf{C}\mathrm{P}^2 \to \mathrm{Gr}_{\mathrm{SL}_3}$ which exhibits $\mathbf{C}\mathrm{P}^2$ as a generating complex, meaning that the 2- and 4-cells of $\mathbf{C}\mathrm{P}^2$ hit the classes a and b, respectively. The ring $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{SL}_3})$ is therefore controlled by the KO_*-module $\mathrm{KO}_*(\mathbf{C}\mathrm{P}^2)$. The key point is that a classical theorem of Wood [Woo] (which we reprove below) gives a KO-module equivalence $\mathrm{KO}[\mathbf{C}\mathrm{P}^2] \simeq \mathrm{KO} \oplus \mathrm{KU}$. In particular, the KO-module $\mathrm{KO}[\mathbf{C}\mathrm{P}^2]$ is not equivalent to $\mathrm{KO} \oplus \Sigma^2\mathrm{KO}$ (unlike $\mathrm{KU}[\mathbf{C}\mathrm{P}^2]$, which is equivalent to $\mathrm{KU} \oplus \Sigma^2\mathrm{KU}$). This implies that $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{SL}_3})$ cannot be isomorphic to $\mathrm{KO}_*[a, b]$.

In fact, this can be generalized: there are equivalences $\mathrm{KO}[\mathbf{C}\mathrm{P}^{2n}] \simeq \mathrm{KO} \oplus \mathrm{KU}^{\oplus n}$ and $\mathrm{KO}[\mathbf{C}\mathrm{P}^{2n+1}] \simeq \mathrm{KO} \oplus \mathrm{KU}^{\oplus n} \oplus \Sigma^{2n+2}\mathrm{KO}$. This implies that $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{SL}_n})$ is not isomorphic to $\mathrm{KO}_*[a_1, \cdots, a_{n-1}]$ for $n > 2$ (and in fact the behavior of $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{SL}_n})$ will depend on the parity of n, in stark contrast to the way one usually thinks about special linear groups!).

Geometrically, this arises from the fact that the generating complex $\mathbf{C}\mathrm{P}^{2n+1}$ of $\mathrm{Gr}_{\mathrm{SL}_{2n+2}}$ is a spin manifold, while the generating complex $\mathbf{C}\mathrm{P}^{2n}$ of $\mathrm{Gr}_{\mathrm{SL}_{2n+1}}$ is not a spin manifold; and KO is Spin-oriented [ABS] (meaning that spin manifolds admit KO-fundamental classes).

Calculating $\mathrm{KO}_*(\mathrm{Gr}_{\mathrm{SL}_3})$ explicitly is somewhat unpleasant, but let us at least indicate (from the perspective of § 3.3) why there is a KO-module equivalence $\mathrm{KO}[\mathbf{C}\mathrm{P}^2] \simeq \mathrm{KO} \oplus \mathrm{KU}$. To do so, let $\widetilde{\mathrm{KU}}_*(\mathbf{C}\mathrm{P}^2)$ denote the reduced KU-homology of $\mathbf{C}\mathrm{P}^2$. One then has the homotopy fixed points spectral sequence

$$\mathrm{E}_2^{s,*} \cong \mathrm{H}^s(\mathbf{Z}/2; \widetilde{\mathrm{KU}}_*(\mathbf{C}\mathrm{P}^2)) \Rightarrow \widetilde{\mathrm{KO}}_{*-s}(\mathbf{C}\mathrm{P}^2), \tag{3.8.8}$$

which we will now calculate. This can be viewed as a special case of the spectral sequence (3.3.2), applied to $k = \mathrm{KO}$ and $\mathcal{F}$ being the pushforward of the constant sheaf along the map $\mathbf{C}\mathrm{P}^2 \to \mathrm{Gr}_{\mathrm{SL}_3}$. To compute (3.8.8), one first observed that the action of complex conjugation on $\mathrm{KU}_*(\mathrm{Gr}_{\mathrm{SL}_3}) \cong \mathbf{Z}[u^{\pm 1}, a, b]$ is given by $u \mapsto -u$, $a \mapsto -a$, and $b \mapsto b + a$. The action on a and b can also be seen from the equivalence

$$\mathrm{Loc}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KO}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}((\mathcal{U}^{\mathrm{reg}}/\check{\mathrm{G}})/\langle\theta\rangle)$$

derived from Proposition 3.8.25, where $\mathcal{U}^{\mathrm{reg}}$ is the regular locus in the unipotent cone of G (and θ acts by inversion on $\mathcal{U}^{\mathrm{reg}}$). The action of complex conjugation on $\mathrm{KU}_0(\mathrm{Gr}_{\mathrm{SL}_3})$ identifies with the action of θ on the centralizer of a regular unipotent element of SL_3, which consists of matrices of the form $\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

This specifies the action of $\mathbf{Z}/2$ on $\widetilde{\mathrm{KU}}_*(\mathbf{C}\mathrm{P}^2) \cong \mathbf{Z}[u^{\pm 1}]\{a, b\}$, from which we find that

$$\mathrm{E}_2^{*,*} \cong \mathrm{E}_2^{0,*} \cong \mathbf{Z}\{\cdots, (2b+a)u^{-2}, au^{-1}, 2b+a, au, (2b+a)u^2, au^3, \cdots\}.$$

The entire spectral sequence is concentrated in a single line, so it automatically degenerates; this implies that $\widetilde{\mathrm{KO}}_*(\mathbf{C}\mathrm{P}^2)$ is isomorphic to $\mathbf{Z}$ in each even degree (and is zero otherwise). There are canonical maps $\mathrm{KO} \to \mathrm{KU}$ and $\mathbf{C}\mathrm{P}^2 \to \mathrm{KU}$, which define a map $\mathrm{KO} \otimes \mathbf{C}\mathrm{P}^2 \to \mathrm{KU}$. The above calculation implies that it induces an isomorphism on homotopy groups, and hence is an equivalence.

It is also possible to describe an analogue of Corollary 3.8.10 with coefficients in the K(1)-local sphere $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0$ (for some fixed prime p). Recall from Definition 3.3.18 that if A is a p-power torsion abelian group and X is a (ind-)finite A-space with even cells, then the ∞-category $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{X}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0)$ is obtained from $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{A}}(\mathrm{X}; \mathrm{KU})$ by taking homotopy $\mathbf{Z}_p^\times$-invariants.

Definition 3.8.28. For $n \geq 0$, let $\tilde{\check{\mathrm{G}}}_{p^n}$ denote the (derived) fiber product

$$\tilde{\check{\mathrm{G}}}_{p^n} := \tilde{\check{\mathrm{G}}} \times_{\mathrm{T}} \mathrm{T}[p^n].$$

That is, $\tilde{\check{\mathrm{G}}}_{p^n}/\check{\mathrm{G}} \cong \mathrm{B}_{p^n}/\check{\mathrm{B}}$, where B_{p^n} is the subgroup of those elements of B whose eigenvalues are all p^nth roots of unity. Similarly, let $\tilde{\check{\mathrm{G}}}^{\mathrm{reg}}_{p^n}$ denote the fiber product

$$\tilde{\check{\mathrm{G}}}^{\mathrm{reg}}_{p^n} := \tilde{\check{\mathrm{G}}}^{\mathrm{reg}} \times_{\mathrm{T}} \mathrm{T}[p^n],$$

There is an action of $\mathbf{Z}_p^\times$ (which factors through an action of $(\mathbf{Z}/p^n)^\times$) on B_{p^n} given by exponentiation; the $\mathbf{Z}_p^\times$-action commutes with the $\check{\mathrm{B}}$-action by conjugation, and hence defines a $\mathbf{Z}_p^\times$-action on the quotient stack $\mathrm{B}_{p^n}/\check{\mathrm{B}} \cong \tilde{\check{\mathrm{G}}}_{p^n}/\check{\mathrm{G}}$.

Proposition 3.8.29. *Let $n \geq 0$. The $\mathbf{Z}_p^\times$-action on $\widetilde{\check{\mathrm{G}}}_{p^n}/\check{\mathrm{G}}$ restricts to an action on $\widetilde{\check{\mathrm{G}}}_{p^n}^{\mathrm{reg}}/\check{\mathrm{G}}$, and there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c[p^n]}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \otimes_{\mathbf{Z}_p} \mathrm{F} \simeq \mathrm{QCoh}((\widetilde{\check{\mathrm{G}}}_{p^n}^{\mathrm{reg}}/\check{\mathrm{G}})/\mathbf{Z}_p^\times).$$

Proof. Base-changing the QCoh(T)-linear equivalence Corollary 3.8.10 along $\mathrm{QCoh}(\mathrm{T}) \to \mathrm{QCoh}(\mathrm{T}[p^n])$ gives an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c[p^n]}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU}_p^\wedge) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}_{p^n}^{\mathrm{reg}}/\check{\mathrm{G}}).$$

Since the $\mathbf{Z}_p^\times$-action on $\mathrm{T}[p^n]$ is given by exponentiation, the strategy of Proposition 3.8.25 shows that the $\mathbf{Z}_p^\times$-action on the left-hand side of the above equivalence via Adams operations on p-completed KU identifies with the $\mathbf{Z}_p^\times$-action on $\widetilde{\check{\mathrm{G}}}_{p^n}^{\mathrm{reg}}/\check{\mathrm{G}}$ described in Definition 3.8.28. Taking homotopy $\mathbf{Z}_p^\times$-invariants of the displayed equivalence then yields the desired statement. □

The equivalences of Proposition 3.8.29 are all compatible in n, and one finds that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c[p^\infty]}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \otimes_{\mathbf{Z}_p} \mathrm{F} \simeq \mathrm{QCoh}((\widetilde{\check{\mathrm{G}}}_{p^\infty}^{\mathrm{reg}}/\check{\mathrm{G}})/\mathbf{Z}_p^\times).$$

Lemma 3.8.30. *There is an isomorphism $\mathrm{B}_{p^\infty} \cong \mathrm{B}[p^\infty]$ over a p-nilpotent ring.*

Proof. Recall that $\mathrm{B}_{p^\infty} = \mathrm{B} \times_{\mathrm{T}} \mathrm{T}[p^\infty]$, so there is a canonical map $\mathrm{B}[p^\infty] \to \mathrm{B}_{p^\infty}$. It suffices to show that if N denotes the unipotent radical of B, then $\mathrm{N} \cong \mathrm{N}[p^\infty]$. This follows by induction on the central series of N (whose quotients are all isomorphic to $\mathbf{G}_a$), and the fact that $\mathbf{G}_a \cong \mathbf{G}_a[p^\infty]$ since we are working over a p-nilpotent base. □

It follows that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c[p^\infty]}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \otimes_{\mathbf{Z}_p} \mathrm{F} \simeq \mathrm{QCoh}((\mathrm{B}[p^\infty]^{\mathrm{reg}}/\check{\mathrm{B}})/\mathbf{Z}_p^\times);$$

similarly, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\tilde{\mathrm{T}}_c[p^\infty]}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{L}_{\mathrm{K}(1)}\mathrm{S}^0) \otimes_{\mathbf{Z}_p} \mathrm{F} \simeq \mathrm{QCoh}((\check{\mathrm{B}}[p^\infty]^{\mathrm{reg}}/\check{\mathrm{B}})/\mathbf{Z}_p^\times).$$

Note that $\check{\mathrm{B}}[p^\infty]^{\mathrm{reg}}/\check{\mathrm{B}}$ is an open substack of $\check{\mathrm{B}}[p^\infty]/\check{\mathrm{B}} \cong \mathrm{colim}_n \mathrm{Map}(\mathrm{B}\mathbf{Z}/p^n, \mathrm{B}\check{\mathrm{B}})$; one might heuristically view the latter as the stack of $\check{\mathrm{B}}$-bundles on the p-adic solenoid.

Finally, let us discuss the question of loop-rotation equivariance. Recall from Definition 3.6.20 the algebra $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ associated to a 1-dimensional group scheme $\mathbf{H}$ over a field F and a root system with torus T and Weyl group W. In the following discussion, we will set $\mathbf{H} = \mathbf{G}_m$, so that $\mathbf{H}_{\mathrm{T}} = \mathrm{T}$; we will also write q to denote the standard character of $\mathrm{S}^1_{\mathrm{rot}}$, so that $\pi_0\mathrm{KU}_{\mathrm{S}^1_{\mathrm{rot}}} \cong \mathbf{Z}[q^{\pm 1}]$. Exactly the same argument as in Theorem 3.6.24 shows the following result; here, G does not need to be simply-laced.

Theorem 3.8.31. *There is an isomorphism of associative $\mathbf{Z}[q^{\pm 1}]$-algebras*

$$\pi_0\mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_{\mathrm{G}})^\vee \cong \mathcal{H}(\mathbf{G}_m, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}}). \tag{3.8.9}$$

Here, $\pi_0\mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_{\mathrm{G}})^\vee$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\pi_0\mathrm{KU}_{\widetilde{\mathrm{T}}_c} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$-algebroids.

Remark 3.8.32. Recall the quotient $\widetilde{\mathrm{T}}/\!\!/\widetilde{\mathrm{W}}$ from Remark 3.6.23. The discussion therein combined with Theorem 3.8.31 gives an equivalence of categories

$$\pi_0\mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl_G})^\vee\text{-mod} \simeq \mathcal{H}(\mathbf{G}_m, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})\text{-mod} \simeq \mathrm{IndCoh}(\widetilde{\mathrm{T}}/\!\!/\widetilde{\mathrm{W}}).$$

It follows, via the argument of Corollary 3.6.32, that $\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl_G}; \mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F}$ is equivalent to the quotient of $\mathrm{QCoh}(\widetilde{\mathrm{T}})$ by the action of $\mathrm{IndCoh}(\widetilde{\mathrm{T}}/\!\!/\widetilde{\mathrm{W}})$.

In future work, we will use the discussion in this section, along with Theorem 6.4.1, to identify a localization of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr_G}; \mathrm{ku})$ with a category built from $\mathrm{IndCoh}(\check{\mathrm{G}}^{q\mathrm{dR}})$, where $\check{\mathrm{G}}^{q\mathrm{dR}}$ denotes the q-de Rham stack of $\check{\mathrm{G}}$ (in the sense of [BL, Dri2]).

3.9 The elliptic story

In this section, we will work over a given algebraically closed field F. For the moment, $\check{\mathrm{G}}$ will be a (split) almost-simple group over F with torsion-free fundamental group. Let E be a (smooth) elliptic curve over k, let $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ denote the moduli stack of $\check{\mathrm{B}}$-bundles on E of degree 0, and let $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ denote the scheme of $\check{\mathrm{T}}$-bundles on E of degree 0. We will also make use of the stack $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ of semistable $\check{\mathrm{G}}$-bundles on E. Our main references for the structure of $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ and $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ will be [Dav, GSB].

Definition 3.9.1. Say that a $\check{\mathrm{B}}$-bundle $\mathcal{P}_{\check{\mathrm{B}}}$ on E is *regular* if $\dim \mathrm{Aut}(\mathcal{P}_{\check{\mathrm{B}}}) = \mathrm{rank}(\check{\mathrm{G}})$. Let $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$ denote the open substack of $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ defined by the regular $\check{\mathrm{B}}$-bundles. Similarly, if $\mathcal{P} \in \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ is a semistable $\check{\mathrm{G}}$-bundle on E, we say that $\mathcal{P}$ is *regular* if $\dim \mathrm{Aut}(\mathcal{P}) = \mathrm{rank}(\check{\mathrm{G}})$. Let $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \subseteq \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ denote the open substack of regular semistable $\check{\mathrm{G}}$-bundles.

Proposition 3.9.2. *The map* $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ *admits a canonical unique section* $\kappa : \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ *landing in* $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$.

Proof. Let $\mathcal{P}$ be a semistable $\check{\mathrm{G}}$-bundle on E. By [Dav, Proposition 4.4.5], the regularity of $\mathcal{P}$ is equivalent to the condition that for any (or some) $\check{\mathrm{B}}$-reduction $\mathcal{P}_{\check{\mathrm{B}}}$ of $\mathcal{P}$ of degree 0, the associated $\check{\mathrm{N}}$-bundle $\mathcal{P}_{\check{\mathrm{B}}}/\check{\mathrm{T}}$ is induced from an $\check{\mathrm{N}}_{\mathcal{P}}$-bundle with nontrivial associated $\check{\mathrm{N}}_\alpha$-bundle for each simple root α in a particular subset of Δ determined by $\mathcal{P}$. Moreover, every geometric fiber of the map $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \to \mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}), \mathrm{E})/\!\!/\mathrm{W}$ to the coarse moduli space of $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ contains a unique regular semistable $\check{\mathrm{G}}$-bundle. Also see [FMW, Proposition 3.9], where a similar result is stated.

Following [Dav, Definition 3.1.7], set

$$\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \cong \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \times_{\mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}),\mathrm{E})/\!\!/\mathrm{W}} \mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}), \mathrm{E}).$$

Let $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$ denote the moduli stack of $\check{\mathrm{B}}$-bundles on E of degree 0. It then follows from the isomorphism $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ of [Dav, Proposition 2.1.11] and the equality $\dim \mathrm{Aut}(\mathcal{P}) = \dim \mathrm{Aut}(\mathcal{P}_{\check{\mathrm{B}}})$ that there is an isomorphism $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$. In particular, every geometric fiber of the map $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \to \mathrm{Hom}(\mathbb{X}^*(\check{\mathrm{T}}), \mathrm{E}) = \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ contains a unique regular $\check{\mathrm{B}}$-bundle of degree 0.

The existence of κ is a consequence of [Dav, Theorem 4.3.2], which is a refinement of [FM2, Theorem 5.1.1]. Since we will not need the full strength of [Dav, Theorem 4.3.2] outside of this proof, we will only briefly recall the necessary notation and statements. In *loc. cit.*, the scheme $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ is denoted by Y. Let $\widetilde{\mathrm{Bun}}_{\check{\mathrm{G}}}(\mathrm{E})$ denote the Kontsevich-Mori compactification of $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$; see [Dav, Definition 2.1.2]. Let Θ denote the theta-line bundle over $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ of [Dav, Corollary 3.2.10], and let $\widetilde{\chi} : \widetilde{\mathrm{Bun}}_{\check{\mathrm{G}}}(\mathrm{E}) \to \Theta^{-1}/\mathbf{G}_m$ denote the map constructed in [Dav, Corollary 3.3.2]. Then, [Dav, Theorem 4.3.2] shows that there is a map $\Theta^{-1} \to \widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ landing in $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}}$ such that the composite

$$\Theta^{-1} \to \widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \xrightarrow{\widetilde{\chi}} \Theta^{-1}/\mathbf{G}_m$$

is the canonical map. Composing with the zero section of Θ^{-1}, we obtain a map

$$\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \cong 0_{\Theta^{-1}} \to \Theta^{-1} \to \widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}).$$

This is the desired map κ. $\square$

Definition 3.9.3. The map $\kappa : \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ from Proposition 3.9.2 will be called the *elliptic Kostant slice.*

The elliptic Kostant slice builds on work of Friedman-Morgan [FM1, FM2, FM3, FMW].

If E is replaced by the constant stack S^1 or by $\mathrm{B}\mathbf{G}_a$, the stack $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ is to be interpreted as $\check{\mathrm{B}}/\check{\mathrm{B}}$ and $\check{\mathfrak{b}}/\check{\mathrm{B}}$, respectively. The analogue of the elliptic Kostant section is given by the maps $f \cdot \check{\mathrm{T}} \to \check{\mathrm{B}}/\check{\mathrm{B}}$ and $f + \check{\mathfrak{t}} \to \check{\mathfrak{b}}/\check{\mathrm{B}}$, respectively.

The following is [Dav, Lemma 3.1.11].

Lemma 3.9.4. *Let* $\mathrm{I} \subseteq \Phi^-$ *be a subset, and let* $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\mathrm{I}}$ *denote the subscheme of* $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ *defined by those bundles* $\mathcal{P}_{\check{\mathrm{T}}}$ *whose* α*-component is trivial precisely for* $\alpha \in \mathrm{I}$*. Let* $\check{\mathrm{N}}_{\mathrm{I}} \subseteq \check{\mathrm{N}}$ *be the smallest unipotent subgroup which is invariant under* $\check{\mathrm{T}}$*-conjugation and which contains* $\check{\mathrm{N}}_\alpha$ *for every* $\alpha \in \mathrm{I}$*. Then the natural map*

$$\mathrm{Bun}^0_{\check{\mathrm{T}}\check{\mathrm{N}}_{\mathrm{I}}}(\mathrm{E}) \times_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})} \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\mathrm{I}} \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \times_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})} \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\mathrm{I}}$$

is an isomorphism.

Example 3.9.5. Suppose that $\mathrm{I} = \emptyset$, so that $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_\emptyset$ denotes the open subscheme of $\check{\mathrm{T}}$-bundles of degree zero whose α-component is nontrivial for every negative root α. The isomorphism $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ implies that the map $\widetilde{\mathrm{Bun}}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ is an isomorphism over $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_\emptyset$. In particular, every point of $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_\emptyset$ has a canonical associated (regular) semistable $\check{\mathrm{G}}$-bundle.

The above results continue to hold if E is replaced by the constant stack S^1 or by $\mathrm{B}\mathbf{G}_a$ (in which case $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ is to be interpreted as $\check{\mathrm{B}}/\check{\mathrm{B}}$ and $\check{\mathfrak{b}}/\check{\mathrm{B}}$, respectively). In the case of S^1, for instance, the semistable $\check{\mathrm{G}}$-bundles obtained in this way from $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_\emptyset$ are precisely those which lie in the regular *semisimple* locus $\check{\mathrm{G}}^{\mathrm{rss}}/\check{\mathrm{G}}$; similarly for the case of $\mathrm{B}\mathbf{G}_a$.

We now turn to the topology of G, so it is connected, almost simple, and simply-laced over $\mathbf{C}$. In this setting, k will be an even 2-periodic $\mathbf{E}_\infty$-ring equipped with an oriented group scheme $\mathbf{G}$ whose underlying classical scheme $\mathbf{G}_0$ over $\pi_0(k)$ is an elliptic curve E. We will continue to fix an algebraically closed field F over $\pi_0(k)$, over which the Langlands dual group $\check{\mathrm{G}}$ will live. As usual, when dealing with the algebraic geometry (as opposed to the topology) of G, we will also view it as living over F; since G is simply-laced, it is isogenous to $\check{\mathrm{G}}$.

Definition 3.9.6. The *elliptic regular centralizer group scheme* $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$ is defined to be the group scheme over $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ given by the fiber product

$$\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}} \cong \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}).$$

Note that this is very slightly (but importantly) different from the definition of $\widetilde{\check{\mathrm{J}}}_\mu$ and $\widetilde{\check{\mathrm{J}}}$; the analogues of the fiber product above would instead be $(f\cdot\check{\mathrm{T}})\times_{\check{\mathrm{B}}/\check{\mathrm{B}}}(f\cdot\check{\mathrm{T}})$ and $(f+\check{\mathfrak{t}})\times_{\check{\mathfrak{b}}/\check{\mathrm{B}}}(f+\check{\mathfrak{t}})$.

In the following discussion, we will consider the $\check{\mathrm{T}}$-equivariant elliptic homology of Gr_{G} (instead of the T-equivariant elliptic homology); this will capture the minor difference between the definitions of $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$ and $\widetilde{\check{\mathrm{J}}}$ mentioned above.

Theorem 3.9.7. *There is an isomorphism of group schemes over* $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \cong \mathcal{M}_{\check{\mathrm{T}},0}$*:*

$$\mathrm{Spec}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}(\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee) \otimes_{\pi_0(k)} \mathrm{F} \cong \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}).$$

Here, $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}(\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee)$ *denotes the relative* Spec *of* $\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ *over* $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$.

As with Theorem 3.6.3 and Theorem 3.8.3, the proof of Theorem 3.9.7 relies on two lemmas.

Lemma 3.9.8. *The projection map* $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ *(onto either factor) is flat.*

Proof. Like in the proof of Lemma 3.6.4, it suffices, by miracle flatness, to show that the fibers of the map $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ have dimension exactly $\mathrm{rank}(\check{\mathrm{G}})$. But this follows from the fact that the map $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ lands in $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$ (see Proposition 3.9.2). ☐

For a root α, let $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\alpha\text{-reg}} \subseteq \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ denote the union of the substacks $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\{\alpha\}}$ and $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\emptyset}$. The next result follows exactly as in Lemma 3.6.6 (using Lemma 3.9.4).

Lemma 3.9.9. *There is an isomorphism*

$$\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}(\check{\mathrm{G}})|_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\alpha\text{-reg}}} \xrightarrow{\sim} \widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}(\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ)|_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\alpha\text{-reg}}}, \tag{3.9.1}$$

where $\mathrm{Z}_{\check{\mathrm{G}}}(x)$ *is the centralizer of some* $x \in \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\alpha\text{-reg}}$ *which lies in* $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})_{\{\alpha\}}$*, and* $\mathrm{Z}_{\check{\mathrm{G}}}(x)^\circ$ *denotes the connected component of the identity.*

Recall that if X is a scheme with subschemes $\mathrm{V} = \mathrm{V}(\mathcal{I}) \subseteq \mathrm{D} = \mathrm{V}(\mathcal{J})$ (so that $\mathcal{J} \subseteq \mathcal{I}$) where D is locally principal, the affine blowup $\mathrm{Bl}^{\mathrm{D}}_{\mathrm{V}}(\mathrm{X})$ is defined to be the complement of $\mathrm{V}_+(\mathcal{J})$ in the blowup $\mathrm{Bl}_{\mathrm{V}}(\mathrm{X})$. That is, it is the relative Spec of the algebra $\mathcal{O}_{\mathrm{X}}[\frac{\mathcal{I}}{\mathcal{J}}]$ of weight zero elements in $\mathrm{Bl}_{\mathcal{I}}(\mathcal{O}_{\mathrm{X}})[\frac{1}{\mathcal{J}}]$, where $\mathrm{Bl}_{\mathcal{I}}(\mathcal{O}_{\mathrm{X}}) = \mathcal{O}_{\mathrm{X}} \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$ is the Rees algebra.

Proof of Theorem 3.9.7. The argument of Theorem 3.6.3 reduces us to checking that the isomorphism of Theorem 3.9.7 holds if G has semisimple rank 1, i.e., is the product of a torus with one of GL_2, SL_2, or PGL_2. Again, it is easy to match up the contributions from the toral factors, so we will assume that G is either GL_2, SL_2, or PGL_2. In this case, we can even replace F by $\pi_0(k)$. The proofs are all rather uniform (as we have seen in Theorem 3.6.3 and Theorem 3.8.3), so we will simply illustrate the argument when $\mathrm{G} = \mathrm{SL}_2$ and $\mathrm{G} = \mathrm{PGL}_2$.

We begin with the case $\mathrm{G} = \mathrm{SL}_2$. Since $\check{\mathrm{T}} = \mathbf{G}_m$, we may identify $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \cong \mathrm{E}$; to emphasize that it plays the role of the base of S^1-equivariant elliptic cohomology, we will denote it by $\mathcal{M}$. Let $\infty \in \mathcal{M} = \mathrm{E}$ denote the identity section. Consider the closed subschemes

$$\mathrm{V} = \{(\infty, 1)\} \subseteq \mathrm{D} = \{\infty\} \times \mathbf{G}_m \subseteq \mathcal{M} \times \mathbf{G}_m.$$

Then, as in Theorem 3.6.3 and Theorem 3.8.3, $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}(\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee)$ identifies with the affine blowup $\mathrm{Bl}^{\mathrm{D}}_{\mathrm{V}}(\mathcal{M} \times \mathbf{G}_m)$.

Since $\check{\mathrm{G}} = \mathrm{PGL}_2$, an S-point of the stack $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ is the data of a degree zero rank 2 vector bundle $\mathcal{V}$ over $\mathrm{S} \times \mathrm{E}$ along with a line subbundle $\mathcal{L} \subseteq \mathcal{V}$ and an isomorphism $\mathcal{V}/\mathcal{L} \cong \mathcal{O}_{\mathrm{S}\times\mathrm{E}}$. In this language, the elliptic Kostant section $\mathcal{M} = \mathrm{E} \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ classifies the unique indecomposable extension $\mathcal{V}$ of $\mathcal{O}_{\mathcal{M}\times\mathrm{E}}$ by the Poincaré line bundle $\mathcal{P}$. (Recall that $\mathcal{P}$ can be identified, for instance, with the line bundle corresponding to the divisor $\Delta - \mathrm{E}\times\{\infty\} - \{\infty\}\times \mathrm{E}$.) This extension is classified by a nonzero section of $\underline{\mathrm{Ext}}^1_{\mathcal{M}\times\mathrm{E}}(\mathcal{O}_{\mathcal{M}\times\mathrm{E}}, \mathcal{P})$.

Let us now compute $\tilde{\tilde{\mathrm{J}}}_{\mathrm{ell}}$. The fiber product $\mathcal{M}\times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})}\mathcal{M}$ is isomorphic (as a group scheme over $\mathcal{M}$) to the subgroup of the constant group scheme $\underline{\check{\mathrm{B}}} := \mathcal{M} \times \check{\mathrm{B}}$ of those $b \in \underline{\check{\mathrm{B}}}$ such that $b \cdot \mathcal{V} = \mathcal{V}$. First, let $\mathrm{U} = (\mathcal{M} - \{\infty\}) \times \mathrm{E}$; then $\mathcal{V}|_{\mathrm{U}}$ splits as $\mathcal{O}_{\mathrm{U}} \oplus \mathcal{P}|_{\mathrm{U}}$. Indeed, the restriction $\mathcal{P}|_{\mathrm{U}}$ is a nontrivial line bundle on U, so its pushforward to $\mathcal{M} - \{\infty\}$ has no cohomology (and hence the extension class is trivial). It follows that $\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V})|_{\mathrm{U}} = \mathcal{M} \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathrm{U}$ can be identified with $\mathrm{U} \times \mathbf{G}_m$.

On the other hand, let $\mathrm{Z} = \{\infty\} \times \mathrm{E}$ denote the complement of U, so that the formal neighborhood $\hat{\mathrm{Z}}$ of Z is isomorphic to $\mathcal{M}^\wedge_\infty \times \mathrm{E} = \hat{\mathbf{A}}^1 \times \mathrm{E}$. Let t denote a coordinate on $\hat{\mathbf{A}}^1$. Then, the restriction of $\mathcal{P}$ to $\hat{\mathrm{Z}}$ is given by the 1-parameter family of line bundles $\mathcal{O}_{\hat{\mathrm{Z}}}(t-\infty)$ over $\hat{\mathbf{A}}^1 \times \mathrm{E}$. The restriction of $\mathcal{V}$ to $\hat{\mathrm{Z}}$ is classified by a map $\mathcal{O}_{\hat{\mathrm{Z}}} \to \mathcal{O}_{\hat{\mathrm{Z}}}(t-\infty)[1]$ which vanishes except at the origin of $\hat{\mathbf{A}}^1$, where it is given by the unique (up to nonzero scalar) nontrivial map $\mathcal{O}_{\mathrm{E}} \to \mathcal{O}_{\mathrm{E}}[1]$.

For instance, $\mathcal{V}|_{\mathrm{Z}}$ is isomorphic to the Atiyah bundle over E from [Ati1] (i.e., the unique indecomposable rank 2 extension of the structure sheaf by itself), so that it can be realized away from $\infty \in \mathrm{E}$ by pairs (f_1, f_2) of regular functions on E; and near ∞ by pairs (f_1, f_2) such that f_1 and $f_1 - zf_2$ are regular, where z is a local coordinate of E. Under this description, $\mathrm{End}(\mathcal{V}|_{\mathrm{Z}}) = \mathrm{End}(\mathcal{V})|_{\mathrm{Z}}$ is spanned by the identity and the map $(f_1, f_2) \mapsto (0, f_1)$. That is, $\mathrm{End}(\mathcal{V})|_{\mathrm{Z}}$ is isomorphic to the group of matrices $\left(\begin{smallmatrix} a & b \\ 0 & a\end{smallmatrix}\right)$, and so $\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V})|_{\mathrm{Z}}$ is isomorphic to $\mathrm{Z} \times \mathbf{G}_a$. It is easy to extend this description to the formal neighborhood of Z, and thereby find that $\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V})|_{\hat{\mathrm{Z}}}$ is isomorphic to the canonical degeneration of $\mathbf{G}_m$ into $\mathbf{G}_a$. In other words, there is an isomorphism

$$\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V})|_{\hat{\mathrm{Z}}} \cong \mathrm{Spec}\,\pi_0(k)[\![t]\!][a^{\pm 1}, \tfrac{a-1}{t}].$$

Gluing this with the description of $\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V})|_{\mathrm{U}}$ from the preceding paragraph, we find that $\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V}) \cong \mathcal{M} \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathcal{M}$ is isomorphic to the affine blowup $\mathrm{Bl}^{\mathrm{D}}_{\mathrm{V}}(\mathcal{M} \times \mathbf{G}_m)$. We will leave it to the reader to verify that the resulting sequence of isomorphisms

$$\mathrm{Aut}_{\check{\mathrm{B}}}(\mathcal{V}) \cong \mathcal{M} \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathcal{M} \cong \mathrm{Bl}^{\mathrm{D}}_{\mathrm{V}}(\mathcal{M} \times \mathbf{G}_m) \cong \mathrm{Spec}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}(\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee)$$

is one of group schemes over $\mathcal{M}$.

The case when $\mathrm{G} = \mathrm{PGL}_2$ is very similar; we only indicate the necessary changes. Let $\mathrm{E}[2] \subseteq \mathrm{E}$ denote the 2-torsion subgroup, and consider the closed subschemes

$$\mathrm{V} = \mathrm{E}[2] \times \mu_2 \subseteq \mathrm{D} = \mathrm{E}[2] \times \mathbf{G}_m \subseteq \mathcal{M} \times \mathbf{G}_m.$$

By arguing as in Theorem 3.6.3 and Theorem 3.8.3, we find that $\mathrm{Spec}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}(\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee)$ identifies with the affine blowup $\mathrm{Bl}^{\mathrm{D}}_{\mathrm{V}}(\mathcal{M} \times \mathbf{G}_m)$. In this case, $\check{\mathrm{G}} = \mathrm{SL}_2$, and the elliptic Kostant section $\mathcal{M} = \mathrm{E} \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ sends a line bundle $\mathcal{L}$ to the trivially filtered SL_2-bundle $\mathcal{O}_{\mathrm{E}} \subseteq \mathcal{O}_{\mathrm{E}} \oplus \mathcal{L}$ if $\mathcal{L}^2 \neq \mathcal{O}_{\mathrm{E}}$; and to the Atiyah extension of $\mathcal{L}$ by itself if $\mathcal{L}^2 \cong \mathcal{O}_{\mathrm{E}}$. This extension is defined by a nontrivial element of $\mathrm{Ext}^1_{\mathrm{E}}(\mathcal{L}, \mathcal{L}^{-1}) \cong \mathrm{H}^1(\mathrm{E}; \mathcal{L}^{-2})$. The calculation of $\mathcal{M} \times_{\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})} \mathcal{M}$ follows exactly the same path as in the case $\mathrm{G} = \mathrm{SL}_2$ studied above. □

Remark 3.9.10. The most classical instantiation of the Atiyah bundle $\mathcal{A}$ is via the Weierstrass functions. The $\mathbf{G}_a$-torsor over E associated to $\mathcal{A}$ is the complement of the section at ∞ of the projective line $\mathbf{P}(\mathcal{A})$. If we work complex-analytically, E^{an} can be identified as the quotient $\mathbf{C}/\Lambda$ for some rank 2 lattice $\Lambda \subseteq \mathbf{C}$. Associated to Λ are two Weierstrass functions defined on $\mathbf{C}$:

$$\wp(z;\Lambda) = \tfrac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left(\tfrac{1}{(z-\lambda)^2} - \tfrac{1}{\lambda^2}\right),$$
$$\zeta(z;\Lambda) = \tfrac{1}{z} + \sum_{\lambda\in\Lambda-\{0\}} \left(\tfrac{1}{z-\lambda} + \tfrac{1}{\lambda} + \tfrac{z}{\lambda^2}\right).$$

Note that $\wp(z;\Lambda)$ is doubly-periodic, i.e., $\wp(z+\lambda;\Lambda) = \wp(z;\Lambda)$ for any $\lambda \in \Lambda$. Alternatively, $\wp$ defines a map $\mathbf{C} \to \mathbf{C}$ which factors through a map $\mathbf{C}/\Lambda = \mathrm{E}^{\mathrm{an}} \to \mathbf{C}$.

Although $\zeta(z;\Lambda)$ is not doubly-periodic, an easy calculation shows that $\wp(z;\Lambda) = -\partial_z\zeta(z;\Lambda)$; so if $\lambda \in \Lambda$, then $\zeta(z+\lambda;\Lambda) - \zeta(z;\Lambda) = c(\lambda)$ for some constant $c(\lambda)$. The function $\lambda \mapsto c(\lambda)$ is evidently additive, and defines a homomorphism $\Lambda \to \mathbf{C}$, which defines a $\mathbf{C}$-bundle over $\mathrm{E}^{\mathrm{an}} = \mathbf{C}/\Lambda$. This $\mathbf{C}$-bundle is precisely the analytification of the $\mathbf{G}_a$-torsor associated to the Atiyah bundle. It follows that although ζ is not defined on E^{an}, this analytification is the universal space over E^{an} on which ζ is well-defined.

This discussion also describes the total space of the rank 2-bundle $\mathcal{A}^{\mathrm{an}}$ purely analytically. For instance, if $q \in \mathbf{C}^\times$ is a unit complex number of modulus < 1, we can identify $\mathrm{Tot}(\mathcal{A}^{\mathrm{an}})$ over the Tate curve $\mathbf{C}^\times/q^{\mathbf{Z}}$ with the quotient

$$\mathrm{Tot}(\mathcal{A}^{\mathrm{an}}) = \left(\mathbf{C}^\times \times \mathbf{C}^2\right) / \left((z,x) \sim (qz, \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)x)\right).$$

The appearance of the Jordan block $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is the basic reason why the Atiyah bundle plays the role of the principal nilpotent element f in the proof of Theorem 3.9.7.

Corollary 3.9.11. *There is an* F*-linear equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}};k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}).$$

Furthermore, the pushforward functor $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}};k) \to \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(*;k)$ *identifies with the pullback functor* $\kappa^* : \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})) \to \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))$.

Proof. By definition, $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}};k)$ is equivalent to the category of comodules over $\pi_0\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ in $\mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{T}},0}) = \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))$. By Theorem 3.9.7, it can be identified the category of quasicoherent sheaves on the quotient stack $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})/\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$. We may view $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$ as a closed subgroup scheme of the constant group scheme $\check{\mathrm{B}} \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$. This gives an isomorphism

$$\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})/\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}} \cong \check{\mathrm{B}}\backslash(\check{\mathrm{B}} \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))/\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}.$$

Let $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})_{\mathrm{triv}}$ denote the scheme whose S-points are of $\check{\mathrm{B}}$-bundles over $\mathrm{S} \times \mathrm{E}$ of degree 0 equipped with a trivialization at $\mathrm{S} \times \{\infty\}$, so that there is a natural map $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})_{\mathrm{triv}} \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$. Let $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}_{\mathrm{triv}}$ denote the restriction of $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})_{\mathrm{triv}}$ to the regular locus $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}} \subseteq \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$. It follows from Davis' work in [Dav] that the $\check{\mathrm{B}}$-orbit of $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ inside $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})_{\mathrm{triv}}$ is precisely the regular locus $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}_{\mathrm{triv}}$. Since $\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$ is by definition the stabilizer of $\kappa : \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$, the quotient $\check{\mathrm{B}}\backslash(\check{\mathrm{B}} \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))/\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}}$ is isomorphic to $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$; so there is an isomorphism $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})/\widetilde{\check{\mathrm{J}}}_{\mu} \cong \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}$. ∎

The equivalence of Corollary 3.9.11 is in fact symmetric monoidal for the convolution tensor structure on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ (described in Remark 3.3.5) and the standard tensor product on $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}})$.

Remark 3.9.12. The work of Gepner and Meier in [GM2, GM1] sets up the theory of G_c-equivariant elliptic cohomology for compact Lie groups G_c. In particular, they describe a scheme $\mathcal{M}_{\mathrm{G}}$ over k with underlying scheme $\mathcal{M}_{\mathrm{G},0}$ over $\pi_0(k)$, such that the global sections of the structure sheaf of $\mathcal{M}_{\mathrm{G}}$ computes G_c-equivariant k-cohomology. Using this setup (and assuming a slight extension of the results of [Dav] replacing the simply-connectedness assumption with the condition of having torsion-free fundamental group), it can be shown that if G is almost simple and simply-laced, and has torsion-free fundamental group, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}}).$$

Here, the left-hand side is defined to be the ∞-category $\mathrm{coLMod}_{\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{G},0}))$, just as in § 3.3. The proof of the displayed equivalence is quite similar to that of Corollary 3.9.11, and in fact can be deduced from it using the observation that $\pi_0(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee) = \pi_0(\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee)^{\mathrm{W}}$ and that the natural map $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}}$ is a (ramified) W-cover. The first statement uses that G is simply-connected, and the second is the elliptic version of Grothendieck-Springer theory studied in [Dav, Proposition 3.1.14].

Restriction of a $\check{\mathrm{B}}$-bundle on E to the zero section defines a map $q : \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \to \mathrm{B}\check{\mathrm{B}} \to \mathrm{B}\check{\mathrm{G}}$, which in turn defines a functor

$$\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F}. \tag{3.9.2}$$

More generally, the map $q : \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \to \mathrm{B}\check{\mathrm{B}} \to \mathrm{B}\check{\mathrm{G}} \times \mathrm{B}\check{\mathrm{T}}$ defines a functor

$$\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{T}}) \to \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}) \simeq \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F}. \tag{3.9.3}$$

If $\mathrm{V} \in \mathrm{Rep}(\check{\mathrm{G}})$, let $\mathcal{S}_k(\mathrm{V})$ denote the corresponding object of $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F}$. The same argument as in Proposition 3.6.13 shows the following, which says that $\mathcal{S}_k(\mathrm{V}) \in \mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ is the associated graded of a particular object $\mathcal{F}_\lambda \in \mathrm{Loc}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ if V is a minuscule $\check{\mathrm{G}}$-representation.

Proposition 3.9.13. *Let $\lambda_\bullet = (\lambda_1, \cdots, \lambda_n)$ be a tuple of dominant minuscule weights of $\check{\mathrm{G}}$, let $|\lambda_\bullet| = \sum_i \lambda_i$, and let $\overline{\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}}$ denote the corresponding convolution variety. Let $\mathcal{F}_{\lambda_\bullet}$ denote the pushforward of the constant sheaf along the canonical map $q : \overline{\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}} \to \overline{\mathrm{Gr}^{|\lambda|}_{\mathrm{G}}} \subseteq \mathrm{Gr}_{\mathrm{G}}$. If V_{λ_i} denotes the irreducible representation of $\check{\mathrm{G}}$ with highest weight λ_i, then there is an isomorphism $\mathcal{S}_k(\bigotimes_i \mathrm{V}_{\lambda_i}) \cong \mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$.*

It would be very interesting to understand whether Proposition 3.9.13 can be extended to other non-minuscule irreducible representations. Again, as in Remark 3.6.14, if λ is a dominant minuscule weight of $\check{\mathrm{G}}$, then the coaction of $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ on $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$ defines a homomorphism

$$\mathrm{Spec}\,\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}})^\vee \to \mathrm{GL}(\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)) \tag{3.9.4}$$

of group schemes over $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$, where $\mathrm{GL}(\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda))$ denotes the group scheme of $\mathcal{O}_{\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})}$-linear automorphisms of the vector bundle $\pi_0\mathcal{F}_{\mathrm{T}}(\mathrm{G}/\mathrm{P}_\lambda)$. Under the isomorphisms of Theorem 3.9.7 and Proposition 3.9.13, this homomorphism factors as the composite

$$\tilde{\check{\mathrm{J}}}_{\mathrm{ell}} \to \check{\mathrm{G}} \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}) \to \mathrm{GL}(\mathrm{V}_\lambda) \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}), \tag{3.9.5}$$

where the second map describes the $\check{\mathrm{G}}$-action on V_λ.

Remark 3.9.14. The statements of Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11 can be packaged into a single statement as follows. Suppose k is a complex-oriented 2-periodic $\mathbf{E}_\infty$-ring, and let $\mathbf{G}$ be an oriented commutative k-group scheme. Let $\mathbf{G}_0$ denote the underlying commutative group scheme over $\pi_0(k)$, and let $\mathbf{G}_0^\vee = \mathrm{Hom}(\mathbf{G}_0, \mathrm{B}\mathbf{G}_m)$ denote its 1-shifted Cartier dual. Let F be an algebraically closed field over $\pi_0(k)$; then there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{B}}^{\mathrm{reg}}_{\mathbf{G}_0}/\check{\mathrm{B}}).$$

Similarly, there is an F-linear equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{G}}^{\mathrm{reg}}_{\mathbf{G}_0}/\check{\mathrm{G}}).$$

Here, the notation is as in Definition 4.3.5 below.

In fact, the arguments of Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11 show that these equivalences are monoidal for the convolution tensor products on $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G}; k)$ and $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; k)$ coming from the $\mathbf{E}_2$-structure on Gr_G, and the ordinary tensor product of quasicoherent sheaves. Moreover, a simple adaptation of the discussion at the end of § 3.8 (as well as the discussion in § 4.4) shows that the above equivalences are canonical: they respect natural symmetries of k coming from endomorphisms of $\mathbf{G}_0$.

The object $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathbf{G}_0^\vee)$ has also appeared previously in the literature in connection to equivariant elliptic cohomology; see, for instance, [ST, MRT]. One could heuristically view $\mathrm{QCoh}(\check{\mathrm{B}}_{\mathbf{G}_0}/\check{\mathrm{B}})$ and $\mathrm{QCoh}(\check{\mathrm{G}}_{\mathbf{G}_0}/\check{\mathrm{G}})$ as the "$\mathbf{G}_0$-Hochschild homology" of $\mathrm{QCoh}(\mathrm{B}\check{\mathrm{B}})$ and $\mathrm{QCoh}(\mathrm{B}\check{\mathrm{G}})$, respectively. (See Definition 4.3.5 below.)

To see that these equivalences do indeed package Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11, note that if $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$, then the 1-shifted Cartier dual of $\mathbf{G}_0$ is $\mathrm{B}\hat{\mathbf{G}}_a$, and $\mathrm{Map}(\mathrm{B}\hat{\mathbf{G}}_a, \mathrm{B}\check{\mathrm{B}}) \cong \check{\mathfrak{b}}/\check{\mathrm{B}}$.[21] Similarly, if $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, then the 1-shifted Cartier dual of $\mathbf{G}_0$ is $\mathrm{B}\mathbf{Z}$, and $\mathrm{Map}(\mathrm{B}\mathbf{Z}, \mathrm{B}\check{\mathrm{B}}) \cong \check{\mathrm{B}}/\check{\mathrm{B}}$. Finally, if $\mathbf{G}_0$ is an elliptic curve E, then its 1-shifted Cartier dual is $\mathrm{Pic}^0(\mathrm{E}) = \mathrm{E}$, so $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathbf{G}_0^\vee) = \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$. In fact, in this language, the calculations of [Dev3] show that the stated equivalence continues to hold if $k = \mathrm{ku}$ (now one must replace $\pi_0(k)$ by $\mathbf{Z}[\beta]$, and F by $\mathrm{F}[\beta]$) and $\mathbf{G}$ is the group scheme $\mathrm{Spec}\, \mathbf{Z}[\beta, x, \frac{1}{1+\beta x}]$ with group law $x + y + \beta xy$.

Observe that if $\mathcal{L}$ is a degree zero line bundle on $\mathbf{G}_0^\vee$, then $\mathrm{H}^*(\mathbf{G}_0^\vee; \mathcal{L})$ vanishes unless $\mathcal{L}$ is trivial, in which case it is isomorphic to an exterior algebra over k on a class in degree 1. Using this, the Kostant slice is straightforward to describe in the semisimple rank 1 cases. For instance, if $\check{\mathrm{G}} = \mathrm{PGL}_2$, the map $\kappa : \mathbf{G}_0 \to \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathbf{G}_0^\vee)$ can be understood as follows. Since $\mathbf{G}_0 = \mathrm{Hom}(\mathbf{G}_0^\vee, \mathrm{B}\mathbf{G}_m)$, a point of $\mathbf{G}_0$ can be viewed as a degree zero line bundle on $\mathbf{G}_0^\vee$. Given such a line bundle $\mathcal{L}$, the map κ sends it to the trivial $\check{\mathrm{B}}$-bundle $\mathcal{L} \subseteq \mathcal{L} \oplus \mathcal{O}_{\mathbf{G}_0^\vee} \twoheadrightarrow \mathcal{O}_{\mathbf{G}_0^\vee}$ if $\mathcal{L}$ is nontrivial, and to the unique nontrivial extension $\mathcal{O}_{\mathbf{G}_0^\vee} \subseteq \mathcal{A} \twoheadrightarrow \mathcal{O}_{\mathbf{G}_0^\vee}$ if $\mathcal{L}$ is trivial. This nontrivial extension comes from a nonzero section of $\mathrm{H}^1(\mathbf{G}_0^\vee; \mathcal{O})$.

Let us now turn to some more concrete consequences of Theorem 3.9.7. Just as with Proposition 3.6.15 and Proposition 3.8.15, the calculation of Theorem 3.9.7 gives an *elliptic*

[21] In fact, this works even if k is an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra. Indeed, the 1-shifted Cartier dual of $\mathbf{G}_a$ over $\mathbf{Z}$ is the classifying stack of $\mathrm{Hom}(\mathbf{G}_a, \mathbf{G}_m) = \widehat{\mathbf{G}_a^\sharp}$; here, $\widehat{\mathbf{G}_a^\sharp}$ denotes the formal scheme $\mathrm{Spf}(\mathbf{Z}\langle x\rangle/\mathrm{I}^{[n]})$ where $\mathbf{Z}\langle x\rangle$ is the divided power algebra on a class x and $\mathrm{I}^{[n]}$ is the ideal generated by elements of $\mathbf{Z}\langle x\rangle$ of degree $\geq n$. Then, $\mathrm{Map}(\mathrm{B}\widehat{\mathbf{G}_a^\sharp}, \mathrm{X})$ is isomorphic to the 1-shifted tangent bundle $\mathrm{T}[-1](\mathrm{X})$, so that $\mathrm{Map}(\mathrm{B}\widehat{\mathbf{G}_a^\sharp}, \mathrm{B}\check{\mathrm{B}}) \cong \check{\mathfrak{b}}/\check{\mathrm{B}}$ even over $\mathbf{Z}$.

version of the Gelfand-Graev action on the affine closure $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$. Taking the affine closure in the naive sense is very destructive in the case of elliptic cohomology. Nevertheless, one can define $\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ to be the relative spectrum over $\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})$ of π_0 of the (*classical*, not derived!) pushforward of the structure sheaf along the quotient morphism

$$(\check{\mathrm{G}} \times \check{\mathrm{T}} \times \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))/\widetilde{\check{\mathrm{J}}}_{\mathrm{ell}} \to \mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}).$$

Proposition 3.9.15 (Elliptic Gelfand-Graev action)**.** *The natural action of* $\check{\mathrm{G}} \times \check{\mathrm{T}}$ *on* $\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ *extends to an action of* $\check{\mathrm{G}} \times (\mathrm{W} \rtimes \check{\mathrm{T}})$*, where* W *is the Weyl group.*

The moment map $\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{G}} \to \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$ is W-equivariant for the trivial action on the target. There is a commutative diagram

$$\begin{tikzcd} \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E}) \arrow[r, hook] \arrow[dr] & \overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}} \arrow[d] \\ & \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}) \end{tikzcd}$$

which relates $\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}$ to the elliptic Grothendieck-Springer resolution [BN]; and via this diagram, the elliptic Gelfand-Graev action is closely related to the Weyl action in elliptic Springer theory.

Remark 3.9.16. The proof of Proposition 3.9.15 generalizes to show that if $\check{\mathrm{P}} \subseteq \check{\mathrm{G}}$ is a parabolic subgroup with Levi quotient $\check{\mathrm{L}}$ and unipotent radical $\mathrm{U}_{\check{\mathrm{P}}}$, then the natural action of $\check{\mathrm{G}} \times \check{\mathrm{L}}$ on the affine closure $\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\mathrm{U}_{\check{\mathrm{P}}})}$ extends to an action of $\check{\mathrm{G}} \times (\mathrm{W}_{\mathrm{L}} \rtimes \check{\mathrm{L}})$, where $\mathrm{W}_{\mathrm{L}} = \mathrm{N}_{\check{\mathrm{G}}}(\check{\mathrm{L}})/\check{\mathrm{L}}$ is the Weyl group.

As in Example 3.6.17 and Example 3.8.17, it is possible to make the action of Proposition 3.9.15 explicit in the case when $\check{\mathrm{G}} = \mathrm{SL}_2$.

Example 3.9.17. Let $\mathcal{O}(\infty)$ denote the inverse of the ideal sheaf cutting out the zero section inside E, and let $\mathcal{F} = (\mathcal{O} \oplus \mathcal{O}(\infty))^{\oplus 2}$. As in Example 3.8.17, $\overline{\mathrm{T}^*_{\mathrm{E}}(\mathrm{SL}_2/\mathbf{G}_a)}$ can be identified with the space of sections $(u, v) = (\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right), (v_1, v_2))$ of the bundle $\mathcal{F} \to \mathrm{E}$ such that the resulting section $\langle u, v\rangle = u_1v_1 + u_2v_2$ of $\mathcal{O}(\infty)$ has vanishing locus given by the zero section of E. Modifying the analysis of Example 3.6.17 shows that if $[-1] : \mathrm{E} \to \mathrm{E}$ denotes the inversion map, the $\mathbf{Z}/2$-action of Proposition 3.9.15 sends

$$(\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right), (v_1, v_2)) \mapsto \left(\left(\begin{smallmatrix} -u_2 \\ u_1 \end{smallmatrix}\right), \alpha(v_2, -v_1)\right),$$

where α is given locally around ∞ by multiplication by $-\frac{[-1](\langle u,v\rangle)}{\langle u,v\rangle}$. (The discussion here makes sense with E replaced by any 1-dimensional group scheme $\mathbf{H}$. When $\mathbf{H} = \mathbf{G}_a$ or $\mathbf{G}_m$, the class $-\frac{[-1](\langle u,v\rangle)}{\langle u,v\rangle}$ is equal to 1 or $\frac{1}{1+\langle u,v\rangle}$, respectively, as expected from Example 3.6.17 and Example 3.8.17.)

One could regard the variety $\overline{\mathrm{T}^*_{\mathrm{E}}(\mathrm{SL}_2/\mathbf{G}_a)}$ of Example 3.9.17 as an elliptic version of Van den Bergh's multiplicative quiver variety $\mathcal{B}(\mathbf{A}^1, \mathbf{A}^2)$ from [Van]. Motivated by this observation, we hope to similarly define a notion of "elliptic quiver varieties" (generalizing the notion of multiplicative quiver variety from [CS]) in future work.

We also have the following analogue of Proposition 3.6.18, whose proof is exactly the same (one only needs to use [Dav, Proposition 3.1.16], which says that $\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}} \hookrightarrow \mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})$ has complement of codimension 2, and similarly for $\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}} \hookrightarrow \mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})$).

Proposition 3.9.18. *Let* $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E})))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{T}}}(\mathrm{E}))$*. Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{\mathrm{B}}}(\mathrm{E})^{\mathrm{reg}}) \to \mathrm{QCoh}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of (3.9.3)*. Furthermore, this functor is* W*-equivariant for the natural action of* $\mathrm{W} = \mathrm{N}_{\mathrm{G}_c}(\check{\mathrm{T}}_c)/\check{\mathrm{T}}_c$ *on the left-hand side and the Gelfand-Graev action of Proposition 3.9.15 on the right-hand side.*

Similarly, suppose G *has torsion-free fundamental group, and let* $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit$ *denote the heart of the t-structure on* $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{coMod}_{\pi_0(\mathcal{F}_{\check{\mathrm{G}}}(\mathrm{Gr}_{\mathrm{G}}))^\vee}(\mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{G}},0}))$ *coming from the standard (homological truncation) t-structure on* $\mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{G}},0})$*. Then, the composite functor*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E})^{\mathrm{reg}}) \to \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))$$

is t-exact, and on hearts, it restricts to a fully faithful functor on the essential image of the functor $\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F}$ *(analogous to* (3.9.2)*).*

Proposition 3.9.18 gives an analogue of [BF, Theorem 4]: namely, if $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{\mathrm{G}}) \to \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))^\heartsuit \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit \otimes_{\pi_0(k)} \mathrm{F}.$$

Similarly, if $\mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$ denotes the essential image of the pullback functor $\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{T}}) \to \mathrm{QCoh}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})$, then there is a fully faithful embedding

$$\mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})^\heartsuit \hookrightarrow \mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit \otimes_{\pi_0(k)} \mathrm{F}.$$

This implies the following result.

Corollary 3.9.19. *Let* $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))^{\mathrm{min},\heartsuit}$ *denote the essential image of* $\mathrm{Rep}_{\mathrm{min}}(\check{\mathrm{G}})$ *under the pullback functor* $\mathrm{Rep}(\check{\mathrm{G}})^\heartsuit \to \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))^\heartsuit$*. Similarly, let* $(\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\pi_0(k)} \mathrm{F})^{\mathrm{min}}$ *denote the idempotent completion of the subcategory of* $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^\heartsuit \otimes_{\pi_0(k)} \mathrm{F}$ *spanned by* $\mathcal{F}^{\mathrm{gr}}_{\lambda_\bullet}$ *ranging over sequences* $\lambda_\bullet$ *of minuscule highest weights. Then there is an equivalence*

$$\mathrm{QCoh}_{\mathrm{free}}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))^{\mathrm{min},\heartsuit} \simeq (\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit \otimes_{\pi_0(k)} \mathrm{F})^{\mathrm{min}}.$$

There is a similar equivalence

$$(\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit \otimes_{\pi_0(k)} \mathrm{F})^{\mathrm{min}} \simeq \mathrm{QCoh}_{\mathrm{free}}(\check{\mathrm{G}}\backslash\overline{\mathrm{T}^*_{\mathrm{E}}(\check{\mathrm{G}}/\check{\mathrm{N}})}/\check{\mathrm{T}})^{\mathrm{min},\heartsuit},$$

where these categories are defined analogously by idempotent completion.

Note, again, that the category $(\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)^\heartsuit \otimes_{\pi_0(k)} \mathrm{F})^{\mathrm{min}}$ is the heart of a degeneration, in the sense of § 3.3, of the similarly-defined category $(\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_k \mathrm{F}[u^{\pm 1}])^{\mathrm{min}}$. Thus Corollary 3.9.19 gives an equivalence between the purely algebraically defined category $\mathrm{QCoh}_{\mathrm{free}}(\mathrm{Bun}^{\mathrm{ss}}_{\check{\mathrm{G}}}(\mathrm{E}))^{\mathrm{min},\heartsuit}$ and a degeneration of the purely topologically defined category

$(\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; k) \otimes_k \mathrm{F}[u^{\pm 1}])^{\min}$. Observe, again, that if $\lambda_\bullet$ and $\mu_\bullet$ are two sequences of dominant minuscule weights of $\check{\mathrm{G}}$, there is an equivalence of k-modules

$$\mathrm{Map}_{(\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G};k)\otimes_k \mathrm{F}[u^{\pm 1}])^{\min}}(\mathcal{F}_{\lambda_\bullet}, \mathcal{F}_{\mu_\bullet}) \simeq \mathcal{F}_{\check{\mathrm{G}}_c}(\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}} \times_{\mathrm{Gr}_\mathrm{G}} \overline{\mathrm{Gr}_\mathrm{G}^{\mu_\bullet}}),$$

so that the category $(\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; k) \otimes_k \mathrm{F}[u^{\pm 1}])^{\min}$ compares to the k-analogue of the category from [CK, Section 3.4].

Let us end this section with a brief comment regarding loop-rotation equivariance. Recall from Definition 3.6.20 the algebra $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ associated to a 1-dimensional group scheme $\mathbf{H}$ over a field F and a root system with torus T and Weyl group W. In the following discussion, we will set $\mathbf{H} = \mathrm{E}$, so that $\mathbf{H}_\mathrm{T} = \mathrm{Bun}^0_\mathrm{T}(\mathrm{E}) = \mathcal{M}_\mathrm{T}$. Exactly the same argument as in Theorem 3.6.24 shows the following result; here, G does not need to be simply-laced.

Theorem 3.9.20. *There is an isomorphism of sheaves of associative algebras over $\mathbf{H}_{\mathbf{G}_m^{\mathrm{rot}}} = \mathrm{E}$:*

$$\pi_0 \mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_\mathrm{G})^\vee \cong \mathcal{H}(\mathrm{E}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}}). \tag{3.9.6}$$

Here, $\pi_0 \mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_\mathrm{G})^\vee$ is equipped with the associative algebra structure coming from convolution. Moreover, the above isomorphism is also one of (cocommutative) Hopf $\mathcal{O}_{\mathcal{M}_{\widetilde{\mathrm{T}},0}} \cong \mathcal{O}_{\mathbf{H}_{\widetilde{\mathrm{T}}}}$-algebroids.

Remark 3.9.21. Recall the quotient $\mathrm{Bun}^0_\mathrm{T}(\mathrm{E})/\!\!/\widetilde{\mathrm{W}}$ from Remark 3.6.23. The discussion therein combined with Theorem 3.9.20 gives an equivalence of categories

$$\pi_0 \mathcal{F}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_\mathrm{G})^\vee\text{-mod} \simeq \mathcal{H}(\mathrm{E}, \widetilde{\mathrm{T}}, \widetilde{\mathrm{W}})\text{-mod} \simeq \mathrm{IndCoh}(\mathrm{Bun}^0_\mathrm{T}(\mathrm{E})/\!\!/\widetilde{\mathrm{W}}).$$

It follows, via the argument of Corollary 3.6.32, that $\mathrm{Loc}^{\mathrm{gr}}_{\widetilde{\mathrm{T}}_c}(\mathrm{Fl}_\mathrm{G}; k) \otimes_{\pi_0(k)} \mathrm{F}$ is equivalent to the quotient of $\mathrm{QCoh}(\mathrm{Bun}^0_\mathrm{T}(\mathrm{E}))$ by the action of $\mathrm{IndCoh}(\mathrm{Bun}^0_\mathrm{T}(\mathrm{E})/\!\!/\widetilde{\mathrm{W}})$.

Assume, again, that G is simply-laced. Just as in § 3.6, one would like to use Theorem 3.9.20 to prove analogues of Corollary 3.6.32 and (3.6.11). However, unlike with Theorem 3.8.31, we do not even have a putative description for the Langlands dual side. By analogy with the K-theoretic case, it is natural to expect that the dual side will be related to elliptic quantum groups à la [Fel]; I am currently exploring this direction of research.

3.10 Comparison to Brylinski-Zhang

In [BZ], Brylinski-Zhang compute the G_c-equivariant complex K-theory of G_c for a connected compact Lie group G_c with torsion-free fundamental group as the ring $\Omega^*_{\mathrm{RU(G)}/\mathbf{Z}} = \Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathbf{Z}}$ of Kähler differentials on the complex representation ring of G. Our goal in this section is to describe the relationship between this calculation and (the proof of) Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11.

We begin by stating an obvious corollary of Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11. Recall that if $\mathbf{G}_0$ is either $\mathbf{G}_a$, $\mathbf{G}_m$, or an elliptic curve E, $\mathcal{M}_{\mathrm{T},0} = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{G}_0)$, and $\mathrm{G}_{\mathbf{G}_0}$ is $\mathfrak{g}$, G, or $\mathrm{Bun}^{\mathrm{ss}}_\mathrm{G}(\mathrm{E}^\vee)$, respectively[22], then there is a Kostant section $\kappa : \mathcal{M}_{\mathrm{T},0} \to \check{\mathrm{B}}_{\mathbf{G}_0}/\check{\mathrm{B}}$ as described in Definition 3.6.1, Definition 3.8.1, and Proposition 3.9.2. Recall that F is an algebraically closed field of characteristic zero containing $\pi_0(k)$.

[22] This is in keeping with the notation introduced below in Definition 4.3.5.

Theorem 3.10.1. *Let* G *be a connected almost simple simply-laced group. Let* k *denote either* $\mathbf{Q}[u^{\pm 1}]$, KU, *or elliptic cohomology, and let* $\mathbf{G}_0$ *be either* $\mathbf{G}_a$, $\mathbf{G}_m$, *or an elliptic curve* E *over* $\pi_0(k)$, *respectively. Then there is an equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{G}_c; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{T}},0} \times_{\check{\mathrm{B}}_{\mathbf{G}_0}/\check{\mathrm{B}}} \mathcal{M}_{\check{\mathrm{T}},0}),$$

where the right-hand side denotes the self-intersection of the Kostant slice.

Proof. Recall from Definition 3.3.4 that

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{G}_c; k) = \mathrm{LMod}_{\pi_0(\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee)}(\mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{T}},0})).$$

In Theorem 3.6.3, Theorem 3.8.3, and Theorem 3.9.7, we showed that $\mathrm{Spec}_{\mathcal{M}_{\check{\mathrm{T}},0}}(\pi_0(\mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee))$ is isomorphic to the self-intersection $\mathcal{M}_{\check{\mathrm{T}},0} \times_{\check{\mathrm{B}}_{\mathbf{G}_0}/\check{\mathrm{B}}} \mathcal{M}_{\check{\mathrm{T}},0}$, so the desired equivalence follows. □

In the same way, if G is further assumed to have torsion-free fundamental group, and $\mathcal{M}_{\mathrm{G},0}$ denotes the moduli *space* of semistable G-bundles on $\mathbf{G}_0^\vee$, there is a Kostant section $\kappa : \mathcal{M}_{\mathrm{G},0} \to \check{\mathrm{G}}_{\mathbf{G}_0}/\check{\mathrm{G}}$. In the additive and multiplicative cases, this follows from Definition 3.6.1, Definition 3.8.1, and in the elliptic case, it can be deduced from [Dav] as in Proposition 3.9.2. Just as in Theorem 3.10.1, there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathcal{M}_{\check{\mathrm{G}},0} \times_{\check{\mathrm{G}}_{\mathbf{G}_0}/\check{\mathrm{G}}} \mathcal{M}_{\check{\mathrm{G}},0}) \tag{3.10.1}$$

where the right-hand side denotes the self-intersection of the Kostant slice. Under this equivalence, the "constant sheaf" in $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c; k)$ is sent to the pushforward of the structure sheaf under the relative diagonal

$$\delta : \mathcal{M}_{\check{\mathrm{G}},0} \to \mathcal{M}_{\check{\mathrm{G}},0} \times_{\check{\mathrm{G}}_{\mathbf{G}_0}/\check{\mathrm{G}}} \mathcal{M}_{\check{\mathrm{G}},0}.$$

In the remainder of this section, we will explain how (3.10.1) implies the calculation of [BZ], as well as the relationship to the Hochschild-Kostant-Rosenberg theorem. (This, of course, is a triple of authors distinct from Hopkins-Kuhn-Ravenel with initials "HKR"!) For simplicity, we will only focus on the case when k is $\mathbf{Q}[u^{\pm 1}]$ or KU (so $\mathbf{G}_0$ is either $\mathbf{G}_a$ or $\mathbf{G}_m$, and $\check{\mathrm{G}}_{\mathbf{G}_0}/\check{\mathrm{G}}$ is either $\check{\mathfrak{g}}/\check{\mathrm{G}}$ or $\check{\mathrm{G}}/\check{\mathrm{G}}$). With a little bit of elbow grease, one can show that most of the results below continue to work for elliptic cohomology, too.

Recall that $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c; k)$ is intended to be an approximation to a k-linear ∞-category of G_c-equivariant local systems on G_c. The algebra of endomorphisms of the constant sheaf in this ∞-category is given by the equivariant cochains $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c)$. This is a quasicoherent sheaf over the spectral k-scheme $\mathcal{M}_{\mathrm{G}}$, and it can be described explicitly as follows. If $\mathfrak{X}_k$ is a spectral prestack over k, let $\mathcal{L}\mathfrak{X}_k$ denote the free loop space of $\mathfrak{X}_k$, i.e., the mapping prestack $\mathrm{Map}(\mathrm{B}\mathbf{Z}, \mathfrak{X}_k)$. Here, $\mathbf{Z}$ is viewed as a constant stack over k. The global sections of the structure sheaf of $\mathcal{L}\mathfrak{X}_k$ computes the Hochschild homology $\mathrm{HH}(\mathfrak{X}_k/k)$.

Proposition 3.10.2. *Assume (for simplicity) that* k *is either* $\mathbf{Q}[u^{\pm 1}]$ *or* KU. *If* G *is connected, then there is an isomorphism of spectral* k*-schemes*

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{G}}}(\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c)) \cong \mathcal{L}\mathcal{M}_{\mathrm{G}}.$$

In particular, there is an isomorphism of $\mathbf{E}_\infty$-k_{G_c}*-algebras*

$$\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) \cong \mathrm{HH}(\mathcal{M}_{\mathrm{G}}/k).$$

Proof. Recall that $\mathrm{B}\mathbf{Z}$ is isomorphic to the constant k-stack S^1, which can be written as the pushout $*\amalg_{*\amalg *}*$. Therefore, since $\mathcal{M}_\mathrm{G} = \operatorname{Spec} k_{\mathrm{G}_c}$ is affine (because k is either $\mathbf{Q}[u^{\pm 1}]$ or KU), we may wite $\mathcal{L}\mathcal{M}_\mathrm{G} = \operatorname{Spec}(k_{\mathrm{G}_c} \otimes_{k_{\mathrm{G}_c}\otimes_k k_{\mathrm{G}_c}} k_{\mathrm{G}_c})$. Since the functor $\mathcal{F}_\mathrm{G} : \mathcal{S}(\mathrm{G}_c)^{\mathrm{op}} \to \mathrm{Mod}_{k_{\mathrm{G}_c}}$ sends finite products of connected finite G-spaces to tensor products, we find that

$$k_{\mathrm{G}_c} \otimes_{k_{\mathrm{G}_c}\otimes_k k_{\mathrm{G}_c}} k_{\mathrm{G}_c} \cong \mathcal{F}_\mathrm{G}(*) \otimes_{\mathcal{F}_{\mathrm{G}\times\mathrm{G}}(*)} \mathcal{F}_\mathrm{G}(*) \cong \mathcal{F}_\mathrm{G}(\mathrm{G}_c),$$

since there is an isomorphism of orbispaces

$$*/\mathrm{G}_c \times_{*/(\mathrm{G}_c\times\mathrm{G}_c)} */\mathrm{G}_c \cong \mathrm{G}_c/\mathrm{G}_c. \quad \square$$

Remark 3.10.3. The approach of Proposition 3.10.2 can be used to compute the equivariant cohomology $\mathcal{F}_\mathrm{G}(\Omega\mathrm{G}_c)$, too. Namely, observe that there is an isomorphism of orbispaces

$$*/\mathrm{G}_c \times_{*/\mathrm{G}_c \times_{*/(\mathrm{G}_c\times\mathrm{G}_c)} */\mathrm{G}_c} */\mathrm{G}_c \cong (\Omega\mathrm{G}_c)/\mathrm{G}_c,$$

so that there is an isomorphism

$$\mathcal{F}_\mathrm{G}(\Omega\mathrm{G}_c) = k_{\mathrm{G}_c} \otimes_{k_{\mathrm{G}_c}\otimes_{k_{\mathrm{G}_c}\otimes_k k_{\mathrm{G}_c}} k_{\mathrm{G}_c}} k_{\mathrm{G}_c}.$$

The right-hand side can be expressed more succinctly as the factorization homology $\int_{\mathrm{S}^2}(k_{\mathrm{G}_c}/k)$.

More generally, observe that if $\mathrm{K}_c \subseteq \mathrm{G}_c$ is a closed subgroup such that $\mathrm{G}_c/\mathrm{K}_c$ is a finite K_c-space (where K_c acts on the left by multiplication), and $\mathrm{L}(\mathrm{G}_c/\mathrm{K}_c)$ denotes the (topological) free loop space of $\mathrm{G}_c/\mathrm{K}_c$, then

$$\mathrm{G}_c\backslash\mathrm{L}(\mathrm{G}_c/\mathrm{K}_c) \simeq \mathrm{K}_c\backslash\Omega(\mathrm{G}_c/\mathrm{K}_c) \simeq (* \times_{*\times_{*/\mathrm{G}_c}*/\mathrm{K}_c} *)/\mathrm{K}_c \simeq */\mathrm{K}_c \times_{*/\mathrm{K}_c\times_{*/\mathrm{G}_c}*/\mathrm{K}_c} */\mathrm{K}_c.$$

It follows that there is an isomorphism

$$\mathcal{F}_\mathrm{G}(\mathcal{L}(\mathrm{G}_c/\mathrm{K}_c)) = k_{\mathrm{K}_c} \otimes_{k_{\mathrm{K}_c}\otimes_{k_{\mathrm{G}_c}} k_{\mathrm{K}_c}} k_{\mathrm{K}_c}.$$

The right-hand side can be expressed more succinctly as the relative Hochschild homology $\mathrm{HH}(\mathcal{M}_\mathrm{K}/\mathcal{M}_\mathrm{G})$, so that there is an isomorphism of spectral k-schemes

$$\operatorname{Spec}_{\mathcal{M}_\mathrm{G}}(\mathcal{F}_\mathrm{G}(\mathrm{G}_c/\mathrm{K}_c)) \cong \mathcal{L}(\mathcal{M}_\mathrm{K}/\mathcal{M}_\mathrm{G}) \cong \mathcal{L}(\mathcal{M}_\mathrm{K}) \times_{\mathcal{L}(\mathcal{M}_\mathrm{G})} \mathcal{M}_\mathrm{G}.$$

The discussion above computing $\mathcal{F}_\mathrm{K}(\Omega\mathrm{K}_c)$ is the special case of the above calculation when $\mathrm{G}_c = \mathrm{K}_c \times \mathrm{K}_c$, with K_c embedded diagonally.

Example 3.10.4. Let $k = \mathbf{Q}[u^{\pm 1}]$. Then the preceding discussion shows that $\mathrm{C}^*_{\mathrm{G}_c}(\Omega\mathrm{G}_c; \mathbf{Q}[u^{\pm 1}])$ is isomorphic to the factorization homology $\int_{\mathrm{S}^2}(k_{\mathrm{G}_c}/k) = \mathrm{HH}(k_{\mathrm{G}_c}/k_{\mathrm{G}_c}\otimes_k k_{\mathrm{G}_c})$. The latter has a Hochschild-Kostant-Rosenberg filtration whose associated graded is given by the 2-periodification $\mathrm{L}\Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/(\mathfrak{t}/\!\!/\mathrm{W}\times_{\operatorname{Spec}\mathbf{Q}}\mathfrak{t}/\!\!/\mathrm{W})}[u^{\pm 1}]$ of the derived Hodge complex of $\mathfrak{t}/\!\!/\mathrm{W}$ embedded diagonally into $\mathfrak{t}/\!\!/\mathrm{W} \times_{\operatorname{Spec}\mathbf{Q}} \mathfrak{t}/\!\!/\mathrm{W}$. Since we are working rationally, the Hochschild-Kostant-Rosenberg filtration splits, and so there is an isomorphism

$$\int_{\mathrm{S}^2}(k_{\mathrm{G}_c}/k) \cong \mathrm{L}\Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/(\mathfrak{t}/\!\!/\mathrm{W}\times_{\operatorname{Spec}\mathbf{Q}}\mathfrak{t}/\!\!/\mathrm{W})}[u^{\pm 1}].$$

Note that if X (like $\mathfrak{t}/\!\!/\mathrm{W}$) is an affine space over a commutative ring R, then $\mathrm{L}\Omega^*_{\mathrm{X}/(\mathrm{X}\times_{\operatorname{Spec}(\mathrm{R})}\mathrm{X})} \cong \Gamma^*(\Omega^1_{\mathrm{X}/\mathrm{R}})$; so the above isomorphism could instead be stated as

$$\int_{\mathrm{S}^2}(k_{\mathrm{G}_c}/k) \cong \operatorname{Sym}_{\mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}}}(\Omega^1_{\mathfrak{t}/\!\!/\mathrm{W}})[u^{\pm 1}] = \mathcal{O}_{\mathrm{T}(\mathfrak{t}/\!\!/\mathrm{W})}[u^{\pm 1}],$$

where $\mathrm{T}(\mathfrak{t}/\!\!/\mathrm{W})$ is the tangent bundle of $\mathfrak{t}/\!\!/\mathrm{W}$. It follows that there is an isomorphism

$$\operatorname{Spec} \mathrm{C}^*_{\mathrm{G}_c}(\Omega \mathrm{G}_c; \mathbf{Q}[u^{\pm 1}]) \cong \mathrm{T}(\mathfrak{t}/\!\!/\mathrm{W}) \times_{\operatorname{Spec}(\mathbf{Q})} \operatorname{Spec}(\mathbf{Q}[u^{\pm 1}]).$$

This recovers the $\hbar = 0$ case of [BF, Theorem 1]. The case with loop-rotation equivariance included, i.e., when $\hbar$ need not be zero, follows from Lemma 3.10.5 below, which recovers the description of $\operatorname{Spec} \mathrm{C}^*_{\mathrm{G}_c \times \mathrm{S}^1_{\mathrm{rot}}}(\Omega \mathrm{G}_c; \mathbf{Q}[u^{\pm 1}])$ as the deformation to the normal cone of the diagonal embedding $\mathfrak{t}/\!\!/\mathrm{W} \hookrightarrow \mathfrak{t}/\!\!/\mathrm{W} \times \mathfrak{t}/\!\!/\mathrm{W}$.

The following statement is essentially Koszul dual to the usual Hochschild-Kostant-Rosenberg theorem describing Hochschild homology with its circle action via the de Rham complex:

Lemma 3.10.5. *Let $\mathrm{X} = \operatorname{Spec}(\mathrm{A})$ be a smooth affine scheme over $\mathbf{Q}$, and let $\operatorname{Def}^{\Delta}_{\hbar}(\mathrm{X})$ denote the deformation to the normal cone of the diagonal embedding $\mathrm{X} \hookrightarrow \mathrm{X} \times \mathrm{X}$. Then there is an isomorphism*

$$\operatorname{Spec} \pi_* \left(\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})\right)^{h\mathrm{S}^1} \cong \operatorname{Def}^{\Delta}_{\hbar}(\mathrm{X})$$

of $\pi_* \mathbf{Q}^{h\mathrm{S}^1} \cong \mathbf{Q}[\![\hbar]\!]$-algebras.

Proof. By standard arguments, it suffices to check the claim when A is a finitely generated polynomial algebra. Let us demonstrate the claim when A is a polynomial algebra on a single class; an easy modification of this argument will prove the claim in general when $\mathrm{A} = \mathbf{Q}[\mathrm{V}]$ for some finite-dimensional $\mathbf{Q}$-vector space V. Let us identify $\mathbf{Q}[x] \otimes \mathbf{Q}[x] = \mathbf{Q}[x, y]$, so that the standard resolution of $\mathbf{Q}[x]$ as a $\mathbf{Q}[x, y]$-algebra identifies

$$\mathbf{Q}[x] \otimes_{\mathbf{Q}[x] \otimes \mathbf{Q}[x]} \mathbf{Q}[x] \cong \mathbf{Q}[x, \sigma(x - y)]/(\sigma(x - y)^2),$$

where $\sigma(x - y)$ is in degree 1. This implies that

$$\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q}) \simeq \mathbf{Q}[x] \otimes_{\mathbf{Q}[x] \otimes_{\mathbf{Q}[x] \otimes \mathbf{Q}[x]} \mathbf{Q}[x]} \mathbf{Q}[x] \cong \mathbf{Q}[x, \sigma^2(x - y)],$$

with $\sigma^2(x-y)$ in degree 2. Note that $\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})$ is an S^1-equivariant $\mathbf{E}_\infty$-$\mathbf{Q}[x, y]$-algebra, so that $\pi_* \left(\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})\right)^{h\mathrm{S}^1}$ is an $\mathbf{E}_\infty$-$\mathbf{Q}[x, y]$-algebra; let us now determine this algebra structure. Since this ring is concentrated in even degrees, the homotopy fixed point spectral sequence computing $\pi_* \left(\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})\right)^{h\mathrm{S}^1}$ degenerates, and we find that $\pi_* \left(\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})\right)^{h\mathrm{S}^1} \cong \mathbf{Q}[\![\hbar]\!][x, \sigma^2(x - y)]$ with $\sigma^2(x - y)$ in weight 2 and $\hbar$ in weight -2. The $\mathbf{Q}[x, y]$-algebra structure is given by the observation that

$$x - y = \hbar \sigma^2(x - y);$$

this relation is true for abstract reasons (as explained, for instance, in [HW, Appendix A]). It follows that there is an isomorphism

$$\pi_* \left(\int_{\mathrm{S}^2} (\mathrm{A}/\mathbf{Q})\right)^{h\mathrm{S}^1} \cong \mathbf{Q}[\![\hbar]\!][x, y, \tfrac{x-y}{\hbar}]$$

of $\mathbf{Q}[x, y]$-algebras. The spectrum of the right-hand side identifies with $\operatorname{Def}^{\Delta}_{\hbar}(\mathbf{A}^1)$, as desired. $\square$

Remark 3.10.6. What is the analogue of the discussion in Example 3.10.4 if we do not rationalize? Suppose first that G is a torus T. Then $k_{\mathrm{T}_c} = \mathrm{C}^*(\mathrm{BT}_c; k)$ can be identified with the free binomial ring $\mathrm{LBin}_k(\mathbb{X}^*(\mathrm{T})[-2])$, with notation as in [KSZ]. It follows that $\int_{\mathrm{S}^2}(k_{\mathrm{T}_c}/k)$ is isomorphic to $\mathrm{LBin}_k(\mathbb{X}^*(\mathrm{T})[-2] \oplus \mathbb{X}^*(\mathrm{T}))$, so that

$$\mathrm{Spec}\, \pi_* \int_{\mathrm{S}^2}(k_{\mathrm{T}_c}/k) \cong \mathfrak{t}(2) \times \overline{\mathbb{X}^*(\mathrm{T})},$$

where $\mathfrak{t}(2)$ denotes $\mathfrak{t}$ placed in weight 2, and $\overline{\mathbb{X}^*(\mathrm{T})}$ is the affinization of the constant scheme $\mathbb{X}^*(\mathrm{T})$. For instance,

$$\pi_* \int_{\mathrm{S}^2}(k_{\mathrm{S}^1}/k) \cong k\left[x, \binom{a}{n}\right]_{n\geq 0},$$

where x is in degree -2 and a is in degree zero. As in Lemma 3.10.5, this implies that

$$\pi_* \int_{\mathrm{S}^2}(k_{\mathrm{S}^1}/k)^{\hbar\mathrm{S}^1} \cong k\left[\hbar, x, y, \tfrac{\prod_{j=0}^{n-1}(x-y-j\hbar)}{n!\hbar^n}\right]_{n\geq 0},$$

where $\hbar$, x, and y all live in degree -2. (The class a corresponds to $\frac{x-y}{\hbar}$.)

The above discussion lets one compute $\int_{\mathrm{S}^2}(k_{\mathrm{G}_c}/k)$ if the map $k_{\mathrm{G}_c} \to k_{\mathrm{T}_c}^{\mathrm{W}}$ is an isomorphism. For instance, one finds that if $k = \mathbf{Z}[1/2]$ and $\mathrm{G} = \mathrm{SL}_2$, then there is an isomorphism

$$\pi_* \int_{\mathrm{S}^2}(k_{\mathrm{SU}(2)}/k)^{\hbar\mathrm{S}^1} \cong k\left[\hbar, x^2, y^2, \tfrac{\prod_{j=0}^{n-1}(x-y+(n-1-2j)\hbar)(x+y+(n-1-2j)\hbar)}{n!\hbar^n}\right]_{n\geq 0}.$$

Let us now discuss the relationship between Proposition 3.10.2 and (3.10.1). Although the cases $k = \mathbf{Q}[u^{\pm 1}]$ and $k = \mathrm{KU}$ can be treated simultaneously, we will present the discussion separately for both for the sake of clarity. The upshot of this discussion is that the approximation to $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c)$ afforded by the degeneration of $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{G}_c; k)$ to $\mathrm{Loc}_{\mathrm{G}_c}^{\mathrm{gr}}(\mathrm{G}_c; k)$ identifies, under Proposition 3.10.2 and (3.10.1), with the Hochschild-Kostant-Rosenberg spectral approximation of $\pi_*\mathrm{HH}(\mathcal{M}_{\mathrm{G}}/k)$ by $\Omega^*_{\mathcal{M}_{\mathrm{G},0}/\pi_0(k)}$.

Lemma 3.10.7. *Let* H *be a smooth affine group scheme over an affine scheme* $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$*, let* $\delta : \mathrm{S} \to \mathrm{H}$ *denote the zero section, and let* $\mathfrak{h}$ *denote its Lie algebra (viewed as a vector bundle over* S*). Then* $\mathrm{End}_{\mathrm{QCoh}(\mathrm{H})}(\delta_*\mathcal{O}_{\mathrm{S}})$ *has a filtration whose associated graded is isomorphic to* $\mathcal{O}_{\mathfrak{h}^*[1]}$*. If* R *is a* $\mathbf{Q}$*-algebra, this filtration splits.*

Proof. The endomorphism algebra $\mathrm{End}_{\mathrm{QCoh}(\mathrm{H})}(\delta_*\mathcal{O}_{\mathrm{S}})$ is isomorphic to the R-linear dual of $\mathcal{O}_{\mathrm{S}\times_{\mathrm{H}}\mathrm{S}}$. The derived scheme $\mathrm{S}\times_{\mathrm{H}}\mathrm{S}$ depends only on the formal completion $\hat{\mathrm{H}}$. Note that $\hat{\mathrm{H}}$ admits a filtration (coming from powers of the ideal sheaf of the zero section of $\hat{\mathrm{H}}$) whose associated graded is isomorphic to $\hat{\mathfrak{h}}$; furthermore, the exponential map defines a splitting of this filtration when R is a $\mathbf{Q}$-algebra. This defines a filtration on $\mathrm{S}\times_{\mathrm{H}}\mathrm{S}$ whose associated graded is isomorphic to $\mathrm{S}\times_{\mathfrak{h}}\mathrm{S} = \mathfrak{h}[-1]$. Therefore, the R-linear dual of $\mathcal{O}_{\mathrm{S}\times_{\mathrm{H}}\mathrm{S}}$ is isomorphic to $\mathcal{O}_{\mathfrak{h}^*[1]}$. □

Example 3.10.8. Suppose $k = \mathbf{Q}[u^{\pm 1}]$, and let $\check{\mathrm{J}} = \mathfrak{t}/\!\!/\mathrm{W} \times_{\check{\mathfrak{g}}^*/\check{\mathrm{G}}} \mathfrak{t}/\!\!/\mathrm{W}$. Then (3.10.1) states that there is an equivalence

$$\mathrm{Loc}_{\mathrm{G}_c}^{\mathrm{gr}}(\mathrm{G}_c; k) \otimes_{\mathbf{Q}} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{J}}),$$

and the "constant sheaf" $\underline{k}^{\mathrm{gr}}$ in $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c;k)$ is sent to the pushforward of the structure sheaf under the identity section $\delta : \mathfrak{t}/\!\!/\mathrm{W} \to \check{\mathrm{J}}$. Taking endomorphisms, we find that

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Q}} \mathrm{F} \cong \mathrm{End}_{\mathrm{QCoh}(\check{\mathrm{J}})}(\delta_* \mathcal{O}_{\mathfrak{t}/\!\!/\mathrm{W}}).$$

By Lemma 3.10.7, the right-hand side admits a (split) filtration whose associated graded is isomorphic to the algebra of functions on $\mathrm{Lie}_{\mathfrak{t}/\!\!/\mathrm{W}}(\check{\mathrm{J}})^*[1]$. By [Ric, Theorem 3.4.2], one finds that the Lie algebra $\mathrm{Lie}_{\mathfrak{t}/\!\!/\mathrm{W}}(\check{\mathrm{J}})$ is isomorphic to the cotangent bundle $\mathrm{T}^*(\mathfrak{t}/\!\!/\mathrm{W})$, so that $\mathrm{Lie}_{\mathfrak{t}/\!\!/\mathrm{W}}(\check{\mathrm{J}})^*[1]$ is isomorphic to $\mathrm{T}[1](\mathfrak{t}/\!\!/\mathrm{W})$. Its ring of functions is precisely the Hodge cohomology $\Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}} = \bigoplus(\Omega^i_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}})[-i]$ of $\mathfrak{t}/\!\!/\mathrm{W}$. Summarizing, we have found that there is an isomorphism

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Q}} \mathrm{F} \cong \Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}}.$$

On the other hand, it follows from the constructions in § 3.3 that there is a filtration on $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) = \mathrm{End}_{\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{G}_c;k)}(\underline{k})$ whose associated graded is $\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c;k)}(\underline{k}^{\mathrm{gr}})[u^{\pm 1}]$. By the above discussion, the latter is $\Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}}$. Proposition 3.10.2 shows that $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) \otimes_k \mathrm{F}[u^{\pm 1}]$ is isomorphic to the Hochschild homology $\mathrm{HH}(\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F})[u^{\pm 1}]$. There is therefore a filtration on $\mathrm{HH}(\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F})[u^{\pm 1}]$ whose associated graded is $\Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}}[u^{\pm 1}]$. This filtration is precisely the Hochschild-Kostant-Rosenberg filtration on Hochschild homology (see, e.g., [Ant, Rak, MRT] for modern references).

Example 3.10.9. Suppose $k = \mathrm{KU}$, and assume G is simply-laced and has torsion-free fundamental group. Let $\check{\mathrm{J}}_\mu = \mathrm{T}/\!\!/\mathrm{W} \times_{\mathrm{G}/\check{\mathrm{G}}} \mathrm{T}/\!\!/\mathrm{W}$. Then (3.10.1) states that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c;\mathrm{KU}) \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{J}}_\mu),$$

and the "constant sheaf" $\underline{\mathrm{KU}}^{\mathrm{gr}}$ in $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{G}_c;\mathrm{KU})$ is sent to the pushforward of the structure sheaf under the identity section $\delta : \mathrm{T}/\!\!/\mathrm{W} \to \check{\mathrm{J}}_\mu$. Taking endomorphisms, we find that

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c;\mathrm{KU})}(\underline{\mathrm{KU}}^{\mathrm{gr}}) \otimes_{\mathbf{Z}} \mathrm{F} \cong \mathrm{End}_{\mathrm{QCoh}(\check{\mathrm{J}}_\mu)}(\delta_* \mathcal{O}_{\mathrm{T}/\!\!/\mathrm{W}}).$$

By Lemma 3.10.7, the right-hand side admits a (split) filtration whose associated graded is isomorphic to the algebra of functions on $\mathrm{Lie}_{\mathrm{T}/\!\!/\mathrm{W}}(\check{\mathrm{J}}_\mu)^*[1]$. There is a multiplicative analogue of [Ric, Theorem 3.4.2] which states the Lie algebra $\mathrm{Lie}_{\mathrm{T}/\!\!/\mathrm{W}}(\check{\mathrm{J}}_\mu)$ is isomorphic to the cotangent bundle $\mathrm{T}^*(\mathrm{T}/\!\!/\mathrm{W})$. In particular, $\mathrm{Lie}_{\mathrm{T}/\!\!/\mathrm{W}}(\check{\mathrm{J}}_\mu)^*[1]$ is isomorphic to $\mathrm{T}[1](\mathrm{T}/\!\!/\mathrm{W})$. Its ring of functions is precisely the Hodge cohomology $\Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}} = \bigoplus(\Omega^i_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}})[-i]$ of $\mathrm{T}/\!\!/\mathrm{W}$. Summarizing, we have found that there is an isomorphism

$$\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c;k)}(\underline{k}^{\mathrm{gr}}) \otimes_{\mathbf{Z}} \mathrm{F} \cong \Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}}.$$

On the other hand, it follows from the constructions in § 3.3 that there is a filtration on $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) = \mathrm{End}_{\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{G}_c;\mathrm{KU})}(\mathrm{KU})$ with associated graded given by $\mathrm{End}_{\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{G}_c;\mathrm{KU})}(\underline{\mathrm{KU}}^{\mathrm{gr}})[u^{\pm 1}]$. By the above discussion, the latter is $\Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}}$. Proposition 3.10.2 shows that $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) \otimes_{\mathrm{KU}} \mathrm{F}[u^{\pm 1}]$ is isomorphic to the Hochschild homology $\mathrm{HH}(\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F})[u^{\pm 1}]$. There is therefore a filtration on $\mathrm{HH}(\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F})[u^{\pm 1}]$ whose associated graded is $\Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}}[u^{\pm 1}]$. Again, this filtration is precisely the Hochschild-Kostant-Rosenberg filtration on Hochschild homology.

Remark 3.10.10. While we are on the topic of the equivariant K-theory of G_c, let us note the relationship between (3.10.1) and the work of Freed-Hopkins-Teleman [FHT2, FHT4, FHT3, FHT1, FT2].[23] We will be brief, since we will not use these results below. Associated to a class $\tau \in \mathrm{H}^4(\mathrm{BG}_c; \mathbf{Z})$ is the "twisted equivariant K-homology" $\mathrm{KU}^{\mathrm{G}}_\tau(\mathrm{G}_c)$. When τ is sufficiently nondegenerete, Freed-Hopkins-Teleman computed that $\pi_*\mathrm{KU}^{\mathrm{G}}_\tau(\mathrm{G}_c)$ is isomorphic to $\mathrm{RU}(\mathrm{G})/\mathrm{I}^\tau$ for a particular ideal I^τ (called the "Verlinde ideal"). The categorification of this isomorphism from [FT2] shows that, associated to τ, there is a map $f : \mathrm{T}/\!\!/\mathrm{W} \cong \operatorname{Spec} \pi_0\mathrm{KU}_{\mathrm{G}} \to \mathbf{A}^1$ such that (under certain hypotheses on τ), there is an isomorphism between $\pi_*\mathrm{KU}^{\mathrm{G}}_\tau(\mathrm{G}_c) \otimes_{\mathbf{Z}} \mathrm{F}$ and the Jacobian ring of f.

This is related to (3.10.1) in the following manner. Below, we will implicitly base-change all rings from $\mathbf{Z}$ to F, to avoid cumbersome notation. Recall from [FHT2, Equation 3] that there is a spectral sequence

$$\mathrm{E}_1^{*,*} \cong \pi_*\mathrm{KU}_{\mathrm{G}} \otimes_{\pi_*\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee} \pi_*\mathrm{KU}_{\mathrm{G}} \Rightarrow \pi_*\mathrm{KU}^{\mathrm{G}}_\tau(\mathrm{G}_c). \tag{3.10.2}$$

The tensor product is derived; moreover, the class τ defines a particular $\pi_*\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee$-module structure on $\pi_*\mathrm{KU}_{\mathrm{G}}$, and one of the tensor factors is given this module structure. (The other tensor factor is given the module structure coming from the augmentation.) Using Theorem 3.8.3, let us view $\operatorname{Spec} \pi_*\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ as the (2-periodification of) $\check{\mathrm{J}}_\mu$. Then τ defines a particular closed subscheme $\mathrm{T}/\!\!/\mathrm{W} \cong \mathrm{L}_\tau \hookrightarrow \check{\mathrm{J}}_\mu$ (which is in fact a Lagrangian), and the E_1-page of this spectral sequence can be identified with (the 2-periodification of) the ring of functions on $\mathrm{L}_\tau \times_{\check{\mathrm{J}}_\mu} \mathrm{T}/\!\!/\mathrm{W}$. If L_τ lies in the formal neighborhood of $\check{\mathrm{J}}_\mu$, then we may replace $\check{\mathrm{J}}_\mu$ in this fiber product by its formal completion $\hat{\check{\mathrm{J}}}_\mu$ at the zero section. Since we have implicitly base-changed everything to the characteristic zero field F, the argument of Lemma 3.10.7 further lets us replace $\hat{\check{\mathrm{J}}}_\mu$ by its Lie algebra, which (as mentioned in Example 3.10.9) is given by $\mathrm{T}^*(\mathrm{T}/\!\!/\mathrm{W})$. Under this replacement, the map $\mathrm{T}/\!\!/\mathrm{W} \cong \mathrm{L}_\tau \to \hat{\check{\mathrm{J}}}_\mu$ becomes identified with the map $df : \mathrm{T}/\!\!/\mathrm{W} \to \mathrm{T}^*(\mathrm{T}/\!\!/\mathrm{W})$, where $f : \mathrm{T}/\!\!/\mathrm{W} \to \mathbf{A}^1$ is the map from [FT2]. The derived fiber product $\mathrm{L}_\tau \times_{\mathrm{T}^*(\mathrm{T}/\!\!/\mathrm{W})} \mathrm{T}/\!\!/\mathrm{W}$ is precisely the Jacobian ring of f; that is to say, the E_1-page of the spectral sequence (3.10.2) identifies with the Jacobian ring of f. If the spectral sequence (3.10.2) degenerates at the E_1-page, then we conclude that $\pi_*\mathrm{KU}^{\mathrm{G}}_\tau(\mathrm{G}_c)$ is isomorphic to the Jacobian ring of f, as desired.

In fact, the Hochschild-Kostant-Rosenberg filtrations on $\mathrm{HH}(\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F})$ and $\mathrm{HH}(\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F})$ from Example 3.10.8 and Example 3.10.9 both split, since F is of characteristic zero and $\mathfrak{t}/\!\!/\mathrm{W}$ and $\mathrm{T}/\!\!/\mathrm{W}$ are smooth schemes. We therefore conclude that there are isomorphisms

$$\begin{aligned} &\mathrm{H}^*_{\mathrm{G}_c}(\mathrm{G}_c; \mathrm{F}[u^{\pm 1}]) \cong \Omega^*_{\mathfrak{t}/\!\!/\mathrm{W}/\mathrm{F}}[u^{\pm 1}], \\ &\mathrm{KU}^*_{\mathrm{G}_c}(\mathrm{G}_c) \otimes_{\mathbf{Z}} \mathrm{F} \cong \Omega^*_{\mathrm{T}/\!\!/\mathrm{W}/\mathrm{F}}[u^{\pm 1}], \end{aligned}$$

the latter for G_c being simply-laced. (This assumption can be removed with further work.) The final isomorphism above recovers (the base-change to F of) the isomorphism of Brylinski-Zhang. Arguing as above, one also finds that if k is an elliptic cohomology theory, G_c is simply-laced and has torsion-free fundamental group, and $i : \operatorname{Spec}(\mathrm{F}[u^{\pm 1}]) \to \mathcal{M}_{\mathrm{G}}$ is a map with F being an algebraically closed field of characteristic zero, there is an isomorphism of quasicoherent sheaves over $\operatorname{Spec}(\mathrm{F}[u^{\pm 1}])$:

$$\pi_* i^* \mathcal{F}_{\mathrm{G}}(\mathrm{G}_c) \cong \Omega^*_{\mathcal{M}_{\mathrm{G},0}/\mathrm{F}}[u^{\pm 1}]. \tag{3.10.3}$$

As stated, (3.10.3) holds if k is $\mathbf{Q}[u^{\pm 1}]$, complex K-theory, or elliptic cohomology. One could ask whether (3.10.3) holds over the sphere spectrum.

[23] Nearly the same perspective can also be found in some of Teleman's talks; e.g., [Tel2].

Remark 3.10.11. At least in the case of classical groups, additive isomorphisms of the form discussed in this section follow from stronger statements about splittings of the suspension spectrum $(\mathrm{G}_c)_+$. Such statements were proved in [Mil]; let us illustrate this when $\mathrm{G} = \mathrm{GL}_n$. For $j \leq n$, let $\mathrm{Gr}_j(\mathbf{C}^n) = \mathrm{U}(n)/(\mathrm{U}(j) \times \mathrm{U}(n-j))$, and let $\mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}$ denote the Thom spectrum of the vector bundle over $\mathrm{Gr}_j(\mathbf{C}^n)$ given by the pulling back the adjoint representation of $\mathrm{U}(j)$ along the map $\mathrm{Gr}_j(\mathbf{C}^n) \to \mathrm{BU}(j)$. Then there is a $\mathrm{U}(n)$-equivariant splitting

$$(\mathrm{G}_c)_+ \simeq \bigoplus_{j=0}^{n} \mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}.$$

This induces a splitting of $\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c)$, and hence of $i^*\mathcal{F}_{\mathrm{G}}(\mathrm{G}_c)$ for any map $i : \mathrm{Spec}(\mathrm{F}[u^{\pm 1}]) \to \mathcal{M}_{\mathrm{G}}$ with F being an algebraically closed field of characteristic zero. One can show that there is an isomorphism

$$\pi_* i^*\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_j(\mathbf{C}^n)^{\mathfrak{u}(j)}) \cong \Omega^j_{\mathcal{M}_{\mathrm{GL}_n,0}/\mathrm{F}}[u^{\pm 1}],$$

so taking the direct sum over $j = 0, \cdots, n$ gives an additive equivalence of the form (3.10.3).

Although such splittings of $(\mathrm{G}_c)_+$ were proved in [Mil] only for classical groups, they can also be extended with some work to the exceptional groups, too. However, one encounters an important difficulty in trying to extend (3.10.3) to a statement about the stable homotopy type of G_c itself. Namely, suppose that (3.10.3) holds for F of arbitrary characteristic; in fact, let us even suppose that the *non*equivariant version of (3.10.3) holds, i.e., that there is an isomorphism

$$\pi_* i^*\mathcal{F}(\mathrm{G}_c) \cong \Omega^*_{\mathcal{M}_{\mathrm{G},0}/\mathrm{F}}[u^{\pm 1}] \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{G},0}}} \mathrm{F} \tag{3.10.4}$$

for any map $i : \mathrm{Spec}(\mathrm{F}[u^{\pm 1}]) \to \mathcal{M}_{\mathrm{G}}$. Motivated by the example of Remark 3.10.11, it is natural to wonder whether this putative splitting could arise from a(n additive) splitting of $(\mathrm{G}_c)_+$ itself, where the summands of $(\mathrm{G}_c)_+$ realize the individual summands $\Omega^j_{\mathcal{M}_{\mathrm{G},0}/\mathrm{F}}[u^{\pm 1}] \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{G},0}}} \mathrm{F}$. Unfortunately, this turns out to be impossible, at least if interpreted naively.

Example 3.10.12. Suppose $\mathrm{G} = \mathrm{G}_2$. Since G_2 is a framed manifold, its top cell stably splits, and so there is a splitting

$$(\mathrm{G}_2)_+ \simeq \mathrm{S}^0 \oplus \mathrm{X} \oplus \mathrm{S}^{\mathfrak{g}_2},$$

where $\mathrm{S}^{\mathfrak{g}_2}$ is the one-point compactification of the adjoint representation of G_2, and X is a finite CW-complex with partial cell diagram shown in Figure 3.1.

Figure 3.1: A partial cell diagram for the stable summand X of $(\mathrm{G}_2)_+$. The dots indicate the cells; starting from the left, the cells lie in dimensions $3, 5, 6, 8, 9$, and 11. The labels represent the action of the Steenrod operations in mod 2 cohomology.

Let us now take F to be a field of characteristic 2. Then $\mathrm{H}^*_{\mathrm{G}_2}(*; \mathrm{F}) \cong \mathrm{F}[w_4, w_6, w_7]$, where the subscript indicates the cohomological degree. This implies that

$$\Omega^*_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})/\mathrm{F}} \otimes_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})} \mathrm{F} \cong \Lambda(dw_4, dw_6, dw_7),$$

where Λ denotes the exterior algebra on the classes dw_4, dw_6, and dw_7. According to (3.10.4), these classes would contribute to $\mathrm{H}^*(\mathrm{G}_2; \mathrm{F})$ in cohomological degrees $3, 5$, and 6 respectively.

First of all, let us observe that the above ring is *not* isomorphic to $\mathrm{H}^*(\mathrm{G}_2;\mathrm{F})$: instead, there is an isomorphism

$$\mathrm{H}^*(\mathrm{G}_2;\mathrm{F}) \cong \mathrm{F}[dw_4, dw_6]/((dw_4)^4, (dw_6)^2).$$

Nevertheless, there is an additive isomorphism between $\Omega^*_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})/\mathrm{F}} \otimes_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})} \mathrm{F}$ and $\mathrm{H}^*(\mathrm{G}_2;\mathrm{F})$, so we can still ask for a stable splitting of $(\mathrm{G}_2)_+$ which realizes the individual summands $\Omega^j_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})/\mathrm{F}} \otimes_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})} \mathrm{F}$. This already fails for $j = 1$. Indeed, the 6-skeleton of X provides a CW-complex Y with a map $\mathrm{Y} \to \mathrm{G}_2$ which realizes the inclusion of the subspace $\Omega^1_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})/\mathrm{F}} \otimes_{\mathrm{H}^*_{\mathrm{G}_2}(*;\mathrm{F})} \mathrm{F} \cong \mathrm{F}\{dw_4, dw_6, dw_7\}$ into $\mathrm{H}^*(\mathrm{G}_2;\mathrm{F})$. However, the map $\mathrm{Y} \to \mathrm{G}_2$ cannot stably split (in particular, the 6-skeleton of X does not stably split off X). This was proved "by hand" in [CP, Theorem 1.10] using Dyer-Lashof operations.

The preceding example is not special to the non-simply-laced case; one can show that a similar result holds for G of type E, too.

Chapter 4

Derived geometric Satake with generalized coefficients

4.1 Full faithfulness

In our discussion below, we will need to assume the existence of a good theory of "genuine" sheaves defined on topological stacks with *compact abelian* stabilizers. This theory has not yet been developed in the literature, but I have been informed that it is work-in-progress by Konovalov-Perunov-Prikhodko and Cnossen-Maegawa-Volpe. In lieu of recalling the construction of such a sheaf theory here, we will instead operate under the assumption that such a theory exists and satisfies the standard properties of a sheaf theory (namely, having a six functor formalism; in fact, the full package of compatibilities afforded by a six functor formalism is not necessary for our discussion below), and such that $\mathrm{Shv}_{\mathrm{T}}(*;\mathrm{Sp})$ is equivalent to the ∞-category Sp_{T} of genuine T-equivariant spectra.

Remark 4.1.1. If we merely asked for *Borel*-equivariant sheaves, then it is easy to define the desired ∞-category $\mathrm{Shv}_{\mathrm{G}}(\mathrm{X};k)^{\mathrm{Bor}}$ as the totalization of the cosimplicial ∞-category $\mathrm{Shv}^0(\mathrm{X}\times \mathrm{G}^\bullet;k)$, where the notation Shv^0 denotes the full subcategory spanned by those sheaves which are locally constant on the fibers of the projection $\mathrm{X}\times\mathrm{G}^\bullet\to\mathrm{X}$.

If G is a general compact Lie group with maximal torus T and X/G is a quotient stack, we will *define*

$$\mathrm{Shv}_{\mathrm{G}}(\mathrm{X};k) := \mathrm{Tot}\,\mathrm{Shv}^0_{\mathrm{T}}(\mathrm{X}\times(\mathrm{G}/\mathrm{T})^\bullet;k) \tag{4.1.1}$$

where the notation $\mathrm{Shv}^0_{\mathrm{T}}$ denotes the full subcategory spanned by those T-equivariant sheaves which are locally constant on the fibers of the projection $\mathrm{X}\times(\mathrm{G}/\mathrm{T})^\bullet\to\mathrm{X}$. In this way, the behaviour of the category $\mathrm{Shv}_{\mathrm{G}}(\mathrm{X};k)$ is essentially determined by the case when G is a compact torus. Even if we only mean Borel-equivariant sheaves, we will occasionally still denote the sheaf category by $\mathrm{Shv}_{\mathrm{G}}(\mathrm{X};k)$; that it means Borel-equivariant sheaves in a particular context will be clear from the choice of genuine equivariant structure on k. (It is not hard to check that in the Borel-equivariant case, the categories appearing on both sides of (4.1.1) are equivalent.)

When X is a point, the category $\mathrm{Shv}_{\mathrm{G}}(*;k)$ identifies with $\mathrm{QCoh}(\mathcal{M}_{\mathrm{G}})$, where $\mathcal{M}_{\mathrm{G}}$ is the spectral stack whose global sections compute G-equivariant k-cohomology. If $q:\mathrm{X}\to *$ denotes the projection map, we will write $\mathcal{F}(\mathrm{X})$ (or $\mathcal{F}(\mathrm{X};k)$ to emphasize k) to denote the quasicoherent sheaf on $\mathcal{M}_{\mathrm{G}}$ given by the image of the constant sheaf on X along the map $q_*:\mathrm{Shv}_{\mathrm{G}}(\mathrm{X};k)\to\mathrm{Shv}_{\mathrm{G}}(*;k)$; this can be viewed as the equivariant cochains on X. If X is a finite G-space, the equivariant chains on X is defined to be the $\mathcal{O}_{\mathcal{M}_{\mathrm{G}}}$-linear dual of $\mathcal{F}(\mathrm{X})$. We will also write $\mathcal{H}^*_{\mathrm{G}}(\mathrm{X};k)$ to denote the quasicoherent sheaf on the stack $\mathcal{M}_{\mathrm{G},0}$ over $\mathrm{Spev}(k)$ given by the associated graded of the (sheafy) Postnikov filtration on $\mathcal{F}(\mathrm{X})$. Similarly for $\mathcal{H}_*^{\mathrm{G}}(\mathrm{X};k)$.

Let k be an $\mathbf{E}_\infty$-ring. Following [Pst], a k-module M will be called *perfect even* if it lies in the smallest subcategory of Mod_k which contains all even shifts of k, and which is closed under extensions and retracts. A similar definition works for the category of quasicoherent sheaves on a spectral stack.

Lemma 4.1.2. *Let k be an even $\mathbf{E}_\infty$-ring, and let* M *and* N *denote two perfect even k-modules. Then any map* $\mathrm{M} \to \Sigma\mathrm{N}$ *is null. In particular, any cofiber sequence* $\mathrm{M}_1 \to \mathrm{M}_2 \to \mathrm{M}_3$ *of perfect even k-modules splits.*

Proof. It suffices to show that $\mathrm{Map}_k(\mathrm{M}, \mathrm{N})$ is even. Since M and N are built from even shifts of k by extensions and retracts, this follows from the assumption that $\mathrm{Map}_k(k, k) = k$ is even. □

Setup 4.1.3. Let T_c be a compact abelian Lie group, and let X be a proper T_c-space with a T_c-equivariant stratification indexed by a (finite) poset P. We will fix a T_c-equivariant even $\mathbf{E}_\infty$-ring k, so that there is a spectral k-stack $\mathcal{M}_\mathrm{T}$ with underlying stack $\mathcal{M}_{\mathrm{T},0}$ over $\pi_*(k)$ such that $k_\mathrm{T} = \Gamma(\mathcal{M}_\mathrm{T}; \mathcal{O})$.

Let X_λ denote the stratum corresponding to $\lambda \in \mathrm{P}$, and let $\mathrm{X}_{\leq\lambda}$ denote its closure in X with complement $\mathrm{X}_{<\lambda}$. Suppose further that each X_λ is a complex affine space of complex dimension n_λ on which T_c acts linearly. Let $j_\lambda : \mathrm{X}_\lambda \hookrightarrow \mathrm{X}$ denote the corresponding open immersion, and let $i_\lambda : \mathrm{X}_{<\lambda} \to \mathrm{X}_{\leq\lambda}$ denote the corresponding closed immersion.

Definition 4.1.4. Let $\mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda; k)_\mathrm{ev}$ denote the smallest subcategory of $\mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda; k)$ which contains all even shifts of the constant sheaf k, and which is closed under extensions and retracts; an object of $\mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda; k)_\mathrm{ev}$ will be called *perfect even.* An object $\mathcal{F} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ will be called $*$-*even* if for each $\lambda \in \mathrm{P}$, the pullback $j_\lambda^*\mathcal{F} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda; k)$ is perfect even. Similarly for being !-*even*; say that $\mathcal{F}$ is *even* if it is both $*$-even and !-even. (We may sometimes refer to such an $\mathcal{F}$ as being "perfect even".)

Our goal is to prove:

Theorem 4.1.5. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ be even. Then the canonical map*

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathcal{H}^*_\mathrm{T}(\mathrm{X};k)}(\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}), \mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{G}))$$

is a graded isomorphism, where we emphasize that the Hom*'s on the right-hand side are taken in the* 1*-category of graded $\mathcal{H}^*_\mathrm{T}(\mathrm{X}; k)$-modules in* $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^\heartsuit$.

The argument for Theorem 4.1.5 is essentially due to Ginzburg [Gin1], and our presentation below closely follows [CMNO, Section 4.7] and [SW, Section 8].

Lemma 4.1.6. *The following conditions on an exact functor* $\mathrm{F} : \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k) \to \mathrm{QCoh}(\mathcal{M}_\mathrm{T})$ *are equivalent:*

a. F *sends $*$-even sheaves to perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-modules;*

b. $\mathrm{F}(j_{\lambda,!}k)$ *is a perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-module for each $\lambda \in \mathrm{P}$.*

Proof. Assume (a). To check (b), it suffices to check that $j_{\lambda,!}k$ is $*$-even. For $\mu \in \mathrm{P}$, the pullback $j_\mu^* j_{\lambda,!}k$ is zero unless $\mu = \lambda$, in which case it is just k itself. For the converse, assume (b). Let $\mathcal{F} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ be an $*$-even sheaf. There is a cofiber sequence

$$j_{\lambda,!}j_\lambda^!\mathcal{F} \to \mathcal{F} \to i_{\lambda,*}i_\lambda^*\mathcal{F}.$$

Since j_λ is an open immersion, $j_\lambda^! = j_\lambda^*$. Applying F to the above cofiber sequence, we obtain another cofiber sequence

$$\mathrm{F}(j_{\lambda,!}j_\lambda^*\mathcal{F}) \to \mathrm{F}(\mathcal{F}) \to \mathrm{F}(i_{\lambda,*}i_\lambda^*\mathcal{F}).$$

By induction on the strata contained in the support of $\mathcal{F}$, we may assume that $\mathrm{F}(i_{\lambda,*}i_\lambda^*\mathcal{F})$ is perfect even. Since $j_\lambda^*\mathcal{F}$ is perfect even by $*$-evenness, the term $\mathrm{F}(j_{\lambda,!}j_\lambda^*\mathcal{F})$ is also perfect even by assumption (b). It follows that $\mathrm{F}(\mathcal{F})$ is an extension between perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-modules, and hence is itself perfct even as desired. $\square$

Corollary 4.1.7. *The functor* $\Gamma : \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k) \to \mathrm{QCoh}(\mathcal{M}_\mathrm{T})$ *sends* $*$*-even sheaves to perfect even* $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$*-modules.*

Proof. By Lemma 4.1.6, it suffices to check that $\Gamma_\mathrm{T}(\mathrm{X}; j_{\lambda,!}k)$ is an perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-module. This follows by our assumption that X_λ is a complex affine space on which T_c acts linearly, and equivariant Poincaré duality. $\square$

Lemma 4.1.8. *If* $\mathcal{G} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ *is* $!$*-even, then the functor*

$$\mathrm{Map}_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(-, \mathcal{G}) : \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_\mathrm{T})$$

sends $*$*-even sheaves to perfect even* $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$*-modules.*

Proof. By Lemma 4.1.6, it suffices to check that $\mathrm{Map}_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(j_{\lambda,!}k, \mathcal{G})$ is a perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-module. By adjunction, we may identify

$$\mathrm{Map}_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(j_{\lambda,!}k, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(k, j_\lambda^!\mathcal{G}) = \Gamma_\mathrm{T}(\mathrm{X}_\lambda; j_\lambda^!\mathcal{G}).$$

Since $\mathcal{G}$ is $!$-even and X_λ is a complex affine space on which T_c acts linearly, this is a perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-module as desired. $\square$

Lemma 4.1.9. *Let* $\mathcal{F} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ *be* $*$*-even, and let* $\mathrm{Y} \subseteq \mathrm{X}$ *be a closed subset containing the support of* $\mathcal{F}$*. Suppose* $i : \mathrm{Z} \hookrightarrow \mathrm{Y}$ *is a* T*-equivariant closed subset, and let* $j : \mathrm{U} \to \mathrm{Y}$ *denote its open complement. If* $\mathrm{F} : \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k) \to \mathrm{QCoh}(\mathcal{M}_\mathrm{T})$ *sends* $*$*-even sheaves to even* $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$*-modules, then there is a split cofiber sequence*

$$\mathrm{F}(j_!j^!\mathcal{F}) \to \mathrm{F}(\mathcal{F}) \to \mathrm{F}(i_*i^*\mathcal{F})$$

of $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$*-modules.*

Proof. There is a cofiber sequence

$$j_!j^!\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F},$$

which gives a cofiber sequence upon applying F. Note that both $i_*i^*\mathcal{F}$ and $j_!j^!\mathcal{F}$ are $*$-even, so Lemma 4.1.8 tells us that the flanking terms are perfect even $\mathcal{O}_{\mathcal{M}_\mathrm{T}}$-modules. Lemma 4.1.2 implies that the boundary map $\mathrm{F}(i_*i^*\mathcal{F}) \to \Sigma\mathrm{F}(j_!j^!\mathcal{F})$ is null, so we obtain the desired split cofiber sequence. $\square$

Corollary 4.1.10. *Let* $\mathcal{F} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ *be* $*$*-even, and let* $\mathcal{G} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ *be* $!$*-even. In the setup of Lemma 4.1.9 there are exact sequences of* $\mathcal{H}_\mathrm{T}^*(\mathrm{X}; k)$*-modules, which split as exact sequences of* $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}\{*\}$*-modules*

$$0 \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; j_!j^!\mathcal{F}) \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; \mathcal{F}) \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; i_*i^*\mathcal{F}) \to 0,$$
$$0 \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; i_!i^!\mathcal{G}) \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; \mathcal{G}) \to \mathcal{H}_\mathrm{T}^*(\mathrm{X}; j_*j^*\mathcal{G}) \to 0.$$

Proof. By Lemma 4.1.9 and Corollary 4.1.7, there is a split cofiber sequence

$$\Gamma_{\mathrm{T}}(\mathrm{X}; j_! j^! \mathcal{F}) \to \Gamma_{\mathrm{T}}(\mathrm{X}; \mathcal{F}) \to \Gamma_{\mathrm{T}}(\mathrm{X}; i_* i^* \mathcal{F})$$

of perfect $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-modules. It therefore induces a split exact sequence of $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}\{*\}$-modules upon taking homotopy groups. The second cofiber sequence follows by a similar argument. ☐

Corollary 4.1.11. *Let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$ be $*$-even, and let $\mathcal{G} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$ be $!$-even. In the setup of Lemma 4.1.9, there is an exact sequence*

$$0 \to \mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{Z};k)}(i^*\mathcal{F}, i^!\mathcal{G}) \to \mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{U};k)}(j^!\mathcal{F}, j^*\mathcal{G}) \to 0$$

of $\mathcal{O}_{\mathcal{M}_{\mathrm{T},0}}\{*\}$-modules.

Proof. By Lemma 4.1.8, the functor $\mathrm{Map}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(-, \mathcal{G})$ sends $*$-even sheaves to even $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-modules, so there is a split cofiber sequence

$$\mathrm{Map}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(i_* i^* \mathcal{F}, \mathcal{G}) \to \mathrm{Map}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Map}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(j_! j^! \mathcal{F}, \mathcal{G})$$

of $\mathcal{O}_{\mathcal{M}_{\mathrm{T}}}$-modules. This implies the desired exact sequence on the level of homotopy by adjunction and evenness. ☐

Proposition 4.1.12. *Suppose V is a complex affine space on which T_c acts linearly, and let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{V}; k)$ be two sheaves such that $\mathcal{F}$ is perfect even. Then there is an isomorphism*

$$\mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{V};k)}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{V};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{V}; \mathcal{F}), \mathcal{H}^*_{\mathrm{T}}(\mathrm{V}; \mathcal{G})),$$

where we emphasize that the Hom*'s on the right-hand side are taken in the 1-category of graded $\mathcal{H}^*_{\mathrm{T}}(\mathrm{V}; k)$-modules in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^{\heartsuit}$.*

Proof. The property that the map

$$\mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{V};k)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{V};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{V}; \mathcal{F}), \mathcal{H}^*_{\mathrm{T}}(\mathrm{V}; \mathcal{G}))$$

is an isomorphism is stable under retracts, extensions and even shifts in the variable $\mathcal{F}$ (and also separately in $\mathcal{G}$, but we will not use this). In particular, to prove that this map is an isomorphism when $\mathcal{F}$ is perfect even, it suffices to prove the claim when $\mathcal{F}$ is the constant sheaf. In this case, the map is obviously an isomorphism. ☐

Proposition 4.1.13. *Let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$ be $*$-even, and let $\mathcal{G} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$ be $!$-even. Suppose that the canonical maps*

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; \mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; j_{\lambda,*} j_{\lambda}^* \mathcal{F})$$
$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; j_{\lambda,!} j_{\lambda}^! \mathcal{G}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; \mathcal{G})$$

are surjective and injective, respectively, for each $\lambda \in \mathrm{P}$. Then the sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}_{<\lambda};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; i_{\lambda,*} i_{\lambda}^* \mathcal{F}), \mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; i_{\lambda,!} i_{\lambda}^! \mathcal{G})) \\ &\to \mathrm{Hom}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; \mathcal{F}), \mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; \mathcal{G})) \\ &\to \mathrm{Hom}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}_{\lambda};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; j_{\lambda,!} j_{\lambda}^! \mathcal{F}), \mathcal{H}^*_{\mathrm{T}}(\mathrm{X}; j_{\lambda,!} j_{\lambda}^! \mathcal{G})) \end{aligned} \tag{4.1.2}$$

is exact on the left and in the middle, where we emphasize that the Hom*'s are taken in the 1-category of graded modules.*

Proof. By assumption on X, the space X_λ is a complex affine space of dimension n_λ on which T_c acts linearly, so that $j_{\lambda,!}j_\lambda^! k) \simeq j_{\lambda,*}j_\lambda^* k[-2n_\lambda]$. One therefore obtains a composite map of sheaves

$$k \to j_{\lambda,*}j_\lambda^* k \xrightarrow{\sim} j_{\lambda,!}j_\lambda^! k[2n_\lambda] \to k[2n_\lambda],$$

and hence a class $[\mathrm{X}_\lambda] \in \pi_{-2n_\lambda}\Gamma_{\mathrm{T}}(\mathrm{X};k)$. It follows that the $[\mathrm{X}_\lambda]$-multiplication map on $\Gamma_{\mathrm{T}}(\mathrm{X};\mathcal{F})$ factors as

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F})(-2n_\lambda) \twoheadrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,*}j_\lambda^*\mathcal{F})(-2n_\lambda) \xrightarrow{\sim} \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{F}) \hookrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}).$$

The first map is surjective by assumption on $\mathcal{F}$, and the final map is an injection by Corollary 4.1.10. In particular, the image of $[\mathrm{X}_\lambda]$-multiplication on $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F})$ is precisely $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{F})$. Moreover, Corollary 4.1.10 implies that the cokernel of $[\mathrm{X}_\lambda]$-multiplication on $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F})$ is $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,*}i_\lambda^*\mathcal{F})$.

Similarly, the $[\mathrm{X}_\lambda]$-multiplication map on $\Gamma_{\mathrm{T}}(\mathrm{X};\mathcal{G})$ factors as

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})(-2n_\lambda) \twoheadrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,*}j_\lambda^*\mathcal{G})(-2n_\lambda) \xrightarrow{\sim} \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{G}) \hookrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G}).$$

The first map is surjective by Corollary 4.1.10, and the final map is an injection by assumption on $\mathcal{G}$. In particular, the image of $[\mathrm{X}_\lambda]$-multiplication on $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})$ is precisely $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{G})$. Moreover, Corollary 4.1.10 implies that the kernel of $[\mathrm{X}_\lambda]$-multiplication on $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})$ is $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,!}i_\lambda^!\mathcal{G})$.

Let us now construct the desired exact sequence (4.1.2). The map

$$\mathrm{Hom}^\bullet_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}_{<\lambda};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,*}i_\lambda^*\mathcal{F}),\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,!}i_\lambda^!\mathcal{G})) \to \mathrm{Hom}^\bullet_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}),\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})) \quad (4.1.3)$$

is given by sending a map $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,*}i_\lambda^*\mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,!}i_\lambda^!\mathcal{G})$ to the composite

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}) \twoheadrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,*}i_\lambda^*\mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,!}i_\lambda^!\mathcal{G}) \hookrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G}),$$

where the surjection and injection indicated above come from Corollary 4.1.10. The map

$$\mathrm{Hom}^\bullet_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}),\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})) \to \mathrm{Hom}^\bullet_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X}_\lambda;k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{F}),\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{G}))$$

is given by sending a $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)$-linear map $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})$ to the map induced on the image of $[\mathrm{X}_\lambda]$-multiplication.

Let us now show that (4.1.2) is exact. First, the map (4.1.3) is injective by construction, so (4.1.2) is exact on the left. For exactness in the middle, suppose $f : \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})$ is a $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)$-linear map which induces the zero map upon $[\mathrm{X}_\lambda]$-multiplication. Then f factors as

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}) \twoheadrightarrow \mathrm{coker}([\mathrm{X}_\lambda]) \to \mathrm{im}([\mathrm{X}_\lambda]) \hookrightarrow \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G}).$$

Our discussion above shows that $\mathrm{coker}([\mathrm{X}_\lambda]) \simeq \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,*}i_\lambda^*\mathcal{F})$, and $\mathrm{im}([\mathrm{X}_\lambda]) \simeq \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};i_{\lambda,!}i_\lambda^!\mathcal{G})$, and so f lies in the image of (4.1.3) as desired. ∎

Proposition 4.1.14. *Let $\mathcal{F} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)$ be $*$-even, and let $\mathcal{G} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)$ be $!$-even. Suppose that the canonical maps*

$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,*}j_\lambda^*\mathcal{F})$$
$$\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};j_{\lambda,!}j_\lambda^!\mathcal{G}) \to \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G})$$

are surjective and injective, respectively, for each $\lambda \in \mathrm{P}$. Then the natural map

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{F}),\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{G}))$$

is an isomorphism.

Proof. Let λ be such that X_λ is open in the union of the supports of $\mathcal{F}$ and $\mathcal{G}$. There is a map of sequences

$$\begin{tikzcd}
\mathrm{Ext}^\bullet_{\mathrm{Shv}_\mathrm{T}(\mathrm{X}_{<\lambda};k)}(i_\lambda^*\mathcal{F}, i_\lambda^!\mathcal{G}) \arrow[r, "f_{<\lambda}"] \arrow[d] & \mathrm{Hom}^\bullet_{\mathcal{H}^*_\mathrm{T}(\mathrm{X}_{<\lambda};k)}(\mathcal{H}^*_\mathrm{T}(\mathrm{X}; i_{\lambda,*}i_\lambda^*\mathcal{F}), \mathcal{H}^*_\mathrm{T}(\mathrm{X}; i_{\lambda,!}i_\lambda^!\mathcal{G})) \arrow[d] \\
\mathrm{Ext}^\bullet_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(\mathcal{F}, \mathcal{G}) \arrow[r, "f"] \arrow[d] & \mathrm{Hom}^\bullet_{\mathcal{H}^*_\mathrm{T}(\mathrm{X};k)}(\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}), \mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{G})) \arrow[d] \\
\mathrm{Ext}^\bullet_{\mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda;k)}(j_\lambda^!\mathcal{F}, j_\lambda^*\mathcal{G}) \arrow[r, "f_\lambda"] & \mathrm{Hom}^\bullet_{\mathcal{H}^*_\mathrm{T}(\mathrm{X}_\lambda;k)}(\mathcal{H}^*_\mathrm{T}(\mathrm{X}; j_{\lambda,!}j_\lambda^!\mathcal{F}), \mathcal{H}^*_\mathrm{T}(\mathrm{X}; j_{\lambda,*}j_\lambda^*\mathcal{G})),
\end{tikzcd}$$

where the leftmost composite is a short exact sequence, and the rightmost composite is left exact (i.e., the first map is injective, and the sequence is exact in the middle) by Proposition 4.1.13. We wish to prove that f is an isomorphism. By induction on λ, we may assume that $f_{<\lambda}$ is a graded isomorphism. Since $\mathcal{F}$ is assumed to be even, $j_\lambda^!\mathcal{F}$ is perfect even; so f_λ is an isomorphism by Proposition 4.1.12, and hence f is also an isomorphism as desired. $\square$

Proof of Theorem 4.1.5. Given Proposition 4.1.14, we only need to show that if $\mathcal{F}$ and $\mathcal{G}$ are even, then the canonical maps

$$\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}) \to \mathcal{H}^*_\mathrm{T}(\mathrm{X}; j_{\lambda,*}j_\lambda^*\mathcal{F})$$
$$\mathcal{H}^*_\mathrm{T}(\mathrm{X}; j_{\lambda,!}j_\lambda^!\mathcal{G}) \to \mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{G})$$

are surjective and injective, respectively, for each $\lambda \in \mathrm{P}$. We will argue the surjectivity of the first map, since the injectivity of the second follows dually. The assumption that T_c acts linearly on $\mathrm{X}_\lambda \cong \mathbf{A}^{n_\lambda}$ implies that the fixed locus $\mathrm{X}_\lambda^{\mathrm{T}_c}$ is just a point $\{x_\lambda\}$. Let us write s_λ to denote the inclusion

$$s_\lambda : \{x_\lambda\} = \mathrm{X}_\lambda^{\mathrm{T}_c} \subseteq \mathrm{X}_\lambda \xrightarrow{j_\lambda} \mathrm{X}.$$

Then there is a composite map

$$\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}) \to \Gamma_\mathrm{T}(\mathrm{X}; j_{\lambda,*}j_\lambda^*\mathcal{F}) \to \Gamma_\mathrm{T}(\mathrm{X}; s_{\lambda,*}s_\lambda^*\mathcal{F}).$$

Since s_λ is a closed inclusion, the map $\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}) \to \Gamma_\mathrm{T}(\mathrm{X}; s_{\lambda,*}s_\lambda^*\mathcal{F})$ is surjective by Corollary 4.1.10. To prove that the map $\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}) \to \Gamma_\mathrm{T}(\mathrm{X}; j_{\lambda,*}j_\lambda^*\mathcal{F})$ is surjective, it therefore suffices to show that the map $\Gamma_\mathrm{T}(\mathrm{X}; j_{\lambda,*}j_\lambda^*\mathcal{F}) \to \Gamma_\mathrm{T}(\mathrm{X}; s_{\lambda,*}s_\lambda^*\mathcal{F})$ is an isomorphism. In fact, since $j_\lambda^*\mathcal{F}$ is perfect even by assumption on $\mathcal{F}$, it suffices to show that $\Gamma_\mathrm{T}(\mathrm{X}_\lambda; \mathcal{K}) \to \Gamma_\mathrm{T}(\mathrm{X}_\lambda; s_{\lambda,*}s_\lambda^*\mathcal{K})$ is an isomorphism for every perfect even $\mathcal{K} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}_\lambda; k)_\mathrm{ev}$. Since $\mathcal{K}$ is built from even shifts of the constant sheaf k by extensions and retracts, it suffices to prove the claim when $\mathcal{K} = k$; but in this case, it is just the equivalence $\Gamma_\mathrm{T}(\mathrm{X}_\lambda; k) \to \Gamma_\mathrm{T}(\{x_\lambda\}; k)$ coming from the fact that the inclusion $\{x_\lambda\} \subseteq \mathrm{X}_\lambda$ is a (T_c-equivariant) homotopy equivalence. $\square$

Theorem 4.1.5 can be extended to the case when X is ind-proper; in this case, one finds that if $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\mathrm{T}(\mathrm{X}; k)$ are even, then the canonical map

$$\mathrm{Ext}^\bullet_{\mathrm{Shv}_\mathrm{T}(\mathrm{X};k)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}^\bullet_{\mathcal{H}_*^\mathrm{T}(\mathrm{X};k)\text{-comod}}(\mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{F}), \mathcal{H}^*_\mathrm{T}(\mathrm{X}; \mathcal{G}))$$

is a graded isomorphism, where we emphasize that the Hom's on the right-hand side are taken in the 1-category of graded $\mathcal{H}_*^\mathrm{T}(\mathrm{X}; k)$-comodules in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^\heartsuit$.

4.2 Degenerations, redux

We now discuss degenerations of sheaf categories.

Notation 4.2.1. If $\mathcal{C}$ is a stable ∞-category and $x \in \mathcal{C}$, let $\langle x \rangle_{\mathcal{C}}$ denote the stable subcategory of $\mathcal{C}$ which is compactly generated by x. Similarly, if $\mathcal{A}$ is an abelian category and $y \in \mathcal{A}$, let $\langle y \rangle_{\mathcal{A}}^{\heartsuit}$ denote the abelian subcategory of $\mathcal{A}$ which is compactly generated by y. If $\mathcal{P} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$, we will write $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}}(\mathrm{X}; k)$ to denote $\langle \mathcal{P} \rangle_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}$.

The Schwede-Shipley theorem implies:

Lemma 4.2.2. *Let* X *be a topological space with an action of a torus* T. *Let* $\mathcal{P} \in \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k)$ *be a compact object. Then, the functor* $\mathrm{Hom}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P}, -)$ *implements an equivalence*

$$\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}}(\mathrm{X}; k) \xrightarrow{\sim} \mathrm{RMod}_{\mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P})}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})).$$

Lemma 4.2.3. *Assume* X *as in Setup 4.1.3. Suppose that* k_{T} *is an even* $\mathbf{E}_\infty$*-ring, that* $\mathcal{P}$ *is even in the sense of Definition 4.1.4, and that* $\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; \mathcal{P})$ *is also even. Then* $\mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P})$ *is concentrated in even degrees.*

Proof. Theorem 4.1.5 gives an equivalence

$$\mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P}, \mathcal{P}) \simeq \mathrm{End}^{\bullet}_{\mathcal{H}_{\mathrm{T}}^*(\mathrm{X};k)}(\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; \mathcal{P})),$$

where the endomorphisms on the right-hand side are taken in the 1-category of graded $\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; k)$-modules. Note that $\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; k)$ is concentrated in even degrees, because X is assumed to have even cells and k_{T} is assumed to have even homotopy. By assumption, $\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; \mathcal{P})$ is also concentrated in even degrees, and so $\mathrm{End}^{\bullet}_{\mathcal{H}_{\mathrm{T}}^*(\mathrm{X};k)}(\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; \mathcal{P}))$ vanishes if $\bullet$ is odd. This implies that $\mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P})$ is concentrated in even degrees, as desired. ☐

It follows from Lemma 4.2.2 that the compactly generated stable ∞-category $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}}(\mathrm{X}; k)$ admits a canonical filtered lift:

Definition 4.2.4. Assume X as in Setup 4.1.3. Suppose that $\mathcal{P}$ is even in the sense of Definition 4.1.4, and that for each even T-equivariant$\mathbf{E}_\infty$-k-algebra $k \to \mathrm{A}$, the base-change $\mathcal{H}_{\mathrm{T}}^*(\mathrm{X}; \mathcal{P} \otimes_k \mathrm{A})$ is also even. Let $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{fil}}(\mathrm{X}; k)$ denote the inverse limit over all even T-equivariant $\mathbf{E}_\infty$-k-algebras $k \to \mathrm{A}$ of the ∞-category of filtered left $\tau_{\geq 2\star} \mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};\mathrm{A})}(\mathcal{P} \otimes_k \mathrm{A})$-modules in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T,fil}})$, so that $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{fil}}(\mathrm{X}; k)$ is naturally a filtered ∞-category. Let $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{gr}}(\mathrm{X}; k)$ denote the graded ∞-category defined as

$$\begin{aligned}\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{gr}}(\mathrm{X}; k) &:= \mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{fil}}(\mathrm{X}; k) \otimes_{\mathrm{Sp}^{\mathrm{fil}}} \mathrm{Sp}^{\mathrm{gr}} \\ &\simeq \lim_{k \to \mathrm{A}} \mathrm{RMod}^{\mathrm{gr}}_{\tau_{[2\bullet, 2\bullet+1]} \mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};\mathrm{A})}(\mathcal{P} \otimes_k \mathrm{A})}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})).\end{aligned}$$

There is a canonical 1-parameter degeneration from $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}}(\mathrm{X}; k)$ to $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{gr}}(\mathrm{X}; k)$, which we will denote by $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}}(\mathrm{X}; k) \rightsquigarrow \mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{gr}}(\mathrm{X}; k)$.

Remark 4.2.5. Below, we will use a slightly different grading on $\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P},\mathrm{gr}}(\mathrm{X}; k)$: namely, we will view it as a graded category by pulling back the original $\mathrm{QCoh}(\mathrm{B}\mathbf{G}_m)$-linear structure along the degree 2 map $\mathrm{B}\mathbf{G}_m \to \mathrm{B}\mathbf{G}_m$. (This is to counteract the fact that we used the double-speed truncation $\tau_{\geq 2\star}$.)

Theorem 4.2.6. *Consider the setup of Definition 4.2.4. Denote by* $\mathcal{A} = \mathrm{Mod}^{\mathrm{gr},\heartsuit}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^{\heartsuit})$*. Then there is an equivalence*

$$\mathrm{Shv}^{\mathcal{P},\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k) \simeq \mathrm{D}(\langle \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P})\rangle^{\heartsuit}_{\mathcal{A}}),$$

where the right-hand side denotes the (unbounded) derived ∞-category.

In other words, there is a 1-parameter degeneration $\mathrm{Shv}^{\mathcal{P}}_{\mathrm{T}}(\mathrm{X};k) \rightsquigarrow \mathrm{D}(\langle \mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P})\rangle^{\heartsuit}_{\mathcal{A}})$.

Proof. Because the sheaves $\{\mathcal{F}_i\}_{i\in\mathrm{I}}$ are all even, the sheaf $\mathcal{P}$ is also even, and Theorem 4.1.5 gives an equivalence

$$\mathrm{Ext}^{\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P},\mathcal{P}) \simeq \mathrm{End}^{\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P})),$$

where the endomorphisms on the right-hand side are taken in the 1-category of graded $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)$-modules. Since $\mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P})$ is even by Lemma 4.2.3, $\tau_{[2\bullet,2\bullet+1]}\mathrm{End}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P})$ is just $\mathrm{Ext}^{2\bullet}_{\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{P},\mathcal{P})$; it follows that there is an equivalence

$$\mathrm{Shv}^{\mathcal{P},\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k) \simeq \mathrm{RMod}^{\mathrm{gr}}_{\mathrm{End}^{2\bullet}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P}))}.$$

This is equivalent to the derived ∞-category of the full subcategory of $\mathrm{Mod}^{\mathrm{gr},\heartsuit}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}$ which is compactly generated by $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P})$, as desired. □

Example 4.2.7. Suppose $\mathcal{P}$ was the constant sheaf. Then $\mathrm{Shv}^{\mathcal{P}}_{\mathrm{T}}(\mathrm{X};k)$ identifies with the category $\mathrm{Loc}_{\mathrm{T}}(\mathrm{X};k)$ of T-equivariant *locally constant* sheaves of k-modules on X. Theorem 4.2.6 says that if $\Gamma_{\mathrm{T}}(\mathrm{X};k)$ is even, then there are equivalences

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k) \simeq \mathrm{RMod}^{\mathrm{gr}}_{\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};k)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})) \simeq \mathrm{coLMod}^{\mathrm{gr}}_{\pi_*\mathcal{F}_{\mathrm{T}}(\mathrm{X};k)^{\vee}}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})),$$

which is exactly Definition 3.3.4. If $\Gamma_{\mathrm{T}}(\mathrm{X};k)$ is not even, then $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k)$ is instead the colimit $\lim_{k\to\mathrm{A}}\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};\mathrm{A})$ where A ranges over even $\mathbf{E}_\infty$-k-algebras. If $k\to\mathrm{A}$ is an even eff cover in the sense of Remark 2.1.6, then $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k)$ is equivalent to $\lim_{\boldsymbol{\Delta}}\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k^{\otimes_{\mathrm{A}}\bullet+1})$. For instance, if $k = \mathrm{ko}, \mathrm{KO}$, or the K(1)-local sphere, then $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k)$ agrees with the constructions of Remark 3.3.16, Definition 3.3.13, and Definition 3.3.18.

One can extend Theorem 4.2.6 to the case when X is not finite and $\mathcal{P}$ is not necessarily compact:

Corollary 4.2.8. *Suppose* X *is an ind-proper* T*-space as in Setup 4.1.3. Suppose that* $\mathcal{P}\in\mathrm{Shv}_{\mathrm{T}}(\mathrm{X};k)$ *is a filtered colimit* $\mathrm{colim}_i\,\mathcal{P}_i$ *of compact sheaves which are even in the sense of Definition 4.1.4, and that each* $\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P}_i)$ *is also even. Denote by* $\mathcal{A} = \mathrm{coMod}^{\mathrm{gr},\heartsuit}_{\mathcal{H}_*^{\mathrm{T}}(\mathrm{X};k)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^{\heartsuit})$*. Then there is an equivalence between* $\mathrm{colim}_i\,\mathrm{Shv}^{\mathcal{P}_i,\mathrm{gr}}_{\mathrm{T}}(\mathrm{X};k)$ *and the colimit of the (unbounded) derived ∞-category of* $\langle\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P}_i)\rangle^{\heartsuit}_{\mathcal{A}}$*. In other words, there is a 1-parameter degeneration*

$$\mathrm{Shv}^{\mathcal{P}}_{\mathrm{T}}(\mathrm{X};k) := \mathrm{colim}_i\,\mathrm{Shv}^{\mathcal{P}_i}_{\mathrm{T}}(\mathrm{X};k) \rightsquigarrow \mathrm{colim}_i\,\mathrm{D}(\langle\mathcal{H}^*_{\mathrm{T}}(\mathrm{X};\mathcal{P}_i)\rangle^{\heartsuit}_{\mathcal{A}}).$$

4.3 Chromatic aberrations

Lemma 4.3.1. *Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism whose nontrivial fibers are unstably cellular, i.e., admit a paving by affine spaces. Then the pushforward $f_* k$ is perfect even.*

Lemma 4.3.2. *Let $\lambda_\bullet$ be a sequence of dominant minuscule weights of $\check{\mathrm{T}}$, and let $|\lambda_\bullet| = \sum_i \lambda_i$. Let $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ denote the corresponding convolution variety, so that there is a map*

$$q : \overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}} \to \overline{\mathrm{Gr}_\mathrm{G}^{|\lambda_\bullet|}} \subseteq \mathrm{Gr}_\mathrm{G}.$$

Let $\mathrm{IC}_{\lambda_\bullet} \in \mathrm{Shv}_\mathrm{T}(\mathrm{Gr}_\mathrm{G}; k)$ denote the pushforward $q_* k[\langle 2\rho, |\lambda|\rangle]$. Then $\mathrm{IC}_{\lambda_\bullet}$ is even.

Proof. Let $j_\mu : \mathrm{Gr}_\mathrm{G}^\mu \to \mathrm{Gr}_\mathrm{G}$ denote the inclusion of a stratum. We need to argue that $j_\mu^* \mathrm{IC}_{\lambda_\bullet}$ and $j_\mu^! \mathrm{IC}_{\lambda_\bullet}$ are perfect even. Since $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ is smooth and proper, it is self-dual, so it suffices to prove that $j_\mu^* \mathrm{IC}_{\lambda_\bullet}$ is perfect even. There is a pullback diagram

$$\begin{tikzcd} q^{-1}(\mathrm{Gr}_\mathrm{G}^\mu) \arrow[r, "j'_\mu"] \arrow[d, "q'"'] & \overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}} \arrow[d, "q"] \\ \mathrm{Gr}_\mathrm{G}^\mu \arrow[r, "j_\mu"'] & \mathrm{Gr}_\mathrm{G}, \end{tikzcd}$$

so by proper base change, we may identify

$$j_\mu^* \mathrm{IC}_{\lambda_\bullet} \cong q'_* k[\langle 2\rho, |\lambda|\rangle],$$

where k denotes the constant sheaf on $q^{-1}(\mathrm{Gr}_\mathrm{G}^\mu)$. To prove that this sheaf is perfect even, Lemma 4.3.1 reduces us to showing that the nonempty fibers of the map $q^{-1}(\mathrm{Gr}_\mathrm{G}^\mu) \to \mathrm{Gr}_\mathrm{G}^\mu$ admit a paving by affine spaces. The result in this case is [Hai, Corollary 1.2]. ☐

Lemma 4.3.3. *Let k be a field of characteristic zero. If μ is a dominant weight of $\check{\mathrm{T}}$ such that $\mu = |\lambda|$ for some sequence of dominant minuscule weights $\lambda_\bullet$, then IC_μ is a retract of $\mathrm{IC}_{\lambda_\bullet}$.*

Proof. This can be proved by appealing to the usual geometric Satake equivalence [MV]: it then amounts to the claim that the irreducible $\check{\mathrm{G}}$-representation V_μ is a summand of $\mathrm{V}_{\lambda_\bullet} = \mathrm{V}_{\lambda_1} \otimes \cdots \otimes \mathrm{V}_{\lambda_n}$, which follows from highest weight theory. ☐

Lemma 4.3.4. *The object $\mathcal{H}_\mathrm{T}^*(\mathrm{X}; \mathrm{IC}_{\lambda_\bullet}) \in \mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})$ is perfect even.*

Proof. We need to show that $\mathcal{H}_\mathrm{T}^*(\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}; k)$ is even. The space $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ is homotopy equivalent to $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_1}} \times \cdots \times \overline{\mathrm{Gr}_\mathrm{G}^{\lambda_n}}$. Since each $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda}}$ is isomorphic to $\mathrm{G}/\mathrm{P}_\lambda$, it has even cells, and hence $\overline{\mathrm{Gr}_\mathrm{G}^{\lambda_\bullet}}$ itself has even cells. ☐

The direct sum $\bigoplus \mathrm{IC}_{\lambda_\bullet}$ as $\lambda_\bullet$ ranges over all sequences of dominant minuscule weights is not finite. To fix this, let us choose a cutoff dominant weight ν and set $\mathcal{P}_\nu = \bigoplus_{|\lambda| \leq \nu} \mathrm{IC}_{\lambda_\bullet}$. Denote by $\mathrm{Shv}_\mathrm{T}^{\min}(\mathrm{Gr}_\mathrm{G}; k)$ the ∞-category $\mathrm{colim}_\nu \mathrm{Shv}_\mathrm{T}^{\mathcal{P}_\nu}(\mathrm{Gr}_\mathrm{G}; k)$. If k is a field of characteristic zero, and every dominant weight of $\check{\mathrm{G}}$ can be written as a sum of miniscule dominant weights, Lemma 4.3.3 implies that $\mathrm{Shv}_\mathrm{T}^{\min}(\mathrm{Gr}_\mathrm{G}; k)$ identifies with $\mathrm{Shv}_\mathrm{T}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{Gr}_\mathrm{G}; k)$.

To describe the degeneration of this category, i.e., the "spectral side", we need to introduce some constructions.

Definition 4.3.5. Let $\mathbf{H}$ be a 1-dimensional commutative (possibly formal) group scheme over stack X. The category-valued stack $\mathrm{Tors}_{\mathbf{H}}$ of torsion sheaves on $\mathbf{H}$ is defined as follows: for a relative affine scheme $\mathrm{Y} \to \mathrm{X}$, the 1-category $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ consists of coherent sheaves $\mathcal{F}$ on $\mathbf{H}_{\mathrm{Y}} = \mathbf{H} \times_{\mathrm{X}} \mathrm{Y}$ whose pushforward to Y is flat and whose support is finite over Y. The commutative structure on $\mathbf{H}$ defines a tensor structure on $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ by convolution.

Let $\mathcal{G}$ be a group scheme over X. Let $\mathcal{G}_{\mathbf{H}}/\mathcal{G}$ denote the stack over X defined by sending a relative affine scheme $\mathrm{Y} \to \mathrm{X}$ to the groupoid of exact symmetric monoidal $\mathrm{QCoh}(\mathrm{X})^{\heartsuit}$-linear functors from $\mathrm{QCoh}(\mathrm{B}_{\mathrm{X}}\mathcal{G})^{\heartsuit}$ to $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$. (More generally, for any Tannakian X-scheme T, one can define the "$\mathbf{H}$-loop space" $\mathcal{L}_{\mathbf{H}}\mathrm{T}$ as the stack over X which sends a relative affine scheme $\mathrm{Y} \to \mathrm{X}$ to the groupoid of exact symmetric monoidal $\mathrm{QCoh}(\mathrm{X})^{\heartsuit}$-linear functors from $\mathrm{QCoh}(\mathrm{T})^{\heartsuit}$ to $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$.) There is a canonical map $\mathcal{G}_{\mathbf{H}}/\mathcal{G} \to \mathrm{B}_{\mathrm{X}}\mathcal{G}$, and we will write $\mathcal{G}_{\mathbf{H}}$ to denote its pullback along the map $\mathrm{X} \to \mathrm{B}_{\mathrm{X}}\mathcal{G}$. If $\mathcal{G}$ is pulled back from a group scheme G over $\mathbf{Z}$ along the canonical map $\mathrm{X} \to \mathrm{Spec}(\mathbf{Z})$, we will instead denote $\mathcal{G}_{\mathbf{H}}$ by $\mathrm{G}_{\mathbf{H}}$.

While this thesis was being written, an article with the preceding construction appeared on the arXiv: see [BK2].

Example 4.3.6. Let $\mathrm{G} = \mathbf{G}_m$. In this case, $(\mathbf{G}_m)_{\mathbf{H}}/\mathbf{G}_m$ sends a relative affine scheme $\mathrm{Y} \to \mathrm{X}$ to the groupoid of exact symmetric monoidal $\mathrm{QCoh}(\mathrm{X})^{\heartsuit}$-linear functors $\mathrm{Rep}(\mathbf{G}_m)^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$. This is equivalent to the category of coherent sheaves $\mathcal{F}$ on $\mathbf{H}_{\mathrm{Y}}$ such that the map $\mathrm{Supp}(\mathcal{F}) \to \mathrm{Y}$ is an isomorphism (i.e., $\mathcal{F}$ is of length 1). Since this implies that $\mathcal{F}$ is a line bundle over Y, we find that $(\mathbf{G}_m)_{\mathbf{H}}/\mathbf{G}_m$ is isomorphic to $\mathbf{H} \times \mathrm{B}\mathbf{G}_m$. It follows that $(\mathbf{G}_m)_{\mathbf{H}} \cong \mathbf{H}$. More generally, if T is a (split) torus, then $\mathrm{T}_{\mathbf{H}} \cong \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{H})$.

Example 4.3.7. Suppose $\mathbf{H} = \mathbf{G}_a$, and suppose for simplicity that $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$. If $\mathrm{Y} = \mathrm{Spec}(\mathrm{R})$, then $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is the category of torsion $\mathrm{R}[x]$-modules. This is the same data as a projective R-module M of finite rank equipped with an endomorphism x, which implies (by the Tannakian formalism) that the groupoid of exact symmetric monoidal $\mathbf{Z}$-linear functors $\mathrm{Rep}(\mathrm{G})^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is isomorphic to $(\mathfrak{g}/\mathrm{G})(\mathrm{Y})$. It follows that $\mathrm{G}_{\mathbf{G}_a} \cong \mathfrak{g}/\mathrm{G}$, so that $\mathrm{G}_{\mathbf{G}_a} = \mathfrak{g}$.

Example 4.3.8. Suppose $\mathbf{H} = \widehat{\mathbf{G}_a}$, and suppose for simplicity that $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$. If $\mathrm{Y} = \mathrm{Spec}(\mathrm{R})$, then $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is the category of torsion $\mathrm{R}[x]$-modules which are set-theoretically supported at the origin. The data of a torsion $\mathrm{R}[x]$-module is the same as a projective R-module M of finite rank equipped with an endomorphism x. The Cayley-Hamilton theorem implies that this module is supported at the zero locus of its characteristic polynomial $\chi(x)$. It is set-theoretically supported at the origin if and only if the non-leading coefficients of $\chi(x)$ are nilpotent. This implies (by the Tannakian formalism) that the groupoid of exact symmetric monoidal $\mathbf{Z}$-linear functors $\mathrm{Rep}(\mathrm{G})^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is isomorphic to $(\mathfrak{g}^{\wedge}_{\mathcal{N}}/\mathrm{G})(\mathrm{Y})$, where $\mathcal{N} \subseteq \mathfrak{g}$ is the nilpotent cone. It follows that $\mathrm{G}_{\widehat{\mathbf{G}_a}} \cong \mathfrak{g}^{\wedge}_{\mathcal{N}}/\mathrm{G}$, so that $\mathrm{G}_{\widehat{\mathbf{G}_a}} = \mathfrak{g}^{\wedge}_{\mathcal{N}}$.

Example 4.3.9. Suppose $\mathbf{H} = \mathbf{G}_m$, and suppose for simplicity that $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$. If $\mathrm{Y} = \mathrm{Spec}(\mathrm{R})$, then $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is the category of torsion $\mathrm{R}[x^{\pm 1}]$-modules. This is the same data as a projective R-module M of finite rank equipped with an invertible endomorphism x, which implies (by the Tannakian formalism) that the groupoid of exact symmetric monoidal $\mathbf{Z}$-linear functors $\mathrm{Rep}(\mathrm{G})^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is isomorphic to $(\mathrm{G}/\mathrm{G})(\mathrm{Y})$. Here, G acts on itself by conjugation. It follows that $\mathrm{G}_{\mathbf{G}_m} \cong \mathrm{G}/\mathrm{G}$, so that $\mathrm{G}_{\mathbf{G}_m} = \mathrm{G}$.

Example 4.3.10. Suppose $\mathbf{H} = \widehat{\mathbf{G}_m}$, and suppose for simplicity that $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$. If $\mathrm{Y} = \mathrm{Spec}(\mathrm{R})$, then $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is the category of torsion $\mathrm{R}[x^{\pm 1}]$-modules which are set-theoretically supported at $x = 1$. A torsion $\mathrm{R}[x^{\pm 1}]$-module is a projective R-module M of finite rank equipped with an invertible endomorphism x. Again, the Cayley-Hamilton theorem implies that such a module is set-theoretically supported at the origin if and only if the non-leading coefficients of the characteristic polynomial of $x - 1$ are nilpotent. This implies (by the Tannakian formalism) that the groupoid of exact symmetric monoidal $\mathbf{Z}$-linear functors $\mathrm{Rep}(\mathrm{G})^{\heartsuit} \to \mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ is isomorphic to $(\mathrm{G}^{\wedge}_{\mathcal{U}}/\mathrm{G})(\mathrm{Y})$, where $\mathcal{U} \subseteq \mathrm{G}$ is the unipotent cone. It follows that $\mathrm{G}_{\widehat{\mathbf{G}_m}} \cong \mathrm{G}^{\wedge}_{\mathcal{U}}/\mathrm{G}$, so that $\mathrm{G}_{\widehat{\mathbf{G}_m}} = \mathrm{G}^{\wedge}_{\mathcal{U}}$.

Example 4.3.11. If $\mathbf{H}$ is an elliptic curve E, then the Fourier-Mukai transform identifies the category $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ with its convolution symmetric monoidal structure with the category of semistable vector bundles on the dual elliptic curve $\mathrm{E}^{\vee}$ of degree zero. The Tannakian formalism implies that the stack G_{E} is isomorphic to the moduli stack of semistable G-bundles over $\mathrm{E}^{\vee}$ of degree zero.

Remark 4.3.12. Let $\mathbf{H}^{\vee} = \mathrm{Hom}(\mathbf{H}, \mathrm{B}\mathbf{G}_m)$. We will say that $\mathbf{H}$ is 1-*dualizable* if for every relative affine morphism $\mathrm{Y} \to \mathrm{X}$, the canonical Fourier-Mukai transform $\mathrm{QCoh}(\mathbf{H}_{\mathrm{Y}}) \to \mathrm{QCoh}(\mathbf{H}^{\vee}_{\mathrm{Y}})$ is an equivalence of categories which swaps the convolution tensor structure on $\mathrm{QCoh}(\mathbf{H}_{\mathrm{Y}})$ with the standard symmetric monoidal structure on $\mathrm{QCoh}(\mathbf{H}^{\vee}_{\mathrm{Y}})$. (The preceding examples are 1-dualizable.) One can verify that under this equivalence, a torsion sheaf on $\mathbf{H}_{\mathrm{Y}}$ is sent to a torsion-free sheaf on $\mathbf{H}^{\vee}_{\mathrm{Y}}$ (and in fact, this restricts to an equivalence of categories between $\mathrm{Tors}_{\mathbf{H}}(\mathrm{Y})$ and semistable torsion-free sheaves of degree zero on $\mathbf{H}^{\vee}_{\mathrm{Y}}$, defined appropriately). This implies that $\mathcal{G}_{\mathbf{H}}/\mathcal{G}$ is isomorphic to a substack of the moduli stack $\mathrm{Bun}_{\mathcal{G}}(\mathbf{H}^{\vee})$ of $\mathcal{G}$-bundles on $\mathbf{H}^{\vee}$, and hence that $\mathcal{G}_{\mathbf{H}}$ is isomorphic to a substack of the moduli stack of $\mathcal{G}$-bundles on $\mathbf{H}^{\vee}$ equipped with a trivialization at the basepoint of $\mathbf{H}^{\vee}$.

Let G be the split (pinned) form of a simply-laced algebraic group (over a stack X), and let $\mathbf{H}$ be a 1-dimensional commutative group scheme over X. Let $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}} = \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathrm{B}_{\mathbf{H}}$, and let $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}$ denote the relative affinization of the map

$$\chi : \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathrm{B}_{\mathbf{H}} \to \mathrm{T}_{\mathbf{H}} = \mathrm{Hom}(\mathbb{X}^*(\mathrm{T}), \mathbf{H}),$$

so that $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}} = \mathrm{Spec}_{\mathrm{T}_{\mathbf{H}}}(\pi_0 \chi_*(\mathcal{O}_{\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}}))$. For instance, if $\mathbf{H} = \mathbf{G}_a$, then $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}$ is the affinization of the Grothendieck-Springer resolution $\widetilde{\check{\mathfrak{g}}} = \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathfrak{b} \cong \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}}$; and if $\mathbf{H} = \mathbf{G}_m$, then $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}$ is the affinization of $\check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathrm{B}$ for the conjugation action of $\check{\mathrm{B}}$ on B.

Given this setup, our calculations in Corollary 3.6.8, Corollary 3.8.10, and Corollary 3.9.11 can be phrased as follows.

Theorem 4.3.13. *Suppose* G *is a connected, almost simple, and simply-laced algebraic group over* $\mathbf{C}$*. Let* $\mathrm{T} \subseteq \mathrm{G}$ *be a maximal torus. Let* k *denote either* 2*-periodified rational cohomology* $\mathbf{Q}[u^{\pm 1}]$*, complex K-theory* KU*, or elliptic cohomology with associated elliptic curve* E*, and let* F *be an algebraically closed field over* $\pi_0(k)$*. Then there is an* $\check{\mathrm{G}}$*-stable open locus* $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{reg}} \subseteq \widetilde{\check{\mathrm{G}}}_{\mathbf{H}}$ *with complement of codimension* 2*, and a* t*-exact equivalence*

$$\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}),$$

where the dual group on the right-hand side is defined over F*. Furthermore, this equivalence is monoidal for the convolution structure on the left-hand side and the standard tensor product on the right-hand side.*

Although our calculations concerned the case of genuine equivariance, it is easy to see that working with Borel-equivariant local systems would have led to the same conclusion, except with $\mathbf{H} = \mathbf{G}_a, \mathbf{G}_m$, or E replaced by their formal completions at the identity.

We will now prove:

Corollary 4.3.14. *Let k denote either 2-periodified rational cohomology $\mathbf{Q}[u^{\pm 1}]$, complex K-theory* KU*, or elliptic cohomology with associated elliptic curve* E*, and let* F *be an algebraically closed field over $\pi_0(k)$. Suppose* G *is a connected, almost simple, and simply-laced algebraic group over $\mathbf{C}$. If k is not the 2-periodification of $\mathbf{Q}$, assume further that* G *is of type* A *or of type* D *(and in the latter case, assume that 2 is a unit in $\pi_0(k)$). Let $\mathrm{T} \subseteq \mathrm{G}$ be a maximal torus. Then there is an equivalence*

$$\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}}),$$

where the dual group on the right-hand side is defined over F*. Furthermore, this equivalence is monoidal for the convolution structure on the left-hand side and the standard tensor product on the right-hand side.*

Proof. By Corollary 4.2.8, the category $\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k)$ is the colimit over ν of the (unbounded) derived ∞-category of $\langle \mathcal{H}_{\mathrm{T}}^*(\mathrm{Gr}_{\mathrm{G}}; \mathcal{P}_\nu)\rangle_{\mathcal{A}}^{\heartsuit}$, where $\mathcal{A} = \mathrm{coMod}^{\mathrm{gr},\heartsuit}_{\mathcal{H}_*^{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}};k)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})^{\heartsuit})$. Notice that $\mathcal{A}$ is precisely the heart of the standard homological t-structure on $\mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)}$ F. It follows from Theorem 4.3.13 that there is an equivalence

$$\mathcal{A} = \mathrm{Loc}_{\mathrm{T}_c}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\heartsuit} \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}}/\check{\mathrm{G}})^{\heartsuit}.$$

Under this equivalence, the sheaf $\mathcal{H}_{\mathrm{T}}^*(\mathrm{Gr}_{\mathrm{G}}; \mathrm{IC}_{\lambda_\bullet})$ is sent to $\mathcal{V}_{\lambda_\bullet} = \bigotimes_j \mathcal{V}_{\lambda_j}$, where $\mathcal{V}_{\lambda_j}$ denotes the pullback of V_{λ_j} along the map $\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}/\check{\mathrm{G}} \to \mathrm{B}\check{\mathrm{G}}$. If we write $\mathcal{V}_{\leq \nu}$ to denote $\bigoplus_{|\lambda| \leq \nu} \mathcal{V}_{\lambda_\bullet}$, it follows that

$$\mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}_\nu,\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq \mathrm{D}\left(\langle \mathcal{V}_{\leq \nu}\rangle^{\heartsuit}_{\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}}/\check{\mathrm{G}})^{\heartsuit}}\right). \tag{4.3.1}$$

The inclusion $j : \widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}} \hookrightarrow \widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}$ has complement of codimension 2, and the vector bundle $\mathcal{V}_{\lambda_\bullet}$ is obtained by restriction along this map. It follows from Hartogs' lemma that the functor

$$j^* : \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}})^{\heartsuit} \to \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}}/\check{\mathrm{G}})^{\heartsuit}$$

is fully faithful on the subcategory spanned by the sheaves $\mathcal{V}_{\lambda_\bullet}$. (In the elliptic case, it follows from [Dav, Proposition 3.1.16] that fiber over each geometric point of $\mathcal{M}_{\mathrm{T},0}$ of the complement of the inclusion $\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}} \to \widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}$ has codimension ≥ 2; one can then use [HK, Proposition 3.5].) In particular, we may replace $\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{reg}}/\check{\mathrm{G}})^{\heartsuit}$ in (4.3.1) by $\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}})^{\heartsuit}$. In other words, (4.3.1) says that $\mathrm{colim}_\nu \mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}_\nu}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{gr}}$ is equivalent to the full subcategory of $\mathrm{D}(\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}})^{\heartsuit})$ which is compactly generated by the quasicoherent sheaves $\mathcal{V}_{\lambda_\bullet}$ as $\lambda_\bullet$ ranges over all sequences of dominant minuscule weights of $\check{\mathrm{G}}$.

By Lemma 4.3.15, the derived category $\mathrm{Rep}(\check{\mathrm{G}})$ is generated by the representations $\mathrm{V}_{\lambda_\bullet}$ as $\lambda_\bullet$ ranges over all sequences of minuscule dominant weights of $\check{\mathrm{G}}$. Thus $\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k) = \mathrm{colim}_\nu \mathrm{Shv}_{\mathrm{T}}^{\mathcal{P}_\nu,\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; k)$ is in fact equivalent to the full subcategory of $\mathrm{D}(\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}})^{\heartsuit}) = \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}_{\mathbf{H}}}^{\mathrm{aff}}/\check{\mathrm{G}})$ which is compactly generated by the quasicoherent sheaves $\mathcal{V}$ as V ranges over all finite-dimensional (irreducible) $\check{\mathrm{G}}$-representations. Since $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}$ is affine and normal, Lemma 4.3.16 implies that $\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k)$ is equivalent to $\mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}/\check{\mathrm{G}})$, as desired. The monoidality can be shown by unwinding the above construction of the equivalence. $\square$

Lemma 4.3.15. *Let* F *be an algebraically closed field of characteristic* p *(possibly* 0*). Suppose* G *is a classical group (i.e., of type* A, B, C, *or* D*); if* G *is of type* B *or* D, *assume that* $p \neq 2$, *and if* G *is of type* C, *assume that* $p > n$. *Then the derived category* $\mathrm{Rep}(\check{\mathrm{G}})$ *is generated by the representations* $\mathrm{V}_{\lambda_\bullet} = \bigotimes_i \mathrm{V}_{\lambda_i}$ *as* $\lambda_\bullet$ *ranges over all sequences* $(\lambda_1, \cdots, \lambda_m)$ *of minuscule dominant weights of* $\check{\mathrm{G}}$.

Proof. Note that $\mathrm{Rep}(\check{\mathrm{G}})$ is generated by tilting modules, so it suffices to see that every tilting module can be generated using the representations $\mathrm{V}_{\lambda_\bullet}$. For each dominant weight μ, let T_μ denote the indecomposable tilting module with highest weight μ. Then $\mathrm{T}_\lambda \cong \mathrm{V}_\lambda$ if λ is dominant minuscule.

When G is of type A, every dominant weight μ of $\check{\mathrm{G}}$ can be written as a sum $\mu = \sum_i \lambda_i$ with each λ_i being dominant minuscule. It follows that T_μ is a direct summand of the tensor product $\mathrm{T}_{\lambda_\bullet} := \bigotimes_i \mathrm{T}_{\lambda_i}$, since the latter is tilting and has highest weight μ.

Next, if G is of type B_n, the fundamental weight ϖ_n is minuscule (corresponding to the spin representation), so one can construct the representation $\mathrm{V}_{\varpi_n}^{\otimes 2}$. As discussed in [JMW, Section 3.6.2], since k has characteristic $\neq 2$, this representation contains the indecomposable tilting modules T_{ϖ_i} associated to all fundamental weights ϖ_i (for $1 \leq i \leq n-1$) as direct summands:

$$\mathrm{V}_{\varpi_n}^{\otimes 2} \cong \mathrm{T}_{2\varpi_n} \oplus \mathrm{T}_{\varpi_{n-1}} \oplus \mathrm{T}_{\varpi_{n-2}} \oplus \cdots .$$

Since every dominant weight μ can be written as a sum of fundamental weights, the indecomposable tilting module T_μ is a direct summand of $\mathrm{T}_{\lambda_\bullet} = \mathrm{V}_{\lambda_\bullet}$ for some sequence of dominant minuscule weights, as desired.

Next, if G is of type C_n, the fundamental weight ϖ_1 is minuscule (corresponding to the standard representation), so one can construct the representations $\mathrm{V}_{\varpi_1}^{\otimes k}$ for $1 \leq k \leq n$. Since k has characteristic $> n$, the exterior power $\wedge^k(\mathrm{V}_{\varpi_1})$ is a direct summand of $\mathrm{V}_{\varpi_1}^{\otimes k}$. As discussed in [JMW, Section 3.6.3], since k has characteristic $> n$, the representation $\wedge^k(\mathrm{V}_{\varpi_1})$ contains the indecomposable tilting module T_{ϖ_k} associated to the fundamental weight ϖ_k (for $1 \leq k \leq n$) as a direct summand:

$$\wedge^k(\mathrm{V}_{\varpi_1}) \cong \mathrm{T}_{\varpi_k} \oplus \mathrm{T}_{\varpi_{k-2}} \oplus \cdots .$$

Since every dominant weight μ can be written as a sum of fundamental weights, the indecomposable tilting module T_μ is a direct summand of $\mathrm{T}_{\lambda_\bullet} = \mathrm{V}_{\lambda_\bullet}$ for some sequence of dominant minuscule weights, as desired.

Finally, if G is of type D_n, the fundamental weights ϖ_{n-1} and ϖ_n are both minuscule (corresponding to the half-spin representations), so one can construct the representations $\mathrm{V}_{\varpi_{n-1}} \otimes \mathrm{V}_{\varpi_n}$ and $\mathrm{V}_{\varpi_n}^{\otimes 2}$. As discussed in [JMW, Section 3.6.4], since k has characteristic $\neq 2$, these representations contain the indecomposable tilting modules T_{ϖ_i} associated to all fundamental weights ϖ_i (for $1 \leq i \leq n-2$) as direct summands. Namely, there are isomorphisms

$$\mathrm{V}_{\varpi_{n-1}} \otimes \mathrm{V}_{\varpi_n} \cong \mathrm{T}_{\varpi_{n-1}+\varpi_n} \oplus \mathrm{T}_{\varpi_{n-3}} \oplus \mathrm{T}_{\varpi_{n-5}} \oplus \cdots ,$$
$$\mathrm{V}_{\varpi_n}^{\otimes 2} \cong \mathrm{T}_{2\varpi_n} \oplus \mathrm{T}_{\varpi_{n-2}} \oplus \mathrm{T}_{\varpi_{n-4}} \oplus \cdots .$$

Since every dominant weight μ can be written as a sum of fundamental weights, the indecomposable tilting module T_μ is a direct summand of $\mathrm{T}_{\lambda_\bullet} = \mathrm{V}_{\lambda_\bullet}$ for some sequence of dominant minuscule weights, as desired. $\square$

Lemma 4.3.16. *Let* H *be a reductive group scheme (over* $\mathbf{Z}$*, say), and let* $f : \mathrm{X} \to \mathrm{S}$ *be an affine morphism of stacks equipped with an action of* H *(i.e., an* S*-linear action of* $\mathrm{H} \times \mathrm{S}$*). Then the objects* $\mathcal{V} \in \mathrm{QCoh}(\mathrm{X}/\mathrm{H})$ *as* V *ranges over all finite-dimensional representations of* H *form a set of compact generators of* $\mathrm{QCoh}(\mathrm{X}/\mathrm{H})$*.*

Proof. Let $\mathrm{X} = \mathrm{Spec}_{\mathrm{S}}(\mathrm{R})$ for some commutative $\mathcal{O}_{\mathrm{S}}$-algebra R, so that an object $\mathcal{F} \in \mathrm{QCoh}(\mathrm{X}/\mathrm{H})$ can be identified with an R-module $\mathrm{M} = f_*\mathcal{F}$ (in $\mathrm{QCoh}(\mathrm{S})$) equipped with an H-action. Since H is reductive, the functor of derived H-invariants is exact, and $\mathrm{Map}_{\mathrm{QCoh}(\mathrm{X}/\mathrm{H})}(\mathcal{V}, \mathcal{F}) \simeq (\mathrm{M} \otimes \mathrm{V}^*)^{\mathrm{H}}$ for all $\mathrm{V} \in \mathrm{Rep}(\mathrm{H})$. If this vanishes for all V, then $\mathrm{M} = 0$ itself, so the quasicoherent sheaves $\mathcal{V}$ generate $\mathrm{QCoh}(\mathrm{X}/\mathrm{H})$ as desired. $\square$

Some standard arguments with the Grothendieck-Springer resolution now imply:

Corollary 4.3.17. *Let* k *denote either 2-periodified rational cohomology* $\mathbf{Q}[u^{\pm 1}]$*, complex K-theory* KU*, or elliptic cohomology with associated elliptic curve* E*, and let* F *be an algebraically closed field over* $\pi_0(k)$*. Suppose* G *is a connected, almost simple, and simply-laced algebraic group over* $\mathbf{C}$ *with torsion-free fundamental group. If* k *is not the 2-periodification of* $\mathbf{Q}$*, assume further that* G *is of type* A *or of type* D *(and in the latter case, assume that 2 is a unit in* $\pi_0(k)$*). Then there is an equivalence*

$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}),$$

where the dual group on the right-hand side is defined over F*. Furthermore, this equivalence is monoidal for the convolution structure on the left-hand side and the standard tensor product on the right-hand side.*

Proof. Since G is assumed to have torsion-free fundamental group, the scheme $\mathrm{G}_{\mathbf{H}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \mathrm{T}_{\mathbf{H}}$ is normal. The map $f : \tilde{\check{\mathrm{G}}}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \mathrm{T}_{\mathbf{H}}$ is proper and birational, so it identifies $\mathrm{G}_{\mathbf{H}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \mathrm{T}_{\mathbf{H}}$ with $\tilde{\check{\mathrm{G}}}_{\mathbf{H}}^{\mathrm{aff}}$ by Zariski's main theorem. On the other hand, descent along the map $\mathrm{T}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}$ identifies with descent from T-equivariant to G-equivariant cohomology. The desired equivalence is then implied by Corollary 4.3.14. $\square$

Example 4.3.18. For instance, if $k = \mathrm{KU}$, then the version of Corollary 4.3.17 for Borel-equivariant sheaves states that there is an equivalence

$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{KU})^{\mathrm{Bor}} \otimes_{\mathbf{Z}} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{G}_{\mathcal{U}}^{\wedge}/\check{\mathrm{G}}),$$

at least if G is of type A, or G is of type D and F is of characteristic $\neq 2$. It is not hard to extend Corollary 4.3.17 to the case $k = \mathrm{KO}$, too; in the Borel-equivariant case, the object appearing on the spectral side is the quotient $(\mathrm{G}_{\mathcal{U}}^{\wedge}/\check{\mathrm{G}})/(\mathbf{Z}/2)$, where $\mathbf{Z}/2$ acts on $\mathrm{G}_{\mathcal{U}}^{\wedge}$ by squaring.

Similarly, if $k = \mathbf{Q}[u^{\pm 1}]$, then the version of Corollary 4.3.17 for Borel-equivariant sheaves states that there is an equivalence

$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{Bor}} \otimes_{\mathbf{Q}} \mathrm{F} \simeq \mathrm{QCoh}(\mathfrak{g}_{\mathcal{N}}^{\wedge}/\check{\mathrm{G}}).$$

Note that this – up to 2-periodification/shearing – is exactly the renormalized form of the derived geometric Satake equivalence [AG]; the "renormalization" here corresponds to the choice of working with Borel-equivariant sheaves (instead of genuine equivariance).

Remark 4.3.19. In Corollary 4.3.14 and Corollary 4.3.17, one can take $k = \mathbf{Q}$ itself (not its 2-periodification). For any algebraically closed field $\mathrm{F} \supseteq \mathbf{Q}$, one then has equivalences

$$\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{F}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{aff}}(2)/\check{\mathrm{G}}),$$
$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{F}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{g}(2)/\check{\mathrm{G}}),$$

where $\widetilde{\check{\mathfrak{g}}}(2) = \mathrm{T}^*(2)(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}} \cong \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathfrak{b}(2)$, and similarly $\mathfrak{g}(2) \cong \check{\mathfrak{g}}^*(2)$.

Corollary 4.3.17 and our discussion in Theorem 3.7.2 suggests the following:

Conjecture 4.3.20. *Suppose* G *is a connected, almost simple, and simply-laced algebraic group over* $\mathbf{C}$ *with torsion-free fundamental group. Let k be an evenly descendable $\mathbf{E}_\infty$-ring, and let $\mathbf{H}$ denote the formal group* $\mathrm{Spev}(k^{\mathbf{C}\mathrm{P}^\infty})$ *over* $\mathrm{Spev}(k)$. *Working with Borel-equivariant sheaves, there is a* $\mathrm{QCoh}(\mathrm{Spev}(k))$*-linear equivalence*

$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl,gr}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{Bor}} \simeq \mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}),$$

where the dual group on the right-hand side is the split form defined over $\mathrm{Spev}(k)$. *Furthermore, this equivalence is monoidal for the convolution structure on the left-hand side and the standard tensor product on the right-hand side.*

In future work, we hope to use Theorem 3.7.2 to prove Conjecture 4.3.20 in the case of an algebraically closed field of arbitrary characteristic (over which a strongly nondegenerate character exists).

Remark 4.3.21. Let us note one interesting consequence of Conjecture 4.3.20. The equivalence must send the delta sheaf at the basepoint of Gr_{G} to the structure sheaf of $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$, which implies that $\mathrm{Spev}(k^{\mathrm{BG}})$ is isomorphic to the derived affinization of $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$. The derived affinizations of $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ and $\mathrm{G}_{\mathbf{H}}/\mathrm{G} = \mathcal{L}_{\mathbf{H}}(\mathrm{BG})$ are isomorphic, so Conjecture 4.3.20 implies that (if G is simply-laced with torsion-free fundamental group) there is an isomorphism

$$\mathrm{Spev}(k^{\mathrm{BG}}) \cong \mathrm{Spec}_{\mathrm{Spev}(k)}(\mathrm{R}\Gamma(\mathcal{L}_{\mathbf{H}}(\mathrm{BG}); \mathcal{O}))$$

over $\mathrm{Spev}(k)$, where the right-hand side denotes the relative Spec. (If one wanted to work in the genuine equivariant setting, then the left-hand side must be replaced by the derived affinization of $\mathcal{M}_{\mathrm{G},0}$.) This would already be very interesting to prove! It exhibits a deep relationship between equivariant cohomology and the $\mathbf{H}$-loop space construction of Definition 4.3.5.

When k is an *ordinary commutative ring*, then $\mathcal{L}_{\mathbf{H}}(\mathrm{BG}) \cong \mathfrak{g}/\mathrm{G}$, and so $\mathrm{Spec}_{\mathrm{Spev}(k)}(\mathrm{R}\Gamma(\mathcal{L}_{\mathbf{H}}(\mathrm{BG}); \mathcal{O}))$ identifies with the derived affinization of $\check{\mathfrak{g}}^*(2)/\check{\mathrm{G}}$ (using the isomorphism $\check{\mathfrak{g}}^* \cong \mathfrak{g}$). Motivated by this, it is reasonable to expect that there is an isomorphism

$$\mathrm{Spev}(k^{\mathrm{BG}}) \cong \mathrm{Spec}(\mathrm{R}\Gamma(\check{\mathfrak{g}}^*(2)/\check{\mathrm{G}}; \mathcal{O}))$$

for an *arbitrary* connected reductive group G (not necessarily simply-laced). Calculations with A. Venkatesh suggest that this isomorphism does indeed hold, and we hope to explore this in the future.

Remark 4.3.22. One can also consider variants of Conjecture 4.3.20; for example, if $\mathrm{I} = \mathrm{G}(\mathcal{O}) \times_{\mathrm{G}} \mathrm{B}$ is the Iwahori subgroup of $\mathrm{G}(\mathcal{O})$, then Conjecture 4.3.20 suggests that there is a $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear equivalence

$$\mathrm{Shv}_{\mathrm{T}}^{\mathrm{I}\text{-cbl,gr}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{Bor}} \simeq \mathrm{QCoh}(\mathrm{B}_{\mathbf{H}}/\check{\mathrm{B}}),$$

where the dual group on the right-hand side is the split form defined over $\mathrm{Spev}(k)$. The quotient stack $\mathrm{B}_{\mathbf{H}}/\check{\mathrm{B}}$ can be identified with $\tilde{\check{\mathrm{G}}}_{\mathbf{H}}/\check{\mathrm{G}}$, where $\tilde{\check{\mathrm{G}}}_{\mathbf{H}} = \check{\mathrm{G}} \times^{\check{\mathrm{B}}} \mathrm{B}_{\mathbf{H}}$. Note that when $\mathbf{H} = \mathbf{G}_a$, the isomorphism $\mathrm{B}_{\mathbf{G}_a} \cong \mathfrak{b} \cong [\check{\mathfrak{b}}, \check{\mathfrak{b}}]^{\perp}$ identifies $\tilde{\check{\mathrm{G}}}_{\mathbf{G}_a}$ with the Grothendieck-Springer resolution $\tilde{\check{\mathfrak{g}}} = \mathrm{T}^*(\check{\mathrm{G}}/[\check{\mathrm{B}}, \check{\mathrm{B}}])/\check{\mathrm{T}}$. The equivalence conjectured above is then the result of Arkhipov-Bezrukavnikov-Ginzburg [ABG].

4.4 Power operations under Langlands duality

We will momentarily review some of the rich theory of power operations in homotopy theory; these force the existence of additional structures on the Langlands dual side of Corollary 4.3.17. Our goal in this section is to describe these structures explicitly. This section is motivated by a discussion with David Treumann.

Warning 4.4.1. Before proceeding, we warn the reader of a terminological mismatch. In [Lon4], Lonergan uses "Steenrod operators" to construct new structures on Coulomb branches (and in particular, on $\check{\mathrm{J}}$). These operators, as we will explain in future work, are better viewed as $\mathbf{E}_3$-*power operations* coming from an $\mathbf{E}_3$-algebra structure on $\mathrm{C}_*^{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{F}_p)$. While these are related to Steenrod operations in the usual sense of the word (as used by algebraic topologists), they are not the same. More generally, $\mathbf{E}_3$-power operations on $\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^{\vee}$ are closely related to, but distinct from, the power operations we will describe below. These $\mathbf{E}_3$-power operations will be described in future work (also see Remark 4.4.11 below); there, we will prove a generalization to other $\mathbf{E}_\infty$-rings of the "Azumaya property" of crystalline differential operators in characteristic p.

Let k be an $\mathbf{E}_\infty$-ring; we will momentarily specialize to the case when k is 2-periodic *integral* cohomology, complex K-theory, or elliptic cohomology. The theory of power operations describes the additional structure acquired by k-cohomology from the $\mathbf{E}_\infty$-structure on k. As we will see below, it is closely related to the structure of isogenies on the associated 1-dimensional group scheme. This relationship is not new; we refer the reader to [Str1, And, Rez] for some sources.

Construction 4.4.2. Any $\mathbf{E}_\infty$-ring k admits a *Tate-valued Frobenius* $k \to k^{t\mathbf{Z}/p}$, which is given by the composite of the Tate diagonal $k \to (k^{\otimes p})^{t\mathbf{Z}/p}$ with the $\mathbf{Z}/p$-Tate construction of the multiplication map $k^{\otimes p} \to k$. See, e.g., [NS, Definition IV.1.1] for a modern reference.

If k admits additional structure, then this structure can be refined: namely, if k admits a refinement to a normed algebra in the ∞-category of genuine $\mathbf{Z}/p$-spectra (which will be true in the examples we will study), and $\Phi^{\mathbf{Z}/p}k$ is its geometric fixed points, then the Tate-valued Frobenius $k \to k^{t\mathbf{Z}/p}$ lifts to an $\mathbf{E}_\infty$-map $\varphi : k \to \Phi^{\mathbf{Z}/p}k$. This map is given by taking geometric fixed points of the $\mathbf{Z}/p$-equivariant norm-multiplication map $\mathrm{N}^{\mathbf{Z}/p}k \to k$, where $\mathrm{N}^{\mathbf{Z}/p}k$ is the Hill-Hopkins-Ravenel norm from [HHR].

If X is any (finite) space, let $\mathcal{F}_k(\mathrm{X})$ denote the $\mathbf{E}_\infty$-k-algebra of k-cochains on X, and let $\mathcal{F}_k(\mathrm{X})^{\vee}$ denote the $\mathbf{E}_\infty$-k-coalgebra of k-chains on X. Then φ induces maps

$$\mathcal{F}_k(\mathrm{X}) \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathrm{X}),\ \mathcal{F}_k(\mathrm{X})^{\vee} \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathrm{X})^{\vee}.$$

We will denote either of these maps by φ_{X}, and call them the *decompleted Frobenius*. Sometimes, we will consider the further composites to $\mathcal{F}_{k^{t\mathbf{Z}/p}}(\mathrm{X})$ and $\mathcal{F}_{k^{t\mathbf{Z}/p}}(\mathrm{X})^{\vee}$; these composites exist for any $\mathbf{E}_\infty$-ring k, even if it does not lift to a normed algebra in genuine $\mathbf{Z}/p$-spectra.

In the above context, one should view $\Phi^{\mathbf{Z}/p}k$ as a decompletion of $k^{t\mathbf{Z}/p}$; we will see this in Example 4.4.5 below.

Remark 4.4.3. Let I_{tr} denote the transfer ideal in $\pi_0\mathcal{F}_k(\mathrm{X} \times \mathrm{B}\mathbf{Z}/p)$, given by the image of the map $\pi_0\mathcal{F}_k(\mathrm{X}) \to \pi_0\mathcal{F}_k(\mathrm{X} \times \mathrm{B}\mathbf{Z}/p)$ induced by the transfer. On π_0, the map $\varphi_{\mathrm{X}} : \mathcal{F}_k(\mathrm{X}) \to \mathcal{F}_{k^{t\mathbf{Z}/p}}(\mathrm{X})$ then factors as a composite

$$\pi_0\mathcal{F}_k(\mathrm{X}) \to \pi_0\mathcal{F}_k(\mathrm{X} \times \mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}} \to \pi_0\mathcal{F}_{k^{t\mathbf{Z}/p}}(\mathrm{X}).$$

The first map in this composite is often referred to as the *total power operation*. We will denote it by $\varphi_{\mathrm{X}}^{\mathrm{tr}}$. It will not be used below in any serious way; we have mentioned it only for completeness.

Remark 4.4.4. Construction 4.4.2 might seem somewhat abstract, but it has very concrete consequences. Suppose, for simplicity, that k is even and 2-periodic, and that $\pi_0\mathcal{F}_k(\mathrm{X} \times \mathrm{B}\mathbf{Z}/p) \cong \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)$. Under the assumption on k, this happens if, for instance, either X is a finite space with even cells, or $\pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)$ is flat over $\pi_0(k)$. The total power operation is then a ring map

$$\varphi_{\mathrm{X}}^{\mathrm{tr}} : \pi_0\mathcal{F}(\mathrm{X}) \to \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}}.$$

In fact, this can be upgraded to a map

$$\pi_0\mathcal{F}(\mathrm{X}) \to \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\Sigma_p)/\mathrm{I}_{\mathrm{tr}}, \tag{4.4.1}$$

where Σ_p is the symmetric group on p letters.

Moreover, under the hypothesis on k, there is an isomorphism $\pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p) \cong \pi_0(k)[\![t]\!]/[p](t)$, where $[p](t)$ is the p-series of the formal group law over $\pi_0\mathcal{F}_k(\mathbf{CP}^\infty) \cong \pi_0(k)[\![t]\!]$.[1] The composite of Remark 4.4.3 can be identified with the map

$$\pi_0\mathcal{F}_k(\mathrm{X}) \xrightarrow{\varphi_{\mathrm{X}}^{\mathrm{tr}}} \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}} \to \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)[1/t].$$

If k admits the structure of a normed algebra in genuine $\mathbf{Z}/p$-spectra, then this composite factors through

$$\pi_0\mathcal{F}_k(\mathrm{X}) \xrightarrow{\varphi_{\mathrm{X}}} \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\Phi^{\mathbf{Z}/p}(k) \to \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)[1/t].$$

It follows, in particular, that $\varphi_{\mathrm{X}}^{\mathrm{tr}}$ and φ_{X} together define a map

$$\pi_0\mathcal{F}_k(\mathrm{X}) \to \pi_0\mathcal{F}_k(\mathrm{X}) \otimes_{\pi_0(k)} \left(\pi_0\Phi^{\mathbf{Z}/p}(k) \times_{\pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)[1/t]} \pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}}\right).$$

The fiber product on the right-hand side does not have any denominators in t, and we will see this explicitly in the examples below.

Example 4.4.5. Let us explicate the preceding remark in two examples.

- Suppose $k = \mathbf{Z}[u^{\pm 1}]$ with u in degree 2. Then $\pi_0\mathcal{F}(\mathrm{B}\mathbf{Z}/p) \cong \mathbf{Z}[\![t]\!]/pt$, and the transfer ideal is simply generated by t. Therefore, $\pi_0\mathcal{F}(\mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}} \cong \mathbf{F}_p[\![t]\!]$. If X is a finite space with even cells, then the map of Remark 4.4.3 can be viewed as an (ungraded) map

$$\mathrm{H}^*(\mathrm{X}; \mathbf{Z}) \xrightarrow{\varphi_{\mathrm{X}}^{\mathrm{tr}}} \mathrm{H}^*(\mathrm{X}; \mathbf{F}_p[\![t]\!]) \to \mathrm{H}^*(\mathrm{X}; \mathbf{F}_p(\!(t)\!)).$$

[1] Unfortunately, this t is common practice in homotopy theory; but it conflicts with the t which is the coordinate of the formal (punctured) disk used to define the affine Grassmannian. We will use the same symbol t to denote both, and the distinction should be clear from context.

The decompleted Frobenius is given by an (ungraded) map

$$\varphi_{\mathrm{X}} : \mathrm{H}^*(\mathrm{X};\mathbf{Z}) \to \mathrm{H}^*(\mathrm{X};\mathbf{F}_p[t^{\pm 1}]).$$

Explicitly, these maps are given on a class $\alpha \in \mathrm{H}^*(\mathrm{X};\mathbf{Z})$ by the formula

$$\alpha \mapsto \sum_{i \geq 0} (-1)^i \mathrm{P}^i(\alpha) t^{(p-1)i}.$$

Here, P^i is the ith Steenrod operation. That is to say, φ_{X} encodes the action of the Steenrod operations on $\mathrm{H}^*(\mathrm{X};\mathbf{Z})$.[2] As expected by Remark 4.4.4, there are no denominators in t in the above formula. For instance, if $\mathrm{X} = \mathbf{C}\mathrm{P}^n$ for any finite n, this map sends $x \in \mathrm{H}^2(\mathbf{C}\mathrm{P}^n;\mathbf{Z})$ to $x - t^{p-1}x^p$.

- Suppose $k = \mathrm{KU}$. Then $\pi_0\mathcal{F}(\mathrm{B}\mathbf{Z}/p) \cong \mathbf{Z}[\![t]\!]/((1+t)^p - 1)$, and the transfer ideal is simply generated by t. Therefore,

$$\pi_0\mathcal{F}(\mathrm{B}\mathbf{Z}/p)/\mathrm{I}_{\mathrm{tr}} \cong \mathbf{Z}[\![t]\!]/\tfrac{(1+t)^p-1}{t} \cong \mathbf{Z}[\zeta_p]^\wedge_t.$$

Here, ζ_p is a primitive pth root of unity and $t = \zeta_p - 1$. Note that since t^{p-1} is a unit multiple of p in $\mathbf{Z}_p[\zeta_p]$, the t-completion above is equivalent to p-completion. The ring $\mathbf{Z}_p[\zeta_p]$ is flat over $\pi_0(k)^\wedge_p = \mathbf{Z}_p$, and so the composite of Remark 4.4.3 can be viewed as a ring map

$$\mathrm{KU}^0(\mathrm{X}) \xrightarrow{\varphi^{\mathrm{tr}}_{\mathrm{X}}} \mathrm{KU}^0(\mathrm{X})[\zeta_p]^\wedge_p \to \mathrm{KU}^0(\mathrm{X})[\zeta_p]^\wedge_p[1/p] \tag{4.4.2}$$

The geometric fixed points $\Phi^{\mathbf{Z}/p}\mathrm{KU}$, on the other hand, has homotopy groups given by

$$\pi_*\Phi^{\mathbf{Z}/p}\mathrm{KU} \cong \mathbf{Z}[q^{\pm 1}, \beta^{\pm 1}][\tfrac{1}{(q-1)\cdots(q^{p-1}-1)}]/(q^p - 1) \cong \mathbf{Z}[\zeta_p, \beta^{\pm 1}][1/p];$$

the final isomorphism comes from noticing that $(\zeta_p - 1)\cdots(\zeta_p^{p-1} - 1)$ is $(-1)^{p-1}p$. The decompleted Frobenius is given by a ring map

$$\varphi_{\mathrm{X}} : \mathrm{KU}^0(\mathrm{X}) \to \mathrm{KU}^0(\mathrm{X})[\zeta_p][1/p].$$

Note that this map is, indeed, a de-p-adic completion of (4.4.2). Both $\varphi^{\mathrm{tr}}_{\mathrm{X}}$ and φ_{X} send a vector bundle V to the pth Adams operation $\psi^p(\mathrm{V}) \in \mathrm{KU}^0(\mathrm{X})$, viewed as a subalgebra of $\mathrm{KU}^0(\mathrm{X})[\zeta_p]^\wedge_p$ and of $\mathrm{KU}^0(\mathrm{X})[\zeta_p][1/p]$. As expected by Remark 4.4.4, there are no denominators in $t = \zeta_p - 1$ in this formula.

In order to understand the interaction between these power operations and Corollary 4.3.17, we will need to port Construction 4.4.2 to the setting of genuine equivariant (co)homology. Namely, we need a decompletion of the map

$$\varphi_{\mathrm{BS}^1} : \mathcal{F}_k(\mathrm{BS}^1) \to \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathrm{BS}^1) \simeq \lim_n \mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathbf{C}\mathrm{P}^n).$$

[2]This is perhaps bad terminology, because the Steenrod algebra does give an endomorphism of integral cohomology. Here, however, we are viewing the Steenrod algebra as acting on a class α in integral cohomology through its mod p reduction $\overline{\alpha}$. Our φ_{X} will only see the action of P^i on $\overline{\alpha}$, and not the operations $\beta\mathrm{P}^i$ (when $p = 2$, this is Sq^{2i+1}). In fact, it turns out that the decompleted Frobenius $\varphi : \mathbf{Z} \to \Phi^{\mathbf{Z}/p}\mathbf{Z}$ factors as an $\mathbf{E}_\infty$-map through the reduction map $\mathbf{Z} \to \mathbf{F}_p$ (so that $\varphi_{\mathrm{X}}(\alpha)$ depends only on $\overline{\alpha}$ in a coherently multiplicative way), but proving this is out of the scope of the present article.

Let us mention that only tracking $\mathrm{P}^i(\overline{\alpha})$ definitely loses some information about the entire Steenrod algebra action. First, since $\overline{\alpha}$ came from the integral class α, its Bockstein $\beta(\overline{\alpha})$ vanishes. It is, however, possible that $\beta\mathrm{P}^i(\overline{\alpha})$ be nonzero despite $\overline{\alpha}$ lifting to integral cohomology. For instance, if we identify $\mathrm{H}^*(\mathbf{R}\mathrm{P}^4 \times \mathbf{R}\mathrm{P}^4;\mathbf{F}_2) = \mathbf{F}_2[x,y]/(x^5, y^5)$, then the class $\overline{\alpha} = xy(x+y)$ lifts to integral cohomology, but $\mathrm{Sq}^3(\overline{\alpha}) = x^2y^2(x^2+y^2) \neq 0$.

First, observe that this map factors through an $\mathbf{E}_\infty$-map

$$\varphi'_{\mathrm{BS}^1} : \mathcal{F}_k(\mathrm{BS}^1) \to \Phi^{\mathbf{Z}/p}k \otimes_k \mathcal{F}_k(\mathrm{BS}^1) \simeq \Phi^{\mathbf{Z}/p}k \otimes_k \lim_n \mathcal{F}_k(\mathbf{C}\mathrm{P}^n);$$

the map from the target to $\mathcal{F}_{\Phi^{\mathbf{Z}/p}k}(\mathrm{BS}^1)$ generally induces a strict inclusion on homotopy[3]. Note that φ'_{BS^1} can be viewed as a homomorphism

$$\hat{\mathbf{G}} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \Phi^{\mathbf{Z}/p}k \to \hat{\mathbf{G}},$$

where $\hat{\mathbf{G}} = \mathrm{Spf}\, \mathcal{F}_k(\mathrm{BS}^1)$.

We will now specialize to the case when k is 2-periodic integral cohomology, complex K-theory, or elliptic cohomology, and let $\mathbf{G}$ denote $\mathbf{G}_a$, $\mathbf{G}_m$, or the spectral elliptic curve E over k (respectively). The choice of $\mathbf{G}$ equips k with a lift to the ∞-category of normed rings in genuine $\mathbf{Z}/p$-spectra. As usual, let $\mathbf{H}$ denote the underlying group scheme over $\pi_0(k)$. (We will also use the notation of Definition 4.3.5 below, in particular, if $\mathcal{G}$ is a group scheme over $\pi_0(k)$, we will use the symbol $\mathcal{G}_{\mathbf{H}}$ to denote the group scheme constructed in Definition 4.3.5.) Our desired decompletion will then be given by a particular homomorphism

$$\varphi : \mathbf{G} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \Phi^{\mathbf{Z}/p}k \to \mathbf{G}. \tag{4.4.3}$$

To describe it, we need to give a moduli-theoretic interpretation of $\Phi^{\mathbf{Z}/p}k$. Let $\mathbf{G}[p]$ denote the p-torsion subgroup of $\mathbf{G}$, so that $\mathbf{G}[p] = \mathrm{Hom}(\mathbf{Z}/p, \mathbf{G})$.

There is a natural action of $\mathbf{F}_p^\times$ on $\mathbf{G}[p]$ given by sending $i \in \mathbf{F}_p^\times$ to the multiplication-by-i map $[i]$. Let $\mathrm{U} \subseteq \mathbf{G}[p]$ denote the open subscheme given by the complement of the closed subscheme

$$\bigcup_{i \in \mathbf{F}_p^\times} \ker(\mathbf{H}[p] \xrightarrow{[i]} \mathbf{H}[p]) \subseteq \mathbf{H}[p].$$

The following is a straightforward consequence of [HM1, Proposition 2.25].

Lemma 4.4.6. *The spectral scheme* $\mathrm{Spec}\, \Phi^{\mathbf{Z}/p}k$ *is isomorphic to* U *over* k.

The spectral scheme $\mathrm{U} \subseteq \mathbf{G}[p]$ is specified by its underlying (classical) scheme $\mathrm{U}_0 \subseteq \mathbf{H}[p]$ over $\pi_0(k)$. If Y is a $\pi_0(k)$-scheme, a map $\mathrm{Y} \to \mathrm{U}_0$ is equivalent to the data of a homomorphism $f : \mathbf{Z}/p \to \mathbf{G}_\mathrm{Y} = \mathbf{G} \times_{\mathrm{Spec}\, \pi_0(k)} \mathrm{Y}$ such that $f(i)$ is not the identity section for $i \in \mathbf{Z}/p - \{0\}$. This implies that f exhibits $\mathbf{Z}/p$ as a closed subgroup scheme of $\mathbf{G}_\mathrm{Y}$ which is isomorphic to the Cartier divisor $\sum_{j \in \mathbf{F}_p} f(j)$.

Construction 4.4.7. Over U_0, there is a universal isogeny $q_0 : \mathbf{H}_{\mathrm{U}_0} \to \mathbf{H}_{\mathrm{U}_0}$ given by quotienting by the subgroup scheme $\mathbf{Z}/p \cong \sum_{j \in \mathbf{F}_p} f(j)$. This isogeny defines an *étale* morphism $\mathcal{O}_{\mathbf{H}_{\mathrm{U}_0}} \to \mathcal{O}_{\mathbf{H}_{\mathrm{U}_0}}$; so [Lur4, Theorem 7.5.0.6] implies that the isogeny q_0 lifts to a map $q : \mathbf{G}_\mathrm{U} \to \mathbf{G}_\mathrm{U}$ over $\mathrm{Spec}\, \Phi^{\mathbf{Z}/p}k$. (In general, q_0 is to be understood as an analogue for $\mathbf{H}$ of the Artin-Schreier map on $\mathbf{G}_a$.) The map (4.4.3) is then given by the composite

$$\mathbf{G}_\mathrm{U} \xrightarrow{q} \mathbf{G}_\mathrm{U} \simeq \mathbf{G} \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \Phi^{\mathbf{Z}/p}k \xrightarrow{\mathrm{pr}} \mathbf{G}.$$

We will denote its underlying map by

$$\varphi_0 : \mathbf{H} \times_{\mathrm{Spec}\, \pi_0(k)} \mathrm{Spec}\, \pi_0(\Phi^{\mathbf{Z}/p}k) \to \mathbf{H}.$$

[3]For instance, take $k = \mathrm{KU}$. Then the map φ_{BS^1} is given on homotopy by the map $\mathbf{Z}[\![t]\!] \to \mathbf{Z}[\zeta_p][1/p][\![t]\!]$ which sends $t \mapsto (1+t)^p - 1$. This factors through a map $\mathbf{Z}[\![t]\!] \to \mathbf{Z}[\zeta_p][\![t]\!][1/p]$; this is the effect of the map φ'_{BS^1} on homotopy. Note that there is a strict inclusion $\mathbf{Z}[\zeta_p][\![t]\!][1/p] \subseteq \mathbf{Z}[\zeta_p][1/p][\![t]\!]$.

Example 4.4.8. Let us explicate Construction 4.4.7 in two examples.

a. Let $k = \mathbf{Z}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$. Then $\mathrm{U}_0 = \operatorname{Spec} \mathbf{F}_p[t^{\pm 1}]$, and the isogeny $q : \mathbf{H}_{\mathrm{U}_0} \to \mathbf{H}_{\mathrm{U}_0}$ is given by the Artin-Schreier map

$$x \mapsto x - t^{p-1}x^p.$$

b. Let $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$ with coordinate y. Then $\mathrm{U}_0 = \operatorname{Spec} \mathbf{Z}[\zeta_p][1/p]$, and $q : \mathbf{H}_{\mathrm{U}_0} \to \mathbf{H}_{\mathrm{U}_0}$ is given by the map

$$y \mapsto 1 + \prod_{j \in \mathbf{F}_p} (y - \zeta_p^j) = y^p.$$

Remark 4.4.9. Let us mention for the sake of completeness that one can interpolate between the two cases in Example 4.4.8, using the group scheme $\mathbf{G}$ over connective complex K-theory ku studied in [Dev3]. (Using this, the results discussed below can be extended to the case $k = \mathrm{ku}$, too, but we will not address this here.) Let $\mathbf{G}_\beta := \mathbf{H}$ denote its underlying group scheme. Explicitly, $\mathbf{G}_\beta$ is the group scheme over $\mathbf{Z}[\beta]$ given by $\operatorname{Spec} \mathbf{Z}[\beta, v^{\pm 1}][\frac{v-1}{\beta}]$, where the group law is determined by $v \mapsto v \otimes v$. In an abuse of notation, we will also write $\mathbf{G}_{f(\beta)}$ for an element $f(\beta) \in \mathbf{Z}[\beta]$ to denote the group scheme given by $\operatorname{Spec} \mathbf{Z}[\beta, v^{\pm 1}][\frac{v-1}{f(\beta)}]$; hopefully this will not cause any confusion to the reader. We will (perhaps unexpectedly) define $t^{-1} := \frac{v-1}{\beta}$, and also define the scheme

$$\mathrm{U}_0 = \operatorname{Spec} \mathbf{Z}[\beta, v^{\pm 1}][\tfrac{v-1}{\beta}, \tfrac{\beta^{p-1}}{(v-1)\cdots(v^{p-1}-1)}]/\tfrac{v^p-1}{\beta}.$$

Note that $v = \zeta_p$ is a primitive pth root of unity, and $\beta = \frac{\zeta_p - 1}{t^{-1}}$. The scheme U_0 is rather remarkable: its fiber over the locus where β is a unit is precisely $\operatorname{Spec} \mathbf{Z}[\zeta_p, \beta^{\pm 1}][1/p]$, while its fiber over $\beta = 0$ is given by $\operatorname{Spec} \mathbf{F}_p[t^{\pm 1}]$. (In homotopy theory, U_0 arises as $\operatorname{Spec} \pi_*(\Phi^{\mathbf{Z}/p}\mathrm{ku})$, where ku is connective complex K-theory.)

Let y denote the invertible coordinate on $\mathbf{G}_{\beta,\mathrm{U}_0}$, and let $x = \frac{y-1}{\beta}$. Then the map $q : \mathbf{G}_{\beta,\mathrm{U}_0} \to \mathbf{G}_{p\beta,\mathrm{U}_0}$ is given by the map $y \mapsto y^p$ and $\beta \mapsto p\beta$, so that it sends

$$q : x = \tfrac{y-1}{\beta} \mapsto \tfrac{y^p-1}{p\beta} = \tfrac{(1+\beta x)^p - 1}{p\beta}.$$

We claim that, as a morphism over $\operatorname{Spec} \mathbf{Z}[\beta]$, this map interpolates between the isogenies of Cases a and b in Example 4.4.8. First, it is obvious that when β is a unit, we simply recover Case b. Next, let us consider the fiber over $\beta = 0$. Recall that $\beta = (\zeta_p - 1)t$, so the binomial theorem gives

$$q(x) = \tfrac{1}{p} \sum_{i=1}^{p} \binom{p}{i} \beta^{i-1} x^i = \sum_{i=1}^{p} \tfrac{(\zeta_p-1)^{i-1}}{p} \binom{p}{i} t^{i-1} x^i.$$

Almost all terms vanish modulo β, except for the terms $i = 1, p$; one is left with

$$q(x) \equiv x + \tfrac{(\zeta_p-1)^{p-1}}{p} t^{p-1} x^p = x + \tfrac{t^{p-1}x^p}{[1]_{\zeta_p} \cdots [p-1]_{\zeta_p}} = x - t^{p-1}x^p \pmod{\beta},$$

as desired. (Here, $[j]_q = \frac{q^j-1}{q-1}$ is the q-integer corresponding to $j \in \mathbf{Z}$; we are using the fact that $[1]_{\zeta_p} \cdots [p-1]_{\zeta_p} \equiv -1 \pmod{\zeta_p - 1}$, which amounts to the fact that $(p-1)! = -1 \in \mathbf{F}_p$.) In general, ku gives a degeneration of power operations/the pth Adams operation on KU to power operations/the Steenrod algebra action on ordinary cohomology; this goes back (albeit not in the form presented above) to [Ati2, Proposition 6.4 and Theorem 6.5].

For any compact torus T_c, we obtain a map

$$\varphi_{\mathrm{T}} : \mathcal{M}_{\mathrm{T}} \times_{\operatorname{Spec} k} \operatorname{Spec} \Phi^{\mathbf{Z}/p} k \to \mathcal{M}_{\mathrm{T}},$$

whose underlying map on classical $\pi_0(k)$-schemes will be denoted by $\varphi_{\mathrm{T},0}$. If X is any (ind-)finite T_c-space X, we then obtain maps

$$\mathcal{F}_{\mathrm{T}}(\mathrm{X}) \to \varphi_{\mathrm{T},*}\varphi_{\mathrm{T}}^*\mathcal{F}_{\mathrm{T}}(\mathrm{X}),\ \mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee \to \varphi_{\mathrm{T},*}\varphi_{\mathrm{T}}^*(\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee).$$

We will denote these maps by $\varphi_{\mathrm{T},\mathrm{X}}$, and call them the T_c-*equivariant decompleted Frobenius.* Note that $\varphi_{\mathrm{T},\mathrm{X}}$ on $\mathcal{F}_{\mathrm{T}}(\mathrm{X})$ is a map of $\mathbf{E}_\infty$-algebras in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$, and similarly $\varphi_{\mathrm{T},\mathrm{X}}$ on $\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee$ is a map of $\mathbf{E}_\infty$-coalgebras in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T}})$. The map $\varphi_{\mathrm{T},\mathrm{X}}$ in fact comes from a functor

$$\varphi_{\mathrm{T},\mathrm{X}} : \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; k) \to \mathrm{Shv}_{\mathrm{T}}(\mathrm{X}; \Phi^{\mathbf{Z}/p} k),$$

which in turn restricts to a functor

$$\varphi_{\mathrm{T},\mathrm{X}} : \mathrm{Loc}_{\mathrm{T}}(\mathrm{X}; k) \to \mathrm{Loc}_{\mathrm{T}}(\mathrm{X}; \Phi^{\mathbf{Z}/p} k).$$

Remark 4.4.10. It is easy to see that if $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W})$ denotes the nil-Hecke algebra from Definition 3.6.20 associated to a root system with torus T and Weyl group W, then the map $\varphi_{\mathrm{T},0}$ induces a map $\mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W}) \to \mathcal{H}(\mathbf{H}, \mathrm{T}, \mathrm{W}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p} k)$. This map is very interesting, but we will postpone a detailed study of its combinatorial implications to a future article. When $\mathbf{H} = \mathbf{G}_a$, for instance, this map describes the total Steenrod operation on the nil-Hecke algebra; similar ideas are explored in [Kit, BC].

Remark 4.4.11. Let us clarify Warning 4.4.1: if X is a(n ind-)finite space, then the Frobenius on k induces a map $\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee \to \varphi_{\mathrm{T},*}\varphi_{\mathrm{T}}^*(\mathcal{F}_{\mathrm{T}}(\mathrm{X})^\vee)$. When X is the affine Grassmannian, this in turn induces a map $\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee \to \varphi_{\mathrm{G},*}\varphi_{\mathrm{G}}^*(\mathcal{F}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}})^\vee)$. Working Borel-equivariantly, this may be identified with a map $k[\Omega\mathrm{G}]^{h\mathrm{G}} \to k[\Omega\mathrm{G}]^{h\mathrm{G}} \otimes_k k^{t\mathbf{Z}/p}$. This, however, is *not* Lonergan's map from [Lon4].

Rather, the latter can be constructed as follows. Recall that $k[\Omega\mathrm{G}]^{h\mathrm{G}}$ is equivalent to the $\mathbf{E}_2$-Hochschild cohomology $\mathrm{HC}_{\mathbf{E}_2}(k[\Omega\mathrm{G}]/k)$, so that the $\mathbf{E}_2^{\mathrm{fr}}$-algebra structure on $k[\Omega\mathrm{G}]^{h\mathrm{G}}$ upgrades to an S^1-equivariant $\mathbf{E}_3$-algebra structure (more precisely, an $\mathbf{E}_2^{\mathrm{fr}} \otimes \mathbf{E}_1$-algebra structure). If A is *any* $\mathbf{E}_2^{\mathrm{fr}} \otimes \mathbf{E}_1$-$k$-algebra, then the homotopy fixed points $\mathrm{A}^{h\mathrm{S}^1}$ (and similarly for $\mathrm{A}^{h\mathbf{Z}/p}$) will only be an $\mathbf{E}_1$-$k^{h\mathrm{S}^1}$-algebra, such that base-changing along $k^{h\mathrm{S}^1} \to k$ (i.e., killing the equivariant parameter $\hbar$) produces the $\mathbf{E}_3$-algebra A. If A and k are even, then $\pi_*(\mathrm{A}^{h\mathrm{S}^1})$ is therefore a deformation quantization of the graded Poisson algebra $\pi_*(\mathrm{A})$ along the map $\pi_*(k^{h\mathrm{S}^1}) \to \pi_*(k)$. (When $k = \mathbf{C}$, this was studied in [BBB$^+$, But1, But2].)

However, $\mathbf{E}_2^{\mathrm{fr}} \otimes \mathbf{E}_1$-$k$-algebras A admit much more structure: namely, there is a Frobenius $\varphi : \mathrm{A} \to \mathrm{A}^{t\mathbf{Z}/p}$ constructed as the composite

$$\mathrm{A} \to \mathrm{THH}(\mathrm{A}) \xrightarrow{\varphi} \mathrm{THH}(\mathrm{A})^{t\mathbf{Z}/p} \to \mathrm{A}^{t\mathbf{Z}/p}.$$

Here, the map $\varphi : \mathrm{THH}(\mathrm{A}) \to \mathrm{THH}(\mathrm{A})^{t\mathbf{Z}/p}$ is the cyclotomic Frobenius on the topological Hochschild homology of A, and the final map $\mathrm{THH}(\mathrm{A})^{t\mathbf{Z}/p} \to \mathrm{A}^{t\mathbf{Z}/p}$ is induced by the S^1-equivariant augmentation $\mathrm{THH}(\mathrm{A}) \to \mathrm{A}$ from [DHL$^+$]. The results therein also show that the map $\mathrm{A} \to \mathrm{A}^{t\mathbf{Z}/p}$ is central, in that it exhibits $\mathrm{A}^{t\mathbf{Z}/p}$ as an $\mathbf{E}_1$-A-algebra. (Note that $\varphi : \mathrm{A} \to \mathrm{A}^{t\mathbf{Z}/p}$ is linear for the Tate-valued Frobenius $\varphi : k \to k^{t\mathbf{Z}/p}$, and also that if A was in fact an $\mathbf{E}_\infty$-k-algebra, then the map $\varphi : \mathrm{A} \to \mathrm{A}^{t\mathbf{Z}/p}$ is just the Tate-valued Frobenius.) When $k = \mathbf{Z}$ and $\pi_*(\mathrm{A})$ is p-torsion-free, for instance, the map $\varphi : \pi_*(\mathrm{A}) \to \pi_*(\mathrm{A}^{t\mathbf{Z}/p})$ exhibits

$\pi_*(\mathrm{A}^{h\mathrm{S}^1}/p)$ as a *Frobenius-constant quantization* of $\pi_*(\mathrm{A}/p)$ (in the sense of [BK1]); for other $\mathbf{E}_\infty$-rings k, one obtains an interesting generalization of this notion which we will explain in future work.

For instance, when applied to $\mathrm{A} = \mathbf{Z}[\Omega\mathrm{G}]^{h\mathrm{G}}$, the resulting Frobenius-constant quantization structure on $\pi_*(\mathrm{A}^{h\mathrm{S}^1}/p) = \mathrm{C}_*^{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G};\mathbf{F}_p)$ recovers Lonergan's construction from [Lon4]. Similarly, when $\mathrm{G} = \mathbf{G}_m$ and $\mathrm{A} = k[\Omega\mathrm{G}]^{h\mathrm{G}} = \mathrm{HC}_{\mathbf{E}_2}((\mathbf{G}_m)_k/k)$ (where k is an arbitrary even $\mathbf{E}_\infty$-ring), the Frobenius $\varphi : \pi_*(\mathrm{A})\otimes_{\pi_*(k)}\pi_*(k^{t\mathbf{Z}/p}) \to \pi_*(\mathrm{A}^{t\mathbf{Z}/p})$ is exactly the "$\langle p\rangle_\mathrm{F}$-curvature map" from Remark 7.4.6. If k is an ordinary commutative ring, then this generalizes as follows. If R is a smooth k-algebra and $\mathrm{A} = \mathrm{HC}_{\mathbf{E}_2}(\mathrm{R}/k)$, then A is even, and furthermore there are isomorphisms

$$\pi_*(\mathrm{A}) \cong \mathrm{Sym}_\mathrm{R}(\mathrm{T}_{\mathrm{R}/k}(-2)) \cong \mathcal{O}_{\mathrm{T}^*(2)(\mathrm{Spec}(\mathrm{R})/\mathrm{Spec}(k))},$$
$$\pi_*(\mathrm{A}^{h\mathrm{S}^1}) \cong \mathcal{D}^\hbar_{\mathrm{R}/k} = k[\hbar]\{f,\xi\}/(\xi f - f\xi = \hbar\xi(f)),$$

where the object appearing on the final line is the Rees construction (with respect to the variable $\hbar$) of the algebra of k-linear differential operators on Spec(R) equipped with its order filtration. In this case, the Frobenius $\varphi : \pi_*(\mathrm{A}) \to \pi_*(\mathrm{A}^{t\mathbf{Z}/p})$ is the usual p-curvature map $\mathrm{Sym}_\mathrm{R}(\mathrm{T}_{\mathrm{R}/k}(-2)) \to \mathcal{D}_{\mathrm{R}/k}[\hbar^{\pm1}]/p$.

We will now study the functor $\varphi_{\mathrm{T},\mathrm{X}} : \mathrm{Shv}_\mathrm{T}(\mathrm{X};k) \to \mathrm{Shv}_\mathrm{T}(\mathrm{X};\Phi^{\mathbf{Z}/p}k)$ the case $\mathrm{X} = \mathrm{Gr}_\mathrm{G}$, where G is connected, almost simple, and simply-laced over $\mathbf{C}$. For notational simplicity, we will write $\mathrm{Shv}^{\mathrm{gr}}_\mathrm{T}(\mathrm{Gr}_\mathrm{G};\Phi^{\mathbf{Z}/p}k)$ to denote the tensor product $\mathrm{Shv}^{\mathrm{gr}}_\mathrm{T}(\mathrm{Gr}_\mathrm{G};k)\otimes_{\pi_0 k}\pi_0(\Phi^{\mathbf{Z}/p}k)$. The T_c-equivariant decompleted Frobenius on $\mathcal{F}_\mathrm{T}(\mathrm{Gr}_\mathrm{G})^\vee$ induces a functor

$$(\varphi_{\mathrm{T},\mathrm{Gr}_\mathrm{G}})_* : \mathrm{Shv}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G};k) \to \mathrm{Shv}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_\mathrm{G};\Phi^{\mathbf{Z}/p}k). \tag{4.4.4}$$

Moreover, the homomorphism $\varphi_{\check{\mathrm{T}},0}$ induces a map in the opposite direction on Cartier duals, and hence a morphism $\varphi_{\mathrm{T},0} : \mathrm{B}_\mathbf{H} \to \mathrm{B}_\mathbf{H}$, which can be viewed as a map

$$\varphi_{\mathrm{T},0} : \widetilde{\check{\mathrm{G}}}_\mathbf{H} \to \widetilde{\check{\mathrm{G}}}_\mathbf{H}. \tag{4.4.5}$$

Remark 4.4.12. In fact, it follows from Remark 4.4.4 that one can replace $\pi_0(\Phi^{\mathbf{Z}/p}k)$ above by the fiber product $\pi_0(\Phi^{\mathbf{Z}/p}k)\times_{\pi_0\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p)[1/t]}\pi_0(\mathcal{F}_k(\mathrm{B}\mathbf{Z}/p))/\mathrm{I}_{\mathrm{tr}}$, which has the effect of working in a "t-lattice" inside $\pi_0(\Phi^{\mathbf{Z}/p}k)$. For simplicity, we will ignore this point below, and just work with $\pi_0(\Phi^{\mathbf{Z}/p}k)$.

The following says that the map (4.4.5) is precisely the effect of the T_c-equivariant decompleted Frobenius under Langlands duality.

Theorem 4.4.13. *Suppose* G *is a reductive group whose derived subgroup is simply-laced. Then the functor* $(\varphi_{\check{\mathrm{T}},\mathrm{Gr}_\mathrm{G}})_*$ *of* (4.4.4) *identifies with the functor given by pullback along the map* (4.4.5)*. That is, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{F}\otimes_{\pi_0(k)}\mathrm{Loc}_\mathrm{T}(\mathrm{Gr}_\mathrm{G};k) \arrow[r, "(\varphi_{\mathrm{T},\mathrm{Gr}_\mathrm{G}})_*"] \arrow[d, "\sim"'] & \mathrm{F}\otimes_{\pi_0(k)}\mathrm{Loc}_\mathrm{T}(\mathrm{Gr}_\mathrm{G};\Phi^{\mathbf{Z}/p}k) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}_\mathbf{H}/\check{\mathrm{G}}) \arrow[r, "\varphi^*_{\check{\mathrm{T}},0}"'] & \mathrm{QCoh}(\widetilde{\check{\mathrm{G}}}^{\mathrm{reg}}_\mathbf{H}/\check{\mathrm{G}})\otimes_{\pi_0(k)}\pi_0(\Phi^{\mathbf{Z}/p}k).
\end{tikzcd}$$

If G *is of type* A *or type* D *(in the latter case, assume that 2 is a unit in* $\pi_0(k)$*), then under the equivalence of Corollary 4.3.17 (which continues to hold true in the case* $k = \mathbf{Z}[u^{\pm1}]$*, at least* *upon inverting enough primes), the functor $(\varphi_{\check{T},\mathrm{Gr_G}})_*$ of (4.4.4) identifies with the functor given by pullback along the map (4.4.5). That is, the following diagram commutes:*

$$\begin{tikzcd}
F \otimes_{\pi_0(k)} \mathrm{Shv}^{\mathrm{min,gr}}_{T}(\mathrm{Gr_G}; k) \arrow[r, "(\varphi_{T,\mathrm{Gr_G}})_*"] \arrow[d, "\sim"'] & F \otimes_{\pi_0(k)} \mathrm{Shv}^{\mathrm{min,gr}}_{T}(\mathrm{Gr_G}; \Phi^{\mathbf{Z}/p} k) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{aff}}_{\mathbf{H}}/\check{G}) \arrow[r, "\varphi^*_{\check{T},0}"'] & \mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{aff}}_{\mathbf{H}}/\check{G}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p} k).
\end{tikzcd}$$

Proof. The argument is essentially that of Proposition 3.8.25, so we only give a sketch. Let us begin by observing that if $\kappa : \mathcal{M}_{\check{T},0} \to \widetilde{\check{G}}_{\mathbf{H}}/\check{G}$ denotes the Kostant section, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\check{T},0} \times_{\mathrm{Spec}\,\pi_0(k)} \mathrm{Spec}\,\pi_0(\Phi^{\mathbf{Z}/p} k) \arrow[r, "\varphi_{\check{T},0}"] \arrow[d, "\kappa"'] & \mathcal{M}_{\check{T},0} \arrow[d, "\kappa"] \\
\widetilde{\check{G}}_{\mathbf{H}}/\check{G} \times_{\mathrm{Spec}\,\pi_0(k)} \mathrm{Spec}\,\pi_0(\Phi^{\mathbf{Z}/p} k) \arrow[r, "\varphi_{\check{T},0}"'] & \widetilde{\check{G}}_{\mathbf{H}}/\check{G}.
\end{tikzcd}$$

The proof of Corollary 4.3.14 shows that it suffices to prove that under the isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\check{T},0}}(\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr_G})^\vee) \cong \mathcal{M}_{\check{T},0} \times_{\widetilde{\check{G}}_{\mathbf{H}}/\check{G}} \mathcal{M}_{\check{T},0}, \tag{4.4.6}$$

the $\check{T}_c$-equivariant decompleted Frobenius on $\mathcal{F}_{\check{T}}(\mathrm{Gr_G})^\vee$ identifies with the effect of the map $\varphi_{\check{T},0}$ on the right-hand side. For brevity, we will phrase this condition as the "Frobenius-equivariance" of (4.4.6).

Let $\mathcal{M}^{\mathrm{gen}}_{\check{T},0} \subseteq \mathcal{M}_{\check{T},0}$ denote the complement of $\bigcup_\alpha \mathcal{M}_{\check{T}_\alpha,0}$ as α ranges over the roots of $\check{G}$, and $\check{T}_\alpha$ denotes the kernel of the map $\alpha : \check{T} \to \mathbf{G}_m$. Since both sides of (4.4.6) are flat over $\mathcal{M}_{\check{T},0}$, their sheaves of functions inject into the corresponding localizations along the map $\mathcal{M}^{\mathrm{gen}}_{\check{T},0} \subseteq \mathcal{M}_{\check{T},0}$. It therefore suffices to show that when restricted to $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, the isomorphism of (4.4.6) is Frobenius-equivariant.

By Lemma 3.2.11, there is an isomorphism

$$\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr_G})^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}} \cong \pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr_T})^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}} \cong \mathcal{O}_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}}[\mathbb{X}_*(T)].$$

Under this isomorphism, the $\check{T}_c$-equivariant decompleted Frobenius is given simply by the Frobenius on $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, and acts trivially on $\mathbb{X}_*(T)$. Similarly, there is an isomorphism

$$(\mathcal{M}_{\check{T},0} \times_{\widetilde{\check{G}}_{\mathbf{H}}/\check{G}} \mathcal{M}_{\check{T},0}) \times_{\mathcal{M}_{\check{T},0}} \mathcal{M}^{\mathrm{gen}}_{\check{T},0} \cong \mathcal{M}^{\mathrm{gen}}_{\check{T},0} \times \check{T}.$$

Under this isomorphism, the action of $\varphi_{\check{T},0}$ is given simply by the Frobenius on $\mathcal{M}^{\mathrm{gen}}_{\check{T},0}$, and acts trivially on $\check{T}$. It is clear that this matches with the Frobenius on $\pi_0 \mathcal{F}_{\check{T}}(\mathrm{Gr_G})^\vee|_{\mathcal{M}^{\mathrm{gen}}_{\check{T},0}}$, as desired. □

The entire discussion above can be adapted without much difficulty to the setting of G_c-equivariant sheaves, if G is almost simple and simply-laced, and has torsion-free fundamental group. Indeed, the analogue of Theorem 4.4.13 states that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{F} \otimes_{\pi_0(k)} \mathrm{Loc}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k) \arrow[r, "(\varphi_{\mathrm{G},\mathrm{Gr}_{\mathrm{G}}})_*"] \arrow[d, "\sim"'] & \mathrm{F} \otimes_{\pi_0(k)} \mathrm{Loc}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; \Phi^{\mathbf{Z}/p} k) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}) \arrow[r, "\varphi^*_{\check{\mathrm{G}},0}"'] & \mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p} k).
\end{tikzcd}$$

Similarly, if G is of type A or type D (in the latter case, assume that 2 is a unit in $\pi_0(k)$), then under the equivalence of Corollary 4.3.17 (which continues to hold true in the case $k = \mathbf{Z}[1/2, u^{\pm 1}]$), the following diagram commutes:

$$\begin{tikzcd}
\mathrm{F} \otimes_{\pi_0(k)} \mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \arrow[r, "(\varphi_{\mathrm{G},\mathrm{Gr}_{\mathrm{G}}})_*"] \arrow[d, "\sim"'] & \mathrm{F} \otimes_{\pi_0(k)} \mathrm{Shv}_{\mathrm{G}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; \Phi^{\mathbf{Z}/p} k) \arrow[d, "\sim"] \\
\mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}) \arrow[r, "\varphi^*_{\check{\mathrm{G}},0}"'] & \mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}) \otimes_{\pi_0(k)} \pi_0(\Phi^{\mathbf{Z}/p} k).
\end{tikzcd} \tag{4.4.7}$$

Under the equivalence of Conjecture 4.3.20, one expects the diagram (4.4.7) to commute for arbitrary simply-laced G.

Remark 4.4.14. Recall that there is a closed immersion $\mathrm{Spec}\, \pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee \hookrightarrow \check{\mathrm{G}} \times \mathcal{M}_{\mathrm{T},0}$. One can try to extend the action of the decompleted Frobenius to $\check{\mathrm{G}} \times \mathcal{M}_{\mathrm{T},0}$ itself, but such an extension will *not* be canonical (and seems to be essentially useless in studying $\check{\mathrm{G}}$).

Remark 4.4.15. Let λ be a dominant minuscule weight for $\check{\mathrm{G}}$, and let $\mathrm{G}/\mathrm{P}_\lambda$ denote the corresponding flag variety for G as in Table 3.1. The decompleted Frobenius acts on $\pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{G}/\mathrm{P}_\lambda)$, and does so compatibly with its action on $\pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee$, in the sense that the action of $\mathrm{Spec}\, \pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{Gr}_{\mathrm{G}})^\vee$ on $\pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{G}/\mathrm{P}_\lambda)$ is equivariant for the decompleted Frobenius. It would be interesting to understand, in some uniform manner, the action of the decompleted Frobenius on $\pi_0 \mathcal{F}_{\check{\mathrm{T}}}(\mathrm{G}/\mathrm{P}_\lambda)$. In the case of ordinary cohomology, this amounts to understanding Steenrod operations on $\mathrm{H}^*(\mathrm{G}/\mathrm{P}_\lambda; \mathbf{Z})$. This is already interesting in the case when G is of type A (i.e., in the case of Grassmannians), where it was studied, for instance, in [BS3, BS2, Lan1].

Let us now explicate Theorem 4.4.13 in some examples. Since the description in the case of elliptic cohomology is not much more explicit than the statement of Theorem 4.4.13 – that is, that the decompleted Frobenius on $\mathrm{Bun}_{\check{\mathrm{G}}}^{\mathrm{ss}}(\mathrm{E})$ is induced by the degree p étale isogeny $\mathrm{E} \to \mathrm{E}$ over $\mathrm{Spec}\, \pi_0(\Phi^{\mathbf{Z}/p} k)$ – we will mostly focus on the cases of ordinary cohomology and complex K-theory below for simplicity. We will also briefly discuss the example of "Tate K-theory", where one can also make the decompleted Frobenius explicit at the level of isomorphism classes of objects of $\mathrm{Bun}_{\check{\mathrm{G}}}^{\mathrm{ss}}(\mathrm{E})$.

Before proceeding, we warn the reader that our discussion above only shows that the decompleted Frobenius is canonically defined on the stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$, and not necessarily on a uniformization. For instance, when $\mathbf{H} = \mathbf{G}_a$, so that $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}} = \mathfrak{g}/\check{\mathrm{G}}$, we will often compute the decompleted Frobenius as a map on $\mathfrak{g}$; but the resulting formulas are only unique up to $\check{\mathrm{G}}$-conjugation.

Example 4.4.16. Let $k = \mathbf{Z}[u^{\pm 1}]$, $\mathbf{G} = \mathbf{G}_a$, and invert $\mathrm{N} \gg 0$ so that the equivalence of Corollary 4.3.14 continues to hold: that is, so that there is an equivalence $\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}}; k) \simeq$

$\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}^{\mathrm{aff}}/\check{\mathrm{G}})$. This can be proved by showing that the isomorphism of Theorem 3.6.3 over $\operatorname{Spec}\mathbf{Z}[1/\mathrm{N}]$ for some $\mathrm{N} \gg 0$. In fact, [YZ2] shows that one can take N to be the integer n_{G} from [YZ2, Remark 5.8].[4] Under the identification $\widetilde{\check{\mathrm{G}}}_{\mathbf{H}}/\check{\mathrm{G}} \cong \mathfrak{b}/\check{\mathrm{B}}$, the map (4.4.5) is given (for $p \nmid \mathrm{N}$) by the map

$$\varphi_{\mathrm{T},0} : (\mathfrak{b} \times_{\operatorname{Spec}\mathbf{Z}[1/\mathrm{N}]} \operatorname{Spec}\mathbf{F}_p[t^{\pm 1}])/\check{\mathrm{B}} \to \mathfrak{b}/\check{\mathrm{B}}$$

which is the $\check{\mathrm{B}}$-quotient of the map

$$\mathfrak{b} \times_{\operatorname{Spec}\mathbf{Z}[1/\mathrm{N}]} \operatorname{Spec}\mathbf{F}_p[t^{\pm 1}] \to \mathfrak{b},\ (x,t) \mapsto x - t^{p-1}x^{[p]}.$$

Here, $x^{[p]}$ denotes the restricted Lie operation on $\mathfrak{b}$. It follows from Theorem 4.4.13 that this map implements the action of the decompleted Frobenius/Steenrod operations on $\mathrm{Shv}_{\mathrm{T}}^{\mathrm{min,gr}}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Z}[u^{\pm 1}])$ (upon inverting $\mathrm{N} \gg 0$).

Similarly, under the identification $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}} \cong \mathfrak{g}/\check{\mathrm{G}}$, the analogue of the map (4.4.5) is given (for $p \nmid \mathrm{N}$) by the map

$$\varphi_{\mathrm{G},0} : (\mathfrak{g} \times_{\operatorname{Spec}\mathbf{Z}[1/\mathrm{N}]} \operatorname{Spec}\mathbf{F}_p[t^{\pm 1}])/\check{\mathrm{G}} \to \mathfrak{g}/\check{\mathrm{G}}$$

which is the $\check{\mathrm{G}}$-quotient of the map

$$\mathfrak{g} \times_{\operatorname{Spec}\mathbf{Z}[1/\mathrm{N}]} \operatorname{Spec}\mathbf{F}_p[t^{\pm 1}] \to \mathfrak{g},\ (x,t) \mapsto x - t^{p-1}x^{[p]}. \tag{4.4.8}$$

Again, this map implements the action of the decompleted Frobenius/Steenrod operations on $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Z}[1/\mathrm{N},u^{\pm 1}])$ under the equivalence between $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}};\mathbf{Z}[1/\mathrm{N},u^{\pm 1}])$ and $\mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}/\check{\mathrm{G}})$.

For instance, suppose $\mathrm{G} = \mathrm{SL}_2$, and assume $p > 2$. When restricted to the Kostant slice $f + \mathfrak{g}^e = \{\left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right)\} \subseteq \mathfrak{g} = \mathfrak{sl}_2$, the map $\varphi_{\check{\mathrm{G}},0}$ sends

$$(\left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right), t) \mapsto \begin{pmatrix} 0 & x - t^{p-1}x^{(p+1)/2} \\ 1 - t^{p-1}x^{(p-1)/2} & 0 \end{pmatrix}.$$

This is conjugate to the matrix $\begin{pmatrix} 0 & x(1-t^{p-1}x^{(p-1)/2})^2 \\ 1 & 0 \end{pmatrix}$, so we find that $\varphi_{\mathrm{G},0}$ is given in coordinates by the map

$$\varphi_{\check{\mathrm{G}},0} : x \mapsto x - 2t^{p-1}x^{(p+1)/2} + t^{2(p-1)}x^p = \prod_{j \in \mathbf{F}_p} (x - j^2t^2)$$

on $f + \mathfrak{g}^e$. Under the isomorphism $f + \mathfrak{g}^e \cong \operatorname{Spec}\mathrm{H}^*_{\mathrm{SU}(2)}(*;\mathbf{Z})$, the coordinate x identifies with the first Pontryagin class p_1; and $\varphi_{\mathrm{G},0}(x)$ is exactly the total Steenrod operation on this class, as expected. Similarly, if $\mathrm{G} = \mathrm{PGL}_2$, one could conjugate the above description of $\varphi_{\mathrm{G},0}$ to find that the decompleted Frobenius acts on a binary quadratic form $q(x,y) \in \mathfrak{pgl}_2 = \mathrm{Sym}^2(\mathbf{A}^2)$ by

$$\varphi_{\mathrm{SL}_2,0} : q(x,y) \mapsto (1 - t^{p-1}\det(q)^{(p-1)/2})q(x,y),$$

where $\det(q)$ is the discriminant of q.

Example 4.4.16 has the following algebraic consequence. This result is not new, and can be found in the literature as [Jan1, Section 4.1]; it also holds in the non-simply-laced case. (The proof below is a thinly veiled topological analogue of Jantzen's argument.)

Proposition 4.4.17. *The map $x \mapsto x^{[p]}$ is zero on the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$ if p is at least the Coxeter number of $\check{\mathrm{G}}$.*

[4] In the setup at hand, one can even take $\mathrm{N} = 1$.

Proof. The map $\varphi_{\mathrm{G},0}$ from (4.4.8) is given by taking affine closures of the map

$$\mathfrak{g}^{\mathrm{reg}} \times_{\mathrm{Spec}\,\mathbf{Z}[1/\mathrm{N}]} \mathrm{Spec}\,\mathbf{F}_p[t^{\pm 1}] \to \mathfrak{g}^{\mathrm{reg}},\ (x,t) \mapsto x - t^{p-1}x^{[p]}.$$

It suffices to show that $\varphi_{\check{\mathrm{G}},0}|_{\mathcal{N}^{\mathrm{reg}}}$ sends $x \mapsto x$. If we identify $\mathcal{N}^{\mathrm{reg}} = \check{\mathrm{G}}/\mathrm{Z}_{\check{\mathrm{G}}}(e)$ and $\mathrm{Z}_{\check{\mathrm{G}}}(e) = \mathrm{Spec}\,\mathrm{H}_*(\mathrm{Gr}_\mathrm{G};\mathbf{Z}[1/\mathrm{N}])$ by Theorem 3.6.3 (or, [YZ2, Theorem 6.1]), then $\varphi_{\check{\mathrm{G}},0}|_{\mathcal{N}^{\mathrm{reg}}}$ is induced by the map

$$\mathrm{Z}_{\check{\mathrm{G}}}(e) \times_{\mathrm{Spec}\,\mathbf{Z}[1/\mathrm{N}]} \mathrm{Spec}\,\mathbf{F}_p[t^{\pm 1}] \to \mathrm{Z}_{\check{\mathrm{G}}}(e)$$

coming from the decompleted Frobenius/total Steenrod operation on $\mathrm{H}_*(\mathrm{Gr}_\mathrm{G};\mathbf{Z}[1/\mathrm{N}])$. It therefore suffices to show that the decompleted Frobenius acts by the identity on $\mathrm{H}_*(\mathrm{Gr}_\mathrm{G};\mathbf{Z}[1/\mathrm{N}])$.

This can be proved using the generating complexes from [Bot], as elaborated upon in [LM]. Namely, recall that if X is a homotopy commutative H-space and $f : \mathrm{Y} \to \mathrm{X}$ is a map from a CW-complex into X, then f is said to exhibit Y as a generating complex for X if f induces a surjection $\mathrm{Sym}(\mathrm{H}_*(\mathrm{Y};\mathbf{Z}[1/\mathrm{N}])) \twoheadrightarrow \mathrm{H}_*(\mathrm{X};\mathbf{Z}[1/\mathrm{N}])$. In [LM], it was shown that if θ denotes the highest (short) coroot of G, then the Schubert variety $\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}}$ corresponding to the antidominant weight $-\theta$ is a generating complex for Gr_G. Since $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{Z}[1/\mathrm{N}])$ generates $\mathrm{H}_*(\mathrm{Gr}_\mathrm{G};\mathbf{Z}[1/\mathrm{N}])$ as a ring, and the decompleted Frobenius is a ring map, it suffices to show that the decompleted Frobenius/total Steenrod operation on $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{Z}[1/\mathrm{N}])$ sends $x \mapsto x$. Equivalently, it suffices to show that all Steenrod operations P^i act trivially on $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{Z}[1/\mathrm{N}])$ for $i > 0$.

To see this, observe that the dimension of $\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}}$ is given by $2(h-1)$, where h is the Coxeter number of $\check{\mathrm{G}}$. The operation P^i sends a class in $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{Z}[1/\mathrm{N}])$ in homological degree j to a class in $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{F}_p)$ in homological degree $j-2i(p-1)$. Since $\mathrm{H}_*(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{F}_p)$ is concentrated in nonnegative degrees and $p \geq h$, we see that P^i could only possibly act nontrivially when $p = h$ and $i = 1$, and that too only on classes in $\mathrm{H}_{2(h-1)}(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{Z}[1/\mathrm{N}])$. However, P^1 applied to such a class would land in $\mathrm{H}_0(\overline{\mathrm{Gr}_\mathrm{G}^{-\theta}};\mathbf{F}_p)$. This implies that it is zero: any Steenrod operation landing in $\mathrm{H}_0(\mathrm{X};\mathbf{F}_p)$ necessarily vanishes if X is a connected space. ∎

Example 4.4.18. Running the argument of Proposition 4.4.17 backwards tells us that if the map $x \mapsto x^{[p]}$ is not zero on the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$, then the decompleted Frobenius/total Steenrod operation on $\mathrm{H}_*(\mathrm{Gr}_\mathrm{G};\mathbf{Z})$ must be nontrivial. (The following example was shown to me by David Treumann, and was my impetus for more generally exploring the decompleted Frobenius.) Indeed, suppose (for simplicity) that $\mathrm{G} = \mathrm{SL}_3$ and $p = 2$. Then the map $x \mapsto x^{[2]}$ is not zero on the nilpotent cone in $\mathfrak{sl}_3$, and in fact the map $\varphi : x \mapsto x - t^{p-1}x^{[p]}$ sends the principal nilpotent $e = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}$ to the principal nilpotent $\begin{pmatrix} 0&1&t\\0&0&1\\0&0&0 \end{pmatrix}$. This is conjugate to e itself by the matrix $n_e = \begin{pmatrix} 1&t&0\\0&1&0\\0&0&0 \end{pmatrix}$. Conjugating the centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(e)$ by n_e sends

$$\begin{pmatrix} 1&a&b\\0&1&a\\0&0&1 \end{pmatrix} \mapsto \begin{pmatrix} 1&a&b+at\\0&1&a\\0&0&1 \end{pmatrix}. \tag{4.4.9}$$

Indeed, this is exactly how the decompleted Frobenius acts on $\mathrm{H}_*(\mathrm{Gr}_{\mathrm{SL}_3};\mathbf{Z}) = \mathbf{Z}[a,b]$. (One can verify this by observing that the generating complex in this case is given by the map $\mathbf{C}\mathrm{P}^2 \to \mathrm{Gr}_{\mathrm{SL}_3}$. The 2- and 4-cells of $\mathbf{C}\mathrm{P}^2$ give the classes a and b, respectively, and they are connected by the Steenrod square Sq^2.) Note that since the action of the decompleted Frobenius on $\mathrm{Z}_{\check{\mathrm{G}}}(e)$ is just conjugation by n_e, one can extend it to an action on all of $\check{\mathrm{G}}$. However, the element n_e is not canonical, and a different choice of n_e will act differently on $\check{\mathrm{G}}$.

Recall from Remark 4.4.14 that if λ denotes a dominant minuscule weight for PGL_3, the action of $\operatorname{Spec} \mathrm{H}_*(\mathrm{Gr}_\mathrm{G}; \mathbf{Z})$ on $\mathrm{H}^*(\mathrm{PGL}_3/\mathrm{P}_\lambda; \mathbf{Z})$ must be equivariant for the decompleted Frobenius. Let us quickly verify this in the case when λ is the fundamental weight: in this case, $\mathrm{PGL}_3/\mathrm{P}_\lambda = \mathbf{C}\mathrm{P}^2$, and if we write $\mathrm{H}^*(\mathrm{PGL}_3/\mathrm{P}_\lambda; \mathbf{Z}) = \mathbf{Z}\{x, y, z\}$, the decompleted Frobenius sends $y \mapsto y+tz$. (Indeed, $\mathrm{H}^*(\mathbf{C}\mathrm{P}^2; \mathbf{Z}) \cong \mathbf{Z}[w]/w^3$, and the total Steenrod operation sends $w \mapsto w + tw^2$. Writing $x = w^0$, $y = w$, and $z = w^2$ gives the desired claim.) It is straightforward to see that the action of $\mathrm{Z}_{\check{\mathrm{G}}}(e)$ on $\mathbf{Z}\{x, y, z\}$ is equivariant for the decompleted Frobenius as described in (4.4.9).

Example 4.4.19. Let $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$. Under the identification $\widetilde{\check{\mathrm{G}}}_\mathbf{H}/\check{\mathrm{G}} \cong \mathrm{B}/\check{\mathrm{B}}$, the map (4.4.5) is given by the $\check{\mathrm{B}}$-quotient of the pth power map on B. That is, if F is an algebraically closed field, then under the equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G}; \mathrm{KU}) \otimes_\mathbf{Z} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{B}}^{\mathrm{reg}}/\check{\mathrm{B}})$$

of Corollary 3.8.10, the decompleted Frobenius on the left-hand side (which encodes the pth Adams operation on KU) identifies with the pth power map on $\check{\mathrm{B}}^{\mathrm{reg}}$. Similarly, under the equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; \mathrm{KU}) \otimes_\mathbf{Z} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{G}}^{\mathrm{reg}}/\check{\mathrm{G}}),$$

the decompleted Frobenius on the left-hand side (which encodes the pth Adams operation on KU) identifies with the pth power map on $\check{\mathrm{G}}^{\mathrm{reg}}$.

For instance, suppose $\check{\mathrm{G}} = \mathrm{SL}_2$. When restricted to the Kostant slice inside $\check{\mathrm{G}} = \mathrm{SL}_2$ of matrices of the form $\left(\begin{smallmatrix} x-1 & x-2 \\ 1 & 1 \end{smallmatrix}\right)$, the map $\varphi_{\check{\mathrm{G}},0}$ is given by raising to the pth power. It turns out that

$$\left(\begin{smallmatrix} x-1 & x-2 \\ 1 & 0 \end{smallmatrix}\right)^p \text{ is conjugate to } \kappa(x) = \left(\begin{smallmatrix} \mathrm{L}_p(x)-1 & \mathrm{L}_p(x)-2 \\ 1 & 1 \end{smallmatrix}\right),$$

where $\mathrm{L}_n(x)$ is the nth “Lucas polynomial”, given by

$$\mathrm{L}_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \tfrac{n}{n-j} \binom{n-j}{j} x^{n-2j} = \mathrm{D}_n(x, 1).$$

Here, $\mathrm{D}_n(x, \alpha)$ is the “Dickson polynomial” from [Dic]. We therefore find that $\varphi_{\check{\mathrm{G}},0}$ is given on $g \in \mathrm{SL}_2$ by the map

$$\varphi_{\check{\mathrm{G}},0}(g) = \mathrm{L}_p(g).$$

Under the isomorphism between the Kostant slice for SL_2 and $\operatorname{Spec} \pi_0 \mathrm{KU}_{\mathrm{SU}(2)}$, the coordinate x identifies with the KU-theoretic Pontryagin class; and $\varphi_{\check{\mathrm{G}},0}(x)$ is exactly the pth Adams operation on this class.

Remark 4.4.20. In fact, one can interpolate between Example 4.4.16 and Example 4.4.19 using the results of [Dev3] and Remark 4.4.9. To state the result, we will use notation from Remark 4.4.9. Namely, the aforementioned results imply that there are equivalences

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{T}}_c}(\mathrm{Gr}_\mathrm{G}; \mathrm{ku}) \otimes_\mathbf{Z} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{B}}^{\mathrm{reg}}_\beta/\check{\mathrm{B}}) \tag{4.4.10}$$

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_\mathrm{G}; \mathrm{ku}) \otimes_\mathbf{Z} \mathrm{F} \simeq \mathrm{QCoh}(\check{\mathrm{G}}^{\mathrm{reg}}_\beta/\check{\mathrm{G}}), \tag{4.4.11}$$

where, for a group scheme H, we define H_β to be (the stacky quotient by $\mathbf{G}_m$ of) the 1-parameter degeneration of H into its Lie algebra. Explicitly, $\mathrm{H}_\beta = \mathrm{Hom}(\mathfrak{D}(\mathbf{H}), \mathrm{H})$, where $\mathfrak{D}(\mathbf{H})$ is the Cartier dual of the 1-dimensional group scheme over $\operatorname{Spec}(\pi_*(\mathrm{ku}))/\mathbf{G}_m = \operatorname{Spec}(\mathbf{Z}[\beta])/\mathbf{G}_m$

from Remark 4.4.9. For instance, if $\mathrm{H} = \mathrm{SL}_n$, then $\mathrm{SL}_{n,\beta}$ consists of (the stacky quotient by $\mathbf{G}_m$ of) the group scheme of those $n \times n$-matrices x such that $\frac{\det(\mathrm{id}+\beta x)-1}{\beta} = 0$.

For simplicity, let us focus on the equivalence (4.4.11) above. The decompleted Frobenius on the topological side of (4.4.11) interpolates between the pth Adams operation and the total Steenrod operation, and it identifies with pullback along the map $(\check{\mathrm{G}}_\beta \times_{\mathrm{Spec}\,\mathbf{Z}[\beta]} \mathrm{U}_0)/\check{\mathrm{G}} \to \check{\mathrm{G}}_{p\beta}/\check{\mathrm{G}}$ given by the $\check{\mathrm{G}}$-quotient of the map on $\check{\mathrm{G}}_\beta$ defined by

$$x \mapsto \tfrac{(1+\beta x)^p - 1}{p\beta}.$$

Using an argument similar to Remark 4.4.9, one finds that when $\beta = 0$, the above map reduces to the Artin-Schreier map on $\check{\mathfrak{g}}$ from Example 4.4.16. In the case $\check{\mathrm{G}} = \mathrm{SL}_2$, for instance, the decompleted Frobenius on the Kostant slice is given by $x \mapsto f_p(x)$, where $f_n(x)$ is the polynomial given by[5]

$$f_n(x) = \sum_{j=0}^{n-1} (-1)^j \tfrac{2n}{2n-j} \binom{2n-j}{j} \beta^{2(n-j)-2} x^{n-j} = \beta^{-2}(\mathrm{D}_{2n}(\beta x^{1/2}, 1) - 2),$$

where $\mathrm{D}_n(x, \alpha)$ is the "Dickson polynomial" from [Dic]. Elementary arithmetic manipulations confirm that the polynomial $f_p(x)$ indeed computes the decompleted Frobenius on $\mathrm{Spec}\,\pi_0\mathrm{ku}_{\mathrm{SU}(2)}$, and furthermore that upon writing $\beta = (\zeta_p - 1)t$ in $\mathcal{O}_{\mathrm{U}_0}$, we have

$$\tfrac{f_p(x)}{(\zeta_p-1)^{2(p-1)}} = \sum_{j=0}^{p-1} \tfrac{(-1)^j}{(\zeta_p-1)^{2j}} \tfrac{2p}{2p-j} \binom{2p-j}{j} t^{2(p-1-j)} x^{p-j}.$$

Suppose $p > 2$ (the situation is much easier to analyze when $p = 2$). Upon reducing modulo $\zeta_p - 1$, only the terms indexed by $j = 0, \frac{p-1}{2}$, and $p-1$ survive. When $j = \frac{p-1}{2}$, the coefficient of $t^{p-1}x^{(p+1)/2}$ is

$$\tfrac{(-1)^{(p-1)/2}}{(\zeta_p-1)^{p-1}} \tfrac{4p}{3p+1} \binom{(3p+1)/2}{(p-1)/2} \equiv -2 \pmod{(\zeta_p - 1)},$$

so that

$$\tfrac{f_p(x)}{(\zeta_p-1)^{2(p-1)}} \equiv x - 2t^{p-1}x^{(p+1)/2} + t^{2(p-1)}x^p \pmod{(\zeta_p - 1)}.$$

This is exactly as expected from Example 4.4.16.

Example 4.4.21. For the sake of completeness, let us explain the analogue of the calculation in Example 4.4.18 for KU, so that $\mathrm{G} = \mathrm{SL}_3$ and $p = 2$. The map $\varphi : x \mapsto x^2$ sends $e = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to $e^2 = \left(\begin{smallmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. This is conjugate to e itself by the matrix $n_e = \left(\begin{smallmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Conjugating the centralizer $\mathrm{Z}_{\check{\mathrm{G}}}(e)$ by n_e sends

$$\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & 2a & a+4b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{smallmatrix}\right). \tag{4.4.12}$$

Indeed, this is exactly how the decompleted Frobenius acts on $\mathrm{KU}_0(\mathrm{Gr}_{\mathrm{SL}_3}) = \mathbf{Z}[a, b]$. (One can verify this by observing that the generating complex in this case is given by the map $\mathbf{CP}^2 \to \mathrm{Gr}_{\mathrm{SL}_3}$. The 2- and 4-cells of $\mathbf{CP}^2$ give the classes a and b, respectively, and the Adams operation ψ^2 sends $a \mapsto 2a$ and $b \mapsto a + 4b$.) Again, since the action of the decompleted

[5] For instance, $f_2(x) = 4x - \beta^2x^2$, $f_3(x) = \beta^4x^3 - 6\beta^2x^2 + 9x$, and $f_5(x) = \beta^8x^5 - 10\beta^6x^4 + 35\beta^4x^3 - 50\beta^2x^2 + 25x$.

Frobenius on $Z_{\check{G}}(e)$ is just conjugation by n_e, one can extend it to an action on all of $\check{G}$; but the element n_e is not canonical, and a different choice of n_e will act differently on $\check{G}$.

Recall from Remark 4.4.14 that if λ denotes a dominant minuscule weight for PGL_3, the action of $\mathrm{Spec}\,\mathrm{KU}_0(\mathrm{Gr}_G)$ on $\mathrm{KU}^0(\mathrm{PGL}_3/\mathrm{P}_\lambda)$ must be equivariant for the decompleted Frobenius. Let us quickly verify this in the case when λ is the fundamental weight: in this case, $\mathrm{PGL}_3/\mathrm{P}_\lambda = \mathbf{C}\mathrm{P}^2$, and if we write $\mathrm{KU}^0(\mathrm{PGL}_3/\mathrm{P}_\lambda) = \mathbf{Z}\{x, y, z\}$, the decompleted Frobenius sends $y \mapsto z + 2y$ and $z \mapsto 4z$. (Indeed, $\mathrm{KU}^0(\mathbf{C}\mathrm{P}^2) \cong \mathbf{Z}[w]/w^3$, and the Adams operation ψ^2 is given by the ring map sending $w \mapsto w^2 + 2w$. Writing $x = w^0$, $y = w$, and $z = w^2$ gives the desired claim.) It is straightforward to see that the action of $Z_{\check{G}}(e)$ on $\mathbf{Z}\{x, y, z\}$ is equivariant for the decompleted Frobenius as described in (4.4.12).

Example 4.4.22. Let k denote *Tate K-theory* [AHS, Section 2.7], so that $k = \mathrm{KU}(\!(q)\!)$ and $\mathbf{G}$ is a lift to k of the Tate elliptic curve $\mathbf{H} = \mathrm{Tate}(q)$ over $\mathbf{Z}(\!(q)\!) = \pi_0(k)$. (See [Lur1, Section 4.3] for a sketch of the construction of $\mathbf{G}$.) As usual, we will identify $\mathrm{Tate}(q)^\vee$ with $\mathrm{Tate}(q)$. Take $\mathrm{F} = \mathbf{C}$, and let q be a point in the punctured open unit disk, so that it defines a continuous embedding $\mathbf{Z}(\!(q)\!) \hookrightarrow \mathbf{C}$. Then there are equivalences

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; \mathrm{KU}(\!(q)\!)) \otimes_{\mathbf{Z}(\!(q)\!)} \mathbf{C} \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathrm{Tate}(q))^{\mathrm{reg}})$$
$$\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathrm{KU}(\!(q)\!)) \otimes_{\mathbf{Z}(\!(q)\!)} \mathbf{C} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))^{\mathrm{reg}}).$$

If $\check{G}$ is simply-laced with torsion-free fundamental group, then there is an equivalence

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathrm{KU}(\!(q)\!)) \otimes_{\mathbf{Z}(\!(q)\!)} \mathrm{F} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))).$$

The ring $\pi_0\Phi^{\mathbf{Z}/p}k$ and the Frobenius $\varphi : \pi_0(k) \to \pi_0\Phi^{\mathbf{Z}/p}k$ can be computed explicitly using Lemma 4.4.6. We will not review the precise description here; instead, we only note that φ sends $q \mapsto q^p$ on homotopy, and refer the reader to [And, Section 6.3] and [Hua, Theorem 3.5] for a description of the degree p-isogeny $\varphi^*\mathrm{Tate}(q) \to \varphi^*\mathrm{Tate}(q)$. This isogeny defines a map $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\varphi^*\mathrm{Tate}(q)) \to \mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))$, pullback along which identifies (by Theorem 4.4.13) with the decompleted Frobenius $\mathrm{Shv}^{\mathrm{min,gr}}_{\check{G}_c}(\mathrm{Gr}_G; \mathrm{KU}(\!(q)\!)) \to \mathrm{Shv}^{\mathrm{min,gr}}_{\check{G}_c}(\mathrm{Gr}_G; \Phi^{\mathbf{Z}/p}\mathrm{KU}(\!(q)\!))$.

In [BG], Baranovsky and Ginzburg explicitly describe the set of $\mathbf{C}$-points of $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))$. Namely, define the q-twisted conjugation action $G(\!(z)\!)$ on itself as follows:

$$\mathrm{Ad}^q_{h(z)}(g(z)) := h(qz)g(z)h(z)^{-1}.$$

Then, there is a natural bijection between $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))(\mathbf{C})$ and the set of those q-twisted conjugacy classes in $G(\!(z)\!)$ which contain an element of $G[\![z]\!]$. Under this bijection, one can show that the decompleted Frobenius on $\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathrm{Tate}(q))(\mathbf{C})$ can be identified with the effect of the map

$$g(z) \mapsto g(q^{p-1}z)g(q^{p-2}z)\cdots g(qz)g(z)$$

on q-twisted conjugacy classes in $G(\!(z)\!)$.

The structures imposed by Theorem 4.4.13 are quite rigid. For instance, there is an action of $\mathrm{Loc}^{\mathrm{gr}}_{\check{G}_c}(\mathrm{Gr}_G; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{\check{T}_c}(\mathrm{Gr}_G; k)$ by convolution, which defines an action of $\mathrm{QCoh}(\check{G}^{\mathrm{reg}}_{\mathbf{H}}/\check{G})$ on $\mathrm{QCoh}(\check{B}^{\mathrm{reg}}_{\mathbf{H}}/\check{B})$. This action is given by pullback along the map $\check{B}_{\mathbf{H}}/\check{B} \to \check{G}_{\mathbf{H}}/\check{G}$, and it is compatible with power operations.

Example 4.4.23. When $k = \mathbf{Z}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$ (where we again invert some $\mathrm{N} \gg 0$ so that the equivalence of Corollary 3.6.8 holds), the action of $\mathrm{Loc}^{\mathrm{gr}}_{G_c}(\mathrm{Gr}_G; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{T_c}(\mathrm{Gr}_G; k)$

by convolution identifies with the action of $\mathrm{QCoh}(\check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}})$ on $\mathrm{QCoh}(\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{\mathrm{B}})$ via pullback along the map $\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{\mathrm{B}} \to \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}}$. It follows from Example 4.4.16 that this map is compatible with the decompleted Frobenius/Steenrod operations.

The composite map

$$\check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{\mathrm{N}} \to \check{\mathfrak{n}}^{\perp,\mathrm{reg}}/\check{\mathrm{B}} \to \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}}$$

can be realized as the $\check{\mathrm{G}}$-quotient of the restriction to regular loci of the moment map $\mu : \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}}) \to \check{\mathfrak{g}}^*$. The action of the decompleted Frobenius/Steenrod operations on the regular locus of $\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})$ in fact extends to all of $\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})$ itself (and hence on its affine closure $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$), and the moment map μ is equivariant for this action. The action of the decompleted Frobenius on $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$ commutes with the Gelfand-Graev action of the Weyl group from Proposition 3.6.15; this can be seen by reducing to the rank 1 case described below (with a bit of care in keeping track of the difference between $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$).

An explicit description of this action when $\check{\mathrm{G}} = \mathrm{SL}_2$ is as follows. If we identify $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})} = \mathrm{T}^*(\mathbf{A}^2)$ with coordinates $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$, then the total power operation is given by the map

$$\varphi : (u, v) \mapsto (u, v - t^{p-1} v \langle u, v\rangle^{p-1}).$$

Since the moment map $\mathrm{T}^*(\mathbf{A}^2) \to \mathfrak{sl}_2^* \cong \mathfrak{pgl}_2$ sends $(u, v) \mapsto \left(\begin{smallmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{smallmatrix}\right)$, it is easy to check that this map is compatible with the action of the decompleted Frobenius on $\mathfrak{sl}_2^*$ as described in Example 4.4.16.

Example 4.4.24. When $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, the action of $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ on $\mathrm{Loc}^{\mathrm{gr}}_{\mathrm{T}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ by convolution identifies with the action of $\mathrm{QCoh}(\mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}})$ on $\mathrm{QCoh}(\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}})$ via pullback along the map $\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}} \to \mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}}$. It follows from Example 4.4.16 that this map is compatible with the decompleted Frobenius/pth Adams operation. The composite map

$$\mathrm{B}^{\mathrm{reg}}/\check{\mathrm{N}} \to \mathrm{B}^{\mathrm{reg}}/\check{\mathrm{B}} \to \mathrm{G}^{\mathrm{reg}}/\check{\mathrm{G}}$$

can be realized as the $\check{\mathrm{G}}$-quotient of the restriction to regular loci of the multiplicative moment map $\mu : \check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B} \to \mathrm{G}$. The action of the decompleted Frobenius/pth Adams operation on the regular locus of $\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}$ in fact extends to all of $\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}$ itself (and hence on its affine closure $\overline{\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}}$), and the moment map μ is equivariant for this action. The action of the decompleted Frobenius on $\overline{\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}}$ commutes with the Gelfand-Graev action of the Weyl group from Proposition 3.8.15; this can be seen by reducing to the rank 1 case described below (with a bit of care in keeping track of the difference between $\mathbf{A}^2$ and $(\mathbf{A}^2)^*$).

An explicit description of the action of the decompleted Frobenius when $\check{\mathrm{G}} = \mathrm{SL}_2$ is as follows. As in Example 3.8.17, we may identify $\overline{\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}}$ with an open subset of $\mathrm{T}^*(\mathbf{A}^2)$ with coordinates $(u, v) \in \mathbf{A}^2 \oplus (\mathbf{A}^2)^*$. The total power operation is then given by the map

$$\varphi : (u, v) \mapsto \left(u, v \tfrac{(1+\langle u, v\rangle)^p - 1}{\langle u, v\rangle}\right).$$

Since the moment map $\overline{\check{\mathrm{G}} \times^{\check{\mathrm{N}}} \mathrm{B}} \to \mathrm{PGL}_2$ sends $(u, v) \mapsto \left(\begin{smallmatrix} 1+u_1 v_1 & u_1 v_2 \\ u_2 v_1 & 1+u_2 v_2 \end{smallmatrix}\right)$, it is easy to check that this map is compatible with the action of the pth power map on PGL_2 as described in Example 4.4.19. In checking that the total power operation is compatible with the Gelfand-Graev action as described in Example 3.8.17, the basic input is the identity $q^{-1}[p]_{q^{-1}} = q^{-p}[p]_q$ applied to $q = 1 + \langle u, v\rangle$ (where $[p]_q = \frac{q^p - 1}{q-1}$).

4.5 Comparison to Hopkins-Kuhn-Ravenel

The calculations of this article (more precisely, the perspective of Remark 3.9.14) were motivated by the work of Hopkins-Kuhn-Ravenel [HKR2], who study the case of finite groups. In this section, we will describe a relationship to their work. Our discussion will be rather heuristic, and we will sweep a few details under the rug to keep the exposition readable.

Before proceeding, the first thing to note is that while the present article only discusses *connected* compact Lie groups, Hopkins-Kuhn-Ravenel only study *discrete* compact Lie groups (that is, finite groups). Next, the work of [HKR2] only deals with Borel-equivariant cohomology. This means that one does *not* need to assume that the complex-oriented 2-periodic $\mathbf{E}_\infty$-ring k is equipped with an oriented commutative k-group $\mathbf{G}$; recall from § 3.2 that the purpose of $\mathbf{G}$ is to provide a decompletion of Borel-equivariant cohomology for compact abelian Lie groups. All that is needed is the formal completion $\hat{\mathbf{G}}$ of $\mathbf{G}$ at the identity section. Note that this is not extra data associated to k, since $\hat{\mathbf{G}} = \operatorname{Spf} k^{\mathbf{C}\mathrm{P}^\infty_+}$. Let $\hat{\mathbf{H}}$ denote the underlying 1-dimensional formal group over $\pi_0(k)$.

In fact, an even more stringent condition is required of k in [HKR2]: not only is it required to be complex-oriented and 2-periodic, but $\pi_0(k)$ is required to be a complete local Noetherian domain with maximal ideal $\mathfrak{m}$ whose residue field $\pi_0(k)/\mathfrak{m}$ is of characteristic $p > 0$, such that p is not nilpotent in $\pi_0(k)$. Let n denote the height of the formal group $\hat{\mathbf{H}}$ base-changed along $\pi_0(k) \to \pi_0(k)/\mathfrak{m}$. In the following discussion, we will simply write $k^0(\mathrm{X})$ to denote π_0 of the the k-cochains on X (instead of the more cumbersome notation $\pi_0\mathcal{F}(\mathrm{X})$).

Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$, and choose a continuous embedding $\pi_0(k) \to \mathcal{O}_{\mathbf{C}_p}$. The base-change of $\hat{\mathbf{H}}$ to $\mathcal{O}_{\mathbf{C}_p}$ defines a formal group law on the maximal ideal of $\mathcal{O}_{\mathbf{C}_p}$; assume that the base-change of $\hat{\mathbf{H}}$ along the map $\pi_0(k) \to \pi_0(k)/\mathfrak{m}$ has finite height. Then, there exists an exponential isomorphism

$$e : (\mathbf{Q}_p/\mathbf{Z}_p)^n \xrightarrow{\sim} (\mathfrak{m}_{\mathcal{O}_{\mathbf{C}_p}}, +_{\hat{\mathbf{H}}}), \tag{4.5.1}$$

where n is the height of $\hat{\mathbf{H}}$. The basic calculation driving the results of [HKR2] is the following.

Proposition 4.5.1. *There is an isomorphism*

$$k^0(\mathrm{B}\mathbf{Z}/p^j) \cong \pi_0(k)[\![t]\!]/[p^j](t),$$

where $[p^j](t) \in \pi_0(k)[\![t]\!]$ is the p^j-series of the formal group law $\hat{\mathbf{H}}$, and t is the first Chern class of the standard character $\mathbf{Z}/p^j \cong \mu_{p^j} \subseteq \mathrm{S}^1$. That is, there is an isomorphism $\operatorname{Spf} k^0(\mathrm{B}\mathbf{Z}/p^j) \cong \hat{\mathbf{H}}[p^j]$.

Construction 4.5.2. Proposition 4.5.1 and the discussion preceding it gives an isomorphism

$$\operatorname{Spf}(k^0(\mathrm{B}\mathbf{Z}/p^j)) \otimes_{\operatorname{Spf} \pi_0(k)} \operatorname{Spec} \mathbf{C}_p \cong \tfrac{1}{p^j}\mathbf{Z}/\mathbf{Z},$$

where the right-hand side denotes the constant group scheme over $\mathbf{C}_p$. A choice of generator (e.g., $\frac{1}{p^j}$) of this group therefore gives a map $k^0(\mathrm{B}\mathbf{Z}/p^j) \to \mathbf{C}_p$. Now let F be a finite group, and let $f : \mathbf{Z}_p^n \to \mathrm{F}$ be a homomorphism. Then f factors as a map $\mathbf{Z}_p^n \to (\mathbf{Z}/p^j)^n \to \mathrm{F}$ for some j, so there is a map $k^0(\mathrm{BF}) \to k^0(\mathrm{B}(\mathbf{Z}/p^j)^n)$. Taking the product of the maps $k^0(\mathrm{B}\mathbf{Z}/p^j) \to \mathbf{C}_p$ described above gives a map $k^0(\mathrm{B}(\mathbf{Z}/p^j)^n) \to \mathbf{C}_p$, which finally defines a composite map

$$k^0(\mathrm{BF}) \to k^0(\mathrm{B}(\mathbf{Z}/p^j)^n) \to \mathbf{C}_p.$$

This composite depends only on the conjugacy class of f, and so this construction defines a map $\operatorname{Hom}(\mathbf{Z}_p^n, \mathrm{F})/\!\!/\mathrm{F} \to \operatorname{Map}(k^0(\mathrm{BF}), \mathbf{C}_p)$, whose adjoint is a map $k^0(\mathrm{BF}) \to \operatorname{Map}(\operatorname{Hom}(\mathbf{Z}_p^n, \mathrm{F})/\!\!/\mathrm{F}, \mathbf{C}_p)$. Here, F acts on $\operatorname{Hom}(\mathbf{Z}_p^n, \mathrm{F})$ by conjugation.

In the discussion below, F will be a finite group. For simplicity, we will further assume that $k^*(\mathrm{BF})$ is concentrated in even degrees (so, by the 2-periodicity of k, it is completely determined by $k^0(\mathrm{BF})$). If X is an F-space, the homotopy orbits of X will be denoted $\mathrm{X}_{h\mathrm{F}}$, while the ordinary quotient of X by the F-action will be denoted $\mathrm{X}/\!\!/\mathrm{F}$.

Theorem 4.5.3 (Hopkins-Kuhn-Ravenel)**.** *The map from Construction 4.5.2 defines an isomorphism*

$$k^0(\mathrm{BF}) \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})/\!\!/\mathrm{F}, \mathbf{C}_p).$$

The quotient $\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})/\!\!/\mathrm{F}$ *can be replaced by the homotopy orbits* $\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})_{h\mathrm{F}}$*, since* F *is a finite group and its order is invertible in* $\mathbf{C}_p$*.*

Note that the homotopy orbits $\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})_{h\mathrm{F}}$ can be identified with $\mathrm{Map}(\mathrm{BT}_p^n, \mathrm{BF})$, where $\mathrm{T}_p^n = (\mathbf{Q}_p/\mathbf{Z}_p)^n$ is the p-adic n-torus. One can use a ring smaller than $\mathbf{C}_p$ in Theorem 4.5.3; essentially, one only needs to extend scalars to the rationalization of the smallest ring containing $\pi_0(k)$ over which the exponential isomorphism (4.5.1) holds.

In [Lur8], Lurie observes that the isomorphism of Theorem 4.5.3 can be categorified, at least if one assumes the data of a decompletion $\mathbf{G}$ of $\hat{\mathbf{G}}$. (We refer the reader to [Lur8] for further details, since the specific setup will not concern us much below.) Namely, if F is a finite group, Lurie defines an ∞-category $\mathrm{Loc}_{\mathrm{F}}(*; k)$ (denoted by $\mathrm{LocSys}_{\mathbf{G}}(\mathrm{BF})$ in *loc. cit.*), and proves the following as (a consequence of) [Lur8, Theorem 6.4.1]:

Theorem 4.5.4 (Lurie)**.** *Fix an embedding* $\pi_0(k) \to \mathbf{C}_p$*, so it defines an* $\mathbf{E}_\infty$*-map* $k \to \mathbf{C}_p[u^{\pm 1}]$*. There is a symmetric monoidal fully faithful embedding*

$$\mathrm{Loc}_{\mathrm{F}}(*; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{Loc}(\mathrm{Map}(\mathrm{BT}_p^n, \mathrm{BF}); \mathbf{C}_p[u^{\pm 1}]).$$

The essential image of the above embedding is described in [Lur8, Theorem 6.5.13].

Let us examine the isomorphism Theorem 4.5.3 and the embedding Theorem 4.5.4 further; we will rephrase the right-hand sides of both results as algebro-geometric objects. To do this, note that the exponential isomorphism between $\hat{\mathbf{H}} \otimes_{\pi_0(k)} \mathbf{C}_p$ and $(\mathbf{Q}_p/\mathbf{Z}_p)^n$ defines an isomorphism between $\mathbf{D}(\hat{\mathbf{H}}) \otimes_{\pi_0(k)} \mathbf{C}_p$ and $\mathbf{Z}_p^n$. Here, $\mathbf{D}(\hat{\mathbf{H}}) = \mathrm{Hom}(\hat{\mathbf{H}}, \mathbf{G}_m)$ is the Cartier dual of $\hat{\mathbf{H}}$. Note that the 1-shifted Cartier dual $\hat{\mathbf{H}}^\vee$ can be identified with the classifying stack of $\mathbf{D}(\hat{\mathbf{H}})$.

View the finite group F as defining a constant group scheme $\underline{\mathrm{F}}$ over $\mathbf{C}_p$. Since $\hat{\mathbf{H}}^\vee \otimes_{\pi_0(k)} \mathbf{C} \cong \mathbf{Z}_p^n$, the mapping stack $\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})$ is the quotient of the discrete scheme $\underline{\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})}$ by the constant group scheme $\underline{\mathrm{F}}$ acting by conjugation. It follows that the $\mathbf{C}_p$-algebra $\mathrm{Map}(\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})/\!\!/\mathrm{F}, \mathbf{C}_p)$ is the algebra of functions on the mapping stack $\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})$. (Not that since the order of F is invertible in $\mathbf{C}_p$, the derived and classical algebras of functions agree.) Similarly, $\mathrm{Loc}(\mathrm{Map}(\mathrm{BT}_p^n, \mathrm{BF}); \mathbf{C}_p)$ can be viewed as the category of quasicoherent sheaves on the mapping stack $\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})$. Therefore, Theorem 4.5.3 and Theorem 4.5.4 can be restated as:

$$\pi_0 k_{\mathrm{F}} \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \Gamma(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}); \mathcal{O}), \tag{4.5.2}$$

$$\mathrm{Loc}_{\mathrm{F}}(*; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})) \otimes_{\mathrm{Mod}_{\mathbf{C}_p}} \mathrm{Mod}_{\mathbf{C}_p[u^{\pm 1}]}. \tag{4.5.3}$$

One can even replace $\hat{\mathbf{H}}$ in the above by $\mathbf{H}$. Observe, now, that $\mathrm{Map}(\mathbf{H}^\vee, \mathrm{B}\underline{\mathrm{F}})$ is simply the stack $\underline{\mathrm{F}}_{\mathbf{H}}/\underline{\mathrm{F}}$.

We can now compare (4.5.2) and (4.5.3) to the discussion in the body of this article. Assume now that k is either $\mathbf{Q}[u^{\pm 1}]$, KU, or elliptic cohomology. If G_c was instead a connected compact Lie group, the analogue of (4.5.2) states that $\pi_0 k_{\mathrm{G}_c} \otimes_{\pi_0(k)} \mathbf{C}$ is the ring of (classical, not derived!) global sections of the structure sheaf on $\mathrm{G}_{\mathbf{H}}/\mathrm{G}$, where G is the complex reductive group corresponding to G_c. This is clear when $\mathbf{H}$ is $\mathbf{G}_a$ (and $k = \mathbf{Q}[u^{\pm 1}]$) or $\mathbf{G}_m$ (and $k = \mathrm{KU}$). In the case when $\mathbf{H}$ is an elliptic curve, this is essentially part of the definition of equivariant elliptic cohomology as sketched in [Lur1] and constructed in [GM2, GM1].

Let us continue to assume that G_c is a connected compact Lie group, and further impose that it is simply-laced and almost simple. We will now a give a heuristic argument suggesting that Conjecture 4.3.20 can be viewed as an analogue of (4.5.3).

Indeed, the rephrasing of Remark 3.9.12 from Remark 3.9.14 states that there is an equivalence

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathbf{C} \simeq \mathrm{QCoh}(\check{\mathrm{G}}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}}). \tag{4.5.4}$$

The regular locus $\check{\mathrm{G}}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}}$ is an open substack of $\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}$ (whose complement has codimension ≥ 2, as proved in [Dav, Proposition 3.1.16]), and so there is a fully faithful embedding $\mathrm{QCoh}(\check{\mathrm{G}}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}}) \hookrightarrow \mathrm{QCoh}(\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}})$. That is, there is a fully faithful embedding

$$\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k) \otimes_{\pi_0(k)} \mathbf{C} \hookrightarrow \mathrm{QCoh}(\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}). \tag{4.5.5}$$

Assume for the moment that (4.5.5) holds if $\check{\mathrm{G}}$ is a finite group $\check{\mathrm{F}}$ (and replace $\mathbf{C}$ above by $\mathbf{C}_p$). Of course, it is not clear what the Langlands dual F of $\check{\mathrm{F}}$ should mean; but it is reasonable to believe that, whatever it is, F should be a finite group (or perhaps a finite group scheme). In any case, Gr_{F} will just be a point, so the left-hand side of (4.5.5) is simply $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{F}}}(*; k) \otimes_{\pi_0(k)} \mathbf{C}$. It is reasonable to expect that, thanks to a formality-type statement, the 2-periodification of the category $\mathrm{Loc}^{\mathrm{gr}}_{\check{\mathrm{F}}}(*; k) \otimes_{\pi_0(k)} \mathbf{C}$ is equivalent to $\mathrm{Loc}_{\check{\mathrm{F}}}(*; k) \otimes_k \mathbf{C}[u^{\pm 1}]$.

Turning to the right-hand side of (4.5.5), note that with these translations made (so the left-hand side of (4.5.5) is replaced by $\mathrm{Loc}_{\check{\mathrm{F}}}(*; k) \otimes_k \mathbf{C}[u^{\pm 1}]$, and the right-hand side by the 2-periodification of $\mathrm{QCoh}(\underline{\check{\mathrm{F}}}_{\mathbf{H}}/\underline{\check{\mathrm{F}}})$), (4.5.5) is precisely of the form (4.5.3), as claimed.

Remark 4.5.5. The above comparison between the quotient $\mathrm{Gr}_{\mathrm{G}}/\mathrm{G}[\![t]\!]$ for a connected compact Lie group G_c and the classifying space BF for a finite group F can be made more precise by noting that $\mathrm{Gr}_{\mathrm{G}}/\mathrm{G}[\![t]\!]$ is homotopy equivalent to the mapping space $\mathrm{Map}(\mathrm{S}^2, \mathrm{BG}_c) = \mathrm{Bun}_{\mathrm{G}_c}(\mathrm{S}^2)$, and that if F is a finite group, then $\mathrm{Bun}_{\mathrm{F}}(\mathrm{S}^2) = \mathrm{BF}$.

The work of Hopkins-Kuhn-Ravenel in fact proves a statement which is much more general than Theorem 4.5.3 (and similarly, Lurie's work in [Lur8] yields a much stronger statement than Theorem 4.5.4). Namely, they prove the following.

Theorem 4.5.6 (Hopkins-Kuhn-Ravenel)**.** *Let* F *be a finite group, and let* X *be a finite* F*-space. For each homomorphism* $\alpha : \mathbf{Z}_p^n \to \mathrm{F}$*, let* X^α *denote the fixed locus of* $\mathrm{im}(\alpha)$*. Then there is an isomorphism*

$$k^*(\mathrm{X}_{h\mathrm{F}}) \otimes_{\pi_0(k)} \mathbf{C}_p \xrightarrow{\cong} \mathrm{H}^*\left(\left(\amalg_{\alpha \in \mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})} \mathrm{X}^\alpha\right) /\!\!/ \mathrm{F}; \mathbf{C}_p[u^{\pm 1}]\right).$$

The isomorphism of Theorem 4.5.3 is the special case when X is a point. In [Lur8], Lurie shows that Theorem 4.5.6 is a consequence of a more general statement. If X is a finite F-space, Lurie defines an ∞-category $\mathrm{Loc}_{\mathrm{F}}(\mathrm{X}; k)$ (denoted by $\mathrm{LocSys}_{\mathbf{G}}(\mathrm{X}//\mathrm{F})$ in *loc. cit.*), and proves the following as [Lur8, Theorem 6.4.1]:

Theorem 4.5.7 (Lurie)**.** *There is a symmetric monoidal fully faithful embedding*

$$\mathrm{Loc}_{\mathrm{F}}(\mathrm{X}; k) \otimes_k \mathbf{C}_p[u^{\pm 1}] \hookrightarrow \mathrm{Loc}(\mathrm{Map}(\mathrm{BT}_p^n, \mathrm{X}_{h\mathrm{F}}); \mathbf{C}_p[u^{\pm 1}]).$$

The essential image of the above embedding is described in [Lur8, Theorem 6.5.13]. For the reader interested in chasing down references: specifically, Theorem 4.5.7 generalizes [Lur8, Theorem 4.3.2]; the latter implies Theorem 4.5.6 by [Lur8, Corollary 4.3.4]. The basic observation is that the mapping space $\mathrm{Map}(\mathrm{BT}_p^n, \mathrm{X}_{h\mathrm{F}})$ is equivalent to $\left(\amalg_{\alpha\in\mathrm{Hom}(\mathbf{Z}_p^n,\mathrm{F})}\mathrm{X}^\alpha\right)_{h\mathrm{F}}$. Note that the homotopy quotient $\mathrm{Hom}(\mathbf{Z}_p^n, \mathrm{F})_{h\mathrm{F}}$ can be written as a disjoint union $\amalg_{[\alpha]}\mathrm{BZ}(\alpha)$ ranging over conjugacy classes of homomorphisms $\alpha : \mathbf{Z}_p^n \to \mathrm{F}$; here $\mathrm{Z}(\alpha)$ denotes the centralizer of the image of α. Similarly, the homotopy orbits $\left(\amalg_{\alpha\in\mathrm{Hom}(\mathbf{Z}_p^n,\mathrm{F})}\mathrm{X}^\alpha\right)_{h\mathrm{F}}$ can be rewritten as the disjoint union $\amalg_{[\alpha]}\mathrm{X}^\alpha_{h\mathrm{Z}(\alpha)}$.

Remark 4.5.8. One could contemplate a variant of Theorem 4.5.6 and Theorem 4.5.7 which replaces $\mathbf{C}_p$ by other $\mathbf{E}_\infty$-k-algebras (e.g., over which the base-change of $\hat{\mathbf{H}}$ is not necessarily isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^n$, but over which it has $(\mathbf{Q}_p/\mathbf{Z}_p)^j$ as a summand for some $j < n$). The analogues of Theorem 4.5.6 and Theorem 4.5.7 in this generality were proved in [Sta1, Sta2] and [Lur8].

Given the analogy between Theorem 4.5.4 and Conjecture 4.3.20, it is natural to ask for an analogue of Theorem 4.5.7 for connected compact Lie groups. In the following discussion, we suggest an analogy: namely, one could view the k-theoretic variant (described for $k = \mathrm{ku}$ in [Dev3]) of the local unramified relative Langlands conjecture of [BZSV] as an analogue of the aforementioned results.

To understand this, let us again massage Theorem 4.5.6 and Theorem 4.5.7 to a form more suited to algebro-geometric considerations. We will continue to assume for simplicity that $k^*(\mathrm{BF})$ is concentrated in even degrees. Theorem 4.5.6 describes how, under the isomorphism of Theorem 4.5.3, the $k^0(\mathrm{BF}) \otimes_{\pi_0(k)} \mathbf{C}_p$-module $k^*(\mathrm{X}_{h\mathrm{F}}) \otimes_{\pi_0(k)} \mathbf{C}_p$ decomposes as a module over $\Gamma(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}); \mathcal{O})$. Similarly, Theorem 4.5.7 says that there is an explicit $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}))$-module category $\widetilde{\mathcal{C}}_\mathrm{X}$ and a fully faithful $\mathrm{Loc}_\mathrm{F}(*; k) \otimes_k \mathbf{C}_p[u^{\pm1}]$-linear embedding

$$\mathrm{Loc}_\mathrm{F}(\mathrm{X}; k) \otimes_k \mathbf{C}_p[u^{\pm1}] \hookrightarrow \widetilde{\mathcal{C}}_\mathrm{X} \otimes_{\mathbf{C}_p} \mathbf{C}_p[u^{\pm1}].$$

Note that one source of $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}))$-module categories are maps $\check{\mathrm{L}} \to \mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})$: namely, $\mathrm{QCoh}(\check{\mathrm{L}})$ is a $\mathrm{QCoh}(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}))$-module category. That is, one could imagine that $\widetilde{\mathcal{C}}_\mathrm{X}$ is of the form $\mathrm{QCoh}(\check{\mathrm{L}})$ for some such $\check{\mathrm{L}}$ as above which is associated to X. (While one can give a somewhat *ad hoc* definition of $\check{\mathrm{L}}$ in terms of the fixed point spaces X^α and their (co)homology[6], it should be rather interesting to intrinsically understand the algebro-geometric properties of $\check{\mathrm{L}}$ directly.)

More generally, recall that the data of a k-linear ∞-category with F-action is just a $\mathrm{Fun}(\mathrm{BF}, \mathrm{Mod}_k)$-module category. Since $\mathrm{Fun}(\mathrm{BF}, \mathrm{Mod}_k)$ is a completion of the ∞-category $\mathrm{Loc}_\mathrm{F}(*; k)$, one might view the data of a $\mathrm{Loc}_\mathrm{F}(*; k)$-module category $\mathcal{C}$ as a decompletion of the notion of a k-linear ∞-category with F-action. One example of such a category is $\mathrm{Loc}_\mathrm{F}(\mathrm{X}; k)$ for a finite F-space X. If $\mathbf{1}_\mathcal{C}$ is a distinguished object of $\mathcal{C}$, then $\mathrm{End}_\mathcal{C}(\mathbf{1}_\mathcal{C}) \otimes_{\pi_0(k)} \mathbf{C}_p$ is a $k^0(\mathrm{BF}) \otimes_{\pi_0(k)} \mathbf{C}_p$-module, and hence a $\Gamma(\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}}); \mathcal{O})$-module. One could now ask for a description of this module structure; when $\mathcal{C} = \mathrm{Loc}_\mathrm{F}(\mathrm{X}; k)$ and $\mathbf{1}_\mathcal{C}$ is the constant sheaf therein, this is precisely answered by Theorem 4.5.6. Similarly, one could ask for an analogue of Theorem 4.5.7 in this generalized context. Summarizing, both Theorem 4.5.6 and Theorem 4.5.7 can be understood as describing how a $\mathrm{Loc}_\mathrm{F}(*; k)$-module category decomposes over the mapping stack $\mathrm{Map}(\hat{\mathbf{H}}^\vee, \mathrm{B}\underline{\mathrm{F}})$.

[6]For instance, take $\check{\mathrm{L}}$ to be the stack $\amalg_{[\alpha]} \mathrm{Spec}(\mathrm{H}^*(\mathrm{X}^\alpha; \mathbf{C}_p))/\underline{\mathrm{Z}(\alpha)}$.

Let G_c be a connected, almost simple, simply-laced compact Lie group. Then, as discussed above, the analogue of $\mathrm{Loc}_{\mathrm{F}}(*; k)$ is the ∞-category $\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$. Moreover, the analogue of the tensor product on $\mathrm{Loc}_{\mathrm{F}}(*; k)$ is the *convolution* tensor product on $\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ coming, for instance, from the $\check{\mathrm{G}}_c$-equivariant $\mathbf{E}_2$-space structure on $\mathrm{Gr}_{\mathrm{G}} \cong \Omega\mathrm{G}_c$. As mentioned in Remark 3.9.14, the equivalence of (4.5.4) is monoidal for the convolution tensor product on $\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ and the ordinary tensor product of quasicoherent sheaves on $\check{\mathrm{G}}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}$.

Based on the discussion above, one can interpret the following question as an analogue of Theorem 4.5.6 and Theorem 4.5.7: how does a $\mathrm{Loc}_{\mathrm{F}}(*; k)$-module category decompose over $\check{\mathrm{G}}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}$? More precisely, any finite G_c-space X should:

a. define a $\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$-module category $\mathcal{C}_{\mathrm{X}}$; this is the analogue of the $\mathrm{Loc}_{\mathrm{F}}(*; k)$-module category $\mathrm{Loc}_{\mathrm{F}}(\mathrm{X}; k)$.

b. define a fully faithful embedding $\mathcal{C}_{\mathrm{X}} \hookrightarrow \widetilde{\mathcal{C}}_{\mathrm{X}}$ into an explicit $\mathrm{QCoh}(\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}})$-module category $\widetilde{\mathcal{C}}_{\mathrm{X}}$; this is the analogue of the fully faithful embedding $\mathrm{Loc}_{\mathrm{F}}(\mathrm{X}; k) \hookrightarrow \bigoplus_{[\alpha]} \mathrm{Loc}(\mathrm{X}^{\alpha}_{h\mathrm{Z}(\alpha)}; \mathbf{C}_p)$ from Theorem 4.5.7.

In the following discussion, we will quietly replace $\mathrm{Loc}_{\check{\mathrm{G}}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ by $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ for conceptual simplicity; this, of course, changes the quasicoherent side, but to avoid getting into more detail than is necessary, we will pretend that the dual side remains unchanged[7]. To describe a candidate for $\mathcal{C}_{\mathrm{X}}$, recall that the quotient $\mathrm{Gr}_{\mathrm{G}}/\mathrm{G}[\![t]\!]$ is homotopy equivalent to the mapping space $\mathrm{Map}(\mathrm{S}^2, \mathrm{BG}_c) = \mathrm{Bun}_{\mathrm{G}_c}(\mathrm{S}^2)$. This, in turn, can be described as the double coset stack $\mathrm{G}_c\backslash(\mathrm{LG}_c)/\mathrm{G}_c$, where LG_c denotes the (topological) free loop space of G_c. Any G_c-space X defines an LG_c-space LX, and the stack $\mathrm{G}_c\backslash(\mathrm{LG}_c)/\mathrm{G}_c$ acts on $(\mathrm{LX})/\mathrm{G}_c$ by convolution. That is, the ∞-category $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{Gr}_{\mathrm{G}}; k)$ with its convolution tensor product acts on $\mathrm{Loc}_{\mathrm{G}_c}(\mathrm{LX}; k)$. One could therefore regard the latter category as a candidate for $\mathcal{C}_{\mathrm{X}}$, and further ask for the following strengthening of (a) and (b) above:

- there should be a stack $\check{\mathrm{L}}^{\mathrm{reg}}$ equipped with a map $\check{\mathrm{L}}^{\mathrm{reg}} \to \check{\mathrm{G}}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}$ such that there is an equivalence
$$\mathcal{C}_{\mathrm{X}} = \mathrm{Loc}_{\mathrm{G}_c}(\mathrm{LX}; k) \simeq \mathrm{QCoh}(\check{\mathrm{L}}^{\mathrm{reg}}).$$
- the stack $\check{\mathrm{L}}^{\mathrm{reg}}$ should be an open substack of a larger stack $\check{\mathrm{L}}$, and the map $\check{\mathrm{L}}^{\mathrm{reg}} \to \check{\mathrm{G}}_{\mathbf{H}}^{\mathrm{reg}}/\check{\mathrm{G}}$ extends to a map $\check{\mathrm{L}} \to \check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}$. This gives a fully faithful embedding
$$\mathcal{C}_{\mathrm{X}} \hookrightarrow \widetilde{\mathcal{C}}_{\mathrm{X}} := \mathrm{QCoh}(\check{\mathrm{L}}).$$

Note that $\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}$ is the quotient of $\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{triv}}$ by $\check{\mathrm{G}}$, so one could equivalently view $\check{\mathrm{L}}$ as the data of a $\check{\mathrm{G}}$-stack $\check{\mathrm{M}}$ equipped with a $\check{\mathrm{G}}$-equivariant map
$$\mu : \check{\mathrm{M}} \to \check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{triv}}.$$
The relation between $\check{\mathrm{L}}$ and $\check{\mathrm{M}}$ is that $\check{\mathrm{L}} = \check{\mathrm{M}}/\check{\mathrm{G}}$. (There is more to say, regarding shifted symplectic structures [PTVV], but we refer the reader to Conjecture 5.2.20 and [Dev3, Section 5.2] for further discussion.)

Example 4.5.9. If $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{H} = \mathbf{G}_a$, then $\check{\mathrm{M}}$ is simply a $\check{\mathrm{G}}$-stack equipped with a $\check{\mathrm{G}}$-equivariant map $\mu : \check{\mathrm{M}} \to \check{\mathfrak{g}}^*$. Similarly, if $k = \mathrm{KU}$ and $\mathbf{H} = \mathbf{G}_m$, then $\check{\mathrm{M}}$ is simply a $\check{\mathrm{G}}$-stack equipped with a $\check{\mathrm{G}}$-equivariant map $\mu : \check{\mathrm{M}} \to \mathrm{G}$.

[7] If $k = \mathbf{Q}[u^{\pm 1}]$ and $\mathbf{G} = \mathbf{G}_a$, the object $\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}} = \check{\mathfrak{g}}/\check{\mathrm{G}}$ must be replaced by $\check{\mathfrak{g}}^*/\check{\mathrm{G}} = \mathfrak{g}/\check{\mathrm{G}}$; and similarly, if $k = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, the object $\check{\mathrm{G}}_{\mathbf{H}}/\check{\mathrm{G}} = \check{\mathrm{G}}/\check{\mathrm{G}}$ must be replaced by $\mathrm{G}/\check{\mathrm{G}}$.

Suppose X is the analytification of an affine G-variety $X_{\mathbf{C}}$. In [BZSV], Ben-Zvi–Sakellaridis–Venkatesh study (under certain additional conditions on $X_{\mathbf{C}}$) the full ∞-category $\mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!); \mathbf{C})$ as a module over $\mathrm{Shv}_{G[\![t]\!]}(\mathrm{Gr}_G; \mathbf{C})$. The local unramified geometric conjecture of [BZSV] (see [BZSV, Conjecture 7.5.1]) says – up to the issue of shearing, which we will ignore here – that associated to $X_{\mathbf{C}}$ is a Hamiltonian $\check{G}$-stack $\check{M}$ such that there is an equivalence of categories $\mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!); \mathbf{C}) \simeq \mathrm{QCoh}(\check{M}/\check{G})$. The data of a Hamiltonian $\check{G}$-structure on $\check{M}$ gives, in particular, an $\check{G}$-equivariant moment map $\check{M} \to \check{\mathfrak{g}}^*$ which makes $\mathrm{QCoh}(\check{M}/\check{G})$ into a $\mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$-module category. Moreover, under certain assumptions on $X_{\mathbf{C}}$, there is a fully faithful embedding $\mathrm{Loc}_{G_c}(LX; \mathbf{C}) \hookrightarrow \mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!); \mathbf{C})$. Putting this together, we find a picture exactly like the one described in the preceding paragraph: namely, assuming [BZSV, Conjecture 7.5.1], there is a fully faithful embedding

$$\mathrm{Loc}_{G_c}(LX; \mathbf{C}) \hookrightarrow \mathrm{Shv}_{G[\![t]\!]}(X_{\mathbf{C}}(\!(t)\!); \mathbf{C}) \simeq \mathrm{QCoh}(\check{M}/\check{G})$$

of $\mathrm{Loc}_{G_c}(LX; \mathbf{C})$ into an explicit $\mathrm{QCoh}(\check{\mathfrak{g}}^*/\check{G})$-module category. Therefore, one could view (the 2-periodification of) [BZSV, Conjecture 7.5.1] as a conjectural analogue for connected compact Lie groups and $k = \mathbf{C}[u^{\pm 1}]$ of Theorem 4.5.6 and Theorem 4.5.7.[8] Motivated by this discussion, we propose in the next few sections (see Conjecture 5.2.20) that there should be a variant of [BZSV, Conjecture 7.5.1] for sheaves with coefficients in other $\mathbf{E}_\infty$-rings (like connective complex K-theory ku or elliptic cohomology).

4.6 Loop rotation equivariance

In this section, we discuss the question of a loop-rotation equivariant analogue of Conjecture 4.3.20. I do not have any definitive answers yet (other than in the case of tori), but nevertheless I hope that formulating some ideas will be helpful to the interested reader. Our discussion will primarily be on the spectral side, where loop-rotation equivariance amounts to a deformation quantization; we will use some results from Part II (in particular, Theorem 6.4.1). (Using the ideas outlined in Remark 4.4.11, one can also incorporate a Frobenius into these quantizations by studying $\mathbf{E}_3 \rtimes S^1$-algebra structures, but we will leave this to future work.) Let us begin with discussing Conjecture 4.3.20 in the case when k is an ordinary commutative ring, so that $\mathbf{H} = \widehat{\mathbf{G}_a}(2)$, and the conjecture states that there is an equivalence

$$\mathrm{Shv}_G^{G[\![t]\!]\text{-cbl,gr}}(\mathrm{Gr}_G; k)^{\mathrm{Bor}} \simeq \mathrm{QCoh}^{\mathrm{gr}}((\check{\mathfrak{g}}^*)^\wedge_{\check{\mathcal{N}}}(2)/\check{G}), \tag{4.6.1}$$

where the dual group on the right-hand side is the split form.

As mentioned in the introduction, the picture of Betti local geometric Langlands [BZN] makes its own prediction for the category $\mathrm{Shv}_G^{G[\![t]\!]\text{-cbl}}(\mathrm{Gr}_G; k)$, namely that there is an equivalence

$$\mathrm{Shv}_G^{G[\![t]\!]\text{-cbl}}(\mathrm{Gr}_G; k)^{\mathrm{Bor}} \simeq \mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\} \times_{\check{G}} \{1\})/\check{G}), \tag{4.6.2}$$

where again the dual group on the right-hand side is the split form. (This equivalence should be monoidal, where the left-hand side is equipped with the convolution monoidal structure coming from the affine Grassmannian, and the right-hand side is equipped with the monoidal structure coming from convolution.) Note that this is a statement about the category of sheaves itself, and *not* about a degeneration of it. In particular, the right-hand side is not

[8] Of course, since F is a finite group, Theorem 4.5.6 and Theorem 4.5.7 are contentless if $k = \mathbf{C}[u^{\pm 1}]$; so what we mean by the analogy between [BZSV, Conjecture 7.5.1] and Theorem 4.5.7 is that the latter admits a conjectural generalization to connected compact Lie groups, and that the resulting statement specialized to $k = \mathbf{C}[u^{\pm 1}]$ is still interesting and bears analogy to [BZSV, Conjecture 7.5.1].

naturally graded. The way (4.6.2) relates to (4.6.1) is via the even stack construction of Definition 2.1.1. Namely:

Proposition 4.6.1. *Let* $\mathrm{A} = \mathcal{O}_{\{1\}\times_{\check{\mathrm{G}}}\{1\}}$*. Then the associated even stack* $\mathrm{Spev}(\mathrm{A})$ *is isomorphic to* $\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)$*, so that there is a* 1*-parameter degeneration*

$$\mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}}) \rightsquigarrow \mathrm{IndCoh}^{\mathrm{gr}}_{\check{\mathcal{N}}}(\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)/\check{\mathrm{G}}) \simeq \mathrm{QCoh}^{\mathrm{gr}}((\check{\mathfrak{g}}^*)^\wedge_{\check{\mathcal{N}}}(2)/\check{\mathrm{G}}).$$

Proof. Because the map $k \to \mathcal{O}_{\check{\mathrm{G}}}$ is an even eff cover, the same is true of the map $\mathrm{A} \to k$. By Lemma 2.1.5, one can identify

$$\operatorname{colim}_{\boldsymbol{\Delta}} \mathrm{Spec}(\pi_*(k^{\otimes_{\mathrm{A}}\bullet+1}))/\mathbf{G}_m \xrightarrow{\sim} \mathrm{Spev}(\mathrm{A}).$$

We claim that there is an isomorphism $\mathrm{Spec}(\pi_*(k\otimes_{\mathrm{A}} k))/\mathbf{G}_m \cong \check{\mathfrak{g}}^\sharp(-2)$ of groupoid schemes over $\mathrm{Spec}(k)/\mathbf{G}_m$; this implies that $\mathrm{Spev}(\mathrm{A}) \cong \mathrm{B}\check{\mathfrak{g}}^\sharp(-2)$. Explicitly, this means that there is an isomorphism

$$\pi_*(k\otimes_{k\otimes_{\mathcal{O}_{\check{\mathrm{G}}}} k} k) \cong \Gamma(\check{\mathfrak{g}}^*(2))$$

of graded Hopf algebras over k. This is an easy calculation, which we leave to the reader.

The final claim is a consequence of the following more general statement: if V is an affine space over k, and $\mathrm{Z} \subseteq \mathrm{V}$ is a closed conical subset. Then there is a Koszul duality equivalence between $\mathrm{IndCoh}^{\mathrm{gr}}_{\mathrm{Z}}(\mathrm{BV}^\sharp(-2))$ and $\mathrm{QCoh}^{\mathrm{gr}}((\mathrm{V}^*)^\wedge_{\mathrm{Z}}(2))$. □

The degeneration of Proposition 4.6.1 should fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}};k)^{\mathrm{Bor}} \arrow[r, rightsquigarrow] \arrow[d, "\sim"] & \mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl,gr}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}};k)^{\mathrm{Bor}} \arrow[d, "\sim"] \\
\mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}}) \arrow[r, rightsquigarrow] & \mathrm{QCoh}^{\mathrm{gr}}((\check{\mathfrak{g}}^*)^\wedge_{\check{\mathcal{N}}}(2)/\check{\mathrm{G}}).
\end{tikzcd} \tag{4.6.3}$$

The degeneration of Proposition 4.6.1 helps in understanding the effect of incorporating loop-rotation equivariance. To explain this, let us recall (see, e.g., [MRT] for the case of schemes, as well as [Ant, Rak, MRT]) that if X is an lci k-scheme, then there is a degeneration from the free loop stack $\mathcal{L}\mathrm{X} = \mathrm{Map}_k(\mathrm{S}^1, \mathrm{X})$ of X into the stack $\mathrm{Map}_k(\mathrm{B}\mathbf{G}_a(2)^\vee, \mathrm{X})$. Here, $\mathbf{G}_a(2)^\vee$ denotes the Cartier dual of $\mathbf{G}_a(2)$, and the degeneration in question comes from the degeneration of $\mathrm{S}^1 = \mathrm{B}\mathbf{Z}$ into $\mathrm{B}\mathbf{G}_a(2)^\vee$ which is the (1-shifted) Cartier dual to the degeneration of $\mathbf{G}_m$ into $\mathbf{G}_a(2)$. Since X is Tannakian, there is an isomorphism $\mathrm{Map}_k(\mathrm{B}\mathbf{G}_a(2)^\vee, \mathrm{X}) \cong \mathrm{T}[-1](-2)(\mathrm{X})$, and so its (inverse) shearing is the stack $\mathrm{BT}^\sharp_{\mathrm{X}}(-2)$, where $\mathrm{T}^\sharp_{\mathrm{X}}$ denotes the divided power hull of the zero section of the tangent bundle of X.[9] This geometrizes the Hochschild-Kostant-Rosenberg filtration on the Hochschild homology $\mathrm{HH}(\mathrm{X}/k)$. We will simply say below that there is a degeneration from the free loop stack $\mathcal{L}\mathrm{X} = \mathrm{Map}_k(\mathrm{S}^1, \mathrm{X})$ of X into $\mathrm{BT}^\sharp_{\mathrm{X}}(-2)$, sweeping the intermediate step of shearing under the rug. (This was, of course, implicit in Proposition 4.6.1 as well.) It is then easy to show:

Lemma 4.6.2. *Under the identifications*

$$(\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}} \cong \check{\mathrm{G}}\backslash(\mathcal{L}\check{\mathrm{G}})/\check{\mathrm{G}},$$
$$\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)/\check{\mathrm{G}} \cong \check{\mathrm{G}}\backslash(\mathrm{BT}^\sharp_{\check{\mathrm{G}}}(-2))/\check{\mathrm{G}},$$

[9] This comes from an isomorphism between $\{0\}\times_{\mathrm{V}(-2)}\{0\}$ and the shearing of $\mathrm{BV}^\sharp(-2)$ for a (pointed) affine space V. The isomorphism in question is a combination of two facts: $\mathrm{BV}^\sharp(-2)$ is an affine stack with $\mathrm{R}\Gamma(\mathrm{BV}^\sharp(-2);\mathcal{O}) \cong \mathrm{L}\,\mathrm{Sym}_k(\mathrm{V}^*[-1](2)) \cong \wedge^\bullet(\mathrm{V}^*)[-\bullet](2\bullet)$, while $\{0\}\times_{\mathrm{V}(-2)}\{0\} \cong \mathrm{Spec}(\wedge^\bullet(\mathrm{V}^*)[\bullet](2\bullet))$.

the degeneration of $(\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}}$ *into* $\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)/\check{\mathrm{G}}$ *from Proposition 4.6.1 identifies with the* $\check{\mathrm{G}}\times\check{\mathrm{G}}$*-equivariant degeneration of* $\mathcal{L}\check{\mathrm{G}}$ *into* $\mathrm{BT}^\sharp_{\check{\mathrm{G}}}(-2)$ *coming from the Hochschild-Kostant-Rosenberg filtration. In particular, the action of* S^1 *on* $(\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}}$ *by loop rotation degenerates to the action of* $\mathrm{B}\mathbf{G}_a^\sharp(-2)$ *on* $\mathrm{BT}^\sharp_{\check{\mathrm{G}}}(-2)$ *via the de Rham differential.*

The equivalence (4.6.2) is expected to be $\mathrm{S}^1_{\mathrm{rot}}$-equivariant. The category of S^1-equivariant objects in $\mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\}\times_{\check{\mathrm{G}}}\{1\})/\check{\mathrm{G}})$ degenerates to the category of $\mathrm{B}\mathbf{G}_a^\sharp$-equivariant objects in $\mathrm{IndCoh}_{\check{\mathcal{N}}}(\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)/\check{\mathrm{G}})$. Under the identification

$$\mathrm{B}\check{\mathfrak{g}}^\sharp(-2)\cong\mathrm{Spec}(\Gamma(\check{\mathfrak{g}}^*[-1](-2)))\cong\mathrm{Spec}(\wedge^\bullet(\check{\mathfrak{g}}^*)[-\bullet](-2\bullet)),$$

the $\mathrm{B}\mathbf{G}_a^\sharp(-2)$-action on the left-hand side amounts to the Chevalley-Eilenberg differential on $\wedge^\bullet(\check{\mathfrak{g}}^*)[-\bullet](-2\bullet)$. The Koszul duality between the filtered Chevalley-Eilenberg complex $(\mathrm{dR}^{\geq\star}_{\check{\mathrm{G}}})^{\check{\mathrm{G}}}$ of $\check{\mathfrak{g}}$ and the PBW-filtered universal enveloping algebra of $\check{\mathfrak{g}}$, along with the square (4.6.3), suggests that there is an equivalence

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl,gr}}_{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)^{\mathrm{Bor}}\simeq\mathrm{LMod}^{\mathrm{fil}}_{\mathrm{U}(\check{\mathfrak{g}})}(\mathrm{Rep}(\check{\mathrm{G}}))_{\check{\mathcal{N}}}.$$

Here, $\mathrm{LMod}^{\mathrm{fil}}_{\mathrm{U}(\check{\mathfrak{g}})}(\mathrm{Rep}(\check{\mathrm{G}}))$ is the category of filtered (left) modules in $\mathrm{Rep}(\check{\mathrm{G}})$ over the PBW-filtered universal enveloping algebra of $\check{\mathfrak{g}}$, and the subscript $\check{\mathcal{N}}$ denotes the full subcategory spanned by those objects on which the invariants $\mathrm{R}\Gamma(\check{\mathrm{G}};\mathrm{U}_\hbar(\check{\mathfrak{g}}))$ acts locally nilpotently.

Similarly, if one worked with genuine (as opposed to Borel) equivariant sheaves, then our discussion above suggests that there is an equivalence

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl,gr}}_{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}};k)\simeq\mathrm{IndCoh}^{\mathrm{fil}}(\check{\mathrm{G}}\backslash\mathrm{Spec}(\mathrm{dR}^{\geq\star}_{\check{\mathrm{G}}})/\check{\mathrm{G}})\simeq\mathrm{LMod}^{\mathrm{fil}}_{\mathrm{U}(\check{\mathfrak{g}})}(\mathrm{Rep}(\check{\mathrm{G}})).$$

Note that the stack $\mathrm{Spec}(\mathrm{dR}^{\geq\star}_{\check{\mathrm{G}}})$ over $\mathbf{A}^1/\mathbf{G}_m$ can be identified with $\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{HH}(\check{\mathrm{G}}/k))$. At least in characteristic zero, the suggested equivalence of categories does indeed hold, and is proved in [BF].

We now turn to the case of a general $\mathbf{E}_\infty$-ring k, which, for simplicity, we will assume to be even and connective. (It is easy to extend our discussion to the case when k is only evenly descendable.) To set the stage, let us begin with the case when $\mathrm{G}=\mathrm{T}$ is a torus. Recall from Theorem 3.2.20 that there is an equivalence

$$\mathrm{Shv}^{\mathrm{T}[\![t]\!]\text{-cbl}}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}};k)^{\mathrm{Bor}}\simeq\mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\}\times_{\check{\mathrm{T}}}\{1\})/\check{\mathrm{T}}),$$

where $\check{\mathrm{T}}=\mathrm{Spec}\,k[\mathbb{X}^*(\check{\mathrm{T}})]$ is a lift of the Langlands dual torus to k. This relates to Conjecture 4.3.20 (which is easy to check in the case when G is a torus) via the following analogue of Proposition 4.6.1:

Proposition 4.6.3. *Let* $\mathrm{A}=\mathcal{O}_{\{1\}\times_{\check{\mathrm{T}}}\{1\}}$*, and let* $\mathbf{H}$ *denote the canonical* 1*-dimensional formal group over* $\mathrm{Spev}(k)$*. Then the associated even stack* $\mathrm{Spev}(\mathrm{A})$ *is isomorphic to* $\mathrm{BT}^\vee_{\mathbf{H}}$*, where* $\mathrm{T}^\vee_{\mathbf{H}}$ *denotes the Cartier dual of* $\mathrm{T}_{\mathbf{H}}$*. In particular, there is a* 1*-parameter degeneration*

$$\mathrm{IndCoh}_{\check{\mathcal{N}}}((\{1\}\times_{\check{\mathrm{T}}}\{1\})/\check{\mathrm{T}})\rightsquigarrow\mathrm{IndCoh}^{\mathrm{gr}}_{\check{\mathcal{N}}}(\mathrm{BT}^\vee_{\mathbf{H}}/\check{\mathrm{T}})\simeq\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{T}_{\mathbf{H}}/\check{\mathrm{T}}).$$

(Note that $\mathrm{IndCoh}_{\check{\mathcal{N}}}$ in this case is simply QCoh; we will use the latter notation below.)

Proof. Indeed, observe that $\{1\}\times_{\check{\mathrm{T}}}\{1\}\cong\mathrm{Spec}(k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})])$. Since the map $k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]\to k$ is an even eff cover, it follows that

$$\mathrm{colim}_{\mathbf{\Delta}}\,\mathrm{Spec}(\pi_*(k^{\otimes_{k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]}\bullet+1}))/\mathbf{G}_m\xrightarrow{\sim}\mathrm{Spev}(k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]).$$

We claim that there is an isomorphism $\mathrm{Spec}(\pi_*(k\otimes_{k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]}k))/\mathbf{G}_m \cong \mathrm{T}^\vee_{\mathbf{H}}$ of groupoid schemes over $\mathrm{Spec}(k)/\mathbf{G}_m$; this implies that $\mathrm{Spev}(k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]) \cong \mathrm{BT}^\vee_{\mathbf{H}}$ as desired. To prove the claim, observe that since T is homotopy equivalent to $\mathrm{B}\mathbb{X}_*(\mathrm{T}) \cong \mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})$, there is an equivalence of $\mathbf{E}_\infty$-k-algebras

$$k \otimes_{k[\mathrm{B}\mathbb{X}^*(\check{\mathrm{T}})]} k \cong k[\mathrm{B}^2\mathbb{X}^*(\check{\mathrm{T}})] = k[\mathrm{BT}];$$

this implies that $\mathrm{Spec}(\pi_*(k[\mathrm{BT}]))/\mathbf{G}_m \cong \mathrm{T}^\vee_{\mathbf{H}}$ by Cartier duality, because $\mathrm{Spf}(\pi_*(k^{\mathrm{BT}}))/\mathbf{G}_m \cong \mathrm{T}_{\mathbf{H}}$ (by Remark 2.2.2). □

Just as in (4.6.3), one then has a commutative diagram of 1-parameter degenerations

$$\begin{tikzcd} \mathrm{Shv}^{\mathrm{T}[\![t]\!]\text{-cbl}}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}};k)^{\mathrm{Bor}} \arrow[r, squiggly] \arrow[d, "\sim"] & \mathrm{Shv}^{\mathrm{T}[\![t]\!]\text{-cbl,gr}}_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{T}};k)^{\mathrm{Bor}} \arrow[d, "\sim"] \\ \mathrm{QCoh}((\{1\}\times_{\check{\mathrm{T}}}\{1\})/\check{\mathrm{T}}) \arrow[r, squiggly] & \mathrm{QCoh}^{\mathrm{gr}}(\mathrm{T}_{\mathbf{H}}/\check{\mathrm{T}}), \end{tikzcd} \tag{4.6.4}$$

where the equivalence on the right-hand side is Conjecture 4.3.20. If we identify $(\{1\}\times_{\check{\mathrm{T}}}\{1\})/\check{\mathrm{T}}$ with $\check{\mathrm{T}}\backslash(\mathcal{L}\check{\mathrm{T}})/\check{\mathrm{T}} = \check{\mathrm{T}}\backslash\mathrm{Spec}(\mathrm{HH}(\check{\mathrm{T}}/k))/\check{\mathrm{T}}$, then the preceding discussion tells us that one can identify

$$\check{\mathrm{T}}\backslash\mathrm{Spev}(\mathrm{HH}(\check{\mathrm{T}}/k))/\check{\mathrm{T}} \cong \mathrm{BT}^\vee_{\mathbf{H}}/\check{\mathrm{T}}.$$

Koszul dually, one can identify $\mathrm{QCoh}(\mathrm{T}_{\mathbf{H}}/\check{\mathrm{T}})$ as a degeneration of $\mathrm{LMod}_{\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)}(\mathrm{Rep}(\check{\mathrm{T}}\times\check{\mathrm{T}}))$, where $\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)$ is the $\mathbf{E}_2$-Hochschild cohomology of $\check{\mathrm{T}}$; in other words, that $\mathcal{O}_{\mathrm{T}_{\mathbf{H}}}$ is a degeneration of $\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)^{\check{\mathrm{T}}}$.

It is now clear how one must incorporate loop-rotation equivariance into the spectral side Conjecture 4.3.20, at least in the case of a torus: namely, one should consider the degeneration

$$\mathrm{QCoh}((\{1\}\times_{\check{\mathrm{T}}}\{1\})/\check{\mathrm{T}})^{h\mathrm{S}^1} \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{T}}\backslash\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{HH}(\check{\mathrm{T}}/k))/\check{\mathrm{T}}),$$

with notation as in Definition 2.1.8. The right-hand side can in turn be identified with graded modules over $\pi_*(\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)^{h\mathrm{S}^1})$ in $\mathrm{Rep}(\check{\mathrm{T}}\times\check{\mathrm{T}})$. Observe that $\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)^{h\mathrm{S}^1} \cong (k[\Omega\mathrm{T}]^{h\mathrm{T}})^{h\mathrm{S}^1}$, and we computed the homotopy groups of the latter in Proposition 3.5.4.[10] Namely, the category $\mathrm{LMod}_{\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{T}}/k)^{h\mathrm{S}^1}}(\mathrm{Rep}(\check{\mathrm{T}}\times\check{\mathrm{T}}))$ degenerates to $\mathrm{LMod}^{\mathrm{gr}}_{\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}}(\mathrm{Rep}(\check{\mathrm{T}}\times\check{\mathrm{T}}))$. In summary, there is a degeneration

$$\mathrm{Shv}^{\mathrm{T}[\![t]\!]\text{-cbl}}_{\mathrm{T}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{T}};k)^{\mathrm{Bor}} \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{T}}\backslash\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{HH}(\check{\mathrm{T}}/k))/\check{\mathrm{T}}) \simeq \mathrm{LMod}^{\mathrm{gr}}_{\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}}(\mathrm{Rep}(\check{\mathrm{T}}\times\check{\mathrm{T}})) \tag{4.6.5}$$

which is a loop-rotation equivariant version of Conjecture 4.3.20 in the case of a torus T.

Example 4.6.4. Suppose $k = \mathrm{ku}$, so that $\mathbf{H}$ is the group scheme $\mathbf{G}_\beta$ with group law $x+y+\beta xy$ over $\mathrm{Spev}(\mathrm{ku}) = \mathbf{A}^1_\beta/\mathbf{G}_m$. If $\mathrm{T} = \mathbf{G}_m$, for instance, there is an isomorphism

$$\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}} \cong \left(\mathbf{Z}[\beta,\hbar,\tfrac{1}{1+\beta\hbar}]\{x,a^{\pm1}\}[\tfrac{1}{1+\beta x}]/([x,a] = a\hbar(1+\beta x))\right)^\wedge_{(\beta\hbar,x)};$$

see Remark 5.3.20. (The completion at $\beta\hbar$ is because we are only working Borel $\mathrm{S}^1_{\mathrm{rot}}$-equivariantly.) Upon killing β, this is simply the Rees construction (with respect to the

[10] The Koszul dual question of computing $\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{HH}(\check{\mathrm{T}}/k))$, or rather of the cohomology of its structure sheaf (which identifies with $\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{T}}/k)$), was studied in unpublished work by A. Raksit.

variable $\hbar$) of the order filtration on the Weyl algebra of $\check{\mathrm{T}} = \mathbf{G}_m$. Moreover, if y denotes the element $x\hbar^{-1}$ in $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}[\hbar^{-1}]$, then there is a relation

$$ya - qay = a,$$

so that y acts as the q-derivative $a\partial_a^q$. That is to say, $\mathcal{D}^{\mathbf{H}}_{\check{\mathrm{T}}}[\hbar^{-1}]$ is a completion of the $\mathbf{Z}[\![q-1]\!]$-algebra

$$\mathcal{D}^q_{\check{\mathrm{T}}} := \mathbf{Z}[\![q-1]\!][\hbar^{\pm 1}]\{a^{\pm 1}, y\}/(ya - qay = a).$$

Similarly, there is a filtration on $\mathrm{HP}(\check{\mathrm{T}}/\mathrm{ku}) = \mathrm{HH}(\check{\mathrm{T}}/\mathrm{ku})^{h\mathrm{S}^1}[1/\hbar]$ whose zeroth associated graded piece is given by $q\mathrm{dR}_{\check{\mathrm{T}}}$. All of these isomorphisms are $\check{\mathrm{T}} \times \check{\mathrm{T}}$-equivariant (for the action of $\check{\mathrm{T}} \times \check{\mathrm{T}}$ on $\check{\mathrm{T}}$ by left and right translation), so one has a degeneration

$$\mathrm{Shv}^{\mathrm{T}[\![t]\!]\text{-cbl}}_{\mathrm{T}\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{T}; \mathrm{ku})^{\mathrm{Bor}} \rightsquigarrow \mathrm{Mod}_{q\mathrm{dR}_{\check{\mathrm{T}}}}(\mathrm{Rep}(\check{\mathrm{T}} \times \check{\mathrm{T}})) \simeq \mathrm{LMod}^{\mathrm{nil}}_{\mathcal{D}^q_{\check{\mathrm{T}}}}(\mathrm{Rep}(\check{\mathrm{T}} \times \check{\mathrm{T}})).$$

Here, the superscript nil means that the Euler q-difference operators in $\mathcal{D}^q_{\check{\mathrm{T}}}$ (such as $y = a\partial_a$ above) act locally nilpotently.

Let us now turn to the case of a more general reductive group G; at this point, our discussion will be entirely speculative. The group scheme $\check{\mathrm{G}}$ does not lift as a group object in $\mathbf{E}_\infty$-schemes over the sphere spectrum (see § 3.4), but as discussed in Remark 3.4.3, the shearing $\check{\mathrm{G}}[2\rho]$ of $\check{\mathrm{G}}$ with respect to the cocharacter $2\rho : \mathbf{G}_m \to \check{\mathrm{G}}$ *does* lift to a group object in $\mathbf{E}_2$-schemes over the sphere spectrum. As mentioned in Remark 3.4.3, we hope to show in future work that $\check{\mathrm{G}}$ itself admits a lifting to a group object in $\mathbf{E}_2^{\mathrm{fr}}$-schemes over the sphere spectrum. In order to continue our discussion, we will *assume* below that such a lift exists, and denote it by $\check{\mathrm{G}}_\mathrm{S}$. This then allows one to make sense of the Hochschild homology of $\check{\mathrm{G}}_k := \check{\mathrm{G}}_\mathrm{S} \times_{\mathrm{Spec(S)}} \mathrm{Spec}(k)$ relative to k as an S^1-equivariant $\mathbf{E}_1$-k-algebra. By analogy to (4.6.5), one might hope for the following (significantly less precise) loop-rotation equivariant analogue of Conjecture 4.3.20.

Conjecture 4.6.5 (Vague)**.** *Let* G *be a connected reductive group over* $\mathbf{C}$ *(note, unlike Conjecture 4.3.20, it is not necessarily simply-laced!). There is a 1-parameter degeneration*

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{G}; k)^{\mathrm{temp},\mathrm{Bor}} \rightsquigarrow \mathrm{LMod}^{\mathrm{gr}}_{\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)}(\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})).$$

Here, the superscript temp *on the left-hand side denotes the "tempered" subcategory;* $\mathrm{HH}(\check{\mathrm{G}}_k/k)$ *denotes the Hochschild homology of the aforementioned (conjectural) lift* $\check{\mathrm{G}}_k$ *of* $\check{\mathrm{G}}$ *to* k *as a group object in* $\mathbf{E}_2^{\mathrm{fr}}$*-$k$-schemes; and* $\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}$ *denotes the associated graded of the* S^1*-equivariant even filtration of [HRW, Definition 1.2.2].*

Even assuming the existence of the lift $\check{\mathrm{G}}_k$ to k, it is not at all clear that the right-hand side of Conjecture 4.6.5 is well-defined! Because $\check{\mathrm{G}}_k$ is a group object in $\mathbf{E}_2^{\mathrm{fr}}$-$k$-schemes, the main result of [DHL+] gives an S^1-equivariant augmentation $\mathrm{HH}(\check{\mathrm{G}}_k/k) \to \mathcal{O}_{\check{\mathrm{G}}_k}$, which is a map of $\mathbf{E}_1$-k-coalgebras, and which exhibits $\mathcal{O}_{\check{\mathrm{G}}_k}$ as a pointed $\mathrm{HH}(\check{\mathrm{G}}_k/k)$-module. Using this, one can make sense of $\mathrm{HH}(\check{\mathrm{G}}_k/k)$ as an object of $\mathrm{Rep}(\check{\mathrm{G}}_k \times \check{\mathrm{G}}_k)$, and hence of $\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)$ as an object of $\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})$. However, the augmentation $\mathrm{HH}(\check{\mathrm{G}}_k/k) \to \mathcal{O}_{\check{\mathrm{G}}_k}$ is not necessarily a map of $\mathbf{E}_1$-k-algebras, and so it is not at all clear that $\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)$ is an $\mathbf{E}_1$-*algebra* object of $\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})$. Nevertheless, I hope to construct such an $\mathbf{E}_1$-algebra structure in future work, which would allow one to make sense of the right-hand side of Conjecture 4.6.5.

The relationship between Conjecture 4.6.5 and Conjecture 4.3.20 is not immediately clear; let us spell it out. (It is similar to the relationship between the two existing approaches to the derived geometric Satake equivalence, which are Bezrukavnikov-Finkelberg-Ginzburg's "top-down" affinization-based approach [BF, Gin2], which we adapted in Corollary 4.3.17, and Gaitsgory-Lurie-Campbell-Raskin's "bottom-up" Whittaker-based approach [CR1], which we adapted in Conjecture 4.6.5.) The Hochschild-Kostant-Rosenberg filtration on $\mathrm{HH}(\check{\mathrm{G}}_k/k)$ defines a filtration on the $\mathbf{E}_2$-Hochschild cohomology $\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k) = \mathrm{End}_{\mathrm{HH}(\check{\mathrm{G}}_k/k)}(\mathcal{O}_{\check{\mathrm{G}}_k})$ (which is an S^1-equivariant $\mathbf{E}_3$-k-algebra by the Deligne conjecture). The latter can easily be checked to be even, by base-change along the map $k \to \pi_0(k)$. The right-hand side of Conjecture 4.6.5 can equivalently be written as $\mathrm{LMod}^{\mathrm{gr}}_{\pi_*(\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k)^{h\mathrm{S}^1})}(\mathrm{Rep}(\check{\mathrm{G}}\times\check{\mathrm{G}}))$. In particular, upon killing the equivariant parameter $\hbar$, the right-hand side of Conjecture 4.6.5 becomes equivalent to $\mathrm{LMod}^{\mathrm{gr}}_{\pi_*(\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k))}(\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}}))$.

In contrast, (the tempered analogue of) Conjecture 4.3.20 says that if G is simply-laced with torsion-free fundamental group, there is a 1-parameter degeneration

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{temp,Bor}} \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\widehat{\mathrm{G}_{\mathbf{H}}}/\check{\mathrm{G}}),$$

where $\widehat{\mathrm{G}_{\mathbf{H}}}$ is the completion of $\mathrm{G}_{\mathbf{H}}$ at the zero section. Comparing to the preceding paragraph, one is led to expect:

Conjecture 4.6.6. *Suppose* G *is simply-laced with torsion-free fundamental group. Assuming the existence of the* $\mathbf{E}_2$*-*k*-scheme* $\check{\mathrm{G}}_k$*, there is a* $\check{\mathrm{G}} \times \check{\mathrm{G}}$*-equivariant isomorphism over* $\mathrm{Spev}(k)$*:*

$$\mathrm{Spf}\, \pi_*(\mathrm{HC}_{\mathbf{E}_2}(\check{\mathrm{G}}_k/k)) \cong \check{\mathrm{G}} \times \widehat{\mathrm{G}_{\mathbf{H}}}. \tag{4.6.6}$$

When G is a torus, this follows from our calculations above; and when k is an ordinary commutative ring, it amounts (by the Hochschild-Kostant-Rosenberg theorem for $\mathbf{E}_2$-Hochschild cohomology) to the $\check{\mathrm{G}}\times\check{\mathrm{G}}$-equivariant isomorphism between $\widehat{\mathrm{T}^*}(2)(\check{\mathrm{G}}) \cong \check{\mathrm{G}}\times\widehat{\check{\mathfrak{g}}}^*(2)$ and $\check{\mathrm{G}}\times\widehat{\mathfrak{g}}(2)$ coming from the isomorphism $\check{\mathfrak{g}}^* \cong \mathfrak{g}$.

When $k = \mathrm{ku}$, the degeneration of Conjecture 4.6.5, or rather the analogous degeneration

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}}; k)^{\mathrm{temp,Bor}}[\hbar^{-1}] \rightsquigarrow \mathrm{LMod}_{\mathrm{gr}^0_{\mathrm{ev},t\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)}(\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})), \tag{4.6.7}$$

makes a concrete prediction involving q-de Rham cohomology:

Example 4.6.7. Suppose $k = \mathrm{ku}$, and fix an odd prime p. The $\mathbf{E}_1$-ring $\mathrm{gr}^\star_{\mathrm{ev},t\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)$ is the associated graded of a filtration on $\mathrm{HP}(\check{\mathrm{G}}_k/k)$. Corollary 6.4.2 gives an equivalence

$$\mathrm{HP}(\check{\mathrm{G}}_{\mathrm{ku}}/\mathrm{ku})^\wedge_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q^{1/p}-1]\!] \cong \mathrm{TP}(\check{\mathrm{G}}_{\mathbf{Z}_p[\zeta_p]}/\mathrm{S}[\![q^{1/p}-1]\!]),$$

and a motivically filtered version thereof identifies $\mathrm{gr}^0_{\mathrm{ev},t\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_{\mathrm{ku}}/\mathrm{ku})^\wedge_p \otimes_{\mathbf{Z}_p[\![q-1]\!]} \mathbf{Z}_p[\![q^{1/p}-1]\!]$ with the Frobenius twist of the q-de Rham complex $q\mathrm{dR}_{\check{\mathrm{G}}}$ of the reductive group $\check{\mathrm{G}}$. This in fact descends to an equivalence $\mathrm{gr}^0_{\mathrm{ev},t\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_{\mathrm{ku}}/\mathrm{ku})^\wedge_p \cong q\mathrm{dR}_{\check{\mathrm{G}}}$ of $\mathbf{E}_1$-$\mathbf{Z}_p[\![q-1]\!]$-algebras. Our discussion above suggests that the action of $\check{\mathrm{G}} \times \check{\mathrm{G}}$ on $\check{\mathrm{G}}$ by left- and right-translation defines an action of $\check{\mathrm{G}} \times \check{\mathrm{G}}$ on $q\mathrm{dR}_{\check{\mathrm{G}}}$. Furthermore, (4.6.7) then says that $q\mathrm{dR}_{\check{\mathrm{G}}}$ upgrades to an $\mathbf{E}_1$-algebra object of $\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})$, and that there is a degeneration

$$\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{ku})^{\mathrm{temp,Bor}}[\hbar^{-1}]^\wedge_p \rightsquigarrow \mathrm{LMod}_{q\mathrm{dR}_{\check{\mathrm{G}}}}(\mathrm{Rep}(\check{\mathrm{G}} \times \check{\mathrm{G}})).$$

When G is a torus, this was shown in Example 4.6.4. Although I do not know how to construct the desired action of $\check{\mathrm{G}} \times \check{\mathrm{G}}$ on $q\mathrm{dR}_{\check{\mathrm{G}}}$, or prove the above degeneration, this seems to be a much more accessible problem than the general (4.6.7) because the theory of q-de Rham/prismatic cohomology is rather well-developed. (Of course, one might wonder about a relationship between $q\mathrm{dR}_{\check{\mathrm{G}}}$ and the quantum group $\mathrm{U}_q(\check{\mathrm{G}})$, but this would take us too far afield.)

One could also consider $\mathbf{E}_\infty$-rings k with *nontrivial* S^1-action in the degeneration of Conjecture 4.6.5, such as $k = \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p} - 1]\!])$ (which we identify with a Frobenius twist of ku in Theorem 6.4.1 below). Then, the $\mathbf{E}_1$-ring $\mathrm{gr}^\star_{\mathrm{ev},h\mathrm{S}^1}\mathrm{HH}(\check{\mathrm{G}}_k/k)$ would identify with the Nygaard-filtered q-de Rham complex $\mathcal{N}^{\geq\star} q\mathrm{dR}_{\check{\mathrm{G}}}\{\star\}$. Proving Conjecture 4.6.5 for any of these examples of k would be very interesting!

Chapter 5

Relative Langlands duality

5.1 Spherical varieties

In this section, we will review some of the theory of spherical varieties. Since the examples we will study in this thesis are rather simple (from the perspective of representation theory), we do not, strictly speaking, need the general theory. However, the recollections of this section will nevertheless be useful in placing basic phenomena that we will observe later into a broader context (see § 5.2).

We will not give any proofs in this section, but instead refer to [BLV, LV, Tim, BZSV] for details; in particular, this section is *not* intended to be an introduction to the theory of spherical varieties or to the theory of their Hamiltonian duals. (Instead, the reader should see [Per] for a very readable introduction to spherical varieties.) The base field in this section will always be the complex numbers, G will always be a connected reductive algebraic group over $\mathbf{C}$, $\mathrm{B} \subseteq \mathrm{G}$ will denote a chosen Borel subgroup, and N will be its unipotent radical.

Definition 5.1.1. A subgroup $\mathrm{H} \subseteq \mathrm{G}$ is called *spherical* if any of the following equivalent conditions are satisfied:

a. For any G-variety X and any H-fixed point $x \in \mathrm{X}$, the closure $\overline{\mathrm{G} \cdot x}$ contains finitely many G-orbits.

b. There are finitely many H-orbits in the flag variety G/B of G.

c. There is an open H-orbit in G/B.

d. The action of B on G/H has an open dense orbit.

An irreducible G-variety X is called *spherical* if it is normal and admits a dense open B-orbit $\mathring{\mathrm{X}} \subseteq \mathrm{X}$. In this case, X also contains an open G-orbit given by $\mathrm{G} \cdot \mathring{\mathrm{X}}$. If $x \in \mathring{\mathrm{X}}$ and H is its stabilizer, there is an isomorphism $\mathring{\mathrm{X}} = \mathrm{G}/\mathrm{H}$, and H is a spherical subgroup of G.

Before delving into examples, let us mention that the condition of being a spherical G-variety is relevant for our purposes because of the following result:

Theorem 5.1.2 ([GN, Theorem 3.2.1])**.** *Let $\mathrm{H} \subseteq \mathrm{G}$ be a subgroup. Then the following conditions are equivalent:*

a. G/H *is a spherical* G*-variety.*

b. The group $\mathrm{H}(\mathbf{C}(\!(t)\!))$ *acts on* $\mathrm{Gr}_{\mathrm{G}}(\mathbf{C}) = \mathrm{G}(\mathbf{C}(\!(t)\!))/\mathrm{G}(\mathbf{C}[\![t]\!])$ *with countably many orbits.*

c. The group $\mathrm{G}(\mathbf{C}[\![t]\!])$ *acts on* $\mathrm{G}(\mathbf{C}(\!(t)\!))/\mathrm{H}(\mathbf{C}(\!(t)\!))$ *with countably many orbits.*

Remark 5.1.3. We refer the reader to [GN] for a proof of Theorem 5.1.2, but since the argument is so short, let us recall why (b) implies (a). Suppose $\lambda : \mathbf{G}_m \to \mathrm{G}$ is a subgroup, so that we obtain a point $x_\lambda \in \mathrm{Gr}_\mathrm{G}(\mathbf{C})$. Then the G-orbit $\mathrm{X}_\lambda = \mathrm{G} \cdot x_\lambda \subseteq \mathrm{Gr}_\mathrm{G}$ is a flag variety of G, and by (b), the number of $\mathrm{H}(\mathbf{C}(\!(t)\!))$-orbits intersecting X_λ is countable. This implies that there is an $\mathrm{H}(\mathbf{C}(\!(t)\!))$-orbit which intersects X_λ in an open set. If we choose a point $y \in \mathrm{X}_\lambda$ in this open set, this implies that there is a surjection $\mathfrak{h} \twoheadrightarrow \mathrm{T}_y\mathrm{X}_\lambda$. If $\mathfrak{p}_y$ is the Lie algebra of the parabolic subgroup of G stabilizing y, the tangent space $\mathrm{T}_y\mathrm{X}_\lambda$ can be identified with $\mathfrak{g}/\mathfrak{p}_y$. In particular, if we choose λ to be regular, $\mathfrak{p}_y$ is isomorphic to a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$, and hence there is a surjection $\mathfrak{h} \twoheadrightarrow \mathfrak{g}/\mathfrak{b}$. But this implies that H has an open orbit in G/B, so H is spherical.

There are a lot of examples of spherical varieties: it includes the class of flag varieties, symmetric spaces (essentially by the Iwasawa decomposition), and toric varieties.

Example 5.1.4. The quotient $\mathrm{GL}_n/\mathrm{GL}_{n-1}$ is an affine spherical GL_n-variety; it is isomorphic to the variety $\{(x, \mathrm{V}) \in \mathbf{C}^{n+1} \times \mathrm{Gr}_n(\mathbf{C}^{n+1}) | x \notin \mathrm{V}\}$. The fact that the $\mathbf{C}$-points of $\mathrm{GL}_n/\mathrm{GL}_{n-1}$ is homotopy equivalent to S^{2n-1} motivates the terminology "spherical".

Example 5.1.5. As mentioned above, any symmetric space is a spherical variety. In particular, since G is the fixed points of the involution on $\mathrm{G} \times \mathrm{G}$ which swaps the two factors, we see that $\mathrm{G} \cong (\mathrm{G} \times \mathrm{G})/\mathrm{G}^\Delta$ is a spherical $\mathrm{G} \times \mathrm{G}$-variety. This will often be called the *group case*.

Example 5.1.6. Suppose $\mathrm{G} = \mathrm{PGL}_2$. Since the flag variety of G is isomorphic to $\mathbf{P}^1$, a subgroup $\mathrm{H} \subseteq \mathrm{PGL}_2$ is spherical if and only if it has an open orbit in $\mathbf{P}^1$. This is equivalent to saying that it is a subgroup of positive dimension. It is not difficult to see that all positive-dimensional subgroups of PGL_2 can be conjugated either to PGL_2 itself, the diagonal torus $\mathbf{G}_m \subseteq \mathrm{PGL}_2$, its normalizer $\mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m) \cong \mathrm{PO}_2 \subseteq \mathrm{PGL}_2$, or $\mathrm{S} \cdot \mathrm{N} \subseteq \mathrm{PGL}_2$, where N is the strictly upper-triangular matrices and $\mathrm{S} \subseteq \mathbf{G}_m$. In general, a spherical subgroup $\mathrm{H} \subseteq \mathrm{G}$ is called *horospherical* if H contains the unipotent radical of the Borel $\mathrm{B} \subseteq \mathrm{G}$; the motivation for this term being, of course, that horocycles in $\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2$ are orbits of the subgroup of strictly upper-triangular matrices in $\mathrm{SL}_2(\mathbf{R})$. These kinds of spherical varieties are *not* considered in the present article.

Warning 5.1.7. If G is a semisimple algebraic group and $\mathrm{T} \subseteq \mathrm{G}$ is a maximal torus, the quotient G/T is generally *not* a spherical G-variety. Indeed, there generally will not be an open dense T-orbit in G/B, since $|\Phi^-|$ is often larger than rank(T), where Φ^- is the set of negative roots of G. For instance, although the quotient $\mathrm{SL}_2/\mathbf{G}_m$ is a spherical SL_2-variety, the quotient SL_3/T is not a spherical SL_3-variety.

Remark 5.1.8. There is a finite list of closed connected spherical subgroups of simple algebraic groups: see [KR, Kra].

Example 5.1.9. Let G be a torus T. Then a T-variety X is spherical if it is normal and contains a dense orbit, and hence is precisely an affine toric variety. Let Λ denote the monoid of weights of T. Note that $\mathcal{O}_\mathrm{X}$ is a T-submodule of $\mathcal{O}_\mathrm{T}$, and so $\mathcal{O}_\mathrm{X} = \bigoplus_{\lambda \in \mathrm{S}_\mathrm{X}} \mathbf{C}_\lambda$ for some subset $\mathrm{S}_\mathrm{X} \subseteq \Lambda$. A standard fact from the theory of affine toric varieties is that a subset $\mathrm{S}_\mathrm{X} \subseteq \Lambda$ arises from an affine toric variety if and only if $\mathrm{S}_\mathrm{X} = \mathrm{C} \cap \Lambda$ for some convex cone $\mathrm{C} \subseteq \Lambda_\mathbf{R}$ generated by finitely many elements of Λ which span $\Lambda_\mathbf{R}$. Equivalently, if $\check{\mathrm{C}} \subseteq \check{\Lambda}$ denotes the dual cone, one observes that C spans $\Lambda_\mathbf{R}$ if and only if $\check{\mathrm{C}}$ is strictly convex (i.e., contains no line). Therefore, affine toric varieties are classified by strictly convex rational polyhedral cones of $\check{\Lambda}_\mathbf{R}$.

Example 5.1.9 is the first indication that certain spherical varieties admit interesting combinatorial data. In particular, this combinatorial data will be useful in defining the *Langlands dual group* to a spherical variety. We will recall some generalities on defining this dual group below, and then explain its manifestation in examples.

To define this dual group following [SV], let us now suppose that X is a homogeneous quasi-affine spherical G-variety. In this case, if $\mathring{\mathrm{X}} \subseteq \mathrm{X}$ is the open B-orbit, we will write H to be the stabilizer of a point $\mathring{\mathrm{X}}(\mathbf{C})$, so that $\mathrm{X} = \mathrm{G}/\mathrm{H}$ and $\mathrm{B} \cdot \mathrm{H} \subseteq \mathrm{G}$ is open.

Construction 5.1.10. Let $\mathrm{Frac}(\mathcal{O}_\mathrm{X})$ denote the fraction field of $\mathcal{O}_\mathrm{X}$, and let $\mathrm{Frac}(\mathcal{O}_\mathrm{X})^{(\mathrm{B})}$ denote the subset of $\mathrm{Frac}(\mathcal{O}_\mathrm{X}) - \{0\}$ consisting of the nonzero rational B-eigenfunctions. Then the lattice $\mathcal{X}_\mathrm{X}$ is simply the group of B-eigencharacters, and there is an exact sequence

$$1 \to \mathbf{C}^\times \to \mathrm{Frac}(\mathcal{O}_\mathrm{X})^{(\mathrm{B})} \to \mathcal{X}_\mathrm{X} \to 1;$$

in other words, for a fixed $\lambda \in \mathcal{X}_\mathrm{X}$, the functions $f \in \mathrm{Frac}(\mathcal{O}_\mathrm{X})^{(\mathrm{B})}$ which are χ-eigenvectors are all proportional by a scalar in $\mathbf{C}^\times$ (this follows from X being spherical). Let Λ_X denote the dual lattice to $\mathcal{X}_\mathrm{X}$. Then Λ_X defines a torus T_X, and we will write $\mathfrak{t}_\mathrm{X}$ to denote $\Lambda_\mathrm{X} \otimes \mathbf{Q}$. The rank of the lattice Λ_X (which is also the rank of $\mathcal{X}_\mathrm{X}$) is called the *rank* of X.

Remark 5.1.11. Suppose $\mathrm{X} = \mathrm{G}/\mathrm{H}$ is a homogeneous quasi-affine G-variety, and let $\mathcal{X}_\mathrm{X} = \mathrm{Frac}(\mathcal{O}_\mathrm{X})^{(\mathrm{B})}/\mathbf{C}^\times$ as above. It is not difficult to see that X is spherical if and only if $\mathcal{X}_\mathrm{X}$ is a lattice of finite rank. If K is a maximal compact subgroup of $\mathrm{G}(\mathbf{C})$, [Akh] shows that

$$\mathrm{rank}(\mathrm{X}) = \dim(\mathrm{K}\backslash\mathrm{X}(\mathbf{C})).$$

This is a purely topological description of the rank of X.

Construction 5.1.12. The stabilizer of the open B-orbit $\mathring{\mathrm{X}} \subseteq \mathrm{X}$ is a parabolic subgroup $\mathrm{P}(\mathrm{X})$. We will write $\mathrm{L}(\mathrm{X})$ to denote the Levi quotient of $\mathrm{P}(\mathrm{X})$; it will often be viewed as a subgroup of $\mathrm{P}(\mathrm{X})$ when convenient. Let T be a maximal torus of $\mathrm{B} \cap \mathrm{L}(\mathrm{X})$; then the torus T_X from above can be identified with $\mathrm{T}/(\mathrm{T} \cap \mathrm{B})$. The T_X-orbit of a point in the open B-orbit $\mathring{\mathrm{X}}(\mathbf{C})$ defines an embedding $\mathrm{T}_\mathrm{X} \hookrightarrow \mathring{\mathrm{X}}(\mathbf{C})$. In other words, the B-action on $\mathring{\mathrm{X}}$ defines a T-action on $\mathring{\mathrm{X}}/\!\!/\mathrm{N} = \mathrm{Spec}\, \mathcal{O}_\mathrm{X}^\mathrm{N}$, and this T-action factors through the quotient $\mathrm{T} \twoheadrightarrow \mathrm{T}_\mathrm{X}$.

Remark 5.1.13. In [Kno1, Lemma 3.1], Knop showed that if X is quasi-affine, the set of coroots in the span of $\Delta_{\mathrm{L}(\mathrm{X})}$ in Λ is precisely the set of coroots $\check{\alpha} \in \Lambda$ which are perpendicular to Λ_X.

Construction 5.1.14. Suppose $v : \mathrm{Frac}(\mathcal{O}_\mathrm{X})^\times \to \mathbf{Q}$ is a discrete valuation which is trivial on $\mathbf{C}^\times$. Then the restriction of v to $\mathrm{Frac}(\mathcal{O}_\mathrm{X})^{(\mathrm{B})}$ defines a homomorphism $\Lambda_\mathrm{X} \to \mathbf{Q}$, i.e., a point of $\mathfrak{t}_\mathrm{X}$. It is known that the map from G-invariant valuations to $\mathfrak{t}_\mathrm{X}$ is an injection, and so we will write $\mathcal{V} \subseteq \mathfrak{t}_\mathrm{X}$ to denote the subspace of G-invariant valuations. Let $\check{\Lambda}_\mathrm{X}^+$ denote the intersection $\Lambda_\mathrm{X} \cap \mathcal{V}$ of G-invariant $\mathbf{Z}$-valued valuations.

It turns out that the subset $\mathcal{V} \subseteq \mathfrak{t}_\mathrm{X}$ is a fundamental domain for the Weyl group W_X of a root system in Λ (where the weight lattice is Λ_X). In other words, the reflections over faces of $\mathcal{V}$ of codimension 1 generate a finite reflection subgroup $\mathrm{W}_\mathrm{X} \subseteq \mathrm{GL}(\mathfrak{t}_\mathrm{X})$, and this Weyl group W_X is called the *little Weyl group* of X. One can canonically identify W_X with a subgroup of W which normalizes the Weyl group $\mathrm{W}_{\mathrm{L}(\mathrm{X})}$ of $\mathrm{L}(\mathrm{X})$ (with respect to the chosen torus T).

Remark 5.1.15. The definition of the little Weyl group given above does not immediately relate to the microlocal nature of X. In [Kno1, Kno2], Knop gave an alternative construction of W_X using the Hamiltonian G-action on T^*X. Very briefly, let us review this construction. The quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defines an inclusion $(\mathfrak{g}/\mathfrak{h})^* \subseteq \mathfrak{g}^*$, and we will denote this by $\mathfrak{h}^\perp$ (it can be viewed as a subspace of $\mathfrak{g}$ via the isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ given by the Killing form). Consider the moment map $\mu : T^*X \cong (G \times \mathfrak{h}^\perp)/H \to \mathfrak{g}^*$ of the Hamiltonian G-action on T^*X. Composing with the characteristic polynomial map $\mathfrak{g}^* \to \mathfrak{g}^*/\!\!/G \cong \mathfrak{t}^*/\!\!/W$ defines a map $T^*X \to \mathfrak{t}^*/\!\!/W$. Observe also that the quotient map $T \twoheadrightarrow T_X$ induces an inclusion $\mathfrak{t}_X^* \hookrightarrow \mathfrak{t}^*$.

Fix a character $\chi : T_X \to \mathbf{G}_m$. Then, there is a $(P(X), \chi)$-eigenfunction $f_\chi \in \mathcal{O}_{\mathring{X}}$ (unique up to scalar multiplication) defines a section $d\log(f_\chi) : \mathring{X} \to T^*\mathring{X}$. This section is independent of the choice of f_χ, since f_χ is unique up to scalar multiplication. Ranging over all characters χ, one obtains a map $\mathfrak{t}_X^* \times \mathring{X} \to T^*\mathring{X}$. If $\mathcal{P}$ denotes the set of conjugates of the parabolic subgroup $P(X)$, we further obtain a map $\mathfrak{t}_X^* \times (\mathcal{P} \times \mathring{X}) \to T^*X$. Knop showed that the image of this map is dense, and that there is an isomorphism $(T^*X)/\!\!/G \cong \mathfrak{t}_X^*/\!\!/W_X$. Said slightly differently, the fiber product $T^*X \times_{\mathfrak{t}^*/\!\!/W} \mathfrak{t}^*$ generally has multiple irreducible components. If C is an irreducible component which dominates T^*X, we obtain a covering $C \to T^*X$, and W_X is the Galois group of this covering. In particular, note that this construction describes W_X as a subquotient of W. (However, there is in fact a canonical embedding $W_X \hookrightarrow W$.)

In [Kno2], Knop reinterpreted the above construction as follows: if $\mathcal{O}_B(X)$ is the set of B-orbits in X, Knop constructed an action of W on $\mathcal{O}_B(X)$. There is a canonical bijection between $\mathcal{O}_B(X)$ and the set of irreducible components of $T^*X \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ (given by taking the conormal bundle), where $\widetilde{\mathfrak{g}}$ is the Grothendieck-Springer resolution. The action of W on $\mathcal{O}_B(X)$ can be understood as arising from the action of the Steinberg scheme $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ by convolution and the isomorphism of [CG, Theorem 3.4.1]. In any case, the stabilizer of the open B-orbit is isomorphic to $W_X \ltimes W_{L(X)}$. A related result was proved in [Res]: namely, if $H \subseteq G$ is a reductive spherical subgroup and $X = G/H$, the Weyl group of H can be recovered as the stabilizer inside W of a(ny) minimal rank B-orbit on X viewed as an element of $\mathcal{O}_B(X)$.

Remark 5.1.16. Continuing Theorem 5.1.2, one can show (see [LV, Proposition 4.10] or [GN, Theorem 3.2.1]) that the $G(\mathbf{C}[\![t]\!])$-orbits on $(G/H)(\mathbf{C}(\!(t)\!))$ are in bijection with $H(\mathbf{C}(\!(t)\!))$-orbits on $\mathrm{Gr}_G(\mathbf{C})$, which in turn are in bijection with $\check{\Lambda}_X/W_X \cong \check{\Lambda}_X^+$. This generalizes the Cartan decomposition, in the sense that when applied to the group case of Example 5.1.5, it recovers the standard parametrization of the $G(\mathbf{C}[\![t]\!])$-orbits on Gr_G. The bijection between $G(\mathbf{C}[\![t]\!])$-orbits on $(G/H)(\mathbf{C}(\!(t)\!))$ and $\check{\Lambda}_X^+$ sends a map $\lambda : \mathcal{O}_{G/H} \to \mathbf{C}(\!(t)\!)$ to the valuation given by the composite

$$\mathcal{O}_{G/H} \to \mathcal{O}_{G/H} \otimes_{\mathbf{C}} \mathcal{O}_G \xrightarrow{\lambda} \mathcal{O}_G(\!(t)\!) \xrightarrow{v_t} \mathbf{Z}.$$

This is a G-invariant discrete valuation of $\mathcal{O}_{G/H}$,

Construction 5.1.17. Let $\mathcal{V}^\perp$ denote the cone $\{\chi \in \mathfrak{t}_X^* | \langle \chi, v\rangle \leq 0 \text{ for each } v \in \mathcal{V}\}$. Let Σ_X denote the set of generators of intersections of extremal rays of $\mathcal{V}^\perp$ with Λ_X. It turns out that the elements of Σ_X are linearly independent; they are known as the *spherical roots* of X. In fact, they form the set of simple roots of the based root system mentioned in Construction 5.1.14.

Remark 5.1.18. It turns out that for each spherical root $\gamma \in \Sigma_X$, there is some element $n \in \{\frac{1}{2}, 1, 2\}$ such that $\gamma' = n\gamma$ is either a positive root of G, or is the sum $\alpha + \beta$ of two positive roots which are orthogonal to each other and α and β are elements of some system of simple roots. These simple roots need not correspond to the choice of B! Let Δ_X denote the set $\{\gamma' | \gamma \in \Sigma_X\}$; then Δ_X is called the set of *normalized spherical roots*. Moreover, if Φ_X

denotes the set of $\mathrm{W_X}$-translates of Δ_X, it is shown in [SV, Proposition 2.2.1] that the pair $(\Phi_\mathrm{X}, \mathrm{W_X})$ defines a root system (called the *normalized spherical root system* of X) where Δ_X forms a set of simple roots. Let $(\check{\Phi}_\mathrm{X}, \mathrm{W_X})$ denote the dual root system, and $\check{\Delta}_\mathrm{X}$ the set of simple coroots.

Theorem 5.1.19 ([SV, Proposition 2.2.2], [KS2])**.** *Suppose that Σ_X does not contain any elements of the form 2α for α being a root of* G*. Then, $(\Lambda_\mathrm{X}, \Phi_\mathrm{X}, \check{\Lambda}_\mathrm{X}, \check{\Phi}_\mathrm{X})$ forms a root datum, with associated split complex reductive group $\mathrm{G_X}$.*

Definition 5.1.20. Let $\check{\mathrm{G}}_\mathrm{X}$ denote the complex reductive group with maximal torus $\check{\mathrm{T}}_\mathrm{X}$ with root datum given by the dual of that of Theorem 5.1.19. We will refer to $\check{\mathrm{G}}_\mathrm{X}$ as the *(Langlands) dual group* of X. It admits a morphism to $\check{\mathrm{G}}$. Also see [GN, KS2].

Example 5.1.21. As in Example 5.1.5, if $\mathrm{X} = \mathrm{G}$ is viewed as a spherical $\mathrm{G} \times \mathrm{G}$-variety, the group $\check{\mathrm{G}}_\mathrm{X}$ is simply the Langlands dual $\check{\mathrm{G}}$ of G itself.

Example 5.1.22 (Spherical PGL_2-varieties)**.** Recall the classification of spherical subgroups $\mathrm{H} \subseteq \mathrm{PGL}_2$ from Example 5.1.6. Let us describe the root datum of $\mathrm{X} = \mathrm{PGL}_2/\mathrm{H}$ from Theorem 5.1.19 in each case.

a. If $\mathrm{H} = \mathrm{PGL}_2$, the quotient X is a point, and everything is trivial.

b. If $\mathrm{H} = \mathbf{G}_m$, the orbits of B on X are the same as orbits of $\mathbf{G}_m$ on $\mathbf{P}^1$. There are therefore three orbits, given by $\mathbf{G}_m$ (the open orbit) and the points 0 and ∞. To describe the spherical roots, let us instead consider $\mathrm{SL}_2/\mathbf{G}_m \cong (\mathbf{P}^1 \times \mathbf{P}^1) - \mathbf{P}^1_{\mathrm{diag}}$. Note that $\mathcal{O}_{\mathrm{SL}_2/\mathbf{G}_m} = \mathcal{O}_{\mathrm{SL}_2}^{\mathbf{G}_m} \cong \bigoplus_{n \geq 0} \mathrm{V}_{n\alpha}$, where α is the positive root of SL_2 and $\mathrm{V}_{n\alpha}$ is the representation with highest weight n. It follows that $\Lambda_\mathrm{X} \cong \mathbf{Z}$, generated by α. A little calculation implies that $\mathcal{V} \subseteq \mathfrak{t}_\mathrm{X}$ identifies with $\{v \in \mathfrak{t}_\mathrm{X} | \langle v, \alpha \rangle \leq 0\}$. This implies that $\Sigma_\mathrm{X} = \Delta_\mathrm{X} = \{\alpha\}$, and so $\check{\mathrm{G}}_\mathrm{X} = \mathrm{PGL}_2$. If we worked with $\mathrm{PGL}_2/\mathbf{G}_m$ instead, we would find that $\check{\mathrm{G}}_\mathrm{X} = \mathrm{SL}_2$.

c. If $\mathrm{H} = \mathrm{N}_{\mathrm{PGL}_2}(\mathbf{G}_m)$, the sublattice $\Lambda_\mathrm{X} \subseteq \Lambda_{\mathrm{PGL}_2/\mathbf{G}_m}$ has index two. In particular, by (b) above, we see that $\Lambda_\mathrm{X} = \mathbf{Z} \cdot 2\alpha$, and $\Sigma_\mathrm{X} = \{2\alpha\}$. In particular, Theorem 5.1.19 does not apply to this particular case.

d. If $\mathrm{H} = \mathrm{S} \cdot \mathrm{N} \subseteq \mathrm{PGL}_2$, the orbits of B on X are the same as orbits of H on $\mathbf{P}^1$. There are therefore two orbits, given by $\mathbf{A}^1$ (the open orbit) and the point ∞. Let us assume for simplicity that $\mathrm{S} = \{1\}$. Again, $\Lambda_\mathrm{X} \cong \mathbf{Z}$, and one now calculates that Σ_X is empty. One therefore finds that $\check{\mathrm{G}}_\mathrm{X} = \check{\mathrm{T}}$. In general, the dual group of horospherical varieties is the Cartan subgroup.

The cases (b), (c), and (d) above are known as types T, N, and U. The spherical $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-variety PGL_2 (i.e., the group case of Example 5.1.5) is known as type G.

Remark 5.1.23. If α is a simple root of G (or α and β are two orthogonal simple roots of G) and P_α (or $\mathrm{P}_{\alpha\beta}$) is the associated parabolic subgroup, then the spherical variety $\mathring{\mathrm{X}}\mathrm{P}_\alpha/\mathrm{U}_{\mathrm{P}_\alpha}$ is isomorphic to one of $\mathrm{PGL}_2/\mathrm{PGL}_2$, $\mathrm{PGL}_2/\mathrm{T}$ for T being a torus, $\mathrm{PGL}_2/\mathrm{N}_{\mathrm{PGL}_2}(\mathrm{T})$, or $(\mathrm{PGL}_2 \times \mathrm{PGL}_2)/\mathrm{PGL}_2^{\mathrm{diag}}$. Correspondingly, the unique element of Σ_X is a normalized spherical root, and its type is as defined in Example 5.1.22. In particular, the condition of Theorem 5.1.19 asks that X have no normalized spherical root of type N.

Remark 5.1.24. Assume from now on that X does not have any spherical roots of type N. As in [SV, Section 3.6], the embedding $\check{\mathrm{G}}_{\mathrm{X}} \hookrightarrow \check{\mathrm{G}}$ commutes with the image of a principal $\mathrm{SL}_2 \to \check{\mathrm{L}}(\mathrm{X})$. In particular, there is a map $\iota : \check{\mathrm{G}}_{\mathrm{X}} \times \mathrm{SL}_2 \to \check{\mathrm{G}}$ such that upon restriction to the diagonal torus $\mathbf{G}_m \subseteq \mathrm{SL}_2$, the map $\mathbf{G}_m \to \check{\mathrm{L}}(\mathrm{X})$ is given by $2\rho_{\mathrm{L}(\mathrm{X})} = \sum_{\alpha \in \Phi^+_{\mathrm{L}(\mathrm{X})}} \alpha$ (regarded as a coweight of $\check{\mathrm{G}}$). Since we will mainly deal with spherical varieties of rank 1 below, where $\check{\mathrm{G}}_{\mathrm{X}}$ itself will sometimes be SL_2, we will distinguish the SL_2 above with a superscript: namely, we will write it as $\mathrm{SL}_2^{\mathrm{Arth}}$.

5.2 Review of the relative Langlands conjectures

In this section, we will review the notion of Whittaker induction (following [BZSV, Section 3.4]), and the statement of [BZSV, Conjecture 7.5.1]. This construction takes as input a map $\mathrm{H} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$ and produces a functor from Hamiltonian H-spaces to Hamiltonian G-spaces. We warn the reader that our notation will differ slightly from that of [BZSV, Section 3.4].

Recollection 5.2.1. A *Hamiltonian* G*-space* is a smooth symplectic variety M (with symplectic form ω) equipped with a Hamiltonian G-action (i.e., the map $i : \mathfrak{g} \to \mathrm{T}_{\mathrm{M}}$ given by the derivative of the G-action lands in the subspace of Hamiltonian vector fields on M). The moment map $\mu : \mathrm{M} \to \mathfrak{g}^*$ is characterized by the property that for each $x \in \mathfrak{g}$, we have $d\langle \mu, x\rangle = \langle i(x), \omega\rangle$. We will often simply specify a Hamiltonian G-space as the pair (M, ω) along with its moment map. There will frequently be a grading present, which we encode by an action of $\mathbf{G}_{m,\mathrm{rot}}$ on M, G, and ω. We will say that $(\mathrm{M}, \omega, \mu : \mathrm{M} \to \mathfrak{g}^*)$ is a *graded* Hamiltonian G-space (for a given $\mathbf{G}_{m,\mathrm{gr}}$-action on G) if M has a $\mathbf{G}_{m,\mathrm{gr}}$-action which acts on ω with weight 2, and the moment map μ is $\mathbf{G}_{m,\mathrm{gr}}$-equivariant.

As described in [Saf1], the notion of a Hamiltonian G-space can be phrased entirely in terms of shifted symplectic geometry [PTVV]: the quotient stack $\mathfrak{g}^*/\mathrm{G}$ is (canonically!) a 1-shifted symplectic stack, and the map $\mathrm{M}/\mathrm{G} \to \mathfrak{g}^*/\mathrm{G}$ is a Lagrangian therein.

Let us review the basic example of Whittaker induction.

Example 5.2.2. Let G be a connected reductive group (over $\mathbf{C}$), and let $e \in \mathfrak{g}$ be a principal nilpotent element, so that the Jacobson-Morozov theorem produces a map $\mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$. Let H be the trivial group, and let M denote the trivial Hamiltonian H-space. Then the Whittaker induction of M along the map $\iota : \{1\} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$ is given by $\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M}) = (\psi + \mathfrak{n}^\perp) \times^{\mathrm{N}} \mathrm{G}$. Note that there is an isomorphism

$$\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})/\mathrm{G} \cong (\psi + \mathfrak{n}^\perp)/\mathrm{N} \cong \mathfrak{g}^* /\!\!/ \mathrm{G}.$$

Let us now describe the construction in general.

Construction 5.2.3. Suppose we are given a map $\mathrm{H} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$ of reductive algebraic groups over $\mathbf{C}$ such that H centralizes the map $\mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$. Let $f \in \mathfrak{g}$ be the image of $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2^{\mathrm{Arth}}$ inside $\mathfrak{g}$. The action of $\mathbf{G}_m^{\mathrm{Arth}} \subseteq \mathrm{SL}_2^{\mathrm{Arth}}$ on $\mathfrak{g}$ defines a decomposition

$$\mathfrak{g} = \mathfrak{z}^{\mathrm{Arth}} \oplus \overline{\mathfrak{n}} \oplus \mathfrak{n}_0 \oplus \mathfrak{n},$$

where $\mathfrak{z}^{\mathrm{Arth}}$ is the centralizer of $\mathfrak{sl}_2^{\mathrm{Arth}} \to \mathfrak{g}$, and $\overline{\mathfrak{n}}$, $\mathfrak{n}_0$, and $\mathfrak{n}$ are the negative, zero, and positive weight spaces. Let N denote the associated unipotent subgroup of G. Note that all the weights of the $\mathbf{G}_m^{\mathrm{Arth}}$-action on $\mathfrak{g}$ are integers, and that $e \in \mathfrak{n}$. Note that the orthogonal complement to $\mathfrak{z}^{\mathrm{Arth}} \subseteq \mathfrak{g}$ is a Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$. Let $\mathrm{L} \subseteq \mathrm{G}$ denote the associated subgroup.

Let $\mathfrak{n}_+$ denote the subspace of $\mathfrak{n}$ of elements with weight ≥ 2, and let N_+ denote the associated unipotent subgroup. One can then equip $\mathfrak{n}/\mathfrak{n}_+$ with the structure of a Hamiltonian HN-space. There is an H-invariant symplectic form ω on $\mathfrak{n}/\mathfrak{n}_+$, given by $\omega(x,y) = \langle f, [x,y]\rangle$.[1] Since H preserves ω, we obtain a homomorphism $\mathrm{H} \to \mathrm{Sp}(\mathfrak{n}/\mathfrak{n}_+)$, and hence a map $\mathfrak{h} \to \mathfrak{sp}_{\mathfrak{n}/\mathfrak{n}_+}$. The group H acts on $\mathfrak{n}/\mathfrak{n}_+$ by the adjoint action. Moreover, the group N acts on $\mathfrak{n}/\mathfrak{n}_+ \cong \mathrm{N}/\mathrm{N}_+$ via translation. The moment map $\mu : \mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{h}^* \oplus \mathfrak{n}^*$ is defined as follows:

- The map $\mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{h}^*$ is adjoint to the map

$$\mathfrak{n}/\mathfrak{n}_+ \oplus \mathfrak{h} \to \mathfrak{n}/\mathfrak{n}_+ \oplus \mathfrak{sp}_{\mathfrak{n}/\mathfrak{n}_+} \xrightarrow{(x,g)\mapsto \frac{1}{2}\omega(gx,x)} \mathfrak{g}_a.$$

- The map $\mathfrak{n}/\mathfrak{n}_+ \to \mathfrak{n}^*$ is given by the composite

$$\mathfrak{n}/\mathfrak{n}_+ \xrightarrow{\omega} (\mathfrak{n}/\mathfrak{n}_+)^* \xrightarrow{x\mapsto f+x} \mathfrak{n}^*.$$

  Here, f is viewed as an element of $\mathfrak{n}^*$ via the identification $\mathfrak{n}^* \cong \mathfrak{n}$. Under this isomorphism, the image of $\mathfrak{n}/\mathfrak{n}_+$ inside $\mathfrak{n}$ is simply $f + \mathfrak{n}_1$, where $\mathfrak{n}_1$ is the weight 1 eigenspace.

Remark 5.2.4. There is a natural grading defined on $\mathfrak{n}/\mathfrak{n}_+$, as well as a natural $\mathbf{G}_{m,\mathrm{gr}}$-action on N via the conjugation action of $\mathbf{G}_m^{\mathrm{Arth}}$. If H is equipped with the trivial $\mathbf{G}_{m,\mathrm{gr}}$-action, the Hamiltonian HN-space $\mathfrak{n}/\mathfrak{n}_+$ from Construction 5.2.3 can be viewed as a graded Hamiltonian HN-space.

Definition 5.2.5. Fix a map $\iota : \mathrm{H} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$ of reductive algebraic groups over $\mathbf{C}$ such that H centralizes the map $\mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$. The conjugation action of $\mathbf{G}_m^{\mathrm{Arth}}$ on G composed with the square character equips G with a grading (which we will think of as a $\mathbf{G}_{m,\mathrm{gr}}$-action). Let M be a graded Hamiltonian H-space. Then the *Whittaker induction* $\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})$ is defined as

$$\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M}) = (\mathrm{M} \times \mathfrak{n}/\mathfrak{n}_+) \times^{\mathrm{HN}}_{\mathfrak{h}^*\oplus\mathfrak{n}^*} (\mathrm{T}^*\mathrm{G}),$$

where $\mathrm{T}^*\mathrm{G}$ is regarded as a Hamiltonian HN-space via restriction along $\mathrm{HN} \subseteq \mathrm{G}$. There is a natural grading on $\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})$, coming from the grading on M, the grading on $\mathfrak{n}/\mathfrak{n}_+$ from Remark 5.2.4, and the grading on $\mathrm{T}^*\mathrm{G}$ coming from the $\mathbf{G}_{m,\mathrm{gr}}$-action on G. In particular, note that there is an isomorphism of stacks

$$\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})/\mathrm{G} \cong ((\mathrm{M} \times \mathfrak{n}/\mathfrak{n}_+) \times_{\mathfrak{h}^*\oplus\mathfrak{n}^*} \mathfrak{g}^*)/\mathrm{HN}.$$

The simplest way to understand Whittaker induction in the case when M is a symplectic H-representation is as follows.

Lemma 5.2.6 ([BZSV, Section 3.4.8])**.** *Suppose* M *is a symplectic* H*-representation, and fix an isomorphism* $\mathfrak{g}^* \cong \mathfrak{g}$*. Then there is an isomorphism of stacks*

$$\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})/\mathrm{G} \cong (\mathrm{M} \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}^e))/\mathrm{H}$$

over BG*.*

[1] Note that this symbol is well-defined: if $x \in \mathfrak{n}_+$, then $[x,y]$ lives in weight ≥ 3, so $\langle f, [x,y]\rangle = 0$ since f has weight -2. Moreover, this form is indeed nondegenerate: if $x \in \mathfrak{n}$ is nonzero of weight 1, then $[f,x]$ is a nonzero element of weight -1. This implies that there is some $y \in \mathfrak{n}$ of weight 1 such that $\langle [f,x], y\rangle = \langle f, [x,y]\rangle$ is nonzero, as desired.

Proof. Using [GG1, Lemma 2.1], one obtains an inclusion $f + \mathfrak{g}^e \subseteq f + \mathfrak{n}_+^\perp$ which is a slice of the N-action on $f + \mathfrak{n}_+^\perp$. Therefore, there is an isomorphism

$$\begin{aligned} \mathrm{N} \times (\mathrm{M} \times_{\mathfrak{h}^*} \mathfrak{g}^e) &\to (\mathrm{M} \times \mathfrak{n}/\mathfrak{n}_+) \times_{\mathfrak{h}^* \oplus \mathfrak{n}^*} \mathfrak{g}^* \\ &\cong \{(v, x) \in \mathrm{M} \times (f + \mathfrak{n}_+^\perp) \text{ such that } \mu(v) = x|_{\mathfrak{h}}\}, \end{aligned}$$

sending $(n, v, y) \mapsto (v, n \cdot (f + y))$. This isomorphism is H-equivariant, so it follows that $\mathrm{WInd}_\iota^{\mathrm{G}}(\mathrm{M})/\mathrm{G}$ is isomorphic to $(\mathrm{M} \times_{\mathfrak{h}^*} \mathfrak{g}^e)/\mathrm{H}$ as stacks over BG. This implies the desired claim, since $\mathrm{M} \times_{\mathfrak{h}^*} \mathfrak{g}^e \cong \mathrm{M} \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}^e)$. □

Remark 5.2.7. An alternative way to describe Whittaker induction using the language of shifted symplectic geometry [PTVV] is as follows. Recall (see [Saf1]) that a Lagrangian morphism $\mathrm{L} \to \mathfrak{h}^*/\mathrm{H}$ is equivalent to the data of a Hamiltonian H-space M; the correspondence sets $\mathrm{L} = \mathrm{M}/\mathrm{H}$. Moreover, intersecting Lagrangian correspondences produces another Lagrangian correspondence. From this perspective, one can describe Whittaker induction as follows. Let $\iota : \mathrm{H} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$ be a map of reductive algebraic groups over **C** such that H centralizes the map $\mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{G}$. Let $(\psi + \mathfrak{n}^\perp)/\mathrm{N}$ denote the slice associated to the $\mathfrak{sl}_2$-triple; then, there is a Lagrangian correspondence

$$\begin{tikzcd} & (\psi + \mathfrak{n}^\perp)/\mathrm{NH} \arrow[dl] \arrow[dr] & \\ \mathfrak{h}^*/\mathrm{H} & & \mathfrak{g}^*/\mathrm{G}, \end{tikzcd}$$

and Whittaker induction amounts to intersecting the above Lagrangian correspondence with the Lagrangian $\mathrm{M}/\mathrm{H} \to \mathfrak{h}^*/\mathrm{H}$. (This will produce a Lagrangian morphism $(\mathrm{M} \times_{\mathfrak{h}^*} (\psi + \mathfrak{n}^\perp)/\mathrm{NH} \to \mathfrak{g}^*/\mathrm{G}$, which is identified with the Hamiltonian G-variety of Definition 5.2.5.)

Let us now recall a statement of [BZSV, Conjecture 7.5.1]; our presentation will follow [BZSV, Section 4.3]. Assume for now that X is an affine spherical G-variety over **C** which is the affine closure of its open G-orbit (for instance, this holds if X is affine and homogeneous).

Definition 5.2.8. A *color* of X is an irreducible B-stable divisor which is not G-stable (if X is homogeneous, this is simply an irreducible B-stable divisor). Following [BZSV, Definition 4.3.4], a standard parabolic $\mathrm{P} \subseteq \mathrm{G}$ is said to be of *even spherical type* if the spherical $\mathrm{P}/\mathrm{U}_\mathrm{P}$-variety $\mathrm{X}^\circ\mathrm{P}/\mathrm{U}_\mathrm{P}$ is isomorphic to either the spherical SO_{2n+1}-variety $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$ or the spherical G_2-variety $\mathrm{G}_2/\mathrm{SL}_3$. (Note that there are diffeomorphisms $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n} \cong \mathrm{S}^{2n}$ and $\mathrm{G}_2/\mathrm{SL}_3 \cong \mathrm{S}^6$.) A color D is said to be of *even spherical type* if it meets $\mathrm{X}^\circ\mathrm{P}$ for a standard parabolic P of even spherical type. Let $\mathcal{C}_\mathrm{X}$ denote the set of colors of X of even spherical type.

Suppose that the elements of $\mathcal{C}_\mathrm{X}$ freely generate a direct summand of $\check{\Lambda}_\mathrm{X}$. Let $\mathcal{D}_\mathrm{X}$ denote the set of dominant W_X-translates of $\mathcal{C}_\mathrm{X} \subseteq \check{\Lambda}_\mathrm{X}$, and let $\mathcal{D}_\mathrm{X}^{\max}$ denote the subset of maximal elements of $\mathcal{D}_\mathrm{X}$ (with respect to the ordering via coroots of $\check{\mathrm{G}}_\mathrm{X}$). Let S_X denote the $\check{\mathrm{G}}_\mathrm{X}$-representation with highest weights $\mathcal{D}_\mathrm{X}^{\max}$. It is expected (see [BZSV, Conjecture 4.3.16]) that S_X admits an $\check{\mathrm{G}}_\mathrm{X}$-invariant symplectic form.

Example 5.2.9 ([BZSV, Example 4.3.9])**.** Consider the example of the spherical GL_2-variety $\mathrm{X} = \mathrm{GL}_2/\mathbf{G}_m$ (in which case $\check{\mathrm{G}}_\mathrm{X} = \check{\mathrm{G}} = \mathrm{GL}_2$). Then $\mathrm{U}\backslash\mathrm{X}^\circ \cong \mathbf{G}_m^2$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (b, d^{-1}\det)$. The colors of X are given by the vanishing loci of b and d, and are both of even spherical type. As explained in [BZSV, Example 4.3.9], this implies that $\mathcal{C}_\mathrm{X}$ is the subset $\{\check{\alpha}_1, -\check{\alpha}_2\}$ of $\check{\Lambda}_\mathrm{X} = \check{\Lambda}$, which in turn implies that $\mathrm{S}_\mathrm{X} = \mathbf{A}^2 \oplus (\mathbf{A}^2)^* \cong \mathrm{T}^*(\mathbf{A}^2)$ as an $\check{\mathrm{G}}_\mathrm{X}$-representation. However, as remarked in [BZSV], the condition that the elements of $\mathcal{C}_\mathrm{X}$ freely generate a direct summand of $\check{\Lambda}_{\mathrm{X}}$ is *not* true in the example of $\mathrm{PGL}_2/\mathbf{G}_m$ (whose dual group is $\check{\mathrm{G}}_{\mathrm{X}} = \check{\mathrm{G}} = \mathrm{SL}_2$). Nevertheless, the variant of Definition 5.2.8 discussed in [BZSV, Section 4.4] shows that S_{X} is the $\check{\mathrm{G}}_{\mathrm{X}}$-representation $\mathrm{T}^*(\mathbf{A}^2)$.

Example 5.2.10. For $n > 2$, the spherical GL_n-variety $\mathrm{X} = \mathrm{GL}_n/\mathrm{GL}_{n-1}$ still has $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{GL}_2$, but the representation S_{X} is zero. (I am very grateful to Justin Hilburn and Yiannis Sakellaridis for this point.) For instance, when $n = 3$, the Whittaker induction $\mathrm{WInd}_\iota^{\mathrm{GL}_3}\mathrm{S}_{\mathrm{X}}$ along the map $\iota : \mathrm{GL}_2 \times \mathrm{SL}_2^{\mathrm{Arth}} \to \mathrm{GL}_3$ of Remark 5.1.24 can be identified with $\mathrm{T}^*(\mathrm{GL}_3/\mathrm{GL}_2)$ using Lemma 5.2.6.

Example 5.2.11. Consider the example of the spherical SO_4/μ_2-variety $\mathrm{SO}_4/\mu_2 \cdot \mathrm{SO}_3$ (in which case $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{SL}_2$). Since $\mathrm{Spin}_4 \cong \mathrm{SL}_2 \times \mathrm{SL}_2$, there is an isomorphism $\mathrm{SO}_4/\mu_2 \cong \mathrm{SO}_3 \times \mathrm{SO}_3$, under which the embedding of SO_3 into SO_4/μ_2 is given by the diagonal. Therefore, there is an isomorphism $\mathrm{SO}_4/\mu_2\mathrm{SO}_3 \cong \mathrm{SO}_3$, and this spherical SO_4/μ_2-variety can be understood as the group case for SO_3. Using this, one can show that $\check{\mathrm{G}}\backslash\mathrm{WInd}_\iota^{\check{\mathrm{G}}}\mathrm{S}_{\mathrm{X}} \cong \mathfrak{sl}_2/\check{\mathrm{G}}_{\mathrm{X}}$.

The following is a slight variant of [BZSV, Conjecture 7.5.1].

Conjecture 5.2.12. *Suppose* X *is a smooth affine spherical* G*-variety over* $\mathbf{C}$ *which is the affine closure of its open* G*-orbit, and with no roots of type N. Let* $\iota : \check{\mathrm{G}}_{\mathrm{X}} \times \mathrm{SL}_2^{\mathrm{Arth}} \to \check{\mathrm{G}}$ *denote the map of Remark 5.1.24. Suppose that* S_{X} *admits an* $\check{\mathrm{G}}_{\mathrm{X}}$*-invariant symplectic form, and let* $\check{\mathrm{M}}$ *denote* $\mathrm{WInd}_\iota^{\check{\mathrm{G}}}\mathrm{S}_{\mathrm{X}}$*. Then:*

- *There is an equivalence*[2]

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{X}(\!(t)\!); \mathbf{Q}) \cong \mathrm{QCoh}(\mathrm{sh}^{1/2}\check{\mathrm{M}}/\check{\mathrm{G}}(-2\rho)).$$

- *This equivalence is equivariant for the actions of* $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]\times\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!); \mathbf{Q})$ *and* $\mathrm{QCoh}(\check{\mathfrak{g}}^*[2-2\rho]/\check{\mathrm{G}}[-2\rho])$ *under the equivalence of Theorem 3.1.4.*

Remark 5.2.13. One of the requirements for the equivalence of Conjecture 5.2.12 is the "pointing" of [BZSV, Section 7.5.2]. Namely, the pushforward of the constant sheaf along $i : \mathrm{X}[\![t]\!] \to \mathrm{X}(\!(t)\!)$ must be sent under the equivalence of Conjecture 5.2.12 to the structure sheaf of $\mathrm{sh}^{1/2}\check{\mathrm{M}}/\check{\mathrm{G}}$. This implies, in particular, that

$$\mathrm{End}_{\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{X}(\!(t)\!);\mathbf{Q})}(i_*\underline{\mathbf{Q}}_{\mathrm{X}[\![t]\!]}) \simeq \mathcal{O}^{\check{\mathrm{G}}}_{\mathrm{sh}^{1/2}\check{\mathrm{M}}}.$$

The left-hand side is simply $\mathrm{C}^*_{\mathrm{G}[\![t]\!]}(\mathrm{X}[\![t]\!]; \mathbf{Q}) \simeq \mathrm{C}^*_{\mathrm{G}}(\mathrm{X}; \mathbf{Q})$, while the right-hand side is $\mathcal{O}_{\mathrm{sh}^{1/2}\check{\mathrm{M}}/\!\!/\check{\mathrm{G}}}$. Therefore, the "pointing" requirement can be restated as the existence of an equivalence of $\mathbf{E}_1$-$\mathbf{Q}$-algebras $\mathrm{C}^*_{\mathrm{G}}(\mathrm{X}; \mathbf{Q}) \simeq \mathcal{O}_{\mathrm{sh}^{1/2}\check{\mathrm{M}}/\!\!/\check{\mathrm{G}}}$. If $\mathrm{X} = \mathrm{G}/\mathrm{H}$, the left-hand side is exactly $\mathrm{C}^*_{\mathrm{H}}(*; \mathbf{Q}) \simeq \mathrm{sh}^{1/2}\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Q})$, so this equivalence can be rephrased as a graded isomorphism

$$\check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \cong \mathrm{Spec}\,\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Q}) \cong \check{\mathfrak{h}}^*(2)/\!\!/\check{\mathrm{H}}. \tag{5.2.1}$$

Using Lemma 5.2.6, one can identify $\check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \cong (\mathrm{S}_{\mathrm{X}} \oplus (\check{\mathfrak{g}}_{\mathrm{X}}^\perp \cap \check{\mathfrak{g}}^e))/\!\!/\check{\mathrm{G}}_{\mathrm{X}}$; it might be possible to prove the resulting identification with $\check{\mathfrak{h}}^*(2)/\!\!/\check{\mathrm{H}}$ in a direct manner (without having first established Conjecture 5.2.12). One approach to proving (5.2.1) is to construct the Cartan $\mathfrak{t}_{\mathrm{H}}$ and the Weyl group W_{H} of H from $\check{\mathrm{M}}$.

[2] The ∞-category on the left-hand side is defined as the full subcategory of $\mathrm{Shv}^c_{\mathrm{G}[\![t]\!]}(\mathrm{X}(\!(t)\!); \mathbf{Q})$ generated by IC_0 under the action of $\mathrm{Shv}^c_{(\mathrm{G}\times\mathrm{G})[\![t]\!]}(\mathrm{G}(\!(t)\!); \mathbf{Q})$.

Remark 5.2.14. What happens beyond $\mathbf{Q}$-coefficients? Let k be a (discrete) commutative ring. Following the philosophy of Corollary 4.3.17, one should expect that there is a 1-parameter degeneration

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{X}(\!(t)\!); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}/\check{\mathrm{G}}(-2\rho)).$$

If the left-hand side satisfies the requisite evenness conditions, then the desired degeneration should be provided by Definition 4.2.4. This is closely related to the integrality questions from [BZSV, Section 5.3]. We will explore some examples of such a degeneration (over $k = \mathbf{Z}$) later.

There are several variants of Conjecture 5.2.12, e.g., where one allows for some ramification. For instance, in the case of tame ramification, local geometric Langlands suggests the following (which is closely related to [FGT, Conjecture 1.1.3]):

Conjecture 5.2.15. *Let $\mathrm{I} \subseteq \mathrm{G}[\![t]\!]$ be an Iwahori subgroup associated to a Borel $\mathrm{B} \subseteq \mathrm{G}$. Suppose X is a smooth affine spherical G-variety over $\mathbf{C}$ which is the affine closure of its open G-orbit, and with no roots of type N. Let $\check{\mathrm{M}}$ denote its dual Hamiltonian $\check{\mathrm{G}}$-space à la [BZSV]. Then:*

- *There is an equivalence*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{I}}(\mathrm{X}(\!(t)\!); \mathbf{Q}) \simeq \mathrm{QCoh}(\mathrm{sh}^{1/2}(\check{\mathrm{M}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{\mathrm{G}}),$$

  and the image of $\mathrm{IC}_0 = i_! \underline{\mathbf{Q}}$ under the above equivalence should be the structure sheaf of $\mathrm{sh}^{1/2}(\check{\mathrm{M}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{\mathrm{G}}$. Here, $\widetilde{\check{\mathfrak{g}}} \cong \mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})/\check{\mathrm{T}}$ denotes the Grothendieck-Springer resolution over $\check{\mathfrak{g}}^*$; and $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{I}}(\mathrm{X}(\!(t)\!); \mathbf{Q})$ denotes the full subcategory of $\mathrm{Shv}^{c}_{\mathrm{I}}(\mathrm{X}(\!(t)\!); \mathbf{Q})$ generated by IC_0 under the action of $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{I}\times\mathrm{I}}(\mathrm{G}(\!(t)\!); \mathbf{Q})$ via convolution.

- *This equivalence should be equivariant for the natural action of*

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{I}\times\mathrm{I}}(\mathrm{G}(\!(t)\!); \mathbf{Q}) \simeq \mathrm{QCoh}(\mathrm{sh}^{1/2}(\widetilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}^*} \widetilde{\check{\mathfrak{g}}})/\check{\mathrm{G}}).$$

  on both sides. This equivalence is provided by [Bez].

There is an obvious variant of Conjecture 5.2.15 for standard parahorics, where the relevant replacement of the equivalence of [Bez] is proved in [CD].

Remark 5.2.16. Suppose, for instance, that $\mathrm{X} = \mathrm{G}/\mathrm{H}$. As in Remark 5.2.13, the first part of Conjecture 5.2.15 then implies that there should be an isomorphism

$$(\check{\mathrm{M}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\!\!/\check{\mathrm{G}} \cong \operatorname{Spec} \mathrm{H}^*_{\mathrm{T}}(\mathrm{G}/\mathrm{H}; \mathbf{Q}) \cong \check{\mathfrak{t}}^*(2) \times_{\check{\mathfrak{g}}^*(2)/\!\!/\check{\mathrm{G}}} \check{\mathfrak{h}}^*(2)/\!\!/\check{\mathrm{H}}.$$

This might be easier to prove than the isomorphism described in (5.2.1).

Remark 5.2.17. Following the philosophy of Corollary 4.3.17, one expects (as in Remark 5.2.14) that for a general (discrete) commutative ring k, there is a 1-parameter degeneration

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{I}}(\mathrm{X}(\!(t)\!); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}((\check{\mathrm{M}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{\mathrm{G}}(-2\rho)).$$

Again, if the left-hand side satisfies the requisite evenness conditions, then the desired degeneration should be provided by Definition 4.2.4.

Remark 5.2.18. In general, there is a graded isomorphism

$$\check{\mathrm{M}}/\check{\mathrm{G}} \cong \check{\mathrm{Y}}/\check{\mathrm{G}}_{\mathrm{X}} \times \text{“Normalization”}$$

of stacks, which comes from an $\check{\mathrm{G}}_{\mathrm{X}}$-equivariant isomorphism between $\mathrm{V}_{\mathrm{X}} = \mathrm{S}_{\mathrm{X}} \oplus (\check{\mathfrak{g}}_{\mathrm{X}}^{\perp} \cap \check{\mathfrak{g}}^{e})$ and $\check{\mathrm{Y}} \times$ “Normalization”. This is at least somewhat surprising: for instance, when $\mathrm{X} = \mathrm{GL}_{n+1}/\mathrm{GL}_n$ (so $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{GL}_2$), we have $\mathrm{S}_{\mathrm{X}} = \mathrm{T}^*(\mathbf{A}^2)$ when $n = 1$ by Example 5.2.9; but $\mathrm{S}_{\mathrm{X}} = 0$ for $n \geq 2$. Nevertheless, $\check{\mathrm{Y}}$ always identifies with $\mathrm{T}^*(\mathbf{A}^2)$ as GL_2-schemes. In general, the “normalization” term above can be identified with $\mathfrak{l}_{\mathrm{X}}^{\wedge}/\!\!/\mathrm{L}_{\mathrm{X}}^{\wedge}$, where $\mathrm{L}_{\mathrm{X}}^{\wedge}$ is the subgroup of $\check{\mathrm{G}}$ from [KS2]. (This is *not* quite the Langlands dual of the Levi subgroup $\mathrm{L}(\mathrm{X})$.) However, I do not know how to prove this in general; any general statement would be very interesting (it is closely related to Remark 5.3.9).

To conclude this section, let us sketch a generalization of the preceding picture to the case of more general coefficient $\mathbf{E}_\infty$-rings. For this, we will assume Conjecture 4.3.20, which we recall proposes a 1-parameter degeneration from $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{Gr}_{\mathrm{G}}; k)$ to $\mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$. Here, k is an $\mathbf{E}_\infty$-ring and $\mathbf{H}$ denotes the canonical 1-dimensional formal group defined over $\mathrm{Spev}(k)$. (If k admits a genuine equivariant refinement, and we consider genuine equivariant sheaves (instead of Borel-equivariant ones), then $\mathbf{H}$ should instead be taken to be the group scheme associated to the algebraization of the formal group scheme $\mathrm{Spf}(k^{\mathbf{C}\mathrm{P}^\infty})$ over k.)

If X is a smooth affine (spherical) G-variety over $\mathbf{C}$, then one can define a category $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{X}(\!(t)\!); k)$. This category admits an action of $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{Gr}_{\mathrm{G}}; k)$ by convolution, which allows us to define $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl,Sat}}(\mathrm{X}(\!(t)\!); k)$ as the full subcategory generated by the δ-sheaf at $\mathrm{X}[\![t]\!] \subseteq \mathrm{X}(\!(t)\!)$ under the convolution action of $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{Gr}_{\mathrm{G}}; k)$.

Motivated by Conjecture 4.3.20 and Conjecture 5.2.12, one expects a spectral/Langlands dual description of a degeneration of the category $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl,Sat}}(\mathrm{X}(\!(t)\!); k)$. We now define the type of objects appearing on the spectral side. To motivate it, recall from [Saf1] that the datum of a Hamiltonian $\check{\mathrm{G}}$-space $\check{\mathrm{M}}$ is *equivalent* to the datum of a Lagrangian morphism $\check{\mathrm{M}}/\check{\mathrm{G}} \to \check{\mathfrak{g}}^*/\check{\mathrm{G}}$, where the quotient stack $\check{\mathfrak{g}}^*/\check{\mathrm{G}} = \mathrm{T}^*[1](\mathrm{B}\check{\mathrm{G}})$ is equipped with its canonical 1-shifted symplectic structure. Similarly, a graded Hamiltonian $\check{\mathrm{G}}$-space is equivalent to the datum of a Lagrangian morphism $\check{\mathrm{M}}/\check{\mathrm{G}} \to \check{\mathfrak{g}}^*(2)/\check{\mathrm{G}}$ over $\mathrm{B}\mathbf{G}_m$. Conjecture 4.3.20 suggests:

Definition 5.2.19. The stack $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ from Conjecture 4.3.20 admits a natural 1-shifted symplectic structure, coming from the isomorphism $\mathfrak{g} \cong \check{\mathfrak{g}}^*$ and the fact that the tangent fibers of $\mathrm{G}_{\mathbf{H}}$ identify with the Lie algebra $\mathfrak{g}$. Turning [Saf1] on its head, we will define an $\mathbf{H}$-*Hamiltonian* $\check{\mathrm{G}}$-*space* to be the datum of a Lagrangian morphism $\check{\mathrm{L}} \to \mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$. Note that this morphism defines a $\check{\mathrm{G}}$-space via $\check{\mathrm{M}}_{\mathbf{H}} = \check{\mathrm{L}} \times_{\mathrm{B}\check{\mathrm{G}}} \mathrm{Spev}(k)$, and we will generally refer to $\check{\mathrm{M}}_{\mathbf{H}}$ itself as a $\mathbf{H}$-Hamiltonian space.

When $k = \mathrm{ku}$ and $\mathbf{H} = \mathbf{G}_\beta$, we referred to an $\mathbf{H}$-Hamiltonian space as a ku-*Hamiltonian space* in [Dev3]. If $k = \mathrm{KU}$, for instance, then this is essentially the data of a *quasi-Hamiltonian space* in the sense of [AMM].

Conjecture 5.2.20. *Suppose* X *is a smooth affine spherical* G*-variety over* $\mathbf{C}$ *which is the affine closure of its open* G*-orbit, and with no roots of type N. Let k be an evenly descendable* $\mathbf{E}_\infty$*-ring, and let* $\mathbf{H}$ *denote the formal group* $\mathrm{Spev}(k^{\mathbf{C}\mathrm{P}^\infty})$ *over* $\mathrm{Spev}(k)$*. Then there is a* $\check{\mathrm{G}}$*-space* $\check{\mathrm{M}}_{\mathbf{H}}$ *over* $\mathrm{Spev}(k)$ *such that:*

- *There is a* 1*-parameter degeneration*

$$\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl,Sat}}(\mathrm{X}(\!(t)\!); k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}}),$$

under which the δ-sheaf at $\mathrm{X}[\![t]\!] \subseteq \mathrm{X}(\!(t)\!)$ degenerates to the structure sheaf on the right-hand side. This degeneration is linear over the degeneration of Mod_k *into* $\mathrm{QCoh}(\mathrm{Spev}(k))$.

- *Suppose* G *is simply-laced with torsion-free fundamental group. Then* $\check{\mathrm{M}}_{\mathbf{H}}$ *admits the structure of an* $\mathbf{H}$*-Hamiltonian* $\check{\mathrm{G}}$*-space, and the preceding degeneration is equivariant for the actions of* $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{Gr}_{\mathrm{G}}; k)$ *and* $\mathrm{QCoh}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$ *under the equivalence of Conjecture 4.3.20.*

There is a similar variant for genuine equivariant sheaves. Note that we have not specified the degeneration in question for the general case, but it ought to be analogous to the degenerations discussed in § 4.2. (We will see some examples of Conjecture 5.2.20 below.)

Remark 5.2.21. Suppose, for instance, that $\mathrm{X} = \mathrm{G}/\mathrm{H}$. As in Remark 5.2.13, the first part of Conjecture 5.2.20 implies that there should be an isomorphism

$$\check{\mathrm{M}}_{\mathbf{H}} /\!\!/ \check{\mathrm{G}} \cong \mathrm{Spev}_{\mathcal{M}_{\mathrm{G},0}} \mathcal{H}^0_{\mathrm{G}}(\mathrm{G}/\mathrm{H}; k) \cong \mathcal{M}_{\mathrm{H},0}.$$

If G is simply-laced with torsion-free fundamental group, then the map $\check{\mathrm{M}}_{\mathbf{H}} /\!\!/ \check{\mathrm{G}} \to \mathrm{G}_{\mathbf{H}} /\!\!/ \check{\mathrm{G}}$ induced by the $\mathbf{H}$-Hamiltonian structure on $\check{\mathrm{M}}_{\mathbf{H}}$ identifies with the map $\mathcal{M}_{\mathrm{H},0} \to \mathcal{M}_{\mathrm{G},0}$ induced by the inclusion $\mathrm{H} \subseteq \mathrm{G}$.

Example 5.2.22. Conjecture 5.2.20 also makes sense for Whittaker-twisted variants of X, where one uses the Kirillov model [GL] to make sense of the category of Whittaker-twisted sheaves. If $\mathrm{X} = \mathrm{G}/(\mathrm{N}, \psi)$, for instance, then the $\mathbf{H}$-Hamiltonian $\check{\mathrm{G}}$-space in question is just $\mathrm{B}\check{\mathrm{G}}$ (which, when G is simply-laced with torsion-free fundamental group, admits a Lagrangian map to $\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}}$ via the identity section of $\mathrm{G}_{\mathbf{H}}$). Conjecture 5.2.20 states that there is a 1-parameter degeneration

$$\mathrm{Whit}(\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl,Sat}}(\mathrm{G}(\!(t)\!); k)) = \mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; k) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\mathrm{B}\check{\mathrm{G}}(-2\rho)).$$

Although I do not know how to show this directly (it is closely related to Remark 3.4.3 and Conjecture 4.6.6), I expect that this can in fact be proved by showing (using parity considerations) that in the universal case when k is the sphere spectrum S, the degeneration $\mathrm{Whit}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S})$ of $\mathrm{Whit}(\mathrm{Gr}_{\mathrm{G}}; \mathrm{S})$ identifies (as a category over $\mathrm{Spev}(\mathrm{S}) = \mathcal{M}_{\mathrm{fg}}$) with the pullback of $\mathrm{Whit}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{Z})$ over $\mathrm{B}\mathbf{G}_m$ along the canonical map $\mathcal{M}_{\mathrm{fg}} \to \mathrm{B}\mathbf{G}_m$. Since $\mathrm{Whit}^{\mathrm{gr}}(\mathrm{Gr}_{\mathrm{G}}; \mathbf{Z}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathrm{B}\check{\mathrm{G}}_{\mathbf{Z}}(-2\rho))$ by the usual geometric Casselman-Shalika equivalence, the desired degeneration follows.

Remark 5.2.23. In Remark 5.2.18, we said that the stack $\check{\mathrm{M}}/\check{\mathrm{G}}$ splits as the product of $\check{\mathrm{Y}}/\check{\mathrm{G}}_{\mathrm{X}}$ and $\mathfrak{l}^{\wedge}_{\mathrm{X}} /\!\!/ \mathrm{L}^{\wedge}_{\mathrm{X}}$. In the context of Conjecture 5.2.20, this will generally only happen for $\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}}$ when k is an *ordinary* commutative ring. This phenomenon shows up clearly in Example 5.5.24.

Remark 5.2.24. Even when k is an ordinary commutative ring, Conjecture 5.2.20 "explains" some subtleties in the proposal for the relative Langlands program from [BZSV] related to issues of spectral quantization. For instance, if $\check{\mathrm{M}}$ is equivariantly polarized, in the sense that it is $\check{\mathrm{G}}$-equivariantly isomorphic to a (twisted) cotangent bundle, then one expects an extension of Conjecture 5.2.20/our interpretation of [BZSV, Conjecture 7.5.1]. If $\check{\mathrm{M}} = \mathrm{T}^*(\check{\mathrm{X}})$ where $\check{\mathrm{X}}$ is an affine $\check{\mathrm{G}}$-space, for example, this extension predicts an *equivalence* (not just a degeneration) of k-linear ∞-categories of the form

$$\mathrm{Shv}_{\mathrm{G}(\mathcal{O})}(\mathrm{X}(\mathrm{F}); k) \simeq \mathrm{IndCoh}(\mathcal{L}(\check{\mathrm{X}})/\check{\mathrm{G}}),$$

where $\mathcal{L}(\check{X}) = \mathrm{Map}(S^1, \check{X})$ is the free loop space of $\check{X}$. The Koszul duality between Hochschild homology and $\mathbf{E}_2$-Hochschild cohomology allows us to rewrite the above equivalence as

$$\mathrm{Shv}_{G(\mathcal{O})}(X(F); k) \simeq \mathrm{QCoh}(\mathrm{Spec}(\mathrm{HC}_{\mathbf{E}_2}(\check{X}/k))/\check{G}),$$

where $\mathrm{HC}_{\mathbf{E}_2}(\check{X}/k)$ is the $\mathbf{E}_2$-Hochschild cohomology of $\check{X}$ (which is an $\mathbf{E}_2^{\mathrm{fr}} \otimes \mathbf{E}_1$-algebra by the Deligne conjecture).

This relates to Conjecture 5.2.20 as follows: by the Hochschild-Kostant-Rosenberg theorem (and shearing), the right-hand side of this equivalence admits a 1-parameter degeneration into the category $\mathrm{IndCoh}^{\mathrm{gr}}(\mathrm{T}[1](\check{X})/\check{G})$, which is indeed equivalent by Koszul duality to $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{T}^*(\check{X})/\check{G})$. However, if $\check{M}$ is not equivariantly polarized, then it is not clear how to provide a "Koszul dual" lift of $\mathrm{QCoh}^{\mathrm{gr}}(\check{M}/\check{G})$ along the even filtration; this is called the issue of "spectral quantization". Said differently, the issue of spectral quantization precisely amounts to the issue of constructing an $\mathbf{E}_3$-k-algebra (analogous to $\mathrm{HC}_{\mathbf{E}_2}(\check{X}/k)$) whose associated graded under the even filtration and Koszul duality is isomorphic to $\mathcal{O}_{\check{M}}$ as a (graded) Poisson algebra. Conjecture 5.2.20 bypasses this issue by only asking for a degeneration of $\mathrm{Shv}_{G(\mathcal{O})}(X(F); k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\check{M}/\check{G})$, as opposed to an equivalence of categories.

5.3 The G-equivariant ku-(co)homology of $\mathcal{L}(G/H)$

Fix a compact Lie group G, and let $H \subseteq G$ be a closed subgroup. Throughout this section, we will always assume that H and G are connected, and also (for simplicity) that G/H has finite fundamental group (so that $\Omega(G/H)$ has finitely many connected components). We will abusively write $G(\!(t)\!)$ or $G[\![t]\!]$ below to mean $G_{\mathbf{C}}(\!(t)\!)$ or $G_{\mathbf{C}}[\![t]\!]$, respectively.

We will now discuss some general statements about the G-equivariant k-(co)homology of the free loop space $\mathcal{L}(G/H) = \mathrm{Map}(S^1, G/H)$. The following basic result is an analogue of the algebro-geometric fact that $G\backslash \mathrm{T}^*(G/H) \cong H\backslash(\mathfrak{g}/\mathfrak{h})^*$, or its homotopic analogue that $(G/H)_+^{hG} \simeq (S^{\mathfrak{g}/\mathfrak{h}})^{hH}$.

Proposition 5.3.1. *Let H act on G/H, and hence on $\Omega(G/H)$, by conjugation (equivalently, left-translation). Then the G-space $\mathcal{L}(G/H)$ is G-equivariantly homotopy equivalent to $\mathrm{Ind}_H^G \Omega(G/H)$. In particular, there is an equivalence of orbispaces*

$$G\backslash\mathcal{L}(G/H) \simeq H\backslash\Omega(G/H).$$

Proof. It is a classical fact that the map $m : G \times \Omega G \to \mathcal{L}G$ sending (g, γ) to the loop $\gamma_g : t \mapsto g\gamma(t)$ is a homotopy equivalence. The left action of G on $\mathcal{L}G$ is simply given by

$$G \ni g' : \gamma_g(t) \mapsto \gamma_{g'g}(t),$$

which allows us to identify $\mathcal{L}G \simeq \mathrm{Ind}_{\{1\}}^G \Omega G$. Recall that there is a principal fibration

$$H \to G \to G/H,$$

which gives equivalences $\Omega(G/H) \simeq \Omega G/\Omega H$ and $\mathcal{L}(G/H) \simeq \mathcal{L}G/\mathcal{L}H$. Since and the diagram

$$\begin{tikzcd} H \times \Omega H \arrow[r] \arrow[d, "m"'] & G \times \Omega G \arrow[d, "m"] \\ \mathcal{L}H \arrow[r] & \mathcal{L}G \end{tikzcd}$$

commutes, we find that there is an equivalence of G-spaces

$$\mathcal{L}G/\mathcal{L}H \simeq (\mathrm{Ind}_{\{1\}}^{G}\Omega G)/(\mathrm{Ind}_{\{1\}}^{H}\Omega H) \simeq \mathrm{Ind}_{H}^{G}\Omega(G/H),$$

as desired. Alternatively, this also follows from (5.3.1) using that $* \times_{* \times_{*/G} */H} * \simeq \Omega(G/H)$. $\square$

Let us now state our key assumption.

Hypothesis 5.3.2. We will assume that $H_{\mathbf{C}} \subseteq G_{\mathbf{C}}$ is a connected reductive subgroup such that if G (resp. H) denotes the maximal compact subgroup of $G(\mathbf{C})$ (resp. $H(\mathbf{C})$), there is a homotopy equivalence of orbifolds

$$G_{\mathbf{C}}[\![t]\!]\backslash G_{\mathbf{C}}(\!(t)\!)/H_{\mathbf{C}}(\!(t)\!) \simeq G\backslash\mathcal{L}(G/H).$$

We will also assume that the $G[\![t]\!]$-action on $G_{\mathbf{C}}(\!(t)\!)/H_{\mathbf{C}}(\!(t)\!)$ is placid, in the sense that there is a presentation $G_{\mathbf{C}}(\!(t)\!)/H_{\mathbf{C}}(\!(t)\!) = \mathrm{colim}_j X^j$ where each X^j is an inverse limit $\lim_n X^j_n$ such that each X^j_n is a $G_{\mathbf{C}}[\![t]\!]$-scheme of finite type where the $G_{\mathbf{C}}[\![t]\!]$-action on X^j_n factors through $G_{\mathbf{C}}[\![t]\!]/t^{m_n}$ for some $m_n \gg 0$ compatibly in n, and such that the maps $X^j_n \to X^j_{n'}$ are $G_{\mathbf{C}}[\![t]\!]/t^{m_n}$-equivariant affine smooth surjections.

Thanks to Hypothesis 5.3.2, one can make sense of the equivariant (co)homology of the stack $G_{\mathbf{C}}[\![t]\!]\backslash G_{\mathbf{C}}(\!(t)\!)/H_{\mathbf{C}}(\!(t)\!)$, and furthermore identify $\mathcal{F}_{G_{\mathbf{C}}[\![t]\!]}(G_{\mathbf{C}}(\!(t)\!)/H_{\mathbf{C}}(\!(t)\!); k)$ with $\mathcal{F}_G(\mathcal{L}(G/H); k)$, etc. We will assume Hypothesis 5.3.2 for the remainder of this section. In the case of symmetric varieties, the part about homotopy equivalences was proved in [Mit].

Remark 5.3.3. There is a multiplicative presentation of $\Omega(G/H)$ as $\mathrm{colim}\, X_\lambda$ via finite H-spaces X_λ, and the induced G-spaces $\mathrm{Ind}_H^G X_\lambda$ defines a presentation of $\mathcal{L}(G/H)$ by finite G-spaces. It follows that there is an equivalence $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!)) \cong \mathcal{F}_H(\Omega(G/H))$ of $\mathbf{E}_\infty$-algebras in $\mathrm{QCoh}(\mathcal{M}_G)$, where the right-hand side is viewed as an $\mathbf{E}_\infty$-algebra in $\mathrm{QCoh}(\mathcal{M}_G)$ via pushforward along the map $\mathcal{M}_H \to \mathcal{M}_G$.

Warning 5.3.4. Although there is an equivalence $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!)) \cong \mathcal{F}_H(\Omega(G/H))$ of $\mathbf{E}_\infty$-algebras in $\mathrm{QCoh}(\mathcal{M}_G)$, there is *not* an equivalence $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!))^\vee \cong \mathcal{F}_H(\Omega(G/H))^\vee$ in $\mathrm{QCoh}(\mathcal{M}_G)$. Indeed, $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!))^\vee$ denotes the $\mathcal{O}_{\mathcal{M}_G}$-*linear* dual of $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!))$, while $\mathcal{F}_H(\Omega(G/H))^\vee$ denotes the $\mathcal{O}_{\mathcal{M}_H}$-*linear* dual of $\mathcal{F}_H(\Omega(G/H))$. (To clarify, this dual is not taken in the naive sense: rather, if $\Omega(G/H) = \mathrm{colim}\, X_\lambda$ as in Remark 5.3.3, $\mathcal{F}_H(\Omega(G/H))^\vee$ means $\mathrm{colim}_\lambda \mathcal{F}_H(X_\lambda)^\vee$.)

Remark 5.3.5. Proposition 5.3.1 breaks the natural symmetry on $G\backslash\mathcal{L}(G/H)$. Namely, since the action of G on $\mathcal{L}(G/H)$ is defined via the G-action on G/H, the orbispace $G\backslash\mathcal{L}(G/H)$ has an action of the circle S^1_{rot} given by rotating loops. However, this structure is not naturally visible on the orbispace $\Omega(G/H)/H$. Indeed, the proof of Proposition 5.3.1 used the splitting $G \times \Omega G \xrightarrow{\sim} \mathcal{L}G$; but this splitting is *not* S^1_{rot}-equivariant.

A slight variant of Proposition 5.3.1 lets us describe the G-equivariant k-*co*homology of $\mathcal{L}(G/H)$. The following result is proved in § 3.10.

Proposition 5.3.6. *There is an* S^1_{rot}*-equivariant equivalence of* $\mathbf{E}_\infty$*-algebras in* $\mathrm{QCoh}(\mathcal{M}_G)$*:*

$$\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!)) \simeq \mathrm{HH}(\mathcal{M}_H/\mathcal{M}_G),$$

where the right-hand side denotes the relative Hochschild homology of the $\mathcal{M}_H \to \mathcal{M}_G$ *(equipped with its natural* S^1*-action).*

Proof. Since G/H is itself the fiber product $* \times_{*/\mathrm{G}} */\mathrm{H}$ in orbispaces, there is an equivalence

$$\mathrm{G}\backslash\mathcal{L}(\mathrm{G}/\mathrm{H}) \simeq \mathrm{G}\backslash(* \times_{\mathcal{L}(*/\mathrm{G})} \mathcal{L}(*/\mathrm{H})) \simeq */\mathrm{G} \times_{\mathcal{L}(*/\mathrm{G})} \mathcal{L}(*/\mathrm{H}).$$

But $\mathcal{L}(*/\mathrm{G}) \simeq */\mathrm{G} \times_{*/\mathrm{G}\times */\mathrm{G}} */\mathrm{G}$, where the two maps $*/\mathrm{G} \to */\mathrm{G} \times */\mathrm{G}$ are both given by the diagonal. Therefore, we can identify

$$\begin{aligned}\mathrm{G}\backslash\mathcal{L}(\mathrm{G}/\mathrm{H}) &\simeq */\mathrm{G} \times_{*/\mathrm{G}\times_{*/\mathrm{G}\times */\mathrm{G}} */\mathrm{G}} (*/\mathrm{H} \times_{*/\mathrm{H}\times */\mathrm{H}} */\mathrm{H}) \\ &\simeq */\mathrm{H} \times_{*/\mathrm{H}\times_{*/\mathrm{G}} */\mathrm{H}} */\mathrm{H}. \end{aligned} \tag{5.3.1}$$

By construction, it follows that there is an equivalence of $\mathbf{E}_\infty$-algebras in $\mathrm{QCoh}(\mathcal{M}_\mathrm{G})$:

$$\mathcal{F}_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!)) \simeq \mathcal{O}_{\mathcal{M}_\mathrm{H}} \otimes_{\mathcal{O}_{\mathcal{M}_\mathrm{H}} \otimes_{\mathcal{O}_{\mathcal{M}_\mathrm{G}}} \mathcal{O}_{\mathcal{M}_\mathrm{H}}} \mathcal{O}_{\mathcal{M}_\mathrm{H}} = \mathrm{HH}(\mathcal{M}_\mathrm{H}/\mathcal{M}_\mathrm{G}).$$

Moreover, the equivalence of (5.3.1) is manifestly S^1-equivariant, so we obtain the desired claim. □

Taking k to be an ordinary commutative ring in Proposition 5.3.6 and using the Hochschild-Kostant-Rosenberg theorem (in the form proved in [Rak, MRT]), one finds:

Corollary 5.3.7. *Let k be an ordinary commutative ring, and let $\mathfrak{c}^*_{\check{\mathrm{H}}} = \operatorname{Spev} \mathrm{C}^*_\mathrm{H}(*; k) = \mathcal{M}_{\mathrm{H},0}$ and $\mathfrak{c}^*_{\check{\mathrm{G}}} = \operatorname{Spev} \mathrm{C}^*_\mathrm{G}(*; k) = \mathcal{M}_{\mathrm{G},0}$ are the Chevalley bases for $\check{\mathrm{H}}$ and $\check{\mathrm{G}}$, respectively (these were studied in Theorem 3.7.7). Then there is a filtration on $\mathrm{C}^*_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k)$ whose associated graded is given by the Hodge cohomology $\mathrm{L}\Omega^*_{\mathfrak{c}^*_{\check{\mathrm{H}}}/\mathfrak{c}^*_{\check{\mathrm{G}}}}$. In other words, there is an isomorphism*

$$\operatorname{Spev} \mathrm{C}^*_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k) \cong \mathrm{T}[1](\mathfrak{c}^*_{\check{\mathrm{H}}}/\mathfrak{c}^*_{\check{\mathrm{G}}}).$$

This equivalence identifies the loop rotation action of $\mathbf{G}_m^{\mathrm{rot}}$ on the left-hand side with the de Rham differential on the right-hand side; this implies that there is an isomorphism

$$\operatorname{Spev} \mathrm{C}^*_{\mathrm{G}[\![t]\!]\rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k) \cong \mathrm{Def}_\hbar(\mathfrak{c}^*_{\check{\mathrm{H}}}/\mathfrak{c}^*_{\check{\mathrm{G}}}),$$

where the right-hand side denotes the deformation to the normal cone of the morphism $\mathfrak{c}^*_{\check{\mathrm{H}}} \to \mathfrak{c}^*_{\check{\mathrm{G}}}$ in the sense of [GR1, Section 9.2], living over the base $\operatorname{Spev} \mathrm{C}^*_{\mathbf{G}_m^{\mathrm{rot}}}(*; k) \cong \mathbf{A}^1_\hbar(2)$.

Example 5.3.8. For instance, in the group case (so $\mathrm{G} = \mathrm{H} \times \mathrm{H}$ with H embedded diagonally), the first isomorphism of Corollary 5.3.7 says that there is an isomorphism

$$\operatorname{Spev} \mathrm{C}^*_{\mathrm{H}[\![t]\!]}(\mathrm{Gr}_\mathrm{H}; k) \cong \mathrm{T}[1](\mathfrak{c}^*_{\check{\mathrm{H}}}/\mathfrak{c}^*_{\check{\mathrm{H}}\times\check{\mathrm{H}}}) \cong \mathrm{T}(\mathfrak{c}^*_{\check{\mathrm{H}}}/\operatorname{Spec}(k)).$$

Similarly, there is an isomorphism

$$\operatorname{Spev} \mathrm{C}^*_{\mathrm{H}[\![t]\!]\rtimes\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{H}; k) \cong \mathrm{Def}_\hbar(\mathfrak{c}^*_{\check{\mathrm{H}}}/\mathfrak{c}^*_{\check{\mathrm{H}}\times\check{\mathrm{H}}});$$

since $\mathfrak{c}^*_{\check{\mathrm{H}}\times\check{\mathrm{H}}} \cong \mathfrak{c}^*_{\check{\mathrm{H}}} \times_{\operatorname{Spec}(k)} \mathfrak{c}^*_{\check{\mathrm{H}}}$, this recovers and generalizes the isomorphism of [BF, Theorem 1]. We have already discussed this perspective above in § 3.10.

Remark 5.3.9. Suppose G/H is an affine spherical G-variety, and assume that (5.2.1) of Remark 5.2.13 holds for G/H (which would follows from Conjecture 5.2.12). Then Corollary 5.3.7 implies that if $\mu : \check{\mathrm{M}} \to \check{\mathfrak{g}}^*$ denotes the moment map, there is a filtration on

$C^*_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k)$ with associated graded given by $L\Omega^*_{\check{M}/\!\!/\check{G}/\mathfrak{c}^*_{\check{G}}}$. In fact, more is true: taking cohomology defines a functor

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k) \xrightarrow{C^*_G(-;k)} \mathrm{Mod}(C^*_G(\mathcal{L}G/\mathcal{L}H; k)).$$

By the preceding discussion, the right-hand side admits a 1-parameter degeneration to the ∞-category of graded modules over $L\Omega^*_{\mathfrak{c}^*_{\check{H}}/\mathfrak{c}^*_{\check{G}}}$, i.e., the ∞-category of perfect complexes over the (-1)-shifted tangent bundle $T[1](\mathfrak{c}^*_{\check{H}}/\mathfrak{c}^*_{\check{G}})$. Under the isomorphism $C^*_G(\mathcal{L}G/\mathcal{L}H; k) \cong C^*_H(\Omega(G/H); k)$, the $C^*_G(*; k)$-module structure on $C^*_G(\mathcal{L}G/\mathcal{L}H; k)$ factors through the canonical map $C^*_G(*; k) \to C^*_H(*; k)$. This defines a factorization

$$\begin{tikzcd}
\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k) \arrow[r, "C^*_G(-;k)"] \arrow[d, dashed] & \mathrm{Mod}(C^*_G(\mathcal{L}G/\mathcal{L}H; k)) \arrow[d, "\text{forget}"] \arrow[dl, dashed] \\
\mathrm{Mod}(C^*_H(*; k)) \arrow[r, "\text{restriction}"'] & \mathrm{Mod}(C^*_G(*; k)),
\end{tikzcd} \tag{5.3.2}$$

which makes the triangles commute.

By Conjecture 5.2.12 (or rather, the generalization from Remark 5.2.17), $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!); k)$ admits a !-parameter degeneration to the ∞-category $\mathrm{Perf}(\check{M}/\check{G})$. In particular, there is a natural map $C^*_H(*; k) \to C^*_G(\mathcal{L}G/\mathcal{L}H; k)$, which gives a functor

$$\mathrm{Perf}(\check{M}/\check{G}) \to \mathrm{Perf}(T[1](\mathfrak{c}^*_{\check{H}}/\mathfrak{c}^*_{\check{G}})) \to \mathrm{Perf}(\mathfrak{c}^*_{\check{H}}).$$

When $G = H \times H$, so that $\check{M} = T^*\check{H}$, this is precisely the Kostant functor of [BF, Section 2.6]. This functor is compatible with the commutative diagram (5.3.2), in that the following diagram is its analogue on the spectral side:

$$\begin{tikzcd}
\mathrm{Perf}(\check{M}/\check{G}) \arrow[r, "\text{“Kostant functor”}"] \arrow[d, dashed, "\kappa_{\check{M}}"'] & \mathrm{Perf}(T[1](\mathfrak{c}^*_{\check{H}}/\mathfrak{c}^*_{\check{G}})) \arrow[d] \arrow[dl, dashed, "\text{zero section}^*"] \\
\mathrm{Perf}(\mathfrak{c}^*_{\check{H}}) \arrow[r, "\text{restriction}"'] & \mathrm{Perf}(\mathfrak{c}^*_{\check{G}}),
\end{tikzcd}$$

where we will now describe the dotted map denoted $\kappa_{\check{M}}$.

Let us ignore gradings in the following discussion. It is natural to expect that the above analogue of the Kostant functor is induced by pullback along a certain map

$$\kappa_{\check{M}} : \mathfrak{c}^*_{\check{H}} \to \check{M}/\check{G}.$$

For instance, when $G = H \times H$, so that $\check{M} = T^*\check{H}$, the map κ is simply the Kostant slice for $\check{H}$. Moreover, in the general case, the compatibility of the equivalence of Conjecture 5.2.12 with the action of the Satake category implies that there is a commutative square

$$\begin{tikzcd}
\mathfrak{c}^*_{\check{H}} \arrow[r, "\kappa_{\check{M}}"] \arrow[d] & \check{M}/\check{G} \arrow[d, "\mu"] \\
\mathfrak{c}^*_{\check{G}} \arrow[r, "\kappa"] & \check{\mathfrak{g}}^*/\check{G}.
\end{tikzcd}$$

Therefore, (5.2.1) and Corollary 5.3.7 together make the following concrete prediction:

Conjecture 5.3.10. *On the spectral side of Conjecture 5.2.12, if* $\check{\mathrm{M}}$ *is the Hamiltonian* $\check{\mathrm{G}}$*-space dual to* G/H*, there is an isomorphism*

$$\check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \cong \mathfrak{c}^*_{\check{\mathrm{H}}}.$$

In general, if $\check{\mathrm{M}}$ *is the Hamiltonian* $\check{\mathrm{G}}$*-space which is dual to an affine (not necessarily homogeneous) spherical* G*-variety, there is a "Kostant section"* $\kappa_{\check{\mathrm{M}}} : \check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \to \check{\mathrm{M}}/\check{\mathrm{G}}$ *which makes the following square commute:*

$$\begin{tikzcd} \check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \arrow[r, "\kappa_{\check{\mathrm{M}}}"] \arrow[d] & \check{\mathrm{M}}/\check{\mathrm{G}} \arrow[d, "\mu"] \\ \mathfrak{c}^*_{\check{\mathrm{G}}} \arrow[r, "\kappa"] & \check{\mathfrak{g}}^*/\check{\mathrm{G}}. \end{tikzcd}$$

Furthermore, $\kappa_{\check{\mathrm{M}}}$ *can be refined (non-uniquely) to a map* $\check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \to \check{\mathrm{M}}$ *such that the algebra of regular functions on its* $\check{\mathrm{G}}$*-orbit is isomorphic to* $\mathcal{O}_{\check{\mathrm{M}}}$*.*

Motivated by [Kno1, Kno2] (see Remark 5.1.15 for a brief summary), we further expect the following. Suppose that $\check{\mathrm{M}}$ *can be written as* $\mathrm{T}^*\check{\mathrm{X}}$ *for a spherical* $\check{\mathrm{G}}$*-variety* $\check{\mathrm{X}}$*, and let* $\mathcal{O}_{\check{\mathrm{B}}}(\check{\mathrm{X}})$ *(resp.* $\mathcal{O}_{\mathrm{B}}(\mathrm{G}/\mathrm{H})$*) denote the poset of* $\check{\mathrm{B}}$*-orbit closures in* $\check{\mathrm{X}}$ *(resp.* B*-orbit closures in* G/H*) equipped with the Bruhat order. Then there is a bijection* $\mathcal{O}_{\check{\mathrm{B}}}(\check{\mathrm{X}}) \leftrightarrow \mathcal{O}_{\mathrm{B}}(\mathrm{G}/\mathrm{H})$ *which is equivariant for the action of the Weyl group* $\mathrm{W}_{\check{\mathrm{G}}} \cong \mathrm{W}_{\mathrm{G}}$ *on either side described in [Kno2].*[3] *This bijection furthermore sends a minimal rank* B*-orbit* $\mathrm{V}_{\min}$ *on* G/H *to the closure* $\check{\mathrm{X}}$ *of the open* $\check{\mathrm{B}}$*-orbit* $\check{\mathrm{X}}^\circ$ *in* $\check{\mathrm{X}}$*. For a* B*-orbit* V *in* G/H*, let* $\check{\mathrm{V}}$ *denote the corresponding* $\check{\mathrm{B}}$*-orbit in* $\check{\mathrm{X}}$*, and let* $\mathrm{W}_{\mathrm{V}} \subseteq \mathrm{W}$ *denote the stabilizer of* $\mathrm{V} \in \mathcal{O}_{\mathrm{B}}(\mathrm{G}/\mathrm{H})$*. If* $\mathbb{X}^*(\mathrm{V})$ *denotes the lattice of weights of* B*-eigenfunctions in the field of rational functions on* V*, there is furthermore a* W_{V}*-action on* $\mathbb{X}^*(\mathrm{V})$ *(resp. on* $\mathbb{X}^*(\check{\mathrm{V}})$*) and a* W_{V}*-equivariant exact sequence*

$$0 \to \mathbb{X}^*(\mathrm{V}) \to \mathbb{X}^*(\mathrm{T}) \cong \mathbb{X}_*(\check{\mathrm{T}}) \to \mathbb{X}_*(\check{\mathrm{V}}) \to 0;$$

here, T *is a maximal torus of* G *(and* $\check{\mathrm{T}}$ *is its dual torus).*

Some brief comments regarding Conjecture 5.3.10:

- The *same* conjecture should hold even in the setting of the generalized relative Langlands duality of Conjecture 5.2.20. Namely, there should be a (W-equivariant) bijection between the set of B-orbit closures on G/H and the set of irreducible components of the fiber product $\check{\mathrm{M}}_{\mathbf{H}} \times_{\mathrm{G}_{\mathbf{H}}} \mathrm{N}_{\mathbf{H}}$. (Here, N denotes the unipotent radical of the Borel subgroup $\mathrm{B} \subseteq \mathrm{G}$.)

- Just as the Kostant slice plays a crucial role in the geometric Langlands program, we expect the Kostant section $\kappa_{\check{\mathrm{M}}}$ to play a central role in the story of relative geometric Langlands.

- Since $\check{\mathrm{M}} = \mathrm{Ind}_{\check{\mathrm{G}}_{\mathrm{X}}}^{\check{\mathrm{G}}}(\mathrm{S}_{\mathrm{X}} \oplus (\check{\mathfrak{g}}_{\mathrm{X}}^{\perp} \cap \check{\mathfrak{g}}^e))$ by Lemma 5.2.6, the first part of Conjecture 5.3.10 is equivalent to the statement that

  $$(\mathrm{S}_{\mathrm{X}} \oplus (\check{\mathfrak{g}}_{\mathrm{X}}^{\perp} \cap \check{\mathfrak{g}}^e))/\!\!/\check{\mathrm{G}}_{\mathrm{X}} \cong \mathfrak{c}^*_{\check{\mathrm{H}}}.$$

  In large enough characteristic, the latter is a polynomial ring (by Chevalley restriction and Chevalley-Shephard-Todd); so Conjecture 5.3.10 forces in particular that in large enough characteristic, the action of $\check{\mathrm{G}}_{\mathrm{X}}$ on $\mathrm{S}_{\mathrm{X}} \oplus (\check{\mathfrak{g}}_{\mathrm{X}}^{\perp} \cap \check{\mathfrak{g}}^e)$ must be *coregular*.

[3] Said differently, there is a W-equivariant bijection between the sets of irreducible components of $\mathrm{T}^*(\mathrm{G}/\mathrm{H}) \times_{\mathfrak{b}^*} \{0\}$ and $\check{\mathrm{M}} \times_{\check{\mathfrak{b}}^*} \{0\}$, where $\mathrm{T}^*(\mathrm{G}/\mathrm{H}) \to \mathfrak{b}^*$ (resp. $\check{\mathrm{M}} \to \check{\mathfrak{b}}^*$) is the moment map for the B-(resp. $\check{\mathrm{B}}$-)action.

- The final paragraph of Conjecture 5.3.10 clearly generalizes to the case when $T^*(G/H)$ is replaced by a more general Hamiltonian G-space. In fact, this generalization of the penultimate part of Conjecture 5.3.10 appears as [FGT, Conjecture 1.1.1]; I am grateful to Akshay Venkatesh and Zhiwei Yun for informing me of this paper.

- The bijection of Conjecture 5.3.10 should be a starting point for proving Conjecture 5.2.15.

- As we now explain, the final paragraph of Conjecture 5.3.10 would provide a "geometric reason" for the expected isomorphism $(T^*\check{X})/\!\!/\check{G} \cong \operatorname{Spev} C^*_H(*;k)$, at least in large enough characteristic. Indeed, let $V_{\min}$ denote a minimal rank B-orbit in G/H, and let $\check{X}^\circ$ denote the the open $\check{B}$-orbit in $\check{X}$. Since $P(\check{X}) = \check{B}$ (so $W_{L(\check{X})} = 0$), [Kno2] identifies the stabilizer of $\check{X}^\circ$ with $W_{\check{X}}$. Moreover, [Res] identifies the stabilizer of $V_{\min}$ with W_H. (See Remark 5.1.15 for a summary of the results from [Kno2, Res].) It follows from Conjecture 5.3.10 that there is an isomorphism $W_{\check{X}} \cong W_H$.

  Let T_H denote a maximal torus of H, and let $T_{\check{X}}$ denotes the Cartan of $\check{X}$. The results of [Res] show that the quotient of $\mathbb{X}^*(T)$ by $\mathbb{X}^*(V_{\min})$ can be identified with $\mathbb{X}^*(T_H)$. Therefore, the exact sequence of Conjecture 5.3.10 imply that there is a canonical $W_H \cong W_{\check{X}}$-equivariant isomorphism $T_H \cong \check{T}_{\check{X}}$. Finally, as described in Remark 5.1.15, Knop has shown that $(T^*\check{X})/\!\!/\check{G} \cong \mathfrak{t}^*_{\check{X}}/\!\!/W_{\check{X}}$; the preceding discussion identifies this with $\mathfrak{t}_H/\!\!/W_H$, which is $\operatorname{Spev} C^*_H(*;k)$ in large enough characteristic, as desired.

Corollary 5.3.11. *The $\mathbf{E}_1$-$\mathcal{O}_{\mathcal{M}_G}$-algebra structure obtained via the $\mathbf{E}_\infty$-map $\mathcal{M}_H \to \mathcal{M}_G$ on the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega(G/H))$ – which is* not *$\mathcal{F}_H(\Omega(G/H))^\vee$ – refines to an $\mathbf{E}_2$-ku_G-algebra structure.*

Proof. Taking the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of the right-hand side of Proposition 5.3.6 produces the Hochschild *co*homology $\mathrm{HC}(\mathcal{M}_H/\mathcal{M}_G)$. By the Deligne conjecture (in the form proved in [Lur4, Section 5.3]), this admits the structure of an $\mathbf{E}_2$-$\mathcal{O}_{\mathcal{M}_G}$-algebra. On the other hand, by Proposition 5.3.1, the right-hand side of Proposition 5.3.6 can be identified with the equivariant cohomology $\mathcal{F}_H(\Omega(G/H))$. The desired result follows. ∎

Remark 5.3.12. One can also identify the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega G)$ with the $\mathbf{E}_2$-centralizer of the map $\mathcal{M}_H \to \mathcal{M}_G$. The $\mathbf{E}_2$-structure on the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is essentially the reason for the $\mathbf{E}_2$-monoidal structure on the relative Langlands category $\mathrm{Shv}^{c,\mathrm{Sat}}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!);k)$ from [BZSV, Remark 7.5.12 and Section 16].

In the special case when $G = H \times H$ and H is embedded diagonally, one can identify $\mathrm{HC}(\mathcal{M}_H/\mathcal{M}_G)$ with the $\mathbf{E}_2$-*Hochschild cohomology* $\mathrm{HC}_{\mathbf{E}_2}(\mathcal{M}_H/\operatorname{Spec}(k))$. The Deligne conjecture therefore equips the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega H)$ with an $\mathbf{E}_3$-algebra structure, and again this is essentially the source of the folklore $\mathbf{E}_3$-monoidal structure on the spherical Hecke category $\mathrm{Shv}^{c,\mathrm{Sat}}_{H[\![t]\!]\times H[\![t]\!]}(H(\!(t)\!);k)$.

Warning 5.3.13. The reader should keep Warning 5.3.4 in mind: the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is *not* equivalent to the $\mathcal{O}_{\mathcal{M}_G}$-linear dual of $\mathcal{F}_{G[\![t]\!]}(G(\!(t)\!)/H(\!(t)\!))$. In fact, as mentioned in Corollary 5.3.11, the $\mathcal{O}_{\mathcal{M}_H}$-linear dual of $\mathcal{F}_H(\Omega(G/H))$ is also not equivalent to the equivariant homology $\mathcal{F}_H(\Omega(G/H))^\vee$; the former is only a *completion* of the latter.

Remark 5.3.14. There are, of course, many mild variants of Corollary 5.3.11. For instance, suppose $K \subseteq H$ is a closed subgroup. Then $\mathcal{O}_{\mathcal{M}_K}$ is an $\mathbf{E}_\infty$-$\mathcal{O}_{\mathcal{M}_H}$-algebra; in particular, it is a $\mathcal{O}_{\mathcal{M}_H}$-bimodule in $\mathcal{O}_{\mathcal{M}_G}$-modules. Therefore, one can consider the Hochschild cohomology $\mathrm{HC}(\mathcal{M}_H/\mathcal{M}_G;\mathcal{O}_{\mathcal{M}_K})$ with coefficients in the bimodule $\mathcal{O}_{\mathcal{M}_K}$. Just as in Corollary 5.3.11, one can identify $\mathrm{HC}(\mathcal{M}_H/\mathcal{M}_G;\mathcal{O}_{\mathcal{M}_K})$ with the $\mathcal{O}_{\mathcal{M}_K}$-linear dual of $\mathcal{F}_K(\Omega(G/H))$. It follows, for instance, that $\pi_*\mathrm{HC}(C^*_H(*;\mathbf{Z})/C^*_G(*;\mathbf{Z});C^*_K(*;\mathbf{Z}))$ is a completion of $H^K_*(\Omega(G/H);\mathbf{Z})$.

Example 5.3.15. Let us illustrate Corollary 5.3.11, or rather, the identification of the $\mathcal{O}_{\mathcal{M}_{\mathrm{H}}}$-linear dual of $\mathcal{F}_{\mathrm{H}}(\Omega(\mathrm{G}/\mathrm{H}))$ with Hochschild cohomology in the case when $k = \mathbf{Z}$ (and $k = \mathbf{Z}[1/2]$ in the second example) in two simple cases:

a. Let $\mathrm{H} = \mathrm{SU}(n-1) \subseteq \mathrm{SU}(n) = \mathrm{G}$. Then $\mathrm{G}/\mathrm{H} \simeq \mathrm{S}^{2n-1}$, and so there is an isomorphism

$$\pi_* \mathcal{F}_{\mathrm{H}}(\Omega(\mathrm{G}/\mathrm{H}))^\vee \cong \mathrm{H}_*(\Omega \mathrm{S}^{2n-1}; \mathbf{Z}) \cong \mathbf{Z}[y],$$

where y lives in weight $2n-2$. On the other hand, the map $\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z}) \to \mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z})$ identifies with the map

$$\mathbf{Z}[c_1, \cdots, c_n] \to \mathbf{Z}[c_1, \cdots, c_{n-1}]$$

sending $c_n \mapsto 0$, where the ith Chern class c_i lives in weight $-2i$. Taking Hochschild homology along this map identifies

$$\mathrm{HH}(\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z})/\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z})) \simeq \mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{HH}(\mathbf{Z}/\mathbf{Z}[c_n]).$$

But $\pi_* \mathrm{HH}(\mathbf{Z}/\mathbf{Z}[c_n])$ is isomorphic to the divided power algebra $\mathbf{Z}\langle \sigma^2(c_n) \rangle$, where σ denotes "suspension", so that $\sigma^2(c_n)$ lives in degree 2 and weight $-2n$; it follows that there is an isomorphism

$$\pi_* \mathrm{HH}(\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z})/\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z})) \cong \mathbf{Z}[c_1, \cdots, c_{n-1}]\langle \sigma^2(c_n) \rangle.$$

This in turn implies that there is an isomorphism

$$\pi_* \mathrm{HC}(\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z})/\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z})) \cong \mathbf{Z}[c_1, \cdots, c_{n-1}][\![y]\!]$$

where the class y in weight $2n-2$ is dual to $\sigma^2(c_n)$. Killing $c_1, \cdots, c_n$ (i.e., base-changing along $\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}) \to \mathbf{Z}$) precisely recovers a completion of $\mathrm{H}_*(\Omega \mathrm{S}^{2n-1}; \mathbf{Z})$.

b. Let $\mathrm{H} = \mathrm{SO}_{2n} \subseteq \mathrm{SO}_{2n+1} = \mathrm{G}$ with $n > 0$, and recall that we are replacing $\mathbf{Z}$ by $\mathbf{Z}' = \mathbf{Z}[1/2]$. Then $\mathrm{G}/\mathrm{H} \simeq \mathrm{S}^{2n}$, and so a standard argument with the Serre spectral sequence shows that there is an isomorphism

$$\pi_* \mathcal{F}_{\mathrm{H}}(\Omega(\mathrm{G}/\mathrm{H}))^\vee \cong \mathrm{H}_*(\Omega \mathrm{S}^{2n}; \mathbf{Z}') \cong \mathbf{Z}'[y, z]/z^2,$$

where z lives in weight $2n-1$ and y lives in weight $4n-2$. On the other hand, the map $\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z}') \to \mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}')$ identifies with the map

$$\mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n] \to \mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n^{1/2}]$$

sending $p_n \mapsto (p_n^{1/2})^2$, where the ith Pontryagin class p_i lives in weight $-4i$ and the Euler class $p_n^{1/2}$ lives in weight $-2n$. Taking Hochschild homology along this map identifies

$$\mathrm{HH}(\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}')/\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z}')) \simeq \mathbf{Z}'[p_1, \cdots, p_{n-1}] \otimes_{\mathbf{Z}'} \mathrm{HH}(\mathbf{Z}'[p_n^{1/2}]/\mathbf{Z}'[p_n]),$$

and so computing the Hochschild cohomology from Corollary 5.3.11 amounts to computing the Hochschild cohomology $\mathrm{HC}(\mathbf{Z}'[p_n^{1/2}]/\mathbf{Z}'[p_n])$. Lemma 5.3.16 implies that there is an isomorphism

$$\pi_* \mathrm{HC}(\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}')/\mathrm{H}^*_{\mathrm{G}}(*; \mathbf{Z}')) \cong \mathbf{Z}'[p_1, \cdots, p_{n-1}, p_n^{1/2}][\![w]\!]/p_n^{1/2} w,$$

with w in in weight $4n-2$. Upon killing $p_1, \cdots, p_{n-1}, p_n^{1/2}$ (i.e., base-changing along $\mathrm{H}^*_{\mathrm{H}}(*; \mathbf{Z}) \to \mathbf{Z}$), one precisely recovers a completion of $\mathrm{H}_*(\Omega \mathrm{S}^{2n}; \mathbf{Z}')$.

Lemma 5.3.16. *Let x be a class in weight n, and let $j \geq 1$. Then there is an isomorphism*

$$\pi_* \mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[x^j]) \cong \mathbf{Z}[x][\![w]\!]/jx^{j-1}w,$$

where w lives in degree -2 and weight $-2nj$.

Proof. Let us first work in the ungraded setting; fix a nonconstant polynomial $g(x) \in \mathbf{Z}[x]$, and consider $\mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[g])$. There is an isomorphism

$$\mathbf{Z}[x] \otimes_{\mathbf{Z}[g]} \mathbf{Z}[x] \cong \mathbf{Z}[x, x']/(g(x) - g(x')) \cong \mathbf{Z}[x, z]/zf,$$

where $z = x' - x$ and $f = \frac{g(x)-g(x+z)}{z}$. (If x has weight n and g is homogeneous of degree j, the class z lives in degree 0 and weight n, and f lives in degree 0 and weight $n(j-1)$.) Our goal is to compute $\pi_* \operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, where the map $\mathbf{Z}[x, z]/zf \to \mathbf{Z}[x]$ sends $z \mapsto 0$. There are several ways to compute this: one is to note that there is a presentation

$$\mathbf{Z}[x] \simeq (\mathbf{Z}[x, z, u]\langle v\rangle/(zf, u^2), d(u) = z, d(v) = uf)$$

of $\mathbf{Z}[x]$ as a $\mathbf{Z}[x, z]/zf$-algebra. If x has weight n and g is homogeneous of degree j, the class u is in degree 1 and weight n, and v is a divided power class in degree 2 and weight nj. This implies that there is an equivalence

$$\operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x]) \simeq (\mathbf{Z}[x, u'][\![w]\!]/u'^2, d(u') = f(z = 0)w),$$

where u' is dual to u and w is dual to v. If x has weight n and g is homogeneous of degree j, the class u' is in degree -1 and weight $-n$, and w is in degree -2 and weight $-nj$. It follows that there is a class $w \in \pi_{-2} \operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$ such that $f(z = 0)w = 0 \in \pi_{-2} \operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, which gives an isomorphism

$$\pi_* \mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[g]) = \pi_* \operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x]) \cong \mathbf{Z}[x][\![w]\!]/g'(0)w.$$

If x has weight n and g is homogeneous of degree j, the class w lives in $\pi_{-2,-nj} \operatorname{End}_{\mathbf{Z}[x,z]/zf}(\mathbf{Z}[x])$, and we obtain a graded isomorphism

$$\pi_* \mathrm{HC}(\mathbf{Z}[x]/\mathbf{Z}[x^j]) \cong \mathbf{Z}[x][\![w]\!]/jx^{j-1}w,$$

which gives the desired calculation. $\square$

Remark 5.3.17. As in Lemma 5.3.16, one can also compute $\pi_* \mathrm{HC}(\pi_* \mathrm{ku}_{S^1}/\pi_* \mathrm{ku}_{\mathrm{SU}(2)})$ to obtain the following:

$$\pi_* \mathrm{HC}(\pi_* \mathrm{ku}_{S^1}/\pi_* \mathrm{ku}_{\mathrm{SU}(2)}) \cong \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}][\![w]\!]/w(x - \overline{x}).$$

Here, w lives in degree 0 and weight 2, and $\overline{x} = -\frac{x}{1+\beta x}$ is the negative of x under the group law on $\mathbf{G}_\beta = \operatorname{Spec} \pi_* \mathrm{ku}_{S^1}$. When $\beta = 0$, this recovers Lemma 5.3.16 for $j = 2$ and $n = -1$. There is a spectral sequence whose E_1-page is $\pi_* \mathrm{HC}(\pi_* \mathrm{ku}_{S^1}/\pi_* \mathrm{ku}_{\mathrm{SU}(2)})$ which converges to $\pi_* \mathrm{HC}(\mathrm{ku}_{S^1}/\mathrm{ku}_{\mathrm{SU}(2)})$; this spectral sequence degenerates. Since the 2-series of x is $[2](x) = (1 + \beta x)(x - \overline{x})$, we find that

$$\pi_* \mathrm{HC}(\mathrm{ku}_{S^1}/\mathrm{ku}_{\mathrm{SU}(2)}) \cong \mathbf{Z}[\beta, x, \tfrac{1}{1+\beta x}][\![w]\!]/w[2](x),$$

where w lives in degree 2.

Remark 5.3.18. The reader might observe that one can analyze $\mathrm{HH}(\pi_*\mathrm{ku_H}/\pi_*\mathrm{ku_G})$ essentially using the combinatorics of the weight lattices and Weyl groups of H and G. More generally, therefore, let $\mathrm{W}_1 \to \mathrm{W}_2$ be a homomorphism of finite groups acting on vector spaces $\mathrm{V}_1 \to \mathrm{V}_2$ over a field k (possibly of nonzero characteristic). Then there is a map $\mathrm{V}_1/\!\!/\mathrm{W}_1 \to \mathrm{V}_2/\!\!/\mathrm{W}_2$, and hence one can consider the Hochschild homology $\mathrm{HH}(\mathrm{V}_1/\!\!/\mathrm{W}_1/\mathrm{V}_2/\!\!/\mathrm{W}_2)$. This should be an interesting invariant associated to homomorphisms of finite groups, but it is likely only well-behaved if the map $\mathrm{V}_1/\!\!/\mathrm{W}_1 \to \mathrm{V}_2/\!\!/\mathrm{W}_2$ is an affine bundle.

Example 5.3.19. Consider the subgroup $\mathrm{G}^{\mathrm{diag}} \subseteq \mathrm{G}\times\mathrm{G}$, so that $(\mathrm{G}\times\mathrm{G})/\mathrm{G}^{\mathrm{diag}} \simeq \mathrm{G}$ (this is the "group case" of Example 5.1.5). Then Proposition 5.3.6 says that there is an S^1-equivariant equivalence of $\mathbf{E}_\infty$-$\mathrm{ku_G}$-algebras

$$\mathcal{F}_{\mathrm{G}\times\mathrm{G}}(\mathcal{L}\mathrm{G}) \simeq \mathrm{HH}(\mathrm{ku_G}/\mathrm{ku_{G\times G}}).$$

By construction of equivariant ku, there is an equivalence $\mathrm{ku_{G\times G}} \simeq \mathrm{ku_G}\otimes_{\mathrm{ku}}\mathrm{ku_G}$, so that the right-hand side can be identified with the factorization homology

$$\mathrm{HH}(\mathrm{ku_G}/\mathrm{ku_G}\otimes_{\mathrm{ku}}\mathrm{ku_G}) \simeq \int_{\mathrm{S}^2}\mathrm{ku_G}/\mathrm{ku}.$$

Note that by Proposition 5.3.1, the left-hand side can be identified with $\mathcal{F}_{\mathrm{G}}(\Omega\mathrm{G})$, so Proposition 5.3.6 describes the G-equivariant ku-*co*homology of the affine Grassmannian:

$$\mathrm{ku}^*_{\mathrm{G}}(\Omega\mathrm{G}) \simeq \pi_*\int_{\mathrm{S}^2}\mathrm{ku_G}/\mathrm{ku}. \tag{5.3.3}$$

Note that there is an equivalence

$$\mathrm{HH}(\mathrm{ku_G}/\mathrm{ku_G}\otimes_{\mathrm{ku}}\mathrm{ku_G}) \simeq \mathrm{ku_G}\otimes_{\mathrm{ku}}\mathrm{HH}(\mathrm{ku}/\mathrm{ku_G}).$$

Upon killing the Bott class β, (5.3.3) implies that

$$\mathrm{C}^*_{\mathrm{G}}(\Omega\mathrm{G};\mathbf{Z}) \simeq \int_{\mathrm{S}^2}\mathrm{C}^*_{\mathrm{G}}(*;\mathbf{Z})/\mathbf{Z}.$$

As argued in § 3.10, this recovers [BF, Theorem 1] and [Gin2, Section 1.7] upon rationalization.

Remark 5.3.20. Unlike Proposition 5.3.1, Proposition 5.3.6 gives an $\mathrm{S}^1_{\mathrm{rot}}$-equivariant equivalence. In particular, it allows us to calculate the $\mathrm{S}^1_{\mathrm{rot}}$-equivariant cohomology $\mathrm{ku}^*_{\mathrm{G}\times\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\mathcal{L}\mathrm{G}) \simeq \mathrm{ku}^*_{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{G})$. We will discuss this in a future article, since addressing loop rotation in the detail it deserves will take us too far afield.

However, since it is not very difficult to make explicit, let us explicate Proposition 5.3.6 (or rather, its variant for Hochschild cohomology describing $\mathrm{ku}_*^{\mathrm{G}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{G})$) in the case when $\mathrm{G} = \mathrm{T}$ is a torus. (At the beginning of this section, we asked that G be simply-connected; this is obviously not true for a torus, but that assumption was necessary only when doing computations with Hochschild (co)homology. We will *not* use this perspective below.)

As in Proposition 3.5.4, the associative graded ring $\mathrm{ku}_*^{\mathrm{T}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{T})$ can be identified with the algebra of $\mathbf{G}_\beta$-differential operators on the dual torus $\check{\mathrm{T}}$. This is an analogue of the algebra of (asymptotic) differential operators. Let us assume for simplicity that T is of rank 1; then the algebra $\mathrm{ku}_*^{\mathrm{T}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{T})$ is the F-Weyl algebra $\mathrm{F}\mathcal{D}_{\square,\mathbf{G}_m}$ of Remark 7.4.6 (see also [DM, Definition 4.4.1]) for $\mathrm{F}(x,y) = x+y+\beta xy$, at least up to completion. Explicitly, when $\mathrm{T} = \mathrm{S}^1$, we have

$$\mathrm{ku}_*^{\mathrm{T}\times\mathrm{S}^1_{\mathrm{rot}}}(\Omega\mathrm{T}) \cong \mathbf{Z}[\beta,\hbar,\tfrac{1}{1+\beta\hbar}]\{x,a^{\pm1}\}[\tfrac{1}{1+\beta x}]/([x,a] = a\hbar(1+\beta x)).$$

Here, the curly brackets denotes the free associative algebra generated by the elements enclosed within. The classes $\hbar$ and x live in weight -2 (they are the S^1-equivariant Chern classes for ku), β lives in weight 2, and a lives in weight zero. Let us note two specializations of this associative algebra:

a. If $\beta = 0$, the right-hand side above simply becomes $\mathbf{Z}[\hbar]\{x, a^{\pm 1}\}/([x,a] = \hbar a)$, which is precisely the algebra of asymptotic differential operators on $\check{T} = \operatorname{Spec} \mathbf{Z}[a^{\pm 1}]$ over $\mathbf{Z}$. Namely, $x = \hbar a \partial_a$; see [DM, Example 4.4.2].

b. If β is inverted, all elements can be pushed to degree zero. Namely, let $q = 1 + \beta\hbar$ and $\Theta = 1 + \beta x$. Then there is an isomorphism

$$\mathbf{Z}[\beta^{\pm 1}, \hbar, \tfrac{1}{1+\beta\hbar}]\{x, a^{\pm 1}\}[\tfrac{1}{1+\beta x}]/([x,a] - a\hbar(1+\beta x)) \cong \mathbf{Z}[\beta^{\pm 1}, q^{\pm 1}]\{\Theta^{\pm 1}, a^{\pm 1}\}/(\Theta a - qa\Theta),$$

since

$$\begin{aligned} \Theta a &= (1+\beta x)a = a + \beta x a = a + \beta a(x + \hbar + \beta\hbar x) \\ &= a(1+\beta x)(1+\beta\hbar) = qa\Theta. \end{aligned}$$

In particular, $\mathrm{ku}_*^{T\times S^1_{\mathrm{rot}}}(\Omega T)[1/\beta] = \mathrm{KU}_*^{T\times S^1_{\mathrm{rot}}}(\Omega T)$ can be identified with the q-Weyl algebra of $\check{T} = \operatorname{Spec} \mathbf{Z}[a^{\pm 1}]$. Namely, $\Theta = q^{a\partial_a}$.

In general, $\mathrm{ku}_*^{T\times S^1_{\mathrm{rot}}}(\Omega T)$ interpolates between the algebra of asymptotic differential and q-difference operators on $\check{T}$. Note that if one inverts $\hbar$ instead, then

$$\mathrm{ku}_*^{T\times S^1_{\mathrm{rot}}}(\Omega T)[\hbar^{-1}] \cong \mathbf{Z}[\hbar^{\pm 1}, q^{\pm 1}]\{y, a^{\pm 1}\}[\tfrac{1}{1+(q-1)y}]/(ya - qay = a),$$

where $y = x\hbar^{-1}$. The final commutation relation shows that y is precisely the operator $a\partial_a^q$, where ∂_a^q is the q-derivative. That is to say, the weight zero piece of $\mathrm{ku}_*^{T\times S^1_{\mathrm{rot}}}(\Omega T)[\hbar^{-1}]$ is exactly the q-Weyl algebra of $\mathbf{G}_m$ generated by $a^{\pm 1}$ and ∂_a^q.

5.4 Using the regular centralizer

If G is a compact Lie group, we will abusively write $G(\!(t)\!)$ or $G[\![t]\!]$ below to mean $G_{\mathbf{C}}(\!(t)\!)$ or $G_{\mathbf{C}}[\![t]\!]$, respectively. We will also assume Hypothesis 5.3.2 in this section.

Definition 5.4.1. Let k be a G-equivariant $\mathbf{E}_\infty$-ring. Let $\mathrm{Shv}_G^{G[\![t]\!]\text{-cbl}}(G(\!(t)\!)/H(\!(t)\!); k)$ denote the ∞-category of G-equivariant sheaves of k-modules on $G(\!(t)\!)/H(\!(t)\!)$ which are constructible for the $G[\![t]\!]$-orbit stratification on $G(\!(t)\!)/H(\!(t)\!)$. One can heuristically view this as being the category of sheaves of k-modules on the stack $G[\![t]\!]\backslash G(\!(t)\!)/H(\!(t)\!) \cong \mathrm{Gr}_G/H(\!(t)\!)$. This category admits a monoidal structure given by convolution on the affine Grassmannian Gr_G.

We will study two examples of subgroups $H \subseteq G$. First, suppose that $X = G/H$ is a symmetric variety, so that there is an equivalence $\mathrm{Shv}_G^{G[\![t]\!]\text{-cbl}}(G(\!(t)\!)/H(\!(t)\!); k) \simeq \mathrm{Shv}_{G_{\mathbf{R}}}^{G_{\mathbf{R}}(\mathbf{R}[\![t]\!])\text{-cbl}}(\mathrm{Gr}_{G_{\mathbf{R}}}; k)$ by (a mild extension of) the main result of [CN], where $\mathrm{Gr}_{G_{\mathbf{R}}}$ is the real Grassmannian. The $G[\![t]\!]$-orbits on $G(\!(t)\!)/H(\!(t)\!)$ are discrete and in bijection with the $G_{\mathbf{R}}(\mathbf{R}[\![t]\!])$-orbits on $\mathrm{Gr}_{G_{\mathbf{R}}}$. They are indexed by the dominant weights $\check{\Lambda}_X^+$ of the dual group $\check{G}_X$. If $\lambda_\bullet$ is a sequence of dominant minuscule weights of $\check{G}_X$, let $|\lambda_\bullet| = \sum_i \lambda_i$. Suppose that $\mathrm{Gr}_{G_{\mathbf{R}}}^{\lambda_i}$ are the corresponding $G_{\mathbf{R}}(\mathbf{R}[\![t]\!])$-orbits on $\mathrm{Gr}_{G_{\mathbf{R}}}$, and let $\mathrm{Gr}_{G_{\mathbf{R}}}^{\lambda_\bullet}$ denote the corresponding convolution variety. Let $\mathrm{IC}_{\lambda_\bullet}$ denote the pushforward of the constant sheaf along the map $\mathrm{Gr}_{G_{\mathbf{R}}}^{\lambda_\bullet} \to \mathrm{Gr}_{G_{\mathbf{R}}}^{|\lambda_\bullet|} \subseteq \mathrm{Gr}_{G_{\mathbf{R}}}$. If

$\mathcal{P} = \mathrm{colim}_\nu \bigoplus_{|\lambda_\bullet| \leq \nu} \mathrm{IC}_{\lambda_\bullet}$, then we will write $\mathrm{Shv}^{\min}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ to denote $\mathrm{Shv}^{\mathcal{P}}_{\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$. Its degeneration in the sense of Definition 4.2.4 will be denoted $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$.[4]

The other class of examples we will study are of the form $\mathrm{G} \times \mathrm{H}$ acting on $\mathrm{X} = \mathrm{G}$ by left- and right-multiplication, where $\mathrm{H} \subseteq \mathrm{H} \times \mathrm{G}$ is a spherical subgroup. Then the category of interest is $\mathrm{Shv}^{\mathrm{H}[\![t]\!]\text{-cbl}}_{\mathrm{H}}(\mathrm{Gr}_{\mathrm{G}}; k)$. Again, the $\mathrm{H}[\![t]\!]$-orbits on Gr_{G} are discrete and indexed by the dominant weights $\check{\Lambda}^+_{\mathrm{X}}$ of the dual group $\check{\mathrm{G}}_{\mathrm{X}}$. We will again define $\mathrm{Shv}^{\min}_{\mathrm{H}}(\mathrm{Gr}_{\mathrm{G}}; k)$ using the convolution varieties $\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}$ built from sequences $\lambda_\bullet$ of dominant minuscule weights of $\check{\mathrm{G}}_{\mathrm{X}}$. Its degeneration in the sense of Definition 4.2.4 will be denoted $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{H}}(\mathrm{Gr}_{\mathrm{G}}; k)$.

Remark 5.4.2. We restrict to these cases because they give examples of subgroups $\mathrm{H} \subseteq \mathrm{G}$ such that $\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k) \simeq \mathrm{Shv}^{\mathcal{S}}_{\mathrm{K}}(\mathrm{Y}; k)$ for some group scheme K whose $\mathbf{C}$-points are homotopy equivalent to $\mathrm{H}_{\mathbf{C}}$, and some ind-K-scheme Y equipped with a K-equivariant stratification $\mathcal{S}$. This is a categorification of the homotopy equivalence between $\mathrm{G}\backslash\mathcal{L}\mathrm{G}/\mathcal{L}\mathrm{H}$ and $\mathrm{H}\backslash\Omega(\mathrm{G}/\mathrm{H})$ studied in the preceding section. In the cases at hand, the $\mathbf{C}$-points of the ind-scheme Y is homotopy equivalent to $\Omega(\mathrm{G}/\mathrm{H})$.

The conditions in the theorem below are definitely not "optimal" (there are many examples which do not satisfy them, and it is quite plausible that a mild modification of the argument of Theorem 5.4.3 will work in the general case); they are just the easiest conditions under which Theorem 4.1.5 can be applied.

Theorem 5.4.3. *Let $\mathrm{H}_{\mathbf{C}} \subseteq \mathrm{G}_{\mathbf{C}}$ be a closed connected reductive subgroup which is either a symmetric subgroup (called "Case I" below) or is a spherical subgroup of the form $\mathrm{H}_{\mathbf{C}} \subseteq \mathrm{H}_{\mathbf{C}} \times \mathrm{K}_{\mathbf{C}}$ for an inclusion $\mathrm{H}_{\mathbf{C}} \subseteq \mathrm{K}_{\mathbf{C}}$ into another closed connected reductive group $\mathrm{K}_{\mathbf{C}}$ (called "Case II" below). Let $\mathrm{X} = \mathrm{G}_{\mathbf{C}}/\mathrm{H}_{\mathbf{C}}$, and let Y denote either $\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$ in Case I, or Gr_{K} in Case II. Suppose that:*

- *if T denotes (a maximal compact subgroup of) a maximal torus of $\mathrm{H}_{\mathbf{C}}$, then Y satisfies the hypotheses of Setup 4.1.3, i.e., the $\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])$-orbit stratification (resp. $\mathrm{H}[\![t]\!]$-orbit stratification, in Case II) $\{\mathrm{Y}^\mu\}$ of Y has a T-equivariant refinement where each stratum is a complex affine space on which T acts linearly.*
- *the varieties Y^{λ_i} have even cells for all dominant minuscule weights λ_i of $\check{\mathrm{G}}_{\mathrm{X}}$.*
- *the nonempty fibers of $\mathrm{Y}^{\lambda_\bullet} \to \overline{\mathrm{Y}^{|\lambda_\bullet|}} \subseteq \mathrm{Y}$ have affine pavings.*

Let k be denote an algebraically closed field, KU, or an elliptic cohomology theory $\mathrm{Ell}_{\mathrm{E}}$. If k is not an ordinary commutative ring, assume that $\check{\mathrm{G}}_{\mathrm{X}}$ is of type A, B, C, or D; if $\check{\mathrm{G}}_{\mathrm{X}}$ is of type B or D, assume that 2 is a unit in $\pi_0(k)$, and if $\check{\mathrm{G}}_{\mathrm{X}}$ is of type C_n, assume that $n!$ is a unit in $\pi_0(k)$. Let $\mathcal{C}_{\mathrm{X}}$ denote either $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ or $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{H}}(\mathrm{Gr}_{\mathrm{K}}; k)$ (corresponding to the two choices of $\mathrm{H}_{\mathbf{C}}$ above).

Let $\check{\mathrm{V}}_{\mathbf{H}}$ denote a normal affine $\check{\mathrm{G}}_{\mathrm{X}}$-space over $\mathrm{Spev}(k)$ (as prescribed by [BZSV] if k is a discrete commutative ring and $\mathrm{H}_{\mathbf{C}}$ is a spherical subgroup). Suppose that there is a "Kostant section" $\kappa_{\check{\mathrm{V}}} : \mathcal{M}_{\mathrm{H},0} \hookrightarrow \check{\mathrm{V}}_{\mathbf{H}}$ (see, e.g., Conjecture 5.3.10) such that:

a. *Let $\check{\mathrm{J}}'_{\mathrm{X}}$ denote the (possibly non-flat) group scheme $\mathcal{M}_{\mathrm{H},0} \times_{\check{\mathrm{V}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}}} \mathcal{M}_{\mathrm{H},0}$ over $\mathcal{M}_{\mathrm{H},0}$, and let*
$$\check{\mathrm{V}}^{\mathrm{reg}}_{\mathbf{H}} := (\mathcal{M}_{\mathrm{H},0} \times \check{\mathrm{G}}_{\mathrm{X}})/\check{\mathrm{J}}'_{\mathrm{X}}.$$
Then the map $\check{\mathrm{V}}^{\mathrm{reg}}_{\mathbf{H}} \hookrightarrow \check{\mathrm{V}}_{\mathbf{H}}$ is an open immersion with complement of codimension at least 2.

[4]Note that if $\check{\mathrm{G}}_{\mathrm{X}}$ is not of type F_4, G_2, or E_8, then every dominant weight can be written as a sum of dominant minuscule coweights.

b. Define

$$\check{\mathrm{J}}_{\mathrm{X}} := \mathrm{Spec}_{\mathcal{M}_{\mathrm{H},0}} \mathcal{H}^{\mathrm{H}}_{*}(\Omega(\mathrm{G}/\mathrm{H}); k). \tag{5.4.1}$$

Then there is an isomorphism of graded group schemes over $\mathcal{M}_{\mathrm{H},0}$*:*

$$\check{\mathrm{J}}_{\mathrm{X}} \cong \mathcal{M}_{\mathrm{H},0} \times_{\check{\mathrm{V}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}}} \mathcal{M}_{\mathrm{H},0} = \check{\mathrm{J}}'_{\mathrm{X}}.$$

Then there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(k))$*-linear* ∞*-categories*

$$\mathcal{C}_{\mathrm{X}} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{V}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}}). \tag{5.4.2}$$

Moreover, this equivalence fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{\mathrm{X}} \arrow[r, "\sim"] \arrow[d, "\text{cohomology}"'] & \mathrm{QCoh}(\check{\mathrm{V}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}}) \arrow[d, "\kappa^*"] \\
\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{H}}(*; k) \arrow[r, "\sim"'] & \mathrm{QCoh}(\mathcal{M}_{\mathrm{H},0}),
\end{tikzcd}$$

where the cohomology functor $\mathcal{C}_{\mathrm{X}} \to \mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}}(*; k)$ *factors through the canonical functor* $\mathrm{Shv}^{\mathrm{gr}}_{\mathrm{H}}(*; k) \to \mathrm{Shv}^{\mathrm{gr}}_{\mathrm{G}}(*; k)$ *by* (5.3.2).

Proof. In the discussion below, we will let T denote a maximal torus of a maximal compact subgroup of $\mathrm{G}_{\mathbf{R}}$ in Case I, or a maximal torus of H in Case II. First, observe that the objects $\mathrm{IC}_{\lambda_\bullet}$ are perfect even. This follows as in Lemma 4.3.2 using the assumption that the nonempty fibers of $\mathrm{Y}^{\lambda_\bullet} \to \mathrm{Y}^{|\lambda_\bullet|}$ have affine pavings. Note also that $\mathcal{H}^*_{\mathrm{T}}(\mathrm{Y}; \mathrm{IC}_{\lambda_\bullet})$ is perfect even, because $\mathrm{Y}^{\lambda_\bullet}$ is homotopy equivalent to $\mathrm{Y}^{\lambda_1} \times \cdots \times \mathrm{Y}^{\lambda_n}$, and each Y^{λ_i} was assumed to have even cells. Let $\mathcal{A} = \mathrm{coMod}^{\mathrm{gr},\heartsuit}_{\mathcal{H}^{\mathrm{H}}_*(\mathrm{Y};k)}(\mathrm{QCoh}(\mathcal{M}_{\mathrm{H},0})^\heartsuit)$. Then the conditions (a) and (b) give an equivalence

$$\mathcal{A} \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{V}}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}})^\heartsuit.$$

Under this equivalence, $\mathcal{H}^*_{\mathrm{H}}(\mathrm{Y}; \mathrm{IC}_{\lambda_\bullet})$ is sent to $\mathcal{V}_{\lambda_\bullet} = \bigotimes_j \mathcal{V}_{\lambda_j}$, where $\mathcal{V}_{\lambda_j}$ is the pullback of the representation V_{λ_j} along the map $\check{\mathrm{V}}^{\mathrm{reg}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}} \to \mathrm{B}\check{\mathrm{G}}_{\mathrm{X}}$. If $\mathrm{IC}_{\leq \nu} = \bigoplus_{|\lambda_\bullet| \leq \nu} \mathrm{IC}_{\lambda_\bullet}$, then it follows from the H-equivariant version of Corollary 4.2.8 that

$$\mathcal{C}_{\mathrm{X}} \simeq \mathrm{colim}_\nu \, \mathrm{D}\left(\langle \mathcal{H}^*_{\mathrm{H}}(\mathrm{Y}; \mathrm{IC}_{\leq \nu}) \rangle_{\mathcal{A}}\right).$$

The same argument as in Corollary 4.3.14 now identifies the right-hand side with $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{V}}_{\mathbf{H}}/\check{\mathrm{G}}_{\mathrm{X}})$, as desired. ☐

Remark 5.4.4. Let $\check{\mathrm{M}}_{\mathbf{H}} = \check{\mathrm{G}} \times^{\check{\mathrm{G}}_{\mathrm{X}}} \check{\mathrm{V}}_{\mathbf{H}}$. If k is not an ordinary commutative ring, assume that G is simply-laced with torsion-free fundamental group. There is a canonical homomorphism

$$\check{\mathrm{J}}_{\mathrm{X}} = \mathrm{Spec}_{\mathcal{M}_{\mathrm{H},0}} \mathcal{H}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k) \to \mathcal{M}_{\mathrm{H},0} \times_{\mathcal{M}_{\mathrm{G},0}} \mathrm{Spec}_{\mathcal{M}_{\mathrm{G},0}} \mathcal{H}^{\mathrm{G}}_*(\Omega\mathrm{G}; k) = \mathcal{M}_{\mathrm{H},0} \times_{\mathcal{M}_{\mathrm{G},0}} \check{\mathrm{J}}$$

of group schemes over $\mathcal{M}_{\mathrm{H},0}$. Suppose that the conditions (a) and (b) of Theorem 5.4.3 are satisfied, and that there is a commutative diagram

$$\begin{tikzcd}
\check{\mathrm{J}}_{\mathrm{X}} \arrow[r] \arrow[d] & \check{\mathrm{G}}_{\mathrm{X}} \times \mathcal{M}_{\mathrm{H},0} \arrow[d] \\
\check{\mathrm{J}} \arrow[r] & \check{\mathrm{G}} \times \mathcal{M}_{\mathrm{G},0}.
\end{tikzcd}$$

Then the argument of Theorem 5.4.3 gives an $\check{\mathrm{G}}_{\mathrm{X}}$-equivariant map $\check{\mathrm{V}}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}}$, i.e., a $\check{\mathrm{G}}$-equivariant map $\mu : \check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}}$ (where the target should be interpreted as $\check{\mathfrak{g}}^*$ if k is an ordinary commutative ring). When k is an ordinary commutative ring, the map $\mu : \check{\mathrm{M}}_{\mathbf{G}_a} \to \check{\mathfrak{g}}^*$ is the moment map for the Hamiltonian $\check{\mathrm{G}}$-action on $\check{\mathrm{M}} = \check{\mathrm{M}}_{\mathbf{G}_a}$ posited by [BZSV]. It follows that the equivalence (5.4.2) is equivariant (under the degeneration of Conjecture 4.3.20) for the action of the (degenerated) spherical Hecke category on $\mathcal{C}_{\mathrm{X}}$ and the action of $\mathrm{QCoh}^{\mathrm{gr}}(\mathrm{G}_{\mathbf{H}}/\check{\mathrm{G}})$ on $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$ by pullback along μ.

Remark 5.4.5. Even if the conditions of Theorem 5.4.3 are not satisfied, one can nevertheless "reverse-engineer" a construction of $\check{\mathrm{M}}_{\mathbf{H}}$ associated to some connected reductive subgroup $\mathrm{H}_{\mathbf{C}} \subseteq \mathrm{G}_{\mathbf{C}}$. Namely, there is *always* a homomorphism

$$\begin{aligned}\check{\mathrm{J}}_{\mathrm{X}} = \mathrm{Spec}_{\mathcal{M}_{\mathrm{H},0}} \mathcal{H}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k) \to \mathcal{M}_{\mathrm{H},0} \times_{\mathcal{M}_{\mathrm{G},0}} \mathrm{Spec}_{\mathcal{M}_{\mathrm{G},0}} \mathcal{H}^{\mathrm{G}}_*(\Omega\mathrm{G}; k) = \mathcal{M}_{\mathrm{H},0} \times_{\mathcal{M}_{\mathrm{G},0}} \check{\mathrm{J}} \\ \xrightarrow{\text{Corollary 4.3.17}} \check{\mathrm{G}} \times \mathcal{M}_{\mathrm{H},0},\end{aligned}$$

and one can *define* $\check{\mathrm{M}}_{\mathbf{H}}$ to be the affinization of the quotient $(\check{\mathrm{G}} \times \mathcal{M}_{\mathrm{H},0})/\check{\mathrm{J}}_{\mathrm{X}}$. Of course, if the conditions of Theorem 5.4.3 are not satisfied, then it will not be clear that there is a degeneration of (the spherically generated subcategory of) $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$. Nevertheless, $\check{\mathrm{M}}_{\mathbf{H}}$ defined in this way at least presents a candidate for the Langlands dual side in such a degeneration, and it turns out to be extremely interesting as an independent object of study.

As such, we will make the following notational distinction: we will write

$$\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}} := \text{affinization of the quotient } (\check{\mathrm{G}} \times \mathcal{M}_{\mathrm{H},0})/\check{\mathrm{J}}_{\mathrm{X}};$$

and we will write $\check{\mathrm{M}}_{\mathbf{H}}$ to denote the (putative) $\check{\mathrm{G}}$-space over $\mathrm{Spev}(k)$ such that there is a degeneration of (a subcategory of) $\mathrm{Shv}_{\mathrm{G}}^{\mathrm{G}[\![t]\!]\text{-cbl}}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!); k)$ into $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}})$. Conjecturally, of course, one expects that $\check{\mathrm{M}}_{\mathbf{H}} = \check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}$, and Theorem 5.4.3 provides some criteria under which this is true. The utility of providing a more "intrinsic" description of $\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}$ (namely as $\check{\mathrm{M}}_{\mathbf{H}}$) is that it allows one to calculate match some finer structures under geometric Langlands duality, like (constructible and coherent) singular support.

Conjecture 5.4.6. *Suppose* G/H *is an affine spherical* G*-variety. Then there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k) \arrow[r] \arrow[d] & \mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{G}; \mathbf{Q}) \arrow[r, "\sim"] \arrow[d] & \check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \\ \check{\mathrm{G}}_{\mathrm{X}}(-2\rho_{\check{\mathrm{G}}}) \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \arrow[r] & \check{\mathrm{G}}(-2\rho_{\check{\mathrm{G}}}) \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) & \end{tikzcd}$$

of graded group schemes over $\mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)$*, where the homomorphism* $\check{\mathrm{G}}_{\mathrm{X}} \to \check{\mathrm{G}}$ *is that of Definition 5.1.20, and the vertical maps are closed immersions.*

Remark 5.4.7. Conjecture 5.4.6 should in some sense follow from the *abelian* Satake equivalence of [GN] via the Tannakian formalism (as in [YZ2]). Namely, let $\mathbf{Q}(\mathrm{Z})$ denote the tensor category studied in [GN]. If Hypothesis 5.3.2 is satisfied (for example, G/H is a symmetric variety for G), Proposition 5.3.1 shows that equivariant homology defines a functor from $\mathbf{Q}(\mathrm{Z})$ to the abelian 1-category $\mathrm{coMod}_{\mathrm{H}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H});\mathbf{Q})}(\mathrm{QCoh}(\check{\mathfrak{h}}^*(2)/\!\!/\check{\mathrm{H}}))$. There is a symmetric monoidal equivalence $\mathbf{Q}(\mathrm{Z}) \simeq \mathrm{Rep}(\check{\mathrm{G}}_{\mathrm{X,GN}})$ by [GN], where $\check{\mathrm{G}}_{\mathrm{X,GN}}$ is the Gaitsgory-Nadler dual group. If $\check{\mathrm{G}}_{\mathrm{X,GN}} \cong \check{\mathrm{G}}_{\mathrm{X}}$, and there is an analogue of [YZ2, Lemma 2.2] in this context, the Tannakian formalism would give a homomorphism $\mathrm{Spec}\, \mathrm{H}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); \mathbf{Q}) \to \check{\mathrm{G}}_{\mathrm{X}} \times \check{\mathfrak{h}}^*(2)/\!\!/\check{\mathrm{H}}$.

Remark 5.4.8. In the setting of Definition 5.4.1, suppose that $\mathrm{X} = \mathrm{G}/\mathrm{H}$ is a symmetric variety. One can then also consider the following mild variant of the category $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$. Let $\lambda_\bullet$ be a sequence of dominant minuscule weights of $\check{\mathrm{G}}$ (not of $\check{\mathrm{G}}_{\mathrm{X}}$!), and let $\widetilde{\mathrm{IC}}_{\lambda_\bullet}$ denote the image of $\mathrm{IC}_{\lambda_\bullet} \in \mathrm{Shv}^{\min}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k)$ along the real nearby cycles functor $\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k) \to \mathrm{Shv}^{\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])\text{-cbl}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ of [Nad]. If $\widetilde{\mathcal{P}} = \mathrm{colim}_\nu \bigoplus_{|\lambda_\bullet| \leq \nu} \widetilde{\mathrm{IC}}_{\lambda_\bullet}$, then we will write $\widetilde{\mathrm{Shv}}^{\min}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ to denote $\mathrm{Shv}^{\widetilde{\mathcal{P}}}_{\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$. Its degeneration in the sense of Definition 4.2.4 will be denoted $\widetilde{\mathrm{Shv}}^{\min,\mathrm{gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$.

Note that because the convolution varieties $\mathrm{Gr}^{\lambda_\bullet}_{\mathrm{G}}$ have even cells, each $\mathcal{H}^*_{\mathrm{T}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; \widetilde{\mathrm{IC}}_{\lambda_\bullet})$ is perfect even. Recall that the nearby cycles functor can be thought of as $*$-pushforward along a map $\mathrm{Gr}_{\mathrm{G}} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$. Up to homotopy, this identifies with the canonical map $\Omega\mathrm{G} \to \Omega(\mathrm{G}/\mathrm{H})$, whose homotopy fiber is $\Omega\mathrm{H}$, which does admit an even cell structure. This suggests that the image in $\mathrm{Shv}^{\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])\text{-cbl}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ of a perfect even object of $\mathrm{Shv}^{\mathrm{G}[\![t]\!]\text{-cbl}}_{\mathrm{G}}(\mathrm{Gr}_{\mathrm{G}}; k)$ remains perfect even, i.e., that the objects $\widetilde{\mathrm{IC}}_{\lambda_\bullet}$ are all perfect even in $\mathrm{Shv}^{\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])\text{-cbl}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$. Were this true, and were the equivariant homology $\mathcal{H}^{\mathrm{T}}_*(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k)$ also (ind-)perfect even in $\mathrm{QCoh}(\mathcal{M}_{\mathrm{T},0})$, then the argument of Theorem 5.4.3 would imply that if $\check{\mathrm{G}}$ is of type A, B, C, or D (with the standard assumptions on 2 or $n!$ being a unit in $\pi_0(k)^\times$ in types B, D or C_n respectively), there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear ∞-categories

$$\widetilde{\mathrm{Shv}}^{\min,\mathrm{gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}/\check{\mathrm{G}}). \tag{5.4.3}$$

Here, $\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}$ is defined as in Remark 5.4.5.

The perspective on $\check{\mathrm{M}}$ as being $\check{\mathrm{M}}^{\ddagger}$ (defined by Remark 5.4.5) leads to several interesting and nontrivial structures. For instance, G/H has an action of its G-equivariant automorphism group $\mathrm{N}_{\mathrm{G}}(\mathrm{H})/\mathrm{H}$, and hence $\mathrm{N}_{\mathrm{G}}(\mathrm{H})/\mathrm{H}$ acts on $\check{\mathrm{J}}_{\mathrm{X}}$. The above construction of $\check{\mathrm{M}}^{\ddagger}$ therefore shows that there is a natural $\mathrm{N}_{\mathrm{G}}(\mathrm{H})/\mathrm{H}$-action on $\check{\mathrm{M}}^{\ddagger}$, and hence an expected $\mathrm{N}_{\mathrm{G}}(\mathrm{H})/\mathrm{H}$-action on $\check{\mathrm{M}}$, which commutes with its Hamiltonian $\check{\mathrm{G}}$-action. This action is highly interesting; for instance, when $\mathrm{H} = \mathrm{T} \subseteq \mathrm{G}$, $\check{\mathrm{M}}^{\ddagger}$ is the affine closure of $\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})$; the above action of $\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T} \cong \mathrm{W}$ turns out to be Gelfand-Graev action of the Weyl group on $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\check{\mathrm{N}})}$ (as described by Ginzburg-Kazhdan in [GK]).

Remark 5.4.9. The group scheme $\mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k)$ can be described in terms of the regular centralizer group schemes $\check{\mathrm{J}}_{\check{\mathrm{G}}}$ and $\check{\mathrm{J}}_{\check{\mathrm{H}}}$ for $\check{\mathrm{G}}$ and $\check{\mathrm{H}}$. There is a fiber sequence of $\mathbf{E}_1$-spaces

$$\Omega\mathrm{H} \to \Omega\mathrm{G} \to \Omega(\mathrm{G}/\mathrm{H}),$$

which gives an equivalence

$$\mathrm{C}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k) \simeq \mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{G}; k) \otimes_{\mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{H}; k)} \mathrm{C}^*_{\mathrm{H}}(*; k)$$

of $\mathbf{E}_1$-k-algebras. It follows from Theorem 3.6.3, for instance, that if the map $\mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{H}; k) \to \mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{G}; k)$ is evenly faithfully flat, there is an isomorphism

$$\mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); \mathbf{Q}) \cong (\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)) \times_{\check{\mathrm{J}}_{\check{\mathrm{H}}}} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)$$

of group schemes over $\mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)$. Therefore, the study of the H-action on $\Omega(\mathrm{G}/\mathrm{H})$ is closely related to understanding the map $\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \to \check{\mathrm{J}}_{\check{\mathrm{H}}}$ (which plays an important role in Langlands transfer).

For instance, let k be an ordinary commutative ring, $\mathrm{G} = \mathrm{SL}_2$ and $\mathrm{H} = \mathbf{G}_m$; then, $\check{\mathrm{J}}_{\check{\mathrm{H}}} \cong \mathrm{T}^*\mathbf{G}_m$, while $\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev\,C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev\,C}^*_{\mathrm{H}}(*;k)$ is isomorphic to the affine blowup $(\mathrm{T}^*\mathbf{G}_m)[\frac{e^x-1}{2x}]$ of $\mathrm{T}^*\mathbf{G}_m$. It follows that

$$\begin{aligned}\mathrm{Spev\,C}^{\mathrm{H}}_*(\Omega(\mathrm{G/H});k) &\cong (\mathrm{T}^*\mathbf{G}_m)[\tfrac{e^x-1}{2x}] \times_{\mathrm{T}^*\mathbf{G}_m} \mathfrak{g}_m \\ &\cong (\mathrm{T}^*\mathbf{G}_m)[\tfrac{e^x-1}{2x}] \times_{\mathbf{G}_m} \{1\} \cong \mathrm{Spec}\,k[x, \tfrac{e^x-1}{2x}]/(2x \cdot \tfrac{e^x-1}{x}).\end{aligned}$$

One can verify that this isomorphism holds by computing $\mathrm{H}^{\mathrm{H}}_*(\Omega(\mathrm{G/H});k)$ independently.

Remark 5.4.10 (Singular support)**.** It is natural to ask a criterion for when an object of $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!);k)$ is compact in terms of the degeneration of (5.4.2). If $\mathcal{F}$ is a compact object of $\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!);k)$, the equivariant cohomology $\mathrm{C}^*_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!);\mathcal{F})$ is a perfect k-module. If $\Phi(\mathcal{F}) \in \mathrm{Perf}(\check{\mathrm{M}}/\check{\mathrm{G}}(-2\rho))$ denotes the corresponding object under the degeneration of (5.4.2), Theorem 5.4.3 implies that the set-theoretic support $\mathrm{Supp}(\Phi(\mathcal{F}))$ intersects the image of κ in a zero-dimensional scheme. We expect that $\mathcal{F}$ is compact if and only if it is set-theoretically supported on the nullcone $\mathcal{U}_{\check{\mathrm{M}}} := \check{\mathrm{M}}_{\mathbf{H}} \times_{\mathrm{Spev\,C}^*_{\check{\mathrm{H}}}(*;k)} \{0\}$ of $\check{\mathrm{M}}$; in other words, that

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{G}[\![t]\!]}(\mathrm{G}(\!(t)\!)/\mathrm{H}(\!(t)\!);k)^\omega \rightsquigarrow \mathrm{Perf}_{\mathcal{U}_{\check{\mathrm{M}}}}(\check{\mathrm{M}}_{\mathbf{H}}/\check{\mathrm{G}}). \tag{5.4.4}$$

Suppose G is simply-laced and has torsion-free fundamental group. Let $\mu : \check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{G}_{\mathbf{H}}$ denote the moment map (where the target is to be interpreted as $\check{\mathfrak{g}}^*$ if k is an ordinary commutative ring and G is not necessarily simply-laced), and let $\mathcal{U}_{\mathbf{H}} \subseteq \mathrm{G}_{\mathbf{H}}$ denote the nullcone of $\mathrm{G}_{\mathbf{H}}$. Then there is a canonical map $\mathcal{U}_{\check{\mathrm{M}}} \to \mu^{-1}(\mathcal{U}_{\mathbf{H}})$, where $\mu^{-1}(\mathcal{U}_{\mathbf{H}})$ is the *derived* preimage of $\mathcal{U}_{\mathbf{H}}$ under the moment map. It turns out that the map $\mathcal{U}_{\check{\mathrm{M}}} \to \mu^{-1}(\mathcal{U}_{\mathbf{H}})$ *nearly* induces an isomorphism on reduced schemes (it is an "Artinian" thickening[5]); so the singular support in (5.4.4) cannot quite be replaced with singular support contained in $\mu^{-1}(\mathcal{U}_{\mathbf{H}})$.

Let us now shift gears somewhat. The following result is related to the discussion in [Sak, Section 5.1.5] and to [Tel1, Section 5.2].

Proposition 5.4.11. *Let* $\mathrm{H} \subseteq \mathrm{G}$ *be a closed subgroup. Then there is a Lagrangian correspondence (interpreted in a derived sense)*

$$\begin{tikzcd} & \check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev\,C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev\,C}^*_{\mathrm{H}}(*;k) \arrow[dl] \arrow[dr] & \\ \check{\mathrm{J}}_{\check{\mathrm{H}}} & & \check{\mathrm{J}}_{\check{\mathrm{G}}}, \end{tikzcd}$$

[5] An easy way to see this is as follows. There is a Cartesian square

$$\begin{tikzcd} \mathcal{U}_{\check{\mathrm{M}}} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mu^{-1}(\mathcal{U}_{\mathbf{H}}) \arrow[r] & \check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \{0\}, \end{tikzcd}$$

which follows from writing $\mathcal{U}_{\check{\mathrm{M}}} = \check{\mathrm{M}}_{\mathbf{H}} \times_{\check{\mathrm{M}}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \{0\}$, $\mu^{-1}(\mathcal{U}_{\mathbf{H}}) = \check{\mathrm{M}}_{\mathbf{H}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \{0\}$, and $\{0\} = \check{\mathrm{M}}_{\mathbf{H}}/\!\!/\check{\mathrm{G}} \times_{\check{\mathrm{M}}_{\mathbf{H}}/\!\!/\mathrm{G}} \{0\}$. The fiber product $\check{\mathrm{M}}_{\mathbf{H}}/\!\!/\check{\mathrm{G}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \{0\}$ is an Artinian thickening of $\{0\}$, which implies the desired claim. In fact, since $\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}} \cong \mathrm{Spev\,C}^*_{\check{\mathrm{G}}}(*;k)$ (see Remark 3.7.9) and Conjecture 5.3.10 says that $\check{\mathrm{M}}_{\mathbf{H}}/\!\!/\check{\mathrm{G}} \cong \mathrm{Spev\,C}^*_{\check{\mathrm{H}}}(*;k)$, one expects an isomorphism $\check{\mathrm{M}}_{\mathbf{H}}/\!\!/\check{\mathrm{G}} \times_{\mathrm{G}_{\mathbf{H}}/\!\!/\check{\mathrm{G}}} \{0\} \cong \mathrm{Spev\,C}^*(\mathrm{G/H};k)$. One again sees that this fiber product is necessarily an Artinian thickening of a point, this time because G/H is a finite CW-complex, so $\mathrm{C}^*(\mathrm{G/H};k)$ is a perfect k-module.

where the left map restricts to the zero section of $\check{\mathrm{J}}_{\check{\mathrm{H}}}$ when pulled back to the identity section of $\check{\mathrm{J}}_{\check{\mathrm{G}}}$.

Proof. The desired claim follows from the analogous statement at the level of Lie algebras. It is a well-known fact (discussed in § 3.10) that the Lie algebra of $\check{\mathrm{J}}_{\check{\mathrm{G}}}$ can be identified with the cotangent bundle $\mathrm{T}^*(2)(\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k))$, and similarly for $\check{\mathrm{J}}_{\check{\mathrm{H}}}$. We therefore need to see that there is a Lagrangian correspondence

$$\begin{tikzcd}
& \mathrm{T}^*(\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)) \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \arrow[dl] \arrow[dr] & \\
\mathrm{T}^*(\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)) & & \mathrm{T}^*(\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)).
\end{tikzcd}$$

More generally, if $\mathrm{Y} \to \mathrm{Z}$ is a map between schemes, there is a Lagrangian correspondence

$$\begin{tikzcd}
& \mathrm{T}^*(\mathrm{Z}) \times_{\mathrm{Z}} \mathrm{Y} \arrow[dl] \arrow[dr] & \\
\mathrm{T}^*\mathrm{Y} & & \mathrm{T}^*\mathrm{Z}.
\end{tikzcd}$$

This is of course well-known if $\mathrm{Y} \to \mathrm{Z}$ is a smooth map of smooth schemes, but the same continues to hold in general (see, e.g., [Cal, Theorem 2.8]). Taking $\mathrm{Y} \to \mathrm{Z}$ to be the map $\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \to \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)$, we win. ☐

Remark 5.4.12. The left map in Proposition 5.4.11 is precisely the one of Remark 5.4.9. Note that $\check{\mathrm{J}}_{\mathrm{X}}$ is the kernel of the homomorphism $\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \to \check{\mathrm{J}}_{\check{\mathrm{H}}}$. Concretely, there is a commutative diagram

$$\begin{tikzcd}
& \check{\mathrm{J}}_{\mathrm{X}} \arrow[dl] \arrow[dr] & & \\
\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \arrow[dr] & & \check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \arrow[dl] \arrow[dr] & \\
& \check{\mathrm{J}}_{\check{\mathrm{H}}} & & \check{\mathrm{J}}_{\check{\mathrm{G}}},
\end{tikzcd} \tag{5.4.5}$$

where the square is Cartesian. This implies that the map $\check{\mathrm{J}}_{\mathrm{X}} \to \check{\mathrm{J}}_{\check{\mathrm{G}}}$ is Lagrangian (in a derived sense). Moreover, it implies that there is an isomorphism

$$\mathrm{Lie}(\check{\mathrm{J}}_{\mathrm{X}}) \cong \mathrm{T}^*[1](\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)/\,\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)),$$

where the right-hand side denotes the 1-shifted cotangent bundle. The formula of Remark 5.4.5 also shows that

$$\begin{aligned}
\check{\mathrm{M}}^{\ddagger} &\cong \overline{(\check{\mathrm{J}}_{\check{\mathrm{H}}} \times \check{\mathrm{G}})/(\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k))} \\
&\cong \overline{(\check{\mathrm{J}}_{\check{\mathrm{H}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{T}^*(\check{\mathrm{G}}/_{\psi}\check{\mathrm{N}}))/(\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k))}.
\end{aligned}$$

The final isomorphism comes from the identification $\mathrm{T}^*(\check{\mathrm{G}}/_{\psi}\check{\mathrm{N}}) \cong \check{\mathrm{G}} \times \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)$ via Theorem 3.7.7. If $\check{\mathrm{G}}$ has trivial center, for instance, the group scheme $\check{\mathrm{J}}_{\check{\mathrm{G}}}$ is connected, and so we find that if k is a field of characteristic zero, then Remark 5.4.5 can be rewritten to describe $\mathcal{O}_{\check{\mathrm{M}}^\ddagger}$ as the Poisson centralizer

$$\mathcal{O}(\check{\mathrm{M}}^\ddagger) \cong \mathcal{O}(\check{\mathrm{J}}_{\check{\mathrm{H}}} \times_{\check{\mathfrak{g}}^*/\!\!/\check{\mathrm{G}}} \mathrm{T}^*(\check{\mathrm{G}}/_\psi \check{\mathrm{N}}))^{\mathcal{O}(\check{\mathfrak{g}}^*/\!\!/\check{\mathrm{G}})}.$$

This is a formula analogous to [GK, Theorem 1.3.3]. Again, if one defines $\check{\mathrm{M}}^\ddagger$ in this way, the question of proving Conjecture 5.2.12 (when $\mathrm{H} \subseteq \mathrm{G}$ is spherical) now becomes about identifying $\check{\mathrm{M}}^\ddagger$ with the prescription of [BZSV].

Furthermore, using the main result of [BFM] and Theorem 5.4.3 (all of which is related to [Tel1, Theorem 5.3]), the above diagram (5.4.5) can be identified with

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega(\mathrm{G}/\mathrm{H}); k) \arrow[dl] \arrow[dr] & & \\
\mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \arrow[dr] & & \mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{G}; k) \arrow[dl] \arrow[dr] & \\
 & \mathrm{Spev}\, \mathrm{C}^{\mathrm{H}}_*(\Omega\mathrm{H}; k) & & \mathrm{Spev}\, \mathrm{C}^{\mathrm{G}}_*(\Omega\mathrm{G}; k).
\end{tikzcd}$$

The long composite on the right-hand side of the above diagram will be Lagrangian, hence coisotropic (but this has to be interpreted in a derived sense; for example, it need not be a closed immersion!).

Proposition 5.4.11 has an interesting consequence.

Construction 5.4.13. The homomorphisms

$$\begin{aligned}
&\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \to \check{\mathrm{G}} \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k), \\
&\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \to \check{\mathrm{J}}_{\check{\mathrm{H}}} \to \check{\mathrm{H}} \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)
\end{aligned}$$

define a closed immersion

$$\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \to \check{\mathrm{G}} \times \check{\mathrm{H}} \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)$$

of group schemes over $\mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k)$. Let $\check{\mathcal{M}}^\ddagger$ denote the affinization

$$\check{\mathcal{M}}^\ddagger = \overline{(\check{\mathrm{G}} \times \check{\mathrm{H}} \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k))/(\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k))},$$

so that

$$\begin{aligned}
\dim(\check{\mathcal{M}}^\ddagger) &= \dim(\check{\mathrm{G}}) + \dim(\check{\mathrm{H}}) + \mathrm{rank}(\check{\mathrm{H}}) - \mathrm{rank}(\check{\mathrm{G}}) \\
&= 2\left(\dim(\check{\mathrm{H}}/\mathrm{N}_{\check{\mathrm{H}}}) + \dim(\check{\mathrm{G}}/\mathrm{B}_{\check{\mathrm{G}}})\right),
\end{aligned}$$

where $\mathrm{N}_{\check{\mathrm{H}}}$ is the unipotent radical of a Borel subgroup of $\check{\mathrm{H}}$, and $\mathrm{B}_{\check{\mathrm{G}}}$ is a Borel subgroup of $\check{\mathrm{G}}$. It can be shown that $\check{\mathcal{M}}^\ddagger$ admits the structure of a Hamiltonian $\check{\mathrm{G}} \times \check{\mathrm{H}}$-space (in fact, this is a consequence of the second part of Corollary 5.4.14 below and [Saf1]).

Corollary 5.4.14. *Define* $\check{\mathrm{M}}^\ddagger$ *as in Remark 5.4.5, and let* $\check{\mathrm{M}}^{\ddagger,\mathrm{reg}} = (\check{\mathrm{G}} \times \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k))/\check{\mathrm{J}}_{\mathrm{X}}$ *denote the* $\check{\mathrm{G}}$*-orbit of the map* $\kappa_{\check{\mathrm{M}}^\ddagger} : \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k) \to \check{\mathrm{M}}^\ddagger$*. Let* $\check{\mathcal{M}}^\ddagger$ *denote the Hamiltonian* $\check{\mathrm{G}} \times \check{\mathrm{H}}$*-space of Construction 5.4.13, and define* $\check{\mathcal{M}}^{\ddagger,\mathrm{reg}}$ *similarly. Then there is an isomorphism*

$$\check{\mathcal{M}}^{\ddagger,\mathrm{reg}}/\check{\mathrm{H}} \cong \check{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\mathrm{Spev}\, \mathrm{C}^*_{\check{\mathrm{G}}}(*;k)} \mathrm{Spev}\, \mathrm{C}^*_{\mathrm{H}}(*; k),$$

and a diagram

$$\begin{tikzcd}
& \check{\mathrm{M}}^{\ddagger,\mathrm{reg}}/\check{\mathrm{G}} \arrow[dl] \arrow[dr] & & \\
\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \arrow[dr, "\kappa"'] & & \check{\mathcal{M}}^{\ddagger,\mathrm{reg}}/(\check{\mathrm{H}}\times\check{\mathrm{G}}) \arrow[dl] \arrow[dr] & \\
& \check{\mathfrak{h}}^{*,\mathrm{reg}}/\check{\mathrm{H}} & & \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}},
\end{tikzcd} \tag{5.4.6}$$

where the square is Cartesian, the long composite on the right-hand side is a Lagrangian morphism, and the span at the bottom of the diagram is a (1-shifted) Lagrangian correspondence. In particular, there is a Cartesian square

$$\begin{tikzcd}
\check{\mathrm{M}}^{\ddagger,\mathrm{reg}} \arrow[r] \arrow[d] & \check{\mathcal{M}}^{\ddagger,reg} \arrow[d] \\
\check{\mathfrak{h}}^*/\!\!/\check{\mathrm{H}} \arrow[r, "\kappa"'] & \check{\mathfrak{h}}^{*,\mathrm{reg}}.
\end{tikzcd} \tag{5.4.7}$$

Proof. By Construction 5.4.13,

$$\check{\mathcal{M}}^{\ddagger,\mathrm{reg}} \cong (\check{\mathrm{G}}\times\check{\mathrm{H}}\times \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k))/(\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)),$$

so that

$$\begin{aligned}
\check{\mathcal{M}}^{\ddagger,\mathrm{reg}}/\check{\mathrm{H}} &\cong (\check{\mathrm{G}}\times \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k))/(\check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)) \\
&\cong ((\check{\mathrm{G}}\times \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k))/\check{\mathrm{J}}_{\check{\mathrm{G}}}) \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \\
&\cong \check{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k),
\end{aligned}$$

as desired. It follows from this identification that the desired diagram (5.4.6) then becomes

$$\begin{tikzcd}
& \check{\mathrm{M}}^{\ddagger,\mathrm{reg}}/\check{\mathrm{G}} \arrow[dl] \arrow[dr] & & \\
\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k) \arrow[dr, "\kappa"'] & & (\check{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k))/\check{\mathrm{G}} \arrow[dl] \arrow[dr] & \\
& \check{\mathfrak{h}}^{*,\mathrm{reg}}/\check{\mathrm{H}} & & \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}},
\end{tikzcd}$$

which satisfies the desired properties since it is obtained by taking classifying stacks of the diagram in Proposition 5.4.11 via the identifications

$$\mathrm{B}_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)}\check{\mathrm{J}}_{\check{\mathrm{G}}} \cong \check{\mathfrak{g}}^{*,\mathrm{reg}}/\check{\mathrm{G}},\ \mathrm{B}_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)}\check{\mathrm{J}}_{\check{\mathrm{H}}} \cong \check{\mathfrak{h}}^{*,\mathrm{reg}}/\check{\mathrm{H}},\ \mathrm{B}_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k)}\check{\mathrm{J}}_{\mathrm{X}} \cong \check{\mathrm{M}}^{\ddagger,\mathrm{reg}}/\check{\mathrm{G}}.$$

□

Remark 5.4.15. Recall that

$$\mathrm{Spev}\,\mathrm{C}^{\mathrm{H}}_*(\Omega((\mathrm{G}\times\mathrm{H})/\mathrm{H});\mathbf{Q}) \cong \check{\mathrm{J}}_{\check{\mathrm{G}}} \times_{\mathrm{Spev}\,\mathrm{C}^*_{\mathrm{G}}(*;k)} \mathrm{Spev}\,\mathrm{C}^*_{\mathrm{H}}(*;k).$$

Based on Theorem 5.4.3, one expects that if $\mathrm{H}^{\mathrm{diag}} \subseteq \mathrm{G} \times \mathrm{H}$ is a spherical subgroup, $\check{\mathcal{M}}^{\ddagger}$ is isomorphic to the Hamiltonian $\check{\mathrm{G}} \times \check{\mathrm{H}}$-space $\check{\mathcal{M}}$ dual to $\mathrm{H}^{\mathrm{diag}} \subseteq \mathrm{G} \times \mathrm{H}$. Note that similarly to the Cartesian square (5.4.7), the quotient $\mu^{-1}(0)/\mathrm{H}$ identifies with $\mathrm{T}^*(\mathrm{G}/\mathrm{H})$, where $\mu^{-1}(0)$ is defined via the Cartesian square

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[r] \arrow[d] & \mathrm{T}^*((\mathrm{G} \times \mathrm{H})/\mathrm{H}^{\mathrm{diag}}) \arrow[d, "\mu"] \\
\{0\} \arrow[r] & \mathfrak{h}^*.
\end{tikzcd} \tag{5.4.8}$$

In other words, the diagram analogous to (5.4.6) in this case is the restriction to regular loci of

$$\begin{tikzcd}
& \mathrm{T}^*(\mathrm{G}/\mathrm{H})/\mathrm{G} \cong (\mathfrak{g}/\mathfrak{h})^*/\mathrm{H} \arrow[dl] \arrow[dr] & & \\
\mathrm{BH} \arrow[dr, "\{0\}"'] & & \mathrm{T}^*((\mathrm{G} \times \mathrm{H})/\mathrm{H}^{\mathrm{diag}})/(\mathrm{G} \times \mathrm{H}) \cong \mathfrak{g}^*/\mathrm{H} \arrow[dl] \arrow[dr] & \\
& \mathfrak{h}^*/\mathrm{H} & & \mathfrak{g}^*/\mathrm{G},
\end{tikzcd}$$

where again the square is Cartesian, the long composite on the right-hand side is the moment map for $\mathrm{T}^*(\mathrm{G}/\mathrm{H})$, and the span at the bottom of the diagram is a (1-shifted) Lagrangian correspondence.

Remark 5.4.16. Assume now that $\check{\mathcal{M}}^{\ddagger} \cong \check{\mathcal{M}}$, and similarly $\check{\mathrm{M}}^{\ddagger} \cong \check{\mathrm{M}}$. The square of (5.4.7) then says that the Whittaker reduction of the $\check{\mathrm{H}}$-action on $\check{\mathcal{M}}$ identifies with $\check{\mathrm{M}}$. Since the dual to $\mathrm{T}^*((\mathrm{G} \times \mathrm{H})/\mathrm{H}^{\mathrm{diag}})$ is $\check{\mathcal{M}}$, and the dual to $\mathrm{T}^*(\mathrm{G}/\mathrm{H})$ is $\check{\mathrm{M}}$, the squares (5.4.7) and (5.4.8) showcase the Langlands duality between "symplectic reduction at 0" and "Whittaker reduction". In the language of quantum field theories, this is the duality between the Dirichlet and Neumann boundary conditions.

Example 5.4.17. If H is a Levi subgroup (spherical or not!) of G with associated parabolic P and unipotent radical $\mathrm{N_P}$, for instance, one can identify $\check{\mathcal{M}}^{\ddagger}$ with the affine closure of $\mathrm{T}^*(\check{\mathrm{G}}/\mathrm{N}^-_{\check{\mathrm{P}}})$. The span at the bottom of the diagram (5.4.6) identifies with the restriction to regular loci of the Lagrangian correspondence

$$\begin{tikzcd}
& \tilde{\check{\mathfrak{g}}}_{\check{\mathrm{P}}}/\check{\mathrm{G}} \cong \mathrm{T}^*(\check{\mathrm{G}}/\mathrm{N}^-_{\check{\mathrm{P}}})/(\check{\mathrm{G}} \times \check{\mathrm{H}}) \arrow[dl] \arrow[dr] & \\
\check{\mathfrak{h}}^*/\check{\mathrm{H}} & & \check{\mathfrak{g}}^*/\check{\mathrm{G}}
\end{tikzcd}$$

coming from the parabolic Grothendieck-Springer resolution (see [Saf2]). This span extends to the affine closure $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\mathrm{N}^-_{\check{\mathrm{P}}})}$, i.e., there is a span

$$\begin{tikzcd}
& \overline{\mathrm{T}^*(\check{\mathrm{G}}/\mathrm{N}^-_{\check{\mathrm{P}}})}/(\check{\mathrm{G}} \times \check{\mathrm{H}}) \arrow[dl] \arrow[dr] & \\
\check{\mathfrak{h}}^*/\check{\mathrm{H}} & & \check{\mathfrak{g}}^*/\check{\mathrm{G}}.
\end{tikzcd}$$

Let us make the following pleasant observation: all constructions on the topological side depend only on the choice of Levi $\mathrm{H} \subseteq \mathrm{G}$, and *not* on the parabolic P. Although the first span *does* rely on the choice of parabolic to even define $\widetilde{\check{\mathfrak{g}}}_{\check{\mathrm{P}}}$, the formula for $\check{\mathcal{M}}^{\ddagger}$ shows that $\overline{\mathrm{T}^*(\check{\mathrm{G}}/\mathrm{N}_{\check{\mathrm{P}}}^-)}$ does not depend on the choice of parabolic.

5.5 Examples

We will now record a few examples of the calculation from § 5.4. In these cases, Hypothesis 5.3.2 does indeed hold (in the case of symmetric varieties, this is due to Quillen-Mitchell [Mit] and [CY]). For simplicity and to be explicit, we will typically only illustrate the calculations for $k = \mathrm{ku}$. In this case, the group scheme $\mathbf{H}$ over $\mathrm{Spev}(\mathrm{ku}) = \mathbf{A}^1_\beta/\mathbf{G}_m$ is given by $\mathrm{Spec}\,\mathbf{Z}[\beta, x, \frac{1}{1+\beta x}]/\mathbf{G}_m$, and $\mathrm{G}_{\mathbf{H}} = \mathrm{Hom}(\mathbf{H}, \mathrm{G})$. If $\mathrm{G} = \mathrm{SL}_n$, for instance, then $\mathrm{G}_{\mathbf{H}}$ identifies with the group scheme of those $n \times n$-matrices A such that $\frac{\det(\mathrm{I}+\beta\mathrm{A})-1}{\beta} = 0$.

5.5.1 Hyperboloids

Our goal in this section is to study the example of hyperboloids. Namely, we will study the example of the symmetric variety $\mathrm{H}_{\mathbf{C}} = \mathrm{SO}_{2n-1} \subseteq \mathrm{PSO}_{2n} = \mathrm{G}_{\mathbf{C}}$ associated to the real group $\mathrm{G}_{\mathbf{R}} = \mathrm{PSO}_{2n-1,1}$. In this case, $\mathrm{X} = \mathrm{PSO}_{2n}/\mathrm{SO}_{2n-1}$ is a hyperboloid (homotopy equivalent to $\mathbf{R}\mathrm{P}^{2n-1}$), and the dual group $\check{\mathrm{G}}_{\mathrm{X}}$ is SL_2, embedded into $\check{\mathrm{G}} = \mathrm{Spin}_{2n}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & \cdots & 0 & b \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ c & 0 & \cdots & 0 & d \end{pmatrix}. \tag{5.5.1}$$

We will write $\mathrm{N} = 2n-2$.

We will now check that the hypotheses of Theorem 5.4.3 are satisfied. The minuscule representation $\mathbf{A}^2$ of $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{SL}_2$ corresponds to the convolution variety $\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}} := \mathrm{S}^{2n-2} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$; this map is homotopic to the canonical map $\mathrm{S}^{2n-2} \to \Omega\mathbf{R}\mathrm{P}^{2n-1}$ which is adjoint to the quotient map $\mathrm{S}^{2n-1} \to \mathbf{R}\mathrm{P}^{2n-1}$. Clearly, the second condition of Theorem 5.4.3 is satisfied. Let us check the third condition. The following result was shown in [CO, Section 4], whose argument we exposit below.

Lemma 5.5.1. *The nonempty fibers of the convolution map $\mu_j : (\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$ have affine pavings.*

Proof. Note that the map μ_j is $\mathrm{G}_{\mathbf{R}}[\![t]\!]$-equivariant, so it suffices to show that the preimage $\mu_j^{-1}(t^\nu)$ is paved by affine spaces for any dominant weight ν of $\check{\mathrm{G}}_{\mathrm{X}}$. The claim is clear when $j=1$. If $j>1$, the map μ_j can be factored as the composite

$$(\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j} \xrightarrow{(\mu_{j-1}, \mathrm{id})} \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}} \widetilde{\times} \mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}} \xrightarrow{\mu_2} \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}.$$

It follows that $\mu_j^{-1}(t^\nu)$ is the preimage of $\mu_2^{-1}(t^\nu) \subseteq \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}} \widetilde{\times} \mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}}$ under the map (μ_{j-1}, id).

One basic observation (which I learned from [CMNO]) is that $\mu_j^{-1}(t^\nu) \subseteq (\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j}$ maps isomorphically onto its image under the map $\mathrm{pr} : (\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j} \to (\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j-1}$ given by the projection onto the first $(j-1)$ factors. Similarly, $\mu_2^{-1}(t^\nu) \subseteq \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}} \widetilde{\times} \mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}}$ maps isomorphically onto its image under the projection map $\mathrm{pr} : \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}} \widetilde{\times} \mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$. This implies that $\mathrm{pr}(\mu_j^{-1}(t^\nu)) \subseteq (\mathrm{Gr}^\lambda_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j-1}$ is the preimage of $\mathrm{pr}(\mu_2^{-1}(t^\nu)) \subseteq \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$ under the convolution map $\mu_{k-1} : (\mathrm{Gr}^{\lambda}_{\mathrm{G}_{\mathbf{R}}})^{\widetilde{\times} j-1} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$. Since the nonempty fibers of the convolution map μ_{j-1} have affine pavings (by the induction hypothesis), it follows that the nonempty fibers of the map $\mu_{j-1} : \mathrm{pr}(\mu_j^{-1}(t^\nu)) \to \mathrm{pr}(\mu_2^{-1}(t^\nu))$ – hence of the map $(\mu_{j-1}, \mathrm{id}) : \mu_j^{-1}(t^\nu) \to \mu_2^{-1}(t^\nu)$ – also have affine pavings.

The space $\mathrm{pr}(\mu_2^{-1}(t^\nu)) \subseteq \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$ has an affine paving. Indeed, $\mu_2^{-1}(t^\nu)$ is the t^ν-translation of $\mu_2^{-1}(t^0) = \mathrm{Gr}^{-\lambda}_{\mathrm{G}_{\mathbf{R}}} \widetilde{\times} \mathrm{Gr}^{\lambda}_{\mathrm{G}_{\mathbf{R}}}$, so $\mathrm{pr}(\mu_2^{-1}(t^0)) = t^\nu \cdot \mathrm{Gr}^{-\lambda}_{\mathrm{G}_{\mathbf{R}}}$. But $\mathrm{Gr}^{-\lambda}_{\mathrm{G}_{\mathbf{R}}}$ is isomorphic to S^{2n-2}, so it has an affine paving: it is the union $\mathbf{R}^{2n-2} \cup \{\infty\}$. Both of these cells are orbits of a unipotent subgroup of $\mathrm{G}_{\mathbf{R}}[\![t]\!]$, so again using $\mathrm{G}_{\mathbf{R}}[\![t]\!]$-equivariance and the inductive hypothesis that the fibers of μ_{j-1} have affine pavings, it follows that the map $\mu_{j-1} : \mathrm{pr}(\mu_j^{-1}(t^\nu)) \to \mathrm{pr}(\mu_2^{-1}(t^\nu))$ is in fact a trivial fibration over each cell of the target. This, finally, implies that $\mu_j^{-1}(t^\nu)$ has an affine paving as desired. ☐

One can also verify that the $\mathrm{G}_{\mathbf{R}}(\mathbf{R}[\![t]\!])$-orbit stratification $\{\mathrm{Gr}^{\mu}_{\mathrm{G}_{\mathbf{R}}}\}$ of $\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$ has a T_c-equivariant refinement where each stratum is a complex affine space on which T_c acts linearly. We are therefore reduced to calculating $\check{\mathrm{V}}_{\mathbf{H}}$. By Theorem 5.4.3, this can be identified with the affinization

$$\check{\mathrm{V}}_{\mathbf{H}} \cong \overline{(\check{\mathrm{G}}_{\mathrm{X}} \times \mathcal{M}_{\mathrm{H},0}) / \operatorname{Spec} \mathcal{H}_*^{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathbf{R}}; k)}.$$

In this case, $\mathrm{G}_{\mathbf{R}}$ is homotopy equivalent to $\mathrm{H} = \mathrm{SO}_{2n-1}$, and $\mathrm{Gr}_{\mathbf{R}}$ is homotopy equivalent to $\Omega\mathbf{R}\mathrm{P}^{2n-1}$. Our most complete calculations in this case are when $k = \mathbf{Z}' := \mathbf{Z}[1/2]$. Let us therefore fix this choice of k, and calculate $\mathrm{H}_*^{\mathrm{SO}_{2n-1}}(\Omega\mathbf{R}\mathrm{P}^{2n-1}; k)$. Arguing as in Theorem 3.6.3 using the (degenerating) Serre spectral sequence and Atiyah-Bott localization shows:

Proposition 5.5.2. *Let* W *be the Weyl group of* SO_{2n-1}*, acting on the Lie algebra* $\mathfrak{t} \cong \mathbf{A}^{n-1}$ *in the obvious way. Then there is an isomorphism of Hopf algebras*

$$\mathrm{H}_*^{\mathrm{SO}_{2n-1}}(\Omega\mathbf{R}\mathrm{P}^{2n-1}; k) \cong k[x_1, \cdots, x_n, a^{\pm 1}, \tfrac{a^2-1}{x_1 \cdots x_{n-1}}]^{\mathrm{W}},$$

where the coproduct on the right-hand side is uniquely specified by the statement that a *is grouplike (i.e.,* $a \mapsto a \otimes a$*).*

Define $\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}} := \mathfrak{sl}_2^*(\mathrm{N} - \mathrm{N}\rho_{\check{\mathrm{G}}_{\mathrm{X}}}) \times \mathfrak{so}^*_{2n-3}(2)/\!\!/\mathrm{SO}_{2n-3}$, so that there is an action of $\check{\mathrm{G}}_{\mathrm{X}} = \mathrm{SL}_2$ on the first factor. There is a Kostant slice

$$\begin{aligned} \kappa : \operatorname{Spev} \mathrm{C}^*_{\mathrm{H}}(*; k) \cong \mathfrak{so}_{2n-1}(2)/\!\!/\mathrm{SO}_{2n-1} &\to \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}, \\ (p_1, \cdots, p_{n-1}) &\mapsto \left(\begin{smallmatrix} 0 & 1 \\ p_{n-1} & 0 \end{smallmatrix}\right), (p_1, \cdots, p_{n-2}). \end{aligned}$$

It follows immediately from Proposition 5.5.2 and Theorem 3.6.3 that there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{SO}_{2n-1}}(\Omega\mathbf{R}\mathrm{P}^{2n-1}; k) \cong \mathfrak{so}_{2n-1}(2)/\!\!/\mathrm{SO}_{2n-1} \times_{\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}/\mathrm{SL}_2} \mathfrak{so}_{2n-1}(2)/\!\!/\mathrm{SO}_{2n-1},$$

and furthermore that the SL_2-orbit of κ is $\mathfrak{sl}_2^{*,\mathrm{reg}} \times \mathfrak{so}_{2n-3}(2)/\!\!/\mathrm{SO}_{2n-3}$. It follows that there is an isomorphism $\check{\mathrm{V}}_{\mathbf{H}} \cong \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}$, and so we conclude:

Theorem 5.5.3. *Let* $\mathrm{G}_{\mathbf{R}}$ *denote the real group* $\mathrm{PSO}_{2n-1,1}$*, and let* k *be an algebraically closed field of characteristic* $\neq 2$*. There is an equivalence of graded* k*-linear* ∞*-categories*

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{sl}_2^*(\mathrm{N} - \mathrm{N}\rho_{\check{\mathrm{G}}_{\mathrm{X}}})/\mathrm{SL}_2(\mathrm{N}\rho_{\check{\mathrm{G}}_{\mathrm{X}}}) \times \mathfrak{so}^*_{2n-3}(2)/\!\!/\mathrm{SO}_{2n-3}).$$

The vector bundle over $\mathfrak{sl}_2^*(\mathrm{N} - \mathrm{N}\rho_{\check{\mathrm{G}}_{\mathrm{X}}})/\mathrm{SL}_2(\mathrm{N}\rho_{\check{\mathrm{G}}_{\mathrm{X}}}) \times \mathfrak{so}^*_{2n-3}(2)/\!\!/\mathrm{SO}_{2n-3}$ *given by pulling back the standard representation of* SL_2 *identifies with the pushforward of the constant sheaf along the inclusion* $\mathrm{S}^{2n-2} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$*.*

This result was previously observed in [CO]; there, the proof of the key Proposition 5.5.2 was replaced by an argument along the lines of [YZ2] using Nadler's real analogue of geometric Satake equivalence [Nad]. Although we did not discuss an analogue of Theorem 5.5.3 for more general coefficients, the argument provided above will generalize easily (while there is no analogue of [Nad] available in this more general context).

5.5.2 Hecke period

Fix the embedding $\mathrm{GL}_{n-1} \subseteq \mathrm{GL}_n$ which on matrices is given by including the top $(n-1) \times (n-1)$-block. Our goal in this section is to study Theorem 5.4.3 in the case of the subgroup $\mathrm{H}_{\mathbf{C}} = \mathrm{GL}_{n-1} \subseteq \mathrm{GL}_{n-1} \times \mathrm{GL}_n = \mathrm{H}_{\mathbf{C}} \times \mathrm{G}_{\mathbf{C}}$. Let $\mathrm{X} = \mathrm{GL}_n$ viewed as a $\mathrm{GL}_{n-1} \times \mathrm{GL}_n$-space via left and right multiplication. In this case, the dual group $\check{\mathrm{G}}_{\mathrm{X}}$ is $\mathrm{GL}_{n-1} \times \mathrm{GL}_n = \check{\mathrm{H}} \times \check{\mathrm{G}}$, and the $\mathrm{H}[\![t]\!]$-orbits on Gr_{G} are parametrized by pairs (λ, μ) of sequences $(\lambda_1 \geq \cdots \geq \lambda_{n-1})$ and $(\mu_1 \geq \cdots \geq \mu_n)$ of integers. One representative for this $\mathrm{H}[\![t]\!]$-orbit is given by the lattice

$$(t^{-\mu_1} + \cdots + t^{-\mu_n})\mathbf{C}[\![t]\!] \oplus \bigoplus_{i=0}^{n-1} t^{-(\lambda_i+\mu_i)}\mathbf{C}[\![t]\!] \subseteq \mathbf{C}(\!(t)\!)^{\oplus n}.$$

One can verify that the $\mathrm{H}[\![t]\!]$-orbit stratification $\{\mathrm{Gr}_{\mathrm{G}}^{(\lambda,\mu)}\}$ of Gr_{G} has a T_c-equivariant refinement where each stratum is a complex affine space on which T_c acts linearly. The convolution map $\overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_{n-1}}} \widetilde{\times} \overline{\mathrm{Gr}^{\mu}_{\mathrm{GL}_n}} \to \mathrm{Gr}_{\mathrm{GL}_n}$ has image contained inside the closure of the orbit $\mathrm{Y}^{(\lambda,\nu)}$.

Lemma 5.5.4. *Let (λ, μ) be a dominant minuscule weight of $\check{\mathrm{G}}_{\mathrm{X}}$. Then the convolution variety $\mathrm{Y}^{(\lambda,\mu)}$ has even cells.*

Proof. In this case, the convolution map $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_{n-1}} \widetilde{\times} \mathrm{Gr}^{\mu}_{\mathrm{GL}_n} \to \mathrm{Gr}_{\mathrm{GL}_n}$ is an isomorphism onto its image, which is $\mathrm{Y}^{(\lambda,\nu)}$. The claim now follows from the fact that $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_{n-1}} \widetilde{\times} \mathrm{Gr}^{\mu}_{\mathrm{GL}_n}$ has even cells since λ and μ are dominant weights of GL_{n-1} and GL_n respectively. □

An inductive argument similar to Lemma 5.5.1 implies the following; see also [BFT, Lemma 2.4.1].

Lemma 5.5.5. *Let $(\lambda_\bullet, \nu_\bullet)$ be a sequence of dominant minuscule weights of $\check{\mathrm{G}}_{\mathrm{X}}$. Then the fibers of the map $\mathrm{Y}^{(\lambda_\bullet,\nu_\bullet)} \to \overline{\mathrm{Y}^{(|\lambda_\bullet|,|\nu_\bullet|)}}$ have affine pavings.*

Our goal is now to calculate $\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}$. Recall, as mentioned in the beginning of § 5.5, that we will illustrate the calculations in question by taking $k = \mathrm{ku}$.

Lemma 5.5.6. *There is an isomorphism*

$$\mathcal{M}_{\mathrm{GL}_{n-1},0} \cong \operatorname{Spec} \mathbf{Z}[\beta, c_1, \cdots, c_{n-1}, \tfrac{1}{1+\beta c_1 + \cdots + \beta^{n-1} c_{n-1}}]/\mathbf{G}_m.$$

Define

$$\check{\mathrm{M}}_{\mathbf{H}} = \{(u,v) \in \mathrm{T}^* \operatorname{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) | \mathrm{id} + \beta uv \in \mathrm{GL}_{n,\beta}\},$$

and represent a point of $\check{\mathrm{M}}_{\mathbf{H}}$ as a sequence $\mathbf{A}^n \xrightarrow{v} \mathbf{A}^{n-1} \xrightarrow{u} \mathbf{A}^n$. There is an action of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ on $\check{\mathrm{M}}_{\mathbf{H}}$ where $(\mathrm{A}, \mathrm{B}) \in \mathrm{GL}_n \times \mathrm{GL}_{n-1}$ sends $(u, v) \mapsto (\mathrm{A}u\mathrm{B}^{-1}, \mathrm{B}v\mathrm{A}^{-1})$. There is also a $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$-equivariant map $\check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{GL}_{n,\beta} \times \mathrm{GL}_{n-1,\beta}$ sending $(u, v) \mapsto (vu, uv)$.

Proposition 5.5.7. *There is a $\mathrm{GL}_{n-1} \times \mathrm{GL}_n$-equivariant isomorphism $\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}} \cong \check{\mathrm{M}}_{\mathbf{H}}$.*

Proof. There is a canonical homomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \mathcal{H}_*^{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_n}; k) \to \mathrm{Spec}_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \mathcal{H}_*^{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_{n-1}}; k) \tag{5.5.2}$$

induced by the map $\mathrm{Gr}_{\mathrm{GL}_{n-1}} \to \mathrm{Gr}_{\mathrm{GL}_n}$. Recall from (the ku-variant of) Corollary 4.3.17 that there is an isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{GL}_m,0}} \mathcal{H}_*^{\mathrm{GL}_m}(\mathrm{Gr}_{\mathrm{GL}_m}; k) \cong \check{\mathrm{J}}_\beta(\mathrm{GL}_m).$$

The map (5.5.2) therefore identifies with a homomorphism

$$\check{\mathrm{J}}_\beta(\mathrm{GL}_n)|_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \to \check{\mathrm{J}}_\beta(\mathrm{GL}_{n-1}),$$

which sends an $n \times n$-matrix to its top $(n-1) \times (n-1)$-block. One therefore obtains a homomorphism

$$\check{\mathrm{J}}_\beta(\mathrm{GL}_n)|_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \to \check{\mathrm{J}}_\beta(\mathrm{GL}_n)|_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \times_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \check{\mathrm{J}}_\beta(\mathrm{GL}_{n-1}) \hookrightarrow \mathrm{GL}_n \times \mathrm{GL}_{n-1} \times \mathcal{M}_{\mathrm{GL}_{n-1},0}. \tag{5.5.3}$$

By Theorem 5.4.3, $\check{\mathrm{M}}^\ddagger_{\mathbf{H}}$ can be identified with the affinization of the quotient

$$\check{\mathrm{V}}^{\ddagger,\mathrm{reg}}_{\mathbf{H}} = (\mathrm{GL}_n \times \mathrm{GL}_{n-1} \times \mathcal{M}_{\mathrm{GL}_{n-1},0})/\, \mathrm{Spec}_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \mathcal{H}_*^{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_n}; k).$$

The homomorphism (5.5.2) defines a map

$$(\mu_n, \mu_{n-1}) : \check{\mathrm{V}}^\ddagger_{\mathbf{H}} \to \mathrm{GL}_{n,\beta} \times \mathrm{GL}_{n-1,\beta};$$

this will be the ku-moment map of $\check{\mathrm{M}}_{\mathbf{H}}$.

Since $\mathbf{H}$ is the group scheme $\mathbf{G}_\beta$ over $\mathrm{Spev(ku)} = \mathbf{A}^1_\beta/\mathbf{G}_m$, we may identify $\mathrm{GL}_{n,\mathbf{H}}$ with the group scheme $\mathrm{GL}_{n,\beta}$ of $n \times n$-matrices A such that $\mathrm{I} + \beta \mathrm{A}$ is invertible. Let us now define a map $\check{\mathrm{V}}^{\ddagger,\mathrm{reg}}_{\mathbf{H}} \to \check{\mathrm{M}}_{\mathbf{H}}$. For this, we define a map $\kappa_{\check{\mathrm{V}}} : \mathcal{M}_{\mathrm{GL}_{n-1},0} \to \check{\mathrm{M}}_{\mathbf{H}}$ using the coordinates of Lemma 5.5.6 as follows: if $\vec{c} = (c_1, \cdots, c_{n-1}) \in \mathcal{M}_{\mathrm{GL}_{n-1},0}$, set

$$\kappa_{\check{\mathrm{M}}}(\vec{c}) = \left(\mathbf{A}^n \xrightarrow{\kappa(\vec{c})} \mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n\right).$$

Here, $\kappa(\vec{c})$ denotes the image of $\vec{c}$ under the composite $\mathcal{M}_{\mathrm{GL}_{n-1},0} \to \mathcal{M}_{\mathrm{GL}_n,0} \to \mathrm{GL}_{n,\beta}$, which is a matrix of rank $n-1$; explicitly,

$$\kappa(\vec{c}) = \begin{pmatrix} c_1+\beta c_2+\cdots+\beta^{n-2}c_{n-1} & 1 & \beta & \cdots & \beta^{n-1} \\ c_2+\cdots+\beta^{n-3}c_{n-1} & 0 & 1 & \cdots & \beta^{n-2} \\ \vdots & & & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

so that there are $(n-1)$ rows and n columns. The action of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ on $\check{\mathrm{M}}_{\mathbf{H}}$ extends $\kappa_{\check{\mathrm{M}}}$ to a map

$$\mathrm{GL}_n \times \mathrm{GL}_{n-1} \times \mathcal{M}_{\mathrm{GL}_{n-1},0} \to \check{\mathrm{M}}_{\mathbf{H}},$$

which we claim factors through the quotient by $\mathrm{Spec}_{\mathcal{M}_{\mathrm{GL}_{n-1},0}} \mathcal{H}_*^{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_n}; k)$.

If $(\mathrm{A}, \mathrm{B}) \in \mathrm{GL}_n \times \mathrm{GL}_{n-1} \times \mathcal{M}_{\mathrm{GL}_{n-1},0}$ stabilizes $\kappa_{\check{\mathrm{M}}}(\vec{c})$, then it must also stabilize $\mu_n(\kappa_{\check{\mathrm{M}}}(\vec{c})) \in \mathrm{GL}_{n,\beta}$ and $\mu_{n-1}(\kappa_{\check{\mathrm{M}}}(\vec{c})) \in \mathrm{GL}_{n-1,\beta}$. But $\mu_{n-1}(\kappa_{\check{\mathrm{M}}}(\vec{c}))$ identifies with the image of $\vec{c} \in \mathcal{M}_{\mathrm{GL}_{n-1},0}$ under the Kostant slice $\mathcal{M}_{\mathrm{GL}_{n-1},0} \to \mathrm{GL}_{n-1,\beta}$; so $\mathrm{B} \in \check{\mathrm{J}}_\beta(\mathrm{GL}_{n-1})$. Similarly,

$A \in \check{J}_\beta(\mathrm{GL}_n)|_{\mathcal{M}_{\mathrm{GL}_{n-1},0}}$. However, since (A, B) must stabilize the inclusion $\mathbf{A}^{n-1} \hookrightarrow \mathbf{A}^n$ too (by construction of $\kappa_{\check{M}}$), the matrix B must be the top $(n-1)\times(n-1)$-block of A. It follows that (A, B) must be in the image of the homomorphism (5.5.3), which gives the desired factorization.

We therefore obtain a $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$-equivariant map $\check{V}_{\mathbf{H}}^{\ddagger,\mathrm{reg}} \to \check{M}_{\mathbf{H}}$. Since the target is affine, it refines to a $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$-equivariant map $\check{V}_{\mathbf{H}}^{\ddagger} \to \check{M}_{\mathbf{H}}$. Note that $\check{M}_{\mathbf{H}}$ is normal and irreducible, so to check that this map is an isomorphism, it suffices to do so over an open locus whose complement is of codimension at least 2. This, in turn, follows from standard linear algebra. □

Finally, Theorem 5.4.3 gives:

Corollary 5.5.8. *Let $k = \mathrm{ku}$, and let*

$$\check{M}_{\mathbf{H}} = \{(u, v) \in T^* \operatorname{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) | \mathrm{id} + \beta uv \in \mathrm{GL}_{n,\beta}\}.$$

Then, there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear ∞-categories

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_n}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{M}_{\mathbf{H}}/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})). \tag{5.5.4}$$

The vector bundle over $\check{M}_{\mathbf{H}}$ given by pulling back the standard representation of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ along the map $\check{M}_{\mathbf{H}} \to \check{M}_{\mathbf{H}}/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})$ identifies with the pushforward of the constant sheaf along the map $\mathbf{C}P^{n-1} \times \mathbf{C}P^{n-2} \to \mathrm{Gr}_{\mathrm{GL}_n}$ which identifies with the composite

$$\mathbf{C}P^{n-1} \times \mathbf{C}P^{n-2} \to \Omega\mathrm{GL}_n \times \Omega\mathrm{GL}_{n-1} \xrightarrow{\text{product}} \Omega\mathrm{GL}_n.$$

Under the equivalence of Corollary 4.3.17, (5.5.4) is compatible with the action of

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{GL}_{n-1}\times\mathrm{GL}_n}(\mathrm{Gr}_{\mathrm{GL}_{n-1}\times\mathrm{GL}_n}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}((\mathrm{GL}_{n,\mathbf{H}} \times \mathrm{GL}_{n-1,\mathbf{H}})/(\mathrm{GL}_n \times \mathrm{GL}_{n-1}))$$

on the left-hand side by convolution, and on the right-hand side via the moment map $\mu : \check{M}_{\mathbf{H}} \to \mathrm{GL}_{n,\mathbf{H}} \times \mathrm{GL}_{n-1,\mathbf{H}}$.

5.5.3 Quaternionic period

Our goal in this section is to generalize the main result of [CMNO]. Namely, we will study the example of the symmetric variety $H_{\mathbf{C}} = \mathrm{Sp}_{2n} \subseteq \mathrm{GL}_{2n} = G_{\mathbf{C}}$ associated to the real group $G_{\mathbf{R}} = \mathrm{GL}_{n,\mathbb{H}}$. In this case, $X = \mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$, and the dual group $\check{G}_X$ is GL_n, embedded into $\check{G} = \mathrm{GL}_{2n}$ by $A \mapsto \mathrm{diag}(A, A)$. The hypotheses of Theorem 5.4.3 are satisfied: one can verify that the $G_{\mathbf{R}}(\mathbf{R}[\![t]\!])$-orbit stratification $\{\mathrm{Gr}^\mu_{G_{\mathbf{R}}}\}$ of $\mathrm{Gr}_{G_{\mathbf{R}}}$ has a T_c-equivariant refinement where each stratum is a complex affine space on which T_c acts linearly; for the other hypotheses, see [CMNO, Lemma 4.11], and Lemma 5.5.1 for a review of the argument therein. We are therefore reduced to calculating $\check{V}_{\mathbf{H}}$. By Theorem 5.4.3, this can be identified with the affinization

$$\check{V}_{\mathbf{H}} \cong \overline{(\check{G}_X \times \mathcal{M}_{H,0})/\operatorname{Spec}\mathcal{H}^{G_{\mathbf{R}}}_*(\mathrm{Gr}_{G_{\mathbf{R}}}; k)}.$$

In this case, $G_{\mathbf{R}}$ is homotopy equivalent to $H = \mathrm{Sp}_{2n}$, and $\mathrm{Gr}_{G_{\mathbf{R}}}$ is homotopy equivalent to $\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n})$, so we need to calculate $\mathcal{H}^{\mathrm{Sp}_{2n}}_*(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$. Recall, again, that we will take $k = \mathrm{ku}$.

Lemma 5.5.9. *There are isomorphisms*

$$\pi_*(\mathrm{ku}_{\mathrm{GL}_{2n}}) \cong \mathbf{Z}[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1+\cdots+\beta^n c_n}],$$
$$\pi_*(\mathrm{ku}_{\mathrm{Sp}_{2n}}) \cong \mathbf{Z}[\beta, p_1, \cdots, p_n],$$

and the map $\mathrm{ku}_{\mathrm{GL}_{2n}} \to \mathrm{ku}_{\mathrm{Sp}_{2n}}$ *induced by the inclusion* $\mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ *sends*

$$c_k \mapsto \sum_{0\leq 2j\leq k} \binom{k-j}{j} \beta^{k-2j} p_{k-j}.$$

Here, the classes c_i *live in weight* $-2i$*, and the classes* p_i *live in weight* $-4i$*.*

Proposition 5.5.10. *There is an isomorphism*

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{Sp}_{2n},0}} \mathcal{H}_*^{\mathrm{Sp}_{2n}}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k) \cong (\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times_{\mathfrak{gl}_n(4)/\mathrm{GL}_n} \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n) \times \mathbf{A}^1_\beta$$

of group schemes over $\mathcal{M}_{\mathrm{Sp}_{2n},0} \cong \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta$*.*

Proof sketch. This can be done by first calculating $\mathcal{H}_*^{\mathrm{T}^n}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$ using Proposition 3.2.15. The Weyl group $\mathrm{W}_{\mathrm{Sp}_{2n}}$ of Sp_{2n} acts on $\mathcal{H}_*^{\mathrm{T}^n}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$, and taking invariants for the subgroup $(\mathbf{Z}/2)^n$ produces $\mathcal{H}_*^{\mathrm{Sp}_2^n}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$. The output of Proposition 3.2.15 in this case is exactly the fiber product $(f+\mathfrak{t}) \times_{\mathfrak{b}/\mathrm{B}} (f+\mathfrak{t})$ where $\mathfrak{t} = \mathbf{A}^n(4)$ and B denotes the (upper-triangular) Borel subgroup of GL_n. In other words, there is an isomorphism

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{Sp}_2^n,0}} \mathcal{H}_*^{\mathrm{Sp}_2^n}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k) \simeq ((f+\mathfrak{t}) \times_{\mathfrak{b}/\mathrm{B}} (f+\mathfrak{t})) \times \mathbf{A}^1_\beta$$

over $\mathcal{M}_{\mathrm{Sp}_2^n,0} \cong \mathbf{A}^n(4) \times \mathbf{A}^1_\beta \cong \mathfrak{t}(4) \times \mathbf{A}^1_\beta$. One can moreover verify that this isomorphism is equivariant for the action of $\mathrm{W}_{\mathrm{Sp}_{2n}}/(\mathbf{Z}/2)^n \cong \Sigma_n$. This, then, implies the desired claim by taking Σ_n-invariants.

Alternatively/equivalently, one can use Atiyah-Bott localization as in the proof of Theorem 4.3.13 to reduce the calculation of $\mathcal{H}_*^{\mathrm{Sp}_2^n}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$ to the case $n = 2$. In this case, there is a homotopy equivalence $\mathrm{GL}_4/\mathrm{Sp}_4 \simeq \mathrm{S}^5$, so we need to calculate $\mathcal{H}_*^{\mathrm{SL}_2}(\Omega\mathrm{S}^5; k)$. This can be done "by hand" similarly to Theorem 3.6.3 and Theorem 3.8.3. □

Since the affine closure of the GL_n-orbit of the Kostant slice $\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \to \mathfrak{gl}_n(4)$ is the entirety of $\mathfrak{gl}_n(4)$, and the stabilizer of the Kostant slice is precisely $\mathrm{Spec}_{\mathcal{M}_{\mathrm{Sp}_{2n},0}} \mathcal{H}_*^{\mathrm{Sp}_{2n}}(\Omega(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}); k)$ by Proposition 5.5.10, we find from Theorem 5.4.3 that:

Theorem 5.5.11. *Let* $k = \mathrm{ku}$*, and let*

$$\check{\mathrm{M}}_{\mathbf{H}} = (\mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n(4)) \times \mathbf{A}^1_\beta,$$

where GL_n *is embedded into* GL_{2n} *via* $\mathrm{A} \mapsto \mathrm{diag}(\mathrm{A}, \mathrm{A})$*. Let* $\mathrm{G}_{\mathbf{R}} = \mathrm{GL}_n(\mathbf{H})$*. Then, there is an equivalence of* $\mathrm{QCoh}(\mathrm{Spev}(k))$*-linear* ∞*-categories*

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{GL}_{2n}). \tag{5.5.5}$$

The vector bundle over $\check{\mathrm{M}}_{\mathbf{H}}$ *given by pulling back the standard representation of* GL_n *along the map* $\check{\mathrm{M}}_{\mathbf{H}} \to \check{\mathrm{M}}_{\mathbf{H}}/\mathrm{GL}_{2n} \cong \mathfrak{gl}_n(4)/\mathrm{GL}_n \times \mathbf{A}^1_\beta$ *identifies with the pushforward of the constant sheaf along the inclusion* $\mathbf{H}\mathrm{P}^{n-1} \to \mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}$*.*

Under the equivalence of Corollary 4.3.17, (5.5.5) is compatible with the action of

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{GL}_{2n}}(\mathrm{Gr}_{\mathrm{GL}_{2n}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathrm{GL}_{2n,\mathbf{H}}/\mathrm{GL}_{2n})$$

on the left-hand side by convolution, and on the right-hand side via the moment map $\mu : \check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{GL}_{2n,\mathbf{H}}$ *sending* $x \in \mathfrak{gl}_n$ *to* $\left(\begin{smallmatrix} \beta x & \mathrm{id}_n \\ x & 0 \end{smallmatrix}\right)$. *Under the identifications* $\check{\mathrm{M}}_{\mathbf{H}} /\!\!/ \mathrm{GL}_{2n} \cong \operatorname{Spec} \pi_*(\mathrm{ku}_{\mathrm{Sp}_{2n}})$ *and* $\mathrm{GL}_{2n,\mathbf{H}} /\!\!/ \mathrm{GL}_{2n} \cong \operatorname{Spec} \pi_*(\mathrm{ku}_{\mathrm{GL}_{2n}})$, *the induced map* $\mu : \check{\mathrm{M}}_{\mathbf{H}} /\!\!/ \mathrm{GL}_{2n} \to \mathrm{GL}_{2n,\mathbf{H}} /\!\!/ \mathrm{GL}_{2n}$ *identifies with the map from Lemma 5.5.9.*

Note the rather surprising fact that the ku-theoretic dual $\check{\mathrm{M}}_{\mathbf{H}}$ to GL_{2n} acting on $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$ is $(\mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n(4)) \times \mathbf{A}^1_\beta$ (albeit with a somewhat exotic ku-Hamiltonian structure), and is therefore a *trivial* deformation of $\check{\mathrm{M}}_{\mathbf{G}_a} \cong \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n(4)$ along the Bott class β!

Remark 5.5.12. In fact, there is also a QCoh(Spev(ko))-linear equivalence

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; \mathrm{ko}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{GL}_{2n}).$$

Here, $\mathbf{H}$ is the canonical 1-dimensional group scheme over Spev(ko) which is the descent of $\mathbf{G}_\beta$ along the covering map Spev(ku) → Spev(ko), and the symbol $\check{\mathrm{M}}_{\mathbf{H}}$ still denotes $\mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n(4)$ viewed as living over Spev(ko). If we invert β for simplicity, so that ku is replaced by KU, this is essentially because complex conjugation on KU acts trivially on $\check{\mathrm{M}}_{\mathbf{H}}$. This is easy to see algebraically: using Definition 3.8.22 and Remark 3.8.26, this follows from the observation that if $x \in \mathfrak{gl}_n$, then $(\mathrm{id}_{2n} + \beta\mu(x))^{-1} = \left(\begin{smallmatrix} \mathrm{id}_n + \beta^2 x & \beta \mathrm{id}_n \\ \beta x & \mathrm{id}_n \end{smallmatrix}\right)^{-1}$ is conjugate to $\mathrm{id}_{2n} + \beta\mu(x)$. However, the triviality of complex conjugation in this case also has a topological explanation. Namely, the quotient $\mathrm{GL}_{2n}[\![t]\!] \backslash \mathrm{GL}_{2n}(\!(t)\!)/\mathrm{Sp}_{2n}(\!(t)\!)$ is homotopy equivalent to the quaternionic affine Grassmannian $\mathrm{GL}_n(\mathbf{H}[\![t]\!]) \backslash \mathrm{Gr}_{\mathrm{GL}_n(\mathbf{H})}$. The map $\mathbf{H}\mathrm{P}^{n-1} \to \mathrm{Gr}_{\mathrm{GL}_n(\mathbf{H})}$ exhibits $\mathbf{H}\mathrm{P}^{n-1}$ as a generating complex for $\mathrm{Gr}_{\mathrm{GL}_n(\mathbf{H})}$. Since $\mathbf{H}\mathrm{P}^{n-1}$ is a Spin-manifold, it is KO-oriented (see [ABS]), which implies that complex conjugation on KU acts trivially on $\mathrm{KU}^*(\mathbf{H}\mathrm{P}^{n-1})$ (and hence on $\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{GL}_{2n}[\![t]\!]}(\mathrm{GL}_{2n}(\!(t)\!)/\mathrm{Sp}_{2n}(\!(t)\!); \mathrm{KU})$).

5.5.4 $\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}$ and $\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$

Let $\mathrm{G} = \mathrm{GL}_{2n+1}$, and let $\mathrm{X} = \mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}$. This example is quite special in the context of (ordinary) relative Langlands duality: the Hamiltonian GL_{2n+1}-space dual to $\mathrm{T}^*(\mathrm{X})$ is $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$. In particular, the dual Hamiltonian space is the cotangent bundle of another affine homogeneous GL_{2n+1}-space. This is a rare occurrence! It allows us to check that $\mathrm{T}^*(\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n})$ and $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$ are indeed *swapped* under (ordinary) relative Langlands duality. This, however, will not be true in the generalized relative Langlands duality of Conjecture 5.2.20: we will see that the ku-theoretic relative Langlands dual to $\mathrm{T}^*(\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n})$ is (perhaps surprisingly) $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$, but that the ku-theoretic relative Langlands dual to $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$ is not $\mathrm{T}^*(\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n})$.

Let us begin by calculating that the ku-theoretic relative Langlands dual to $\mathrm{T}^*(\mathrm{X})$ is indeed $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$. The conditions of Theorem 5.4.3 do not quite apply to this example, although I expect that it fits into the modified context of Remark 5.4.8. We can nevertheless compute $\check{\mathrm{V}}^\ddagger_{\mathbf{H}}$ as in Remark 5.4.5. For this, we need to calculate $\mathcal{H}^{\mathrm{Sp}_{2n}}_*(\Omega\mathrm{X}; k)$ (where, as usual, we take $k = \mathrm{ku}$). To do this, we note:

Lemma 5.5.13. *There is an isomorphism* $\mathrm{GL}_{2n+2}/\mathrm{Sp}_{2n+2} \cong \mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}$ *of* GL_{2n+1}-*spaces.*

Let $\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}} = \mathrm{T}^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n+1}) \times \mathbf{A}^1_\beta$, viewed as the space of pairs of maps $\mathbf{A}^n \to \mathbf{A}^{n+1}$ and $\mathbf{A}^{n+1} \to \mathbf{A}^n$, so that it has a natural $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$-action. Equip it with the grading where the coordinates $(x_0, \cdots, x_n)$ of $\mathbf{A}^{n+1}$ have weights $\mathrm{wt}(x_i) = -4i$, and the coordinates $(y_1, \cdots, y_n)$ of $\mathbf{A}^n$ have weights $\mathrm{wt}(y_i) = -4i$. Let $\mu_1 : \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}} \to \mathfrak{gl}_{n+1}$ denote the map sending a pair $(u : \mathbf{A}^n \to \mathbf{A}^{n+1}, v : \mathbf{A}^{n+1} \to \mathbf{A}^n)$ to the composite vu. Let $\kappa : \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta \to \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}$ denote the map sending

$$\kappa_{\check{\mathrm{V}}}(\vec{p}) = \left(\mathbf{A}^{n+1} \xrightarrow{\kappa(\vec{p})} \mathbf{A}^n \hookrightarrow \mathbf{A}^{n+1}\right).$$

Here, $\kappa(\vec{p})$ denotes the matrix

$$\kappa(\vec{p}) = \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ p_n & 0 & 0 & \cdots & 1 \end{pmatrix},$$

so that there are n rows and $(n+1)$ columns.

Proposition 5.5.14. *There is an isomorphism*

$$\mathrm{Spec}_{\mathcal{M}_{\mathrm{Sp}_{2n},0}} \mathcal{H}^{\mathrm{Sp}_{2n}}_*(\Omega \mathrm{X}; k) \cong (\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta) \times_{\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1})} (\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta)$$

of group schemes over $\mathcal{M}_{\mathrm{Sp}_{2n},0} \cong (\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n) \times \mathbf{A}^1_\beta$.

Proof. Note that the embedding $\mathcal{M}_{\mathrm{Sp}_{2n},0} \to \mathcal{M}_{\mathrm{Sp}_{2n+2},0}$ identifies with the inclusion $\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \to \mathfrak{gl}_{n+1}(4)/\!\!/\mathrm{GL}_{n+1}$ which is the zero locus of the top Pontryagin class p_{n+1}. The preceding lemma and Proposition 5.5.10 thus give isomorphisms

$$\begin{aligned} \mathrm{Spec}\, \mathcal{H}^{\mathrm{Sp}_{2n}}_*(\Omega \mathrm{X}; k) &\cong \mathrm{Spec}(\mathcal{H}^{\mathrm{Sp}_{2n+2}}_*(\Omega \mathrm{X}; k)) \times_{\mathcal{M}_{\mathrm{Sp}_{2n+2},0}} \mathcal{M}_{\mathrm{Sp}_{2n},0}) \\ &\cong (\mathfrak{gl}_{n+1}(4)/\!\!/\mathrm{GL}_{n+1} \times_{\mathfrak{gl}_{n+1}(4)/\mathrm{GL}_{n+1}} \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n) \times \mathbf{A}^1_\beta. \end{aligned}$$

The Kostant slice $\mathfrak{gl}_{n+1}(4)/\!\!/\mathrm{GL}_{n+1} \to \mathfrak{gl}_{n+1}(4)/\mathrm{GL}_{n+1}$ lands in the regular locus $\mathfrak{gl}^{\mathrm{reg}}_{n+1}(4)/\mathrm{GL}_{n+1}$, and sends $\mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n$ to the locus of those elements with vanishing determinant. A regular $(n+1)\times(n+1)$-matrix with vanishing determinant has rank exactly n (indeed, it is conjugate to a companion matrix, and the companion matrix of a degree $n+1$ polynomial which is divisible by t must have rank exactly n). Using this observation, one can check that there is a commutative diagram

$$\begin{tikzcd} \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta \arrow[r, "\kappa_{\check{\mathrm{V}}}"] \arrow[d] & \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}) \arrow[d, "\mu_1"] \\ \mathfrak{gl}_{n+1}(4)/\!\!/\mathrm{GL}_{n+1} \times \mathbf{A}^1_\beta \arrow[r, "\kappa"'] & \mathfrak{gl}_{n+1}(4)/\mathrm{GL}_{n+1} \times \mathbf{A}^1_\beta, \end{tikzcd}$$

which is a Cartesian square when restricted to the regular locus $\mathfrak{gl}^{\mathrm{reg}}_{n+1}(4)/\mathrm{GL}_{n+1} \subseteq \mathfrak{gl}_{n+1}(4)/\mathrm{GL}_{n+1}$. The desired isomorphism of group schemes then follows. $\square$

Just as in Corollary 5.5.8, one can check that the affinization of the $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$-orbit of $\kappa_{\check{\mathrm{V}}} : \mathfrak{gl}_n(4)/\!\!/\mathrm{GL}_n \times \mathbf{A}^1_\beta \to \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}$ is in fact the entirety of $\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}$. The stabilizer of the map $\kappa_{\check{\mathrm{V}}}$ is precisely $\mathrm{Spec}_{\mathcal{M}_{\mathrm{Sp}_{2n},0}} \mathcal{H}^{\mathrm{Sp}_{2n}}_*(\Omega \mathrm{X}; k)$, so (following Theorem 5.4.3), we expect that if

$$\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}} = \mathrm{GL}_{2n+1} \times^{\mathrm{GL}_n \times \mathrm{GL}_{n+1}} \check{\mathrm{V}}^{\ddagger}_{\mathbf{H}} \cong \mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1})) \times \mathbf{A}^1_\beta,$$

then there is an equivalence of QCoh(Spev(ku))-linear ∞-categories

$$\widetilde{\mathrm{Shv}}^{\mathrm{min,gr}}_{\mathrm{GL}_{2n+1}}(\mathrm{GL}_{2n+1}(\!(t)\!)/\mathrm{Sp}_{2n}(\!(t)\!);\mathrm{ku}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}/\mathrm{GL}_{2n+1}(2\rho_{\mathrm{GL}_{2n+1}})).$$

Here, we are using notation as in Remark 5.4.8.

The moment map $\mu : \check{\mathrm{M}}^{\ddagger}_{\mathbf{H}} \to \mathrm{GL}_{2n+1,\mathbf{H}}$ is induced from a $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$-equivariant map $\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}} \cong \mathrm{T}^*\,\mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n+1}) \times \mathbf{A}^1_{\beta} \to \mathrm{GL}_{2n+1,\mathbf{H}}$; although it is certainly possible given the preceding discussion, I have not computed this map explicitly yet. It is, however, easy to compute the fiber of this map at the locus where $\beta = 0$, i.e., the $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$-equivariant map $\mathrm{T}^*\,\mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n+1}) \to \mathfrak{gl}_{2n+1}$. It sends an $n \times (n+1)$-matrix u and an $(n+1) \times n$-matrix v to the "checkerboard" matrix $\mu(u, v)$ given by

$$\mu(u,v)_{s,t} = \begin{cases} u_{i,j} & s = 2i-1, t = 2j, \\ v_{i,j} & s = 2i, t = 2j+1, \\ 0 & \text{else}. \end{cases}$$

For example, if $n = 2$, then the map $\mathrm{T}^*\,\mathrm{Hom}(\mathbf{A}^2, \mathbf{A}^3) \to \mathfrak{gl}_5$ sends

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix} \mapsto \begin{pmatrix} 0 & u_{11} & 0 & u_{12} & 0 \\ v_{11} & 0 & v_{12} & 0 & v_{13} \\ 0 & u_{21} & 0 & u_{22} & 0 \\ v_{21} & 0 & v_{22} & 0 & v_{23} \\ 0 & u_{31} & 0 & u_{32} & 0 \end{pmatrix}.$$

Note the rather surprising fact that the ku-theoretic dual $\check{\mathrm{M}}^{\ddagger}_{\mathbf{H}}$ to GL_{2n+1} acting on $\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}$ is $\mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1})) \times \mathbf{A}^1_{\beta}$ (albeit with a somewhat exotic ku-Hamiltonian structure), and is therefore a *trivial* deformation of $\check{\mathrm{M}}^{\ddagger}_{\mathbf{G}_a} \cong \mathrm{T}^*(\mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$ along the Bott class β!

Let us now turn to the question of placing $\mathrm{Y} = \mathrm{GL}_{2n+1}/(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$ on the automorphic side; the dual group $\check{\mathrm{G}}_{\mathrm{Y}}$ in this case is Sp_{2n}. Again, the conditions of Theorem 5.4.3 do not quite apply to this example, but one can nevertheless compute $\check{\mathrm{V}}^{\ddagger}_{\mathbf{H}}$ as in Remark 5.4.5. For this, we need to calculate $\mathcal{H}^{\mathrm{GL}_n \times \mathrm{GL}_{n+1}}_*(\Omega \mathrm{Y}; k)$. (It is not so hard to essentially reduce this to calculating $\mathcal{H}^{\mathrm{GL}_n \times \mathrm{GL}_n}_*(\Omega(\mathrm{GL}_{2n}/(\mathrm{GL}_n \times \mathrm{GL}_n)); k)$.) However, we will not do this calculation here, because I have been informed that (when k is a commutative $\mathbf{Q}$-algebra) it is work in-progress by Chen-Macerato-Nadler-O'Brien. The expected answer is that there is an Sp_{2n}-equivariant isomorphism

$$\check{\mathrm{V}}_{\mathbf{G}_a} \cong \wedge^2(\mathbf{A}^{2n}) \times \mathrm{T}^*(\mathbf{A}^{2n}) \times \mathbf{A}^1,$$

where the moment map $\check{\mathrm{M}}_{\mathbf{G}_a} \cong \mathrm{T}^*(\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n}) \to \mathfrak{gl}^*_{2n+1}$ sends

$$\wedge^2(\mathbf{A}^{2n}) \times \mathrm{T}^*(\mathbf{A}^{2n}) \times \mathbf{A}^1 \ni (\omega', u, v, x) \mapsto (\omega' + vu) \oplus x \in \mathfrak{gl}_{2n} \oplus \mathfrak{gl}_1 \subseteq \mathfrak{gl}_{2n+1}.$$

Note that there is an isomorphism $\check{\mathrm{M}}_{\mathbf{G}_a} /\!\!/ \mathrm{GL}_{2n+1} \cong \mathbf{A}^n \times \mathbf{A}^{n+1}$ (as predicted by Remark 5.2.13), coming from an isomorphism

$$(\wedge^2(\mathbf{A}^{2n}) \times \mathrm{T}^*(\mathbf{A}^{2n})) /\!\!/ \mathrm{Sp}_{2n} \cong \mathbf{A}^n \times \mathbf{A}^n.$$

In fact, this isomorphism implies that the ku-theoretic dual $\check{\mathrm{M}}_{\mathbf{H}} = \mathrm{GL}_{2n+1} \times^{\mathrm{Sp}_{2n}} \check{\mathrm{V}}_{\mathbf{H}}$ *cannot* be isomorphic to $\mathrm{T}^*(\mathrm{GL}_{2n+1}/\mathrm{Sp}_{2n})$. Indeed, Remark 5.2.21 would give an isomorphism

$$\check{\mathrm{M}}_{\mathbf{H}} /\!\!/ \mathrm{GL}_{2n+1} \cong \check{\mathrm{V}}_{\mathbf{H}} /\!\!/ \mathrm{Sp}_{2n} \cong \mathrm{Spec}\,\pi_*(\mathrm{ku}_{\mathrm{GL}_n \times \mathrm{GL}_{n+1}}),$$

and the latter is not isomorphic to $\mathbf{A}^n \times \mathbf{A}^{n+1}$. That is to say, $\check{\mathrm{V}}_{\mathbf{H}}$ is not isomorphic to $\wedge^2(\mathbf{A}^{2n}) \times \mathrm{T}^*(\mathbf{A}^{2n}) \times \mathbf{A}^1$; it would be interesting to calculate $\check{\mathrm{V}}_{\mathbf{H}}$ using the prescription of Remark 5.4.5.

5.5.5 Triple-product period

In this section, we will study the *triple product period*, corresponding to the case $\mathrm{H} = \mathrm{SO}_3 \subseteq \mathrm{G} = \mathrm{SO}_3 \times \mathrm{PSO}_4 = \mathrm{SO}_3^3$. This is an example of a spherical subgroup, and in particular it fits into the context of Definition 5.4.1. The dual group $\check{\mathrm{G}}_{\mathrm{X}}$ in this case is given by $\check{\mathrm{G}} = \mathrm{SL}_2 \times \mathrm{Spin}_4$; note that this is isomorphic to SL_2^3. If μ is the unique dominant minuscule weight of SO_3, then the corresponding Schubert variety is given by $\mathbf{P}^1 \to \mathrm{Gr}_{\mathrm{SO}_3}$; similarly if ν denotes the unique dominant minuscule weight of Spin_4, the corresponding Schubert variety is given by $\mathbf{P}^1 \widetilde{\times} \mathbf{P}^1 \to \mathrm{Gr}_{\mathrm{PSO}_4}$. It follows that if (μ, ν) is the unique dominant minuscule weight of $\check{\mathrm{G}} = \mathrm{SL}_2 \times \mathrm{Spin}_4$, then the corresponding $\mathrm{H}[\![t]\!]$-orbit on $\mathrm{Gr}_{\mathrm{PSO}_4}$ is given by the convolution map

$$m : \mathbf{P}^1 \widetilde{\times} (\mathbf{P}^1 \widetilde{\times} \mathbf{P}^1) \to \mathrm{Gr}_{\mathrm{PSO}_4}.$$

More generally, one obtains a convolution map

$$m : \mathrm{Y}_3^{\lambda_\bullet} := (\mathbf{P}^1)^{\widetilde{\times} 3i} \to \mathrm{Gr}_{\mathrm{PSO}_4}$$

associated to a sequence $\lambda_\bullet$ of dominant minuscule weights of $\check{\mathrm{G}}$. These convolution varieties $\mathrm{Y}_3^{\lambda_\bullet}$ satisfy the conditions of Theorem 5.4.3.

Even more generally, we may replace $\mathrm{Gr}_{\mathrm{PSO}_4} = \mathrm{Gr}_{\mathrm{SO}_3}^{\times 2}$ by $\mathrm{Gr}_{\mathrm{SO}_3}^{\times n}$, and consider the convolution map

$$m : \mathrm{Y}_n^{\lambda_\bullet} = (\mathbf{P}^1)^{\widetilde{\times}(n+1)i} \to \mathrm{Gr}_{\mathrm{SO}_3}^{\times n}$$

associated to a sequence $\lambda_\bullet$ of dominant minuscule weights of SL_2^{n+1}. Similarly to Lemma 5.5.1 (or the argument in [BFT, Lemma 2.4.1]), one has:

Lemma 5.5.15. *The convolution varieties $\mathrm{Y}_n^{\lambda_\bullet}$ have even cells, and the nonempty fibers of m have affine pavings.*

Using Lemma 5.5.15, one can define categories $\mathrm{Shv}^{\min}_{\mathrm{SO}_3[\![t]\!]}(\mathrm{Gr}_{\mathrm{SO}_3}^{\times n})$ and $\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{SO}_3[\![t]\!]}(\mathrm{Gr}_{\mathrm{SO}_3}^{\times n})$ in the same way as in Definition 5.4.1, using the varieties $\mathrm{Y}_n^{\lambda_\bullet}$ in place of $\mathrm{Gr}_{\mathrm{G}}^{\lambda_\bullet}$. One can verify that if $\lambda_{\bullet,1}, \cdots, \lambda_{\bullet,m}$ is a sequence of dominant minuscule weights of SL_2^{n+1}, then the stratification of $\mathrm{Gr}_{\mathrm{SO}_3^n}$ by the image of the convolution varieties $\mathrm{Y}_n^{\lambda_{\bullet,1}} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Y}_n^{\lambda_{\bullet,m}}$ in $\mathrm{Gr}_{\mathrm{SO}_3^n}$ has a T_c-equivariant refinement where each stratum is a complex affine space on which T_c acts linearly. Using this, our main result is the following:

Theorem 5.5.16. *Let k be an algebraically closed field. Suppose n is a positive even integer, and let $\mathfrak{C}_n$ denote the affine cone on the secant variety of lines on the Segre embedding $(\mathbf{P}^1)^{n+1} \to \mathbf{P}^{2^{n+1}-1}$. There is an embedding $\mathfrak{C}_n \hookrightarrow (\mathbf{A}^2)^{\otimes n+1}$, and the Hamiltonian SL_2^{n+1}-action on $(\mathbf{A}^2)^{\otimes n+1}$ defines an SL_2^{n+1}-equivariant map $\mu : \mathfrak{C}_n \to (\mathfrak{sl}_2^*)^{n+1}$. There is a $\mathbf{G}_m$-action on $\mathbf{P}^1$ given by $[x : y] \mapsto [\lambda^2 x : y]$, and hence a $\mathbf{G}_m$-action on $(\mathbf{P}^1)^{n+1}$ and thus on $\mathfrak{C}_n$. Then there is an equivalence of categories*

$$\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{SO}_3}(\mathrm{Gr}_{\mathrm{SO}_3}^{\times n}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{C}_n/\mathrm{SL}_2^{n+1}).$$

Under the equivalence of Corollary 4.3.17, it is compatible with the action of

$$\mathrm{Shv}^{\min,\mathrm{gr}}_{\mathrm{SO}_3^{n+1}}(\mathrm{Gr}_{\mathrm{SO}_3}^{\times n+1}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}((\mathfrak{sl}_2^*)^{n+1}/\mathrm{SL}_2^{n+1})$$

on the left-hand side by convolution, and on the right-hand side via the map $\mu : \mathfrak{C}_n \to (\mathfrak{sl}_2^*)^{n+1}$.

Proof. Let $\mathrm{H} = \mathrm{SO}_3$ and $\mathrm{X} = \mathrm{SO}_3^{n+1}/\mathrm{SO}_3^{\mathrm{diag}}$, so that $\check{\mathrm{J}}_{\mathrm{X}} = \operatorname{Spec} \mathrm{H}_*^{\mathrm{SO}_3}(\Omega\mathrm{X}; k)$. We will also write $\check{\mathrm{J}}$ to denote the regular centralizer group scheme for SL_2 acting on $\mathfrak{sl}_2^* \cong \mathfrak{so}_3$. It follows from the Serre spectral sequence that there is an exact sequence of group schemes

$$\check{\mathrm{J}}_{\mathrm{X}} \to \check{\mathrm{J}}^{\times_{\mathcal{M}_{\mathrm{SO}_3,0}} n+1} \to \check{\mathrm{J}},$$

so that $\check{\mathrm{J}}_{\mathrm{X}} \cong \check{\mathrm{J}}^{\times_{\mathcal{M}_{\mathrm{SO}_3,0}} n}$, embedded inside $\mathrm{SL}_2^{n+1} \times \mathcal{M}_{\mathrm{SO}_3,0}$ via $(g_1, \cdots, g_n) \mapsto (g_1, \cdots, g_n, g_1^{-1} \cdots g_n^{-1})$. The argument of Theorem 5.4.3 reduces us to computing that the scheme $\mathfrak{C}_n$ is isomorphic to the affinization of the quotient $(\mathrm{SL}_2^{n+1} \times \mathcal{M}_{\mathrm{SO}_3,0})/\check{\mathrm{J}}^{\times_{\mathcal{M}_{\mathrm{SO}_3,0}} n}$.

In fact, a much more general statement is true. Let $\sigma \subseteq \Sigma_{n+1}$ be a cycle type corresponding to a partition $d_1 + \cdots + d_m = n+1$. Let $\check{\mathrm{J}}[\sigma]$ denote the subgroup scheme of $\mathrm{SL}_2^m \times \mathcal{M}_{\mathrm{SO}_3,0}$ given by the kernel of the homomorphism

$$\check{\mathrm{J}}^{\times_{\mathcal{M}_{\mathrm{SO}_3,0}} m} \xrightarrow{(g_1, \cdots, g_m) \mapsto \prod_{j=1}^m g_j^{d_j}} \check{\mathrm{J}}. \tag{5.5.6}$$

Let $\nu_n : \mathbf{P}^1 \hookrightarrow \mathbf{P}^n$ denote the embedding of the rational normal curve. Let $\mathfrak{C}_\sigma$ denote the affine cone of the secant variety of lines on the projective variety

$$\prod_{j=1}^m \mathbf{P}^1 \xrightarrow{\nu_{d_j}} \prod_{j=1}^m \mathbf{P}(\mathrm{Sym}^{d_j}(\mathbf{A}^2)) \hookrightarrow \mathbf{P}\left(\bigotimes_{j=1}^m \mathrm{Sym}^{d_j}(\mathbf{A}^2)\right),$$

where the final map is the Segre embedding. Then, the affinization of $(\mathrm{SL}_2^m \times \mathcal{M}_{\mathrm{SO}_3,0})/\check{\mathrm{J}}[\sigma]$ is isomorphic to $\mathfrak{C}_\sigma$.

The proof for general σ is very similar to the case when σ is represented by a cyclic permutation, except that it is more combinatorially involved; so for simplicity (and to illustrate the main point), let us assume σ is the trivial partition $(n+1)$ of $n+1$. It suffices to show that the affine closure of the quotient $(\mathrm{SL}_2 \times \mathfrak{sl}_2^*(2)/\!\!/\mathrm{SL}_2)/\check{\mathrm{J}}[n+1]$ is isomorphic to the affine cone $\mathfrak{C}_n$ on the rational normal curve $\nu_{n+1} : \mathrm{C} = \mathbf{P}^1 \hookrightarrow \mathbf{P}^{n+1}$. Recall from [Har, Example 9.6] that the secant variety $\mathrm{Sect}(\mathrm{C})$ is the determinantal variety inside $\mathbf{P}^{n+1} = \mathbf{P}(\mathrm{Sym}^{n+1}(\mathbf{A}^2))$ (which has dimension $\min(3, n+1)$), which is an SL_2-stable subscheme cut out by the condition that the Hankel matrix built from the coefficients of a binary form of degree n has rank ≤ 2. Note that the assumption $n \geq 2$ guarantees that the affine cone $\mathcal{C}(\mathrm{Sect}(\mathrm{C}))$ is 4-dimensional. Let $\kappa : \mathbf{A}^1/\!\!/(\mathbf{Z}/2) = \operatorname{Spec} k[a^2] \to \mathrm{Sym}^{n+1}(\mathbf{A}^2)$ denote the closed immersion sending

$$a^2 \mapsto \begin{cases} \frac{1}{2}((y+ax)^{n+1} + (y-ax)^{n+1}) & n \text{ odd}; \\ \frac{1}{2a}((y+ax)^{n+1} - (y-ax)^{n+1}) & n \text{ even}. \end{cases} \tag{5.5.7}$$

One can now check:

- This map in fact lands inside the determinantal variety, which gives the desired map $\kappa : \mathbf{A}^1/\!\!/(\mathbf{Z}/2) \to \mathcal{C}(\mathrm{Sect}(\mathrm{C}))$. In fact, $\mathcal{C}(\mathrm{Sect}(\mathrm{C}))/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1/\!\!/(\mathbf{Z}/2)$, and κ is a section of the above map.
- The stabilizer of κ is isomorphic to $\check{\mathrm{J}}[n+1]$. One way to see this is that if $a^2 \neq 0$, the matrix $\left(\begin{smallmatrix} a & -a \\ 1 & 1 \end{smallmatrix}\right)$ "diagonalizes" $\kappa(a^2)$: namely, it sends $\kappa(a^2)$ to the binary form $x^{n+1} - y^{n+1}$. It is a nice exercise to directly verify that the stabilizer of $x^{n+1} - y^{n+1}$ is $\mu_{n+1} \subseteq \mathbf{G}_m \subseteq \mathrm{SL}_2$; this is the only place in the argument where one needs to use the fact that $n \geq 2$ is even. Since conjugating $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \in \mu_{n+1}$ by $\left(\begin{smallmatrix} a & -a \\ 1 & 1 \end{smallmatrix}\right)$ produces the matrix $\frac{1}{2}\begin{pmatrix} \alpha+\alpha^{-1} & a^2 \cdot \frac{\alpha-\alpha^{-1}}{a} \\ \frac{\alpha-\alpha^{-1}}{a} & \alpha+\alpha^{-1} \end{pmatrix}$, one finds that the stabilizer of $\kappa : \mathbf{A}^1 \to \mathrm{Sym}^{n+1}(\mathbf{A}^2)$ is $\check{\mathrm{J}}[n+1]$.

- The SL_2-orbit of κ has complement of codimension 2. Indeed, one finds by calculation that the SL_2-orbit of κ is the complement of the cone on C inside the cone on Sect(C), and this does indeed have codimension 2.

- The affine cone $\mathcal{C}(\mathrm{Sect}(\mathrm{C}))$ is normal and irreducible (so that the affine closure of the SL_2-orbit of κ is indeed $\mathcal{C}(\mathrm{Sect}(\mathrm{C}))$). This is classical (but in fact, one can prove much more; see [ENP, Theorem 1.1]). $\square$

We hope to prove a ku-theoretic analogue of the preceding result in future work.

Remark 5.5.17. Let $\mathbf{C}(\!(t^\sigma)\!)$ denote the degree $n+1$ extension of $\mathbf{C}(\!(t)\!)$ given by $\prod_{j=1}^m \mathbf{C}(\!(t^{1/d_j})\!)$, so that $\mathbf{C}(\!(t^\sigma)\!)$ admits an action of $\prod_{j=1}^m \Sigma_j$. Similarly, let $\mathbf{C}[\![t^\sigma]\!]$ denote $\prod_{j=1}^m \mathbf{C}[\![t^{1/d_j}]\!]$, so that $\mathrm{Spf}\,\mathbf{C}[\![t^\sigma]\!] \to \mathrm{Spf}\,\mathbf{C}[\![t]\!]$ is an $(n+1)$-fold cover with symmetry group $\prod_{j=1}^m \Sigma_j$. Then Theorem 5.5.16 should generalize, to at least give a fully faithful functor

$$\mathrm{QCoh}^{\mathrm{gr}}(\mathfrak{C}_\sigma/\mathrm{SL}_2^m) \hookrightarrow \mathrm{Shv}^{\mathrm{gr}}_{\mathrm{SO}_3[\![t^\sigma]\!]}(\mathrm{SO}_3(\!(t^\sigma)\!)/\mathrm{SO}_3(\!(t)\!); k);$$

in fact, the same argument will work, as long as the right-hand side is defined correctly.

Remark 5.5.18. Let $n \geq 2$ be an even integer. The Hamiltonian SL_2^{n+1}-scheme $\mathfrak{C}_n$ can be understood as an explicit model for the *Moore-Tachikawa* variety $\eta_{\mathrm{SL}_2}(\mathbf{C}\mathrm{P}^1 - \{0, 1, \cdots, n-1, \infty\})$ of SL_2 associated to an $(n+1)$-punctured 2-sphere. (One could view the more general schemes $\mathfrak{C}_\sigma$ as "twisted" versions of these Moore-Tachikawa varieties.) Such a model did not seem to be previously known: one usually declared $\mathfrak{C}_n$ to be the output of a procedure involving the Hamiltonian reduction of products of $\eta_{\mathrm{SL}_2}(\mathbf{C}\mathrm{P}^1 - \{0, 1, \infty\})$. It is natural to wonder whether a variant of Theorem 5.5.16 can be used to explicitly describe the Moore-Tachikawa varieties associated to an $(n+1)$-punctured 2-sphere for a general connected reductive group G. I have a conjectural description of this object using a construction similar to the secant variety, except with the projective line being replaced by the flag variety of G.

Example 5.5.19. Recall that the expected dimension of the secant variety of an m-dimensional reduced and irreducible projective scheme $\mathrm{X} \subseteq \mathbf{P}^d$ is $\min\{d, 2m+1\}$ (the $2m$ comes from specifying the two points on X which are to be connected by the secant line, and the 1 comes from specifying points on the secant line). Applied to the secant variety of $(\mathbf{P}^1)^{n+1} \subseteq \mathbf{P}^{2^{n+1}-1}$, we find that the expected dimension of $\mathfrak{C}_n$ is $\min\{2^{n+1}, 2n+4\} = 2n+4$ (since $n \geq 2$). In particular, when $n = 2$, one has $2^{n+1} = 2n+4 = 8$, and indeed $\mathfrak{C}_2 = (\mathbf{A}^2)^{\otimes 3}$. (This is the only case when $\mathfrak{C}_n$ is smooth.)

The map $\mu : \mathfrak{C}_2 \to (\mathfrak{sl}_2^*)^3$ is just the moment map for the Hamiltonian SL_2^3-action on $(\mathbf{A}^2)^{\otimes 3}$, which can be computed explicitly as follows. A $2\times 2\times 2$-cube A defines $\binom{3}{2} = 3$ pairs $(\mathrm{M}_i, \mathrm{N}_i)$ of 2×2-matrices given by pairs of opposite faces of the cube. The map $\mu : (\mathbf{A}^2)^{\otimes 3} \to (\mathfrak{sl}_2^*)^3$ sends

$$\mathrm{A} \mapsto (\det(\mathrm{M}_1 x + \mathrm{N}_1 y), \det(\mathrm{M}_2 x + \mathrm{N}_2 y), \det(\mathrm{M}_3 x + \mathrm{N}_3 y)).$$

Furthermore, the invariant-theoretic quotient map $\mathfrak{C}_2 \to \mathfrak{C}_2/\!\!/\mathrm{SL}_2^3 \cong \mathbf{A}^1$ identifies with the Cayley hyperdeterminant (see [Cay]) $\det(\mathcal{C})$ on a $2\times 2\times 2$-cube $\mathcal{C}$.

The above map $\mu : (\mathbf{A}^2)^{\otimes 3} \to (\mathfrak{sl}_2^*)^3$ has already appeared in the literature: if we identify $\mathfrak{sl}_2^* \cong \mathrm{Sym}^2(\mathbf{A}^2)$ as the space of binary quadratic forms, then it is precisely Bhargava's construction [Bha1] of three quadratic forms from a $2\times 2\times 2$-cube. In [Bha1], this was used to describe Gauss composition; and reading Bhargava's argument shows that it reduces exactly to the observation from Theorem 5.5.16 that the stabilizer of the Kostant slice $\kappa_{\check{\mathrm{M}}} : \mathfrak{sl}_2^*/\!\!/\mathrm{SL}_2 \to (\mathbf{A}^2)^{\otimes 3}$ under the SL_2^3-action is given by $\check{\mathrm{J}} \times_{\mathfrak{sl}_2^*/\!\!/\mathrm{SL}_2} \check{\mathrm{J}}$. (Torsors for this stabilizer group scheme therefore exactly parametrize pairs of elements in the class group of a quadratic extension.) However, the interpretation of the symplectic vector space $(\mathbf{A}^2)^{\otimes 3}$ as the affine cone on the secant variety of $(\mathbf{P}^1)^3 \hookrightarrow \mathbf{P}^7$ does not directly play any role in Bhargava's work.

Remark 5.5.20. Example 5.5.19 can be used to understand the relative Langlands dual to some *non-affine* spherical SO_3-varieties. For example, consider the Hamiltonian SO_3-variety $\mathrm{T}^*(\mathbf{P}^1 \times \mathbf{P}^1)$, which is obtained by restricting the Hamiltonian SO_3^2-action on $\mathrm{T}^*(\mathbf{P}^1 \times \mathbf{P}^1)$ along the diagonal embedding. Since the relative Langlands dual to $\mathrm{T}^*(\mathbf{P}^1 \times \mathbf{P}^1)$ viewed as a Hamiltonian SO_3^2-space is the Hamiltonian SL_2^2-space $\mathrm{T}^*(\mathrm{SL}_2^2/\mathbf{G}_a^2)$ by the Eisenstein period (in the local geometric context, this is [ABG]), it follows that the relative Langlands to $\mathrm{T}^*(\mathbf{P}^1 \times \mathbf{P}^1)$ as a Hamiltonian SO_3-variety is given by the Hamiltonian SL_2-space

$$\check{\mathrm{M}} = (\mathrm{T}^*(\mathrm{SL}_2^2/\mathbf{G}_a^2) \times_{(\mathfrak{sl}_2^*)^2} (\mathbf{A}^2)^{\otimes 3})/\mathrm{SL}_2^2 \cong (\mathbf{A}^2)^{\otimes 3}/\!\!/_0 \mathbf{G}_a^2.$$

Here, the notation $/\!\!/_0$ denotes Hamiltonian reduction at $(0,0) \in (\mathfrak{g}_a^*)^2$. That is to say, $\check{\mathrm{M}}$ is the (stacky!) quotient of the degree 4 closed subscheme in $(\mathbf{A}^2)^{\otimes 3}$ of those $2 \times 2 \times 2$-cubes such that $\det(\mathrm{M}_1) = \det(\mathrm{M}_2) = 0$ by the action of $\mathbf{G}_a^2$ (where the notation is as in Example 5.5.19).

Example 5.5.21. Like in Example 5.5.19, one can take the extremal case $\sigma = (3)$ for $n = 2$. Then $\mathfrak{C}_\sigma$ is the affine cone on the secant variety of the twisted cubic $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$, which is in fact all of $\mathbf{A}^4 = \mathrm{Sym}^3(\mathbf{A}^2)$. The map $\mu : \mathrm{Sym}^3(\mathbf{A}^2) \to \mathfrak{sl}_2^* \cong \mathrm{Sym}^2(\mathbf{A}^2)$ is exactly the *quadratic resolvent* construction, sending

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 \mapsto (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2.$$

This example was also studied by Bhargava in [Bha1]: it again reduces to the observation from Theorem 5.5.16 that the stabilizer of the Kostant slice $\kappa_{\check{\mathrm{M}}} : \mathfrak{sl}_2^*/\!\!/\mathrm{SL}_2 \to \mathrm{Sym}^3(\mathbf{A}^2)$ (sending $a \mapsto ax^3 + 3xy^2$) under the SL_2-action is given by $\check{\mathrm{J}}[3]$. (Torsors for this stabilizer group scheme therefore exactly parametrize 3-torsion elements in the class group of a quadratic extension.)

Example 5.5.22. Like in the preceding example, one can take the case $\sigma = (2,1)$ for $n = 2$. Then $\mathfrak{C}_\sigma$ is the affine cone on the secant variety of

$$\mathbf{P}^1 \times \mathbf{P}^1 \xrightarrow{\mathrm{id}\times\nu_2} \mathbf{P}^1 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^5,$$

which is in fact all of $\mathbf{A}^6 = \mathbf{A}^2 \otimes \mathrm{Sym}^2(\mathbf{A}^2)$. The map $\mu : \mathbf{A}^2 \otimes \mathrm{Sym}^2(\mathbf{A}^2) \to (\mathfrak{sl}_2^*)^2$ sends a pair $(f_1, f_2) = (ax^2 + 2bxy + cy^2, dx^2 + 2exy + fy^2)$ of binary quadratic forms to another pair of binary quadratic forms with the same discriminant:

$$\mu(f_1, f_2) = ((ac - b^2)x^2 + (2be - af - cd)xy + (df - e^2)y^2, (ae - bd)x^2 + (cd - af)xy + (bf - ce)y^2).$$

This example was also studied by Bhargava in [Bha1]: it again reduces to the observation from Theorem 5.5.16 that the stabilizer of the Kostant slice $\kappa_{\check{\mathrm{M}}} : \mathfrak{sl}_2^*/\!\!/\mathrm{SL}_2 \to \mathbf{A}^2 \otimes \mathrm{Sym}^2(\mathbf{A}^2)$ (sending $a \mapsto (ax^2 + y^2, 2xy)$) under the SL_2-action is given by $\check{\mathrm{J}}$. (Torsors for this stabilizer group scheme therefore exactly parametrize elements in the class group of a quadratic extension.)

The scheme $\mathfrak{C}_\sigma$ associated to a cycle type $\sigma \subseteq \Sigma_{n+1}$ is very interesting. Although it is generally singular, it only has symplectic singularities. Moreover, the canonical inclusion

$$\text{Affine cone on } (\mathbf{P}^1)^m \hookrightarrow \mathfrak{C}_\sigma$$

in fact exhibits the source as a *Lagrangian* subvariety of the target. (Ordinary) relative Langlands duality does not just predict a matching of (suitable) Hamiltonian G-spaces and Hamiltonian $\check{\mathrm{G}}$-spaces, but also a matching of equivariant Lagrangian correspondences between Hamiltonian spaces. For instance, if $\psi_0, \cdots, \psi_n$ are additive characters of $\mathbf{G}_a$ which sum to zero, one can verify that the Lagrangian in $\mathfrak{C}_n$ (corresponding to the cycle type $\sigma = (1, \cdots, 1)$) given by the affine cone on $(\mathbf{P}^1)^{n+1}$ is Langlands dual to the SO_3^{n+1}-equivariant Lagrangian correspondence between $\mathrm{T}^*(\mathrm{SO}_3^{n+1}/\mathrm{SO}_3^{\mathrm{diag}}) = \mathrm{SO}_3^{n+1} \times^{\mathrm{SO}_3^{\mathrm{diag}}} (\mathfrak{so}_3^*)^n$ and $\mathrm{T}^*(\mathrm{SO}_3/(\mathbf{G}_a, \psi_0) \times \cdots \times \mathrm{SO}_3/(\mathbf{G}_a, \psi_n)) \cong \mathrm{SO}_3^{n+1} \times^{\mathbf{G}_a^{n+1}} \prod_{i=0}^n (\psi_i + \mathfrak{g}_a^\perp)$ given by $\mathrm{SO}_3^{n+1} \times^{\mathbf{G}_a^{\mathrm{diag}}} \prod_{i=0}^{n-1} (\psi_i + \mathfrak{g}_a^\perp)$. We will explore this, and other related phenomena, in future work jointly with D. Ben-Zvi.

5.5.6 Rank 1 of type T

In this brief subsection, we study the case of the rank 1 homogeneous affine spherical varieties whose spherical root is of type T. The examples in question are $\mathrm{PGL}_{n+1}/\mathrm{GL}_n$, $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$, $\mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$, $\mathrm{F}_4/\mathrm{Spin}_9$, and $\mathrm{G}_2/\mathrm{SL}_3$. The conditions of Theorem 5.4.3 does not quite apply to these examples, although I expect that it fits into the modified context of Remark 5.4.8. Here, we will just describe the calculation of $\check{\mathrm{M}}_{\mathbf{H}}$. (We remind the reader that 2 is always assumed to be invertible in our choice of coefficients.) The basic calculation that applies to all these examples is the following:

Lemma 5.5.23. *Let $\kappa : \mathbf{A}^1 \to \mathrm{T}^*(\mathbf{A}^2)$ denote the map sending $x \mapsto (x, 0), (1, 0)$. Then the stabilizer $\check{\mathrm{J}}_0 := \mathbf{A}^1 \times_{\mathrm{T}^*(\mathbf{A}^2)/\mathrm{SL}_2} \mathbf{A}^1$ is isomorphic (as a group scheme over $\mathbf{A}^1$) to* $\mathrm{Spec}\, \mathbf{Z}[x, b]/bx$. *Furthermore, the affinization of $(\mathrm{SL}_2 \times \mathbf{A}^1)/\check{\mathrm{J}}_0$ is isomorphic to $\mathrm{T}^*(\mathbf{A}^2)$ via this map.*

In the discussion below, we will write $\mathbf{A}^2(i, j)$ to denote the product $\mathbf{A}^1(i) \times \mathbf{A}^1(j)$.

Example 5.5.24. The case of $\mathrm{X} = \mathrm{PGL}_{n+1}/\mathrm{GL}_n$, which is the complement of the diagonal $\mathbf{P}^n \subseteq \mathbf{P}^n \times \mathbf{P}^n$, follows from the calculation of Corollary 5.5.8 by Whittaker reduction. Namely, define

$$\check{\mathrm{V}}_{\mathbf{H}}^\ddagger \subseteq \mathfrak{gl}_{n-1}^*(2)/\!\!/\mathrm{GL}_{n-1} \times \mathrm{T}^*(2n)(\mathbf{A}^2(2n, 0))$$

to be the locus of those tuples $(c_1, \cdots, c_{n-1}, u, v)$ such that $1 + \beta c_1 + \cdots + \beta^{n-1} c_{n-1} + \beta^n \langle u, v \rangle$ is a unit. Then, $\check{\mathrm{M}}_{\mathbf{H}}^\ddagger \cong \mathrm{SL}_{n+1} \times^{\mathrm{SL}_2} \check{\mathrm{V}}_{\mathbf{H}}^\ddagger$, where SL_2 acts on $\check{\mathrm{V}}_{\mathbf{H}}^\ddagger$ only through the factor $\mathrm{T}^*(\mathbf{A}^2)$. Here, SL_2 is embedded into SL_{n+1} via (5.5.1). As mentioned above, this description of $\check{\mathrm{M}}_{\mathbf{H}}^\ddagger$ follows from the calculation of Corollary 5.5.8; but it can also be proved directly, by computing that there is an isomorphism

$$\mathrm{Spec}\, \mathcal{H}_*^{\mathrm{GL}_n}(\Omega(\mathrm{PGL}_{n+1}/\mathrm{GL}_n); \mathrm{ku}) \cong \mathbf{Z}[\beta, c_1, \cdots, c_n, \tfrac{1}{1+\beta c_1 + \cdots + \beta^n c_n}, b]/bc_n,$$

where b is in weight $2n$. In particular, one expects an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(\mathrm{ku}))$-linear ∞-categories

$$\mathrm{Shv}_{\mathrm{PGL}_{n+1}}^{\mathrm{min,gr}}(\mathrm{PGL}_{n+1}(\!(t)\!)/\mathrm{GL}_n(\!(t)\!); \mathrm{ku}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}^\ddagger/\mathrm{SL}_{n+1}(2\rho_{\mathrm{SL}_{n+1}})).$$

This equivalence does indeed hold, as follows from Corollary 5.5.8 (by extending the $\mathrm{GL}_n[\![t]\!]$-equivariance therein to $\mathrm{GL}_n(\!(t)\!)$-equivariance, which amounts on the spectral side to a Whittaker reduction, i.e., pulling back the moment map from Corollary 5.5.8 along the multiplicative Kostant slice $(\mathrm{T}^n)_\beta/\!\!/\Sigma_n \to \mathrm{GL}_{n,\beta}$). Similarly, there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(\mathrm{ku}))$-linear ∞-categories

$$\mathrm{Shv}_{\mathrm{GL}_{n+1}}^{\mathrm{min,gr}}(\mathrm{GL}_{n+1}(\!(t)\!)/\mathrm{GL}_n(\!(t)\!); \mathrm{ku}) \simeq \mathrm{QCoh}^{\mathrm{gr}}((\mathrm{GL}_{n+1} \times^{\mathrm{GL}_2} \check{\mathrm{V}}_{\mathbf{H}}^\ddagger)/\mathrm{GL}_{n+1}(2\rho_{\mathrm{GL}_{n+1}})).$$

Note that the $\beta = 0$ fiber of $(\mathrm{GL}_{n+1} \times^{\mathrm{GL}_2} \check{\mathrm{V}}_{\mathbf{H}}^{\ddagger})/\mathrm{GL}_{n+1} \cong \check{\mathrm{V}}_{\mathbf{H}}^{\ddagger}/\mathrm{GL}_2$ splits as $\mathrm{T}^*(\mathbf{A}^2)/\mathrm{GL}_2 \times \mathfrak{gl}_{n-1}^* /\!\!/ \mathrm{GL}_{n-1}$; but $\check{\mathrm{V}}_{\mathbf{H}}^{\ddagger}/\mathrm{GL}_2$ itself does not split in this way.

Example 5.5.25. Suppose $\mathrm{X} = \mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}$, and let k be an algebraically closed field of characteristic $\neq 2$. Then there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{SO}_{2n}}(\Omega(\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n}); k) \cong k[p_1, \cdots, p_{n-1}, c_n, b]/bc_n,$$

where b is in weight $4n-2$. This follows from the fact that $\mathrm{SO}_{2n+1}/\mathrm{SO}_{2n} \cong \mathrm{S}^{2n}$, with the maximal torus $\mathrm{T}^n \subseteq \mathrm{SO}_{2n}$ acting by the one-point compactification of its standard complex n-dimensional representation. It follows that if we define

$$\check{\mathrm{V}}_{\mathbf{H}}^{\ddagger} = \mathfrak{sp}_{2n-2}^*(2) /\!\!/ \mathrm{Sp}_{2n-2} \times \mathrm{T}^*(2n)(\mathbf{A}^2(4n-2, 0)),$$

with SL_2 acting on the second factor, then $\check{\mathrm{M}}_{\mathbf{H}}^{\ddagger} \cong \mathrm{Sp}_{2n} \times^{\mathrm{SL}_2} \check{\mathrm{V}}_{\mathbf{H}}^{\ddagger}$, where again SL_2 is embedded into Sp_{2n} via (5.5.1). In particular, one expects an equivalence of graded k-linear ∞-categories

$$\widetilde{\mathrm{Shv}}_{\mathrm{SO}_{2n+1}}^{\mathrm{min,gr}}(\mathrm{SO}_{2n+1}(\!(t)\!)/\mathrm{SO}_{2n}(\!(t)\!); k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{Sp}_{2n}(2\rho_{\mathrm{Sp}_{2n}})),$$

where we are using notation as in Remark 5.4.8. This equivalence would follow from the above calculation.

Example 5.5.26. Suppose $\mathrm{X} = \mathrm{Sp}_{2n}/(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2})$. Then there is an isomorphism

$$\operatorname{Spec} \mathcal{H}_*^{\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}}(\Omega\mathrm{X}; \mathrm{ku}) \cong \mathbf{Z}[\beta, p_1', p_1, \cdots, p_{n-1}, b]/bp_{n-1},$$

where b is in weight $4n-2$. Let $\mathrm{L}_{\mathrm{X}}^{\wedge} = \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-4}$. It follows that if we define

$$\check{\mathrm{V}}_{\mathbf{H}}^{\ddagger} = (\mathrm{L}_{\mathrm{X}}^{\wedge})_{\mathbf{H}} /\!\!/ \mathrm{L}_{\mathrm{X}}^{\wedge} \times \mathrm{T}^*(16)(\mathbf{A}^2(22, 0)),$$

with SL_2 acting on the second factor, then $\check{\mathrm{M}}_{\mathbf{H}}^{\ddagger} \cong \mathrm{F}_4 \times^{\mathrm{SL}_2} \check{\mathrm{V}}_{\mathbf{H}}^{\ddagger}$. In particular, one expects an equivalence of graded k-linear ∞-categories

$$\widetilde{\mathrm{Shv}}_{\mathrm{Sp}_{2n}}^{\mathrm{min,gr}}(\mathrm{Sp}_{2n}(\!(t)\!)/(\mathrm{Sp}_2(\!(t)\!) \times \mathrm{Sp}_{2n-2}(\!(t)\!)); \mathrm{ku}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{SO}_{2n+1}(2\rho_{\mathrm{SO}_{2n+1}})),$$

where we are using notation as in Remark 5.4.8. This equivalence would follow from the above calculation.

Example 5.5.27. Suppose $\mathrm{X} = \mathrm{F}_4/\mathrm{Spin}_9$, and let k be an algebraically closed field of characteristic $\neq 2$. Then there is an isomorphism

$$\operatorname{Spec} \mathrm{H}_*^{\mathrm{Spin}_9}(\Omega(\mathrm{F}_4/\mathrm{Spin}_9); k) \cong k[p_1, p_2, p_3, p_4, b]/bp_4,$$

where b is in weight 22. This follows from the fact that $\mathrm{F}_4/\mathrm{Spin}_9 \cong \mathbf{O}\mathrm{P}^2$. It follows that if we define

$$\check{\mathrm{V}}_{\mathbf{H}}^{\ddagger} = \mathfrak{sp}_6^*(2) /\!\!/ \mathrm{Sp}_6 \times \mathrm{T}^*(16)(\mathbf{A}^2(22, 0)),$$

with SL_2 acting on the second factor, then $\check{\mathrm{M}}_{\mathbf{H}}^{\ddagger} \cong \mathrm{F}_4 \times^{\mathrm{SL}_2} \check{\mathrm{V}}_{\mathbf{H}}^{\ddagger}$. In particular, one expects an equivalence of graded k-linear ∞-categories

$$\widetilde{\mathrm{Shv}}_{\mathrm{F}_4}^{\mathrm{min,gr}}(\mathrm{F}_4(\!(t)\!)/\mathrm{Spin}_9(\!(t)\!); k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{F}_4(2\rho_{\mathrm{F}_4})),$$

where we are using notation as in Remark 5.4.8. This equivalence would follow from the above calculation.

Example 5.5.28. Suppose $\mathrm{X} = \mathrm{G}_2/\mathrm{SL}_3$, and let us identify

$$\mathbf{G}_\beta^2 /\!\!/ \Sigma_3 \cong \operatorname{Spec} \mathbf{Z}[\beta, c_2, c_3].$$

Then there is an isomorphism

$$\operatorname{Spec} \mathcal{H}_*^{\mathrm{SL}_3}(\Omega(\mathrm{G}_2/\mathrm{SL}_3); \mathrm{ku}) \cong (\mathcal{O}_{\mathbf{G}_\beta^2 /\!\!/ \Sigma_3} \otimes \mathbf{Z}[b])/bc_3,$$

where b is in weight 10. This follows from the fact that $\mathrm{G}_2/\mathrm{SL}_3 \cong \mathrm{S}^6$. Let $\mathbf{G}_a(-10)$ act on $\mathbf{A}^1(-4) \times (\mathbf{G}_\beta^2 /\!\!/ \Sigma_3)$ by $(z, c_2, c_3) \mapsto (z - bc_3, c_2, c_3)$. It follows that if we define $\check{\mathrm{V}}_{\mathbf{H}}^\ddagger$ as the affinization of

$$\check{\mathrm{V}}_{\mathbf{H}}^{\ddagger,\mathrm{reg}} = \mathrm{SL}_2(-10\rho_{\mathrm{SL}_2}) \times^{\mathbf{G}_a(-10)} (\mathbf{A}^1(-4) \times \mathbf{G}_\beta^2 /\!\!/ \Sigma_3),$$

then $\check{\mathrm{M}}_{\mathbf{H}}^\ddagger \cong \mathrm{G}_2 \times^{\mathrm{SL}_2} \check{\mathrm{V}}_{\mathbf{H}}^\ddagger$. In particular, one expects an equivalence of QCoh(Spev(ku))-linear ∞-categories

$$\widetilde{\mathrm{Shv}}_{\mathrm{G}_2}^{\mathrm{min,gr}}(\mathrm{G}_2(\!(t)\!)/\mathrm{SL}_3(\!(t)\!); \mathrm{ku}) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{G}_2(2\rho_{\mathrm{G}_2})),$$

where we are using notation as in Remark 5.4.8. This equivalence would follow from the above calculation.

5.6 Power operations under relative Langlands duality

Continuing § 4.4, one can study power operations in the context of relative Langlands duality, too. Recall that our mildly refined version (see Remark 5.2.17) of the local version of the conjectures from [BZSV] state that if X is an affine spherical G-variety, then there exists a graded affine Hamiltonian $\check{\mathrm{G}}$-variety $\check{\mathrm{M}} = \check{\mathrm{M}}_{\mathbf{G}_a}$ over $\mathbf{Z}$ (possibly with an integer $\mathrm{N} \gg 0$ inverted) with moment map $\mu : \check{\mathrm{M}} \to \check{\mathfrak{g}}^*$ such that there is 1-parameter degeneration

$$\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{X}(\!(t)\!); \mathbf{Z}) \rightsquigarrow \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}/\check{\mathrm{G}}).$$

Moreover, under a $\mathbf{Z}$-linear analogue of the derived geometric Satake equivalence, the natural action of $\mathrm{Shv}_{\mathrm{G}[\![t]\!]}(\mathrm{Gr}_\mathrm{G}; \mathbf{Z})$ on the left-hand side by convolution should degenerate to the action of $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathfrak{g}}^*(2)/\check{\mathrm{G}})$ on $\mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}/\check{\mathrm{G}})$ via pullback along the moment map.

Following the discussion in § 4.4, the left-hand side will admit an action of the decompleted Frobenius/Steenrod operations, and so one expects the right-hand side to also admit such a structure. That is to say, $\check{\mathrm{M}}$ should admit an action of the decompleted Frobenius, and the moment map $\mu : \check{\mathrm{M}} \to \check{\mathfrak{g}}^*$ should be compatible with this action; here, $\check{\mathfrak{g}}^*$ is equipped with the action of the decompleted Frobenius described in Example 4.4.16. It is worth remarking that this picture of relative Langlands duality only predicts that the decompleted Frobenius/Steenrod operations only act canonically on the *stack* $\check{\mathrm{M}}/\check{\mathrm{G}}$, and that any formula one writes on $\check{\mathrm{M}}$ will not be canonical. This will be abundantly clear in the examples below, where it is obvious that the formulas we write are not unique (but any other choice will be an $\check{\mathrm{G}}$-translate of our formulas). In any case, these extra symmetries on $\check{\mathrm{M}}/\check{\mathrm{G}}$ are very interesting, and we expect them to play an important role in positive-characteristic analogues of the relative Langlands program.

In Conjecture 5.2.20, we proposed a version of this picture for sheaves with coefficients in arbitrary $\mathbf{E}_\infty$-rings k, and provided several examples in the case $k = \mathrm{ku}$. If $k = \mathrm{KU}$, for instance, the main difference from [BZSV] is that $\check{\mathrm{M}}$ must be replaced by a *quasi-Hamiltonian* $\check{\mathrm{G}}$-variety in the sense of [AMM], so that one has a "multiplicative" moment map $\check{\mathrm{M}} \to \mathrm{G}$. Again, as above, $\check{\mathrm{M}}$ (or more canonically, $\check{\mathrm{M}}/\check{\mathrm{G}}$) should admit an action of the decompleted

Frobenius/pth Adams operation on KU, and the multiplicative moment map $\check{\mathrm{M}} \to \mathrm{G}$ should be compatible with this action, where the action of the decompleted Frobenius on G is as described in Example 4.4.19. Outside of simple cases like Example 4.4.24, the structure of power operations on quasi-Hamiltonian varieties can be quite complicated.

Let us present two explicit and nontrivial examples of "Frobenius compatibility" in the context of relative Langlands. The simplest is perhaps the following example, which generalizes Example 4.4.23 and Example 4.4.24.

Example 5.6.1 (Mirabolic Satake)**.** This example is concerned with the relative Langlands dual to $\mathrm{G} = \mathrm{GL}_n \times \mathrm{GL}_{n-1}$ acting on $\mathrm{G}/\mathrm{GL}_{n-1}^{\mathrm{diag}} = \mathrm{GL}_n$. In [BFGT], it was shown that there is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{GL}_{n-1}[\![t]\!]}(\mathrm{Gr}_{\mathrm{GL}_n}; \mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{T}^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})),$$

where, if we identify $\mathrm{T}^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$ with $\mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) \oplus \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$, the moment map $\mu : \mathrm{T}^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1}) \to \mathfrak{gl}_n^* \times \mathfrak{gl}_{n-1}^*$ sends

$$\mu : (f, g) \mapsto (fg, gf).$$

The equivalence of categories above will continue to hold over $\mathbf{Z}$ (in the sense of there being a 1-parameter degeneration from the left-hand side to the right-hand side), so we may consider the decompleted Frobenius for arbitrary p. Unwinding the proof of the above equivalence shows that the decompleted Frobenius/Steenrod algebra acts on $\mathrm{T}^* \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1})$ via

$$\varphi : (f, g) \mapsto (f, g - t^{p-1} g(fg)^{p-1}).$$

It is easy to check that the moment map is indeed Frobenius-equivariant.

There is also a multiplicative version of this picture. Namely, it follows from Corollary 5.5.8 that if $k = \mathrm{ku}$ and we define

$$\check{\mathrm{M}}_{\mathbf{H}} = \{(u, v) \in \mathrm{T}^* \mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) | \mathrm{id} + \beta uv \in \mathrm{GL}_{n,\beta}\},$$

then there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear ∞-categories

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{GL}_{n-1}}(\mathrm{Gr}_{\mathrm{GL}_n}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/(\mathrm{GL}_n \times \mathrm{GL}_{n-1})).$$

where $\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1})^{\mathrm{reg}}$ is a particular open subset inside Van den Bergh's variety from [Van]:

$$\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1}) = \{(f, g) \in \mathrm{Hom}(\mathbf{A}^{n-1}, \mathbf{A}^n) \oplus \mathrm{Hom}(\mathbf{A}^n, \mathbf{A}^{n-1}) | \mathrm{id} + fg \in \mathrm{GL}_n\}.$$

The multiplicative moment map $\mu : \check{\mathrm{M}}_{\mathbf{H}} \to \mathrm{GL}_{n,\mathbf{H}} \times \mathrm{GL}_{n-1,\mathbf{H}}$ sends

$$\mu : (f, g) \mapsto (fg, gf).$$

The decompleted Frobenius/pth Adams operation acts on $\mathcal{B}(\mathbf{A}^n, \mathbf{A}^{n-1})$ via

$$\varphi : (f, g) \mapsto (f, f^{-1} \tfrac{(\mathrm{id} + \beta fg)^p - \mathrm{id}}{\beta}),$$

and again, the multiplicative moment map is Frobenius-equivariant.

Example 5.6.2. In [BFGT], it was also shown that there is an equivalence

$$\mathrm{Shv}^{c,\mathrm{Sat}}_{\mathrm{GL}_n[\![t]\!]}(\mathrm{Gr}_{\mathrm{GL}_n} \times \mathbf{A}^n(\!(t)\!); \mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\mathrm{T}^*\mathfrak{gl}_n/(\mathrm{GL}_n \times \mathrm{GL}_n)),$$

where, if we identify $\mathrm{T}^*\mathfrak{gl}_n$ with $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$, the moment map $\mu : \mathrm{T}^*\mathfrak{gl}_n \to \mathfrak{gl}_n^* \times \mathfrak{gl}_n^*$ sends

$$\mu : (f, g) \mapsto (fg, gf).$$

Such an equivalence will continue to hold over $\mathbf{Z}$ (in the sense of there being a 1-parameter degeneration from the left-hand side to the right-hand side), so we may consider the decompleted Frobenius for at any prime p. Unwinding the proof of the above equivalence shows that, just as in Example 5.6.1, the decompleted Frobenius/Steenrod algebra acts on $\mathrm{T}^*\mathfrak{gl}_n$ via

$$\varphi : (f, g) \mapsto (f, g - t^{p-1} g(fg)^{p-1}).$$

Again, it is easy to check that the moment map is indeed Frobenius-equivariant.

Example 5.6.3 (Symplectic period)**.** The "quaternionic" Satake equivalence is concerned with the relative Langlands dual to $\mathrm{G} = \mathrm{GL}_{2n}$ acting on $\mathrm{GL}_{2n}/\mathrm{Sp}_{2n}$. The main result of [CMNO] says that there is an equivalence

$$\mathrm{Shv}^{c}_{\mathrm{GL}_{2n}[\![t]\!]}(\mathrm{GL}_{2n}(\!(t)\!)/\mathrm{Sp}_{2n}(\!(t)\!); \mathbf{Q}) \simeq \mathrm{Perf}^{\mathrm{sh}}(\check{\mathrm{M}}/\mathrm{GL}_{2n}),$$

where $\check{\mathrm{M}} \cong \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n^*[4]$ is equipped with a particular Hamiltonian structure. (Here, GL_n sits diagonally inside GL_{2n}.) Such an equivalence will continue to hold over $\mathbf{Z}$ (in the sense of there being a 1-parameter degeneration from the left-hand side to the right-hand side), so we may consider the decompleted Frobenius for all p. In particular, we will assume $p > 2$. The moment map $\check{\mathrm{M}} \to \mathfrak{gl}_{2n}^*$ is induced by the inclusion $\mathfrak{gl}_n^* \to \mathfrak{gl}_{2n}^*$ sending

$$\mu : x \mapsto \left(\begin{smallmatrix} 0 & \mathrm{id}_n \\ x & 0 \end{smallmatrix}\right).$$

Unwinding the proof of [CMNO] shows that the decompleted Frobenius/Steenrod algebra acts on $\check{\mathrm{M}}$ via the map

$$\varphi : x \mapsto x - 2t^{p-1} x^{(p+1)/2} + t^{2(p-1)} x^p = \prod_{j \in \mathbf{F}_p} (x - j^2 t^2 \mathrm{id}_n)$$

on $\mathfrak{gl}_n^*$. (Observe that the formula for φ is a matrix version of the total Steenrod operation on $\mathrm{H}^*_{\mathrm{SU}(2)}(*; \mathbf{F}_p)$.) If $x \in \mathfrak{gl}_n^*$, it is *not* true that $\varphi(\mu(x)) = \mu(\varphi(x))$; but these two elements of $\mathfrak{gl}_{2n}^*$ are conjugate, from which it follows that the moment map $\check{\mathrm{M}}/\mathrm{GL}_{2n} \to \mathfrak{gl}_{2n}^*/\mathrm{GL}_{2n}$ is equivariant for the action of the decompleted Frobenius.

There is a ku-theoretic variant of the preceding discussion. Recall from Theorem 5.5.11 that if $\mathrm{G}_{\mathbf{R}} = \mathrm{GL}_n(\mathbf{H})$, then there is an equivalence of $\mathrm{QCoh}(\mathrm{Spev}(k))$-linear ∞-categories

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{G}_{\mathbf{R}}}(\mathrm{Gr}_{\mathrm{G}_{\mathbf{R}}}; k) \simeq \mathrm{QCoh}^{\mathrm{gr}}(\check{\mathrm{M}}_{\mathbf{H}}/\mathrm{GL}_{2n}).$$

Here, $\check{\mathrm{M}}_\beta \cong \mathrm{GL}_{2n} \times^{\mathrm{GL}_n} \mathfrak{gl}_n$ (with GL_n sitting diagonally inside GL_{2n}). Recall that the multiplicative moment map $\mu : \check{\mathrm{M}}_\beta \to \mathrm{GL}_{2n,\beta}$ is induced by the inclusion $\mathfrak{gl}_n \to \mathrm{GL}_{2n,\beta}$ sending

$$\mu : x \mapsto \left(\begin{smallmatrix} \beta x & \mathrm{id}_n \\ x & 0 \end{smallmatrix}\right).$$

In this case, the decompleted Frobenius/pth Adams operation acts on $\check{\mathrm{M}}_\beta$ via the map $\varphi(x) = f_p(x)$ on $\mathfrak{gl}_n$, with $f_p(x)$ as in Remark 4.4.20. For $x \in \mathfrak{gl}_n$, the elements $\varphi(\mu(x))$ and $\mu(\varphi(x))$ of GL_{2n} are conjugate, so the moment map $\check{\mathrm{M}}_\beta/\mathrm{GL}_{2n} \to \mathrm{GL}_{2n,\beta}/\mathrm{GL}_{2n}$ is equivariant for the action of the decompleted Frobenius.

In the language of [BZSV, Dev3], the preceding discussion says that the stack which is relative Langlands dual to the Hamiltonian GL_{2n}-space $\mathrm{T}^*(\mathrm{GL}_{2n}/\mathrm{Sp}_{2n})$ is isomorphic to $\mathfrak{gl}_n(4)/\mathrm{GL}_n$ with coefficients in both ordinary cohomology *and* complex/real K-theory. However, this will no longer be true for elliptic cohomology. Geometrically, this is because elliptic cohomology is not Spin-oriented, but is only "String-oriented" [AHR]; and $\mathbf{H}\mathrm{P}^{n-1}$ is a generating complex for the quaternionic affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_n(\mathbf{H})}$, but it is not a String-manifold.[6]

There are some examples where the generating complex for the real Grassmannian $\mathrm{Gr}_{\mathrm{G}_\mathbf{R}}$ *is* orientable for elliptic cohomology, such as the case of (the simply-connected form of) E_6 equipped with the involution whose fixed subgroup is F_4. (This is the Cartan symmetric space EIV, and in the parlance of relative Langlands duality, it corresponds to the "octonionic" Satake equivalence of [CO] and [Dev3, Remark 3.6.5].) In this case, the generating complex for $\mathrm{Gr}_{(\mathrm{E}_6)_\mathbf{R}}$ is given by the octonionic projective plane $\mathbf{O}\mathrm{P}^2$, which is indeed a String-manifold (and hence is orientable for elliptic cohomology). It is possible to use this observation to compute the relative Langlands dual to the Hamiltonian E_6-space $\mathrm{T}^*(\mathrm{E}_6/\mathrm{F}_4)$ with coefficients in elliptic cohomology, but the calculations become very intricate (so we will leave it to future work).

All of the examples so far have been concerned with cases of G-spaces X where the group $\mathrm{L}_\mathrm{X}^\wedge$ from [KS2] is trivial. As discussed in Remark 5.2.18, the quotient stack $\check{\mathrm{M}}/\check{\mathrm{G}}$ splits as $\check{\mathrm{Y}}/\check{\mathrm{G}}_\mathrm{X} \times \mathfrak{l}_\mathrm{X}^\wedge /\!\!/ \mathrm{L}_\mathrm{X}^\wedge$ for some $\check{\mathrm{G}}_\mathrm{X}$-space $\check{\mathrm{Y}}$; however, although the map $\mathfrak{l}_\mathrm{X}^\wedge /\!\!/ \mathrm{L}_\mathrm{X}^\wedge \to \check{\mathrm{Y}}/\check{\mathrm{G}}_\mathrm{X}$ is always closed under Frobenius, the Frobenius on $\check{\mathrm{M}}/\check{\mathrm{G}}$ forces the factors of $\check{\mathrm{Y}}/\check{\mathrm{G}}_\mathrm{X}$ and $\mathfrak{l}_\mathrm{X}^\wedge /\!\!/ \mathrm{L}_\mathrm{X}^\wedge$ to interact in a very interesting way. Since the Steenrod operations are related to a presentation of $\mathbf{F}_p$ as an algebra over the sphere spectrum, this interaction between the various factors is in some sense "explained" by the failure of an analogous splitting to hold for $\check{\mathrm{M}}_\mathbf{H}/\check{\mathrm{G}}$ for a general $\mathbf{E}_\infty$-ring k (see Remark 5.2.23).

Example 5.6.4. Consider the example of $\mathrm{PGL}_{n+1}/\mathrm{GL}_n$ (with coefficients in an ordinary commutative ring where $n!$ is a unit). Then $\check{\mathrm{M}}/\check{\mathrm{G}} \cong \mathrm{T}^*(2n)(\mathbf{A}^2)/\mathrm{SL}_2 \times \mathfrak{gl}_{n-1}^*(2)/\!\!/\mathrm{GL}_{n-1}$. However, the action of Frobenius on $\check{\mathrm{M}}$ does not factor through an action on each individual factor, because this already fails to happen at the level of invariant-theoretic quotients. Namely, $\check{\mathrm{M}}/\!\!/\check{\mathrm{G}} \cong \mathfrak{gl}_n(2)/\!\!/\mathrm{GL}_n$, and the factor $\mathrm{T}^*(2n)(\mathbf{A}^2)/\!\!/\mathrm{SL}_2 \cong \mathbf{A}^1$ corresponds to the coordinate c_n on $\mathfrak{gl}_n(2)/\!\!/\mathrm{GL}_n$. The Frobenius forces c_n to interact with the complementary factor $\mathfrak{gl}_{n-1}(2)/\!\!/\mathrm{GL}_{n-1}$: if $n = 2$ and $p = 3$, for instance, the Frobenius sends

$$\begin{aligned} c_1 &\mapsto c_1 - c_1^3 t^2 \\ c_2 &\mapsto c_2 + (2c_1^2 - c_1^2 c_2)t^2 + c_2^3 t^4. \end{aligned}$$

Finally, let us discuss the Frobenius for a non-polarized example. The most famous example of this is the Gan-Gross-Prasad period, which, in the parlance of [BZSV], is concerned with the relative Langlands dual to the homogeneous spherical $\mathrm{G} = \mathrm{SO}_{2n-1} \times \mathrm{SO}_{2n}$-variety given by $\mathrm{G}/\mathrm{SO}_{2n-1}^{\mathrm{diag}}$. This dual is given by the Hamiltonian $\check{\mathrm{G}} = \mathrm{Sp}_{2n-2} \times \mathrm{SO}_{2n}$-space $\mathrm{std}_{2n-2} \otimes \mathrm{std}_{2n}$. It was studied geometrically in [BFT]. The following is one of the simplest nontrivial cases of the Gan-Gross-Prasad period:

Example 5.6.5 (Triple product period)**.** The triple product period, studied geometrically in Theorem 5.5.16 and Example 5.5.19, is concerned with the relative Langlands dual to $\mathrm{G} = \mathrm{PGL}_2^{\times 3}$ acting on $\mathrm{X} = \mathrm{G}/\mathrm{PGL}_2^{\mathrm{diag}}$. (This can be regarded as a special case of the Gan-Gross-Prasad period, because $\mathrm{PGL}_2 \cong \mathrm{SO}_3$ and $\mathrm{PGL}_2^{\times 2} \cong \mathrm{PSO}_4$.) The dual Hamiltonian

[6] Note that the TMF-homology of $\mathbf{H}\mathrm{P}^n$ is described explicitly in [Mei, Proposition 7.5].

$\check{\mathrm{G}} = \mathrm{SL}_2^{\times 3}$-variety in this case is given by the 8-dimensional symplectic vector space $(\mathbf{A}^2)^{\otimes 3}$, with each factor of SL_2 in $\check{\mathrm{G}}$ acting on the corresponding tensor factor. One can easily compute the action of the Frobenius on $(\mathbf{A}^2)^{\otimes 3}/\mathrm{SL}_2^{\times 3}$, at least for $p > 2$. We will only describe the *completed* Frobenius, i.e., the functor

$$\mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{PGL}_2}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k) \to \mathrm{Shv}^{\mathrm{min,gr}}_{\mathrm{G}[\![t]\!]}(\mathrm{PGL}_2^{\times 3}(\!(t)\!)/\mathrm{PGL}_2(\!(t)\!); k^{t\mathbf{Z}/p}),$$

which, when $k = \mathbf{Z}[u^{\pm 1}]$, identifies with the functor given by pullback along a map

$$\varphi : (\mathbf{A}^2)^{\otimes 3}/\mathrm{SL}_2^{\times 3} \times_{\mathrm{Spec}(\mathrm{F})} \mathrm{Spec}(\mathrm{F}(\!(t)\!)) \to (\mathbf{A}^2)^{\otimes 3}/\mathrm{SL}_2^{\times 3}.$$

To describe it, pick a basis $e_1, e_2 \in \mathbf{A}^2$, and equip $\mathbf{A}^2$ with the $\mathbf{Z}[1/3]$-grading where e_1 has weight $2/3$ and e_2 has weight $-1/3$. This equips $(\mathbf{A}^2)^{\otimes 3}$ with an $\mathbf{Z}$-grading, and one can then show that the Frobenius map is given by scaling the cube by its natural $\mathbf{G}_m$-action with respect to the scalar $\delta = 1 - t^{p-1}\det(\mathcal{C})^{(p-1)/2}$. In other words, it is given by multiplying each coordinate $\mathcal{C}_{ijk}$ of a cube $\mathcal{C}$ by $\delta^{|\mathcal{C}_{ijk}|}$, where $|\mathcal{C}_{ijk}|$ is the weight of $\mathcal{C}_{ijk}$. Note that some coordinates will have negative weight, and in this case one must interpret δ^{-1} as $\sum_{n\geq 0} t^{n(p-1)}\det(\mathcal{C})^{n(p-1)/2}$; ensuring convergence of this power series is why we elected to work with the completed Frobenius in the present example.

It might be interesting to describe the (de)completed Frobenius explicitly for the general case of the Gan-Gross-Prasad period, as well as for other non-polarized examples.

Remark 5.6.6. Just as in Warning 4.4.1, there is a slight variant of the Frobenius acting on $\check{\mathrm{M}}/\check{\mathrm{G}}$ which comes from the $\mathbf{E}_3$-Tate Frobenius from Remark 4.4.11. Namely, it follows from Remark 4.4.11 and general properties of the relative Langlands duality (see, e.g., [BZSV, Section 17]) that $\mathcal{O}_{\check{\mathrm{M}}_{\mathbf{G}_a}}$ admits a *Frobenius-constant* quantization in the sense of [BK1]. The underlying deformation quantization comes from imposing loop-rotation equivariance on the automorphic side, much as in § 4.6. As mentioned in Remark 4.4.11, one obtains an interesting generalization of the notion of Frobenius-constant quantizations for other $\mathbf{E}_\infty$-rings k, which we will explain in future work. Under the generalized relative Langlands duality conjectures of Conjecture 5.2.20, $\mathcal{O}_{\check{\mathrm{M}}_{\mathbf{H}}}$ admits such a generalized Frobenius-constant quantization, which again comes from imposing loop-rotation equivariance on the automorphic side. We will explain this in future work. (When $\mathbf{H} = \mathbf{G}_m$, for instance, this generalized Frobenius-constant quantization gives a q-deformation of $\mathcal{O}_{\check{\mathrm{M}}_{\mathbf{G}_m}}$. In the case of Corollary 5.5.8, for instance, this q-deformation is essentially the quantized multiplicative quiver variety of [Jor], and the generalized Frobenius-constant quantization structure is closely related to that of [GJS].)

Part II

Spherochromatism in arithmetic geometry

Chapter 6

Topological Hochschild homology of $\mathbf{Z}_p$

6.1 Statement of main results

Our goal in this section, whose content is joint work [DR] with Arpon Raksit, is to give a calculation of $\mathrm{THH}(\mathbf{Z}_p)$ as a cyclotomic spectrum and present several applications of this calculation. Throughout this section, we will fix an odd prime p and implicitly p-complete all objects involved.

To motivate our main result, let us recall a description of $\mathrm{THH}(\mathbf{F}_p)$ following [NS]. A classical theorem of Bökstedt [Bok], reinterpreted using a result of Hopkins-Mahowald [Mah], states that there is an equivalence $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{F}_p[\Omega \mathrm{S}^3]$ of $\mathbf{E}_1$-$\mathbf{F}_p$-algebras. This result can be refined to provide a description of $\mathrm{THH}(\mathbf{F}_p)$ as a cyclotomic $\mathbf{E}_\infty$-ring; to state it, we need a few constructions.

Construction 6.1.1. Let X be a connective cyclotomic spectrum. Let $\mathrm{X}^{(-1)}$ denote the cyclotomic spectrum whose underlying S^1-spectrum is $\tau_{\geq 0}(\mathrm{X}^{(-1)})$, and whose cyclotomic Frobenius is given by taking the $\mathbf{Z}/p$-Tate construction of the map $\mathrm{X} \to \tau_{\geq 0}(\mathrm{X}^{t\mathbf{Z}/p})$ which factors the cyclotomic Frobenius on X. The functor $\mathrm{CycSp}_{\geq 0} \to \mathrm{CycSp}$ sending $\mathrm{X} \mapsto \mathrm{X}^{(-1)}$ is lax symmetric monoidal.

Similarly, if Y is any spectrum, then $\mathrm{Y}^{\mathrm{triv}}$ denotes the cyclotomic spectrum whose underlying S^1-spectrum is Y with the trivial action, and whose cyclotomic Frobenius is given by the composite $\mathrm{Y} \to \mathrm{Y}^{h\mathbf{Z}/p} \to \mathrm{Y}^{t\mathbf{Z}/p}$. Again, the functor $\mathrm{Sp} \to \mathrm{CycSp}$ sending $\mathrm{Y} \mapsto \mathrm{Y}^{\mathrm{triv}}$ is lax symmetric monoidal.

Theorem 6.1.2 ([NS, AMMN])**.** *Write $\mathbf{Z}_p$ to denote the cyclotomic $\mathbf{E}_\infty$-ring $\mathbf{Z}_p^{\mathrm{triv}}$.*

a. There is a canonical equivalence $\mathrm{THH}(\mathbf{F}_p) \simeq \mathbf{Z}_p^{(-1)}$.

b. The cofiber of the canonical map $\mathbf{Z}_p \to \mathbf{Z}_p^{(-1)}$ is S^1-nilpotent (meaning that for any S^1-spectrum X, the map $\mathbf{Z}_p \otimes \mathrm{X} \to \mathbf{Z}_p^{(-1)} \otimes \mathrm{X}$ induces an equivalence on S^1-Tate constructions).

In this equivalence, the unit map $\mathbf{Z}_p \to \mathrm{THH}(\mathbf{F}_p)$ is adjoint to the unit map $\mathrm{K}(\mathbf{F}_p)_p^\wedge \cong \mathbf{Z}_p \to \mathrm{TC}(\mathbf{F}_p)$.

Our goal is to prove an analogous result describing $\mathrm{THH}(\mathbf{Z}_p)$, *at least for* $p > 2$. Interestingly, this turns out to be related to a very classical object from algebraic topology, known as the *image of J* spectrum [Ada2]. In modern language, this object can be defined as follows.

Definition 6.1.3. Let p be an odd prime. The $\mathbf{E}_\infty$-ring KU_p of p-complete complex K-theory admits a continuous action of $\mathbf{Z}_p^\times$ by Adams operations. Then the K(1)-local sphere $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}$, which we will denote by J_p in the present text, is defined to be the homotopy fixed points $\mathrm{KU}_p^{h\mathbf{Z}_p^\times}$. This $\mathbf{E}_\infty$-ring is equivalent to the homotopy fixed points of KU_p by the subgroup $\mathbf{F}_p^\times \times \mathbf{Z} \subseteq \mathbf{Z}_p^\times$, where $\mathbf{Z}$ is generated by the element $1+p$. Finally, let $j_p = \tau_{\geq 0}(\mathrm{J}_p)$ denote the connective cover of J_p.

Theorem 6.1.4 (Joint with A. Raksit)**.** *Let $p > 2$, and write j_p to denote the cyclotomic $\mathbf{E}_\infty$-ring j_p^{triv}.*

a. There is a canonical equivalence $\mathrm{THH}(\mathbf{Z}_p) \simeq j_p^{(-1)}$, as well as a commutative diagram of cyclotomic $\mathbf{E}_\infty$-rings

$$\begin{tikzcd} j_p \arrow[r] \arrow[d] & \mathrm{THH}(\mathbf{Z}_p) \arrow[d] \\ \mathbf{Z}_p \arrow[r] & \mathrm{THH}(\mathbf{F}_p). \end{tikzcd}$$

b. The cofiber of the canonical map $j_p \to j_p^{(-1)}$ is S^1-nilpotent (meaning that for any S^1-spectrum X, the map $j_p \otimes \mathrm{X} \to j_p^{(-1)} \otimes \mathrm{X}$ induces an equivalence on S^1-Tate constructions).

Remark 6.1.5. There are a few variants of Theorem 6.1.4 which can be proved, with varying levels of effort. Let $\mathrm{ku}_p^{\mathbf{Z}/p^n} = \tau_{\geq 0}(\mathrm{ku}_p^{h\mathbf{Z}/p^n})$; note that this is *not* the same as the genuine equivariant version of ku_p studied in Part I. The ring $\pi_*(\mathrm{ku}_p^{\mathbf{Z}/p^n})$ is isomorphic to $\mathbf{Z}_p[\![q^{1/p^{n-1}} - 1]\!][\beta]/(q-1)$. There is an action of the group $(1+p^n\mathbf{Z}_p)^\times$ on $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}}$, and we will write $j_{p,n-1} = \tau_{\geq 0}((\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}})^{h(1+p^n\mathbf{Z}_p)^\times})$. Note that there is a residual action of $\mathrm{S}^1 = \mathrm{S}^1/(\mathbf{Z}/p^{n-1})$ on $j_{p,n-1}$. In fact, this S^1-action refines to a cyclotomic structure which is inherited from a cyclotomic structure on $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}}$.[1] It turns out that there is a map $j_{p,n-1} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_{p^n}])$ of cyclotomic $\mathbf{E}_\infty$-rings which is an equivalence on $\mathbf{Z}/p$-Tate constructions, and such that the induced map $j_{p,n-1}^{(-1)} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_{p^n}])$ is an equivalence of cyclotomic $\mathbf{E}_\infty$-rings.

Before proving Theorem 6.1.4, we will explain several consequences of this calculation. First, Theorem 6.1.4 allows for a refinement of the main result of [PV]:

Corollary 6.1.6. *If $\mathcal{C}$ is a dualizable $\mathbf{Z}_p$-linear ∞-category, there is a natural lax symmetric monoidal equivalence*

$$\mathrm{TP}(\mathcal{C} \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \simeq \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p).$$

In particular, this equivalence is $\mathrm{TP}(\mathbf{F}_p) \simeq \mathbf{Z}_p^{t\mathrm{S}^1}$-linear.

Theorem 6.1.4 can also be used to provide a recalculation of $\mathrm{TC}(\mathbf{Z}_p)$, refining the results of [BM]. To state this refinement, recall that Bökstedt-Hsiang-Madsen proved in [BHM] that there is a canonical equivalence of spectra

$$\mathrm{TC}(\mathrm{S}) \simeq \mathrm{S} \oplus \Sigma\mathrm{S} \oplus \mathrm{fib}(\overline{\mathrm{tr}}),$$

where $\overline{\mathrm{tr}} : \Sigma\mathbf{C}\mathrm{P}^\infty \to \mathrm{S}$ is the *reduced* S^1-transfer map. Using Theorem 6.1.4, we show:

[1] This cyclotomic structure can be defined as follows. The unit map $\mathrm{ku}_p \to \mathrm{ku}_p^{h\mathbf{Z}/p}$ induces a map $\mathrm{ku}_p^{h\mathbf{Z}/p^{n-1}} \to (\mathrm{ku}_p^{h\mathbf{Z}/p})^{h\mathbf{Z}/p^{n-1}}$, which (by precomposing with the connective cover map $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}} \to \mathrm{ku}_p^{h\mathbf{Z}/p^{n-1}}$) produces a map $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}} \to (\mathrm{ku}_p^{h\mathbf{Z}/p})^{h\mathbf{Z}/p^{n-1}}$. Identifying the target with $(\mathrm{ku}_p^{h\mathbf{Z}/p^{n-1}})^{h\mathbf{Z}/p}$ shows that this refines to a map $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}} \to (\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}})^{h\mathbf{Z}/p}$. Composing with the map to $(\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}})^{t\mathbf{Z}/p}$ then produces the desired cyclotomic structure on $\mathrm{ku}_p^{\mathbf{Z}/p^{n-1}}$.

Corollary 6.1.7. *There is a canonical equivalence of spectra*

$$\mathrm{TC}(\mathbf{Z}_p) \simeq j_p \oplus \Sigma j_p \oplus \mathrm{X},$$

where X *is noncanonically equivalent to* $\bigoplus_{0\leq k\leq p, k\neq 1,p-1} \ell_p[2k-1]$*. Moreover, the unit map* $\mathrm{TC}(\mathrm{S}) \to \mathrm{TC}(\mathbf{Z}_p)$ *is diagonalizable with respect to the decompositions stated above (i.e., it is the direct sum of the unit map* $\mathrm{S} \to j_p$*, the shift of the unit map* $\mathrm{S} \to j_p$*, and a particular map* $\mathrm{fib}(\overline{\mathrm{tr}}) \to \mathrm{X}$.

Since it is quite technical, the proof of this corollary will be deferred to my forthcoming joint paper with A. Raksit.

Finally, as in [AMMN], Theorem 6.1.4 can be used to provide a refinement of the Beilinson fiber square. Recall that this result states that if R is a connective $\mathbf{E}_1$-ring, then there is a commutative diagram

$$\begin{tikzcd} \mathrm{TC}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R}\otimes \mathbf{F}_p) \arrow[d] \\ \mathrm{HC}^-(\mathrm{R}\otimes \mathbf{Z}_p/\mathbf{Z}_p) \arrow[r] & \mathrm{TP}(\mathrm{R}\otimes \mathbf{F}_p), \end{tikzcd}$$

which is Cartesian upon rationalization. Here, the tensor products are taken over the sphere spectrum. (This is slightly different from the statement one finds in [AMMN].) In the same way, we show:

Corollary 6.1.8. *Let* R *be a connective* $\mathbf{E}_1$*-ring, and let* $\mathrm{F}(-)$ *denote either* p*-complete* TC *or algebraic K-theory. Then there is a Cartesian square*

$$\begin{tikzcd} \mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\pi_0\mathrm{R}) \arrow[d] \\ \mathrm{TC}^-(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}) \arrow[r] & \mathrm{TP}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}). \end{tikzcd}$$

In particular, the fiber of the top horizontal map is $\Sigma\mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})_{h\mathrm{S}^1}$*. When* F *is algebraic K-theory, the term* $\mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\pi_0\mathrm{R})$ *can be replaced by* $\mathrm{L}_{\mathrm{K}(1)}\mathrm{F}(\pi_0\mathrm{R}[1/p])$.

Remark 6.1.9. The preceding result is closely related to a "quantitative" version of the statement of purity of K(1)-local algebraic K-theory as proved in [LMMT] (for $p > 2$): indeed, if R is a p-complete connective $\mathbf{E}_1$-ring, there are equivalences

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{R})) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{R})[1/p]) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_{\mathrm{K}(0)}\mathrm{R});$$

the final equivalence comes from the fact that K(1)-local algebraic K-theory is truncating on $\mathbf{E}_1$-$\mathbf{Z}$-algebras. The Cartesian square of Corollary 6.1.8 can therefore be rewritten as a fiber sequence

$$\Sigma\mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})_{h\mathrm{S}^1} \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_{\mathrm{K}(0)}\mathrm{R}).$$

It follows that $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R})$ has a filtration whose graded pieces can be computed only using $\mathrm{L}_{\mathrm{K}(0)}\mathrm{R}$ and $\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}$; this is closely related to the purity theorem of [LMMT], which asserts that the map $\mathrm{R} \to \mathrm{L}_1\mathrm{R}$ induces an equivalence $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}) \xrightarrow{\sim} \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_1\mathrm{R})$.

Note that if we use this identification, the preceding fiber sequence becomes

$$\Sigma\mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})_{h\mathrm{S}^1} \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_1\mathrm{R}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{L}_{\mathrm{K}(0)}\mathrm{R}),$$

which is our desired "quantitative" version of purity for K(1)-local algebraic K-theory. It may be the case that a statement of this form holds at all chromatic heights and at all primes: namely, that if $n \geq 1$ and R is a p-complete connective $\mathbf{E}_1$-ring, there is a Cartesian square

$$\begin{tikzcd}
\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathrm{R}) \simeq \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathrm{L}_{\mathrm{T}(n-1)\oplus\mathrm{T}(n)}\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathrm{L}_{\mathrm{T}(n-1)}\mathrm{R}) \arrow[d] \\
\mathrm{TC}^{-}(\mathrm{L}_{\mathrm{T}(n)}\mathrm{R}) \arrow[r] & \mathrm{TP}(\mathrm{L}_{\mathrm{T}(n)}\mathrm{R})
\end{tikzcd}$$

for some mysterious map $\mathrm{L}_{\mathrm{T}(n)}\mathrm{K}(\mathrm{L}_{\mathrm{T}(n-1)}\mathrm{R}) \to \mathrm{TP}(\mathrm{L}_{\mathrm{T}(n)}\mathrm{R})$.

Corollary 6.1.8 gives a very explicit calculation of the K(1)-local algebraic K-theory of some chromatically interesting ring spectra. For instance, we will show:

Example 6.1.10. The maps $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{S}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(j_p) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{J}_p)$ are all equivalences. Moreover, if R is any connective complex oriented $\mathbf{E}_1$-ring such that $\pi_0(\mathrm{R}) \cong \mathbf{Z}_p$, then there is an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})_{h\mathrm{S}^1}.$$

In particular, there is an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{MU}_p) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{L}_{\mathrm{K}(1)}\mathrm{MU}[\mathrm{SU}]_{h\mathrm{S}^1}.$$

Furthermore, there is an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{KU}_p) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p) \oplus \Sigma\mathrm{KU}_p[\mathbf{C}\mathrm{P}^\infty].$$

We will not explain these examples here, because their deduction from Corollary 6.1.8 relies on the details of our proof of Corollary 6.1.7, which I will defer to my forthcoming joint paper with A. Raksit.

There is an interesting pattern appearing in these examples: if $\mathrm{R} = \mathrm{S}, \mathrm{KU}_p, \mathbf{Q}_p$, then there is an equivalence

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}) \simeq \mathrm{A} \oplus \Sigma\mathrm{A} \oplus \Sigma\mathrm{B},$$

where $\mathrm{A} = \mathrm{J}_p, \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathbf{Z}_p), \mathrm{J}_p$ (respectively) and $\mathrm{B} = \mathrm{L}_{\mathrm{K}(1)}\,\mathrm{fib}(\overline{\mathrm{tr}}), \mathrm{KU}_p[\mathbf{C}\mathrm{P}^\infty], \mathrm{KU}_p$ (respectively). I do not have a non-calculational explanation for why such decompositions hold, but it is likely related to some putative version of higher chromatic variants of Tate duality along the lines of Rognes [Rog3] (at least for $\mathrm{R} = \mathrm{KU}_p, \mathbf{Q}_p$).

6.2 Calculating $\mathrm{THH}(\mathbf{Z}_p)$

In order to prove Theorem 6.1.4, we first need to calculate $\pi_*(j_p^{t\mathbf{Z}/p})$. We first need the following simple calculation:

Lemma 6.2.1. *There are isomorphisms*

$$\begin{aligned}
\pi_*(\mathrm{ku}_p^{t\mathrm{S}^1}) &\cong \mathbf{Z}_p[\![q-1]\!][\beta, \hbar]/(\beta\hbar = q-1), \\
\pi_*(\mathrm{ku}_p^{t\mathrm{S}^1}) &\cong \mathbf{Z}_p[\![q-1]\!][\hbar^{\pm 1}], \\
\pi_*(\mathrm{ku}_p^{t\mathbf{Z}/p}) &\cong \mathbf{Z}_p[\zeta_p][\hbar^{\pm 1}], \\
\pi_*(\ell_p^{t\mathbf{Z}/p}) &\cong \mathbf{Z}_p[\hbar^{\pm 1}],
\end{aligned}$$

where β lives in weight 2 and $\hbar$ lives in weight -2. Furthermore, the action of ψ^{1+p} on $\pi_{2n}(\ell_p^{t\mathbf{Z}/p})$ and on $\pi_{2n}(\mathrm{ku}_p^{t\mathbf{Z}/p})$ is given by multiplication by $(p+1)^n$.

Proof. The displayed isomorphisms are all straightforward; they all follow from the first claim, which is implied by the isomorphism $\pi_*(\mathrm{ku}_p) \cong \mathbf{Z}_p[\beta]$. For the claim about the action of ψ^{1+p}, it suffices to study the case of $\pi_{2n}(\mathrm{ku}_p^{t\mathbf{Z}/p})$. It is clear that $\psi^{1+p}(\zeta_p) = \zeta_p$, because the Adams operation ψ^{1+p} sends $q \mapsto q^p$. To prove the claim, it therefore suffices to show that $\psi^{1+p}(\hbar) = (p+1)^{-1}\hbar$. This, however, is forced on us because $\beta\hbar = \zeta_p - 1$ and ψ^{1+p} acts on β by multiplication by $p+1$. □

Proposition 6.2.2. *There are isomorphisms*

$$\pi_n(j_p^{t\mathbf{Z}/p}) \cong \begin{cases} \mathbf{Z}_p & n = 0, \\ \mathbf{Z}_p/k & n = 2k-1 \text{ for } k \in \mathbf{Z}, \\ 0 & \text{else.} \end{cases}$$

Proof. Let $\Gamma_0 = \mathbf{Z} \subseteq \mathbf{Z}_p^\times$ denote the subgroup generated by $1+p$. Then there is a Cartesian square

$$\begin{tikzcd} j_p \arrow[r] \arrow[d] & \ell_p^{h\Gamma_0} \arrow[d] \\ \mathbf{Z}_p \arrow[r] & \mathbf{Z}_p^{h\Gamma_0}, \end{tikzcd}$$

which gives a Cartesian square

$$\begin{tikzcd} j_p^{t\mathbf{Z}/p} \arrow[r] \arrow[d] & (\ell_p^{t\mathbf{Z}/p})^{h\Gamma_0} \arrow[d] \\ \mathbf{Z}_p^{t\mathbf{Z}/p} \arrow[r] & (\mathbf{Z}_p^{t\mathbf{Z}/p})^{h\Gamma_0}, \end{tikzcd} \tag{6.2.1}$$

since taking Γ_0-homotopy fixed points is a finite limit. It follows from Lemma 6.2.1 that

$$\pi_n((\ell_p^{t\mathbf{Z}/p})^{h\Gamma_0}) \cong \begin{cases} \mathbf{Z}_p & n = 0, -1, \\ \mathbf{Z}_p/pk & n = 2k-1, \\ 0 & \text{else.} \end{cases}$$

Similarly, $\pi_n((\mathbf{Z}_p^{t\mathbf{Z}/p})^{h\Gamma_0})$ is isomorphic to $\mathbf{F}_p$ in every degree. Running the long exact sequence in homotopy groups for the square (6.2.1) leads to the claimed calculation of $\pi_*(j_p^{t\mathbf{Z}/p})$. □

Proof of the first part of Theorem 6.1.4. We first construct an $\mathbf{E}_\infty$-map $j_p \to \mathrm{THH}(\mathbf{Z}_p)$. This is equivalent to constructing an $\mathbf{E}_\infty$-ring map $j_p \to \mathrm{TC}(\mathbf{Z}_p)$. For this, observe that there is a canonical map $\mathrm{J}_p \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p)$ since J_p is the K(1)-local sphere. Since the map $\mathrm{TC}(\mathbf{Z}_p) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p)$ is an equivalence in degrees ≥ 2 by the Lichtenbaum-Quillen conjecture, and the map $\mathrm{S} \to j_p$ is an equivalence in degrees ≤ 1, it follows that there is a unique way to lift the composite $j_p \to \mathrm{J}_p \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p)$ to an $\mathbf{E}_\infty$-map $j_p \to \mathrm{TC}(\mathbf{Z}_p)$.

The map $j_p \to \mathrm{THH}(\mathbf{Z}_p)$ induces a map $j_p^{(-1)} \to \mathrm{THH}(\mathbf{Z}_p)^{(-1)}$. The canonical map $\mathrm{THH}(\mathbf{Z}_p) \to \mathrm{THH}(\mathbf{Z}_p)^{(-1)}$ is an equivalence by the "Segal conjecture", so it suffices to show that the map $j_p \to \mathrm{THH}(\mathbf{Z}_p)$ induces an equivalence on $\mathbf{Z}/p$-Tate constructions. It follows from Proposition 6.2.2 that $\pi_n(j_p^{t\mathbf{Z}/p}/p)$ is isomorphic to $\mathbf{Z}/p$ if $n = 2kp$ or $2nkp - 1$ with $k \in \mathbf{Z}$, and that the map $j_p^{t\mathbf{Z}/p}/p \to \mathbf{Z}_p^{t\mathbf{Z}/p}$ is an equivalence on even homotopy groups. The same statements are true with j_p replaced by $\mathrm{THH}(\mathbf{Z}_p)$ and the map $j_p^{t\mathbf{Z}/p}/p \to \mathbf{Z}_p^{t\mathbf{Z}/p}$ replaced by the map $\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}/p \to \mathrm{THH}(\mathbf{F}_p)^{t\mathbf{Z}/p}$. There is a commutative diagram

$$\begin{tikzcd} j_p \arrow[r] \arrow[d] & \mathrm{THH}(\mathbf{Z}_p) \arrow[d] \\ \mathbf{Z}_p \arrow[r] & \mathrm{THH}(\mathbf{F}_p), \end{tikzcd}$$

and the bottom horizontal map induces an equivalence on $\mathbf{Z}/p$-Tate constructions. This implies that the map $j_p^{t\mathbf{Z}/p}/p \to \mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}/p$ induces an equivalence on even homotopy groups, and hence (by the mod p Bockstein) an isomorphism on π_1. Both $\pi_*(j_p^{t\mathbf{Z}/p}/p)$ and $\pi_*(\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}/p)$ are naturally graded rings (for $j_p^{t\mathbf{Z}/p}/p$ because p is odd, and for $\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}/p$ because $\mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}$ is a $\mathbf{Z}$-algebra), which are isomorphic to an exterior algebra on a class in degree 1 and a Laurent polynomial algebra on a generator in degree 2. It follows from the ring structure that the map $j_p^{t\mathbf{Z}/p}/p \to \mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}/p$ induces an isomorphism on homotopy groups, and hence is an equivalence, as desired. □

To prove the second part of Theorem 6.1.4, we need to analyze the map $j_p \to j_p^{(-1)}$ further. Again, our handle on this map will come from analyzing the map $\mathrm{ku}_p \to \mathrm{ku}_p^{(-1)}$. The basic tool in proving S^1-nilpotence is the following.

Lemma 6.2.3. *Let* R *be a complex oriented* $\mathbf{E}_\infty$*-ring, with Euler class* $\hbar \in \pi_{-2}(\mathrm{R}^{h\mathrm{S}^1})$*. Then an* S^1*-equivariant* R*-module* M *is* S^1*-nilpotent if and only if* $\hbar$ *acts nilpotently on* $\mathrm{M}^{h\mathrm{S}^1}$*.*

We will need:

Lemma 6.2.4. *There are isomorphisms*

$$\pi_*(\mathrm{ku}_p^{(-1)})^{h\mathrm{S}^1} \cong \mathbf{Z}_p[\![q^{1/p}-1]\!][\hbar^{-1}, t]/(t\hbar^{-1} = [p]_{q^{1/p}}),$$
$$\pi_*(\mathrm{ku}_p^{(-1)})^{t\mathrm{S}^1} \cong \mathbf{Z}_p[\![q^{1/p}-1]\!][t^{\pm 1}],$$

where $\hbar^{-1}$ *lives in weight* 2 *and* t *lives in weight* -2*. Furthermore, the map* $\pi_*(\mathrm{ku}_p^{h\mathrm{S}^1}) \to \pi_*(\mathrm{ku}_p^{(-1)})^{h\mathrm{S}^1}$ *is given by the map*

$$\mathbf{Z}_p[\![q-1]\!][\beta, \hbar]/(\beta\hbar = q-1) \to \mathbf{Z}_p[\![q^{1/p}-1]\!][\hbar^{-1}, t]/(t\hbar^{-1} = [p]_{q^{1/p}})$$

which sends $q \mapsto (q^{1/p})^p$*,* $\beta \mapsto (q^{1/p}-1)\hbar^{-1}$*, and* $\hbar \mapsto t$*.*

Using this, one finds:

Lemma 6.2.5. *The map*

$$\tau_{\geq 2}(\mathrm{ku}_p) \otimes_{\mathrm{ku}_p} \mathrm{ku}_p^{(-1)} \to \tau_{\geq 2}((\tau_{\geq 2}\mathrm{ku}_p)^{t\mathbf{Z}/p})$$

is an equivalence.

Proof. The map $\pi_*(\mathrm{ku}_p^{t\mathbf{Z}/p}) \to \pi_*(\mathbf{Z}_p^{t\mathbf{Z}/p})$ is given by the map $\mathbf{Z}_p[\zeta_p][\hbar^{\pm 1}] \to \mathbf{F}_p[\hbar^{\pm 1}]$ which kills $\zeta_p - 1$. It follows that $\pi_*((\tau_{\geq 2}\mathrm{ku}_p)^{t\mathbf{Z}/p}) \cong (\zeta_p - 1)\mathbf{Z}_p[\zeta_p][\hbar^{\pm 1}]$. This easily implies the claimed isomorphism. □

Construction 6.2.6. The work of Lurie [Lur7, Section 6.5] gives an equivalence $\mathrm{KU}_p^{h\mathrm{S}^1} \cong \mathrm{KU}_p[\![q-1]\!]$ of $\mathbf{E}_\infty$-KU_p-algebras, and hence a map $\mathrm{S}[\![q-1]\!] \to \mathrm{KU}_p^{h\mathrm{S}^1}$. (The equivalence $\mathrm{KU}_p^{h\mathrm{S}^1} \cong \mathrm{KU}_p[\![q-1]\!]$ could also be constructed by taking cobar constructions of the equivalence $\mathrm{KU}_p^{\mathrm{S}^1} \simeq \mathrm{KU}_p[\mathrm{S}^1]$ of $\mathbf{E}_\infty$-KU_p-algebras which arises from the Bott class being a *strict* unit.) This, in turn, can be regarded as an S^1-equivariant $\mathbf{E}_\infty$-map $\mathrm{S}[\![q-1]\!] \to \mathrm{KU}_p$. Such a map necessarily factors through the connective cover to give an S^1-equivariant $\mathbf{E}_\infty$-map $\mathrm{S}[\![q-1]\!] \to \mathrm{ku}_p$, which gives an $\mathbf{E}_\infty$-map $\mathrm{S}[\![q-1]\!] \to \mathrm{ku}_p^{h\mathrm{S}^1}$ hitting $q-1$ on homotopy groups. It is not hard to see that this map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{S}[\![q-1]\!] \arrow[r] \arrow[d] & \mathrm{ku}_p \arrow[d] \\
\mathrm{S}[\![q^{1/p}-1]\!] \arrow[r] & \mathrm{ku}_p^{h\mathbf{Z}/p},
\end{tikzcd}$$

so that the canonical map $\mathrm{ku}_p \to \tau_{\geq 0}(\mathrm{ku}_p^{t\mathbf{Z}/p})$ upgrades to an $\mathrm{S}[\![q^{1/p}-1]\!]$-linear map $\phi'_{\mathrm{ku}} : \mathrm{S}[\![q^{1/p}-1]\!] \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{ku}_p \to \mathrm{ku}_p^{(-1)}$.

Proposition 6.2.7. *Let* C *denote the cofiber of the map* $\phi'_{\mathrm{ku}} : \mathrm{S}[\![q^{1/p}-1]\!] \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{ku}_p \to \mathrm{ku}_p^{(-1)}$. *Then* $\mathrm{C}/(p,\beta)$ *is* S^1*-nilpotent.*

Proof. It follows from Lemma 6.2.4 that the map $\phi^{h\mathrm{S}^1}$ is given on homotopy by the map

$$\mathbf{Z}_p[\![q^{1/p}-1]\!][\beta,\hbar]/(\beta\hbar = (q^{1/p})^p - 1) \to \mathbf{Z}_p[\![q^{1/p}-1]\!][\hbar^{-1},t]/(t\hbar^{-1} = [p]_{q^{1/p}})$$

sending $\beta \mapsto (q^{1/p}-1)\hbar^{-1}$ and $\hbar \mapsto t$. In particular, the map $\phi/(p,\beta)$ is given by

$$\mathbf{F}_p[\![q^{1/p}-1]\!][\hbar]/((q^{1/p})^p - 1) \to \mathbf{F}_p[\![q^{1/p}-1]\!][\hbar^{-1},t]/(t\hbar^{-1} = [p]_{q^{1/p}}, (q^{1/p}-1)\hbar^{-1}).$$

This implies that

$$\pi_*(\mathrm{C}^{h\mathrm{S}^1}/(p,\beta)) \cong \{\hbar^{-1},\hbar^{-2},\cdots\} \cdot \mathbf{F}_p[\![q^{1/p}-1]\!]/(q^{1/p}-1).$$

Notice that t acts by zero, because $\hbar^{-1}t \equiv (q^{1/p}-1)^{p-1} \pmod p$. This implies that $\mathrm{C}/(p,\beta)$ is S^1-nilpotent. $\square$

We now turn to proving the second part of Theorem 6.1.4. It will be technically convenient to perform our calculations with the following variant of j_p and then deduce the desired result. We thank Lurie for indicating the convenience of this object.

Notation 6.2.8. We define an $\mathbf{E}_\infty$-ring $j_{p,0} := \tau_{\geq 0}(\mathrm{KU}_p^{h\Gamma_0})$. Noting that the canonical map $\pi_2(j_{p,0}/p) \to \pi_2(\mathrm{KU}_p/p)$ is an isomorphism, we abusively write $\beta \in \pi_2(j_{p,0}/p)$ to denote the unique preimage under this map of the reduction of the Bott class β. The class β is classified by a map of $j_{p,0}$-modules $\Sigma^2 j_{p,0}/p \to j_{p,0}/p$, which we will abusively also denote by β.

Lemma 6.2.9. *Let* M *be a* $j_{p,0}$*-module. Then the spectrum* $\mathrm{M}/(p,\beta)$ *naturally admits an* $\mathbf{F}_p$*-module structure.*

Proof. A calculation shows that $j_{p,0}/(p,\beta)$ is 1-truncated, so that the $j_{p,0}$-module structure on $j_{p,0}/(p,\beta)$ is canonically restricted from a $\tau_{\leq 1}(j_{p,0})$-module structure. It follows that

$$\mathrm{M}/(p,\beta) = \mathrm{M} \otimes_{j_{p,0}} j_{p,0}/(p,\beta) \cong (\mathrm{M} \otimes_{j_{p,0}} \tau_{\leq 1}(j_{p,0})) \otimes_{\tau_{\leq 1}(j_{p,0})} j_{p,0}/(p,\beta)$$

naturally admits a $\tau_{\le 1}(j_{p,0})$-module structure. Since p is odd, $\tau_{\le 1}(j_{p,0})^{h\mathbf{F}_p^\times} \cong \mathbf{Z}_p$, so that $\mathrm{M}/(p,\beta)$ naturally admits a $\mathbf{Z}_p$-module structure. Again using that p is odd, we know that the action of p on M/p, and hence on $\mathrm{M}/(p,\beta)$, is naturally zero, so that the $\mathbf{Z}_p$-module structure naturally factors through an $\mathbf{F}_p$-module structure as desired. □

In the following discussion, the action of an element $u \in \mathbf{Z}_p^\times$ on ku_p will be denoted by ψ^u; similarly, its action on $\mathrm{S}[\![q-1]\!]$ and $\mathrm{S}[\![q'-1]\!]$ will be denoted by Ψ^u.

Construction 6.2.10. If X is a connective spectrum with trivial S^1-action, let us write ϕ^0_{X} to denote the canonical map $\mathrm{X} \to \mathrm{X}^{(-1)}$. We construct a commutative diagram of S^1-equivariant $j_{p,0}$-modules

$$\begin{tikzcd}
j_{p,0} \arrow[rr, "\phi^0_{j_{p,0}}"] \arrow[d] & & j_{p,0}^{(-1)} \arrow[d] \\
\mathrm{ku}_p \arrow[r] \arrow[d, "\psi^{1+p}_\circ"] & \mathrm{ku}_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!] \arrow[r, "\phi'_{\mathrm{ku}_p}"] \arrow[d, "(\psi^{1+p}\otimes\Psi^{1+p})_\circ"] & \mathrm{ku}_p^{(-1)} \arrow[d, "\psi^{1+p}_{\circ\circ}"] \\
\tau_{\ge 2}(\mathrm{ku}_p) \arrow[r] & \tau_{\ge 2}(\mathrm{ku}_p) \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!] \arrow[r, "\phi'_{\tau_{\ge 2}(\mathrm{ku}_p)}"] & \tau_{\ge 2}(\tau_{\ge 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p})
\end{tikzcd}$$

in which the left and right columns are fiber sequences. Here, the notation is as follows.

The map $\psi^{1+p}_\circ$ is the unique map lifting the endomorphism $\psi^{1+p} - \mathrm{id}$ of ku_p along the canonical map $\tau_{\ge 2}(\mathrm{ku}_p) \to \mathrm{ku}_p$. Both existence and uniqueness follow from the fiber sequence $\tau_{\ge 2}(\mathrm{ku}_p) \to \mathrm{ku}_p \to \mathbf{Z}_p$, and the fact that ψ^{1+p} induces the identity map on $\pi_0(\mathrm{ku}_p) \cong \mathbf{Z}_p$. The maps $(\psi^{1+p} \otimes \Psi^{1+p})_\circ$ and $\psi^{1+p}_{\circ\circ}$ are defined similarly, the former arising from the endomorphism $\psi^{1+p} \otimes \Psi^{1+p} - \mathrm{id}$ of $\mathrm{ku}_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!]$ and the latter arising from the map $\tau_{\ge 0}(\mathrm{ku}_p^{t\mathbf{Z}/p}) \to \tau_{\ge 0}(\tau_{\ge 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p})$ induced by $\psi^{1+p}_\circ$.

The map $\phi'_{\mathrm{ku}_p^{\mathrm{triv}}}$ is as defined in Construction 6.2.6. The map $\phi'_{\tau_{\ge 2}(\mathrm{ku}_p)}$ is defined similarly using the map $\phi^2_{\tau_{\ge 2}(\mathrm{ku}_p)} : \tau_{\ge 2}(\mathrm{ku}_p) \to \tau_{\ge 2}(\tau_{\ge 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p})$. Note that Lemma 6.2.5 implies that $\phi'_{\tau_{\ge 2}(\mathrm{ku}_p)}$ is just the two-fold suspension of ϕ'_{ku_p}.

Lemma 6.2.11. *Let* K *denote the total cofiber of the commutative square of* S^1*-equivariant* $j_{p,0}$*-modules*

$$\begin{tikzcd}
\mathrm{ku}_p \arrow[r] \arrow[d, "\psi^{1+p}_\circ"] & \mathrm{ku}_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!] \arrow[d, "(\psi^{1+p}\otimes\Psi^{1+p})_\circ"] \\
\tau_{\ge 2}(\mathrm{ku}_p) \arrow[r] & \tau_{\ge 2}(\mathrm{ku}_p) \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!]
\end{tikzcd} \tag{6.2.2}$$

contained in the commutative diagram of Construction 6.2.10. Then $\mathrm{K}/(p,\beta)$ *is* S^1*-nilpotent.*

Proof. Lemma 6.2.9 says that $\mathrm{K}/(p,\beta)$ admits an S^1-equivariant $\mathbf{F}_p$-module structure, so Lemma 6.2.3 applies. Under the isomorphism

$$\pi_*(\mathrm{ku}_p^{h\mathrm{S}^1}) \cong \mathbf{Z}_p[\![q-1]\!][\beta,\hbar]/(\beta\hbar - (q-1)),$$

the map

$$\pi_*(\tau_{\ge 2}(\mathrm{ku}_p)^{h\mathrm{S}^1}) \to \pi_*(\mathrm{ku}_p^{h\mathrm{S}^1})$$

is injective. If $n \leq 0$, its image on π_{2n} is $(q-1)\hbar^n \cdot \mathbf{Z}_p[\![q-1]\!]$, while its image on π_{2n} for $n > 0$ is $\beta^n \cdot \mathbf{Z}_p[\![q-1]\!]$. Similar statements hold for the map

$$\pi_*((\tau_{\geq 2}(\mathrm{ku}_p) \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q^{1/p}-1]\!])^{h\mathrm{S}^1}) \to \pi_*((\mathrm{ku}_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q'-1]\!])^{h\mathrm{S}^1}) :$$

one just has to tensor up along $\mathbf{Z}_p[\![q-1]\!] \to \mathbf{Z}_p[\![q^{1/p}-1]\!]$.

In terms of these identifications, applying $\pi_*(-)^{h\mathrm{S}^1}$ to (6.2.2) and reducing modulo (p, β) produces in the commutative square of $\mathbf{F}_p$-modules:

$$\begin{tikzcd}
\mathbf{F}_p[\hbar] \arrow[r] \arrow[d, "\psi_\circ^{1+p}"] & \mathbf{F}_p[\![q^{1/p}-1]\!][\hbar]/(q-1) \arrow[d, "(\psi^{1+p}\otimes\Psi^{1+p})_\circ"] \\
\beta \cdot \mathbf{F}_p \oplus \left(\{\hbar, \hbar^2, \cdots\} \cdot \frac{(q-1)\mathbf{F}_p[\![q-1]\!]}{(q-1)^2}\right) \arrow[r] & \beta \cdot \mathbf{F}_p[\![q^{1/p}-1]\!]/(q-1) \oplus \left(\{\hbar, \hbar^2, \cdots\} \cdot \frac{(q-1)\mathbf{F}_p[\![q^{1/p}-1]\!]}{(q-1)^2}\right).
\end{tikzcd}$$

Let us compute the vertical maps. On $\pi_*(\mathrm{ku}_p^{h\mathrm{S}^1})$, we have

$$\begin{aligned}
\psi^{1+p}(\beta) &= (1+p)\beta \equiv \beta \pmod p, \\
\psi^{1+p}(q) &= q^{1+p}.
\end{aligned}$$

Since $\beta\hbar = q-1$, we find that

$$\psi^{1+p}(\hbar) \equiv \frac{q^{1+p}-1}{q-1}\hbar \equiv \left(\frac{q^{1+p}-q}{q-1}+1\right)\hbar \equiv (q(q-1)^{p-1}+1)\hbar \equiv \hbar \pmod{p, (q-1)^2}.$$

(Note that $p-1 \geq 2$ because p is odd.) Because $\Psi^{1+p}(q') = (q')^{1+p} = q'q$, we find

$$\begin{aligned}
(\psi^{1+p} \otimes \Psi^{1+p})_\circ(\hbar^n (q')^m) &= \psi^{1+p}(\hbar^n)\Psi^{1+p}((q')^m) - \hbar^n (q')^m \\
&\equiv \hbar^n (q')^m q^m - \hbar^n (q')^m \pmod{p, (q-1)^2} \\
&\equiv \hbar^n (q')^m (q^m - 1) \pmod{p, (q-1)^2}.
\end{aligned}$$

Note that $q^m - 1$ is nonzero in $\frac{(q-1)\mathbf{F}_p[\![q^{1/p}-1]\!]}{(q-1)^2}$ for $1 \leq m \leq p-1$. We may now compute the total cofiber of the above square of $\mathbf{F}_p$-modules by taking horizontal cofibers first and then the vertical cofiber. Doing so results in the isomorphism

$$\pi_*(\mathrm{K}^{h\mathrm{S}^1}/(p,\beta)) \cong \{\beta q', \cdots, \beta(q')^{p-1}\} \cdot \mathbf{F}_p,$$

concentrated in π_2. Clearly, $\hbar$ must act by zero. ☐

Proposition 6.2.12. *Let $\mathrm{K}_{j_{p,0}}$ denote the cofiber of the map $j_{p,0} \to j_{p,0}^{(-1)}$. Then $\mathrm{K}_{j_{p,0}}/(p,\beta)$ is S^1-nilpotent.*

Proof. Using the commutative diagram of Construction 6.2.10, the claim follows from combining Proposition 6.2.7 and Lemma 6.2.11. ☐

Proof of the second part of Theorem 6.1.4. Recall that the class v_1 in $\pi_{2(p-1)}(j_{p,0})$ is given by β^{p-1}. Proposition 6.2.12 then implies that $\mathrm{K}_{j_{p,0}}/(p, v_1)$ is S^1-nilpotent. If K_{j_p} denotes the cofiber of the map $j_p \to j_p^{(-1)}$, then the desired S^1-nilpotence of K_{j_p} follows from this and the fact that $\mathrm{K}_{j_p} \cong \mathrm{K}_{j_{p,0}}^{h\mathbf{F}_p^\times}$ is a retract of $\mathrm{K}_{j_{p,0}}$. (Here, we have used that $p-1 = |\mathbf{F}_p^\times|$ is a p-adic unit, so that it acts invertibly on $j_{p,0}$, $j_{p,0}^{(-1)}$, and $\mathrm{K}_{j_{p,0}}$.) ☐

6.3 Applications of the main theorem

In this section, we explain some of the applications of Theorem 6.1.4.

6.3.1 Noncommutative crystalline-de Rham comparison

We begin by proving Corollary 6.1.6.

Definition 6.3.1. Let R be an S^1-equivariant $\mathbf{E}_\infty$-ring. Let $\mathcal{I}_{\mathrm{R}} \subseteq \mathrm{Mod}_{\mathrm{R}}(\mathrm{Sp}^{\mathrm{BS}^1})$ denote the full subcategory spanned by those S^1-equivariant R-modules M such that $\mathrm{M}/(p, v_1)$ is nilpotent. Define

$$(\mathrm{Mod}_{\mathrm{R}}^{t\mathrm{S}^1})^\wedge_{(p,v_1)} := \mathrm{Mod}_{\mathrm{R}}(\mathrm{Sp}^{\mathrm{BS}^1})/\mathcal{I}_{\mathrm{R}}$$

to be the Verdier quotient. Since $\mathcal{I}_{\mathrm{R}}$ is a thick tensor ideal, both $(\mathrm{Mod}_{\mathrm{R}}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}$ and the associated quotient functor $\mathrm{Mod}_{\mathrm{R}}(\mathrm{Sp}^{\mathrm{BS}^1}) \to (\mathrm{Mod}_{\mathrm{R}}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}$ carry canonical symmetric monoidal structures. Essentially by construction, the lax symmetric monoidal functor

$$((-)^{t\mathrm{S}^1})^\wedge_{(p,v_1)} : \mathrm{Mod}_{\mathrm{R}}(\mathrm{Sp}^{\mathrm{BS}^1}) \to \mathrm{Sp}$$

factors uniquely through the functor $\mathrm{Mod}_{\mathrm{R}}(\mathrm{Sp}^{\mathrm{BS}^1}) \to (\mathrm{Mod}_{\mathrm{R}}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}$.

A map $\mathrm{R} \to \mathrm{R}'$ of S^1-equivariant $\mathbf{E}_\infty$-rings induces an adjunction

$$(\mathrm{Mod}_{\mathrm{R}}^{t\mathrm{S}^1})^\wedge_{(p,v_1)} \rightleftarrows (\mathrm{Mod}_{\mathrm{R}'}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}. \tag{6.3.1}$$

Lemma 6.3.2. *Let $f : \mathrm{R} \to \mathrm{R}'$ be a map of S^1-equivariant $\mathbf{E}_\infty$-rings. Suppose that $\mathrm{fib}(f) \in \mathcal{I}_{\mathrm{R}}$. Then the adjunction (6.3.1) is an equivalence.*

The following result is immediate from Theorem 6.1.2 and Theorem 6.1.4.

Proposition 6.3.3. *Consider the diagram of S^1-equivariant $\mathbf{E}_\infty$-j_p-algebras*

$$\begin{tikzcd} j_p \arrow[r, "\alpha"] \arrow[d] & \mathrm{THH}(\mathbf{Z}_p) \arrow[d] \arrow[dl] \\ \mathbf{Z}_p \arrow[r, "\overline{\alpha}"] & \mathrm{THH}(\mathbf{F}_p), \end{tikzcd}$$

where the left hand vertical map and diagonal map are the truncation maps and the right hand vertical map is induced by the reduction map $\mathbf{Z}_p \to \mathbf{F}_p$. After applying the quotient functor

$$\mathrm{Mod}_{j_p}^{\mathrm{BS}^1} \to (\mathrm{Mod}_{j_p}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}$$

the maps α and $\overline{\alpha}$ become equivalences, and the preceding diagram commutes.

Corollary 6.1.6 now follows from:

Corollary 6.3.4. *Let M be an S^1-equivariant $\mathrm{THH}(\mathbf{Z}_p)$-module. Then there is a natural p-complete equivalence*

$$(\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathbf{Z}_p)^{t\mathrm{S}^1} \cong (\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathrm{THH}(\mathbf{F}_p))^{t\mathrm{S}^1},$$

which is lax symmetric monoidal. When $\mathrm{M} = \mathrm{THH}(\mathbf{Z}_p)$, this map recovers the equivalence $\mathbf{Z}_p^{t\mathrm{S}^1} \cong \mathrm{TP}(\mathbf{F}_p)$ induced by the map $\overline{\alpha}$.

Proof. It follows from Lemma 6.3.2 and Proposition 6.3.3 that $\overline{\alpha}$ defines an equivalence between the images of $\mathbf{Z}_p$ and $\mathrm{THH}(\mathbf{F}_p)$ in $\mathrm{CAlg}((\mathrm{Mod}^{t\mathrm{S}^1}_{\mathrm{THH}(\mathbf{Z}_p)})^{\wedge}_{(p,v_1)})$. The claim then follows. $\square$

Remark 6.3.5. If $\mathcal{C}$ is a stable ∞-category and R is an $\mathbf{E}_\infty$-ring, let $\mathcal{C}_{\mathrm{R}}$ denote the base-change $\mathcal{C} \otimes \mathrm{R}$. Then a similar argument to the one given above shows that there is a natural lax symmetric monoidal equivalence

$$\mathrm{HH}(\mathcal{C}_{j_p}/j_p) \simeq \mathrm{THH}(\mathcal{C}) \otimes j_p \simeq \mathrm{THH}(\mathcal{C}) \otimes \mathrm{THH}(\mathbf{Z}_p) \simeq \mathrm{THH}(\mathcal{C}_{\mathbf{Z}_p})$$

in $(\mathrm{Sp}^{t\mathrm{S}^1})^{\wedge}_{(p,v_1)}$. In particular, there is a natural lax symmetric monoidal equivalence

$$\mathrm{HP}(\mathcal{C}_{j_p}/j_p) \simeq \mathrm{TP}(\mathcal{C}_{\mathbf{Z}_p}).$$

It would be interesting to know under what conditions this equivalence could hold when $\mathcal{C}_{j_p}$ is replaced by an arbitrary j_p-linear ∞-category $\mathcal{D}$ and $\mathcal{C}_{\mathbf{Z}_p}$ is replaced by its base-change $\mathcal{D} \otimes_{j_p} \mathbf{Z}_p$; this would be a higher chromatic analogue of Corollary 6.1.6.

6.3.2 A $\mathrm{K}(1)$-local Beilinson fiber square

We first formulate a result for the topological cyclic homology of general bounded below cyclotomic spectra. As above, let $\alpha : j_p \to \mathrm{THH}(\mathbf{Z}_p)$ denote the map of Theorem 6.1.4. The S^1-nilpotence of Theorem 6.1.4 says that if M is a bounded below S^1-equivariant spectrum, then $(\mathrm{M} \otimes \mathrm{fib}(\alpha))^{t\mathrm{S}^1} \cong 0$.

Theorem 6.3.6. *For* M *a bounded below cylclotomic spectrum, there is a natural map of spectra* $\mathrm{TC}(\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p)) \to (\mathrm{M} \otimes j_p)^{t\mathrm{S}^1}$ *making the following square commute*

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{M} \otimes j_p) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p)) \arrow[d] \\
(\mathrm{M} \otimes j_p)^{h\mathrm{S}^1} \arrow[r] & (\mathrm{M} \otimes j_p)^{t\mathrm{S}^1}.
\end{tikzcd} \tag{6.3.2}$$

The upper horizontal map is induced by α*, and the left vertical and lower horizontal maps are the canonical ones. Moreover, upon* $\mathrm{K}(1)$*-localization, this square becomes Cartesian, and the map* $\mathrm{TC}(\mathrm{M}) \to \mathrm{TC}(\mathrm{M} \otimes j_p)$ *induced by the unit map* $\mathrm{S} \to j_p$ *becomes an equivalence.*

Proof. The map and commutative square are obtained immediately from the commutative diagram

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{M} \otimes j_p) \arrow[r, "\alpha"] \arrow[d] & \mathrm{TC}(\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p)) \arrow[d] \\
(\mathrm{M} \otimes j_p)^{h\mathrm{S}^1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{can}"'] & (\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p))^{h\mathrm{S}^1} \arrow[d, "\mathrm{can}"] \\
(\mathrm{M} \otimes j_p)^{t\mathrm{S}^1} \arrow[r, "\alpha"] & (\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p))^{t\mathrm{S}^1}
\end{tikzcd} \tag{6.3.3}$$

and the fact that the lowest horizontal map is an equivalence.

The claim that the diagram of the theorem is Cartesian follows from two observations about the diagram (6.3.3):

- The upper square is cartesian. This follows from considering the variant of (6.3.3) in which the can maps are replaced by $\mathrm{can} - \varphi$; there the columns are fiber sequences, and the lowest horizontal map remains an equivalence.
- The map $\mathrm{can} : (\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p))^{h\mathrm{S}^1} \to (\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p))^{t\mathrm{S}^1}$ is a K(1)-local equivalence. Indeed, the fiber identifies with a shift of $(\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p))_{h\mathrm{S}^1}$, which is a colimit of $\mathbf{Z}$-modules (so it vanishes K(1)-locally).

That the map $\mathrm{TC}(\mathrm{M}) \to \mathrm{TC}(\mathrm{M} \otimes j_p)$ is a K(1)-local equivalence follows from the facts that the canonical map $\mathrm{TC}(\mathrm{M}) \otimes j_p \to \mathrm{TC}(\mathrm{M} \otimes j_p)$ is an equivalence [AMMN, Remark 2.4] and the unit map $\mathrm{S} \to j_p$ is a K(1)-local equivalence. ☐

Corollary 6.3.7. *For* M *a bounded-below cyclotomic spectrum, there is a natural fiber sequence of spectra*

$$\Sigma \mathrm{L}_{\mathrm{K}(1)}(\mathrm{M}_{h\mathrm{S}^1}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{M}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{M} \otimes \mathrm{THH}(\mathbf{Z}_p)),$$

where the second map is induced by the unit of $\mathrm{THH}(\mathbf{Z}_p)$.

We now apply Theorem 6.3.6 to the study of K(1)-localized TC and K-theory of ring spectra. We begin with the following result for j_p-algebras, parallel to the result [AMMN, Theorem 2.12] for $\mathbf{Z}$-algebras.

Theorem 6.3.8. *For* R *a connective* $\mathbf{E}_1$-j_p*-algebra, there is a natural commutative square of spectra*

$$\begin{tikzcd} \mathrm{TC}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{TC}(\mathrm{R} \otimes_{\mathrm{S}} \mathbf{Z}) \arrow[d] \\ \mathrm{TC}^{-}(\mathrm{R}/j_p) \arrow[r] & \mathrm{TP}(\mathrm{R}/j_p) \end{tikzcd}$$

in which all maps except the right vertical one are the canonical ones. Again, this square becomes Cartesian upon K(1)*-localization.*

Proof. This follows from combining Theorem 6.3.6 (applied to $\mathrm{M} = \mathrm{THH}(\mathrm{R})$) with the commutative diagram

$$\begin{tikzcd} (\mathrm{THH}(\mathrm{R}) \otimes j_p)^{h\mathrm{S}^1} \arrow[r, "\mathrm{can}"] \arrow[d] & (\mathrm{THH}(\mathrm{R}) \otimes j_p)^{t\mathrm{S}^1} \arrow[d] \\ \mathrm{TC}^{-}(\mathrm{R}/j_p) \arrow[r, "\mathrm{can}"] & \mathrm{TP}(\mathrm{R}/j_p) \end{tikzcd}$$

induced by the canonical S^1-equivariant map $\mathrm{THH}(\mathrm{R}) \otimes j_p \to \mathrm{THH}(\mathrm{R}/j_p)$ and noting that this square becomes cartesian after K(1)-localization. The last claim follows from considering the map induced on horizontal fibers, as the aforementioned S^1-equivariant map is a K(1)-local equivalence, and hence so too is the map obtained from this by applying $(-)_{h\mathrm{S}^1}$. ☐

We now turn to Corollary 6.1.8. The proof will rely on the following result on the K(1)-local algebraic K-theory of $\mathbf{Z}$-algebras.

Theorem 6.3.9 (Bhatt–Clausen–Mathew [BCM, Theorem 1.1]; Land–Meier–Mathew–Tamme [LMMT, Corollary 4.23])**.** *For* A *a connective* $\mathbf{E}_1$-$\mathbf{Z}$*-algebra, the canonical maps*

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{A}) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{A})) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{A})[1/p])$$

are equivalences.

In fact, [LMMT, Corollary 4.23] is more general than Theorem 6.3.9: it is shown there that the statement holds for A any connective, K(1)-acyclic $\mathbf{E}_1$-ring.

Lemma 6.3.10. *For* M *an* S^1*-equivariant spectrum, there is a natural fiber square of spectra*

$$\begin{tikzcd} \mathrm{L}_{\mathrm{K}(1)}((\mathrm{M}\otimes j_p)^{h\mathrm{S}^1}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}((\mathrm{M}\otimes j_p)^{t\mathrm{S}^1}) \arrow[d] \\ (\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{h\mathrm{S}^1} \arrow[r] & (\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{t\mathrm{S}^1} \end{tikzcd}$$

in which the horizontal maps are the canonical ones.

Proof. The localization map $\mathrm{M}\otimes j_p \to \mathrm{L}_{\mathrm{K}(1)}(\mathrm{M}\otimes j_p) \cong \mathrm{L}_{\mathrm{K}(1)}\mathrm{M}$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{L}_{\mathrm{K}(1)}((\mathrm{M}\otimes j_p)^{h\mathrm{S}^1}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}((\mathrm{M}\otimes j_p)^{t\mathrm{S}^1}) \arrow[d] \\ \mathrm{L}_{\mathrm{K}(1)}((\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{h\mathrm{S}^1}) \arrow[r] & \mathrm{L}_{\mathrm{K}(1)}((\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{t\mathrm{S}^1}), \end{tikzcd}$$

which is Cartesian because the induced map on horizontal fibers is an equivalence. To finish the proof, note that $(\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{h\mathrm{S}^1}$ is already K(1)-local and that the canonical map $(\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{t\mathrm{S}^1} \to \mathrm{L}_{\mathrm{K}(1)}((\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{t\mathrm{S}^1})$ is an equivalence. The latter is true because the same statement holds when $(-)^{t\mathrm{S}^1}$ is replaced by $(-)^{h\mathrm{S}^1}$ (because $(\mathrm{L}_{\mathrm{K}(1)}\mathrm{M})^{h\mathrm{S}^1}$ is already K(1)-local) or by $(-)_{h\mathrm{S}^1}$ (by writing $\mathrm{L}_{\mathrm{K}(1)}(-) \cong (\mathrm{L}_1(-))^\wedge_p$ and using that L_1 is smashing). □

Proof of Corollary 6.1.8. The claim follows from considering the commutative diagram

$$\begin{tikzcd} \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{R}\otimes_{\mathrm{S}}\mathbf{Z}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{R})) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\pi_0(\mathrm{R})[1/p]) \\ \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{R}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{R}\otimes_{\mathrm{S}}\mathbf{Z}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\pi_0(\mathrm{R})) & \\ \mathrm{L}_{\mathrm{K}(1)}((\mathrm{THH}(\mathrm{R})\otimes j_p)^{h\mathrm{S}^1}) \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)}((\mathrm{THH}(\mathrm{R})\otimes j_p)^{t\mathrm{S}^1}) \arrow[d] & & \\ \mathrm{TC}^-(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}) \arrow[r] & \mathrm{TP}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R}) & & \end{tikzcd}$$

which can be described as follows:

- In the first two rows, the horizontal maps are induced by the canonical maps of $\mathbf{E}_1$-rings (noting that $\pi_0(\mathrm{R}\otimes_{\mathrm{S}}\mathbf{Z}) \cong \pi_0(\mathrm{R})$) and the vertical maps are given by the cyclotomic trace. By the Dundas–Goodwillie–McCarthy theorem, the two squares formed by these rows are cartesian. By Theorem 6.3.9, the second and third maps in the first row are equivalences, and by cartesianness it follows that the second map in the second row is also an equivalence.
- The square formed by the second and third rows is the fiber square of Theorem 6.3.6 (applied to $\mathrm{M} = \mathrm{THH}(\mathrm{R})$).
- The square formed by the third and fourth rows is the fiber square of Lemma 6.3.10 (applied to $\mathrm{M} = \mathrm{THH}(\mathrm{R})$), noting that we have a natural equivalence $\mathrm{L}_{\mathrm{K}(1)}\mathrm{THH}(\mathrm{R}) \cong \mathrm{THH}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{R})$ (by writing $\mathrm{L}_{\mathrm{K}(1)}(-) \cong (\mathrm{L}_1(-))^\wedge_p$ and using that L_1 is smashing). □

6.4 Application to q-de Rham cohomology

In this section, I record some mild extensions of Theorem 6.1.4; I am very grateful to J. Lurie for suggesting the first part of Theorem 6.4.1 and for permitting me to include it here. Since this is not part of my joint work with A. Raksit, any mistakes in the arguments below are only mine. Our goal is to prove:

Theorem 6.4.1. *Let $p > 2$, and view $\mathbf{Z}_p[\zeta_p]$ as an $\mathrm{S}[\![q^{1/p}-1]\!]$-algebra via the map $q^{1/p} \mapsto \zeta_p$.*

a. There is a $\mathbf{Z}_p^\times$-equivariant equivalence of cyclotomic $\mathbf{E}_\infty$-$\mathrm{S}[\![q-1]\!]$-algebras

$$\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!]).$$

Furthermore, these are equivalent to $\mathrm{ku}_p^{(-1)}$ as $\mathrm{S}^1 \times \mathbf{Z}_p^\times$-equivariant $\mathbf{E}_\infty$-$\mathrm{S}[\![q-1]\!]$-algebras. These equivalences fit into a commutative diagram of cyclotomic $\mathbf{E}_\infty$-rings

$$\begin{tikzcd}
j_{p,0} \arrow[r] \arrow[d] & j_{p,0}^{(-1)} \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \arrow[d] \\
\mathrm{ku}_p \arrow[r] \arrow[d] & \mathrm{ku}_p^{(-1)} \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!]) \arrow[d] \\
\mathbf{Z}_p \arrow[r] & \mathbf{Z}_p^{(-1)} \simeq \mathrm{THH}(\mathbf{F}_p)
\end{tikzcd}$$

extending that of Theorem 6.1.4.

b. The cofiber of the canonical map

$$\mathrm{ku}_p \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q^{1/p}-1]\!] \to \mathrm{ku}_p^{(-1)} \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!])$$

is S^1-nilpotent.

A. Raksit communicated to me that Nikolaus had previously proved an equivalence of S^1-equivariant $\mathbf{E}_1$-rings between $\mathrm{ku}_p^{(-1)}$ and $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!])$; see [MW, Theorem 3.18] for an argument. Let us describe some applications of Theorem 6.4.1 (two of which are relatively simple). First, arguing as in Corollary 6.1.6 shows:

Corollary 6.4.2. *If $\mathcal{C}$ is a dualizable stable ∞-category and R is an $\mathbf{E}_\infty$-algebra, let $\mathcal{C}_\mathrm{R}$ denote the base-change $\mathcal{C} \otimes \mathrm{R}$. Then there is a natural $\mathbf{Z}_p^\times$-equivariant lax symmetric monoidal equivalence*

$$\mathrm{HP}(\mathcal{C}_{\mathrm{ku}_p}/\mathrm{ku}_p) \otimes_{\mathrm{S}[\![q-1]\!]} \mathrm{S}[\![q^{1/p}-1]\!] \simeq \mathrm{TP}(\mathcal{C}_{\mathbf{Z}_p[\zeta_p]}/\mathrm{S}[\![q^{1/p}-1]\!]).$$

This statement admits a Frobenius untwist: in the above setting, there is a natural $\mathbf{Z}_p^\times$-equivariant lax symmetric monoidal transformation

$$\mathrm{TC}^-(\mathcal{C}_{\mathbf{Z}_p[\zeta_p]}/\mathrm{S}[\![q^{1/p}-1]\!])[\mu^{-1}] \to \mathrm{HP}(\mathcal{C}_{\mathrm{ku}_p}/\mathrm{ku}_p) \tag{6.4.1}$$

which exhibits the source as a completion of the target; here, μ is a generator of $\pi_2\mathrm{TC}^-(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!])$. In this way, one finds that the associated graded of the S^1-equivariant even filtration on $\mathrm{HH}(\mathrm{ku}_p[x]/\mathrm{ku}_p)^{t\mathbf{Z}/p}$ identifies with the q-de Rham complex of the affine line. (A. Raksit has observed that it is more generally true that if A is the connective cover of an even-periodic

$\mathbf{E}_\infty$-ring, then the associated graded of the S^1-equivariant even filtration on $\mathrm{HH}(\mathrm{A}[x]/\mathrm{A})^{t\mathbf{Z}/p}$ identifies with the F-de Rham complex $\mathrm{F}\Omega_{\mathbf{A}^1}$ of the affine line in the sense of Definition 7.4.3. When $\mathrm{A}[x]$ is replaced by $\mathrm{A}[x^{\pm 1}]$, this, of course, is Koszul dual to our calculations in § 3.5.)

Our next application is the following:

Corollary 6.4.3. *Let* X *be a* p*-adic formal scheme over* $\mathbf{Z}_p[\zeta_p]$*, and let* X_η *denote its adic generic fiber. Denote by* $\mathrm{X}_\eta^{\mathrm{cyc}}$ *the base-change of* X_η *along the map* $\mathbf{Q}_p(\zeta_p) \to \mathbf{Q}_p^{\mathrm{cyc}}$*. Then there is a* $\mathbf{Z}_p^\times$*-equivariant isomorphism*

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{X}/\mathrm{S}[\![q^{1/p}-1]\!]) \cong \mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{X}_\eta^{\mathrm{cyc}}).$$

Proof. Indeed, Theorem 6.4.1 implies that the left-hand side identifies with $\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathrm{X})\otimes_{\mathrm{KU}_p^{h(1+p\mathbf{Z}_p)^\times}} \mathrm{KU}_p$. By [BCM], this can in turn be identified with $\mathrm{L}_{\mathrm{K}(1)}\mathrm{K}(\mathrm{X}_\eta^{\mathrm{cyc}})$, as claimed. □

On the level of motivic associated graded pieces, the preceding result says that the *relative* syntomic cohomology $\mathbf{Z}_p(*)^{\mathrm{Syn}}(\mathrm{X}/\mathbf{Z}_p[\![q^{1/p}-1]\!])[v_1^{-1}]$ is isomorphic to $\mathbf{Z}_p(*)(\mathrm{X}_\eta^{\mathrm{cyc}})$. D. Manam has pointed out to me that an analogous isomorphism holds more generally for arbitrary prisms replacing the q-de Rham prism. Note also that the preceding result gives a $\mathbf{Z}_p^\times$-equivariant identification

$$\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p}-1]\!]) \simeq \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cyc}}) \simeq \mathrm{KU}_p \oplus \Sigma\mathrm{KU}_p \oplus \Sigma\mathrm{Y},$$

where Y is a spectrum which is noncanonically equivalent to $\mathrm{Map}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathrm{KU}_p)$.

For the third application, we need a construction.

Construction 6.4.4. The augmentation $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \to \mathbf{Z}_p[\zeta_p]$ and the map $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \to \mathrm{THH}(\mathbf{F}_p)$ both exhibit $\mathbf{Z}_p[\zeta_p]$ and $\mathrm{THH}(\mathbf{F}_p)$ as S^1-equivariant $\mathbf{E}_\infty$-$\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$-algebras. In the symmetric monoidal category $(\mathrm{Mod}^{t\mathrm{S}^1}_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])})^\wedge_{(p,v_1)}$, there is a canonical map $\mathrm{THH}(\mathbf{F}_p) \to \mathbf{Z}_p[\zeta_p]$: indeed, it follows from Theorem 6.1.2 and Theorem 6.1.4 (and the remark following it) that the unit maps $j_{p,0} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ and $\mathbf{Z}_p \to \mathrm{THH}(\mathbf{F}_p)$ are equivalences in $(\mathrm{Sp}^{t\mathrm{S}^1})^\wedge_{(p,v_1)}$. The desired map then follows by noting that the unit map $j_{p,0} \to \mathbf{Z}_p[\zeta_p]$ factors uniquely through the map $j_{p,0} \to \mathbf{Z}_p$ (and this is true already in $\mathrm{Sp}^{\mathrm{BS}^1}$). It follows that if M is an S^1-equivariant $\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$-module, then there is a natural map (which is lax symmetric monoidal in M)

$$(\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathrm{THH}(\mathbf{F}_p))^{t\mathrm{S}^1} \to (\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathbf{Z}_p[\zeta_p])^{t\mathrm{S}^1},$$

where (as usual) both sides are implicitly p-completed. In particular, there is a natural map (which is lax symmetric monoidal in M)

$$(\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathrm{THH}(\mathbf{F}_p))^{t\mathrm{S}^1} \otimes_{\mathbf{Z}_p^{t\mathrm{S}^1}} \mathbf{Z}_p[\zeta_p]^{t\mathrm{S}^1} \to (\mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathbf{Z}_p[\zeta_p])^{t\mathrm{S}^1}.$$

For instance, if $\mathrm{M} = \mathrm{THH}(\mathcal{C})$ for a $\mathbf{Z}_p[\zeta_p]$-linear ∞-category $\mathcal{C}$ with spcial fiber $\mathcal{C}_0 = \mathcal{C} \otimes_{\mathbf{Z}_p[\zeta_p]} \mathbf{F}_p$, then the preceding construction gives a natural $\mathbf{Z}_p[\zeta_p]^{t\mathrm{S}^1}$-linear map $\mathrm{TP}(\mathcal{C}_0)\otimes_{\mathbf{Z}_p^{t\mathrm{S}^1}} \mathbf{Z}_p[\zeta_p]^{t\mathrm{S}^1} \to \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])$. This map is generally *not* an equivalence (because of Nygaard completion issues: the divided powers of $(\zeta_p - 1)$ do not grow in p-adic valuation).

The following result is a noncommutative version of a result announced by Lurie [Lur9] (and by lifting all of our arguments to the category of synthetic cyclotomic spectra [AR], it also implies Lurie's result).

Corollary 6.4.5. *Let $\mathcal{C}$ be a $\mathbf{Z}_p[\zeta_p]$-linear ∞-category (such as $\mathrm{QCoh}(\mathrm{X})$ for a p-adic formal scheme over $\mathrm{Spf}(\mathbf{Z}_p[\zeta_p])$), and let $\mathcal{C}_0$ denote its special fiber. Let $\mathrm{F}(\mathcal{C})$ denote the total fiber of the following commutative square:*

$$\begin{tikzcd}
\mathrm{TC}(\mathcal{C}_0) \times \mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \arrow[r] \arrow[d] & \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \arrow[d] \\
\mathrm{HC}^-(\mathcal{C}_0/\mathbf{F}_p) \arrow[r] & \mathrm{HP}(\mathcal{C}_0/\mathbf{F}_p),
\end{tikzcd}$$

where the map $\mathrm{TC}(\mathcal{C}_0) \to \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])$ is induced via the map of Construction 6.4.4. Then there is a natural lax symmetric monoidal equivalence

$$\mathrm{F}(\mathcal{C}) \simeq \mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \mathbf{Z}_p.$$

In particular, the natural map from $\mathrm{TC}(\mathcal{C})$ to $\mathrm{F}(\mathcal{C})$ has a filtration

$$\mathrm{TC}(\mathcal{C}) \to \cdots \to \mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \tau_{\leq 4}(j_{p,0}) \to \mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \tau_{\leq 2}(j_{p,0}) \to \mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \tau_{\leq 0}(j_{p,0}) = \mathrm{F}(\mathcal{C}),$$

where the fiber of each map $\mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \tau_{\leq 2n}(j_{p,0}) \to \mathrm{F}(\mathcal{C})$ is killed by $p^n n!$.

Proof. Let us first dispense of the final claim: this follows immediately from the fact that $j_{p,0}$ is connective with no even homotopy groups above degree zero, and $\pi_{2n-1}(j_{p,0}) \cong \mathbf{Z}/np$ if $n \geq 1$. The claim about identifying $\mathrm{F}(\mathcal{C})$ is equivalent to the statement that there is a Cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathcal{C}) \otimes_{j_{p,0}} \mathbf{Z}_p \arrow[r] \arrow[d] & \mathrm{TC}(\mathcal{C}_0) \arrow[d] \\
\mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \arrow[r] & \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p]) \times_{\mathrm{HP}(\mathcal{C}_0/\mathbf{F}_p)} \mathrm{HC}^-(\mathcal{C}_0/\mathbf{F}_p).
\end{tikzcd}$$

We will in fact prove the following more general claim. Suppose $\mathrm{M} \in \mathrm{Mod}_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])}(\mathrm{CycSp})$; define

$$\begin{aligned}
\rho_{\mathrm{dR}}(\mathrm{M}) &= \mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathbf{Z}_p[\zeta_p], \\
\rho_{\mathrm{crys}}(\mathrm{M}) &= \mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathrm{THH}(\mathbf{F}_p), \\
\overline{\rho_{\mathrm{dR}}(\mathrm{M})} &= \mathrm{M} \otimes_{\mathrm{THH}(\mathbf{Z}_p[\zeta_p])} \mathbf{F}_p.
\end{aligned}$$

Then, there is a Cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{M}) \otimes_{j_{p,0}} \mathbf{Z}_p \arrow[r] \arrow[d] & \mathrm{TC}(\rho_{\mathrm{crys}}(\mathrm{M})) \arrow[d] \\
\rho_{\mathrm{dR}}(\mathrm{M})^{h\mathrm{S}^1} \arrow[r] & \rho_{\mathrm{dR}}(\mathrm{M})^{t\mathrm{S}^1} \times_{\overline{\rho_{\mathrm{dR}}(\mathrm{M})}^{t\mathrm{S}^1}} \overline{\rho_{\mathrm{dR}}(\mathrm{M})}^{h\mathrm{S}^1}.
\end{tikzcd} \tag{6.4.2}$$

To see this, observe that tensor product $\mathbf{Z}_p \otimes_{\mathrm{ku}_p} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p} - 1]\!])$ identifies with the cofiber of multiplication by β on $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p} - 1]\!])$. Under the isomorphism $\pi_*\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p} - 1]\!]) \cong \mathbf{Z}_p[\zeta_p][u]$ with u in weight 2, the class β identifies with $(\zeta_p - 1)u$. It follows that there are ($\mathbf{F}_p^\times$-equivariant) equivalences of S^1-equivariant $\mathbf{E}_\infty$-rings:

$$\mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \otimes_{j_{p,0}} \mathbf{Z}_p \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q^{1/p} - 1]\!]) \otimes_{\mathrm{ku}_p} \mathbf{Z}_p \cong \mathrm{THH}(\mathbf{F}_p) \times_{\mathbf{F}_p} \mathbf{Z}_p[\zeta_p]. \tag{6.4.3}$$

Since the left-hand side is a cyclotomic $\mathbf{E}_\infty$-ring, this in particular equips $\mathrm{THH}(\mathbf{F}_p)\times_{\mathbf{F}_p}\mathbf{Z}_p[\zeta_p]$ with the structure of a cyclotomic $\mathbf{E}_\infty$-ring such that the projection to $\mathrm{THH}(\mathbf{F}_p)$ is a map of cyclotomic $\mathbf{E}_\infty$-rings. It also follows that there is an S^1-equivariant equivalence

$$\mathrm{M}\otimes_{j_{p,0}}\mathbf{Z}_p\simeq\rho_{\mathrm{crys}}(\mathrm{M})\times_{\overline{\rho_{\mathrm{dR}}(\mathrm{M})}}\rho_{\mathrm{dR}}(\mathrm{M}).$$

There is a map of fiber sequences

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{M})\otimes_{j_{p,0}}\mathbf{Z}_p \arrow[r]\arrow[d] & \mathrm{TC}(\rho_{\mathrm{crys}}\mathrm{M}) \arrow[d]\\
(\rho_{\mathrm{crys}}(\mathrm{M})\times_{\overline{\rho_{\mathrm{dR}}(\mathrm{M})}}\rho_{\mathrm{dR}}(\mathrm{M}))^{h\mathrm{S}^1} \arrow[r]\arrow[d, "\mathrm{can}-\varphi"'] & \rho_{\mathrm{crys}}(\mathrm{M})^{h\mathrm{S}^1} \arrow[d, "\mathrm{can}-\varphi"]\\
(\rho_{\mathrm{crys}}(\mathrm{M})\times_{\overline{\rho_{\mathrm{dR}}(\mathrm{M})}}\rho_{\mathrm{dR}}(\mathrm{M}))^{t\mathrm{S}^1} \arrow[r] & \rho_{\mathrm{crys}}(\mathrm{M})^{t\mathrm{S}^1}.
\end{tikzcd}\tag{6.4.4}$$

The cyclotomic Frobenius on $\mathrm{THH}(\mathbf{Z}_p[\zeta_p])\otimes_{j_{p,0}}\mathbf{Z}_p$ factors as

$$\mathrm{THH}(\mathbf{Z}_p[\zeta_p])\otimes_{j_{p,0}}\mathbf{Z}_p\to\mathrm{THH}(\mathbf{Z}_p[\zeta_p])^{t\mathbf{Z}/p}\otimes_{j_{p,0}^{t\mathbf{Z}/p}}\mathbf{Z}_p^{t\mathbf{Z}/p}\to(\mathrm{THH}(\mathbf{Z}_p[\zeta_p])\otimes_{j_{p,0}}\mathbf{Z}_p)^{t\mathbf{Z}/p}.$$

Since the map $j_{p,0}\to\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ is an equivalence on $\mathbf{Z}/p$-Tate constructions by Theorem 6.1.4, the middle term is just $\mathbf{Z}_p^{t\mathbf{Z}/p}\simeq\mathrm{THH}(\mathbf{F}_p)^{t\mathbf{Z}/p}$. That is to say, under the identification (6.4.3), the cyclotomic Frobenius on $\mathrm{THH}(\mathbf{F}_p)\times_{\mathbf{F}_p}\mathbf{Z}_p[\zeta_p]$ factors through the projection onto $\mathrm{THH}(\mathbf{F}_p)$. There is an S^1-equivariant fiber sequence

$$\mathrm{fib}(\rho_{\mathrm{dR}}(\mathrm{M})\to\overline{\rho_{\mathrm{dR}}(\mathrm{M})})\to\rho_{\mathrm{crys}}(\mathrm{M})\times_{\overline{\rho_{\mathrm{dR}}(\mathrm{M})}}\rho_{\mathrm{dR}}(\mathrm{M})\to\rho_{\mathrm{crys}}(\mathrm{M}),$$

which describes the fibers of the bottom two horizontal maps in the diagram (6.4.4). Since the cyclotomic Frobenius on $\mathrm{THH}(\mathbf{F}_p)\times_{\mathbf{F}_p}\mathbf{Z}_p[\zeta_p]$ factors through the projection onto $\mathrm{THH}(\mathbf{F}_p)$, the resulting map on fibers

$$\mathrm{fib}(\rho_{\mathrm{dR}}(\mathrm{M})\to\overline{\rho_{\mathrm{dR}}(\mathrm{M})})^{h\mathrm{S}^1}\to\mathrm{fib}(\rho_{\mathrm{dR}}(\mathrm{M})\to\overline{\rho_{\mathrm{dR}}(\mathrm{M})})^{t\mathrm{S}^1}$$

is just the canonical map from homotopy fixed points to the Tate construction. A small diagram chase now implies the desired Cartesian square (6.4.2). □

Remark 6.4.6. Rationalizing Corollary 6.4.5 produces a Cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathcal{C})_{\mathbf{Q}} \arrow[r]\arrow[d] & \mathrm{TC}(\mathcal{C}_0)_{\mathbf{Q}} \arrow[d]\\
\mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}} \arrow[r] & \mathrm{HP}(\mathcal{C}/\mathbf{Z}_p[\zeta_p])_{\mathbf{Q}},
\end{tikzcd}$$

which reproduces the Beilinson fiber square of [AMMN]. In other words, Corollary 6.4.5 produces (sharp) torsion bounds for the failure of the above square to be Cartesian before rationalization.

Remark 6.4.7. It follows from the diagram (6.4.2) in the proof of Corollary 6.4.5 (and the general theory of prismatic stacks developed below in § 7.1) that an evident analogue of the square therein holds for *any* F-gauge $\mathrm{M} \in \mathrm{Perf}(\mathbf{Z}_p[\zeta_p]^{\mathrm{Syn}})$: the term $\mathrm{TC}(\mathrm{M}) \otimes_{j_{p,0}} \mathbf{Z}_p$ must be replaced by the (global sections of) the pullback of $\mathrm{M}\{*\} = \bigoplus_n \mathrm{M}\{n\}$ to the p-adic formal stack $\mathbf{Z}_p[\zeta_p]^{\mathrm{Syn}} \times_{\mathrm{Spev}(j_{p,0})} \mathrm{B}\mathbf{G}_m$.

We now proceed to the proof of Theorem 6.4.1; first, we need some preliminary observations.

Lemma 6.4.8. *There is a map $\delta : \Sigma\mathbf{F}_p \to \mathfrak{gl}_1(j_{p,0})$ which fits into a commutative diagram*

$$\begin{array}{ccc} \Sigma\mathbf{F}_p & \longrightarrow & \mathfrak{gl}_1(j_{p,0}) \\ \downarrow & & \downarrow \\ \Sigma^2\mathbf{Z}_p & \longrightarrow & \mathfrak{gl}_1(\mathrm{ku}_p), \end{array}$$

where the map $\Sigma^2\mathbf{Z}_p \to \mathfrak{gl}_1(\mathrm{ku}_p)$ extends the map $\mathrm{S}^2 \to \mathfrak{gl}_1(\mathrm{ku}_p)$ detecting $\beta \in \pi_2(\mathrm{ku}_p)$, and the map $\Sigma\mathbf{F}_p \to \Sigma^2\mathbf{Z}_p$ is the Bockstein.

Proof. It is well-known that the map $\Sigma^2\mathbf{Z}_p \to \mathfrak{gl}_1(\mathrm{ku}_p)$ is equivariant for the standard action of $\mathbf{Z}_p^\times$ on ku_p and for the standard/cyclotomic action of $\mathbf{Z}_p^\times$ on $\mathbf{Z}_p$. It therefore induces a map $\Sigma^2\mathbf{Z}_p^{h(1+p\mathbf{Z}_p)^\times} \to \mathfrak{gl}_1(\mathrm{ku}_p)^{h(1+p\mathbf{Z}_p)^\times}$. A standard calculation shows that there is an equivalence $\mathbf{Z}_p^{h(1+p\mathbf{Z}_p)^\times} \simeq \Sigma^{-1}\mathbf{F}_p$, under which the map $\mathbf{Z}_p^{h(1+p\mathbf{Z}_p)^\times} \to \mathbf{Z}_p$ identifies with the Bockstein $\Sigma^{-1}\mathbf{F}_p \to \mathbf{Z}_p$. In particular, we obtain a map $\Sigma\mathbf{F}_p \to \mathfrak{gl}_1(\mathrm{ku}_p)^{h(1+p\mathbf{Z}_p)^\times}$. The source is connective, and so it factors through the connective cover of $\mathfrak{gl}_1(\mathrm{ku}_p)^{h(1+p\mathbf{Z}_p)^\times}$, which can be identified with $\mathfrak{gl}_1(\tau_{\geq 0}(\mathrm{ku}_p^{h(1+p\mathbf{Z}_p)^\times})) = \mathfrak{gl}_1(j_{p,0})$. This defines the desired map $\delta : \Sigma\mathbf{F}_p \to \mathfrak{gl}_1(j_{p,0})$. It is clear from the construction that the claimed properties of δ hold. $\square$

Remark 6.4.9. The map $j_{p,0} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ of cyclotomic $\mathbf{E}_\infty$-rings is equivalent to the data of an $\mathbf{E}_\infty$-map $j_{p,0} \to \mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. One way to construct this map is as follows. First note that there is a $\mathbf{Z}_p^\times$-equivariant $\mathbf{E}_\infty$-map $\mathrm{ku}_p \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p^{\mathrm{cyc}})$, which induces an $\mathbf{F}_p^\times$-equivariant $\mathbf{E}_\infty$-map $\mathrm{ku}_p^{h(1+p\mathbf{Z}_p)} \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. The map $\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$ is an equivalence in degrees ≥ 2, so it suffices to construct a factorization of the map

$$j_{p,0} \to \tau_{\geq 0}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \tau_{[0,1]}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \tag{6.4.5}$$

through the map $\tau_{[0,1]}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \tau_{[0,1]}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. To do so, we may replace $j_{p,0}$ by its truncation $\tau_{\leq 1}j_{p,0}$.

Lemma 6.4.8 gives an $\mathbf{E}_\infty$-map $\mathrm{S}[\mathrm{B}\mathbf{Z}/p] \to j_{p,0}$, and hence an $\mathbf{E}_\infty$-map $\tau_{\leq 1}\mathrm{S}[\mathrm{B}\mathbf{Z}/p] \to \tau_{\leq 1}j_{p,0}$. Inspection on homotopy groups shows that this map is an equivalence. The composite (6.4.5) is therefore determined by an $\mathbf{E}_\infty$-map $\mathrm{S}[\mathrm{B}\mathbf{Z}/p] \to \tau_{[0,1]}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. Observe that this composite is determined by the inclusion $\mathbf{Z}/p \subseteq \mathrm{GL}_1(\mathbf{Z}_p[\zeta_p])$, which defines an $\mathbf{E}_\infty$-map $\mathrm{S}[\mathrm{B}\mathbf{Z}/p] \to \mathrm{K}(\mathbf{Z}_p[\zeta_p])$, and hence an $\mathbf{E}_\infty$-map

$$\mathrm{S}[\mathrm{B}\mathbf{Z}/p] \to \mathrm{K}(\mathbf{Z}_p[\zeta_p]) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \tau_{[0,1]}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]).$$

This composite clearly factors through the map $\tau_{[0,1]}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \tau_{[0,1]}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$, as desired.

It follows from Lemma 6.4.8 that the composite map

$$\Sigma\mathbf{Z} \to \Sigma\mathbf{F}_p \xrightarrow{\delta} \mathfrak{gl}_1(j_{p,0}) \to \mathfrak{gl}_1(\mathrm{ku}_p)$$

is null-homotopic. The map $\Sigma\mathbf{Z} \to \mathfrak{gl}_1(j_{p,0})$ is adjoint to an $\mathbf{E}_\infty$-map $\mathrm{S}[\mathrm{S}^1] \to j_{p,0}$ which we will denote by $\overline{\delta}$. The above observation implies that that the map $\mathrm{S}[\mathrm{S}^1] \to j_{p,0} \to \mathrm{ku}_p$ factors through the augmentation $\mathrm{S}[\mathrm{S}^1] \to \mathrm{S}$ as a map of $\mathbf{E}_\infty$-rings.

Lemma 6.4.10. *Each square in the following commutative diagram is a pushout of $\mathbf{E}_\infty$-rings:*

$$\begin{tikzcd} \mathrm{S}[\mathrm{S}^1] \arrow[r] \arrow[d] & \mathrm{S}[\mathrm{B}\mathbf{Z}/p] \arrow[r] \arrow[d] & j_{p,0} \arrow[d] \\ \mathrm{S} \arrow[r] & \mathrm{S}[\mathbf{C}\mathrm{P}^\infty] \arrow[r] & \mathrm{ku}_p. \end{tikzcd}$$

The top horizontal composite is the map $\overline{\delta} : \mathrm{S}[\mathrm{S}^1] \to j_{p,0}$, and the map $\mathrm{S}[\mathbf{C}\mathrm{P}^\infty] \to \mathrm{ku}_p$ is the natural lift of the $\mathbf{E}_\infty$-map $\mathrm{S}[\mathbf{C}\mathrm{P}^\infty] \to \mathrm{KU}_p$ which detects that the Bott class is a strict unit of KU_p *(i.e., is the map from Snaith's theorem on* KU*).*

Proof. It suffices to check that the resulting $\mathbf{E}_\infty$-ring map $j_{p,0} \otimes_{\mathrm{S}[\mathrm{S}^1]} \mathrm{S} \to \mathrm{ku}_p$ induces an isomorphism on homotopy modulo p. Since p is an odd prime, S/p admits an $\mathbf{A}_2$-algebra structure; in particular, $j_{p,0}/p = j_{p,0} \otimes_{\mathrm{S}} \mathrm{S}/p$ is an $\mathbf{A}_2$-ring, so that its homotopy groups acquire the structure of a graded ring. (This graded ring is necessarily commutative.) Observe that the action of the topological generator $1+p \in (1+p\mathbf{Z}_p)^\times$ sends the generator $\beta^n \in \pi_{2n}(\mathrm{ku}_p)$ to $(1+p)^n\beta^n$. This, along with the cofiber sequence

$$j_{p,0} \to \mathrm{ku}_p \xrightarrow{\psi_\circ^{1+p}} \tau_{\geq 2}(\mathrm{ku}_p) \tag{6.4.6}$$

implies that there is an isomorphism of graded (commutative) rings

$$\pi_*(j_{p,0}/p) \xrightarrow{\cong} \mathbf{F}_p[\overline{\beta}] \otimes_{\mathbf{F}_p} \Lambda(\overline{\delta}).$$

Here, $\overline{\beta}$ lives in degree 2, and its image under the map $j_{p,0}/p \to \mathrm{ku}_p/p$ is the Bott class; similarly, $\overline{\delta}$ lives in degree 1, and it is the image of $\overline{\delta} : \mathrm{S}^1 \to j_{p,0}$ under the map $j_{p,0} \to j_{p,0}/p$. It follows immediately that there is an isomorphism

$$\pi_*(j_{p,0}/p \otimes_{\mathrm{S}[\mathrm{S}^1]} \mathrm{S}) \xrightarrow{\cong} \mathbf{F}_p[\overline{\beta}].$$

This implies that the map $j_{p,0} \otimes_{\mathrm{S}[\mathrm{S}^1]} \mathrm{S} \to \mathrm{ku}_p$ induces an isomorphism on homotopy modulo p, as desired. ∎

Lemma 6.4.11. *The following commutative diagram is a pushout of cyclotomic $\mathbf{E}_\infty$-rings:*

$$\begin{tikzcd} j_{p,0}^{\mathrm{triv}} \arrow[r] \arrow[d] & j_{p,0}^{(-1)} \arrow[d] \\ \mathrm{ku}_p^{\mathrm{triv}} \arrow[r] & \mathrm{ku}_p^{(-1)}. \end{tikzcd}$$

Proof. Since the diagram obviously commutes as one of cyclotomic $\mathbf{E}_\infty$-rings, it suffices to prove that the underlying (nonequivariant) diagram of $\mathbf{E}_\infty$-rings is a pushout. By Lemma 6.4.10, we just need to show that the following diagram is a pushout of $\mathbf{E}_\infty$-rings:

$$\begin{tikzcd} \mathrm{S}[\mathrm{S}^1] \arrow[r] \arrow[d] & j_{p,0}^{(-1)} \arrow[d] \\ \mathrm{S} \arrow[r] & \mathrm{ku}_p^{(-1)}. \end{tikzcd} \tag{6.4.7}$$

Here, the map $\mathrm{S}[\mathrm{S}^1] \to j_{p,0}^{(-1)}$ is the composite of $\overline{\delta} : \mathrm{S}[\mathrm{S}^1] \to j_{p,0}$ with the canonical map $j_{p,0} \to j_{p,0}^{(-1)}$. The proof that (6.4.7) is a pushout is exactly as in Lemma 6.4.10: namely, we will show that the $\mathbf{E}_\infty$-ring map

$$j_{p,0}^{(-1)} \otimes_{\mathrm{S}[\mathrm{S}^1]} \mathrm{S} \to \mathrm{ku}_p^{(-1)} \tag{6.4.8}$$

induces an isomorphism on homotopy. To do this, recall that the map $j_{p,0}^{\mathrm{triv}} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ induces an equivalence on $-^{(-1)}$, and exhibits an equivalence $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq j_{p,0}^{(-1)}$ of cyclotomic $\mathbf{E}_\infty$-rings (by Theorem 6.1.4). In particular, $j_{p,0}^{(-1)}$ admits the structure of an $\mathbf{E}_\infty$-$\mathbf{Z}_p[\zeta_p]$-algebra; we will check that the map (6.4.8) induces an isomorphism on homotopy upon base-changing along $\mathbf{Z}_p[\zeta_p] \to \mathbf{F}_p$.

Recall that there is an isomorphism $\pi_*(\mathrm{ku}_p^{(-1)}) \simeq \mathbf{Z}_p[\zeta_p][u]$ with u in degree 2. (The class u was denoted by $\hbar^{-1}$ in the preceding section.) This implies that $\pi_*(\mathrm{ku}_p^{(-1)} \otimes_{\mathbf{Z}_p[\zeta_p]} \mathbf{F}_p) \xrightarrow{\cong} \mathbf{F}_p[\overline{u}]$, where $\overline{u}$ is the reduction of u. Similarly, there is a cofiber sequence

$$j_{p,0}^{(-1)} \to \mathrm{ku}_p^{(-1)} \xrightarrow{\psi_{\circ\circ}^{1+p}} \tau_{\geq 2}(\tau_{\geq 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p}),$$

and the map $\psi_{\circ\circ}^{1+p}$ is given on homotopy by the map $\mathbf{Z}_p[\zeta_p][u] \to u\mathbf{Z}_p[\zeta_p][u]$ sending $u \mapsto \frac{p}{\zeta_p - 1} u$. Upon base-changing along the map $\mathbf{Z}_p[\zeta_p] \to \mathbf{F}_p$, we therefore find that there is an isomorphism

$$\pi_*(j_{p,0}^{(-1)} \otimes_{\mathbf{Z}_p[\zeta_p]} \mathbf{F}_p) \xrightarrow{\cong} \mathbf{F}_p[u] \otimes_{\mathbf{F}_p} \Lambda(\epsilon)$$

with ϵ in degree 1. The base-change of the map (6.4.8) along $\mathbf{Z}_p[\zeta_p] \to \mathbf{F}_p$ will therefore induce an isomorphism on homotopy once we show that the map

$$j_{p,0}/p \to j_{p,0}^{(-1)} \otimes_{\mathbf{Z}_p[\zeta_p]} \mathbf{F}_p$$

sends $\overline{\delta} \mapsto \epsilon$ on π_1. To see this, we will use the commutative diagram

$$\begin{tikzcd} j_{p,0} \arrow[r] \arrow[d] & \tau_{\geq 0}(j_{p,0}^{t\mathbf{Z}/p}) \arrow[d] \\ \mathrm{ku}_p \arrow[r] \arrow[d, "\psi_{\circ}^{1+p}"] & \tau_{\geq 0}(\mathrm{ku}_p^{t\mathbf{Z}/p}) \arrow[d, "\psi_{\circ\circ}^{1+p}"] \\ \tau_{\geq 2}(\mathrm{ku}_p) \arrow[r] & \tau_{\geq 2}(\tau_{\geq 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p}) \end{tikzcd}$$

The map $\mathrm{ku}_p \to \tau_{\geq 0}(\mathrm{ku}_p^{t\mathbf{Z}/p})$ is given on homotopy by the map $\mathbf{Z}_p[\beta] \to \mathbf{Z}_p[\zeta_p][u]$ sending $\beta \mapsto (\zeta_p - 1)u$. This implies that the map $\tau_{\geq 2}(\mathrm{ku}_p) \to \tau_{\geq 2}(\tau_{\geq 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p})$ in the above diagram is given on π_2 by the map $\mathbf{Z}_p \cdot \beta \to \mathbf{Z}_p[\zeta_p] \cdot u$ sending $\beta \mapsto u$. Since the maps $\pi_2\tau_{\geq 2}(\mathrm{ku}_p) \to \pi_1 j_{p,0}$ and $\pi_2\tau_{\geq 2}(\tau_{\geq 2}(\mathrm{ku}_p)^{t\mathbf{Z}/p}) \to \pi_1\tau_{\geq 0}(j_{p,0}^{t\mathbf{Z}/p})$ are surjections which send $\beta \mapsto \overline{\delta}$ and $u \mapsto \epsilon$, respectively, it follows that $\overline{\delta} \mapsto \epsilon$, as desired. $\square$

Remark 6.4.12. One of the key inputs into Lemma 6.4.11 was the observation that the map $\pi_1(j_{p,0}/p) \to \pi_1(j_{p,0}^{(-1)} \otimes_{\mathbf{Z}_p[\zeta_p]} \mathbf{F}_p)$ sends $\overline{\delta} \mapsto \epsilon$. The analogous claim fails for j itself: namely, there is a class $\alpha_1 \in \pi_{2p-3}j/p$, but its image under the map $\pi_{2p-3}(j/p) \to \pi_{2p-3}(\tau_{\geq 0}(j^{t\mathbf{Z}/p}) \otimes_{\mathbf{Z}_p} \mathbf{F}_p)$ is zero, since the target is $\mathbf{Z}_p/(p-1) \cong 0$.

Construction 6.4.13. Recall that if X is a connected space, there is a map of cyclotomic spectra

$$\mathrm{S}[\mathrm{X}]^{\mathrm{triv}} \simeq \mathrm{S}[\mathrm{X}]^{\mathrm{triv}} \otimes \mathrm{THH}(\mathrm{S}) \to \mathrm{THH}(\mathrm{S}[\Omega \mathrm{X}]),$$

which is referred to as the *assembly map*. When $\mathrm{X} = \mathrm{S}^1$, we will identify $\mathrm{S}[\Omega\mathrm{X}] \simeq \mathrm{S}[q^{\pm 1}]$; we then obtain a map $\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \to \mathrm{THH}(\mathrm{S}[q^{\pm 1}])$ of cyclotomic $\mathbf{E}_\infty$-rings. This can be viewed as an $\mathbf{E}_\infty$-map $\mathrm{S}[\mathrm{S}^1] \to \mathrm{TC}(\mathrm{S}[q^{\pm 1}])$, whose underlying $\mathbf{E}_1$-map is just specified by a class $d\log(q) \in \pi_1\mathrm{TC}(\mathrm{S}[q^{\pm 1}])$.

The following is easy:

Lemma 6.4.14. *There is a pushout diagram of cyclotomic $\mathbf{E}_\infty$-rings*

$$\begin{tikzcd} \mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{THH}(\mathrm{S}[q^{\pm 1/p}]) \arrow[d] \\ \mathrm{S} \arrow[r] & \mathrm{S}[q^{\pm 1/p}]^{\mathrm{triv}}. \end{tikzcd}$$

Lemma 6.4.15. *Let* X *be a spectrum concentrated in degrees $\leq 2p-3$. Then any map $\Sigma\mathbf{Z} \to \mathrm{X}$ is determined by the composite $\mathrm{S}^1 \to \Sigma\mathbf{Z} \to \mathrm{X}$.*

Proof. Since $\mathrm{X} \simeq \tau_{\leq 2p-3}\mathrm{X}$ and

$$\tau_{\leq 2p-3}(\Sigma\mathbf{Z}) \simeq \Sigma\tau_{\leq 2p-4}(\mathbf{Z}) \simeq \Sigma\tau_{\leq 2p-4}\mathrm{S} \simeq \tau_{\leq 2p-3}(\mathrm{S}^1),$$

we obtain equivalences

$$\mathrm{Map}(\Sigma\mathbf{Z}, \mathrm{X}) \simeq \mathrm{Map}(\tau_{\leq 2p-3}(\Sigma\mathbf{Z}), \mathrm{X}) \simeq \mathrm{Map}(\tau_{\leq 2p-3}(\mathrm{S}^1), \mathrm{X}) \simeq \mathrm{Map}(\mathrm{S}^1, \mathrm{X}),$$

as desired. $\square$

Lemma 6.4.16. *Let* R *be an $\mathbf{E}_\infty$-ring which is concentrated in degrees $\leq 2p-3$, and let $f : \mathrm{S}[\mathrm{S}^1] \to \mathrm{R}$ be an $\mathbf{E}_\infty$-map, so it determines a class $\epsilon \in \pi_1\mathrm{R}$. Then f is determined (as an $\mathbf{E}_\infty$-map) by ϵ; equivalently, f is determined by its underlying $\mathbf{E}_1$-map.*

Proof. Note that f is determined by a map $f : \Sigma\mathbf{Z} \to \mathfrak{gl}_1(\mathrm{R})$. Since R is concentrated in degrees $\leq 2p-3$, the spectrum $\mathfrak{gl}_1(\mathrm{R})$ is also concentrated in degrees $\leq 2p-3$. The desired claim therefore follows from Lemma 6.4.15. $\square$

Lemma 6.4.17. *Let $p > 2$. There is a commutative diagram of $\mathbf{E}_\infty$-rings:*

$$\begin{tikzcd} \mathrm{S}[\mathrm{S}^1] \arrow[r, "\overline{\delta}"] \arrow[d, "d\log(q^{1/p})"] & j_{p,0} \arrow[d] \\ \mathrm{TC}(\mathrm{S}[q^{\pm 1/p}]) \arrow[r] & \tau_{\leq 1}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]). \end{tikzcd}$$

Proof. By Lemma 6.4.16 (and the fact that $2p-3 \geq 1$), any $\mathbf{E}_\infty$-map $f : \mathrm{S}[\mathrm{S}^1] \to \tau_{\leq 1}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$ is completely determined by its underlying $\mathbf{E}_1$-map; in other words, two such $\mathbf{E}_\infty$-maps agree if they detect the same class in $\pi_1\tau_{\leq 1}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) = \pi_1\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. We therefore need to check that $\overline{\delta} = d\log(q^{1/p}) \in \pi_1\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$. Recall that the trace map $\mathrm{K}(\mathbf{Z}_p[\zeta_p]) \to \mathrm{TC}(\mathbf{Z}_p[\zeta_p])$ is an equivalence in degrees ≥ 0, so we equivalently need to check that $\overline{\delta} = d\log(q^{1/p}) \in \pi_1\mathrm{K}(\mathbf{Z}_p[\zeta_p])$. But this is true by construction; see Remark 6.4.9. $\square$

To ensure that we do not have any ± 1-issues in the proof of Proposition 6.4.20 below, we record:

Lemma 6.4.18. *Let R be a p-complete L_n-local $\mathbf{E}_\infty$-ring, and let F denote the fiber of the map $\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R})$. Then $\mathrm{F} \simeq \tau_{\leq n+1}(\mathrm{F})$, and $\pi_{n+1}(\mathrm{F})$ is torsion-free. Moreover, the map $\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})$ is injective on π_{n+1}.*

Proof. By [AHR, Theorem 4.11], the fiber F' of the map $\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})$ satisfies $\mathrm{F}' \simeq \tau_{\leq n}(\mathrm{F}')$, and furthermore $\pi_*(\mathrm{F}')$ is torsion. Since there is a long exact sequence

$$\pi_{n+1}(\mathrm{F}') \cong 0 \to \pi_{n+1}(\mathrm{R}) \to \pi_{n+1}\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}) \to \pi_n(\mathrm{F}'),$$

it follows that the map $\pi_{n+1}(\mathrm{R}) \to \pi_{n+1}\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})$ is injective as claimed.

There is a commutative diagram

$$\begin{tikzcd} \mathfrak{gl}_1(\mathrm{R}) \arrow[r] \arrow[d, equal] & \mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}) \arrow[d] \\ \mathfrak{gl}_1(\mathrm{R}) \arrow[r] & \mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R}), \end{tikzcd}$$

so that if F'' denotes the fiber of the map $\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R})$, then there is a fiber sequence

$$\mathrm{F}' \to \mathrm{F} \to \mathrm{F}''.$$

The following is a pullback square of endofunctors of spectra:

$$\begin{tikzcd} \mathrm{L}_n \arrow[r] \arrow[d] & \mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)} \arrow[d] \\ \mathrm{L}_{\mathbf{Q}} \arrow[r] & \mathrm{L}_{\mathbf{Q}}\mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}; \end{tikzcd}$$

so F'' is equivalently the fiber of the map $\mathrm{L}_{\mathbf{Q}}\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{L}_{\mathbf{Q}}\mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R})$. Since this is a map between $\mathbf{Q}$-modules, the homotopy groups of F'' are $\mathbf{Q}$-vector spaces. Because $\pi_{n+1}(\mathrm{F}')$ vanishes, we find that there is an exact sequence

$$0 \to \pi_{n+1}(\mathrm{F}) \hookrightarrow \pi_{n+1}(\mathrm{F}'') \to \pi_n(\mathrm{F}').$$

In particular, $\pi_{n+1}(\mathrm{F})$ injects into a $\mathbf{Q}$-vector space, so it is torsion-free.

Let us now show that $\mathrm{F}'' \simeq \tau_{\leq n+1}(\mathrm{F}'')$. The natural transformation $\mathrm{L}_n \to \mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}$ is given by p-completion. Since $\pi_i(\mathrm{F}') = 0$ for $i \geq n+1$, there are isomorphisms

$$\pi_i(\mathrm{R}) \cong \pi_i(\mathfrak{gl}_1(\mathrm{R})) \cong \pi_i(\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}))$$

for $i \geq n+2$. The homotopy groups $\pi_i(\mathrm{R})$ are derived p-complete because R is assumed to be p-complete, so the same is true of $\pi_i(\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}))$ for $i \geq n+2$. By general properties of derived p-completion, there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}_p/\mathbf{Z}_p, \pi_i\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})) \to \pi_i\mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R}) \to \mathrm{Map}_{\mathbf{Z}}(\mathbf{Q}_p/\mathbf{Z}_p, \pi_{i-1}\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})) \to 0.$$

The first term is precisely derived p-completion, so it agrees with $\pi_i\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R})$ as long as $i \geq n+2$; moreover, the final term vanishes for $i \geq n+3$. It follows that the map $\pi_i\mathrm{L}_n\mathfrak{gl}_1(\mathrm{R}) \to \pi_i\mathrm{L}_{\mathrm{K}(1)\vee\cdots\vee\mathrm{K}(n)}\mathfrak{gl}_1(\mathrm{R})$ is an isomorphism for $i \geq n+3$ and is injective for $i = n+2$. This implies that $\pi_i(\mathrm{F}'')$ vanishes for $i \geq n+2$, as desired. $\square$

Recall that if R is an $\mathbf{E}_\infty$-ring, the space $\mathbf{G}_m(\mathrm{R})$ is defined by $\mathrm{Map}_{\mathrm{Sp}}(\mathbf{Z}, \mathfrak{gl}_1(\mathrm{R}))$.

Lemma 6.4.19. *If* R *is a* $\mathrm{K}(n)$*-local* $\mathbf{E}_\infty$*-ring, then* $\mathbf{G}_m(\mathrm{R})$ *has homotopy concentrated in degrees* $\leq n+1$*, and* $\pi_{n+1}\mathbf{G}_m(\mathrm{R})$ *is torsion-free. If* R *is* $\mathrm{K}(1)$*-local, there is also a fiber sequence*

$$\mathbf{G}_m(\mathrm{R}) \to \mathrm{GL}_1(\mathrm{R}) \xrightarrow{\theta} \Omega^\infty\mathrm{R},$$

where $\theta : \mathrm{R} \to \mathrm{R}$ *denotes the* $\mathrm{K}(1)$*-local power operation.*

Proof. That $\mathbf{G}_m(\mathrm{R})$ has homotopy concentrated in degrees ≤ 2 and $\pi_2\mathbf{G}_m(\mathrm{R})$ is torsion-free follow from Lemma 6.4.18. Next, since $\mathbf{G}_m(\mathrm{R}) \cong \mathrm{Map}_{\mathbf{E}_\infty}(\mathrm{S}[q^{\pm 1}], \mathrm{R})$, and R is K(1)-local, we may identify $\mathbf{G}_m(\mathrm{R}) \cong \mathrm{Map}_{\mathbf{E}_\infty}(\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}[q^{\pm 1}], \mathrm{R})$. But $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}[q^{\pm 1}]$ is the $\mathbf{E}_\infty$-quotient of the free $\mathbf{E}_\infty$-algebra $\mathrm{L}_{\mathrm{K}(1)}\mathrm{S}\{q^{\pm 1}\}$ by the class $\theta(q)$, which implies that $\mathbf{G}_m(\mathrm{R})$ is the homotopy fiber of the map $\mathrm{GL}_1(\mathrm{R}) \to \Omega^\infty\mathrm{R}$ given by θ. $\square$

Proposition 6.4.20. *Let* $p > 2$*. There is a commutative diagram of cyclotomic* $\mathbf{E}_\infty$*-rings*

$$\begin{tikzcd}
\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \arrow[r, "\overline{\delta}"] \arrow[d, "\mathrm{dlog}(q^{1/p})"] & j_{p,0}^{\mathrm{triv}} \arrow[d] \\
\mathrm{THH}(\mathrm{S}[q^{\pm 1/p}]) \arrow[r] & \mathrm{THH}(\mathbf{Z}_p[\zeta_p]).
\end{tikzcd}$$

Proof. It suffices to show that there is a commutative diagram of $\mathbf{E}_\infty$-rings

$$\begin{tikzcd}
\mathrm{S}[\mathrm{S}^1] \arrow[r, "\overline{\delta}"] \arrow[d, "\mathrm{dlog}(q^{1/p})"] & j_{p,0} \arrow[d] \\
\mathrm{TC}(\mathrm{S}[q^{\pm 1/p}]) \arrow[r] & \mathrm{TC}(\mathbf{Z}_p[\zeta_p]).
\end{tikzcd}$$

The Lichtenbaum-Quillen conjecture (which is a theorem) says that the map $\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \to \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$ is an equivalence in degrees ≥ 2, so that the following square is Cartesian:

$$\begin{tikzcd}
\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \arrow[r] \arrow[d] & \tau_{\geq 0}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \arrow[d] \\
\tau_{\leq 1}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]) \arrow[r] & \tau_{\leq 1}\tau_{\geq 0}\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]).
\end{tikzcd}$$

By Lemma 6.4.17, we are reduced to proving the commutativity of the following diagram of $\mathbf{E}_\infty$-rings:

$$\begin{tikzcd} \mathrm{S}[\mathrm{S}^1] \arrow[rr, "\overline{\delta}"] \arrow[d, "d\log(q^{1/p})"] & & j_{p,0} \arrow[d] \\ \mathrm{TC}(\mathrm{S}[q^{\pm 1/p}]) \arrow[r] & \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]). & \end{tikzcd}$$

For notational simplicity, let us write $\mathrm{R} = \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$, so we need to check that two maps $\Sigma\mathbf{Z} \to \mathfrak{gl}_1(\mathrm{R})$ of spectra are homotopic. In other words, we need to check that two classes in $\pi_1\mathbf{G}_m(\mathrm{R})$ agree. The fiber sequence of Lemma 6.4.19 gives a long exact sequence

$$\pi_3\mathbf{G}_m(\mathrm{R}) \to \pi_3\mathrm{GL}_1(\mathrm{R}) \xrightarrow{\theta} \pi_3\mathrm{R} \to \pi_2\mathbf{G}_m(\mathrm{R}) \to \pi_2\mathrm{GL}_1(\mathrm{R}) \xrightarrow{\theta} \pi_2\mathrm{R} \to \pi_1\mathbf{G}_m(\mathrm{R}) \to \pi_1\mathrm{GL}_1(\mathrm{R}).$$

Note that $\pi_3\mathbf{G}_m(\mathrm{R}) \cong 0$ by Lemma 6.4.19. We will argue that the map $\pi_1\mathbf{G}_m(\mathrm{R}) \to \pi_1\mathrm{GL}_1(\mathrm{R})$ is injective, so that we only need to verify that the two classes $\overline{\delta}, d\log(q^{1/p})$ agree in $\pi_1\mathrm{GL}_1(\mathrm{R})$; but this was proved in Lemma 6.4.17.

The injectivity of the map $\pi_1\mathbf{G}_m(\mathrm{R}) \to \pi_1\mathrm{GL}_1(\mathrm{R})$ follows once we show that $\theta : \pi_2\mathrm{GL}_1(\mathrm{R}) \to \pi_2\mathrm{R}$ is an isomorphism. Recall (see [HM2]) that

$$\mathrm{R} = \mathrm{J}_0 \oplus \Sigma\mathrm{J}_0 \oplus \Sigma\mathrm{KU}_p^{\oplus(p-1)}.$$

This implies that $\pi_2\mathrm{GL}_1(\mathrm{R})$ and $\pi_2\mathrm{R}$ are both isomorphic to $\pi_2(\Sigma\mathrm{J}_0) \cong \mathbf{Z}/p$. To prove that $\theta : \pi_2\mathrm{GL}_1(\mathrm{R}) \to \pi_2\mathrm{R}$ is an isomorphism, it suffices to argue the map is injective. For this, it suffices to show that $\pi_2\mathbf{G}_m(\mathrm{R}) \cong 0$. This in turn follows from the claim that the map $\theta : \pi_3\mathrm{GL}_1(\mathrm{R}) \to \pi_3\mathrm{R}$ is an isomorphism: the long exact sequence above then implies that there is an injection $\pi_2\mathbf{G}_m(\mathrm{R}) \hookrightarrow \pi_2\mathrm{GL}_1(\mathrm{R})$; but Lemma 6.4.19 says that $\pi_2\mathbf{G}_m(\mathrm{R})$ is torsion-free, while $\pi_2\mathrm{GL}_1(\mathrm{R}) \cong \mathbf{Z}/p$, so in fact $\pi_2\mathbf{G}_m(\mathrm{R}) \cong 0$ as desired.

It follows from the long exact sequence above that the map $\theta : \pi_3\mathrm{GL}_1(\mathrm{R}) \to \pi_3\mathrm{R}$ is injective (since $\pi_3\mathbf{G}_m(\mathrm{R}) \cong 0$ by Lemma 6.4.19), and that its cokernel is torsionfree (since it is a subgroup of $\pi_2\mathbf{G}_m(\mathrm{R})$, which is torsion-free by Lemma 6.4.19). The description of R shows that $\pi_3\mathrm{GL}_1(\mathrm{R})$ and $\pi_3\mathrm{R}$ are both isomorphic to $\mathbf{Z}/2p \oplus \mathbf{Z}_p^{\oplus(p-1)}$. The map θ must be an isomorphism on the torsion subgroup of $\pi_3\mathrm{GL}_1(\mathrm{R})$ (since its cokernel is torsionfree); but it must also be an isomorphism on the torsionfree piece (indeed, since it is injective, its cokernel is necessarily a torsion $\mathbf{Z}_p$-module, hence zero by torsionfreeness). Thus $\theta : \pi_3\mathrm{GL}_1(\mathrm{R}) \to \pi_3\mathrm{R}$ is an isomorphism, and we win. $\square$

Remark 6.4.21. The square of Proposition 6.4.20 in fact commutes as a diagram of $\mathbf{Z}_p^\times$-equivariant cyclotomic $\mathbf{E}_\infty$-rings, where the $\mathbf{Z}_p^\times$-action on $\mathrm{S}[\mathrm{S}^1]$ (implicitly p-completed) is by the cyclotomic action on $(\mathrm{S}^1)^\wedge_p \simeq \mathrm{B}\mathbf{Z}_p$, the action on $j_{p,0}$ and on $\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ is via the quotient $\mathbf{Z}_p^\times \to \mathbf{F}_p^\times$, and the action on $\mathrm{THH}(\mathrm{S}[\![q^{1/p}-1]\!])$ is via the standard $\mathbf{Z}_p^\times$-action where $g \in \mathbf{Z}_p^\times$ sends $q^{1/p} \mapsto q^{g/p}$. I had a rather convoluted argument for this, but F. Wagner showed me the following simple argument, and I am grateful to him for allowing me to include it here.

Following the proof of Proposition 6.4.20 shows that we only need to prove that two maps $\Sigma\mathbf{Z}_p \to \mathfrak{gl}_1(\mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p]))$ are $\mathbf{Z}_p^\times$-equivariantly homotopic. As in the proof of Proposition 6.4.20, let us write $\mathrm{R} = \mathrm{L}_{\mathrm{K}(1)}\mathrm{TC}(\mathbf{Z}_p[\zeta_p])$ for notational simplicity. The action of $\mathbf{Z}_p^\times$ on the target factors through the quotient $\mathbf{Z}_p^\times \to \mathbf{F}_p^\times$, so we only need to show that two maps $\Sigma(\mathbf{Z}_p)_{h(1+p\mathbf{Z}_p)} \simeq \Sigma\mathbf{F}_p \to \mathfrak{gl}_1(\mathrm{R})$ are $\mathbf{F}_p^\times$-equivariantly homotopic. But

$$\pi_*(\mathrm{Map}(\Sigma\mathbf{F}_p, \mathfrak{gl}_1(\mathrm{R}))^{h\mathbf{F}_p^\times}) \simeq \pi_*(\mathrm{Map}(\Sigma\mathbf{F}_p, \mathfrak{gl}_1(\mathrm{R})))^{\mathbf{F}_p^\times},$$

so it suffices to show that two maps $\Sigma\mathbf{F}_p \to \mathfrak{gl}_1(\mathrm{R})$ are nonequivariantly homotopic. Since the two maps $\Sigma\mathbf{Z}_p \to \mathfrak{gl}_1(\mathrm{R})$ are homotopic (by Proposition 6.4.20), we only need to check that the nullhomotopies of the two maps

$$\Sigma\mathbf{Z}_p \xrightarrow{p} \Sigma\mathbf{Z}_p \to \mathfrak{gl}_1(\mathrm{R})$$

are also homotopic. But the difference between these nullhomotopies lies in the group $\pi_2\mathbf{G}_m(\mathrm{R})$, which vanishes (as established in the course of proving Proposition 6.4.20).

Note that the argument above in fact shows that there is a commutative diagram of cyclotomic $\mathbf{E}_\infty$-rings

$$\begin{tikzcd}
\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \arrow[r] \arrow[d, "d\log(q^{1/p})"] & \mathrm{S}[\mathrm{B}\mathbf{Z}/p]^{\mathrm{triv}} \arrow[r, "\overline{\delta}"] \arrow[d, "d\log(q^{1/p})"] & j_{p,0}^{\mathrm{triv}} \arrow[d] \\
\mathrm{THH}(\mathrm{S}[\![q^{1/p}-1]\!]) \arrow[r] & \mathrm{THH}(\mathrm{S}[\![q^{1/p}-1]\!]/(q-1)) \arrow[r] & \mathrm{THH}(\mathbf{Z}_p[\zeta_p]),
\end{tikzcd}$$

where the outer square commutes $\mathbf{Z}_p^\times$-equivariantly, and the rightmost square commutes $\mathbf{F}_p^\times$-equivariantly.

Proof of Theorem 6.4.1. The equivalence $\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{ku}_p^{(-1)}$ follows from Lemma 6.4.11 and the fact (Theorem 6.1.4) that the map $j_{p,0} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ exhibits an equivalence $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq j_{p,0}^{(-1)}$ of cyclotomic $\mathbf{E}_\infty$-rings. It remains to prove that there is an equivalence $\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[q^{\pm 1}])$. By Lemma 6.4.10, there is an equivalence

$$\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{S} \otimes_{\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]).$$

of cyclotomic $\mathbf{E}_\infty$-rings. By Proposition 6.4.20, the map $\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \to \mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ factors through the map $\mathrm{S}[\mathrm{S}^1]^{\mathrm{triv}} \to \mathrm{THH}(\mathrm{S}[q^{\pm 1}])$. Finally, Lemma 6.4.14 lets us further identify

$$\mathrm{ku}_p \otimes_{j_{p,0}} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]) \simeq \mathrm{S}[q^{\pm 1/p}] \otimes_{\mathrm{THH}(\mathrm{S}[q^{\pm 1/p}])} \mathrm{THH}(\mathbf{Z}_p[\zeta_p]);$$

the latter is precisely $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[q^{\pm 1/p}])$, as desired. The $\mathbf{Z}_p^\times$-equivariance of these equivalences follows from the observation that all the results cited above are in fact $\mathbf{Z}_p^\times$-equivariant (the trickiest part is the $\mathbf{Z}_p^\times$-equivariance of Proposition 6.4.20, but this is handled by Remark 6.4.21). The second part of Theorem 6.4.1 was already proved in Proposition 6.2.7. $\square$

Chapter 7

Prismatic cohomology

7.1 Prismatization of ring spectra

In this section, I will review some joint work [DHRY] with J. Hahn, A. Raksit, and A. Yuan, whose aim is to extend some of the recent work of Bhatt-Lurie [BL, Bha3] and Drinfeld [Dri2] to the setting of ring spectra. This is project is rather sprawling, so we will defer most of the details to the forthcoming paper; here, we will instead give an overview of the theory and explain particular explicit calculations which play an important role. Throughout this section, we will use the "double-step" variant of the Spev construction from Remark 2.1.2. One interesting component of our work is that it involves some (very basic) genuine equivariant homotopy theory: this is because the standard picture of the even filtration/Bhatt-Morrow-Scholze motivic filtration on noncommutative invariants like TC^- and TP only recover *Nygaard-completed* (Nygaard-filtered) prismatic cohomology. Genuine equivariance allows us to overcome the issue of completions, by introducing a slight variant of the ∞-category of cyclotomic spectra.[1]

Definition 7.1.1. A *decompleted cyclotomic spectrum* is the data of a cyclotomic spectrum X (i.e., a spectrum X with S^1-action along with an S^1-equivariant map $\varphi : \mathrm{X} \to \mathrm{X}^{t\mathbf{Z}/p}$) along with another S^1-spectrum $\Phi\mathrm{X}$ equipped with maps $\mathrm{X} \to \Phi\mathrm{X}$ and $\Phi\mathrm{X} \to \mathrm{X}^{t\mathbf{Z}/p}$ which factor the cyclotomic Frobenius on X. Let $\mathrm{CycSp}_{\mathbb{A}}$ denote the ∞-category of decompleted cyclotomic spectra. The basic idea is that $(\Phi\mathrm{X})^{h\mathrm{S}^1}$ can be regarded as a decompletion of $(\mathrm{X}^{t\mathbf{Z}/p})^{h\mathrm{S}^1}$, which, if X is bounded-below or is a $\mathbf{Z}$-module, is equivalent to $\mathrm{X}^{t\mathrm{S}^1}$. There is a canonical symmetric monoidal structure on $\mathrm{CycSp}_{\mathbb{A}}$ where Φ is symmetric monoidal.

Note that there is a fully faithful lax symmetric monoidal functor $\beta : \mathrm{CycSp} \to \mathrm{CycSp}_{\mathbb{A}}$ sending (X, φ) to the decompleted cyclotomic spectrum where $\Phi\mathrm{X} = \mathrm{X}^{t\mathbf{Z}/p}$. A decompleted cyclotomic spectrum which is in the essential image of this functor will be called *Borel-complete*. Similarly, there is a forgetful functor $\mathrm{CycSp}_{\mathbb{A}} \to \mathrm{Sp}_{\mathbf{Z}/p}$ sending a decompleted cyclotomic spectrum X to the $\mathbf{Z}/p$-equivariant spectrum whose underlying spectrum with $\mathbf{Z}/p$-action is X, and whose geometric fixed points is $\Phi\mathrm{X}$. Finally, there is a functor $\mathrm{CycSp}_{\mathbb{A}} \to \mathrm{Fun}(\mathrm{BS}^1, \mathrm{Sp})$ sending $\mathrm{X} \mapsto \mathrm{X}^{\mathbf{Z}/p} = \Phi\mathrm{X} \times_{\mathrm{X}^{t\mathbf{Z}/p}} \mathrm{X}^{h\mathbf{Z}/p}$.

Construction 7.1.2. Let X be a decompleted cyclotomic spectrum whose underlying spectrum is bounded-below. Then there are two maps $(\mathrm{X}^{\mathbf{Z}/p})^{h\mathrm{S}^1} \to (\Phi\mathrm{X})^{h\mathrm{S}^1}$:

- The *canonical* map is induced by taking homotopy S^1-fixed points of the projection map $\mathrm{X}^{\mathbf{Z}/p} \to \Phi\mathrm{X}$.

[1] The content of this section is work in progress; so the definition of "decompleted cyclotomic spectrum" below will certainly evolve in the future. This will change some results below, like Theorem 7.1.11 (albeit not in a significant way), but it will not change comparison results like Theorem 7.1.9.

- The *Frobenius* map uses the decompleted cyclotomic structure: it is given by the composite
$$(\mathrm{X}^{\mathbf{Z}/p})^{h\mathrm{S}^1} \to (\mathrm{X}^{h\mathbf{Z}/p})^{h\mathrm{S}^1} \simeq \mathrm{X}^{h\mathrm{S}^1} \to (\Phi\mathrm{X})^{h\mathrm{S}^1},$$
where the latter map is induced by the decompleted Frobenius $\mathrm{X} \to \Phi\mathrm{X}$.

There is a large class of decompleted cyclotomic ring spectra coming from arithmetic geometry:

Example 7.1.3. If R is a $\mathbf{Z}_p$-algebra, let $\mathrm{THH}_{\Delta}(\mathrm{R})$ denote the decompleted cyclotomic $\mathbf{E}_\infty$-ring whose underlying cyclotomic spectrum is THH(R), and whose decompleted structure is given by declaring $\Phi\mathrm{THH}_{\Delta}(\mathrm{R}) = \mathrm{THH}(\mathrm{R}) \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathrm{THH}(\mathbf{Z}_p)^{t\mathbf{Z}/p}$.

Another extremely important example is the following:

Example 7.1.4. Let $i_\mathrm{W} : \mathrm{ku} \to \mathrm{ku}^\mathrm{H}$ denote the map associated to a virtual complex H-representation W which sends a complex vector space V to $\mathrm{V} \otimes \mathrm{W}$. The composite of i_W with the map $\mathrm{ku}^\mathrm{H} \to \mathrm{ku}$ which sends a virtual complex H-representation to its fixed points is just the map $\mathrm{ku} \to \mathrm{ku}$ given by multiplication by $\dim_\mathbf{C}(\mathrm{W}^\mathrm{H})$. If $\mathrm{MUP}^{[n]}$ denotes the Thomification of the degree n map $\mathrm{ku} \to \mathrm{ku}$, the map i_W defines a map $\mathrm{MUP}^{[\dim_\mathbf{C}(\mathrm{W}^\mathrm{H})]} \to \Phi^\mathrm{H}\mathrm{MUP}$. If $\mathrm{H} = \mathbf{Z}/p$ and λ is its standard 1-dimensional complex representation, then taking $\mathrm{W} = \frac{\lambda^p - 1}{\lambda - 1}$ produces a map $\varphi : \mathrm{MUP} \to \Phi^{\mathbf{Z}/p}\mathrm{MUP}$ which can be viewed as the geometric fixed points of the norm map.

Fix a virtual complex $\mathbf{Z}/p$-representation W. Choosing a virtual complex S^1-representation $\widetilde{\mathrm{W}}$ lifting W such that $\widetilde{\mathrm{W}}^{\mathbf{Z}/p}$ has trivial $\mathrm{S}^1/\mathbf{Z}/p$-action then determines Borel S^1-equivariant structure on the map $i_\mathrm{W} : \mathrm{ku} \to \mathrm{ku}^{\mathbf{Z}/p}$, which in turn determines a Borel S^1-equivariant map $\mathrm{MUP}^{[\dim_\mathbf{C}(\widetilde{\mathrm{W}}^{\mathbf{Z}/p})]} \to \Phi^{\mathbf{Z}/p}\mathrm{MUP}$ lifting the map φ. In particular, for each $1 \leq i \leq p$, let $\widetilde{\mathrm{W}}_i = \frac{\lambda^i - 1}{\lambda - 1}$. Then the restriction of the $\widetilde{\mathrm{W}}_i$ to $\mathbf{Z}/p$ form a basis for the representation ring of $\mathbf{Z}/p$, and furthermore $\widetilde{\mathrm{W}}_i^{\mathbf{Z}/p}$ is the trivial 1-dimensional $\mathrm{S}^1/\mathbf{Z}/p$-representation. One therefore obtains Borel S^1-equivariant maps $\mathrm{MUP}^{\otimes p} \to \Phi^{\mathbf{Z}/p}\mathrm{MUP}$, which can easily be seen to be an equivalence. It follows that $\Phi^{\mathbf{Z}/p}\mathrm{MUP}$ in fact has *trivial* Borel-equivariant S^1-action.

The above discussion determines the structure of a decompleted cyclotomic $\mathbf{E}_\infty$-ring on MUP, where $\Phi\mathrm{MUP} = \Phi^{\mathbf{Z}/p}\mathrm{MUP}$, the map $\mathrm{MUP} \to \Phi^{\mathbf{Z}/p}\mathrm{MUP}$ is given by $\iota_{\widetilde{\mathrm{W}}_p}$, and the map $\Phi^{\mathbf{Z}/p}\mathrm{MUP} \to \mathrm{MUP}^{t\mathbf{Z}/p}$ is induced by the natural map from geometric fixed points to the $\mathbf{Z}/p$-Tate construction. In particular, the composite $\mathrm{MUP} \to \Phi^{\mathbf{Z}/p}\mathrm{MUP} \to \mathrm{MUP}^{t\mathbf{Z}/p}$ is the Tate-valued Frobenius on MUP, so it follows that MUP equipped with the trivial S^1-action and the cyclotomic Frobenius given by the Tate-valued Frobenius is in fact a cyclotomic spectrum.

A similar argument works with MUP replaced by MU. Given the above setup, we can finally construct the desired stacks:

Definition 7.1.5. An object $\mathrm{X} \in \mathrm{CycSp}_{\Delta}$ will be called *even* if its image under the functor $\mathrm{CycSp}_{\Delta} \to \mathrm{Sp}_{\mathbf{Z}/p}$ is even in the sense of Definition 2.1.11. If $\mathrm{A} \in \mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$, then define
$$\begin{aligned}
\mathrm{A}^{\mathrm{conj}} &= \mathrm{colim}_{\mathrm{A}\to\mathrm{B}}\, \mathrm{Spev}(\mathrm{B}),\\
\widehat{\mathrm{A}^{\mathrm{Nyg}}} &= \mathrm{colim}_{\mathrm{A}\to\mathrm{B}}\, \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{B}),\\
\mathrm{A}^{\mathrm{HT}} &= \mathrm{colim}_{\mathrm{A}\to\mathrm{B}}\, \mathrm{Spev}(\Phi\mathrm{B}),\\
\mathrm{A}^{\Delta} &= \mathrm{colim}_{\mathrm{A}\to\mathrm{B}}\, \mathrm{Spev}_{\mathrm{S}^1}(\Phi\mathrm{B}),
\end{aligned}$$

where the colimit in all cases is taken over maps $A \to B$ in $\mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$ with B being even. We will refer to these as the *conjugate*, *completed Nygaard*, *Hodge-Tate*, and *prismatic* stacks of A. If R is an $\mathbf{E}_\infty$-ring and $\mathrm{THH}_{\Delta}(R)$ is a chosen decompletion of $\mathrm{THH}(R) \in \mathrm{CycSp}$, we will abusively write $R^?$ to denote $\mathrm{THH}_{\Delta}(R)^?$.

Note that we have not yet defined the Nygaard stack of A; we will do so momentarily.

Remark 7.1.6. Suppose A is an $\mathbf{E}_\infty$-algebra in decompleted cyclotomic spectra which admits an eff cover $A \to B$ by an even decompleted cyclotomic $\mathbf{E}_\infty$-ring B such that the map $\Phi A \to \Phi B$ is an eff cover of S^1-equivariant $\mathbf{E}_\infty$-rings. Then there are isomorphisms

$$A^{\mathrm{conj}} \cong \mathrm{Spev}(A),\ \widehat{A^{\mathrm{Nyg}}} \cong \mathrm{Spev}_{S^1}(A),\ A^{\mathrm{HT}} \cong \mathrm{Spev}(\Phi A),\ A^{\Delta} \cong \mathrm{Spev}_{S^1}(\Phi A).$$

One would like to define the Nygaard stack of $A \in \mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$ as $\mathrm{colim}_{A\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p})$, where again the colimit in all cases is taken over maps $A \to B$ in $\mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$ with B being even. However, this does not quite recover the Nygaard stack of Bhatt-Lurie-Drinfeld; instead, we must modify this procedure as follows.

Construction 7.1.7. If $A \in \mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$, then the colimit $\mathrm{colim}_{A\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p})$ is a stack over $\mathrm{colim}_{S\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p})$; here, all colimits are indexed by maps in $\mathrm{CAlg}(\mathrm{CycSp}_{\Delta})$. Using Remark 7.1.6, Example 7.1.4, and Example 2.2.11, one finds that $\mathrm{colim}_{S\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p})$ can be identified with the completion of the moduli stack of S^1-equivariant formal groups $\mathcal{M}_{\mathrm{fg}}^{S^1}$ along the locus $\mathcal{M}_{\mathrm{fg}}^{\mathbf{Z}/p} \subseteq \mathcal{M}_{\mathrm{fg}}^{S^1}$ of $\mathbf{Z}/p$-equivariant formal groups. In other words, there is a canonical map $\mathrm{colim}_{A\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p}) \to (\mathcal{M}_{\mathrm{fg}}^{S^1})^\wedge_{\mathcal{M}_{\mathrm{fg}}^{\mathbf{Z}/p}}$.

The stack $\mathcal{M}_{\mathrm{fg}}^{\mathbf{Z}/p}$ has two irreducible components, one corresponding to the moduli stack of 1-dimensional formal groups itself; we will denote the other component by $\mathcal{M}_{\mathrm{fg}}^{\mathbf{v}_0}$. The *Nygaard stack* of A is defined to be the completion of $\mathrm{colim}_{A\to B}\,\mathrm{Spev}_{S^1}(B^{\mathbf{Z}/p})$ along the preimage of the locus $\mathcal{M}_{\mathrm{fg}}^{\mathbf{v}_0\text{-frml}} := (\mathcal{M}_{\mathrm{fg}}^{S^1})^\wedge_{\mathcal{M}_{\mathrm{fg}}^{\mathbf{v}_0}}$ (which we call the moduli stack of "$\mathbf{v}_0$-formal groups"). If R is an $\mathbf{E}_\infty$-ring and $\mathrm{THH}_{\Delta}(R)$ is a chosen decompletion of $\mathrm{THH}(R) \in \mathrm{CycSp}$, we will abusively write $R^?$ to denote $\mathrm{THH}_{\Delta}(R)^?$.

One can check that the Frobenius map $(A^{\mathbf{Z}/p})^{hS^1} \to (\Phi A)^{hS^1}$ upgrades to a map $F : A^{\Delta} \to A^{\mathrm{Nyg}}$. Similarly, the canonical map $A^{\mathbf{Z}/p} \to \Phi A$ upgrades to a map $\mathrm{can} : A^{\Delta} \to A^{\mathrm{Nyg}}$.

Definition 7.1.8. Let A be a decompleted cyclotomic $\mathbf{E}_\infty$-ring. Then the *syntomification* A^{Syn} is defined to be the pushout

$$\begin{tikzcd}
A^{\Delta} \amalg A^{\Delta} \arrow[r, "\mathrm{can} \amalg \varphi"] \arrow[d, "\mathrm{fold}"] & A^{\mathrm{Nyg}} \arrow[d] \\
A^{\Delta} \arrow[r] & A^{\mathrm{Syn}}.
\end{tikzcd}$$

Again, if R is an $\mathbf{E}_\infty$-ring and $\mathrm{THH}_{\Delta}(R)$ is a chosen decompletion of $\mathrm{THH}(R) \in \mathrm{CycSp}$, we will abusively write $R^?$ to denote $\mathrm{THH}_{\Delta}(R)^?$.

We note the following consistency, which is proved by quasisyntomic descent:

Theorem 7.1.9. *Let R be a p-quasisyntomic discrete commutative ring, and let $\mathrm{THH}_{\Delta}(R)$ denote the decompletion of Example 7.1.3. Then the stacks R^{conj}, R^{HT}, and R^{Δ} agree with the stacks constructed by Bhatt-Lurie [BL, Bha3] and Drinfeld [Dri2].*

Remark 7.1.10. Let $p > 2$. When applied to $\mathrm{R} = \mathbf{Z}_p$, the preceding result combined with Theorem 6.1.4 gives a description of the stacks $\mathbf{Z}_p^?$ purely in terms of the stack $\mathrm{Spev}(j_p)$ from Example 2.2.6. For instance, there is a natural map $\mathbf{Z}_p^{\mathrm{Syn}} \to \mathrm{Spev}(j_p)$. Moreover, the 1-dimensional formal group over $\mathrm{Spev}(j_p)$ classified by the map $\mathrm{Spev}(j_p) \to \mathrm{Spev}(\mathrm{S}) = \mathcal{M}_{\mathrm{fg}}$ admits a decompletion $\mathbf{G}_j$. The complement $\mathbf{G}_j - \{0\}$ of the zero section of this decompletion identifies with $\mathbf{Z}_p^{\Delta}$, and the pullback of $\mathbf{G}_j$ along the structure map $\mathbf{Z}_p^{\Delta} \to \mathrm{Spev}(j_p)$ identifies with a decompletion of the Drinfeld formal group. (One might be able to give a similar, but more complicated, construction of $\mathbf{Z}_p^{\mathrm{Nyg}}$ using $\mathbf{G}_j$.) The Hodge-Tate divisor $\mathbf{Z}_p^{\mathrm{HT}} \hookrightarrow \mathbf{Z}_p^{\Delta}$ corresponds to the inclusion of the p-torsion subgroup $\mathbf{G}_j[p] \subseteq \mathbf{G}_j$. These results were also observed independently by Lurie.

In particular, since the map $\mathbf{Z}_p^{\Delta} \to \mathrm{Spev}(j_p)$ factors through the inclusion $\mathbf{Z}_p^{\Delta} \to \mathbf{Z}_p^{\mathrm{Syn}}$, it follows that the Drinfeld formal group descends to $\mathbf{Z}_p^{\mathrm{Syn}}$ and admits a decompletion there. This result was also proved in [Man] through different methods (and the resulting descent of the decompletion of the Drinfeld formal group to $\mathbf{Z}_p^{\mathrm{Syn}}$ agrees with the one described above).

The picture above becomes particularly simple after pulling back along the map $\mathrm{Spev}(\mathrm{ku}_p) \cong \mathbf{A}^1(-1)/\mathbf{G}_m \to \mathrm{Spev}(j_p)$. Indeed, writing $\pi_{2*}(\mathrm{ku}_p) \cong \mathbf{Z}_p[\beta]$, the decompleted formal group over $\mathrm{Spev}(\mathrm{ku}_p)$ is given by the scheme

$$\mathbf{G}_{\mathrm{ku}} := \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathbf{Z}_p[\![q-1]\!][\beta, \hbar]/(\beta\hbar = q-1), (p, q-1))$$

with $\hbar$ in weight -1, where the group law is given by $\hbar + \hbar' + \beta\hbar\hbar'$. (See also [Man].) As expected from the preceding paragraph, the complement of the zero section (cut out by $\hbar = 0$) in $\mathbf{G}_{\mathrm{ku}}$ is precisely the q-de Rham point $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!])$. Moreover, the pullback of $\mathbf{G}_{\mathrm{ku}}$ along the map $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!]) \to \mathrm{Spev}(\mathrm{ku}_p)$ is isomorphic to $\mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!][x], (p, q-1))$, with group law given by $x + y + (q-1)xy$. In particular, this group scheme acquires a canonical nonvanishing section.

Using Example 2.2.11, Example 7.1.4, and Remark 7.1.6, one finds:

Theorem 7.1.11. *The various stacks associated to the sphere spectrum can be identified as follows:*

a. *The stack* $\mathrm{S}^{\mathrm{conj}}$ *is isomorphic to* $\mathcal{M}_{\mathrm{fg}}$*;*

b. *The stack* S^{HT} *is also isomorphic to* $\mathcal{M}_{\mathrm{fg}}$*;*

c. *The stack* S^{Δ} *can be identified with the universal* 1*-dimensional formal group* $\widehat{\mathbf{G}}_{\mathrm{univ}}$ *over* $\mathcal{M}_{\mathrm{fg}}$ *(so it classifies* 1*-dimensional formal groups equipped with a section);*

d. *The stack* $\mathrm{S}^{\mathrm{Nyg}}$ *is isomorphic to* $\mathcal{M}_{\mathrm{fg}}^{\mathbf{v}_0\text{-frml}} := (\mathcal{M}_{\mathrm{fg}}^{\mathrm{S}^1})^{\wedge}_{\mathcal{M}_{\mathrm{fg}}^{\mathbf{v}_0}}$*;*

e. *The canonical map* $\mathrm{can} : \mathrm{S}^{\Delta} \to \mathrm{S}^{\mathrm{Nyg}}$ *sends a* 1*-dimensional formal group* $\mathbf{H}$ *with a section* s *to the* $\mathbf{v}_0$*-formal group* (C, α) *given by the pushout*

$$\begin{tikzcd} p\underline{\mathbf{Z}} \arrow[r, "p \mapsto s"] \arrow[d] & \mathbf{H} \arrow[d] \\ \underline{\mathbf{Z}} \arrow[r, "\alpha"'] & \mathrm{C}. \end{tikzcd}$$

f. *The Frobenius map* $\varphi : \mathrm{S}^{\Delta} \to \mathrm{S}^{\mathrm{Nyg}}$ *sends a* 1*-dimensional formal group* $\mathbf{H}$ *with a section* s *to the* $\mathbf{v}_0$*-formal group* $\underline{\mathbf{Z}} \to \mathbf{H}$ *(sending* $1 \mapsto s$*).*

Note that both $\mathrm{S}^{\mathrm{conj}}$ and S^{HT} are isomorphic to $\mathcal{M}_{\mathrm{fg}}$. This can be regarded as a version of the Segal conjecture, which identifies $\mathrm{S}^{t\mathbf{Z}/p}$ with the p-completion of the sphere spectrum. In fact, it is not so hard to see that $\mathrm{S}^{\mathrm{conj}} \cong \mathrm{S}^{\mathrm{HT}}$. They are computed as the geometric realizations of $\mathrm{Spev}_{\mathrm{S}^1}(\mathrm{MU}^{\otimes \bullet+1})$ and $\mathrm{Spev}_{\mathrm{S}^1}(\Phi\mathrm{MU}^{\otimes \bullet+1})$, but (as discussed in Example 7.1.4) $\Phi\mathrm{MU}^{\otimes \bullet+1}$ just identifies with the tensor product of $\mathrm{MU}^{\otimes \bullet+1}$ with some copies of MUP, which cancel out upon taking geometric realizations.

Corollary 7.1.12. *Let $i : \mathrm{S}^{\mathrm{HT}} \cong \mathcal{M}_{\mathrm{fg}} \to \mathrm{S}^{\triangle} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$ denote the inclusion of the Hodge-Tate locus. Then the $\mathbf{v}_0$-formal group $(i^*\mathrm{C}, \alpha : \underline{\mathbf{Z}} \to i^*\mathrm{C})$ classified by the composite map*

$$\mathrm{S}^{\mathrm{HT}} \to \mathrm{S}^{\triangle} \xrightarrow{\mathrm{can}} \mathrm{S}^{\mathrm{Nyg}}$$

fits into an exact sequence of group schemes

$$0 \to \underline{\mathbf{Z}/p} \xrightarrow{\alpha} i^*\mathrm{C} \to \widehat{\mathbf{G}}_{\mathrm{univ}} \to 0.$$

Proof. The pushout square from the penultimate part of Theorem 7.1.11 pulls back to a pushout square

$$\begin{tikzcd}
\underline{p\mathbf{Z}} \arrow[r, "p \mapsto s"] \arrow[d] & i^*\widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}} \arrow[d] \\
\underline{\mathbf{Z}} \arrow[r, "\alpha"] & i^*\mathrm{C}.
\end{tikzcd}$$

over $\mathrm{S}^{\mathrm{HT}} = \mathcal{M}_{\mathrm{fg}}$. The map $i : \mathcal{M}_{\mathrm{fg}} \to \widehat{\mathbf{G}}_{\mathrm{univ}}$ classifies the zero section of the universal formal group over $\mathcal{M}_{\mathrm{fg}}$, so there is an isomorphism $i^*\widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$. The homomorphism $\underline{p\mathbf{Z}} \to \widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}}$ over $\mathrm{S}^{\triangle}$ sends p to the tautological section of $\widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}}$; since this tautological section pulls back along i to the zero section of $\widehat{\mathbf{G}}_{\mathrm{univ}}$, the homomorphism

$$\underline{p\mathbf{Z}} \to i^*\widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$$

is zero. It follows that the above pushout square can be expanded to a diagram where each square is a pushout:

$$\begin{tikzcd}
\underline{p\mathbf{Z}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & i^*\widehat{\mathbf{G}}_{\mathrm{S}^{\triangle}} \cong \widehat{\mathbf{G}}_{\mathrm{univ}} \arrow[d] \\
\underline{\mathbf{Z}} \arrow[r] & \underline{\mathbf{Z}/p} \arrow[r, "\alpha"] & i^*\mathrm{C}.
\end{tikzcd}$$

This implies that the map $\alpha : \underline{\mathbf{Z}/p} \to i^*\mathrm{C}$ is a closed immersion, and its quotient is $\widehat{\mathbf{G}}_{\mathrm{univ}}$. $\square$

One can also use Example 2.2.10 to calculate the various stacks associated to MU. However, our understanding of this stack and the conceptual role of the moduli problems they classify does not seem nearly as extensive as Theorem 7.1.11, so we will defer discussion of this example to our forthcoming paper.

7.2 The Drinfeld formal group

The Drinfeld formal group plays a very important role in prismatic cohomology; for instance, it is crucial in defining a theory of prismatic Chern classes. Let us briefly review its construction due to Drinfeld [Dri1]:

Construction 7.2.1. There is a natural map $\mathbf{G}_m \times \mathbf{Z}_p^{\Delta} \to \mathbf{G}_m^{\Delta}$ coming from a δ-structure on $\mathbf{G}_m$. Let $\mathrm{G}_{\mathbf{Z}_p^{\Delta}}$ denote the kernel of this map, and let $\mathrm{H}_{\mathbf{Z}_p^{\Delta}}$ denote its Cartier dual. Then $\mathrm{H}_{\mathbf{Z}_p^{\Delta}}$ is a 1-dimensional formal group equipped with a section (given by the image of 1 under the map $\underline{\mathbf{Z}} \to \mathrm{H}_{\mathbf{Z}_p^{\Delta}}$ which is Cartier dual to the map $\mathrm{G}_{\mathbf{Z}_p^{\Delta}} \to \mathbf{G}_m \times \mathbf{Z}_p^{\Delta}$). This formal group $\mathrm{H}_{\mathbf{Z}_p^{\Delta}}$ over $\mathbf{Z}_p^{\Delta}$ is called the *Drinfeld formal group*. There is also a canonical homomorphism $\log_{\Delta} : \mathrm{G}_{\mathbf{Z}_p^{\Delta}} \to \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$ called the *prismatic logarithm*, which results from a canonical isomorphism between the Lie algebra of $\mathrm{H}_{\mathbf{Z}_p^{\Delta}}$ and the Breuil-Kisin line bundle $\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{-1\}$.

Remark 7.2.2. Suppose $\mathrm{Y} \to \mathrm{X}$ is a $\mathbf{G}_m$-torsor (i.e., is associated to a line bundle) of p-adic formal schemes. Then there is a map $\mathrm{X} \to \mathrm{B}\mathbf{G}_m$ classifying Y, and hence a map $\mathrm{X}^{\Delta} \to \mathrm{B}\mathbf{G}_m^{\Delta}$ over $\mathbf{Z}_p^{\Delta}$. The Drinfeld formal group defines a canonical map $\mathrm{B}\mathbf{G}_m^{\Delta} \to \mathrm{B}^2\mathrm{G}_{\mathbf{Z}_p^{\Delta}}$ (which measures the obstruction for the map $\mathrm{Y}^{\Delta} \to \mathrm{X}^{\Delta}$ to be induced from a $\mathbf{G}_m$-torsor over X^{Δ}). The homomorphism $\mathrm{G}_{\mathbf{Z}_p^{\Delta}} \to \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$ therefore yields a map

$$\mathrm{X}^{\Delta} \to \mathrm{B}\mathbf{G}_m^{\Delta} \to \mathrm{B}^2\mathrm{G}_{\mathbf{Z}_p^{\Delta}} \xrightarrow{\log_{\Delta}} \mathrm{B}^2\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\},$$

which classifies a class in $\mathrm{H}^2(\mathrm{X}^{\Delta}; \mathcal{O}\{1\})$. This is precisely the *first prismatic Chern class* of the $\mathbf{G}_m$-torsor $\mathrm{Y} \to \mathrm{X}$.

Our goal in this brief section is to observe that the Drinfeld formal group over $\mathbf{Z}_p^{\Delta}$ is in fact pulled back from the universal formal group over $\mathrm{S}^{\Delta} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$ via the map $\mathbf{Z}_p^{\Delta} \to \mathrm{S}^{\Delta}$ induced by the unit $\mathrm{S} \to \mathbf{Z}_p$.

Construction 7.2.3. View $\mathrm{S}[\mathbf{Z}]$ as a decompleted cyclotomic $\mathbf{E}_\infty$-ring with $\mathrm{S}[\mathbf{Z}] = \Phi\mathrm{S}[\mathbf{Z}]$ and such that the map $\Phi\mathrm{S}[\mathbf{Z}] \to \mathrm{S}[\mathbf{Z}]^{t\mathbf{Z}/p}$ identifies with the Tate-valued Frobenius. Then there is a map $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}]$ of Borel S^1-equivariant $\mathbf{E}_\infty$-rings which upgrades to a genuine $\mathbf{Z}/p$-equivariant map. (Note that with this choice of genuine $\mathbf{Z}/p$-equivariant structure on $\mathrm{S}[\mathbf{Z}]$, the augmentation $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}]$ is *not* a map of decompleted cyclotomic $\mathbf{E}_\infty$-rings.) Similarly to Example 2.2.10, one can check that $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}]$ is faithfully evenly projective. Note that $\mathrm{S}[\mathbf{Z}]^{\Delta} \cong \mathrm{Spev}_{\mathrm{S}^1}(\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}]))$ by Remark 7.1.6, and similarly $\mathrm{Spev}_{\mathrm{S}^1}(\Phi\mathrm{S}[\mathbf{Z}])$ is isomorphic to $\mathbf{G}_m \times \mathrm{S}^{\Delta} \cong \mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}}$. It follows that there is a homomorphism $\mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}} \to (\mathbf{G}_m)_{\mathrm{S}}^{\Delta}$ of group stacks over $\mathrm{S}^{\Delta} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$.

Theorem 7.2.4. *Let $(\mathbf{H}, s)$ denote the universal 1-dimensional formal group with a section, so that $\mathbf{H}$ is the formal group over $\widehat{\mathbf{G}}_{\mathrm{univ}}$ classified by the structure map $\widehat{\mathbf{G}}_{\mathrm{univ}} \to \mathcal{M}_{\mathrm{fg}}$. View s as a homomorphism $s : \underline{\mathbf{Z}} \to \mathbf{H}$. Then $\mathbf{H}$ is the Cartier dual of the kernel of the homomorphism $\mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}} \to (\mathbf{G}_m)_{\mathrm{S}}^{\Delta}$. Moreover, the map $s : \underline{\mathbf{Z}} \to \mathbf{H}$ identifies with the Cartier dual of the map from the kernel of the homomorphism $\mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}} \to (\mathbf{G}_m)_{\mathrm{S}}^{\Delta}$ to $\mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}}$.*

The basic idea behind this proof is that the kernel of the homomorphism $\mathbf{G}_m \times \widehat{\mathbf{G}}_{\mathrm{univ}} \to (\mathbf{G}_m)_{\mathrm{S}}^{\Delta}$ is computed, essentially, by $\mathrm{S}[\mathbf{Z}] \otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \mathrm{S} = \mathrm{HH}(\mathrm{S}/\mathrm{S}[\mathbf{Z}])$, which is equivalent to $\mathrm{S}[\mathbf{CP}^\infty]$. The claim then follows from the observation that the even stack $\mathrm{Spev}(\mathrm{S}[\mathbf{CP}^\infty])$ is precisely the Cartier dual of the universal 1-dimensional formal group over $\mathcal{M}_{\mathrm{fg}}$.

Proof. Let $\mathrm{G}_{\mathrm{S}^{\Delta}}$ denote the kernel of the homomorphism $\mathbf{G}_m \times \mathrm{S}^{\Delta} \to (\mathbf{G}_m)_{\mathrm{S}}^{\Delta}$. Let $\mathrm{B} = \mathrm{MU}[\mathbf{Z}]$ (for notational simplicity), so that the composite $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}] \to \mathrm{B}$ is an evenly descendable map. Therefore,

$$(\mathbf{G}_m)_{\mathrm{S}}^{\Delta} = \operatorname{colim}_{\mathbf{\Delta}} \mathrm{Spev}_{\mathrm{S}^1} \Phi(\mathrm{B}^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \bullet+1}).$$

Since the map $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{S}[\mathbf{Z}]$ is faithfully evenly projective, the map $\mathrm{S}[\mathbf{Z}] \to \mathrm{B} \otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \mathrm{S}[\mathbf{Z}]$ is an evenly descendable cover, and so

$$\mathbf{G}_m \times \mathrm{S}^{\mathbb{A}} \cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}(\Phi(\mathrm{B}^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])}\bullet+1}) \otimes_{\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \Phi\mathrm{S}[\mathbf{Z}])).$$

Write $\mathrm{S}^{\mathbb{A}} = \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1} \Phi\mathrm{MU}^{\otimes \bullet+1}$. Then

$$\mathrm{G}_{\mathrm{S}^{\mathbb{A}}} = (\mathbf{G}_m \times \mathrm{S}^{\mathbb{A}}) \times_{(\mathbf{G}_m)^{\mathbb{A}}_{\mathrm{S}}} \mathrm{S}^{\mathbb{A}} \cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}(\Phi(\mathrm{B}^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])}\bullet+1}) \otimes_{\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \Phi\mathrm{S}[\mathbf{Z}])) \times_{\operatorname{Spev}_{\mathrm{S}^1} \Phi(\mathrm{B}^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])}\bullet+1})} \operatorname{Spev}_{\mathrm{S}^1} \Phi\mathrm{MU}^{\otimes \bullet+1}.$$

It is not true in general that $\operatorname{Spev}_{\mathrm{S}^1}$ takes tensor products to fiber products, but this turns out to be true in the present case. It follows that

$$\begin{aligned} \mathrm{G}_{\mathrm{S}^{\mathbb{A}}} &\cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}(\Phi\mathrm{MU}^{\otimes \bullet+1} \otimes_{\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \Phi\mathrm{S}[\mathbf{Z}]) \\ &\cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}(\Phi\mathrm{MU}^{\otimes \bullet+1} \otimes (\mathrm{S} \otimes_{\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \Phi\mathrm{S}[\mathbf{Z}])). \end{aligned}$$

The map $\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \Phi\mathrm{S}[\mathbf{Z}]$ identifies with the map $\mathrm{THH}(\mathrm{S}[\mathbf{Z}])^{\wedge}_p \to \mathrm{S}[\mathbf{Z}]$, so that the tensor product $\mathrm{S} \otimes_{\Phi\mathrm{THH}(\mathrm{S}[\mathbf{Z}])} \Phi\mathrm{S}[\mathbf{Z}]$ identifies with $\mathrm{THH}(\mathrm{S}/\mathrm{S}[\mathbf{Z}])$. There is an S^1-equivariant equivalence

$$\mathrm{THH}(\mathrm{S}/\mathrm{S}[\mathbf{Z}]) \simeq \mathrm{S}[\mathrm{B}^{\lambda}\mathbf{Z}] = \mathrm{S}[\mathrm{B}^2\mathbf{Z}] = \mathrm{S}[\mathbf{CP}^{\infty}],$$

where the identification $\mathrm{B}^{\lambda}\mathbf{Z} \simeq \mathrm{B}^2\mathbf{Z}$ comes from the fact that $\Sigma^{\lambda}\mathbf{Z} \simeq \Sigma^2\mathbf{Z}$ as Borel S^1-equivariant spectra (via the canonical complex orientation of $\mathbf{Z}$). It follows that

$$\mathrm{G}_{\mathrm{S}^{\mathbb{A}}} \cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}((\Phi\mathrm{MU}^{\otimes \bullet+1})[\mathbf{CP}^{\infty}]) = \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}((\Phi\mathrm{MU}^{\otimes \bullet+1})[\mathbf{CP}^{\infty}])^{h\mathrm{S}^1}, (t));$$

but this is easily seen to identify with the Cartier dual of $\mathbf{H}$ over $\widehat{\mathbf{G}}_{\mathrm{univ}}$. □

Corollary 7.2.5. *The line bundle $\mathcal{O}_{\mathrm{S}^{\mathbb{A}}}\{-1\}$, the Lie algebra of $\mathbf{H}$, and the group scheme $\underline{\mathrm{Hom}}(\mathbf{H}^{\vee}, \mathbf{G}_a)$ are all isomorphic as group schemes over $\mathrm{S}^{\mathbb{A}}$. In particular, there is a canonical homomorphism $\mathbf{H}^{\vee} \to \mathcal{O}_{\mathrm{S}^{\mathbb{A}}}\{1\}$, and hence (following Remark 7.2.2) a theory of prismatic Chern classes for $\mathbf{G}_m$-torsors on spectral stacks.*

Corollary 7.2.6. *The Drinfeld formal group over $\mathbf{Z}_p^{\mathbb{A}}$ is the pullback of $\mathbf{H}$ along the canonical map $\mathbf{Z}_p^{\mathbb{A}} \to \mathrm{S}^{\mathbb{A}}$. In particular, the canonical section $v_1 \in \mathrm{H}^0((\mathbf{Z}_p^{\mathbb{A}})_{p=0}; \mathcal{O}\{p-1\})$ from [Bha3, Construction 6.2.1] is the Hasse invariant of the Drinfeld formal group over $\mathbf{Z}_p^{\mathbb{A}}$.*

Proof. By Theorem 7.2.4, it suffices to show that the pullbacks of $\mathbf{G}_m \times \mathrm{S}^{\mathbb{A}}$ and $(\mathbf{G}_m)^{\mathbb{A}}_{\mathrm{S}}$ along the map $\mathbf{Z}_p^{\mathbb{A}} \to \mathrm{S}^{\mathbb{A}}$ identify with $\mathbf{G}_m \times \mathbf{Z}_p^{\mathbb{A}}$ and $\mathbf{G}_m^{\mathbb{A}}$ (and similarly for the natural map between them). This is of course clear for $\mathbf{G}_m \times \mathbf{Z}_p^{\mathbb{A}}$; we will argue the same for $\mathbf{G}_m^{\mathbb{A}}$ (the claim about the natural map between them follows from the proof). For this, note that the map $\mathrm{THH}(\mathrm{S}[\mathbf{Z}]) \to \mathrm{MU}[\mathbf{Z}]$ is evenly descendable by Example 2.2.10 and Example 2.2.11. If we use the evenly descendable map $\mathrm{THH}(\mathbf{Z}_p) \to \mathrm{THH}(\mathbf{Z}_p/\mathrm{MU}) = \mathrm{THH}(\mathbf{Z}_p) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$, then we are attempting to compute the pullback

$$(\mathbf{G}_m)^{\mathbb{A}}_{\mathrm{S}} \times_{\mathrm{S}^{\mathbb{A}}} \mathbf{Z}_p^{\mathbb{A}} \cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1}(\Phi\mathrm{MU}[\mathbf{Z}]^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])}\bullet+1}) \times_{\operatorname{Spev}_{\mathrm{S}^1}(\Phi\mathrm{MU}^{\otimes \bullet+1})} \operatorname{Spev}_{\mathrm{S}^1}(\Phi\mathrm{THH}(\mathbf{Z}_p/\mathrm{MU}^{\otimes \bullet+1})).$$

It is not true in general that $\operatorname{Spev}_{\mathrm{S}^1}$ takes tensor products to fiber products, but this turns out to be true in the present case; so we may identify

$$\begin{aligned} (\mathbf{G}_m)^{\mathbb{A}}_{\mathrm{S}} \times_{\mathrm{S}^{\mathbb{A}}} \mathbf{Z}_p^{\mathbb{A}} &\cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1} \Phi(\mathrm{MU}[\mathbf{Z}]^{\otimes_{\mathrm{THH}(\mathrm{S}[\mathbf{Z}])}\bullet+1} \otimes_{\mathrm{MU}^{\otimes \bullet+1}} \mathrm{THH}(\mathbf{Z}_p/\mathrm{MU}^{\otimes \bullet+1})) \\ &\cong \operatorname{colim}_{\mathbf{\Delta}} \operatorname{Spev}_{\mathrm{S}^1} \Phi(\mathrm{THH}(\mathbf{Z}_p/\mathrm{MU})[\mathbf{Z}]^{\otimes_{\mathrm{THH}(\mathbf{Z}_p[\mathbf{Z}])}\bullet+1}). \end{aligned}$$

The map $\mathrm{THH}(\mathbf{Z}_p[\mathbf{Z}]) \to \mathrm{THH}(\mathbf{Z}_p/\mathrm{MU})[\mathbf{Z}]$ is an evenly descendable cover, so one finds that the colimit above is just $\operatorname{Spev}_{\mathrm{S}^1} \Phi(\mathrm{THH}(\mathbf{Z}_p[\mathbf{Z}])) = \mathbf{G}_m^{\mathbb{A}}$, as desired. □

7.3 Decompleted cyclotomic spectra with trivial S^1-action

One large and interesting class of (decompleted) cyclotomic $\mathbf{E}_\infty$-rings come from the functors $-^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{CycSp}$ and $-^{(-1)} : \mathrm{Sp}_{\geq 0} \to \mathrm{CycSp}$. The resulting stacks and the corresponding interpretation of the maps to the various stacks associated to the sphere spectrum are very interesting. Here, we will explore these stacks in the specific case of decompleted cyclotomic $\mathbf{E}_\infty$-rings of the form $\mathrm{R}^{\mathrm{triv}}$ where R is a p-complete connective even $\mathbf{E}_\infty$-ring equipped with a complex orientation and an isomorphism $\pi_{2*}(\mathrm{R}) \cong \pi_0(\mathrm{R})[\beta]$ with β in degree 2. We will write R_0 to denote $\pi_0(\mathrm{R})$.

Lemma 7.3.1. *Let* X *be a cyclotomic spectrum such that* $\mathrm{X}^{h\mathbf{Z}/p}$ *is even and whose underlying (naive)* $\mathbf{Z}/p$*-spectrum is an* $\mathrm{MU}^{\mathrm{triv}}$*-module. Then* $\beta\mathrm{X}$ *is an even decompleted cyclotomic spectrum.*

Proof. Let us abusively write $\beta\mathrm{X}$ to denote the image of $\beta\mathrm{X}$ under the forgetful functor $\mathrm{CycSp}_{\mathbb{A}} \to \mathrm{Sp}_{\mathbf{Z}/p}$, so that $\beta\mathrm{X}$ is a Borel $\mathbf{Z}/p$-spectrum. We need to show that $\pi_{\mathrm{V}-1}(\beta\mathrm{X})$ vanishes for any virtual complex $\mathbf{Z}/p$-representation V. But $\pi_{\mathrm{V}-1}(\beta\mathrm{X})$ identifies with π_{-1} of the nonequivariant spectrum $(\Sigma^{\mathrm{V}}\beta\mathrm{X})^{\mathbf{Z}/p}$. Since $\beta\mathrm{X}$ is Borel, the natural map

$$(\Sigma^{\mathrm{V}}\beta\mathrm{X})^{\mathbf{Z}/p} \to (\Sigma^{\mathrm{V}}\beta\mathrm{X})^{h\mathbf{Z}/p} = (\Sigma^{\mathrm{V}}\mathrm{X})^{h\mathbf{Z}/p}$$

is an equivalence, so we are reduced to showing that the latter is even for any virtual complex $\mathbf{Z}/p$-representation V.

Note that $(\Sigma^{\mathrm{V}}\mathrm{X})^{h\mathbf{Z}/p}$ depends only on the underlying naive $\mathbf{Z}/p$-spectrum of X. Since X is an $\mathrm{MU}^{\mathrm{triv}}$-module, a choice of MU-Thom class for V gives a $\mathbf{Z}/p$-equivariant equivalence

$$\Sigma^{\mathrm{V}}\mathrm{X} \simeq \Sigma^{\mathrm{V}}\mathrm{MU} \otimes_{\mathrm{MU}} \mathrm{X} = \Sigma^{\dim_{\mathbf{R}}(\mathrm{V})}\mathrm{MU} \otimes_{\mathrm{MU}} \mathrm{X} = \Sigma^{\dim_{\mathbf{R}}(\mathrm{V})}\mathrm{X}.$$

We may therefore identify $(\Sigma^{\mathrm{V}}\mathrm{X})^{h\mathbf{Z}/p} \simeq \Sigma^{\dim_{\mathbf{R}}(\mathrm{V})}\mathrm{X}^{h\mathbf{Z}/p}$. Since $\mathrm{X}^{h\mathbf{Z}/p}$ is even by assumption, and $\dim_{\mathbf{R}}(\mathrm{V})$ is an even integer, we concldue that $(\Sigma^{\mathrm{V}}\mathrm{X})^{h\mathbf{Z}/p}$ is even, as desired. ☐

Lemma 7.3.2. *Suppose that the Tate construction* $\mathrm{R}^{t\mathbf{Z}/p}$ *(for the trivial* $\mathbf{Z}/p$*-action) is even. Then* $\mathrm{R}^{h\mathbf{Z}/p}$ *is also even.*

Proof. The assumption on R tells us that $\pi_*(\mathrm{R}^{h\mathrm{S}^1}) \cong \mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar = \theta)$. Since $\pi_*(\mathrm{R}^{h\mathbf{Z}/p}) \cong \pi_*(\mathrm{R}^{h\mathrm{S}^1})/[p](\hbar)$, where $[p](\hbar)$ denotes the p-series of $\hbar$, the desired evenness follows once we show that $[p](\hbar)$ is not a regular element. Note that $\pi_*(\mathrm{R}^{h\mathrm{S}^1})$ is t-torsionfree, so it injects into $\pi_*(\mathrm{R}^{t\mathrm{S}^1}) \cong \pi_*(\mathrm{R}^{h\mathrm{S}^1})[\hbar^{-1}]$; so to check that $[p](\hbar)$ is a regular element in $\pi_*(\mathrm{R}^{h\mathrm{S}^1})$, it suffices to do so upon inverting $\hbar$. But $\pi_*(\mathrm{R}^{t\mathbf{Z}/p}) \cong \pi_*(\mathrm{R}^{t\mathrm{S}^1})/[p](\hbar)$ is even, so $[p](\hbar)$ is a regular element in $\pi_*(\mathrm{R}^{t\mathrm{S}^1})$, as desired. ☐

Proposition 7.3.3. *Suppose that the Tate construction* $\mathrm{R}^{t\mathbf{Z}/p}$ *(for the trivial* $\mathbf{Z}/p$*-action) is even. Then* $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \cong \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{R}^{t\mathbf{Z}/p})$*, and the maps* $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to \mathrm{Spev}(\mathrm{R})$ *and* $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}} \to \mathrm{Spev}(\mathrm{R})$ *are smooth and affine of relative dimension* 1*.*

Proof. First, note that since $(\beta\mathrm{R}^{\mathrm{triv}})$ is a Borel $\mathbf{Z}/p$-spectrum, we may identify $((\beta\mathrm{R}^{\mathrm{triv}}))^{\mathbf{Z}/p} \cong \mathrm{R}^{h\mathbf{Z}/p}$. This $\mathbf{E}_\infty$-ring is even by the assumption on $\mathrm{R}^{t\mathbf{Z}/p}$ and Lemma 7.3.2. Note, also, that R is complex-oriented, so Lemma 7.3.1 implies that $(\beta\mathrm{R}^{\mathrm{triv}})$ is even as a decompleted cyclotomic $\mathbf{E}_\infty$-ring. Therefore,

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \cong \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}((\mathrm{R}^{t\mathbf{Z}/p})^{h\mathrm{S}^1}), (\hbar)).$$

By the Tate orbit lemma, there is an equivalence $(\mathrm{R}^{t\mathbf{Z}/p})^{h\mathrm{S}^1} \simeq \mathrm{R}^{t\mathrm{S}^1}$, and so

$$\pi_{2*}((\mathrm{R}^{t\mathbf{Z}/p})^{h\mathrm{S}^1}) \cong \pi_{2*}(\mathrm{R}^{t\mathrm{S}^1}) \cong \mathrm{R}_0[\![\theta]\!][\beta,\hbar^{\pm 1}]/(\beta\hbar=\theta). \tag{7.3.1}$$

The topology is generated by the ideal $(p,[p](\hbar))$, where $[p](\hbar)$ denotes the p-series of the formal group law on $\pi_{2*}(\mathrm{R}^{h\mathrm{S}^1})$. Since $\hbar$ is a unit, this is equivalently the topology generated by the ideal $(p,\langle p\rangle(\hbar))$, where $\langle p\rangle(\hbar)=\frac{[p](\hbar)}{\hbar}$ lives in weight 0. The map $f:(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}}\to\mathrm{Spev}(\mathrm{R})$ is therefore given by applying $\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}$ to the continuous graded ring map $\mathrm{R}_0[\beta]\to\mathrm{R}_0[\![\theta]\!][\beta,\hbar^{\pm1}]/(\beta\hbar=\theta)$. The map f is therefore evidently affine and smooth of relative dimension 1.

Let us now turn to $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}$. Since $(\beta\mathrm{R}^{\mathrm{triv}})$ is even as a decompleted cyclotomic $\mathbf{E}_\infty$-ring, we may identify $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}$ with the $\mathbf{v}_0$-completion of

$$\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}((\mathrm{R}^{h\mathbf{Z}/p})^{h\mathrm{S}^1}),(\hbar))\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar=\theta),(p,[p](\hbar))).$$

Since $\mathbf{v}_0=\frac{[p](\hbar)}{\hbar}=\langle p\rangle(\hbar)$, we find upon taking the $\mathbf{v}_0$-completion that

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar=\theta),(p,\langle p\rangle(\hbar)).$$

The map $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}\to\mathrm{Spev}(\mathrm{R})$ is given by $\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}$ to the continuous graded ring map $\mathrm{R}_0[\beta]\to\mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar=\theta)$. This map is therefore evidently affine and smooth of relative dimension 1. ☐

Proposition 7.3.4. *Suppose that the Tate construction $\mathrm{R}^{t\mathbf{Z}/p}$ (for the trivial $\mathbf{Z}/p$-action) is even. Then there are isomorphisms*

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}}\cong\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\theta)),$$
$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar=\theta),(p,\theta)).$$

Proof. It follows from (7.3.1) that there are isomorphisms

$$\begin{aligned}(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}}&\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}((\mathrm{R}^{t\mathbf{Z}/p})^{h\mathrm{S}^1}),(p,\langle p\rangle(\hbar)))\\&\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta,\hbar^{\pm1}]/(\beta\hbar=\theta),(p,\langle p\rangle(\hbar)))\\&\cong\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\langle p\rangle(\hbar))).\end{aligned}$$

Note that here, $\langle p\rangle(\hbar)$ is viewed as an element of $\pi_0(\mathrm{R}^{t\mathbf{Z}/p})^{h\mathrm{S}^1}\cong\mathrm{R}_0[\![\theta]\!]$. Similarly, we already saw in Proposition 7.3.3 that

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}\cong\mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta,\hbar]/(\beta\hbar=\theta),(p,\langle p\rangle(\hbar)).$$

To finish, it suffices to show that $(p,\langle p\rangle(\hbar))$-completion identifies with (p,θ)-adic completion. Since $\mathrm{R}^{t\mathbf{Z}/p}$ is even and $\pi_*(\mathrm{R}^{t\mathbf{Z}/p})\cong\pi_*(\mathrm{R}^{t\mathrm{S}^1})/[p](\hbar)$, we see that $[p](\hbar)$ is nonzero. Therefore, $[p](\hbar)\equiv\lambda\hbar^{p^h}+\nu\mathrm{O}(\hbar^{p^h+1}) \pmod p$, where $\lambda\in\pi_{2p^h-2}(\mathrm{R})\cong(\beta^{p^h-1})$ is of the form $\beta^{p^h-1}\lambda'$ for a unit $\lambda'\in\mathrm{R}_0$, and $\nu\in\pi_{2p^h}(\mathrm{R})\cong(\beta^{p^h})$ is of the form $\nu=\beta^{p^h}\nu'$ with $\nu'\in\mathrm{R}_0$. It follows that

$$\begin{aligned}\langle p\rangle(\hbar)=\frac{[p](\hbar)}{\hbar}&\equiv\hbar^{p^h-1}(\lambda+\nu\mathrm{O}(\hbar))\pmod p\\&\equiv(\beta\hbar)^{p^h-1}(\lambda'+\beta\nu'\mathrm{O}(\hbar))\pmod p.\end{aligned}$$

Because $\beta\hbar=\theta$, λ' is a unit, and $\beta\nu'\mathrm{O}(\hbar)=\nu'\mathrm{O}(\theta)$ is topologically nilpotent, we see that $\langle p\rangle(\hbar)$ is congruent modulo p to a unit multiple of θ^{p^h-1}. This implies the desired claim. ☐

We now compute the Frobenius and canonical maps $(\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to (\beta \mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}$.

Proposition 7.3.5. *Suppose that the Tate construction $\mathrm{R}^{t\mathbf{Z}/p}$ (for the trivial $\mathbf{Z}/p$-action) is even. Under the isomorphisms of Proposition 7.3.4, the map* $\mathrm{can} : (\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to (\beta \mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}$ *is given by the open immersion*

$$\mathrm{can} : \mathrm{Spf}(\mathrm{R}_0[\![\theta]\!], (p, \theta)) \to \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta, \hbar]/(\beta\hbar = \theta), (p, \theta))$$

corresponding to the locus where $\hbar$ is a unit. The map $\varphi : (\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to (\beta \mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}}$ is given by the map

$$\varphi : \mathrm{Spf}(\mathrm{R}_0[\![\theta]\!], (p, \theta)) \to \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][\beta, \hbar]/(\beta\hbar = \theta), (p, \theta))$$

sending $\theta \mapsto \theta \cdot \langle p\rangle(\hbar)$. Here, $\langle p\rangle(\hbar) = \frac{[p](\hbar)}{\hbar}$ is viewed as an element of $\mathrm{R}_0[\![\theta]\!]$.

Proof. Recall that

$$(\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} = \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}(\mathrm{R}^{t\mathrm{S}^1}), (p, \langle p\rangle(\hbar))),$$
$$(\beta \mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}} = \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}(\mathrm{R}^{h\mathrm{S}^1}), (p, \langle p\rangle(\hbar))).$$

The canonical map is induced by the natural map $\mathrm{can} : \mathrm{R}^{h\mathrm{S}^1} \to \mathrm{R}^{t\mathrm{S}^1}$, which inverts t. The Frobenius map is induced by applying homotopy S^1-fixed points to the composite $\mathrm{R} \to \mathrm{R}^{h\mathbf{Z}/p} \to \mathrm{R}^{t\mathbf{Z}/p}$. That is, it sends $\beta \mapsto \beta$ and $\hbar \mapsto [p](\hbar)$; so $\theta = \beta\hbar$ is sent to $\beta[p](\hbar) = \theta\langle p\rangle(\hbar)$. □

Let us now investigate the relationship between these stacks and the analogous stacks for the sphere spectrum. The unit map $\mathrm{S} \to \mathrm{R}$ induces a commutative diagram

$$\begin{tikzcd}
(\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \arrow[r] \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\mathrm{can}"] & \mathrm{S}^{\mathbb{A}} \cong \widehat{\mathbf{G}}_{\mathrm{univ}} \arrow[d, shift right, "\varphi"'] \arrow[d, shift left, "\mathrm{can}"] \\
(\beta \mathrm{R}^{\mathrm{triv}})^{\mathrm{Nyg}} \arrow[r] \arrow[d] & \mathrm{S}^{\mathrm{Nyg}} \arrow[d] \\
\mathrm{Spev}(\mathrm{R}) \arrow[r] & \mathrm{Spev}(\mathrm{S}) \cong \mathcal{M}_{\mathrm{fg}}.
\end{tikzcd}$$

Therefore, $(\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}}$ carries a canonical 1-dimensional formal group law with a section; the underlying formal group law is given by pulling back the formal group law over $\mathrm{Spev}(\mathrm{R})$ along the map $(\beta \mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to \mathrm{Spev}(\mathrm{R})$.

Recollection 7.3.6. Let S be a (formal) pre-algebraic stack in the sense of [Dri1, Section 3.6]. A formal polydisk over S (in the sense of [Dri1, Section 3]) is a pair $(\mathrm{X}, \Sigma : \mathrm{S} \to \mathrm{X})$,where X is a formal scheme over S, and $\Sigma : \mathrm{S} \to \mathrm{X}$ is a section, where this data is locally isomorphic to $\hat{\mathbf{A}}^n_{\mathrm{S}}$ and the zero section $0 : \mathrm{S} \to \hat{\mathbf{A}}^n_{\mathrm{S}}$.

Let $(\Delta, i_0 : \mathrm{S} \to \Delta)$ denote a 1-dimensional formal polydisk over S, and let $\pi : \Delta \to \mathrm{S}$ denote the projection. Then the divisor $\mathrm{D} := i_0(\mathrm{S}) \subseteq \Delta$ defines a section of the line bundle $\mathcal{O}(-\mathrm{D})$ over Δ. As in [Dri1, Section 3.4], we may therefore rescale the pullback $\pi^*\mathrm{X}$ to obtain a new formal polydisk $\widetilde{\mathrm{X}}$ over Δ. If X was a group object in formal polydisks over S (e.g., a 1-dimensional formal group), then $\widetilde{\mathrm{X}}$ will inherit the structure of a group object in formal polydisks over Δ. We will refer to $\widetilde{\mathrm{X}}$ as the rescaling of $\pi^*\mathrm{X}$. There is a canonical map $\widetilde{\mathrm{X}} \to \pi^*\mathrm{X}$ of formal polydisks over Δ.

In particular, if X is itself a 1-dimensional formal polydisk over S, we may set $\Delta = \mathrm{X}$, and consider the rescaling $\widetilde{\mathrm{X}}$ as a formal polydisk over X. There is a canonical map $\mathrm{X} \to \pi^*\mathrm{X}$, which factors through the map $\widetilde{\mathrm{X}} \to \pi^*\mathrm{X}$. Therefore, $\widetilde{\mathrm{X}}$ admits a canonical section $s : \mathrm{X} \to \widetilde{\mathrm{X}}$.

Construction 7.3.7. The map $\mathrm{Spev}(\mathrm{R}) = \mathrm{Spec}(\mathrm{R}_0[u])/\mathbf{G}_m \to \mathcal{M}_{\mathrm{fg}}$ classifies the graded formal group given by

$$\begin{aligned}\widehat{\mathbf{G}} = \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{R}) &= \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\pi_{2*}(\mathrm{R}^{h\mathrm{S}^1}), (\hbar)) \\ &\cong \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta]\!][u,t]/(ut=\theta), (p,t)).\end{aligned}$$

Let $\pi : \widehat{\mathbf{G}} \to \mathrm{Spev}(\mathrm{R})$ denote the projection, and let $i_0 : \mathrm{Spev}(\mathrm{R}) \to \widehat{\mathbf{G}}$ denote the zero section. As in Recollection 7.3.6, we obtain the *rescaling* $\widetilde{\widehat{\mathbf{G}}}_t$ of $\pi^*\widehat{\mathbf{G}}$, equipped with a canonical section $s : \widehat{\mathbf{G}} \to \widetilde{\widehat{\mathbf{G}}}_t$. Explicitly, the underlying formal stack of $\widetilde{\widehat{\mathbf{G}}}_t$ agrees with $\pi^*\widehat{\mathbf{G}}$, so

$$\widetilde{\widehat{\mathbf{G}}}_t \cong \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta,\gamma]\!][\beta,\hbar,t']/(\beta\hbar=\theta, \beta t' = \gamma), (p,\hbar,t')).$$

The section $s : \widehat{\mathbf{G}} \to \widetilde{\widehat{\mathbf{G}}}_t$ then sends

$$s : t' \mapsto \hbar. \tag{7.3.2}$$

The S^1-equivariant map $\mathrm{can} : \mathrm{R} \to \mathrm{R}^{t\mathbf{Z}/p}$ induces a map

$$\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!], (p,\theta)) \cong (\beta\mathrm{R}^{\mathrm{triv}})^{\triangle} = \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{R}^{t\mathbf{Z}/p}) \to \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{R}) = \widehat{\mathbf{G}}, \tag{7.3.3}$$

which factors the map f from Proposition 7.3.3. Note that the map $\mathrm{can} : \mathrm{R} \to \mathrm{R}^{t\mathbf{Z}/p}$ factors through the S^1-equivariant unit map $\mathrm{R} \to \mathrm{R}^{h\mathbf{Z}/p}$, which in turn defines the map $\mathrm{can}^{h\mathrm{S}^1} : \mathrm{R}^{h\mathrm{S}^1} \to \mathrm{R}^{h\mathrm{S}^1}$ induced by the degree-p map on S^1. In particular, $\mathrm{can}^{h\mathrm{S}^1}$ sends $\hbar \mapsto [p](\hbar)$.

Pulling back the data $(\widetilde{\widehat{\mathbf{G}}}_t, s)$ along the map (7.3.3) defines a 1-dimensional formal group with a section $(\widehat{\mathbf{G}}_\theta, s)$ over $\mathrm{R}_0[\![\theta]\!]$. It follows from the discussion above that

$$\begin{aligned}\widehat{\mathbf{G}}_\theta &\cong \mathrm{Spf}_{\mathrm{B}\mathbf{G}_m}(\mathrm{R}_0[\![\theta,\theta']\!][\beta,\hbar^{\pm 1},t']/(\beta\hbar=\theta, \hbar^{-1}t'=\theta'), (p,[p](\hbar),t')) \\ &\cong \mathrm{Spf}(\mathrm{R}_0[\![\theta,\theta']\!], (p,\theta,\theta')).\end{aligned}$$

If $\mathrm{F}(x,y)$ is the formal group law over $\mathrm{Spev}(\mathrm{R}[\beta^{-1}]) \cong \mathrm{Spf}(\mathrm{R}_0,(p))$ arising from a choice of coordinate on on $\widehat{\mathbf{G}}\times_{\mathrm{Spev}(\mathrm{R})}\mathrm{Spev}(\mathrm{R}[\beta^{-1}])$, then the formal group law on $\widehat{\mathbf{G}}_\theta$ over $\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\theta))$ is determined by the formula

$$\widetilde{\mathrm{F}}(x,y) = \tfrac{1}{\theta}\mathrm{F}(\theta x, \theta y).$$

Since can sends $t \mapsto [p](\hbar)$, we find from (7.3.2) that the section $s : \mathrm{Spf}(\mathrm{R}_0[\![\theta]\!]) \to \widehat{\mathbf{G}}_\theta$ sends $t' \mapsto [p](\hbar)$. In terms of θ': the section s sends

$$s : \theta' = t'/\hbar \mapsto [p](\hbar)/\hbar = \langle p\rangle(\hbar) \in \mathrm{R}_0[\![\theta]\!].$$

The preceding discussion implies:

Proposition 7.3.8. *Suppose that the Tate construction $\mathrm{R}^{t\mathbf{Z}/p}$ (for the trivial $\mathbf{Z}/p$-action) is even. Let $\widehat{\mathbf{G}}$ denote the graded formal group over $\mathrm{R}_0[u]$ classified by the map $\mathrm{Spev}(\mathrm{R}) \to \mathcal{M}_{\mathrm{fg}}$. Then the map*

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\triangle} \cong \mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\theta)) \to \mathrm{S}^{\triangle} \cong \widehat{\mathbf{G}}_{\mathrm{univ}}$$

classifies the pair $(\widehat{\mathbf{G}}_\theta, s)$ from Construction 7.3.7.

It follows from Proposition 7.3.8, Theorem 7.1.11, and Corollary 7.1.12 that:

Corollary 7.3.9. *Suppose that the Tate construction $\mathrm{R}^{t\mathbf{Z}/p}$ (for the trivial $\mathbf{Z}/p$-action) is even. Under the identifications of Theorem 7.1.11, the composite map*

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to \mathrm{S}^{\mathbb{A}} \xrightarrow{\varphi} \mathrm{S}^{\mathrm{Nyg}}$$

classifies the $\mathbf{v}_0$-formal group $\widehat{\mathbf{G}}_\theta$ over $\mathrm{R}_0[\![\theta]\!]$ given by $\underline{\mathbf{Z}} \to \widehat{\mathbf{G}}_\theta$ sending $1 \mapsto s$. Similarly, the composite map

$$(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \to \mathrm{S}^{\mathbb{A}} \xrightarrow{\mathrm{can}} \mathrm{S}^{\mathrm{Nyg}}$$

classifies the $\mathbf{v}_0$-formal group C_θ over $\mathrm{R}_0[\![\theta]\!]$ given by the pushout

$$\begin{tikzcd} \underline{p\mathbf{Z}} \arrow[r, "p \mapsto \langle p\rangle(t)"] \arrow[d] & \widehat{\mathbf{G}}_\theta \arrow[d] \\ \underline{\mathbf{Z}} \arrow[r, "\alpha"] & \mathrm{C}_\theta. \end{tikzcd}$$

Moreover, when pulled back along the map

$$i : (\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{HT}} \cong \mathrm{Spev}(\mathrm{R}^{t\mathbf{Z}/p}) \to (\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \cong \mathrm{Spev}_{\mathrm{S}^1}(\mathrm{R}^{t\mathbf{Z}/p}),$$

there is an exact sequence

$$0 \to \underline{\mathbf{Z}/p}_{(\beta\mathrm{R}^{\mathrm{triv}})^{\mathrm{HT}}} \xrightarrow{\alpha} i^*\mathrm{C}_\theta \to i^*\widehat{\mathbf{G}}_\theta \to 0.$$

Of course, C_θ is isomorphic to $\underline{\mathbf{Z}/p}_{(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}}} \times \widehat{\mathbf{G}}_\theta$ as *schemes*; but the *group* structure on C_θ is much more interesting, and we will now describe it. Since C_θ is a commutative group scheme, our task is equivalent to calculating the Cartier dual $\mathrm{C}_\theta^\vee$ as a scheme. We will accomplish this in Theorem 7.4.26.

Example 7.3.10. It follows from Example 7.4.27 below that if $\mathrm{R} = \mathrm{ku}_p$, then the Cartier dual of C_θ over $\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!]) = \mathrm{Spf}(\mathbf{Z}_p[\![q-1]\!])$ is given by

$$\mathrm{C}_\theta^\vee \cong \mathrm{Spf}\,\mathrm{R}[\![t]\!]\left[y^{\pm 1}, \frac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q!}\right]_{n\geq 0}.$$

The latter is a q-deformed version $\mathbf{G}_m^{\sharp,q}$ of the divided power hull of $1 \in \mathbf{G}_m$. We establish the above isomorphism through computational methods in the next section, but it can also be proved using Theorem 6.4.1, which implies that there is an S^1-equivariant equivalence

$$\mathrm{HH}(\mathrm{ku}_p[\mathbf{Z}]/\mathrm{ku}_p)^{t\mathbf{Z}/p} \simeq \mathrm{THH}(\mathbf{Z}_p[\zeta_p][\mathbf{Z}]/\mathrm{S}[\![q^{1/p}-1]\!])[\mu^{-1}].$$

Remark 7.3.11. In this section, we focused mainly on the example of decompleted cyclotomic spectra with trivial S^1-action. One could also run the above analysis with R replaced by $\mathrm{R}^{(-1)}$: here, $\mathrm{R}^{(-1)}$ is viewed as an object of $\mathrm{CycSp}_{\mathbb{A}}$ via $\Phi(\mathrm{R}^{(-1)}) := \mathrm{R}^{t\mathbf{Z}/p}$. Note that the cyclotomic Frobenius $\mathrm{R}^{(-1)} \to (\mathrm{R}^{(-1)})^{t\mathbf{Z}/p}$ does in fact factor through the connective cover map $\mathrm{R}^{(-1)} \to \mathrm{R}^{t\mathbf{Z}/p}$. There is a natural map $\mathrm{R}^{\mathrm{triv}} \to \mathrm{R}^{(-1)}$ of decompleted cyclotomic spectra, so that our calculations above describing the (equivariant) formal group over the various stacks associated to R also determine the corresponding stacks over $\mathrm{R}^{(-1)}$. In fact, at least in the case of the prismatic stack, there is an isomorphism $(\beta\mathrm{R}^{\mathrm{triv}})^{\mathbb{A}} \cong (\mathrm{R}^{(-1)})^{\mathbb{A}}$ as long as $\mathrm{R}^{t\mathbf{Z}/p}$ and $\mathrm{R}^{t\mathbf{Z}/p^2}$ are both even. Since many objects of interest, like $\mathrm{THH}(\mathbf{Z}_p[\zeta_p]/\mathrm{S}[\![q-1]\!])$ and $\mathrm{THH}(\mathbf{Z}_p[\zeta_p])$ (by Theorem 6.1.4 and Theorem 6.4.1), can be expressed as $\mathrm{R}^{(-1)}$ for various $\mathbf{E}_\infty$-rings that (nearly) fit the assumptions made above, the preceding analysis has many concrete implications for arithmetic geometry. We will elaborate on these applications in future work.

7.4 Calculating an S^1-equivariant formal group

Our goal in this section is to describe the Cartier dual of the group scheme C_θ over $\mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\theta))$ from Corollary 7.3.9; see Theorem 7.4.26 for the statement. This is joint work [DM] with M. Misterka. The Cartier dual $\mathrm{C}_\theta^\vee$ sits in a pullback square

$$\begin{tikzcd} \mathrm{C}_\theta^\vee \arrow[r] \arrow[d] & \widehat{\mathbf{G}}_\theta^\vee \arrow[d] \\ \mathbf{G}_m \arrow[r, "\mathrm{Frob}"] & \mathbf{G}_m^{(1)}. \end{tikzcd} \tag{7.4.1}$$

We begin with a brief review of the fiber of C_θ when $\theta = 0$:

Example 7.4.1. The base-change of the group scheme $\widehat{\mathbf{G}}_\theta$ along the map $\mathrm{Spf}(\mathrm{R}_0,(p)) \to \mathrm{Spf}(\mathrm{R}_0[\![\theta]\!],(p,\theta))$ identifies with $\widehat{\mathbf{G}}_a$. Its Cartier dual is therefore the divided power hull $\mathbf{G}_a^\sharp$ of $0 \in \mathbf{G}_a$, so that the diagram (7.4.1) is given by

$$\begin{tikzcd} \mathrm{C}_\theta^\vee|_{\theta=0} \arrow[r] \arrow[d] & \mathbf{G}_a^\sharp \arrow[d] \\ \mathbf{G}_m \arrow[r, "\mathrm{Frob}"'] & \mathbf{G}_m^{(1)}. \end{tikzcd}$$

The right vertical map $\mathbf{G}_a^\sharp \to \mathbf{G}_m^{(1)}$ is given by the map $x \mapsto \exp(px) = \sum_{n\geq 0} \frac{p^n x^n}{n!}$. In [BL, Lemma 3.5.18], it is shown that there is an isomorphism $\mathrm{C}_\theta^\vee|_{\theta=0} \cong \mathbf{G}_m^\sharp$, the divided power hull of $1 \in \mathbf{G}_m$. Under this isomorphism, the map $\mathrm{C}_\theta^\vee|_{\theta=0} \to \mathbf{G}_m$ identifies with the canonical map $\mathbf{G}_m^\sharp \to \mathbf{G}_m$, and the map $\mathrm{C}_\theta^\vee|_{\theta=0} \to \mathbf{G}_a^\sharp$ identifies with the logarithm $\mathbf{G}_m^\sharp \to \mathbf{G}_a^\sharp$. Note that this statement already hides a nontrivial claim: namely, the logarithm $\log(x) \in \mathcal{O}_{\mathbf{G}_m^\sharp} = \mathbf{Z}_p[x^{\pm 1}, \frac{(x-1)^n}{n!}]$ in fact admits divided powers. One way to see this is to note that there is an equality of power series

$$\sum_{n\geq 0} \tfrac{\log(x)^n}{n!} t^n = \exp(t\log(x)) = (1+(x-1))^t = \sum_{n\geq 0} t(t-1)\cdots(t-(n-1))\tfrac{(x-1)^n}{n!}; \tag{7.4.2}$$

equating coefficients then expresses $\frac{\log(x)^n}{n!}$ as an element of $\mathcal{O}_{\mathbf{G}_m^\sharp}$.

Remark 7.4.2. The fact that there is a pullback square

$$\begin{tikzcd} \mathbf{G}_m^\sharp \arrow[r] \arrow[d] & \mathbf{G}_a^\sharp \arrow[d] \\ \mathbf{G}_m \arrow[r, "\mathrm{Frob}"'] & \mathbf{G}_m^{(1)} \end{tikzcd}$$

implies that there is an isomorphism of stacks $\mathbf{G}_m/\mathbf{G}_m^\sharp \cong \mathbf{G}_m^{(1)}/\mathbf{G}_a^\sharp$. The left-hand side identifies with the de Rham stack of $\mathbf{G}_m$, while the right-hand side identifies (by Construction 7.2.1) with the Frobenius twist of the prismatization of $(\mathbf{G}_m)_{\mathbf{F}_p}$. In particular, the preceding pullback square is a stacky version of the comparison between de Rham cohomology of $(\mathbf{G}_m)_{\mathbf{Z}_p}$ and the crystalline cohomology of $(\mathbf{G}_m)_{\mathbf{F}_p}$.

We will perform an analysis similar to Example 7.4.1 to calculate $\mathrm{C}_\theta^\vee$ in general. Below, we will replace the variable θ by t for notational simplicity. We first need some preliminaries:

Definition 7.4.3. Let R be an arbitrary torsionfree commutative ring (we will later specialize to the p-nilpotent setting). Let F be a formal group law over R (in the case at hand, this will be the one associated to the formal group $\widehat{\mathbf{H}} \times_{\mathrm{Spev(R)}} \mathrm{Spev}(\mathrm{R}[\beta^{-1}])$ over $\mathrm{Spev}(\mathrm{R}[\beta^{-1}]) \cong \mathrm{Spf}(\mathrm{R},(p))$), so that F defines a coproduct on the ring $\mathrm{R}[\![t]\!]$. (Here, unlike in the previous sections, t lives in weight zero.) We will use the symbol $\widetilde{\mathrm{F}}$ to denote the new formal group law over $\mathrm{R}[\![t]\!]$ obtained from Construction 7.3.7, so that $\widetilde{\mathrm{F}}(x,y) = \frac{\mathrm{F}(tx,ty)}{t}$. Write $\langle n\rangle_\mathrm{F}$ to denote the quotient $[n](t)/t$, where $[n](t)$ is the n-fold sum of t under the formal group law F. Similarly, let $n!_\mathrm{F} = \langle 1\rangle_\mathrm{F}\langle 2\rangle_\mathrm{F}\cdots\langle n-1\rangle_\mathrm{F}\langle n\rangle_\mathrm{F}$. Let $\mathrm{R}[\![t]\!][\langle\mathbf{N}\rangle^{-1}]$ denote the localization $\mathrm{R}[\![t]\!][\langle 2\rangle_\mathrm{F}^{-1}, \langle 3\rangle_\mathrm{F}^{-1}, \langle 4\rangle_\mathrm{F}^{-1}, \cdots]$. Then, let $\binom{n}{k}_\mathrm{F} = \frac{\langle n\rangle_\mathrm{F}}{\langle k\rangle_\mathrm{F}\langle n-k\rangle_\mathrm{F}}$. Note that this element in fact lies in $\mathrm{R}[\![t]\!]$. This follows by induction on n: the base case ($n=0$) is trivial; for the inductive step, one has

$$\binom{n}{k}_\mathrm{F} = \binom{n-1}{k-1}_\mathrm{F} + \tfrac{\langle n\rangle_\mathrm{F} - \langle k\rangle_\mathrm{F}}{\langle n-k\rangle_\mathrm{F}}\binom{n-1}{k}_\mathrm{F},$$

and the fraction $\frac{\langle n\rangle_\mathrm{F} - \langle k\rangle_\mathrm{F}}{\langle n-k\rangle_\mathrm{F}}$ does in fact lie in $\mathrm{R}[\![t]\!]$.

Define an operator ∇_F on $\mathrm{R}[\![t]\!][x]$ by declaring that it is $\mathrm{R}[\![t]\!]$-linear and is given on monomials by the formula

$$\nabla_\mathrm{F}(x^n) = \langle n\rangle_\mathrm{F} x^{n-1};$$

this is called the F-*derivative*. This, for instance, allows one to construct a two-term complex

$$\mathrm{F}\Omega_{\mathbf{A}^1} := [\mathrm{R}[\![t]\!][x] \xrightarrow{\nabla_\mathrm{F}} \mathrm{R}[\![t]\!][x]d_\mathrm{F}x]$$

called the F-*de Rham complex*; we will write $\mathrm{F}\Omega_{\mathbf{A}^1}$ to denote the underlying graded $\mathrm{R}[\![t]\!]$-module (i.e., where ∇_F is replaced by 0). It is easy to check that the F-de Rham complex behaves quite similarly to the usual de Rham complex. For instance, the Cartier isomorphism holds:

$$\mathrm{H}^*(\mathrm{F}\Omega_{\mathbf{A}^1} \otimes_{\mathrm{R}[\![t]\!]} \mathrm{R}[\![t]\!]/\langle p\rangle_\mathrm{F}) \cong \mathrm{F}\Omega^*_{(\mathbf{A}^1)^{(1)}} \otimes_{\mathrm{R}[\![t]\!]} \mathrm{R}[\![t]\!]/\langle p\rangle_\mathrm{F}.$$

Example 7.4.4. Suppose F is the additive formal group law over $\mathbf{Z}$. Then $\langle n\rangle_\mathrm{F} = n$, and the operator ∇_F is the usual derivative (so $\mathrm{F}\Omega_{\mathbf{A}^1}$ is just the base-change $\Omega^\bullet_{\mathbf{A}^1} \otimes_\mathbf{Z} \mathbf{Z}[\![t]\!]$). If F is the multiplicative formal group law over $\mathbf{Z}$, and we write $t = q-1$, then $\langle n\rangle_\mathrm{F}$ is the q-integer $[n]_q$. It follows that the operator ∇_F is the q-derivative sending $f(x) \mapsto \frac{f(qx)-f(x)}{(q-1)x}$, and $\mathrm{F}\Omega_{\mathbf{A}^1}$ is the corresponding q-de Rham complex. For a general formal group law, it is not so easy to describe the operator ∇_F on a general function (in a manner which is not given monomial-by-monomial).

Remark 7.4.5. We assumed above that R is a torsionfree commutative ring. Since the *universal* case of the Lazard ring is torsionfree, our discussion below will be valid with this choice of R. The universality of L then implies that our formulas below (in particular, the key (7.4.8) and Corollary 7.4.23) will be valid in the p-nilpotent setting as well.

Remark 7.4.6. The operator ∇_F was discovered simultaneously and independently by myself and A. Raksit. The construction of $\mathrm{F}\Omega_{\mathbf{A}^1}$ was first proposed by A. Raksit in connection to the even filtration on $\mathrm{HP}(k[x]/k)$ where k is the connective cover of an even-periodic $\mathbf{E}_\infty$-ring. For me, the operator ∇_F arose in the context of Proposition 3.5.4. Namely, observe that the operator $x\nabla_\mathrm{F}$ satisfies the following commutation rule:

$$(x\nabla_\mathrm{F})(x) = x((x\nabla_\mathrm{F}) +_{\widetilde{\mathrm{F}}} 1).$$

The discussion in Proposition 3.5.4 tells us that if k is the connective cover of an even-periodic $\mathbf{E}_\infty$-ring, $\mathrm{R} = \pi_0(k)$, and $\mathrm{F}\mathcal{D}_{\mathbf{G}_m}$ denotes the "F-Weyl algebra" defined as the associative $\mathrm{R}[\![t]\!]$-algebra generated by $x^{\pm 1}$ and $x\nabla_\mathrm{F}$ subject to the above relation, then there is an isomorphism

$$\mathrm{F}\mathcal{D}_{\mathbf{G}_m} \cong \pi_0(\mathrm{C}_*^{\mathrm{T}\times \mathrm{S}^1_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{T}; k)[1/\hbar]).$$

Here, $\mathrm{T} = \mathbf{G}_m$, we work with Borel-equivariant Borel-Moore k-homology, and $\hbar$ is the generator of $\pi_{-2}(k^{h\mathrm{S}^1_{\mathrm{rot}}})$. The analogue of the Cartier isomorphism in this "Koszul dual" context is explained by calculating that $\pi_0(\mathrm{C}_*^{\mathrm{T}\times (\mathbf{Z}/p)_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{T}; k)[1/\hbar])$ has a large center:

$$\begin{aligned}
\mathrm{Z}(\pi_0(\mathrm{C}_*^{\mathrm{T}\times (\mathbf{Z}/p)_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{T}; k)[1/\hbar])) &\cong \mathrm{Z}(\mathrm{F}\mathcal{D}_{\mathbf{G}_m} \otimes_{\mathrm{R}[\![t]\!]} \mathrm{R}[\![t]\!]/\langle p\rangle_\mathrm{F}) \\
&\cong \mathrm{R}[\![t]\!]\left[x^{\pm p}, \prod_{j=0}^{p-1}((x\nabla_\mathrm{F}) +_{\widetilde{\mathrm{F}}} \langle j\rangle_\mathrm{F})\right]/\langle p\rangle_\mathrm{F}.
\end{aligned}$$

The product appearing on the right-hand side is the usual Artin-Schreier polynomial $(x\nabla_\mathrm{F})^p - x\nabla_\mathrm{F}$ when F is the additive formal group law. The inclusion of $\mathrm{R}[\![t]\!]\left[x^{\pm p}, \prod_{j=0}^{p-1}((x\nabla_\mathrm{F}) +_{\widetilde{\mathrm{F}}} \langle j\rangle_\mathrm{F})\right]/\langle p\rangle_\mathrm{F}$ into $\mathrm{F}\mathcal{D}_{\mathbf{G}_m}/\langle p\rangle_\mathrm{F}$ as the center can be viewed as a "$\langle p\rangle_\mathrm{F}$-curvature" map. The phenomenon that $\pi_0(\mathrm{C}_*^{\mathrm{T}\times (\mathbf{Z}/p)_{\mathrm{rot}}}(\mathrm{Gr}_\mathrm{T}; k)[1/\hbar])$ has a large center admits a homotopy theoretic explanation, coming from the observation that $\mathrm{C}_*^\mathrm{T}(\mathrm{Gr}_\mathrm{T}; k)$ is an S^1-equivariant $\mathbf{E}_3$-k-algebra, and all such have central Tate-valued Frobenius $\mathbf{E}_1$-algebra maps $\mathrm{A} \to \mathrm{A}^{t\mathbf{Z}/p}$ which are linear over the $\mathbf{E}_\infty$-Frobenius $k \to k^{t\mathbf{Z}/p}$. (See Remark 4.4.11.)

Notation 7.4.7. If $\mathbf{Q} \subseteq \mathrm{R}$, then every formal group law $\mathrm{F}(x, y)$ is isomorphic to the additive formal group law via the logarithm. Let $\mathrm{F}_y(x, y) = \partial_y \mathrm{F}(x, y)$; then, the logarithm is given by the integral

$$\ell_\mathrm{F}(x) := \int_0^x \tfrac{dt}{\mathrm{F}_y(t,0)}.$$

We will write $\mathcal{E}_\mathrm{F}(x)$ to denote its compositional inverse, so that $\mathrm{F}(x, y) = \mathcal{E}_\mathrm{F}(\ell_\mathrm{F}(x) + \ell_\mathrm{F}(y))$. Observe that $[n]_\mathrm{F}(t) = \mathcal{E}_\mathrm{F}(n\ell_\mathrm{F}(t))$ for any $n \in \mathbf{Z}$.

Proposition 7.4.8. *Suppose that $\mathbf{Q} \subseteq \mathrm{R}$. Then there is an equality of $\mathrm{R}[\![t]\!]$-linear operators on $\mathrm{R}[\![t]\!][x]$:*

$$\nabla_\mathrm{F} = \tfrac{1}{tx}\mathcal{E}_\mathrm{F}(\ell_\mathrm{F}(t)x\partial_x).$$

Proof. Let ∇'_F denote the expression on the right-hand side. By definition of ∇_F, it suffices to check that $\nabla'_\mathrm{F}(x^m) = \langle m\rangle x^{m-1}$ for every $m \geq 1$. Write $\mathcal{E}_\mathrm{F}(t) = \sum_n a_n t^n$; then

$$\begin{aligned}
\nabla'_\mathrm{F}(x^m) &= \tfrac{1}{xt}\sum_n a_n \ell_\mathrm{F}(t)^n (x\partial_x)^n(x^m) \\
&= \tfrac{1}{xt}\sum_n a_n (m\ell_\mathrm{F}(t))^n x^m \\
&= \tfrac{1}{t}\mathcal{E}_\mathrm{F}(m\ell_\mathrm{F}(t))x^{m-1} = \langle m\rangle_\mathrm{F} x^{m-1},
\end{aligned}$$

as desired. □

Note that for the multiplicative formal group law, $\mathcal{E}_\mathrm{F}(x) = \exp(x) - 1$ and $\ell_\mathrm{F}(x) = \log(x+1)$, so that $\mathcal{E}_\mathrm{F}(\ell_\mathrm{F}(t)x\partial_x) = (1+t)^{x\partial_x} - 1$. Writing $t = q-1$, it follows that $\nabla_\mathrm{F} = \frac{1}{(q-1)x}(q^{x\partial_x} - 1)$, so one recovers the usual formula for the q-derivative as the difference $\frac{1}{(q-1)x}(f(qx) - f(x))$.

We now introduce some polynomials which, for an arbitrary 1-dimensional formal group law, play the role of the polynomials $(x+y)_q^n := (x+y)(x+qy)\cdots(x+q^{n-1}y)$. Just as the polynomials $(x+y)_q^n$ are characterized uniquely by certain identities involving the q-derivative (see, e.g., [KC]), one can analogously characterize our desired polynomials $(x+y)_{\mathrm{F}}^n$ as follows.

Definition 7.4.9. Let $(x+y)_{\mathrm{F}}^n$ denote the unique polynomial in $\mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}][x,y]$ characterized by the following three conditions:

- $(x+y)_{\mathrm{F}}^0 = 1$;
- $(x+(-x))_{\mathrm{F}}^n = 0$ for all $n > 0$;
- $\nabla_{\mathrm{F},x}(x+y)_{\mathrm{F}}^n = \langle n\rangle_{\mathrm{F}}(x+y)_{\mathrm{F}}^{n-1}$.

Note that the symbol $(x+y)_{\mathrm{F}}^n$ is *not* symmetric in x and y, despite the notation.

Lemma 7.4.10. *The symbol $(x+y)_{\mathrm{F}}^n$ in Definition 7.4.9 is well-defined: it exists and is unique. Moreover, $(x+y)_{\mathrm{F}}^n$ is a homogeneous polynomial of degree n.*

Proof. We will use induction on n. For the base case $n = 0$, observe that $(x+y)_{\mathrm{F}}^0 = 1$. For the inductive step, fix $n > 0$, and suppose that for all $k < n$, $(x+y)_{\mathrm{F}}^k$ is well-defined and homogeneous of degree k. Let $\mathrm{I}_{\mathrm{F},x} : \mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}][x,y] \to \mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}][x,y]$ denote the $\mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}]$-linear operator (called the "F-antiderivative") defined by $\mathrm{I}_{\mathrm{F},x}(x^k) = \langle k+1\rangle_{\mathrm{F}}^{-1}x^{k+1}$. By definition, this operator produces polynomials with no term of x-degree 0. Although the F-antiderivative

$$f(x,y) = \mathrm{I}_{\mathrm{F},x}(\langle n\rangle_{\mathrm{F}}(x+y)_{\mathrm{F}}^{n-1})$$

is homogeneous of degree n (since the operator $\mathrm{I}_{\mathrm{F},x}$ increases x-degree by 1) and satisfies the third condition of Definition 7.4.9, it might not equal $(x+y)_{\mathrm{F}}^n$ because it does not have to satisfy the second condition of Definition 7.4.9. But since $f(x,y)$ is homogeneous of degree n, $f(x,-x)$ is a scalar multiple of x^n, say ax^n. Then, the polynomial

$$g(x,y) = f(x,y) - a(-y)^n$$

satisfies

$$\nabla_{\mathrm{F},x}g(x,y) = \nabla_{\mathrm{F},x}f(x,y) = \langle n\rangle_{\mathrm{F}}(x+y)_{\mathrm{F}}^{n-1}$$

and

$$g(x,-x) = f(x,-x) - a(-(-x))^n = ax^n - ax^n = 0.$$

It follows that $(x+y)_{\mathrm{F}}^n$ exists, and one possible value for it is $g(x,y)$, which is homogeneous of degree n.

It remains to show that $(x+y)_{\mathrm{F}}^n$ is unique. We know that any polynomial $h(x,y)$ that satisfies the conditions of $(x+y)_{\mathrm{F}}^n$ must match $g(x,y)$ in every term with positive x-degree, because their F-derivatives with respect to x are both $\langle n\rangle_{\mathrm{F}}(x+y)_{\mathrm{F}}^{n-1}$. Therefore, $h(x,y) - g(x,y)$ is a scalar multiple of y^n, say by^n. Setting $y = -x$ gives $b(-x)^n = h(x,-x) - g(x,-x)$, which is 0 by the second condition of Definition 7.4.9, so $b = 0$. Therefore, $h(x,y) = g(x,y)$, so that $(x+y)_{\mathrm{F}}^n$ is unique and is equal to $g(x,y)$. $\square$

We will momentarily show that $(x+y)_{\mathrm{F}}^n \in \mathrm{R}[\![t]\!][x,y] \subseteq \mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}][x,y]$ below. We will use the following "F-binomial theorem" as input:

Proposition 7.4.11. *As elements of $\mathrm{R}[\![t]\!][\langle \mathbf{N}\rangle^{-1}][x,y]$, we have:*

$$(x+y)_{\mathrm{F}}^n = \sum_{k=0}^{n} \binom{n}{k}_{\mathrm{F}} x^{n-k}y^k(0+1)_{\mathrm{F}}^k.$$

Proof. We will use induction on n, and the inductive step will mainly consist of applying the F-antiderivative $\mathrm{I}_{\mathrm{F},x}$ from the proof of Lemma 7.4.10 to both sides. For the base case, observe that if $n = 0$, both sides are 1. For the inductive step, fix $n > 0$, and assume that Proposition 7.4.11 is true for $n - 1$:

$$[(x+y)_{\mathrm{F}}^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k}_{\mathrm{F}} x^{n-k-1} y^k (0+1)_{\mathrm{F}}^k.$$

Multiplying both sides by $\langle n \rangle_{\mathrm{F}}$, applying $\mathrm{I}_{\mathrm{F},x}$, and using $\mathrm{R}[\![t]\!][\langle \mathbf{N} \rangle^{-1}][y]$-linearity gives:

$$\begin{aligned} \mathrm{I}_{\mathrm{F},x}(\langle n \rangle_{\mathrm{F}} (x+y)_{\mathrm{F}}^{n-1}) &= \langle n \rangle_{\mathrm{F}} \sum_{k=0}^{n-1} \binom{n-1}{k}_{\mathrm{F}} \mathrm{I}_{\mathrm{F},x}(x^{n-k-1}) y^k (0+1)_{\mathrm{F}}^k \\ &= \sum_{k=0}^{n-1} \tfrac{\langle n \rangle_{\mathrm{F}}}{\langle n-k \rangle_{\mathrm{F}}} \binom{n-1}{k}_{\mathrm{F}} x^{n-k} y^k (0+1)_{\mathrm{F}}^k. \end{aligned}$$

Notice that

$$\tfrac{\langle n \rangle_{\mathrm{F}}}{\langle n-k \rangle_{\mathrm{F}}} \binom{n-1}{k}_{\mathrm{F}} = \tfrac{\langle n \rangle_{\mathrm{F}} (n-1)!_{\mathrm{F}}}{\langle n-k \rangle_{\mathrm{F}} k!_{\mathrm{F}} (n-k-1)!_{\mathrm{F}}} = \tfrac{n!_{\mathrm{F}}}{k!_{\mathrm{F}} (n-k)!_{\mathrm{F}}} = \binom{n}{k}_{\mathrm{F}}.$$

This implies that

$$\mathrm{I}_{\mathrm{F},x}(\langle n \rangle_{\mathrm{F}} (x+y)_{\mathrm{F}}^{n-1}) = \sum_{k=0}^{n-1} \binom{n}{k}_{\mathrm{F}} x^{n-k} y^k (0+1)_{\mathrm{F}}^k.$$

The right-hand side is nearly of the correct form, except that the upper limit of the summation is $n - 1$ instead of n. To fix this, add $y^n (0+1)_{\mathrm{F}}^n$ to both sides, giving

$$\mathrm{I}_{\mathrm{F},x}(\langle n \rangle_{\mathrm{F}} (x+y)_{\mathrm{F}}^{n-1}) + y^n (0+1)_{\mathrm{F}}^n = \sum_{k=0}^{n} \binom{n}{k}_{\mathrm{F}} x^{n-k} y^k (0+1)_{\mathrm{F}}^k.$$

We just have to show that the left-hand side is equal to $(x+y)_{\mathrm{F}}^n$.

Let $g(x,y)$ be the left-hand side. Applying $\nabla_{\mathrm{F},x}$ to $g(x,y)$ produces $\langle n \rangle_{\mathrm{F}} (x+y)_{\mathrm{F}}^{n-1}$, because $\mathrm{I}_{\mathrm{F},x}$ is a right inverse of $\nabla_{\mathrm{F},x}$ and $y^n (0+1)_{\mathrm{F}}^n$ is constant with respect to x. This is equal to $\nabla_{\mathrm{F},x}(x+y)_{\mathrm{F}}^n$, so $g(x,y)$ is equal to $(x+y)_{\mathrm{F}}^n$ in all terms with positive x-degree. Both $g(x,y)$ and $(x+y)_{\mathrm{F}}^n$ are homogeneous of degree n, so the only terms with x-degree 0 are the y^n terms. The coefficient of y^n in $g(x,y)$ is $(0+1)_{\mathrm{F}}^n$ because $\mathrm{I}_{\mathrm{F},x}$ never produces terms with x-degree 0. The y^n term of $(x+y)_{\mathrm{F}}^n$ is $(0+y)_{\mathrm{F}}^n$, which is $y^n (0+1)_{\mathrm{F}}^n$ by homogeneity, so the coefficient of y^n in $(x+y)_{\mathrm{F}}^n$ is also $(0+1)_{\mathrm{F}}^n$. Therefore, $g(x,y) = (x+y)_{\mathrm{F}}^n$, so that

$$(x+y)_{\mathrm{F}}^n = \sum_{k=0}^{n} \binom{n}{k}_{\mathrm{F}} x^{n-k} y^k (0+1)_{\mathrm{F}}^k,$$

as desired. $\square$

Proposition 7.4.12. *For all nonnegative integers n, the polynomial $(x+y)_{\mathrm{F}}^n$ lies in $\mathrm{R}[\![t]\!][x,y]$. In particular, the identity of Proposition 7.4.11 holds in $\mathrm{R}[\![t]\!][x,y]$ itself.*

Proof. It suffices to show that $(0+1)^n_{\mathrm{F}} \in \mathrm{R}$ for all nonnegative integers n, because using Proposition 7.4.11, we would find that

$$(x+y)^n_{\mathrm{F}} = \sum_{k=0}^{n} \binom{n}{k}_{\mathrm{F}} x^{n-k} y^k (0+1)^k_{\mathrm{F}} \in \mathrm{R}[\![t]\!][x,y].$$

Setting $x=-1$ and $y=1$ in Proposition 7.4.11, we get

$$0 = ((-1)+1)^n_{\mathrm{F}} = \sum_{k=0}^{n} \binom{n}{k}_{\mathrm{F}} (-1)^{n-k} (0+1)^k_{\mathrm{F}},$$

which produces a recurrence relation

$$(0+1)^n_{\mathrm{F}} = \sum_{k=0}^{n-1} \binom{n}{k}_{\mathrm{F}} (-1)^{n-k-1} (0+1)^k_{\mathrm{F}}. \tag{7.4.3}$$

To prove that $(0+1)^n_{\mathrm{F}} \in \mathrm{R}$, we will use induction on n. For the base case, note that $(0+1)^0_{\mathrm{F}} = 1$ which is an element of R. For the inductive step, note that if $(0+1)^k_{\mathrm{F}} \in \mathrm{R}$ for all $k < n$, then by the recurrence relation (7.4.3) and the fact that $\binom{n}{k}_{\mathrm{F}} \in \mathrm{R}[\![t]\!]$ for all n, k, we find that

$$(0+1)^n_{\mathrm{F}} = \sum_{k=0}^{n-1} \binom{n}{k}_{\mathrm{F}} (-1)^{n-k-1} (0+1)^k_{\mathrm{F}} \in \mathrm{R}[\![t]\!],$$

as desired. □

Lemma 7.4.13 (F-Taylor expansion)**.** *Let* F *be a formal group law over* R*. If* $f(x) \in (\mathrm{R} \otimes \mathbf{Q})[\![t, x-1]\!]$*, there is a Taylor expansion*

$$f(x) = \sum_{n \geq 0} \nabla^n_{\mathrm{F}}(f(x))|_{x=1} \frac{(x-1)^n_{\mathrm{F}}}{n!_{\mathrm{F}}}.$$

Here, $(x-1)^n_{\mathrm{F}}$ *denotes the symbol from Definition 7.4.9 with* $y=-1$*.*

Proof. This is the same argument as in [AL, Proposition 4.4]. First, observe that if $g(x) \in (\mathrm{R} \otimes \mathbf{Q})[\![t, x-1]\!]$ is a function such that $\nabla^n_{\mathrm{F}}(g(x))|_{x=1} = 0$ for all $n \geq 0$, then $g = 0$. Indeed, since ∇_{F} is simply the usual derivative modulo t, we see that $g(x)$ is divisible by t. Write $g(x) = t g_1(x)$; then, $\nabla^n_{\mathrm{F}}(g_1(x))|_{x=1} = 0$ for all $n \geq 0$, so $t \mid g_1(x)$. Continuing, we see that $g(x)$ is infinitely t-divisible, and hence is zero (since t is topologically nilpotent).

We can now apply the above observation to

$$g(x) := f(x) - \sum_{n \geq 0} \nabla^n_{\mathrm{F}}(f(x))|_{x=1} \frac{(x-1)^n_{\mathrm{F}}}{n!_{\mathrm{F}}}.$$

By definition of $(x-1)^n_{\mathrm{F}}$, we know that $\nabla_{\mathrm{F}}(\frac{(x-1)^n_{\mathrm{F}}}{n!_{\mathrm{F}}}) = \frac{(x-1)^{n-1}_{\mathrm{F}}}{(n-1)!_{\mathrm{F}}}$; so $\nabla^n_{\mathrm{F}}(g(x))|_{x=1} = 0$ for all $n \geq 0$, and hence $g = 0$, as desired. □

Corollary 7.4.14 (F-logarithm)**.** *Let* F *be a formal group law over* R*. Consider the function* $\mathrm{Flog}(x) \in (\mathrm{R} \otimes \mathbf{Q})[\![t, x-1]\!]$ *given by* $\frac{t}{\ell_{\mathrm{F}}(t)} \log(x)$*. Then, we have:*

a. $\nabla_{\mathrm{F}}(\mathrm{Flog}(x)) = \frac{1}{x}$*.*

b. $\mathrm{Flog}(xy) = \mathrm{Flog}(x) + \mathrm{Flog}(y)$.

c. There is a series expansion

$$\mathrm{Flog}(x) = \sum_{n\geq 1} \tfrac{\langle -n+1\rangle_{\mathrm{F}}\cdots\langle -1\rangle_{\mathrm{F}}}{n!_{\mathrm{F}}}(x-1)^n_{\mathrm{F}}.$$

Proof. The first statement follows from Proposition 7.4.8. Indeed, write $\mathcal{E}_{\mathrm{F}}(y) = \sum_{n\geq 1} a_n y^n$; the condition that $\mathrm{F}(x,y) \equiv x+y \pmod{(x,y)^2}$ forces $a_1 = 1$. Since

$$(x\partial_x)(\mathrm{Flog}(x)) = \tfrac{t}{\ell_{\mathrm{F}}(t)}(x\partial_x)\log(x) = \tfrac{t}{\ell_{\mathrm{F}}(t)},$$

we see that

$$\begin{aligned} x\nabla_{\mathrm{F}}(\mathrm{Flog}(x)) &= \tfrac{1}{t}\sum_{n\geq 1} a_n \ell_{\mathrm{F}}(t)^n (x\partial_x)^n(\mathrm{Flog}(x)) \\ &= \tfrac{1}{t}\left(\ell_{\mathrm{F}}(t)\cdot\tfrac{t}{\ell_{\mathrm{F}}(t)} + \sum_{n\geq 2} a_n\ell_{\mathrm{F}}(t)^n(x\partial_x)^n(\mathrm{Flog}(x))\right). \end{aligned}$$

The second sum vanishes, since $(x\partial_x)^n(\mathrm{Flog}(x)) = 0$ for $n \geq 2$. The first term cancels out to give $x\nabla_{\mathrm{F}}(\mathrm{Flog}(x)) = 1$, as desired.

The second statement is clear. For the third statement, observe that

$$\nabla^n_{\mathrm{F}}(\mathrm{Flog}(x)) = \nabla^{n-1}_{\mathrm{F}}(1/x) = \langle -n+1\rangle_{\mathrm{F}}\cdots\langle -1\rangle_{\mathrm{F}} x^{-n}.$$

Evaluating at $x = 1$ and using Lemma 7.4.13 gives the desired claim. ∎

Example 7.4.15. When F is the multiplicative formal group law over $\mathbf{Z}$, the function $\mathrm{Flog}(x)$ can be identified with Euler's q-logarithm (see [Eul])

$$\log_q(x) = \sum_{n\geq 1}(-1)^{n+1}q^{-\binom{n}{2}}\tfrac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q} \in \mathbf{Q}[\![q-1, x-1]\!].$$

Indeed, this follows from Corollary 7.4.14(3) and the observation that $\langle -j\rangle_{\mathrm{F}} = [-j]_q = -q^{-j}[j]_q$. See [AL, Section 4] for more on the q-logarithm.

Warning 7.4.16. The F-logarithm $\mathrm{Flog}(x)$ is *not* the same as the logarithm $\ell_{\mathrm{F}}(x)$ associated to the formal group law. This unfortunate terminology stems from attempting to simultaneously emulate the standard terminology "q-logarithm" and the "logarithm of the multiplicative formal group law".

Remark 7.4.17. Corollary 7.4.14 implies that $\mathrm{Flog}(x)$ is a well-defined class in the ring $\mathrm{R}[\![t]\!]\left[x^{\pm 1}, \tfrac{(x-1)^n_{\mathrm{F}}}{n!_{\mathrm{F}}}\right]$; this is the ring of functions on an F-analogue of $\mathbf{G}_m^\sharp$ (see Definition 7.4.21).

We can now turn to understanding the pullback square of (7.4.1).

Definition 7.4.18. Construction 7.3.7 gives a formal group $\hat{\mathbf{G}}_t$ over $\mathrm{Spf}\,\mathrm{R}[\![t]\!]$ whose logarithm is $\widetilde{\ell_{\mathrm{F}}}(x) := \frac{1}{t}\ell_{\mathrm{F}}(tx)$. Let x denote the coordinate on $\hat{\mathbf{G}}_t$, so that its underlying formal scheme is $\mathrm{Spf}\,\mathrm{R}[\![t,x]\!]$. Let $\hat{\mathbf{G}}_t^\vee$ denote the Cartier dual $\mathrm{Hom}(\hat{\mathbf{G}}_t, (\mathbf{G}_m)_{\mathrm{R}[\![t]\!]})$ of $\hat{\mathbf{G}}_t$; see [Dri1, Section 3] for some generalities on Cartier duals of formal groups. The element $x \in \mathcal{O}_{\hat{\mathbf{G}}_t}$ defines a homomorphism $\tau : \hat{\mathbf{G}}_t^\vee \to (\mathbf{G}_a)_{\mathrm{R}[\![t]\!]}$.

Construction 7.4.19. Over $(\mathrm{R} \otimes \mathbf{Q})[\![t]\!]$, the rescaled logarithm $\widetilde{\ell}_{\mathrm{F}}$ defines an isomorphism $\widetilde{\ell}_{\mathrm{F}} : \hat{\mathbf{G}}_t \xrightarrow{\sim} (\hat{\mathbf{G}}_a)_{(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]}$ of formal groups. Therefore, the canonical pairing $\hat{\mathbf{G}}_t \times_{\mathrm{R}[\![t]\!]} \hat{\mathbf{G}}_t^\vee \to (\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}$ fits into a diagram

$$\begin{tikzcd} \hat{\mathbf{G}}_t \times_{\mathrm{R}[\![t]\!]} \hat{\mathbf{G}}_t^\vee \arrow[d, "\widetilde{\ell}_{\mathrm{F}} \times \mathrm{id}"', "\sim"] \arrow[dr, "\mu"] & \\ (\hat{\mathbf{G}}_a)_{(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]} \times_{\mathrm{R}[\![t]\!]} \hat{\mathbf{G}}_t^\vee \arrow[r, "\nu"'] & (\mathbf{G}_m)_{(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]}. \end{tikzcd}$$

Since $\mathrm{R}[\![t]\!]$ is (p,t)-adically complete, the Cartier dual of $(\hat{\mathbf{G}}_a)_{\mathrm{R}[\![t]\!]}$ can be identified with the divided power completion $(\mathbf{G}_a^\sharp)_{\mathrm{R}[\![t]\!]}$. The pairing ν is base-changed from $\mathrm{R}[\![t]\!]$ itself, where it is given by the formula

$$\nu : (x, y) \mapsto \exp(xy).$$

It follows that the pairing μ is given by

$$\mu(x, y) = \exp(\widetilde{\ell}_{\mathrm{F}}(x)y).$$

This can be expanded as a power series in x:

$$\mu(x, y) = \sum_{n\geq 0} \beta_n(y)x^n.$$

Unwinding the definition of the Cartier dual, and using that $\mathrm{R}[\![t]\!]$ is p-torsionfree (using our assumption that R is a p-completely flat $\mathbf{Z}_p$-algebra), we see that the ring of functions on $\hat{\mathbf{G}}_t^\vee$ has underlying $\mathrm{R}[\![t]\!]$-module given by (the (p,t)-adic completion of)

$$\mathcal{O}_{\hat{\mathbf{G}}_t^\vee} = \mathrm{R}[\![t]\!]\{\beta_n(y)\}_{n\geq 0}.$$

Example 7.4.20. When F is the multiplicative formal group law, the function μ is simply

$$\mu(x, y) = \exp\left(\tfrac{y}{q-1}\log(1 + (q-1)x)\right) = (1 + (q-1)x)^{y/(q-1)};$$

its power series expansion is given by

$$\mu(x, y) = \sum_{n\geq 0} \frac{\prod_{j=0}^{n-1}(y - j(q-1))}{n!} x^n.$$

This expression plays an important role in [Dri1].

Definition 7.4.21. Let $\mathbf{G}_m^{\sharp,\mathrm{F}}$ denote the formal scheme over $\mathrm{Spf}\,\mathrm{R}[\![t]\!]$ given by (the (p,t)-adic completion of)

$$\mathbf{G}_m^{\sharp,\mathrm{F}} = \mathrm{Spf}\,\mathrm{R}[\![t]\!]\left[y^{\pm 1}, \tfrac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}\right]_{n\geq 0}.$$

This can be viewed as the "F-divided power hull" of the identity section of $(\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}$. Equip $\mathbf{G}_m^{\sharp,\mathrm{F}}$ with the structure of a group scheme where the coproduct sends $y \mapsto y \otimes y$. It is not immediate that this is well-defined, but we will prove this below in Corollary 7.4.25. There is a canonical homomorphism $\mathrm{can} : \mathbf{G}_m^{\sharp,\mathrm{F}} \to (\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}$.

Note that Remark 7.4.17 implies that $\mathrm{Flog}(y)$ defines an element of the coordinate ring of $\mathbf{G}_m^{\sharp,\mathrm{F}}$, i.e., it defines a map $\mathrm{Flog} : \mathbf{G}_m^{\sharp,\mathrm{F}} \to (\mathbf{G}_a)_{\mathrm{R}[\![t]\!]}$. This is in fact a homomorphism, since $\mathrm{Flog}(y_1y_2) = \mathrm{Flog}(y_1) + \mathrm{Flog}(y_2)$.

Proposition 7.4.22. *Work over the base* $(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]$. *Then, the iterated* F*-derivative of* $\mu(x,\mathrm{Flog}(y))$ *with respect to the variable* y *is given by*

$$\nabla^n_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y)) = \tfrac{x(x+_{\widetilde{\mathrm{F}}}\langle -1\rangle_{\mathrm{F}})\cdots(x+_{\widetilde{\mathrm{F}}}\langle -n+1\rangle_{\mathrm{F}})}{y^n}\mu(x,\mathrm{Flog}(y)). \tag{7.4.4}$$

Proof. Observe that:

$$\begin{aligned}\mu(x,\mathrm{Flog}(y)) &= \sum_{n\geq 0}\beta_n(\mathrm{Flog}(y))x^n = \exp(\mathrm{Flog}(y)\widetilde{\ell}_{\mathrm{F}}(x))\\ &= \exp\left(\tfrac{t}{\ell_{\mathrm{F}}(t)}\log(y)\cdot\tfrac{\ell_{\mathrm{F}}(tx)}{t}\right) = \exp\left(\log(y)\tfrac{\ell_{\mathrm{F}}(tx)}{\ell_{\mathrm{F}}(t)}\right) = y^{\frac{\ell_{\mathrm{F}}(tx)}{\ell_{\mathrm{F}}(t)}};\end{aligned}$$

the third equality used the definition of $\mathrm{Flog}(y)$ via Corollary 7.4.14 and the definition of $\widetilde{\ell}_{\mathrm{F}}(x)$. One can deduce (7.4.4) from this; let us illustrate this rather inefficiently. Write $a=\frac{\ell_{\mathrm{F}}(tx)}{\ell_{\mathrm{F}}(t)}$ for notational simplicity, so that

$$y^a = \sum_{m\geq 0}\tfrac{a(a-1)\cdots(a-(m-1))}{m!}(y-1)^m,$$

and $\partial_y y^a = ay^{a-1}$.

We can now inductively compute the iterated F-derivative using Proposition 7.4.8. We begin with the base case $n=1$. Note that $(y\partial_y)y^a = ay^a$, so that

$$(y\partial_y)^m y^a = a^m y^a \tag{7.4.5}$$

by an easy induction on m. Write $\mathcal{E}_{\mathrm{F}}(z)=\sum_{m\geq 0}b_m z^m$; then using (7.4.5), we have:

$$\begin{aligned}\nabla_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y)) &= \tfrac{1}{yt}\sum_{m\geq 0}b_m\ell_{\mathrm{F}}(t)^m(y\partial_y)^m y^a = \tfrac{1}{yt}\sum_{m\geq 0}b_m\ell_{\mathrm{F}}(t)^m a^m y^a\\ &= \tfrac{y^a}{yt}\sum_{m\geq 0}b_m\ell_{\mathrm{F}}(tx)^m = \tfrac{y^a}{yt}\mathcal{E}_{\mathrm{F}}(\ell_{\mathrm{F}}(tx)) = \tfrac{tx}{yt}\cdot y^a\\ &= \tfrac{x}{y}y^a = \tfrac{x}{y}\mu(x,\mathrm{Flog}(y)),\end{aligned}$$

as desired.

The proof of the iterated F-derivative is similar. Indeed, note that (7.4.5) implies that for any $j\geq 0$, we have:

$$(y\partial_y)^m\left(\tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\right) = (y\partial_y)^m y^{a-j} = (a-j)^m y^{a-j} = (a-j)^m\tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}. \tag{7.4.6}$$

Assume that (7.4.4) holds for n; then:

$$\begin{aligned}\nabla^{n+1}_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y)) &= \nabla_{\mathrm{F},y}\nabla^n_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y))\\ &= x(x+_{\widetilde{\mathrm{F}}}\langle -1\rangle_{\mathrm{F}})\cdots(x+_{\widetilde{\mathrm{F}}}\langle -n+1\rangle_{\mathrm{F}})\nabla_{\mathrm{F},y}\left(\tfrac{\mu(x,\mathrm{Flog}(y))}{y^n}\right).\end{aligned} \tag{7.4.7}$$

The derivative on the right-hand side can be calculated as follows:

$$\begin{aligned}
\nabla_{\mathrm{F},y}\left(\tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\right) &= \tfrac{1}{yt}\sum_{m\geq 0} a_m \ell_{\mathrm{F}}(t)^m (y\partial_y)^m \left(\tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\right) \\
&= \tfrac{1}{yt}\sum_{m\geq 0} a_m \ell_{\mathrm{F}}(t)^m \left(\tfrac{\ell_{\mathrm{F}}(tx)}{\ell_{\mathrm{F}}(t)} - j\right)^m \tfrac{\mu(x,\mathrm{Flog}(y))}{y^j} \\
&= \tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\tfrac{1}{yt}\sum_{m\geq 0} a_m (\ell_{\mathrm{F}}(tx) - j\ell_{\mathrm{F}}(t))^m \\
&= \tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\tfrac{1}{yt}\mathcal{E}_{\mathrm{F}}(\ell_{\mathrm{F}}(tx) - j\ell_{\mathrm{F}}(t)).
\end{aligned}$$

Since

$$\ell_{\mathrm{F}}(tx) - j\ell_{\mathrm{F}}(t) = \ell_{\mathrm{F}}(tx) + \ell_{\mathrm{F}}([-j]_{\mathrm{F}}(t)) = \ell_{\mathrm{F}}(tx +_{\mathrm{F}} [-j]_{\mathrm{F}}(t))),$$

this becomes

$$\begin{aligned}
\nabla_{\mathrm{F},y}\left(\tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\right) &= \tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\tfrac{1}{yt}\mathcal{E}_{\mathrm{F}}(\ell_{\mathrm{F}}(tx +_{\mathrm{F}} [-j]_{\mathrm{F}}(t))) \\
&= \tfrac{\mu(x,\mathrm{Flog}(y))}{y^j}\tfrac{x +_{\widetilde{\mathrm{F}}} \langle -j\rangle_{\mathrm{F}}}{y}.
\end{aligned}$$

Plugging this into (7.4.7), we get that

$$\nabla^{n+1}_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y)) = x(x +_{\widetilde{\mathrm{F}}} \langle -1\rangle_{\mathrm{F}})\cdots(x +_{\widetilde{\mathrm{F}}} \langle -n+1\rangle_{\mathrm{F}})(x +_{\widetilde{\mathrm{F}}} \langle -n\rangle_{\mathrm{F}})\tfrac{\mu(x,\mathrm{Flog}(y))}{y^{n+1}},$$

as desired. □

The following result is the analogue of our observation in Example 7.4.1 that the logarithm has divided powers in $\mathbf{G}_m^\sharp$:

Corollary 7.4.23. *There is a dotted map (which is a homomorphism over* $\mathrm{R}[\![t]\!]$*) filling in the following diagram:*

$$\begin{tikzcd}
 & \hat{\mathbf{G}}_t^\vee \arrow[d, "\tau"] \\
\mathbf{G}_m^{\sharp,\mathrm{F}} \arrow[ur, dashed] \arrow[r, "y\mapsto \mathrm{Flog}(y)"'] & (\mathbf{G}_a)_{\mathrm{R}[\![t]\!]}.
\end{tikzcd}$$

Proof. In the notation of Construction 7.4.19, we need to show that $\beta_n(\mathrm{Flog}(y)) \in \mathcal{O}_{\mathbf{G}_m^{\sharp,\mathrm{F}}}$ for every $n \geq 0$. To prove this, let us work over $(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]$, and expand $\mu(x,\mathrm{Flog}(y))$ as a power series in $\frac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}$ using Lemma 7.4.13. Evaluating (7.4.4) in Proposition 7.4.22 at $y=1$, we obtain

$$\nabla^n_{\mathrm{F},y}\mu(x,\mathrm{Flog}(y))|_{y=1} = x(x +_{\widetilde{\mathrm{F}}} \langle -1\rangle_{\mathrm{F}})\cdots(x +_{\widetilde{\mathrm{F}}} \langle -n+1\rangle_{\mathrm{F}}).$$

It follows from Lemma 7.4.13 that

$$\begin{aligned}
\sum_{n\geq 0}\beta_n(\mathrm{Flog}(y))x^n &= \mu(x,\mathrm{Flog}(y)) \\
&= \sum_{n\geq 0} x(x +_{\widetilde{\mathrm{F}}} \langle -1\rangle_{\mathrm{F}})\cdots(x +_{\widetilde{\mathrm{F}}} \langle -n+1\rangle_{\mathrm{F}})\tfrac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}. \qquad (7.4.8)
\end{aligned}$$

Taking the coefficient of x^n on the right-hand side expresses $\beta_n(\mathrm{Flog}(y))$ as an $(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]$-linear combination of the divided powers $\frac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}$; but since no rational denominators appear, this in fact expresses $\beta_n(\mathrm{Flog}(y))$ as an $\mathrm{R}[\![t]\!]$-linear combination of the divided powers $\frac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}$, as desired. □

Example 7.4.24. The identity (7.4.8) is the analogue of the identity (7.4.2). For instance, in the case of the multiplicative formal group law, it asserts that

$$\sum_{n\geq 0} \tfrac{\log_q(y)(\log_q(y)-(q-1))\cdots(\log_q(y)-(n-1)(q-1))}{n!} x^n \\ = \sum_{n\geq 0} q^{-\binom{n}{2}} x(x-[1]_q)\cdots(x-[n-1]_q) \tfrac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q!}.$$

Indeed, we have

$$x +_{\widetilde{\mathrm{F}}} [-n]_q = x + \tfrac{q^{-n}-1}{q-1} + (q^{-n}-1)x = q^{-n}x + [-n]_q = q^{-n}(x-[n]_q).$$

The above identity with the q-logarithm seems to be new: it was discovered in a discussion with Michael Kural, and it was my motivation for the more general (7.4.8). It is amazing to me that the above identity with the q-logarithm could have been proved by Euler (who introduced the q-logarithm [Eul]!), but falls very naturally out of the theory of prismatic cohomology/equivariant formal groups over connective complex K-theory (via Corollary 7.3.9 and Example 7.3.10).

Corollary 7.4.25. *The group structure on* $\mathbf{G}_m^{\sharp,\mathrm{F}}$ *is well defined.*

Proof. Suppose that y_1 and y_2 both admit F-divided powers $\frac{(y-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}$; we need to show that the same is true of the product y_1y_2. Since $\mathrm{Flog}(y_1y_2) = \mathrm{Flog}(y_1) + \mathrm{Flog}(y_2)$, one can express $\beta_n(\mathrm{Flog}(y_1y_2))$ in terms of $\beta_n(\mathrm{Flog}(y_1))$ and $\beta_n(\mathrm{Flog}(y_2))$. Using the identity (7.4.8) in the proof of Corollary 7.4.23 shows that y_1y_2 must also admit F-divided powers, as desired. □

Finally, we have the desired description of the pullback (7.4.1):

Theorem 7.4.26. *Suppose* R *is* p*-completely flat over* $\mathbf{Z}_p$*. Then there is a Cartesian square of group schemes over* $\mathrm{R}[\![t]\!]$*:*

$$\begin{tikzcd} \mathbf{G}_m^{\sharp,\mathrm{F}} \arrow[r, "y\mapsto \mathrm{Flog}(y)"] \arrow[d, "\mathrm{can}"'] & \hat{\mathbf{G}}_t^\vee \arrow[d, "\langle p\rangle^*"] \\ (\mathbf{G}_m)_{\mathrm{R}[\![t]\!]} \arrow[r, "y\mapsto y^p"'] & (\mathbf{G}_m^{(1)})_{\mathrm{R}[\![t]\!]}. \end{tikzcd} \tag{7.4.9}$$

The right-vertical map is Cartier dual to the homomorphism $p\underline{\mathbf{Z}} \to \hat{\mathbf{G}}_t$ *sending* $p \mapsto \langle p\rangle_{\mathrm{F}}$*. That is to say, the group scheme* $\mathrm{C}_\theta^\vee$ *over* $\mathrm{R}[\![t]\!] \cong \mathrm{R}[\![\theta]\!]$ *from* (7.4.1) *is isomorphic to* $\mathbf{G}_m^{\sharp,\mathrm{F}}$*. In particular, there is an extension*

$$0 \to (\mu_p)_{\mathrm{R}[\![t]\!]} \to \mathbf{G}_m^{\sharp,\mathrm{F}} \xrightarrow{\mathrm{Flog}} \hat{\mathbf{G}}_t^\vee \to 0.$$

Proof. To check that the diagram commutes, we need to check that there is an equality of elements of $\mathcal{O}_{\mathbf{G}_m^{\sharp,\mathrm{F}}}$:

$$y^p = \langle p\rangle^*(\mathrm{Flog}(y)).$$

Since R is p-completely flat over $\mathbf{Z}_p$, there is an injection $\mathrm{R}[\![t]\!] \subseteq (\mathrm{R}\otimes\mathbf{Q})[\![t]\!]$; so it suffices to check the desired identity in $(\mathrm{R}\otimes\mathbf{Q})[\![t]\!]\{\beta_n(y)\}_{n\geq 0}$. By the discussion in Construction 7.4.19,

$\langle p\rangle^*$ can be expressed as

$$\begin{aligned}\langle p\rangle^*(z) = \exp(z\widetilde{\ell}_{\mathrm{F}}(\langle p\rangle_{\mathrm{F}})) &= \exp\left(z\tfrac{\ell_{\mathrm{F}}(t\langle p\rangle_{\mathrm{F}})}{t}\right)\\ &= \exp\left(z\tfrac{\ell_{\mathrm{F}}([p]_{\mathrm{F}}(t))}{t}\right) = \exp\left(p\tfrac{\ell_{\mathrm{F}}(t)}{t}z\right).\end{aligned} \tag{7.4.10}$$

Note that this is also $\exp\left(\frac{\ell_{\mathrm{F}}([p]_{\mathrm{F}}(t))}{t}z\right)$. It follows that

$$\langle p\rangle^*(\mathrm{Flog}(y)) = \exp\left(p\tfrac{\ell_{\mathrm{F}}(t)}{t}\mathrm{Flog}(y)\right) = \exp\left(p\tfrac{\ell_{\mathrm{F}}(t)}{t}\tfrac{t}{\ell_{\mathrm{F}}(t)}\log(y)\right) = \exp(p\log(y)) = y^p,$$

as desired.

To check that the square is Cartesian, first note that the horizontal maps are surjective. This is clear for the Frobenius on $(\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}$. For the map Flog, define $\mathrm{F}\exp(z) := \exp(\frac{\ell_{\mathrm{F}}(t)}{t}z)$, so that $\mathrm{F}\exp(z) = \sum_{n\geq 0}\beta_n(z)$. There is a homomorphism $\hat{\mathbf{G}}_t^\vee \to (\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}$ sending $z \mapsto \mathrm{F}\exp(z) := \exp(\frac{\ell_{\mathrm{F}}(t)}{t}z)$, and $z = \mathrm{Flog}(\mathrm{F}\exp(z))$. Using (7.4.8) with $y = \mathrm{F}\exp(z)$, one sees that $\mathrm{F}\exp$ lands in $\mathbf{G}_m^{\sharp,\mathrm{F}}$, i.e., that $\frac{(\mathrm{F}\exp(z)-1)_{\mathrm{F}}^n}{n!_{\mathrm{F}}}$ is well-defined in $\mathcal{O}_{\hat{\mathbf{G}}_t^\vee}$. This implies that Flog is surjective.

It remains to show that the kernel of $\mathrm{Flog} : \mathbf{G}_m^{\sharp,\mathrm{F}} \to \hat{\mathbf{G}}_t^\vee$ is isomorphic to $(\mu_p)_{\mathrm{R}[\![t]\!]}$. Observe that $\mathrm{Flog}(y) = 0$ implies that $\log(y) = 0$, which happens (by the Cartesian square Example 7.4.1) if and only if $y^p = 1$. Conversely, if $y^p = 1$, then

$$p \cdot \mathrm{Flog}(y) = \mathrm{Flog}(y^p) = 0,$$

which implies that $\mathrm{Flog}(y) = 0$. □

Example 7.4.27. It follows from Theorem 7.4.26 that $\mathbf{G}_m^{\sharp,\mathrm{F}}$ is an extension of $\hat{\mathbf{G}}_t^\vee$ by $(\mu_p)_{\mathrm{R}[\![t]\!]}$. In the case of the multiplicative formal group law over $\mathrm{R} = \mathbf{Z}_p$, this was studied in [Dri1]. By definition of $\mathbf{G}_m^{\sharp,\mathrm{F}}$, we can identify it with

$$\mathbf{G}_m^{\sharp,\mathrm{F}} \cong \mathrm{Spf}\,\mathrm{R}[\![t]\!]\left[y^{\pm 1}, \tfrac{(y-1)(y-q)\cdots(y-q^{n-1})}{[n]_q!}\right]_{n\geq 0} =: \mathbf{G}_m^{\sharp,q}.$$

Now, in [Dri1, Section 5.3.1], it is shown that there is an extension $\widetilde{\mathrm{G}}_{\mathrm{Q}}$ of $(\hat{\mathbf{G}}_{m,q-1})^\vee$ by $(\mu_p)_{\mathbf{Z}_p[\![q-1]\!]}$, given by the functor

$$\widetilde{\mathrm{G}}_{\mathrm{Q}} : \mathrm{R} \mapsto \{(q, x, u) \in \mathrm{R}^\times \times \mathrm{W}(\mathrm{R}) \times \mathrm{R}^\times \mid q-1 \text{ is nilpotent}, 1 + \Phi_p([q])x = [u^p]\}.$$

Here, $\mathrm{W}(\mathrm{R})$ denotes the ring of p-typical Witt vectors of R. Drinfeld shows that the group scheme $\widetilde{\mathrm{G}}_{\mathrm{Q}}$ is isomorphic over $\mathbf{Z}_p[\![q-1]\!]$ to $\mathbf{G}_m^{\sharp,\mathrm{F}}$ (as extensions of $(\hat{\mathbf{G}}_{m,q-1})^\vee$ by $(\mu_p)_{\mathbf{Z}_p[\![q-1]\!]}$).

As shown in [Dri1, Appendix D] (see also [Dev2, Remark C.3] and Example 7.4.20), the Cartier dual $(\hat{\mathbf{G}}_{m,q-1})^\vee$ can be identified with

$$(\hat{\mathbf{G}}_{m,q-1})^\vee = \mathrm{Spf}\,\mathbf{Z}_p[\![q-1]\!]\left[y, \tfrac{\prod_{j=0}^{n-1}(y-j(q-1))}{n!}\right]_{n\geq 0}.$$

By (7.4.10), the homomorphism $\langle p\rangle^*$ corresponds to the invertible element

$$\langle p\rangle^*(z) = \exp\left(p\tfrac{\log(q)}{q-1}z\right) = q^{pz/(q-1)};$$

this element plays an important role in [Dri1]. Note that this can alternatively be written as

$$\langle p\rangle^*(z) = \sum_{n\geq 0} \frac{\prod_{j=0}^{n-1}(z-j(q-1))}{n!}[p]_q^n = \sum_{n\geq 0} \frac{\prod_{j=0}^{n-1}(pz-j(q-1))}{n!}.$$

In this case, Theorem 7.4.26 therefore specializes to give a Cartesian square over $\mathbf{Z}_p[\![q-1]\!]$:

$$\begin{tikzcd} \widetilde{\mathrm{G}}_{\mathrm{Q}} \arrow[r, "\sim"] & \mathbf{G}_m^{\sharp,q} \arrow[r, "y\mapsto \log_q(y)"] \arrow[d, "\mathrm{can}"'] & \hat{\mathbf{G}}_{m,q-1}^\vee \arrow[d, "z\mapsto q^{pz/(q-1)}"] \\ & (\mathbf{G}_m)_{\mathbf{Z}_p[\![q-1]\!]} \arrow[r, "y\mapsto y^p"'] & (\mathbf{G}_m^{(1)})_{\mathbf{Z}_p[\![q-1]\!]}. \end{tikzcd}$$

It follows from this pullback square that there is an isomorphism of stacks $(\mathbf{G}_m)_{\mathbf{Z}_p[\![q-1]\!]}/\mathbf{G}_m^{\sharp,q} \cong (\mathbf{G}_m^{(1)})_{\mathbf{Z}_p[\![q-1]\!]}/\widehat{\mathbf{G}}_{m,q-1}^\vee$. The left-hand side identifies with the q-de Rham stack of $\mathbf{G}_m$, while the right-hand side identifies (by Construction 7.2.1) with the Frobenius twist of the prismatization (relative to the q-de Rham prism) of $(\mathbf{G}_m)_{\mathbf{Z}_p[\zeta_p]}$. In particular, the preceding pullback square is a stacky version of the comparison between q-de Rham cohomology and the prismatic cohomology of $\mathbf{G}_m$.

In fact, in general, there is an isomorphism of stacks $(\mathbf{G}_m)_{\mathrm{R}[\![t]\!]}/\mathrm{C}_\theta^\vee \cong (\mathbf{G}_m^{(1)})_{\mathrm{R}[\![t]\!]}/\widehat{\mathbf{G}}_t^\vee$. The right-hand side is the Frobenius twist of the prismatization of $\mathbf{G}_m$ over R, while the left-hand side can be identified with the stack computing the F-de Rham complex $\mathrm{F}\Omega_{\mathbf{G}_m}$.

Remark 7.4.28. In future work, we will use the results established in this section to construct an F-analogue of Gauss' hypergeometric equation. It is a second-order "F-differential equation", meaning that it is built from the operators ∇_{F} and ∇_{F}^2, on $\mathcal{O}_{\mathbf{P}^1_{\mathrm{R}}-\{0,1,\infty\}} = \mathrm{R}[x^{\pm 1}, \frac{1}{x-1}]$ with regular singularities at $x = 0, \infty$ (and probably also at $x = 1$, in some sense that I do not currently understand) given by

$$(x\nabla_{\mathrm{F}} +_{\widetilde{\mathrm{F}}} \langle\alpha\rangle_{\mathrm{F}})(x\nabla_{\mathrm{F}} +_{\widetilde{\mathrm{F}}} \langle\beta\rangle_{\mathrm{F}}) = \nabla_{\mathrm{F}}(x\nabla_{\mathrm{F}} +_{\widetilde{\mathrm{F}}} \langle\gamma-1\rangle_{\mathrm{F}}).$$

One solution is given by the following F-analogue of the hypergeometric function:

$${}_2^{\mathrm{F}}\varphi_1(\alpha,\beta;\gamma;x) = \sum_{n\geq 0} \frac{\langle\alpha\rangle_{\mathrm{F}}\cdots\langle\alpha+(n-1)\rangle_{\mathrm{F}}\langle\beta\rangle_{\mathrm{F}}\cdots\langle\beta+(n-1)\rangle_{\mathrm{F}}}{\langle\gamma\rangle_{\mathrm{F}}\cdots\langle\gamma+(n-1)\rangle_{\mathrm{F}}} \frac{x^n}{[n]!_{\mathrm{F}}}.$$

When F is the additive formal group law, this is Gauss' hypergeometric function; and when F is the multiplicative formal group law, this is Heine's q-hypergeometric function from [Hei]. We hope to show that when k is a p-complete complex oriented $\mathbf{E}_\infty$-ring with $p > 2$, F is its associated 1-dimensional formal group law over $\mathrm{Spev}(k)$, $\alpha = \beta = 1/2$, and $\gamma = 1$, the hypergeometric F-differential equation above arises as the Picard-Fuchs equation associated to a lift of the *ordinary* locus of the Legendre elliptic curve $\mathrm{E}_x \to \mathbf{P}^1 - \{0,1,\infty\}$ to a spectral $\mathbf{E}_2$-k-scheme. (Combining this calculation for $k = \mathrm{ku}$ with Corollary 6.4.2 would give a positive answer to a question of Scholze [Sch] regarding the q-hypergeometric equation. In this case, the desired result was previously established by Shirai [Shi].)

7.5 Application: Hodge-de Rham degeneration

The theory of prismatic stacks developed above can be used to prove some results about degeneration of the Hodge-de Rham spectral sequence for smooth and proper varieties over

$\mathbf{F}_p$. First, we must mention a caveat: the theory developed above took as input an $\mathbf{E}_\infty$-ring R (and a chosen lift of THH(R) to $\mathrm{CAlg}(\mathrm{CycSp}_{\triangle})$) to produce the stacks $\mathrm{R}^?$. However, one can develop the theory of prismatic cohomology for $\mathbf{E}_2$-rings as well (see [Pst] and the forthcoming work of Pstragowski-Raksit); if the $\mathbf{E}_2$-rings in question admit $\mathbf{E}_3$-structures, then one can also develop an analogous theory of prismatic stacks.

The $\mathbf{E}_3$-rings relevant to the case at hand are the truncated Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$. These are ($p$-complete, by our convention) $\mathbf{E}_3$-MU-algebras with the property that

$$\pi_{2*}(\mathrm{BP}\langle n\rangle) \cong \mathbf{Z}_p[v_1, \cdots, v_n]$$

with v_i in weight $p^i - 1$. They are *not* unique as ring spectra, but it was shown in [HW] that one can make a choice of such a spectrum which admits an $\mathbf{E}_3$-MU-algebra structure. Note that there are natural $\mathbf{E}_3$-algebra maps $\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$ which implement quotienting by the class v_n.

The truncated Brown-Peterson spectra have the property that $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is an even $\mathbf{E}_2$-ring, and the map $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is evenly descendable (in a suitable sense). One can therefore define the various stacks $\mathrm{BP}\langle n\rangle^?$ associated to $\mathrm{BP}\langle n\rangle$ by using the cosimplicial diagram $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}^{\otimes \bullet+1})$ of even $\mathbf{E}_2$-rings. To state it, let W denote the group scheme of p-typical Witt vectors, let $\mathrm{W}^\times$ denote the group scheme of invertible p-typical Witt vectors (so that the Teichmüller lift defines a homomorphism $\mathbf{G}_m \to \mathrm{W}^\times$), let F denote the Frobenius on W, and let $\mathrm{W}[\mathrm{F}^j]$ and $\mathrm{W}^\times[\mathrm{F}^j]$ denote the kernels of Frobenius acting on these group schemes. For instance, $\mathrm{W}^\times[\mathrm{F}^j]$ consists of those invertible Witt vectors x such that $\mathrm{F}^j(x) = 1$. One of our main calculations (whose proof we defer to the forthcoming paper), then, is the following:

Theorem 7.5.1. *Let $0 \leq n \leq \infty$. The stacks $\mathrm{BP}\langle n\rangle^{\mathrm{conj}}$ and $\mathrm{BP}\langle n\rangle^{\mathrm{HT}}$ live over $\mathrm{Spev}(\mathrm{BP}\langle n\rangle) \cong \mathrm{Spec}(\mathbf{Z}_p[v_1, \cdots, v_n])/\mathbf{G}_m$, and there are isomorphisms*

$$\mathrm{BP}\langle n\rangle^{\mathrm{conj}}_{p=\cdots=v_n=0} \cong (\mathbf{G}_a/\mathrm{W}[\mathrm{F}^{n+1}])/\mathbf{G}_m,$$
$$\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0} \cong \mathrm{BW}^\times[\mathrm{F}^{n+1}].$$

Furthermore, these isomorphisms are compatible in n. The structure morphism $\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0} \to \mathrm{B}\mathbf{G}_m$ is induced by the homomorphism $\mathrm{W}^\times[\mathrm{F}^{n+1}] \to \mathrm{W}^\times \to \mathbf{G}_m$.

Although we will not prove this result here, we give a brief indication of the argument. The first key observation is that the map $\mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbf{F}_p) := \mathrm{THH}(\mathrm{BP}\langle n\rangle)/(p, \cdots, v_n) \to \mathrm{THH}(\mathbf{F}_p)$ is evenly descendable. In particular, one can compute

$$\mathrm{BP}\langle n\rangle^{\mathrm{conj}}_{p=\cdots=v_n=0} \cong \mathrm{colim}_{\boldsymbol{\Delta}} \,\mathrm{Spec}\, \pi_{2*}(\mathrm{THH}(\mathbf{F}_p)^{\otimes_{\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbf{F}_p)} \bullet+1})/\mathbf{G}_m.$$

The first term in the simplicial diagram is $\mathrm{Spec}\,\pi_{2*}(\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m \cong \mathbf{G}_a/\mathbf{G}_m$. To compute $\mathrm{BP}\langle n\rangle^{\mathrm{conj}}_{p=\cdots=v_n=0}$, one needs to calculate the term $\mathrm{Spec}\,\pi_{2*}(\mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbf{F}_p)} \mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m$. This, in turn, is a consequence of the following isomorphism of graded $\mathbf{F}_p$-bialgebras:

$$\pi_{2*}\mathrm{HH}(\mathbf{F}_p/\mathrm{BP}\langle n\rangle) \cong \mathcal{O}_{\mathrm{W}[\mathrm{F}^{n+1}]}.$$

One can prove this, essentially, by reducing to the case when $n = \infty$, in which case it is a consequence of an isomorphism of graded $\mathbf{F}_p$-bialgebras

$$\pi_{2*}\mathrm{HH}(\mathbf{F}_p/\mathrm{MU}) \cong \mathcal{O}_{\mathrm{C}^2(\widehat{\mathbf{G}}_a;\mathbf{G}_m)},$$

where $\mathrm{C}^2(\widehat{\mathbf{G}}_a; \mathbf{G}_m)$ is the group scheme of (pointed) symmetric 2-cocycles $\widehat{\mathbf{G}}_a \times \widehat{\mathbf{G}}_a \to \mathbf{G}_m$.

Just as in [BL, Theorem 3.5.8], one finds:

Corollary 7.5.2. *Let $0 \leq n \leq \infty$. Then:*

a. Let $\mathcal{A}_1^{[n+1]}$ denote the algebra of differential operators on $\mathbf{A}^1$ with "partial divided powers", i.e., the associative $\mathbf{F}_p$-algebra generated by $x, \partial_x = \partial_x^{[1]}, \partial_x^{[p]}, \cdots, \partial_x^{[p^n]}$ with relations given by $[\partial_x^{[p^j]}, x] = \partial_x^{[p^{j-1}]}$. Equip $\mathcal{A}_1^{[n+1]}$ with the grading where x has weight 1 and $\partial_x^{[p^j]}$ has weight $-p^j$. Then $\mathrm{QCoh}(\mathrm{BP}\langle n\rangle^{\mathrm{conj}}_{p=\cdots=v_n=0})$ embeds fully faithfully into the category of graded $\mathcal{A}_1^{[n+1]}$-modules such that $\partial_x^{[p^j]}$ acts locally nilpotently for each $0 \leq j \leq n$.

b. There is a fully faithful functor $\mathrm{QCoh}(\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0}) \hookrightarrow \mathrm{QCoh}(\mathrm{W}_{n+1}) = \mathrm{Mod}_{\mathbf{F}_p[\Theta_0,\cdots,\Theta_n]}$ whose essential image consists of those $\mathrm{M} \in \mathrm{QCoh}(\mathrm{W}_{n+1})$ such that $\Theta_i^p - \Theta_i$ acts locally nilpotently for each $0 \leq i \leq n$. The Θ_i are called the (higher) Sen operators. Furthermore, this embedding is symmetric monoidal for the standard tensor product on $\mathrm{QCoh}(\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0})$ and the convolution symmetric monoidal structure on $\mathrm{QCoh}(\mathrm{W}_{n+1})$.

These follow by computing the Cartier duals of $\mathrm{W}[\mathrm{F}^{n+1}]$ and $\mathrm{W}^\times[\mathrm{F}^{n+1}]$. For instance, just as in Example 7.4.1, there is a pushout square

$$\begin{tikzcd} \underline{p^n\mathbf{Z}} \arrow[r] \arrow[d] & \widehat{\mathrm{W}}_{n+1} \arrow[d] \\ \underline{\mathbf{Z}} \arrow[r] & \mathrm{W}^\times[\mathrm{F}^{n+1}]^\vee, \end{tikzcd}$$

which gives the desired description of $\mathrm{QCoh}(\mathrm{BP}\langle n\rangle^{\mathrm{HT}}_{p=\cdots=v_n=0})$.

Example 7.5.3. Suppose X is a smooth scheme over $\mathbf{F}_p$ with Frobenius $\mathrm{F} : \mathrm{X} \to \mathrm{X}^{(1)}$. Then $\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}$ can be viewed as a quasicoherent sheaf over $\mathbf{F}_p^{\mathrm{HT}} \cong \mathrm{Spec}(\mathbf{F}_p)$. As discussed in [BL, Remark 4.7.18], a choice of smooth p-adic formal scheme over $\mathbf{Z}_p$ lifting X defines a lift of $\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ along the map $\mathbf{F}_p^{\mathrm{HT}} \to \mathbf{Z}_p^{\mathrm{HT}}$. In particular, since $(\mathbf{Z}_p^{\mathrm{HT}})_{p=0} \cong \mathrm{BW}^\times[\mathrm{F}] \cong \mathrm{B}\mathbf{G}_m^\sharp$ by Theorem 7.5.1 (which, in this case, is [BL, Theorem 3.4.13]), one obtains an action of $\mathrm{W}^\times[\mathrm{F}]$ on $\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$. (In fact, this comes from an action of $\mathrm{W}^\times[\mathrm{F}]$ on $\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}$.) The subgroup $\mu_p \subseteq \mathrm{W}^\times[\mathrm{F}]$ splits $\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}$ into p summands, where μ_p acts on $\mathcal{H}^i(\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ by weight $-i$. (The action of μ_p is induced by the Sen operator Θ_0 coming from Corollary 7.5.2.) It follows that for any integer i, there is a natural decomposition

$$\tau^{[i,i+p-1]}\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p} \cong \bigoplus_{j=0}^{p-1} \Omega^{i+j}_{\mathrm{X}^{(1)}/\mathbf{F}_p}[-(i+j)].$$

The μ_p-action on $\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}$ therefore provides a refinement of the famous Deligne-Illusie theorem [DI].

In the same way, the inclusion of $\mu_{p^{n+1}}$ into $\mathrm{W}^\times[\mathrm{F}^{n+1}]$ shows that a choice of lift of the quasicoherent sheaf $\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ over $\mathbf{F}_p^{\mathrm{HT}}$ to $\mathrm{BP}\langle n\rangle^{\mathrm{HT}}$ provides an action of $\mu_{p^{n+1}}$ on $\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ which acts on $\mathrm{H}^i(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ by multiplication by $-i$. (Again, the action of μ_{p^n} is induced by the higher Sen operators $\Theta_0, \cdots, \Theta_n$ coming from Corollary 7.5.2.) It follows that for any integer i, there is a natural decomposition

$$\mathrm{R\Gamma}(\mathrm{X}^{(1)}; \tau^{[i,i+p^{n+1}-1]}\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}) \cong \bigoplus_{j=0}^{p^{n+1}-1} \mathrm{R\Gamma}(\mathrm{X}^{(1)}; \Omega^{i+j}_{\mathrm{X}^{(1)}/\mathbf{F}_p}[-(i+j)]).$$

One can show (using [Pst] and the forthcoming work of Pstragowski-Raksit) that a choice of lift of the sheaf $\mathcal{O}_{\mathrm{X}}$ of $\mathbf{F}_p$-algebras to a sheaf of $\mathbf{E}_2$-$\mathrm{BP}\langle n\rangle$-algebras is enough to provide a lift of the quasicoherent sheaf $\mathrm{R}\Gamma(\mathrm{X}^{(1)}; \mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p})$ (with its $\mathbf{E}_1$-algebra structure) over $\mathbf{F}_p^{\mathrm{HT}}$ to $\mathrm{BP}\langle n\rangle^{\mathrm{HT}}$. We therefore conclude:

Corollary 7.5.4. *Suppose* X *is a smooth scheme over* $\mathbf{F}_p$ *which admits a choice of lift of the sheaf* $\mathcal{O}_{\mathrm{X}}$ *of commutative* $\mathbf{F}_p$*-algebras to a sheaf of* $\mathbf{E}_2$-$\mathrm{BP}\langle n\rangle$*-algebras. Then for any integer* i*, there is a natural decomposition*

$$\mathrm{R}\Gamma(\mathrm{X}^{(1)}; \tau^{[i,i+p^{n+1}-1]}\mathrm{F}_*\Omega^\bullet_{\mathrm{X}/\mathbf{F}_p}) \cong \bigoplus_{j=0}^{p^{n+1}-1} \mathrm{R}\Gamma(\mathrm{X}^{(1)}; \Omega^{i+j}_{\mathrm{X}^{(1)}/\mathbf{F}_p}[-(i+j)]).$$

In particular, if X *is further assumed to be proper of dimension* $< p^{n+1}$*, then the Hodge-de Rham spectral sequence*

$$\mathrm{E}_1^{*,*} = \mathrm{H}^*(\mathrm{X}; \Omega^*_{\mathrm{X}/\mathbf{F}_p}) \Rightarrow \mathrm{H}^*_{\mathrm{dR}}(\mathrm{X}/\mathbf{F}_p)$$

degenerates at the E_1*-page. (In fact, to get Hodge-de Rham degeneration, it even suffices to assume that* X *is smooth and proper of dimension* $< p^{n+1}$ *and that the monoidal* $\mathbf{F}_p$*-linear* ∞*-category* QCoh(X) *admits a lift to a monoidal* $\mathrm{BP}\langle n\rangle$*-linear* ∞*-category.)*

Using the trick of Serre duality as in [DI], the Hodge-de Rham degeneration above can be extended to the case when X has dimension p^{n+1} as well. We view Corollary 7.5.4 as a step towards a positive answer of Deligne and Illusie's question in some generality.

Remark 7.5.5. Sasha Petrov recently constructed in [Pet1] a $(p+1)$-dimensional smooth and proper $\mathbf{Z}_p$-scheme $\mathfrak{X}$ such that the Hodge-de Rham spectral sequence for its special fiber $\mathfrak{X}_{p=0}$ does not degenerate at the E_1-page. It follows from Corollary 7.5.4 that $\mathfrak{X}$ provides an example of a scheme over $\mathbf{Z}_p$ such that $\mathcal{O}_{\mathfrak{X}}$ does *not* lift to a sheaf of $\mathbf{E}_2$-ku-algebras.

Remark 7.5.6. Corollary 7.5.4 has the following counter-intuitive consequence: the differentials in the Hodge-de Rham spectral sequence obstruct the lifting of $\mathcal{O}_{\mathrm{X}}$ to a sheaf of $\mathbf{E}_2$-$\mathrm{BP}\langle n\rangle$-algebras. One class of X for which QCoh(X) does satisfy the hypotheses of Corollary 7.5.4 are toric varieties; but in those cases, degeneration was already known for X of arbitrary dimension (since they are F-liftable).

Example 7.5.7. Recall from Remark 3.4.3 that the category $\mathrm{Rep}(\check{\mathrm{G}})$ for any (split) reductive group over $\mathbf{Z}$ admits an $\mathbf{E}_2$-monoidal lift to the sphere spectrum. In particular, $\mathrm{Rep}(\check{\mathrm{G}})$ lifts, as an $\mathbf{E}_2$-monoidal category, to BP. Corollary 7.5.4 implies that if $\check{\mathrm{G}}$ denotes a (split) reductive group over $\mathbf{F}_p$, then the conjugate spectral sequence for $\mathrm{B}\check{\mathrm{G}}$ collapses at the E_2-page. Using the finiteness results of [Jan2, Proposition II.4.10 and Corollary II.4.7] and a standard dimension counting argument, it follows that the Hodge-de Rham spectral sequence

$$\mathrm{E}_1^{*,*} = \mathrm{H}^*(\mathrm{B}\check{\mathrm{G}}; \Omega^*_{\mathrm{B}\check{\mathrm{G}}/\mathbf{F}_p}) \Rightarrow \mathrm{H}^*_{\mathrm{dR}}(\mathrm{B}\check{\mathrm{G}}/\mathbf{F}_p)$$

degenerates at the E_1-page. In fact, there is an equivalence of $\mathbf{E}_2$-$\mathbf{F}_p$*-algebras*

$$\mathrm{R}\Gamma(\mathrm{B}\check{\mathrm{G}}^{(1)}; \mathrm{F}_*\mathrm{dR}_{\mathrm{B}\check{\mathrm{G}}/\mathbf{F}_p}) \cong \mathrm{R}\Gamma(\mathrm{B}\check{\mathrm{G}}^{(1)}; \mathrm{Sym}^*_{\mathbf{F}_p}(\check{\mathfrak{g}}^{(1)})).$$

This was also observed by Petrov [Pet2, Theorem 1.4] (albeit as an equivalence of $\mathbf{E}_1$-algebras, not $\mathbf{E}_2$-algebras) using the observation that $\mathrm{B}\check{\mathrm{G}}$ is in fact F-split; his argument is closely related to the Frobenius contraction functor of [GM3].

There is a noncommutative analogue of Corollary 7.5.4, which is much easier to prove (in the sense that it does not require the full setup of prismatic stacks, etc.); see [Dev1].

Proposition 7.5.8. *Let $n \leq \infty$, and let $\mathcal{C}$ be a smooth and proper $\mathbf{F}_p$-linear ∞-category such that $\pi_j \mathrm{HH}(\mathcal{C}/\mathbf{F}_p) = 0$ for $j \notin [-p^n, p^n]$. If $\mathcal{C}$ lifts to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, then the Tate spectral sequence*

$$\mathrm{E}_2^{*,*} = \hat{\mathrm{H}}^*(\mathrm{BS}^1; \pi_*\mathrm{HH}(\mathcal{C}/\mathbf{F}_p)) \Rightarrow \pi_*\mathrm{HP}(\mathcal{C}/\mathbf{F}_p)$$

collapses at the E_2-page.

This was already known if $\mathcal{C}$ lifts all the way to S^0; see [Mat3, Example 3.5].

Remark 7.5.9. Let $\mathrm{I} = (p^2, v_1^2, \cdots, v_{n-1}^2)$. Were $\mathrm{BP}\langle n-1\rangle/\mathrm{I}$ to admit the structure of an $\mathbf{E}_2$-ring, Proposition 7.5.8 would continue to hold with $\mathrm{BP}\langle n-1\rangle$ replaced by $\mathrm{BP}\langle n-1\rangle/\mathrm{I}$. This is because one can prove that Lemma 7.5.12 continues to hold for $\mathrm{BP}\langle n-1\rangle/\mathrm{I}$.

Some preliminary calculations seem to suggest that Petrov's first Sen class (see [Pet1, Ill]) is related to the obstruction in Hochschild cohomology to lifting a $\mathbf{Z}_p$-scheme $\mathfrak{X}$ along the map $\mathrm{BP}\langle 1\rangle/v_1^2 \to \mathbf{Z}_p$ (and even along the map $\tau_{\leq 2p-3} j \to \mathbf{Z}_p$, where j is the connective complex image-of-J spectrum). For instance, the first k-invariant of $\mathrm{BP}\langle 1\rangle/v_1^2$ is given by the map $\mathbf{Z}_p \to \mathbf{Z}_p[2p-1]$ defined via the composite

$$\mathbf{Z}_p \to \mathbf{F}_p \xrightarrow{\mathrm{P}^1} \mathbf{F}_p[2p-2] \xrightarrow{\beta} \mathbf{Z}_p[2p-1],$$

where P^1 is a Steenrod operation and β is the Bockstein. In other words, $\mathrm{BP}\langle 1\rangle/v_1^2$ is equivalent to the fiber of the above composite. On the other hand, the extension class for $\mathcal{O}_{\mathfrak{X}} \to \mathrm{F}^p\Omega^{\not{D}}_{\mathfrak{X},0} \to \mathrm{L}\Omega^p_{\mathfrak{X}}[-p]$ is computed in [Pet1, Lemma 6.5] to be the composite

$$\mathrm{L}\Omega^p_{\mathfrak{X}}[-p] \to \mathrm{L}\Omega^p_{\mathfrak{X}_{p=0}/\mathbf{F}_p}[-p] \xrightarrow{c_{\mathrm{X},p}} \mathcal{O}_{\mathfrak{X}_{p=0}} \xrightarrow{\beta} \mathcal{O}_{\mathfrak{X}}[1],$$

where the "first Sen class" $c_{\mathrm{X},p}$ can be defined using Steenrod operations on cosimplicial algebras via [Pet1, Theorem 7.1]. We hope to explore this further to obtain a tighter connection between the results in this article and those of Petrov's.

The idea to prove Proposition 7.5.8 is essentially the argument of [Mat3], so we recommend reading that paper first. Recall Bökstedt's calculation that $\pi_*\mathrm{THH}(\mathbf{F}_p) \cong \mathbf{F}_p[\sigma]$, where σ lives in degree 2. By [Mat3, Proposition 3.4], Proposition 7.5.8 is a consequence of:

Proposition 7.5.10. *Let $\mathcal{C}$ be a smooth and proper $\mathbf{F}_p$-linear ∞-category such that $\pi_j \mathrm{HH}(\mathcal{C}/\mathbf{F}_p) = 0$ for $j \notin [-p^n, p^n]$. If $\mathcal{C}$ lifts to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, then $\mathrm{THH}(\mathcal{C})$ is σ-torsionfree.*

To prove Proposition 7.5.10, we need a preliminary result. It follows from [DHL+, Theorem 5.2 and Corollary 2.8] that there is an augmentation $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \mathrm{BP}\langle n-1\rangle$; composing with the map $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$ defines a map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \mathbf{F}_p$.

Proposition 7.5.11. *The map $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ factors, as an $\mathbf{E}_2$-algebra map, as the composite*

$$\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \mathbf{F}_p \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p).$$

We will prove a much stronger version of Proposition 7.5.11 in Corollary 7.5.18 below.

Proof. It evidently suffices to show that the map

$$\tau_{\leq 2p^n-1}(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)\otimes_{\mathrm{BP}\langle n-1\rangle}\mathbf{F}_p)\to\tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$$

factors, as an $\mathbf{E}_2$-algebra map, as the composite

$$\tau_{\leq 2p^n-1}(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)\otimes_{\mathrm{BP}\langle n-1\rangle}\mathbf{F}_p)\to\mathbf{F}_p\to\tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p).$$

There is an $\mathbf{E}_3$-map $\mathrm{BP}\to\mathrm{BP}\langle n\rangle$, which defines an $\mathbf{E}_2$-map

$$\mathrm{THH}(\mathrm{BP})\otimes_{\mathrm{BP}}\mathbf{F}_p\to\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)\otimes_{\mathrm{BP}\langle n-1\rangle}\mathbf{F}_p.$$

This map is an equivalence in degrees $\leq 2p^n-1$.[2] Therefore, it suffices to show that the map $\mathrm{THH}(\mathrm{BP})\otimes_{\mathrm{BP}}\mathbf{F}_p\to\mathrm{THH}(\mathbf{F}_p)$ factors, as an $\mathbf{E}_2$-map, as the composite

$$\mathrm{THH}(\mathrm{BP})\otimes_{\mathrm{BP}}\mathbf{F}_p\to\mathbf{F}_p\to\mathrm{THH}(\mathbf{F}_p);$$

equivalently, that the map $\mathrm{THH}(\mathrm{BP})\to\mathrm{THH}(\mathbf{F}_p)$ factors, as an $\mathbf{E}_2$-map, as the composite

$$\mathrm{THH}(\mathrm{BP})\to\mathrm{BP}\to\mathrm{THH}(\mathbf{F}_p).$$

Here, the map $\mathrm{BP}\to\mathrm{THH}(\mathbf{F}_p)$ is just the composite of the map $\mathrm{BP}\to\mathbf{F}_p$ with the unit $\mathbf{F}_p\to\mathrm{THH}(\mathbf{F}_p)$. Since BP is an $\mathbf{E}_4$-algebra retract of MU (compatibly with their natural maps to $\mathbf{F}_p$), it suffices to replace BP by MU in the above discussion; in fact, we will even show that the map $\mathrm{THH}(\mathrm{MU})\to\mathrm{THH}(\mathbf{F}_p)$ factors, as an $\mathbf{E}_3$-map, as the composite

$$\mathrm{THH}(\mathrm{MU})\to\mathrm{MU}\to\mathrm{THH}(\mathbf{F}_p).$$

Here, the map $\mathrm{MU}\to\mathrm{THH}(\mathbf{F}_p)$ is just the composite of the map $\mathrm{MU}\to\mathbf{F}_p$ with the unit $\mathbf{F}_p\to\mathrm{THH}(\mathbf{F}_p)$.

Recall from [BCS] and [Kla] that there is an equivalence $\mathrm{THH}(\mathrm{MU})\simeq\mathrm{MU}[\mathrm{SU}]$ of $\mathbf{E}_\infty$-MU-algebras, and that the augmentation $\mathrm{THH}(\mathrm{MU})\to\mathrm{MU}$ is given by taking MU-chains of the augmentation $\mathrm{SU}\to *$. The $\mathbf{E}_\infty$-MU-linear map $\mathrm{THH}(\mathrm{MU})\to\mathrm{THH}(\mathbf{F}_p)$ is therefore equivalent to the data of an $\mathbf{E}_\infty$-map $\mathrm{SU}\to\mathrm{GL}_1(\mathrm{THH}(\mathbf{F}_p))$. Since $\mathrm{THH}(\mathbf{F}_p)$ is concentrated in even degrees, $\mathrm{GL}_1(\mathrm{THH}(\mathbf{F}_p))$ is an $\mathbf{E}_\infty$-space with even homotopy. It therefore suffices to prove the following claim: any $\mathbf{E}_3$-map $f:\mathrm{SU}\to\mathrm{X}$ to an $\mathbf{E}_3$-space X with even homotopy factors (as an $\mathbf{E}_3$-map) through the augmentation $\mathrm{SU}\to *$. Indeed, f is equivalent to the data of a map $\mathrm{B}^3f:\mathrm{B}^3\mathrm{SU}\to\mathrm{B}^3\mathrm{X}$. Since $\mathrm{B}^3\mathrm{SU}=\mathrm{BU}\langle 6\rangle$ has an even cell decomposition and $\mathrm{B}^3\mathrm{X}$ has odd homotopy, the map B^3f is necessarily null (so f is null as an $\mathbf{E}_3$-map), as desired. ☐

The proof of the following result is a direct adaptation of that of [Mat3, Proposition 3.7].

Lemma 7.5.12. *Let* M *be a perfect* $\mathrm{THH}(\mathbf{F}_p)$*-module such that* $\pi_i(\mathrm{M})=0$ *for* $i<a$. *If* M *lifts to a perfect* $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$*-module* $\widetilde{\mathrm{M}}$*, then* σ*-multiplication* $\sigma:\pi_{i-2}\mathrm{M}\to\pi_i\mathrm{M}$ *is injective for* $i\leq a+2p^n-1$.

[2] For instance, this follows from [ACH, Proposition 2.9] (see also [Dev2, Remark 2.2.5]), which says that for $n\leq\infty$, there is an isomorphism

$$\pi_*(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)\otimes_{\mathrm{BP}\langle n-1\rangle}\mathbf{F}_p)\cong\mathbf{F}_p[\sigma^2(v_n)]\otimes\Lambda(\sigma(t_1),\cdots,\sigma(t_n)),$$

where $|\sigma^2(v_n)|=2p^n$ and $|\sigma(t_i)|=2p^i-1$.

Proof. To prove the result of the lemma, we can assume without loss of generality that $a = 0$. Then, there is a map

$$\mathrm{M} \to \tau_{\leq 2p^n-1}\widetilde{\mathrm{M}} \otimes_{\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)} \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p),$$

which is an equivalence on $\tau_{\leq 2p^n-1}$. By Proposition 7.5.11, the map $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ factors through $\mathbf{F}_p \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$, so we see that $\tau_{\leq 2p^n-1}\mathrm{M}$ is a free $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$-module on classes in nonnegative degrees. Therefore, σ-multiplication is injective through the stated range. □

Proposition 7.5.10 is now a consequence of the following, whose proof is a direct adaptation of that of [Mat3, Proposition 3.8].

Proposition 7.5.13. *Let* M *be a perfect* $\mathrm{THH}(\mathbf{F}_p)$*-module with Tor-amplitude in* $[-p^n, p^n]$*. If* M *lifts to a perfect* $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$*-module* $\widetilde{\mathrm{M}}$*, then* M *is free.*

Proof. The argument is the same as in [Mat3, Proposition 3.8]. Indeed, M is a direct sum of $\mathrm{THH}(\mathbf{F}_p)$-modules which are free or of the form $\mathrm{M}_{i,j} = \Sigma^i\mathrm{THH}(\mathbf{F}_p)/\sigma^j$ (see [Mat3, Proposition 3.3]). Since $\mathrm{M}_{i,j}$ has Tor-amplitude in $[i, i+2j+1]$, the condition on M implies that $\mathrm{M}_{i,j}$ could appear as a summand of M if and only if $-p^n \leq i \leq i+2j+1 \leq p^n$.

The class $\sigma^{j-1}[i] \in \pi_{i+2j-2}\mathrm{M}_{i,j}$ is killed by σ, so taking $a = -p^n$ in Lemma 7.5.12, we see that

$$i + 2j > -p^n + 2p^n - 1 = p^n - 1.$$

In particular, $i + 2j + 1 > p^n$, which contradicts $i + 2j + 1 \leq p^n$. Therefore, no $\mathrm{M}_{i,j}$ can be a summand of M, so that M is free. □

Let us make a brief remark about an extension of Proposition 7.5.11. If Y is a connected space, $f : \mathrm{Y} \to \mathcal{C}$ is a functor detecting an $\Omega\mathrm{Y}$-action on an object $c \in \mathcal{C}$, and $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is a functor, there is a natural map

$$\mathrm{colim}_{\mathrm{Y}}\, \mathrm{F}f = \mathrm{F}(c)_{h\Omega\mathrm{Y}} \to \mathrm{F}(c_{h\Omega\mathrm{Y}}) = \mathrm{F}(\mathrm{colim}_{\mathrm{Y}}\, f).$$

This is called the *assembly map*.

Let $\mathcal{C}$ denote the ∞-category $\mathrm{Mod}_{\mathrm{Sp}}$, and let $\mathrm{F} : \mathcal{C} \to \mathrm{CycSp}$ denote the functor given by THH. If X is an $\mathbf{E}_1$-space and $f : \mathrm{X} \to \mathrm{Pic}(\mathrm{Sp})$ is an $\mathbf{E}_1$-map, we obtain a map $f : \mathrm{BX} \to \mathrm{B}\,\mathrm{Pic}(\mathrm{Sp}) \subseteq \mathrm{Mod}_{\mathrm{Sp}}$ which defines an X-action on Sp. (A point of X acts on Sp by tensoring with its image under f in $\mathrm{Pic}(\mathrm{Sp})$.) We therefore obtain a map

$$\mathrm{BX} \to \mathrm{Mod}_{\mathrm{Sp}} \xrightarrow{\mathrm{THH}} \mathrm{Sp},$$

which can be thought of as capturing the induced X-action on $\mathrm{THH}(\mathrm{Sp}) = \mathrm{S}$. The above discussion therefore gives an assembly map

$$\mathrm{colim}_{\mathrm{BX}}\, \mathrm{S} = \mathrm{S}_{h\mathrm{X}} \to \mathrm{THH}(\mathrm{colim}_{\mathrm{BX}}\, f) = \mathrm{THH}(\mathrm{Sp}_{h\mathrm{X}}).$$

However, $\mathrm{Sp}_{h\mathrm{X}}$ is just the category $\mathrm{LMod}_{\mathrm{X}^f}$ of modules over the Thom spectrum of f. We therefore obtain an assembly map

$$\mathrm{colim}_{\mathrm{BX}}\, \mathrm{S} \to \mathrm{THH}(\mathrm{X}^f) \tag{7.5.1}$$

in CycSp. Note that the source itself can be viewed as the Thom spectrum of the map $\mathrm{BX} \to \mathrm{Pic}(\mathrm{Sp})$ which captures the action of X on S.

For instance, if f is the null map, then $X^f = S[X]$, and the assembly map is just the standard assembly map $S[BX]^{triv} \to THH(S[X])$; usually, this is viewed as a map $S[BX] \to TC(S[X])$.

Suppose, for simplicity, that X is in fact an infinite loop space (so $X = \Omega^\infty x$ for some connective spectrum x). Then this assembly map can be constructed in a slightly different way as follows. Let λ denote the standard action of S^1 on the $\mathbf{C}$, and let $S(\lambda)$ denote the corresponding unit sphere. We recall the construction of an S^1-equivariant composite

$$S^1 \to S(\lambda)_+ \to S^0. \tag{7.5.2}$$

The second map $S(\lambda)_+ \to S^0$ just crushes $S(\lambda)$. Its dual is a map $S^0 \to (S(\lambda)_+)^\vee$; equivariant Poincaré duality identifies $(S(\lambda)_+)^\vee \simeq \Sigma^{-1}(S(\lambda)_+)$, so we obtain a map $S^0 \to \Sigma^{-1}(S(\lambda)_+)$. This gives a map $S^1 \to S(\lambda)_+$, which is the first map in (7.5.2). Note that since we used Poincaré duality, one could alternatively also view (7.5.3) as the map obtained by tensoring (7.5.2) with x: there is an S^1-equivariant equivalence $S(\lambda)_+ \otimes x \simeq \mathrm{Map}(S(\lambda)_+, \Sigma x)$.

More generally, if G is a compact Lie group, there is a canonical map $G_+ \to S^0$, whose dual is a map $S^0 \to (G_+)^\vee \simeq \Sigma^{-\mathfrak{g}}(G_+)$, so we obtain a map

$$S^{\mathfrak{g}} \to G_+ \to S^0;$$

this is exactly the G-equivariant transfer map. In other words, (7.5.2) is just the S^1-equivariant transfer map; more precisely, the S^1-equivariant map $S^1 \to S^0$ defines a map $\Sigma\mathbf{C}P^\infty_+ \to S^0$, which can be identified with the standard S^1-transfer map. This is why, for instance, its underlying nonequivariant map is just η (as claimed in [BCS]).

Applying the functor $\mathrm{Map}(-, \Sigma x)$ to (7.5.2) defines a map

$$\Sigma x \to \mathrm{Map}(S(\lambda)_+, \Sigma x) \to x. \tag{7.5.3}$$

If $f : x \to \mathrm{Pic}(\mathrm{Sp})$ is a map of spectra with associated Thom spectrum X^f (so it is an $\mathbf{E}_\infty$-ring), then the Thom spectrum of the composite $\mathrm{Map}(S(\lambda)_+, \Sigma x) \to x \to \mathrm{Pic}(\mathrm{Sp})$ acquires an S^1-action, and as such identifies with the $\mathbf{E}_\infty$-ring $THH(X^f)$. One can verify:

Proposition 7.5.14. *If $X = \Omega^\infty x$ is an infinite loop space and $f : x \to \mathrm{Pic}(\mathrm{Sp})$ is a map of spectra, then the source of the assembly map (7.5.1) is given by the Thom spectrum of the composite*

$$\Sigma x \to S(\lambda)_+ \otimes x \to x \xrightarrow{f} \mathrm{Pic}(\mathrm{Sp}).$$

We now claim that the S^1-equivariant transfer map $\Sigma x \to x$ is null if x is an MU-module. It evidently suffices to prove this when $x = \mathrm{MU}$. Since MU has the trivial S^1-action, the S^1-equivariant transfer $\Sigma\mathrm{MU} \to \mathrm{MU}$ is the same as a map $\Sigma\mathrm{MU}_{hS^1} \to \mathrm{MU}$. This map is *canonically* nullhomotopic: indeed, the data of such a nullhomotopy is the data of a complex-orientation. This discussion implies:

Corollary 7.5.15. *If x is an MU-module and $f : x \to \mathrm{Pic}(\mathrm{Sp})$ is a map, then there is a (Borel) S^1-equivariant $\mathbf{E}_\infty$-map $\mathrm{colim}_{BX}\, S = S[BX] \to THH(X^f)$, which we call the (twisted) assembly map. The tensor product $THH(X^f) \otimes_{S[BX]} S$ identifies with the Thom spectrum X^f itself.*

Remark 7.5.16. Since the S^1-equivariant transfer map $\Sigma x \to S(\lambda)_+ \otimes x \to x$ is null, the choice of such a nullhomotopy amounts to a lift of the map $\Sigma x \to S(\lambda)_+ \otimes x$ to a map $\Sigma x \to \Sigma^{\lambda-1} x$. This map is an equivalence (by the MU-module structure on x). One therefore obtains a boundary map $\Sigma^{-1}x \to \Sigma x$ whose cofiber is $S(\lambda)_+ \otimes x$. The map $\Sigma^{-1}x \to \Sigma x$ identifies (under the trivialization $\Sigma^{\lambda-2}x = x$) with the tensor product of $\Sigma^{-1}x$ with the map $a_\lambda : S^0 \to S^\lambda$ (given by inclusion of the fixed points). If x is connected (i.e., $x = \tau_{\geq 1}x$), then the map $f : x \to \mathrm{Pic}(\mathrm{Sp})$ lifts to a map $f : x \to \mathfrak{bgl}_1(S) = \tau_{\geq 1}\mathrm{Pic}(\mathrm{Sp})$. The cofiber sequence

$$\Sigma^{-1}x \xrightarrow{a_\lambda} \Sigma x \to S(\lambda)_+ \otimes x$$

along with the general machinery of Thom spectra implies that there is a commuative diagram, each square of which is a pushout in S^1-equivariant $\mathbf{E}_\infty$-rings:

$$\begin{tikzcd} S[\Omega X] \arrow[r, "a_\lambda"] \arrow[d, "f"'] & S[BX] \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r] & \mathrm{THH}(X^f) \arrow[r] & X^f. \end{tikzcd}$$

Here, we have identified $\Omega^\infty\Sigma^{-1}x = \Omega X$; and the map $f : S[\Omega X] \to S$ is adjoint to the map $\Sigma^{-1}x \to \mathfrak{gl}_1(S)$ coming from $f : x \to \mathfrak{bgl}_1(S)$. Note that although $S[\Omega X]$ and $S[BX]$ both have trivial S^1-action in the preceding pushout square, the *map* a_λ has a nontrivial S^1-equivariant structure: it defines a map $S[\Omega X] \to S[BX]^{hS^1}$ which does not factor through the unit $S[BX]^{hS^1} \to S[BX]$. In fact, the map $a_\lambda : S[\Omega X] \to S[BX]$ is *only* interesting S^1-equivariantly: nonequivariantly, it factors as the composite

$$S[\Omega X] \to S \to S[BX];$$

in this way, one recovers the *non-S^1-equivariant* equivalence $\mathrm{THH}(X^f) \simeq X^f[BX]$ of $\mathbf{E}_\infty$-rings.

Although these observations are very general, it turns out that when $x = \mathrm{ku}$ itself (and f is the J-homomorphism), one can refine the map $S[SU] \to \mathrm{THH}(MU)$ to a map of *cyclotomic* $\mathbf{E}_\infty$-rings; this is special to the case at hand, and need not be true in the general setup above. Again, we defer the proof to our forthcoming paper. The following result gives a complete description of $\mathrm{THH}(MU)$ as an S^1-equivariant $\mathbf{E}_\infty$-ring:

Theorem 7.5.17 (Joint with J. Hahn, A. Raksit, and A. Yuan)**.** *The assembly map for* $\mathrm{THH}(MU)$ *defines a map* $S[SU]^{\mathrm{triv}} \to \mathrm{THH}(MU)$ *of cyclotomic* $\mathbf{E}_\infty$*-rings. Moreover, if* $J : S[U] \to S$ *denotes the map adjoint to the J-homomorphism* $\Sigma^{-1}\mathrm{bu} \to \mathfrak{gl}_1(S)$*, then there is a commutative diagram, each square of which is a pushout in* S^1*-equivariant* $\mathbf{E}_\infty$*-rings:*

$$\begin{tikzcd} S[U]^{\mathrm{triv}} \arrow[r, "a_\lambda"] \arrow[d, "J"'] & S[SU]^{\mathrm{triv}} \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r] & \mathrm{THH}(MU) \arrow[r] & MU. \end{tikzcd}$$

Moreover, the cyclotomic Frobenius on MU *resulting from taking the pushout of the rightmost square in* CycSp *is given by the* $\mathbf{E}_\infty$*-Frobenius on* MU*.*

This has many applications. For instance, it leads to the following:

Corollary 7.5.18. *The map* $\mathrm{THH}(MU) \to \mathrm{THH}(\mathbf{F}_p)$ *factors, as a map of cyclotomic* $\mathbf{E}_\infty$*-rings, as the composite*

$$\mathrm{THH}(MU) \to MU \to \mathbf{Z}_p \to \mathrm{THH}(\mathbf{F}_p).$$

In particular, using the argument of the first part of Proposition 7.5.11, it follows that the map $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ *factors, as an* S^1*-equivariant* $\mathbf{E}_2$*-algebra map, as the composite*

$$\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \tau_{\leq 2p^n-1}\mathrm{BP}\langle n-1\rangle \to \mathbf{Z}_p \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p).$$

Proof. Indeed, by Theorem 7.5.17, it suffices to prove that the map $\mathrm{S}[\mathrm{SU}]^{\mathrm{triv}} \to \mathrm{THH}(\mathrm{MU}) \to \mathrm{THH}(\mathbf{F}_p)$ factors as the composite

$$\mathrm{S}[\mathrm{SU}]^{\mathrm{triv}} \to \mathrm{S} \to \mathbf{Z} \to \mathrm{THH}(\mathbf{F}_p);$$

but this is clear, because any cyclotomic map $\mathrm{S}[\mathrm{SU}]^{\mathrm{triv}} \to \mathrm{THH}(\mathbf{F}_p)$ is given by a map $\mathrm{S}[\mathrm{SU}] \to \mathbf{Z}_p \cong \tau_{\geq 0}\mathrm{TC}(\mathbf{F}_p)$, which necessarily factors through the truncation $\tau_{\leq 0}(\mathrm{S}[\mathrm{SU}]) = \mathbf{Z}$. □

Corollary 7.5.19. *If* $\mathcal{C}_0$ *is an* $\mathbf{F}_p$*-linear category which lifts to* MU *(or* BP*), and* $\mathcal{C}$ *is the base-change of this lift to* $\mathbf{Z}_p$*, then there is a natural equivalence*

$$\mathrm{TC}^-(\mathcal{C}_0) \simeq \mathrm{HC}^-(\mathcal{C}/\mathbf{Z}_p)\widehat{\otimes}_{\mathbf{Z}_p^{h\mathrm{S}^1}}\mathrm{TC}^-(\mathbf{F}_p) \tag{7.5.4}$$

which is lax symmetric monoidal in the lift of $\mathcal{C}_0$ *to* MU*. The tensor product on the right-hand side is* $\hbar$*-completed (where* $\pi_*\mathbf{Z}_p^{h\mathrm{S}^1} \cong \mathbf{Z}_p[\hbar]$*).*

Proof. Indeed, if $\widetilde{\mathcal{C}}$ denotes the lift of $\mathcal{C}_0$ to MU, there are S^1-equivariant equivalences

$$\begin{aligned}\mathrm{THH}(\mathcal{C}_0) &\simeq \mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{THH}(\mathbf{F}_p)\\ &\simeq \mathrm{THH}(\widetilde{\mathcal{C}}) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{F}_p)\\ &\simeq \mathrm{HH}(\widetilde{\mathcal{C}} \otimes_{\mathrm{MU}} \mathbf{Z}_p/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{F}_p) = \mathrm{HH}(\mathcal{C}/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathrm{THH}(\mathbf{F}_p).\end{aligned}$$

The desired claim follows by taking homotopy S^1-fixed points. □

The analogous statement with TC^- and HC^- replaced by TP and HP is also true (see [PV] and Corollary 6.1.6), and it does not require any assumptions on $\mathcal{C}_0$ other than it lifting to $\mathbf{Z}_p$.

Since the factorizations in Corollary 7.5.18 are via $\mathbf{E}_2$-maps, one obtains a similar factorization at the level of prismatic stacks. This, for instance, implies the following analogue of (7.5.4):

Corollary 7.5.20. *Let* $n \geq 0$*. Suppose* X *is a smooth scheme over* $\mathbf{F}_p$ *which admits a choice of lift of the sheaf* $\mathcal{O}_\mathrm{X}$ *of commutative* $\mathbf{F}_p$*-algebras to a sheaf of* $\mathbf{E}_2$*-*$\mathrm{BP}\langle n\rangle$*-algebras, and let* $\mathfrak{X}$ *denote the corresponding* p*-adic formal scheme over* $\mathbf{Z}_p$ *(base-changing along the map* $\mathrm{BP}\langle n\rangle \to \mathbf{Z}_p$*). Then there is a filtered isomorphism*

$$\mathcal{N}^{\geq\star}\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{X}/\mathbf{Z}_p) \cong \mathrm{F}^{\geq\star}_{\mathrm{Hdg}}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} (p^\star)$$

in weights $\leq p^n-1$*. Here,* $(p^\star)$ *denotes the* p*-adic filtration on* $\mathbf{Z}_p$*. (To get this isomorphism, it even suffices to assume that* X *is smooth of dimension* $< p^{n+1}$ *and that the monoidal* $\mathbf{F}_p$*-linear* ∞*-category* QCoh(X) *admits a lift to a monoidal* $\mathrm{BP}\langle n\rangle$*-linear* ∞*-category.)*

In particular, taking $n = \infty$, one finds that the Nygaard filtration on the crystalline cohomology of X is the tensor product of the Hodge filtration on the de Rham cohomology of $\mathfrak{X}$ and the p-adic filtration on $\mathbf{Z}_p$.

Example 7.5.21. It follows from Remark 3.4.3 that if $\check{\mathrm{G}}$ denotes a (split) reductive group over $\mathbf{Z}$ whose fiber over $\mathbf{F}_p$ is $\check{\mathrm{G}}_{\mathbf{F}_p}$, then there is an isomorphism of filtered $\mathbf{E}_2$-$\mathbf{Z}_p$-algebras

$$\mathcal{N}^{\geq\star}\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{B}\check{\mathrm{G}}_{\mathbf{F}_p}/\mathbf{Z}_p) \cong \mathrm{F}^{\geq\star}_{\mathrm{Hdg}}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{B}\check{\mathrm{G}}/\mathbf{Z}) \otimes_{\mathbf{Z}} (p^\star).$$

This generalizes Example 7.5.7. It would be good to have a more direct proof of this fact (which does not wind through geometric Langlands duality), along the lines of [Pet2].

7.6 Relation to the dual Steenrod algebra

Many structural results in the theory of prismatic stacks developed above can be explained using the structure of the dual Steenrod algebra $\mathcal{A}_* = \pi_*(\mathbf{F}_p \otimes \mathbf{F}_p)$. Recall that there are isomorphisms

$$\mathcal{A}_* \cong \begin{cases} \mathbf{F}_2[\tau_0, \tau_1, \cdots] & p = 2, \\ \Lambda_{\mathbf{F}_p}(\tau_0, \tau_1, \cdots) \otimes \mathbf{F}_p[\zeta_1, \zeta_2, \cdots] & p > 2, \end{cases}$$

where τ_i is in weight $2p^i - 1$ and ζ_i is in weight $2p^i - 2$. At the prime 2, we set $\zeta_i = \tau_{i-1}^2$. The subalgebra of $\mathcal{A}_*$ generated by the ζ_i will be denoted $\mathcal{A}_*^{\mathrm{ev}}$; dividing the grading by 2, we will identify $\mathcal{A}_*^{\mathrm{ev}}$ with $\pi_{2*}(\mathbf{F}_p \otimes \mathrm{BP})$. This is not quite standard notation, but leads to more consistent formulas across primes. The following is a famous theorem of Mahowald and Hopkins (see [AB1] for a more modern take, but with ultimately the same argument):

Theorem 7.6.1 (Hopkins-Mahowald [Mah])**.** *The Thom spectrum of the $\mathbf{E}_2$-map $\Omega^2\mathrm{S}^3 \xrightarrow{1+p} \mathrm{BGL}_1(\mathbb{S}_p)$ detecting $1+p$ on the bottom cell of the source is equivalent to $\mathbf{F}_p$ as an $\mathbf{E}_2$-algebra. In other words, $\mathbf{F}_p$ is the free $\mathbf{E}_2$-algebra with a nullhomotopy of p. The Thom isomorphism then identifies $\mathcal{A}_* \cong \pi_*\mathbf{F}_p[\Omega^2\mathrm{S}^3]$.*

This leads to at least two algebro-geometric interpretations of $\mathrm{Spec}(\mathcal{A}_*)$. The first is classical, and we will take this opportunity to explain how it relates to various constructions in homotopy theory. For the moment, we ask that the reader allow us to take Spec of a graded commutative ring which is not necessarily concentrated in even degrees.

Example 7.6.2. For simplicity, take $p = 2$ (the analogue at odd primes is not much more complicated, but it requires working in the $\mathbf{Z}/2$-graded context). Then the group scheme $\mathrm{Spec}(\mathcal{A}_*)$ is isomorphic to the group scheme $\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\widehat{\mathbf{G}}_a(1)/\mathbf{G}_m)$ of graded automorphisms of the additive formal group law; here, one views $\widehat{\mathbf{G}}_a(1)$ as $\mathrm{Spf}(\pi_*(\mathbf{F}_2^{\mathbf{R}\mathrm{P}^\infty}))$. Indeed, $\mathrm{Spec}(\mathcal{A}_*)$ carries the universal automorphism given by the formula $\sum_{i\geq 0} \tau_{i-1}x^{p^i}$, where we have set $\tau_{-1} = 1$. The grading here places the coordinate x of $\widehat{\mathbf{G}}_a$ in weight -1.

This perspective suggests several natural subalgebras of $\mathcal{A}_*$. For instance, the homomorphism $\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\widehat{\mathbf{G}}_a(1)/\mathbf{G}_m) \to \mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\alpha_{p^n}(1)/\mathbf{G}_m)$ induces an isomorphism between $\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\alpha_{p^n}(1)/\mathbf{G}_m)$ and $\mathbf{F}_p[\tau_0, \cdots, \tau_{n-1}]$. The inclusion of $\mathbf{F}_p[\tau_0, \cdots, \tau_{n-1}]$ into $\mathcal{A}_*$ admits a topological realization: it identifies with the map $\pi_*(\mathbf{F}_p \otimes y(n)) \to \pi_*(\mathbf{F}_p \otimes \mathbf{F}_p)$, where $y(n)$ is the Thom spectrum of the composite map

$$\Omega\mathrm{J}_{p^n-1}(\mathrm{S}^2) \to \Omega^2\mathrm{S}^3 \xrightarrow{1-p} \mathrm{BGL}_1(\mathbb{S}_p).$$

Here, $\mathrm{J}_{p^n-1}(\mathrm{S}^2)$ is the (p^n-1)st stage of the James construction on $\Omega\mathrm{S}^3$. Equivalently, using the Thom isomorphism, the inclusion $\mathbf{F}_p[\tau_0, \cdots, \tau_{n-1}] \hookrightarrow \mathcal{A}_*$ identifies with the effect on $\mathbf{F}_p$-homology of the map $\Omega\mathrm{J}_{p^n-1}(\mathrm{S}^2) \to \Omega^2\mathrm{S}^3$. Summarizing, there are isomorphisms

$$\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\alpha_{p^n}(1)/\mathbf{G}_m) \cong \mathrm{Spec}\,\pi_*(\mathbf{F}_p \otimes y(n)) \cong \mathrm{Spec}\,\pi_*(\mathbf{F}_p[\Omega\mathrm{J}_{p^n-1}(\mathrm{S}^2)]).$$

Remark 7.6.3. There is a similar interpretation for the subalgebra $\mathcal{A}_*^{\mathrm{ev}}$ at any prime p. Namely, $\mathrm{Spec}(\mathcal{A}_*^{\mathrm{ev}})$ is also isomorphic to automorphisms of the additive formal group $\widehat{\mathbf{G}_a}$, except that now the formal group $\widehat{\mathbf{G}_a}$ is viewed as $\mathrm{Spf}(\pi_{2*}(\mathbf{F}_p^{\mathbf{C}\mathrm{P}^\infty}))$. Indeed, $\mathrm{Spec}(\mathcal{A}_*^{\mathrm{ev}})$ carries the universal automorphism given by the formula $\sum_{i\geq 0}\zeta_i x^{p^i}$, where we have set $\zeta_0 = 1$. As before, $\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\alpha_{p^n}(1)/\mathbf{G}_m)$ also has a topological interpretation: namely, there is an isomorphism

$$\mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\alpha_{p^n}(1)/\mathbf{G}_m) \cong \mathrm{Spec}\,\pi_{2*}(\mathbf{F}_p \otimes \mathrm{T}(n)),$$

where $\mathrm{T}(n)$ is Ravenel's p-local summand of the $\mathbf{E}_2$-ring $\mathrm{X}(p^n)$.

The perspective that $\mathrm{Spec}(\mathcal{A}_*) \cong \mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\widehat{\mathbf{G}_a}(1)/\mathbf{G}_m)$ therefore relates various topologically realizable filtrations on the dual Steenrod algebra with natural algebro-geometric constructions on the additive formal group. This picture meshes very well with the chromatic worldview. However, we will now discuss another perspective on the dual Steenrod algebra which plays better with the modern approach to p-adic Hodge theory.

Example 7.6.4. The ring $\mathcal{A}_*$ is, up to grading issues, precisely the cohomology of $\mathrm{B}\mathbf{G}_a$ over $\mathbf{F}_p$. More precisely, equip the coordinate μ of $\mathbf{G}_a$ with weight 1, and equip $\mathrm{H}^*(\mathrm{B}(\mathbf{G}_a(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ with the grading where a class in $\mathrm{H}^s(\mathrm{B}(\mathbf{G}_a(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{t\})$ lies in weight $2t - s$. Then there is a graded isomorphism

$$\mathrm{H}^*(\mathrm{B}(\mathbf{G}_a(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathcal{A}_* = \pi_*(\mathbf{F}_p \otimes \mathbf{F}_p). \tag{7.6.1}$$

In fact, there is an equivalence of $\mathbf{E}_2$-$\mathbf{F}_p$-algebras between the shearing of $\mathrm{R}\Gamma(\mathrm{B}(\mathbf{G}_a(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathrm{Sym}_{\mathbf{F}_p}(\mathbf{F}_p[-1](1))$ and $\mathbf{F}_p \otimes \mathbf{F}_p$. (Here, we are using the algebraists' convention for shearing, which sends a module $\mathrm{M}(n)$ in weight n to $\mathrm{M}(n)[2n]$ (as opposed to $\mathrm{M}(n)[n]$, which was the convention used in Part I).)

One can check (7.6.1) directly, but a more natural explanation comes from Theorem 7.6.1. Namely, the map $\mathbf{F}_p \otimes \mathbf{F}_p \to \mathbf{F}_p$ exhibits

$$\mathbf{F}_p \otimes \mathbf{F}_p = \mathrm{Tot}(\mathbf{F}_p^{\otimes_{\mathbf{F}_p \otimes \mathbf{F}_p} \bullet+1}).$$

This cosimplicial diagram is extended from the Hopf algebroid $(\mathbf{F}_p, \mathbf{F}_p \otimes_{\mathbf{F}_p \otimes \mathbf{F}_p} \mathbf{F}_p) = (\mathbf{F}_p, \mathrm{THH}(\mathbf{F}_p))$, which gives a spectral sequence

$$\mathrm{E}_2^{*,*} = \mathrm{Ext}^*_{\pi_*\mathrm{THH}(\mathbf{F}_p)\text{-comod}^{\mathrm{gr}}}(\mathbf{F}_p, \mathbf{F}_p) \Rightarrow \pi_*(\mathbf{F}_p \otimes \mathbf{F}_p). \tag{7.6.2}$$

Here, the E_1-page is the cohomology of the graded Hopf algebroid $(\mathbf{F}_p, \pi_*\mathrm{THH}(\mathbf{F}_p))$. It follows from Theorem 7.6.1 that $\mathrm{THH}(\mathbf{F}_p) \cong \mathbf{F}_p[\Omega \mathrm{S}^3]$, so that $\pi_*\mathrm{THH}(\mathbf{F}_p) \cong \mathbf{F}_p[\mu]$ where μ is primitive (i.e., the coproduct sends μ to $\mu \otimes 1 + 1 \otimes \mu$). The E_2-page of the above spectral sequence is therefore precisely the cohomology of the stack $\mathrm{B}\,\mathrm{Spec}\,\pi_*(\mathrm{THH}(\mathbf{F}_p)) = \mathrm{B}\mathbf{G}_a$. One can now verify that (7.6.2) degenerates at the E_2-page, which gives the isomorphism (7.6.1).

The preceding example can be used to organize various natural constructions around the group scheme $\mathbf{G}_a$ and the dual Steenrod algebra. In the following, if k is an $\mathbf{E}_2$-ring equipped with a map $k \to \mathbf{F}_p$, let us write $\mathcal{A}_*^k$ to denote $\pi_*(\mathbf{F}_p \otimes_k \mathbf{F}_p)$. In all cases of interest, $\mathbf{F}_p \otimes_k \mathbf{F}_p$ is a perfect $\mathbf{F}_p$-module, so that $\pi_*(\mathbf{F}_p \otimes_k \mathbf{F}_p)$ is even a self-dual finite-dimensional Hopf algebra over $\mathbf{F}_p$!

Example 7.6.5. Let $\mathrm{W}(-*)$ denote the graded lift of the group scheme of p-typical Witt vectors where the coordinates μ_i have weights p^i, and equip $\mathrm{H}^*(\mathrm{B}(\mathrm{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ with the grading where a class in $\mathrm{H}^s(\mathrm{B}(\mathrm{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{t\})$ lies in weight $2t-s$. Then there is a graded isomorphism

$$\mathrm{H}^*(\mathrm{B}(\mathrm{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathcal{A}_*^{\mathrm{BP}} = \pi_*(\mathbf{F}_p \otimes_{\mathrm{BP}} \mathbf{F}_p). \tag{7.6.3}$$

In fact, there is an equivalence of $\mathbf{E}_2$-$\mathbf{F}_p$-algebras between the shearing of $\mathrm{R\Gamma}(\mathrm{B}(\mathrm{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ and $\mathbf{F}_p \otimes_{\mathrm{BP}} \mathbf{F}_p$.

Similarly, let $\mathbf{W}(-*)$ denote the graded lift of the group scheme of (integral) Witt vectors where the coordinates have weights i, and equip $\mathrm{H}^*(\mathrm{B}(\mathbf{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ with the grading where a class in $\mathrm{H}^s(\mathrm{B}(\mathbf{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{t\})$ lies in weight $2t-s$. Then there is a graded isomorphism

$$\mathrm{H}^*(\mathrm{B}(\mathbf{W}(-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathcal{A}_*^{\mathrm{MU}} = \pi_*(\mathbf{F}_p \otimes_{\mathrm{MU}} \mathbf{F}_p). \tag{7.6.4}$$

Again, these isomorphisms can either be established calculationally, or can be proved as in Example 7.6.4 by our calculations (see the discussion after Theorem 7.5.1) that there are isomorphisms $\pi_*\mathrm{HH}(\mathbf{F}_p/\mathrm{BP}) \cong \mathcal{O}_{\mathrm{W}(-*)}$ and $\pi_*\mathrm{HH}(\mathbf{F}_p/\mathrm{MU}) \cong \mathcal{O}_{\mathbf{W}(-*)}$ of bialgebras.

Example 7.6.6. Let $\mathbf{G}_a^\sharp(-1)$ denote the graded lift of $\mathbf{G}_a^\sharp$ where the coordinate μ has weight 1, and equip $\mathrm{H}^*(\mathrm{B}(\mathbf{G}_a^\sharp(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ with the grading where a class in $\mathrm{H}^s(\mathrm{B}(\mathbf{G}_a^\sharp(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{t\})$ lies in weight $2t-s$. Then there is a graded isomorphism

$$\mathrm{H}^*(\mathrm{B}(\mathbf{G}_a^\sharp(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathcal{A}_*^{\mathbf{Z}} = \pi_*(\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{F}_p). \tag{7.6.5}$$

In fact, there is an equivalence of $\mathbf{E}_2$-$\mathbf{F}_p$-algebras between the shearing of $\mathrm{R\Gamma}(\mathrm{B}(\mathbf{G}_a^\sharp(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ and $\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{F}_p$. The isomorphism (7.6.5) again follows from the fact that there is an isomorphism $\pi_{2*}\mathrm{HH}(\mathbf{F}_p/\mathbf{Z}) \cong \mathcal{O}_{\mathbf{G}_a^\sharp(-1)}$ of bialgebras.

The preceding examples can be unified: there is a graded isomorphism

$$\mathrm{H}^*(\mathrm{B}(\mathrm{W}[\mathrm{F}^n](-*) \rtimes \mathbf{G}_m); \mathcal{O}\{*\}) \cong \mathcal{A}_*^{\mathrm{BP}\langle n-1\rangle} = \pi_*(\mathbf{F}_p \otimes_{\mathrm{BP}\langle n-1\rangle} \mathbf{F}_p). \tag{7.6.6}$$

The flow of information here can be reversed: using the Tor spectral sequence, isomorphisms like (7.6.6) can be used to calculate $\pi_{2*}\mathrm{HH}(\mathbf{F}_p/\mathrm{BP}\langle n-1\rangle)$, and this is in turn the input into computing the even stack $\mathrm{Spev}(\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \otimes_{\mathrm{BP}\langle n-1\rangle} \mathbf{F}_p)$. For instance, suppose $n=1$ (for simplicity). As discussed after Theorem 7.5.1, the map $\mathrm{THH}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{F}_p \to \mathrm{THH}(\mathbf{F}_p)$ is an even eff cover, so that $\mathrm{Spev}(\mathrm{THH}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{F}_p)$ is the geometric realization of the simplicial diagram

$$\cdots \Rrightarrow \mathrm{Spec}(\pi_{2*}(\mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p} \mathrm{THH}(\mathbf{F}_p)))/\mathbf{G}_m \rightrightarrows \mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m.$$

The final term is just $\mathbf{G}_a(-1)/\mathbf{G}_m$. The preceding term is the groupoid scheme given by the fiber product $\mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m \times_{\mathrm{Spev}(\mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p)} \mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m$, and its fiber over $\mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m$ is precisely $\mathrm{Spec}(\pi_{2*}(\mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p} \mathbf{F}_p))/\mathbf{G}_m$. However, there is an equivalence

$$\mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p} \mathbf{F}_p \simeq \mathrm{THH}(\mathbf{F}_p) \otimes_{\mathrm{THH}(\mathbf{Z}_p)} \mathbf{Z}_p = \mathrm{HH}(\mathbf{F}_p/\mathbf{Z}_p).$$

As we have mentioned, (7.6.5) implies that this is isomorphic to $\mathcal{O}_{\mathbf{G}_a^\sharp}$, so we find that there is an isomorphism

$$\mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m \times_{\mathrm{Spev}(\mathrm{THH}(\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p)} \mathrm{Spec}(\pi_{2*}\mathrm{THH}(\mathbf{F}_p))/\mathbf{G}_m \cong (\mathbf{G}_a(-1) \times \mathbf{G}_a^\sharp(-1))/\mathbf{G}_m.$$

From this, it is easy to conclude that there is an isomorphism

$$(\mathbf{Z}_p^{\mathrm{conj}})_{p=0} = \mathrm{Spev}(\mathrm{THH}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{F}_p) \cong \mathbf{G}_a(-1)/(\mathbf{G}_a^\sharp(-1) \rtimes \mathbf{G}_m).$$

In summary, the perspective that $\mathrm{Spec}(\mathcal{A}_*) \cong \mathrm{Aut}_{\mathrm{B}\mathbf{G}_m}(\widehat{\mathbf{G}_a}(1)/\mathbf{G}_m)$ is more suited to the formal group theoretic perspective on chromatic homotopy theory and is related to the filtration of the map $\mathrm{S} \to \mathbf{F}_p$ (resp. $\mathrm{S} \to \mathrm{BP}$) by the maps $y(n) \to \mathbf{F}_p$ (resp. $\mathrm{T}(n) \to \mathbf{F}_p$), while the perspective that $\mathcal{A}_*$ is isomorphic to $\mathrm{H}^*(\mathrm{B}(\mathbf{G}_a(-1) \rtimes \mathbf{G}_m); \mathcal{O}\{*\})$ is more well-suited to applications to p-adic Hodge theory and is related to the filtration of the map $\mathrm{S} \to \mathbf{F}_p$ through the truncated Brown-Peterson spectra $\mathrm{BP}\langle n-1\rangle \to \mathbf{F}_p$. (These perspectives are closely related, e.g., through a version of Tate duality for the syntomic cohomology of $\mathrm{BP}\langle n-1\rangle$; we hope to explain this in future work.) This has been very useful as an organizational tool for me.

Bibliography

[AB1] O. Antolín-Camarena and T. Barthel. A simple universal property of Thom ring spectra. *J. Topol.*, 12(1):56–78, 2019.

[AB2] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[ABG] S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg. Quantum groups, the loop Grassmannian, and the Springer resolution. *J. Amer. Math. Soc.*, 17(3):595–678, 2004.

[ABS] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl. 1):3–38, 1964.

[ACH] G. Angelini-Knoll, D. Culver, and E. Honing. Topological Hochschild homology of truncated Brown-Peterson spectra I. https://arxiv.org/abs/2106.06785, 2021.

[Ada1] J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[Ada2] J. F. Adams. On the groups J(X). IV. *Topology*, 5:21–71, 1966.

[AG] D. Arinkin and D. Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Math. (N.S.)*, 21(1):1–199, 2015.

[AHR] M. Ando, M. Hopkins, and C. Rezk. Multiplicative orientations of KO-theory and of the spectrum of topological modular forms. http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf, May 2010.

[AHS] M. Ando, M. Hopkins, and N. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001.

[Akh] D. Akhiezer. On the notion of rank of a spherical homogeneous space. *Uspekhi Mat. Nauk*, 43(5(263)):175–176, 1988.

[AL] J. Anschütz and A.-C. Le Bras. The p-completed cyclotomic trace in degree 2. *Ann. K-Theory*, 5(3):539–580, 2020.

[AMM] A. Alekseev, A. Malkin, and E. Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445–495, 1998.

[AMMN] B. Antieau, A. Mathew, M. Morrow, and T. Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 171(18), 2022.

[And] M. Ando. Power operations in elliptic cohomology and representations of loop groups. *Trans. Amer. Math. Soc.*, 352(12):5619–5666, 2000.

[Ant] B. Antieau. Periodic cyclic homology and derived de Rham cohomology. *Ann. K-Theory*, 4(3):505–519, 2019.

[AR] B. Antieau and N. Riggenbach. Cyclotomic synthetic spectra. https://arxiv.org/abs/2411.19929, 2024.

[Ati1] M. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Ati2] M. Atiyah. Power operations in K-theory. *Quart. J. Math. Oxford Ser. (2)*, 17:165–193, 1966.

[BBB+] C. Beem, D. Ben-Zvi, M. Bullimore, T. Dimofte, and A. Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincare*, 21(4):1235–1310, 2020.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BC] A. Beliakova and B. Cooper. Steenrod structures on categorified quantum groups. *Fund. Math.*, 241(2):179–207, 2018.

[BCM] B. Bhatt, D. Clausen, and A. Mathew. Remarks on K(1)-local K-theory. *Sel. Math.*, 26(3), 2020.

[BCS] A. Blumberg, R. Cohen, and C. Schlichtkrull. Topological Hochschild homology of Thom spectra and the free loop space. *Geom. Topol.*, 14(2):1165–1242, 2010.

[Bei] A. Beilinson. How to glue perverse sheaves. In K*-theory, arithmetic and geometry (Moscow, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 42–51. Springer, Berlin, 1987.

[Bez] R. Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123:1–67, 2016.

[BF] R. Bezrukavnikov and M. Finkelberg. Equivariant Satake category and Kostant-Whittaker reduction. *Mosc. Math. J.*, 8(1):39–72, 183, 2008.

[BFGT] A. Braverman, M. Finkelberg, V. Ginzburg, and R. Travkin. Mirabolic Satake equivalence and supergroups. *Compos. Math.*, 157(8):1724–1765, 2021.

[BFM] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. Equivariant homology and K-theory of affine Grassmannians and Toda lattices. *Compos. Math.*, 141(3):746–768, 2005.

[BFN] A. Braverman, M. Finkelberg, and H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Adv. Theor. Math. Phys.*, 22:1071–1147, 2018.

[BFT] A. Braverman, M Finkelberg, and R. Travkin. Orthosymplectic Satake equivalence. *Commun. Number Theory Phys.*, 16(4):695–732, 2022.

[BG] V. Baranovsky and V. Ginzburg. Conjugacy classes in loop groups and G-bundles on elliptic curves. *Internat. Math. Res. Notices*, (15):733–751, 1996.

[Bha1] M. Bhargava. Higher composition laws. I. A new view on Gauss composition, and quadratic generalizations. *Ann. of Math. (2)*, 159(1):217–250, 2004.

[Bha2] B. Bhatt. p-adic derived de Rham cohomology. https://arxiv.org/abs/1204.6560, 2012.

[Bha3] B. Bhatt. Prismatic F-gauges. https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf, 2024.

[BHM] M. Bökstedt, W.-C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3), 1993.

[BK1] R. Bezrukavnikov and D. Kaledin. Fedosov quantization in positive characteristic. *J. Amer. Math. Soc.*, 21(2):409–438, 2008.

[BK2] E. Bouaziz and A. Khan. Elliptic loop spaces. https://arxiv.org/abs/2502.13882, 2025.

[BL] B. Bhatt and J. Lurie. Absolute prismatic cohomology. https://arxiv.org/abs/2201.06120, 2022.

[BLV] M. Brion, D. Luna, and T. Vust. Espaces homogènes sphériques. *Invent. Math.*, 84(3):617–632, 1986.

[BM] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. In K*-theory - Strasbourg, 1992*, number 226 in Astérisque. 1994.

[BMS] B. Bhatt, M. Morrow, and P. Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[BN] D. Ben-Zvi and D. Nadler. Elliptic Springer theory. *Compos. Math.*, 151(8):1568–1584, 2015.

[BO] P. Berthelot and A. Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[Bok] M. Bokstedt. The topological Hochschild homology of $\mathbf{Z}$ and $\mathbf{Z}/p$. https://pi.math.cornell.edu/~dmehrle/notes/references/bokstedt2.pdf, 1985.

[Bot] R. Bott. The space of loops on a Lie group. *Michigan Math. J.*, 5:35–61, 1958.

[BR] R. Bezrukavnikov and S. Riche. Modular affine Hecke category and regular centralizer. https://arxiv.org/abs/2206.03738, 2022.

[Bri] M. Brion. Poincaré duality and equivariant (co)homology. *Michigan Math. J.*, 48:77–92, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[BS1] B. Bhatt and P. Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[BS2] A. Borel and J.-P. Serre. Détermination des p-puissances réduites de Steenrod dans la cohomologie des groupes classiques. Applications. *C. R. Acad. Sci. Paris*, 233:680–682, 1951.

[BS3] A. Borel and J.-P. Serre. Groupes de Lie et puissances réduites de Steenrod. *Amer. J. Math.*, 75:409–448, 1953.

[BTLM] A. Buch, J. Thomsen, N. Lauritzen, and V. Mehta. The Frobenius morphism on a toric variety. *Tohoku Math. J. (2)*, 49(3):355–366, 1997.

[But1] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Foundations. https://arxiv.org/abs/2011.14988, 2020.

[But2] D. Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Gauge theory applications. https://arxiv.org/abs/2011.14978, 2020.

[BZ] J.-L. Brylinski and B. Zhang. Equivariant K-theory of compact connected Lie groups. *K-Theory*, 20(1):23–36, 2000. Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part I.

[BZN] D. Ben Zvi and D. Nadler. Betti geometric Langlands. In *Algebraic geometry: Salt Lake City 2015*, volume 97.2 of *Proc. Sympos. Pure Math.*, pages 3–41. Amer. Math. Soc., Providence, RI, 2018.

[BZSV] D. Ben Zvi, Y. Sakellaridis, and A. Venkatesh. Relative Langlands duality. https://arxiv.org/abs/2409.04677, 2023.

[Cal] D. Calaque. Shifted cotangent stacks are shifted symplectic. *Ann. Fac. Sci. Toulouse Math. (6)*, 28(1):67–90, 2019.

[Cay] A. Cayley. *On the Theory of Linear Transformations*, page 80–94. Cambridge Library Collection - Mathematics. Cambridge University Press, 2009.

[CD] H. Chen and G. Dhillon. A Langlands dual realization of coherent sheaves on the nilpotent cone. https://arxiv.org/abs/2310.10539v1, 2023.

[CG] N. Chriss and V. Ginzburg. *Representation theory and complex geometry.* Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1997 edition.

[CGK] M. Cole, J. P. C. Greenlees, and I. Kriz. Equivariant formal group laws. *Proc. London Math. Soc. (3)*, 81(2):355–386, 2000.

[CK] S. Cautis and J. Kamnitzer. Quantum K-theoretic geometric Satake: the SL_n case. *Compos. Math.*, 154(2):275–327, 2018.

[CMNO] T. H. Chen, M. Macerato, D. Nadler, and J. O'Brien. Quaternionic Satake equivalence. https://arxiv.org/abs/2207.04078, 2022.

[CN] T. H. Chen and D. Nadler. Real groups, symmetric varieties and Langlands duality. https://arxiv.org/abs/2403.13995, 2024.

[CO] T. H. Chen and J. O'Brien. Lorentzian and Octonionic Satake equivalence. https://arxiv.org/abs/2409.03969, 2024.

[CP] F. R. Cohen and F. Peterson. Suspensions of Stiefel manifolds. *Quart. J. Math. Oxford Ser. (2)*, 35(138):115–119, 1984.

[CR1] J. Campbell and S. Raskin. Langlands duality on the Beilinson-Drinfeld Grassmannian. Available at https://gauss.math.yale.edu/~sr2532/, 2023.

[CR2] P.-E. Chaput and M. Romagny. On the adjoint quotient of Chevalley groups over arbitrary base schemes. *J. Inst. Math. Jussieu*, 9(4):673–704, 2010.

[CS] W. Crawley-Boevey and P. Shaw. Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem. *Adv. Math.*, 201(1):180–208, 2006.

[CSY] S. Carmeli, T. Schlank, and L. Yanovski. Ambidexterity in Chromatic Homotopy Theory. https://arxiv.org/abs/1811.02057, 2018.

[CY] T. H. Chen and L. Yi. Slices in the loop spaces of symmetric varieties and the formality conjecture. https://arxiv.org/abs/2310.20006, 2023.

[Dav] D. Davis. The elliptic Grothendieck-Springer resolution as a simultaneous log resolution of algebraic stacks. https://arxiv.org/abs/1908.04140, 2019.

[Dev1] S. Devalapurkar. Lifting to truncated Brown-Peterson spectra and Hodge-de Rham degeneration in characteristic $p > 0$. https://sanathdevalapurkar.github.io/files/higher-dim-hdR-degen.pdf, 2023.

[Dev2] S. Devalapurkar. Topological Hochschild homology, truncated Brown-Peterson spectra, and a topological Sen operator. https://sanathdevalapurkar.github.io/files/thh-Xn.pdf, 2023.

[Dev3] S. Devalapurkar. ku-theoretic spectral decompositions for spheres and projective spaces. https://sanathdevalapurkar.github.io/files/hyperboloid_spectral_decomp.pdf, 2024.

[DFHH] C. Douglas, J. Francis, A. Henriques, and M. Hill. *Topological Modular Forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2014.

[DHL$^+$] S. Devalapurkar, J. Hahn, T. Lawson, A. Senger, and D. Wilson. Examples of disk algebras. https://arxiv.org/abs/2302.11702, 2023.

[DHRY] S. Devalapurkar, J. Hahn, A. Raksit, and A. Yuan. Prismatization of commutative ring spectra. Forthcoming, 2025.

[DI] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Dic] L. E. Dickson. The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group. *Ann. of Math.*, 11(1-6):65–120, 1896/97.

[DM] S. Devalapurkar and M. Misterka. Generalized n-series and de Rham complexes. https://sanathdevalapurkar.github.io/files/fgls-and-dR-complexes.pdf, 2023.

[DR] S. Devalapurkar and A. Raksit. THH($\mathbf{Z}$) and the image of J. Forthcoming, 2025.

[Dri1] V. Drinfeld. A 1-dimensional formal group over the prismatization of Spf $\mathbf{Z}_p$. *Pure Appl. Math. Q.*, 20(1):233–305, 2024.

[Dri2] V. Drinfeld. Prismatization. *Selecta Math. (N.S.)*, 30(3):Paper No. 49, 150, 2024.

[ENP] L. Ein, W. Niu, and J. Park. Singularities and syzygies of secant varieties of nonsingular projective curves. *Invent. Math.*, 222(2):615–665, 2020.

[Eul] L. Euler. Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae. *Novi Commentarii academiae scientiarum Petropolitanae*, 3:86–108, 1753. Available at https://scholarlycommons.pacific.edu/euler-works/190/.

[Fel] G. Felder. Elliptic quantum groups. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pages 211–218. Int. Press, Cambridge, MA, 1995.

[FGKV] E. Frenkel, D. Gaitsgory, D. Kazhdan, and K. Vilonen. Geometric realization of Whittaker functions and the Langlands conjecture. *J. Amer. Math. Soc.*, 11(2):451–484, 1998.

[FGT] M. Finkelberg, V. Ginzburg, and R. Travkin. Lagrangian subvarieties of hyperspherical varieties. https://arxiv.org/abs/2310.19770, 2023.

[FGV] E. Frenkel, D. Gaitsgory, and K. Vilonen. Whittaker patterns in the geometry of moduli spaces of bundles on curves. *Ann. of Math. (2)*, 153(3):699–748, 2001.

[FHT1] D. Freed, M. Hopkins, and C. Teleman. Twisted equivariant K-theory with complex coefficients. *J. Topol.*, 1(1):16–44, 2008.

[FHT2] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory I. *J. Topol.*, 4(4):737–798, 2011.

[FHT3] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory III. *Ann. of Math. (2)*, 174(2):947–1007, 2011.

[FHT4] D. Freed, M. Hopkins, and C. Teleman. Loop groups and twisted K-theory II. *J. Amer. Math. Soc.*, 26(3):595–644, 2013.

[FM1] R. Friedman and J. Morgan. Holomorphic principal bundles over elliptic curves. https://arxiv.org/abs/math/9811130, 1998.

[FM2] R. Friedman and J. Morgan. Holomorphic principal bundles over elliptic curves. II. The parabolic construction. *J. Differential Geom.*, 56(2):301–379, 2000.

[FM3] R. Friedman and J. Morgan. Holomorphic Principal Bundles Over Elliptic Curves III: Singular Curves and Fibrations. https://arxiv.org/abs/math/0108104, 2001.

[FMW] R. Friedman, J. Morgan, and E. Witten. Principal G-bundles over elliptic curves. *Math. Res. Lett.*, 5(1-2):97–118, 1998.

[Fra] J. Francis. *Derived algebraic geometry over $\mathcal{E}_n$-rings*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[FT1] M. Finkelberg and A. Tsymbaliuk. Multiplicative slices, relativistic Toda and shifted quantum affine algebras. In *Representations and nilpotent orbits of Lie algebraic systems*, volume 330 of *Progr. Math.*, pages 133–304. Birkhäuser/Springer, Cham, 2019.

[FT2] D. Freed and C. Teleman. Dirac families for loop groups as matrix factorizations. *C. R. Math. Acad. Sci. Paris*, 353(5):415–419, 2015.

[Gan1] T. Gannon. Classification of nondegenerate G-categories. https://arxiv.org/abs/2206.11247, 2022.

[Gan2] T. Gannon. The coarse quotient for affine Weyl groups and pseudo-reflection groups. https://arxiv.org/abs/2206.00175, 2022.

[Gan3] T. Gannon. The cotangent bundle of G/U_P and Kostant-Whittaker descent. https://arxiv.org/abs/2407.16844, 2024.

[GG1] W. Gan and V. Ginzburg. Quantization of Slodowy slices. *Int. Math. Res. Not.*, (5):243–255, 2002.

[GG2] T. Gannon and V. Ginzburg. Quantization of the universal centralizer and central D-modules. https://arxiv.org/abs/2409.18054, 2024.

[Gin1] V. Ginzburg. Perverse sheaves and $\mathbf{C}^*$-actions. *J. Amer. Math. Soc.*, 4(3):483–490, 1991.

[Gin2] V. Ginzburg. Perverse sheaves on a Loop group and Langlands' duality. https://arxiv.org/abs/alg-geom/9511007, 1995.

[Gin3] V. Ginzburg. Nil-Hecke algebras and Whittaker $\mathcal{D}$-modules. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 137–184. Birkhäuser/Springer, Cham, 2018.

[GJS] I. Ganev, D. Jordan, and P. Safronov. The quantum Frobenius for character varieties and multiplicative quiver varieties. https://arxiv.org/abs/1901.11450, 2019.

[GK] V. Ginzburg and D. Kazhdan. Differential operators on G/U and the Gelfand-Graev action. *Adv. Math.*, 403:Paper No. 108368, 48, 2022.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.

[GKV1] V. Ginzburg, M. Kapranov, and E. Vasserot. Elliptic algebras and equivariant elliptic cohomology. https://arxiv.org/abs/q-alg/9505012, 1995.

[GKV2] V. Ginzburg, M. Kapranov, and E. Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128(1):1–19, 1997.

[GL] D. Gaitsgory and S. Lysenko. Metaplectic Whittaker category and quantum groups : the "small"' FLE. https://arxiv.org/abs/1903.02279, 2019.

[GM1] D. Gepner and L. Meier. On equivariant topological modular forms. https://arxiv.org/abs/2004.10254, 2020.

[GM2] D. Gepner and L. Meier. Equivariant elliptic cohomology with integral coefficients. Forthcoming, 2023.

[GM3] M. Gros and K. Masaharu. Contraction par Frobenius et modules de Steinberg. *Ark. Mat.*, 56(2):319–332, 2018.

[GN] D. Gaitsgory and D. Nadler. Spherical varieties and Langlands duality. *Mosc. Math. J.*, 10(1):65–137, 271, 2010.

[GR1] D. Gaitsgory and N. Rozenblyum. *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[GR2] H. Garland and M. S. Raghunathan. A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott. *Proc. Nat. Acad. Sci. U.S.A.*, 72(12):4716–4717, 1975.

[GR3] V. Ginzburg and S. Riche. Differential operators on G/U and the affine Grassmannian. *J. Inst. Math. Jussieu*, 14(3):493–575, 2015.

[Gro] B. Gross. On minuscule representations and the principal SL_2. *Represent. Theory*, 4:225–244, 2000.

[GSB] I. Grojnowski and N. Shepherd Barron. Del Pezzo surfaces as Springer fibres for exceptional groups. *Proc. Lond. Math. Soc. (3)*, 122(1):1–41, 2021.

[Hai] T. Haines. Equidimensionality of convolution morphisms and applications to saturation problems. *Adv. Math.*, 207(1):297–327, 2006.

[Har] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. A first course.

[Hau] M. Hausmann. Global group laws and equivariant bordism rings. *Ann. of Math. (2)*, 195(3):841–910, 2022.

[Hei] E. Heine. Über die Reihe $1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta}1-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}+\cdots$. (Aus einem Schreiben an Lejeune Dirichlet). *J. Reine Angew. Math.*, 32:210–212, 1846.

[HHH] M. Harada, A. Henriques, and T. Holm. Computation of generalized equivariant cohomologies of Kac-Moody flag varieties. *Adv. Math.*, 197(1):198–221, 2005.

[HHR] M. Hill, M. Hopkins, and D. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[HK] B. Hassett and S. Kovács. Reflexive pull-backs and base extension. *J. Algebraic Geom.*, 13(2):233–247, 2004.

[HKR1] G. Hochschild, B. Kostant, and A. Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.

[HKR2] M. Hopkins, N. Kuhn, and D. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594, 2000.

[HL] M. Hopkins and J. Lurie. Ambidexterity in K(n)-local stable homotopy theory. http://www.math.harvard.edu/~lurie/papers/Ambidexterity.pdf, 2013.

[HM1] M. Hausmann and L. Meier. Invariant prime ideals in equivariant Lazard rings. https://arxiv.org/abs/2309.00850v1, 2023.

[HM2] L. Hesselholt and I. Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[Hop] M. Hopkins. K(1)-local $\mathbf{E}_\infty$-ring spectra. In *Topological Modular Forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2014.

[HRW] J. Hahn, A. Raksit, and D. Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. https://arxiv.org/abs/2206.11208, 2022.

[HS] M. Hovey and H. Sadofsky. Tate cohomology lowers chromatic Bousfield classes. *Proc. Amer. Math. Soc.*, 124:3579–3585, 1996.

[Hua] Z. Huan. Universal Finite Subgroup of the Tate Curve. https://arxiv.org/abs/1708.08637, 2017.

[HW] J. Hahn and D. Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. of Math. (2)*, 196(3):1277–1351, 2022.

[HY] J. Hahn and A. Yuan. Multiplicative structure in the stable splitting of $\Omega\mathrm{SL}_n(\mathbf{C})$. *Adv. Math.*, 348:412–455, 2019.

[Ill] L. Illusie. New advances on de Rham cohomology in positive or mixed characteristic, after Bhatt-Lurie, Drinfeld, and Petrov. https://www.imo.universite-paris-saclay.fr/~luc.illusie/Bruno60-slides.pdf, 2022.

[Jan1] J. Jantzen. Kohomologie von p-Lie-Algebren und nilpotente Elemente. *Abh. Math. Sem. Univ. Hamburg*, 56:191–219, 1986.

[Jan2] J. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, Second edition, 2003.

[JMW] D. Juteau, C. Mautner, and G. Williamson. Parity sheaves and tilting modules. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(2):257–275, 2016.

[Jor] D. Jordan. Quantized multiplicative quiver varieties. *Adv. Math.*, 250:420–466, 2014.

[KC] V. Kac and P. Cheung. *Quantum calculus*. Universitext. Springer-Verlag, New York, 2002.

[Kit] N. Kitchloo. Cohomology operations and the Nil-Hecke ring. https://math.jhu.edu/~nitu/papers/NH.pdf, 2013.

[KK1] B. Kostant and S. Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Proc. Nat. Acad. Sci. U.S.A.*, 83(6):1543–1545, 1986.

[KK2] B. Kostant and S. Kumar. T-equivariant K-theory of generalized flag varieties. *J. Differential Geom.*, 32(2):549–603, 1990.

[Kla] I. Klang. The factorization theory of Thom spectra and twisted non-abelian Poincare duality. *Algebr. Geom. Topol.*, 18(5):2541–2592, 2018.

[Kno1] F. Knop. The asymptotic behavior of invariant collective motion. *Invent. Math.*, 116(1-3):309–328, 1994.

[Kno2] F. Knop. On the set of orbits for a Borel subgroup. *Comment. Math. Helv.*, 70(2):285–309, 1995.

[Kos1] B. Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963.

[Kos2] B. Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

[KR] F. Knop and G. Rohrle. Spherical subgroups in simple algebraic groups. *Compos. Math.*, 151(7):1288–1308, 2015.

[Kra] M. Kramer. Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen. *Compositio Math.*, 38(2):129–153, 1979.

[KS1] M. Kashiwara and P. Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel.

[KS2] F. Knop and B. Schalke. The dual group of a spherical variety. *Trans. Moscow Math. Soc.*, 78:187–216, 2017.

[KSZ] D. Kubrak, G. Shuklin, and A. Zakharov. Derived binomial rings I: integral Betti cohomology of log schemes. https://arxiv.org/abs/2308.01110, 2023.

[Kum] S. Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[KW] V. Kac and B. Weisfeiler. Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic p. *Indag. Math.*, 38(2):136–151, 1976. Nederl. Akad. Wetensch. Proc. Ser. A **79**.

[Lan1] T. Lance. Steenrod and Dyer-Lashof operations on BU. *Trans. Amer. Math. Soc.*, 276(2):497–510, 1983.

[Lan2] P. Landweber. Cobordism operations and Hopf algebras. *Trans. Amer. Math. Soc.*, 129:94–110, 1967.

[LM] P. Littig and S. Mitchell. Generating varieties for affine Grassmannians. *Trans. Amer. Math. Soc.*, 363(7):3717–3731, 2011.

[LMMT] M. Land, A. Mathew, L. Meier, and G. Tamme. Purity in chromatically localized algebraic K-theory. *J. Amer. Math. Soc.*, 37(4), 2024.

[Lon1] G. Lonergan. A remark on descent for Coxeter groups. https://arxiv.org/abs/1707.01156v2, 2017.

[Lon2] G. Lonergan. A Fourier transform for the quantum Toda lattice. *Selecta Math. (N.S.)*, 24(5):4577–4615, 2018.

[Lon3] G. Lonergan. Geometric Satake over KU. Online talk, available at https://www.youtube.com/watch?v=Aazmxfb2i_A, 2021.

[Lon4] G. Lonergan. Steenrod operators, the Coulomb branch and the Frobenius twist. *Compos. Math.*, 157(11):2494–2552, 2021.

[Lur1] J. Lurie. A survey of elliptic cohomology. In *Algebraic Topology*, volume 4 of *Abel. Symp.*, pages 219–277. Springer, 2009.

[Lur2] J. Lurie. Moduli problems for ring spectra. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 1099–1125. Hindustan Book Agency, New Delhi, 2010.

[Lur3] J. Lurie. Rotation invariance in algebraic K-theory. https://www.math.ias.edu/~lurie/papers/Waldhaus.pdf, 2015.

[Lur4] J. Lurie. Higher Algebra. http://www.math.harvard.edu/~lurie/papers/HA.pdf, 2016.

[Lur5] J. Lurie. Spectral Algebraic Geometry. http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf, 2017.

[Lur6] J. Lurie. Elliptic Cohomology I: Spectral Abelian Varieties. http://www.math.harvard.edu/~lurie/papers/Elliptic-I.pdf, 2018.

[Lur7] J. Lurie. Elliptic Cohomology II: Orientations. http://www.math.harvard.edu/~lurie/papers/Elliptic-II.pdf, 2018.

[Lur8] J. Lurie. Elliptic Cohomology III: Tempered Cohomology. https://www.math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf, 2019.

[Lur9] J. Lurie. Rationalized syntomic cohomology. https://youtu.be/wkiBi3I9PSs, 2024.

[LV] D. Luna and T. Vust. Plongements d'espaces homogènes. *Comment. Math. Helv.*, 58(2):186–245, 1983.

[Mah] M. Mahowald. Ring spectra which are Thom complexes. *Duke Math. J.*, 46(3):549–559, 1979.

[Man] D. Manam. On the Drinfeld formal group. https://arxiv.org/abs/2403.02555, 2024.

[Mat1] A. Mathew. The homology of tmf. *Homology Homotopy Appl.*, 18(2):1–20, 2016.

[Mat2] A. Mathew. Examples of descent up to nilpotence. In *Geometric and topological aspects of the representation theory of finite groups*, volume 242 of *Springer Proc. Math. Stat.*, pages 269–311. Springer, Cham, 2018.

[Mat3] A. Mathew. Kaledin's degeneration theorem and topological Hochschild homology. *Geom. Topol.*, 24(6):2675–2708, 2020.

[Mei] L. Meier. Relatively free TMF-modules. https://webspace.science.uu.nl/~meier007/RelativelyFree4.pdf, 2017.

[Mil] H. Miller. Stable splittings of Stiefel manifolds. *Topology*, 24(4):411–419, 1985.

[Mit] S. Mitchell. Quillen's theorem on buildings and the loops on a symmetric space. *Enseign. Math. (2)*, 34(1-2):123–166, 1988.

[MRT] T. Moulinos, M. Robalo, and B. Toën. A universal Hochschild-Kostant-Rosenberg theorem. *Geom. Topol.*, 26(2):777–874, 2022.

[MV] I. Mirkovic and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[MW] S. Meyer and F. Wagner. Derived q-Hodge complexes and refined TC^{-}. https://arxiv.org/abs/2410.23115, 2024.

[Nad] D. Nadler. Perverse sheaves on real loop Grassmannians. *Invent. Math.*, 159(1):1–73, 2005.

[Noc] G. Nocera. A model for the $\mathbf{E}_3$ fusion-convolution product of constructible sheaves on the affine Grassmannian. https://arxiv.org/abs/2012.08504, 2020.

[Nov] S. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967.

[NP1] B. C. Ngô and P. Polo. Résolutions de Demazure affines et formule de Casselman-Shalika géométrique. *J. Algebraic Geom.*, 10(3):515–547, 2001.

[NP2] G. Nocera and M. Porzio. On the homotopy type of the Beilinson–Drinfeld Grassmannian. https://www.ihes.fr/~nocera/Papers/BDGrassmannian.pdf, 2024.

[NS] T. Nikolaus and P. Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[Per] N. Perrin. On the geometry of spherical varieties. *Transform. Groups*, 19(1):171–223, 2014.

[Pet1] A. Petrov. Non-decomposability of the de Rham complex and non-semisimplicity of the Sen operator. https://arxiv.org/abs/2302.11389, 2023.

[Pet2] A. Petrov. Decomposition of de Rham complex for quasi-F-split varieties. https://arxiv.org/abs/2502.13356, 2025.

[Pre] A. Premet. Special transverse slices and their enveloping algebras. *Adv. Math.*, 170(1):1–55, 2002. With an appendix by Serge Skryabin.

[Pst] P. Pstragowski. Perfect even modules and the even filtration. https://arxiv.org/abs/2304.04685, 2023.

[PTVV] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi. Shifted symplectic structures. *Publ. Math. Inst. Hautes Études Sci.*, 117:271–328, 2013.

[PV] A. Petrov and V. Vologodsky. On the periodic topological cyclic homology of dg categories in characteristic p. https://arxiv.org/abs/1912.03246, 2023.

[Qui] D. Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.

[Rak] A. Raksit. Hochschild homology and the derived de Rham complex revisited. https://arxiv.org/abs/2007.02576, 2020.

[Res] N. Ressayre. About Knop's action of the Weyl group on the set of orbits of a spherical subgroup in the flag manifold. *Transform. Groups*, 10(2):255–265, 2005.

[Rez] C. Rezk. Isogenies, power operations, and homotopy theory. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1125–1145. Kyung Moon Sa, Seoul, 2014.

[Ric] S. Riche. Kostant section, universal centralizer, and a modular derived Satake equivalence. *Math. Z.*, 286(1):223–261, 2017.

[Rog1] J. Rognes. Algebraic K-theory of finitely presented ring spectra. https://www.mn.uio.no/math/personer/vit/rognes/papers/red-shift.pdf, 2000.

[Rog2] J. Rognes. *Galois Extensions of Structured Ring Spectra/Stably Dualizable Groups*, volume 192 of *Mem. Amer. Math. Soc.* American Mathematical Society, 2008.

[Rog3] J. Rognes. Algebraic K-theory of strict ring spectra. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1259–1283. Kyung Moon Sa, Seoul, 2014.

[RW] S. Riche and G. Williamson. Smith-Treumann theory and the linkage principle. 136:225–292, 2022.

[Saf1] P. Safronov. Quasi-Hamiltonian reduction via classical Chern-Simons theory. *Adv. Math.*, 287:733–773, 2016.

[Saf2] P. Safronov. Symplectic implosion and the Grothendieck-Springer resolution. *Transform. Groups*, 22(3):767–792, 2017.

[Sak] Y. Sakellaridis. Spherical varieties, functoriality, and quantization. https://arxiv.org/abs/2111.03004, ICM 2022 report, 2021.

[Sch] P. Scholze. Canonical q-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(5):1163–1192, 2017.

[Shi] R. Shirai. q-deformation with (φ, Γ) structure of the de Rham cohomology of the Legendre family of elliptic curves. https://arxiv.org/abs/2006.12310v1, 2020.

[ST] N. Sibilla and P. Tomasini. Equivariant Elliptic Cohomology and Mapping Stacks I. https://arxiv.org/abs/2303.10146, 2023.

[Sta1] N. Stapleton. Transchromatic generalized character maps. *Algebr. Geom. Topol.*, 13(1):171–203, 2013.

[Sta2] N. Stapleton. Transchromatic twisted character maps. *J. Homotopy Relat. Struct.*, 10(1):29–61, 2015.

[Ste] R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[Str1] N. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.

[Str2] N. P. Strickland. Multicurves and equivariant cohomology. *Mem. Amer. Math. Soc.*, 213(1001):vi+117, 2011.

[SV] Y. Sakellaridis and A. Venkatesh. Periods and harmonic analysis on spherical varieties. *Astérisque*, (396):viii+360, 2017.

[SW] W. Soergel and M. Wendt. Perverse motives and graded derived category $\mathcal{O}$. *J. Inst. Math. Jussieu*, 17(2):347–395, 2018.

[Tel1] C. Teleman. Gauge theory and mirror symmetry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1309–1332. Kyung Moon Sa, Seoul, 2014.

[Tel2] C. Teleman. Topological Gauge Theory in low dimensions. https://math.berkeley.edu/~teleman/math/Auckland.pdf, 2018.

[Tim] D. Timashev. *Homogeneous spaces and equivariant embeddings*, volume 138 of *Encyclopaedia of Mathematical Sciences*. Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8.

[Van] M. Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008.

[Woj] Z. Wojciechowski. A study of nil Hecke algebras via Hopf algebroids. https://arxiv.org/abs/2410.08061v3, 2024.

[Woo] R. Wood. Banach algebras and Bott periodicity. *Topology*, 4:371–389, 1965/66.

[YZ1] Y. Yang and G. Zhao. Frobenii on Morava E-theoretical quantum groups. https://arxiv.org/abs/2105.14681, 2021.

[YZ2] Z. Yun and X. Zhu. Integral homology of loop groups via Langlands dual groups. *Represent. Theory*, 15:347–369, 2011.

[Zhu] X. Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017.