MORAVA K-THEORIES AND POINCARÉ DUALITY

Our goal in this talk is to give an introduction to Morava K-theories and the way in which they appear in [AB21]. To motivate our dicussion, let us briefly review the formal setup discussed by Semon last time. Let us begin by remarking that in [AB21], there is a discrete group Π floating around (which is the image of $\pi_1(LM) \to \mathrm{H}_2(M; \mathbf{Z})$); the data of a Π-action will just be extra structure that will behave well with respect to the constructions involved in this lecture. We will therefore ignore Π today.

1. Review of Semon's talk, and a preview

Recall that if $\mathcal{P}$ is a poset, then a *flow category* over $\mathcal{P}$ is a topologically enriched category $\mathcal{M}$ whose objects are elements of $\mathcal{P}$, such that $\mathcal{M}(p, q)$ is empty if $p < q$ and is a point if $p = q$. Let $2^{\mathcal{P}}(p, q)$ be the poset of totally ordered subsets of $\mathcal{P}$ whose elements lie strictly between p and q. Last time, Semon gave examples of the *collared completion* $\hat{\mathcal{M}}$ of a flow category: this is a new topological category with the same objects as $\mathcal{M}$ (i.e., elements of $\mathcal{P}$), but where the morphism spaces $\hat{\mathcal{M}}(p, q)$ for $p, q \in \mathcal{P}$ are described as the homotopy colimit of the diagram $2^{\mathcal{P}}(p, q) \to \mathrm{Top}$ sending $J = (q_1, \cdots, q_n)$ to $\mathcal{M}(J) = \mathcal{M}(p, q_1) \times \cdots \times \mathcal{M}(q_n, q)$. For instance, if $\mathcal{M} = \mathcal{P} = \mathbf{Z}_{>0}$, then $\hat{\mathcal{P}}(i, j) = [0, 1]^{j-i-1}$; this is homeomorphic to a $(j - i - 1)$-simplex whose vertices are the integers $i < k \leq j$.

The collared completion $\hat{\mathcal{M}}$ is another topological category, where $\hat{\mathcal{M}}(p, q)$ has a boundary. The inclusion $\partial\hat{\mathcal{M}}(p, q) \to \hat{\mathcal{M}}(p, q)$ is a Hurewicz cofibration (in particular, is a closed inclusion). For instance, if $\mathcal{M} = \mathcal{P} = \mathbf{Z}_{>0}$, then $\partial\hat{\mathcal{P}}(i, j)$ is the boundary of those faces of $[0, 1]^{j-i-1}$ where at least one coordinate is 1. We then made the following definition:

Definition 1. Let k be a ring spectrum[1], and let $\mathcal{M}$ be a flow category with object poset $\mathcal{P}$. For $q \in \mathcal{P}$, let ℓ_q denote the real line[2] $\mathbf{R}^{\{q\}}$. For $p, q \in \mathcal{M}$, define[3]

$$C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])(p, q) = \begin{cases} C^*(\hat{\mathcal{M}}(p, q), \partial\hat{\mathcal{M}}(p, q); k[-\ell_q]) & p > q \\ k & p = q \\ 0 & p < q. \end{cases}$$

Remark 2. One should just think of $C^*(\hat{\mathcal{M}}(p, q), \partial\hat{\mathcal{M}}(p, q); k[-\ell_q])$ as being a homotopically well-behaved replacement of $C^*(\mathcal{M}(p, q); k[-\ell_q])$. In fact, this is the way in which we will calculate with $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])$ in the next section.

Example 3. If $\mathcal{M} = \mathcal{P} = \mathbf{Z}_{>0}$, the inclusion $\partial\hat{\mathcal{M}}(p, q) \to \hat{\mathcal{M}}(p, q)$ is a homotopy equivalence unless p and q are consecutive integers. In that case, $\hat{\mathcal{M}}(p, q) = *$ (so it has no boundary). It follows that $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])(p, q)$ is 0 unless p and q are successive integers or $p = q$; if p, q are successive, it is $k[-1]$, while if $p = q$, it is k.

The k-modules $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])(p, q)$ assemble into a spectral[4] category $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])$ whose objects are elements of $\mathcal{P}$. The composition in this category is a little complicated to define, so we will not do so here. In any case, our discussion below will only involve the important "base case" where $\mathcal{P}$ has two elements $p < q$ (so there is no nontrivial composition, and there is only one nontrivial mapping k-module $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])(p, q)$).

[1] I believe one only need a homotopy ring structure to make these definitions.

[2] This is the 1-dimensional $\mathbf{R}$-vector space with basis element q. The line ℓ_q is isomorphic to $\mathbf{R}$, of course, but it is useful to know *which* copy of $\mathbf{R}$ we are working with.

[3] The notation used in [AB21] to denote a 1-fold desuspension is Ωk. This notation is a little unsettling to me, so we will write $k[-1]$ instead. If we are desuspending k by the sphere S^{ℓ_q}, we will write $k[-\ell_q]$ instead of $\Omega^{\ell_q} k$. Therefore, the notation used in [AB21] for what we write as $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])$ below is $C^*_{\mathrm{rel}\partial}(\mathcal{M}; \Omega k)$.

[4] Since k is only assumed to be a homotopy ring, there is no notion of k-linear category. This will be true even when we take k to be a Morava K-theory: the notion of a k-linear category is sensible only when k is an $\mathbf{E}_2$-ring.

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Definition 4. Let $\mathcal{M}$ be a flow category, and let k be an $\mathbf{E}_1$-ring (i.e., an A_∞-ring spectrum). A *virtual fundamental chain* for $\mathcal{M}$ is a $\mathbb{S}$-linear functor $\delta : C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1]) \to \mathrm{LMod}_k$ such that $\delta(p)$ is a suspension of k for each $p \in \mathcal{M}$.

Example 5. Let $\mathcal{P}$ be the poset $\{p < q\}$. Then, as mentioned above, $C^*_{\mathrm{rel}\partial}(\mathcal{M}; k[-1])$ has exactly one nontrivial morphism k-module, given by $C^*(\hat{\mathcal{M}}(p,q), \partial\hat{\mathcal{M}}(p,q); k[-\ell_q])$. Suppose δ is a virtual fundamental chain for $\mathcal{M}$; then δ is specified entirely by the k-linear map $\delta(p) \to \delta(q)$. Since both $\delta(p)$ and $\delta(q)$ are just suspensions of k, the k-linear map $\delta(p) \to \delta(q)$ is just a shift $\Sigma^{-d}k$ for some $d \in \mathbf{Z}$. In particular, δ is specified by a k-linear map

$$\delta_{p,q} : C^*(\hat{\mathcal{M}}(p,q), \partial\hat{\mathcal{M}}(p,q); k[-\ell_q]) \to \Sigma^{-d}k.$$

If these relative cochains satisfy some form of Poincaré duality, i.e., are equivalent to relative *chains* up to some shift, then we could just set d to be this shift and crush $(\hat{\mathcal{M}}(p,q), \partial\hat{\mathcal{M}}(p,q))$ to get the desired map $\delta_{p,q}$.

Our goal in the remainder of this talk is to explain how to construct a virtual fundamental chain for a flow category $\mathcal{M}$ as in Example 5, where the morphism spaces $\mathcal{M}(p,q)$ are modeled as in the flow categories which show up in Floer theory. These are given by *Kuranishi structures*, the topic of which will be Ben's talk next week. In Floer theory (thanks to Semon for explaining this to me), the poset $\mathcal{P}$ consists of lifts of time-1 orbits of the Hamiltonian to a cover of the free loop space, and the $\mathcal{M}(p,q)$ are moduli spaces gradient flow lines of the action functional between two such orbits p, q. These moduli spaces are locally modeled on "footprints of Kuranishi structures". We will only give the definition here, and leave Ben with the task of explaining where this structure originates from in Floer theory.

Definition 6. A *Kuranishi chart/structure* is a tuple (X, V, s, G) where G is a finite group, X is a G-manifold which is locally modeled on a G-representation, V is a finite-dimensional G-representation with a G-invariant inner product, and $s : X \to V$ is a G-equivariant map. The *footprint* of the Kuranishi structure is the quotient space $s^{-1}(0)/G$; this is supposed to be reminiscent of a moment map.

Below, the *only* example of a virtual flow category we will consider is the following.

Example 7. Let (X, V, s, G) be a Kuranishi structure. Define the flow category $\mathcal{M}$ to be the topological category whose object poset is $\{p < q\}$ and whose morphism space $\mathcal{M}(p,q)$ is $s^{-1}(0)/G$.

2. Constructing a virtual fundamental chain

Let k be an $\mathbf{E}_1$-ring. We will now attempt to define a virtual fundamental chain for the flow category $\mathcal{M}$ associated to a Kuranishi structure (X, V, s, G) from Example 7. This notation will be fixed for the remainder of this talk. Following Remark 2, we will abusively work with $C^*(\mathcal{M}(p,q); k[-\ell_q])$ instead of $C^*(\hat{\mathcal{M}}(p,q), \partial\hat{\mathcal{M}}(p,q); k[-\ell_q])$. As explained in Example 5, our goal is to construct a k-linear map from $C^*(\mathcal{M}(p,q); k[-\ell_q])$ into some shift of k. Because $\mathcal{M}(p,q)$ is nontrivial only for the unique pair $p < q$, where it is $s^{-1}(0)/G$, we will just write $s^{-1}(0)/G$ instead of $\mathcal{M}(p,q)$. Similarly, we will just write $[-1]$ to denote desuspension by S^{ℓ_q}. In this section, we will attempt to construct a virtual fundamental chain

$$\nu : C^*(s^{-1}(0)/G; k[-1]) \to \Sigma^{-d}k \tag{1}$$

by "brute force", and keep a running tally of conditions on k that will be necessary to get this construction to make sense. We first need a definition.

Definition 8. Let G be a compact Lie group, and let EG denote a contractible space with free G-action. Let X be a topological space with a continuous action of G. Then the *Borel construction*[5] X_{hG} is the topological space defined by $(X \times EG)/G$. If X is a point, for instance, $X_{hG} = BG$. If G acts freely on X, then there is a homotopy equivalence $X_{hG} \simeq X/G$. In general, X_{hG} is also known as the *homotopy quotient of X by the G-action*.

Let k be a spectrum. The *Borel-equivariant homology* $k^G(X)$ of X is defined as $C_*(X_{hG}; k)$. Similarly, the *Borel-equivariant cohomology* $k_G(X)$ of X is defined as $C^*(X_{hG}; k)$. There is a natural way to extend these definitions to *pairs* $(X, Z \subseteq X)$.

If $k = \mathbb{S}$, for instance, then $\mathbb{S}^G(X) = \Sigma^\infty_+(X_{hG})$. It turns out that (by essentially the same construction as above), one can define the homotopy quotient Y_{hG} of any *spectrum* Y with a G-action[6], and that $\Sigma^\infty_+(X_{hG}) \simeq (\Sigma^\infty_+ X)_{hG}$. Then, $\mathbb{S}_G(X)$ is the $\mathbb{S}$-linear dual of $(\Sigma^\infty_+ X)_{hG}$. This linear dual is also known as the *homotopy fixed points* of $\Sigma^\infty_+ X$, and is denoted $(\Sigma^\infty_+ X)^{hG}$. In general, the $\mathbb{S}$-linear dual of the homotopy quotient Y_{hG} of a spectrum Y with G-action is denoted Y^{hG}. If G acts trivially on Y, then $Y_{hG} = C_*(BG; Y)$, while $Y^{hG} = C^*(BG; Y)$.

The construction $X \mapsto X_{hG}$ is a well-defined functor from the homotopy category of spaces with G-action to the homotopy category of spaces. In fact, the homotopy quotient can be viewed as a left adjoint (in the *homotopy-coherent* sense!) to the functor from spaces to G-spaces which sends X to the space with trivial G-action. The ease with which these sort of constructions can be performed in the ∞-categorical setting is one of the main advantages of that technology. In this talk, we will not need to know anything about ∞-categories.

[5] This is denoted "BX" in [AB21], but this notation confused me for a very long time.

[6] Confusingly, this is *not* the same as a "G-spectrum"; that notion is much stronger.

Construction 9. Let us now return to our Kuranishi structure (X, V, s, G) and to the construction of (1). There is a canonical map $s^{-1}(0)_{hG} \to s^{-1}(0)/G$, which gives a k-linear map

$$C^*(s^{-1}(0)/G; k[-1]) \to C^*(s^{-1}(0)_{hG}; k[-1]) = C^*_G(s^{-1}(0); k[-1]).$$

After this point, we will never actually work with $s^{-1}(0)/G$, only with the Borel-equivariant cohomology of $s^{-1}(0)$. Observe now that $s^{-1}(0)$ is a closed subset of X, and so we might hope to apply a version of Poincaré duality to work with Borel-equivariant *homology* instead. This leads to our first desideratum on k:

(a) There should be a Poincaré duality equivalence

$$C^*_G(s^{-1}(0); k[-1]) \simeq C^G_*(X, X - s^{-1}(0); k[d])$$

for some $d \in \mathbf{Z}$. Moreover, because we want this to be true for *any* Kuranishi structure, such a Poincaré duality equivalence should exist for any finite group G (we will not care about whether d can be chosen uniformly in G).

If (a) is satisfied, we can compose with the map on Borel-equivariant homology induced by the map $(X, X - s^{-1}(0)) \to (V, V - \{0\})$ of pairs. Since $C^G_*(V, V - \{0\}; k) \simeq C^G_*(S^V; k)$, we finally get a long composite

$$\begin{aligned} C^*(s^{-1}(0)/G; k[-1]) &\to C^*_G(s^{-1}(0); k[-1]) \\ &\xrightarrow{\text{Poincare}} C^G_*(X, X - s^{-1}(0); k[d]) \\ &\xrightarrow{s} C^G_*(V, V - \{0\}; k[d]) \simeq C^G_*(S^V; k[d]). \end{aligned} \tag{2}$$

We are nearly done, except for two snags. First, $C^G_*(S^V; k)$ is not the same as a suspension of $C^G_*(k) = k_{hG}$; and even if it was, the above construction would produce a map to k_{hG} and not to k. The second snag is easy to fix[7]: since $k_{hG} = C_*(BG; k)$, we can just

[7] The following construction is a very lossy procedure, and it is reasonable to think that there should be a way to carry along the data of the group G in these constructions. As Semon explained to me, there is a belief that one should be able to define "global MU-Floer theory", where "global" is used in the sense of global homotopy theory.

compose with the map $k_{hG} \to k$ which crushes BG. The first snag cannot be fixed so easily for all k, so we list it as another one of our desiderata:

(b) For any finite-dimensional G-representation V, there should be an equivalence $C_*^G(S^V; k) \simeq C_*^G(S^{|\dim V|}; k) = k_{hG}[|\dim V|]$.

The upshot is that via Equation (2), we get a map

$$C^*(s^{-1}(0)/G; k[-1]) \to C_*^G(S^V; k[d]) \simeq k_{hG}[|\dim V| + d] \to k[|\dim V| + d],$$

which is a virtual fundamental chain for $\mathcal{M}$.

Our goal is therefore to find examples of $\mathbf{E}_1$-rings k which satisfy (a) and (b) from Construction 9. Let us first show that there are many examples which satisfy (b):

Proposition 10. *Let k be an $\mathbf{E}_1$-ring which is complex-oriented. Then there is an equivalence $C_*^G(S^V; k) \simeq C_*^G(S^{|\dim V|}; k)$.*

Proof. We may assume without loss of generality that $k = \mathrm{MU}$. We can rewrite the desired equivalence as $(S^V \otimes \mathrm{MU})_{hG} \simeq \mathrm{MU}[|\dim V|]_{hG}$. It therefore suffices to show that there is an equivalence $S^V \otimes \mathrm{MU} \simeq S^{\dim V} \otimes \mathrm{MU}$ of (left) MU-modules with G-action, i.e., that there is an equivalence $\Sigma^{-\dim V} S^V \otimes \mathrm{MU} \simeq \mathrm{MU}$ of MU-modules with G-action. Indeed, a MU-module with G-action is a functor $BG \to \mathrm{Mod}_{\mathrm{MU}}$, where BG is regarded as a category/Kan complex. The MU-module underlying $\Sigma^{-\dim V} S^V \otimes \mathrm{MU}$ is just MU, and so the functor classifying $\Sigma^{-\dim V} S^V \otimes \mathrm{MU}$ can be viewed as a map $BG \to \mathrm{BGL}_1(\mathrm{MU})$. This map is simply the composite

$$BG \xrightarrow{\text{map classifying } V} \mathrm{BU} \xrightarrow{J} \mathrm{BGL}_1(\mathbb{S}) \to \mathrm{BGL}_1(\mathrm{MU}).$$

But this composite is zero, because MU is the Thom spectrum of $J : \mathrm{BU} \to \mathrm{BGL}_1(\mathbb{S})$. Therefore, the functor classifying $\Sigma^{-\dim V} S^V \otimes \mathrm{MU}$ agrees with the *trivial* functor $BG \to \mathrm{Mod}_{\mathrm{MU}}$, i.e., MU itself. ☐

Given this proposition, our goal is to find examples of complex-oriented $\mathbf{E}_1$-rings k which satisfy (a) from Construction 9 (i.e., satisfies G-equivariant Poincaré duality for any finite group G). This is true when $k = \mathrm{H}\mathbf{Q}$, i.e., for *rational* homology; therefore, it makes sense to say that virtual fundamental chains exist in rational homology. In the next section, we will discuss the *Morava K-theories*, which give other examples of such $\mathbf{E}_1$-rings. (We will give an essentially complete proof, modulo technical details, that Morava K-theory satisfies equivariant Poincaré duality. Abouzaid and Blumberg cite/reprove this via the main result of [Che13], but the argument we give below is slightly different.)

3. Morava K-theories

To define Morava K-theory, we need to fix an implicit prime p. Then there is a Morava K-theory defined for every $n \geq 1$, denoted $K(n)$; when $n = 0$, $K(n)$ is often denoted $\mathrm{H}\mathbf{Q}$ (i.e., the Eilenberg-Maclane spectrum). It would be rather unenlightening to give a construction of Morava K-theory in this talk. Instead, we will describe $K(n)$ via the following proposition.

Proposition 11. (a) *For $n \geq 1$ (known as the height), the nth Morava K-theory $K(n)$ is the unique complex-oriented homotopy associative ring spectrum whose homotopy groups are $\pi_* K(n) \cong \mathbf{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$, and whose formal group has height n. Concretely, the second condition means that if $\hbar \in \pi_{-2} C^*(\mathbf{C}P^\infty; K(n))$ is a complex-orientation[8], then $c_1(\mathcal{O}(p)) = v_n \hbar^{p^n}$.*

(b) *Let A be a ring spectrum such that $\pi_* A$ is a graded field, i.e., is concentrated in even degrees and all nonzero elements are invertible. Then A is a $K(n)$-module for some $n \geq 0$.*

[8] This is not standard notation; usually, this is written as x.

(c) *The nth Morava K-theory admits the structure of an $\mathbf{E}_1$-ring spectrum. If $p > 2$, then $K(n)$ admits a homotopy commutative multiplication; this fails if $p = 2$. Moreover, if X is any space, then*

$$K(n)^*(X) = \mathrm{MU}^*(X) \otimes_{\mathrm{MU}_*} \mathbf{F}_p[v_n^{\pm 1}];$$

in particular, $K(n)^*(X)$ is still often a commutative ring even though $K(n)$ need not be homotopy commutative.

Part (b) of Proposition 11 tells us that the Morava K-theories $K(n)$ are the "only fields" in the category of spectra. For instance, the fact that $K(n)_*$ is a field allows us to use the Künneth spectral sequence to conclude that there is a Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

It is quite useful to visualize the $K(n)$ as n varies via the following diagram:

$$\mathrm{H}\mathbf{Q} = K(0) \quad K(1) \quad K(2) \quad \cdots \quad K(\infty) = \mathrm{H}\mathbf{F}_p.$$

There are no maps between these $K(n)$ for varying n, but it *is* true that if X is a finite spectrum such that $K(n)_*(X) = 0$, then $K(m)_*(X) = 0$ for $1 \leq m \leq n$.

Example 12. The only easily accessible example of a Morava K-theory is at $n = 1$. Let KU denote the complex K-theory spectrum, and let KU$/p$ denote its quotient by p (so $\pi_*(\mathrm{KU}/p) \cong \pi_*(\mathrm{KU})/p$). Then $K(1)$ is a p-local summand of KU$/p$. More precisely, the action of the Adams operations on $\mathrm{KU}_{(p)}$ (by integers prime to p) extends to an action of $\mathbf{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbf{Z}_p)$ on $\mathrm{KU}_p^\wedge$, and hence on KU$/p$. The Morava K-theory $K(1)$ is defined to be the (homotopy) invariants $(\mathrm{KU}/p)^{h\mu_{p-1}}$. Since the action of μ_{p-1} on the Bott class $\beta \in \pi_2\mathrm{KU}$ is by multiplication by $(p-1)$st roots of unity, it is easy to check that $\pi_*(\mathrm{KU}/p)^{h\mu_{p-1}} \cong \mathbf{F}_p[\beta^{\pm(p-1)}]$; therefore, $v_1 = \beta^{p-1}$.

The above example explains the term "Morava *K-theory*"; historically, $K(1)$ was the first example. Below, we will often illustrate calculations using $K(1)$. (A general principle in homotopy theory is that nearly anything to do with $K(1)$ is *probably* true for the higher $K(n)$.) In any case, because $K(1)$ is a retract of KU$/p$, any computation with $K(1)$ is really just a calculation with KU$/p$ in disguise.

Let us now turn to condition (a) from Construction 9. We would like to prove that the $K(n)$ do satisfy this condition, i.e., Borel-equivariant Poincaré duality. To prove something like this, we should separate out the *geometric* and *homotopy-theoretic* components of Poincaré duality. Let us begin with the following.

Proposition 13 (Alexander duality)**.** *Let A be a closed subset of a closed manifold M, and let $i : A \hookrightarrow M$ denote the inclusion. If μ is a virtual vector bundle over A, let A^μ denote the Thom spectrum of μ. Then there is a (geometrically defined) equivalence*

$$\mathbf{D}(A^{-i^*T_M}) \simeq \Sigma^\infty(M/(M - A)),$$

where the left-hand side is the Spanier-Whitehead dual of $A^{-i^*T_M}$. If everything in sight admits an action of a finite group G, then this equivalence also respects the G-action.

Example 14. For instance, suppose A is a closed manifold, and choose an embedding $i : A \hookrightarrow \mathbf{R}^n$ for some $n \gg 0$. Then $T_{\mathbf{R}^n}$ is the trivial rank n bundle, so the same is true of $i^*T_{\mathbf{R}^n}$. The Thom spectrum $A^{-i^*T_{\mathbf{R}^n}}$ is therefore $\Sigma_+^{\infty-n}A$. Similarly, the right hand side of Proposition 13 is $\mathbf{R}^n/(\mathbf{R}^n - A) \simeq \Sigma(\mathbf{R}^n - A) \cong \Sigma(S^n - A_+)$. Therefore, Proposition 13 states that there is an equivalence $\Sigma^n\mathbf{D}(\Sigma_+^\infty A) \simeq \Sigma^{\infty+1}(S^n - A_+)$. Desuspending n times, we obtain the usual spectrum-level statement of Alexander duality. In fact, the general statement of Proposition 13 can be proved in this way, too, by embedding the ambient closed manifold M into $\mathbf{R}^n$ for some $n \gg 0$.

Let k be an $\mathbf{E}_1$-ring spectrum, and let G be a finite group acting on everything in sight in Proposition 13; then, we have

$$\begin{aligned}\mathbf{D}(A^{-i^*T_M})_{hG} \otimes k &\xrightarrow{\simeq,\text{Proposition 13}} (\Sigma^\infty(M/(M-A)))_{hG} \otimes k \\ &= C_*(M_{hG},(M-A)_{hG};k) = C_*^G(M, M-A;k).\end{aligned} \tag{3}$$

The left hand side, unfortunately, is *not* the G-equivariant k-cohomology of $A^{-i^*T_M}$. What is the issue? Let us try to unpack what the the G-equivariant k-cohomology of a finite CW-complex X with G-action would be:

$$C_G^*(X;k) = \mathrm{Map}(\Sigma_+^\infty X_{hG}, k) \simeq \mathrm{Map}(\Sigma_+^\infty X, k)^{hG} \simeq (\mathbf{D}(\Sigma_+^\infty X)\otimes k)^{hG} \simeq \mathbf{D}(\Sigma_+^\infty X)^{hG}\otimes k.$$

The second equivalence uses that mapping spaces send homotopy colimits in the first variable to homotopy limits, while the third and fourth equivalences use the finiteness of X. Therefore, in the notation of Proposition 13, we would have

$$C_G^*(A;k) \simeq \mathbf{D}(\Sigma_+^\infty A)^{hG} \otimes k. \tag{4}$$

This *almost* looks like $\mathbf{D}(A^{-i^*T_M})_{hG} \otimes k$, the differences being:

(a) The Spanier-Whitehead dual in Equation (4) is that of $\Sigma_+^\infty A$, whereas we need to understand the Spanier-Whitehead dual of $A^{-i^*T_M}$. This is easily fixed, though. Indeed, observe that

$$\mathbf{D}(A^{-i^*T_M})_{hG} \otimes k \simeq (\mathbf{D}(A^{-i^*T_M}) \otimes k)_{hG} \simeq \mathbf{D}_k(k \otimes A^{-i^*T_M})_{hG},$$

and similarly with $\mathbf{D}(\Sigma_+^\infty A)_{hG} \otimes k$. All we *really* need to understand, therefore, is whether $k \otimes A^{-i^*T_M}$ is equivalent to a shift of $k \otimes \Sigma_+^\infty A$. This is easy enough: such an equivalence exists if the tangent bundle of M is k-orientable. If we assume that k is complex-oriented (which we needed to do anyway, by Proposition 10), then this is true if i^*T_M admits a stably almost complex structure. My understanding is that this condition is satisfied in the examples of Floer-theoretic interest.

(b) Next, Equation (4) used homotopy *fixed points* instead of homotopy *orbits*. This is a much more serious issue than above, and is where "ambidexterity" comes in. Roughly, if we knew that there was an equivalence

$$\mathbf{D}(\Sigma_+^\infty A)^{hG} \otimes k \simeq \mathbf{D}(\Sigma_+^\infty A)_{hG} \otimes k, \tag{5}$$

then we could just use the discussion above to obtain an equivalence $C_G^*(A;k) \simeq \mathbf{D}(A^{-i^*T_M})_{hG} \otimes k$ (and hence get the desired equivariant Poincaré duality by Equation (3)). The condition (5) is satisfied when $k = K(n)$ by the following result:

Proposition 15. *Let N be a finite spectrum with G-action (where G is a finite group). Let $K(n)$ be a Morava K-theory with $0 \leq n < \infty$. Then the canonical "norm" map $\mathrm{Nm}: N_{hG} \to N^{hG}$ induces an equivalence*[9] $K(n) \otimes N_{hG} \xrightarrow{\sim} K(n) \otimes N^{hG}$.

[9] Note that $K(n) \otimes (N_{hG}) \simeq (K(n) \otimes N)_{hG}$ is true for *any* spectrum N (finite or not); similarly, $K(n) \otimes (N^{hG}) \simeq (K(n) \otimes N)^{hG}$ when N is *finite.*

In fact, Proposition 15 is a special case of the following more general result:

Theorem 16 (Ambidexterity for finite groups; Hovey and Sadofsky)**.** *Let $K(n)$ be a Morava K-theory with $0 \leq n < \infty$, and let G be a finite group. Let Y be a $K(n)$-local spectrum (e.g., a left $K(n)$-module). Then the canonically defined "norm" map $\mathrm{Nm}: Y_{hG} \to Y^{hG}$ induces a $K(n)$-local equivalence $Y_{hG} \xrightarrow{\sim} Y^{hG}$.*

In particular, if k is a $K(n)$-local spectrum (we will not define this notion here), then the smash product of k with any *finite* spectrum remains $K(n)$-local. Therefore, Equation (5) holds *after $K(n)$-localization* for any finite CW-complex A with an action of a finite group G and any $K(n)$-local $\mathbf{E}_1$-ring k (such as "Morava E-theory"). More precisely, if $L_{K(n)}$ denotes $K(n)$-localization, then Equation (5) and Equation (3) give a map (when i^*T_M is k-orientable)

(6)
$$\begin{aligned} L_{K(n)}C^*_G(A;k) &\simeq L_{K(n)}(\mathbf{D}(\Sigma^\infty_+ A)^{hG}\otimes k) \\ &\xrightarrow{\simeq, \mathrm{Nm}^{-1}} L_{K(n)}(\mathbf{D}(\Sigma^\infty_+ A)_{hG}\otimes k) \\ &\xrightarrow{\simeq,\text{ up to shift}} L_{K(n)}(\mathbf{D}(A^{-i^*T_M})_{hG}\otimes k) \\ &\xrightarrow{\text{Equation (3)}} L_{K(n)}C^G_*(M, M-A;k). \end{aligned}$$

In general, it is not possible to get rid of the prefix $L_{K(n)}$. However, if k is $K(n)$ (or an $\mathbf{E}_1$-algebra whose underlying spectrum is a $K(n)$-module), then the prefix $L_{K(n)}$ is not needed (because everything is $K(n)$-local), and we get equivariant Poincaré/Alexander duality with coefficients in k.

We will not explain the proof of Theorem 16 today. However, we will explain some parts of Proposition 15. Namely, we will describe a construction of the aforementioned norm map $\mathrm{Nm} : N_{hG} \to N^{hG}$, and then explain why Proposition 15 is true when $G = \mathbf{Z}/p$ and $n = 1$ (i.e., for $K(1)$). Let us begin with Nm. There are a few ways to construct this map, depending on how much technology one wishes to use.

- Recall that if X is a space, then $X_{hG} = (X \times EG)/G$. If N is a spectrum, then $N_{hG} = (N \otimes \Sigma^\infty_+ EG)/G$. Similarly, $N^{hG} = \mathrm{Map}(\Sigma^\infty_+ EG, N)^G$. Now, observe that there is a G-equivariant composite
$$\Sigma^\infty_+ EG \to \mathbb{S} \to \mathrm{Map}(\Sigma^\infty_+ EG, \mathbb{S}),$$
where both maps are given by crushing EG to a point (and observing that $\mathbb{S} \simeq \mathrm{Map}(\mathbb{S}, \mathbb{S})$). Smashing with N, we get a G-equivariant composite
$$N \otimes \Sigma^\infty_+ EG \to N \to N \otimes \mathrm{Map}(\Sigma^\infty_+ EG, \mathbb{S}) \simeq \mathrm{Map}(\Sigma^\infty_+ EG, N).$$
Taking G-fixed points[10], we get a map $(N \otimes \Sigma^\infty_+ EG)^G \to N^{hG}$. This is not quite the norm map, because the source has G-fixed points instead of G-orbits. This is the same kind of problem we encountered above, but the situation in this case is easier: we only need to know that there is an equivalence $(N \otimes \Sigma^\infty_+ EG)^G \simeq (N \otimes \Sigma^\infty_+ EG)_G$ for the specific G-spectrum $N \otimes \Sigma^\infty_+ EG$. Such an equivalence does exist, essentially because $N \otimes \Sigma^\infty_+ EG$ is a "free" G-spectrum (this is similar to the fact in algebra that orbits and fixed points agree for free actions). This equivalence is known as the *Adams isomorphism*. Given this equivalence, we can now define Nm as the composite
$$N_{hG} \xrightarrow{\sim, \text{Adams iso}} (N \otimes \Sigma^\infty_+ EG)^G \to N^{hG}.$$
- The preceding construction relied on knowing some genuine equivariant homotopy theory. However, the construction of Nm can be done entirely in the "naive" world, as we now describe (see the Hopkins-Lurie paper for more details). We will assume everything in sight is homotopy-coherent/∞-categorical. Recall that the functor of homotopy fixed points (resp. homotopy orbits) is right (resp. left) adjoint to the functor $\mathrm{triv}_G : \mathrm{Sp} \to \mathrm{Fun}(BG, \mathrm{Sp})$ which gives a spectrum the trivial G-action. If $f : X \to Y$ is a map of Kan complexes, then there is a functor $f^* : \mathrm{Fun}(Y, \mathrm{Sp}) \to \mathrm{Fun}(X, \mathrm{Sp})$; then, the functor triv_G is just f^* for $f : BG \to *$. Returning to the general case when $f : X \to Y$ is a map of Kan complexes, let f_* (resp. $f_!$) denote the right (resp. left) adjoint of f^*. One might then wish for a generalization of the norm map, given by a natural transformation $\mathrm{Nm}_f : f_! \to f_*$. Such a natural transformation will not exist for arbitrary f, but it will exist if f is the crushing $BG \to *$.

[10] By this, we mean *strict* fixed points. This is not a procedure available if the G-action on N is defined up to homotopy; so N should be a "G-spectrum" for this construction of Nm to work. In the next bullet, we will give a homotopy-invariant construction of Nm that only requires N to admit a G-action.

To describe the construction, let $f : X \to Y$ be a map of Kan complexes, and let $\delta : X \to X \times_Y X$ be the diagonal. Assume that we have already built a "norm" $\mathrm{Nm}_\delta : \delta_! \to \delta_*$, and that Nm_δ is an equivalence. Let $\mathrm{pr}_0, \mathrm{pr}_1 : X \times_Y X \to X$ be the projections; then, we get a natural transformation

$$\mathrm{pr}_0^* \to \delta_* \delta^* \mathrm{pr}_0^* \simeq \delta_* \xrightarrow{\sim, \mathrm{Nm}_\delta^{-1}} \delta_! \simeq \delta_! \delta^* \mathrm{pr}_1^* \to \mathrm{pr}_1^*.$$

Taking adjoints, this is a map $\mathrm{id}_{\mathrm{Sp}} \to (\mathrm{pr}_0)_* \mathrm{pr}_1^*$. But there is an equivalence $(\mathrm{pr}_0)_* \mathrm{pr}_1^* \simeq f^* f_*$, so we get a map $\mathrm{id}_{\mathrm{Sp}} \to f^* f_*$. This adjoints to our desired norm map $\mathrm{Nm}_f : f_! \to f_*$.

Let us try to apply this construction to $f : BG \to *$. Note that the diagonal map $BG \to BG \times BG$ has discrete fibers (because G is *finite*). In order for the hypotheses in the above paragraph to be satisfied, it therefore suffices to know that there is a natural transformation $\mathrm{Nm}_g : g_! \to g_*$ which is an equivalence whenever $g : Z \to W$ is a map of Kan complexes whose homotopy fibers are finite sets. This is indeed true in Sp: for instance, suppose Z is a finite set, and that $g : Z \to *$ is the crushing map. A functor $Z \to \mathrm{Sp}$ is just a Z-indexed collection of spectra $\{V_z\}_{z \in Z}$. The image of $\{V_z\}_{z \in Z}$ under $g_! : \mathrm{Fun}(Z, \mathrm{Sp}) \to \mathrm{Sp}$ is just the coproduct $\coprod_{z \in Z} V_z$, while its image under $g_* : \mathrm{Fun}(Z, \mathrm{Sp}) \to \mathrm{Sp}$ is just the product $\prod_{z \in Z} V_z$. The norm map is then just the canonical map $\coprod_{z \in Z} V_z \to \prod_{z \in Z} V_z$. This map is therefore an equivalence if finite coproducts and finite products agree in Sp (which is true; in fact, finite limits and finite colimits agree in Sp).

The upshot of the preceding discussion is that when there is a canonical natural transformation $\mathrm{Nm} : -_{hG} \to -^{hG}$ of functors $\mathrm{Fun}(BG, \mathrm{Sp}) \to \mathrm{Sp}$, which can be defined without appealing to genuine equivariant homotopy theory.

Let us conclude by explaining why Proposition 15 is true when $G = \mathbf{Z}/p$ and $n = 1$ (i.e., for $K(1)$). Recall the statement we wish to prove: if N is a finite spectrum with G-action, then the norm map $\mathrm{Nm} : N_{hG} \to N^{hG}$ induces an equivalence $K(n) \otimes N_{hG} \xrightarrow{\sim} K(n) \otimes N^{hG}$. Essentially because every finite spectrum is a finite limit/colimit of copies of $\mathbb{S}$, we can reduce to the case when $N = \mathbb{S}$ with the trivial G-action[11]. In this case, we must show that the norm map $K(n)_{hG} \to K(n)^{hG}$ for the trivial G-action on $K(n)$ is an equivalence.

[11] This reduction requires a little bit of work, but it is not too difficult.

Now we will specialize to the case $G = \mathbf{Z}/p$ and $n = 1$, so that $K(1)$ is a summand of KU/p by Example 12. It then suffices to show:

Proposition 17. *The norm map* $(\mathrm{KU}/p)_{h\mathbf{Z}/p} \to (\mathrm{KU}/p)^{h\mathbf{Z}/p}$ *for the trivial* $\mathbf{Z}/p$*-action on* KU/p *is an equivalence.*

Proof. It suffices to show that the cofiber of the norm map is trivial. In general, if N is a spectrum with a G-action, then the cofiber of the norm map $N_{hG} \to N^{hG}$ is known as the *Tate construction*, and is denoted N^{tG}. We therefore need to understand $(\mathrm{KU}/p)^{t\mathbf{Z}/p}$. This is the cofiber of multiplication by p on $\mathrm{KU}^{t\mathbf{Z}/p}$, so let us compute $\mathrm{KU}^{t\mathbf{Z}/p}$ instead. There is a canonical map $\mathrm{KU}^{h\mathbf{Z}/p} \to \mathrm{KU}^{t\mathbf{Z}/p}$, so let us first understand $\mathrm{KU}^{h\mathbf{Z}/p}$. Because $\mathbf{Z}/p$ is acting trivially, we see that $\mathrm{KU}^{h\mathbf{Z}/p} = C^*(B\mathbf{Z}/p; \mathrm{KU})$. There is a fiber sequence

$$B\mathbf{Z}/p \to \mathbf{C}P^\infty \xrightarrow{p} \mathbf{C}P^\infty.$$

The degree p map on $\mathbf{C}P^\infty$ induces a map $\mathrm{KU}^*(\mathbf{C}P^\infty) \to \mathrm{KU}^*(\mathbf{C}P^\infty)$, i.e., a map $\mathbf{Z}[\beta^{\pm 1}, \hbar] \to \mathbf{Z}[\beta^{\pm 1}, \hbar]$, which sends $\hbar$ to $[p](\hbar) = (1 - (\beta\hbar - 1)^p)/\beta$. The Gysin sequence of the aforementioned fibration induces an isomorphism

$$\pi_*(\mathrm{KU}^{h\mathbf{Z}/p}) = \mathrm{KU}^*(B\mathbf{Z}/p) \cong \mathrm{KU}^*(\mathbf{C}P^\infty)/[p](\hbar) = \mathbf{Z}[\beta^{\pm 1}, \hbar]/(1 - (\beta\hbar - 1)^p).$$

One can also prove this via the Atiyah-Segal completion theorem. The effect of the map $\mathrm{KU}^{h\mathbf{Z}/p} \to \mathrm{KU}^{t\mathbf{Z}/p}$ is just inverting $\hbar$ (this requires some work to show). Therefore, we conclude that

$$\pi_*(\mathrm{KU}^{t\mathbf{Z}/p}) \cong \mathbf{Z}[\beta^{\pm 1}, \hbar^{\pm 1}]/(1 - (\beta\hbar - 1)^p) \cong \mathbf{Q}_p(\zeta_p)[\beta^{\pm 1}].$$

Since this ring is rational, we see that $\mathrm{KU}^{t\mathbf{Z}/p}/p \simeq (\mathrm{KU}/p)^{t\mathbf{Z}/p}$ must be zero, as desired. $\square$

References

[AB21] M. Abouzaid and A. Blumberg. Arnold Conjecture and Morava K-theory. https://arxiv.org/abs/2103.01507, 2021. 1, 3

[Che13] M. C. Cheng. Poincare duality in Morava K-theory for classifying spaces of orbifolds. https://arxiv.org/abs/1305.2731, 2013. 4

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