Lecture V: (noncommutative) Hodge-de Rham degeneration, part one

In the previous lecture, we discussed a proof of the Bogomolov-Tian-Todorov theorem for Calabi-Yau varieties over a field $k$ of characteristic zero: the key steps involved were Hodge-de Rham degeneration (which holds for any smooth and proper $k$-variety) and a result from [KKP08] relating degeneration for a $\mathcal{BV}$-algebra $A$ to homotopy abelianness of the differential graded Lie algebra $A[1]$. Our goal in this lecture is to describe a generalization of the first part of this proof to fields of characteristic $p > 0$: namely, we will study Hodge-de Rham degeneration for varieties over $k \supseteq \mathbf{F}_p$ (as well a noncommutative analogue).

The main result along these lines was proven by Deligne and Illusie in [DI87]. In order to state their result, we need to recall a construction:

**Construction 1.** Let $A$ be a commutative ring. Define the commutative ring $W_2(A)$ via the pullback diagram

$$\begin{tikzcd} W_2(A) \arrow[r] \arrow[d] & A \arrow[d, "a \mapsto a"] \\ A \arrow[r, "a \mapsto a^p"'] & A/p. \end{tikzcd}$$

There is a bijection $W_2(A) \simeq A \times A$, sending a pair $(a,b) \in W_2(A)$ to $(a^p + pb, a)$. Note that $a^p + pb \equiv \mathrm{Frob}(a) \pmod p$. Under the identification $W_2(A) \simeq A \times A$, the addition and multiplication are given by

$$(a,b) + (c,d) = \left(a + c, b + d + \frac{a^p + c^p - (a+c)^p}{p}\right),$$
$$(a,b) \cdot (c,d) = (ac, a^p d + c^p b + pbd).$$

If $k = \mathbf{F}_p$, then $W_2(k) \cong \mathbf{Z}/p^2$.

**Theorem 2** (Deligne-Illusie)**.** *Let $X$ be a smooth and proper variety over a field $k$ of characteristic $p > 0$. Suppose that $\dim(X) < p$ and that $X$ lifts to a smooth and proper variety over $W_2(k)$. Then the Hodge-de Rham spectral sequence*

$$E_1^{s,t} = \mathrm{H}^s(X; \Omega^t_{X/k}) \Rightarrow \mathrm{H}^{s+t}_{\mathrm{dR}}(X/k)$$

*degenerates at the $E_1$-page.*

One of our goals in this lecture is to give a proof of Theorem 2. Let us mention that by a spreading-out argument, one can use Theorem 2 to reprove Hodge-de Rham degeneration over a field of characteristic zero.

Let us begin with a simple observation. The spectral sequence in Theorem 2 implies that $\mathrm{H}^*_{\mathrm{dR}}(X/k)$ is a subquotient of $\mathrm{H}^*(X; \Omega^*_{X/k})$, so we must have the inequality

$$\dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k) \leq \sum_{s+t=n} \dim_k \mathrm{H}^s(X; \Omega^t_{X/k}). \tag{1}$$

If the inequality in (1) is an equality, then the conclusion of Theorem 2 holds.

**Proposition 3.** *Suppose that there is a quasi-isomorphism $\bigoplus_{i=0}^{\dim X} \Omega^i_{X^{(p)}/k}[-i] \xrightarrow{\sim} \mathrm{Frob}_* \Omega^\bullet_{X/k}$ of complexes of quasicoherent sheaves on $X^{(p)}$. Then the inequality in Equation (1) is an equality.*

*Proof.* Since $\mathrm{Frob} : X \to X^{(p)}$ is an isomorphism, we have $\mathrm{H}^n_{\mathrm{dR}}(X/k) \cong \mathrm{H}^*(X^{(p)}; \mathrm{Frob}_*\Omega^\bullet_{X/k})$. By the assumption on $\mathrm{Frob}_*\Omega^\bullet_{X/k}$, we see that $\mathrm{H}^n_{\mathrm{dR}}(X/k) \cong \bigoplus_{s+t=n} \mathrm{H}^s(X^{(p)}; \Omega^t_{X^{(p)}/k})$;

in particular,

$$\dim \mathrm{H}^n_{\mathrm{dR}}(X/k) = \sum_{s+t=n} \dim \mathrm{H}^s(X^{(p)}; \Omega^t_{X^{(p)}/k}).$$

Therefore, it suffices to show that $\dim \mathrm{H}^s(X^{(p)}; \Omega^t_{X^{(p)}/k}) = \dim \mathrm{H}^s(X; \Omega^t_{X/k})$. This, however, follows from the fact that the base-change of $\Omega^*_{X/k}$ along the absolute Frobenius (on $\operatorname{Spec} k$) is $\Omega^*_{X^{(p)}/k}$. $\square$

To prove Theorem 2, it therefore suffices to show that if $\dim(X) < p$ and $X$ lifts to a smooth and proper variety over $W_2(k)$, then there is a quasi-isomorphism $\bigoplus_{i=0}^{\dim X} \Omega^i_{X^{(p)}/k}[-i] \xrightarrow{\sim} \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$.

*Proof sketch of Theorem 2.* We begin by constructing a map $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$. Let $\widetilde{X}$ and $\widetilde{X}^{(p)}$ denote the chosen lifts of $X$ and $X^{(p)}$ to $W_2(k)$. First, suppose that the relative Frobenius $\mathrm{Frob} : X \to X^{(p)}$ lifts to a map $F : \widetilde{X} \to \widetilde{X}^{(p)}$ over $W_2(k)$. Define a map $\psi$ as follows: if $x$ is a local section of $\mathcal{O}_{X^{(p)}}$, let $\widetilde{x}$ be a lift to $\mathcal{O}_{\widetilde{X}^{(p)}}$. Then

$$F(\widetilde{x}) = \widetilde{x}^p + p\widetilde{y}$$

for some $\widetilde{y} \in \mathcal{O}_{\widetilde{X}^{(p)}}$. In particular,

$$F(d\widetilde{x}) = p(\widetilde{x}^{p-1} d\widetilde{x} + d\widetilde{y});$$

then, one defines $\psi(dx) = F(d\widetilde{x})/p$. In order for this to be a well-defined map $\Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$, we need $\psi(dx)$ to be killed by the de Rham differential on $\mathrm{Frob}_* \Omega^\bullet_{X/k}$, but this is clear.

Next, one checks (see [DI87] for details) that different choices for $\widetilde{F}$ lead to chain-homotopic maps $\psi$. Therefore, if $\mathrm{Frob} : X \to X^{(p)}$ lifts to $W_2(k)$, then $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ is a well-defined morphism in $D(X^{(p)})$ which is independent of the choice of Frobenius lift. However, not every smooth $k$-scheme has a lift of Frobenius. Nonetheless, every smooth affine $k$-scheme does have a lift of Frobenius to $W_2(k)$ by the yoga of deformation theory; therefore, *Zariski-locally*, smooth $k$-variety has a lift of Frobenius. Using this observation, Deligne and Illusie calculate (using the Čech complex of some covering $\{U_i\}$ of $X$ by smooth affine $k$-schemes) that one can glue the maps $\psi_{U_i}$ to a well-defined morphism $\Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$ even if $X$ itself does not have a lift of Frobenius. This is a little tedious, so we refer the reader to the original paper.

Having obtained the map $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$ for any smooth and proper $X$ which lifts to $W_2(k)$, we must now describe its extension to a map $\Psi : \bigoplus_{i=0}^{\dim X} \Omega^i_{X^{(p)}/k}[-i] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$. In other words, we must describe maps $\Omega^i_{X^{(p)}/k}[-i] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$. For this, one observes that *if* $i < p$, then we may identify $\Omega^i_{X^{(p)}/k} = \wedge^i_{\mathcal{O}_{X^{(p)}}} \Omega^1_{X^{(p)}/k}$ with the *alternating* tensors; in other words, the canonical map $(\Omega^1_{X^{(p)}/k})^{\otimes i} \to \Omega^i_{X^{(p)}/k}$ admits a section. Consequently, when $i < p$ (which is always satisfied if $\dim X < p$, since $i \leq \dim X$), we can define our desired map via the composite

$$\Omega^i_{X^{(p)}/k}[-i] \to (\Omega^1_{X^{(p)}/k}[-1])^{\otimes i} \to (\mathrm{Frob}_* \Omega^\bullet_{X/k})^{\otimes i} \to \mathrm{Frob}_* \Omega^\bullet_{X/k}.$$

We must now show that the map $\Psi : \bigoplus_{i=0}^{\dim X} \Omega^i_{X^{(p)}/k}[-i] \to \mathrm{Frob}_* \Omega^\bullet_{X/k}$ in $D(X^{(p)})$ is an equivalence, i.e., that it induces isomorphisms on each cohomology group. On cohomology, we obtain the map $\mathcal{H}^*(\Psi) : \bigoplus_{i=0}^{\dim X} \Omega^i_{X^{(p)}/k}[-i] \to \bigoplus_{i=0}^{\dim X} \mathcal{H}^i(X^{(p)}; \mathrm{Frob}_* \Omega^\bullet_{X/k})$, which in degree 1 sends $dx$ to $[x^{p-1} dx]$. The isomorphism $\mathcal{H}^*(\Psi)$ is called the *Cartier* *isomorphism*; to check that it is indeed an isomorphism, a standard argument reduces us to the case of $\mathbf{A}^1 = \operatorname{Spec} k[t]$. In this case, we must show that $k[t^p] \oplus k[t^p]d(t^p)$ is isomorphic to $\mathrm{H}^*_{\mathrm{dR}}(\mathbf{A}^1/k)$. This is a straightforward exercise using $dt^p = 0$. □

**Remark 4.** During the course of the proof of Theorem 2, we constructed the $\mathcal{O}_{X^{(p)}}$-linear map $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$, and then extended this in some way to a map $\bigoplus_{i=1}^{p-1} \Omega^i_{X^{(p)}/k}[-i] \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$; note the range of the direct sum. Therefore, it is natural to ask: just as there is an equivalence $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(\Omega^1_{X^{(p)}/k}[1]) \simeq \bigoplus_{i=1}^{\dim X}(\wedge^i_{\mathcal{O}_{X^{(p)}}}\Omega^1_{X^{(p)}/k})[i]$, is there some universal property of $\bigoplus_{i=1}^{p-1} \Omega^i_{X^{(p)}/k}[-i]$ which allows us to build the desired extension $\Psi$? Could this universal property be used to bypass the restriction $i < p$?

To explain this, let us declutter notation, and consider the following situation: let $R$ be a (discrete) commutative ring, and let $M$ be an $R$-module. The symbol $\wedge^n_R M$ then has two meanings: either it could be the quotient of $M^{\otimes_R n}$ by a certain ideal, or it could be the submodule of $M^{\otimes_R n}$ generated by the alternating tensors. To distinguish between the two, let us write $\wedge^n_R M$ to denote the former, and $\mathcal{A}^n_R(M)$ to denote the latter. Then, the canonical map $\mathcal{A}^n_R(M)[n] \to M^{\otimes_R n}[n] \to \mathrm{Sym}^n_R(M[1])$ is an equivalence. Taking $R$-linear duals, we get a canonical equivalence $\mathrm{Sym}^n_R(M[1])^\vee \xrightarrow{\sim} (\mathcal{A}^n_R M)^\vee[-n]$. The linear dual of $\mathrm{Sym}^n_R(M[1])^\vee$ is canonically identified with the divided power module $\Gamma^n_R(M^\vee[-1])$. The linear dual $(\mathcal{A}^n_R M)^\vee$ is canonically equivalent to $\wedge^n_R(M^\vee)$, at least in odd characteristic: indeed, such an identification corresponds to a nondegenerate bilinear form $\mathcal{A}^n_R M \otimes_R \wedge^n_R(M^\vee) \to R$ given by

$$(m_1 \wedge \cdots \wedge m_n, f_1 \wedge \cdots \wedge f_n) \mapsto \det(f_i(m_j)). \tag{2}$$

Therefore, we conclude[1] that $\Gamma^n_R(M^\vee[-1]) \xrightarrow{\sim} \wedge^n_R(M^\vee)[-n]$.

[1] This was a rather roundabout way of reaching the desired conclusion. A simpler approach is to use the (somewhat less familiar) canonical equivalence $\mathrm{Sym}^n_R(M[2]) \simeq (\Gamma^n_R M)[2n]$.

Returning back to the situation of Theorem 2, we see that generalizing the above discussion to the case where $M$ and $R$ are replaced by $T_{X/k}$ and $\mathcal{O}_{X^{(p)}}$ provides an equivalence $\Gamma^n_{\mathcal{O}_{X^{(p)}}}(\Omega^1_{X^{(p)}/k}[-1]) \xrightarrow{\sim} \Omega^n_{X^{(p)}/k}[-n]$. To build the map $\Psi$ (without any dimension restrictions on $X$), we must therefore be able to extend the map $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$ to a map $\widetilde{\Psi} : \Gamma_{\mathcal{O}_{X^{(p)}}}(\Omega^1_{X^{(p)}/k}[-1]) \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$ from the divided power $\mathcal{O}_{X^{(p)}}$-algebra. In general, if $R$ is a discrete commutative ring and $A$ is a commutative $R$-algebra, one cannot extend an $R$-module map $M \to A$ to an $R$-algebra map $\Gamma_R(M) \to A$. Therefore, the desired extension $\widetilde{\Psi}$ can be built in one of two ways: either $\mathrm{Frob}_*\Omega^\bullet_{X/k}$ can be endowed with additional structure, or one can impose restrictions on $X$ guaranteeing that $\mathrm{Sym}_{\mathcal{O}_{X^{(p)}}}(\Omega^1_{X^{(p)}/k}[-1]) \xrightarrow{\sim} \Gamma_{\mathcal{O}_{X^{(p)}}}(\Omega^1_{X^{(p)}/k}[-1])$. The latter is satisfied once $\dim X < p$, which explains the appearance of this condition in Theorem 2.

**Remark 5.** As discussed in Remark 4, one approach to removing the dimension condition on $X$ would be to describe the existence of additional structure on $\mathrm{Frob}_*\Omega^\bullet_{X/k}$. The required structure is not easy to specify, though. Indeed, if $R$ is a discrete commutative ring and $A$ is a commutative $R$-algebra, our goal is to extend an $R$-module map $f : M \to A$ to an $R$-algebra map $\Gamma_R(M) \to A$. Even if we assume for simplicity that $M$ is a free $R$-module on a generator $t$, we see that $\Gamma_R(M) \cong R[t, \gamma_p(t), \cdots]/(t^p, \gamma_p(t)^p, \cdots)$. Therefore, extending the map $f$ to $\Gamma_R(M)$ requires specifying $\gamma_p(t) \in A$ — and this element has no relations with $t$, other than the requirement that its $p$th power be zero. Therefore, the map $\psi : \Omega^1_{X^{(p)}/k}[-1] \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$ from the proof of Theorem 2 wil no longer be sufficient: one would need to specify certain maps $\Omega^{p^n}_{X^{(p)}/k}[-p^n] \to \mathrm{Frob}_*\Omega^\bullet_{X/k}$ for $p^n \leq \dim X$. Note, however, that the proof of Theorem 2 shows that if $X$ and its Frobenius lift to $W_2(k)$, then no dimension assumptions are necessary to get the desired map $\Psi$.

**Remark 6.** Although this remark is not necessary for the broader discussion, let us use this opportunity to clarify a confusing point. We saw in Remark 4 that there is a canonical equivalence $(\mathcal{A}^n_R M)^\vee \simeq \wedge^n_R(M^\vee)$. There is a canonical map $c : \mathcal{A}^n_R M \to M^{\otimes_R n} \to \wedge^n_R M$, but this map is *not* an equivalence once $n \geq p$. Indeed, if $M$ is free (with generators $m_i$, say), then the generators of $\mathcal{A}^n_R M$ are of the form $\sum_{\sigma\in\Sigma_n}(-1)^\sigma m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$, which the composite $c$ sends to $n! m_1 \wedge \cdots \wedge m_n$. This, of course, is zero if $n \geq p$. We conclude that although there is a canonical equivalence $(\mathcal{A}^n_R M)^\vee \simeq \wedge^n_R(M^\vee)$, there will not be a canonical equivalence $\wedge^n_R(M^\vee) \simeq (\wedge^n_R M)^\vee$ if $n \geq p$ (but that there is such an equivalence if $n < p$).

**Remark 7.** It is natural to wonder what $\mathrm{Sym}^n_R(M[-1])$ is in general. Away from characteristic zero, it cannot be canonically identified with $(\wedge^n_R M)[-n]$, as our discussion above shows. In fact, $\mathrm{Sym}^n_R(M[-1])$ is a rather complicated beast: for instance, suppose $M = R$, and $R$ is a perfect field of characteristic $p > 0$. Since $(B\mathbf{G}_a)(S)$ is $S[1] \cong \mathrm{Map}_R(\mathrm{Sym}_R(R[-1]), S)$ for any simplicial commutative $R$-algebra $S$, we see that $\Gamma(B\mathbf{G}_a; \mathcal{O}_{B\mathbf{G}_a}) \cong \mathrm{Sym}_R(R[-1])$. In other words, $\mathrm{Sym}_R(R[-1])$ is the group cohomology of $\mathbf{G}_a$. But it is known that

$$\pi_*\Gamma(B\mathbf{G}_a; \mathcal{O}_{B\mathbf{G}_a}) = \begin{cases} R[x_1, x_2, \cdots] & p = 2 \\ R[y_1, y_2, \cdots] \otimes_R \Lambda_R(x_1, x_2, \cdots) & p > 2, \end{cases}$$

where $|x_i| = -1$ and $|y_i| = -2$ in homological grading. On the other hand, $\pi_*\Gamma(B\mathbf{G}_a; \mathcal{O}_{B\mathbf{G}_a})$ is $\Lambda_R(x_1)$ if $R$ is a field of characteristic zero.

Let us now turn to the noncommutative setting. In the remainder of this lecture, we will explain the objects involved in the statement of the noncommutative Hodge-de Rham spectral sequence; in the next few lectures, we will then set up further technology that will allow us to prove the desired statement.

**Construction 8.** Let $k$ be an $\mathbf{E}_\infty$-ring. We will define a symmetric monoidal functor $\mathrm{HH} : \mathrm{LinCat}_k \to \mathrm{Fun}(BS^1, \mathrm{Mod}_k)$ from $k$-linear stable $\infty$-categories to $k$-modules with $S^1$-action (equipped with the pointwise tensor product). If $\mathcal{C}$ is a $k$-linear stable $\infty$-category, the associated $S^1$-equivariant $k$-module is denoted $\mathrm{HH}(\mathcal{C}/k)$. If $k = \mathbb{S}$ is the sphere spectrum, then $\mathrm{HH}(\mathcal{C}/\mathbb{S})$ is just denoted $\mathrm{THH}(\mathcal{C})$.

To give a precise construction of this functor, we need to introduce some notation. Let $\Lambda_\infty$ be the *paracyclic category*, defined as the 1-category whose objects are indexed by nonnegative integers, and are denoted $[n]_{\Lambda_\infty}$. The set of morphisms between $[n]_{\Lambda_\infty}$ and $[m]_{\Lambda_\infty}$ is the set of $\mathbf{Z}$-equivariant maps $\frac{1}{n}\mathbf{Z} \to \frac{1}{m}\mathbf{Z}$ of partially ordered sets, where the action of $\mathbf{Z}$ on the source and target is by translation. Note that there is a canonical action of the category $B\mathbf{Z}$ on $\Lambda_\infty$. The *cyclic category* $\Lambda$ is defined as the quotient $\Lambda_\infty/B\mathbf{Z}$.

There is a canonical functor $\Delta \to \Lambda_\infty$ sending $[n] = \{0, \cdots, n\}$ to $[n+1]_{\Lambda_\infty}$, which one can prove is final. Moreover, one can prove that $\Lambda_\infty^{\mathrm{op}}$ is sifted, so its geometric realization is contractible. This implies that the geometric realization of $\Lambda$ is the quotient of $|N(\Lambda_\infty^{\mathrm{op}})| \simeq *$ by $B\mathbf{Z} \simeq S^1$; since the action of $\mathbf{Z}$ on the mapping spaces of $N(\Lambda_\infty^{\mathrm{op}})$ is free, we see that $|N(\Lambda^{\mathrm{op}})| \simeq BS^1$.

Let $\mathcal{C}$ be a $k$-linear $\infty$-category. Then $\mathcal{C}$ defines a functor $\mathcal{C}^\natural : \Lambda^{\mathrm{op}} \to \mathrm{Mod}_k$ by the assignment

$$[n]_\Lambda \mapsto \mathrm{colim}_{x_0,\cdots,x_n\in\mathcal{C}} \mathrm{Map}_\mathcal{C}(x_0, x_1) \otimes \cdots \otimes \mathrm{Map}_\mathcal{C}(x_{n-1}, x_n) \otimes \mathrm{Map}_\mathcal{C}(x_n, x_0).$$

Then, one defines $\mathrm{HH}(\mathcal{C}/k)$ as the geometric realization of the functor $\mathcal{C}^\natural|_{\Delta^{\mathrm{op}}}$. By the preceding discussion, $\mathrm{HH}(\mathcal{C}/k)$ is canonically equipped with a $R$-linear $S^1$-action. There are several invariants one can extract from $\mathrm{HH}(\mathcal{C}/k)$:

- The *negative cyclic homology* $\mathrm{HC}^-(\mathcal{C}/k)$ is the homotopy fixed points $\mathrm{HH}(\mathcal{C}/k)^{hS^1}$. If $k = \mathbb{S}$, this is denoted $\mathrm{TC}^-(\mathcal{C})$, and is called *topological negative cyclic homology.*
- The *periodic cyclic homology* $\mathrm{HP}(\mathcal{C}/k)$ is the Tate construction $\mathrm{HH}(\mathcal{C}/k)^{tS^1}$. If $k = \mathbb{S}$, this is denoted $\mathrm{TP}(\mathcal{C})$, and is called *topological periodic cyclic homology.*

**Remark 9.** If $\mathcal{C} = \mathrm{QCoh}(X)$ is the $\infty$-category of quasicoherent sheaves on a $k$-scheme $X$, then we will write $\mathrm{HH}(X/k)$ to denote $\mathrm{HH}(\mathcal{C}/k)$. A more concrete model for $\mathrm{HH}(\mathcal{C}/k)$ is given by the global sections $\Gamma(X; \mathcal{O}_X \otimes_{\mathcal{O}_X \otimes_k \mathcal{O}_X} \mathcal{O}_X)$. To describe the $S^1$-action, observe that since $S^1 \simeq * \amalg_{S^0} *$, we can view $\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes_k \mathcal{O}_X} \mathcal{O}_X$ as the colimit of the $k$-linearization of the (sifted) constant functor $S^1 \to \mathrm{CAlg}(\mathrm{QCoh}(X))$ with value $\mathcal{O}_X$.

The main result that will motivate our discussion is the HKR theorem. Let us first introduce some notation: denote by $\mathrm{sh} : \mathrm{Mod}_k^{\mathrm{gr}} \to \mathrm{Mod}_k^{\mathrm{gr}}$ the (symmetric monoidal) functor sending a graded $k$-module $M_\bullet$ to the graded $k$-module which is $M_n[2n]$ in weight $n$. Moreover, if $X$ is a $k$-scheme, let $L\Omega^*_{X/k}$ denote the graded $\mathcal{O}_X$-module $\wedge^* L_{X/k}[-*]$. Let $\widehat{L\Omega}^*_{X/k}$ denote its completion $\varprojlim L\Omega^*_{X/k}/\mathrm{F}^{\geq n}_{\mathrm{Hdg}} L\Omega^*_{X/k}$ with respect to the Hodge filtration. Finally, let $L\Omega^\bullet_{X/k}$ denote the derived de Rham *complex*.

**Theorem 10** (HKR, but as proved in [Ant18, Rak20])**.** *Let $k$ be a discrete commutative ring, and let $X$ be a qcqs $k$-scheme. Then there is a functorial complete decreasing multiplicative $\mathbf{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(X/k)$ such that:*

(a) *There is a filtered action of $\tau_{\leq \star} k[S^1]$ on $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(X/k)$, which induces an action of $\mathbf{D}_+ := \mathrm{gr}(\tau_{\leq\star} k[S^1])$ on $\mathrm{gr}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(X/k))$.*

(b) *There is a graded $k$-linear equivalence $\mathrm{sh}(\mathrm{gr}(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(X/k)))_* \simeq \Gamma(X; L\Omega^*_{X/k})$, which is equivariant for the action of $\mathrm{sh}(\mathbf{D}_+) = k \oplus k[-1](1)$. The action of $\mathrm{sh}(\mathbf{D}_+)$ on $\mathrm{sh}(\mathrm{gr}^\bullet(\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(X/k)))$ is by part (a), and its action on $\Gamma(X; L\Omega^*_{X/k})$ is via the derived de Rham differential.*

(c) *There are induced filtrations on $\mathrm{HC}^-(X/k)$ and $\mathrm{HP}(X/k)$, denoted $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(X/k)$, such that*

$$\mathrm{gr}^n(\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(X/k)) \simeq \Gamma(X; \widehat{L\Omega}^{\bullet\geq n}_{X/k}[2n])$$
$$\mathrm{gr}^n(\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(X/k)) \simeq \Gamma(X; \widehat{L\Omega}^{\bullet}_{X/k}[2n]).$$

(d) *The pieces $\mathrm{F}^n_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{F}^n_{\mathrm{B}}\mathrm{HP}(X/k)$ have compatible decreasing filtrations, denoted $\mathrm{F}^\star_{\mathbf{C}P}-$, such that the induced filtration on $\mathrm{gr}_{\mathrm{B}}$ is the Hodge filtration.*

(e) *The filtration $\mathrm{F}^\star_{\mathrm{HKR}}$ on $\mathrm{HH}(X/k)$ and the filtration $\mathrm{F}^\star_{\mathrm{B}}$ on $\mathrm{HC}^-(X/k)$ and $\mathrm{HP}(X/k)$ are split if $k$ is a $\mathbf{Q}$-algebra.*

It follows from (c) that there is a spectral sequence

$$E_1^{*,*} = \mathrm{H}^*_{\mathrm{dR}}(X/k) \cong \mathrm{H}^*(X; \widehat{L\Omega}^\bullet_{X/k}) \Rightarrow \pi_*\mathrm{HP}(X/k)$$

corresponding to the filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(X/k)$. Part (d) allows us to form a "commutative" square of spectral sequences:

$$\begin{array}{ccc} \mathrm{H}^*(X; \widehat{L\Omega}^*_{X/k}) & \xRightarrow{\mathrm{Hodge-deRham}} & \mathrm{H}^*_{\mathrm{dR}}(X/k) \\ \Big\Downarrow{\scriptstyle \mathrm{HKR}} & & \Big\Downarrow{\scriptstyle \mathrm{B-filtration}} \\ \hat{\mathrm{H}}^*(BS^1; \pi_*\mathrm{HH}(X/k)) & \xRightarrow{\mathrm{Tate}} & \pi_*\mathrm{HP}(X/k). \end{array}$$

If $k$ is a field and two adjacent arrows collapse on the first page, then all the arrows collapse on the first page. Therefore, the noncommutative analogue of the Hodge-de

Rham spectral sequence is the Tate spectral sequence

$$E_2^{*,*} = \hat{\mathrm{H}}^*(BS^1; \pi_*\mathrm{HH}(\mathcal{C}/k)) \Rightarrow \pi_*\mathrm{HP}(\mathcal{C}/k).$$

If $\hbar$ is the generator of $\mathrm{H}^2(BS^1; k)$, then the $E_2$-page of the Tate spectral sequence may be identified with $\pi_*\mathrm{HH}(\mathcal{C}/k)(\!(\hbar)\!)$. The notational choice was briefly justified in Lecture I; we will discuss it further later.

Let us end this lecture by stating the main result, whose proof we will discuss next time:

**Theorem 11** ([Kal09, Mat20])**.** *Let $k$ be a perfect field, and let $\mathcal{C}$ be a smooth and proper $k$-linear stable $\infty$-category. If $\mathbf{F}_p \subseteq k$, assume that $\mathcal{C}$ lifts to $W_2(k)$ and that $\pi_n\mathrm{HH}(\mathcal{C}/k) = 0$ if $|n| > p$. Then the Tate spectral sequence for $\mathrm{HP}(\mathcal{C}/k)$ degenerates at the $E_2$-page.*

The proof we will discuss is a rephrasing of [Mat20]; we will use this as an opportunity to introduce cyclotomic spectra.

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