Lifting to truncated Brown-Peterson spectra and Hodge-de Rham degeneration in characteristic $p > 0$

S. K. Devalapurkar

Abstract. The goal of this note is to prove that Hodge-de Rham degeneration holds for smooth and proper $\mathbf{F}_p$-schemes X with $\dim(X) < p^n$ as soon as its category of quasicoherent sheaves admits a lift to the truncated Brown-Peterson spectrum $\mathrm{BP}\langle n-1\rangle$, and the Hochschild-Kostant-Rosenberg spectral sequence for X degenerates at the E_2-page. This is obtained from a noncommutative version, whose proof is essentially the same as Mathew's argument in [**Mat20**].

I'm grateful to Ben Antieau and Bhargav Bhatt for suggestions which improved this note.

Recollection 1. Let X be a smooth scheme over a commutative ring k. One then has the HKR and de-Rham-to-HP spectral sequences (see [**ABM21**, Definition 3.1]):

$$E_2^{s,t} = \mathrm{H}^s(X; \wedge^{-t} L_{X/k}) \Rightarrow \pi_{-(s+t)}\mathrm{HH}(X/k),$$
$$E_2^{s,t} = \mathrm{H}^{s-t}_{\mathrm{dR}}(X/k) \Rightarrow \pi_{-(s+t)}\mathrm{HP}(X/k).$$

Fix an $\mathbf{E}_3$-form of the truncated Brown-Peterson spectrum $\mathrm{BP}\langle n-1\rangle$, which exists thanks to [**HW20**, Theorem A]. By construction, $\pi_*\mathrm{BP}\langle n-1\rangle \cong \mathbf{Z}_{(p)}[v_1, \cdots, v_{n-1}]$ for classes v_i in degree $2p^i - 2$. Moreover, $\mathrm{BP}\langle -1\rangle = \mathbf{F}_p$, $\mathrm{BP}\langle 0\rangle = \mathbf{Z}_p$, and $\mathrm{BP}\langle 1\rangle$ can be identified with the connective cover of the Adams summand of p-completed complex K-theory. There is also a tight relationship between $\mathrm{BP}\langle 2\rangle$ and elliptic cohomology.

Our goal in this note is to prove:

Theorem 2. *Let X be a smooth and proper scheme over*[1] $\mathbf{F}_p$ *of dimension* $< p^n$. *Suppose that:*

(a) *The HKR spectral sequence degenerates at the E_2-page; and*
(b) $\mathrm{QCoh}(X)$ *lifts to a smooth and proper left* $\mathrm{BP}\langle n-1\rangle$*-linear* ∞*-category*[2].

Part of this work was done when the author was supported by the PD Soros Fellowship and NSF DGE-2140743.

[1] Here, $\mathbf{F}_p$ could be replaced by any perfect field of characteristic $p > 0$; we only use $\mathbf{F}_p$ to avoid introducing conceptually unnecessary notation.

[2] Recall that at the beginning of this article, we picked an $\mathbf{E}_3$-form of $\mathrm{BP}\langle n-1\rangle$, which exists by [**HW20**, Theorem A]. Then, a "left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category" is simply a left $\mathrm{LMod}_{\mathrm{BP}\langle n-1\rangle}$-module in Pr^{L}, where $\mathrm{LMod}_{\mathrm{BP}\langle n-1\rangle}$ is equipped with the $\mathbf{E}_2$-monoidal structure arising from the $\mathbf{E}_3$-structure on $\mathrm{BP}\langle n-1\rangle$. See [**Lur17**, Variant D.1.5.1].

Then the Hodge-de Rham spectral sequence

$$E_1^{*,*} = \mathrm{H}^*(X; \Omega^*_{X/\mathbf{F}_p}) \Rightarrow \mathrm{H}^*_{\mathrm{dR}}(X/\mathbf{F}_p)$$

*collapses at the E_1-page, and the de-Rham-to-*HP *spectral sequence collapses at the E_2-page.*

Remark 3. When $n = 1$, Theorem 2 is part of the main result of [**DI87**][3]: in this case, condition (b) in Theorem 2 is asking for a lifting to $\mathrm{BP}\langle 0\rangle = \mathbf{Z}_p$. In [**Ill96**, Problem 7.10], Deligne and Illusie asked if the Hodge-de Rham spectral sequence degenerates for a smooth proper $\mathbf{F}_p$-scheme with a lift to $\mathbf{Z}/p^2$ (or even to $\mathbf{Z}_p$), without any dimension assumptions. This remarkable question has recently been answered (in the negative) by Sasha Petrov in [**Pet23**]; he constructed a $(p+1)$-dimensional smooth and proper $\mathbf{Z}_p$-scheme $\mathfrak{X}$ such that the Hodge-de Rham spectral sequence for its special fiber $\mathfrak{X}_{p=0}$ does not degenerate at the E_1-page. If the HKR spectral sequence degenerates at the E_2-page for Petrov's $\mathfrak{X}_{p=0}$, then $\mathrm{QCoh}(\mathfrak{X})$ provides an example of a $\mathbf{Z}_p$-linear ∞-category which cannot lift to a ku-linear ∞-category.

We view Theorem 2 as a step towards a conditional positive answer of Deligne and Illusie's question. Note that condition (b) in Theorem 2 is significantly weaker than asking that X itself admit some sort of lifting as a spectral scheme. Note, also, that we do not prove anything nearly as refined as [**DI87**]: namely, we do not provide any sort of correspondence between liftings and splittings of truncations of the de Rham complex. For instance, it would be very interesting if, for a $\mathbf{Z}_p$-scheme $\mathfrak{X}$, there were a relationship between splittings of the mod p reduction $\widehat{\Omega}^{\not{D}}_{\mathfrak{X},0} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ of the zeroth eigenspace of the diffracted Hodge complex (see [**BL22a**, Remark 4.7.20] for this notion) and liftings of $\mathrm{QCoh}(\mathfrak{X})$ to $\mathrm{BP}\langle 1\rangle$.

Remark 4. Theorem 2 has the following counter-intuitive consequence: if the HKR spectral sequence for X degenerates at the E_2-page, then the differentials in the Hodge-de Rham spectral sequence obstruct the lifting of $\mathrm{QCoh}(X)$ to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category.

The discussion in [**ABM21**, Remark 3.6] implies that if the HKR and Tate spectral sequences both degenerate, then both the Hodge-de Rham and de Rham-to-HP spectral sequences must also degenerate. It therefore suffices to prove the following noncommutative statement[4]:

Proposition 5. *Let $\mathcal{C}$ be a smooth and proper $\mathbf{F}_p$-linear ∞-category such that $\pi_j \mathrm{HH}(\mathcal{C}/\mathbf{F}_p) = 0$ for $j \notin [-p^n, p^n]$. If $\mathcal{C}$ lifts to a smooth and proper left* $\mathrm{BP}\langle n-1\rangle$*-linear ∞-category, then the Tate spectral sequence*

$$E_2^{*,*} = \mathrm{HH}(\mathcal{C}/\mathbf{F}_p)[\hbar^{\pm 1}] \Rightarrow \mathrm{HP}(\mathcal{C}/\mathbf{F}_p)$$

collapses at the E_2-page.

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[3] As the reader may have noticed, the title of this work is a tribute to the inspirational paper [**DI87**].

[4] Our original proof used the higher chromatic topological Sen operators of [**Dev23**] to argue in a manner similar to [**BL22a**, Example 4.7.17], but we soon realized that the argument could be simplified much further. See also the discussion in [**Dev23**, Remark C.14] phrasing an analogue of Proposition 5 in algebraic geometry via the Sen operator of [**BL22a**] and the stack $BW^\times[F^n]$. The expected isomorphism, which we hope to study in joint work with Jeremy Hahn and Arpon Raksit, between $BW^\times[F^n]$ and the stack associated to the motivic filtration of $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)^{t\mathbf{Z}/p}/(p, \cdots, v_{n-1})$ was the original motivation for our result.

Remark 6. One consequence of Proposition 5 is the fact that if $\mathcal{C}$ is a smooth and proper $\mathbf{F}_p$-linear ∞-category which admits a smooth and proper lift to BP, then the Tate spectral sequence collapses at the E_2-page (with no further assumption on $\mathrm{HH}(\mathcal{C}/\mathbf{F}_p)$ vanishing outside a certain range!).

This was already known if $\mathcal{C}$ lifts all the way to S^0; see [**Mat20**, Example 3.5]. In particular, therefore, one class of X for which $\mathrm{QCoh}(X)$ does satisfy the hypotheses of Proposition 5 and Theorem 2 are toric varieties; but in those cases, degeneration was already known for X of arbitrary dimension (since they are F-liftable). Interesting examples of Theorem 2 and Proposition 5 are currently lacking, but one would be most welcome.

Remark 7. Some preliminary calculations seem to suggest that Petrov's first Sen class (see [**Pet23, Ill22**]) is related to the obstruction in Hochschild cohomology to lifting along the map $\mathrm{BP}\langle 1\rangle/v_1^2 \to \mathbf{Z}_p$. We hope to explore this further to obtain a tighter connection between the results in this article and those of Petrov's.

Remark 8. One could also ask the following question: if $n \geq 0$, is there an example of a smooth and proper $\mathbf{F}_p$-linear ∞-category $\mathcal{C}$ which lifts to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, but not to a smooth and proper left $\mathrm{BP}\langle n\rangle$-linear ∞-category?

The idea to prove Proposition 5 is essentially the argument of [**Mat20**], so we recommend reading that paper first. Recall Bökstedt's calculation that $\pi_*\mathrm{THH}(\mathbf{F}_p) \cong \mathbf{F}_p[\sigma]$, where σ lives in degree 2. By [**Mat20**, Proposition 3.4], Proposition 5 is a consequence of:

Proposition 9. *Let $\mathcal{C}$ be a smooth and proper $\mathbf{F}_p$-linear ∞-category such that $\pi_j\mathrm{HH}(\mathcal{C}/\mathbf{F}_p) = 0$ for $j \notin [-p^n, p^n]$. If $\mathcal{C}$ lifts to a smooth and proper left $\mathrm{BP}\langle n-1\rangle$-linear ∞-category, then $\mathrm{THH}(\mathcal{C})$ is σ-torsionfree.*

To prove Proposition 9, we need a lemma. The following result is essentially [**Mat20**, Proposition 3.7]; it could also be proved using the methods of [**Dev23**].

Lemma 10. *Let M be a perfect $\mathrm{THH}(\mathbf{F}_p)$-module such that $\pi_i(M) = 0$ for $i < a$. If M lifts to a perfect $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$-module $\widetilde{M}$, then σ-multiplication $\sigma : \pi_{i-2}M \to \pi_iM$ is injective for $i \leq a + 2p^n - 1$.*

Proof. Let $I_n = (p, \cdots, v_{n-1})$. First, observe that the map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \mathrm{THH}(\mathbf{F}_p)$ factors through a map

$$\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \otimes \mathbf{F}_p \simeq \mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/I_n \to \mathrm{THH}(\mathbf{F}_p).$$

By [**ACH21**, Proposition 2.9] (see also [**Dev23**, Remark 2.2.5]), there is an isomorphism

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/I_n \cong \mathbf{F}_p[\sigma^2(v_n)] \otimes \Lambda(\sigma(t_1), \cdots, \sigma(t_n)).$$

Here, $|\sigma^2(v_n)| = 2p^n$ and $|\sigma(t_i)| = 2p^i - 1$. Moreover, the map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/I_n \to \mathrm{THH}(\mathbf{F}_p)$ induces a map of motivically filtered $\mathbf{E}_2$-ring spectra (see [**HRW22**, Example 4.2.4]), and is given on motivic associated graded by a graded map of rings:

$$\mathbf{F}_p[\sigma^2(v_n)] \otimes \Lambda(\sigma(t_1), \cdots, \sigma(t_n)) \to \mathbf{F}_p[\sigma].$$

Since $\sigma^2(v_n)$ lives in weight p^n, while $\sigma(t_i)$ lives in weight p^i and degree 1, we see that $\sigma(t_i) \mapsto 0$ and $\sigma^2(v_n) \mapsto \sigma^{p^n}$. Therefore, $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to$

$\tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ factors through the map $\tau_{\leq 0}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/I_n \simeq \mathbf{F}_p \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ of ring spectra.[5]

To prove the result of the lemma, we can assume without loss of generality that $a = 0$. Then, there is a map

$$M \to \tau_{\leq 2p^n-1}\widetilde{M} \otimes_{\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)} \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p),$$

which is an equivalence on $\tau_{\leq 2p^n-1}$. But the map $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathrm{BP}\langle n-1\rangle) \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$ factors through $\mathbf{F}_p \to \tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$, so we see that $\tau_{\leq 2p^n-1}M$ is a free $\tau_{\leq 2p^n-1}\mathrm{THH}(\mathbf{F}_p)$-module on classes in nonnegative degrees. Therefore, σ-multiplication is injective through the stated range. □

Proposition 9 is now a consequence of the following, which is essentially [**Mat20**, Proposition 3.8].

Proposition 11. *Let M be a perfect $\mathrm{THH}(\mathbf{F}_p)$-module with Tor-amplitude in $[-p^n, p^n]$. If M lifts to a perfect $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)$-module $\widetilde{M}$, then M is free.*

Proof. The argument is the same as in [**Mat20**, Proposition 3.8]. Indeed, M is a direct sum of $\mathrm{THH}(\mathbf{F}_p)$-modules which are free or of the form $M_{i,j} = \Sigma^i\mathrm{THH}(\mathbf{F}_p)/\sigma^j$ (see [**Mat20**, Proposition 3.3]). Since $M_{i,j}$ has Tor-amplitude in $[i, i+2j+1]$, the condition on M implies that $M_{i,j}$ could appear as a summand of M if and only if $-p^n \leq i \leq i+2j+1 \leq p^n$.

The class $\sigma^{j-1}[i] \in \pi_{i+2j-2}M_{i,j}$ is killed by σ, so taking $a = -p^n$ in Lemma 10, we see that

$$i + 2j > -p^n + 2p^n - 1 = p^n - 1.$$

In particular, $i+2j+1 > p^n$, which contradicts $i+2j+1 \leq p^n$. Therefore, no $M_{i,j}$ can be a summand of M, so that M is free. □

In the remainder of this note, we will clarify the relationship between liftings of X itself and Hodge-de Rham degeneration. First, observe that assumption (b) in Theorem 2 is only a condition on $\mathrm{QCoh}(X)$, which is essentially why Proposition 5 is the more natural noncommutative statement. One might hope that assumption (a) in Theorem 2 could be removed if we strengthened (b) to assume that X itself lifted as a spectral scheme to $\mathrm{BP}\langle n-1\rangle$. Unfortunately, this often does not make sense, since $\mathrm{BP}\langle n-1\rangle$ is generally not an $\mathbf{E}_\infty$-ring [**Law18, Sen17**]. Nevertheless, this is possible if, for instance, $n = 2$ (since $\mathrm{BP}\langle 1\rangle$ is an $\mathbf{E}_\infty$-ring). In this case, requiring that X lift is significantly stronger than the assumptions of Theorem 2, as shown by the following.

Proposition 12. *Let X be a smooth and proper $\mathbf{F}_p$-scheme. If X lifts to a p-adic flat $\mathrm{ku}_p^\wedge$-scheme $\mathfrak{X}$, then the Hodge-de Rham spectral sequence for X degenerates at the E_1-page.*

Proof. The lift $\mathfrak{X}$ defines a lift of X to $\mathbf{Z}_p$ via $\mathfrak{X}_0 := \mathfrak{X} \otimes_{\mathrm{ku}_p^\wedge} \mathbf{Z}_p$. It suffices to show that $\mathfrak{X}_0$ admits a δ-ring structure; then, the Hodge-Tate gerbe over $\mathfrak{X}_0$ (from [**BL22b**, Proposition 5.12]) splits, so that the conjugate (and hence Hodge-de Rham) spectral sequence for X degenerates. The fact that $\mathfrak{X}$ is assumed to be

[5]In fact, the map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle)/I_n \to \mathrm{THH}(\mathbf{F}_p)$ factors through a map $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/I_n \to \mathrm{THH}(\mathbf{F}_p)$, which on homotopy is given by the map $\mathbf{F}_p[\sigma^2(v_n)] \to \mathbf{F}_p[\sigma]$ sending $\sigma^2(v_n) \mapsto \sigma^{p^n}$; see [**Dev23**, Example 4.2.6]. However, the $\mathbf{F}_p$-module $\mathrm{THH}(\mathrm{BP}\langle n-1\rangle/T(n))/I_n$ does not naturally have any ring structure.

flat implies that $\pi_0 L_{K(1)}\mathcal{O}_{\mathfrak{X}} \cong \pi_0\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}_0}$. By [**Hop14**], if R is any $K(1)$-local $\mathbf{E}_\infty$-ring, then $\pi_0(R)$ admits a δ-ring structure (functorially in R). Globalizing, we see that $\pi_0 L_{K(1)}\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}_0}$ has a δ-ring structure, which implies the desired claim. □

Remark 13. It follows from Proposition 12 that lifting an arbitrary-dimensional X to a $\mathrm{ku}_p^\wedge$-scheme suffices to conclude Hodge-de Rham degeneration; in particular, this assumption is significantly stronger than those of Theorem 2. One intermediate between the assumptions of Proposition 12 and Theorem 2 is the following: one could assume that $\mathcal{O}_X$ only admit a lift to a sheaf of $\mathbf{E}_m$-BP$\langle n-1\rangle$-algebras (whenever this makes sense). Proposition 12 corresponds to the case $n=2$ and $m=\infty$, while Theorem 2 roughly corresponds to the case $m=1$ (and n arbitrary). What constraints does such a lifting impose on the Hodge-de Rham spectral sequence for X? For instance, if p is an odd prime, and $\mathcal{O}_X$ admits a flat lift to a sheaf of $\mathbf{E}_{2n+1}$-$\mathrm{ku}_p^\wedge$-algebras, then the general construction of power operations (following [**Hop14**]) shows that $\mathfrak{X}_0$ has a lift of Frobenius modulo p^{n+1}. In particular, if $\mathcal{O}_X$ admits a flat lift to a sheaf of $\mathbf{E}_3$-$\mathrm{ku}_p^\wedge$-algebras, and $\dim(X) < p$, then [**DI87**] implies that the Hodge-de Rham spectral sequence degenerates for X.

Remark 14. Finally, one might wonder whether a lifting of X to BP$\langle n-1\rangle$, or $\mathrm{ku}_p^\wedge$, or even the sphere spectrum can be used to prove that the HKR spectral sequence degenerates. Unfortunately, it seems that there is no clear relationship between HKR degeneration and liftings to the sphere. For instance, the stack $B\mu_p$ over $\mathbf{Z}_p$ lifts to the p-complete sphere spectrum (by writing $\mu_p = \operatorname{Spec} S[\mathbf{Z}/p]$), but the HKR spectral sequence for $B\mu_p$ does not degenerate by [**ABM21**, Theorem 4.6].

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1 Oxford St, Cambridge, MA 02139

Email address: sdevalapurkar@math.harvard.edu, February 23, 2023