G3

FINITE ELEMENT ANALYSIS OF A THIN ELASTIC PLATE

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FINITE ELEMENT ANALYSIS OF A THIN PLATE

AIM: To find the deflections and bending, twisting moments of a thin plate acted upon by a transversal load, f(x) using the variational formulation.

ASSUMPTIONS:

The plate theory used for solution is the Kirchhoff – Love plate theory. This is an extension of the Euler-Bernoulli beam theory. The other major theory is the Reissner – Mindlin Beam theory and other theories are just extensions of the two above mentioned theories.

The assumptions of the Kirchhoff – Love plate theory are:

1) Small deflection (u)
2) Linear isotropic elastic material
3) Straight lines normal to the mid-surface
   (i) remains straight
   (ii) remains normal to the mid-surface after deformation
4) Plate is very thin
5) Vertical strain is ignored

EQUATIONS:

Strong form:

Hooke's law: $\sigma_{ij} = \lambda\Delta u\, \delta_{ij} + \mu\, \chi_{ij}(u)\,,$ i = 1, 2 (1)

Where $\chi_{ij}(u) = u,_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ is the curvature tensor.

Equilibrium equation: $\sigma_{ij},_{ij} = f \qquad in\ \Omega$ (2)

Boundary Conditions: The boundary Γ is partitioned into three parts $\Gamma_i, i = 1,2,3$ depending on the boundary conditions:

$u = \frac{du}{dn} = 0$ on Γ_1 (clamped)

$u = \sigma_{nn} = 0$ on Γ_2 (simply supported)

$\sigma_{nn} = R(\sigma) = 0$ on Γ_3 (free boundary)

VARIATIONAL FORMULATION

The equilibrium equation (2) is multiplied by a test function v and is integrated over Ω where the test function $v \epsilon H^2\ (\Omega)$ satisfies the boundary conditions

$$v = \frac{\partial u}{\partial n} = 0 \quad on\ \ \Gamma_1$$

$$v = 0 \qquad on\ \Gamma_2$$

$\int_\Omega fvdx = \int_\Omega \sigma_{ij,ij}\, vdx = \int_\Gamma \sigma_{ij,j} n_i vds - \int_\Omega \sigma_{ij,j} v_{,i} dx$ (Using Green's formula)

$$= \int_\Gamma \sigma_{ij,j} n_i vds - \int_\Gamma \sigma_{ij} n_j v_{,i} ds + \int_\Omega \sigma_{ij}\chi_{ij}(v)dx \qquad (5)$$

(Using Green's formula on the second term in previous equation)

Since, $v_{,i} = \frac{\partial v}{\partial n} n_i + \frac{\partial v}{\partial t} t_i$

Gives $\sigma_{ij} n_j v_{,i} = \sigma_{nn}\frac{\partial v}{\partial n} + \sigma_{nt}\frac{\partial v}{\partial t}$

Substitute this in equation (5).

$$\int_\Omega \sigma_{ij}\chi_{ij}(v)dx = \int_\Omega fvdx - \int_\Gamma \sigma_{ij,j} n_i vds + \int_\Gamma \sigma_{nn}\frac{\partial v}{\partial n} ds + \int_\Gamma \sigma_{nt}\frac{\partial v}{\partial t} ds \qquad (6)$$

For a smooth boundary $\int_\Gamma \sigma_{nn}\frac{\partial v}{\partial t} ds = -\int_\Gamma \frac{\partial \sigma_{nn}}{\partial t} vds$

Thus equation (6) can be written as

$$\int_\Omega \sigma_{ij}\chi_{ij}(v)dx = \int_\Omega fvdx + \int_\Gamma \sigma_{nn}\frac{\partial v}{\partial n} ds - \int_\Gamma R(\sigma) vds$$

The boundary integrals disappear for the given boundary conditions of the problem

Substituting the value of $\sigma_{ij} = \lambda\Delta u\delta_{ij} + \mu\chi_{ij}(u)$ in the above equation

$$\int_\Omega [\lambda\Delta u\Delta v + \mu\chi_{ij}(u)\chi_{ij}(v)]dx = \int_\Omega fvdx$$

Weak form:

Find $u \in V$ such that

$$a(u,v) = L(v) \qquad \forall\, v \in V$$

where $L(v)$ has the usual definition i.e, $L(v) = \int_\Omega fvdx$

and, $a(u,v) = \int_\Omega [\lambda\Delta u\Delta v + \mu\chi_{ij}(u)\chi_{ij}(v)]dx$

$V = \{v \epsilon H^2(\Omega): v = \frac{\partial v}{\partial n} = 0\ on\ \Gamma_1, v = 0\ on\ \Gamma_2\}$

SOLUTION : PHASE I

In the weak formulation, the order of differentiation of the field variable (u) is two (Δ). So, the trial function used for approximation of the solution in this case is taken to be second order polynomial $P_2(K)$. $P_2(K) = \{ v:v$ is a polynomial of degree ≤ 2 on $K \}$

Therefore, $v(x) = a_{00} + a_{01}x + a_{10}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$

Accordingly the element chosen for discretizing the domain is a triangular element with the values specified at the vertices and midpoint of edges.

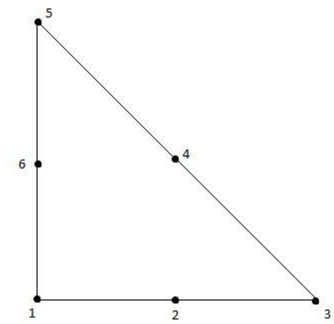


Figure 1

This element has 6 degrees of freedom i.e values at all the specified points which is equal to the dimension of $P_2(K)$.

LOCAL STIFFNESS MATRIX

To find the basis function of v(x) we need to formulate the equation for the surfaces which take the value at the particular nodes and zero at all other nodes.

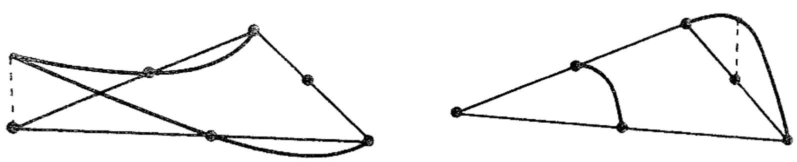


Figure 2

Therefore, the basis functions corresponding to the particular degrees of freedom (i) are mentioned below:

$\phi_1 = 2(x+y-1)(x+y-0.5)$

$\phi_2 = -4x(x+y-1)$

$\phi_3 = 2x(x-0.5)$

$\phi_4 = 4xy$

$\phi_5 = 2y(y-0.5)$

$\phi_6 = -4y(x+y-1)$

v(x) $= \sum_{j=1}^{M} \eta_j \varphi_j(x), \quad \eta_j = v(N_j)$ i.e it takes the value of the trial function at nodes itself.

Substituting the equation of v(x) in the weak form gives

$\sum_j \eta_j a(\varphi_j, \varphi_i)$ = (f,φ_i)

Therefore, the element stiffness matrix can be computed by

$$a(\varphi_i, \varphi_j) = \int_\Omega \left[\lambda \Delta\phi_j \Delta\phi_i + \mu \left(\frac{\partial^2 \varphi_i}{\partial x^2}\, \frac{\partial^2 \varphi_j}{\partial x^2} + 2\, \frac{\partial^2 \varphi_i}{\partial x \partial y}\, \frac{\partial^2 \varphi_j}{\partial x \partial y} + \frac{\partial^2 \varphi_i}{\partial y^2}\, \frac{\partial^2 \varphi_j}{\partial y^2}\right)\right] d\Omega$$

$$\begin{bmatrix} 64, & -64, & 24, & 16, & 24, & -64 \\ -64, & 80, & -32, & -16, & -16, & 48 \\ 24, & -32, & 16, & 0, & 8, & -16 \\ 16, & -16, & 0, & 16, & 0, & -16 \\ 24, & -16, & 8, & 0, & 16, & -32 \\ -64, & 48, & -16, & -16, & -32, & 80 \end{bmatrix}$$

GLOBAL STIFFNESS MATRIX

A rectangular plate of dimensions 3 unit × 1 unit is discretized using the element discussed above into 6 equal parts.

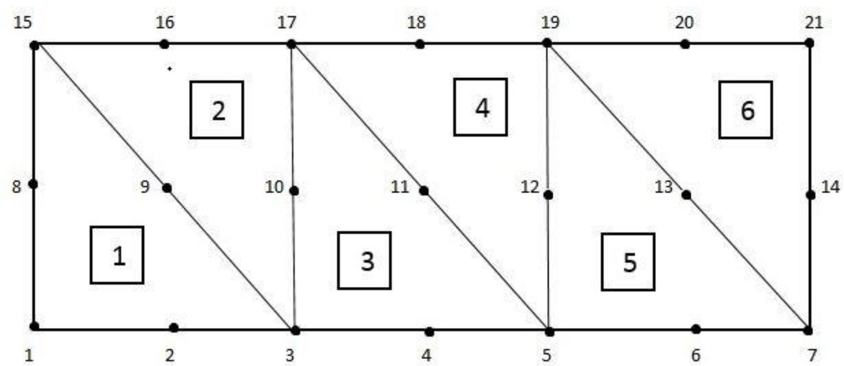


Figure 3

The numbering is shown for the global degrees of freedom. A different method of numbering may be used like the band matrix numbering system. To assemble the global stiffness matrix, the connectivity matrix was calculated to be

$$\begin{bmatrix} 1 & 2 & 3 & 9 & 15 & 8 \\ 17 & 16 & 15 & 9 & 3 & 10 \\ 3 & 4 & 5 & 11 & 17 & 10 \\ 19 & 18 & 17 & 11 & 5 & 12 \\ 5 & 6 & 7 & 13 & 19 & 12 \\ 21 & 20 & 19 & 13 & 7 & 14 \end{bmatrix}$$

The global stiffness matrix computed using the above connectivity matrix is

$$\begin{bmatrix}
64 & -64 & 24 & 0 & 0 & 0 & 0 & -64 & 16 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 \\
-64 & 80 & -32 & 0 & 0 & 0 & 0 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & -32 & 96 & -64 & 24 & 0 & 0 & -16 & 0 & -96 & 16 & 0 & 0 & 0 & 16 & -16 & 48 & 0 & 0 & 0 & 0 \\
0 & 0 & -64 & 80 & -32 & 0 & 0 & 0 & 0 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 \\
0 & 0 & 24 & -32 & 96 & -64 & 24 & 0 & 0 & -16 & 0 & -96 & 16 & 0 & 0 & 0 & 16 & -16 & 48 & 0 & 0 \\
0 & 0 & 0 & 0 & -64 & 80 & -32 & 0 & 0 & 0 & 0 & 48 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 \\
0 & 0 & 0 & 0 & 24 & -32 & 32 & 0 & 0 & 0 & 0 & -16 & 0 & -32 & 0 & 0 & 0 & 0 & 16 & -16 & 24 \\
-64 & 48 & -16 & 0 & 0 & 0 & 0 & 80 & -16 & 0 & 0 & 0 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 16 & 0 & 0 & 0 & 0 \\
0 & 0 & -96 & 48 & -16 & 0 & 0 & 0 & -16 & 160 & -16 & 0 & 0 & 0 & -16 & 48 & -96 & 0 & 0 & 0 & 0 \\
0 & 0 & 16 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 16 & 0 & 0 \\
0 & 0 & 0 & 0 & -96 & 48 & -16 & 0 & 0 & 0 & -16 & 160 & -16 & 0 & 0 & 0 & -16 & 48 & -96 & 0 & 0 \\
0 & 0 & 0 & 0 & 16 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 32 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & -16 & 80 & 0 & 0 & 0 & 0 & -16 & 48 & -64 \\
24 & -16 & 16 & 0 & 0 & 0 & 0 & -32 & 0 & -16 & 0 & 0 & 0 & 0 & 32 & -32 & 24 & 0 & 0 & 0 & 0 \\
0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 48 & 0 & 0 & 0 & 0 & -32 & 80 & -64 & 0 & 0 & 0 & 0 \\
0 & 0 & 48 & -16 & 16 & 0 & 0 & 0 & 16 & -96 & 0 & -16 & 0 & 0 & 24 & -64 & 96 & -32 & 24 & 0 & 0 \\
0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 48 & 0 & 0 & 0 & 0 & -32 & 80 & -64 & 0 & 0 \\
0 & 0 & 0 & 0 & 48 & -16 & 16 & 0 & 0 & 0 & 16 & -96 & 0 & -16 & 0 & 0 & 24 & -64 & 96 & -32 & 24 \\
0 & 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & -16 & 48 & 0 & 0 & 0 & 0 & -32 & 80 & -64 \\
0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 16 & -64 & 0 & 0 & 0 & 0 & 24 & -64 & 64
\end{bmatrix}$$

CASES UNDER CONSIDERATION: In this solution we will be considering the cases where the plate is simply supported at the boundary

I) Point load at the center of the Plate (node 11)

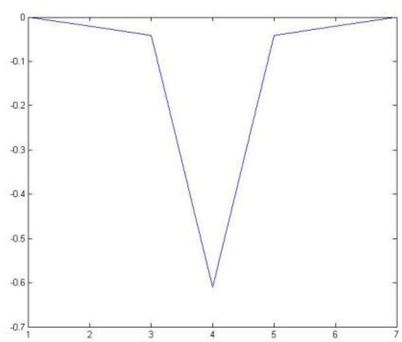


Figure 4

II) Point load at off center (node 10)

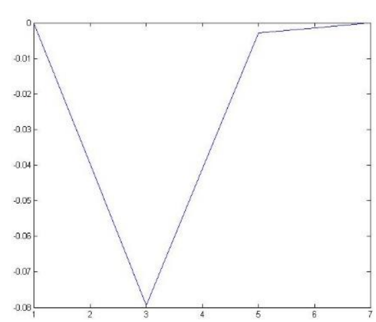


Figure 5

For our first solution, we have considered 6 element mesh for the plate. This solution is not accurate but it explains the graph obtained as the deflection shown is maximum at the node on which load is applied. The accuracy of a solution can be increased by increasing number of node points and elements which represent the mesh such that the solution converges to the actual solution. We will attempt to refine the mesh in the next report.

SOLUTION : PHASE II

In the second phase of the solution, we will consider a generalized uniform mesh and generalized loading and boundary conditions. We must notice that for a boundary condition other than simply supported, the element must support C^1 continuity. The test function $v \in H^2(\Omega)$ satisfies the following boundary conditions

$$v = \frac{\partial v}{\partial n} = 0 \qquad \text{on} \qquad \Gamma_1 \text{ (clamped)}$$

$$v = 0 \qquad \text{on} \qquad \Gamma_2 \text{ (simply supported)}$$

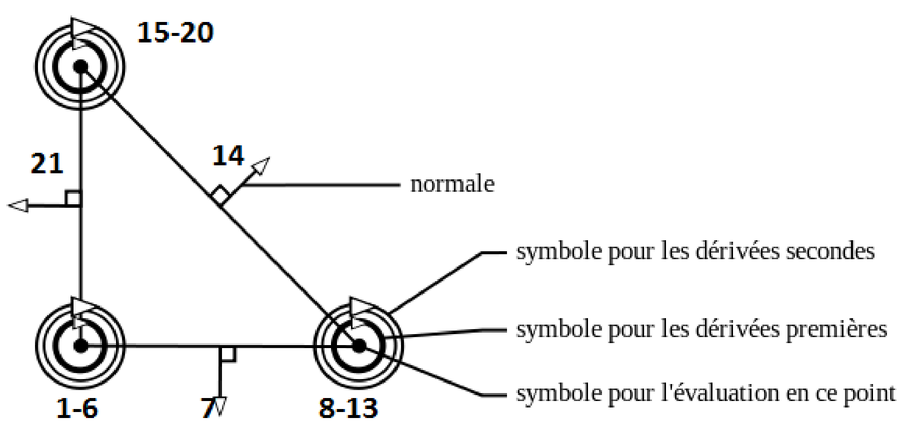
To satisfy the C^1 continuity condition, the finite element space $V_h = \{ v: v|_k \in P_5(K)\}$.

$$v = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{30}x^3 + a_{03}y^3 + a_{12}xy^2 + a_{21}x^2y + a_{40}x^4 + a_{04}y^4 + a_{13}xy^3 + a_{31}x^3y + a_{22}x^2y^2 + a_{50}x^5 + a_{14}xy^4 + a_{41}x^4y + a_{23}x^2y^3 + a_{32}x^3y^2 + a_{05}y^5 .$$

So the degrees of freedom defined for the element is equal to 21. Let K be a triangle with vertices a^i, i = 1,2,3 and the midpoints a^{ij}, i,j = 1,2,3, i<j. The function $v \in P_5(K)$ is uniquely determined by the following degrees of freedom:

$$D^\alpha v(a^i), i = 1,2,3, \ |\alpha| \leq 2,$$

$\frac{\partial v}{\partial n}(a^{ij}), \ i,j = 1,2,3, i < j$ where $\frac{\partial}{\partial n}$ denotes outward normal differentiation on the boundary of K.



his triangle element is known as the Argyris Triangle and has the values of function, the value of the first and the second derivatives specified at vertices and normal derivative specified at midpoints of edges. The numbering [1-6] follows the order :

$1 - v,$ $\quad 2 - \frac{\partial v}{\partial x},$ $\quad 3 - \frac{\partial v}{\partial y},$

$4 - \frac{\partial^2 v}{\partial x^2},$ $\quad 5 - \frac{\partial^2 v}{\partial y^2},$ $\quad 6 - \frac{\partial^2 v}{\partial x \partial y}.$

Similar order is followed for [8-13] and [15-20].

The dof 7, 14 & 21 define outward normal derivatives on the edges $- \frac{\partial v}{\partial n}$

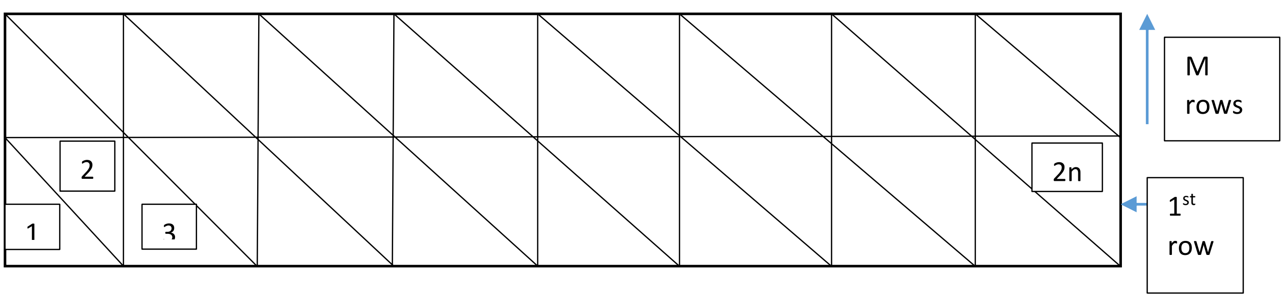
For the 21 degrees of freedom, 21 basis functions are obtained.

```
- 6*x^5 + 15*x^4 + 30*x^3*y^2 - 10*x^3 + 30*x^2*y^3 - 30*x^2*y^2 - 6*y^5 + 15*y^4 - 10*y^3 + 1
- 3*x^5 + 8*x^4 + x^3*y^2 - 6*x^3 - 10*x^2*y^3 + 10*x^2*y^2 - 8*x*y^4 + 18*x*y^3 - 11*x*y^2 + x
- 8*x^4*y - 10*x^3*y^2 + 18*x^3*y + x^2*y^3 + 10*x^2*y^2 - 11*x^2*y - 3*y^5 + 8*y^4 - 6*y^3 + y
- x^5/2 + (3*x^4)/2 + (3*x^3*y^2)/2 - (3*x^3)/2 + x^2*y^3 - (3*x^2*y^2)/2 + x^2/2
x^3*y^2 + (3*x^2*y^3)/2 - (3*x^2*y^2)/2 - y^5/2 + (3*y^4)/2 - (3*y^3)/2 + y^2/2
- 2*x^4*y - 6*x^3*y^2 + 5*x^3*y - 6*x^2*y^3 + 10*x^2*y^2 - 4*x^2*y - 2*x*y^4 + 5*x*y^3 - 4*x*y^2 + x*y
- 16*x^4*y - 32*x^3*y^2 + 32*x^3*y - 16*x^2*y^3 + 32*x^2*y^2 - 16*x^2*y
6*x^5 - 15*x^4 - 15*x^3*y^2 + 10*x^3 - 15*x^2*y^3 + 15*x^2*y^2
- 3*x^5 + 7*x^4 + (7*x^3*y^2)/2 - 4*x^3 + (7*x^2*y^3)/2 - (7*x^2*y^2)/2
- 8*x^4*y - (37*x^3*y^2)/2 + 14*x^3*y - (27*x^2*y^3)/2 + (37*x^2*y^2)/2 - 5*x^2*y
x^5/2 - x^4 - (x^3*y^2)/4 + x^3/2 - (x^2*y^3)/4 + (x^2*y^2)/4
- (3*x^3*y^2)/4 - (5*x^2*y^3)/4 + (5*x^2*y^2)/4
2*x^4*y + (7*x^3*y^2)/2 - 3*x^3*y + (5*x^2*y^3)/2 - (7*x^2*y^2)/2 + x^2*y
8*2^(1/2)*x^3*y^2 + 8*2^(1/2)*x^2*y^3 - 8*2^(1/2)*x^2*y^2
- 15*x^3*y^2 - 15*x^2*y^3 + 15*x^2*y^2 + 6*y^5 - 15*y^4 + 10*y^3
- (27*x^3*y^2)/2 - (37*x^2*y^3)/2 + (37*x^2*y^2)/2 - 8*x*y^4 + 14*x*y^3 - 5*x*y^2
(7*x^3*y^2)/2 + (7*x^2*y^3)/2 - (7*x^2*y^2)/2 - 3*y^5 + 7*y^4 - 4*y^3
- (5*x^3*y^2)/4 - (3*x^2*y^3)/4 + (5*x^2*y^2)/4
- (x^3*y^2)/4 - (x^2*y^3)/4 + (x^2*y^2)/4 + y^5/2 - y^4 + y^3/2
(5*x^3*y^2)/2 + (7*x^2*y^3)/2 - (7*x^2*y^2)/2 + 2*x*y^4 - 3*x*y^3 + x*y^2
- 16*x^3*y^2 - 32*x^2*y^3 + 32*x^2*y^2 - 16*x*y^4 + 32*x*y^3 - 16*x*y^2
```

The element stiffness matrix is obtained as a 21×21 matrix.

MESHING

The plate is meshed using parameters m and n such that the total number of elements is 2mn.



CONNECTIVITY MATRIX

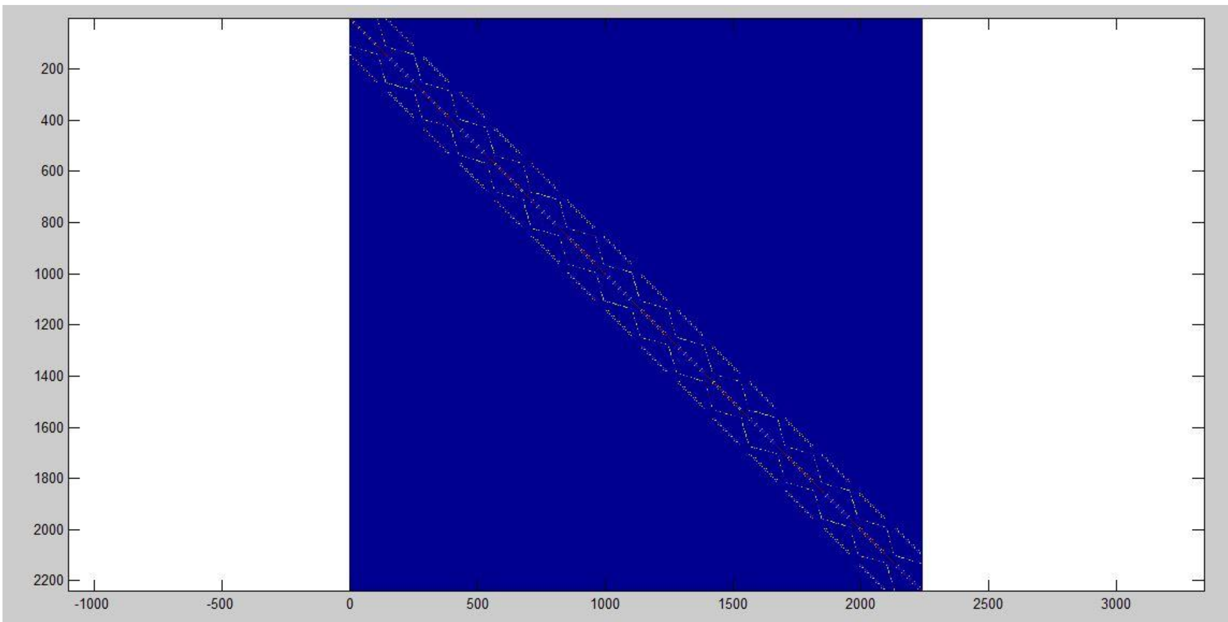
For demonstration we have taken m = n =15. The connectivity matrix has the dimension on [$2mn \times 21$] where 2mn is the no. of elements and 21 is the degrees of freedom on an element.

So, for m = n = 15, the dimensions of the connectivity matrix is 450 × 21.

1	2	3	4	5	6	7	8	9	10	11	12	13	113
150	151	152	153	154	155	149	143	144	145	146	147	148	113
8	9	10	11	12	13	14	15	16	17	18	19	20	115
157	158	159	160	161	162	156	150	151	152	153	154	155	115
.													
.													
.													
.													
.													
2236	2237	2238	2239	2240	2241	2235	2229	2230	2231	2232	2233	2234	2129

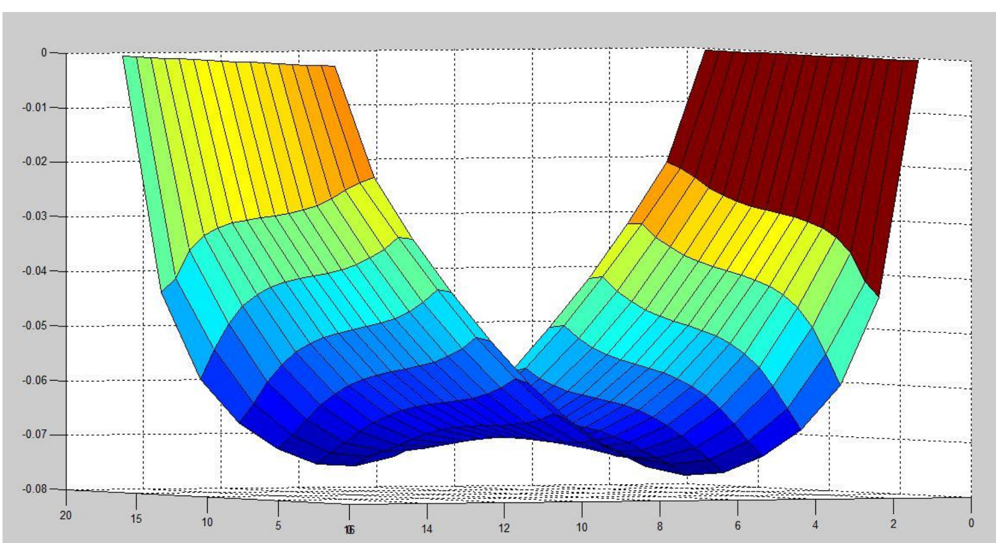
GLOBAL STIFFNESS MATRIX

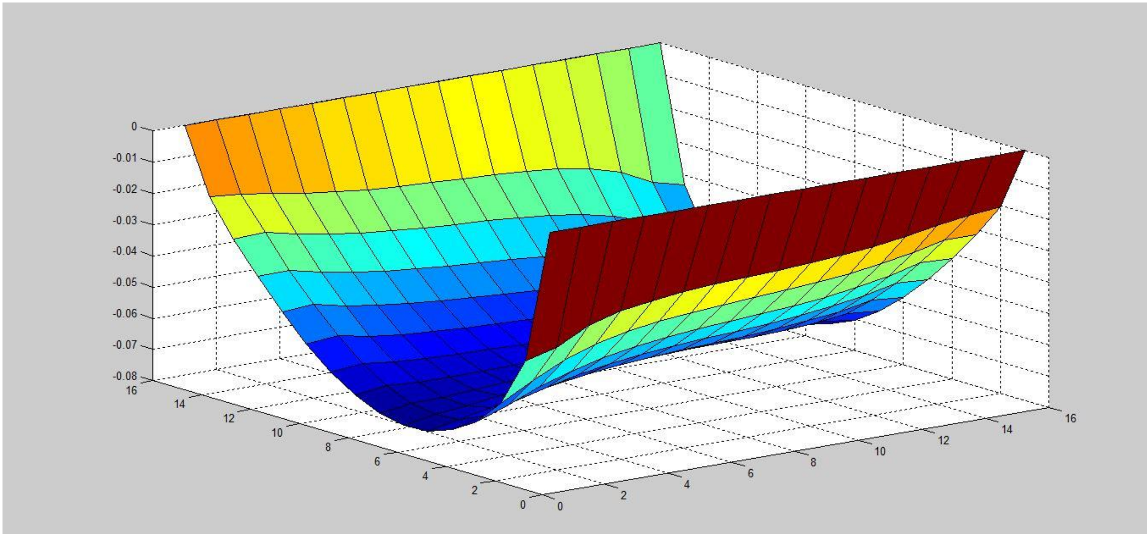
The Global Stiffness Matrix is a sparse matrix with values contained in three diagonal chunks. The dimension of the matrix is 2241 × 2241. The Heat Map is shown below.



SOLUTION

Uniform loading is applied on the plate. The solution was obtained for two simply supported edges.





The maximum deflection obtained is 0.075 units.

CONCLUSION

The solution is expected to be symmetric for symmetric boundary conditions. Unfortunately, our results do not match the expected output. We did obtain the maximum deflection in the middle of the plate. We are working towards obtaining a more accurate solution. More details will be shared during the presentation.