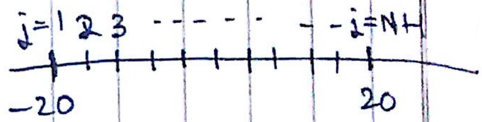
$$u_t + 6uu_x + u_{xxx} = 0 \quad , -20 < x < 20 \, , t > 0$$

IC : $u(x,0) = A\,\text{sech}^2(kx)$

BC : $u(-20,t) = u(20,t)$

Writing the PDE in its conservation form :



$$u_t + 6\left(\frac{u^2}{2}\right)_x + u_{xxx} = 0$$

$$\Rightarrow u_t + 6\,[f(u)]_x + u_{xxx} = 0$$

Considering backward in time & central in space discretization :

$$\therefore \; u_t = \frac{U_j^{n+1} - U_j^n}{\Delta t} \qquad [f(u)]_x = \frac{f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})}{2h}$$

$$u_{xxx} = \frac{-\frac{1}{2}U_{j-2}^{n+1} + U_{j-1}^{n+1} - U_{j+1}^{n+1} + \frac{1}{2}U_{j+2}^{n+1}}{h^3}$$

$\Rightarrow$ The scheme becomes:

$$\frac{U_j^{n+1}-U_j^{n}}{\Delta t} + \frac{6}{2h}\left[f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})\right] + \frac{1}{h^3}\left[-\frac{1}{2}U_{j-2}^{n+1} + U_{j-1}^{n+1} - U_{j+1}^{n+1} + \frac{1}{2}U_{j+2}^{n+1}\right] = 0$$

$\rightarrow$ Giving us a non-linear system of equations [as f is non linear]

So, by using Newton's method: $U^{n+1} = U^{n} - J^{-1}F$ $\quad$ J: Jacobian.

Now, writing:

$$F(j) = U_j^{n+1} - U_j^{n} + \frac{3\Delta t}{h}\left[f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})\right] + \frac{\Delta t}{h^3}\left[-\frac{1}{2}U_{j-2}^{n+1} + U_{j-1}^{n+1} - U_{j+1}^{n+1} + \frac{1}{2}U_{j+2}^{n+1}\right] = 0$$

let $\boxed{\mu = \frac{3\Delta t}{h}}$, $\boxed{\tau = \frac{\Delta t}{h^3}}$

At $j=1$:

$$f(1) = U_1^{n+1} - U_1^{n} + \mu\left[f(U_2^{n+1}) - f(\overset{U_N^{n+1}}{\cancel{U_0^{n+1}}})\right] + \tau\left[-\frac{1}{2}\overset{U_{N-1}^{n+1}}{\cancel{U_{-1}^{n+1}}} + \overset{U_N^{n+1}}{\cancel{U_0^{n+1}}} - U_2^{n+1} + \frac{1}{2}U_3^{n+1}\right] = 0$$

At $j=2$:

$$F(2) = U_2^{n+1} - U_2^{n} + \mu\left[f(U_3^{n+1}) - f(U_1^{n+1})\right] + \tau\left[-\frac{1}{2}U_N^{n+1} + U_1^{n+1} - U_3^{n+1} + \frac{1}{2}U_4^{n+1}\right] = 0$$

For $j=3:N-1$

$$F(j) = U_j^{n+1} - U_j^{n} + \mu\left[f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})\right] + \tau\left[-\frac{1}{2}U_{j-2}^{n+1} + U_{j-1}^{n+1} - U_{j+1}^{n+1} + \frac{1}{2}U_{j+2}^{n+1}\right] = 0$$

end.

At $j=N$:

$$F(N) = U_N^{n+1} - U_N^{n} + \mu\left[f(U_{N+1}^{n+1}) - f(U_{N-1}^{n+1})\right] + \tau\left[-\frac{1}{2}U_{N-2}^{n+1} + U_{N-1}^{n+1} - U_{N+1}^{n+1} + \frac{1}{2}U_2^{n+1}\right] = 0$$

At $j=N+1$:

$$F(N+1) = U_{N+1}^{n+1} - U_{N+1}^{n} + \mu\left[f(U_2^{n+1}) - f(U_N^{n+1})\right] + \tau\left[-\frac{1}{2}U_{N-1}^{n+1} + U_N^{n+1} - U_2^{n+1} + \frac{1}{2}U_3^{n+1}\right] = 0$$

$$
J = \begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & \cdots & N-2 & N+1 & \cancel{N-1}\,N & \cancel{N+1}\,N+1 \\
\hline
i=1 & 1 & \mu f'(U_2^{n+1})-\tau & \frac{\tau}{2} & 0 & \cdots & 0 & -\frac{\tau}{2} & -\mu f'(U_N^{n+1})+\tau & 0 \\
2 & -\mu f'(U_1^{n+1})+\tau & 1 & \mu f'(U_3^{n+1})-\tau & \frac{\tau}{2} & 0 \cdots & & 0 & \cancel{-\frac{\tau}{2}}\ -\frac{\tau}{2} & 0 \\
\cancel{3}\ 3 & -\tau/2 & -\mu f'(U_2^{n+1})+\tau & 1 & \mu f'(U_4^{n+1})-\tau & \frac{\tau}{2} \cdots & & 0 & 0 & 0 \\
4 & 1 & 2 & 3 & 4 & 5 & & N-1 & N & N+1 \\
\vdots & & & & & & & & & \\
N-\cancel{3} & 1 & 2 & 3 & & & N-2 & N-1 & N & N+1 \\
N & 0 & \tau/2 & 0 & \cdots & 0 & -\tau/2 & -\mu f'(U_{N-1}^{n+1})+\tau & 1 & \mu f'(U_{N+1}^{n+1})-\tau \\
N+1 & 0 & \mu f'(U_2)-\tau & \tau/2 & \cdots & 0 & 0 & -\tau/2 & -\mu f'(U_N)+\tau & 1
\end{array}
$$