CRANK NICOLSON SCHEME:-

$$u_t + 6uu_x + u_{xxx} = 0 \quad , \quad -20 < x < 20 \, , \; t > 0$$

IC: $u(x,0) = A\,\mathrm{sech}^2(kx)$

BC: $u(-20,t) = u(20,t)$

Writing out the PDE in its conservation form:-

$$u_t + 6\left(\frac{u^2}{2}\right)_x + u_{xxx} = 0$$

$$\Rightarrow u_t + 6\,(f(u))_x + u_{xxx} = 0 \quad , \quad f(u) = \frac{u^2}{2}$$

Now, in Crank Nicolson scheme,

$$u_t = \frac{U_j^{n+1} - U_j^n}{\Delta t}$$

$$(f(u))_x = \frac{1}{2}\left(\frac{f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})}{2h} + \frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2h}\right)$$

$$= \frac{1}{4h}\left(f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1}) + f(U_{j+1}^n) - f(U_{j-1}^n)\right)$$

$$u_{xxx} = \frac{1}{4h^3}\left(U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - U_{j-2}^{n+1} + U_{j+2}^n - 2U_{j+1}^n + 2U_j{}_{-1}^n - U_{j-2}^n\right)$$

$\Rightarrow$ The scheme becomes.

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{6}{4h}\Big(f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1})\Big) + f(U_{j+1}^n) + f(U_{j-1}^n)\Big)\Big)$$
$$+ \frac{1}{4h^3}\left(U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - U_{j-2}^{n+1} + U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n\right)$$
$$= 0$$

Giving us a non-Linear system of equation :- So using Newton Raphson Method.

$$U^{n+1} = U^n - J^{-1} F \qquad J : \text{Jacobian of "F"}$$

Now, writing :-

$$F(j) = U_j^{n+1} - U_j^n + \frac{3\Delta t}{2h}\left(f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1}) + f(U_{j+1}^n) - f(U_{j-1}^n) \right)$$
$$+ \frac{\Delta t}{4h^3}\left(\begin{array}{l} U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - U_{j-2}^{n+1} + \\ U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n \end{array} \right)$$

$$\boxed{\mu = \frac{3\Delta t}{2h}} \qquad \boxed{\tau = \frac{\Delta t}{4h^3}}$$

j=1

$$F(1) = U_1^{n+1} - U_1^n + \mu\left(f(U_2^{n+1}) - f(\overset{U_N^{n+1}}{\cancel{U_0^{n+1}}}) + f(U_2^n) - f(\overset{U_N^n}{\cancel{U_0^n}}) \right)$$
$$+ \tau\left(\begin{array}{l} U_3^{n+1} - 2U_2^{n+1} + 2U_N^{n+1} - U_{N-1}^{n+1} - \\ U_3^n - 2U_2^n + 2U_N^n - U_{N-1}^n \end{array} \right)$$

j=2

$$F(2) = U_2^{n+1} - U_2^n + \mu\left(f(U_3^{n+1}) - f(U_1^{n+1}) + f(U_2^n) - f(U_1^n) \right)$$
$$+ \tau\left(U_4^{n+1} - 2U_3^{n+1} + 2U_1^{n+1} - U_N^{n+1} + U_4^n - 2U_3^n + 2U_1^n - U_N^n \right)$$

for j=3 : N-1

$$F(j) = U_j^{n+1} - U_j^n + \mu\left(f(U_{j+1}^{n+1}) - f(U_{j-1}^{n+1}) + f(U_{j+1}^n) - f(U_{j-1}^n) \right)$$
$$+ \tau\left(\begin{array}{l} U_{j+2}^{n+1} - 2U_{j+1}^{n+1} + 2U_{j-1}^{n+1} - U_{j-2}^{n+1} + \\ U_{j+2}^n - 2U_{j+1}^n + 2U_{j-1}^n - U_{j-2}^n \end{array} \right)$$

end

j = N

$$F(N) = U_N^{n+1} - U_N^n + \mu\left(f(U_{N+1}^{n+1}) - f(U_{N-1}^{n+1}) + f(U_{N+1}^n) - f(U_{N-1}^n) \right)$$
$$+ \tau\left(\begin{array}{l} \underset{U_2^{n+1}}{\cancel{U_{N+2}^{n+1}}} - 2U_{N+1}^{n+1} + 2U_{N-1}^{n+1} - U_{N-2}^{n+1} + \\ \underset{U_2^n}{\cancel{U_{N+2}^n}} - 2U_{N+1}^n + 2U_{N-1}^n - U_{N-2}^n \end{array} \right)$$

j=N+1

$$F(N+1) = U_{N+1}^{n+1} - U_{N+1}^n + \mu\left(f(U_2^{n+1}) - f(U_N^{n+1}) + f(U_2^n) - f(U_N^n) \right)$$
$$+ \tau\left(\begin{array}{l} U_3^{n+1} - 2U_2^{n+1} + 2U_N^{n+1} - U_{N-1}^{n+1} \\ + U_3^n - 2U_2^n + 2U_N^n - U_{N-1}^n \end{array} \right)$$

$$J = \left[\begin{array}{cccccccc}
1 & \mu f'(U_2^{n+1})-2\tau & \tau & \cdots & \cdots & -\tau & \left(2\tau - \mu f'(U_N^{n+1})\right) & 0 \\
-\mu f'(U_1^{n+1})+2\tau & 1 & \mu f'(U_3^{n+1})-2\tau & \tau & \cdots & 0 & -\tau & 0 \\
-\tau & -\mu f'(U_2^{n+1})+2\tau & 1 & \mu f'(U_4^{n+1})-2\tau & \tau \cdots & 0 & 0 & 0 \\
① & ② & ③ & ④ & ⑤ & \textcircled{N-1} & \textcircled{N} & \textcircled{N+1} \\
 & & & & & & & \tau \\
① & ② & ③ & & \textcircled{N-2} & \textcircled{N-1} & \textcircled{N} & \textcircled{N+1} \\
0 & \tau & 0 & \cdots & -\tau & -\mu f'(U_{N-1}^{n+1})+2\tau & 1 & \mu f'(U +2\tau \\
① & ② & ③ & & \textcircled{N-2} & \textcircled{N-1} & \textcircled{N} & \textcircled{N+1} \\
0 & -2\tau+\mu f'(U_2^{n+1}) & \tau & \cdots & 0 & -\tau & -\mu f'(U_N)+2\tau & 1
\end{array}\right.$$

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