ZABRUSKY KRUSKAL FOR KdV.

Consider the following IBVP:-

$$u_t + 6uu_x + u_{xxx} = 0 \quad , \quad -20 < x < 20 \; , \; t > 0$$

IC: $u(x,0) = A \operatorname{sech}^2(kx)$, $k = \sqrt{\frac{A}{2}}$ and

BC: $u(-20,t) = u(20,t)$

(Periodic).

t

x_{-1} x_0 x_1 x_2 . . . . . . x_k . . . x_{N-1} x_N x_{N+1} x_{N+2} x_{N+3}

$(u_1$ u_2 . . . . . u_k . . . u_{N-1} u_N $u_{N+1})$

With the periodic boundary conditions, we have

$u_0 = u_N$, $u_{-1} = u_{N-1}$

$u_2 = u_{N+2}$ $u_3 = u_{N+3}$

The Zabrusky Kruskal scheme is given by

$$\underbrace{u_n^{m+1} = u_n^{m-1}} - \frac{\cancel{6}^{2}}{\cancel{3}} \frac{\Delta t}{\Delta x} (u_{n+1}^m + u_n^m + u_{n-1}^m)(u_{n+1}^m - u_{n-1}^m)$$

[Centered on m] $$- \frac{\Delta t}{(\Delta x)^3} (u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m)$$

where.

$u_n^m = u(n\Delta x, m\Delta t)$

As we have three levels in time, we use the uncentered scheme to find the first level

$$u_n^1 = u_n^0 - \frac{\cancel{2}\Delta t}{\cancel{2}\Delta x} (u_{n+1}^0 + u_n^0 + u_{n-1}^0)(u_{n+1}^0 - u_{n-1}^0)$$

$$- \frac{\Delta t}{2(\Delta x)^3} (u_{n+2}^0 - 2u_{n+1}^0 + 2u_{n-1}^0 - u_{n-2}^0)$$

For the initial time step, due to the periodic BCs

n=1

$$u_1^1 = u_1^0 - \frac{\Delta t}{\Delta x}\left(u_2^0 + u_1^0 + \cancel{u_0^0} \to u_N^0\right)\left(u_2^0 - \cancel{u_0^0} \to u_N^0\right) - \frac{\Delta t}{2(\Delta x)^3}\left(u_3^0 - 2u_2^0 + 2\cancel{u_0^0} \to u_N^0 - \cancel{u_{-1}^0} \to u_{N-1}^0\right)$$

n=2

$$u_2^1 = u_2^0 - \frac{\Delta t}{\Delta x}\left(u_3^0 + u_2^0 + u_1^0\right)\left(u_3^0 - u_1^0\right) - \frac{\Delta t}{2(\Delta x)^3}\left(u_4^0 - 2u_3^0 + 2u_1^0 - \overset{u_N^0}{\cancel{u_0^0}}\right)$$

for $n = 3 : N-1$

$$u_n^1 = u_n^0 - \frac{\Delta t}{\Delta x}\left(u_{n+1}^0 + u_n^0 + u_{n-1}^0\right)\left(u_{n+1}^0 - u_{n-1}^0\right) - \frac{\Delta t}{2(\Delta x)^3}\left(u_{n+2}^0 - 2u_{n+1}^0 + 2u_{n-1}^0 + u_{n-2}^0\right)$$

end

for $n=N$

$$u_N^1 = u_N^0 - \frac{\Delta t}{\Delta x}\left(u_{N+1}^0 + u_N^0 + u_{N-1}^0\right)\left(u_{N+1}^0 - u_{N-1}^0\right) - \frac{\Delta t}{2(\Delta x)^3}\left(\cancel{u_{N+2}^0} \to u_2^0 - 2u_{N+1}^0 + 2u_{N-1}^0 - u_{N-2}^0\right)$$

$n = N+1$

$$u_{N+1}^1 = u_{N+1}^0 - \frac{\Delta t}{\Delta x}\left(u_2^0 + u_{N+1}^0 + u_N^0\right)\left(u_2^0 - u_N^0\right) - \frac{\Delta t}{2(\Delta x)^3}\left(\cancel{u_{N+3}^0} \to u_3^0 - 2\cancel{u_{N+2}} \to u_2^0 + 2u_N^0 - u_{N-1}^0\right)$$

For subsequent time steps, we use the centered scheme

for $m = 2 : M$

n=1

$$u_1^{m+1} = u_1^{m-1} - \frac{2\Delta t}{\Delta x}\left(u_2^m + u_1^m + u_N^m\right)\left(u_2^m - u_N^m\right) - \frac{\Delta t}{(\Delta x)^3}\left(u_3^m - 2u_2^m + 2u_N^m - u_{N-1}^m\right)$$

n=2

$$u_2^{m+1} = u_2^{m-1} - \frac{2\Delta t}{\Delta x}\left(u_3^m + u_2^m + u_1^m\right)\left(u_3^m - u_1^m\right) - \cancel{\frac{\Delta t}{2(\Delta x)^3}}$$

$$- \frac{\Delta t}{(\Delta x)^3}\left(u_4^m - 2u_3^m + 2u_1^m - u_N^m\right)$$

for $n = 3 : N-1$

$$u_n^{m+1} = u_n^{m-1} - \frac{2\Delta t}{\Delta x}\left(u_{n+1}^m - u_n^m + u_{n-1}^m\right)\left(u_{n+1}^m - u_{n-1}^m\right)$$

$$- \frac{\Delta t}{(\Delta x)^3}\left(u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m\right)$$

end

$n = N$

$$u_N^{m+1} = u_N^{m-1} - \frac{2\Delta t}{\Delta x}\left(u_{N+1}^m - u_N^m + u_{N-1}^m\right)\left(u_{N+1}^m - u_{N-1}^m\right)$$

$$- \frac{\Delta t}{(\Delta x)^3}\left(\overset{u_2^m}{\cancel{u_{N+2}^m}} - 2u_{N+1}^m + 2u_{N-1}^m - u_{N-2}^m\right)$$

$n = N+1$

$$u_{N+1}^{m+1} = u_{N+1}^{m-1} - \frac{2\Delta t}{\Delta x}\left(u_2^m + u_{N+1}^m + u_N^m\right)\left(u_2^m - u_N^m\right)$$

$$- \frac{\Delta t}{(\Delta x)^3}\left(u_3^m - 2u_2^m + 2u_N^m - u_{N-1}^m\right)$$