UNIT-IV

DEFINITE INTEGRAL AND ITS PROPERTIES

CHAPTER-4 *Definite Integral and its Properties*

CHAPTER-4

4.1 INTRODUCTION

In the previous sections, we studied about the indefinite integrals and discussed methods of finding them including integrals of some special functions. In this section, shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)dx$, where 'a' is called the lower limit and is called the upper limit of the integral. If F be the anti derivative of 'f' in the interval then its value is the difference between the values of F at the end points, i.e. F(b) – F(a)

4.2 DEFINITION

Let F(x) be the primitive or anti-derivative of a function f(x) defined on [a,b] $\frac{d}{dx}(F(x)) = f(x)$. Then the definite integral of f(x) over [a, b] is denoted by $\int_a^b f(x)dx$ and is defined as F(b) – F(a). i.e.,

$$\int_a^b f(x)dx = [F(x)]_{x=a}^{x=b} = F(b) - F(a)$$

This is also called Newton-Leibnitz formula.

Here the numbers a and b are called the limits of integration a is called the lower limit and b is called the upper limit.

NOTE

To evaluate the definite integral there is no need to keep the constant of integration, because if $\int f(x)dx = F(x) + c$, then

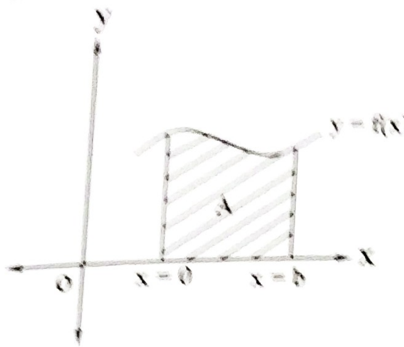
$$\int_a^b f(x)dx = [F(x)+c]_a^b = (F(b)+c) - (F(a)+c)$$
$$= F(b) - F(a)$$

Working Rule to Evaluate $\int_a^b f(x)dx$

Evaluate the indefinite integral $\int f(x)dx$ as usual. Then take difference between function values at upper and lower limits.

4.3 GEOMETRICAL REPRESENTATION OF DEFINITE INTEGRAL

The definite integral $\int_a^b f(x)dx$ represents the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.



FIG

4.4 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f(x) be a continuous function defined on [a, b] and F(x) be an anti-derivative of f(x) then

$$\boxed{\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)}$$

SOLVED EXAMPLES

EXAMPLE-1

Evaluate

(i) $\int_{-1}^{2} x\,dx$

July 2021

(ii) $\int_0^1 (x^2 + 1)dx$

Oct. 2015, 2013

(iii) $\int_1^2 (x-1)(x+2)dx$ (or) $\int_1^2 (x^2 + x - 2)dx$

Apr. 2013

(iv) $\int_0^1 (2x+3)^2 dx$

Apr. 2018 ; Oct. 2012

Solution :

(i) $$\int_{-1}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{2} = \frac{2^2}{2} - \frac{(-1)^2}{2}$$

$$= \frac{4}{2} - \frac{1}{2}$$

$$= \frac{3}{2}$$

(ii) $$\int_{0}^{1}(x^2+1)dx = \left[\frac{x^3}{3}+x\right]_{x=0}^{1}$$

$$= \left(\frac{1^3}{3}+1\right) - \left(\frac{0^3}{3}+0\right)$$

$$= \frac{4}{3} - 0$$

$$= \frac{4}{3}$$

(iii) $$\int_{1}^{2}(x-1)(x+2)dx = \int_{1}^{2}(x^2+x-2)dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{2}-2x\right]_{1}^{2}$$

$$= \left[\frac{2^3}{3}+\frac{2^2}{2}-2(2)\right] - \left[\frac{1^3}{3}+\frac{1^2}{2}-2(1)\right]$$

$$= \left(\frac{8}{3}+2-4\right) - \left(\frac{1}{3}+\frac{1}{2}-2\right)$$

$$= \left(\frac{8}{3}-\frac{2}{1}\right) - \left(\frac{2+3-12}{6}\right)$$

$$= \frac{2}{3}+\frac{7}{6}$$

$$= \frac{11}{6}$$

(iv) $$\int_0^1 (2x+3)^2\,dx = \frac{1}{2}\left[\frac{(2x+3)^3}{3}\right]_0^1$$

$$= \frac{1}{6}[(2(1)+3)^3 - (2\cdot(0)+3)^3]$$

$$= \frac{1}{6}[5^3 - 3^3]$$

$$= \frac{98}{6}$$

$$= \frac{49}{3}$$

EXAMPLE-2

Evaluate

(i) $\displaystyle\int_{-1}^{1} \frac{1}{x+2}\,dx$ **Apr. 2011 ; Oct. 2017**

(ii) $\displaystyle\int_0^1 e^{2x+3}\,dx$ **Oct. 2017**

(iii) $\displaystyle\int_0^{\pi/2} \cos x\,dx$ **Oct. 2018 ; Apr. 2019**

(iv) $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$ **Oct. 2017**

Solution :

(i) $$\int_{-1}^{1} \frac{1}{x+2}\,dx = [\log|x+2|]_{-1}^{1}$$

$$= \log(1+2) - \log(-1+2)$$

$$= \log 3 - \log 1$$

$$= \log 3 \qquad [\because \log 1 = 0]$$

(ii) $\int_0^1 e^{2x+3}dx = \left[\frac{e^{2x+3}}{2}\right]_{x=0}^{1}$

$$= \frac{e^{2(1)+3}}{2} - \frac{e^{2(0)+3}}{2}$$

$$= \frac{1}{2}[e^5 - e^3]$$

(iii) $\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2}$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

(iv) $\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2} dx$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - 0\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

EXAMPLE-3

Evaluate

(i) $\int_0^{\pi/2} \sqrt{1 - \sin 2x}\, dx$ Apr. 2018

(ii) $\int_0^{\pi/2} \sqrt{1 + \cos 2x}\, dx$

(iii) $\int_0^{\frac{\pi}{2}} \sqrt{1 + \sin x}\, dx$

Solution :

(i) $$\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx = \int_0^{\pi/2} \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx$$

$$= \int_0^{\pi/2} \sqrt{(\sin x - \cos x)^2}\,dx$$

$$= \pm\int_0^{\pi/2} (\sin x - \cos x)dx$$

$$= \pm[-\cos x - \sin x]_0^{\pi/2}$$

$$= \mp[\cos x + \sin x]_0^{\pi/2}$$

$$= \pm\left[\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (\cos 0 + \sin 0)\right]$$

$$= \pm[(0+1)-(1+0)]$$

$$= 0$$

(ii) $$\int_0^{\pi/2} \sqrt{1+\cos 2x}\,dx = \int_0^{\pi/2} \sqrt{2\cos^2 x}\,dx \qquad [\because 1+\cos 2A = 2\cos^2 A]$$

$$= \sqrt{2}\cdot\int_0^{\pi/2} \cos x\,dx$$

$$= \sqrt{2}[\sin x]_0^{\pi/2}$$

$$= \sqrt{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \sqrt{2}[1-0]$$

$$= \sqrt{2}$$

(iii) $$\int_0^{\pi/2} \sqrt{1+\sin x}\,dx = \int_0^{\pi/2} \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$\left[\because \sin^2\theta + \cos^2\theta = 1 \;\&\; \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$$

$$= \int_0^{\pi/2} \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}\,dx$$

$$= \pm \int_0^{\pi/2} \left(\sin\frac{x}{2} + \cos\frac{x}{2} \right) dx$$

$$= \pm \left[-\frac{\cos\frac{x}{2}}{\frac{1}{2}} + \frac{\sin\frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2}$$

$$= \pm 2\left[\sin\frac{x}{2} - \cos\frac{x}{2} \right]_0^{\pi/2}$$

$$= \pm 2\left[\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4} \right) - \left(\sin\frac{0}{2} - \cos\frac{0}{2} \right) \right]$$

$$= \pm 2\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - (0-1) \right]$$

$$= \pm 2$$

EXAMPLE-4

Evaluate

(i) $\int_0^{\pi/2} \cos^3 x \, dx$

(ii) $\int_0^{\pi/2} \sin^4 x \, dx$

Solution :

(i) We know that

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\Rightarrow \quad 4\cos^3 A = 3\cos A + \cos 3A$$

$$\Rightarrow \quad \cos^3 A = \frac{1}{4}[3\cos A + \cos 3A]$$

$$\therefore \quad \int_0^{\pi/2} \cos^3 x \, dx = \int_0^{\pi/2} \frac{1}{4}[3\cos x + \cos 3x] dx$$

$$= \frac{1}{4}\left[3\sin x + \frac{\sin 3x}{3} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\left(3\sin\frac{\pi}{2} + \frac{\sin 3\frac{\pi}{2}}{3}\right) - \left(3\sin 0 + \frac{\sin 3(0)}{3}\right)\right]$$

$$= \frac{1}{4}\left[\left(3(1) + \frac{(-1)}{3}\right) - 0\right]$$

$$= \frac{1}{4}\left[3 - \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{9-1}{3}\right]$$

$$= \frac{1}{\cancel{4}}\left[\frac{\cancel{8}^{2}}{3}\right]$$

$$= \frac{2}{3}$$

(ii) Consider

$$\sin^4 x = (\sin^2 x)^2$$

$$= \left[\frac{1-\cos 2x}{2}\right]^2$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{2 - 4\cos 2x + 1 + \cos 4x}{2}\right]$$

$$= \frac{1}{8}\left[3 - 4\cos 2x + \cos 4x\right]$$

$$\therefore \int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \frac{1}{8}[3 - 4\cos 2x + \cos 4x]\,dx$$

$$= \frac{1}{8}\left[3x - \frac{\overset{2}{\cancel{4}}\sin 2x}{\cancel{2}} + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left\{\left[3\frac{\pi}{2} - 2\sin 2\left(\frac{\pi}{2}\right) + \frac{\sin \cancel{4}^{2}\left(\frac{\pi}{\cancel{2}}\right)}{4}\right] - \left[3(0) - 2\sin(0) + \frac{\sin 0}{4}\right]\right\}$$

$$= \frac{1}{8}\left[\left(\frac{3\pi}{2} - 0 + 0\right) - (0 - 0 + 0)\right] \qquad [\because \sin\pi = \sin 2\pi = 0]$$

$$= \frac{3\pi}{16}$$

EXAMPLE-5

Evaluate

(i) $\int_0^{\pi} \sin\frac{x}{2}\cos\frac{x}{2}\, dx$

(ii) $\int_0^{\pi/2} \sin 5x \cos 3x\, dx$

Solution : **(Oct. 2015)**

(i) $$\int_0^{\pi} \sin\frac{x}{2}\cos\frac{x}{2}dx = \frac{1}{2}\int_0^{\pi} 2\sin\frac{x}{2}\cos\frac{x}{2}dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin x\, dx \qquad \left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$$

$$= \frac{1}{2}[-\cos x]_0^{\pi}$$

$$= \frac{1}{2}[-\cos\pi + \cos 0]$$

$$= \frac{1}{2}(-(-1)+1) \qquad [\because \cos\pi = -1 \text{ and } \cos 0 = 1]$$

$$= \frac{1}{2}(1+1)$$

$$= 1$$

(ii) $$\int_0^{\pi/2} \sin 5x \cos 3x\, dx = \frac{1}{2}\int_0^{\pi/2} 2\sin 5x \cos 3x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\sin(5x+3x) + \sin(5x-3x)]dx$$

$$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{2}\left[\left(\frac{\cos 8\left(\frac{\pi}{2}\right)}{8} + \frac{\cos 2\left(\frac{\pi}{2}\right)}{2}\right) - \left(\frac{\cos 0}{8} + \frac{\cos 0}{2}\right)\right]$$

$$= -\frac{1}{2}\left[\left(\frac{\cos 4\pi}{8} + \frac{\cos \pi}{2}\right) - \left(\frac{1}{8} + \frac{1}{2}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{1}{8} - \frac{1}{2} - \frac{1}{8} - \frac{1}{2}\right] \qquad \left[\begin{array}{l}\because \cos \pi = -1 \\ \cos 4\pi = 1\end{array}\right]$$

$$= -\frac{1}{2}[-1]$$

$$= \frac{1}{2}$$

EXAMPLE-6

Evaluate

(i) $\int_0^1 \frac{1}{1+x^2}\,dx$

Oct. 2018, 2010 ; Apr. 2018, 2016, 2012

(ii) $\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

Oct. 2014, 2013 ; Feb 2021

(iii) $\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx$

Apr. 2012 ; Oct. 2017

(iv) $\int_0^a \sqrt{a^2-x^2}\,dx$

Oct. 2015 ; Apr. 2012, 2011

Solution :

(i) $\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1}(x)]_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$

(ii) $\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1}(x)]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

(iii) $\int_1^2 \frac{1}{x\sqrt{x^2-1}} dx = [\sec^{-1}(x)]_1^2$

$= \sec^{-1}(2) - \sec^{-1}(1)$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}$

(iv) $\int_0^a \sqrt{a^2 - x^2}\, dx = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$

$= \left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right) - \left(0\sqrt{a^2 - 0^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{0}{a}\right)\right)\right]$

$= 0 + \frac{a^2}{2}\sin^{-1}(1) - 0$

$= \frac{a^2}{2} \cdot \frac{\pi}{2}$

$= \frac{\pi a^2}{4}$

4.5 EVALUATION OF DEFINTE INTEGRALS BY SUBSTITUTION

To evaluate $\int_a^b f(x)dx$, by substitution method follow the steps.

Step-1: Consider the integral without limits and reduce the given integral into a standard integral by taking a suitable substitution.

Step-2: Compute the upper and lower limits for new variable

Step-3: Integrate the new integrand with respect to new variable and substitute the new limits.

SOLVED EXAMPLES

EXAMPLE-1

Evaluate

(i) $\int_0^4 x\sqrt{x^2+1}\,dx$ **(Nov. 2022)**

(ii) $\int_0^1 \frac{x^3}{1+x^8}dx$ **Oct. 2018, 2017, 2009 ; Apr. 2008**

(iii) $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}dx$ **Apr. 2018 ; Oct. 2008**

(iv) $\int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx$ **Oct. 2018 ; Apr. 2011**

(v) $\int_0^\infty \frac{dx}{(1+e^x)(1+e^{-x})}$

(vi) $\int_0^{\pi/2} e^{\sin^2 x}\sin 2x\,dx$

Solution :

(i) Put $x^2+1=t$, so that

$2x\,dx = dt$

$\Rightarrow \quad x\,dx = \frac{dt}{2}$

Limits

Lower limit, when $x = 0$, $t = 0^2 + 1 = 1$

Upper limit when $x = 4$, $t = 4^2 + 1 = 17$

$$\therefore \quad \int_0^4 x\sqrt{x^2+1}\,dx = \int_1^{17} \sqrt{t}\cdot\frac{dt}{2}$$

$$= \frac{1}{2}\cdot\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{t=1}^{17}$$

$$= \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{17}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left[t^{3/2}\right]_1^{17}$$

$$= \frac{1}{3}[(17)^{3/2} - 1^{3/2}]$$

$$= \frac{1}{3}[(17)^{3/2} - 1]$$

(ii) Put $x^4 = t$, so that

$$4x^3\,dx = dt$$

$$\Rightarrow \quad x^3\,dx = \frac{dt}{4}$$

Limits

L.L when $x = 0$, $t = 0^4 = 0$

U.L when $x = 1$, $t = 1^4 = 1$

$$\therefore \quad \int_0^1 \frac{x^3}{1+x^8}\,dx = \int_0^1 \frac{x^3}{1+(x^4)^2}\,dx = \int_0^1 \frac{1}{1+t^2}\cdot\frac{dt}{4}$$

$$= \frac{1}{4}\cdot\int_0^1 \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{4}[\tan^{-1}(t)]_0^1$$

$$= \frac{1}{4}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{16}$$

(iii) Put $1 + \cos x = t$,

$$-\sin x \, dx = dt$$

$$\Rightarrow \quad \sin x \, dx = -dt$$

Limits

Lower limit when $x = 0$, $t = 1 + \cos 0 = 1 + 1 = 2$

Upper limit when $x = \frac{\pi}{2}$, $t = 1 + \cos\frac{\pi}{2} = 1 + 0 = 1$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx = \int_2^1 \frac{1}{t}(-dt)$$

$$= -[\log t]_2^1$$

$$= -[\log 1 - \log 2]$$

$$= -[0 - \log 2] \qquad [\because \log 1 = 0]$$

$$= \log 2$$

(iv) Put $\sin^{-1}(x) = t$, so that

$$\frac{1}{\sqrt{1-x^2}} dx = dt$$

Limits

Upper limit, when $x = 0$, $t = \sin^{-1}(0) = 0$

Upper limit, when $x = 1$, $t = \sin^{-1}(1) = \frac{\pi}{2}$

$$\therefore \int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} dx = \int_0^{\pi/2} t \cdot dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{x^2}{8}$$

(v) Consider $\int_0^{\infty} \frac{dx}{(1+e^x)(1+e^{-x})}$

$$= \int_0^{\infty} \frac{1}{(1+e^x)\left(1+\frac{1}{e^x}\right)} dx$$

$$= \int_0^{\infty} \frac{1}{(1+e^x)\left(\frac{e^x+1}{e^x}\right)} dx$$

$$= \int_0^{\infty} \frac{e^x}{(1+e^x)(e^x+1)} dx$$

$$= \int_0^{\infty} \frac{e^x}{(1+e^x)^2} dx$$

Put $1 + e^x = t$, so that $e^x \, dx = dt$

Limits :

U.L. when $x = 0 \Rightarrow t = 1 + e^0 = 2$

U.L. when $x = \infty \Rightarrow = 1 + e^{\infty} = \infty$

$$\therefore \int_0^{\infty} \frac{1}{(1+e^x)(1+e^{-x})} dx = \int_0^{\infty} \frac{e^x}{(1+e^x)^2} dx$$

$$= \int_2^{\infty} \frac{1}{t^2} dt$$

$$= \left[-\frac{1}{t}\right]_2^{\infty}$$

$$= -\frac{1}{\infty} - \left(-\frac{1}{2}\right)$$

$$= 0 + \frac{1}{2}$$

$$= \frac{1}{2}$$

(vi) Put $\sin^2 x = t$, so that

$2 \sin x \cos x\, dx = dt \Rightarrow \sin 2x\, dx = dt$

Limits

L.L. when $x = 0,\ t = \sin^2(0) = 0$

U.L. when $x = \frac{\pi}{2},\ t = \sin^2 \frac{\pi}{2} = 1$

$$\therefore \int_0^{\pi/2} e^{\sin^2 x} \sin 2x\, dx = \int_0^1 e^t\, dt$$

$$= [e^t]_0^1$$

$$= e^1 - e^0$$

$$= e - 1$$

EXAMPLE-2

Evaluate $\int_0^{\pi/2} \frac{1}{4+5\cos x}\, dx$ **Oct. 2009**

Solution :

Put $\tan\frac{x}{2} = t$, so that

$dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

Limits

L.L. when $x = 0,\ t = \tan\left(\frac{0}{2}\right) = 0$

U.L when $x = \frac{\pi}{2},\ t = \tan\frac{\pi}{4} = 1$

$$\therefore \int_0^{\pi/2} \frac{1}{4+5\cos x}\, dx = \int_0^1 \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{(1+t^2)}$$

$$= \int_0^1 \frac{1}{\frac{4(1+t^2)+5(1-t^2)}{\cancel{(1+t^2)}}} \cdot \frac{2dt}{\cancel{(1+t^2)}}$$

$$= 2\int_0^1 \frac{1}{9-t^2}\,dt$$

$$= 2\cdot\int_0^1 \frac{1}{3^2-t^2}\,dt$$

$$= 2\left[\frac{1}{2.3}\log\left|\frac{3+t}{3-t}\right|\right]_0^1 \qquad \left[\because \int\frac{1}{a^2-x}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c\right]$$

$$= 2\cdot\frac{1}{6}\left[\log\left(\frac{3+1}{3-1}\right)-\log\left(\frac{3+0}{3-0}\right)\right]$$

$$= \frac{1}{3}\left[\log\left(\frac{4}{2}\right)-\log 1\right]$$

$$= \frac{1}{3}\log 2 \qquad [\because \log 1 = 0]$$

EXAMPLE-3

Evaluate

(i) $\int_0^1 x^2 e^{2x}\,dx$

(ii) $\int_1^e \log x\,dx$

(iii) $\int_0^{\pi/2} x^2 \cos x\,dx$ **Apr. 2016; Oct. 20**

(iv) $\int_0^1 x\sin^{-1}(x)\,dx$ **Oct. 20**

Solution :

(i) $$\int_0^1 x^2 e^{2x}\,dx = \left[x^2\frac{e^{2x}}{2} - \cancel{2}x\frac{e^{2x}}{\cancel{4}_2} + \cancel{2}\frac{e^{2x}}{\cancel{8}_4}\right]_{x=0}^1 \qquad [\because \text{By Bernoulli's Rule}]$$

$$= \left(\frac{1^2 e^{2(1)}}{2} - \frac{1.e^{2(1)}}{2} + \frac{e^{2(1)}}{4}\right) - \left(0\frac{e^0}{2} - 0\frac{e^0}{2} + \frac{e^0}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{1}{4}$$

$$= \frac{e^2 - 1}{4}$$

(ii) $\int_1^e \log x \, dx = \int_1^e \log x \cdot 1 dx$

$$= [\log x \cdot \int x \, dx]_0^e - \int_1^e \left(\frac{d}{dx}(\log x) \cdot \int 1 dx \right) dx \quad [\because \textbf{By Integration by parts}]$$

$$= [x \log x]_1^e - \int_1^e \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$$

$$= (e \log e - 1 \cdot \log 1) - [x]_1^e \qquad \left[\because \begin{array}{l} \log 1 = 0 \\ \log e = 1 \end{array}\right]$$

$$= (e(1) - 1 \cdot 0) - (e - 1)$$

$$= (e - 0) - (e - 1)$$

$$= e - e + 1$$

$$= 1$$

(iii) $\int_0^{\pi/2} x^2 \cos x \, dx = [x^2 \sin x - (2x)(-\cos x) + 2(-\sin x)]_0^{\pi/2}$ $\quad [\because$ **By Bernoulli's Rule**]

$$= [x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} + \cancel{2} \cdot \frac{\pi}{\cancel{2}} \cos\frac{\pi}{2} - 2\sin\frac{\pi}{2}\right] - \left[0^2 \sin 0 + 2(0)\cos 0 - 2\sin 0\right]$$

$$= \frac{\pi^2}{4}(1) + \pi(0) - 2(1) - 0$$

$$= \frac{\pi^2}{4} - 2$$

$$= \frac{\pi^2 - 8}{4}$$

(iv) Put $x = \sin\theta$, so that $dx = \cos\theta \, d\theta$

Limits

Lower limits, when $x = 0, \ \theta = \sin^{-1}(0) = 0$

Upper limits, when $x = 1, \ \theta = \sin^{-1}(1) = \frac{\pi}{2}$

$$\therefore \int_0^1 x\sin^{-1}(x)\,dx = \int_0^{\pi/2} \sin\theta\cdot\theta\cdot\cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \theta(\sin\theta\,\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \theta\sin 2\theta\, d\theta$$

$$= \frac{1}{2}\left[\theta\left(\frac{-\cos 2\theta}{2}\right) - 1\cdot\left(\frac{-\sin 2\theta}{4}\right)\right]_0^{\pi/2} \qquad \left[\because \int uv\,dx = uv_1 - u'v_2 + u''v_3\right]$$

$$= \frac{1}{2}\left[\frac{-\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi\cos \cancel{2}(\pi/\cancel{2})}{2\cdot 2} + \frac{\sin \cancel{2}(\pi/2)}{4}\right) - \left(\frac{-0\cdot\cos 0}{2} + \frac{\sin 0}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{-\pi}{4}\cos\pi + \frac{\sin\pi}{4}\right) - 0\right]$$

$$= \frac{1}{2}\left[-\frac{\pi}{4}(-1) + 0\right] \qquad [\because \cos\pi = -1, \sin\pi = 0]$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8}$$

4.6 PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (i.e., the value of the definite integral depends on the limits and not on the variable of integration).

PROOF :

Let $\int f(x)\,dx = F(x) + c$

By definition

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \qquad \text{.............. (1)}$$

$$\int f(t)dx = F(t) + c$$

Also

$$\therefore \quad \int_a^b f(t)dt = [(F(t)]_a^b = F(b) - F(a) \qquad \text{.............. (2)}$$

From (1) and (2), we have

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

2. (i) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(ii) $\int_a^a f(x)dx = 0$

PROOF :

Let $\int f(x)dx = F(x) + c$

(i) $\therefore$

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

$$= -[F(a) - F(b)] \qquad \text{.............. (1)}$$

and

$$\int_b^a f(x)dx = [F(x)]_b^a = F(a) - F(b) \qquad \text{.............. (2)}$$

From (1) and (2), we get

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(ii) $\int_a^a f(x)dx = [f(x)]_a^a = F(a) - F(a) = 0$

3. If $a < c < b$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

PROOF :

Let $\int f(x)dx = F(x) + c$, then by definition

$$\int_a^c f(x)dx = [F(x)]_a^c = F(c) - F(a)$$

$$\int_c^b f(x)dx = [F(x)]_c^b = F(b) - F(c)$$

$$\therefore \int_a^c f(x)dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= \int_a^b f(x)dx$$

In general,

If $a < c_1 < c_2 < c_3 < \ldots < c_n < b$ then

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_n}^b f(x)dx$$

4. (i) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(ii) $\int_a^b K \cdot f(x)dx = K\int_a^b f(x)dx$,

where 'K' is a constant

PROOF :

(i) Let $\int f(x)dx = F(x)+c$

and $\int g(x)dx = G(x)+c$

$$\therefore \int_a^b [f(x)+g(x)]dx = [F(x)+G(x)]_a^b$$

$$= [F(b)+G(b)]-[F(a)+G(a)]$$

$$= F(b) - F(a) + G(b) - G(a)$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

(iii) $\int_a^b K f(x)dx = [K \cdot F(x)]_a^b$

$$= K \cdot F(b) - KF(a)$$

$$= K[F(b) - F(a)]$$

$$= K \cdot \int_a^b f(x)dx$$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

PROOF :

Let $\quad a - x = t$, then $-dx = dt$

$\Rightarrow \qquad dx = -dt$

Limit

Lower limit when $x = 0,\quad t = a - 0 = a$

Upper limit when $x = a$, $t = a - a = 0$

$$\therefore \quad \int_0^a f(a-x)\,dx = \int_a^0 f(t)(-dt)$$

$$= -\int_a^0 f(t)\,dt$$

$$= \int_0^a f(t)\,dt \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_0^a f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$\therefore \quad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

6. $\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,dx$

PROOF :

$$\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a 2f(x)\,dx$$

$$= \frac{1}{2}\int_0^a [f(x) + f(x)]\,dx$$

$$= \frac{1}{2}\left[\int_0^a f(x)\,dx + \int_0^a f(x)\,dx\right]$$

$$= \frac{1}{2}\left[\int_0^a f(x)\,dx + \int_0^a f(a-x)\,dx\right] \qquad [\because \text{ from property 5}]$$

$$= \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,dx$$

NOTE

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

7. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

PROOF :

From property 3, we have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

Put $x = 2a - t$ in the second integral on RHS.

So that $dx = -dt$

L.L. when $x = a, \Rightarrow t = 2a - a = a$

U.L. When $x = 2a \Rightarrow t = 2a - 2a = 0$

$$\therefore \int_a^{2a} f(x)dx = \int_a^0 f(2a-t)(-dt)$$

$$= -\int_a^0 f(2a-t)dt$$

$$= \int_0^a f(2a-t)dt$$ [By property

$$= \int_0^a f(2a-x)dx, \text{ using property1.}$$

Substituting (2), in (1), we get

$$\int_a^{2a} f(x)dx = \int_a^a f(x)dx + \int_0^a f(2a-x)dx$$

8. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } (2a-x) = -f(x) \end{cases}$

PROOF :

By properties 7, we have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)dx$$

Case-(i) :

If $f(2x - x) = f(x)$, then

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)\,dx$$

Case-(ii) :

If $f(2a - x) = -f(x)$, then

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a -f(x)dx$$

$$= \int_0^a f(x)\,dx - \int_0^a f(x)dx$$

$$= 0$$

$$\therefore \quad \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Even Function :

A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x.

For example, $f(x) = \cos x$ is an even function. Since $f(-x) = \cos(-x) = \cos x = f(x)$

Similarly x^2, x^4, sec x etc are even functions.

Odd Function :

A function $f(x)$ is said to be an odd if $f(-x) = -f(x)$ for all x.

For example, $f(x) = \sin x$ is an odd function. Since $f(-x) = \sin(-x) = -\sin x = -f(x)$.

Similarly, x, x^3 etc are odd functions.

9. $$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

PROOF :

From property 3, we have

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \qquad \text{.............} \quad (1)$$

Put $x = -t$ in the first integral on the RHS then $dx = -dt$

Lower limit, when $x = -a$, $t = -(-a) = a$

Upper limit, when $x = 0$, $t = -(0) = 0$

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)\,dt$$

$$= \int_{0}^{a} f(-t)\,dt \qquad \left[\because \int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx\right]$$

$$= \int_{0}^{a} f(-x)\,dx, \text{ by property 1.}$$

Substituting (2), in (1) we get,

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{if f is an even function} \\ 0, & \text{if f is an odd function} \end{cases}$$

SOLVED EXAMPLES

EXAMPLE-1

Evaluate

(i) $\int_{-5}^{5} x\,dx$

(ii) $\int_{-1}^{1} (3x^2 + 5)\,dx$

(iii) $\int_{-2}^{2} (x^2 - 4x + 3)\,dx$

(iv) $\int_{-\pi/2}^{\pi/2} x^2 \sin x\,dx$

(v) $\int_{-\pi/2}^{\pi/2} x^2 \cos x\,dx$

(vi) $\int_{-\pi/4}^{\pi/4} \log\left(\frac{1-\sin x}{1+\sin x}\right)dx$

Apr. 2014 : Oct. 2007

Solution :

(i) Let $f(x) = x$. Then $f(-x) = -x = -f(x)$

$\therefore$ $f(x)$ is an odd function

We know that

$$\therefore \int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd} \end{cases}$$

$$\therefore \int_{-5}^{5} x\,dx = 0$$

(ii) Let $f(x) = 3x^2 + 5$ Then

$$f(-x) = 3(-x)^2 + 5 = 3x^2 + 5 = f(x)$$

$\therefore$ $f(x)$ is an even function

$\therefore$ By property 9,

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is odd} \end{cases}$$

$$\therefore \int_{-1}^{1} (3x^2+5)dx = 2\cdot\int_0^1 (3x^2+5)dx$$

$$= 2\cdot\left[\frac{\cancel{3}x^3}{\cancel{3}} + 5x\right]_0^1$$

$$= 2\left[[1^3 + 5(1)] - [0^3 + 5(0)]\right]$$

$$= 2\,(6)$$

$$= 12$$

(iii) $$\int_{-2}^{2} (x^2 - 4x + 3)dx = \int_{-2}^{2} x^2 dx - 4\int_{-2}^{2} x\,dx + 3\cdot\int_{-2}^{2} dx$$

$$= 2\int_0^2 x^2\,dx - 4(0) + 3(2)\int_0^2 dx \qquad [\because x^2, 1 \text{ are even and } x \text{ is odd}]$$

$$= 2\left(\frac{x^3}{3}\right)_0^2 + 6[x]_0^2$$

$$= \frac{2}{3}(2^3 - 0^3) + 6(2-0)$$

$$= \frac{16}{3} + 12$$

$$= \frac{52}{3}$$

(iv) Let $f(x) = x^2 \sin x$, then

$$f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$$

$\therefore$ f(x) is an odd function

$$\therefore \int_{-\pi/2}^{\pi/2} x^2 \sin x \, dx = 0$$

(v) Let $f(x) = \cos x$, then

$$f(-x) = \cos(-x) = \cos x = f(x)$$

$\therefore$ f(x) is an even function

$\therefore$ By property 9,

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if f is even} \\ 0, & \text{if f is odd} \end{cases}$$

$$\therefore \int_{-\pi/2}^{\pi/2} \cos x \, dx = 2\int_0^{\pi/2} \cos x \, dx$$

$$= 2(\sin x)_0^{\pi/2}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(1-0)$$

$$= 2$$

(vi) Let $f(x) = \log\left(\frac{1-\sin x}{1+\sin x}\right)$, then

$$f(-x) = \log\left(\frac{1-\sin(-x)}{1+\sin(-x)}\right)$$

$$= \log\left(\frac{1+\sin x}{1-\sin x}\right)$$

$$= \log\left(\frac{1-\sin x}{1+\sin x}\right)^{-1}$$

$$= -\log\left(\frac{1-\sin x}{1+\sin x}\right) \qquad [\because \log a^n = n \log a]$$

$$= -f(x)$$

f (x) is an odd function.

We know that $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if f is even} \\ 0, & \text{if f is odd} \end{cases}$

$$\therefore \int_{-\pi/4}^{\pi/4} \log\left(\frac{1-\sin x}{1+\sin x}\right)dx = 0$$

PLE-2

valuate

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}dx$

Oct. 2018, 2012, 2011; Apr. 2012, 2010

(ii) $\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx$

Oct. 2018, 2008 ; Apr. 2015

(iii) $\int_0^{\pi/2} \frac{1}{1+\tan x}dx$

Oct. 2017, 2018, 2011 2005 ; Apr. 2010, 2009

(iv) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Apr. 2019, 2018, 2012, 2011 ; Oct. 2018, 2017, 2013, 2012

(v) $\int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx$

Feb. 2021

n :

Let $\quad I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}dx \quad \ldots\ldots\ldots\ldots (1)$

$$= \int_0^{\pi/2} \frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x+\sin^n x}dx \quad \ldots\ldots\ldots\ldots (2)$$

Adding (1) and (2), we have

$$I + I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x} dx$$

$$= \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx$$

$$= \int_0^{\pi/2} 1 \cdot dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$$

(ii) Let

$$I = \int_0^{\pi/2} \frac{1}{1 + \tan x} dx$$

$$= \int_0^{\pi/2} \frac{1}{1 + \frac{\sin x}{\cos x}} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \qquad \text{.............. (1)}$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx \qquad \text{.............. (2)}$$

Adding (1) and (2), we get

$$I + I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \left(\frac{\cos x}{\cos x + \sin x} + \frac{\sin x}{\sin x + \cos x} \right) dx$$

$$= \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} 1 dx$$

$$= [x]_0^{\pi/2}$$

$$2 \cdot I = \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \quad \int_0^{\pi/2} \frac{1}{1 + \tan x} dx = \frac{\pi}{4}$$

An alternative method will discus for (iii)

(iii) $$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{1}{2} \int_0^{\pi/2} \left[\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} \right] dx$$

$$\left[\because \int_0^a f(x) dx = \frac{1}{2} \int_0^a [f(x) + f(a - x)] dx \right]$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right] dx$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 1dx$$

$$= \frac{1}{2}[x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4}$$

(iv) $$\int_0^{\pi/2}\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}dx = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}+\frac{e^{\sin\left(\frac{\pi}{2}-x\right)}}{e^{\sin\left(\frac{\pi}{2}-x\right)}+e^{\cos\left(\frac{\pi}{2}-x\right)}}\right]dx$$

$$\left[\because \int_0^a f(x)+dx = \frac{1}{2}\int_0^a [f(x)+f(a-x)]dx\right.$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{\sin x}}{e^{\sin x}+e^{\cos x}}+\frac{e^{\cos x}}{e^{\cos x}+e^{\sin x}}\right]dx$$

$$= \frac{1}{2}\left(\int_0^{\pi/2}\frac{e^{\sin x}+e^{\cos x}}{e^{\sin x}+e^{\cos x}}\right)dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\,dx$$

$$= \frac{1}{2}[x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4}$$

EXAMPLE-3

Show that

(i) $\int_0^{\pi/2} \frac{\sqrt{tanx}}{\sqrt{tanx}+\sqrt{cot\,x}}dx = \frac{\pi}{4}$ (ii) $\int_0^{\pi/2} \frac{sec\,x}{secx + cosec\,x}dx = \frac{\pi}{4}$

Solution :

(i) Let

$$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx \quad \text{.............. (1)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\tan x\left(\frac{\pi}{2}-x\right)}}{\sqrt{\tan\left(\frac{\pi}{2}-x\right)}+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx \quad \text{.............. (2)}$$

Adding (1) & (2), we get,

$$I + I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx + \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$$

$$2I = \int_0^{\pi/2}\left[\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} + \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\right]dx$$

$$= \int_0^{2}\left(\frac{\sqrt{\tan}+\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}\right)dx$$

$$= \int_0^{\pi/2} 1\cdot dx$$

$$= [x]_0^{\pi/2}$$

$$2\cdot I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx = \frac{\pi}{4}$$

(ii) $$\int_0^{\pi/2} \frac{\sec x}{\sec x + \operatorname{cosec} x} dx = \int_0^{\pi/2} \left[\frac{\sec x}{\sec x + \operatorname{cose} x} + \frac{\sec\left(\frac{\pi}{2} - x\right)}{\sec\left(\frac{\pi}{2} - x\right) + \operatorname{cosec}\left(\frac{\pi}{2} - x\right)} \right] dx$$

$$\left[\because \int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x) + f(a-x)]\,dx \right]$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\sec x}{\sec x + \operatorname{cose} x} + \frac{\operatorname{cosec} x}{\operatorname{cosec} x + \sec x} \right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\sec x + \operatorname{cosec} x}{\sec x + \operatorname{cosec} x} \right] dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 dx$$

$$= \frac{1}{2}[x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4}$$

EXAMPLE-4

Evaluate $\int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} dx$

Solution :

Put $x = a \sin \theta$, so that $dx = a \cos\theta \; d\theta$

Limits :

L.L when $x = 0 \Rightarrow \theta = \sin^{-1}\left(\frac{a}{a}\right) = 0$

U.L when $x = a, \Rightarrow \theta = \sin^{-1}\left(\frac{a}{a}\right) = \sin^{-1}(1) = \frac{\pi}{2}$

$$\therefore \int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} dx = \int_0^{\pi/2} \frac{1}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + a\cdot\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\cancel{a}\cos\theta}{\cancel{a}[\sin\theta+\cos\theta]}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\cos\theta}{\sin\theta+\cos\theta} + \frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}\right] d\theta$$

$$\left[\because \int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x)+f(a-x)]\,dx\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\cos\theta}{\sin\theta+\cos\theta} + \frac{\sin\theta}{\cos\theta+\sin\theta}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{\cos\theta+\sin\theta}{\sin\theta+\cos\theta}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\,d\theta$$

$$= \frac{1}{2}\cdot[\theta]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4}$$

EXAMPLE-5

Evaluate $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Solution :

Let

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \quad \text{............ (1)}$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{\cancel{a}-\cancel{a}+x}}\,dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \quad \ldots\ldots\ldots(2)$$

Adding (1) and (2), we get,

$$I + I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$$

$$2I = \int_0^a \left[\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\right]dx$$

$$= \int_0^a \left[\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\right]dx$$

$$= \int_0^a 1\,dx$$

$$= [x]_0^a$$

$$2I = a$$

$$I = \frac{a}{2}$$

$$\therefore \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx = \frac{a}{2}$$

EXAMPLE-6

Evaluate

(i) $\displaystyle\int_0^{\pi/2} \log \tan x\, dx$

(ii) $\displaystyle\int_0^{\pi/2} \log\left[\frac{2+3\sin x}{2+3\cos x}\right]dx$

Solution :

(i) We know that

$$\int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x)+f(a-x)]\,dx$$

$$\therefore \int_0^{\pi/2} \log\tan x\,dx = \frac{1}{2}\int_0^{\pi/2}\left[\log\tan x + \log\tan\left(\frac{\pi}{2}-x\right)\right]dx$$

$$= \frac{1}{2}\int_0^{\pi/2}[\log\tan x + \log\cot]\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\log[\tan x\cdot\cot x]\,dx \qquad [\because \log ab = \log a + \log b]$$

$$= \frac{1}{2}\int_0^{\pi/2}\log\left(\cancel{\tan x}\,\frac{1}{\cancel{\tan x}}\right)dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\log 1\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 0\,dx \qquad [\because \log 1 = 0]$$

$$= 0$$

(ii) $$\int_0^{\pi/2}\log\left[\frac{2+3\sin x}{2+3\cos x}\right]dx = \frac{1}{2}\int_0^{\pi/2}\left[\log\left(\frac{2+3\sin x}{2+3\cos x}\right)+\log\left(\frac{2+3\sin\left(\frac{\pi}{2}-x\right)}{2+3\cos\left(\frac{\pi}{2}-x\right)}\right)\right]dx$$

$$\left[\because \int_0^a f(x)dx = \frac{1}{2}\int_0^a [f(x)+f(a-x)]dx\right]$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\log\left(\frac{2+3\sin x}{2+3\cos x}\right)+\log\left(\frac{2+3\cos x}{2+3\sin x}\right)\right]dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\log\left(\frac{\cancel{2+3\sin x}}{\cancel{2+3\cos x}}\cdot\frac{\cancel{2+3\cos x}}{\cancel{2+\sin x}}\right)dx \qquad [\because \log ab = \log a + \log b]$$

$$= \frac{1}{2}\int_0^{\pi/2} \log 1\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 0\, dx \qquad [\because \log 1 = 0]$$

$$= 0$$

EXAMPLE-7

Evaluate $\int_0^{\pi/2} \log \sin x\, dx$

Solution :

Let
$$I = \int_0^{\pi/2} \log \sin x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\log \sin x + \log \sin\left(\frac{\pi}{2} - x\right)\right] dx$$

$$\left[\because \int_0^a f(x)\,dx = \frac{1}{2}\int_0^a f(x) + f(a-x)\,dx\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \log(\sin x \cdot \cos x)\, dx \qquad [\because \log ab = \log a + \log b]$$

$$= \frac{1}{2}\int_0^{\pi/2} \log\left(\frac{2\sin x \cdot \cos x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \log \sin 2x\, dx - \frac{1}{2}\int_0^{\pi/2} \log 2\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \log \sin 2x\, dx - \frac{1}{2}\log 2 \cdot [x]_0^{\pi/2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \log\sin 2x\,dx - \frac{1}{2}\log 2\cdot\left(\frac{\pi}{2}-0\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \log\sin 2x\,dx - \frac{\pi}{4}\log 2 \qquad \text{.............. (1)}$$

Put $2x = t$ in the first integral on the RHS. So that $2dx = dt \Rightarrow dx = \frac{dt}{2}$

Limits : L.L. when $x = 0,\ t = 0$

U.L when $x = \frac{\pi}{2},\ t = 2\cdot\frac{\pi}{2} = \pi$

$$\therefore \int_0^{\frac{\pi}{2}} \log\sin 2x\,dx = \int_0^{\pi} \log\sin t\cdot\frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{\pi} \log\sin t\,dt$$

$$= \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \log\sin t\,dt \qquad \left[\because \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}\right]$$

$$= \int_0^{\pi/2} \log\sin x\,dx \qquad \text{.............. (2)}$$

[$\because$ **by property (1)**]

Substituting (2) in (1), we get,

$$I = \frac{1}{2}\int_0^{\pi/2} \log\sin x\,dx - \frac{\pi}{4}\log 2$$

$$= \frac{1}{2}I - \frac{\pi}{4}\log 2$$

$$I - \frac{1}{2}I = -\frac{\pi}{4}\log 2$$

$$I\left(1-\frac{1}{2}\right) = -\frac{\pi}{4}\log 2$$

$$\frac{1}{2}I = -\frac{\pi}{4}\log 2$$

$$I = -\frac{\pi}{2}\log 2$$

EXAMPLE-8

Evaluate

(i) $\int_{0}^{\pi/2} \frac{x\sin x}{1+\sin x}dx$

Oct. 2011

(ii) $\int_{0}^{\pi} \left[\frac{x\sin x}{1+\cos^2 x}\right]dx$

Oct. 2009, 2012, 2014 ; Feb. 2021

Solution :

(i) Let

$$I = \int_{0}^{\pi} \frac{x\sin x}{1+\sin x}dx$$

$$= \int_{0}^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx \qquad \left[\because \int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx\right]$$

$$= \int_{0}^{\pi} \frac{(\pi-x)\sin x}{1+\sin x}dx$$

$$= \int_{0}^{\pi} \frac{\pi\sin x}{1+\sin x}dx - \int_{0}^{\pi} \frac{x\sin x}{1+\sin x}dx$$

$$= \pi\int_{0}^{\pi} \frac{\sin x}{1+\sin x}dx - I$$

$$\Rightarrow \quad I + I = \pi\int_{0}^{\pi} \frac{1+\sin x - 1}{1+\sin x}dx$$

$$2I = \pi\int_{0}^{\pi}\left(\frac{1+\sin x}{1+\sin x} - \frac{1}{1+\sin x}\right)dx$$

$$= \pi\left[\int_{0}^{\pi} 1dx - \int_{0}^{\pi} \frac{1}{1+\sin x}dx\right]$$

$$= \pi\left\{[x]_0^{\pi} - \int_{0}^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}dx\right\}$$

$$= \pi\left\{(\pi-0) - \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx\right\}$$

$$= \pi\left[\pi - \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx\right]$$

$$= \pi^2 - \pi\left[\int_0^{\pi}\left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx\right]$$

$$= \pi^2 - \pi\int_0^{\pi}(\sec^2 x - \sec x \tan x)\,dx$$

$$= \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi^2 - \pi[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)]$$

$$= \pi^2 - \pi[0-(-1)-(0-1)]$$

$$= \pi^2 - \pi(1+1) = \pi^2 - 2\pi$$

$\therefore$ $$I = \frac{1}{2}(\pi^2 - 2\pi)$$

$\therefore$ $$\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx = \frac{\pi}{2}(\pi-2)$$

(ii) Let $$I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{1}{2}\int_0^{\pi}\left[\frac{x\sin x}{1+\cos^2 x} + \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\right]dx$$

$$\left[\because \int_0^{a} f(x)\,dx = \frac{1}{2}\int_0^{a}[f(x)+f(a-x)]\,dx\right]$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{x\sin x}{1+\cos^2 x} + \frac{(\pi-x)\sin x}{1+\cos^2 x}\right]dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\cancel{x\sin x} + \pi\sin x - \cancel{x\sin x}}{1+\cos^2 x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = t$, so that $-\sin x\,dx = dt \Rightarrow \sin dx = -dt$

L.L when $x = 0$, $t = \cos 0 = 1$

U.L when $x = \pi$, $t = \cos \pi = -1$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_1^{-1} \frac{1}{1+t^2}(-dt)$$

$$= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+t^2}dt$$

$$= \frac{\pi}{2}[\tan^{-1}(t)]_{t=-1}^{1}$$

$$= \frac{\pi}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

$$\therefore \quad \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$$

4.7 MODULUS FUNCTION (OR) ABSOLUTE VALUE FUNCTION

A function $f : R \to R$, defined as

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called modulus function or absolute value function.

4.8 INTEGRATION OF MODFUNCTIONS

Working rule to evaluate $\int_a^b |f(x)|dx$

Step-1 : Solve the equation $f(x) = 0$, Let α and β be the roots of $f(x) = 0$, such that $a < \alpha < \beta < b$.

Step-2 : Divide the interval (a, b) into subinterals $(a,\alpha),(\alpha,\beta)$ and (β,b)

Step-3 : Use the property (3),

$$\int_a^b |f(x)|dx = \int_a^\alpha |f(x)|dx + \int_\alpha^\beta |f(x)|dx + \int_\beta^b |f(x)|dx$$

compute, with proper sign of $|f(x)|$ in each subintervals $(a,\alpha),(\alpha,\beta)$ and (β,b)

SOLVED EXAMPLES

EXAMPLE-1

Evaluate

(i) $\int_{-1}^{1} x|x|dx$

(ii) $\int_{-2}^{3} |1-x^2|dx$

(iii) $\int_{0}^{2\pi} |\sin x|dx$

Nov. 2022

Solution :

(i) Let $f(x) = x\,|x|$

$$f(-x) = (-x)\,|-x| = -x\,|x| = -f(x)$$

$\therefore$ f(x) is an odd function.

We know that $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if f is an even function} \\ 0, & \text{if f is an odd function} \end{cases}$

$$\therefore \quad \int_{-1}^{1} x|x|dx = 0$$

(ii) Let $f(x) = 1 - x^2$, then $f(x) = 0$

$\Rightarrow \quad 1 - x^2 = 0$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

$\therefore$ f(x) is – ve in (–2, –1) and (1, 3) and f(x) is + ve in (–1, 1).

$$\therefore \int_{-2}^{3}|f(x)|dx = \int_{-2}^{-1}|f(x)|dx + \int_{-1}^{1}|f(x)|dx + \int_{1}^{3}|f(x)|dx$$

$$\therefore \int_{-2}^{3}|1-x^2|dx = \int_{-2}^{-1}-(1-x^2)dx + \int_{-1}^{1}(1-x^2)dx + \int_{1}^{3}-(1-x^2)dx$$

$$= -\int_{-2}^{-1}(1-x^2)dx + \int_{-1}^{1}(1-x^2)dx - \int_{1}^{3}(1-x^2)dx$$

$$= \left(x-\frac{x^3}{3}\right)_{-2}^{-1} + \left(x-\frac{x^3}{3}\right)_{-1}^{1} - \left(x-\frac{x^3}{3}\right)_{1}^{3}$$

$$= -\left[\left((-1)-\frac{(-1)^3}{3}\right)-\left((-2)-\frac{(-2)^3}{3}\right)\right]+\left[\left((1)-\frac{(1)^3}{3}\right)-\left((-1)-\frac{(-1)^3}{3}\right)\right]$$

$$-\left[\left(3-\frac{3^3}{3}\right)-\left(1-\frac{(1)^3}{3}\right)\right]$$

$$= -\left[\left(-1+\frac{1}{3}\right)-\left(-2+\frac{8}{3}\right)\right]+\left[\left(1-\frac{1}{3}\right)-\left(-+\frac{1}{3}\right)\right]-\left[(3-9)-\left(1-\frac{1}{3}\right)\right]$$

$$= \frac{28}{3}$$

(iii) Since $\sin x > 0$ in $[0,\pi]$ and $\sin x < 0$ in $[\pi, 2\pi]$

$$\therefore \int_{0}^{2\pi}|\sin x|dx = \int_{0}^{\pi}|\sin x|dx + \int_{\pi}^{2\pi}|\sin x|dx$$

$$= \int_{0}^{\pi}\sin x\,dx - \int_{\pi}^{2\pi}\sin x\,dx$$

$$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$$

$$= [-\cos\pi + \cos 0] + [\cos 2\pi - \cos\pi]$$

$$= -(-1) + 1 + 1 - (-1)$$

$$= 1 + 1 + 1 + 1$$

$$= 4$$

REVIEW QUESTIONS

1. Define definite integral.
2. State the fundamental theorem of integral calculus. . (Feb. 2021)
3. State any one property of definte integral. (Feb. 2021)
4. Define an even function.
5. Define an odd function.

EXERCISE

I. Evaluate

1. (i) $\int_{-4}^{5} x^2 dx$ **(Apr. 2014)** (ii) $\int_{0}^{1} (x^3+1) dx$ **(Apr. 2014, 2016)**

(iii) $\int_{0}^{1} (x^4+1) dx$ **(Oct. 2013, 2019)** (iv) $\int_{0}^{1} (x^5+1) dx$

(v) $\int_{0}^{1} (x^{10}+1) dx$ **(Apr. 2015)** (vi) $\int_{1}^{2} (x^2-1) dx$

(vii) (a) $\int_{0}^{1} (x+1)(x+2) dx$ (b) $\int_{0}^{1} (x+1)(x+3) dx$

(viii) (a) $\int_{0}^{\pi} \sin x \, dx$ (b) $\int_{0}^{\pi} \sin 3x \, dx$

(ix) $\int_{0}^{2} x^2(x^3+1) dx$ (x) $\int_{0}^{1} (x+1)(2x-1) dx$ **(Oct. 2018)**

2. (i) $\int_{0}^{1} \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) dx$ (ii) $\int_{1}^{2} \frac{1}{\sqrt{3x-2}} dx$

(iii) $\int_{2}^{4} \frac{x^4+3x^2-1}{x^2} dx$ **(Apr. 2016)** (iv) $\int_{-1}^{1} \frac{1}{x+4} dx$

(v) $\int_{0}^{\pi/4} \tan x \, dx$ (vi) $\int_{-1}^{1} e^{2x+3} dx$

(vii) $\int_{0}^{1} e^{-x} dx$ **(Nov. 2022)**

3. (i) $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ **(Apr. 2016)** (ii) $\int_{1/2}^1 \frac{1}{\sqrt{1-x^2}} dx$

(Oct. 2013)

(iii) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2} dx$

(Apr. 2018, 2017 ; Oct. 2017, 2010)

(iv) $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$

(Oct. 2017 ; Apr. 2012)

(v) $\int_1^2 \frac{1}{x\sqrt{x^2-1}} dx$

(vi) $\int_0^4 \sqrt{16-x^2}dx$ (vii) $\int_0^2 \sqrt{4-x^2}dx$

4. (i) $\int_0^2 \sqrt{4-x^2}dx$ **(Oct. 2018, 2011)**

(ii) $\int_0^{\pi/4} \tan^2 x\, dx$ **(Oct. 2017, 2013, 2011, 2010 ; Apr. 2012)**

(iii) $\int_0^{\pi/2} \cos^2 x\, dx$ **(Apr. 2011)** (iv) $\int_0^{\pi/2} \sin^2 x\, dx$

(v) $\int_0^{\pi/2} \cos^4 x\, dx$ **(Oct. 2011)**

5. (i) $\int_0^{\pi/2} \sin 3x \cos x\, dx$ **(Apr. 2019, 2011)** (ii) $\int_0^{\pi/2} \cos 4x \cos 3x\, dx$

(iii) $\int_0^{\pi} \sin 3x \sin x\, dx$

6. (i) $\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx$

(Oct. 2018, 2015, 2011 ; Apr. 2011)

(ii) $\int_0^{\pi/2} \sqrt{1-\sin x}\, dx$ (iii) $\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx$

(iv) $\int_0^{\pi/4} \frac{1}{1-\sin x} dx$

7. (i) $\int_0^1 \frac{x^2-1}{x^2+1}dx$ (ii) $\int_0^1 \frac{1-x^2}{x^2+1}dx$ **(Apr. 2001)**

(iii) $\int_0^2 \frac{5x+1}{x^2+4}dx$

8. (i) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ (ii) $\int_1^4 x\sqrt{x^2-1}\,dx$

(iii) $\int_0^{\pi/3} \frac{\sin x}{3+4\cos x}$ (iv) $\int_0^{\pi/2} \frac{\cos x}{2+\sin x}$

(v) $\int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x}dx$

9. (i) $\int_0^{\pi/2} \frac{1}{5+4\cos x}dx$ (ii) $\int_0^{\pi} \sin^3 x\cos^4 x\,dx$

10. (i) $\int_0^1 \frac{5x^3}{\sqrt{1-x^8}}dx$ (ii) $\int_0^1 \frac{[\tan^{-1}(x)]^2}{1+x^2}dx$ **(Oct. 2019)**

(iii) $\int_0^1 \frac{[\sin^{-1}(x)]^2}{\sqrt{1-x^2}}dx$ (iv) $\int_0^{\pi/4} \frac{\sec^2 x}{(1+\tan x)^2}dx$

(v) $\int_0^1 \frac{x^3}{x^2+1}dx$ (vi) $\int_0^1 x(1-x)^7\,dx$

11. (i) $\int_0^1 \frac{1}{e^x+e^{-x}}dx$ (ii) $\int_0^{\infty} \frac{dt}{t^2+2t+2}$ **(Apr. 2018, 2011)**

12. (i) $\int_0^{\pi} x\cos x\,dx$ (ii) $\int_0^{\pi/2} x\cos x\,dx$

(iii) $\int_0^{\pi} x^2\cos x\,dx$ (iv) $\int_0^{\pi/4} x^2\sec^2 x\,dx$ **(Apr. 2018)**

(v) $\int_0^1 x^2\tan^{-1}(x)$ (vi) $\int_0^1 x^2\tan^{-1}(x)\,dx$

(vii) $\int_0^1 \sin^{-1}(x)\,dx$ (viii) $\int_0^2 x^2e^{-2x}\,dx$

(ix) $\int_3^4 x\log x\,dx$

II. Evaluate using Properties of Definite Integrals

13. (i) $\int_{-1}^{1} x\,dx$ (ii) $\int_{-\pi/2}^{\pi/2} x\cos x\,dx$

(iii) $\int_{-5}^{5} x^3\,dx$ (iv) $\int_{-5}^{5} x^2\,dx$

(v) $\int_{-1}^{1} \log\left|\frac{5-x}{5+x}\right| dx$ (vi) $\int_{-1}^{1} \log\left|\frac{3-x}{3+x}\right| dx$

(vii) $\int_{-1}^{1} (x^2-3x+1)\,dx$ (viii) $\int_{-2}^{2} (4x+3x^2+7x^5+12x^2)\,dx$

(ix) $\int_{0}^{\pi} \cos^5 x\,dx$ (x) $\int_{-2}^{2} x\sqrt{7-x^2}\,dx$

(xi) $\int_{-2}^{2} x\sqrt{7-x^2}\,dx$

14. (i) $\int_{0}^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx$ **(Apr. 2016, 2013, 2011, 2008, 2004 ; Oct. 2010)**

(or) $\int_{0}^{\pi/2} \frac{1}{1+\cot x}dx$

(ii) $\int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ (iii) $\int_{0}^{\pi/2} \frac{\cos^n x}{\cos^n x+\sin^n x}dx$

(iv) $\int_{0}^{\pi/4} \frac{\sin^2 x}{\sin^2 x+\cos^2 x}dx$ **(Apr. 2010)**

(v) $\int_{0}^{\pi/2} \frac{1}{1+\tan^3 x}dx$ (OR) $\int_{0}^{\pi/2} \frac{\cos^3 x}{\cos^3 x+\sin^3 x}dx$

(vi) $\int_{0}^{\pi/2} \frac{\sin^4 x}{\sin^4 x+\cos^4 x}dx$ **(Apr. 2011)** (vii) $\int_{0}^{\pi/2} \frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx$

(viii) $\int_0^{\pi/2} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx$ (Oct. 2018; 2013)

(ix) $\int_0^{\pi/2} \frac{\sin^{12} x}{\sin^{12} x + \cos^{12} x} dx$ (Oct. 2013)

(x) $\int_0^{\pi/2} \frac{\sin^{20} x}{\sin^{20} x + \cos^{20} x} dx$ (Apr. 2013 ; Oct. 2010)

(xi) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$ (Oct. 2017; Apr. 2009, 2008, 2003)

15. (i) $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$ (Apr. 2018)

(ii) $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\cot x}} dx$ (Oct. 2015, 2010 ; Apr. 2012)

(iii) $\int_0^{\pi/2} \frac{\operatorname{cosec} x}{\operatorname{cosec} x + \cot x} dx$

16. $\int_0^{3} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{3-x}} dx$ (Apr. 2016, 2013, 2012)

17. $\int_0^{\pi/2} \log \cot x \, dx$

18. (i) $\int_0^{\pi/2} \log \cos x \, dx$ (ii) $\int_0^{1} \frac{\sin^{-1}(x)}{x} dx$ [Put $x = \sin\theta$] (Oct. 2015, 2009, 2008 ; Apr. 2010, 2009)

19. Show that $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

(or) Evaluate $\int_0^{\pi/4} \log(1+\tan x)\, dx$ (Apr. 2008)

20. $\int_0^{\pi/2} \sin 2x \log \tan x \, dx$ (Oct. 2017, 2006; Apr. 2018)

21. $\int_0^{\pi} \frac{1}{1+\sin x} dx$

III. Integration of Mod Function

22. (i) $\int_{-1}^{1} |x|\,dx$ (ii) $\int_{0}^{2} |1-x|\,dx$

(iii) $\int_{-2}^{2} |1-x|\,dx$ (iv) $\int_{0}^{4} |x-2|\,dx$

(v) $\int_{-1}^{1} e^{|x|}\,dx$

ANSWERS

1. (i) 63 (ii) $\frac{5}{4}$ (iii) $\frac{6}{5}$ (iv) $\frac{7}{6}$

(v) $\frac{12}{21}$ (vi) $\frac{4}{3}$ (vii) (a) $\frac{23}{6}$ (b) $\frac{16}{3}$ (viii) (a) 2 (b) 2

(ix) $\frac{40}{3}$ (x) $\frac{1}{6}$

2. (i) $\frac{7}{2}+\log 2$ (ii) $\frac{2}{3}$ (iii) $\frac{293}{12}$ (iv) $\log\frac{5}{3}$

(v) $\log\sqrt{2}$ (vi) $\frac{1}{2}\left[e^5 - e\right]$ (vii) $1-e^{-1}$

3. (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{12}$

(v) $\frac{\pi}{3}$ (vi) 4π

4. (i) 1 (ii) $1-\frac{\pi}{4}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{2}{3}$

(v) $\frac{3\pi}{16}$

5. (i) $\frac{1}{2}$ (ii) $-\frac{3}{7}$ (iii) 0

6. (i) ± 2 (ii) $\pm 2\ (1-\sqrt{2})$ (iii) $\sqrt{2}$ (iv) $\sqrt{2}$

7. (i) $1-\frac{\sqrt{\pi}}{2}$ (ii) $\frac{\sqrt{\pi}}{2}-1$ (iii) $\frac{5}{4}\log 2+\frac{\pi}{8}$

8. (i) $2(e-1)$ (ii) (a) $5\sqrt{15}$ (b) $\log\left(1+\frac{\sqrt{3}}{2}\right)$ (iii) $\frac{-1}{4}\log\frac{5}{7}$

(iv) $\log\frac{3}{2}$ (v) $\frac{\pi}{2}-1$

9. (i) $\frac{\pi}{3}$

10. (i) $\frac{5\pi}{8}$ (ii) $\frac{\pi^3}{192}$ (iii) $\frac{\pi^3}{24}$ (iv) $\frac{1}{2}$

(v) $\frac{1}{2}(1-\log 2)$ (vi) $\frac{1}{72}$

11. (i) $\tan^{-1}(e)-\frac{\pi}{4}$ (ii) $\frac{1}{2}$

12. (i) -2 (ii) $\frac{\pi}{2}-1$ (iii) -2π

(iv) $\frac{\pi}{4}-\log\sqrt{2}$ (v) $\frac{\pi-2}{4}$ (vi) $\frac{\pi}{12}-\frac{1}{6}+\frac{1}{6}\log 2$

(vii) $\frac{\pi}{4}$ (viii) $\frac{-1}{4}(13e^{-4}-1)$

(ix) $8\log 4-\frac{9}{2}\log 3-\frac{7}{4}$

13. (i) 0 (ii) 0 (iii) 0 (iv) $\frac{250}{3}$

(v) 0 (vi) 0 (vii) 3 (viii) 16

Correct the problem as $\int_{-2}^{2}(4x+3x^2+7x^5+12x^7)\,dx$

(ix) 0 (x) 0 (xi) 0

14. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{4}$

(v) $\frac{\pi}{4}$ (vi) $\frac{\pi}{4}$ (vii) $\frac{\pi}{4}$ (viii) $\frac{\pi}{4}$

(ix) $\frac{\pi}{4}$ (x) $\frac{\pi}{4}$ (xi) $(a+b)\frac{\pi}{4}$

15. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{4}$

16. $\frac{3}{2}$

17. 0

18. (i) $\frac{-\pi}{2}\log 2$ (ii) $\frac{\pi}{2}\log 2$

20. 0

21. π

22. (i) 1 (ii) 1 (iii) 5 (iv) 4

(v) $2(e-1)$