Principal Component Analysis (PCA)

In this note, I provie an example for principal component analysis. The analysis is carried out using SAS language.

Suppose we have X number of variables and we put them in the following matrix form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_p \end{pmatrix}$$

The Variance-Covariance Matrix is defind as follows:

$$\Sigma = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22}^2 & \dots & \sigma_{2p} \\ \dots & \dots & \dots & \dots \\ \sigma_{p1}^2 & \sigma_{2p} & \dots & \sigma_{pp}^2 \end{pmatrix}$$

The principal components are the linear combination of x-variables and the one that has maximum variance (among all linear combinations) is considered the first principal component, which accounts for as much variation in the data as possible.

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_p \end{pmatrix} = \begin{pmatrix} e_{11}X_1 + e_{12}X_1 + \dots + e_{1p}X_1 \\ e_{21}X_2 + e_{22}X_2 \dots + e_{2p}X_2 \\ \dots \\ e_{p1}^2X_p + e_{2p}X_p + \dots + e_{pp}X_p \end{pmatrix}$$

The e values are eigenvectors or we can consider them as the coefficients in the linear regression:

e_i = eigenvectors

λ_i = eigenvalues

We can re-write Σ in terms of eigenvectors and eigenvalues. $\Sigma = \sum_{i=1}^{p} \lambda_i e_i e_i'$

trace(Σ) = $\sigma_1^2 + \sigma_2^2 + \dots + \sigma_p^2 = \lambda_1 + \lambda_2 + \dots + \lambda_p$

Where Proportion of the total variation for every i,

$$\frac{\lambda_i}{\lambda_1 + \lambda_2 + \dots + \lambda_p}$$

The total proportion of variation due to the first k components: $\frac{\lambda_1 + \lambda_2 + \dots + \lambda_k}{\lambda_1 + \lambda_2 + \dots + \lambda_p} \sim 1$

if this is large enough then we don't have to consider any more. The first K components would suffice. We must find k in such a way that the above ratio is cloes to one.

Suppose a test case where we want to rank a few locations. However, the ranking can be done using several different variables such as :

Climate and Terrain, Housing, Health Care & Environment, Crime, Transportation, Education, The Arts, Recreation, Economics

In the following SAS code, we will try to carry out the Principal Component Procedure.

First read the data from the stored location:

In [ ]:
```
options ls=78;
title "PCA - Covariance Matrix - Ranks";
data places;
  infile "Storage/Folder/data.txt";
```

Then define a column for each nine variable as well as an ID column to identify the location we're looking at:

In [ ]:
```
input climate housing health crime trans educate arts recreate econ id
```

Then normalize the data by taking the logarithm of all variables. The reason being that the values tend to be skewed:

In [ ]:
```
  climate=log10(climate);
  housing=log10(housing);
  health=log10(health);
  crime=log10(crime);
  trans=log10(trans);
  educate=log10(educate);
  arts=log10(arts);
  recreate=log10(recreate);
  econ=log10(econ);
  run;
```

The princomp procedue will then carry out the principal component analysis. Make sure to pass on the "cov" option so that SAS uses the covariance-variance matrix to do the analysis as opposed to correlation matrix:

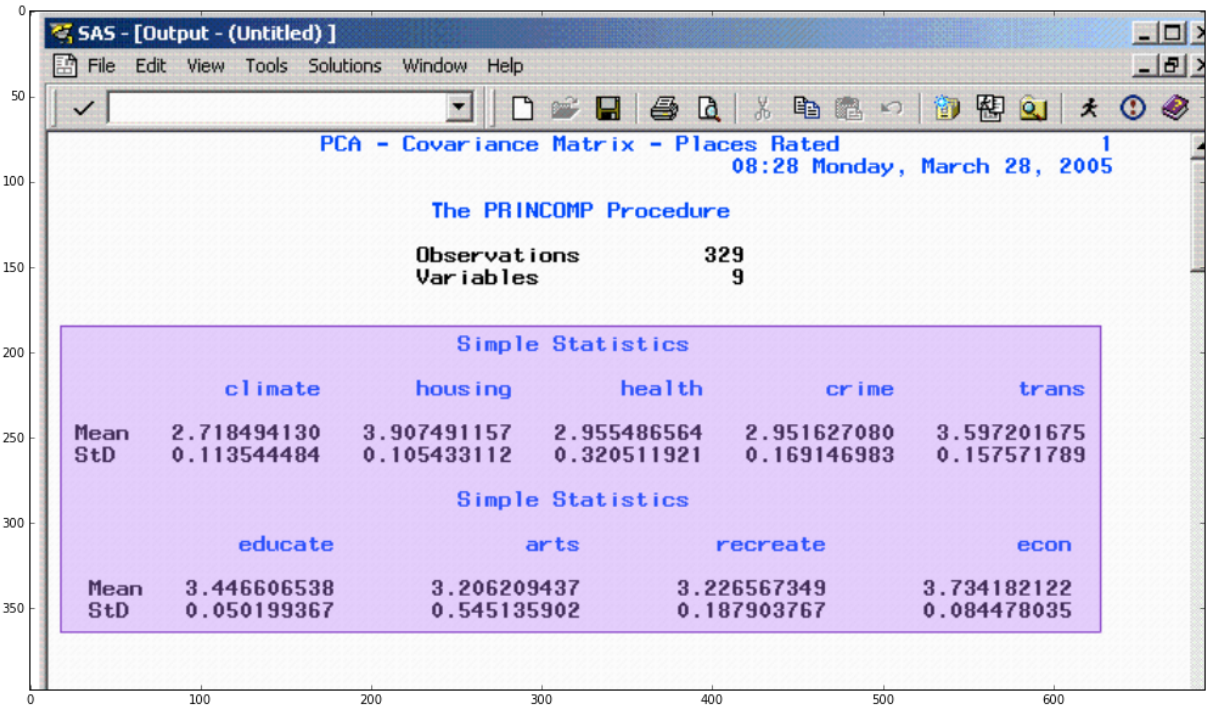
In [ ]:
```
proc princomp cov out=a;
  var climate housing health crime trans educate arts recreate econ;
  run;
```

The first result that shows up after running this piece of code is the following. It shows the

Mean and starndard deviation of all nine variables:

In [4]:

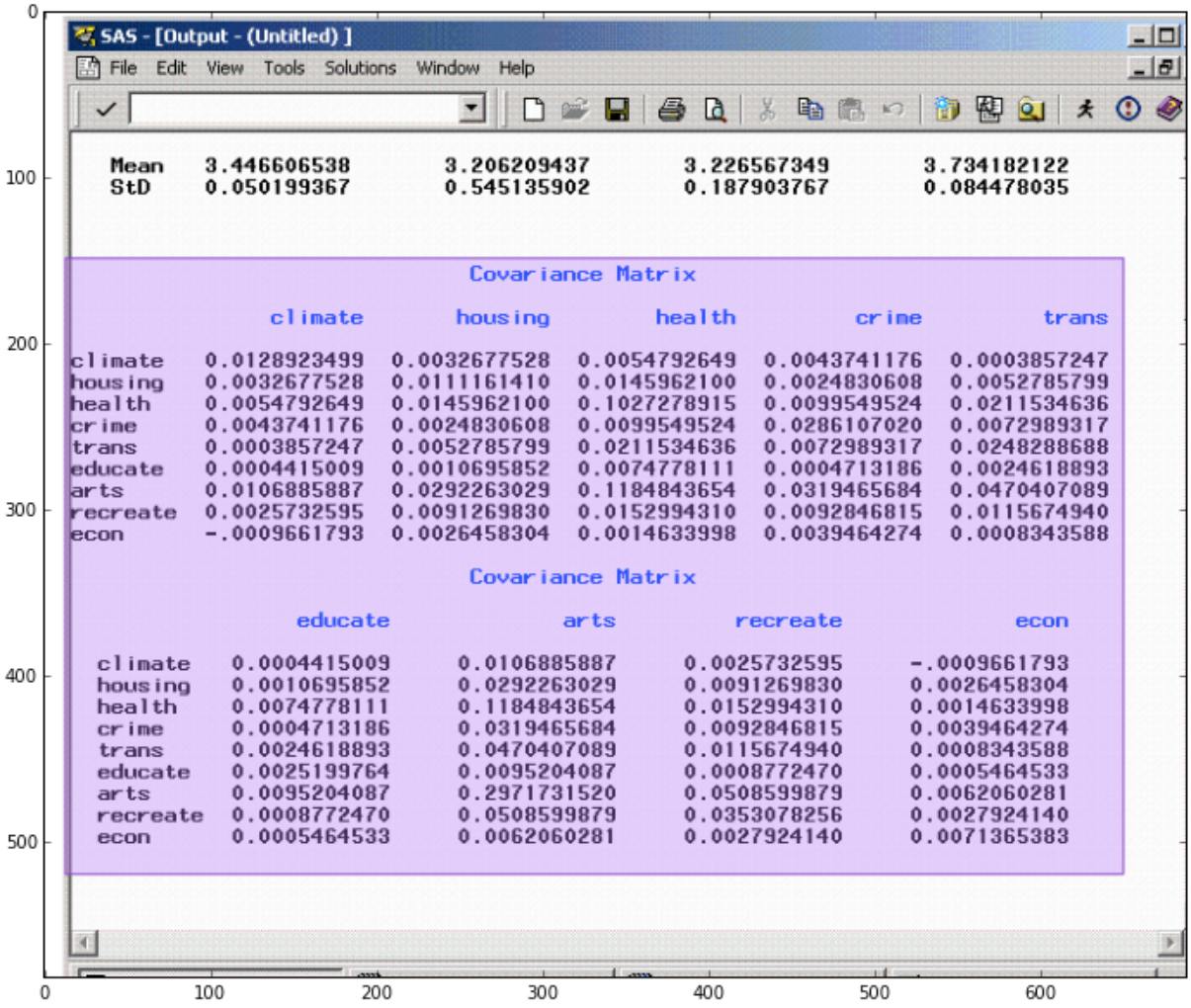
Out[4]: <matplotlib.image.AxesImage at 0x106eb2c90>



Followed by the variance-covariance matrix:

In [5]:

Out[5]: <matplotlib.image.AxesImage at 0x107264710>



In the image shown below:

The important part however, is the eigenvalue and eigenvectors, the following shows them in descending order. If you add all of the eigen values you get the total variance of 0.5223.

The proportion of variation is also given by each λ_i divided by the total $\sum \lambda$ which is given in the third column. So for example, aboute 72% of the variation is explained by the first eigenvalue (0.377).

The cumulative percentage is shown in the fourth column of Eigenvalues matrix. It represents the Proportion of every new eigen vector added to the cumulative of the previous variable. It goes up until it reaches one.

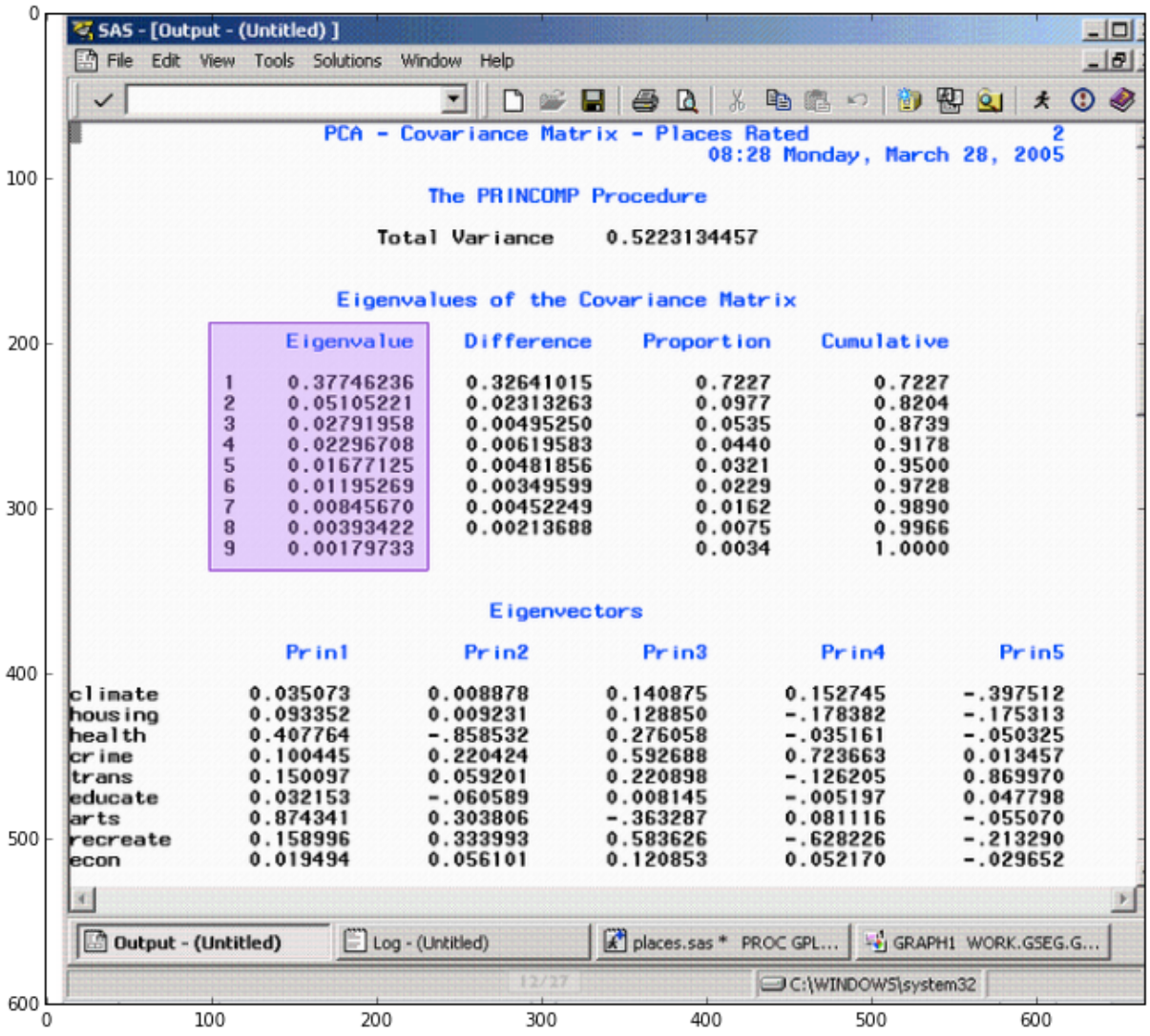
Another way to put it is that, for instance 72% of the variation is explained by the first eigenvalue, then 82% is represented by the first two eigenvalues together and so on...

In the second column, we look at the difference between every eigenvalue and it's next value. We are specially looking for sharp drops. This is the case in between the first and second eigenvalues.

In the eigenvector table or matrix, each column represents the eigenvectors or coefficients for that linear combinations. we use this to compute the scores of the corresponding principal components (first, second,etc)

In [7]:

Out[7]: <matplotlib.image.AxesImage at 0x107797f90>



Next :

we use the "CORR" procedure to find the correlation between the principal component scores and the original variables:

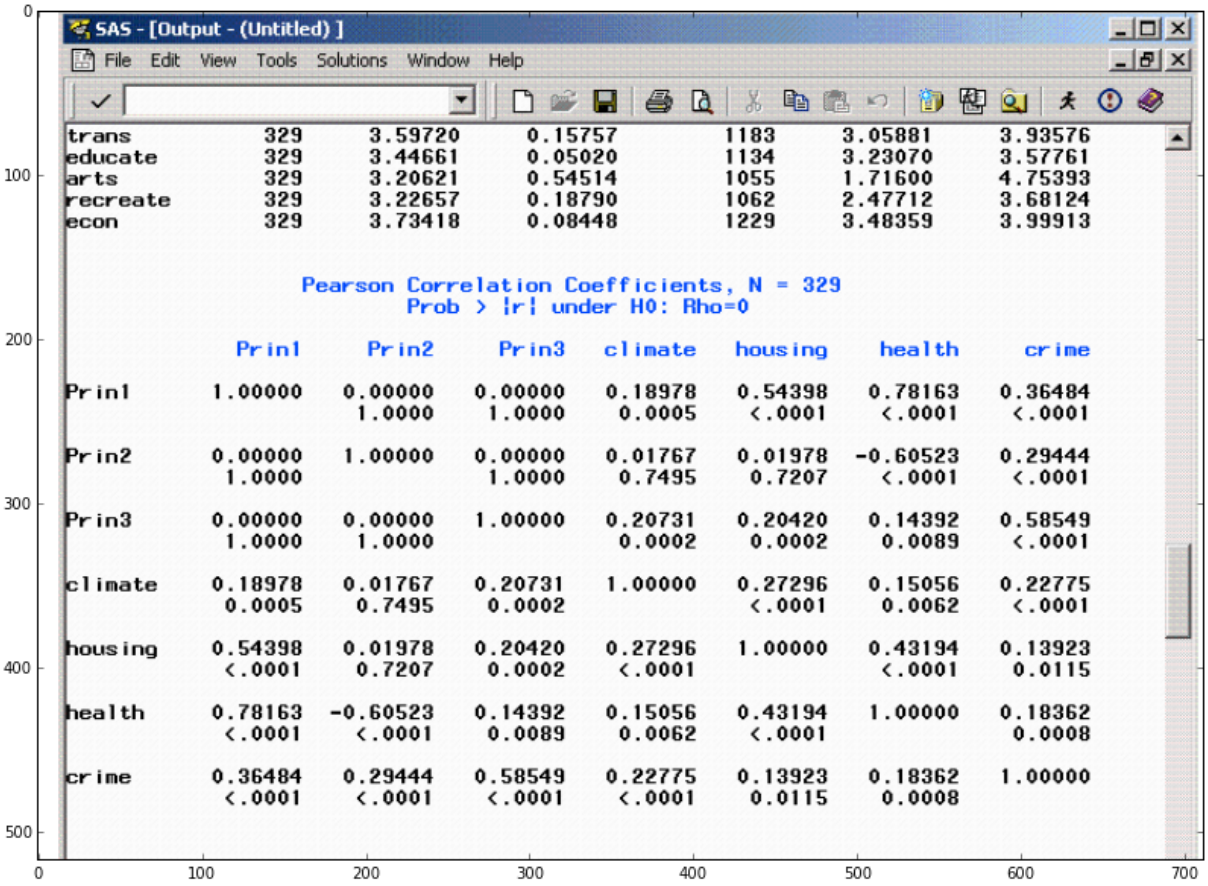
Note that we are passing on all the nine variables we originally defined as well as the first three principapl components prin1-3. The relations are given in the image that follows.

Principal components 1 to 3, represent 87% of the variation in the data and that is deemed sufficient in this study. Note that these choices are subjective and vary from one case to another.

In [ ]:
```
proc corr;
  var prin1 prin2 prin3 climate housing health crime trans educate art
      recreate econ;
  run;
```

In [8]:
```
```

Out[8]: <matplotlib.image.AxesImage at 0x108bc6a90>



In order to interpret the results of the above correlation table we need to first define what is the threshold of importance. In this case we use 0.5 and above to be important and mark the important results in the following table:

Variable	PC1	PC2	PC3
Climate	0.190	0.017	0.207
Housing	**0.544**	0.020	0.204
Health	**0.782**	**-0.605**	0.144
Crime	0.365	0.294	**0.585**

Transportation	**0.585**	0.085	0.234
Education	0.394	-0.273	0.027
Arts	**0.985**	0.126	-0.111
Recreation	**0.520**	0.402	**0.519**
Economy	0.142	0.150	0.239

This means that the first principal component is strongly correlated with Housing,Healthy, Arts, and Recreation. This means that with one increasing, the others will also increase.

The second principal component (PC2) is negatively correlated with Health, meaning that the principal component increases with decreasing health. This can be a measure of how poor health care is in certain location.

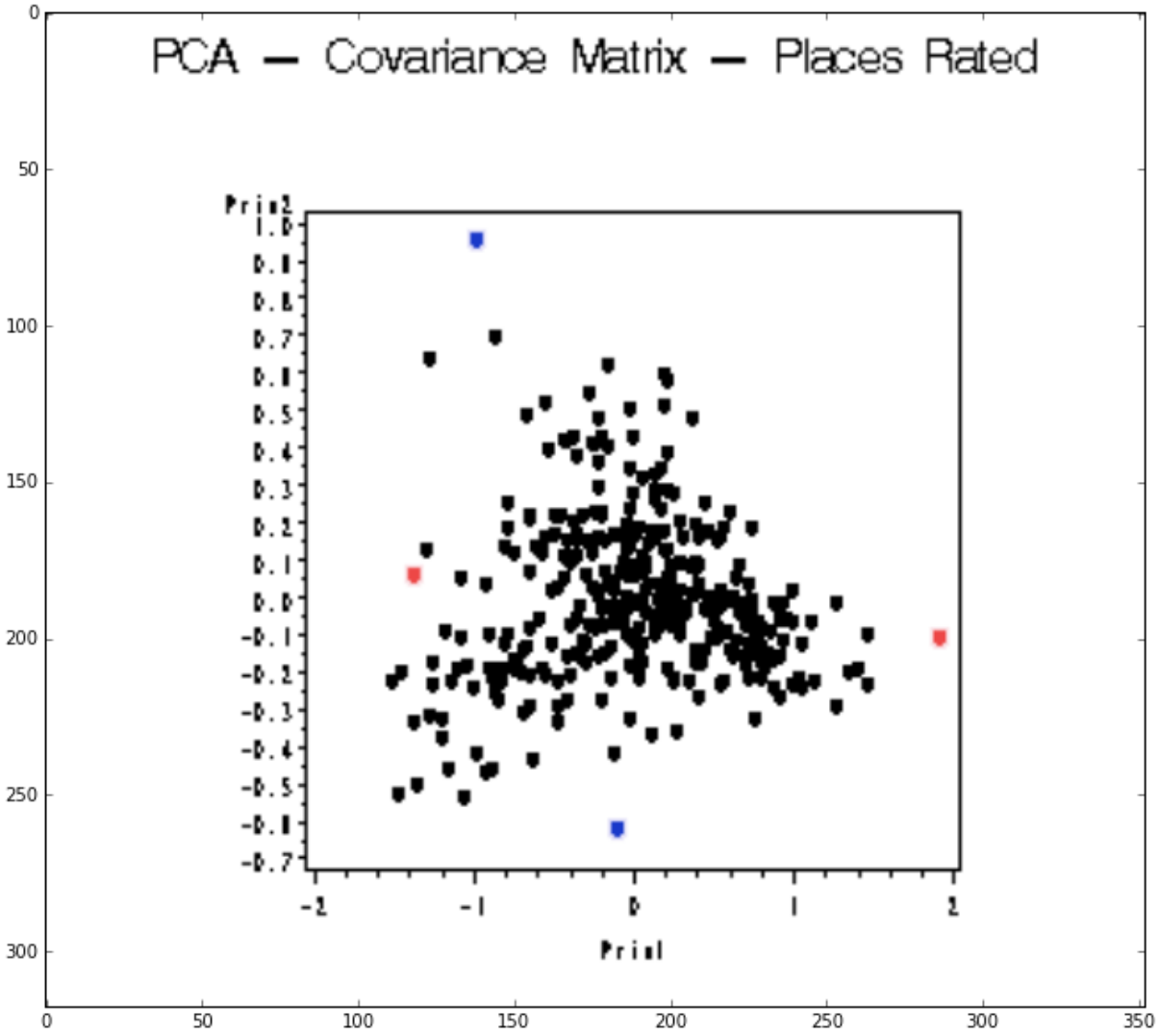
The third principal components is positively correlated with the recreation and crime. This means that these two variables increase together, so one possible conclusion can be that places with higher crime rate also have higher recreation facilities.

One more step in our analysis can be to produce a scatter plot of the first two principal components. This block of code will plot first versus second principal components;

In [ ]:
```
proc gplot;
  axis1 length=5 in;
  axis2 length=5 in;
  plot prin2*prin1 / vaxis=axis1 haxis=axis2;
  symbol v=J f=special h=2 i=none color=black;
  run;
```

In [9]:

Out[9]: <matplotlib.image.AxesImage at 0x108c312d0>



Every black data point is one of the communities we were originally trying to rank. The are a few outliers that are separated by colors.

NOTE: this is a practice case just to demonstrate my understanding of the PCA analysis using SAS (citation (https://onlinecourses.science.psu.edu/stat505/)).