1) Assume $n = 2^K = K = \log_2 n$

$T(n) = T\left(\frac{n}{2}\right) + O(1)$

$= \left[T\left(\frac{n}{2^2}\right) + O(1)\right] + O(1)$

$= \left(T\left(\frac{n}{2^2}\right) + 2 \times O(1)\right]$

$= \left[T\left(\frac{n}{2^3}\right) + O(1)\right] + 2 \times O(1)$

$= T\left(\frac{n}{2^3}\right) + 3 \times O(1)$

$\vdots$

$= T\left(\frac{n}{2^K}\right) + K \times O(1)$

$= T(1) + \log_2 n \times O(1)$

$= O(\log n)$

2) Assume $n = 2^K = K = \log_2 n$

$T(n) = 2T\left(\frac{n}{2}\right) + 17$

$= 2\left[2T\left(\frac{n}{2^2}\right) + 17\right] + 17$

$= 2^2 T\left(\frac{n}{2^2}\right) + 2^1 \times 17 + 2^0 \times 17$

$= 2^2\left[2T\left(\frac{n}{2^3}\right) + 17\right] + 2^1 \times 17 + 2^0 \times 17$

$= 2^3 T\left(\frac{n}{2^3}\right) + 2^2 \times 17 + 2^1 \times 17 + 2^0 \times 17$

$= 2^3 T\left(\frac{n}{2^3}\right) + 17 \times (2^2 + 2^1 + 2^0)$

$\vdots$

$= 2^k T\left(\frac{n}{2^k}\right) + 17 \times (2^{k-1} + 2^{k-2} + \ldots 2^2 + 2^1 + 2^0)$

$= 2^k T(1) + 17 \times 2^k$

$= 18 \times 2^k$ assuming $T(1) = 1$

$= 18n$

$= O(n)$

Which is also $O(n \log n)$ according to definition of big-oh

3) $O(2^n)$

It generates complete binary tree with height n with work 2^i on ith level

$T(n) = 2^0 + 2^1 + 2^2 + \ldots 2^{n-1}$

$= 2^{n-1}$

$= O(2^n)$

4) ~~Re~~cursion Tree

$T(n)$

$T(n/2+2)$ $T(n/2+2)$ $T(n/2+2)$ $T(n/2+2)$

$T(n/4+4)$ $T(n/4+4)$ $T(n/4+4)$ .... 4 times

each level of recursion tree represent 4 subproblems of size n, there are 4 subproblems at each level, each of size $\frac{n}{2}+2$

~~Assume~~ $T(n) = O(f(n))$ where $f(n)$ is asymptotic upper bound we want to determine; substitute assumption into recurrence relation.

$$T(n) = 4T\left(\frac{n}{2}+2\right)+n$$

Assume $T(n) \leq cf(n)$ for some constant $c>0$

$$cf(n) \leq 4c\left(\frac{n}{2}+2\right)+n$$

Simplify; $cf(n) \leq 2cn + 8c + n$

$$cf(n) \leq (2c+1)n + 8c$$

ans. 4 continued

Let's verify this soln using substitution method

Base case : Assume $T(K) \leq cK$ for constant $c > 0$ for all K such that $0 \leq K \leq n$

Inductive case : assume $T(m) \leq cm$ for all m with $0 \leq m \leq n$; consider $T(n)$

$$T(n) = 4T\left(\frac{n}{2}+2\right)+n$$

$$\leq 4c\left(\frac{n}{2}+2\right)+n$$

$$\leq (2c+1)n+8c$$

solution verified.

5) a) $T_n = 2T\left(\frac{n}{2}\right) + n\log n$ $\therefore a = 2, b = 2$

$\log_2 2 = 1$

~~is [illegible]~~

$\therefore n^{\log_2 2} = n^1 < n\log n \quad \forall n \geq 3$

ie $f(n) = n^{\log_2 (2+\epsilon)} = n\log n$

$\therefore T(n) = \Theta(f(n)) = \Theta(n\log n)$

b) $T(n) = 2T\left(\frac{n}{4}\right) + n^{0.51}$ here $a = 2, b = 4$

$\log_4 2 = 0.5$

$\therefore n^{\log_4 2} = n^{0.5} > n^{0.51}$

ie $f(n) = n^{\log_4 (2-\epsilon)} = n^{0.51}$

$\therefore T(n) = \Theta(n^{\log_4 2}) = \Theta(n^{0.5})$

c) $T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$ here $a = 0.5$, $b = 2$

Now $\log_2 0.5 = -1$

$\therefore n^{\log_2 0.5} = n^{-1} = \frac{1}{n}$

ie $f(n) = n^{\log_2(0.5)} = \frac{1}{n}$

$\therefore T(n) = \Theta(n^{\log_2 0.5} \times \log n) = \Theta\left(\frac{1}{n} \times \log n\right) = \Theta\left(\frac{\log n}{n}\right)$

d) $T(n) = 16T(n/4) + n!$

Here $a = 16$, $b = 4$, Now $\log_4 16 = 2$

$\therefore n^{\log_4 16} = n^2 > f(n)$

$\therefore T(n) = \Theta(n^{\log_4 16}) = \Theta(n^2)$

e) $T(n) = 3T(n/2) + n$

Here $a = 3$, $b = 2$ Now $\log_2^3 = 1.585$

$\therefore n^{\log_2^3} = n^{1.585} > n$, ie $f(n) = n^{\log_2(3-\epsilon)} = n$

$\therefore T(n) = \Theta(n^{\log_2^3}) = \Theta(n^{1.585})$

~~e) ...~~

f) $T(n) = 3T(n/3) + \sqrt{n}$, here $a = 3$, $b = 3$

now $\log_3^3 = 1$, $\therefore n^{\log_3^3} = n^1 > \sqrt{n}$

ie $f(n) = n^{\log_3(3-\epsilon)} = \sqrt{n}$

$\therefore T(n) = \Theta(n^{\log_3^3}) = \Theta(n)$