DD MM AA

TALLER TRANSFORMADA DE FOURIER - Santiago Burgos Salazar

1)

a) $F\{e^{-a|t|}\}$; $a \in \mathbb{R}^+$ $\quad F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} e^{-a|t|} \cdot e^{-j\omega t}\, dt$$

Definimos el valor absoluto

$$|t| = \begin{cases} t & si \quad t \geq 0 \\ -t & si \quad t < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{-a(-t)}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{(-a-j\omega)t}\, dt$$

$$= \frac{1}{(a-j\omega)}\, e^{(a-j\omega)t} \Big|_{-\infty}^{0} + \frac{1}{(-a-j\omega)}\, e^{(-a-j\omega)t} \Big|_{0}^{\infty}$$

$$= \left[\frac{e^{(a-j\omega)(0)}}{a-j\omega} - \cancel{\frac{e^{(a-j\omega)(-\infty)}}{a-j\omega}}^{\,0}\right] + \left[\cancel{\frac{e^{(-a-j\omega)(\infty)}}{-a-j\omega}}^{\,0} - \frac{e^{(-a-j\omega)(0)}}{-a-j\omega}\right]$$

$$= \frac{1}{a-j\omega} - \frac{1}{-a-j\omega} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

b) $F\{\cos(\omega_c t)\}$; $\omega_c \in \mathbb{R}$

$$= \int_{-\infty}^{\infty} \cos(\omega_c t)\, e^{-j\omega t}\, dt \qquad \therefore \cos(\omega_c t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$X(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_c t} + e^{-j\omega_c t}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j\omega_c t} \cdot e^{-j\omega t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega_c t}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{j(\omega_c - \omega)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{j(-\omega_c - \omega)t}\, dt$$

Usando la propiedad de Delta dirac aplicamos

$$\therefore \int_{-\infty}^{\infty} e^{-j\alpha t}\, dt = 2\pi\,\delta(\alpha)$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_c)t}\, dt + \int_{-\infty}^{\infty} e^{-j(\omega+\omega_c)t}\, dt\right]$$

Aplicando

$$= \frac{1}{2}\left[2\pi\,\delta(\omega-\omega_c) + 2\pi\,\delta(\omega+\omega_c)\right]$$

$$\boxed{x(\omega) = \pi\left[\delta(\omega+\omega_c) + \delta(\omega+\omega_c)\right]}$$

c) $F\{\operatorname{sen}(\omega_c t)\}; \quad \omega_c \in \mathbb{R}$

$$= \int_{-\infty}^{\infty} \operatorname{sen}(\omega_c t)\, e^{-j\omega t}\, dt \qquad \therefore \operatorname{sen}(\omega_c t) = \frac{1}{2j}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} \left(e^{j\omega_c t} - e^{-j\omega_c t}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j\omega_c t}\, e^{-j\omega t}\, dt - \int_{-\infty}^{\infty} e^{-j\omega_c t}\, e^{-j\omega t}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j(\omega_c-\omega)t}\, dt - \int_{-\infty}^{\infty} e^{-j(\omega_c+\omega)t}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{-j(\omega-\omega_c)t}\, dt - \int_{-\infty}^{\infty} e^{-j(\omega+\omega_c)t}\, dt\right]$$

Aplicando propiedad de delta de dirac

$$= \frac{1}{2j}\left[2\pi\,\delta(\omega-\omega_c) - 2\pi\,\delta(\omega+\omega_c)\right]$$

$$x(\omega) = \frac{1}{j}\left[\pi\,\delta(\omega-\omega_c) - \pi\,\delta(\omega+\omega_c)\right]$$

$$x(\omega) = \frac{j}{j}\,\frac{1}{j}\left[\pi\,\delta(\omega-\omega_c) - \pi\,\delta(\omega+\omega_c)\right]$$

$$\boxed{x(\omega) = j\pi\left[\delta(\omega+\omega_c) - \delta(\omega-\omega_c)\right]}$$

d) $\mathcal{F}\{f(t)\cos(\omega_c t)\}, \quad \omega_c \in \mathbb{R}; \quad f(t) \in \mathbb{R}, \mathbb{C}$

$$x(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega_c t)\, e^{-j\omega t}\, dt \qquad \therefore \cos(\omega_c t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$x(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(e^{j\omega_c t} + e^{-j\omega_c t}\right) e^{-j\omega t}\, dt$$

$$x(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{j(\omega_c-\omega)t}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{j(-\omega_c-\omega)t}\, dt\right]$$

$$x(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_c)t}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega+\omega_c)t}\, dt\right]$$

$$\omega' = \omega - \omega_c, \qquad \omega'' = \omega + \omega_c$$

$$x(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega' t}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-j\omega'' t}\, dt\right]$$

$$x(\omega) = \frac{1}{2}\left[\mathcal{F}\{f(t)\} + \mathcal{F}\{f(t)\}\right]$$

$$x(\omega) = \frac{1}{2}\left[F(\omega') + F(\omega'')\right]$$

$$\boxed{x(\omega) = \frac{1}{2}\left[F(\omega-\omega_c) + F(\omega+\omega_c)\right]}$$

e) $\mathcal{F}\{e^{-a|t|^2}\}, \quad a \in \mathbb{R}^+$

$$x(\omega) = \int_{-\infty}^{\infty} e^{-a|t|^2} \cdot e^{-j\omega t}\, dt$$

$$|t| = \begin{cases} t & \text{si } t \geq 0 \\ -t & \text{si } t < 0 \end{cases}$$

$$x(\omega) = \int_{-\infty}^{0} e^{-a(-t)^2} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$x(\omega) = \int_{-\infty}^{0} e^{-at^2} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$x(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$x(\omega) = \int_{-\infty}^{\infty} e^{(-at^2 - j\omega t)}\, dt$$

$$x(\omega) = \int_{-\infty}^{\infty} e^{-a\left(t^2 + \frac{j\omega t}{a}\right)}\, dt \qquad \text{Completamos cuadrado}$$

$$-a\left(t^2 + \frac{j\omega t}{a}\right) \Rightarrow -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2\right]$$

$$\Rightarrow -a\left(t + \frac{j\omega}{2a}\right)^2 + a\left(\frac{j\omega}{2a}\right)^2 \Rightarrow -a\left(t + \frac{j\omega}{2a}\right)^2 + a\frac{j^2\omega^2}{4a^2}$$

$$\Rightarrow -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\omega^2}{4a} \qquad \text{Volviendo a la integral}$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a\left[t + \frac{j\omega}{2a}\right]^2} \cdot e^{\frac{-\omega^2}{4a}} dt$$

$$X(\omega) = e^{\frac{-\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(t + \frac{j\omega}{2a}\right)^2} dt \qquad u = t + \frac{j\omega}{2a}$$

$$du = dt$$

$$X(\omega) = e^{\frac{-\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-au^2} du \Longrightarrow \text{Integral de Gauss}$$

$$X(\omega) = e^{\frac{-\omega^2}{4a}} \sqrt{\frac{\pi}{a}} \Longrightarrow \boxed{X(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}}$$

f) $F\{A\,\mathrm{rect}_d(t)\}$, $A, d \in \mathbb{R}$

$$X(\omega) = \int_{-\infty}^{\infty} A\,\mathrm{rect}_d(t)\, e^{-j\omega t} dt$$

$$X(\omega) = A \int_{-d/2}^{d/2} e^{-j\omega t} dt$$

$$X(\omega) = A\left(\frac{e^{-j\omega t}}{-j\omega}\right)\Bigg|_{-d/2}^{d/2}$$

$$X(\omega) = A\left[\frac{e^{-j\omega d/2}}{-j\omega} - \frac{e^{-j\omega(-d/2)}}{-j\omega}\right]$$

$$X(\omega) = \frac{-A}{j\omega}\left[e^{-j\omega d/2} - e^{j\omega d/2}\right]$$

$$X(\omega) = \frac{2A}{\omega}\left[\frac{e^{j\omega d/2} - e^{-j\omega d/2}}{2j}\right]$$

$$X(\omega) = \frac{2A}{\omega} \operatorname{sen}\left(\frac{\omega d}{2}\right)$$

$$X(\omega) = \frac{d/2}{d/2} \frac{2A}{\omega} \operatorname{sen}\left(\omega \frac{d}{2}\right) \qquad \therefore \frac{\operatorname{sen}\left(\omega \frac{d}{2}\right)}{d/2} = \operatorname{senc}\left(\omega \frac{d}{2}\right)$$

$$\boxed{X(\omega) = A d \operatorname{senc}\left(\omega \frac{d}{2}\right)}$$

2) Aplique las propiedades de la transformada de Fourier para resolver

a) $F\{e^{-j\omega_1 t} \cos(\omega_c t)\}$, $\omega_1, \omega_c \in \mathbb{R}$

$$F\{x(t) \cdot y(t)\} = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

$$X(\omega) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-j\omega_1 t} e^{-j\omega t} dt + \int_{-\infty}^{\infty} \cos(\omega_c t)\, e^{-j\omega t} dt\right]$$

$$x(\omega) = \frac{1}{2\pi}\left[F\left\{ e^{-j\omega_1 t} \right\} * F\left\{ \cos(\omega_c t) \right\} \right]$$

$$x(\omega) = \frac{1}{2\pi}\left[2\pi\,\delta(\omega-\omega_1) * \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]\right]$$

$$x(\omega) = \frac{2\pi * \pi}{2\pi}\left[\delta(\omega-\omega_1) * \left(\delta(\omega+\omega_c) + \delta(\omega+\omega_c)\right)\right]$$

$$x(\omega) = \pi\left[\delta(\omega-\omega_1) * \delta(\omega-\omega_c) + \delta(\omega-\omega_1)\,\delta(\omega+\omega_c) \right]$$

$$\boxed{x(\omega) = \pi\left(\delta(\omega-\omega_1-\omega_c) + \delta(\omega+\omega_c-\omega_1)\right)}$$

b) $F\left\{ u(t)\cos^2(\omega_c t) \right\}$, $\omega_c \in \mathbb{R}$

$\therefore \cos^2(\omega_c t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)$

Entonces

$$x(\omega) = F\left\{ u(t) * \left(\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)\right)\right\}$$

$$F\left\{ \frac{1}{2}u(t) + \frac{1}{2}u(t)\cos(2\omega_c t)\right\}$$

Aplicando la propiedad de linealidad.

$$\frac{1}{2}F\left\{u(t)\right\} + \frac{1}{2}F\left\{u(t)\cos(2\omega_c t)\right\}$$

Primera

$$F\left\{u(t)\right\} \Rightarrow \boxed{u(\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}}$$

Segundo

$$F\left\{u(t)\cos(2\omega_c t)\right\} = \frac{1}{2\pi}\left[F\left\{u(t)\right\} * F\left\{\cos(2\omega_c t)\right\}\right]$$

$$= \frac{1}{2\pi}\left[u(\omega) * \left[\pi\left(\delta(\omega-2\omega_c) + \delta(\omega+2\omega_c)\right)\right]\right]$$

Aplicando propiedad de convolucion con Delta Dirac

$$= \frac{1}{2\cancel{\pi}}\left[\cancel{\pi}\left[u(\omega-2\omega_c) + u(\omega+2\omega_c)\right]\right]$$

$$= \frac{1}{2}\left[u(\omega-2\omega_c) + u(\omega+2\omega_c)\right]$$

Entonces

$$F\left\{u(t)\cos^2(\omega_c t)\right\} = \frac{1}{2}u(\omega) + \frac{1}{2}\left(\frac{1}{2}\left[u(\omega-2\omega_c) + u(\omega+2\omega_c)\right]\right)$$

$$= \frac{\pi\,\delta(\omega)}{2} + \frac{1}{2j\omega} + \frac{1}{4}\left[\pi\,\delta(\omega-2\omega_c) + \frac{1}{j(\omega-2\omega_c)} + \pi\delta(\omega+2\omega_c) + \frac{1}{j(\omega+2\omega_c)}\right]$$

c) $F^{-1}\left\{\frac{7}{\omega^2+6\omega+45} * \frac{10}{(8+j\omega/3)^2}\right\} = z(t)$

$$F^{-1}\left\{X(\omega) * Y(\omega)\right\} = 2\pi\, x(t) \cdot y(t)$$

Entonces

$F^{-1}\left\{\frac{7}{\omega^2+6\omega+45}\right\}$ completamos cuadrados

$$\omega^2+6\omega+45 = (\omega+3)^2 - 3^2 + 45 = (\omega+3)^2 + 6^2$$

$$F^{-1}\left\{\frac{7}{(\omega+3)^2+6^2}\right\} = \frac{7}{6} F^{-1}\left\{\frac{6}{(\omega+3)^2+6^2}\right\}$$

Aplicando la formula

$$F^{-1}\left\{\frac{b}{(\omega+a)^2+b^2}\right\} = \pi e^{-at}\, sen(bt)\, u(t)$$

$$x(t) = \frac{7}{6}\pi e^{-3t} sen(6t)\, u(t)$$

Ahora

$$F^{-1}\left\{\frac{10}{(8+j\omega/3)^2}\right\} = F^{-1}\left\{\frac{10}{\left(\frac{24+j\omega}{3}\right)^2}\right\} = F^{-1}\left\{\frac{10}{\frac{(24+j\omega)^2}{9}}\right\}$$

$$F^{-1}\left\{\frac{90}{(24+j\omega)^2}\right\}$$

Aplicando la formula

$$F^{-1}\left\{\frac{1}{(j\omega+a)^2}\right\} = t\, e^{-at}\, u(t)$$

$$y(t) = 90\, t\, e^{-24t} u(t)$$

Ahora entonces

$$z(t) = 2\pi\left(\frac{7}{6}\pi e^{-3t} sen(6t)\, u(t)\right)\left(90 t e^{-24t} u(t)\right)$$

$$\boxed{z(t) = 2\pi\, u(t)\left[\frac{7}{6}\pi e^{-3t} sen(6t) \cdot 90\, t\, e^{-24t}\right]}$$