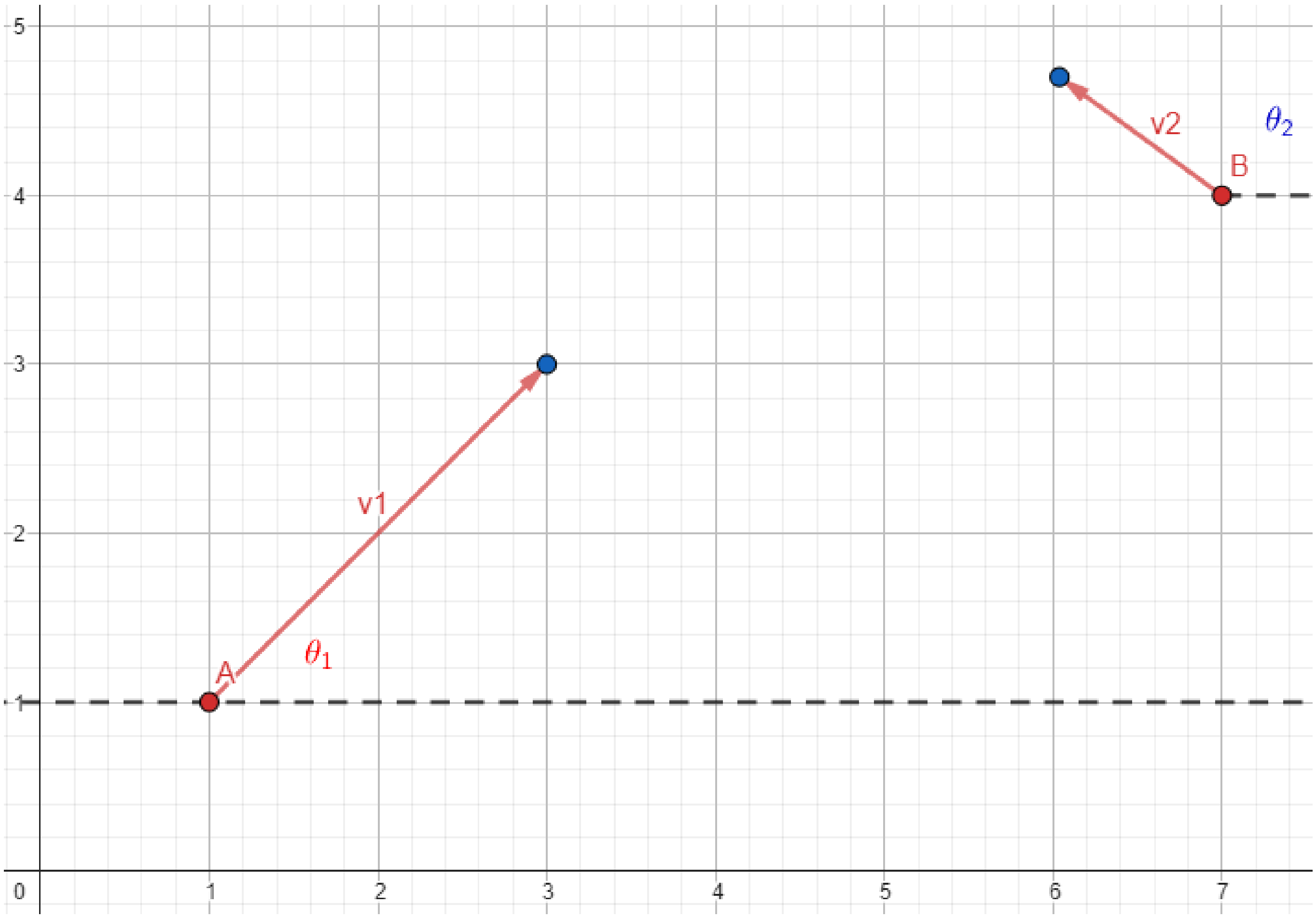
Projectile Intercept

Basic Info



Known:

1. Coordinates of A
2. Magnitude and direction of $\vec{v_1}$ (given θ_1)
3. Coordinates of B
4. Magnitude of $\vec{v_2}$

Want:

1. Direction of $\vec{v_2}$ (meaning, θ_2) in order to intercept v1

Assumptions:

- No acceleration, |v1| and |v2| are constant

- Projectiles 1 and 2 are launched at the same instant t_{start}

Approach

To find the angle at which projectile 2 should be launched (since we know the direction and magnitude of the velocity of projectile 1, and we know the magnitude of the velocity of projectile 2), we establish the system of equations that state that the (x, y) coordinates of the two projectiles have to be the exact same at some instant t_{final} (I won't label t as t_{final} in the equations to avoid cluttering). Using the named values of the image above, we can thus write:

$$\begin{aligned} A_x + v_{1,x}t = B_x + v_{2,x}t \\ A_y + v_{1,y}t = B_y + v_{2,y}t \end{aligned} \tag{1}$$

Notice that Equation 1 has 3 unknowns:

$$Unkowns : v_{2,x}, v_{2,y}, t$$

We are trying to solve for t. We can also say that, since $\vec{v_2}$ is a vector, $v_{2,x}$ and $v_{2,y}$ are related by knowing $|\vec{v_2}|$.

$$\begin{aligned} |\vec{v_2}| = \sqrt{v_{2,x}^2 + v_{2,y}^2} \\ v_{2,y} = \sqrt{|\vec{v_2}|^2 - v_{2,x}^2} \end{aligned} \tag{2}$$

If we plug Equation 2 into the second part of Equation 1, we get:

$$\begin{aligned} A_x + v_{1,x}t = B_x + v_{2,x}t \\ A_y + v_{1,y}t = B_y + t\sqrt{|\vec{v_2}|^2 - v_{2,x}^2} \end{aligned} \tag{3}$$

Equation 3 now only has 2 unknowns:

$$Unknowns : v_{2,x}, t$$

With 2 unknowns and 2 equations, we can solve the problem.

The Algebra

There is likely a neater way to solve this with linear algebra, but I'm going to write it all out explicitly for clarity.
To solve for t, we need to write one of the equations in Equation 3 in terms of a single unknown. To do this, we can write one of our unknowns -- $v_{2,x}$ -- in terms of the other unknown -- t. From the first part of Equation 3, put $v_{2,x}$ alone on one side:

$$v_{2,x} = v_{1,x} + \frac{A_x - B_x}{t} \tag{4}$$

We can then plug Equation 4 into Equation 3:

$$A_y + v_{1,y}t = B_y + t\sqrt{|\vec{v_2}|^2 - v_{1,x}^2 - \frac{2v_{1,x}(A_x - B_x)}{t} - \frac{(A_x - B_x)^2}{t^2}} \tag{5}$$

Let's now put the square-root to one side on its own in Equation 5:

$$\frac{(A_y - B_y)}{t} + v_{1,y} = \sqrt{|\vec{v_2}|^2 - v_{1,x}^2 - \frac{2v_{1,x}(A_x - B_x)}{t} - \frac{(A_x - B_x)^2}{t^2}} \tag{6}$$

Let's now square both sides to get rid of the square root:

$$\begin{aligned} \left[\frac{(A_y - B_y)}{t} + v_{1,y}\right]^2 &= |\vec{v_2}|^2 - v_{1,x}^2 - \frac{2v_{1,x}(A_x - B_x)}{t} - \frac{(A_x - B_x)^2}{t^2} \\ \frac{(A_y - B_y)^2}{t^2} + \frac{2v_{1,y}(A_y - B_y)}{t} + v_{1,y}^2 &= |\vec{v_2}|^2 - v_{1,x}^2 - \frac{2v_{1,x}(A_x - B_x)}{t} - \frac{(A_x - B_x)^2}{t^2} \end{aligned} \tag{7}$$

Let's multiple Equation 7 by t^2 to put everything into a more recognizable polynomial:

$$\begin{aligned} \left[\frac{(A_y - B_y)^2}{t^2} + \frac{2v_{1,y}(A_y - B_y)}{t} + v_{1,y}^2\right] * t^2 &= \left[|\vec{v_2}|^2 - v_{1,x}^2 - \frac{2v_{1,x}(A_x - B_x)}{t} - \frac{(A_x - B_x)^2}{t^2}\right] * t^2 \\ (A_y - B_y)^2 + 2v_{1,y}(A_y - B_y)t + v_{1,y}^2 t^2 &= (|\vec{v_2}|^2 - v_{1,x}^2)t^2 - 2v_{1,x}(A_x - B_x)t - (A_x - B_x)^2 \end{aligned}$$

Now put everything in Equation 8 onto one side by subtracting the left side of the equation from both sides, grouping like-terms together:

$$0 = (|\vec{v_2}|^2 - v_{1,x}^2 - v_{1,y}^2)t^2 + (-2v_{1,x}(A_x - B_x) - 2v_{1,y}(A_y - B_y))t + \left[-(A_x - B_x)^2 - (A_y - B_y)^2\right]$$

And now you have a polynomial that can be solved using the quadratic formula. Do some variable substitution as:

$$\begin{aligned} A &= |\vec{v_2}|^2 - v_{1,x}^2 - v_{1,y}^2 \\ B &= (-2v_{1,x}(A_x - B_x) - 2v_{1,y}(A_y - B_y)) \\ C &= \left[-(A_x - B_x)^2 - (A_y - B_y)^2\right] \\ 0 &= At^2 + Bt + C \end{aligned} \tag{10}$$

All values in Equation 10 are known. Remembering the quadratic formula as:

$$t_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We can plug our substitutions from Equation 10 into the quadratic formula to solve for t_1 and t_2. Now we have solved for t, so what do we do with it? Important to note: for a Python script, using t_1 is best.

$$t_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Finding the angle of $\vec{v_2}$ from t

Going back to Equation 1, we can plug our solved time t into either equation to solve for one of the components of $\vec{v_2}$:

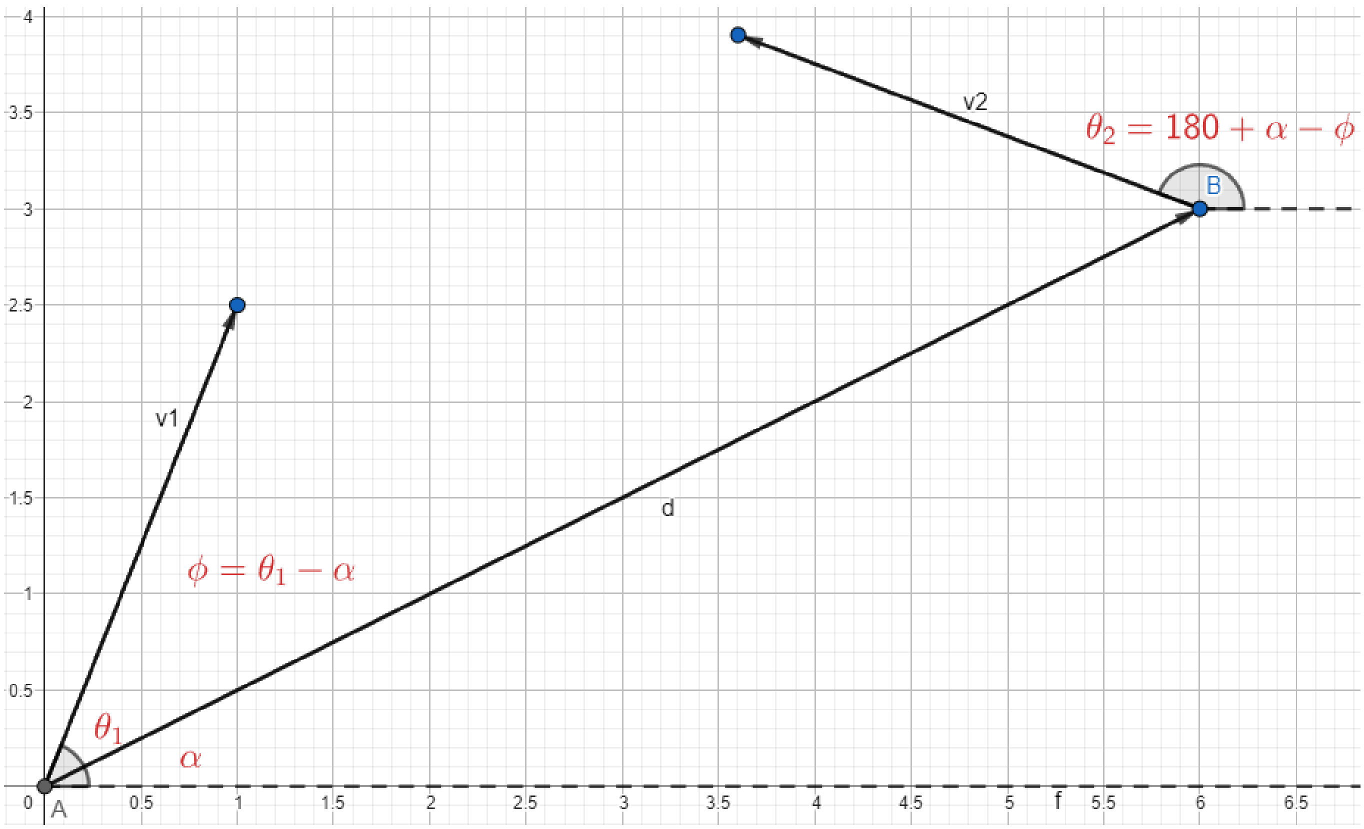
$$\begin{aligned} &\text{If using t1:} \\ A_x + v_{1,x}t_1 &= B_x + v_{2,x}t_1 \\ v_{2,x} &= \frac{(A_x - B_x)}{t_1} + v_{1,x} \\ \\ A_y + v_{1,y}t_1 &= B_y + v_{2,y}t_1 \\ v_{2,y} &= \frac{A_y - B_y}{t_1} + v_{1,y} \end{aligned} \tag{11}$$

And since we know the components of $\vec{v_2}$, we can finally find its angle:

$$\theta_2 = \arctan_2(v_{2,y}, v_{2,x}) \tag{12}$$

Special Case

A special case arises when both projectiles have the same speed (the same $|\vec{v}|$). The solution presented here does not accurately solve for that, as the A value in Equation 10 becomes zero, which leads to an undefined value for $t_{1,2}$.
However, this special case can be solved with simple geometry. Consider points A and B are the launcher positions, vector $\vec{d}$ is the vector from launcher A to launcher B, and vectors $\vec{v_1}$ and $\vec{v_2}$ are the bullet velocity vectors. Like before, we already know the full definition of $\vec{v_1}$ (including the value of θ_1), but we don't yet know the direction of $\vec{v_2}$. If the speeds of bullet 1 and bullet 2 are the same (meaning, if $|\vec{v_1}| = |\vec{v_2}|$), then finding the angle θ_2 is fairly simple without needing difficult calculations.



To summarize the important values in the image:

$$\begin{aligned}
\alpha &= \text{Angle between vector } \vec{d} \text{ and the horizontal} \\
\theta_1 &= \text{Angle between } \vec{v_1} \text{ and the horizontal} \\
\phi &= \text{Angle between } \vec{v_1} \text{ and } \vec{d} \\
\theta_2 &= \text{Angle between } \vec{v_2} \text{ and the horizontal}
\end{aligned}$$