How does it work?

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Important note: This file is still under construction. Feel free to contact me at sancharsharma@gmail.com if you have suggestions for additions.

Let's start with a monomial, think of an arbitrary one, say $a^2a^{\dagger,3}a^5a^\dagger aa^{\dagger,6}$. Its very difficult to interpret many of such terms. However, one can write them in a standard form that has only one annihilation or creation operator and the pre-factor is a function of number operator. The above expression can be written as

$$n_a\left(1+n_a\right)^2\left(2+n_a\right)^2\left(3+n_a\right)\left(4+n_a\right)^2a^{\dagger,2}.$$

This is far more transparent, ain't it? It is also useful because this way we can combine terms in a random polynomial properly. For instance, knowing the above and also,

$$a^3a^{\dagger,3}a^2a^{\dagger,4}a^2a^{\dagger,2}=n_a\left(1+n_a\right)^2\left(2+n_a\right)^2\left(3+n_a\right)\left(n_a-1\right)a^{\dagger,2},$$

it is much easier to simplify

$$a^2a^{\dagger,3}a^5a^\dagger aa^{\dagger,6}-a^3a^{\dagger,3}a^2a^{\dagger,4}a^2a^{\dagger,2}=n_a\left(1+n_a\right)^2\left(2+n_a\right)^2\left(3+n_a\right)\left(n_a^2+7n_a+17\right)a^{\dagger,2}.$$

The question still remains on how to find the standard form? Doing it manually is too laborious. So, I wrote a code.

1 Standard form

The code is a simple divide-and-conquer on monomials. Consider a monomial of the form

$$B_{\boldsymbol{rs}}=b^{s_0}b^{\dagger,r_0}\dots b^{s_{M-1}}b^{\dagger,r_{M-1}}.$$

Obviously, $r_{M-1}=0$ takes care of the case of the last term being an annihilation operator. Now, we want to separate the number functions. For $d=\sum r_m-\sum s_m$, we have two cases depending on the sign of d,

$$B_{\boldsymbol{rs}}=p_{\boldsymbol{rs}}(b^\dagger b)b^{\dagger,d},\ B_{\boldsymbol{rs}}=p_{\boldsymbol{rs}}(b^\dagger b)b^{-d},$$

where $p_{\boldsymbol{rs}}$ are polynomials. Elementary case of $|\boldsymbol{r}|=|\boldsymbol{s}|=1$ is

$$r>s,\ b^sb^{\dagger,r}=b^sb^{\dagger,s}b^{\dagger,r-s}=(n+1)(n+2)\dots(n+s)b^{\dagger,r-s}.$$

$s \geq r,\ b^s b^{\dagger,r} = b^{s-r} b^{\dagger,r} b^{\dagger,r} = b^{s-r}(n{+}1)(n{+}2)\dots(n{+}r) = (n{+}s{-}r{+}1)(n{+}s{-}r{+}1)\dots(n{+}s)b^{s-r}.$

Thus, the polynomial is

$$p_{\{r\}\{s\}} = \prod_{1+\max(0,s-r)}^{s} (n+i).$$

Out of curiosity

$$r > s,\ b^{\dagger,r} b^s = b^{\dagger,r-s} n(n{-}1)\dots(n{-}s{+}1) = (n{-}r{+}s)(n{-}r{+}s{-}1)\dots(n{-}r{+}1)b^{\dagger,r-s}$$

$$r < s,\ b^{\dagger,r} b^s = b^{\dagger,r} b^r b^{s-r} = n(n-1)\dots(n-r+1)b^{s-r}$$

giving the polynomial

$$\prod_{1-r}^{\min(0,s-r)} (n+i).$$

If $|\boldsymbol{r}| = |\boldsymbol{s}| > 1$, we divide the list into two parts $\boldsymbol{r} = \{\boldsymbol{r}_<, \boldsymbol{r}_>\}$ and same for $\boldsymbol{s}$. Recursively, we have four cases depending on the sign of $d_>$ and $d_<$.

$$d_< > 0,\ d_> > 0:\ B_{\boldsymbol{rs}} = p_<(n)b^{\dagger,d_<} p_>(n)b^{\dagger,d_>} = p_<(n)p_>(n-d_<)b^{\dagger,d_<+d_>}.$$

$$d_< > 0,\ d_> \leq 0:\ B_{\boldsymbol{rs}} = p_<(n)b^{\dagger,d_<} p_>(n)b^{-d_>} = p_<(n)p_>(n-d_<)b^{\dagger,d_<} b^{-d_>}.$$

$$d_< \leq 0, d_> > 0:\ B_{\boldsymbol{rs}} = p_<(n)b^{-d_<} p_>(n)b^{\dagger,d_>} = p_<(n)p_>(n-d_<)b^{-d_<} b^{\dagger,d_>}$$

$$d_< \leq 0, d_> \leq 0:\ B_{\boldsymbol{rs}} = p_<(n)b^{-d_<} p_>(n)b^{-d_>} = p_<(n)p_>(n-d_<)b^{-d_<-d_>}$$

The first and last example are already in standard form. The second and third examples require require a last step of putting $b^{\dagger,d_<} b^{-d_>}$ or $b^{-d_<} b^{\dagger,d_>}$, quite simple to do.

1.1 Code details

The first function NumberForm takes in an expression, and find the standard form in a relatively unoptimized form. Given a random expression, say

$$e = 2m^\dagger a^{\dagger,4} g\left(n_a\right) a^3 f\left(n_m\right) m^2 - a^2 m^{\dagger,5} a^{\dagger,3} m q^8,$$

first, we separate the terms and run NumberForm on each of them,

$$NF(e) = NF\left(2m^\dagger a^{\dagger,4} g\left(n_a\right) a^3 f\left(n_m\right) m^2\right) - NF\left(a^2 m^{\dagger,5} a^{\dagger,3} m q^8\right).$$

A simple function figures out all the bosons involved in an expression, i.e. in the first expression $\{m, a\}$ and in the second $\{m, a, q\}$. Then, we want to separate each term into factors which are independent of each other. Taking the first expression, we want to write,

$$2m^\dagger a^{\dagger,4} g\left(n_a\right) a^3 f\left(n_m\right) m^2 = 2 \times f\left(n_m - 1\right) g\left(n_a - 4\right) \times a^{\dagger,4} a^3 \times m^\dagger m^2.$$

This can be done by traversing the expression from left and keeping a track of all signed powers of all bosons. I am defining signed powers via $\mathrm{pow}(m^{\dagger,4}) = 4$, $\mathrm{pow}(m^7) = -7$, $\mathrm{pow}(m^3 m^{\dagger,2} m^8) = -9$ and so on. For multiple vectors, of course, pow is a dictionary like

$$\mathrm{pow}(m^7 a^{\dagger,12} m^{\dagger,2}) = \{a : 12, m : -5\}.$$

Finally, the variable 'ncomm_anncre' is another dictionary storing the relevant operators,

$$\mathrm{anncre} = \{a : a^{\dagger,4} a^3, m : m^\dagger m^2\}.$$

Then, we can simply apply the above algorithm to $a^{\dagger,4} a^3$ and $m^\dagger m^2$ and get

$$2m^\dagger a^{\dagger,4} a^3 f(n_m) m^2 = 2\times f(n_m - 1) g(n_a - 4)(n_a - 1)(n_a - 2)(n_a - 3) n_m a^\dagger m,$$

a nice standard form. This is easily scaleable.

1.2 Normal form

There are some who love normal form where all the annihilation operators are to the right of creation operators. There is an algorithmic way to find those too. First of all, we need a normal ordered expansion, assuming $l \geq k$, of

$$b^k b^{\dagger,l} = \sum_{p=0}^{k} \binom{k}{p} \binom{l}{p} p! b^{\dagger,l-p} b^{k-p}.$$

To prove this by induction, assume the formula for $b^x b^{\dagger,y}$ for $x + y < k + l$. Then,

$$b b^{k-1} b^{\dagger,l} = \sum_{p=0}^{k-1} \binom{k-1}{p} \binom{l}{p} p! b b^{\dagger,l-p} b^{k-1-p}$$

Using

$$\left[b, b^{\dagger,y}\right] = b^\dagger \left[b, b^{\dagger,y-1}\right] + b^{\dagger,y-1} = y b^{\dagger,y-1},$$

$$b b^{k-1} b^{\dagger,l} = \sum_{p=0}^{k-1} \binom{k-1}{p} \binom{l}{p} p! \left((l-p) b^{\dagger,l-p-1} b^{k-1-p} + b^{\dagger,l-p} b^{k-p}\right).$$

A coefficient for p is

$$\binom{k-1}{p}\binom{l}{p} p! + \binom{k-1}{p-1}\binom{l}{p-1}(p-1)!(l-p+1) = \frac{(k-1)!}{p!\,(k-1-p)!}\frac{l!}{(l-p)!} + \frac{(k-1)!}{(p-1)!\,(k-p)!}\frac{l!}{(l-p)!}$$

$$= \frac{k!}{p!\,(k-p)!}\frac{l!}{(l-p)!}$$

exactly what we want! For $l \leq k$,

$$b^k b^{\dagger,l} = \left(b^l b^{\dagger,k}\right)^\dagger = \left(\sum_{p=0}^{l} \binom{k}{p}\binom{l}{p} p! b^{\dagger,k-p} b^{l-p}\right)^\dagger = \sum_{p=0}^{l} \binom{k}{p}\binom{l}{p} p! b^{\dagger,l-p} b^{k-p}.$$

Exactly same as before except that the summation goes only until l. So,

$$b^k b^{\dagger,l} = \sum_{p=0}^{\min(k,l)} \binom{k}{p}\binom{l}{p} p! b^{\dagger,l-p} b^{k-p}.$$

Now, back to the problem at hand. We mostly don't care about the leading $b^\dagger$ term and the final b term. For sequences $\boldsymbol{r}, \boldsymbol{s}$,

$$B_{\boldsymbol{rs}} = b^{s_0} b^{\dagger,r_0} \dots b^{s_{M-1}} b^{\dagger,r_{M-1}}.$$

Let's define the generalized Stirling's numbers via the two possibilities,

$$B_{\boldsymbol{rs}} = b^{\dagger,d(\boldsymbol{r},\boldsymbol{s})} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k,\ B_{\boldsymbol{rs}} = \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k b^{-d(\boldsymbol{r},\boldsymbol{s})},$$

where $d(\boldsymbol{r},\boldsymbol{s}) = \sum r_m - \sum s_m$. If $d < 0$, we get

$$B^\dagger_{\boldsymbol{rs}} = B_{\bar{\boldsymbol{s}}\bar{\boldsymbol{r}}} \Rightarrow b^{\dagger,-d(\boldsymbol{r},\boldsymbol{s})} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = b^{\dagger,d(\bar{\boldsymbol{s}},\bar{\boldsymbol{r}})} \sum_{k=0}^{\infty} \frac{S_{\bar{\boldsymbol{s}}\bar{\boldsymbol{r}}}(k)}{k!} b^{\dagger,k} b^k \Rightarrow S_{\boldsymbol{rs}} = S_{\bar{\boldsymbol{s}}\bar{\boldsymbol{r}}}.$$

For $|\boldsymbol{r}| = 1$, we have

$$d > 0 : S(k) = \frac{s_0! r_0!}{(s_0 - k)!\,(r_0 - s_0 + k)!},\ d < 0 : S(k) = \frac{s_0! r_0!}{(s_0 - r_0 + k)!\,(r_0 - k)!}$$

Now, there is a chance that we don't need to start from $k = 0$. In reality, for $d > 0$,

$$k_{\min} \geq \sum_{m=T}^{M-1} (s_m - r_m)\, \forall T \in \{0, \dots, M-1\}$$

It feels intuitive that $k_{\min}$ should just be the maximum of all these numbers (unless they are all negative) but can't put my finger on the proof. Further, the maximum should be $k_{\max} = \sum s_m$ with the coefficient 1 because of Wick's theorem. Now, let's try to find the recursion. We need the sequence $\boldsymbol{r}_- = \{r_1, r_2, \dots, r_{M-1}\}$ and $\boldsymbol{s}_-$.

Let $d(\boldsymbol{r},\boldsymbol{s}) \geq 0$ and $d(\boldsymbol{r}_-,\boldsymbol{s}_-) \geq 0$, then,

$$b^{\dagger,d} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = b^{s_0} b^{\dagger,r_0} b^{\dagger,d_-} \sum_{k=0}^{\infty} \frac{S_-(k)}{k!} b^{\dagger,k} b^k.$$

$r_0 + d_- - s_0 = d \geq 0$, so

$$b^{\dagger,d} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = \sum_{k=0}^{\infty} \frac{S_-(k)}{k!} \sum_{p=0}^{s_0} \binom{s_0}{p}\binom{r_0 + d_- + k}{p} p! b^{\dagger,r_0+d_-+k-p} b^{s_0+k-p}.$$

$$\sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = \sum_{k=0}^{\infty} \frac{S_-(k)}{k!} \sum_{p=0}^{s_0} \binom{s_0}{p}\binom{s_0 + d + k}{s_0 - p} (s_0 - p)! b^{\dagger,k+p} b^{k+p}.$$

$$S_{\boldsymbol{rs}}(k) = \sum_{p=0}^{\min[s_0,k]} \frac{k! S_-(k-p)}{(k-p)!} \binom{s_0}{p} \binom{s_0+d+k-p}{s_0-p} (s_0-p)!.$$

Let $d(\boldsymbol{r},\boldsymbol{s}) \geq 0$ and $d(\boldsymbol{r}_-,\boldsymbol{s}_-) \leq 0$, then, [calling $d_- = -d(\boldsymbol{r}_-,\boldsymbol{s}_-)$]

$$b^{\dagger,d} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = b^{s_0} b^{\dagger,r_0} \sum_{k=0}^{\infty} \frac{S_-(k)}{k!} b^{\dagger,k} b^k b^{d_-}.$$

Now, $r_0 - s_0 = d_- + d \geq 0$,

$$b^{\dagger,d} \sum_{k=0}^{\infty} \frac{S_{\boldsymbol{rs}}(k)}{k!} b^{\dagger,k} b^k = \sum_{k=0}^{\infty} \frac{S_-(k)}{k!} \sum_{p=0}^{s_0} \binom{s_0}{p} \binom{r_0+k}{p} p! b^{\dagger,r_0+k-p} b^{s_0+k+d_--p}.$$

$$S_{\boldsymbol{rs}}(k) = \sum_{p=0}^{\min[s_0,k-d_-]} \frac{k! S_-(k-p-d_-)}{(k-p-d_-)!} \binom{s_0}{p} \binom{s_0+d+k-p}{s_0-p} (s_0-p)!.$$

This actually finishes the recursion! The case of $d < 0$ can be easily handled by Hermitian conjugating. Then, the recursion continues.

1.3 Sophisticated simplification

There are apparently far more sophisticated methods to simplify such non-commutative polynomials involving not only creation and annihilation operators but also displacement and squeezing operators and what not. It seems that there is a systematic method to find the Gröbner basis involving symbols satisfying any affine commutation relations (not just a Lie algebra).