

**Claim:** No smallest positive real number can exist

**Proof:** 1. **Claim:**  $\frac{1}{2} > 0$

2. **Proof:** By the Trichotomy Property, exactly 1 of the following hold:  
 $\frac{1}{2} > 0$  or  $\frac{1}{2} = 0$  or  $\frac{1}{2} < 0$ .

3. Let,  $\frac{1}{2} < 0$  or  $0 > \frac{1}{2}$  holds 4.  $2 > 0$  (Thm 2.1.8(c))  
5.  $0 \cdot 2 > \frac{1}{2} \cdot 2$  (Thm 2.1.7(c)) 6.  $\frac{1}{2} \cdot 2 = 1$  (M4) 7.  $0 \cdot 2 > 1$  (Substitution of eq 6 in 5)  
8.  $0 \cdot 2 = 2 \cdot 0$  (M1) 9.  $2 \cdot 0 = 0$  (Thm 2.1.2(c))  
10.  $0 \cdot 2 = 0$  (Transitivity of eq on 8, 9) 11.  $0 > 1$  (Substitution of eq 10 in 7)  
12.  $0 < 1$  (Thm 2.1.8(b)) . By the Trichotomy property, 11 is a contradiction

13.  $\therefore \frac{1}{2} < 0$  is False

14. Let,  $\frac{1}{2} = 0$  holds 15.  $\frac{1}{2} \cdot 2 = 0 \cdot 2$  (Substitution of eq 14)  
16.  $\frac{1}{2} \cdot 2 = 0$  (Transitivity of eq on 15, 10) 17.  $0 = \frac{1}{2} \cdot 2$  (Symmetry of eq 16)  
18.  $0 = 1$  (Transitivity of eq 17, 6) 19.  $0 \neq 1$  (M3)  
 $\therefore$  18 is a contradiction, and  $\frac{1}{2} = 0$  is F

$\therefore$  By the Trichotomy property,  $\frac{1}{2} > 0$ . (20)

21.  $a > 0$  (Given) 22.  $\frac{1}{2} \cdot a > 0 \cdot a$  (Thm 2.1.7(c)) 23.  $0 \cdot a = a \cdot 0$  (M1)  
24.  $a \cdot 0 = 0$  (Thm 2.1.2(c)) 25.  ~~$0 \cdot a = 0$~~   $0 \cdot a = 0$  (Transitivity of eq on 23, 24)  
26.  $\frac{1}{2} \cdot a > 0$  (Substitution of eq 25 in 22)

27. **Claim:**  $1 > \frac{1}{2}$

28. **Proof:** By the Trichotomy Property, exactly 1 of the following hold

$1 > \frac{1}{2}$  or  $1 = \frac{1}{2}$  or  $1 < \frac{1}{2}$   
29. Let,  $1 < \frac{1}{2}$  hold. 30.  $1 \cdot 2 < \frac{1}{2} \cdot 2$  (Thm 2.1.7(c), use 4 and 29)  
31.  $1 \cdot 2 = 2$  (M3) 32.  ~~$2 < \frac{1}{2} \cdot 2$~~   $2 < \frac{1}{2} \cdot 2$  (Substitute eq 31 on 30)  
33.  $2 < 1$  (Substitution of eq 6 in 32) 34.  $2 + (-1) < 1 + (-1)$   
(Thm 2.1.7(b))