d) $\sum_{i=0}^{n} i^2 = \Theta(n^3)$ $\sum_{i=0}^{n} i^2 = 0^2+1^2+2^2+3^2+\ldots\ldots+n^2$

$\geq n^2+n^2+n^2+\ldots\ldots+n^2 = n^2 \cdot n = n^3$ (↳ n times)

$\therefore \sum_{i=0}^{n} i^2 \geq n^3$ for all $n \geq 0, n \in N$

$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (Known result) $= \frac{(n^2+n)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$

$= \frac{n^3}{2}+\frac{n^2}{2}+\frac{n}{6}$

Let, $\frac{n^3}{2}+\frac{n^2}{2}+\frac{n}{6} \geq \frac{n^3}{2}$ $\therefore \frac{n^2}{2}+\frac{n}{6} \geq 0 \quad \forall\, n \geq 0$

$\frac{n^3}{2} \leq \sum_{i=0}^{n} i^2 \leq n^3$ for $n \geq 1$ $\quad c_1 = \frac{1}{2}, c_2 = 1, n_0 = 1$

$\therefore \sum_{i=0}^{n} i^2 = \Theta(n^3)$

e) $\sum_{i=0}^{n} i^3 = \Theta(n^4)$ $\sum_{i=0}^{n} i^3 = 0^3+1^3+2^3+\ldots\ldots+n^3 \leq n^3+n^3+n^3+\ldots\ldots+n^3$

$= n^3 \cdot n = n^4$ (↳ n times)

$\therefore \sum_{i=0}^{n} i^3 \leq n^4$ for all $n \geq 0, n \in N$.

$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n^2+2n+1)}{6} = \frac{n^4}{6}+\frac{2n^3}{6}+\frac{n^2}{6} \geq \frac{n^4}{6}$ for all $n \geq 0, n \in N$.

(Known result)

$\frac{n^4}{6} \leq \sum_{i=0}^{n} i^3 \leq n^4$ for $n \geq 1$ $\quad c_1 = \frac{1}{6}, c_2 = 1, n_0 = 1$

$\therefore \sum_{i=0}^{n} i^3 = \Theta(n^4)$

f) $n^{2^n}+6 \cdot 2^n = \Theta(n^{2^n})$ Let, $n^{2^n}+6 \cdot 2^n \geq 2n^{2^n}$ hold,

or, $6 \cdot 2^n \geq n^{2^n}$.

Claim: $6 \cdot 2^n \leq n^{2^n}$ for $n \geq 3, n \in N$

Proof: **Base case:** $n=3$, $6 \cdot 2^3 = 48$ $n^{2^n} = 3^{2^3} = 3^8$ $48 \leq 3^8$

$\therefore$ Base Case holds

I.H: $n=K$, $6 \cdot 2^K \leq K^{2^K}$. **Induction Step:** $6 \cdot 2^{K+1} = 2 \cdot (6 \cdot 2^K)$

$\leq 2K^{2^K}$ (By I.H.)