$\boxed{\text{Prob 21:}}$ a) Prove that p^2-1 is divisible by 24 if p is a prime number greater than 3.

$\because p$ is prime, $\therefore p$ is not divisible by 2, 3 and 6.

• By division algorithm, p can be $6K+1$ or $6K+5$ for $K \in N$.
($6K+2$, $6K+4$ is div by 2 while $6K+3$ is div by 3, $6K$ is div by 6)

If $p = 6K+1$, $p^2-1 = (6K+1)^2-1 = 36K^2+12K = 12(3K^2+K)$

• By division algorithm, K can be $2K_1$ or $2K_1+1$, $K_1 \in N$

If $K = 2K_1$, $12(3K^2+K) = 12(3\cdot(2K_1)^2+2K_1) = \boxed{24(6K_1^2+K_1)}$

If $K = 2K_1+1$, $12(3K^2+K) = 12(3\cdot(2K_1+1)^2+2K_1+1) = 12(12K_1^2+12K_1+3+2K_1+1)$
$= \boxed{24(6K_1^2+7K_1+2)}$ $\therefore 24 \mid (p^2-1)$ when $p = 6K+1$

• If $p = 6K+5$, $p^2-1 = (6K+5)^2-1 = 36K^2+60K+24 = 12(3K^2+5K+2)$

• By division algorithm, K can be $2K_1$ or $2K_1+1$, $K_1 \in N$

If $K = 2K_1$, $12(3K^2+5K+2) = 12(3\cdot(2K_1)^2+5\cdot 2K_1+2) = 24(6K_1^2+5K_1+1)$

If $K = 2K_1+1$, $12(3K^2+5K+2) = 12(3\cdot(2K_1+1)^2+5(2K_1+1)+2)$
$= 12(12K_1^2+12K_1+3+10K_1+5+2) = 24(6K_1^2+11K_1+5)$

$\therefore 24 \mid (p^2-1)$ when $p = 6K+5$

So, p^2-1 is div by 24 when p is a prime no. greater than 3

b) Prove that p^2-q^2 is divisible by 24 if p and q are prime no.s greater than 3.

• By similar reasoning as previous problem, there are four choices:
① $p = 6K_1+1$, $q = 6K_2+1$ ② $p = 6K_1+5$, $q = 6K_2+1$ ③ $p = 6K_1+1$, $q = 6K_2+5$
④ $p = 6K_1+5$, $q = 6K_2+5$ $\boxed{K_1, K_2 \in Z}$

$\boxed{\text{Case 1:}}$ $p^2-q^2 = \underbrace{(36K_1^2+12K_1)}_{\text{div. by 24}} - \underbrace{(36K_2^2+12K_2)}_{\text{div by 24.}}$ (shown prev).

$\boxed{\text{Case 2:}}$ $p^2-q^2 = \underbrace{(36K_1^2+60K_1+24)}_{\text{div. by 24}} - \underbrace{(36K_2^2+12K_2)}_{\text{div. by 24}}$