• $p=10k+9$ $\quad 4p^2+1=4(10k+9)^2+1=400k^2+720k+325$
$=5(80k^2+144k+65)$
↳ not prime

• $p=2$ $\quad 4p^2+1=17$ $\quad 6p^2+1=25$ → not prime

$\boxed{\cdot\ p=5 \quad 4p^2+1=101 \quad 6p^2+1=151}$ → only soln.

$\boxed{\text{Prob 43:}}$ Prove that a^3+b^3+4 is not a perfect cube for any natural numbers a and b. (Deserves a more concise and elegant soln.)

• By division algorithm, a can be $2x$ or $2x+1$, b can be $2y$ or $2y+1$, for some $x, y \in \mathbb{N}$.

• $a=2x, b=2y$ $\quad a^3+b^3+4=8x^3+8y^3+4=4(2x^3+2y^3+1)=4(\underbrace{2(x^3+y^3)+1}_{\text{not div by 2.}})$

$\therefore 8 \nmid a^3+b^3+4$, and it's not a perfect cube here

• $a=2x, b=2y+1$. $a^3+b^3+4=8x^3+8y^3+12y^2+6y+1+4=8(x^3+y^3)+(12y^2+6y+5)$

$\therefore$ For a^3+b^3+4, to be a perfect cube, $8 \mid (12y^2+6y+5)$

But, $12y^2+6y+5=\boxed{2(6y^2+3y+2)+1}$
↳ not div. by 2 itself

$\therefore a^3+b^3+4$ is not a perfect cube here

• $a=2x+1, b=2y$ (Symmetric to prev case)

• $a=2x+1, b=2y+1$ $\quad a^3+b^3+4=(2x+1)^3+(2y+1)^3+4=8x^3+12x^2+6x+8y^3+12y^2+6y+6$

$=8(x^3+y^3)+6(2x^2+x+2y^2+y+1)$

We will now try to see that $6(2x^2+x+2y^2+y+1)$ is div by 8 or not

$6(2x^2+x+2y^2+y+1)=4(3x^2+3y^2)+6(x+y+1)$

• By div. algorithm, x can be $2p_1$ or $2p_1+1$, y can be $2p_2$ or $2p_2+1$.

• $x=2p_1, y=2p_2$ $\quad 4(3x^2+3y^2)=8(6x^2+6y^2)$ → div by 8

$6(x+y+1)=6(2p_1+2p_2+1)=6(2(p_1+p_2)+1)$
↳ not div. by 8