In each of the cases, z^2 gives a remainder 2 when divided by 3.

$\because$ z exists, By division algorithm, let $z = 3K$ or $3K+1$ or $3K+2$

- $z = 3K$, $z^2 = 9K^2 = \boxed{3(3K^2)+0}$
- $z = 3K+1$, $z^2 = 9K^2+6K+1 = 3(3K^2+2K)+1$
- $z = 3K+2$, $z^2 = 9K^2+12K+4 = 3(3K^2+4K+1)+1$

$\therefore$ z^2 either leaves a remainder of 0 or 1 when divided by 3.

→ (Contradiction)

$\therefore$ Cases 1–4, z^2 can't be a perfect square, so in such cases, it's not possible that both x and y are not divisible by 3.

$\boxed{\therefore \text{ Atleast one of } x \text{ and } y \text{ is divisible by } 3}$ (Proved)

→ Great obs

Prob 23: Given natural numbers a and b such that a^2+b^2 is divisible by 21, prove that the same sum of squares is divisible by 441.

- $\boxed{21 = 3 \cdot 7}$. We will try showing a^2+b^2 is divisible by 3^2 and 7^2.
- By the division algorithm, $a = 3K_1$ or $3K_1+1$ or $3K_1+2$ and $b = 3K_2, 3K_2+1, 3K_2+2$, where $K_1, K_2 \in \mathbb{N}$.

Case 1: $a = 3K_1$, $b = 3K_2$ $\quad a^2+b^2 = 9K_1^2+9K_2^2 = 9(K_1^2+K_2^2) = 3(3K_1^2+3K_2^2)$

Case 2: $a = 3K_1$, $b = 3K_2+1$ $\quad a^2+b^2 = 9K_1^2+9K_2^2+6K_2+1 = 3(3K_1^2+3K_2^2+2K_2)+1$

$3 \nmid a^2+b^2$

→ Not possible

Case 3: $a = 3K_1$, $b = 3K_2+2$ $\quad a^2+b^2 = 9K_1^2+9K_2^2+12K_2+4 = 3(3K_1^2+3K_2^2+4K_2+1)+1$

$3 \nmid a^2+b^2$

→ Not possible

Case 4: $a = 3K_1+1$ $\quad b = 3K_2$ (Symmetrical to Case 2, $\therefore$ Not possible)

Case 5: $a = 3K_1+1$ $\quad b = 3K_2+1$ $\quad a^2+b^2 = 9K_1^2+6K_1+1+9K_2^2+6K_2+2$

$= 3(3K_1^2+2K_1+3K_2^2+2K_2)+2$