$\therefore$ r can take up the value from the set $\{0,1,2,\ldots\ldots,1986\}$ ↳ The cardinality is 1987

$\therefore$ In the worst case, we can have 1987 powers of 2 (arbitrarily chosen), which give one occurence each of the elements of the above set.

∴ So, in any 1988 arbitrarily chosen powers of 2, $\exists$ two powers of 2 which give the same remainder when divided by 1987 (Pigeonhole Principle)

∴ Let, ~~these~~ those numbers be 2^{k_1} and 2^{k_2}, $k_1 < k_2$, $k_1, k_2 \in \mathbb{N}$ ↳ wlog

$$2^{k_1} = 1987q_1 + r \qquad \boxed{0 \leq r < 1987}$$
$$2^{k_2} = 1987q_2 + r$$

→ By division algorithm

$$2^{k_2} - 2^{k_1} = 1987(q_2 - q_1)$$

$$\boxed{\therefore 1987 \mid (2^{k_2} - 2^{k_1})}$$ (Proved)

$\boxed{\text{Prob 20:}}$ Prove that of any 52 integers, two can always be found such that the difference of their squares is divisible by 100. (Write properly and rigorously)

∴ Suppose, we are given an arbitrary set of integers: $S = \{x_1, x_2, \ldots\ldots, x_{52}\}$

∴ We have to show $\exists \boxed{x_i, x_j}$, $i<j$, such that $100 \mid (x_j^2 - x_i^2)$ ↳ $\in S$

i.e. $100 \mid (x_j + x_i)(x_j - x_i)$

• Any arbitrary integer n when divided by 100 gives the remainder r in the range, $0 \leq r < 100$, $r \in \mathbb{Z}$ (By division algorithm)

• I will split and write down the remainders in a convenient way.

$$\begin{array}{lccccc} & a_{11} & a_{12} & a_{13} & & a_{1,49} \\ S_1 & 1 & 2 & 3 & \cdots\cdots & 49 \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ S_2 & 99 & 98 & 97 & \cdots\cdots & 51 \\ & a_{21} & a_{22} & a_{23} & & a_{2,49} \end{array}$$

∴ $\boxed{\text{Observation:}}$ Sum of any two downward adjacent pairs is 100, i.e. $n_1 = 100q_1 + u_{a_{1n}}$, $n_2 = 100q_2 + u_{a_{2n}}$

$$n_1 + n_2 = 100(q_1+q_2) + \underbrace{(u_{a_{1n}} + u_{a_{2n}})}_{100}$$