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STAT2203 – Probability and Statistics for Engineering

STAT2203 Lecture Notes

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Affine Combinations of Normals Example

Exercise: Let $X_1, ..., X_n \sim N(u,o^2)$ represent repeated measurements. Find is the distribution of the average measurement

$$Y = \frac{X_1 \cdots + X_n}{n}$$

Sums of Independent Random Variables

Law of Large Numbers and the **Central Limit Theorem**. Both theorems deal with **Sums of Independent Random Variables**. They arise for example in the following situations:

1) We flip a (biased) coin infinitely many times. Let $X_i = 1$ if the ith flip is "heads" and $X_i = 0$ otherwise. In general we do not know $p = P(X_i = 1)$. However, using the outcomes $x_1, ..., x_n$, we could estimate p by $(x_1 + ... + x_n)/n$
2) A certain machine needs to work continuously. The machine has one component that is very unreliable. This component is replaced immediately upon failure. Suppose there are n such (spare) components. If we denote the component lifetimes by $X_1, ..., X_n$, then the lifetime of the machine is given by $X_1 + ... + X_n$.
3) We weigh 20 randomly selected people. The average weight of the group is $(X_1 + ... + X_{20})/20$

Let $X_1, ..., X_n$ be independent and identically distributed random variables. For each n let

$$Sn = X1 \cdots + Xn$$

Let $EX_i = u$ and $Var(X_i) = o^2$ (assuming that these are finite).

Some easy results are:

$$\mathbb{E}S_n = n\mathbb{E}X_1 = n\mu$$

and, by the independence of the summands,

$$Var(S_n) = n\ Var(X_1) = n\sigma^2$$

If we know the pdf or pmf of X_i, then we can (in principle) determine the pdf or pmf of S_n. The easiest way is to use **transform** techniques (Laplace transform, Characteristic function, etc).
An important property of these transforms is that the transform of the **sum** of independent random variables is equal to the **product** of the individual transforms.

Example

Example: Suppose each $X_i \sim Exp(lambda)$. The Laplace transform of X_i, say L is given by

$$L(s) = \mathbb{E}e^{-sX_i} = \frac{\lambda}{\lambda + s}$$

The Laplace transform of S_n, is given by

$$\begin{aligned}\mathbb{E}e^{-sS_n} &= \mathbb{E}e^{-s(X_1+\cdots+X_n)} \\ &= \mathbb{E}e^{-sX_1}\cdots\mathbb{E}e^{-sX_n} = (L(s))^n \\ &= \left(\frac{\lambda}{\lambda+s}\right)^n\end{aligned}$$

Using the uniqueness of Laplace transforms, this shows that S_n has a Gamma(n, lambda) distribution (Erlang distribution)

Law of Large Numbers

Consider the coin flip example. We expect that S_n/n is close to the unknown p for large n.
We know this happens "empirically".

In general, we expect S_n/n to be close to u. Does this happen in our mathematical model?
By *Chebyshev's inequality* we have for all $e > 0$,

$$\mathbb{P}\left(\mid \frac{S_n}{n} - \mu \mid > \epsilon \right) \leq \frac{Var(S_n/n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0$$

as $n \to$ *infinite*.

In other words the probability that S_n/n is more than *e* away from *u* can be made arbitrarily small by choosing *n* large enough.
This is the **Weak Law of Large Numbers**.

There is also a **Strong Law of Large Numbers:**

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu \right) = 1$$

as $n \to$ *infinite*

Central Limit Theorem

The Central Limit Theorem states, roughly, this: The **sum** of a large number of **iid** random variables has **approximately** a **Gaussian** distribution.
More precisely, it states that for all *x*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x \right) = \Phi(x)$$

where *Phi* is the cdf of the standard normal distribution.

Approximating Binomial by Normal

Using the CLT we thus find the following important approximation:
Let *X ~ Bin(n, p)*. For large *n*, we have

$$\mathbb{P}(X \leq k) \approx \mathbb{P}(Y \leq k)$$

where *Y ~ N(np, np(1 - p))*.

As a rule of thumb, the approximation is accurate if both *np* and *n(1 - p)* are larger than 5.

We can improve on this somewhat by using a continuity correction, as illustrated by the following graph for the pmf of the *Bin(10, 1/2)* distribution.

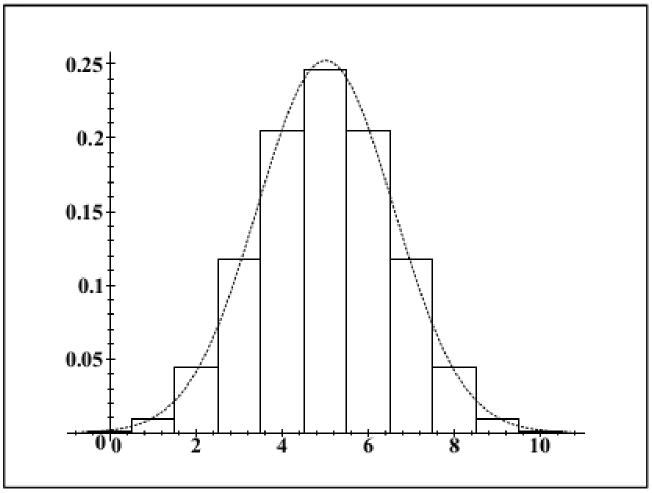

Figure 1: Approximating Binomial by Normal

For example,

$$\mathbb{P}(X = k) \approx \mathbb{P}(k - \frac{1}{2} \leq Y \leq k + \frac{1}{2})$$

Example

Example: Let *X ~ Bin(200, 0.51)*, and suppose we wish to to calculate *P(X <= 99)*.
Let *Y ~ N(200 x 0.51, 200 x 0.51 x 0.49)*, and let *Z* be standard normal. Using the CLT we have

$$\begin{aligned}\mathbb{P}(X \le 99) &\approx \mathbb{P}(Y \le 99)\\ &= \mathbb{P}\left(\frac{Y-102}{\sqrt{49.98}} \le \frac{99-102}{\sqrt{49.98}}\right)\\ &= \mathbb{P}(Z \le -0.4243) = 1 - \mathbb{P}(Z \le 0.4243)\\ &= 0.3357\end{aligned}$$

Using the continuity correction we find

$$\mathbb{P}(X \le 99) \approx \mathbb{P}(Y \le 99 + \frac{1}{2}) = 0.3618$$

Approximating via the CLT

Exercise: The number of calls X arriving at a call centre during an hour has a *Poi(100)* distribution.
Show, using probability generating functions, that X has the same distribution as $X_1 + ... + X_{100}$, where $X_1, ..., X_{100}$ are independent *Poi(1)*-distributed random variables.
Use this fact to approximate (with the CLT) the probability that there are more than 130 arrivals during an hour

Extremes of Independent Random Variables

In addition to the average behaviour of iid variates $X_1, ..., x_n$, we are often interested in the extremes – that is, how the largest (or smallest) variate behaves.

If $M = max\{X_1, ..., X_n\}$, we have seen (by example) that

$$\begin{aligned}F_M(m) &= \mathbb{P}(M \le m) = \mathbb{P}(X_1 \le m, \ldots, X_n \le m)\\ &= \mathbb{P}(X_1 \le m)^n = (F_X(m))^n\end{aligned}$$

What distribution does M have, as $n \to$ *infinite*

Remark: It turns out that, when M is suitably shifted and scaled, there are essentially three possibilities (listed here for completeness). The Gumbel distribution (*u element R, o > 0*):

$$f(x) = \frac{1}{\sigma} exp\left[-\frac{x-\mu}{\sigma} - exp\left[-\frac{x-\mu}{\sigma}\right]\right], x \in \mathbb{R}$$

The Frechet distribution (*u element R, o > 0, alpha > 0*):

$$f(x) = \frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-\alpha-1} exp\left[-\left(\frac{x-\mu}{\alpha}\right)^{-\alpha}\right], x > 0$$

The reversed Weibull distribution (*u element R, o > 0, alpha > 0*):

$$f(x) = \frac{\alpha}{\sigma}\left(\frac{\mu-x}{\sigma}\right)^{\alpha-1} exp\left[-\left(\frac{\mu-x}{\sigma}\right)^{\sigma}\right], x < \mu$$

Similarly, if $M = min\{X_1, ..., X_n\}$, we have that

$$\begin{aligned}F_M(m) &= \mathbb{P}(M \le m) = 1 - \mathbb{P}(M > m)\\ &= 1 - \mathbb{P}(X_1 > m, \ldots, X_n > m)\\ &= 1 - \mathbb{P}(X_1 > m)^n = 1 - (1 - F_X(m))^n\end{aligned}$$

Remark: It turns out that, when M is suitably shifted and scaled, there are again essentially three possibilities as $n \to$ *infinite*, being the distribution of $Y = -X$, where X is one of the three listed for the largest extreme value.

Summary

- Law of Large Numbers: statement, weak, strong
- Central Limit Theorem: statement, approximation of sums via CLT, examples
- Extreme Value Distributions: calculation (finite n), limiting behaviour (statement)

Statistics, Likelihood, and Estimation

Sums and Extremes of Independent Random Variables

- Law of Large Numbers; statement, weak, strong
- Central Limit Theorem; statement, approximation of sums via CLT, examples
- Extreme Value Distributions; calculation (finite n), limiting behaviour (statement)

Statistics

Data *x* is viewed as the outcome of a random variable *X* described by a probabilistic model. Usually, model is specified up to a (multidimensional) parameter: $X \sim F(.;\theta)$ for some element in Θ. In classical (frequentist) statistics, purely concerned with the model and in particular with the parameter $\varnothing$.
For example, given data, we may wish to

- estimate the parameter,
- perform statistical tests on that parameter, or
- validate the model

In Bayesian statistics, concerned with distribution of parameter $\theta \sim F(\theta)$.

Any real- or vector-valued function of data *x* or *X* is called a **statistic** of the data.
For example, the sample mean is a statistic:

$$T = T(x) = \frac{1}{N}\sum_{i=1}^{N} x_i$$

given an outcome of *X*, or as a random variable

$$T = T(X) = \frac{1}{N}\sum_{i=1}^{N} X_i$$

Often, we will view data as a series of independent outcomes from the same random experiment: $X = (X_1, ..., X_N)$, where $X_1, ..., X_N$ are iid from $F(.;\theta)$. $\{X_1, ..., X_N\}$ is called a **random sample** (from $F(.;\theta)$ or from *X*).
Therefore, the joint cdf of a random sample is given by

$$F(x;\theta) = \prod_{k=1}^{N} F(x_k;\theta)$$

and so the joint pdf/pmf is of the same form:

$$f(x;\theta) = \prod_{k=1}^{N} f(x_k;\theta)$$

Likelihood

When viewed as a function of θ, then point pdf/pmf of a random sample is called the **Likelihood**:

$$L(\theta;x) = f(x;\theta)$$

The (natural) logarithm of the likelihood

$$l(\theta;s) = lnL(\theta;x)$$

is called the **log-likelihood**

Likelihood Example

Example: Model $X_{1, 2}, ..., X_N \sim iid\ Bin(m, p)$; *m* known, *p* unknown, in $\Theta = (0, 1)$
pmf:

$$f(x;p) = \binom{m}{x} p^x (1-p)^{m-x}, x \in \{0, 1, \ldots, m\}$$

Therefore, the likelihood can be written as

$$L(p;X) = \prod_{i=1}^{N} \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i}$$

$$= p^{\sum_{i=1}^{N} x_i} (1-p)^{Nm - \sum_{i=1}^{N} x_i} \prod_{i=1}^{N} \binom{m}{x_i}$$

Maximum Likelihood Estimation

How do we find "good" estimators for model parameters? Given data and a parametric model, how to find a member of that family (point estimate) from which the data is "most likely" to have come?
Given data *x*, one approach is to maximize the likelihood in θ – that is, find

$$\hat{\theta} \in \Theta$$

for which

$$L(\hat{\theta};x) \geq L(\theta;x), \theta \in \Theta$$

A maximizer

$$\hat{\theta} \equiv \hat{\theta}(x)$$

of L is called a **maximum likelihood estimate** (MLE). The corresponding random variable &hat;θ(X) is called a maximum likelihood estimator (also MLE).
Remark: A maximiser of *l* equivalent to a maximiser of *L*

ML Estimation Example: Binomial Probability

Example: Continuing our example, recall that we found

$$L(p; X) = \prod_{i=1}^{N} \binom{m}{x_i} p^{x_i}(1-p)^{m-x_i}$$

$$= p^{\sum_{i=1}^{N} x_i}(1-p)^{Nm-\sum_{i=1}^{N} x_i} \prod_{i=1}^{N} \binom{m}{x_i}$$

How do we find an MLE?
Maximisation Strategy: Since *L* is a continuous function of *p*, find *p* such that

$$\frac{d}{dp}L(p; x) = 0$$

Working directly with *L* appears cumbersome; obtain the log-likelihood and work with that instead.
Taking the natural logarithm of *L*, we obtain the log-likelihood:

$$l(p; X) = ln(p)\sum_{i=1}^{N} x_i ln(1-p)\left(Nm - \sum_{i=1}^{N} x_i\right) + \sum_{i=1}^{N} ln\left(\binom{m}{x_i}\right)$$

First Derivative with respect to *p*:

$$\frac{d}{dp}l(p; X) = \frac{1}{p}\sum_{i=1}^{N} x_i - \frac{1}{1-p}\left(Nm - \sum_{i=1}^{N} x_i\right)$$

Set to zero and rearrange to find critical point:

$$(1-p)\sum_{i=1}^{N} x_i = p\left(Nm - \sum_{i=1}^{N} x_i\right)$$

Unique solution:

$$\hat{p} = \frac{1}{Nm}\sum_{i=1}^{N} x_i$$

What type of critical point is this?
Second Derivative with respect to *p*:

$$h(p) = \frac{d^2}{dp^2}l(p; X) = -\frac{1}{p^2}\sum_{i=1}^{N} x_i - \frac{1}{(1-p)^2}\left(Nm - \sum_{i=1}^{N} x_i\right) < 0$$

Therefore &hat;p is a local maximiser.
Moreover, *l(p;X)* → ∞ as *p* → *0* or *p* → *1* (boundary of Θ). Thus &hat;p is in fact a global maximiser. Therefore, we have the Maximum Likelihood Estimator:

$$\hat{p} = \frac{1}{Nm}\sum_{i=1}^{N} X_i$$

Summary

- Statistics; definition, example
- Likelihood and log-likelihood; definition, binomial example
- Maximum Likelihood Estimation; definition, examples, bias, consistency

Confidence Intervals and Hypothesis Testing

Statistics, Likelihood, and Estimation

- Statistics; definition, example
- Likelihood and log-likelihood; definition, binomial example
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Confidence Intervals

Last time, we were introduced to maximum likelihood estimation, which provided a systematic way of obtaining estimates and estimators &hat;θ of unknown parameters contained in θ ∈ Θ
How can we gauge the accuracy of &hat;θ?
Confidence intervals (sometimes called interval estimates) provide a precise way of describing the uncertainty of &hat;θ

Formally, given random variables $X_1, ..., X_n$ whose joint distribution depends on some unknown θ ∈ Θ, a **(1 - α) stochastic confidence interval** is a pair of statistics

$$T_1(X_1, \ldots, X_n) \text{and} T_2(X_1, \ldots, X_n)$$

with the property that

$$\mathbb{P}(T_1 < \theta < T_2) \geq 1 - \alpha, \text{ for all } \theta \in \Theta$$

for some α ∈ [0, 1]
That is, (T_1, T_2) is a random interval, based only on the (as yet to be observed) outcomes $X_1, ..., X_n$, that contains the unknown θ with probability at least $1 - \alpha$.

A realisation of the random interval, say (t_1, t_2), is called a **(1 - α) numeric confidence interval** for θ.

Remark: Whilst stochastic confidence intervals contain the unknown θ with probability at least $1 - \alpha$, their numerical counterparts either contain θ or they do not. It may be helpful to think of a Bernoulli analogy, where "success" occurs with probability (at least) $1 - \alpha$ – then outcomes are either "successes" or "failures"

Confidence Interval Example

Example: Model: $X_1, X_2, ..., X_n$ ~iid $N(\mu, \sigma^2)$; σ^2 known, μ unknown, in Θ = R.
We have seen that

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad \tilde{N}(\mu, \frac{\sigma^2}{N})$$

Therefore,

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{N}}}\tilde{N}(0, 1)$$

Hence,

$$\mathbb{P}\left(z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{N}}} \leq z_{1-\alpha/2}\right) = 1 - \alpha$$

where z_γ is the γ-quantile of the standard normal distribution.

Rearranging, we have

$$\mathbb{P}\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{N}} \leq \mu \leq \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right) = 1 - \alpha$$

Note that, by symmetry, the quantiles satisfy $-z_{\alpha/2} = z_{1-\alpha/2}$. Hence a stochastic $1 - \alpha$ confidence interval for μ in this case is

$$\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{N}}, \bar{X}z_{1-\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$$

which is often abbreviated to

$$\bar{X} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{N}}$$

Approximate Confidence Intervals

When

$$\mathbb{P}(T_1 < \theta < T_2) \geq 1 - \alpha, \text{ or all } \theta \in \Theta$$

only holds approximately, we call (T_1, T_2) an **approximate (1 - α) confidence interval** for θ.

Remark: We can often employ the central limit theorem to construct such approximate confidence intervals, as we shall see next.

Approximate Confidence Interval Example

Example: Model $X_1, X_2, ..., X_N$ ~iid $Bin(m, p)$; m known, p unknown, in Θ = (0, 1), with MLE for p:

$$\hat{p} = \frac{1}{Nm}\sum_{i=1}^{N} X_i$$

Notice that $Y = \Sigma^N_{i=1} X_i$ can be thought of as $Y \sim Bin(Nm, p)$, and so by the central limit theorem,

$$Y \ _{approx}\mathbf{N}(Nmp, Nmp(1-p))$$

or equivalently

$$\hat{p} \ _{approx}\mathbf{N}\left(p, \frac{p(1-p)}{Nm}\right)$$

Therefore, we have

$$\mathbb{P}\left(z_{\alpha/2} \leq \frac{\hat{p}-p}{\frac{\sqrt{p(1-p)}}{\sqrt{Nm}}} \leq z_{1-\alpha/2}\right) \approx 1-\alpha$$

By the law of large numbers, &hat;p≈p, so we may replace p in the denominator to obtain

$$\mathbb{P}\left(z_{\alpha/2} \leq \frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})}/\sqrt{Nm}} \leq z_{1-\alpha/2}\right) \approx 1-\alpha$$

Rearranging, and using the symmetry of standard normal quantiles, we have

$$\mathbb{P}\left(\hat{p} - z_{1-\alpha/2}\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{Nm}} \leq p \leq \hat{p} z_{1-\alpha/2}\frac{\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{Nm}}\right) \approx 1-\alpha$$

which is an approximate $1 - \alpha$ confidence interval for p:

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{Nm}}$$

Hypothesis Testing

Closely related to the notion of confidence intervals is that of hypothesis tests. In hypothesis testing, given data, we wish to determine which of two competing hypotheses $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$ holds true. H_0 is called the **null hypothesis** and contains the "status quo" statement, whereas H_1 is called the **alternative hypothesis** which is unlikely to have occurred if H_0 were true.
Remark: Usually, $\Theta_0 \cap \Theta_1 = \varnothing$

Outcomes of hypothesis tests are decisions as to whether to accept the "status quo" H_0, or reject the "status quo" in favour of the alternative H_1. As such, we seek a decision rule based on the outcome of a statistic T.

- Decision Rule 1: Reject H_0 if T falls in some critical region C
- Decision Rule 2: Reject H_0 if $P(T \in C)$ is less than some critical p-value p_c.

Remark: Common critical regions are one-sided ($C = (-\infty, c]$, $C = [c, \infty)$), or two-sided ($C = (-\infty, c_1] \cup [c_2, \infty)$, $c_1 <= c_2$)

Regardless of which type of decision rule is employed, we can make two types of error.

Decision	H_0 True	H_1 True
Retain H_0	Correct	Type II Error
Reject H_0	Type I Error	Correct

Remark: We can think of Type I error as a "false positive" and Type II error as a "false negative".
In classical statistics, Type I error is considered more serious, and so decision rules are designed to control this type of error.

We will denote the probability of a Type I error by α, and the probability of a Type II error by β.
Remark: The **power** of a statistical test is the probability of correctly rejecting the null, $1 - \beta$
We will design our decision rules around a predetermined **significance level** α, which describes the acceptable level of Type I error for our test. In this framework, the two types of decision rule are equivalent:

- Decision Rule 2: Reject H_0 if $P(T \in C_\alpha) <= \alpha$
- Decision Rule 1: Reject H_0 if T falls in C_α

Hypothesis Testing Example

Example: Model $X_1, X_2, ..., X_N \sim iid\ N(\mu, \sigma^2)$; σ^2 known, μ unknown, in $\Theta = R$. We can readily adapt our previous work to form a hypothesis test together with a decision rule about the unknown μ.
Let $H_0: \mu = \mu_0$ and $H_1: \mu \neq \mu_0$;
Under the null hypothesis H_0,

$$T = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{N}} \ N(0,1)$$

and so

$$C_\infty = (-\infty, z_{\alpha/2}] \cup [z_{1-\alpha/2}, \infty)$$

is a critical region satisfying

$$\mathbb{P}_{H_0}(T \in C_\alpha) \leq \alpha$$

Therefore, we reject H_0 if our observed statistic t falls in C_α

Summary

- Confidence intervals; definition, stochastic, numerical, approximate, examples
- Hypothesis testing; decision rules, null and alternative hypotheses, Type I and II error, significance level, power, critical region, critical *p*-value, one- and two-sided regions (hence tests).

Confidence Intervals and Hypothesis Testing II

Sample Variance

For a single normal random sample with known variance σ^2, we have seen that the sample mean (X bar) is normally distributed, and can therefore construct confidence intervals and hypothesis tests for the unknown mean μ

How can we proceed when σ^2 is unknown?
First, we will determine an appropriate estimator for σ^2, and state its distribution for a normal random sample.

Recall that we defined the sample variance of data as

$$\hat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \bar{x}^2$$

For a random sample, is the associated random variable an unbiased estimator for σ^2?
We have

$$\begin{aligned}
\mathbb{E}\hat{\sigma^2} &= \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[X_i^2] - \mathbb{E}[\bar{X}^2] \\
&= \mathbb{E}[X_1^2] - \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)^2\right] \\
&= \mathbb{E}[X_1^2] - \mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\right] \\
&= \mathbb{E}[X_1^2] - \frac{1}{N^2}\mathbb{E}\left[\sum_{i=1}^{N} X_i^2 + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} X_i X_j\right] \\
&= \mathbb{E}[X_1^2] - \frac{1}{N}\mathbb{E}[X_1^2] - \frac{N(N-1)}{N^2}\mathbb{E}[X_1]^2 \\
&= \frac{N-1}{N}(\mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2) = \frac{N-1}{N}\sigma^2
\end{aligned}$$

Therefore, &hat;σ^2 is a biased (but consistent) estimator for σ^2.
Remark: &hat;σ^2 is the MLE of σ^2 for a normal random sample.

We can easily correct for the bias in the **(bias corrected) sample variance:**

$$\begin{aligned}
S^2 &= \frac{N}{N-1}\hat{\sigma^2} \\
&= \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N} X_i^2 - \frac{N}{N-1}\bar{X}^2
\end{aligned}$$

For a normal random sample, it turns out that

$$(N-1)\frac{S^2}{\sigma^2}\tilde{X}^2_{N-1} \equiv \text{Gamma}(\frac{N-1}{2},\frac{1}{2})$$

Remark: The fact that the degrees of freedom is *N - 1* comes from the fact that there are only *N - 1* linearly independent elements of

$$\begin{pmatrix} X_1-\bar{X} \\ . \\ . \\ . \\ X_N-\bar{X} \end{pmatrix}$$

Sample Variance Example

Example: For a normal random sample $X_1, ..., X_N$ *~iid* $N(\mu, \sigma^2)$ with unknown mean μ and variance σ^2, find a $1 - \alpha$ (stochastic) confidence interval for σ^2.

Since $(N - 1)S^2/\sigma^2 \sim X_{N-1}{}^2$, we have by definition

$$\mathbb{P}\left(X^2_{N-1;\alpha/2} \le (N-1)\frac{S^2}{\sigma^2} \le X^2_{N-1;1-\alpha/2}\right) = 1-\alpha$$

where $X_{N-1;\gamma}{}^2$ denotes the γ-quantile of this chi-squared distribution

Since $\sigma^2 > 0$ and $S^2 > 0$, we rearrange as follows:

$$\mathbb{P}\left(\frac{1}{X^2_{N-1;\alpha/2}} \ge \frac{\sigma^2}{(N-1)S^2} \ge \frac{1}{X^2_{N-1;1-\alpha/2}}\right) = 1-\alpha$$

giving

$$\mathbb{P}\left(\frac{(N-1)S^2}{X^2_{N-1;1-\alpha/2}} \le \sigma^2 \le \frac{(N-1)S^2}{X^2_{N-1;\alpha/2}}\right) = 1-\alpha$$

Hence, a stochastic $1 - \alpha$ confidence interval for σ^2 for a normal random sample is

$$\left(\frac{(N-1)S^2}{X^2_{N-1;1-\alpha/2}}, \frac{(N-1)S^2}{X^2_{N-1;\alpha/2}}\right)$$

We can easily construct hypothesis tests at significance level α.

If $H_0 : \sigma^2 = \sigma_0{}^2$ and $H_1 : \sigma^2 \ne \sigma_0{}^2$, then our test statistic is

$$T = (N-1)\frac{S^2}{\sigma_0^2}$$

which (under H_0) has a $X^2{}_{N-1}$ distribution

Therefore, we reject H_0 in favour of H_1 if T falls in the (two-sided) critical region

$$(-\infty, X^2_{N-1;\alpha/2}] \cup [X^2_{N-1;1-\alpha/2}, \infty)$$

Similarly, if $H_0 : \sigma^2 = \sigma_0{}^2$ and $H_1 : \sigma^2 > \sigma_0{}^2$, we reject H_0 in favour of H_1 if T falls in the (right one-sided) critical region

$$[X^2_{N-1;1-\alpha}, \infty)$$

and if $H_0 : \sigma^2 = \sigma_0{}^2$ and $H_1 : \sigma^2 < \sigma_0{}^2$, we reject H_0 in favour of H_1 if T falls in the (left one-sided) critical region

$$(-\infty, X^2_{N-1;\alpha}]$$

Sample Mean with Unknown Variance

We have seen how to construct confidence intervals and hypothesis tests for a normal random sample with known variance σ^2. How does this change when σ^2 is unknown, and must instead be replaced by an estimate?

Recall that $X_1, X_2, ..., X_N$ *~iid* $N(\mu, \sigma^2)$, and consider the hypothesis test $H_0 : \mu = \mu_0$ and $H_1 : \mu \ne \mu_0$. Our test statistic in this case simply replaces the known σ with its unbiased estimator *S = &sqrt;S*^{2}, giving

$$T = \frac{\bar{X}-\mu_0}{S/\sqrt{N}}$$

If H_0 is true, then it turns out that T has a **(Student's) t** distribution, with *N - 1* degrees of freedom, which we will write as t_{N-1}. We will not concern ourselves with the particulars of this distribution, other than to note a few salient points:

- A *t*-distribution random variable is continuous, symmetric around zero, and has non-zero pdf over R (just like the standard normal distribution)

- As with any other distribution, we may compute γ-quantiles for a t_N-distributed random variable, which we will denote by $t_{N;\gamma}$.
  - Like the standard normal distribution, we will rely on tables or numerical computation for quantiles and probabilities.
- As $N \to \infty$, t_N converges in distribution to N(0, 1). (Moreover, t_1 is the Cauchy distribution)

Accepting that $T \sim t_{N-1}$, we construct a two-sided critical region at significance level α:

$$(-\infty, t_{N-1;\alpha/2}] \cup [t_{N-1;1-\alpha/2}, \infty)$$

and we reject H_0 if the outcome of our test statistic falls in this region. Similarly, critical regions for one-sided tests are easily constructed:

- $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$. Critical region: $[t_{N-1;1-\alpha}, \infty)$
- $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$. Critical region: $(-\infty, t_{N-1;\alpha}]$

Moreover, confidence intervals for the mean are straight-forwardly constructed from T:

$$\left(\bar{X} - t_{N-1;1-\alpha/2}\frac{S}{\sqrt{N}}, \bar{X} - t_{N-1;\alpha/2}\frac{S}{\sqrt{N}}\right)$$

or more compactly, by the symmetry of this distribution around zero:

$$\bar{X} \pm t_{N-1;1-\alpha/2}\frac{S}{\sqrt{N}}$$

Summary

- Sample variance; bias and correction, confidence intervals and hypothesis tests for normal population.
- Sample mean with unknown variance; Student's *t* distribution (briefly), confidence intervals and hypothesis tests for normal population

Confidence Intervals and Hypothesis Testing III

- Sample variance; bias and correction, confidence intervals and hypothesis tests for normal population
- Sample mean with uknown variance; Student's *t* distribution (briefly), confidence intervals and hypothesis tests for normal population

Two Sample Inference

Previously, we have seen how to construct confidence intervals and hypothesis tests for unknown parameters for a single random sample. However, in many cases we are interested in inference regarding the unknown parameters of two random samples. How does the construction of confidence intervals and hypothesis tests extend to this situation?

Two Sample Inference Example

Example: Model $X_1, \ldots, X_M \sim iid\ N(\mu_X, \sigma^2_X)$ independent of $Y_1, \ldots, Y_N \sim iid\ N(\mu_Y, \sigma^2_Y)$, with known variances σ^2_X and σ^2_Y, but unknown means μ_X and μ_Y.
Construct a $1 - \alpha$ stochastic confidence interval for the difference in means, $\mu_X - \mu_Y$
Firstly, notice that *&bar;X ~ N*(μ_X, σ^2_X/M) independent of *&bar;Y ~ N*(μ_Y, σ^2_Y/N).
Therefore, *&bar;X - &bar;Y ~ N*($\mu_X - \mu_Y$, $\sigma_X^2/M + \sigma_Y^2/N$), and so

$$Z = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{M} + \frac{\sigma_Y^2}{N}}} \tilde{N}(0,1)$$

Hence, by definition,

$$\mathbb{P}(z_{\alpha/2} \le Z \le z_{1-\alpha/2}) = 1 - \alpha$$

Rearranging as usual, we obtain an output which can be put more compactly (and using the symmetry of normal quantiles)

$$(\bar{X} - \bar{Y}) \pm z_{1-\alpha/2}\sqrt{\sigma_X^2/M + \sigma_Y^2/N}$$

as a $1 - \alpha$ stochastic confidence interval for the difference in means.

Remark: If each random sample has a common known variance $\sigma_X^2 = \sigma_Y^2 = \sigma^2$, then this confidence interval reduces to

$$(\bar{X} - \bar{Y}) \pm z_{1-\alpha/2}\sigma\sqrt{\frac{1}{M} + \frac{1}{N}}$$

This work can be extended to create hypothesis tests in the usual way, as follows. For the two-sided test, with a pair of normal random samples with known variances σ_X^2 and σ_Y^2, we have $H_0 : (\mu_X - \mu_Y) = \delta_0$ and $H_1 : (\mu_X - \mu_Y) \neq \delta_0$.
Under H_0

$$T = \frac{(\bar{X} - \bar{Y}) - \delta_0}{\sqrt{\frac{\sigma_X^2}{M} + \frac{\sigma_Y^2}{N}}} \tilde{N}(0,1)$$

and so the critical region for a test with significance level α is

$$C_\alpha = (-\infty, z_{\alpha/2}] \cup [z_{1-\alpha/2}, \infty)$$

One-sided tests, and tests with common variance σ^2 can be constructed in the same way.

Two Sample Inference with Unknown Variance

How does this change when the variances of the samples are unknown?
There are two possibilities:

- The unknown variances are not assumed to be the same
- The unknown variances are assumed to be the same

In the first case, we may estimate σ_X^2 by S_X^2, and σ_Y^2 by S_Y^2.
Then we may construct the same intervals and tests as before, replacing each variance by its estimator. This will yield approximate confidence intervals, and approximate hypothesis tests, which become more exact as both of the sample sizes become large.
In the second case, we need to estimate the common variance. The (uncorrected) pooled sample variance would just be

$$\hat{\sigma_p^2} = \frac{1}{M+N} \left(\sum_{i=1}^{M} (X_i - \bar{X})^2 + \sum_{j=1}^{N} (Y_j - \bar{Y})^2 \right)$$

However, as we have seen before, this is a biased estimator. Here, we can easily compute

$$\mathbb{E}[\hat{\sigma_p^2}] = \frac{M - 1 + N - 1}{M+N} \sigma^2$$

so the (bias corrected) pooled sample variance is just

$$S_p^2 = \frac{1}{M+N-2} \left(\sum_{i=1}^{M} (X_i - \bar{X})^2 + \sum_{j=1}^{N} (Y_j - \bar{Y})^2 \right)$$

Therefore, we can use our previous work, and note that

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{S_p \sqrt{\frac{1}{M} + \frac{1}{N}}} \tilde{t}_{M+N-2}$$

Hence, a $1 - \alpha$ stochastic confidence interval for $(\mu_X - \mu_Y)$ with unknown common variance is

$$(\bar{X} - \bar{Y}) \pm t_{M+N-2;1-\alpha/2} S_p \sqrt{\frac{1}{M} + \frac{1}{N}}$$

For the two-sided test, with a pair of normal random samples with unknown common variance σ^2, we have $H_0 : (\mu_X - \mu_Y) = \delta_0$ and $H_1 : (\mu_X - \mu_Y) \neq \delta_0$. Under H_0,

$$T = \frac{(\bar{X} - \bar{Y}) - \delta_0}{S_p \sqrt{\frac{1}{M} + \frac{1}{N}}} \tilde{t}_{M+N-2}$$

and so the critical region for a test with significance level α is

$$C_\alpha = (-\infty, t_{M+N-2;\alpha/2}] \cup [t_{M+N-2;1-\alpha/2}, \infty)$$

One-sided tests are simply constructed as seen previously

Approximate Intervals and Tests

We can readily adapt the confidence intervals and tests described so far to give approximate results by appealing to the central limit theorem.

Exercise: If $X \sim Bin(M, p_X)$ independently of $Y \sim Bin(N, P_Y)$, show that an approximate $1 - \alpha$ stochastic confidence interval for $p_X - p_Y$ is

$$(\hat{p_X} - \hat{p_Y}) \pm z_{1-\alpha/2} \sqrt{\frac{\hat{p_X}(1-\hat{p_X})}{M} + \frac{\hat{p_Y}(1-\hat{p_Y})}{N}}$$

where

$$\hat{p_X} = \frac{X}{M}, \qquad \hat{p_Y} = \frac{Y}{N}$$

Two Sample Inference for Variances

How can we construct confidence intervals and hypothesis tests for the unknown variances of two random samples?

Last time, we stated that for a normal random sample, $X_1, \ldots, X_M$ ~*iid* $N(\mu_X, \sigma_X^2)$,

$$(M-1)\frac{S_X^2}{\sigma_X^2}\tilde{X}^2_{M-1} \equiv \mathrm{Gamma}(\frac{M-1}{2}, \frac{1}{2})$$

This time, we will state that if we have two independent normal random samples $X_1, \ldots, X_M$ ~*iid* $N(\mu_X, \sigma_X^2)$ and $Y_1, \ldots, Y_N$ ~*iid* $N(\mu_Y, \sigma_Y^2)$,

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}\tilde{F}_{M-1,N-1}$$

where $F_{m,n}$ is the *F*-distribution with *m* and *n* degrees of freedom

Remark: As with the *t*-distribution, we will not go into details regarding the *F*-distribution, but simply accept this and rely on numerical computation or tabulation of its quantiles.

Using this fact, we may write by definition

$$\mathbb{P}\left(F_{N-1,M-1;\alpha/2} \leq \frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} \leq F_{N-1,M-1;1-\alpha/2}\right) = 1-\alpha$$

Rearranging, we have a stochastic $1 - \alpha$ confidence interval for the ratio of the unknown population variances:

$$\mathbb{P}\left(F_{N-1,M-1;\alpha/2}\frac{S_X^2}{S_Y^2} \leq \frac{\sigma_X^2}{\sigma_Y^2} \leq F_{N-1,M-1;1-\alpha/2}\frac{S_X^2}{S_Y^2}\right) = 1-\alpha$$

We may use this to construct hypothesis tests: $H_0 : \sigma_X^2 = \sigma_Y^2$ vs $H_1 : \sigma_X^2 \neq \sigma_Y^2$.

Under H_0,

$$\frac{S_X^2}{S_Y^2}\tilde{F}_{M-1,N-1}$$

and so an appropriate critical region at the α significance level is

$$C_\alpha = (-\infty, F_{M-1,N-1;\alpha/2}] \cup [F_{M-1,N-2;1-\alpha/2}, \infty)$$

One-sided tests can be constructed as seen before.

Summary

- Two-sample difference of means; confidence intervals and hypothesis tests for normal population, known and unknown (common and not) variance.
- Two-sample ratio of variances; *F* distribution (briefly), confidence intervals and hypothesis tests for normal population.