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Problem: $y' = y\ln\left(\frac{1}{y}\right)$, $y(0) = e^{-2}$

Guessing: $u = \frac{1}{y}$

We have

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

Substituting

$$-y^2 u' = \frac{1}{u}\ln(u)$$

$$-\left(\frac{1}{u}\right)^2 u' = \frac{1}{u}\ln(u)$$

$$u' = -u\ln(u)$$

Separable equation:

$$\int \frac{1}{u\ln(u)}\,du = \int -1\,dx$$

$v = \ln(u) \Rightarrow \frac{dv}{du} = \frac{1}{u}$

$$\int \frac{1}{v}\,dv = -x + C_0$$

$$\ln(v) = -x + C_1$$

$$\ln(\ln(u)) = -x + C_1$$

If $\ln(\ln(u)) = -x + C_1$

Then

$$\ln(u) = C_2 e^{-x}$$

$$u = C_3 e^{e^{-x}}$$

Substitute $u = \frac{1}{y}$

$$\frac{1}{y} = C_3 e^{e^{-x}}$$

$$y = \frac{1}{C_3 e^{e^{-x}}}$$

Root

$$y(0) = \frac{1}{C_3 e} = e^{-2}$$

$$\boxed{C_3 = e^{1}}$$

Particular solution:

$$y(x) = \frac{1}{e \cdot e^{e^{-x}}} = \frac{1}{e^{e^{-x}+1}}$$